REDUCTION OF UNBOUNDED DOMAINS TO BOUNDED DOMAINS FOR PARTIAL DIFFERENTIAL EQUATION PROBLEMS

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Abstract

Many boundary value problems which arise in applied mathematics are given in unbounded domains. Here we develop a theory for the imposition of boundary conditions at an artificial boundary which lead to finite domain problems that are equivalent to the unbounded domain problems from which they come. By considering the Cauchy problem with initial data in the appropriate space of functions on the artificial boundary, we show that satisfaction of the boundary conditions at infinity is equivalent to satisfaction of a certain projection condition at the artificial boundary. This leads to an equivalent finite problem. The solvability of the finite problem is discussed and estimates of the solution in terms of the inhomogeneous data are given.

Applications of our reduction to problems whose coefficients are independent of the unbounded coordinate are considered first. For a class of problems we shall term 'separable', solutions in the tail can be developed in an eigenfunction expansion. These expansions are used to write down an explicit representation of the projection, which is useful in computations. Specific problems considered here include elliptic equations in cylindrical domains. Spatially unbounded parabolic and hyperbolic problems are also discussed. Here, the eigenfunction expansions must include continuous transform variables.

We use these 'constant tail' results to develop a perturbation theory for the case when the coefficients depend upon the unbounded coordinate. This theory is based on Duhamel's principle and is seen to be especially useful when the 'limiting' problem possesses an exponential dichotomy. We apply our results to the Helmholtz equation, perturbed hyperbolic systems and nonlinear problems. We present a numerical solution of the Bratu problem in a semi-infinite, two-dimensional, stepped channel to illustrate our method.

TABLE OF CONTENTS

Introduction

Many of the partial differential equation problems arising in applied mathematics are given on unbounded domains. Discrete approximations of these, suitable for automatic computation, must, however, be given on bounded domains. One is thus led to consider the possibility of finding finite problems which are equivalent, or at least approximately equivalent, to the infinite ones. There are two distinct approaches to this problem. The first is to map the original unbounded domain to a bounded one and solve the often singular new equation that results from the mapping. The second is to create an 'artificial' boundary and impose boundary conditions there. Our approach will be the latter.

Various investigators have addressed the analagous questions for the case of ordinary differential equations. De Hoog and Weiss [28,29] suggest mappings of semi-infinite intervals to finite ones. They analyze the irregular singular endpoint problem which results from their mapping and develop some numerical techniques to solve it. Keller and Lentini [32] and Jepson and Keller [30], on the other hand, introduce an artificial boundary. They prove the existence of a boundary condition there which leads to a finite problem which is equivalent to the original. They also present methods of approximating this condition.

Moving on to partial differential equations, we note that the Helmholtz equation has been discussed by many authors. For exterior domains, this becomes the problem of finding artificial boundary conditions which are equivalent to the imposition of a radiation condition at infinity. Kriegsmann and Morawetz [33], Goldstein [19], Bayliss, Gunzburger and Turkel [3], Aziz, Dorr and Kellogg [2] and Guderley [21] all develop methods based on the known representations of outgoing solutions. Guderley's formulation, based on application of Green's theorem in the discarded region, has the advantage that it can be applied to a wide variety of boundary configurations. Aziz and coworkers note that if the Helmholtz equation holds throughout the region, the artificial boundary can be taken to be the real boundary and expansion coefficients associated with basic outgoing solutions can be calculated from the boundary conditions there. They go on to analyze a numerical scheme based on such an approach. Finally, Bayliss and coworkers develop a hierarchy of local conditions based on an asymptotic expansion of outgoing solutions. We discuss these in greater detail in section 5. Generalizations of these methods to problems in other geometries, where decaying and growing modes as well as radiating ones might be present, are made by Fix and Marin [17], Goldstein [20] and Bayliss and Turkel [5].

Another problem which has been the subject of many papers is that of boundary conditions for hyperbolic equations. Engquist and Majda [15] suggest that a principle of no reflection be imposed at all artificial boundaries. That is, that the incoming characteristic variables be set equal to zero. They go on to develop local approximations to their conditions based on the theory of pseudodifferential operators. Hedstrom [25] generalizes some of these ideas for application to nonlinear problems. Gustafsson and Kreiss [23] point out that non-reflecting artificial boundary conditions do not, in general, lead to problems which are equivalent to the unbounded domain problems from which they come.They discuss the form of the correct conditions and suggest a method based on Laplace transformation which, in some cases, can be used to approximate them. A practical application of their ideas can be found in Ferm and Gustafsson [16]. Finally, Bayliss and Turkel [4] present boundary conditions for the wave equation exterior to a body. These are closely related to those they developed for the Helmholtz problem.

Our purpose is twofold; first, we develop a theory describing the exact boundary conditions satisfied on an artificial boundary by the solutions of a problem on an unbounded domain and, second, we find computationally useful approximations to these theoretical conditions. As such, our work is most closely related to the work on ordinary differential equations by Keller and Lentini [32] and Jepson and Keller [30] and that on partial differential equations by Gustafsson and Kreiss [16].

In section 1 we consider the Cauchy problem for ordinary differential equations in Banach space, which we find to be a convenient formulation of the general partial differential equation problem in a cylindrical domain. (For us, cylindrical domains include cones, channels and exterior domains as well as, of course, cylinders.) An exact reduction theorem is proved for problems on semi-infinite domains, leading to an artificial boundary condition of projection type. We develop the notion of a dichotomy for our abstract equation and use it to analyze the finite boundary value problem resulting from our reduction. Dichotomies turn out to be very useful in the analysis of the ill-posed elliptic Cauchy problems that our reduction theory leads us to consider.

In sections 2 and 3 we discuss the use of eigenfunction expansions to represent our theoretical conditions. We now assume that the problem is autonomous in the unbounded coordinate-the constant tail case. Using some results of Agmon and Nirenberg [1], we show that such representations are possible for a wide class of problems, including elliptic boundary value problems in semi-infinite cylinders. We generalize these results to time-dependent problems in section 4. Our approach is to Laplace transform in time, use the results of the preceding sections to find boundary conditions in the transform variable and invert the transform to obtain a condition in the real variables. We apply these results to both hyperbolic and parabolic problems. We note that the use of eigenfunction expansions to find approximate boundary conditions in the constant tail case was first suggested by Gustafsson and Kreiss [16].

In section 5 we discuss the effect of bounded perturbations of the operator appearing in our abstract equation. We show that, under certain assumptions on the unperturbed operator, the functions and operators appearing in our boundary conditions for the perturbed problem can be expressed as the solution of an integral equation. From the integral equation we are able to construct general asymptotic expansions of the boundary conditions. This method is applicable to non-constant tail problems whose limiting problem can be analyzed by the eigenfunction expansion techniques of the preceding sections. Their specialization to the case of ordinary differential equations reproduces many of the expansions of Jepson and Keller [30]. In the final section, we apply the perturbation theory to nonlinear problems. This leads us to consider nonlinear boundary conditions. To illustrate our methods, we present a numerical solution of the Bratu problem in a semi-infinite, two dimensional stepped channel.

1. Exact Reduction for Linear Problems

We consider abstract boundary value problems in the form:

$$\begin{aligned} &a)\ \frac{du}{dx} = A(x)u + f(x),\ 0 < x < \infty\,; \\ &b)\ B_0 u(0) = \gamma_0\,; \qquad\qquad (1.1) \\ &c)\ \lim_{x\to\infty} B_\infty u(x) = \gamma_\infty\,. \end{aligned}$$

In addition we may impose:

$$d)\ \|u(x)\|\ \textit{bounded as}\ x \to \infty\,.$$

For some Banach space, $\mathbf{B}$, we seek $u(x) \in \mathbf{B}$ for $x \in [0,\infty)$. We suppose that $A(x)$, B_0, and B_∞ are linear operators with domain in $\mathbf{B}$, to which we also constrain the range of $A(x)$. Finally, $f(x) \in \mathbf{B}$.

Problems of form (1.1) follow from general partial differential equation problems in cylindrical domains. Specifically we consider:

$$\left(\sum_{j=0}^{n} P_j\left(\underset{\sim}{y}, x, \frac{\partial}{\partial \underset{\sim}{y}}\right) \frac{\partial^j}{\partial x^j}\right) w = g(x, \underset{\sim}{y}) \qquad (1.2)$$

on the cylindrical domain

$$(x, \underset{\sim}{y}) \in [0,\infty) \times \Omega\,,\ \Omega \subset \mathbf{R}^n.$$

Homogeneous boundary conditions are imposed on $\partial\Omega$ involving w and its normal derivatives.

$$\left(\sum_{j=0}^{m} B_{\Omega,j}\left(\underset{\sim}{y}, \frac{\partial^j}{\partial \underset{\sim}{n}^j}\right)\right) w = 0. \qquad (1.3)$$

We further suppose that, subject to these boundary conditions:

$$\left[P_n\left(x,\underset{\sim}{y},\frac{\partial}{\partial \underset{\sim}{y}}\right)\right]^{-1}$$

exists for all x. This can be loosely described as a condition that the x=constant surfaces be non-characteristic. Now we solve (1.2) and rewrite it in the form (1.1) by introducing:

$$u = \begin{bmatrix} \frac{\partial^{n-1} w}{\partial x^{n-1}} \\ \frac{\partial^{n-2} w}{\partial x^{n-2}} \\ \vdots \\ w \end{bmatrix},$$

$$f = \begin{bmatrix} P_n^{-1} g \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{1.4}$$

$$A = \begin{bmatrix} -P_n^{-1}P_{n-1} & -P_n^{-1}P_{n-2} & \cdots & & -P_n^{-1}P_0 \\ 1 & 0 & \cdots & & 0 \\ 0 & 1 & 0 \;\cdots & & \cdot \\ \cdot & 0 & 1 \;\cdots & & \cdot \\ \cdot & \cdot & 0 \;\cdots & & \cdot \\ \cdot & \cdot & \cdots & & \cdot \\ 0 & \cdot & \cdots & 1 & 0 \end{bmatrix}.$$

The boundary conditions at $x=0$ and $x=\infty$ are transformed analagously. The space, **B**, is some space of n-tuples of functions on Ω which satisfy the homogeneous boundary conditions, (1.3). It is necessary to eliminate inhomogeneous conditions on $\partial\Omega$ in order to reduce the problem to the abstract form. This can be accomplished by subtracting a function that satisfies the inhomogeneous condition. We note that the operators, $B_{\Omega,j}$, affect P_n^{-1} and, ultimately, $A(x)$.

Returning to (1.1), we choose some finite point, $x=\tau$, and attempt to reduce the infinite problem on $[0,\infty)$ to a finite one on $[0,\tau)$. We consider the following homogeneous Cauchy problem on the tail, $[\tau,\infty)\times\Omega$:

$$\begin{aligned} &a)\ \frac{dv}{dx}=A(x)v,\ \ \tau< x<\infty; \\ &b)\ v(\tau)=v_0. \end{aligned} \tag{1.5}$$

Here $v(x)$ is required to be in $\mathbf{B}$. Now we define $\mathbf{A}(\tau)$, the admissible space of Cauchy data at τ, as that set leading to solutions, v, which satisfy appropriate conditions at infinity. More precisely we have:

Definition 1.6

The set $\mathbf{A}(\tau)\subset\mathbf{B}$, the admissible space at $x=\tau$, is the set of all $v_0\in\mathbf{B}$ such that (1.5) has a solution, $v(x)$, satisfying:

$$\lim_{x\to\infty} B_\infty v(x)=0. \tag{1.6}$$

If (1.1d) is imposed then we also require

$$\|v(x)\|\ \textit{bounded as}\ x\to\infty. \tag{1.7}$$

Note that $\mathbf{A}(\tau)$ is obviously a subspace of $\mathbf{B}$ by the linearity of the Cauchy problem (1.5). It can be used to obtain the following basic reduction theorem.

Theorem 1.8

Problem (1.1) has a solution if and only if the following two problems have solutions:

There exists a particular solution, $u_p(x)$, satisfying:

$$\begin{aligned} &a)\ \frac{du_p}{dx}=A(x)u_p+f(x),\ \ \tau< x<\infty; \\ &b)\ \lim_{x\to\infty} B_\infty u_p(x)=\gamma_\infty; \end{aligned} \tag{1.8}$$

and, if (1.1d) is imposed,

$$c)\ \|u_p(x)\|\ \textit{bounded as } x \to \infty .$$

There exists a solution to the finite problem:

$$\begin{aligned} &a)\ \frac{dw}{dx} = A(x)w + f(x),\ \ 0 < x < \tau; \\ &b)\ \ B_o w(0) = \gamma_o; \qquad\qquad (1.9) \\ &c)\ \ w(\tau) - u_p(\tau) \in \mathbf{A}(\tau). \end{aligned}$$

Furthermore, whenever (1.1) has a solution, $u(x)$, then (1.9) has a solution which is identical to u on $[0,\tau]$.

Proof:

Assume that a solution, $u(x)$, of (1.1) exists. Then the set of particular solutions in the tail, satisfying (1.8), is obviously not empty. Choose any function in this set and call it $u_p(x)$. By assumption, $u(x)$ satisfies

$$\begin{aligned} &a)\ \ \frac{du}{dx} = A(x)u + f(x),\ \ 0 < x < \tau; \\ &b)\ \ B_o u(0) = \gamma_o . \end{aligned}$$

On the tail, set

$$v(x) = u(x) - u_p(x),\ \ \tau < x < \infty . \qquad (1.10)$$

Clearly, $v(x)$ satisfies (1.5) and (1.6). By definition, then,

$$v(\tau) \in \mathbf{A}(\tau). \qquad (1.11)$$

Hence, the restriction of u to $[0,\tau]$ satisfies (1.9).

Now suppose that (1.8) and (1.9) have solutions. Set

$$v(\tau) = w(\tau) - u_p(\tau).$$

Since $v(\tau) \in \mathbf{A}(\tau)$, there exists a $v(x)$ satisfying (1.5) and (1.6). Define

$$u^+(x) = v(x) + u_p(x), \quad \tau < x < \infty.$$

This satisfies:

$$a) \quad \frac{du^+}{dx} = A(x)u^+ + f(x), \quad \tau < x < \infty;$$

$$b) \quad \lim_{x \to \infty} B_\infty u^+(x) = \gamma_\infty;$$

$$c) \quad u^+(\tau) = w(\tau),$$

and, if (1.1d) is imposed:

$$d) \quad \|u^+(x)\| \text{ bounded as } x \to \infty.$$

Hence:

$$u(x) \equiv \begin{cases} w(x) & 0 \le x \le \tau, \\ u^+(x) & \tau < x < \infty, \end{cases}$$

is a solution to (1.1), completing the proof.

Corollary

Suppose that for all $v_0 \in \mathbf{A}(\tau)$, solutions to (1.5) are unique. Then (1.1) has a unique solution if and only if (1.8) does.

Proof:

Assuming uniqueness of solutions to (1.1) immediately yields uniqueness for (1.8). In the other direction, note that the assumption on problem (1.5) guarantees the uniqueness of $u^+(x)$. This, combined with the uniqueness of w, implies the uniqueness of u.

Note that if there exists a projection operator, $Q(\tau)$, associated with the admissible space, $\mathbf{A}(\tau)$, it is possible to rewrite (1.9) in the form:

$$
\begin{aligned}
&a)\ \frac{dw}{dx} = A(x)w + f(x),\ 0 < x < \tau, \\
&b)\ B_o w(0) = \gamma_o; \qquad (1.12) \\
&c)\ (I - Q(\tau))(w(\tau) - u_p(\tau)) = 0.
\end{aligned}
$$

By the projection theorem, $Q(\tau)$ will exist whenever $\mathbf{A}(\tau)$ is closed and $\mathbf{B}$ is a Hilbert space. Many of our examples fall into this category and, in general, we shall assume that $Q(\tau)$ exists.

We further note that Theorem (1.8) essentially becomes the reduction theorem of Jepson and Keller [30] when $A(x)$ is a matrix (i.e. for ordinary differential equations). Uniqueness is then easily reduced to a matrix condition.

Finally, we note that the theory presented above is readily adapted to problems which are unbounded in both directions; that is, when $x \in (-\infty, \infty)$. It is simply necessary to choose two finite boundary points, τ_- and τ_+, and find the two admissible spaces and particular solutions associated with the two discarded tails. The basic reduction theorem can be obviously extended to include this case.

The questions of existence, uniqueness and asymptotic behavior (or stability) for problem (1.5), which are obviously crucial to the reduction program, have not been answered in such great generality as have the analagous questions for ordinary differential equations. An exception to this is the case when the operator $A(x)$ is bounded, which is extensively discussed in the books of Massera and Schaffer [35] and Daletskiy and Krein [12]. Indeed, this situation is not much different from the finite dimensional one. For example, existence and uniqueness can be established by application of a contraction principle. In the works above, stability is investigated by use of the familiar concepts from the study of ordinary differential equations: Ljapunov and Bohl exponents and ordinary and exponential dichotomies. We find the latter of these to be very useful in our study of problems with unbounded $A(x)$.

Equations with constant, unbounded A are considered in Hille and Phillips [27]. They use the theory of semi-groups to examine the connection between existence of solutions and continuous dependence on Cauchy data. As such, their main interest is well-posed Cauchy problems. In many of the important cases we consider, however, (1.5) is not well-posed. This is true, for example, when the underlying partial differential equation is of elliptic type.

More applicable to the problems we study are the results of Agmon and Nirenberg [1]. They, too, study the constant coefficient case but do not restrict themselves to well-posed Cauchy problems. Under certain conditions on A, or more properly on its resolvent, they develop asymptotic expansions of the solutions in terms of generalized eigenvectors of A. We discuss these results in greater detail in sections 2 and 3.

We now study existence of solutions to the finite problem, (1.9), and derive bounds on its norm. (These bounds are, of course, useful for estimating errors caused by approximations to $u_p(\tau)$ and $Q(\tau)$.) Throughout we assume that solutions to (1.5) are unique and that the projector, $Q(\tau)$, exists.

The solution operator, $S(x_1,x_2;A)$, is defined in the following way:

Definition 1.13

Let $v_0 \in \mathbf{B}$. If there exists a solution, $v(x)$, to the problem,

$$\begin{aligned} &a)\ \frac{dv}{dx} = A(x)v\,,\ \ x_0 < x \le x_1 \text{ or } x_1 \le x < x_0; \\ &b)\ v(x_0) = v_0, \end{aligned} \qquad (1.13)$$

then

$$S(x_1,x_0;A)v_0 = v(x_1). \tag{1.14}$$

Otherwise, v_0 is not in the domain of $S(x_1,x_0;A)$.

The linearity of (1.13) clearly implies the linearity of S. The stated uniqueness of solutions implies the consistency of the definition. Note that the need to restrict the domain of S certainly arises for ill-posed problems. For proof one need only consider Hadamard's famous example of the Cauchy problem for Laplace's equation in a half-space. (Garabedian [18]) Whenever S exists, however, it does have the familiar semi-group properties:

$$\begin{aligned} &a)\ \ S(x_1,x^*;A)S(x^*,x_0;A) = S(x_1,x_0;A); \\ &b)\ \ S(x_0,x_0;A) = I. \end{aligned} \tag{1.15}$$

The notion of dichotomies is very useful in what follows. First we present definitions of exponential and ordinary dichotomies. These are adapted from Daletskiy and Krein [12], with some modifications required by the possible non-existence of solutions.

Definition 1.16

We say that (1.5) has an exponential dichotomy if, for any $x^* \in [0,\infty)$, the space $\mathbf{B}$ can be decomposed into a direct sum of subspaces $\mathbf{B}_-(x^*)$ and $\mathbf{B}_+(x^*)$ such that:

If $v \in \mathbf{B}_-(x^*)$ then

$$\begin{aligned} &i)\ \ S(x,x^*;A)v \textit{ exists } \text{for } \textit{any } x \geq x^* \\ &\text{and} \\ &ii)\ \ \|S(x,x^*;A)v\| \leq N_- e^{-\alpha_-(x-x^*)}\|v\| \end{aligned} \tag{1.16}$$

for some N_- and $\alpha_- \geq 0$.

If $v \in \mathbf{B}_+(x^*)$ then

$$i)\ S(x,x^*;A)v \text{ exists for any } x \leq x^*$$

and (1.17)

$$ii)\ \|S(x,x^*;A)v\| \leq N_+ e^{-\alpha_+(x^*-x)}\|v\|$$

for some N_+ and $\alpha_+ \geq 0$.

There exists $\gamma > 0$, independent of x^*, such that

$$inf\|u_+ + u_-\| \geq \gamma$$

for $u_\pm \in \mathbf{B}_\pm(x^*)$ and $\|u_\pm\| = 1$. (1.18)

(This infemum is typically called the angular distance between $\mathbf{B}_+(x^*)$ and $\mathbf{B}_-(x^*)$.) An ordinary dichotomy is defined as above except that $\alpha_\pm = 0$ is allowed.

In the definitions above no 'continuity' of the spaces as functions of x^* is required. In general we impose a sort of continuity in the form of the following 'no-mixing' condition:

Definition 1.19

The dichotomy (1.16-18) satisfies the no-mixing condition if whenever

$$a)\ Q(x) \text{ is the projection operator into } \mathbf{B}_-(x)$$

and

$$b)\ S(x_1,x_o;A)v \text{ exists}, \tag{1.19}$$

then

$$c)\ Q(x_1)S(x_1,x_o;A)v = S(x_1,x_o;A)Q(x_o)v.$$

Now we can state and prove an existence theorem for the finite boundary value problem (1.12).

Theorem 1.20

Suppose that:

a) solutions to all Cauchy problems for (1.5a) with data given at $x \in [0,\tau)$ are unique

and that

b) (1.5a) has a non-mixing ordinary dichotomy on $[0,\tau)$ with projector, $Q(x)$, into $\mathbf{B}_-(x)$. ($\mathbf{B}_-(\tau) = \mathbf{A}(\tau)$, the admissible space).

Then (1.12) has a solution for arbitrary $f(x)$, $u_p(\tau)$ and γ_0 in the range of B_0 if and only if the operator

$$\Phi u \equiv \begin{Bmatrix} (I - Q(0))u \\ B_0 u \end{Bmatrix} \tag{1.20}$$

has an inverse with domain containing all vectors of the form:

$$\begin{bmatrix} 0 \\ \gamma \end{bmatrix}, \quad \gamma \in Range(B_0). \tag{1.21}$$

Proof:

We use the ordinary dichotomy defined by $Q(x)$ to solve certain initial value problems. Let

$$u_+(x) = S(x,\tau;A)(I - Q(\tau))u_p(\tau) + \int_\tau^x S(x,w;A)(I - Q(w))f(w)dw. \tag{1.22}$$

This exists for all x on $[0,\tau)$ by the definition of Q. If we seek solutions to (1.12) in the form

$$u(x) = u_+(x) + u_-(x) \tag{1.23}$$

then u is a solution if and only if u_- solves

$$\begin{aligned} &a)\ \ \frac{du_-}{dx} = A(x)u_- + Q(x)f(x); \\ &b)\ \ B_0 u_-(0) = \gamma_0 - B_0 u_+(0); \\ &c)\ \ (I - Q(\tau))u_-(\tau) = 0. \end{aligned} \tag{1.24}$$

We write $u_-(x)$ in the form:

$$u_-(x) = S(x,0;A)u_-(0) + \int_0^x S(x,w;A)Q(w)f(w)dw. \tag{1.25}$$

The right-hand side again exists by the definition of Q, so that this representation is valid for any solution of (1.24a). By (1.19c) and (1.24c) we have:

$$0 = (I - Q(\tau))u_-(\tau) = S(\tau,0;A)(I - Q(0))u_-(0). \tag{1.26}$$

which, by the uniqueness of solutions to the Cauchy problem, implies:

$$(I - Q(0))u_-(0) = 0. \tag{1.27}$$

Hence, we can find a solution to (1.24) if and only if we can simultaneously solve:

$$(I - Q(0))u_-(0) = 0.$$

$$B_0 u_-(0) = \gamma_0 - B_0 u_+(0),$$

which, in vector form, is:

$$\Phi u_-(0) = \begin{bmatrix} 0 \\ \gamma_0 - B_0 u_+(0) \end{bmatrix}, \tag{1.28}$$

completing the proof. Note that when this (restricted) inverse is unique and bounded, the boundary value problem (1.12) is well-posed.

We note the similarity of the condition given in (1.20) and (1.21) to that for well-posedness of the mixed initial-boundary value problem for hyperbolic systems. Then, Q is a projector into an 'incoming' characteristic space and our condition is that the incoming variables can be calculated from the outgoing ones. For a more detailed discussion of the hyperbolic problem, including characterizations of these conditions in terms of Fourier variables, the reader is referred to Hersh [26].

The final topic in our consideration of the general finite problem (1.12) is the development of estimates of the solution in terms of the inhomogeneous data. These, in turn, can be used to estimate the errors which result from inaccurate evaluation of $Q(\tau)$ and $u_p(\tau)$. We now state the basic theorem.

Theorem 1.29

Assume that problem (1.12) satisfies the hypotheses of Theorem (1.20). Assume further that:

$$\begin{aligned} &a)\ \|S(x,w;A)Q(w)\| \le K_-(x,w),\ \ 0 \le w \le x \le \tau; \\ &b)\ \|S(x,w;A)(I-Q(w)\| \le K_+(x,w),\ \ 0 \le x \le w \le \tau; \\ &c)\ \|\Phi^{-1}\begin{pmatrix}0\\ \gamma\end{pmatrix}\| \le K_\varphi \|\gamma\|,\ \ \gamma \in Range(B_o); \\ &d)\ \|B_o\| \le K_o. \end{aligned} \tag{1.29}$$

Then we have:

$$\begin{aligned} \|u(x)\| \le{} & K_-(x,0)K_\varphi\|\gamma_o\| + \max_{x\in[0,\tau)}\|f(x)\|\int_0^x K_-(x,w)dw \\ & + \max_{x\in[0,\tau)}\|f(x)\|(\int_x^\tau K_+(x,w)dw + K_-(x,0)K_\varphi K_o\int_0^\tau K_+(0,w)dw) \\ & + (K_+(x,\tau) + K_-(x,0)K_\varphi K_o K_+(0,\tau))\|u_p(\tau)\|. \end{aligned} \tag{1.30}$$

Proof:

Note that

$$u(x) = u_+(x) + u_-(x),$$

where $u_+(x)$ is given by (1.22), $u_-(x)$ is given by (1.25) and $u_-(0)$ is given by (1.28). Using (1.29b) we have:

$$||u_+(x)|| \le K_+(x,\tau)||u_p(\tau)|| + \max_{x\in[0,\tau]}||f(x)||\int_x^\tau K_+(x,w)dw.$$

By (1.29a) we have:

$$||u_-(x)|| \le K_-(x,0)||u_-(0)|| + \max_{x\in[0,\tau]}||f(x)||\int_x^\tau K_-(x,w)dw.$$

Finally, by (1.29c,d) we obtain:

$$||u_-(0)|| \le K_\varphi||\gamma_o|| + K_\varphi K_o||u_+(0)||.$$

Combining all these results in (1.30).

We specialize these results to estimate the errors caused by approximations to $Q(\tau)$ and $u_p(\tau)$. Suppose we solve the following finite problem instead of (1.12):

$$a)\ \ \frac{du_a}{dx} = A(x)u_a + f(x),\ \ 0 < x < \tau;$$

$$b)\ \ B_o u_a(0) = \gamma_o; \tag{1.31}$$

$$c)\ \ (I - Q^*(\tau))u_a(\tau) = (I - Q^*(\tau))u_p^*(\tau).$$

where $Q^*(\tau)$ and $u_p^*(\tau)$ differ from $Q(\tau)$ and $u_p(\tau)$. We define the error, $e(x)$, by

$$e(x) \equiv u(x) - u_a(x).$$

We find that:

$$a)\ \frac{de}{dx}=A(x)e,\ \ 0\le x\le \tau;$$

$$b)\ B_0 e(0)=0;$$

$$c)\ (I-Q(\tau))e(\tau)=(I-Q(\tau))(u_p(\tau)-u_p^*(\tau)) \qquad (1.32)$$

$$+(Q(\tau)-Q^*(\tau))(u_p^*(\tau)-u_a(\tau))$$

$$\equiv \Delta(\tau).$$

Note that $\Delta(\tau)$, by construction, is in the range of $(I-Q(\tau))$. (We assume of course, that $u_a(x)$ exists.) Therefore we have

$$(I-Q(\tau))\Delta(\tau)=\Delta(\tau).$$

Problem (1.32) is now in the form of (1.12) and we use (1.30) to write:

$$\|e(x)\|\le (K_+(x,\tau)+K_-(x,0)K_\varphi K_0 K_+(0,\tau))\|\Delta(\tau)\|. \qquad (1.33)$$

Further specializing to the case of an exponential dichotomy this becomes:

$$\|e(x)\|\le (N_+e^{-\alpha_+(\tau-x)}+N_-e^{-\alpha_- x}K_\varphi K_0 N_+e^{-\alpha_+\tau})\|\Delta(\tau)\|. \qquad (1.34)$$

That is, the large part of the error decays exponentially off the artificial boundary.

2. Problems with Constant Tails: Abstract Theory

In this section we examine the situation when the operator, A, of equation (1.1) is independent of x in the tail. We show that, for many important examples, explicit representations of the projection operator, $Q(\tau)$, and particular solution, $u_p(x)$, can be found in terms of the eigenfunctions of A. We begin with a basic definition.

Definition 2.1

The operator, $A(x)$, is called separable in the tail if

a) There exists $\tau \in [0,\infty)$ such that $A(x) = A_\infty$ for $x \geq \tau$, A_∞ independent of x.

(b) There exists a countable set of pairs, (λ_n, u_n), with $\lambda_n \in \mathbb{C}$, $u_n \in \mathbf{B}$ and 0 not an accumulation point of $\{\lambda_n\}$ and there exist adjoint pairs, (λ_n^*, v_n) with $v_n \in Dual(\mathbf{B})$, satisfying:

$$i)\quad A_\infty u_n = \lambda_n u_n;$$

$$ii)\quad A_\infty^* v_n = \lambda_n^* v_n; \tag{2.1}$$

$$iii)\quad (v_m, u_n) = \delta_{mn}.$$

(c) Any function, $u \in \mathbf{B}$, can be uniquely written in the form:

$$u = \sum_{n=1}^{\infty} c_n u_n;$$

$$c_n = (v_n, u). \tag{2.2}$$

Note that this is the obvious generalization of the usual notion of separability as it allows us to write solutions to (1.1) in the tail as:

$$u(x) = \sum_{n=1}^{\infty} c_n(x) u_n. \tag{2.3}$$

Now we use an expansion like that of (2.3) to construct the admissible space, $\mathbf{A}(\tau)$. Recall we are now considering the homogeneous problem in the tail. Also, we retain the boundedness condition, (1.1d).

$$a)\ \frac{dv}{dx} = A_\infty v,\ \ \tau < x < \infty;$$

$$b)\ \lim_{x \to \infty} B_\infty v = 0; \tag{2.4}$$

$$c)\ \|v(x)\|\ \textit{bounded as}\ x \to \infty.$$

Representing $v(x)$ in an eigenfunction expansion:

$$v(x) = \sum_{n=1}^{\infty} c_n(x) u_n \tag{2.5}$$

and taking the inner product of (2.4a) and an adjoint eigenfunction, v_n, yields (for $\frac{dv}{dx} \in \mathbf{B}$):

$$\frac{dc_n}{dx} = \lambda_n c_n, \tag{2.6}$$

which can be trivially solved:

$$c_n(x) = e^{\lambda_n (x - \tau)} c_n(\tau). \tag{2.7}$$

Hence, we have

$$v(x) = \sum_{n=1}^{\infty} c_n(\tau) e^{\lambda_n (x - \tau)} u_n,\ \ \tau \le x < \infty. \tag{2.8}$$

Before solving (2.4b,c), we write $\mathbf{B}$ as the direct sum of three closed subspaces:

$$\begin{aligned}
&a)\ \mathbf{B} = \mathbf{B}^{+} \oplus \mathbf{B}^{-} \oplus \mathbf{B}^{0};\\
&b)\ \mathbf{B}^{+} \equiv \left\{ span\ u_n\ with\ \mathrm{Re}\,\lambda_n > 0 \right\};\\
&c)\ \mathbf{B}^{-} \equiv \left\{ span\ u_n\ with\ \mathrm{Re}\,\lambda_n < 0 \right\};\\
&d)\ \mathbf{B}^{0} \equiv \left\{ span\ u_n\ with\ \mathrm{Re}\,\lambda_n = 0 \right\}.
\end{aligned} \tag{2.9}$$

that $\mathbf{B}$ is the direct sum of these subspaces is obvious. Their closure follows from the fact that any Cauchy sequence of elements of $\mathbf{B}$ must have a Cauchy sequence of expansion coefficients by formula (2.2). We further divide $\mathbf{B}^0$ in the following way, assuming each eigenvalue has finite multiplicity:

$$\mathbf{B}^0 = \sum_i \mathbf{B}_i^0;$$

$$\mathbf{B}_i^0 \equiv \left\{ span\ u_n\ with\ \lambda_n = i\alpha_i \right\}; \tag{2.10}$$

$$\alpha_i\ real,\ \alpha_i \neq \alpha_j \text{ for } i \neq j.$$

Finally, we relate the nullspace of B_∞ to the important subspaces we have defined above.

Definition 2.11

Let A_∞ be given with associated spectral representation satisfying (2.1) and (2.2) and let the subspaces $\mathbf{B}^+$, $\mathbf{B}^-$ and $\mathbf{B}_i^0$ be defined by (2.9) and (2.10). Then, if $\mathbf{N}(B_\infty) \subset \mathbf{B}$ is the nullspace of B_∞, the A_∞*-restricted* nullspace of B_∞, $\mathbf{NR}(B_\infty; A_\infty) \subset \mathbf{B}$, is given by:

$$\mathbf{NR}(B_\infty; A_\infty) \equiv \sum_i (\mathbf{B}_i^0 \cap \mathbf{N}(B_\infty)). \tag{2.11}$$

Using this definition we obtain:

Theorem 2.12

If $A(x)$ is separable in the tail, then the admissible space, $\mathbf{A}(\tau)$, determined by the homogeneous problem (2.4), is given by:

$$\mathbf{A}(\tau) = \mathbf{NR}(B_\infty ; A_\infty) \oplus \mathbf{B}_- . \tag{2.12}$$

First we prove a lemma which will also be used in the proof of theorem (2.27).

Lemma 2.13

Suppose that the series

$$\sum_{j=1}^{\infty} w_j , \quad w_j \in \mathbf{B},$$

is absolutely convergent and that

$$\lim_{x \to \infty} \sum_{j=1}^{\infty} e^{i\alpha_j x} w_j = L \tag{2.13}$$

for some sequence

$$\{\alpha_j\}, \qquad \textit{with } \alpha_j \neq \alpha_k \text{ for } j \neq k , \quad \{\alpha_j\} \textit{ bounded away from zero.}$$

Then

$$\begin{aligned} &a)\ L = 0. \\ &b)\ w_j = 0, \ j = 1, 2, \cdots . \end{aligned} \tag{2.14}$$

Proof:

For arbitrary real t, integrate equation (2.13) from x to $x+t$. This yields:

$$\sum_{j=1}^{\infty} \frac{e^{i\alpha_j t} - 1}{i\alpha_j} w_j e^{i\alpha_j x} = Lt + e(x)t ,$$

where $\|e(x)\| \to 0$ as $x \to \infty$. As the original summation was absolutely convergent and the α_j's are bounded away from zero, the norm of the left-hand side of the equation above is bounded independent of t by some constant, K_o. That is,

$$K_o + t\,||e(x)|| \geq t\,||L||,$$

which implies

$$||L|| \leq \frac{K_o}{t} + ||e(x)||.$$

Letting t and x tend to infinity then implies $L=0$;

$$||L|| \leq \lim_{x,t \to \infty} \frac{K_o}{t} + ||e(x)|| = 0.$$

To derive (2.14b) simply multiply (2.13) by $e^{-i\alpha_k x}$ and note that this implies

$$\lim_{x \to \infty} \sum_{\substack{j=1 \\ j \neq k}}^{\infty} e^{i(\alpha_j - \alpha_k)x} w_j = -w_k.$$

Application of the first part of the proof then yields the desired result.

Proof of Theorem 2.12:

We write an arbitrary solution to (2.4a) as

$$v(x) = v^+(x) + v^-(x) + \sum_j v_j^o(x),$$

where

$$v^+(\tau) \in \mathbf{B}^+,$$

$$v^-(\tau) \in \mathbf{B}^-,$$

$$v_j^o(\tau) \in \mathbf{B}_j^o.$$

The boundedness requirement, (2.4c), combined with (2.7) clearly implies that $v^+(\tau)$ and, hence, $v^+(x)$, is identically zero. (2.7) also implies that the rest of

$v(x)$ is bounded and that

$$\lim_{x \to \infty} v^-(x) = 0.$$

Hence, we have

$$\lim_{x \to \infty} B_\infty v(x) = \lim_{x \to \infty} \sum_j B_\infty v_j^o(x)$$

$$= \lim_{x \to \infty} \sum_j e^{i\alpha_j(x-\tau)} B_\infty v_j^o(\tau).$$

If $v_j^o \in \mathbf{N}(B_\infty)$ then the limit is clearly zero. Otherwise, the inequality of the α_j's and lemma (2.13) show that the limit cannot exist. Hence, (2.4) is satisfied if and only if

$$v(\tau) \in \mathbf{NR}(B_\infty; A_\infty) \oplus \mathbf{B}^-,$$

completing the proof.

We can now represent $Q(\tau)$, the projection operator into $\mathbf{A}(\tau)$, in terms of our eigenfunctions. As the subspace $\mathbf{B}_j^o$ is finite dimensional, the projection operator for $(\mathbf{B}_j^o \cap \mathbf{N}(B_\infty))$ has a finite matrix representation in the space of expansion coefficients for elements of $\mathbf{B}_j^o$. That is, if

$$\mathbf{B}_j^o \equiv \left\{ span\ (u_1^{o,j}, u_2^{o,j}, \cdots, u_{n_j}^{o,j}) \right\} \tag{2.15}$$

then there exists an $n_j \times n_j$ projection matrix, Q_j^o, such that

$$a)\ \ u_j^o \equiv \sum_{k=1}^{n_j} c_k u_k^{o,j} \in (\mathbf{B}_j^o \cap \mathbf{N}(B_\infty))$$

if and *only* if (2.16)

$$b)\ \ (I - Q_j^o) \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n_j} \end{bmatrix} = \underset{\sim}{0}.$$

We use this to write the following corollary to theorem (2.12).

Corollary

The projection operator, $Q(\tau)$, for the admissible space, $\mathbf{A}(\tau)$, is given by:

$$Q(\tau)u = \sum_{n=1}^{\infty} d_n u_n, \tag{2.17}$$

where

$$\begin{aligned} &a)\ \ d_n = 0 \ \text{ if } u_n \in \mathbf{B}^+; \\ &b)\ \ d_n = (v_n, u)\ \text{ if } u_n \in \mathbf{B}^-; \\ &c)\ \ d_n \equiv d_j^{o,i} = \sum_{k=1}^{n_i} (Q_i^o)_{jk} (v_k^{o,i}, u) \\ &\qquad \text{if } u_n \equiv u_j^{o,i} \in \mathbf{B}_i^o. \end{aligned} \tag{2.18}$$

It is the representation (2.18) or, more properly, approximations to it which we use in computations. We note that more complicated conditions of the form,

$$\lim_{x \to \infty} B_\infty(x) u(x) = 0,$$

can be included in our analysis. Then it may be necessary to include parts of $\mathbf{B}^-$ in the restricted nullspace, **NR**. Also, (2.4c) could be dropped, giving rise to even more possibilities for **NR**.

Eigenfunction representations for the particular solution, $u_p(x)$, can also be found. Recall that u_p satisfies

$$\begin{aligned} &a)\ \ \frac{du_p}{dx} = A_\infty u_p + f(x),\ \ \tau < x < \infty; \\ &b)\ \ \lim_{x \to \infty} B_\infty u_p(x) = \gamma_\infty; \\ &c)\ \ \|u_p(x)\|\ \textit{bounded as}\ x \to \infty. \end{aligned} \tag{2.19}$$

We assume that $\|f(x)\|$ is integrable and Lipschitz continuous on $[\tau,\infty)$ and f has an eigenfunction expansion:

$$f(x) = \sum_{n=1}^{\infty} f_n(x) u_n. \tag{2.20}$$

The integrability and Lipschitz continuity of $f(x)$ imply the integrability and continuity of $f_n(x)$. Any solution of equation (2.19a) can be written in an eigenfunction expansion. In particular, if $u_p(x)$, the solution of (2.19), exists we write it as:

$$a)\ \ u_p(x) = \sum_{n=1}^{\infty} c_n^p(x) u_n;$$

$$\textit{where the coefficients, } c_n^p, \textit{ must satisfy:} \tag{2.21}$$

$$b)\ \ \frac{dc_n^p}{dx} = \lambda_n c_n^p + f_n(x),\ \ \tau < x < \infty.$$

The general solution of (2.21b) is

$$c_n^p(x) = c_n^p(\tau) e^{\lambda_n (x-\tau)} + \int_\tau^x e^{\lambda_n (x-p)} f_n(p)\, dp. \tag{2.22}$$

Formula (2.22) has different implications for coefficients of eigenfunctions in the different subspaces $\mathbf{B}^+$, $\mathbf{B}^-$ and $\mathbf{B}^o$. For $u_n \in \mathbf{B}^+$, that is $\mathrm{Re}\lambda_n > 0$, the boundedness condition (2.19c) requires

$$e^{\lambda_n (x-\tau)} (c_n^p(\tau) + \int_\tau^x e^{-\lambda_n (p-\tau)} f_n(p)\, dp)\ \textit{bounded as}\ x \to \infty.$$

This implies:

$$c_n^p(\tau) + \int_\tau^\infty e^{-\lambda_n (p-\tau)} f_n(p)\, dp = 0. \tag{2.23}$$

Hence, (2.22) can be rewritten:

$$c_n^p(x) = -\int_x^\infty e^{\lambda_n(x-p)} f_n(p)dp, \quad when \ \mathrm{Re}\,\lambda_n > 0. \tag{2.24}$$

For absolutely integrable $f_n(x)$, (2.22) and (2.24) clearly yield bounded $c_n^p(x)$. The boundedness of $u_p(x)$ further requires a certain decay in n of the expansion coefficients which, in turn, restrains the initial data. This restraint is stated in the following theorem.

Theorem 2.25

Suppose that for some $\alpha > 0$, $\beta > 0$ and positive integrable $K(x)$:

$$\begin{aligned} &a) \quad \frac{|f_n(x)|}{K(x)} = O\left(\frac{1}{n^\alpha}\right), \ n \to \infty, \ uniformly \ on \ \tau < x < \infty, \\ &b) \quad |c_n^p(\tau)| = O\left(\frac{1}{n^\beta}\right), \ n \to \infty, \ \mathrm{Re}\,\lambda_n \le 0. \end{aligned} \tag{2.25}$$

($c_n^p(\tau)$ is given by (2.23) for $\mathrm{Re}\,\lambda_n > 0$.)

Then $c_n^p(x)$ given by (2.22) and (2.24) satisfies

$$\begin{aligned} &a) \quad |c_n^p(x)| = O\left(\frac{1}{n^\gamma}\right), \ n \to \infty, \ uniformly \ on \ \tau < x < \infty, \\ &b) \quad \gamma = \min(\alpha, \beta). \end{aligned} \tag{2.26}$$

Proof:

We first consider the case of $\mathrm{Re}\,\lambda_n \le 0$. Then, from (2.22) we have:

$$|c_n^p(x)| \le |c_n^p(\tau)| + \max_{s \in [\tau,\infty)} \left| \frac{f_n(s)}{K(s)} \right| \int_\tau^\infty K(p)dp.$$

From (2.25b) the first term is $O\left(\frac{1}{n^\beta}\right)$ and from (2.25a) the second is $O\left(\frac{1}{n^\alpha}\right)$, yielding the desired result.

For $\mathrm{Re}\lambda_n > 0$, we use (2.24) to write:

$$\|c_n^p(x)\| \le \max_{s\in[\tau,\infty)} \frac{|f_n(s)|}{K(s)} \int_\tau^\infty K(p)\,dp$$

$$= O\left(\frac{1}{n^\alpha}\right),$$

completing the proof.

Finally, we state a theorem on the existence of solutions to (2.19).

Theorem 2.27

There exists a solution, $u_p(x)$, to problem (2.19) for absolutely integrable Lipschitz continuous $f(x)$ and A_∞ with a spectrum satisfying (2.1) and (2.2) if and only if there exists w_o satisfying:

$$\begin{aligned} &a)\ \ w_o \in \mathbf{N}(A_\infty); \\ &b)\ \ B_\infty w_o = \gamma_\infty. \end{aligned} \tag{2.27}$$

If so, one particular solution, $u_p(x)$, is given by:

$$a)\ \ u_p(x) = \sum_{\substack{n \\ \mathrm{Re}\lambda_n \neq 0}} a_n^p(x) u_n + w_o;$$

$$b)\ \ a_n^p(x) = \begin{cases} -\int_x^\infty e^{\lambda_n(x-p)} f_n(p)\,dp\,, & \mathrm{Re}\lambda_n \ge 0,\ \lambda_n \ne 0; \\ \int_\tau^x e^{\lambda_n(x-p)} f_n(p)\,dp\,, & \mathrm{Re}\lambda_n < 0. \end{cases} \tag{2.28}$$

Note that $u_p(x)$ satisfies (2.22) and (2.24).

We begin by proving a lemma.

Lemma 2.29

The function

$$w(x) = \sum_{n=1}^{\infty} c_n^p(x) u_n$$

with $c_n^p(x)$ given by (2.22) and (2.24) satisfies

$$a)\ \lim_{x \to \infty} (u(x) - u_o(x)) = 0;$$

where (2.29)

$$b)\ u_o(x) = \sum_{\substack{n \\ \mathrm{Re}\lambda_n = 0}} c_n^o(x) u_n;$$

$$c)\ c_n^o(x) = e^{\lambda_n(x-\tau)}(c_n^p(\tau) + \int_\tau^\infty e^{-\lambda_n(p-\tau)} f_n(p)\,dp).$$

Proof:

Let

$$e(x) = u(x) - u_o(x)$$

$$= \sum_{n=1}^{\infty} e_n(x) u_n .$$

By (2.22), (2.24) and (2.29b,c) we have:

$$e_n(x) = \begin{cases} -\int_x^\infty e^{\lambda_n(x-p)} f_n(p)\,dp, & \mathrm{Re}\lambda_n \geq 0; \\ e^{\lambda_n(x-\tau)}(c_n^p(\tau) + \int_\tau^x e^{-\lambda_n(p-\tau)} f_n(p)\,dp), & \mathrm{Re}\lambda_n < 0. \end{cases} \tag{2.30}$$

The continuity and absolute integrability of $f_n(x)$ implies that

$$\lim_{x \to \infty} |f_n(x)| = 0,$$

which allows us to conclude that the first integral in (2.30) approaches zero as $x \to \infty$. For the second term,

$$\lim_{x \to \infty} e^{\lambda_n (x-\tau)} c_n^p(\tau) = 0$$

is obvious. For the integral part we have:

$$| e^{\lambda_n}(x-\tau) \int_{\tau}^{x} e^{-\lambda_n (p-\tau)} f_n(p) dp | \le$$

$$e^{\lambda_n (x-x_0)} | \int_{\tau}^{x_0} e^{\lambda_n (x_0 - p)} f_n(p) dp | + \max_{s \in [x_0, \infty)} | f_n(s) | \int_{x_0}^{x} e^{\lambda_n (x-p)} dp$$

$$\le e^{\lambda_n (x-x_0)} \int_{\tau}^{\infty} | f_n(p) | dp + \frac{1}{\lambda_n} \max_{s \in [x_0, \infty)} | f_n(s) |; \text{ for } x_0 \in [\tau, \infty).$$

Hence

$$\lim_{x \to \infty} | e^{\lambda_n (x-\tau)} \int_{\tau}^{x} e^{-\lambda_n (p-\tau)} f_n(p) dp | \le \frac{1}{\lambda_n} \max_{s \in [x_0, \infty)} | f_n(s) |.$$

As $|\lambda_n|$ is bounded away from zero, choosing x_0 sufficiently large we can make the limit as small as we choose. Hence, the limit is zero. We have now shown that $\|e(x)\| \to 0$ as $x \to \infty$, completing the proof of the lemma. We now prove the main theorem.

Proof of Theorem 2.27:

We first assume that w_0 exists and let $u_p(x)$ be given by $w(x)$ of (2.27). By use of lemma (2.29) we see that

$$\lim_{x \to \infty} u_p(x) = w_0.$$

This immediately yields that

$$\lim_{x \to \infty} B_\infty u_p(x) = B_\infty w_0 = \gamma_\infty$$

Now we suppose that $u_p(x)$ exists. By lemma (2.29) again we have:

$$\lim_{x \to \infty} B_\infty u_p(x) = \lim_{x \to \infty} B_\infty u_o(x).$$

Hence, we have

$$\lim_{x \to \infty} \sum_{\substack{n \\ \lambda_n \text{ imaginary}}} e^{\lambda_n(x-\tau)} B_\infty (c_n^p(\tau) + \int_\tau^\infty e^{-\lambda_n(p-\tau)} f_n(p)dp)$$

$$+ \sum_{\substack{n \\ \lambda_n = 0}} B_\infty (c_n^p(\tau) + \int_\tau^\infty f_n(p)dp)$$

$$\equiv \lim_{x \to \infty} \sum_{\substack{n \\ \lambda_n \text{ imaginary}}} e^{\lambda_n(x-\tau)} B_\infty w_n + B_\infty w_o = \gamma_\infty \qquad (2.31)$$

$$\textit{where} \quad w_n \equiv c_n^p(\tau) + \int_\tau^\infty e^{-\lambda_n(p-\tau)} f_n(p)dp,$$

$$w_o \equiv \sum_{\substack{n \\ \lambda_n = 0}} (c_n^p(\tau) + \int_\tau^\infty f_n(p)dp) \in \mathrm{N}(A_\infty).$$

By lemma (2.13) and the assumption that the left-hand limit in (2.31) exists we have:

$$\lim_{x \to \infty} \sum_{\substack{n \\ \lambda_n \text{ imaginary}}} e^{\lambda_n(x-\tau)} B_\infty w_n = 0.$$

Hence, (2.31) becomes

$$B_\infty w_o = \gamma_\infty, \qquad (2.32)$$

completing the proof.

We note that the solvability conditions on γ_∞ would be altered if the integrability and continuity conditions on $f(x)$ were replaced by a simple boundedness requirement. In particular, the $\mathbf{B}^+$ and $\mathbf{B}^-$ components of $u_p(x)$ might no longer approach zero as $x \to \infty$ and, hence, might contribute to the limit of $B_\infty u_p(x)$.

The eigenfunction expansions discussed above can also be used to represent the solution operator, $S(x_1,x_o;A_\infty)$, for the constant operator, A_∞. It is:

$$a)\ S(x_1,x_o;A_\infty)u = \sum_{n=1}^{\infty} e^{\lambda_n(x_1-x_o)} c_n u_n;$$

$$b)\ c_n = (v_n,u). \tag{2.33}$$

The spaces $\mathbf{B}^+$, $\mathbf{B}^-$ and $\mathbf{B}^o$ can be used to define dichotomies for the equation

$$\frac{dv}{dx} = A_\infty v,\ \ \tau < x < \infty. \tag{2.34}$$

In particular we have:

Theorem 2.35

a) If $\mathbf{B}^o$ is the empty set; that is, if A_∞ has no eigenvalues with zero real part, then (2.34) has an exponential dichotomy. The spaces $\mathbf{B}_+(x^*)$ and $\mathbf{B}_-(x^*)$ of definition (1.16) are given by:

$$\mathbf{B}_+(x^*) = \mathbf{B}^+;$$

$$\mathbf{B}_-(x^*) = \mathbf{B}^-. \tag{2.35}$$

b) If $\mathbf{B}^o = \mathbf{B}^o_+ \oplus \mathbf{B}^o_-$ then (2.34) has an ordinary dichotomy. The spaces $\mathbf{B}_+(x^*)$ and $\mathbf{B}_-(x^*)$ are given by:

$$\mathbf{B}_+(x^*) = \mathbf{B}^+ \oplus \mathbf{B}^o_+;$$

$$\mathbf{B}_-(x^*) = \mathbf{B}^- \oplus \mathbf{B}^o_-. \tag{2.36}$$

Proof:

The definition of the solution operator, (2.33), combined with the definitions of $\mathbf{B}^+$, $\mathbf{B}^-$ and $\mathbf{B}^o$ immediately yield the inequalities of (1.16-18).

We have shown that an extensive analysis of the constant tail problem is possible whenever the operator, A_∞, has a complete set of eigenfunctions. We now present a discussion of problems where this does not hold. The main results come from the work of Agmon and Nirenberg [1]. We only consider the case where **B** is a Hilbert space, though they also prove weaker theorems in the Banach space setting. Restating the problem to conform to their situation, we write:

$$\frac{du}{dx} = iAu, \quad 0 < x < \infty \tag{2.37}$$

and seek solutions which decay at infinity sufficiently fast that

$$\|u(x)\| \in L_2([0,\infty)). \tag{2.38}$$

Let $R(\lambda)$ be the resolvent operator associated with A:

$$R(\lambda) \equiv (\lambda I - A)^{-1}. \tag{2.39}$$

Recall that the poles of R as a complex function of λ occur at the eigenvalues of A with residues equal to eigenfunctions (or generalized eigenfunctions). For technical reasons involved with the reduction of higher order equations to first order, we use $R_S(\lambda)$, the restriction of R to a closed subspace, S, of **B** in the statement of the theorem. For our equation (1.4), S would be the image of **B** under the matrix projection operator

$$\begin{bmatrix} 1 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

Lastly we associate an eigenfunction expansion with an L_2 solution, $u(x)$, in the following way: let $v(x)$ be such that

$$(\frac{d}{dx} - iA)v \equiv f \in S;$$

$$v(0) = 0;$$

$$u(x) - v(x) = 0, x \geq 1;$$

Assume that such a v can always be chosen which satisfies

$$\|f\| \leq \kappa \|u\|$$

for some κ independent of u. Since u, and, hence, v for $x > 1$ satisfies (2.37), f vanishes for $x > 1$ and its Fourier transform, $\hat{f}(\lambda)$, is an entire function of exponential type. The exponential solution, $u_j(x)$, associated with the eigenvalue λ_j is given by:

$$u_j(x) = \sqrt{2\pi} i \operatorname*{Residue}_{\lambda=\lambda_j}(\exp\{i\lambda x\} R_S(\lambda)\hat{f}(\lambda)). \tag{2.40}$$

An eigenfunction expansion of u is then given by:

$$u \sim \sum_j u_j(x) \tag{2.41}$$

which need not be convergent. Note that this is an expansion in the eigenfunctions and generalized eigenfunctions belonging to the eigenvalues of A with positive imaginary part. These, of course, are the eigenvalues of iA (the analogue of our A_∞) with negative real part. Here, without proof, is the fundamental theorem.

Theorem 2.42 (Agmon and Nirenberg [1,p.155])

Suppose that $R_S(\lambda)$ is meromorphic in $\operatorname{Im}\lambda \geq 0$ and satisfies:

$$\|R_S(\lambda)\| = O(1), \quad |\lambda| \to \infty,$$

in every strip $0 \leq \operatorname{Im}\lambda < \alpha$. Then, if $u(x)$ is a solution of (2.37) and (2.38), the eigenfunction expansion (2.41) is an asymptotic expansion of u in the following sense:

If $u_k(x)$ is an exponential solution of index m_k then, for any $\varepsilon > 0$

$$\left|(1+x)^{\frac{1}{2}-m_k-\varepsilon} \exp\left\{\operatorname{Im}\lambda_k\right\} \|u(x) - \sum_{j=1}^{k-1} u_j(x)\|\right|_{L_2[0,\infty)} < \infty . \tag{2.42}$$

Note that (2.42) does not tell us what the admissible space, $\mathbf{A}(\tau)$, actually is. However, it does show that the space spanned by the eigenfunctions and generalized eigenfunctions associated with the eigenvalues of A with positive imaginary part does approximate the image under $S(\tau,0;iA)$ of $\mathbf{A}(0)$ for $x > 0$. It is also used to show that all L_2 solutions decay exponentially. Applications of this result to elliptic and parabolic partial differential equations are given in Section 3.

We close the section with a discussion of some problems which do not have constant tails, but for which eigenfunction expansion techniques of the sort discussed above can lead to exact expressions for $Q(\tau)$ and $u_p(x)$. The first problem we shall consider is:

$$\frac{du}{dx} = x^{\alpha} A_{\infty} u + f(x); \quad \tau \leq x < \infty; \tag{2.43}$$

which has the associated homogeneous problem

$$\frac{dv}{dx} = x^{\alpha} A_{\infty} v; \quad \tau \leq x < \infty; \tag{2.44}$$

We assume that A_{∞} is separable in the tail. Representing v in an eigenfunction expansion, (2.5), we derive the following analogue of equation (2.6):

$$\frac{dc_n}{dx} = x^{\alpha}\lambda_n c_n; \tag{2.45}$$

which has the solution:

$$c_n(x) = \begin{cases} e^{\frac{\lambda_n}{\alpha+1}(x-\tau)^{\alpha+1}} c_n(\tau), \ \alpha \neq -1, \\ \left(\frac{x}{\tau}\right)^{\lambda_n} c_n(\tau), \ \alpha = -1. \end{cases} \tag{2.46}$$

Assuming, for example, that $\mathbf{B}^0$ is null, we have that $\mathbf{B}^+$ and $\mathbf{B}^-$ induce an ordinary dichotomy for (2.44) which, for $\alpha > -1$, is an exponential dichotomy. For $\alpha \geq -1$, all bounded solutions decay at infinity and $Q(\tau)$ is given by (2.18a,b). Integral formulas like (2.22) can also be derived for use in the calculation of $u_p(x)$.

Another non-constant case where we can obtain exact results is that of equations whose operator, $A(x)$, is periodic. Specifically we consider:

$$\begin{aligned} &\frac{du}{dx} = A(x)u + f(x), \ \tau \leq x < \infty; \\ &A(x+T) = A(x); \end{aligned} \tag{2.47}$$

and its associated homogeneous problem,

$$\frac{dv}{dx} = A(x)v, \ \tau \leq x < \infty. \tag{2.48}$$

We reduce consideration of the behavior of solutions of (2.48) to the consideration of an eigenvalue problem by use of Floquet theory. An account of Floquet theory for ordinary differential equations can be found, for example, in Cesari [7], while a generalization to the Banach space setting is made by Daletskiy and Krein [12]. Here is the basic theorem.

Theorem 2.49

Suppose that solutions to (2.48) are unique and let $S(x_1,x_o,A)$ be the solution operator for $x_o, x_1 \geq \tau$. Let

$$M = S(\tau + T, \tau; A). \tag{2.49}$$

Then, for all $x \geq \tau$,

$$S(x + T, \tau; A) = S(x, \tau; A)M. \tag{2.50}$$

Proof:

The proof follows immediately, as in the ordinary differential case, from the semigroup properties of S and the fact that

$$S(x + T, \tau + T; A) = S(x, \tau; A).$$

Corollary

Let $x \geq \tau$ be given by

$$x = \alpha T + e + \tau, \quad 0 \leq e < T;$$

then

$$S(x, \tau; A) = S(e + \tau, \tau; A)M^{\alpha}. \tag{2.51}$$

From an analysis of the Floquet operator, M, we develop an exact theory of the behavior of solutions of (2.48) We make:

Assumption 2.50

The operator M is separable with spectrum bounded away from the unit circle.

We now prove that (2.48) has an exponential dichotomy.

Theorem 2.51

Suppose that $\|S(x_1,x_o;A)v_o\|$ is bounded by K_+ for $0 \leq x_1 - x_o < T$ and $v_o \in \mathbf{B}^+(x_o)$

and is bounded by K_- for $0 \leq x_o - x_1 < T$ and $v_o \in \mathbf{B}^-(x_o)$. An exponential dichotomy for (2.48) is induced by $\mathbf{B}^+(x)$ and $\mathbf{B}^-(x)$ given by:

$$\begin{aligned} \mathbf{B}^+(x) &\equiv \textit{Image under } S(\tau+e, T+\tau; A) \textit{ of } \mathbf{B}^+(\tau); \\ \mathbf{B}^-(x) &\equiv \textit{Image under } S(\tau+e, T+\tau; A) \textit{ of } \mathbf{B}^-(\tau); \end{aligned} \tag{2.52}$$

where

$$\begin{aligned} & x = \alpha T + e, \quad 0 \leq e < T; \\ & \mathbf{B}^+(\tau) \equiv span \left\{ u_n ; |\lambda_n| > 1 \right\}; \\ & \mathbf{B}^-(\tau) \equiv span \left\{ u_n ; |\lambda_n| < 1 \right\}. \end{aligned} \tag{2.53}$$

Proof:

We immediately have the following estimates:

$$\|S(x_1, x_o; A) v_o\| \leq K_+ m_+^{-\alpha} \|v_o\|;$$

$$\text{for } v_o \in \mathbf{B}^+(x_o), \ \ x_o = x_1 + \alpha T + e;$$

and

$$\|S(x_1, x_o; A) v_o\| \leq K_- m_-^{\alpha} \|v_o\|;$$

$$\text{for } v_o \in \mathbf{B}^-(x_o), \ \ x_1 = x_o + \alpha T + e;$$

where

$$m_+ \equiv \inf_{|\lambda_n| > 1} |\lambda_n|,$$

$$m_- \equiv \sup_{|\lambda_n| < 1} |\lambda_n|.$$

These yield

$$\|S(x_1,x_o;A)v_o\| \le K_+^* e^{-\log m_+ \frac{(x_o - x_1)}{T}} \|v_o\|;$$

$$\text{for } x_o \ge x_1, \ \ v_o \in \mathbf{B}^+(x_o);$$

and

$$\|S(x_1,x_o;A)v_o\| \le K_-^* e^{-\log m_- \frac{(x_1 - x_o)}{T}} \|v_o\|;$$

$$\text{for } x_1 \ge x_o, \ \ v_o \in \mathbf{B}^-(x_o);$$

where

$$K_\pm^* = \frac{K_\pm}{\alpha_\pm};$$

completing the proof.

From this, a representation of $Q(\tau)$ in terms of the eigenfunctions of M is possible. Calculation of the particular solution would require a knowledge of S over a period.

3. Problems with Constant Tails: Applications

We now apply the general results of the preceding section to some specific problems. We begin by mentioning two simple cases.

The theory of ordinary differential equations with constant tails on semi-infinite intervals is developed by Lentini and Keller [32] and Jepson and Keller [30]. In this case the Banach space, **B**, is finite dimensional and the operator A_∞, is a matrix. The problem is 'separable' in our sense whenever A_∞ is diagonalizable. Then their theory is the same as ours. They are able to extend their theory to the more general case-that is, when the Jordan form of A_∞ has non-trivial Jordan blocks. Here we could apply the Agmon-Nirenberg results, at least when no eigenvalues have zero real part (The Lentini-Keller and Jepson- Keller results, however, are exact, not asymptotic).

Moving to the realm of partial differential equations, we consider a problem of parabolic type on a bounded spatial domain.

$$\frac{du}{dt} = Lu, \quad (\underset{\sim}{y},t) \in \Omega \times [0,\infty), \quad \Omega \subset \mathbf{R}^n; \tag{3.1}$$

where L is a formally self-adjoint 2mth order elliptic operator. The function u is also supposed to satisfy the following boundary conditions:

$$\sum_{|\alpha| < 2m} b_{\alpha_j}(\underset{\sim}{y})\partial^\alpha u = 0, \quad j = 1, \cdots, m; \qquad \underset{\sim}{y} \in \partial\Omega; \tag{3.2}$$

where

$$\partial^\alpha \equiv \frac{\partial^{\alpha_1}}{\partial y_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial y_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_p}}{\partial y_p^{\alpha_p}}, \quad \sum_{j=1}^{p} \alpha_j = |\alpha|.$$

Then, if the boundary conditions cover L, (see Berezanskii [6,p.207] for a definition) and various smoothness conditions on the boundary and the coefficients are met, an expansion in the generalized eigenfunctions of L is complete in $L_2(\Omega)$. (Berezanskii [6,Ch.6]). That is, the problem (3.1,2) is 'separable' in the sense of the preceeding section. Furthermore, all eigenvalues, λ_n, are real and all but finitely many are negative. Hence, if $\lambda = 0$ is not an eigenvalue of L, there is an exponential dichotomy with a finite dimensional (perhaps null) space of initial data leading to exponentially growing solutions.

Although this parabolic example nicely fits our theory, it is the mixed initial-boundary value problem rather than the boundary value problem on the cylindrical domain which most often arises in applications. A more important application of our results is made in the case of elliptic boundary value problems in cylindrical domains.

The question of reduction of such problems to finite domains, as a prelude to numerical approximation, has been discussed by other authors. The most general work is that of Kreiss and Gustafsson [23], who describe a method which is essentially the same as ours. A later paper by Goldstein [20] dealing with the Helmholtz equation for cylindrical waveguides proposes conditions which are specializations of these. Also, Bayliss and Turkel [5] give some idea of the general approach. Their work is discussed in greater detail in section 5.

We begin with the following problem, with conditions at infinity so far unspecified:

$$\begin{aligned} &a)\ \ u_{xx} + \nabla_{\underset{\sim}{y}}^2 u + a(\underset{\sim}{y})u = f(x,\underset{\sim}{y}); \\ &b)\ \ (x,\underset{\sim}{y}) \in [0,\infty) \times \Omega\,,\ \ \Omega \subset \mathbb{R}^n; \qquad\qquad (3.3) \\ &c)\ \ u = 0,\ \underset{\sim}{y} \in \partial\Omega\,;\ \ a_o(\underset{\sim}{y})u_x + b_o(\underset{\sim}{y})u = g_o(\underset{\sim}{y}),\ x = 0; \end{aligned}$$

where $\nabla^2_{\tilde{y}}$ is the Laplacian in $\mathbf{R}^n$. Rewriting the problem in first order form we have:

$$\frac{d\tilde{w}}{dx} = A\tilde{w} + \tilde{F}(x);$$
$$\tilde{w} = \begin{pmatrix} u_x \\ u \end{pmatrix}, \; A = \begin{pmatrix} 0 & -\nabla^2_{\tilde{y}} - a(\tilde{y}) \\ 1 & 0 \end{pmatrix}, \; \tilde{F} = \begin{pmatrix} f \\ 0 \end{pmatrix}; \tag{3.4}$$

where we choose $\mathbf{B}$ to be the Hilbert space of 2-vectors with components in the closed subspace of $L_2(\Omega)$, $L_2^0(\Omega)$, with zero Dirichlet data. From the standard theory of elliptic operators (see, e.g.,Courant-Hilbert [11]), we know that:

There exist $\{\lambda_n\}$, λ_n real and $M > 0$ such that $\lambda_n > 0$ for $n > M$.

There also exist $u_n \in L_2^0(\Omega)$ such that:

$$a)\;\; -\nabla^2_{\tilde{y}} u_n - a(\tilde{y})u_n = \lambda_n u_n;$$
$$b)\;\; (u_n, u_m)_{L_2^0(\Omega)} = \delta_{nm}; \tag{3.5}$$
$$c)\;\; \{u_n\} \textit{ complete in } L_2^0(\Omega).$$

We use this expansion to develop an eigenfunction expansion for A. Suppose that:

$$A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Then we have

$$u_1 = \lambda u_2;$$
$$-(\nabla^2_{\tilde{y}} + a(\tilde{y}))u_2 = \lambda^2 u_2.$$

From (3.5) we see that

$$\lambda^2 = \lambda_n . \tag{3.6}$$

That is

$$\lambda = \pm \sqrt{\lambda_n} \tag{3.7}$$

with eigenfunctions

$$\begin{bmatrix} \pm \sqrt{\lambda_n} u_n \\ u_n \end{bmatrix} . \tag{3.8}$$

We use the expansion theorem, (3.5b), to show that, under certain conditions, the eigenfunctions of A are complete in **B**.

Theorem 3.9

If $\lambda = 0$ is not an eigenvalue of (3.5a), then the eigenfunction expansion associated with A with eigenvalues and eigenvectors given by (3.7) and (3.8) is complete in **B**.

Proof:

Let

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbf{B}.$$

Then, by (3.5), we have

$$u_1 = \sum_{n=1}^{\infty} c_n^1 u_n ;$$

$$u_2 = \sum_{n=1}^{\infty} c_n^2 u_n ;$$

Hence

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \sum_{j=1}^{\infty} c_j^+ \begin{bmatrix} \sqrt{\lambda_j} u_j \\ u_j \end{bmatrix} + \sum_{j=1}^{\infty} c_j^- \begin{bmatrix} -\sqrt{\lambda_j} u_j \\ u_j \end{bmatrix};$$

where (3.9)

$$c_j^+ = \frac{\sqrt{\lambda_j} c_j^2 + c_j^1}{2\sqrt{\lambda_j}};$$

$$c_j^- = \frac{\sqrt{\lambda_j} c_j^2 - c_j^1}{2\sqrt{\lambda_j}}.$$

Note that the coefficients $c_j^{\pm}$ are determined by taking inner products with adjoint eigenfunctions.

$$c_j^{\pm} = \int_{\Omega} \left(\frac{\pm u_j}{2\sqrt{\lambda_j}} \quad \frac{u_j}{2} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} d\underset{\sim}{y}; \tag{3.10}$$

where the adjoint eigenfunctions are given by:

$$v_j^{\pm} = \begin{bmatrix} \pm \dfrac{u_j}{2\sqrt{\lambda_j}} \\ \dfrac{u_j}{2} \end{bmatrix}, \tag{3.11}$$

and satisfy

$$\begin{bmatrix} 0 & 1 \\ -\nabla_{\underset{\sim}{y}}^2 - a(\underset{\sim}{y}) & 0 \end{bmatrix} v_j^{\pm} = \pm \sqrt{\lambda_j} v_j^{\pm}. \tag{3.12}$$

Hence, A is separable in the sense of section 2.

We briefly mention the excluded case where there are zero eigenvalues. For simplicity we assume that one eigenvalue of (3.5) is zero. The two-dimensional subspase of **B** which is associated with the corresponding eigenvalue of A is spanned by

$$\begin{bmatrix} u_o \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ u_o \end{bmatrix},$$

where

$$-(\nabla_{\underset{\sim}{y}}^2 + \alpha(\underset{\sim}{y}))u_o = 0, \quad (u_o, u_o) = 1.$$

Only $\begin{bmatrix} 0 \\ u_o \end{bmatrix}$, however, is an eigenfunction of A. The other, $\begin{bmatrix} u_o \\ 0 \end{bmatrix}$ is a generalized eigenfunction satisfying:

$$A^2 \begin{bmatrix} u_o \\ 0 \end{bmatrix} = 0.$$

That is, the restriction of A to this finite dimensional subspace leads to a matrix which is not diagonalizable, but which has a non-trivial Jordan form. This possibility could be included in our analysis. The algebra would be similar to the case of ordinary differential equations (Lentini and Keller [32]).

Further analysis of problem (3.3) requires additional knowledge of the spectrum of (3.5). There are two district cases we shall consider: firstly, the case when $\lambda_n > 0$ for all n and, secondly, the case when finitely many λ_n are negative.

We begin, then, with the assumption

$$\lambda_n > 0, \quad n = 1,2, \cdots ; \tag{3.13}$$

which can be shown to hold, for example, whenever $\alpha(\underset{\sim}{y})$ is strictly negative. (This is a consequence of the variational principle for the eigenvalues. See Courant-Hilbert [11]). The eigenvalues of A, given by (3.7), are, then, all real.

Applying Theorem (2.35a) we have that the spaces $\mathbf{B}^+$ and $\mathbf{B}^-$ given by:

$$\mathbf{B}^+ \equiv \left\{ span \begin{bmatrix} \sqrt{\lambda_n} u_n \\ u_n \end{bmatrix} \right\};$$
$$\mathbf{B}^- \equiv \left\{ span \begin{bmatrix} -\sqrt{\lambda_n} u_n \\ u_n \end{bmatrix} \right\}; \tag{3.14}$$

induce an exponential dichotomy for (3.3). Hence, by Theorems (2.12) and (2.26) the only boundary conditions of our form which can be applied are:

$$\|u(x)\| \ bounded \ as \ x \to \infty. \tag{3.15}$$

The admissible space, $\mathbf{A}(\tau)$, is $\mathbf{B}^-$ for all τ and its projector, $Q(\tau)$ is given by:

$$Q(\tau)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \tag{3.16}$$

where

$$v_i = \sum_{n=1}^{\infty} c_n^i u_n,$$

$$w_i = \sum_{n=1}^{\infty} d_n^i u_n, \tag{3.17}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{\lambda_n}}{2} \\ \frac{-1}{2\sqrt{\lambda_n}} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} c_n^1 \\ c_n^2 \end{bmatrix} = \begin{bmatrix} d_n^1 \\ d_n^2 \end{bmatrix}.$$

Similarly, Theorem (2.26) can be used to write down a particular solution for absolutely integrable, Lipschitz continuous inhomogeneous term, $f(x, \underset{\sim}{y})$.

Now we drop the assumption of equation (3.13) and suppose that finitely many λ_n are negative. That is,

$$\lambda_n < 0, \; n = 1,2, \cdots, m; \qquad \lambda_n > 0, \; n = m+1, m+2, \cdots \tag{3.18}$$

The operator, A, now has not only the real eigenvalues,

$$\lambda = \pm\sqrt{\lambda_n}, \; n = m+1, m+2, \cdots \; ; \tag{3.19}$$

but, also, the purely imaginary eigenvalues,

$$\lambda = \pm i\sqrt{-\lambda_n}, \; n = 1,2, \cdots, m. \tag{3.20}$$

By Theorems (2.12) and (2.26) we now see that, in addition to the boundedness condition (3.15), an additional condition of the form:

$$\lim_{x\to\infty} B_\infty u(x) = 0, \tag{3.21}$$

can be imposed. This can be, by (2.12), only a condition on the (finite dimensional) subspace spanned by the eigenvectors of eigenvalues with zero real part. Assuming, for simplicity, that the negative eigenvalues are distinct, we have that the spaces, $\mathbf{B}_i^0$, of definition (2.11) are one dimensional and spanned by the vectors:

$$w_n^\pm \equiv \begin{bmatrix} \pm i\sqrt{-\lambda_n}u_n \\ u_n \end{bmatrix}, \quad n = 1,2, \cdots, m. \tag{3.22}$$

Hence, we are allowed to set the expansion coefficients associated with any of the eigenfunctions $w_n^\pm$ to zero at infinity. The expression for $Q(\tau)$ will be as given (3.16) and (3.17) for $\lambda_n > 0$. For the finite set of of negative λ_n, the projection matrix of equation (3.17) is replaced by

$$a)\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ if } \textit{both modes are set to zero};$$

$$b)\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ if } \textit{neither is set to zero}; \tag{3.23}$$

$$c)\ \begin{bmatrix} \frac{1}{2} & \pm\frac{i\sqrt{-\lambda_n}}{2} \\ -\frac{\pm i}{2\sqrt{-\lambda_n}} & \frac{1}{2} \end{bmatrix} \text{ if } w_n^{\mp} \textit{ is set to zero but } w_n^{\pm} \textit{ is not.}$$

In any case, the admissible space, $\mathbf{A}(\tau)$, and it is complement induce an ordinary dichotomy for (3.3) by Theorem (2.35b). Note that our freedom in choosing the boundary condition at infinity is restrained by the conditions for solvability of the finite problem given in Theorem (1.20).

An example of a problem of the type discussed above is that of the Helmholtz equation in a cylindrical waveguide:

$$-(\nabla^2 + k^2)u = f\,,\quad (x, \underset{\sim}{y}) \in [0,\infty) \times \Omega\,;$$
$$u = 0,\quad \underset{\sim}{y} \in \partial\Omega\,;$$

which is extensively analyzed by Goldstein [20]. A natural condition at infinity is to demand "outgoing" radiation; that is, to set to zero at infinity the coefficients of eigenfunctions whose imaginary eigenvalues have negative real part. Application of such a condition results in the projection matrix of equation (3.23c). Goldstein applies conditions which are specializations of ours to the case $f = 0$, as he assumes that the support of f is contained within the finite, computational domain.

We finally note that all of the considerations discussed above go over to the case of more general boundary conditions on $\partial\Omega$. All that is necessary is that the completeness assumption on the cross-sectional eigenvalue problem, (3.5), still holds.

There are various possibilities for the use of (3.16), (3.17) and (3.23) in combination with an appropriate expression for $u_p(\tau)$, such as (2.28), in a computation of problem (3.3). (Of course, we are considering a computation on a finite domain with the right boundary condition, (1.12c).) One possibility is to expand the solution at the boundary in the eigenfunctions, u_n, and directly apply the various formulas above. A drawback of this approach, though it is the approach which we use, is that one must be content with using only finitely many of them. That is, we approximate the operator $Q(\tau)$ by $Q^*(\tau)$ which is given by:

$$
\begin{aligned}
& Q^*(\tau)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}; \\
& v_i = \sum_{n=1}^{\infty} c_n^i u_n; \\
& w_i = \sum_{n=1}^{L} d_n^i u_n; \\
& Q_n \begin{pmatrix} c_n^1 \\ c_n^2 \end{pmatrix} = \begin{pmatrix} d_n^1 \\ d_n^2 \end{pmatrix}, \quad n = 1,2, \cdots ,L;
\end{aligned}
\tag{3.24}
$$

where Q_n is the projection matrix of (3.17) or (3.23). Similarly, $u_p(\tau)$ must be approximated by a $u_p^*(\tau)$ involving only these first L eigenfunctions. By the error analysis of section 1, the resulting errors depend linearly on:

$$
\|u_p(\tau) - u_p^*(\tau)\|
$$

and (3.25)

$$
\|(Q(\tau) - Q^*(\tau))(u_p^*(\tau) - u_a(\tau))\|;
$$

where $u_a(\tau)$ is the solution to the approximate boundary value problem. We use this to prove the following theorem.

Theorem 3.26

Suppose that $u_a(x)$ is a solution of the approximate problem, (1.31) (coming from (3.3)), with Q^* and u_p^* given above. Then, if

$$a)\ u_a(\tau) = \sum_{n=1}^{\infty} c_n^{a\pm} w_n^{\pm};$$

$$w_n^{\pm} = \begin{bmatrix} \pm\sqrt{\lambda_n} u_n \\ u_n \end{bmatrix}, \tag{3.26}$$

and

$$b)\ c_n^{a\pm} = O(n^{-\alpha}),\ n \to \infty;$$

and the expansion of the particular solution, $u_p(x)$, satisfies a similar estimate with exponent β, then the error, $e(x) \equiv u(x) - u_a(x)$, satisfies

$$\max_{x\in[0,\tau]} \|e(x)\| = O(L^{-\gamma}),\ L \to \infty;$$

$$\gamma = \min(\alpha - 1, \beta - 1). \tag{3.27}$$

Proof:

By equation (1.33) we need only estimate $\|\Delta(\tau)\|$ which, in turn, requires that we estimate (3.25). However,

$$u_p(\tau) - u_p^*(\tau) = \sum_{n=L+1}^{\infty} c_n^{p\pm}(\tau) w_n^{\pm},$$

which implies that

$$\|u_p(\tau) - u_p^*(\tau)\| = O(L^{-\beta+1}),\ L \to \infty,$$

while

$$(Q(\tau) - Q^*(\tau))(u_p^*(\tau) - u_a(\tau)) = \sum_{n=L+1}^{\infty} c_n^{a\pm}(\tau) Q_n w_n^{\pm};$$

which yields

$$\|(Q(\tau)-Q^*(\tau))(u_p^*(\tau)-u_a(\tau))\| = O(L^{-\alpha+1}), \ \ L\to\infty .$$

Combining these gives the desired result. Note that the decay rates depend on the smoothness of the solution and the dimension of the cross-section. They increase with increasing smoothness and decreasing dimension.

In some cases the cutoff approximation, (3.24), to the projection operator, $Q(\tau)$, might be unnecessary. From (3.17) and (3.23) we see that, for $w_i(\underset{\sim}{y}) \in L_2^0(\Omega)$,

$$Q(\tau)\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \int_\Omega \hat{Q}(\underset{\sim}{y},\underset{\sim}{p},\tau)\begin{bmatrix} w_1(\underset{\sim}{p}) \\ w_2(\underset{\sim}{p}) \end{bmatrix} d\underset{\sim}{p}; \tag{3.28}$$

where

$$\hat{Q}(\underset{\sim}{y},\underset{\sim}{p},\tau) = \sum_{n=1}^{\infty} u_n(\underset{\sim}{y}) u_n(\underset{\sim}{p}) Q_n(\tau). \tag{3.29}$$

Hence, if (3.29) could be evaluated exactly, so could (3.28). However, this is not possible in general.

We note that, no matter how it is applied, the boundary condition, (1.12c), is non-local. This does not, though, significantly increase the computational effort. Suppose, for example, we were solving our finite reduction of problem (3.3) by a centered finite difference method and that the dimension of Ω was one. Then, if we ordered the points in a typical fashion, an interior row would look like this:

$$(0 \cdots 0\, a_1 \underbrace{0 \cdots 0}_{O(n)} a_2\, a_d\, a_{-2} \underbrace{0 \cdots 0}_{O(n)} a_1\, 0 \cdots 0),$$

where a_d is the diagonal element and n is the number of grid points in a cross-section. Application of standard, local boundary conditions preserves this structure at the ends. Our condition can be represented in the form

$$Q_1 \underset{\sim}{u}_x(\tau) = Q_2 \underset{\sim}{u}(\tau) + \underset{\sim}{g}, \tag{3.30}$$

where $\underset{\sim}{u}_x(\tau)$, $\underset{\sim}{u}(\tau)$ are n-vectors of values of $u_x(\tau,\underset{\sim}{y})$ and $u(\tau,\underset{\sim}{y})$ at the gridpoints and Q_1 and Q_2 are nxn matrices. Choosing τ half-way between our last two grid-points in the x direction and differencing the derivative we write:

$$Q_1 \frac{\underset{\sim}{u}_{N+1} - \underset{\sim}{u}_N}{(x_{N+1} - x_N)} = Q_2 \frac{\underset{\sim}{u}_{N+1} + \underset{\sim}{u}_N}{2} + \underset{\sim}{g}. \tag{3.31}$$

This fills the last $2n$ columns of the last n rows. The system can be solved, for example, by combining a standard Laplace solver with a bordering technique for the last n rows and columns. The added work is, then, swamped by the work needed to invert the "big" part of the Laplace matrix, which is unaffected by the boundary conditions.

Elliptic problems in conical domains can also be treated by the theory of section 2. Consider the following problem:

$$\left[\frac{1}{x^\alpha}\frac{\partial}{\partial x}(x^\alpha \frac{\partial u}{\partial x}) + \frac{1}{x^2}(L^* + a(\underset{\sim}{\theta}))u\right] = 0;$$
$$(x,\underset{\sim}{\theta}) \in [\tau,\infty) \times \Omega; \tag{3.32}$$

where L^* is the Laplace-Beltrami operator for the α dimensional cross-section and $\underset{\sim}{\theta}$ are the angular coordinates. Consider the change of variables

$$w_1 = x^\alpha \frac{\partial u}{\partial x};$$

$$w_2 = x^{\alpha-1} u.$$

This tansforms (3.32) to

$$\frac{\partial}{\partial x}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \frac{1}{x}\begin{bmatrix} 0 & -L^{*}-\alpha \\ 1 & \alpha-1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}; \tag{3.33}$$

which is of the form (2.44). Using the results presented there, we can represent the admissible space in terms of the eigenfunctions of:

$$\begin{bmatrix} 0 & -L^{*}-\alpha \\ 1 & \alpha-1 \end{bmatrix}. \tag{3.34}$$

This, in turn, leads us to consider an analogue of (3.5) where the Laplacian is replaced by the Laplace-Beltrami operator, L^{*}.

The eigenvalue problem, (2.1), for problems arising from partial differential equations can, in general, be written in the form of a partial differential equation eigenvalue problem-though not so simple a problem as (3.5). Specializing (1.2) to problems which are independent of x, we see that (2.1) becomes:

$$\begin{bmatrix} -P_n^{-1}P_{n-1} & -P_n^{-1}P_{n-2} & \cdots & -P_n^{-1}P_o \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \vdots & \vdots \\ \vdots & \vdots & 1 & 0 \end{bmatrix}\begin{bmatrix} w_i^{(n-1)} \\ w_i^{(n-2)} \\ \vdots \\ w_i^{(0)} \end{bmatrix} = \lambda_i \begin{bmatrix} w_i^{(n-1)} \\ w_i^{(n-2)} \\ \vdots \\ w_i^{(0)} \end{bmatrix};$$

Solving this we have:

$$\begin{aligned} &a)\ \left[\sum_{j=0}^{n} P_j(\underset{\sim}{y}, \frac{\partial}{\partial \underset{\sim}{y}})\lambda_i^j\right] w_i = 0,\ \ \underset{\sim}{y} \in \Omega\,; \\ &b)\ \left[\sum_{j=0}^{m} B_{\Omega\,j}(\underset{\sim}{y}, \frac{\partial^j}{\partial n^j})\right] w_i = 0,\ \ \underset{\sim}{y} \in \partial\Omega\,. \end{aligned} \tag{3.35}$$

That is, we consider the eigenvalue problem arising from Laplace transformation in x. (The Laplace transform approach is also suggested by Gustafsson and Kreiss [23] as a generalization of the direct separation of variable method). The applicability of the formulas of the preceding section depend on the completeness properties of the eigenfunctions of (3.33); properties which are not known in general. Whenever they are complete, however, the computational considerations we have just discussed are valid for this more general case.

Lastly, we present some results of Agmon and Nirenberg [1]: applications of their general asymptotic expansion theorems, Theorem (2.42) and its variants, to partial differential equations. Again, we give the theorems without proofs, for which the reader is referred to their paper. We begin with a definition of the class of problems to be considered.

Definition 3.34 (Agmon and Nirenberg, [1,pp.202-204])

The boundary value problem:

$$\begin{aligned} &a)\ \ Au \equiv \left[\sum_{j=0}^{l-1} A_{l-j}(\underset{\sim}{y}, \frac{\partial}{\partial \underset{\sim}{y}}) \frac{1}{i^j} \frac{\partial^j}{\partial x^j} + \frac{1}{i^l} \frac{\partial^l}{\partial x^l}\right] u = 0; \\ &b)\ \ (x, \underset{\sim}{y}) \in [0, \infty) \times \Omega\,; \\ &c)\ \ B_j(\underset{\sim}{y}, \frac{\partial}{\partial \underset{\sim}{y}}) u = 0, \ \ \underset{\sim}{y} \in \partial\Omega\,, \ j = 1, \cdots, m; \end{aligned} \tag{3.36}$$

is called a regular weighted elliptic boundary value problem of order type $(2m, l)$ if the following hold:

i) $A_k(\underset{\sim}{y}, \frac{\partial}{\partial \underset{\sim}{y}})$ is a differential operator of order s_k with:

$$s_l = 2m;$$

$$s_j \leq \frac{2m_j}{l}.$$

ii)If $A^{\#}(\underset{\sim}{y}, \frac{\partial}{\partial \underset{\sim}{y}}, \frac{1}{i} \frac{\partial}{\partial x})$, the weighted principal part of A, is given by:

$$A^{\#} = \left[\sum_{j=0}^{l-1} A^{\#}_{l-j} \frac{1}{i^j} \frac{\partial^j}{\partial x^j} + \frac{1}{i^l} \frac{\partial^l}{\partial x^l}\right],$$

where $A_{l-j}^{\#}$ is that part of A_{l-j} which is exactly of order $\frac{2m_j}{j}$, then

$$A^{\#}(\underset{\sim}{y},\xi\tau) \neq 0$$

for all real vectors $(\xi\tau)$ and $\underset{\sim}{y} \in \Omega$.

iii) B_j is a normal differential operator of order m_j; $m_j \leq 2m-1$, principal part, B_j', satisfying:

The polynomials in s

$$B_j'(\underset{\sim}{y},\xi + s\nu), \;\; j = 1, \cdots, m;$$

where ξ is parallel to $\partial\Omega$ and ν is normal, are linearly independent modulo the polynomial:

$$\prod_{k=1}^{m} (s - s_k^{+}(\xi\,\tau));$$

where $s_k^{+}(\xi\,\tau)$ are the m roots of $A^{\#}(\underset{\sim}{y},\xi + s\nu\,\tau)$ with positive imaginary parts.

iv) The coefficients of the various operators and the cross-sectional domain, Ω , are bounded and sufficiently smooth.

Note that a regular weighted elliptic boundary value problem of order type $(2m, 2m)$ is simply an elliptic boundary value problem. Problems of different order type include some parabolic equations, also. The authors go on to show that such a problem, when reduced to abstract form, satisfies the conditions of their asymptotic expansion theorems.

Recall that (3.36) can be written in abstract form:

$$\frac{dU}{dx} = iAU; \;\; U = \begin{pmatrix} u \\ \frac{1}{i}\frac{du}{dx} \\ \vdots \\ \frac{1}{i^{l-1}}\frac{d^{l-1}u}{dx^{l-1}} \end{pmatrix}; \tag{3.37}$$

A solution, $u(x,y)$ is said to be of the class $L^{*}_{w,q}$ for some real w and $1 \leq q \leq \infty$ if its companion, $U(x)$, is such that

$$e^{wx} \| U(x) \| \in L_q(0,\infty).$$

for each u in this class associate a formal eigenfunction expansion,

$$u \sim \sum_k e_k \tag{3.38}$$

in the following way;

$$\begin{bmatrix} e_k \\ \frac{1}{i}\frac{de_k}{dx} \\ \vdots \\ \frac{1}{i^{l-1}}\frac{d^{l-1}e_k}{dx^{l-1}} \end{bmatrix} \equiv E_k(x);$$

$$E_k(x) = exp(i\lambda_k x)P_k(x) = \operatorname*{Res}_{\lambda=\lambda_k} R(\lambda;A)U(0). \tag{3.39}$$

Then we have:

Theorem 3.40 (Agmon and Nirenberg [1,p.219])

Let $u(x)$ be a solution (3.34) belonging to some class $L^{*}_{w,q}$, then

i) u as a function of x with values in $H_{2m,L_p}(\Omega;[B_j])$, (the subspace of the Sobolev space $H_{2m,L_p}(\Omega)$ satisfying the proper boundary conditions, (3.34c)), is analytic $x > 0$. Morever, there exists $\delta > 0$ depending on A so that u can be analytically continued into the angle $|arg\ x| < \delta$.

ii) The eigenfunction expansion, (3.36), is an asymptotic expansion in the following sense: Let $\alpha > w$ and $\lambda_1, \lambda_2, \cdots, \lambda_N$ be the eigenvalues of A in the strip $w < \operatorname{Im}\lambda < \alpha$. Let $\varepsilon > 0$ be such that $\operatorname{Im}\lambda_k < \alpha - \varepsilon$ for $k = 1, \cdots, N$. Then, for

$x > 1$, the following holds:

$$\left\| \frac{\partial^j}{\partial x^j}\left(u(x,\underset{\sim}{y}) - \sum_{k=1}^{N} e_k(x,\underset{\sim}{y})\right)\right\|_{2m,L_p(\Omega)} \leq$$

$$constant \times \left(\sum_{i=0}^{l-1} \left\| \frac{\partial^j u(0,\underset{\sim}{y})}{\partial x^j} \right\|_{2m(1-\frac{i+1}{l}),L_p}\right) exp\,(-(a-\varepsilon)x). \tag{3.40}$$

for all j. The constant is independent of u.

For A with eigenvalues bounded off the real axis, the theorem above establishes the exponential decay of bounded solutions. Then, the image under S of the admissible space, $\mathbf{A}(0)$, is approximated, for x sufficiently large, by the span of the eigenfunctions (and generalized eigenfunctions) associated with eigenvalues of iA with positive imaginary part. Application of this result in numerical computations leads to the same considerations as for the other examples of this section.

We finally note that, in our discussion of numerical approximations, we have ignored the problem of finding the eigenvalues and eigenfunctions. For a non-standard problem of the form (3.33), this might be difficult.

4. Time Dependent Problems on Unbounded Spatial Domains

The theory of time dependent problems on unbounded spatial domains presents more difficulties than the spatial problem discussed in the preceding sections. The reduction to first order, abstract form given in section 1 requires the cross-section, Ω, to include the unboundedness (for non-periodic problems) of the time variable. This, combined with the fact that an initial value problem rather than a boundary value problem is posed, often leads to operators, A, which do not have the nice spectral properties used above.

In this section we eschew the abstract formulation and pursue, instead, a more direct one based on Laplace transforms. (The formulations are, in fact, equivalent.) For simplicity we begin with a problem in one space dimension, but later generalize to higher dimensional problems.

Consider the following mixed initial-boundary value problem on the tail:

$$\begin{aligned}
&a)\ \frac{\partial^n u}{\partial x^n}+\sum_{j=0}^{n-1} L_j\left(\frac{\partial}{\partial t}\right)\frac{\partial^j u}{\partial x^j}=f(x,t);\ \ \tau< x<\infty,\ t>0;\\
&b)\ L_j\left(\frac{\partial}{\partial t}\right)=\sum_{k=0}^{m} a_{kj}\frac{\partial^k}{\partial t^k};\\
&c)\ \frac{\partial^l u}{\partial t^l}(x,0)=g_l(x),\ \ l=1,\cdots,m-1;\\
&d)\ \lim_{x\to\infty} u(x,t)=0.
\end{aligned} \tag{4.1}$$

Define:

$$\hat{u}(x,s)=\int_0^\infty e^{-st}u(x,t)\,dt. \tag{4.2}$$

Then, from (4.1), $\hat{u}$ satisfies the following ordinary differential equation in x:

$$\begin{aligned} a)\ & \frac{\partial^n \hat{u}}{\partial x^n} + \sum_{j=0}^{n-1} L_j(s) \frac{\partial^j \hat{u}}{\partial x^j} = \\ & \hat{f}(x,s) + \sum_{j=0}^{n-1} \left\{ \sum_{k=1}^{m} a_{k,j} \left(\sum_{l=0}^{k-1} s^{(k-1-l)} \frac{\partial^j g_l(x)}{\partial x^j} \right) \right\}, \\ & \tau \le x < \infty; \\ b)\ & \lim_{x \to \infty} \hat{u}(x,s) = 0. \end{aligned} \tag{4.3}$$

(This reduction is, in fact, valid in greater generality than is the Laplace transform formula (4.2). For details see Mikusinski [36].)

An operator function, $g(s)$, is called a logarithm if there is a solution to the operator differential equation:

$$\frac{dw}{dx}(x,s) = g(s) w(x,s);$$

in which case the solution is given by:

$$w(x,s) = e^{g(s)x}. \tag{4.4}$$

(The reader is referred to Mikusinski [36,Ch.2] for details.)

The formal solutions of the homogeneous problem associated with (4.3a) are given by:

$$w_p(x,s) = e^{\alpha_p(s)x}; \tag{4.5}$$

where $\alpha_p(s)$ is a root of the characteristic equation:

$$\alpha_p^n + \sum_{j=0}^{n-1} L_j(s) \alpha_p^j = 0. \tag{4.6}$$

The question of whether or not $\alpha_p(s)$ is a logarithm is answered by the following theorem of Mikusinski [36,pp.310-311], which we state without proof.

Theorem 4.7

The n roots, $\alpha_p(s)$, of equation (4.6) can all be written in the form:

$$\alpha_p(s) = \beta_k^p s^{\frac{k}{q}} + \beta_{k-1}^p s^{\frac{k-1}{q}} + \cdots + \beta_1^k s + \beta_o + L; \tag{4.7}$$

where k is a non-negative integer, q is a positive integer and L can be represented as a convergent series of integral operators (powers of $1/s$). The root α_p is a logarithm if,

$$a)\ \beta_r^p = 0,\ \frac{r}{q} > 1;$$

and (4.8)

$$b)\ \beta_r^p \text{ is real},\ \frac{r}{q} = 1.$$

We suppose that all solutions of (4.6) are logarithms and that, for each p, least one β_k^p, $k > 0$, is non-zero and the coefficient of the highest power of s is real. Then, fixing s as a complex number with real part sufficiently large, the theory of Keller and Lentini [32] for ordinary differential equations yields that $\hat{u}(x,s)$ is a solution of (4.3) if and only if:

$$\hat{w}(x,s) \equiv \begin{pmatrix} \dfrac{\partial^{n-1}\hat{u}}{\partial x^{n-1}} \\ \vdots \\ \hat{u} \end{pmatrix};$$

satisfies:

$$(I - \hat{Q}(s))\hat{w}(\tau,s) = (I - \hat{Q}(s))\hat{w}_p(\tau,s). \tag{4.9}$$

Here, $\hat{Q}(s)$ is the nxn matrix projection operator into the invariant subspace generated by the eigenvalues with negative real part of:

$$A(s) \equiv \begin{bmatrix} -L_{n-1}(s) & -L_{n-2}(s) & \cdots & -L_0(s) \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \vdots & \vdots \\ \vdots & \vdots & 1 & 0 \end{bmatrix}.$$

Note that the eigenvalues of $A(s)$ are the roots of the characteristic equation, (4.6), and those with negative real part (for $|s|$ sufficiently large in a right half-plane) are the ones whose leading coefficient, β_k^p, is negative. The particular solution vector $\hat{w}_p(\tau,s)$, is given by:

$$\hat{w}_p(\tau,s) = -\int_\tau^\infty e^{A(s)(\tau-p)}(I - \hat{Q}(s)) \begin{bmatrix} \hat{g}(p,s) \\ 0 \\ \vdots \\ 0 \end{bmatrix};$$

$$\hat{g}(x,s) \equiv \hat{f}(x,s) + \sum_{j=0}^{n-1} \left\{ \sum_{k=1}^{m} a_{k,j} \left(\sum_{l=0}^{k-1} s^{(k-1-l)} \frac{\partial^j g_l(x)}{\partial x^j} \right) \right\}. \tag{4.10}$$

Using these we have the following theorem.

Theorem 4.11

Assume that:

(i) All solutions, $\alpha_p(s)$, of (4.6) are logarithms.

(ii) The expansion, (4.7), for each α_p is such that some β_k^p, $k > 0$ is non-vanishing and the non-vanishing coefficient with highest index, k, is real.

(iii) The functions $g_l(x)$, $l = 1, \cdots, m$, are absolutely integrable and Lipschitz continuous and $e^{-\gamma t} f(x,t)$ is absolutely integrable and Lipschitz continuous in x uniformly in t for some γ.

Then if $u(x,t)$ with exponentially bounded growth in time (uniformly in x) is a solution of (4.1), its Laplace transform, $\hat{u}(x,s)$, satisfies (4.3) and (4.9).

Proof:

The equivalence of (4.1a,b,c) and (4.3a), whenever the partial derivatives of the inverse transform of $\hat{u}(x,s)$ appearing in (4.1a) exist, is established by Mikusinski [36,p.306]. Hence, if $u(x,t)$ solves (4.1) and has (uniform in x) exponentially bounded temporal growth, its transform must satisfy (4.3a). Furthermore, as there exists some γ such that

$$\lim_{x \to \infty} e^{-\gamma t} u(x,t) = 0$$

uniformly in time, then

$$\lim_{x \to \infty} \hat{u}(x,s) = 0$$

uniformly in the half-plane Re $s > \gamma$. Therefore, $\hat{u}(x,s)$ satisfies (4.9).

It is somewhat unsatisfying that the conclusion of the preceding theorem is given in one direction only. In general, to check that the inverse transform of a solution, $\hat{u}(x,s)$, of (4.3) and (4.9) satisfies (4.1), it is necessary to check that its partial derivatives are continuous and that (4.1d) is satisfied. This might be accomplished by direct inversion or by application of the inversion theorems appearing in Churchill [8,Ch.6].

We note that direct application of (4.9) in the time domain requires the Laplace inversion of the known functions of s, $\hat{Q}(s)$ and $\hat{\underset{\sim}{u}}_p(\tau,s)$. This yields:

$$\int_0^t (I - Q(t-p))\underset{\sim}{u}(\tau,p)\,dp = \int_0^t (I - Q(t-p))\underset{\sim}{u}^p(\tau,p)\,dp \equiv B(t). \tag{4.11}$$

The implementation of (4.11) in a numerical computation requires that $\underset{\sim}{u}(\tau,t)$, ($u(\tau,t)$ and its first n-1 partial x derivatives), be stored for all time. Note that the form of the condition is that of an integral over time, the cross-section variable, which is similar to the form found for spatial problems. Lastly, we mention that the analysis above is easily extended to systems of equations. The only difference is in the form of the first order ordinary differential equation to which the Keller-Lentini theory is applied.

Hyperbolic Problems

We now examine some applications of the preceding theorem. We first consider a hyperbolic system:

$$\frac{\partial \underset{\sim}{w}}{\partial x} = \Lambda \frac{\partial \underset{\sim}{w}}{\partial t} + M\underset{\sim}{w}, \quad \tau < x < \infty; \qquad \underset{\sim}{w}(x,0) = \underset{\sim}{g}(x). \tag{4.12}$$

Here $\underset{\sim}{w}$ is an n-vector and Λ and M are nxn matrices. The matrix Λ is given by:

$$\Lambda = \begin{bmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{bmatrix};$$

$$\Lambda^+ = diag_{(k \times k)}[\lambda_i^+], \quad \Lambda^- = diag_{(l \times l)}[-\lambda_i^-]. \tag{4.13}$$

$$\lambda_i^{\pm} > 0 \text{ and } distinct, \quad k + l = n.$$

The vector $\underset{\sim}{w}$ can be written:

$$\underset{\sim}{w} = \begin{bmatrix} \underset{\sim}{w}^+ \\ \underset{\sim}{w}^- \end{bmatrix}; \tag{4.14}$$

where $\underset{\sim}{w}^+$ is a k-vector of 'incoming' variables and $\underset{\sim}{w}^-$ is an l-vector of 'outgoing' variables. We impose the boundary condition at infinity:

$$\lim_{x \to \infty} \underset{\sim}{w}^+(x,t) = \underset{\sim}{0} \tag{4.15}$$

That is, there are no incoming waves at infinity.

The constraction of artificial boundary conditions at $x = \tau$ for such problems has been considered by other authors. Engquist and Majda [15] develop a general theory of non-reflecting conditions. These are designed so that no incoming waves propagate from the point $x = \tau$ into the region $x < \tau$. For (4.12) this becomes:

$$\underset{\sim}{w}^{+}(\tau,t) = \underset{\sim}{0} \tag{4.16}$$

Gustafsson and Kreiss [23] point out, however, that the non-reflecting condition (4.16) is not in general, equivalent to the condition, (4.15), at infinity. They suggest a method for constant coefficient problems to which ours reduces in that case.

The transformed equation corresponding to (4.12) is:

$$\frac{d\hat{w}}{dx} = (s\Lambda + M)\hat{\underset{\sim}{w}} - \Lambda g. \tag{4.17}$$

As pointed out by Gustafsson and Kreiss [23], for Res sufficiently large the matrix

$$s\Lambda + M$$

will have k eigenvalues with positive real part and l with negative real part. Hence, the boundary condition in s-space, given by (4.9), will consist of k relations. In general it can be written in the form:

$$\hat{\underset{\sim}{w}}^{+}(\tau,s) = \hat{B}(s)\hat{\underset{\sim}{w}}^{-}(\tau,s) + \hat{\underset{\sim}{g}}(\tau,s); \tag{4.18}$$

where $\hat{B}(s)$ is a $k \times l$ matrix. Inverting the transform yields:

$$\underset{\sim}{w}(\tau,t) = \int_0^t B(t-p)\underset{\sim}{w}^{-}(\tau,p)\,dp + \underset{\sim}{g}(\tau,t). \tag{4.19}$$

The integral term measures the contribution to the incoming waves made by all the outgoing waves at previous times. It would be absent if the coupling matrix, M, were zero (or of some special non-coupling form.) The function $\underset{\sim}{g}(\tau,t)$, gives the contribution of the initial conditions in the tail.

In cases where $\underset{\sim}{g}(x)$ (or its asymptotic structure) is simple and the dimension is not too large, it should be possible to calculate B and q exactly. As an example we specialize to the following problem:

$$\begin{aligned} \underset{\sim}{w} &= \begin{bmatrix} w^+ \\ w^- \end{bmatrix}; \\ \Lambda &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \\ \underset{\sim}{g}(x) &= \underset{\sim}{0} \\ M &= \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}. \end{aligned} \tag{4.20}$$

The projection condition, (4.7), then becomes;

$$\begin{bmatrix} s+\sqrt{s^2+ab} & a \\ b & -s+\sqrt{s^2+ab} \end{bmatrix} \begin{bmatrix} \hat{w}_+ \\ \hat{w}_- \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.21}$$

or

$$\hat{w}_+(s,\tau) = \frac{s-\sqrt{s^2+ab}}{b}\hat{w}_-(s,\tau). \tag{4.22}$$

Inverting (4.22) yields:

$$w^+(t,\tau) = -\sqrt{\frac{a}{b}}\int_0^t \frac{J_1(\sqrt{ab}\,(t-p))}{(t-p)} w^-(p,\tau)dp. \tag{4.23}$$

We note that, as a tends to zero, the ingoing and outgoing variables decouple and (4.23) becames the no-reflection condition, (4.16)

For the general problem, (4.12), the Laplace inversion step leading from (4.18) to the boundary condition in the time domain, (4.19), might be difficult to perform analytically. It is pointed out by Gustafsson and Kreiss [23], however, that an asymptotic expansion of the solutions of (4.17), valid for large $|s|$, can be easily obtained. The inversion of such an expansion leads to a small time expansion in the time domain. We state the following theorem on this which can be found, for example, in Smith [38,p.97].

Theorem 4.24

Suppose $f(t)$ has a Laplace transform, $\hat{f}(s)$, and satisfies:

$$t^{-\rho} f(t) = a_o + a_1 t + \cdots + a_{n-1} t^{n-1} + t^n A_n(t);$$
$$|A_n(t)| \le A \text{ for } 0 \le t \le t_o. \tag{4.24}$$

Then

$$\hat{f}(s) = \sum_{k=0}^{n-1} a_k \{(\rho+k)!\} s^{-\rho-k-1} + B s^{-\rho-n-1} + O(e^{-st_o});$$
$$B \le (\rho+n)! A. \tag{4.25}$$

Hence, if we assume that f has an expansion of the form (4.24), we can invert, term by term, an expansion of the form (4.25) for its transform. Given the assumed distinctness of the eigenvalues of Λ, the "perturbed" matrix, $\Lambda + \frac{1}{s}M$, will have eigenvalues and associated eigenprojections which are holomorphic functions of $(1/s)$ for $|s|$ sufficiently large. (See Kato [31,Ch.2].) This implies the holomorphicity of the eigenvectors of $\Lambda + \frac{1}{s}M$ and its adjoint and, hence,the existence of a holomorphic diagonalizing similarity transformation. Using this fact, we outline a procedure for finding expansions in $1/s$ of the projection operator, $\hat{Q}(s)$, and the eigenvalues of $\Lambda + \frac{1}{s}M$. These, in turn, could be used to generate expansions of $\hat{u}_p(s,\tau)$ using (4.10).

We suppose that $T(s)$ is a matrix which is non-singular in some neighborhood of infinity and which also satisfies:

$$T(s)(\Lambda + \frac{1}{s}M)T^{-1}(s) = D(s),\ \textit{diagonal};$$
$$\lim_{|s| \to \infty} T(s) = I. \tag{4.26}$$

We write $T(s)$ in the form:

$$T(s) = I + \frac{1}{s}T_1 + \frac{1}{s^2}T_2 + \cdots ; \tag{4.26}$$

which can be inverted to give:

$$T^{-1} = I - \frac{1}{s}T_1 + \frac{1}{s^2}(T_1^2 - T_2) + \cdots . \tag{4.28}$$

Substituting (4.27) and (4.28) into (4.26) and noting that $D(s)$ must have an expansion:

$$D(s) = \Lambda + \frac{1}{s}D_1 + \frac{1}{s^2}D_2 + \cdots , \tag{4.29}$$

we have a hierarchy of equations:

$$\begin{aligned} &T_1\Lambda - \Lambda T_1 + M = D_1, \\ &T_2\Lambda - \Lambda T_2 + \Lambda T_1^2 \\ &+ T_1 M - M T_1 - T_1 \Lambda T_1 = D_2, \\ &\vdots \end{aligned} \tag{4.30}$$

These can be solved in order. The first, for example, yields (where $\lambda_i = \lambda_i^+$, $i \leq k$; $\lambda_i = -\lambda_p^-$, $i = p + k$):

$$(T_1)_{ij}(\lambda_j - \lambda_i) + M_{ij} = (d_i)\delta_{ij};$$

which implies:

$$T_{ij} = (\frac{1}{\lambda_i - \lambda_j}) M_{ij}; \qquad d_i = M_{ii}. \tag{4.31}$$

The ith eigenvalue becomes:

$$\lambda_i + \frac{1}{s} M_{ii} + O(\frac{1}{s^2}). \tag{4.32}$$

The projection matrix, $Q(s)$, is given by

$$\begin{aligned} \hat{Q}(s) &= T^{-1} \begin{bmatrix} 0 & 0 \\ 0 & I_{(l \times l)} \end{bmatrix} T(s) \\ &= \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + \frac{1}{s} \left\{ \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} T_1 - T_1 \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \right\} \\ &+ O(\frac{1}{s^2}). \end{aligned} \tag{4.33}$$

If we write M and T_1 in the form:

$$M = \begin{bmatrix} M^1_{(k \times k)} & M^2_{(k \times l)} \\ M^3_{(l \times k)} & M^4_{(l \times l)} \end{bmatrix}, \quad T_1 = \begin{bmatrix} T_1^1 & T_1^2 \\ T_1^3 & T_1^4 \end{bmatrix}$$

we have

$$\hat{Q}(s) = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + \frac{1}{s} \begin{bmatrix} 0 & -T_1^2 \\ T_1^3 & 0 \end{bmatrix} + O(\frac{1}{s^2}). \tag{4.34}$$

Specializing to the case of zero initial data in the tail, we use (4.34) to approximate the boundary condition, (4.19), by:

$$\underset{\sim}{w}^+(\tau,t) = -T_1^2 \int_0^t \underset{\sim}{w}^-(\tau,p)\,dp + O(t^2). \tag{4.35}$$

Equation (4.35) shows (for small t) how the value of the incoming variable on the boundary depends on the coupling term, M^2, which is connected to T_1^2 by (4.31).

Parabolic Problem

Our method is not, of course, restricted to equations of hyperbolic type. As an example, we apply the general construction to the one dimensional heat equation. The relevant problem in the tail, $[\tau,\infty)$, is:

$$\begin{aligned} &a)\ u_t = u_{xx} + f(x,t),\ \ \tau < x < \infty; \\ &b)\ u(x,0) = g(x),\ \ \tau < x < \infty; \\ &c)\ \lim_{x\to\infty} u(x,t) = 0. \end{aligned} \tag{4.36}$$

We assume, as usual, that f and g are Lipschitz continuous and absolutely integrable in x and further assume that these hold uniformly in t for the function f. The characteristic equation, (4.6), for this problem is:

$$\lambda^2 - s = 0;$$

which has multivalued solution:

$$\lambda = s^{\frac{1}{2}}.$$

By (4.8), this solution is a logarithm. Denoting by $\pm\sqrt{s}$ the branches of $s^{\frac{1}{2}}$ with non-negative and non-positive real parts, condition (4.19) becomes:

$$\begin{pmatrix} 1 & \sqrt{s} \\ \frac{1}{\sqrt{s}} & 1 \end{pmatrix} \begin{pmatrix} \hat{u}_x(\tau,s) \\ \hat{u}(\tau,s) \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{s} \\ \frac{1}{\sqrt{s}} & 1 \end{pmatrix} \begin{pmatrix} \hat{w}_1^p(\tau,s) \\ \hat{w}_2^p(\tau,s) \end{pmatrix}; \tag{4.37}$$

where, by (4.10):

$$\hat{w}_1^p(\tau,s) = \int_\tau^\infty e^{s^{\frac{1}{2}}(\tau-p)}(\hat{f}(p,s)+g(p))dp\,,$$
$$\hat{w}_2^p(\tau,s) = \frac{1}{\sqrt{s}}\hat{w}_1^p(\tau,s). \tag{4.38}$$

Inverting this yields:

$$\int_0^t \frac{1}{\sqrt{\pi(t-p)}}u_x(\tau,p)dp + u(\tau,t) = b(t); \tag{4.39}$$

where $\hat{b}(s)$, the Laplace transform of $b(t)$, is given by:

$$\hat{b}(s) = \frac{2}{\sqrt{s}}\int_\tau^\infty e^{s^{\frac{1}{2}}(\tau-p)}(\hat{f}(p,s)+g(p))dp. \tag{4.40}$$

In this case, the condition measures the contribution of what has, in the past, diffused out the boundary into the tail and of the initial conditions there. Its computational implementation is similar to the implementation of the hyperbolic conditions.

Higher Dimensions

We now generalize our method to problems in more than one space dimension. Specifically we consider:

$$\begin{aligned}
&a)\ \frac{\partial^n u}{\partial x^n}+\sum_{j=0}^{n-1} L_j(\frac{\partial}{\partial t},\underset{\sim}{y},\frac{\partial}{\partial \underset{\sim}{y}})\frac{\partial^j u}{\partial x^j}=f(x,\underset{\sim}{y},t),\ \ (x,\underset{\sim}{y})\in[\tau,\infty)\times\Omega\,;\ t\geq 0;\\
&b)\ L_j(\frac{\partial}{\partial t},\underset{\sim}{y},\frac{\partial}{\partial \underset{\sim}{y}})=\sum_{k=0}^{m} a_{k,j}(\underset{\sim}{y},\frac{\partial}{\partial \underset{\sim}{y}})\frac{\partial^k}{\partial t^k};\\
&c)\ \frac{\partial^l u}{\partial t^l}(x,\underset{\sim}{y},0)=g_l(x,\underset{\sim}{y}),\ \ l=1,\cdots,m-1;\\
&d)\ B_\Omega u(x,\underset{\sim}{y},t)=0,\ \ \underset{\sim}{y}\in\partial\Omega\,;\\
&e)\ \lim_{x\to\infty} u(x,\underset{\sim}{y},t)=0.
\end{aligned} \tag{4.41}$$

Introducing the temporal Laplace transform of $u(x,\underset{\sim}{y},t)$ as in (4.2), we derive, in analogy with (4.3), the following partial differential equation:

$$\begin{aligned}
&a)\ \frac{\partial^n \hat{u}}{\partial x^n}+\sum_{j=0}^{n-1} L_j(s,\underset{\sim}{y},\frac{\partial}{\partial \underset{\sim}{y}})\hat{u}=\\
&\qquad \hat{f}(x,\underset{\sim}{y},s)+\sum_{j=0}^{n-1}\left\{\sum_{k=1}^{m} a_{k,j}(\underset{\sim}{y},\frac{\partial}{\partial \underset{\sim}{y}})[\sum_{l=0}^{k-1} s^{(k-1-l)}\frac{\partial^j g_l}{\partial x^j}(x,\underset{\sim}{y})]\right\};\\
&\qquad (x,\underset{\sim}{y})\in[\tau,\infty)\times\Omega\,;\\
&b)\ B_\Omega \hat{u}(x,\underset{\sim}{y},s)=0,\ \ \underset{\sim}{y}\in\partial\Omega\,;\\
&c)\ \lim_{x\to\infty}\hat{u}(x,\underset{\sim}{y},s)=0.
\end{aligned} \tag{4.42}$$

For fixed s in an appropriate right half-plane, problem (4.42) can be treated by the methods discussed in the preceding sections. In particular, equation (3.33) leads to the following generalization of the characteristic equation, (4.6):

$$\begin{aligned}
&a)\ \left\{\lambda_i^n+\sum_{j=0}^{n-1} L_j(s,\underset{\sim}{y},\frac{\partial}{\partial \underset{\sim}{y}})\lambda_i^j\right\}w_i(\underset{\sim}{y},s)=0,\ \ \underset{\sim}{y}\in\Omega\,;\\
&b)\ B_\Omega w_i(\underset{\sim}{y},s)=0,\ \ \underset{\sim}{y}\in\partial\Omega\,.
\end{aligned} \tag{4.43}$$

That is, we consider an eigenvalue problem in the spatial cross-section. Assuming that the eigenvalues defined above are logarithms when considered as functions of s and that the eigenfunctions satisfy certain completeness properties, the results of section 2 can be combined with those of this section to yield a boundary condition at $x = \tau$. We note that this will be a condition on functions of $\underset{\sim}{y}$ and t, involving integrals over $\Omega \times [0,t)$.

We do not attempt to discover general conditions which guarantee that the method outlined above gives correct conditions. Even if it does, the general solution of the s-dependent eigenvalue problem, (4.43), might be difficult to obtain. Gustafsson and Kreiss [23], however, point out a class of problems where this is not the case. For our general equation, (4.41), this class corresponds to equations which satisfy:

Assumption 4.44

The operators $a_{kj}(\underset{\sim}{y}, \frac{\partial}{\partial \underset{\sim}{y}})$ satisfy:

$$a_{kj} = constant, \quad k \text{ or } j \neq 0. \tag{4.44}$$

Equation (4.43) than becomes:

$$\begin{aligned} a) \quad & \left\{ \lambda_i^n + \sum_{j=1}^{n-1} L_j(s) \lambda_i^j + \sum_{k=1}^{m} a_{k,o} s^k \right\} w_i(\underset{\sim}{y}, s) \\ & + a_{o,o}(\underset{\sim}{y}, \frac{\partial}{\partial \underset{\sim}{y}}) w_i(\underset{\sim}{y}, s) = 0, \quad \underset{\sim}{y} \in \Omega ; \\ b) \quad & B_\Omega w_i(\underset{\sim}{y}, s) = 0, \quad \underset{\sim}{y} \in \partial\Omega . \end{aligned} \tag{4.45}$$

Then, if $(\gamma_p, w_p(\underset{\sim}{y}))$ are the eigenvalue-eigenfunction pairs satisfying:

$$\begin{aligned} & a_{o,o}(\underset{\sim}{y}, \frac{\partial}{\partial \underset{\sim}{y}}) w_p(\underset{\sim}{y}) = \gamma_p w_p(\underset{\sim}{y}), \quad \underset{\sim}{y} \in \Omega ; \\ & B_\Omega w_p(\underset{\sim}{y}) = 0, \quad \underset{\sim}{y} \in \partial\Omega ; \end{aligned} \tag{4.46}$$

equation (4.45) is reduced to the algebraic equation:

$$\lambda_i^n + \sum_{j=1}^{n-1} L_j(s)\lambda_i^j + \sum_{k=1}^{m} a_{k,o} s^k + \gamma_p = 0. \tag{4.47}$$

This has n solutions:

$$\lambda_i(s,\gamma_p), \quad i = 1, \cdots, n. \tag{4.48}$$

Expanding $\hat{u}(x,s,\underset{\sim}{y})$ in sums of the eigenfunctions, $w_p(\underset{\sim}{y})$, leads to an uncoupled, infinite sequence of one-dimensional problems of the type considered at the start of this section. That is, if

$$\hat{u}(x,s,\underset{\sim}{y}) = \sum_{l=1}^{\infty} \hat{c}_l(x,s) w_l(\underset{\sim}{y}); \tag{4.49}$$

and the operators $\lambda_i(s,\gamma_l)$, given by (4.48), are all logarithms, we use (4.9) to write:

$$(I - \hat{Q}_l(s)) \begin{bmatrix} \dfrac{\partial^{n-1}\hat{c}_l}{\partial x^n}(\tau,s) \\ \vdots \\ \hat{c}_l(\tau,s) \end{bmatrix} = (I - \hat{Q}_l(s))\hat{\underset{\sim}{w}}^p(\tau,s). \tag{4.50}$$

Here $\hat{\underset{\sim}{w}}^p(\tau,s)$ is given by (4.10) with $\hat{g}\ (x,s)$ replaced by:

$$\hat{g}_l(x,s) \equiv (w_l(\underset{\sim}{y}), \hat{f}(x,\underset{\sim}{y},s) + \sum_{j=0}^{n-1} \left\{ \sum_{k=1}^{m} a_{k,j} (\sum_{l=0}^{k-1} s^{(k-1-l)} \frac{\partial^j g_l}{\partial x^j}(x,\underset{\sim}{y})) \right\};$$

Inverting the transform finally yields:

$$\int_0^t (I - Q_l(t-p)) \begin{bmatrix} (w_l(\underset{\sim}{y}), \dfrac{\partial^{n-1} u}{\partial x^{n-1}}(\tau,\underset{\sim}{y},p))_\Omega \\ \vdots \\ (w_l(\underset{\sim}{y}), u(\tau,\underset{\sim}{y},p))_\Omega \end{bmatrix} dp$$

$$= F_l(\tau,t); \quad l = 1,2,3, \cdots . \tag{4.51}$$

Here F_l is the inverse Laplace transform of the right-hand side of (4.50). If (4.47) is simple enough to be solved analytically and the functions $f(x, \underset{\sim}{y}, t)$ and $g_l(x, \underset{\sim}{y})$ are also sufficiently simple, (4.46) can be solved numerically to yield approximations to (4.52).

We close the section by mentioning the work of other authors. Guderley [22] develops conditions for linearized subsonic flow equations. In particular, he analyzes:

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} - 2M(1-M^2)^{-1}\varphi_{xt} - (1-M^2)^{-1}\varphi_{tt} = 0.$$

His conditions are based on application of Green's theorem in 4-dimensional space-time. As such, he is free to choose the shape of his artificial boundary. He needs to use basic exact solutions at infinity and assumes zero initial data in the tail. If his region were of the cylindrical shape we consider and his artificial boundary of our form, the methods would be equivalent.

In many numerical calculations, time-dependent problems are solved in order to find time-independent (or steady) solutions. For such cases, steady boundary conditions such as those discussed in sections 2 and 3 might be appropriate. Ferm and Gustafsson [16], for example, calculate steady solutions of the Euler equations in a two-dimensional channel using steady boundary conditions of the type we have previously discussed.

5. Perturbation Theory and Asymptotic Boundary Conditions

In the previous three sections we found useful representations of the projection operator, $Q(\tau)$, of the admissible space and of the particular solution, $u_p(x)$, for equations of the form (1.1) with constant tails. In the present section we relax this assumption and replace it with:

Assumption 5.1

$$\lim_{x \to \infty} A(x) = A_\infty \tag{5.1}$$

We proceed to show that if the constant tail problem with operator A_∞ posseses an exponential dichotomy, then useful representations (for large τ) of the projection operator and particular solution for the perturbed problem in the tail,

$$\begin{aligned} &\frac{du}{dx} = A(x)u + f(x), \quad \tau < x < \infty; \\ &\lim_{x \to \infty} u(x) = 0; \end{aligned} \tag{5.2}$$

can be obtained. We then apply our results to various particular problems, including some where the dichotomy is absent.

Abstract Perturbation Theory

It is known for the case of ordinary differential equations that an exponential dichotomy is stable under small perturbations of the coefficient matrix. A proof which allows the unperturbed matrix to be unbounded is given by Coppel [10,Ch. 4]. We adapt his methods to prove the following theorem.

Theorem 5.3

Assume that the operator $A_\infty(x)$ posseses a non-mixing exponential dichotomy as given by Definition (1.16) with $Q_\infty(x)$ a projection operator for $\mathbf{B}^-(x)$. Assume further that

$$a)\ A(x) = A_\infty(x) + B(x);$$

$$b)\ \|B(x)\| < \varepsilon,\ \tau \leq x < \infty. \tag{5.3}$$

Then, if ε is sufficiently small and $f(x)$ is absolutely integrable and Lipschitz continuous, the projection operator, $Q(\tau)$, for the admissible space and the particular solution, $u_p(x)$, for problem (5.2) exist and are given by the solutions of the following integral equations.

$$a)\ Y_1(x) = S(x,\tau;A_\infty)Q_\infty(\tau) + \int_\tau^x S(x,r;A_\infty)Q_\infty(r)B(r)Y_1(r)dr$$

$$-\int_x^\infty S(x,r;A_\infty)(I - Q_\infty(r))B(r)Y_1(r)dr;$$

$$b)\ Q(\tau) = Y_1(\tau); \tag{5.4}$$

$$c)\ u_p(x) = \int_\tau^x S(x,r;A_\infty)Q_\infty(r)(B(r)u_p(r) + f(r))dr$$

$$-\int_x^\infty S(x,r;A_\infty)(I - Q_\infty(r)(B(r)u_p(r) + f(r))dr.$$

Before proving the main theorem we state, without proof, the following lemma.

Lemma 5.4 (Coppel [10,p.29])

Let $\varphi(x)$ be a bounded, continuous real valued function such that

$$\varphi(x) \leq Ke^{-\alpha(x-\tau)} + \theta\alpha\int_\tau^\infty e^{-\alpha|x-p|}\varphi(p)dp,\ \ x \geq \tau;$$

where K, α, and θ are positive constants; $\theta < \frac{1}{2}$. Then

$$\varphi(x) \leq \rho Ke^{-\beta(x-\tau)},\ \ x \geq \tau,$$

where

$$\beta = \alpha(1-2\theta)^{\frac{1}{2}}, \ \rho = \theta^{-1}\{1-(1-2\theta)^{\frac{1}{2}}\}.$$

Proof of Theorem 5.3

We recall that, by (1.16), for some positive constants K and α,

$$\begin{aligned} &a) \ \|S(x,\tau;A_\infty)Q_\infty(\tau)\| \le Ke^{-\alpha(x-\tau)}, \ x \ge \tau; \\ &b) \ \|S(x,\tau;A_\infty)(I-Q_\infty(\tau))\| \le Ke^{-\alpha(\tau-x)}, \ \tau \ge x. \end{aligned} \tag{5.5}$$

Let $\mathbb{M}$ be the Banach space of bounded operator valued functions of x with norm given by

$$\|Y\| = \sup_{x \ge \tau} \|Y(x)\|.$$

Let $Y(x) \in \mathbb{M}$ and define $O(Y(x))$ by:

$$O(Y(x)) = S(x,\tau;A_\infty)Q_\infty(\tau) + \int_\tau^x S(x,\tau;A_\infty)Q_\infty(\tau)B(\tau)Y(\tau)d\tau$$

$$-\int_x^\infty S(x,\tau;A_\infty)(I-Q_\infty(\tau))B(\tau)Y(\tau)d\tau.$$

Then, using (5.5) we have:

$$\|O(Y)\| \le K + \frac{2K\varepsilon}{\alpha}\|Y\|;$$

$$\|O(Y)-O(Y^*)\| \le \frac{2K\varepsilon}{\alpha}\|Y-Y^*\|.$$

Hence, $O(Y) \in \mathbb{M}$ and is a contraction mapping if

$$\varepsilon < \frac{\alpha}{2K}. \tag{5.6}$$

In this case O has a unique fixed point which we denote by $Y_1(x)$. We note that the continuity and differentiability of $S(x,\tau;A_\infty)$ imply the continuity and differentiability of Y_1. By Duhamel's formula, Y_1 is an operator solution of the homogeneous differential equation

$$\frac{dY_1}{dx} = A(x)Y_1, \quad \tau \le x \le \infty. \tag{5.7}$$

Therefore, if $S(x,\tau;A(x))$ is the solution operator for (5.7), Y_1 satisfies:

$$\begin{aligned} Y_1(x) &= S(x,\tau;A(x))Y_1(\tau) \\ &\equiv S(x,\tau;A(x))Q(\tau). \end{aligned} \tag{5.8}$$

Left multiplying equation (5.4b) by $Q_\infty(\tau)$ and using the no-mixing condition we have:

$$Q_\infty(\tau)Q(\tau) = Q_\infty(\tau).$$

Hence, $Y_1(x)Q(\tau)$ is also a fixed point of O, which implies:

$$Q(\tau)Q(\tau) = Q(\tau).$$

That is, $Q(\tau)$ is a projection operator. From (5.4a) we also have

$$\|Y_1(x)\| \le Ke^{-\alpha(x-\tau)} + \left(\frac{K\varepsilon}{\alpha}\right)\alpha\int_\tau^\infty e^{-\alpha|x-r|}\|Y_1(r)\|dr;$$

which, combined with Lemma (5.4) and equation (5.8) yields:

$$\|Y_1(x)\| \le \rho Ke^{-\beta(x-\tau)}, \quad x \ge \tau.$$

That is, $Q(\tau)$ projects into the admissible space for (5.2). We must also show that the range of $Q(\tau)$ includes all the admissible space. Let $v(x)$ be a solution of the homogeneous problem which satisfies:

$$\lim_{x \to \infty} v(x) = 0;$$

$$(I - Q(\tau))v(\tau) = v(\tau).$$

Then define $\rho(x)$ by

$$\rho(x) = v(x) + \int_x^\infty S(x, r; A_\infty)(I - Q_\infty(r))B(r)v(r)dr$$

$$-\int_\tau^x S(x, r; A_\infty)Q_\infty(r)B(r)v(r)dr.$$

The function $\rho(x)$ is a solution of the unperturbed equation. From its definition we conclude

$$\lim_{x \to \infty} \rho(x) = 0.$$

This implies

$$(I - Q_\infty(\tau))\rho(\tau) = 0.$$

However,

$$(I - Q_\infty(\tau))\rho(\tau) = (I - Q_\infty(\tau))v(\tau) + \int_\tau^\infty S(x, r; A_\infty)(I - Q_\infty(r))B(r)v(r)dr$$

$$= (I - Q(\tau))v(\tau) + \int_\tau^\infty S(x, r; A_\infty)(I - Q_\infty(r))B(r)v(r)dr$$

$$= \rho(\tau).$$

Hence we conclude that ρ is identically zero. The definition of ρ then yields:

$$\max_{x \geq \tau} \|v(x)\| \leq \max_{x \geq \tau} \| \int_x^\infty S(x,r;A_\infty)(I - Q_\infty(r))B(r)v(r)dr \|$$

$$+ \max_{x \geq \tau} \| \int_\tau^x S(x,r;A_\infty)Q_\infty(r)B(r)v(r)dr \|$$

$$\leq \frac{2K\varepsilon}{\alpha} \max_{x \geq \tau} \|v(x)\|.$$

This can only be true if

$$v(x) = 0.$$

We use similar arguments to establish the formula for the particular solution. Let **N** be the Banach space of bounded **B**–*valued* functions which tend to zero at infinity. We define $P(w(x))$ by:

$$P(w(x)) = \int_\tau^x S(x,r;A_\infty)Q_\infty(r)(B(r)w(r) + f(r))dr$$

$$- \int_x^\infty S(x,r;A_\infty)(I - Q_\infty(r))(B(r)w(r) + f(r))dr$$

$$\equiv u_p^o(x) + \int_\tau^x S(x,r;A_\infty)Q_\infty(r)B(r)w(r)dr$$

$$- \int_x^\infty S(x,r;A_\infty)(I - Q_\infty(r))B(r)w(r)dr.$$

The function $P(w) \in \mathbf{N}$ whenever $w(x)$ is. The operator P satisfies:

$$\|P(w) - P(w^*)\| \leq \frac{2K\varepsilon}{\alpha} \|w - w^*\|.$$

Hence, P is a contraction mapping and a unique fixed point, $u_p(x)$, exists. Differentiating (5.4c) yields that u_p is a solution of (5.2). This completes the proof.

Equation (5.4) can be used to calculate iterative approximations to $Q(\tau)$ and $u_p(x)$. For example, consider the following scheme based on (5.4a):

$$a)\ \ Y_1^0(x) = S(x,\tau;A_\infty)Q_\infty(\tau);$$

$$b)\ \ Y_1^{n+1}(x) = S(x,\tau;A_\infty)Q_\infty(\tau) + \int_\tau^x S(x,r;A_\infty)Q_\infty(r)B(r)Y_1^n(r)dr \qquad (5.9)$$

$$-\int_x^\infty S(x,r;A_\infty)(I - Q_\infty(r))B(r)Y_1^n(r)dr.$$

By the contraction properties of the operator, O, Y_1^n converges to Y_1 and the error at the nth iterate is given by:

$$\max_{x \geq \tau} \|Y_1(x) - Y_1^n(x)\| \leq \frac{\delta^{n-1}}{1-\delta} \max_{x \geq \tau} \|Y_1^1(x) - Y_1^0(x)\|$$

$$\leq \frac{\delta^n}{1-\delta} \max_{x \geq \tau} \|Y_1^0(x)\|;$$

where

$$\delta = \frac{2K\varepsilon}{\alpha}. \qquad (5.11)$$

In particular, (5.10) applies to our nth approximation to $Q(\tau)$,

$$Q^n(\tau) \equiv Y_1^n(\tau);$$

$$\|Q(\tau) - Q^n(\tau)\| \leq \frac{\delta^n}{1-\delta} \max_{x \geq \tau} \|Y_1^0(x)\| \leq \frac{K\delta^n}{1-\delta}. \qquad (5.12)$$

Using (5.4c) in an analagous fashion to approximate $u_p(x)$, we find that $\Delta(\tau)$, the term appearing in the error estimates, (1.33) and (1.34), satisfies:

$$\|\Delta(\tau)\| = O(\delta^n), \quad \delta \to 0. \qquad (5.13)$$

Using the spectral representations of section 2, approximations to the iterates defined by equation (5.9) can be calculated. For example, assume that A_∞ is constant and separable and that $B(x)$ is given by:

$$B(x) = \frac{1}{x} B_o, \quad x \geq \tau;$$

$$\|B(x)\| < \infty .$$

Further assume that the limiting equation,

$$\frac{dv}{dx} = A_\infty v, \quad \tau < x < \infty; \tag{5.14}$$

has an exponential dichotomy. Then, if the spectral representation associated with A_∞ is given by definition (2.1), we define matrix elements, B_{mn}, of the operator B_o in the following way:

$$B_o u = \sum_m (\sum_n B_{mn} c_n) u_m; \qquad c_n = (v_n, u). \tag{5.15}$$

Note that the boundedness of B_o implies the boundedness of its matrix elements. Using (2.18) and (2.33), we begin the iteration described in (5.9). The first term is given by:

$$Y_1^1(x)u = \sum_{\substack{n \\ u_n \in \mathbf{B}^-}} e^{\lambda_n (x-\tau)} c_n u_n + \sum_{\substack{m \\ u_m \in \mathbf{B}^-}} \sum_{\substack{n \\ u_n \in \mathbf{B}^-}} B_{mn} c_n u_m \int_\tau^x e^{\lambda_m (x-\tau)} e^{\lambda_n (\tau - \tau)} \frac{1}{p} dp$$

$$- \sum_{\substack{m \\ u_m \in \mathbf{B}^+}} \sum_{\substack{n \\ u_n \in \mathbf{B}^-}} B_{mn} c_n u_m \int_x^\infty e^{\lambda_m (x-\tau)} e^{\lambda_n (\tau - \tau)} \frac{1}{p} dp; \tag{5.16}$$

$$c_n = (v_n, u).$$

Using integration by parts, the integrals can be approximated to yield:

$$Y_1^1(x)u = \sum_{\substack{m \\ u_n \in \mathbf{B}^-}} e^{\lambda_n(x-\tau)} c_n u_n (1 + B_{nn} \log \frac{x}{\tau})$$

$$+ \sum_{\substack{m \\ u_m \in \mathbf{B}^-}} \sum_{\substack{n \\ u_n \in \mathbf{B}^-}} \frac{B_{mn} c_n u_m}{\lambda_n - \lambda_m} \left[\frac{e^{\lambda_n(x-\tau)}}{x} - \frac{e^{\lambda_m(x-\tau)}}{\tau} \right] \tag{5.17}$$

$$+ \sum_{\substack{m \\ u_m \in \mathbf{B}^+}} \sum_{\substack{n \\ u_n \in \mathbf{B}^-}} \frac{B_{mn} c_n u_m}{\lambda_n - \lambda_m} \frac{e^{\lambda_m(x-\tau)}}{\tau} + O(\frac{1}{\tau^2});$$

Note that although the correction to the diagonal term, $B_{nn} log(\frac{x}{\tau})$, is large compared to one for large x, it gives a $O(\frac{1}{\tau})$ correction in the maximum norm;

$$\max_{x \geq \tau} | e^{\lambda_n(x-\tau)} log(\frac{x}{\tau}) | < \frac{-1}{\tau \lambda_n}, \ \lambda_n < 0.$$

Higher order corrections could be calculated in the same manner. Note that the $O(\frac{1}{\tau^2})$ correction requires both the next approximations to the integrals in (5.16) and first order approximations to $Y_1(x)$. Formula (5.17) assumes that the λ_n are distinct. If not, more logarithm terms are introduced. From (5.17) we obtain the following approximation to $Q(\tau)$:

$$Q(\tau)u = \sum_{\substack{n \\ u_n \in \mathbf{B}^-}} c_n u_n + \frac{1}{\tau} \sum_{\substack{m \\ u_m \in \mathbf{B}^+}} \sum_{\substack{n \\ u_n \in \mathbf{B}^-}} \frac{B_{mn} c_n u_m}{\lambda_n - \lambda_m} + O(\frac{1}{\tau^2});$$

$$c_n = (v_n, u). \tag{5.18}$$

Similarly, we approximate $u_p(x)$. Let $f(x)$ be given by:

$$f(x) = \frac{1}{x} f^1 + \frac{1}{x^2} f^2 + \cdots ,$$

$$f^i = \sum_n c_n^i u_n. \tag{5.19}$$

(This f does not satisfy the integrability conditions previously required. Nonetheless, the results of theorem (5.4) still hold.) The first approximation to u_p is given by:

$$u_p^o(x) = \sum_{\substack{n \\ u_n \in \mathbf{B}^-}} c_n^i u_n \int_\tau^x e^{\lambda_n(x-p)} \frac{1}{p} dp$$
$$- \sum_{\substack{m \\ u_m \in \mathbf{B}^+}} c_m^i u_m \int_x^\infty e^{\lambda_m(x-r)} \frac{1}{p} dp. \tag{5.20}$$

Integration by parts yields:

$$u_p^o(x) = \sum_{\substack{n \\ u_n \in \mathbf{B}^-}} \frac{c_n^i u_n}{\lambda_n} \left(\frac{e^{\lambda_n(x-\tau)}}{\tau} - \frac{1}{x} \right)$$
$$- \sum_{\substack{m \\ u_m \in \mathbf{B}^+}} \frac{c_m^i u_m}{\lambda_m x} + O\left(\frac{1}{\tau^2}\right). \tag{5.21}$$

We can now combine (5.18) and (5.21) in an approximation of the boundary condition (1.12c). The error term, $\Delta(\tau)$, appearing in (1.33) and (1.34) is $O(\frac{1}{\tau^2})$. We note that more terms in the expansion could easily be calculated. Furthermore, the expansions of $B(x)$ and $f(x)$ need only be asymptotic and could be of a different form. In the latter case, of course, the formulas above would be altered.

We now extend these methods to problems whose limit at infinity does not possess an exponential dichotomy, but has, instead, an ordinary dichotomy. We consider

$$a) \quad \frac{du}{dx} = A_\infty(x)u + B(x)u + f(x), \quad \tau < x < \infty;$$
$$b) \quad \lim_{x \to \infty} B_\infty u(x) = 0; \tag{5.22}$$

where $f(x)$ is absolutely integrable and Lipschitz continuous. We assume that the unperturbed, homogeneous problem:

$$\begin{aligned} &a)\ \frac{dv}{dx} = A_\infty(x)v,\ \ \tau < x < \infty; \\ &b)\ \lim_{x \to \infty} B_\infty v(x) = 0; \end{aligned} \tag{5.23}$$

has an ordinary dichotomy with $Q_\infty(x)$ the projector for a space of initial data which can be propagated forward. We further assume that solutions, v, of (5.23a) satisfying (5.23b) must also satisfy:

$$(I - Q_\infty(\tau))v(\tau) = 0. \tag{5.24}$$

That is, Q_∞ is a projection operator for the admissible space. We note that we are now assuming that the boundary condition at infinity is compatible with the unperturbed operator. For a characterization of this compatibility condition for separable problems, the reader is referred to section 2. We replace the condition on the perturbation, (5.3b), with:

$$\int_\tau^\infty \|B(x)\| dx < \varepsilon. \tag{5.25}$$

We now prove an analogue of Theorem (5.3).

Theorem 5.26

Suppose that $A_\infty(x)$ has an ordinary dichotomy as given in definition (1.16) with constant, K. Suppose further that the space, $\mathbf{B}^-(x)$, is also the admissible space for (5.23) and has projection operator $Q_\infty(x)$. If ε defined by (5.25) satisfies:

$$\delta \equiv 2K\varepsilon < 1; \tag{5.26}$$

then the projector, $Q(\tau)$, for the admissible space of problem (5.22) and a particular solution are given by (5.4).

Proof:

Defining the operator, O, as in the proof of Theorem 5.3, the same contraction estimates can be obtained by use of (5.26). The solution of the integral equation (5.4a) is now an operator solution of the homogeneous part of (5.22) by application of Duhamel's formula. All that remains is to show that (5.22b) is satisfied. We note that

$$\lim_{x\to\infty}\|B_\infty S(x,r;A_\infty)Q_\infty(r)\|=0,\ \ \tau\leq r,\ \ r\ fixed.$$

Hence, we have the following for any $x_0 > \tau$:

$$\lim_{x\to\infty}\|B_\infty Y_1(x)\|\leq \lim_{x\to\infty}\Big\{\|B_\infty S(x,\tau,A_\infty)Q_\infty(\tau)\|$$

$$+\max_{\tau\leq x_0\leq r}\|B_\infty S(x,r;A_\infty)Q_\infty(r)\|\varepsilon\max_{x\geq\tau}\|Y_1(x)\|$$

$$+2\|B_\infty\|K\max_{x\geq\tau}\|Y_1(x)\|\int_{x_0}^{\infty}\|B(x)\|dx\Big\}$$

$$=2\|B_\infty\|K\max_{x\geq\tau}\|Y_1(x)\|\int_{x_0}^{\infty}\|B(x)\|dx.$$

Choosing x_0 sufficiently large, we can make the remaining integral arbitrarily small, leading to the desired result, that the limit is zero. The same arguments can be applied to establish the expression for the particular solution. Finally, we show that the range of $Q(\tau)$ contains the entire admissible space. Suppose that $w(x)$ is a solution of the homogeneous part of (5.22) which also satisfies:

$$(I-Q(\tau))w(\tau)=w(\tau).$$

Define $\rho(x)$ by:

$$\rho(x)=w(x)+\int_{\tau}^{x}S(x,r;A_\infty)Q_\infty(r)B(r)w(r)dr-\int_{x}^{\infty}S(x,r;A_\infty)(I-Q_\infty(r))B(r)w(r)dr.$$

Then, ρ is a solution of (5.23). Using the no-mixing properties of the dichotomy and the fact that

$$(I - Q_\infty(\tau))(I - Q(\tau)) = (I - Q(\tau));$$

we have:

$$0 = (I - Q_\infty(\tau))\rho(\tau)$$

$$= \rho(\tau) = \rho(x);$$

which implies:

$$\max_{x \geq \tau} \|w(x)\| \leq \delta \max_{x \geq \tau} \|w(x)\|;$$

yielding

$$w(x) = 0.$$

This completes the proof.

This theorem justifies the use of the iteration scheme, (5.9), for the calculation of approximate boundary conditions in the case when the limiting equation has an ordinary dichotomy. The error estimate, (5.13), is unchanged. The integrability requirement for the perturbation, (5.25), prohibits $O(\frac{1}{x})$ terms of the type analyzed in the dichotomous case. If this requirement is met, however, it is always possible to choose τ sufficiently large that the contraction inequality, (5.26), holds true. To illustrate the result, we assume that the operator A_∞ is constant and separable and that the perturbation is of the form:

$$B(x) = \frac{1}{x^2} B_o .$$

We further assume B_0 to have the matrix representation, (5.15). Now A_∞ has eigenvalues with positive, negative and zero real part. Let the spaces $\mathbf{B}^+$, $\mathbf{B}^-$ and $\mathbf{B}_i^o$ be defined as in (2.9) and (2.10). The projection operator, $Q_\infty(\tau)$, for the admissible space is given by (2.17) and (2.18). Incorporating all of this into equation (5.9) yields the following $O(\frac{1}{\tau})$ correction:

$$Y_1^1(x)u = \sum_{\substack{n \\ u_n \in \mathbf{B}^-}} e^{\lambda_n(x-\tau)} c_n u_n + \sum_i \sum_{\substack{n \\ u_n \in \mathbf{B}_i^o}} e^{\lambda_n(x-\tau)} \sum_{k=1}^{n_i} (Q_i^o)_{nk} c_k u_n$$

$$+ \sum_i \sum_{\substack{n \\ u_n \in \mathbf{B}_i^o}} \sum_{\substack{m \\ u_m \in \mathbf{B}_i^o}} \sum_{k=1}^{n_i} \sum_{l=1}^{n_i} (Q_i^o)_{nk} B_{km} (Q_i^o)_{ml} c_l u_n e^{\lambda_n(x-\tau)} (\frac{1}{\tau} - \frac{1}{x}) \qquad (5.27)$$

$$+ \sum_i \sum_{\substack{n \\ u_n \in \mathbf{B}_i^o}} \sum_{\substack{m \\ u_m \in \mathbf{B}_i^o}} \sum_{k=1}^{n_i} \sum_{l=1}^{n_i} (I - Q_i^o)_{nk} B_{km} (Q_i^o)_{ml} c_l u_n e^{\lambda_n(x-\tau)} \tau;$$

The projector $Q(\tau)$ becomes:

$$Q(\tau)u = \sum_{\substack{n \\ u_n \in \mathbf{B}^-}} c_n u_n \sum_i \sum_{\substack{n \\ u_n \in \mathbf{B}_i^o}} \sum_{k=1}^{n_i} (Q_i^o)_{nk} c_k u_n$$

$$+ \frac{1}{\tau} \sum_i \sum_{\substack{n \\ u_n \in \mathbf{B}_i^o}} \sum_{\substack{m \\ u_m \in \mathbf{B}_i^o}} \sum_{k=1}^{n_i} \sum_{l=1}^{n_i} (I - Q_i^o)_{nk} B_{km} (Q_i^o)_{ml} c_l u_n \qquad (5.28)$$

$$+ O(\frac{1}{\tau^2});$$

$$c_n = (v_n, u).$$

The universality of the forms of the expansions given above makes their automatic implementation a practical possibility. We also note that there are many circumstances not discussed above when formula (5.4) is valid. Later in the section we analyze such a problem, the Helmholtz equaticn exterior to a finite, two-dimensional body.

Applications

Specific applications of these results include those problems whose "limit" at infinity is one of the differential equations discussed in section 3. For ordinary differential equations, the general problem of finding asymptotic expansions on semi-infinite intervals is addressed by Wasow [39]. Jepson and Keller [30] find expansions of the projection operator, $Q(\tau)$. In the finite dimensional case, our expression reduces to theirs.

Another application is to elliptic partial differential equations in cylindrical domains. For example, replace the function $a(\underset{\sim}{y})$ in problem (3.3) by $a(x,\underset{\sim}{y})$ satisfying:

$$a(x,\underset{\sim}{y}) \sim a_o(\underset{\sim}{y}) + \frac{1}{x}a_1(\underset{\sim}{y}) + \cdots, \quad x >> 1.$$

Then, if the cross-sectional eigenvalue problem

$$a) \quad -\nabla^2_{\underset{\sim}{y}} - a_o(\underset{\sim}{y})Y_n = \alpha_n Y_n;$$

$$b) \quad (Y_n, Y_m)_{L_2^o(\Omega)} = \delta_{nm};$$

$$c) \quad \{Y_n\} \textit{ complete in } L_2^o(\Omega);$$

has only positive eigenvalues, the limiting problem possesses an exponential dichotomy and the formulas above can be applied. In particular, the first correction to $Q(\tau)$, given by equation (5.18), becomes:

$$Q(\tau)\begin{bmatrix} u_x(\tau,\underset{\sim}{y}) \\ u(\tau,\underset{\sim}{y}) \end{bmatrix} = \sum_{n=1}^{\infty} c_n \begin{bmatrix} -\sqrt{\alpha_n} \\ 1 \end{bmatrix} Y_n(\underset{\sim}{y})$$
$$- \frac{1}{\tau}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{B_{mn}c_n}{\sqrt{\alpha_m}+\sqrt{\alpha_n}} \begin{bmatrix} \sqrt{\alpha_m} \\ 1 \end{bmatrix} Y_m(\underset{\sim}{y}). \tag{5.29}$$

Here the coefficients are given by:

$$c_n = \int_{\Omega} u(\tau, \underset{\sim}{y}) Y_n(\underset{\sim}{y}) d\underset{\sim}{y};$$

$$B_{mn} = -\int_{\Omega} a_1(\underset{\sim}{y}) Y_n(\underset{\sim}{y}) Y_m(\underset{\sim}{y}) d\underset{\sim}{y}. \tag{5.30}$$

Note that the numerical implementation of the new conditions adds no new non-zero elements to the Laplace matrix; the correction term affects only the bottom right-hand block which was filled by the first order approximation.

The theory can be at least formally applied to the time dependent problems of section 4. In particular, we apply it to a perturbed version of the transformed problem, (4.42), for fixed complex number s in an appropriate right half-plane. The applicability of the results depends, of course, on the behaviour of the error terms as functions of s. As the upcoming discussion of hyperbolic problems will show, we can not expect the expansions to be uniformly valid in time.

We begin with an analysis of the following hyperbolic system:

$$\frac{\partial \underset{\sim}{u}}{\partial x} = \begin{bmatrix} \Lambda^+ & 0 \\ 0 & -\Lambda^- \end{bmatrix} \frac{\partial \underset{\sim}{u}}{\partial t} + \frac{1}{x} \begin{bmatrix} M^1 & M^2 \\ M^3 & M^4 \end{bmatrix} \underset{\sim}{u}; \quad \tau < x < \infty; \tag{5.31}$$

where

$$\Lambda^+ = diag\ \{\lambda_1^+, \lambda_2^+, \cdots, \lambda_k^+\}; \ \ \Lambda^- = diag\ \{\lambda_1^-, \lambda_2^-, \cdots, \lambda_p^-\}; \ \ k + p = n; \ \ \lambda_j^{\pm} > 0, \ distinct.$$

For simplicity, we take

$$\lim_{x \to \infty} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \underset{\sim}{u}(x, t) = \underset{\sim}{0};$$

$$\underset{\sim}{u}(x, 0) = \underset{\sim}{0} \tag{5.32}$$

Problem (5.31) is similar to problem (4.12); the difference is that the coupling between the incoming and outgoing variables decays, in this case, as x approaches infinity, while in (4.12) it did not. Hence, we expect the absorbing boundary condition,

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \underset{\sim}{u}(\tau,t) = \underset{\sim}{0} \tag{5.33}$$

to be a first approximation to the exact condition. Letting $\hat{\underset{\sim}{u}}(x,s)$ denote the temporal Laplace transform of $\underset{\sim}{u}(x,t)$, we have, by (4.3):

$$\frac{d\hat{\underset{\sim}{u}}}{dx} = \begin{bmatrix} s\Lambda^+ & 0 \\ 0 & -s\Lambda^- \end{bmatrix} \hat{\underset{\sim}{u}} + \frac{1}{x} \begin{bmatrix} M^1 & M^2 \\ M^3 & M^4 \end{bmatrix} \hat{\underset{\sim}{u}}$$

$$\lim_{x \to \infty} \hat{\underset{\sim}{u}}(x,s) = 0. \tag{5.34}$$

where s is taken to be a complex number with positive real part. The limiting problem at infinity is given by:

$$\frac{d\hat{\underset{\sim}{u}}}{dx} = \begin{bmatrix} s\Lambda^+ & 0 \\ 0 & -s\Lambda^- \end{bmatrix} \hat{\underset{\sim}{u}} \tag{5.35}$$

The constant tail problem, (5.35), clearly has an exponential dichotomy, with $\hat{Q}_\infty(\tau,s)$, the projector for the space of initial data of given by:

$$\hat{Q}_\infty(\tau,s) = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}. \tag{5.36}$$

This yields a first order approximation to the boundary condition at $x = \tau$ which is given by the absorbing condition above. The first correction to $\hat{Q}(\tau,s)$, given by

(5.18), is:

$$\hat{Q}(\tau,s) = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + \frac{1}{s\tau}\begin{bmatrix} 0 & M_*^2 \\ 0 & 0 \end{bmatrix};$$

$$(M_*^2)_{ij} = -\frac{M_{ij}^2}{\lambda_j^- + \lambda_i^+}; \tag{5.37}$$

where the error is $O(\frac{1}{s^2\tau^2})$. The approximate boundary condition in time is:

$$u_+(\tau,t) = \frac{1}{\tau}\int_0^t M_*^2 u_-(\tau,p)\,dp\,; \tag{5.38}$$

where u_+ is the k-vector of the first k components of u and u_- is the p-vector of the last p. We note the similarity of condition (5.38) and condition (4.35). Higher order approximations to (5.38) have the form:

$$u_+(\tau,t) = \sum_{l=1}^{q} \frac{1}{\tau^l}\int_0^t C_l u_-(\tau,p)(t-p)^{l-1}dp\,; \tag{5.39}$$

where C_l is a $k \times p$ matrix. The error is $O(\frac{1}{s^{q+1}\tau^{q+1}})$ in terms of the Laplace variable.

From the form of the error, we see that our expansions in transform space are valid only for $s\tau$ sufficiently small. Hence, we expect the boundary conditions, (5.39), to be valid for τ large and t small; that is $\frac{t}{\tau}$ small. Suppose our error in transform space were proportional to $\frac{1}{(s\tau)^q}$. Then, in real space we would have:

$$||Q(\tau,t) - Q_{ap.}(\tau,t)|| \propto \frac{t^{q-1}}{\tau^q}. \tag{5.40}$$

That is, our approximation degrades in time. An heuristic approach to this problem would be to take time derivatives of the boundary condition, as such derivatives decrease the power in time with which the error grows. For equation (5.38) this yields:

$$\frac{\partial}{\partial t}\underset{\sim}{u}_+(\tau,t) = \frac{1}{\tau}M_*^2\underset{\sim}{u}_-(\tau,t). \tag{5.41}$$

We discuss this possibility later on.

A problem of physical interest which can be put in the form (5.34) is the two-dimensional wave equation exterior to a finite body. The tail problem, written in cylindrical coordinates, is:

$$\begin{gathered}\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}=\frac{\partial^2\varphi}{\partial t^2};\ \tau< r<\infty,\ t>0,\ \theta\in[0,2\pi);\\ \varphi(r,0,\theta)=0;\ \ \varphi\ \mathit{periodic\ in}\ \theta;\ \ \varphi\ \mathit{outgoing\ at\ infinity}.\end{gathered} \tag{5.42}$$

Note that (5.42) satisfies assumption (4.44). Then if $\varphi_n(r,t)$ is defined by:

$$\varphi_n(r,t)=\frac{1}{2\pi}\int_0^{2\pi}\varphi(r,t,\theta)\begin{cases}\sin n\theta\\ \cos n\theta\end{cases}d\theta,$$

φ_n satisfies:

$$\begin{gathered}\frac{\partial^2\varphi_n}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi_n}{\partial r}-\frac{n^2}{r^2}\varphi_n=\frac{\partial^2\varphi_n}{\partial t^2};\ \ \tau< r<\infty,\ \ t>0;\\ \varphi_n(r,0)=0,\ \ \varphi_n\ \mathit{outgoing\ at\ infinity}.\end{gathered} \tag{5.43}$$

If we make the change of variables:

$$\begin{gathered}w_n=\frac{u_n}{r^{\frac{1}{2}}},\\ v_n=r\frac{\partial w_n}{\partial r}+r\frac{\partial w_n}{\partial t}+w_n;\end{gathered} \tag{5.44}$$

equation (5.43) becomes

$$\frac{\partial}{\partial r}\begin{bmatrix} v_n \\ w_n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\partial}{\partial t}\begin{bmatrix} v_n \\ w_n \end{bmatrix} + \frac{1}{r}\begin{bmatrix} 0 & \frac{4n^2-1}{4} \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_n \\ w_n \end{bmatrix};$$

$$\lim_{r \to \infty} v_n(r,t) = 0; \tag{5.45}$$

$$v_n(r,0) = w_n(r,0) = 0.$$

Using formula (5.38), our first two approximations to the boundary condition are:

$$a)\ v_n(\tau,t) = 0;$$

$$b)\ v_n(\tau,t) = \frac{1}{\tau}\int_0^t \left(\frac{1-4n^2}{8}\right) w_n(\tau,p)\,dp. \tag{5.46}$$

Transforming back to the original variables these become:

$$a)\ \frac{\partial \varphi_n}{\partial r}(\tau,t) + \frac{\partial \varphi_n}{\partial t}(\tau,t) + \frac{1}{2\tau}\varphi_n(\tau,t) = 0;$$

$$b)\ \frac{\partial \varphi_n}{\partial r}(\tau,t) + \frac{\partial \varphi_n}{\partial t}(\tau,t) + \frac{1}{2\tau}\varphi_n(\tau,t) = \frac{1-4n^2}{8\tau^2}\int_0^t \varphi_n(\tau,p)\,dp. \tag{5.47}$$

Equation (5.47b) can be replaced by its time-differentiated version:

$$b)'\ \frac{\partial^2 \varphi_n}{\partial r\,\partial t}(\tau,t) + \frac{\partial^2 \varphi_n}{\partial t^2}(\tau,t) + \frac{1}{2\tau}\frac{\partial \varphi_n}{\partial t}(\tau,t) + \frac{(4n^2-1)}{8\tau^2}\varphi_n(\tau,t) = 0.$$

It is not necessary to make the transformation (5.44) in order to derive (5.47). We did so to illustrate the general results.

Problem (5.42) has been considered by other authors. Engquist and Majda [15,p.637] derive a hierarchy of absorbing boundary conditions. The first two on their list are:

$$a)\ \left(\frac{\partial}{\partial r}+\frac{\partial}{\partial t}+\frac{1}{2\tau}\right)\varphi(\tau,t,\theta)=0.$$

$$b)\ \left(\frac{\partial^3}{\partial r\,\partial t^2}+\frac{\partial^3}{\partial t^3}-\frac{1}{2\tau^2}\frac{\partial^3}{\partial t\,\partial\theta^2}+\frac{1}{2\tau}\frac{\partial^2}{\partial t^2}+\frac{1}{2\tau^3}\frac{\partial^2}{\partial\theta^2}\right)\varphi(\tau,t,\theta)=0. \qquad (5.48)$$

Bayliss and Turkel [4] also treat (5.42). Their conditions are based on the following expansion of outgoing solutions:

$$\varphi(r,t,\theta)\sim\sum_{j=1}^{\infty}\frac{f_j(t-r,\theta)}{r^{\frac{2j-1}{2}}},\ r>>1. \qquad (5.49)$$

They suggest a sequence of boundary conditions based on the term-by-term annihilation of the expansion (5.49). In particular, their first two conditions are:

$$a)\ \left(\frac{\partial}{\partial r}+\frac{\partial}{\partial t}+\frac{1}{2\tau}\right)\varphi(\tau,t,\theta)=0.$$

$$b)\ \left(\frac{\partial}{\partial r}+\frac{\partial}{\partial t}+\frac{5}{2r}\right)\left(\frac{\partial}{\partial r}+\frac{\partial}{\partial t}+\frac{1}{2r}\right)\varphi(\tau,t,\theta)=0. \qquad (5.50)$$

The connection between the Bayliss-Turkel conditions, (5.50), and ours can be developed in the following way. An asymptotic expansion of the Laplace transform of $\varphi_n(r,t)$ is given by:

$$\hat{\varphi}_n(r,s)=\sqrt{\frac{\pi}{2rs}}e^{-rs}\{1+\frac{n-1}{8rs}+\cdots\},\ rs>>1.$$

This is simply the expansion of the modified Bessel function of order n, $K_n(rs)$, which is the solution of the transform of equation (5.42) which decays as r approaches infinity. A projection condition, such as we use, must relate the r-derivative of the solution with the solution itself. In fact, the relation derived by differentiating the expansion above leads to our condition. Note, however, that the expansion can also be annihilated term-by-term by the operators:

$$\hat{B}_m \equiv \prod_{k=1}^{m} \left(\frac{\partial}{\partial r} + s + \frac{4k-3}{2r} \right).$$

Inversion of these leads to the Bayliss-Turkel hierarchy. We note that their conditions are local. This is possible because the form of the expansion of K_n is independent of n. Our conditions, (5.47), could also be made local by replacing n^2 by $-\frac{\partial^2}{\partial \theta^2}$. In general, this step is impossible. Localization can only be achieved by use of pseudodifferential operator expansions as suggested by Engquist and Majda [15]; a procedure which can introduce smooth, but not small, errors.

We now compare the various conditions. Note that all of the first approximations are the same. This is simply the absorbing condition for a purely cylindrical wave. Note that the second absorbing condition, (5.48b), is just the second time derivative of (5.48a) in the absence of angular dependence. We recall that an outgoing solution of (5.42), which is independent of angle, is given by:

$$\varphi(r,t) = \begin{cases} 0 & t \leq r \\ (t^2 - r^2)^{-\frac{1}{2}} & t > r \end{cases} ; \qquad (5.51)$$

Tabulated below are the residuals which are left after applying the various approximate boundary conditions to the exact solution, (5.51). We assume that $t > r$ and let

$$w = t - r.$$

In the left-hand column we give the equation number of the condition to which the rest of the row refers. We present both the maximum error for $w > 0$ and an expansion of the error for $\frac{w}{\tau}$ small.

Table (5.52)

Condition	Residual: w/τ Small	Maximum Residual
(5.47a)	$\frac{w^{1/2}}{4\sqrt{2}\,\tau^{5/2}}$	$\frac{1}{2\sqrt{27}\,\tau^{2}}$
(5.47b)	$\frac{(9-2\sqrt{2})w^{3/2}}{48\sqrt{2}\,\tau^{7/2}}$	∞
(5.47b')	$\frac{w^{1/2}}{4\tau^{7/2}}$	$\frac{.076}{\tau^{3}}$
(5.48b)	$\frac{1}{16\sqrt{2}\,w^{3/2}\tau^{5/2}}$	$\frac{1}{\sqrt{3}\,\tau^{4}}$
(5.50b)	$\frac{3w^{3/2}}{16\sqrt{2}\,\tau^{9/2}}$	$\frac{\sqrt{3}}{100\sqrt{5}\,\tau^{3}}$

We note the change in the τ dependence of the residual for (5.50b), the Bayliss-Turkel condition, as w goes from being $O(1)$ to $O(\tau)$. From the form of (5.49), we would expect the $O(1/\tau^{\frac{9}{2}})$ estimate to be valid for all time. The degradation is a manifestation of the non-uniformity in time of the expansion, as suggested by the form of the expansion in the Laplace variable. This loss of accuracy has not been noted in the literature. Similar considerations also hold for conditions

(5.47a,b,b'). We note that our differentiated condition, (5.47b'), appears to have superior error characteristics to (5.47b). However, the residual can be a misleading measure of the accuracy of a condition. From the results of section 1, we know that it is the subspaces into which boundary data is projected that is important. Hence, for example, the Engquist-Majda condition, (5.48b), can not be better than (5.47a), though their residuals are much different. Consider the following ordinary differential equation.

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} - \frac{1}{x^2}u = 0, \quad 1 < x < \infty;$$

$$\lim_{x \to \infty} u(x) = 0; \quad u(1) = 1.$$

Its solution is:

$$u = \frac{1}{x}.$$

Let the following hierarchy of boundary conditions be applied at $x = \tau$; note the decay of the residual resulting from their application to the exact solution.

$$i) \quad u(\tau) = 0; \quad Residual = \tau^{-1};$$

$$ii) \quad \frac{du}{dx}(\tau) = 0; \quad Residual = -\tau^{-2};$$

$$iii) \quad \frac{d^2u}{dx^2}(\tau) = 0; \quad Residual = 2\tau^{-3}.$$

The solutions to these approximate problems are, respectively:

$$i) \quad \frac{1}{x(1 - \frac{1}{\tau^2})} - \frac{x}{\tau^2 - 1};$$

$$ii) \quad \frac{1}{x(1 + \frac{1}{\tau^2})} + \frac{x}{\tau^2 + 1};$$

$$iii) \quad x.$$

Note there is little difference between the first and the second and that the third, coming from the condition with the smallest residual, is completely wrong.

Our last topic in this section is that of radiation boundary conditions for the Helmholtz equation exterior to a finite, two-dimensional body. The relevant problem in the tail, written in cylindrical coordinates, is:

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+k^2 u = f\,(r,\theta);$$
$$\tau < r < \infty,\quad \theta \in [0,2\pi); \tag{5.53}$$

We impose the boundary conditions:

$$a)\ u \text{ periodic in } \theta,$$
$$b)\ \lim_{r\to\infty} r\left(\frac{\partial u}{\partial r}+iku\right)=0, \tag{5.54}$$

and also require:

$$r^{\frac{1}{2}} f\,(r,\theta) \text{ absolutely integrable.}$$

Using (5.54a), we immediately reduce (5.53) to an infinite system of ordinary differential equations by introducing:

$$c_0(r)=\frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)d\,\theta,$$

$$\begin{Bmatrix} c_n(r) \\ d_n(r) \end{Bmatrix} = \frac{1}{\pi}\int_0^{\pi} u(r,\theta)\begin{Bmatrix}\cos n\theta \\ \sin n\theta\end{Bmatrix} d\,\theta,\quad n=1,\cdots .$$

The problem for the expansion coefficients c_n (and d_n) is:

$$a)\quad \frac{d^2c_n}{dr^2}+\frac{1}{r}\frac{dc_n}{dr}-\frac{n^2}{r^2}c_n+k^2c_n=f_n(r),\quad \tau<r<\infty;$$
$$b)\quad \lim_{r\to\infty} r\left(\frac{dc_n}{dr}+ikc_n\right)=0. \tag{5.55}$$

We rewrite (5.55) in first order form:

$$w_n=\begin{bmatrix}\frac{dc_n}{dr}\\ c_n\end{bmatrix};$$

$$\frac{dw_n}{dr}=\begin{bmatrix}0 & -k^2\\ 1 & 0\end{bmatrix}w_n+\frac{1}{r}\begin{bmatrix}-1 & 0\\ 0 & 0\end{bmatrix}w_n+\frac{1}{r^2}\begin{bmatrix}0 & n^2\\ 0 & 0\end{bmatrix}w_n+\begin{bmatrix}f_n\\ 0\end{bmatrix}; \tag{5.56}$$

$$\lim_{r\to\infty} r^{\frac{1}{2}}\begin{bmatrix}1 & ik\\ 1 & ik\end{bmatrix}w_n=0.$$

Note that (5.56) does not fit the specifications of the various perturbation theorems we have stated. In the first place, the boundary condition at infinity depends on r; although this causes only a slight modification to the theory of section 1. Secondly, the limiting problem has an ordinary dichotomy, but the first perturbation term is not integrable. Hence, if we are to apply any of the preceeding theorems, it is necessary to try to include the $(1/r)$ term in the leading order operator. We therefore seek an asymptotic solution of the form:

$$w_n^{\pm}=r^{\alpha}e^{\pm ikr}\begin{bmatrix}\pm ik+\frac{\alpha}{r}\\ 1\end{bmatrix}.$$

The function $w_n^{\pm}$ satisfies the differential equation:

$$\frac{dw_n^{\pm}}{dr}=\begin{bmatrix}\frac{2\alpha}{r} & -k^2-\frac{(\alpha^2+\alpha)}{r^2}\\ 1 & 0\end{bmatrix}w_n^{\pm}. \tag{5.57}$$

As we are trying to match (5.56), we choose $\alpha = \frac{-1}{2}$. We take (5.57) to be our limiting equation. From (5.56) the perturbation is:

$$\frac{1}{\tau^2}\begin{bmatrix} 0 & \frac{-4n^2-1}{4} \\ 0 & 0 \end{bmatrix}. \tag{5.58}$$

The solution of the limiting problem, (5.57), which satisfies the boundary condition at infinity, (5.55b), is $w_n^-(\tau)$. Hence, the first approximation to the projector into the admissible space, $\mathbf{A}(\tau)$, is given by:

$$Q_\infty(\tau) = \frac{i}{2k}\begin{bmatrix} -ik - \frac{1}{2\tau} & -k^2 - \frac{1}{4\tau^2} \\ 1 & -ik + \frac{1}{2\tau} \end{bmatrix}. \tag{5.59}$$

We now apply the ideas of the first part of the section. We note that our new limiting problem does not possess a dichotomy. The operator

$$S(\tau, p; A_\infty)(I - Q_\infty(p)), \;\; p \geq \tau > T;$$

has a norm which increases like $p^{\frac{1}{2}}$ as p increases. However, the perturbation, $B(\tau)$, is $O(\frac{1}{\tau^2})$, so that the combination

$$S(\tau, p; A_\infty)(I - Q_\infty(p))B(p)$$

is integrable. This allows us to apply our contraction argument for τ sufficiently large. The first approximation to the boundary condition is given by:

$$\frac{dc_n}{d\tau} + (ik + \frac{1}{2\tau})c_n = \int_\tau^\infty \frac{i}{2k}\frac{p^{\frac{1}{2}}}{\tau^{\frac{1}{2}}} e^{ik(\tau - p)} f_n(p)(2ik + \frac{1}{2\tau} - \frac{1}{2p})dp; \tag{5.60}$$

Note that the integral exists by the assumption made on the integrability of f. We can now apply the iteration scheme, (5.9), to approximate the solutions of (5.4). For example, the next approximation to $Q(\tau)$ is:

$$Q(\tau)=\frac{i}{2k}\begin{bmatrix} -ik-\frac{1}{2\tau} & -k^2-\frac{1}{4\tau^2} \\ 1 & -ik+\frac{1}{2\tau} \end{bmatrix} - \frac{1}{4k^2}\begin{bmatrix} (1-ik)(\frac{4n^2-1}{4\tau^2}) & (-ik-k^2)(\frac{4n^2-1}{4\tau^2}) \\ \frac{1-4n^2}{4\tau^2} & -ik(\frac{1-4n^2}{4\tau^2}) \end{bmatrix}.$$

When f is zero, the formula above leads to the following approximation to the boundary condition:

$$\frac{dc_n}{dr}+ikc_n+\frac{c_n}{2\tau}+\frac{(1-4n^2)}{4\tau^2}c_n=0. \tag{5.61}$$

Bayliss, Gunzburger and Turkel [3] develop a hierarchy of boundary conditions from problem (5.53-54). There approach is much the same as the one used by Bayliss and Turkel [4] in their analysis of the wave equation. They note the asymptotic expansion of the outgoing solutions;

$$u(r,\theta)\sim\frac{e^{ikr}}{r^{\frac{1}{2}}}(f_o(\theta)+\frac{f_1(\theta)}{r}+\cdots) \tag{5.62}$$

This can be annihilated term-by-term by the operators:

$$B_m=\prod_{l=1}^{m}(\frac{\partial}{\partial r}+ik-\frac{(4l-3)}{2r}); \tag{5.63}$$

which are their suggested boundary operators. A comparison of these conditions and ours is similar to the comparison made in the case of the wave equation. As before, the locality of their conditions is due to the special nature of the problem, and could be extended to our conditions by the replacement:

$$n^2=-\frac{\partial^2}{\partial\theta^2}.$$

Formula (5.63) can be derived from a separation of variables solution and is based on the asymptotic expansion of the cylindrical Hankel functions. Our method can be derived by using this expansion to relate the solution and its r derivative; theirs comes from annihilating the expansion term-by-term.

Another work which develops boundary conditions for the exterior Helmholtz equation is that of Guderley [21]. His conditions are based on an application of Green's theorem in the tail. In fact, he works in three dimensions, but his methods are easily specialized to two. Let Γ be the artificial boundary and w^- an outgoing solution. Then, any outgoing solution, u, must satisfy:

$$\int_{\Gamma}\left(\frac{\partial w^-}{\partial n}u - \frac{\partial u}{\partial n}w^-\right)ds = 0.$$

In the special case that Γ is a circle, this condition is formally equivalent to ours.

Finally, Fix and Marin [17] look at the Helmholtz problem in three dimensions with axial symmetry. Their method is based on asymptotic expansions of proper solutions at infinity. For their problem, our conditions reduce to theirs.

6. Applications to Nonlinear Problems

In this section we extend our theory to include some nonlinear problems. A general account of the stability theory of nonlinear differential equations in a Banach space can be found in Daletskiy and Krein [12,Ch. 7]. They restrict themselves, however, to the case of bounded linearized operators. Halilov [24] removes the boundedness restriction, but requires a well-posed linearized Cauchy problem. Domslak [13,14] examines the connection between the solvability of inhomogeneous problems associated with well-posed homogeneous Cauchy problems and the existence of dichotomies. He goes on to apply this to nonlinear problems. It is the assumption of well-posedness which we wish to remove.

Perturbation Theory for Nonlinear Problems

The problem we shall consider is:

$$\begin{aligned} &a)\ \frac{du}{dx}=F(u), \quad \tau< x <\infty; \\ &b)\ \lim_{x\to\infty} u(x)=u_\infty; \qquad (6.1) \\ &c)\ F(u_\infty)=0; \end{aligned}$$

where $u(x)$ is an element of some Banach space, **B**, and F is a nonlinear operator with domain and range in **B**. Letting $v=u-u_\infty$, we rewrite (6.1):

$$\begin{aligned} &a)\ \frac{dv}{dx}=F_u(u_\infty)v+R(v), \quad \tau< x\ \infty; \\ &b)\ \lim_{x\to\infty} v(x)=0; \qquad (6.2) \\ &c)\ R(v)\equiv F(u_\infty+v)-F_u(u_\infty)(u_\infty)v. \end{aligned}$$

We generalize the notion of the admissible space to be applicable to problem (6.2).

Definition (6.3)

The set $\mathbf{A}(\tau) \subset \mathbf{B}$, the admissible set at $x = \tau$, is the set of all $v_0 \in \mathbf{B}$ such that there exists a solution to (6.2) satisfying:

$$v(\tau) = v_0. \tag{6.3}$$

The set $\mathbf{A}(\tau)$ is no longer a subspace of $\mathbf{B}$. We do not, in general, expect to be able to characterize it by a projection type operator. However, if $||v||$ is sufficiently small, (6.2d) implies that (6.2a) is nearly linear. We treat this case with the perturbation theory of the preceding section, finding useful representations of the intersection of $\mathbf{A}(\tau)$ with neighborhoods of the origin in $\mathbf{B}$. Note that by (6.2b), all solutions of (6.2) are eventually arbitrarily small and must, hence, satisfy the conditions we derive.

We make two additional assumptions. The first is that the operator $F_u(u_\infty)$ is such that the linearized problem in the tail,

$$\frac{dw}{dx} = F_u(u_\infty)w, \quad \tau < x < \infty; \tag{6.4}$$

has an exponential dichotomy. We denote the projector into $\mathbf{B}_-(x)$ by $Q_\infty(x)$. We also assume that there exists $\gamma_0 > 0$ so that, if v_1 and v_2 are elements of $\mathbf{B}$ with norm less than or equal to γ_0 we have, for some monotone increasing real-valued function c:

$$\begin{aligned}
&a)\ \ ||R(v_i)|| < c(||v_i||)||v_i||;\\
&b)\ \ ||R(v_1) - R(v_2)|| < c(\gamma)||v_1 - v_2||;\\
&c)\ \ c(\gamma) \to 0,\ \ \gamma \to 0;\\
&d)\ \ \gamma = \max(||v_1||, ||v_2||).
\end{aligned} \tag{6.5}$$

Problems which satisfy this assumption are sometimes referred to as quasilinear.

We now state the basic theorem.

Theorem 6.6

Given the assumptions made above, there exists $\delta > 0$ and idempotent, nonlinear operator $Q(\tau)$ such that any $v(x)$ satisfying (6.2a) and

$$\begin{aligned} &a)\ \|v(\tau)\| < \delta; \\ &b)\ \max_{x \geq \tau}\|v(\tau)\| < \delta^*(\delta); \end{aligned} \tag{6.6}$$

satisfies (6.2b) if and only if:

$$(I - Q(\tau))v(\tau) = 0. \tag{6.7}$$

(The constant δ^* will be defined in the proof. It satisfies $\delta^* = O(\delta)$.) Furthermore, $Q(\tau)$ is given by the solution of the following integral equation:

$$\begin{aligned} a)\ Y_1(x) = {} & S(x,\tau;F_u(u_\infty))Q_\infty(\tau) + \int_\tau^x S(x,r;F_u(u_\infty))Q_\infty(r)R(r)Y_1(r)dr \\ & - \int_x^\infty S(x,r;F_u(u_\infty))(I - Q_\infty(r))R(r)Y_1(r)dr. \end{aligned} \tag{6.8}$$

$$b)\ Q(\tau) = Y_1(\tau).$$

Proof:

For any $\varphi > 0$, define $\mathbf{M}_\varphi$ as a Banach space of bounded operators on the subset of elements of **B** with norm less than or equal to φ. For $Y \in \mathbf{M}_\varphi$, $\|Y\|$ is defined by:

$$\|Y\|_{\mathbf{M}_\varphi} = \sup_{\substack{p \in \mathbf{B} \\ \|p\| \leq \varphi}} \|Yp\|.$$

Let $\mathbf{M}_\varphi^*$ be the Banach space of $\mathbf{M}_\varphi$*–valued* functions on $[\tau,\infty)$ with the norm:

$$\|Y\|_{\mathbf{M}_\varphi^*} = \sup_{x \geq \tau}\|Y(x)\|_{\mathbf{M}_\varphi}.$$

Define the operator O by:

$$OY(x) = S(x,\tau;F_u(u_\infty))Q_\infty(\tau) + \int_\tau^x S(x,r;F_u(u_\infty))Q_\infty(r)R(r)Y(r)dr$$

$$-\int_x^\infty S(x,r;F_u(u_\infty))(I - Q_\infty(r))R(r)Y(r)dr.$$

By (6.5) and the dichotomy inequalities (1.16-18), we have the following:

If

$$\|Y_1(x)\|_\varphi \le \delta^*, \quad \delta^* \le \gamma_o;$$

then

$$\begin{aligned} &a)\ \|OY_1\|_\varphi \le K\varphi + \frac{2K}{\alpha}c(\delta^*)\delta^*; \\ &b)\ \|O(Y_1 - Y_2)\|_\varphi \le \frac{2K}{\alpha}c(\delta^*)\|Y_1 - Y_2\|_\varphi. \end{aligned} \tag{6.9}$$

Choose φ by:

$$\varphi = \max_{0<\delta^* \le \gamma_o} \frac{(1 - \frac{2K}{\alpha}c(\delta^*))}{K}\delta^*.$$

We note that this maximum must be attained for positive δ^* which also satisfies:

$$\frac{2K}{\alpha}c(\delta^*) < 1.$$

Hence, for this pair of φ and δ^*, (6.9) implies that O maps elements of $\mathbb{M}_\varphi^*$ with norm less than or equal to δ^* to other elements whose norm satisfies the same inequality and is a contraction mapping for this closed subset. This further implies the existence of a unique fixed point, $Y_1(x) \subset \mathbb{M}_\varphi^*$, with norm less than or equal to δ^*. By the definition of O, Y_1 satisfies (6.8a) and is an operator solution of (6.2a). Application of lemma (5.4) yields:

$$||Y_1(x)|| \le \delta^* e^{-\beta(x-\tau)}, \quad \beta > 0.$$

That is, if $v(x)$ is given by:

$$v(x) = Y_1(x)v_o; \qquad ||v_o|| \le \varphi; \tag{6.10}$$

then v is an exponentially decaying solution of (6.2a,b). Furthermore, for all $v_o \in \mathbf{B}$ with $||v_o|| < \varphi$ and $||Q(\tau)v_o|| < \varphi$, the arguments used in the proof of Theorem (5.3) can be applied to yield:

$$Q(\tau)Q(\tau)v_o = Q(\tau)v_o.$$

From the fixed point equation we have, again for any v_o with norm less than or equal to φ,

$$||Q(\tau)v_o|| \le \frac{K||v_o||}{1 - \frac{2K}{\alpha}c(\delta)}.$$

Hence, with domain taken as the set of all v_o satisfying:

$$a)\ ||v_o|| < \delta; \qquad b)\ \delta = \frac{(1 - \frac{2K}{\alpha}c(\delta^*))}{K}\varphi. \tag{6.11}$$

the operator $Q(\tau)$ is idempotent. We have now shown that if v_o satisfies (6.11a) and

$$Q(\tau)v_o = v_o, \tag{6.12}$$

then there exists an exponentially decaying solution of (6.2) with initial value v_o

and having norm which is always less than or equal δ^*.

Now suppose that $w(x)$ is any solution of (6.2) with $\|w(x)\| < \delta^*$ and $\|w(\tau)\| < \delta$. Let $w^*(x)$ be the solution satisfying:

$$w^*(\tau) = Q(\tau)w(\tau).$$

Let $\rho(x)$ and $\rho^*(x)$ be defined by using $w(x)$ and $w^*(x)$, respectively, in:

$$\rho^{(*)} = w^{(*)}(x) + \int_x^\infty S(x,r;F_u(u_\infty))(I - Q_\infty(r))R(r)w^{(*)}(r)dr$$

$$-\int_\tau^x S(x,r;F_u(u_\infty))Q_\infty(r)R(r)w^{(*)}(r)dr.$$

Using Duhamel's formula and the differential equation satisfied by $w^{(*)}$ they are both decaying solutions of the linearized problem as, therefore, is their difference. This implies:

$$0 = (I - Q_\infty(\tau))(\rho(\tau) - \rho^*(\tau))$$

$$= (I - Q_\infty(\tau))(w(\tau) - w^*(\tau))$$

$$+ \int_\tau^\infty S(\tau,r;F_u(u_\infty))(I - Q_\infty(r))(Rw - Rw^*)dr.$$

By definition,

$$w(\tau) - w^*(\tau) = (I - Q(\tau))w(\tau);$$

and we also have:

$$(I - Q_\infty(\tau))(I - Q(\tau)) = (I - Q(\tau)).$$

This yields:

$$0 = (w(\tau) - w^*(\tau)) + \int_\tau^\infty S(\tau,r;F_u(u_\infty))(I - Q_\infty(r))(Rw - Rw^*)dr.$$

Since the expression above is simply equal to $\rho(\tau)-\rho^*(\tau)$ and solutions to the linear problem are unique we have:

$$0=\rho(x)-\rho^*(x)$$

$$=(w(x)-w^*(x))-\int_\tau^x S(x,r;F_u(u_\infty))Q_\infty(r)(Rw-Rw^*)dr$$

$$+\int_x^\infty S(x,r;F_u(u_\infty))(I-Q_\infty(r))(Rw-Rw^*)dr.$$

From (6.5b) we derive the estimate:

$$\|w(x)-w^*(x)\|\le \frac{2Kc(\delta^*)}{\alpha}\max_{x\in[\tau,\infty)}\|w(x)-w^*(x)\|.$$

Since $\frac{2Kc(\delta^*)}{\alpha}$, the factor on the right-hand side, is less than one, this implies:

$$w(x)=w^*(x);$$

completing the proof.

This theorem justifies the use of an iteration scheme analagous to (5.9) for the nonlinear problem; where B is replaced by R. The error estimates now come in powers of ε, which is defined by:

$$\varepsilon=\frac{2K}{\alpha}c(\delta^*). \tag{6.13}$$

We note that the expression for the boundary condition, (6.7), given above holds only for v sufficiently small. Translated to the original problem, we require the solution, u, to be close to its asymptotic value, u_∞. Hence, in practical computations, we must choose τ large enough that this is true. Furthermore, since ε decreases with δ^*, the error in our approximations also decreases as u approaches u_∞. These considerations will be viewed more concretely when we do some specific examples. We also note that the theorem above reduces to one very similar to that of Halilov [24] in the case of a well-posed linearized Cauchy problem.

We now use the spectral representations of section 2 in conjunction with the perturbation theory developed above. We assume that (6.4) has an exponential dichotomy and that $F_u(u_\infty)$ is separable and has a spectral representation given by definition (2.1). We also assume that R can be written in the form:

$$R \sim A_1 vv + A_2 vvv + \cdots ; \quad \|v\| << 1.$$

In terms of the eigenfunction expansion this yields:

$$Rv = \sum_m u_m \Big(\sum_i \sum_j \alpha_{ij}^m c_j c_i + \sum_i \sum_j \sum_k \beta_{ijk}^m c_i c_j c_k + \cdots \Big) ; \tag{6.14}$$

$$c_n = (v_n, v).$$

Equation (6.14) indicates that $c(\gamma)$ appearing in (6.5) is linear in γ with a constant depending on the coefficients α_{ij}^m. Applying (5.9), our first approximation to $Y_1(x)$ is:

$$Y_1^1(x)v = \sum_{\substack{n \\ u_n \in \mathbf{B}^-}} e^{\lambda_n (x-\tau)} c_n u_n + \sum_{\substack{m,i,j \\ u_{m,i,j} \in \mathbf{B}^-}} \alpha_{ij}^m c_i c_j u_m \int_\tau^x e^{\lambda_m (x-p) + (\lambda_i + \lambda_j)(p-\tau)} dp$$

$$- \sum_{\substack{m \\ u_m \in \mathbf{B}^+}} \sum_{\substack{i,j \\ u_{i,j} \in \mathbf{B}^-}} \alpha_{ij}^m c_i c_j u_m \int_x^\infty e^{\lambda_m (x-p) + (\lambda_i + \lambda_j)(p-\tau)} dp \; ; \tag{6.15}$$

$$c_n = (v_n, u).$$

We have only used the first term in (6.14) since the form of the expansion indicates an error of $O(\|v\|^2)$. The next approximation should include both the second iterate defined in (5.9) and the contribution of the second term in (6.14) to the first iterate. The integrals can be done exactly. The final expression for the operator $Q(\tau)$ which appears in the boundary condition at τ is:

$$Q(\tau)v = \sum_{\substack{n \\ u_n \in \mathbf{B}^-}} c_n u_n - \sum_{\substack{m \\ u_m \in \mathbf{P}^+}} \sum_{\substack{i,j \\ u_{i,j} \in \mathbf{B}^-}} \frac{\alpha_{ij}^m c_i c_j}{\lambda_m - \lambda_i - \lambda_j};$$
$$c_n = (v_n, u). \tag{6.16}$$

Again, the form of the expansion suggests an error on the order of the cube of the norm of v. From the discussion of linear problems with periodic opearators which appears at the end of section 2, we note that our method could be applied to nonlinear problems with periodic solutions at infinity.

We apply the results above to nonlinear elliptic equations in cylindrical domains. Specifically, we consider:

$$a)\ u_{xx} + \nabla_{\underset{\sim}{y}}^2 u + a(\underset{\sim}{y})u = f(u,\underset{\sim}{y}),\ (x,\underset{\sim}{y}) \in [\tau,\infty) \times \Omega;$$
$$b)\ \lim_{x \to \infty} u(x,\underset{\sim}{y}) = u_\infty(\underset{\sim}{y}); \tag{6.17}$$
$$c)\ B_\Omega u = 0,\ \underset{\sim}{y} \in \partial\Omega.$$

We require that:

$$\nabla_{\underset{\sim}{y}}^2 u_\infty + a(\underset{\sim}{y})u_\infty = f(u_\infty,\underset{\sim}{y}),\ \underset{\sim}{y} \in \Omega;$$
$$B_\Omega u_\infty = 0,\ \underset{\sim}{y} \in \partial\Omega. \tag{6.18}$$

Finally, we assume that the cross-sectional eigenvalue problem:

$$\begin{aligned}
&a)\ -\nabla_{\underset{\sim}{y}}^2 Y_n - \hat{a}(\underset{\sim}{y}) Y_n = \alpha_n Y_n; \\
&b)\ (Y_n, Y_m)_{L_2^0(\Omega)} = \delta_{nm}; \\
&c)\ \{Y_n\}\ \textit{complete in}\ L_2^0(\Omega); \\
&d)\ \hat{a}(\underset{\sim}{y}) = a(\underset{\sim}{y}) - f_u(u_\infty(\underset{\sim}{y}), \underset{\sim}{y});
\end{aligned} \tag{6.19}$$

has only positive eigenvalues. Then, by the results of section 3, the linearized problem has an exponential dichotomy and the formulas above can be applied. The first correction to $Q(\tau)$, given by (6.16), is:

$$\begin{aligned}
Q(\tau)v = &\sum_{n=1}^{\infty} c_n \begin{bmatrix} -\sqrt{\alpha_n} \\ 1 \end{bmatrix} Y_n(\underset{\sim}{y}) \\
&- \sum_{m=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\alpha_{ij}^m c_i c_j}{\sqrt{\alpha_m} + \sqrt{\alpha_i} + \sqrt{\alpha_j}} \begin{bmatrix} \sqrt{\alpha_m} \\ 1 \end{bmatrix} Y_m(\underset{\sim}{y}).
\end{aligned} \tag{6.20}$$

The coefficients are given by:

$$\begin{aligned}
c_n &= \int_\Omega u(\tau, \underset{\sim}{y}) Y_n(\underset{\sim}{y}) d\underset{\sim}{y}, \\
\alpha_{ij}^n &= \frac{1}{2} \int_\Omega f_{uu}(u_\infty, \underset{\sim}{y}) Y_i(\underset{\sim}{y}) Y_j(\underset{\sim}{y}) Y_n(\underset{\sim}{y}).
\end{aligned} \tag{6.21}$$

Note that the implementation of these conditions in a numerical computation using Newton's method does not lead to any more nonzero elements in the Laplace matrix than the linear conditions did. The general form of the expansion makes its automatic computation possible. The main limitaticn is the increasing cost of evaluating the additions to the Jacobian which result when higher order approximations are used. An existence theory in the tail for two-dimensional nonlinear elliptic problems is developed in appendix A. Estimates of the constants appearing in the constraints on v, (6.6), are made.

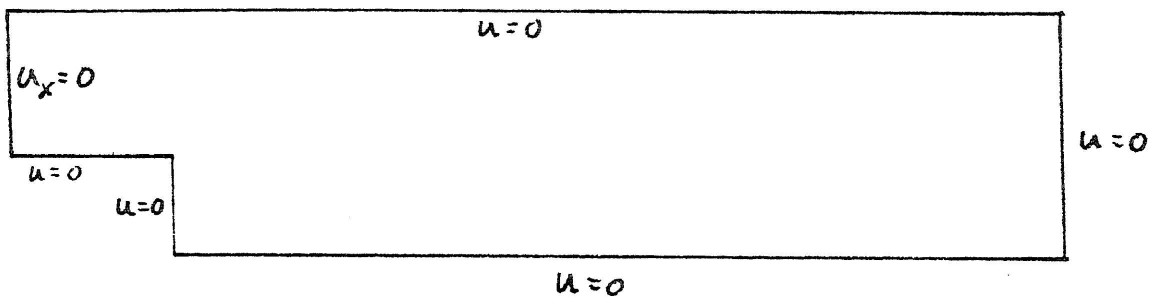
The remainder of the section is devoted to the application of the theory presented above to the numerical solution of the Bratu problem in a semi-infinite, two-dimensional stepped channel. Specifically we consider:

$$-u_{xx} - u_{yy} = \lambda e^{u}; \tag{6.22}$$

on a variety of domains of the type shown below:

$$y \in [0,1], \; x > 0;$$

$$y \in [a,1], \; -b < x < 0;$$



The boundary conditions are as shown in the drawing. The tail is always taken to have width one. At infinity we require:

$$a) \; \lim_{x \to \infty} u(x,y) = u_\infty(y);$$

$$b) \; \frac{d^2 u_\infty}{dy^2} = -\lambda e^{u_\infty}; \tag{6.23}$$

$$c) \; u_\infty(0) = u_\infty(1) = 0.$$

This problem is clearly one of the form discussed above.

Before proceeding to the numerical results, we discuss the theory of the Bratu problem in further detail. We note that the nonlinearity is of positive, monotone type. Problems of the form:

$$-Lu = \lambda f(u) \tag{6.24}$$

on a finite domain with L a second order uniformly elliptic operator and f a positive, monotone function of u are extensively discussed by Cohen and Keller [9]. They mention various applications to equilibrium problems in the theory of nonlinear heat generation. Cohen and Keller present results on the existence of positive solutions on finite domains. We state, without proof, two of their theorems.

Theorem 6.24 (Cohen and Keller [9])

Suppose a positive solution of (6.24) exists for some positive λ_o on a finite domain, Ω, and satisfies the Dirichlet boundary conditions:

$$u = 0, \quad \underset{\sim}{x} \in \partial\Omega .$$

Then a positive solution on Ω, satisfying the Dirichlet conditions, exists for all $0 < \lambda < \lambda_o$.

Theorem 6.25 (Cohen and Keller [9])

A positive solution to (6.24) exists on the finite domain Ω satisfying Dirichlet boundary conditions on $\partial\Omega$ if and only if the sequence $\{u_n(\lambda;x)\}$ defined below converges.

$$\begin{aligned} &u_o = 0; \\ &-Lu_{n+1} = \lambda f(u_n), \quad \underset{\sim}{x} \in \Omega ; \\ &u_{n+1} = 0; \quad \underset{\sim}{x} \in \partial\Omega . \end{aligned} \tag{6.25}$$

When it converges, it converges to the minimal positive solution.

In particular, the theorems presented above apply to the one dimensional problem for the asymptotic state, (6.23b). In fact, (6.23b) can be solved analytically. For $\lambda < \lambda_c$, $\lambda_c = 3.51 \cdots$, two positive solutions exist, while for $\lambda > \lambda_c$ there are none. It can also be shown that the linearized eigenvalue problem, (6.19), associated with the minimal solution has only positive eigenvalues while the problem associated with the other solution has one which is negative. Hence, we shall always take the minimal solution as our u_∞.

As we are interested in solutions on a variety of domains, we establish some facts concerning the behavior of the critical value of λ, λ_c, as the domain, Ω, varies. (The value of λ_c is the least upper bound of the set of all λ for which positive solutions exist.) This result is for the Dirichlet problem on a finite domain.

Theorem (6.26)

Suppose that $\Omega' \subset \Omega$. Then if λ is such that a positive solution exists to the problem

$$-Lu = \lambda f(u), \quad \underset{\sim}{x} \in \Omega;$$

$$u = 0, \quad \underset{\sim}{x} \in \partial\Omega;$$

there also exists a positive solution to the problem:

$$-Lu = \lambda f(u), \quad \underset{\sim}{x} \in \Omega';$$

$$u = 0, \quad \underset{\sim}{x} \in \partial\Omega'.$$

Proof:

Let $\{u_n(\lambda;x)\}$ be the sequence defined by (6.25) for the domain Ω and let $\{v_n(\lambda;x)\}$ be the sequence for the domain Ω'. Let

$$w_n(\underset{\sim}{x},\lambda) = u_n(\underset{\sim}{x},\lambda) - v_n(\underset{\sim}{x},\lambda), \quad \underset{\sim}{x} \in \Omega'.$$

The function w_1 satisfies:

$$-Lw_1 = \lambda f(u_o) - \lambda f(v_o) = 0; \ \underset{\sim}{x} \in \Omega';$$

$$w_1 = u_1, \ \underset{\sim}{x} \in \partial\Omega'.$$

Since u_1 is non-negative, the maximum principle implies that w_1 is non-negative. (See Protter and Weinberger [37,Ch. 2] for a discussion of maximum principles.) Now suppose that w_n is non-negative. Then w_{n+1} satisfies:

$$-Lw_{n+1} = \lambda(f(u_n) - f(v_n)), \ \underset{\sim}{x} \in \Omega';$$

$$w_{n+1} = u_n, \ \underset{\sim}{x} \in \partial\Omega'.$$

Since w_n is non-negative and f is monotone, the right-hand side of the equation for w_{n+1} is non-negative, as is its boundary value. The maximum principle again implies that w_{n+1} is non-negative. Hence, by induction, w_n is non-negative for all n. That is:

$$v_n \le u_n.$$

Therefore, the sequence (6.25), for the v_n's on Ω', is bounded above and must converge to the minimal positive solution by Theorem (6.25). This completes the proof.

We now present some results from our computations. All were performed on a rectangular domain obtained from the original domain by a Schwarz-Christoffel transformation. A uniform mesh was employed and derivatives were approximated by three-point centered differences. The nonlinear difference equations, including when applicable the nonlinear boundary conditions, were solved by Newton's method. The solution of the linear system combined the IMSL banded matrix solver with a bordering technique as discussed in section 2. The asymptotic solution, u_∞, was also found by a finite difference solution of

(6.23b). It was then used in a finite approximation of the cross-sectional eigenvalue problem, (6.19). The number of gridpoints in a cross-section, the number used to calculate u_∞ and the number used to solve (6.19) were always the same. Approximations to the boundary condition at $x=\tau$ were made up to second order, employing formulas (6.20) and (6.21). In the mapped plane, the left and right boundaries are curved. Therefore, it was necessary to use interpolation to apply the conditions there. Single precision arithmetic was always employed and the Newton iterations were judged to be converged when the absolute value of the residual was less than 10^{-5}. All of this was done on the timesharing VAX of the California Institute of Technology.

We solved (6.22) on a variety of domains for many values of λ and orders of approximation to the boundary condition. As suggested (though not, of course guaranteed) by the theorems earlier discussed, solutions exist whenever the step goes into the channel ($\alpha > 0$). The choice of boundary condition had no apparent effect on the number of Newton iterations required.

We present two sequences of calculations to illustrate our results. The first is for $\lambda = 3.4$ and a step size equal to .4 times the channel width. Figures (6.27-31) show u=*constant* curves for calculations in the domains shown. In the mapped plane, the left boundary was located at -3.0 and the right boundary at 2.0, 1.0, 0.5, 0.25 and 0.1 respectively. In the unmapped plane, this yields a left boundary location of -1.57 and right boundary locations of 2.26, 1.26, .764, .520 and .301. The step coordinate is -.0055. The mesh size was .05 yielding grids of 19X101, 19X81, 19X71, 19X66 and 19X63 respectively. The values of u on the plotted contours range between .025 and .894. All of these calculations employ the approximation to the right boundary condition given by (6.20).

A comparison of figures (6.27-31) yields that the solution is little changed by moving the artificial boundary towards the step. That is, we can solve the problem on the semi-infinite interval almost as accurately with the boundary at 0.1 as we can with the boundary at 2.0, with a resulting decrease in the size of the grid. This conclusion is supported by figure (6.32), which shows the superposition of (6.27) and (6.31). Figure (6.33) shows the superposition of (6.27) and a solution on the small domain using a zeroth order approximation to the artificial boundary condition, $u_x(\tau)=0$. The error is relatively large near the boundary, but decays as one moves inside. This behavior is in line with the linear error analysis for problems with exponential dichotomies given in section 1. Finally, table (6.34) lists the maximum absolute errors as calculated using the solution on the largest domain. The approximation of order zero is $u_x(\tau)=0$ while those of first and second order are the linear and quadratic approximations given by our theory. The quadratic approximation is consistently the best, but the simpler, linear condition is not too much worse. Note that both yield approximations on the smallest domain that are superior to the zeroth order approximation on the largest.

Figures (6.35-41) and table (6.42) show the same things for calculations in a slightly different domain and for λ near the critical value, $\lambda=3.51$. The step is now .3 times the channel width. The left boundary has been moved from -3.0 to -1.0 in the mapped plane, while all else has remained the same. The grids are now of the sizes 19X61, 19X41, 19X31, 19X26 and 19X23. In the unmapped plane, the left boundary is at -.512, the step at -.062 and the right boundaries at 2.2, 1.2, .71, .40 and .32 respectively. The results are not significantly different from the preceding example. However, we do see the effect of the near-zero eigenvalue as manifested in larger errors for the small domain calculations and much slower decay of these errors off the boundary.

Figure (6.27)

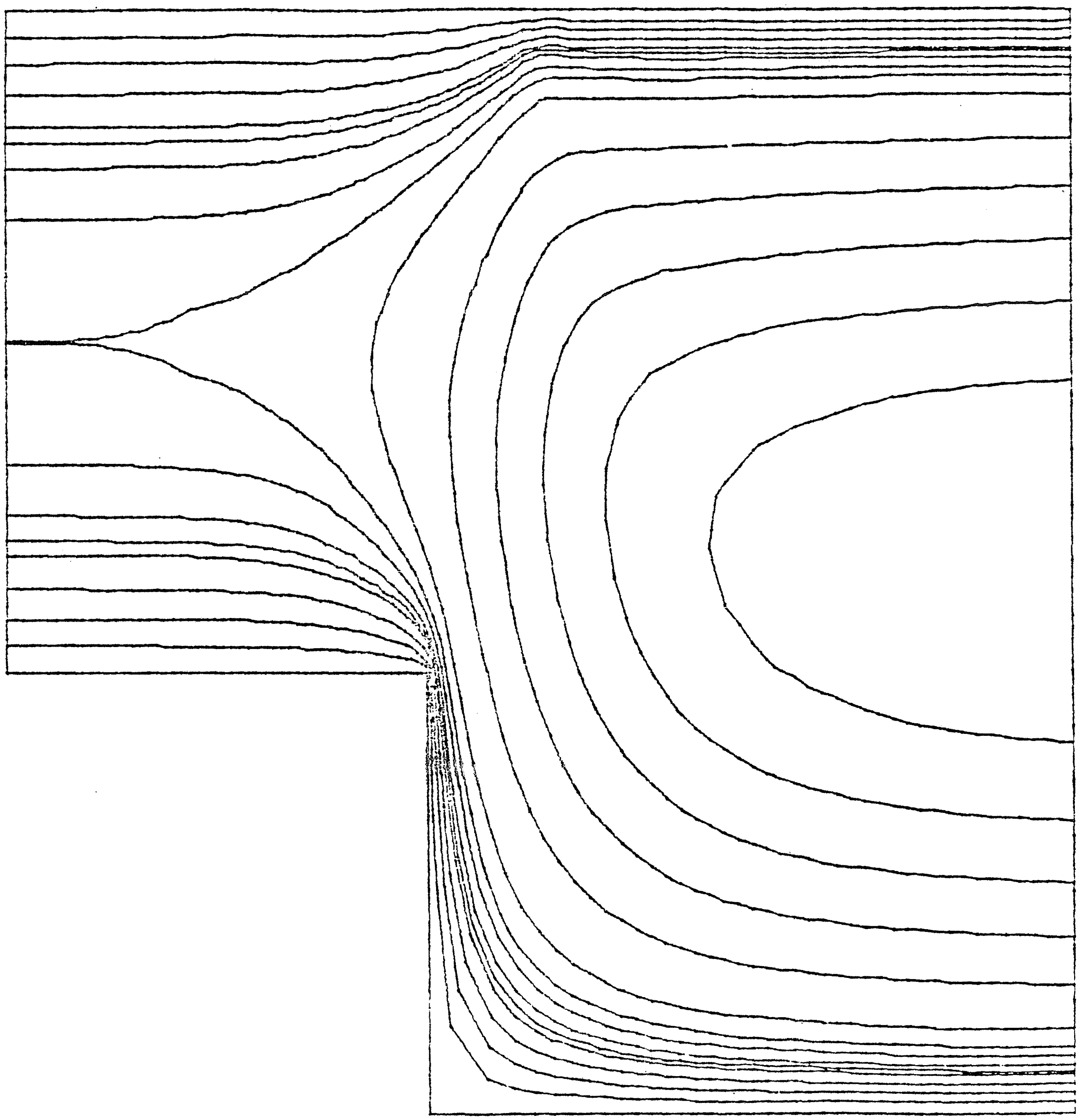


Figure (6.28)

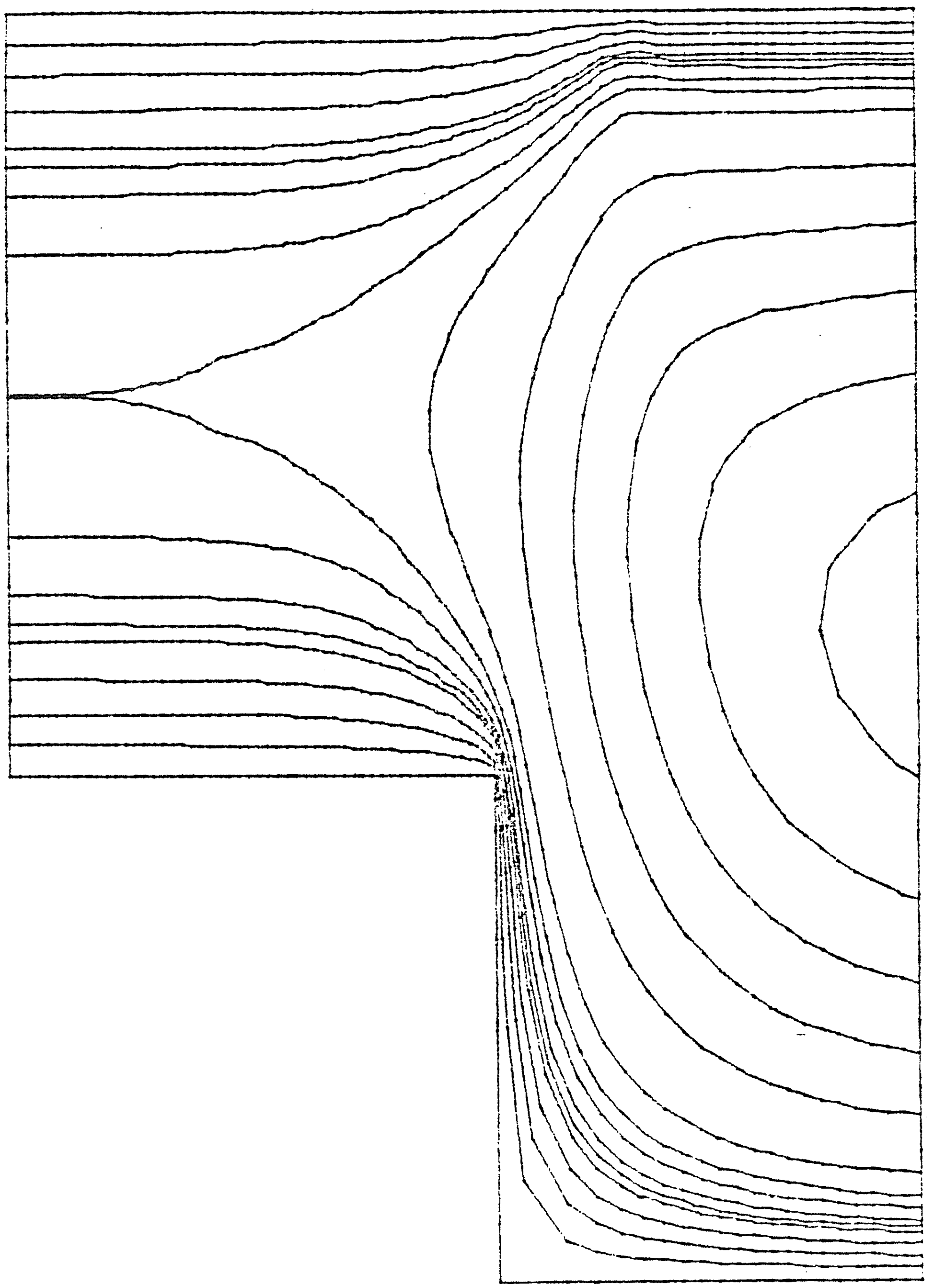


Figure (6.29)

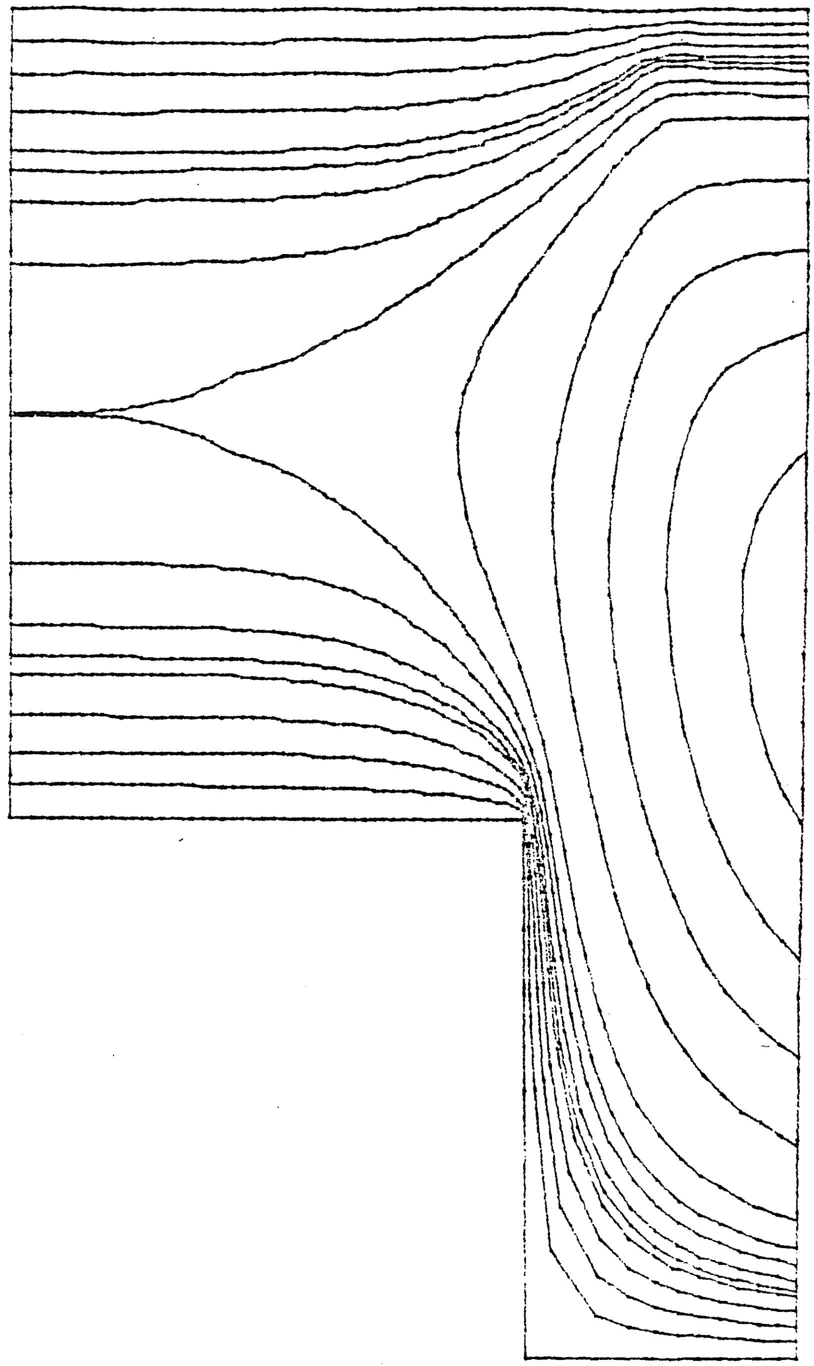


Figure (6.30)

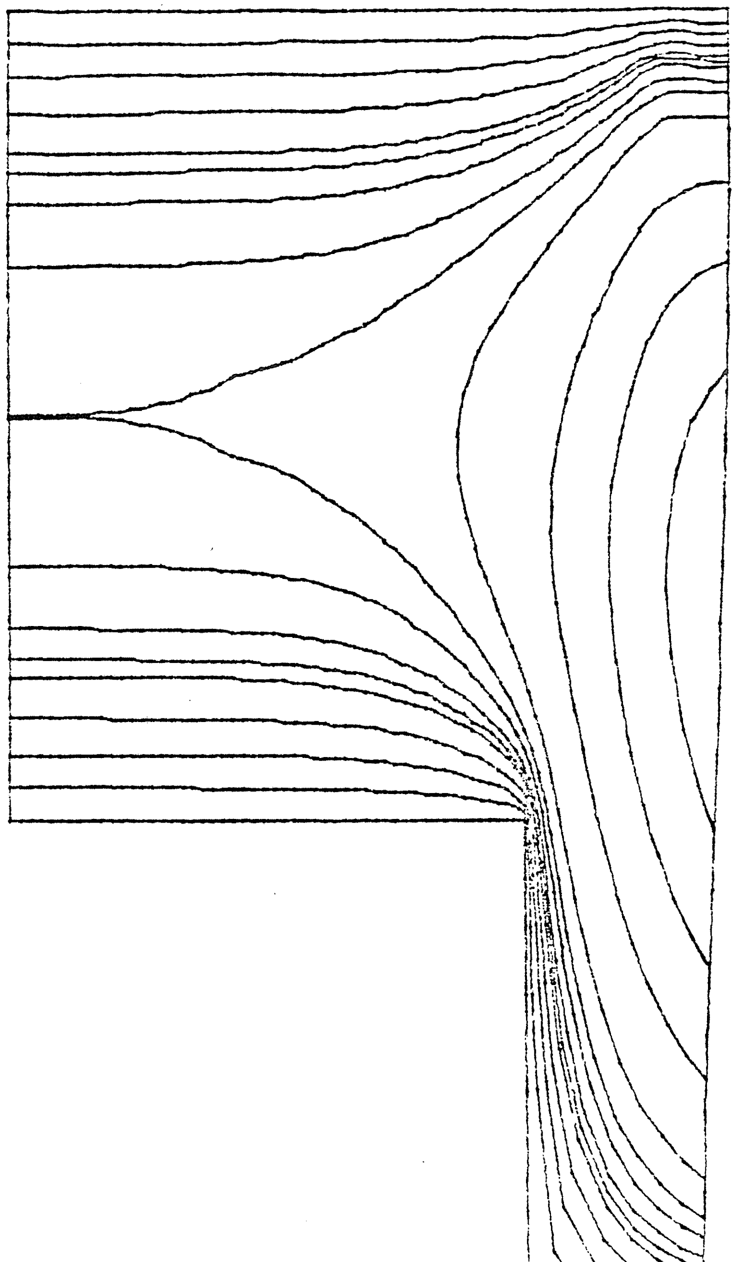


Figure (6.31)

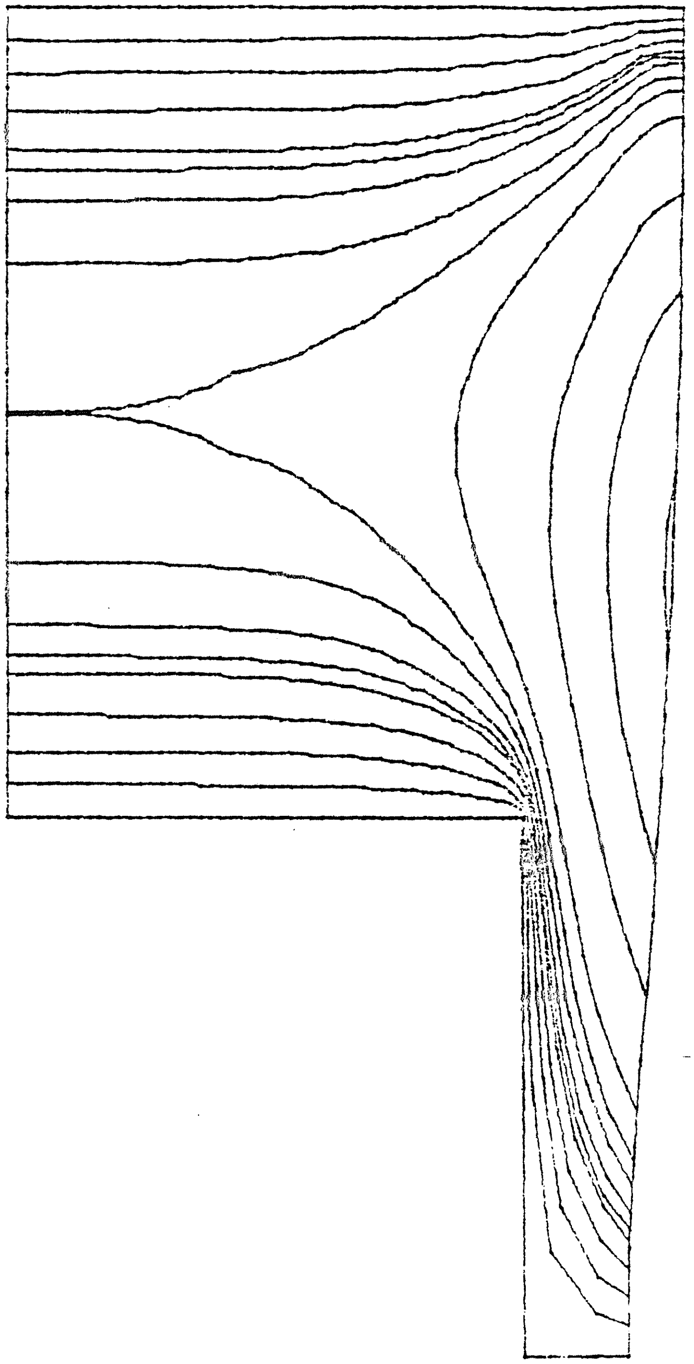


Figure (6.32)

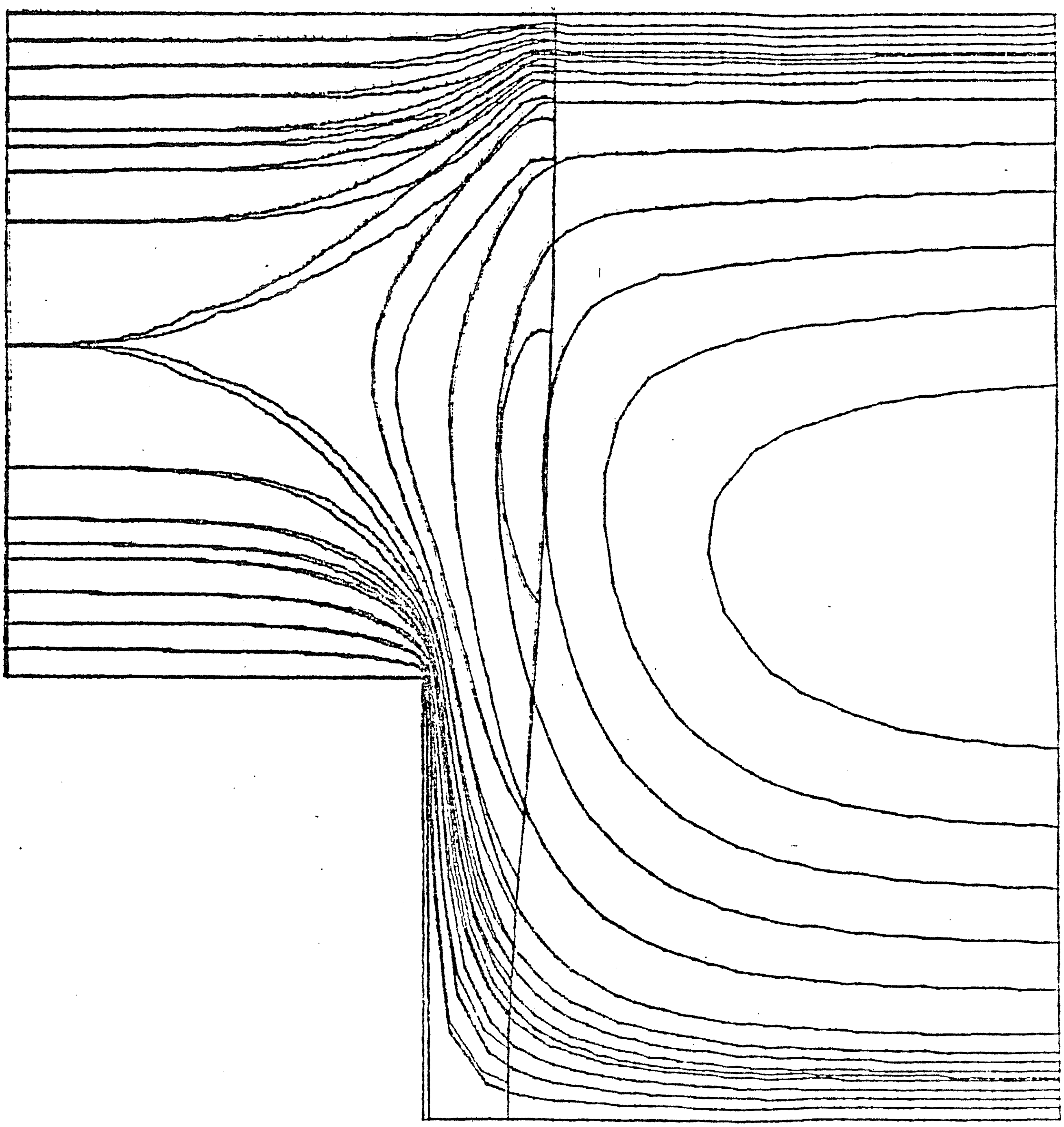


Figure (6.33)

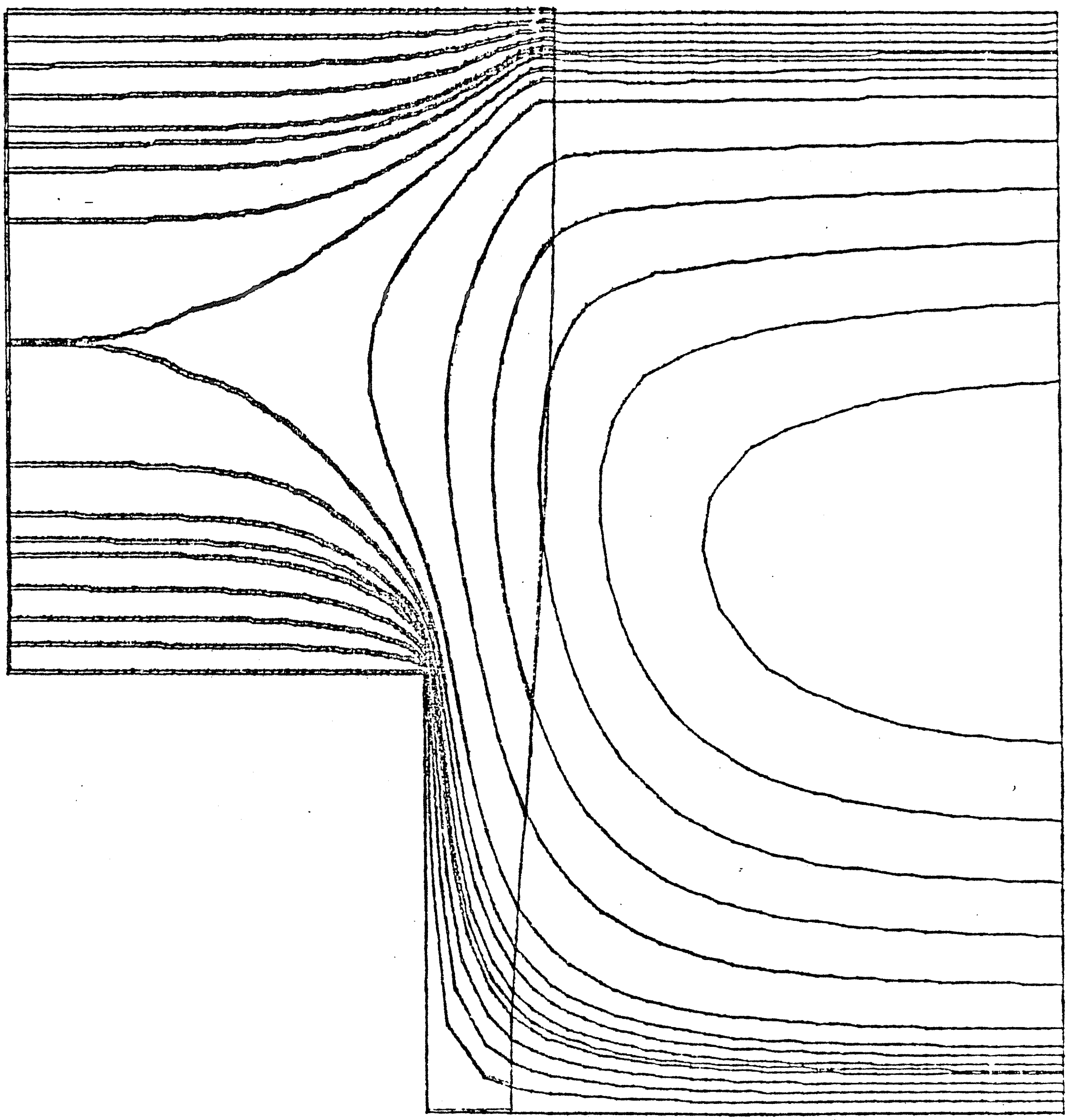


Table (6.34)

τ (mapped plane)	Order of Approximation	$\lvert u-u_\infty \rvert_{max}$ at the Boundary	Maximum Absolute Error
1.0	0	.076	.0743
1.0	1	.076	.0030
1.0	2	.076	.0015
0.5	0	.177	.1472
0.5	1	.177	.0082
0.5	2	.177	.0018
0.25	0	.269	.1936
0.25	1	.269	.0154
0.25	2	.269	.0051
0.1	0	.341	.2131
0.1	1	.341	.0213
0.1	2	.341	.0105

Figure (6.35)

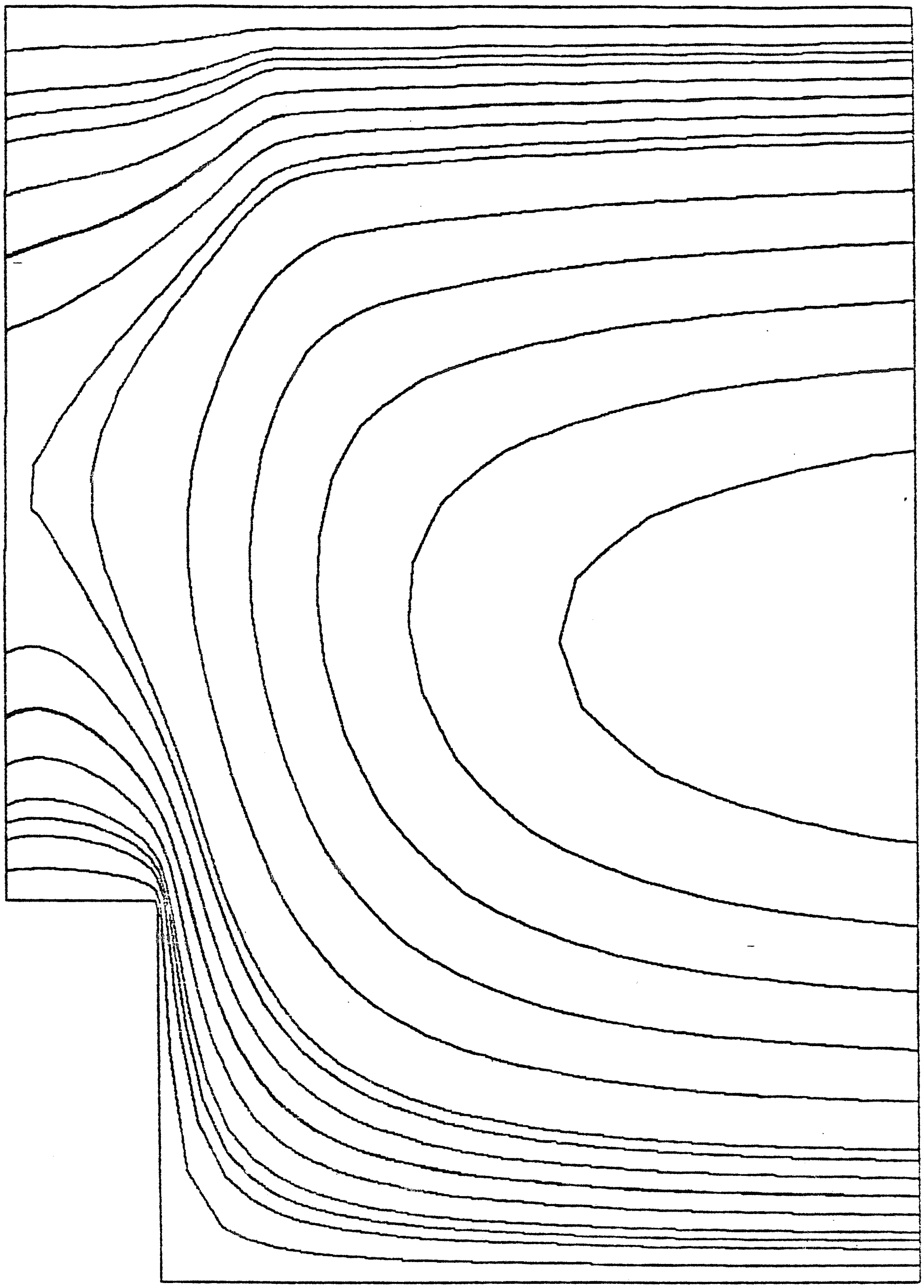


Figure (6.36)

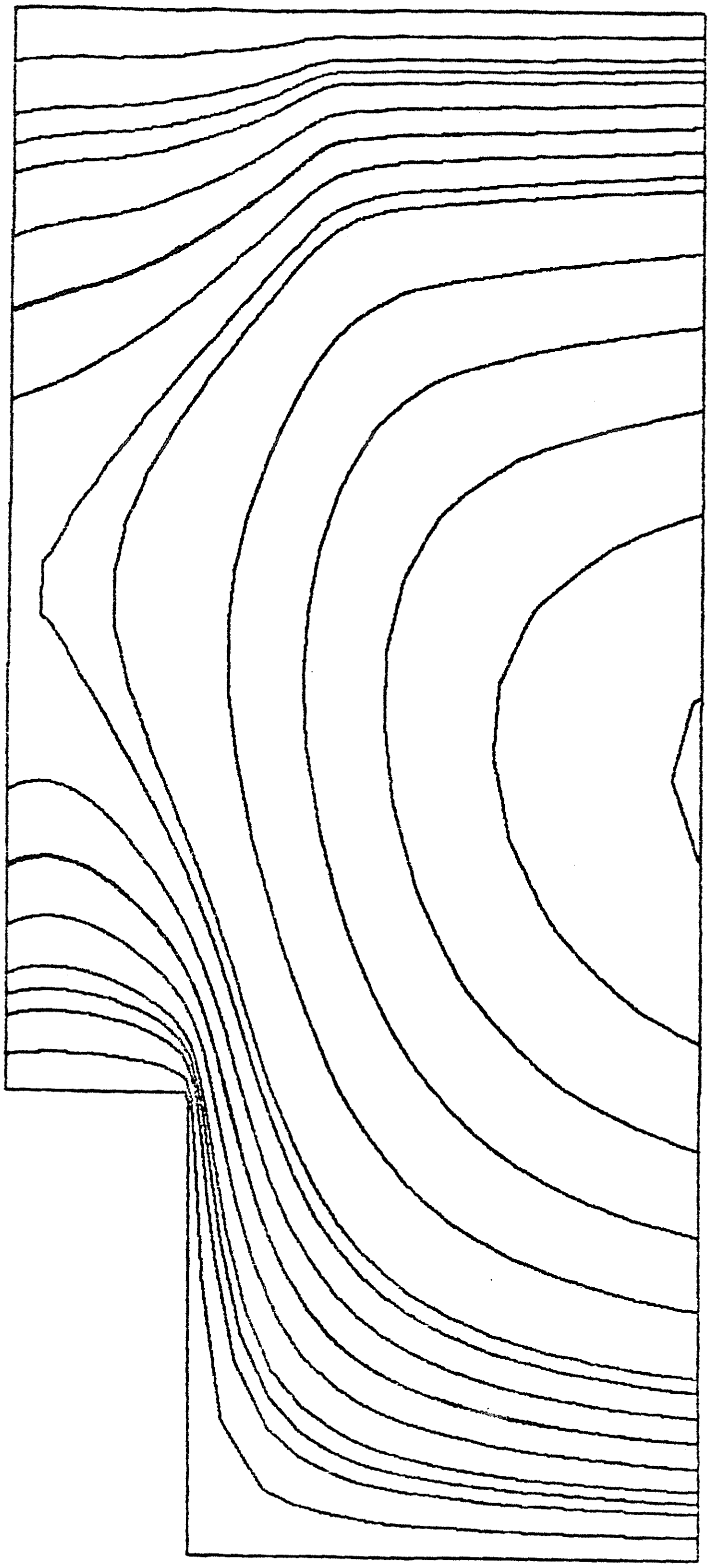


Figure (6.37)

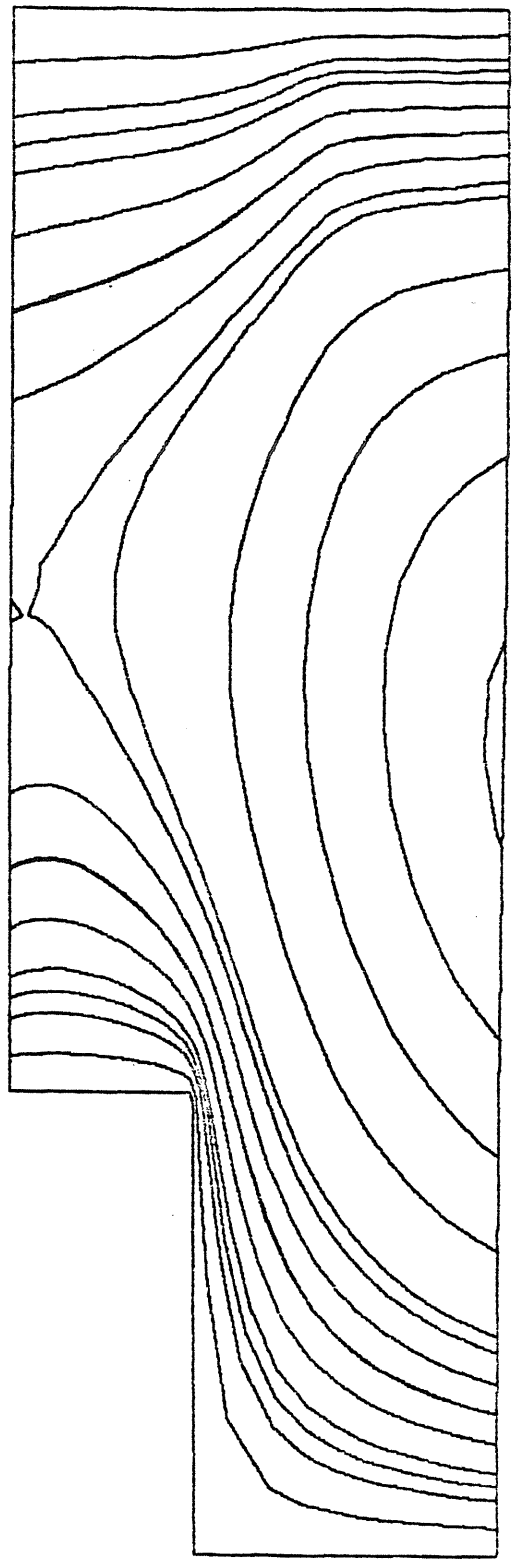


Figure (6.38)

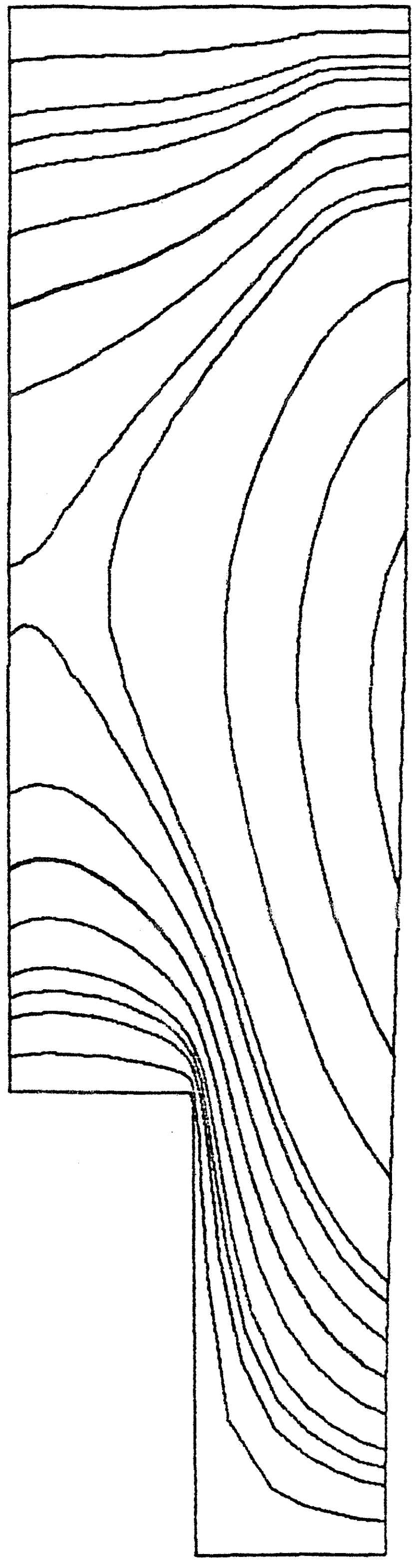


Figure (6.39)

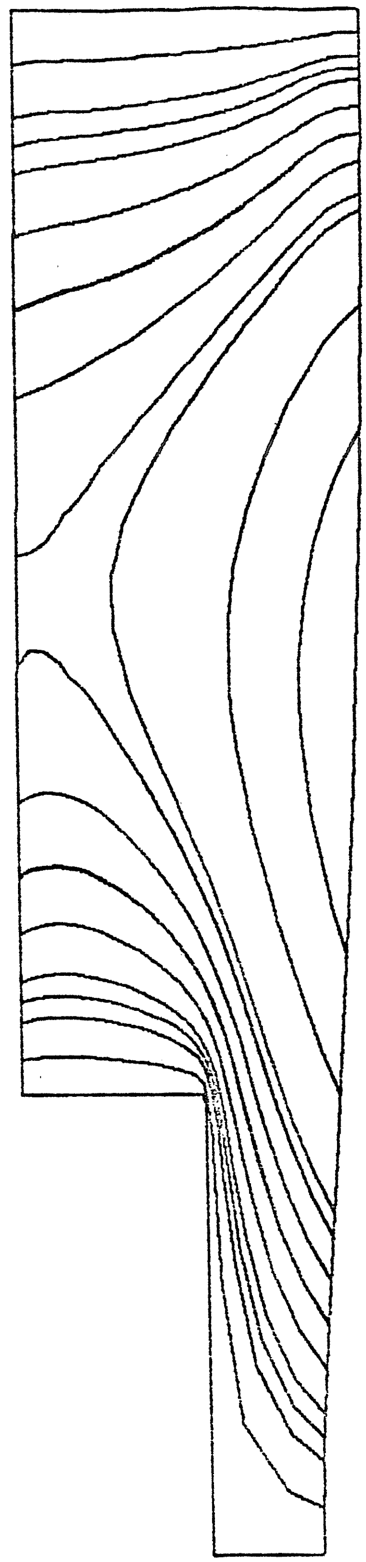


Figure (6.40)

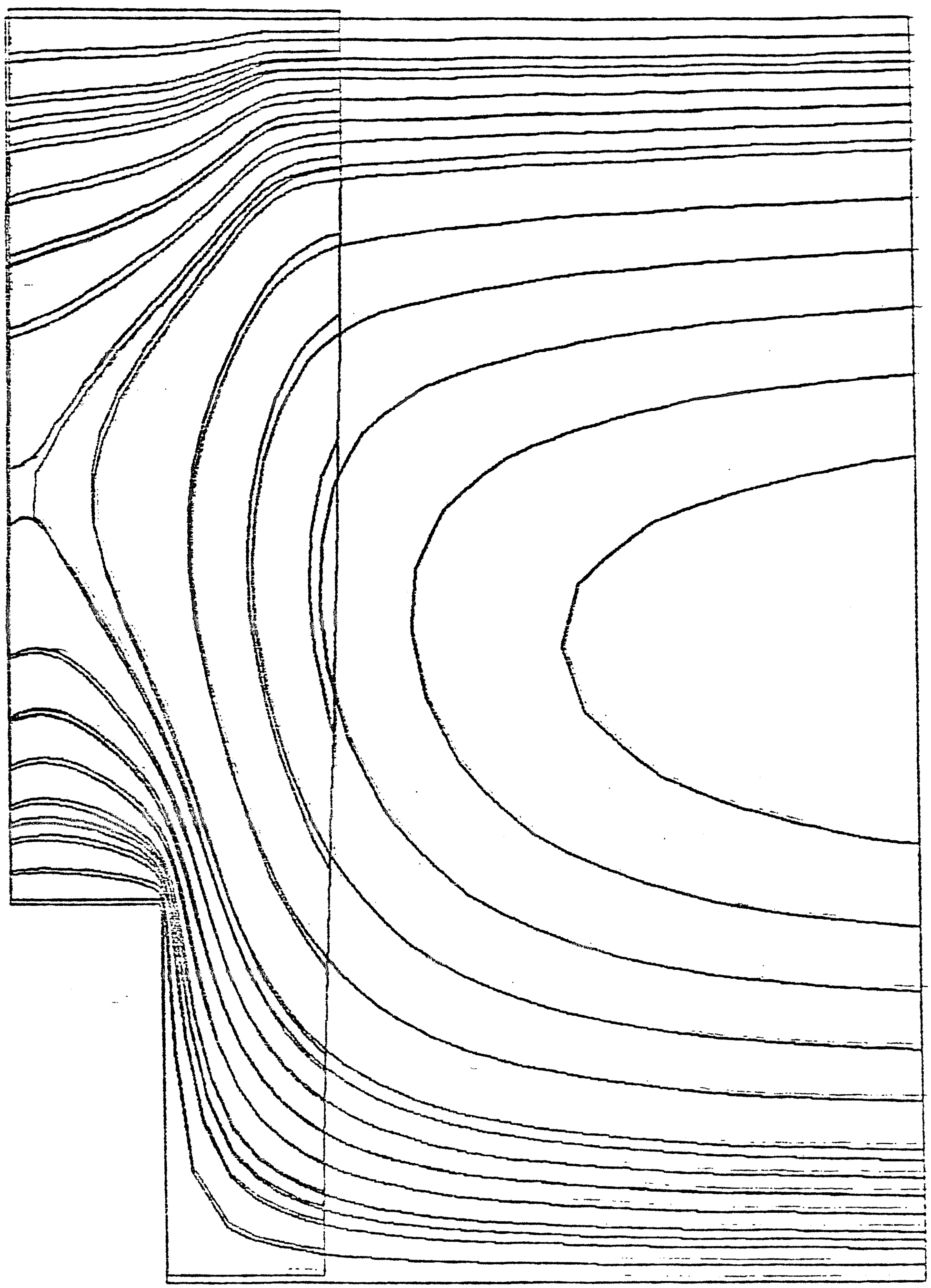


Figure (6.41)

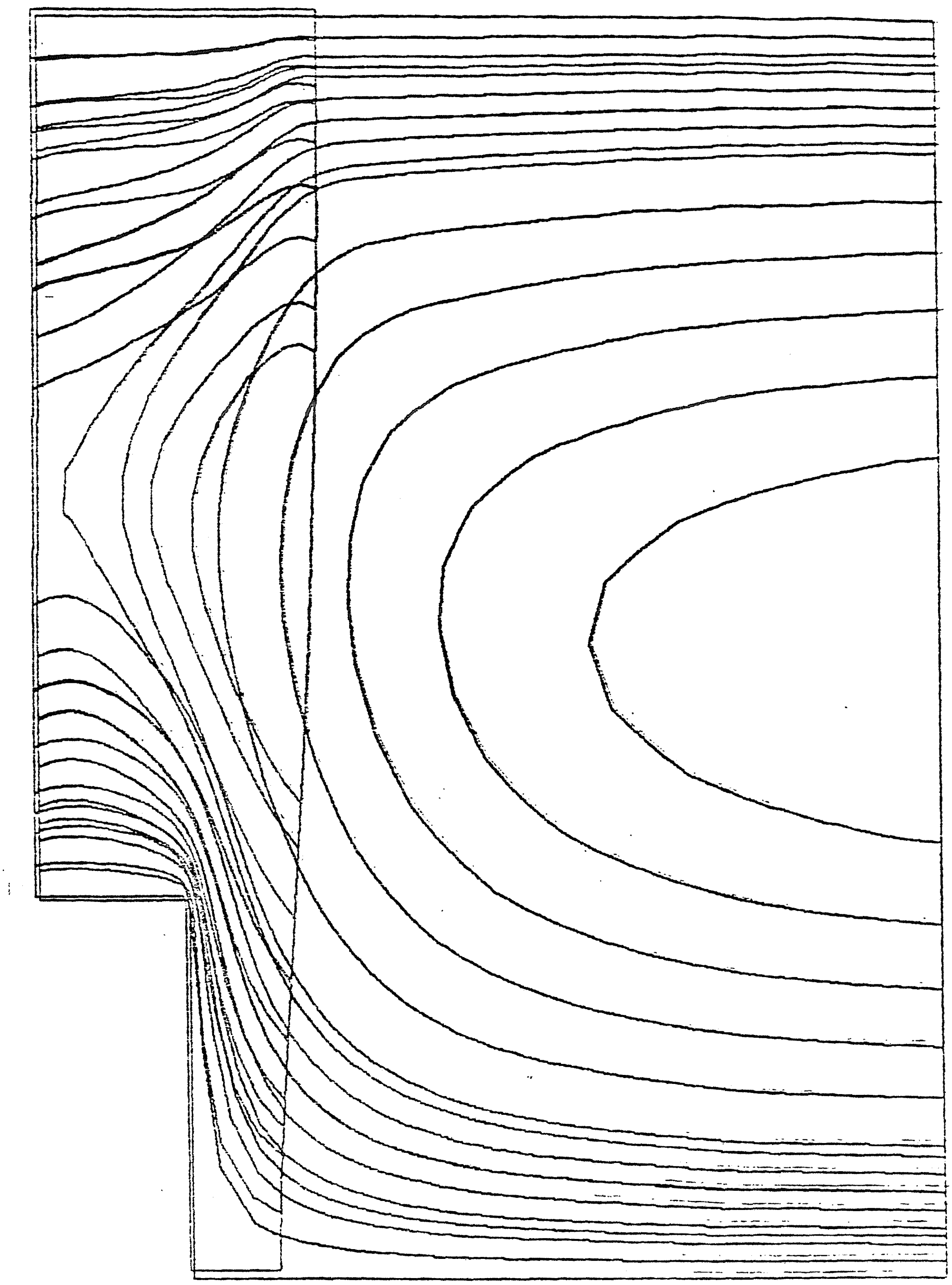


Table (6.42)

γ (mapped plane)	Order of Approximation	$\lvert u-u_{\infty}\rvert_{max}$ at the Boundary	Maximum Absolute Error
1.0	0	.106	.117
1.0	1	.106	.029
1.0	2	.106	.0018
0.5	0	.232	.183
0.5	1	.232	.059
0.5	2	.232	.014
0.25	0	.334	.224
0.25	1	.334	.083
0.25	2	.334	.027
0.1	0	.409	.242
0.1	1	.409	.097
0.1	2	.409	.039

Appendix A: Existence of Solutions for Some Second Order Problems

In this section we develop an alternative theory to the one presented in Section 6 for the following problem on the tail:

$$\begin{aligned} &a)\ u_{xx} + u_{yy} + a(y)u = f(u,y);\ \ x \in [\tau,\infty),\ y \in [0,1]; \\ &b)\ \alpha_{0,1}u_y + \beta_{0,1}u = 0,\ \ y = 0,1; \qquad\qquad \text{(A1)} \\ &c)\ \lim_{x\to\infty} u(x,y) = u_\infty(y). \end{aligned}$$

The function u_∞ satisfies:

$$\begin{aligned} &(u_\infty)_{yy} + a(y)u_\infty = f(u_\infty,y),\ \ y \in [0,1]; \\ &\alpha_{0,1}u_{\infty_y} + \beta_{0,1}u_\infty = 0,\ \ y = 0,1. \end{aligned} \qquad \text{(A2)}$$

We also assume that the eigenvalue problem, (6.19), has only positive eigenvalues and the eigenfunction expansion is as given there. We then have the following theorem.

Theorem A3

Suppose there exists a $K > 0$ such that, for d sufficiently small

$$f_{uu}(u_\infty + d,y) < K.$$

Then there exists $\varepsilon > 0$ such that for any sufficiently smooth $v_0(y)$ satisfying the boundary conditions (A1b) $\hat{v}_0(x,y)$, the bounded solution of the linear problem on the tail:

$$\hat{v}_{o\,xx} + \hat{v}_{o\,yy} + (a(y) - f_u(u_\infty, y))\hat{v}_o = 0;$$

$$\hat{v}_o(\tau, y) = v_o(y);$$

$$\alpha_{0,1}\hat{v}_{o\,y} + \beta_{0,1}\hat{v}_o = 0, \;\; y = 0,1;$$

also satisfying

$$\max_{\substack{x \in [\tau,\infty) \\ y \in [0,1]}} |\hat{v}_o| < \varepsilon,$$

there exists a solution, $u(x,y)$, to problem (A1). This solution satisfies:

$$a)\;\; u(\tau, y) = u_\infty(y) + v_o(y);$$

$$b)\;\; \int_0^1 dy\, \frac{\partial v}{\partial x}(\tau, y)\, Y_n(y) = -\sqrt{\alpha_n} \int_0^1 dy\, v(\tau, y)\, Y_n(y)$$

$$- \int_\tau^\infty ds\, e^{-\sqrt{\alpha_n}(s-\tau)} \int_0^1 dy\, R(u_\infty, v, y)\, Y_n(y), \;\; n = 1, \cdots ; \qquad \text{(A3)}$$

$$c)\;\; v(x,y) = u(x,y) - u_\infty(x,y);$$

$$d)\;\; R(u_\infty, v, y) = f(u_\infty + v, y) - f(u_\infty, y) - f_u(u_\infty, y)v.$$

The constant, ε, is defined by:

$$\varepsilon \equiv \frac{1}{2K_o K_1 K_2^2}; \;\; K_o \equiv \max_{\substack{|d| \le 3\varepsilon \\ y \in [0,1]}} |f_{uu}(d + u_\infty, y)|;$$

$$K_1 \equiv \sum_{n=1}^{\infty} \frac{1}{\alpha_n}; \;\; K_2 \equiv \max_{\substack{n \\ y \in [0,1]}} |Y_n(y)|. \qquad \text{(A4)}$$

Note that (A3b) is equivalent to the condition resulting from (6.8). The existence of finite K_1 and K_2 is guaranteed by the results of Sturm-Liouville theory; see, for example, Levitan and Sargsjan [34].

Proof:

We seek a solution of the form $v(x,y)=\hat{v}_o(x,y)+\hat{v}(x,y)$. We have that $\hat{v}_o(x,y)$ is given by:

$$\hat{v}_o(x,y)=\sum_{n=1}^{\infty} c_n(\tau)e^{-\sqrt{\alpha_n}(x-\tau)}Y_n(y);$$

$$\hat{v}_o(\tau,y)=\sum_{n=1}^{\infty} c_n(\tau)Y_n(y).$$

We represent $\hat{v}(x,y)$ in the form:

$$\hat{v}(x,y)=\sum_{n=1}^{\infty}\hat{c}_n(x)Y_n(y);$$

where the expansion coefficients must satisfy:

$$\begin{aligned} &a)\ \ \frac{d^2\hat{c}_n}{dx^2}-\alpha_n\hat{c}_n=\int_0^1 dy\ R(\hat{v}_o+\hat{v},y)Y_n(y); \\ &b)\ \ \hat{c}_n(\tau)=0. \end{aligned} \tag{A5}$$

We define the operator $F(\hat{v}(x,y))$ by the following:

$$F(\hat{v}(x,y))=\sum_{n=1}^{\infty}F_n(\hat{v}(x,y))Y_n(y);$$

$$F_n(\hat{v}(x,y)); =\frac{e^{\sqrt{\alpha_n}(x-\tau)}-e^{-\sqrt{\alpha_n}(x-\tau)}}{2\sqrt{\alpha_n}}\int_x^{\infty}ds\ e^{-\sqrt{\alpha_n}(s-\tau)}\int_0^1 dy\ R(\hat{v}_o+\hat{v},y)Y_n(y) \tag{A6}$$

$$+\frac{e^{-\sqrt{\alpha_n}(x-\tau)}}{2\sqrt{\alpha_n}}\int_{\tau}^{x}ds\ (e^{\sqrt{\alpha_n}(s-\tau)}-e^{-\sqrt{\alpha_n}(s-\tau)})\int_0^1 dy\ R(\hat{v}_o+\hat{v},y)Y_n(y).$$

We note that a fixed point of F is simply a Green's function solution of (A5). For

$\hat{v}(x,y)$ bounded in the maximum norm, we can estimate the expressions above:

$$\max_{x \in [\tau,\infty)} |F_n(\hat{v})| \leq \frac{K_2}{\alpha_n} \max_{\substack{x \in [\tau,\infty) \\ y \in [0,1]}} |R(\hat{v}_o + \hat{v};y)|;$$

$$\max_{x \in [\tau,\infty)} |F_n(\hat{v}_1) - F_n(\hat{v}_2)| \leq \frac{K_2}{\alpha_n} \max_{\substack{x \in [\tau,\infty) \\ y \in [0,1]}} |R(\hat{v}_o + \hat{v}_1,y) - R(\hat{v}_o + \hat{v}_2,y)|.$$

Similarly, from the definition of R we have:

$$\max_{x,y} |R(\hat{v}_o + \hat{v},y)| \leq \frac{K_o}{2} \max_{x,y} (\hat{v}_o + \hat{v})^2;$$

$$\max_{x,y} |R(\hat{v}_o + \hat{v}_1,y) - R(\hat{v}_o + \hat{v}_2,y)| \leq K_o \max_{x,y} |\frac{1}{2}(\hat{v}_1 + \hat{v}_2) + \hat{v}_o| \max_{x,y} |\hat{v}_1 - \hat{v}_2|.$$

Using these and supposing that we are restricting ourselves to the set of functions whose maximum is less than ε, the contraction requirement becomes:

$$\varepsilon < \frac{1}{2K_o K_1 K_2^2}.$$

Also, the condition that F maps functions of norm less than ε to other such functions is:

$$\varepsilon < \frac{1}{2K_o K_1 K_2^2}.$$

Hence, the condition on ε in (A4) suffices to guarantee the existence of a solution. The boundary condition (A3b) is obtained by differentiating the Green's function representation, which completes the proof.

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