DOUBLEWELL TUNNELING VIA THE FEYNMAN-KAC FORMULA

Thesis by

Askell Hardarson

In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1988

(Submitted September 14, 1987)

Acknowledgements

I am most grateful to my advisor, Barry Simon who suggested my thesis problem, for his encouragement, patience and great support.

I would like to thank W.A.J. Luxemburg for his support and very pleasant discussions and I wish to thank him and Barry for their enjoyable courses in functional and asymptotic analysis and in mathematical physics.

It is a pleasure to thank the IHES and David Ruelle for their hospitality during the months of the preparation of this thesis and to thank Tom Spencer and others for many helpful discussions over there.

I also would like to thank the Alfred P. Sloan Foundation for awarding me Dissertation Fellowship, 1986-1987.

Finally I would like to thank my friends, above all Matt Newlin, Pierre Baldi and Clemens Glaffig for keeping me company and to thank Lillian Chappelle, Valarie Steppes-Glisson and Nancy O'Connor for the expert typing of this thesis.

Abstract

We discuss asymptotics of the heat kernel $e^{-T\frac{H(\lambda)}{\lambda}}(x,y)$ and its x-derivatives when T, $\lambda \to \infty$ and $\frac{T}{\lambda} \to 0$ where $H(\lambda) = -\frac{\Delta}{2} + \lambda^2 V$ and where V is a double well. When the groundstate is localized in both wells for λ large we derive, by the Feynman-Kac formula, W.K.B. expansions of the groundstate, the first excited state and their gradients.

As a consequence we get a general asymptotic formula for the splitting of the two lowest eigenvalues, $E_0(\lambda)$ and $E_1(\lambda)$.

This formula allows us, in principle, always to go beyond the leading order given by $\frac{\log(E_1(\lambda)-E_0(\lambda))}{\lambda} \to -C$ where C is the action of a classical instanton.

Table of Contents

1. Introduction.

1. Introduction.

§1.1. The statement of the problem.

Let $H(\lambda) = -\frac{\Delta}{2} + \lambda^2 V$ on $L^2(\mathbb{R}^d)$ d = 1, 2, 3, ... with $\lambda > 0$ where $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian and V is a double well defined by:

(1.1) $V \in C^\infty(\mathbb{R}^d, \mathbb{R})$, $V \geq 0$, $V(x) = 0 \Leftrightarrow x \in \{a, b\}$

where $a \neq b$, the Hessian matrices $V''(a)$

and $V''(b)$ are strictly positive and $\varliminf_{|x| \to \infty} V(x) > 0$

As λ grows $H(\lambda)$ looks more and more like that if V is replaced by the sum of its quadratic approximations at a and b. More precisely (Simon [1]): Fix an integer n. Then for λ large enough $H(\lambda)$ has at least (n+1) eigenvalues $0 < E_0(\lambda) \leq E_1(\lambda) \leq \cdots \leq E_n(\lambda)$ below its continuous spectrum and

(1.2) $\frac{E_n(\lambda)}{\lambda} \to e_n$ where $0 < e_0 \leq e_1 \leq \cdots$ are the eigenvalues of $h(a) \oplus h(b)$

with

(1.3) $h(c) := -\frac{\Delta}{2} + \frac{1}{2}\langle V''(c)\, x, x\rangle$ on $L^2(\mathbb{R}^d)$, for $c \in \{a, b\}$.

For $E_n(\lambda)$, as above, we let $\Omega_n(x,\lambda)$ denote some corresponding eigenfunction with

$$\|\Omega_n\|_2 = 1 + 0(\lambda^{-\frac{1}{2}}) \text{ as } \lambda \to \infty .$$

The classical example of a double well is the symmetric double well in one dimension: $V(x) = \frac{1}{2}(x+\frac{1}{2})^2(x-\frac{1}{2})^2$ (e.g., Landau-Lipschitz [1], Kac-Thompson [1], Reed-Simon [1] and Harrell [1]). Here V is even, the eigenfunctions Ω_{2n}'s are even and Ω_{2n+1}'s are odd, for n = 0, 1, ... Moreover, up to a sign, (1.4) $$\Omega_0(x,\lambda) = \frac{\lambda^{\frac{1}{4}}}{\sqrt{2}}\,\varphi_0(\lambda^{\frac{1}{2}}(x+\tfrac{1}{2})) + \frac{\lambda^{\frac{1}{4}}}{\sqrt{2}}\,\varphi_0(\lambda^{\frac{1}{2}}(x-\tfrac{1}{2})) + r_0(x\,,\,\lambda)$$

$$\Omega_1(x,\lambda) - \frac{\lambda^{\frac{1}{4}}}{\sqrt{2}}\,\varphi_0(\lambda^{\frac{1}{2}}(x+\tfrac{1}{2})) + \frac{\lambda^{\frac{1}{4}}}{\sqrt{2}}\,\varphi_0(\lambda^{\frac{1}{2}}(x-\tfrac{1}{2})) + r_1(x\,,\,\lambda)$$

where $\|r_i(\cdot\,,\,\lambda)\|_2 = 0(\lambda^{-\frac{1}{2}})$ as $\lambda \to \infty$ for $i \in \{0\,,\,1\}$. φ_0 is a normalized groundstate of $h(a) = h(b)$, defined by (1.3) with $a = -\frac{1}{2}$ and $b = \frac{1}{2}$.

Hence if we define, for some small $\varepsilon > 0$ and $c \in \{a\,,\,b\}$,

(1.5) $$j_c(x) := \begin{cases} 1 & \text{if } |x-c| \le \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

we have $\|j_a\Omega_0\|_2\ \|j_b\Omega_0\| = \frac{1}{2} + 0(\lambda^{-\frac{1}{2}})$ as $\lambda \to \infty$,

which says the groundstate is living in both wells as $\lambda \to \infty$.

Similarly, in this L^2-biology, $(\Omega_0+\Omega_1)$ lives near $a = -\frac{1}{2}$ and $\Omega_0 - \Omega_1$ near $b = \frac{1}{2}$.

Applying the time evolution $e^{-itH(\lambda)}$ to $(\Omega_0+\Omega_1)$ we get

$$e^{-itH(\lambda)}(\Omega_0+\Omega_1) = e^{-itE_0(\lambda)}(\Omega_0+e^{-it(E_1(\lambda)-E_0(\lambda))}\Omega_1),$$

so after time $t = {}^{\pi}/(E_1(\lambda) - E_0(\lambda))$ a state living near $a = -\frac{1}{2}$ has evolved to a state living near $b = \frac{1}{2}$. Hence the splitting $E_1(\lambda) - E_0(\lambda)$ tells how long it takes to tunnel through the classically forbidden region between the wells.

Since "Every child knows that the amptitude for transmission obeys the W.K.B. formula" (Coleman [1], p. 806-807) we expect $E_1(\lambda) - E_0(\lambda)$ to be of the order $\exp\left(-\lambda \int_{x_1(\lambda)}^{x_2(\lambda)} \sqrt{2(V(x) - \frac{E_0(\lambda)}{\lambda^2})}\, dx\right)$ where $a = -\frac{1}{2} \leq x_1(\lambda) < x_2(\lambda) \leq \frac{1}{2} = b$ are the classical turning points at energy $\frac{E_0(\lambda)}{\lambda^2}$. This suggests

$$\frac{\log(E_1(\lambda)-E_0(\lambda))}{\lambda} \to -\int_a^b \sqrt{2V(x)}\, dx \text{ as } \lambda \to \infty$$

which follows from:

<u>Theorem</u> (Simon [2]). If $\varliminf_{\lambda\to\infty} \|j_a\Omega_0\|_2 \, \|j_b\Omega_0\| > 0$ then

$$\lim_{\lambda\to\infty} \frac{\log(E_1(\lambda)-E_0(\lambda))}{\lambda} = -\rho(a,b) \tag{1.7}$$

where, for $x, y \in \mathbb{R}^d$,

$$\rho(x\,,\,y) := \inf\left\{\int_0^1 \sqrt{2V(\gamma(s))}\, |\dot\gamma(s)| ds \colon \gamma(0) = x \text{ and } \gamma(1) = y\right\} \tag{1.8}$$

is the Agmon distance between x and y.

Note if $d = 1$ and $a < b$ then $\rho(a,b) = \int_a^b \sqrt{2V(x)}\, dx$.

This theorem was originally proven using Feynman-Kac formula and path integrals and the proof allows a weaker hypothesis:

$$\begin{cases} \|j_a\Omega_0\|_2 \, \|j_b\Omega_0\|_2 \geq \text{constant } \lambda^{-m} > 0 \\ \qquad\qquad \text{for some } m \geq 0 \end{cases} \tag{1.9}$$

Assuming (1.9) *we want to go beyond the leading order in* (1.7) *using path integrals.* When there are finitely many Agmon geodesics between the wells, satisfying a nondegeneracy condition we get (see (1.56) below)

$$E_1(\lambda) - E_0(\lambda) = c_0 \lambda^{\frac{3}{2}} e^{-\lambda\rho(a,b)} (1 + \frac{c_1}{\lambda} + 0(\lambda^{-\frac{3}{2}}))$$

$$\text{as } \lambda \to \infty .$$

In this case Helffer and Sjöstrand [1] have expansions to all orders in $\frac{1}{\lambda}$. One can obtain this much more precise result by using our method, but now we are mainly concerned with error terms.

We give a more general formula for $E_1(\lambda) - E_0(\lambda)$ (see (1.50) below) which also follows from the discussion before Theorem 6.6 in Helffer and Sjöstrand [1]. What is new here is the method. We make asymptotic expansions of the heatkernel $e^{-T\frac{H(\lambda)}{\lambda}}$ (x,y) and its x-derivatives at T, $\lambda \to \infty$, $T^4 \leq \lambda$ and we use those to get the asymptotics for Ω_0 , Ω_1 and their derivatives that we will use, as we describe now.

§1.2. <u>Reduction to asymptotics of eigenfunctions and their gradients</u>.

Following Harrell [2] we use

$$(1.10) \quad E_1(\lambda) - E_0(\lambda) = \frac{\int_{\partial w} \left(\Omega_1 \frac{\partial\Omega_0}{\partial n} - \Omega_0 \frac{\partial\Omega_1}{\partial n}\right) dS(x)}{2 \int_w \Omega_0\Omega_1 \, dx}$$

where w is basically (explained later) $\{x : \rho(x,b) < \rho(x,a)\}$ and here n is the outward normal. Now the problem reduces to find asymptotics of Ω_0 , Ω_1 and their gradients as $\lambda \to \infty$.

§1.3. L^2-asymptotics, from Perturbation theory, suffice near the bottom of the wells.

To find asymptotics of $\int_w \Omega_0(x,\lambda)\ \Omega_1(x,\lambda)\ dx$ as $\lambda \to \infty$ for any w, that contains a neighborhood of b and does not intersect some neighborhood of a, it suffices to have L^2-asymptotic expansions of Ω and Ω_1 from perturbation theory. From Simon [1], when V is polynomially bounded and more generally from Helffer and Sjöstrand [1] (see also Combes-Duclos and Seiler [1]) follows:

There are functions α , $\beta : (0,\infty) \to (0,1)$ with

(1.11) $\alpha^2(\lambda) + \beta^2(\lambda) = 1$

and $\alpha(\lambda) = \|j_a \Omega_0\|_2\ (1+0(\lambda^{-\frac{1}{2}}))$, $\beta(\lambda) = \|j_b \Omega_0\|(1+0(\lambda^{-\frac{1}{2}}))$

so under assumption (1.9) we have

(1.12) $\alpha(\lambda)\ \beta(\lambda) \geq c_1\ \lambda^{-m} > 0$ for λ large.

Moreover, for $c \in \{a,b\}$ and any integer $k \geq 1$ there are polynomials $P_{1,c}$, $P_{2,c}$, ... , $P_{k,c}$, where $P_{1,c}$ is odd, such that:

(1) $$\Omega_0(x,\lambda) = \alpha(\lambda)(\psi_{k,a}(x,\lambda) + r_{k,a}(x,y)) + \beta(\lambda)(\psi_{k,b}(x,\lambda) + r_{k,b}(x,\lambda))$$

and

(2) $$\Omega_1(x,\lambda) = -\beta(\lambda)(\psi_{k,a}(x,\lambda) + r_{k,a}(x,\lambda)) + \alpha(\lambda)(\psi_{k,b}(x,\lambda) + r_{k,b}(x,\lambda))$$

where

$$(1.13)\quad \psi_{k,c}(x,\lambda) = \lambda^{\frac{d}{4}} \varphi_c(\lambda^{\frac{1}{2}}(x-c))$$

$$\left(1 + \frac{P_{1,c}(\lambda^{\frac{1}{2}}(x-c))}{\lambda^{\frac{1}{2}}} + \cdots + \frac{P_{k,c}(\lambda^{\frac{1}{2}}(x-c))}{\lambda^{\frac{k}{2}}}\right).$$

Here φ_c is a normalized (Gaussian) groundstate of $h(c) = -\frac{\Delta}{2} + \frac{1}{2}\langle V''(c)\, x,x\rangle$

for $c \in \{a,b\}$ and the errors satisfy $\|r_{k,c}(\cdot, \lambda)\|_2 = 0\left(\lambda^{-\frac{(k+1)}{2}}\right)$ as $\lambda \to \infty$.

Those expansions imply the Rayleigh-Schrödinger series (Reed-Simon [1]) for $E_1(\lambda)$ and $E_0(\lambda)$ are the same and are asymptotic series (Simon [1]): There are constants $\{e_0^{(n)}\}_{n=0}^{\infty}$ such that for any $\ell \geq 0$

$$(1.14)\quad \frac{E_1(\lambda)}{\lambda} = e_0^{(0)} + \frac{e_0^{(1)}}{\lambda} + \cdots + \frac{e_0^{(\ell-1)}}{\lambda^{(\ell-1)}} + 0\left(\frac{1}{\lambda^{\ell}}\right) = \frac{E_0(\lambda)}{\lambda} \quad \text{as } \lambda \to \infty .$$

By (1.2) $e_0^{(0)} = e_0 = e_1$ is the groundstate energy of h(a) and of h(b).

If (1.13) holds with $k = 2$ we get, since $P_{1,c}$ is odd,

$$(1.15)\quad \int_W \Omega_0\, \Omega_1\, dx = \alpha(\lambda)\, \beta(\lambda)\left(1 + \frac{\text{constant}}{\lambda} + 0\left(\lambda^{-\frac{3}{2}}\right)\right) \text{ for any w as above.}$$

From now on we will assume:

There are functions α , β : $(0,\infty) \rightarrow (0,1)$ satisfying (1.11) and (1.12). With m as in (1.12) there are polynomials $P_{1,c}, \dots, P_{2m+2,c}$

(∗) for $c \in \{a,b\}$, where $P_{1,c}$ is odd, so that (1.13) holds with $k = 2m + 2$. Moreover (1.14) holds for $\ell = 2$.

Now we can concentrate on finding asymptotics of "$\int_{\partial w}$" in (1.10).

§1.4. General upperbounds for eigenfunction and their gradients.

Upper bounds for $|\Omega_n(x,\lambda)|$ uniformly on compacts (u.o.c.) give upper bounds (Simon [6], Theorem C.2.5) for $|\nabla\Omega_n(x,\lambda)|$ u.o.c., see also Remark (9.5). To get upper bounds for $|\Omega_n(x,\lambda)|$ u.o.c. (or at infinity) one can use (Simon [2], Proposition 3.1)

(1.16) $$e^{-T\frac{E_n(\lambda)}{\lambda}} |\Omega_n(x,\lambda)|^2 \leq e^{-T\frac{H(\lambda)}{\lambda}}(x,x) .$$

The Feynman-Kac formula (see §5.1) and large deviations (§5.2), yield:

Theorem (Simon [2]).

(1.17) $$\lim_{\lambda\to\infty} \frac{\log e^{-T\frac{H(\lambda)}{\lambda}}(x,y)}{\lambda} = -A(x,y,T) \quad \text{u.o.c. in } \mathbb{R}^d \times \mathbb{R}^d \times (0,\infty) \text{ where}$$

(1.18) $$A(x,y,T) = \inf\{\int_0^T (\tfrac{1}{2}\dot{\gamma}^2 + V(\gamma))\, dt \mid \gamma(0) = x \text{ and } \gamma(T) = y\}$$

is the classical action for $-V$.

(1.17) together with (1.2) and

(1.19) $$A(x,y,T) \rightarrow \min\{\rho(x,a) + \rho(a,y), \rho(x,b) + \rho(b,y)\} \text{ u.o.c. in } \mathbb{R}^d \times \mathbb{R}^d \text{ as}$$

$T \rightarrow \infty$ gives the first part of

Theorem (Simon [2]).

(1) For each n

$$\overline{\lim_{\lambda\to\infty}} \frac{\log|\Omega_n(x,\lambda)|}{\lambda} \leq -\min\{\rho(x,a), \rho(x,b)\} \text{ and} \tag{1.20}$$

$$\overline{\lim_{\lambda\to\infty}} \frac{\log|\nabla\Omega_n(x,\lambda|}{\lambda} \leq -\min\{\rho(x,a), \rho(x,b)\} \qquad \text{u.o.c.}$$

(2) There is $R > 0$ s.t. for each n there are c_2 and $c_3 > 0$ with

$$|\Omega_n(x,\lambda)| < c_3\, e^{-\lambda c_2|x|} \text{ if } |x| \geq R \text{ and } \lambda \geq \text{some } \lambda_0(n) . \tag{1.21}$$

§1.5. Finer analysis near K, the set of midpoints of Agmon geodesics between and and b.

Thinking of ∂w as $\{x : \rho(x,a) = \rho(x,b)\}$ we see from (1.20) that the main contribution in "$\int_{\partial w}$" in (1.10) is from a neighborhood of the set of midpoints of Agmon geodesics between a and b, given by

$$K = \{x : \rho(x,a) = \rho(x,b) = \frac{\rho(a,b)}{2}\} . \tag{1.22}$$

By (1.15) going beyond the leading order in (1.10) reduces to go beyond the leading order in (1.20) near K.

To make sense out of this we need (Carmona-Simon [1])

$$\rho(x,y) = \inf_{T>0} A(x,y,T) . \tag{1.23}$$

For (1.23) to attain minimum we may have to take $T \to \infty$. If $x \notin \{a,b\}$ and $c \in \{a,b\}$ then (Simon [2])

(1.24) $\rho(x,c) = \inf\{\int_0^\infty (\frac{1}{2}\dot\gamma^2 + V(\gamma))dt : \gamma(0) = x, \gamma(\infty) = c\}$

and

(1.25) $\rho(a,b) = \inf\{\int_{-\infty}^\infty (\frac{1}{2}\dot\gamma^2 + V(\gamma)dt) : \gamma(-\infty) = a, \gamma(\infty) = b\}$

We will denote by $g^{x,b}$ the minimizing paths of (1.24). The minimizing paths for (1.25) are called instantons (Coleman [1]). If g minimizes (1.25) then so do all time translates of g. For $x \in K = \{x : \rho(x,c) = \rho(x,b) = \frac{\rho(a,b)}{2}\}$ we let g_x be the instanton with $g_x(0) = x$. The g_x's are reparametrizations on $[-\infty,\infty]$ of the Agmon geodesics between a and b minimizing (1.8), that are parametrized with respect to arclength.

Moreover,

Lemma 1. K is compact and has a neighborhood U such that

(1.27) For $x \in U$ and $c \in \{a,b\}$ that is a unique $g^{x,c}$ minimizing (1.24), the second variation (see (2.11)) of it is positive definite.

Moreover

(1.28) $\rho(x,a)$ and $\rho(x,b)$ are C^∞ on U and the gradient $(\rho'(x,a) - \rho'(x,b)) \neq 0$ on U.

For $x \in K$ $(\rho'(x,a)-\rho'(x,b)) = 2\dot g_x(0) = \frac{2d}{dt} g_x(t)|_{t=0}$.

(Proof in Chapter 13.)

By (1.28)

$Z := \{x : \rho(x,a) = \rho(x,b)\} \cap U$ defines a hypersurface containing K and for $x \in K$ Z is transversal to g_x , at $x = g_x(0)$.

If w is a bounded open set with smooth boundary such that $\partial w \cap U \subset Z$ and $\partial w \setminus U \subseteq \{x : \rho(x,a) \geq \frac{\rho(a,b)}{2} + \varepsilon_0 , \rho(x,b) \geq \frac{\rho(a,b)}{2} + \varepsilon_0\}$, for some $\varepsilon_0 > 0$, then by (1.20)

$$(1.30)\quad \int_{\partial w} \left(\Omega_1 \frac{\partial\Omega_0}{\partial n} - \Omega_0 \frac{\partial\Omega_1}{\partial n}\right) ds(x) = \int_Z \left(\Omega_1 \frac{\partial\Omega_0}{\partial n} - \Omega_0 \frac{\partial\Omega_1}{\partial n}\right) ds(x)$$

$$+0\left(e^{-\lambda(\rho(a,b)+\varepsilon_0)}\right).$$

To obtain refinements of (1.20) near K we use

$$(1.31)\quad \Omega_i(x,\lambda) = e^{T\frac{E_i(\lambda)}{\lambda}} \int e^{-T\frac{H(\lambda)}{\lambda}} (x,y)\, \Omega_i(y,\lambda)\, dy$$

and

$$(1.32)\quad \frac{\partial}{\partial x_j} \Omega_i(x,\lambda) = e^{T\frac{E_i(\lambda)}{\lambda}} \int \frac{\partial}{\partial x_j} e^{-T\frac{H(\lambda)}{\lambda}} (x,y)\, \Omega_i(y,\lambda)\, dy$$

if $i \in \{0,1\}$, $j \in \{1,\dots,d\}$ and make asymptotic expansions

of $e^{-T\frac{H(\lambda)}{\lambda}}(x,y)$ and of $\frac{\partial}{\partial x_j} e^{-T\frac{H(\lambda)}{(\lambda)}}(x,y)$

(refinements of (1.17)):

§1.6. Heat kernel asymptotics.

Theorem A. Let V be a double well and assume $|V'(x)| = 0(e^{A|x|^2})$ for statements about derivatives. If $x_0 \in K = \{x : \rho(x,a) = \rho(x,b) = \frac{\rho(a,b)}{2}\}$ then there exists $\delta > 0$ and,

1. There is $\gamma > 0$ such that if $\bar{\delta} \in (0,\delta]$ then

$$|\frac{1}{\lambda}\frac{\partial}{\partial x_j} e^{-T\frac{H(\lambda)}{\lambda}}(x,y))|,\ e^{-T\frac{H(\lambda)}{\lambda}}(x,y) =$$

$$(1.33)\quad 0(\min\{e^{-Te_0(a)}\,e^{-\lambda\rho(x,a)}\,,\,e^{-Te_0(b)}\,e^{-\lambda\rho(x,b)}\}\,e^{-\lambda\gamma\bar{\delta}^2})$$

uniformly for $x \in B(x_0\,,\,\delta)$ $y \notin B(a,\bar{\delta}) \cup B(b,\bar{\delta})$ as $T, \lambda \to \infty$ and $T/\lambda \to 0$.

Here $e_0(c) := \frac{\mathrm{trace}\sqrt{V''(c)}}{2}$ for $c \in\{a,b\}$.

2. Assume

(1.34) $V''(b) = \Omega^2$ where $\Omega = \mathrm{diag}(\omega_1\,,\dots,\,\omega_d)$ with $0 < \omega_1 \le \cdots \le \omega_d$ then there are positive constants $\delta_1 \le \frac{\omega_1}{2}$, δ_2, c_4, and T_0 and functions $a_0^T(x,y), a_1^T(x,y), a_{1,j}^T(x,y)$ on $B(x_0\,,\,\delta) \times B(b,\delta) \times [T_0\,,\,\infty)$ for $i \in \{1,\dots,d\}$ with $a_0^T(x,y), a_1^T(x,y), a_{1,j}^T(x,y) \in C^\infty(B(x_0\,,\,\delta) \times B(b,\delta))$ for $T \ge T_0$ and $j \in \{1,\dots,d\}$ such that

$$(1.35)\quad e^{-T\frac{H(\lambda)}{\lambda}}(x,y) = \pi^{-\frac{d}{2}}\lambda^{\frac{d}{2}}(\prod_{i=1}^{d}\omega_i)^{\frac{1}{2}}\,a_0^T(x,y)\exp(-\lambda A(x,y,T))\,e^{-Te_0(b)}$$

$$\{1 + \frac{a_1^T(x,y)}{\lambda} + 0(\frac{T^4}{\lambda^2}) + 0(\frac{T^2}{\lambda}\exp(-\lambda\frac{c_4}{T^4}))\}$$

and

$$(1.36)\quad -\frac{1}{\lambda}\frac{\partial}{\partial x_j} e^{-T\frac{H(\lambda)}{\lambda}}(x,y) = \pi^{-\frac{d}{2}}\lambda^{\frac{d}{2}}(\prod_{i=1}^{d}\omega_i)^{\frac{1}{2}}\exp(-\lambda\,A(x,y,T))$$

$$\left\{\frac{\partial A(x,y,T)}{\partial x_j}\,a_0^T(x,y) + \frac{a_{1,j}^T(x,y)}{\lambda} + 0(\frac{T^4}{\lambda^2}) + 0(\frac{T^2}{\lambda}\exp(-\lambda\frac{c_4}{T^4}))\right\}$$

uniformly for $x \in B(x_0,\delta)$, $y \in B(b,\delta)$ as T, $\lambda \to \infty$ and $T^4 \leq \lambda$

where

(1.37) $a_0^T(x,y) = a_0(x)\,(1 + 0(e^{-\delta_1 T} + |y-b|))$

with $a_0(x) = (\det X^{x,b}(0))^{-\frac{1}{2}} \geq$ constant > 0 and

$X^{x,b}$ as described in Theorem B below.

(1.38) $A(x,y,T) = \rho(x,b) + \rho(b,y) + 0(e^{-\omega_1 T})$

(1.39) $\rho(b,y) = \frac{1}{2}\sum_{i=1}^{n} \omega_i (y_i - b_i)^2 + 0(|y-b|^3)$

(1.40) $\dfrac{\partial A(x,y,T)}{\partial x_j} = \dfrac{\partial \rho(x,b)}{\partial x_j} + 0(e^{-\delta_1 T})$

and

(1.41) $a_1^T(x,y) = 0(T) = a_{1,j}^T(x,y)$ uniformly on $B(x_0\,,\,\delta) \times B(b,\delta) \times [T_0\,,\,\infty\}$.

The proof of this theorem, especially the uniformity for large T (see chapter 6), is the issue here.

We use large deviations for T in a fixed time interval and then the semigroup property to get large T's. Since the total time the instantons spend outside given neighborhoods of a and b is bounded from above, we can show it suffices to do finer analysis near a and near b. There V is close to its quadratic approximations and we can do the analysis.

We use (1.34), (1.20), and (1.21) to estimate the contribution in (1.31) and (1.32) from $\mathbb{R}^d \backslash (B(a\,,\,\frac{\delta}{2}) \cup B(a\,,\,\frac{\delta}{2}))$ for δ as Theorem A. Then we use (1.35) (resp. (1.36)) and the L^2-asymptotic expansions in (*) when integrating over $B(a,\frac{\delta}{2}) \cup B(b,\frac{\delta}{2})$ in (1.31) (resp (1.32)). There we use the method of stationary phase (Hörmander [1], Theorem 7.7.5) which requires improvements of (1.37)-(1.41) (see Chapter 4, lemmas 6 and 7). After using the asymptotics of $E_1(\lambda)$ and $E_0(\lambda)$ in (*) we put T = constant log λ for some positive constant (idea from Breen [1]) and we get:

§1.7. <u>W.K.B. expansions near K.</u>

<u>Theorem B</u>. If $|V'(x)| = 0(e^{A|x|^2})$ as $|x| \to \infty$ and (*) holds then there is a neighborhood U of $K = \{x : \rho(x,a) = \rho(x,b) = \frac{\rho(a,b)}{2})\}$ and functions f_c (see (1.44)) $f_{1,c}$ and $f_{1,c,j} \in C^{\infty}(U,\mathbb{R})$ for $c \in \{a,b\}$ and for $j \in \{1,\dots,d\}$ such that if, for $c \in \{a,b\}$ and $j \in \{1,\dots,d\}$,

$$F_c(x,\lambda) := f_c(x) + \frac{f_{1,c}(x)}{\lambda} \tag{1.42}$$

$$F_{c,j}(x,\lambda) := \frac{\partial\rho(x,c)}{\partial x_j} f_c(x) + \frac{f_{1,c,j}(x)}{\lambda}$$

then (compare with (1.13))

$$\Omega_0(x,\lambda) = \lambda^{\frac{d}{4}} \alpha(\lambda)(F_a(x,\lambda) + (\lambda^{-\frac{3}{2}}))\, e^{-\lambda\rho(x,a)}$$

$$+ \lambda^{\frac{d}{4}} \beta(\alpha)(F_b(x,\lambda)+0(\lambda^{-\frac{3}{2}}))\, e^{-\lambda\rho(x,b)}$$

$$\Omega_1(x,\lambda) = -\lambda^{\frac{d}{4}} \beta(\lambda)(F_a(x,\lambda) + 0(\lambda^{-\frac{3}{2}}))\, e^{-\lambda\rho(x,a)}$$

$$+ \lambda^{\frac{d}{4}} \alpha(\lambda)(F_b(x,\lambda)+0(\lambda^{-\frac{3}{2}}))\, e^{-\lambda\rho(x,b)}$$

(1.43) $$-\frac{1}{\lambda}\frac{\partial}{\partial x_j}\,\Omega_0(x,\lambda) = \lambda^{\frac{d}{4}}\,\alpha(\lambda)(F_{a,j}(x,\lambda) + 0(\lambda^{-\frac{3}{2}}))\,e^{-\lambda\rho(x,a)}$$

$$+\ \lambda^{\frac{d}{4}}\,\beta(\lambda)(F_{b,j}(x,\lambda) + 0(\lambda^{-\frac{3}{2}}))\,e^{-\lambda\rho(x,b)}$$

and

$$-\frac{1}{\lambda}\frac{\partial}{\partial x_j}\,\Omega_1(x,\lambda) = -\lambda^{\frac{d}{4}}\,\beta(\lambda)(F_{a,j}(x,\lambda) + 0(\lambda^{-\frac{3}{2}}))\,e^{-\lambda\rho(x,a)}$$

$$+\ \lambda^{\frac{d}{4}}\,\alpha(\lambda)(F_{b,j}(x,\lambda) + 0(\lambda^{-\frac{3}{2}}))\,e^{-\lambda\rho(x,b)}$$ uniformly for $x \in U$ as $\lambda \to \infty$.

For $x \in$ and $c \in \{a,b\}$.

(1.44) $$f_c(x) = \pi^{-\frac{d}{4}}\left(\det\sqrt{V''(c)}\right)^{\frac{1}{2}}\left(\det X^{x,c}(0)\right)^{-\frac{1}{2}}$$

where $X^{x,c}$ is the unique nonsingular matrix solution on $[0\ ,\ \infty)$

of the Jacobi equation:

(1.45) $\ddot{X}(t) = V''(g^{x,c})(t)\,X(t)$ satisfying

(1.46) $X^{x,c}(t) = (I + 0(e^{-\delta_1 t}))\,e^{-\sqrt{V''(c)}\,t}$ as $t \to \infty$ where $\delta_1 > 0$.

(Proof in chapter 4 .)

<u>Remarks</u>. (1) Equations (1.43) hold near any point x_0 such that:

If $c \in (a,b)$ and $\rho(x_0\ ,\ c) = \min\{\rho(x_0\ ,\ a),\ \rho(x_0\ ,\ b)\}$

(1.47) then there is a unique Agmon geodesic $g^{x_0, c}$ and its second variation is positive definite (holds for $x_0 \in K$ by (1.27).

If for x_0 with $\rho(x_0, b) < \rho(x_0, a)$ say we put f_a, $f_{1,a}$ and $f_{1,a,j}$ all equal to zero.

So we only need to worry about the Agmon geodesics to the closest well(s) (because the eigenfunctions are living in both wells). In particular (1.43) holds near the set of Agmon geodesics $\bigcup_{x \in K} \{g_x(t) \mid t \in [\infty, \infty]\}$.

We have, for example, for some $\delta > 0$

$$(1.48) \quad \Omega_0(x,\lambda) = \lambda^{\frac{d}{4}} \beta(\lambda)(f_b(\lambda) + \frac{f_{1,b}(x)}{\lambda} + 0(\lambda^{-\frac{3}{2}}))\, e^{-\lambda\rho(x,b)}$$

uniformly for $x \in B(b,\delta)$. Moreover we have $f_b(b) = \pi^{-\frac{d}{4}} \left(\det\sqrt{V''(b)}\right)^{\frac{1}{2}}$ and $\rho(x,b) = \frac{1}{2}\langle(V''(b))^{\frac{1}{2}}(x-b), (x-b)\rangle + 0(|x-b|^3)$.

Hence (1.13) width $k = 2m + 2$ looks like a Taylor-expansion in $\lambda^{\frac{1}{2}}(x-b)$ of the result in (1.48).

Without assumption (1.47) one can *presumably* often get expansion similar to those in (1.43). With new f_c , $f_{1,c}$ and $f_{1,c,j}$'s and the prefactor $\lambda^{\frac{d}{4}}$ replaced by $\lambda^{\frac{d}{4}} P_c(\lambda)$ where $1 \le P_c(\lambda) \le \lambda^{\frac{(d-1)}{2}}$, e.g., $P_c(\lambda) = \lambda^s(\log \lambda)^k$.

§1.8. <u>The asymptotics of $E_1(\lambda) - E_0(\lambda)$, as $\lambda \to \infty$</u> .

Theorem B and lemma 1 yield (see chapter 3)

Theorem C. Let U be a neighborhood of $K = \{x : \rho(x,a) = \rho(x,b) = \frac{\rho(a,b)}{2}\}$ as in Theorem B and lemma 1. Assume (*) and $|V'(x)| = 0(e^{A|x|^2})$ as $|x| \to \infty$. Then there are functions $F_0(x) > 0$ (see (1.52) below) and $F_1(x) \in C^\infty(U,\mathbb{R})$ such that

$$(1.50) \quad E_1(\lambda) - E_0(\lambda) = \frac{\lambda^{\frac{d+2}{2}}}{2\alpha(\lambda)\beta(\lambda)(1+\frac{\text{const}}{\lambda}+0(\lambda^{-\frac{3}{2}}))}$$

$$\int_{Z=\{x,\rho(x,a)=\rho(x,b)\}\cap U} (F_0(x) + \frac{F_1(x)}{\lambda} + 0(\lambda^{-\frac{3}{2}}))e^{-\lambda(\rho(x,a)+\rho(x,b))}ds(x)$$

where $0(\lambda^{-\frac{3}{2}})$ is uniform on Z.

If $x \in K$

$$(1.51) \quad m_x := \text{nullity of the Hessian of } (\rho(a,x) + \rho(x,b)) \text{ along } Z, \text{ at } x$$

$$= \dim\{f \in L^2(\mathbb{R},\mathbb{R}^d) : \ddot{f}(t) = V''(g_x(t))\, f(t) \text{ and } <\dot{g}_x(0), f(0)> = 0\}$$

$$= \dim\{f \in L^2(\mathbb{R},\mathbb{R}^d) : \ddot{f}(t) = V''(g_x(t))\, f(t)\} - 1.$$

(1.52) $F_0(x) = |\rho'(x,a) - \rho'(x,b)|\, f_a(x)\, f_b(x) > 0$ for $x \in U$ where $f_a(x)$ and $f_b(x)$ are defined in (1.36).

Remarks. (1) Observe the simple dependence of $\alpha(\lambda)$ and $\beta(\lambda)$. Recall $\alpha(\lambda) = \|j_a \Omega_0\|_2 \left(1 + 0\left(\frac{1}{\lambda^{\frac{1}{2}}}\right)\right)$ and $\beta(\lambda) = \|j_b \Omega_0\|_2\left(1 + 0\left(\frac{1}{\lambda^{\frac{1}{2}}}\right)\right)$

(2) For $x \in K$

(1.53) $|\rho'(x,a)-\rho'(x,b)| = 2\sqrt{2V(x)}$ see lemma 2 below.

(3) Note $\rho(x,a) + \rho(x,b) \geq \rho(a,b)$ with equality only on K which makes the nullity of the Hessian of $\rho(x,a) + \rho(x,b)$ along Z at $x \in K$ important. $\dot{g}_x$ is L^2-solution of the Jacobi equations.

(1.54) $\ddot{f}(t) = V''(g_x(t)f, t \in \mathbb{R}$

and (1.51) is saying we should only look at the subspace of

solutions of (1.54) orthogonal to g_x .

Now we look at a few cases influenced from reading papers on short time asymptotic of diffusion processes on a Riemannian manifold (see Molchanov [1] and C. Bellaiche [1]).

Case 1. $K = \{x_0\}$ and $m_{x_0} = 0$. Then there are coordinates $x = (x_1 , ..., x_d)$ near x_0 such that $x_d = 0$ defines Z and positive constants $\alpha_1 , ..., \alpha_d$ such that

(1.55) $$\rho(a,x) + \rho(x,b) = \rho(a,b) + \sum_{i=1}^{d} \alpha_i x_i^2 + 0(|x|^3) \text{ for } x \in Z.$$

Hence (Erdelyi [1]) there are constants c_0 and c_1 such that

(1.56) $$E_1(\lambda) - E_0(\lambda) = \frac{C_0}{2\alpha(\lambda)\beta(\lambda)} \lambda^{\frac{3}{2}} (1 + \frac{c_1}{\lambda} + 0(\lambda^{-\frac{3}{2}}))e^{-\lambda\rho(a,b)} \text{ as } \lambda \to \infty .$$

Remark. (1.57) Taking straight line segments as trial paths for the Agmon distance, we get:

There exist $\delta > 0$ and $C > 0$ such that $\rho(y,x_0) \leq C|y-x_0|$ for all $y \in B(x_0, \delta)$. Hence by (1.55) and (1.28) we have in case 1;

(1.58) There is a unique Agmon geodesic g_{x_0} between a and b and there is a hypersurface Z that intersects g_{x_0} at x_0 transversally and a neighborhood U of x_0 and a constant $\overline{c}$ such that $\rho(a,x) + \rho(b,x) \geq \rho(a,b) + \frac{1}{\overline{c}} \rho(x,x_0)$, for all $x \in U \cap Z$.

This nondegeneracy assumption was made by Helffer and Sjöstrand [1] in a multiwell situation when they proved, as mentioned earlier,

$$(1.59) \quad E_1(\lambda) - E_0(\lambda) = c_0 \lambda^{\frac{3}{2}} e^{-\lambda\rho(a,b)} (1 + \tfrac{c_1}{\lambda} + \cdots + \tfrac{c_n}{\lambda^n} + 0(\lambda^{-(n+1)}))$$

for all n, as $\lambda \to \infty$.

Case 2. $K = \{x_0\}$, $m_{x_0} = 1$ (so $d \geq 2$) and in some coordinates near x_0 $\rho(a,x) + \rho(x,b) = \rho(a,b) + x_1^{2p} + \sum_{\ell=2}^{d-1} x_\ell^2$ if $x \in Z$, for some $p \geq 2$. Then there are constants $c_1 \cdots c_{3p-1}$ so that (Erdelyi [1] or Olver [1])

$$E_1(\lambda) - E_0(\lambda) =$$

$$\frac{C_0}{2\alpha(\lambda)\beta(\lambda)} \lambda^{(2-\frac{1}{2p})} (1 + \frac{c_1}{\lambda^{\frac{1}{2p}}} + \cdots + \frac{c_{3p-1}}{\lambda^{(\frac{3}{2}-\frac{1}{2p})}} + 0(\lambda^{-\frac{3}{2}}))e^{-\lambda\rho(a,b)}$$

as $\lambda \to \infty$.

Case 3. $K = \{x_0\}$ with $m_{x_0} = 2$ (so $d \geq 3$) and in some coordinates near x_0 $\rho(a,x) + \rho(x,b) = \rho(a,b) + x_1^4 + x_1^2 x_2^2 + x_2^{2P} + \sum_{\ell=3}^{d-1} x_\ell^2$ with $p \geq 2$. The Laplace integrals needed are in C. Bellaiche [1] and we have: If $p = 2$ $E_1(\lambda) - E_0(\lambda) = \frac{C_0}{2\alpha(\lambda)\beta(\lambda)} \lambda^2 e^{-\lambda\rho(a,b)} (1 + o(1))$ as $\lambda \to \infty$ and if $p > 2$ $E_1(\lambda) - E_0(\lambda) = \frac{C_0}{2\alpha(\lambda)\beta(\lambda)} \lambda^2 \log \lambda \, e^{-\lambda\rho(a,b)} (1 + o(1))$! as $\lambda \to \infty$.

Case 4. K is a submanifold and $k = \dim K(\leq d - 1)$ and for $x \in K$ $m_x = k$ (resp. $m_x = k + 1$ so $k \leq d - 2$) and in some coordinates in Z $\rho(a,x) + \rho(x,b) = \rho(a,b) + \sum_{\ell=k+2}^{d-1} x_\ell^2$ (resp. $\rho(a,x) + \rho(x,b) = \rho(a,b) + x_{k+1}^{2P} + \sum_{\ell=k+1}^{d-1} x_\ell^2$, for $P \geq 2$) then there are constants c_0, $c_1 \cdots$ such that $E_1(\lambda) - E_0(\lambda) = \frac{C_0}{2\alpha(\lambda)\beta(\lambda)} \lambda^{(\frac{3+k}{2})} (1 + \frac{c_1}{\lambda} + 0(\lambda^{-\frac{3}{2}}))e^{-\lambda\rho(a,b)}$, as $\lambda \to \infty$ (resp. $E_1(\lambda) - E_0(\lambda) = \frac{C_0}{2\alpha(\lambda)\beta(\lambda)} \lambda^{(\frac{3+k+1}{2} - \frac{1}{2p})} (1 + \frac{c_1}{\lambda^{\frac{1}{2p}}} + \cdots + \frac{c_{3p-1}}{\lambda^{(\frac{3}{2} - \frac{1}{2p})}} + 0(\lambda^{-\frac{3}{2}}))e^{-\lambda\rho(a,b)}$ as $\lambda \to \infty$.

Estimating $\rho(x,a) + \rho(x,b) \geq \rho(a,b)$ on Z gives a general upper bound

$$E_1(\lambda) - E_0(\lambda) = 0\left(\frac{\lambda^{\frac{(d+2)}{2}}}{\alpha(\lambda)\beta(\lambda)} e^{-\lambda\rho(a,b)}\right)$$

as $\lambda \to \infty$. (See Hellfer and Sjöstrand [1], Theorem 6.6).

§1.9. The plan of the paper.

The main difficulty here is the uniformity for large T in Theorem A. Our method is to use large derivations on an interval $[T_0 , 2T_0]$ where T_0 is a fixed (large!) number. Then we use the semigroup property of the heatkernel for $T = (n + 1) T_1$ where $T_1 \in [T_0 , 2T_0)$. In both cases and in proving Theorem B using (1.31) and (1.32) we need to analyze the minimal action, the Agmon distance, their paths, the related Jacobi fields and to compare those. We state these results in chapters 2 and 4 but their proofs or sketches of proofs are in chapters 11 to 16.

We proof Theorem C (given Theorem B) in chapter 3, Theorem B (given Theorem A) in chapter 4 and in chapters 5 to 10 we prove Theorem A, the central part of the proof is in chapter 6.

2. The Agmon Distance and the Minimal Action.

§2.1. An Introduction - Proposition 1.

The Agmon distance defined in (1.8) by

$$\rho(x,y) = \inf\{\int_0^1 \sqrt{2V(\gamma(s))}\, |\dot\gamma(x)|\, ds : \gamma(0) = x,\ \gamma(1) = y\}$$

is the geodesic distance in the metric $2V(x)dx^2$ which (and for more general V's) is called the "Agmon metric" since the studies of Agmon [1] of exponential decay of L^2-solutions of $(-\Delta + v)\, u = 0$, at ∞. For $V \geq$ constant > 0, at ∞, as we consider, it was earlier used by Lithner [1] for the same purpose and (1.21) follows from his work.

Carmona and Simon [1] proved Agmon's upper bound was a lower bound for the ground state, at ∞. They used Feynman-Kac formula (see Simon [5]) and (1.23) that says $\rho(x,y) = \inf_{T>0} A(x,y,T)$, where $A(x,y,T)$ is the classical action given by (1.18)

$$A(x,y,T) = \inf_\gamma \{\int_0^T (\tfrac{1}{2}\dot\gamma^2 + V(\gamma))\, dt : \gamma(0) = x,\ \gamma(T) = y\}$$

In tunneling problems the Agmon distance was introduced by Simon [2] when proving

(1.7) by reducing to the upper bound in (1.20) and (1.21) and

$$(2.1)\qquad \lim_{\lambda\to\infty} \frac{\log \Omega_0(x,\lambda)}{\lambda} = -\min\{\rho(x,a)\,,\ \rho(x,b)\}.$$

Helffer and Sjöstrand [1], [2], and [3] (see also Sjöstrand [1]) use the Agmon distance and perturbations of it extensively their studies of multiwells, and Dirichlet's problems that they reduce to.

In the analysis of $e^{-T\frac{H(\lambda)}{\lambda}}(x,y)$ and $\frac{\partial}{\partial x_j} e^{-T\frac{H(\lambda)}{\lambda}}(x,y)$ (see (1.17), (1.19 and Theorem A) the Agmon distance enters mainly through

<u>Proposition 1</u>.

(2.2) $\lim_{T\to\infty} A(x,y,T) = \min\{\rho(x,a) + \rho(a,y), \rho(x,b) + \rho(b,y)\} + 0(e^{-\alpha T})$

uniformly on compacts on $\mathbb{R}^d \times \mathbb{R}^d$, for some $\alpha > 0$ (depending on the compact) as $T \to \infty$. (A Proof in Chapter 12.)

We will study $\rho(x,c)$ and $\rho(a,b)$ given by (1.24)

$$\rho(x,c) = \inf\{\int_0^\infty(\tfrac{1}{2}\dot\gamma^2 + V(\gamma))dt : \gamma(0) = x, \rho(\infty) = c\}$$

if $x \notin (a,b)$ and $c \in \{a,b\}$ and (1.25)

$$\rho(a,b) = \inf\{\int_{-\infty}^\infty(\tfrac{1}{2}\dot\gamma^2 + V(\gamma))dt : \gamma(-\infty) = a, \gamma(\infty) = b\}$$

which implies that minimizing paths $g_T^{x,y}$ for $A(x,y,T)$, $g^{x,c}$ for $\rho(x,c)$ in (1.24) and g_x for $\rho(a,b)$ in (1.25), if $x \in K$, all satisfy the same Euler-Lagrange equations.

(2.3) $\ddot\gamma(t) = V'(\gamma(t))$.

That is convenient when we compare $g^{x,b}(t)$ to $g_T^{x,y}(t)$ on $[0, \frac{T}{2}]$ and $g^{y,b}(t)$ to $g_T^{x,y}(T - t)$ on $[0, \frac{T}{2}]$

if all have those paths are unique and, say, (2.4) $\rho(x,b) + \rho(b,y) < \rho(x,a) + \rho(a,y)$.

Note that to avoid the potential term in

$$A(x,y,T) = \int_0^T (\tfrac{1}{2}(\dot{g}_T^{x,y})^2 + V(g_T^{x,y}))\, dt$$

to grow as $T \to \infty$ the minimal action paths have to run into the wells and stay there most of the time. Conditions like (2.4) make the $g_T^{x,y}$'s for large T, go near b and not near a.

§2.2. <u>The second variation and the solutions of Jacobi equations.</u>

In the rest of this chapter we look at x_0 satisfying (recall lemma 1)

(2.5) $\rho(x_0,b) < \rho(x_0\ , a) + \rho(a,b)$

(2.6) There is a unique Agmon geodesic $g^{x_0,b}$ minimizing $\rho(x_0\ , b)$ in (1.24)

and

(2.7) the second variation of $g^{x_0,b}$ is positive definite (see (2.11) below).

It follows (see lemma 2 below) that these conditions are satisfied for x near x_0. Moreover for x near x_0, y near b and large T there is a unique $g_T^{x,y}$ and its second variation is positive definite. Moreover (2.5) implies (2.4) for x near x_0 and y near b and so we know we should compare $g_T^{x,y}$ with $g^{x,b}$ and $g^{y,b}$. To describe what (2.7) means, let $J = [0,\infty)$ or $J = [0,T]$ and $Q(t)$ a symmetric continuous $(d \times d)$-matrix on J then

Definition (2.8). The system

(2.9) $-\ddot{f}(t) + Q(t)\, f(t) = 0$ on J is *nonconjugate* if no nonzero solution vanishes twice on J. This is equivalent to (Hartman [1], Coppel [1]) .

For any subinterval $[t_1 , t_2] \subset J$,

$$\text{(2.10)}\qquad I(\eta;t_1,t_2) := \int_{t_1}^{t_2} (\langle\dot\eta,\dot\eta\rangle + \langle Q\eta,\eta\rangle)\, dt \geq 0 ,$$

$\forall \eta \in D_0(t_1,t_2) := \{\eta \in AC([t_1,t_2], \mathbb{R}^d) : \dot\eta \in L^2(t_1,t_2), \eta(t_1) =$

$= 0 = \eta(t_2)\}$ and $I(\eta,t_1,t_2) = 0$ only if $\eta = 0$.

Definitions. The second variation of

(2.11) $g^{x,b}$ is positive iff (2.10) holds with $J = [0,\infty)$ and $Q(t) = V''(g^{x,b}(t))$.

The second variation of

(2.12) $g_T^{x,y}$ is positive definite iff (2.10) holds with $J = [0,T]$ and $Q(t) = V''(g_T^{x,y}(t))$.

Hence (2.7) is telling us the Jacobi system (1.4) $-\ddot{f}(t) + V''(g^{x_0,b}(t))\, f(t) = 0$ is nonconjugate on $(0,\infty)$ meaning that (1.54) has no nonzero solution vanishing twice on $[0,\infty)$.

Remark (2.13). (See A. Bellaiche [1] and Kobayashi [1]). On a Riemannian manifold m of class C^r for $r \geq 2$ one defines the cutlocus of m, by $c(m) := \{(x,y) \in m \times m$ there are at least 2 minimal geodesics between x and y, or x and y are conjugate along a minimal geodesic (second variation is not positive definite)$\}$. Moreover one defines the cutlocus of $x \in m$ by $c(x) := \{y : (x,y) \in c(m)\}$. Then $c(m)$ is a closed subset of $m \times m$, $c(x)$ is a closed subset of measure zero in m and the energy $E = \frac{1}{2} d^2(x,y)$ is C^{r-1} on $m \times m \backslash c(m)$. Moreover if g is a minimal geodesic between x and y then x and any point z on g before y are in c(m).

Note the metric tensor $g_{ij}(x) = 2V(x)\, \delta_{ij}$, for the Agmon distance vanishes at a and b. Therefore we cannot quote those results and therefore $e^{-T\frac{H(\lambda)}{\lambda}}$ behaves as described in Theorem A and not like the free one: $(\frac{\lambda}{2\pi T})^{\frac{d}{2}} \exp(-\frac{\lambda(x-y)^2}{2T})$. We mainly make statements about $\rho(x,b)$, $\rho(b,y)$, $g^{x,b}$ and $g^{y,b}$ for x near x_0 and y near b and not about $\rho(x,y)$ and Agmon geodesics between x and y.

§2.3. Asymptotics of Agmon geodesics, minimal action paths and solutions of related Jacobi equations.

We will assume (1.33) $V''(b) = \Omega^2$ where $\Omega = \mathrm{diag}(\omega_1, ..., \omega_d)$ with $0 < \omega_1 \leq \omega_2 \leq \cdots \leq \omega_d$.

If (2.5), (2.6) and (2.7) hold then

Lemma 2. There exists δ_0, δ_1 and $T_0 > 0$ such that

1. If $x \in B(x_0, \delta_0)$ then there is a unique Agmon geodesic $g^{x,b}$. It's second variation is positive definite and it depends C^∞-ly on $x \in B(x_0, \delta_0)$. For each $\alpha \in \mathbb{N}_0^d$

$$(2.14) \qquad \frac{\partial^{|\alpha|}}{\partial x^\alpha}(g^{x,b}(t) - b) = 0(e^{-\omega_1 t}) = \frac{\partial^{|\alpha|}}{\partial x^\alpha}\, \dot{g}^{x,b}(t)$$

uniformly on $B(x_0, \delta_0) \times [0,\infty)$.

(2.15) For $y \in B(b,\delta_0)$ and $i \in \{1,\ldots,d\}$

$$(g^{y,b}(t)-b)_i = (y-b)_i\, e^{-\omega_i t} + 0(|y-b|^2)\, e^{-\min(\omega_i,2\omega_1)t} = -\omega_i^{-1}(\dot g^{y,b}(t))_i.$$

For $x \in B(x_0,\delta_0)$ the Jacobi system $-\frac{d^2}{dt^2} U(t) + V''(g^{x,b}(t))\, U(t) = 0$ for $t \in [0,\infty)$ has matrix solutions $X^{x,b}(t)$ and $Y^{x,b}(t)$ satisfying

(2.16) $$X^{x,b}(t) = (I + 0(e^{-\delta_1 t}))\, e^{-\Omega t} = -\dot X^{x,b}(t)\, \Omega^{-1}$$

$$Y^{x,b}(t) = (I + 0(e^{-\delta_1 t}))\, e^{\Omega t} = \dot Y^{x,b}(t)\, \Omega^{-1}$$

uniform on $B(x_0\, ,\, \delta_0) \times [0,\infty)$

which are infinitely differentiable in $B(x_0\, ,\, \delta_0)$ with

For each $\alpha \in \mathbb{N}_0^d$ $|\alpha| \geq 1$

(2.17) $$\frac{\partial^{|\alpha|}}{\partial x^\alpha} X^{x,b}(t) = 0(e^{-\delta_1 t})\, e^{-\Omega t} = \frac{\partial^{|\alpha|}}{\partial x^\alpha} \dot X^{x,b}(t)$$

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} Y^{x,b}(t) = 0(e^{-\delta_1 t})\, e^{\Omega t} = \frac{\partial^{|\alpha|}}{\partial x^\alpha} \dot Y^{x,b}(t)$$

uniformly on $B(x_0\, ,\, \delta_0) \times [0,\infty)$

and

(2.18) $$\det(X^{x,b}(t)) \geq \text{constant}\; e^{-t(\sum_{i=1}^{d} \omega_i)} > 0$$

uniformly on $B(x_0\, ,\, \delta_0) \times [0,\infty)$.

Moreover

(2.19) $$\frac{\partial}{\partial x_j}\left(\frac{d}{dt}\right)^i g^{x,b}(t) = (((\frac{d}{dt})^i X^{x,b}(t))(X^{x,b}(0))^{-1} e_j$$

if $i \in \{0,1\}$ and $j \in \{1,2,\ldots,d\}$ where

$e_j = (\delta_{ij})_{i=1}^d$ and δ_{ij} is the Kronecker delta

and

(2.20) $$\frac{\partial \rho(x,b)}{\partial x_j} = -(\dot{g}^{x,b}(0))_j \text{ for } j = 1, \ldots, d$$

2. If $(x,y,T) \in B(x_0, \delta_0) \times B(b,\delta_0) \times [T_0, \infty)$ there is a unique minimal action path $g_T^{x,y}$ and it has positive second variation (as defined in (1.12)), $g_T^{x,y}(t)$, $\dot{g}_T^{x,y}(t)$, $A(x,y,T)$ are infinitely differentiable with respect to $(x,y) \in B(x_0, \delta_0) \times B(b_0, \delta_0)$ (see (2.26) below).

The Jacobi system

(2.21) $$\left(-\frac{d^2}{dt^2} + V''(g_T^{x,y})\right) U(t) = 0 \text{ for } t \in [0,T]$$

has solutions $X^{x,y,T}$ and $Y^{x,y,T}$ satisfying

(2.22) $$X^{x,y,T}(t) = (I + 0(e^{-\delta_1 t} + |y-b|))\, e^{-\Omega t} = \dot{X}^{x,y,T}(t)\, \Omega^{-1}$$

$$Y^{x,y,T}(t) = (I + 0(e^{-\delta_1 t} + |y-b|))\, e^{\Omega t} = \dot{Y}^{x,y,T}(t)\, \Omega^{-1}$$

uniformly $(x,y,T,t) \in B(x_0, \delta_0) \times B(b,\delta_0) \times [T_0, \infty) \times [0,T]$

and

(2.23) $$\det X^{x,y,T}(t) \geq (\text{const})\, e^{-t(\sum_{i=1}^{d} \omega_i)} > 0$$

uniformly $(x,y,T,t) \in B(x_0, \delta_0) \times B(b,\delta_0) \times [T_0, \infty) \times [0,T]$

The $X^{x,y,T}$, $\dot{X}^{x,y,T}$, $Y^{x,y,T}$ and $\dot{Y}^{x,y,T}$'s are infinitely differentiable and satisfy

(2.24) For each α , $\beta \in N_0^d$

$$\frac{\partial^{|\alpha|+|\beta|}}{\partial x^\alpha \partial y^\beta} X^{x,y,T}(t) = 0(1)\, e^{-\Omega t} = \frac{\partial^{|\alpha|+|\beta|}}{\partial x^\alpha \partial y^\beta} \dot{X}^{x,y,T}(t)$$

$$\frac{\partial^{|\alpha|+|\beta|}}{\partial x^\alpha \partial y^\beta} Y^{x,y,T}(t) = 0(1)\, e^{\Omega t} = \frac{\partial^{|\alpha|+|\beta|}}{\partial x^\alpha \partial y^\beta} \dot{Y}^{x,y,T}(t)$$

uniformly for $(x,y,T,t) \in B(x_0 , \delta) \times B(b,\delta_0) \times [T_0 , \infty) \times [0,T]$.

With $X^{x,b}$ and $Y^{x,b}$ as in (2.15) we have

(2.25) $(X^{x,y,T}(t) - X^{x,b}(t))\, e^{\Omega t} =$

$$= 0(e^{-\omega_1 \frac{T}{2}} + |y-b|) = (\dot{X}^{x,y,T}(t) - \dot{X}^{x,b}(t))\, e^{\Omega t}$$

and $(Y^{x,y,T}(t) - Y^{x,y}(t))\, e^{-\Omega t}$

$$= 0(e^{-\omega_1 \frac{T}{2}} + |y-b|) = (\dot{Y}^{x,y,T}(t) - \dot{Y}^{x,b}(t))\, e^{-\Omega t}$$

uniformly for $(x,y,T,t) \in B(x_0 , \delta_0) \times B(b,\delta_0) \times [T_0 , \infty] \times [0,T]$.

Moreover, with $g = g_T^{x,y}$, $X = X^{x,y,T}$ and $Y = Y^{x,y,T}$ we have

$$(2.26)\qquad \frac{\partial}{\partial y_i} A(x,y,T) = (\dot g(T))_i$$

$$\frac{\partial A(x,y,T)}{\partial x_i} = -(\dot g(0))_i$$

$$\frac{\partial A(x,y,T)}{\partial T} = \frac{1}{2} (\dot g(T)^2 + V(y)$$

$$(\tfrac{d}{dt})^j \frac{\partial}{\partial x_i} g(t) = \frac{\partial}{\partial x_i} (\tfrac{d}{dt})^j g(t)$$

$$= (\tfrac{d}{dt})^j \Big(X(t)-Y(t)\,(Y(T)^{-1}X(T)\Big)\Big(X(0)-Y(0)\,(Y(T))^{-1}X(T)\Big)^{-1} e_i$$

$$(\tfrac{d}{dt})^j \frac{\partial}{\partial y_i} g(t) = \frac{\partial}{\partial y_i} (\tfrac{d}{dt})^j g(t) =$$

$$= (\tfrac{d}{dt})^j \Big(Y(t)-X(t)(X(0))^{-1}Y(0)\Big)\Big(Y(T)-X(T)\,(X(0))^{-1}Y(0)\Big)^{-1} e_i$$

for $j \in \{0,1\}$ and $i \in \{1,\dots,d\}$

and finally

If $\alpha \in \mathbb{N}_0^d$ then

$$(2.27)\qquad \frac{\partial^{|\alpha|}}{\partial y^\alpha} V(g_T^{x,y}(t)) = 0(1) \text{ uniformly for}$$

$(x,y,T,t) \in B(x_0\,,\,\delta_0) \times B(b,\delta_0) \times [T_0\,,\,\infty) \times [0,T]$.

<u>Remark</u>. By the Euler-Lagrange equation (2.3) $\ddot g_T^{x,y}(t) = \nabla V(g_T^{x,y}(t))$ if $t \in [0,T]$ we have constant energy along $g_T^{x,y}(t)$ given by

$$(2.28)\qquad E_T(x,y) := \tfrac{1}{2}(\dot g_T^{x,y}(0))^2 - V(g_T^{x,y}(0)) = \tfrac{1}{2}(\dot g_T^{x,y}(t))^2 - V(g_T^{x,y}(t))$$

for all $t \in [0,T]$.

Similar 0 energy along $g^{x,b}$ by (2.3) and (2.14)

$$(2.29)\qquad \tfrac{1}{2}(\dot{g}^{x,b}(t))^2 - V(g^{x,b}(t)) = \int_t^\infty (\ddot{g}^{x,b}(s)^2 - \nabla V(g^{x,b}(s))\, \dot{g}^{x,b}(s)\, ds = 0\,.$$

Using (2.26) and (2.28) we get the Hamilton-Jacobi equation (Gelfand-Fomin [1])

$$\tfrac{1}{2}(\nabla_x A(x,y,T))^2 + V(x) = \frac{\partial A(x,y,T)}{\partial T}$$

$$\tfrac{1}{2}(\nabla_x A(x,y,T))^2 + V(y) = \frac{\partial A(x,y,T)}{\partial T}\quad .$$

§2.4. The positivity of the second variation of $g_T^{x,y}$.

In the next lemma we describe in terms of inequalities what "the second variation of $g_T^{x,y}$ is positive definite" means. That is useful when we go beyond the leading order in

$$\frac{\log e^{-T\frac{H(\lambda)}{\lambda}}(x,y)}{\lambda} = -A(x,y,T) \text{ as } \lambda \to \infty\,.$$

Some information about the Agmon distance, the minimal action and the comparison of different minimal action paths, that we need later will follow.

Lemma 3. With x_0 , δ_0 , δ_1 , and T_0 as in lemma 2

1. There exists a constant $k_1 > 0$ such that if $\|\gamma - g_T^{x,y}\|_{L^\infty(0,T)} \le \delta_0$ then

$$(2.30)\qquad \int_0^T (\dot{\eta}^2 + \langle V''(\gamma)\eta,\eta\rangle)\, dt \ge k_1 \|\eta\|^2_{L^\infty[0,T]} \quad \text{for all } \eta \in D_0(0,T)$$

where

(2.31) $D_0(0,T) := \{\eta \in AC([0,T],\mathbb{R}^d) : \eta(0) = 0 = \eta(T) \text{ and } \dot{\eta} \in L^2[0,T]\}$

with

(2.32) $$\|\eta\|_{L^\infty[0,T]} = \sup_{0\le t\le T} \Big(\sum_{i=1}^{d} \eta_i^2(t)\Big)^{\frac{1}{2}}.$$

Also

(2.33) $$\int_0^T \big(\tfrac{1}{2}(\dot{g}_T^{x,y} + \dot{\eta})^2 + V((g_T^{x,y} + \eta))dt - \int_0^T \big(\tfrac{1}{2}(\dot{g}_T^{x,y})^2 + V(g_T^{x,y})\big)dt$$

$$\ge k_1 \min\Big(\delta_0^2 \,,\, \|\eta\|^2_{L^\infty[0,T]}\Big) \text{ for all } \eta \in D_0(0,T),$$

uniformly for $(x,y,T) \in B(x_0 , \delta_0) \times B(b , \delta_0) \times [T_0 , \infty)$.

2. Uniformly on compacts in

(2.34) $\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : \rho(x,b) + \rho(b,y) < \rho(x,a) + \rho(a,y)\}$

$$|\dot{g}_T^{x,y}(t)| \,,\, |g_T^{x,y}(t) - b| = 0\Big(e^{-\omega_1 t} + \min(|y-b| \,,\, \delta_0)e^{-\omega_1(T-t)}\Big),$$

$$E_T(x,y) = \tfrac{1}{2}(\dot{g}_T^{x,y}(t))^2 - V(g_T^{x,y}(t)) = 0(e^{-\omega_1 T}) \text{ (see (2.28))}.$$

(2.35) $A(x,y,T) = \rho(x,b) + \rho(b,y) + 0(e^{-\omega_1 T})$

where

(2.36) $\rho(y,b) = \frac{1}{2} \sum_{i=1}^{d} \omega_i(y_i - b_i)^2 + 0(|y-b|^3)$ uniformly on $B(b , \delta_0)$.

3. If $(x_n , y_n , T_n) \to (x,y,T) \in B(x_0 , \delta_0) \times B(b , \delta_0) \times [T_0 , \infty)$

then

(2.37) $$\|g_{T_n}^{x_n,y_n}(\cdot\ \tfrac{T_n}{T}) - g_T^{x,y}\|_{L^\infty[0,T]} \to 0 \text{ as } n \to \infty .$$

4. There exists $k_2 \geq 1$ such that

(2.38) $$|g_T^{x_1,y_1}(t) - g_T^{x_2,y_2}(t)|,\ |\dot g_T^{x_1,y_1}(t) - \dot g_T^{x_2,y_2}(t)| \leq$$

$$k_2(|x_1-x_2|e^{-\omega_1 t} + |y_1-y_2|e^{-\omega_1(T-t)})$$

for $t \in [0,T]$, $(x_1 , x_2) \in (B(x_0 , \delta_0))^2$, $(y_1 , y_2) \in (B(b,\delta_0))^2$

and $T \in [T_0 ,\infty)$,

(2.39) $$g_T^{x,y}(t) \in B(b,\tfrac{\delta_0}{2}) \text{ if } t \in [\tfrac{T_0}{2} , \tfrac{T-T_0}{2}]$$

and

(2.40) $$|g_T^{x,b}(t) - g^{x,b}(t)| \leq k_2\, e^{-\omega_1(T-t)}\, e^{-\omega_1 T} \text{ if } t \in [0,T] \text{ for all}$$

$(x,y,T) \in B(x_0 , \delta_0) \times B(b,\delta_0) \times [T_0 , \infty)$.

Finally if $x_0 = b$

(2.41) $$|g_T^{x_1,y_1}(t) - g_T^{x_2,y_2}| \leq \max(|x_1-x_2|, |y_1-y_2|).$$

(A proof in chapter 16.)

Note (2.38) and (2.40) imply

(2.42) $$|g_T^{x,y}(t) - g^{x,b}(t)| \leq k_2(|y-b| + e^{-\omega_1 T})\, e^{-\omega_1(T-t)}.$$

§2.5. The Green's matrix $G^{x,y,T}(t,s) = \left(-\frac{d^2}{dt^2} + V''(g_T^{x,y}(t))\right)^{-1}(t,s)$; some asymptotics.

When analyzing the heat kernel $e^{-T\frac{H(\lambda)}{\lambda}}$ we will change a measure on the path space and instead of using the Brownian bridge measure whose covariance operator (see Simon [5], chapter 2) is $(-\frac{d^2}{dt^2}$ on [0,T] with Dirichlet's boundary conditions$)^{-1}$ we use the measure associated with the mean zero Gaussian process whose covariance operator is $((-\frac{d^2}{dt^2} + V''(g_T^{x,y}(t))$ on [0,T] with Dirichlets boundary conditions$)^{-1}$.

We need to analyze its integral kernel which we call the Green's matrix $G^{x,y,T}(s,t)$. Observe if $x = y = b$ then

$$(2.43)\qquad G^{b,b,T}(s,t)= [e^{-\Omega|t-s|} - (e^{2\Omega T} - I)^{-1}\{e^{\Omega(s+t)} - e^{\Omega(s-t)} - e^{\Omega(t-s)} + e^{\Omega(2T-s-t)}\}]\,(2\Omega)^{-1}$$

where

$\Omega = \mathrm{diag}(\omega_1,...,\omega_d) = \sqrt{V''(b)}$ and we have a d-dimensional oscillator process tied down at 0 and T. For certain error terms we need small perturbation of $G^{x,y,T}(t,s)$.

Lemma 4. With x_0, δ_0, δ_1 and T_0 as in lemma 2 let $W_T^r(t)$ for $t \in [0,T]$ if $(r,T) \in [0, r_1] \times [T_0, \infty)$, $r_1 > 0$, be continuous symmetric $(d \times d)$-matrices with

(2.44) $\int_0^T |W_T^r(t)|\, dt = 0(r)$ as $r \downarrow o$ uniformly for $T \in [T_0, \infty]$.

1. Then there exists r_0 such that

(2.45) $\ddot{U} = (V''(g_T^{x,y}) + W_T^r(t)\, U$ has solutions $X_r^{x,y,T}(t)$ and $Y_r^{x,y,T}(t)$

on $[0,T]$ for $(x,y,T,r) \in B(x_0, \delta_0) \times B(b,\delta_0) \times [T_0, \infty) \times [0,r_0]$

with

(2.46) $X_r^{x,y,T}(t) - X^{x,y,T}(t) =$

$0(\int_0^t e^{-\delta_1(t-s)} |W_T^r(s)| ds + \int_t^T |W_T^r(s)|\, ds)\, e^{-\Omega t} =$

$(\dot{X}_r^{x,y,T}(r) - \dot{X}^{x,y,T})$

and

(2.47) $Y_r^{x,y,T}(t) - Y^{x,y,T}(t) = 0(\int_0^t e^{-\delta_1(t-s)} |W_T^r(s)|\, ds$

$+ \int_t^T |W_T^r(s)| ds)\, e^{\Omega t} = (\dot{Y}_r^{x,y,T}(t) - \dot{Y}^{x,y,T}(t))$

uniformly for those (x,y,T,r)'s

(2.48) $X^{x,y,T}(t) = X_r^{x,y,T}(t)|_{r=0}$ and $Y^{x,y,T}(t) = Y_r^{x,y,T}(t)|_{r=0}$

are those in (2.22).

2. If $(-\frac{d^2}{dt^2} + V''(g_T^{x,y}(t)) + W_T^r(t))$ is the Dirichlet operator on $[0,T]$ there are constants k_3 and k_4 such that

(2.49) $\quad (-\frac{d^2}{dt^2} + V''(g_T^{x,y}) + W_T^r(t)) \geq \frac{k_3}{\sqrt{T}} > 0$ (recall (2.30)) on $D_0(0,T)$ (defined in (2.31)) uniformly for $(x,y,T,r) \in B(x_0 , \delta_0) \times B(b_1 , \delta_0) \times [T_0 , \infty) \times [0,r_0]$ and its Green's matrix is given by width $X = X_r^{x,y,T}$ and $Y = Y_r^{x,y,T}$, (see Heimes [1])

(2.50) $\quad G_r^{x,y,T}(t,s) =$

$[X(t)-(Y(t)-X(t)X^{-1}(0)Y(0))(Y(T)-X(T)X^{-1}(0)Y(0))^{-1}X(T)]X^{-1}(s)$

$(Y(s)-X(s)X^{-1}(0)Y(0))(\dot{Y}(s)-\dot{X}(s)X^{-1}(s)Y(s))^{-1}$ if $0 \leq s \leq t \leq T$

and

$G_r^{x,y,T}(t,s) =$

$(Y(t)-X(t)X^{-1}(0)Y(0)) \{I-(Y(T)-X(T)X^{-1}(0)Y(0))^{-1}$

$X(T)X^{-1}(s)(Y(s)-X(s)X^{-1}(0)Y(0))\}(\dot{Y}(s)-\dot{X}(s)X^{-1}(s)Y(s))^{-1}$

if $0 \leq t \leq s \leq T$.

The Green's matrix satisfies

(2.51) $\quad |G_r^{x,y,T}(t,s)| \leq k_4 \, e^{-\omega_1|t-s|}$ if $0 \leq t, s \leq T$

and

(2.52) $\quad |G_r^{x,y,T}(t_1,s) - G_r^{x,y,T}(t_2,s)| \leq k_4 \, e^{-\omega_1|t_1-t_2|}$ if $0 \leq t_1 , t_2 , s \leq T$

uniformly for those (x,y,t,r) 's .

3. For $G^{x,y,T}(t,s) = G_r^{x,y,T}(t,s)|_{r=0}$ we have:

For each $\alpha \in \mathbb{N}_0^d$

$$(2.53)\qquad |\frac{\partial^{|\alpha|}}{\partial y^\alpha} G^{x,y,T}(t,s)|\,,\ |\frac{\partial^{|\alpha|}}{\partial x^\alpha} G^{x,y,T}(t,s)| = 0(e^{-\omega_1|t-s|})\ \text{if } 0 \le t,\, s \le T$$

uniformly for $(x,y,T) \in B(x_0\,,\,\delta_0) \times B(b,\delta_0) \times [T_0\,,\,\infty)$.

[A sketch of a proof in chapter 16.]

(2.52) shows the process with covariance operator $(-\frac{d^2}{dt^2} + V''(g_T^{x,y}(t)) + W_r^T(t))^{-1}$ will have continuous sample paths (see Simon [5], Theorem 5.1) and (2.51) shows the paths won't go far away for large T (by Donsker-Varadhan, see Stroock [1], Cor. (3.50)).

§2.6. Finer asymptotics of $G^{x,y,T}(t,s)$ when y is exponentially close to b.

Finally we need a little bit finer analysis when $y = \overline{y}$ satisfying

$$(2.54)\qquad \overline{y} = \overline{y}(x,T) = b + 0(e^{-\omega_1 T})\ \text{uniformly for}$$

$(x,t) \in B(x_0\,,\,\delta_0) \times [T_0\,,\,\infty)$.

Lemma 5. With x_0 , δ_0 , δ_1 and T_0 as in lemma 2 and $\overline{y}$ satisfying (2.54) we have

1. For each $\alpha \in \mathbb{N}_0^d$ there are diagonal matrices $A(\alpha,\cdot)$, $B(\alpha,\cdot) \in C_b^\infty([0,\infty),\mathbb{R}^{d^2})$ (i.e., each derivative is uniformly bounded on $\mathbb{R}$), independent of x and T, such that

$$(2.55)\qquad \frac{\partial^{|\alpha|}}{\partial y^\alpha} X^{x,y,T}(T-x)|_{y=\overline{y}} = (A(\alpha,v) + 0(e^{-\delta_1(T-v)}))\, e^{-\Omega(T-v)}$$

$$= -\frac{\partial^{|\alpha|}}{\partial y^\alpha}\frac{d}{dt} X^{x,y,T}(t)|_{\substack{y=\overline{y}\\ t=T-v}}\ \Omega^{-1}$$

and

$$(2.56)\qquad \frac{\partial^{|\alpha|}}{\partial y^\alpha} Y^{x,y,T}(T-v)|_{y=\bar{y}} = (B(\alpha,v) + 0(e^{-\delta_1(T-v)}))\, e^{\Omega(T-v)}$$

$$= \frac{\partial^{|\alpha|}}{\partial y^\alpha} \frac{d}{dt} Y^{x,y,T}(t)|_{\substack{y=\bar{y}\\ t=T-v}} \quad \Omega^{-1}$$

uniformly for $(x,T) \in B(x_0\,,\,\delta_0) \times [T_0\,,\,\infty)$.

2. $G^{x,\bar{y}T}(t,s) = G_1^{x,b}(t,s) + G_2^T(t,s) + R^{x,T}(t,s)$ where ($X^{x,b}$ and $Y^{x,b}$ as in (2.16))

$$(2.57)\qquad G_1^{x,b}(t,s) = X^{x,b}(t)(X^{x,b}(s))^{-1}(Y^{x,b}(s)-X^{x,b}(s)(X^{x,b}(0))^{-1}\, Y^{x,b}(0))$$
$$(\dot{Y}^{x,b}(s)-\dot{X}^{x,b}(s)(X^{x,b}(s))^{-1}\, Y^{-x,b}(s))^{-1}$$

if $0 \leq s \leq t \leq T$

and

$$G_1^{x,b}(t,s) = (Y^{x,b}(t)-X^{x,b}(t)(X^{x,b}(0))^{-1}\, Y^{x,b}(0))$$
$$(\dot{Y}^{x,b}(s)-\dot{X}^{x,b}(s)(X^{x,b}(s))^{-1}\, Y^{x,b}(s))^{-1}$$

if $0 \leq t \leq s \leq T$

and satisfies : $\forall \alpha \in \mathbb{N}_0^d$

$$(2.58)\qquad \frac{\partial^{|\alpha|}}{\partial x^\alpha} (G_1^{x,b}(t,s) - e^{-\Omega|t-s|}(2\Omega)^{-1}) = 0(e^{-\delta_1 \min(s,t)}\, e^{-\omega_1|t-s|})$$

uniformly for $(x,t,s) \in B(x_0\,,\,\delta_0) \times (0,\infty) \times (0,\infty)$

$$(2.59)\qquad G_2^T(t,s) = -e^{-\Omega(2T-s-t)}\, (2\Omega)^{-1}$$

and $R^{x,T}(t,s)$ satisfies

$$(2.60)\qquad \int_0^T \int_0^T |R^{x,T}(t,s)|\,dtds = 0(e^{-\delta_1 T}) = |R^{x,T}(t,s)| \text{ if } 0 \le s,\, t \le T$$

uniformly for $(x,T) \in B(x_0\,,\,\delta_0) \times [T_0\,,\,\infty)$.

Moreover with the same uniformity

$$(2.61)\qquad \frac{\partial G^{x,y,T}}{\partial y_i}(T-v,T-u)|_{y=\bar{y}} = \{A(e_i,v) - A(e_i,u) + B(e_i,u)\}(2\Omega)^{-1} e^{-\Omega(u-v)}$$

$$-\{B(e_i,v) - B(e_i,u) - B(e_i,0) + A(e_i,0) - A(e_i,u)\}(2\Omega)^{-1} e^{-\Omega(u+v)}$$

$$+ 0(e^{-\delta_1(T-u)} e^{-\omega_1(u-v)}) \text{ if } 0 \le v \le u \le T$$

and

$$\frac{\partial G^{x,y,T}}{\partial y_i}(T-v,T-u)|_{y=\bar{y}} = B(e_i,v)(2\Omega)^{-1} e^{-\Omega(v-u)}$$

$$-\{B(e_i,v) - B(e_i,u) - B(e_i,0) + A(e_i,0) - A(e_i,u)\}$$

$$(2\Omega)^{-1} e^{-\Omega(u+v)} + 0(e^{-\delta_1(T-v)} e^{-\omega_1(v-u)}) \text{ if } 0 \le u \le v \le T\,.$$

3. Proof of Theorem C

Recall (1.10)

$$E_1(\lambda) - E_0(\lambda) = \frac{\int_{\partial W} (\Omega_1 \frac{\partial \Omega_0}{\partial n} - \Omega_0 \frac{\partial \Omega_1}{\partial n})\, dS(x)}{2\int_W \Omega_0\, \Omega_1\, dx}$$

where we take W to be an open bounded set containing a neighborhood of b and not intersecting some neighborhood of a. Moreover with a smooth boundary ∂W such that for an open neighborhood V of $K = \{x\colon \rho(x,a) = \rho(x,b) = \frac{\rho(a,b)}{2}\}$, as described in Proposition 1 and and Theorem B

$$\partial W \cap U = \{x : \rho(x,a) = \rho(x,b)\} \cap U = : Z$$

and

(3.1) $\quad \partial W \backslash U \subseteq \{x\colon \rho(x,a) \geq \frac{\rho(a,b)}{2} + \epsilon_0,\ \rho(x.b) \geq \frac{\rho(a,b)}{2} + \varepsilon_0\}$, for some $\varepsilon_0 > 0$.

By Lemma 1.2 defines a smooth hypersurface and for $x \in K$, Z intersects g_x transversally at $x = g_x(0)$, i.e. $\dot{g}_x(0)$ is a normal of Z of $x \in K$. By hypothesis (*) we have (1.15). By (1.20) we can bound Ω_0, $|\frac{\partial \Omega_0}{\partial n}|$, $|\Omega_1|$ and $|\frac{\partial \Omega_1}{\partial n}|$ by $0\left(e^{-\lambda(\min(\rho(x,a),\ \rho(x,b)) - \varepsilon_0)}\right)$ uniformly on ∂ W, since $\overline{W}$ is compact.

Thus by (3.1) we have (1.31):

$$\int_{\partial W} (\Omega_1 \frac{\partial \Omega_0}{\partial n} - \Omega_0 \frac{\partial \Omega_1}{\partial n})\, dS(x) = \int_Z (\Omega_1 \frac{\partial \Omega_0}{\partial n} - \Omega_0 \frac{\partial \Omega_1}{\partial n})\, dS(x) + e^{-\lambda(\rho(a,b) + \epsilon_0)}$$ as

$\lambda \to \infty$.

Now we use the formulas for Ω_0, Ω_1 and their gradients is Theorem B. Since $\alpha^2(\lambda) + \beta^2(\lambda) = 1$, $F_a(x)\ F_b(x)$, $F_{a_j}(x)$ and $F_{b_j}(x)$ are $0(1)$ and $\rho(x,a) = \rho(x\ b)$ on Z, we get:

$$\Omega_1 \frac{\partial\Omega_0}{\partial x_j} - \Omega_0 \frac{\partial\Omega_1}{\partial x_j}$$

$$= \lambda^{\left(\frac{d+2}{2}\right)} \Bigg\{ \left(\frac{\partial\rho(x,b)}{\partial x_j} - \frac{\partial\rho(x,a)}{\partial x_j} \right) f_a(x)\ f_b(x)$$

$$+ \frac{\left(\frac{\partial\rho(x,b)}{\partial x_j} f_a\ f_b\ f_{1,a} - \frac{\partial\rho(x,a)}{\partial x_j} f_a\ f_b\ f_{1,b} + f_{1,b,j}\ f_a - f_{1,a,j}\ f_b \right)}{\lambda}$$

$$+ 0(\lambda^{-\frac{3}{2}}) \Bigg\}\ e^{-\lambda(\rho(x,a) + \rho(x,b))}$$

for $x \in Z$.

W contains point with $\rho(a,x) - \rho(b,x) > 0$ near Z and the outward normal is

$$\frac{\rho'(x,b) - \rho'(x,a)}{|\rho'(x,b) - \rho'(x,a)|}.$$

Therefore

$$(\Omega_1 \frac{\partial\Omega_0}{\partial n} - \Omega_0 \frac{\partial\Omega_1}{\partial n}) = \lambda^{\frac{(d+2)}{2}} (F_0(x) + \frac{F_1(x)}{\lambda} + 0(\lambda^{-\frac{3}{2}}))\ e^{-\lambda(\rho(x,a) + \rho(x,b))}$$

uniformly on Z where $F_0(x) = |\ \rho'(x,b) - \rho'(x,a)|\ f_a(x)\ f_b(x)$ and F_0 and F_1 are C^∞ on Z by Theorem B and lemma 1.

This proves (1.50) of Theorem C, (1.52) follows from above and (1.53) from (2.20) and (2.29).

To prove (1.51) of Theorem C we note if

$$(3.1)\qquad \begin{cases} \ddot{f}(t) = V''(g_x(t))\, f(t) \text{ on } \mathbf{R} \\ \text{and } \int_{-\infty}^{\infty} |f|^2\, dt < \infty \end{cases}$$

then, since $g^{x,b} = g_x|_{[0,\infty)}$, $f|_{[0,\infty)}$ is a bounded solution of

$$(3.2)\qquad \ddot{\gamma}(t) = V''(g^{x,b}(t))\, \gamma(t) \text{ on } (0,\infty).$$

Recall from Theorem B that $X^{x,b}(t)$ is a nonsingular matrix solution of (3.2) with $X^{x,b}(t) = (I + 0(e^{-\delta_1 t})\, e^{-\sqrt{V''(b)}\, t}$ as $t \to \infty$. Any bounded solution of (3.2) (see Coppel [1] Chapter 2, proposition 1) is of the form $\gamma(t) = X^{x,b}(t)\,(X^{x,b}(0))^{-1}\gamma(0)$ (other solutions grow exponentially at ∞) .

Therefore f as in (3.1) is given by

$$(3.3)\qquad \begin{cases} f|_{[0,\infty)}(t) = X^{xb}(t)\left(X^{x,b}(0)\right)^{-1} f(0) \\ f|_{(-\infty,0]}(-t) = X^{x,a}(t)\,(X^{x,a}(0)^{-1}\, f(0) \end{cases}$$

Hence if f satisfies (3.1) and $h \in C_o^\infty(\mathbf{R}, \mathbf{R}^d)$ then

$$(3.4) \quad \int_{-\infty}^{\infty} \langle(-\ddot{f} + V''(g_x) f), h\rangle \, dt = \int_{-\infty}^{\infty} (\langle\dot{f}, \dot{h}\rangle + \langle V''(g_x) f, h\rangle) \, dt =$$

$$= (\int_{-\infty}^{0} + \int_{0}^{\infty}) = -\langle(\dot{X}^{a,x}(0)(X^{x,a}(0))^{-1} + \dot{X}^{x,b}(0)(X^{x,b}(0))^{-1}) f(0), h(0)\rangle$$

by (3.1) and (3.3)

$$= \langle(\rho''(x,a) + \rho''(x,b)) f(0), h(0)\rangle$$

by (2.20) and (2.19).

This shows the number of independent L^2- solution of $\ddot{f} = V''(g_x) f$ is equal the nullity of the Hessian of $\rho(x,a) + \rho(x,b)$.

If we restrict ourselves to solutions f of (3.1) with $\langle\dot{g}_x(0), f(0)\rangle = 0$ so f(0) is tangent to Z at $x \in K$ and h with $\langle\dot{g}_x(0), h(0)\rangle = 0$ we get by (3.4) the first part of (1.51). The second part follows from $\ddot{g}_x(t) = V'(g_x(t))$ and $\dot{g}_x$ is in $L^2(\mathbb{R})$ by (2.14).

4. A proof of Theorem B.

§4.1. The setup.

§4.2. The contribution to (1.31) and (1.32) from the complement of neighborhoods of a and b are small.

§4.3. The use of the L^2-asymptotics of the eigenfunctions.

§4.4. A stationary phase theorem from Hörmander [1] and lemmas 6 and 7.

§4.5. Asymptotics of terms not containing the polynomials.

§4.6. The terms with the polynomials.

§4.7. The Remainder.

§4.8. Cancellations and taking T = constant $\log \lambda$ completes the proof.

4. A proof of Theorem B.

§4.1. The setup.

As mentioned earlier, we use (1.33), (1.20) and (1.21) to estimate the contribution in (1.31) and (1.32) from $\mathbb{R}^d \setminus (B(a,\frac{\delta}{2}) \cup B(b,\frac{\delta}{2}))$, for δ as in Theorem A. Then we use (1.35) (resp. (1.36)) and the L^2-asymptotic expansions in (*), when integrating over $B(a,\frac{\delta}{2}) \cup B(b,\frac{\delta}{2})$ in (1.31) (resp. (1.32)).

In these asymptotic expansions and estimates we have "the same kind" of formulas for $e^{-T\frac{H(\lambda)}{\lambda}}(x,y)$ and $\frac{\partial}{\partial x_j} e^{-T\frac{H(\lambda)}{\lambda}}(x,y)$, and for Ω_0 and Ω_1. Therefore we only do the calculations for one of Ω_0, Ω_1, $\frac{\partial}{\partial x_i}\Omega_0$ and $\frac{\partial \Omega_1}{\partial x_i}$, Ω_0 say, and we claim the others are similar

§4.2. The contribution to (1.31) and (1.32) from the complement of neighborhoods of a and b, are small.

If $x_0 \in K$ and with $\delta > 0$ as in Theorem A we write

$$(4.1) \qquad \Omega_0(x,\lambda) = e^{T\frac{E_0(\lambda)}{\lambda}} \int e^{-T\frac{H(\lambda)}{\lambda}}(x,y)\, \Omega_0(y,\lambda)\, dy$$

$$= e^{T\frac{E_0(\lambda)}{\lambda}} \left(\int_{B(a,\frac{\delta}{2})} + \int_{B(b,\frac{\delta}{2})} + \int_{\mathbb{R}^d \setminus (B(a,\frac{\delta}{2}) \cup B(b,\frac{\delta}{2}))} \right)$$

$$e^{-T\frac{H(\lambda)}{\lambda}}(x,y)\, \Omega_0(y,\lambda)\, dy$$

By (1.33) in Theorem A, (1.14), and the assumption (*) in §1.3 we have

$$(4.2)\qquad e^{T\frac{E_0(\lambda)}{\lambda}}\left(\int_{\mathbb{R}^d\setminus(B(a,\frac{\delta}{2})\cup B(b,\frac{\delta}{2}))}\right)$$

$$0(e^{-Te_0}\, e^{-\lambda(\min\{\rho(x,a),\rho(x,b)\}+\frac{\gamma\delta^2}{4})}\qquad \Omega_0(y,\lambda)\,dy$$

$$= 0(e^{-\lambda(\min\{\rho(x,a),\rho(x,b)\}+\frac{\gamma\delta^2}{4})}\int_{\mathbb{R}^d\setminus(B(a,\frac{\delta}{2})\cup B(b,\frac{\delta}{2}))}\Omega_0(y,\lambda)\,dy$$

uniformly for $x \in B(x_0\,,\,\delta)$ when $T,\lambda \to \infty$ and $\frac{T}{\lambda} \to 0$.

By (1.20) and (1.21)

$$\int_{\mathbb{R}^d\setminus(B(a,\frac{\delta}{2})\cup B(b,\frac{\delta}{2}))}\Omega_0(y,\lambda)\,dy \quad =$$

$$\left(\int_{B(0,R)\setminus(B(a,\frac{\delta}{2})\cup B(b,\frac{\delta}{2}))} + \int_{\mathbb{R}^d\setminus B(0,R)}\right)\Omega_0(y,\lambda)\,dy =$$

$$= 0_\varepsilon\left(\int_{B(0,R)\setminus(B(a,\frac{\delta}{2})\cup B(b,\frac{\delta}{2}))} e^{-\lambda\{\min(\rho(y,a),\rho(y,b)-\varepsilon\}}\right)$$

$$+\ 0\left(\int_{\mathbb{R}^d\setminus B(0,R)} e^{-\lambda c_3|y|}\,dy\right)$$

for large R and for any $\varepsilon > 0$.

If $y \notin B(a,\frac{\delta}{2}) \cup B(b,\frac{\delta}{2})$ then $\rho(y,a),\ \rho(y,b) \geq$ some $\varepsilon_0 > 0$. After taking $\varepsilon = \frac{\varepsilon_0}{2}$ we get

$$\int_{\mathbf{R}^d\setminus(B(a,\frac{\delta}{2})\cup B(b,\frac{\delta}{2}))} \Omega_0(y,\lambda)\,dy = 0\left(e^{-\lambda\frac{\varepsilon_0}{2}} + e^{-\lambda c_3 R}\right) = 0(1),$$

which is all we need.

Now we have, by (4.2)

$$(4.3)\qquad e^{T\frac{E_0(\lambda)}{\lambda}} \int_{\mathbf{R}^d\setminus(B(a,\frac{\delta}{2})\cup B(b,\frac{\delta}{2}))} e^{-T\frac{H(\lambda)}{\lambda}}(x,y)\,\Omega_0(y,\lambda)\,dy$$

$$= 0\left(e^{-\lambda(\min(\rho(x,a),\rho(x,b))+\gamma\frac{\delta^2}{4})}\right)$$

uniformly for $x \in B(x_0\ ,\ \delta)$ as $T,\ \lambda \to \infty$ and $\frac{T}{\lambda} \to 0$.

§4.3. The use of the L^2-asymptotics of the eigenfunctions.

If $x_0 \in K$ then, by Theorem A, there is a $\delta > 0$ such that we can expand $e^{-T\frac{H(\lambda)}{\lambda}}(x,y)$ as in (1.35) for $x \in B(x_0\ ,\ \delta)$ and $y \in B(a,\delta) \cup B(b,\delta)$.

Now we look at $\int_{B(b,\delta)} e^{-T\frac{H(\lambda)}{\lambda}}(x,y)\ \Omega_0(y,\lambda)\ dy$, say. After change of variables we have (1.34) $V''(b) = \Omega^2$. By assumption (*), $\Omega_0(x,\lambda) =$

$$\alpha(\lambda)(\psi_{2m+2,a}(x,\lambda) + r_{2m+2,a}(x,\lambda)) + \beta(\lambda)(\psi_{2m+2,b}(x,\lambda) + r_{2m+2,b}(x,\lambda))$$

with

$$\psi_{2m+2,c}(x,\lambda) = \lambda^{\frac{d}{2}} \varphi_c(\lambda^{\frac{1}{2}}(x-c))\Big(1 + \frac{P_{1,c}(\lambda^{\frac{1}{2}}(x-c))}{\lambda^{\frac{1}{2}}} + \cdots + \frac{P_{2m+2,c}(\lambda^{\frac{1}{2}}(x-c))}{\lambda^{(m+1)}}\Big)$$

and $\|r_{2m+2}(\,\cdot\,,\lambda)\|_2 = 0(\lambda^{-(m+\frac{3}{2})})$.

Here φ_c is the Gaussian groundstate of

$$h(c) = -\tfrac{\Delta}{2} + \tfrac{1}{2}\langle V''(c)x,x\rangle \text{ on } L^2(\mathbb{R}^d),$$

$P_{1,c}, \ldots, P_{2m+2,c}$ are polynomials,

$P_{1,c}$ is odd, $\alpha^2 + \beta^2 = 1$ and $\alpha(\lambda)\beta(\lambda) \geq c_1\,\lambda^{-m} > 0$ as $\lambda \to \infty$.

Now we write this as

$$\text{(4.4)} \qquad \Omega_0(x,\lambda) = \beta(\lambda)\Big\{\lambda^{\frac{d}{4}}\varphi_b(\lambda^{\frac{1}{2}}(x-b))$$

$$\Big(1 + P_{1,b}(\lambda^{-\frac{1}{2}}(x-b))\lambda^{\frac{1}{2}} + P_{2,b}(\lambda^{\frac{1}{2}}(x-b))\lambda^{-1} + \tilde{r}_{2,b}(x,\lambda)\Big)\Big\}$$

where $\varphi_b(x) = (\det \Omega)^{\frac{1}{4}}\,\pi^{-\frac{d}{4}}\,e^{-\frac{1}{2}\langle\Omega x,x\rangle}$

and $\|\tilde{r}_{2,b}(\,\cdot\,,\lambda)\|_{L^2(B(b,\delta))} = 0(\lambda^{-\frac{3}{2}})$ as $\lambda \to \infty$

where we used the $L^2(B(b,\delta))$-norm of $\lambda^{\frac{d}{4}}\ \beta(\lambda)\ \varphi_b(\lambda^{\frac{1}{2}}(x-b))\ \dfrac{P_{i,b}(\lambda^{\frac{1}{2}}(x-b))}{\lambda^{\frac{i}{2}}}$ is $0(\beta(\lambda)\lambda^{-\frac{i}{2}}) = 0(\beta(\lambda)\lambda^{-\frac{3}{2}})$ for $i = 3,\ldots,2m+2$. The $L^2(B(b,\delta))$-norm of

$\beta(\lambda)\ r_{2m+2,b}(x,\lambda)$ is $0(\beta(\lambda)\lambda^{-(m+\frac{3}{2})}) = (\beta(\lambda)\lambda^{-\frac{3}{2}})$ and of $\varphi_a(\lambda^{\frac{1}{2}}(x-b))$ is exponentially small, so it is $0(\beta(\lambda)\ \lambda^{-\frac{3}{2}})$. Finally, the $L^2(B(b,\delta))$-norm of $\alpha(\lambda)\ r_{2m+2,a}(x,\lambda)$ is $0(\alpha(\lambda)\ \lambda^{-(m+\frac{3}{2})}) = 0(\lambda^{-(m+\frac{3}{2})}) = 0(\beta(\lambda)\lambda^{-\frac{3}{2}})$, since $\alpha(\lambda) \leq 1$ and $\lambda^{-m} \leq \alpha(\lambda)\ \beta(\lambda)\ c_1^{-1} \leq \beta(\lambda)\ c_1^{-1}$ where $c_1 > 0$. By (*) we know $e_0 = \frac{\mathrm{trace}(V''(b))^{\frac{1}{2}}}{2}$ and hence by (1.25) $e^{-T\frac{H(\lambda)}{\lambda}}(x,y) = \pi^{-\frac{d}{2}}\ \lambda^{\frac{d}{2}}(\det\ \Omega)^{\frac{1}{2}}\ a_0^T(x,y)$ $e^{-\lambda A(x,y,T)}\ e^{-Te_0}$

$$\{1 + \frac{a_1^T(x,y)}{\lambda} + 0(\frac{T^4}{\lambda^2}) + 0(\frac{T^2}{\lambda}\ \exp(-\lambda\ \frac{c_4}{T^4}))\}$$

Therefore by (4.4) with $P_{0,b}(y) = 1$,

$$(4.5)\qquad \int_{B(b,\frac{\delta}{2})} e^{-T\frac{H(\lambda)}{\lambda}}(x,y)\ \Omega_0(y,\lambda)\ dy =$$

$$\pi^{-\frac{3d}{4}}\ e^{-Te_0}\ \beta(\lambda)\ \lambda^{\frac{3d}{4}}(\det\ \Omega)^{\frac{3}{4}}\left(\sum_{i=0}^{2}\left(\frac{I_{0,i}(x,T,\lambda)}{\lambda^{\frac{i}{2}}} + \frac{I_{1,i}(x,T,\lambda)}{\lambda^{(1+\frac{i}{2})}}\right)\right.$$

$$+ R_1(x,T,\lambda) + R_2(x,T,\lambda)$$

where

$$(4.6)\qquad I_{0,i}(x,T,\lambda) = \int_{B(b,\frac{\delta}{2})} e^{-\lambda(A(x,y,T)+\frac{1}{2}<\Omega(y-b),(y-b)>)}\, a_0^T(x,y)\, P_{i,b}(\lambda^{\frac{1}{2}}(y-b))\, dy \text{ for } i \in \{0,1,2\}$$

$$(4.7)\qquad I_{1,i}(x,T,\lambda) = \int_{B(b,\frac{\delta}{2})} e^{-\lambda(A(x,y,T)+\frac{1}{2}<\Omega(y-b),(y-b)>)}\, a_0^T(x,y)\, a_1^T(x,y)\, P_{i,b}(\lambda^{\frac{1}{2}}(y-b))\, dy \quad \text{for } i \in \{0,1,2\}$$

$$(4.8)\qquad R_1(x,T,\lambda) = \beta(\lambda) \int_{B(b,\frac{\delta}{2})} e^{-T\frac{H(\lambda)}{\lambda}}(x,y)\, \tilde{r}_{2,b}(y,\lambda)\, dy$$

and

$$(4.9)\qquad R_2(x,T,\lambda) = \lambda^{\frac{3d}{4}}\, e^{-Te_0}\, \beta(\lambda)\left\{0\left(\frac{T^4}{\lambda^2}\right) + 0\left(\frac{T^2}{\lambda}\exp\left(-\lambda\frac{c_4}{T^4}\right)\right)\right\}\left(\sum_{i=0}^{2} \frac{I_{0,i}(x,T,\lambda)}{\lambda^{\frac{i}{2}}}\right)$$

For a fixed (x,T) we integrate over y in $B(b,\frac{\delta}{2})$ with help from:

§4.4. A stationary phase theorem from Hörmander [1] and lemmas 6 and 7.

We will frequently quote the following;

Theorem (Hörmander [1], Theorem 7.7.5). Let $K \subset \mathbb{R}^n$ be a compact set, X an open neighborhood of K and k a positive integer. If $u \in C_0^{2k}(K)$, $f \in C^{3k+1}(X)$ and $\operatorname{Im} f \geqq 0$ in X, $\operatorname{Im} f(x_0) = 0$, $f'(x_0) = 0$, $\det f''(x_0) \neq 0$, $f' \neq 0$ in $K\backslash\{x_0\}$ then

$$(4.10) \qquad |\int u(x)e^{i\omega f(x)}\,dx - e^{i\omega f(x_0)}(\det(\omega f''(x_0)/2\pi i))^{-\frac{1}{2}} \sum_{j<k} \omega^{-j} L_j u|$$

$$\leqq C\omega^{-k} \sum_{|\alpha|\leqq 2k} \sup|D^\alpha u|, \qquad \omega > 0.$$

(4.11) Here C is bounded when f stays in a bounded set in $C^{3k+1}(X)$ and $|x-x_0|/|f'(x)|$ has a uniform bound.

With

$$g_{x_0}(x) = f(x) - f(x_0) - \langle f''(x_0)(x-x_0), (x-x_0)\rangle/2$$

which vanishes of third order at x_0 we have

$$L_j u = \sum_{v-\mu=j} \sum_{2v\geqq 3\mu} i^{-j} 2^{-v} \langle f''(x_0)^{-1}D,D\rangle^v (g_{x_0}^\mu u)(x_0)/\mu!\, v!\,.$$

This is a differential operator of order 2j acting on u at x_0. The coefficients are rational homogeneous functions of degree $-j$ in $f''(x_0), \ldots, f^{(2j+2)}(x_0)$ with denominator $(\det f''(x_0))^{3j}$. In every term the total number of derivatives of u and of f'' is at most 2j.

Remark. Here $i = \sqrt{-1}$ and the integration in (4.10) is over K.

We will need a continuation of Theorem A:

Lemma 6. In part 2 of Theorem A we have: For each $(x,T) \in B(x_0,\delta) \times [T_0 , \infty)$ the function

(4.12) $\overline{B(b,\delta)} \ni y \mapsto A(x,y,T) + \frac{1}{2}\langle\Omega(y-b)(y-b)\rangle$ $[y \mapsto A(x,y,T)]$ attains its absolute minimum value at a unique point $\bar{y} = \bar{y}(x,T)$ [resp. $\bar{\bar{y}} = \bar{\bar{y}}(x,T)$] satisfying

(4.13) $\bar{y} - b = 0(e^{-\omega_1 T})$ [resp. $\bar{\bar{y}} - b = 0(e^{-\omega_1 T})$]

uniformly for $(x,T) \in B(x_0 , \delta) \times [T_0 , \infty)$.

(Proof in Chapter 16.)

For $(x,T) \in B(x_0 , \delta) \times [T_0 , \infty)$ we write (here $i = \sqrt{-1}$)

(4.14) $A(x,y,T) + \frac{1}{2}\langle\Omega(y-b), (y-b)\rangle =:$

$A(x,\bar{y},T) + \frac{1}{2}\langle\Omega(\bar{y}-b), (\bar{y}-b)\rangle - i\, f_{x,T}(y)$

and

(4.15) $A(x,y,T) =: A(x,\bar{\bar{y}},T) - i\,\tilde{f}_{x,T}(y)$

which defines $f_{x,T}$ and $\tilde{f}_{x,T}$ on $B(b,\delta)$.

Then

$$\mathrm{Im}\, f_{x,T}(y) \geq 0, \ \mathrm{Im}\, f_{x,T}(y) = 0$$

only if $y = \bar{y}(x,T)$ and $f'_{x,T}(\bar{y}) = 0$, i.e., $\bar{y}(x,T)$ is the only point in $B(b,\delta)$ where $f_{x,T}$ is real and stationary and similar $\bar{\bar{y}}$ for $\tilde{f}_{x,T}$.

To use Hörmander's stationary phase theorem with uniformity (see (4.11)) for $(x,T) \in B(x_0 , \delta) \times [T_0 , \infty)$ we state

Lemma 7. In Theorem A.2 with same notation as there and $\overline{y} = \overline{y}(x,T)$ as in lemma 6 there exist positive constants $c_{0,\beta}$ for $\beta \in N_0^d$ with $|\beta| \leq 2$, $c_{4,\beta}$ for $\beta \in N_0^d$ for $|\beta| \geq 3$, c_5, $c_{6,j}$ for $j \in \{1,...,d\}$, c_7 and $C^\infty(B(x_0, \delta))$ - functions $a_1(x)$ and $a_{1j}(x)$ for $j \in \{1,...,d\}$ such that *uniformly* for $(x,y,T) \in B(x_0, \delta) \times B(b,\delta_0) \times [T_0, \infty)$ and for $(x,T) \in B(x_0, \delta) \times [T_0, \infty)$ when $y = \overline{y}$, we have

1. For each $\alpha \in N_0^d$

(4.16) $$\frac{\partial^{|\alpha|}}{\partial y^\alpha} a_0^T(x,y) = 0(1)$$

and for each $\beta \in N_0^d$ with $|\beta| \leq 2$

(4.17) $$\frac{\partial^{|\beta|}}{\partial y^\beta} a_0^T(x,y,T)|_{y=\overline{y}} = c_{0,\beta}\, a_0(x) + 0(e^{-\delta_1 T})$$

where $c_{0,0} = 1$.

2. For each $\alpha \in N_0^d$

(4.18) $$\frac{\partial^{|\alpha|}}{\partial y^\alpha} A(x,y,T) = 0(1) \text{ and } A''_{yy}(x,y,T) =$$

$$= \Omega + 0(e^{-\delta_1 T} + |y-b|) \geq \text{constant} > 0,$$

(4.19) $$\frac{\partial^{|\beta|}}{\partial y^\beta} A(x,y,T)|_{y=\overline{y}} = c_{4,\beta} + 0(e^{-\delta_1 T}) \text{ if } |\beta| \geq 3$$

and

(4.20) $A(x,\bar{y},T) = \rho(x,b) + 0(e^{-2\omega_1 T}) = A(x,\bar{\bar{y}},T).$

3. For each $\alpha \in N_0^d$

(4.21) $\frac{\partial^{|\alpha|}}{\partial y^\alpha} a_1^T(x,y) = 0(T)$ and

(4.22) $a_1^T(x,y,T)|_{y=\bar{y}} = c_5\, T + a_1(x) + 0(Te^{-\delta_1 T}).$

4. For each $(\alpha,j) \in N_0^d \times \{1,\ldots,d\}$

(4.23) $\frac{\partial^{|\alpha|}}{\partial y^\alpha} a_{1,j}^T(x,y) = 0(T)$

and

(4.24) $a_{1,j}^T(x,\bar{y}) = C_{6,j}\, T + a_{1,j}(x) + 0(Te^{-\delta_1 T})$

5. With $f_{x,T}(y)$ and $\tilde{f}_{x,T}(y)$ defined in (4.14) and (4.15) we have

(4.25) $\frac{|y-\bar{y}|}{|f'_{x,T}(y)|} \le c_7 < \infty$ and $\frac{|y-\bar{\bar{y}}|}{|\tilde{f}'_{x,T}(y)|} \le c_7$ (recall (4.11)).

(A proof in Chapter 11.)

Now we take $X = B(b,\delta)$ and $K = \overline{B(b,\frac{\delta}{2})}$ in Hömander's stationary phase theorem.

§4.5. Asymptotics of terms not containing the polynomials.

Recall from (4.14) we defined for each $(x,T) \in B(x_0, \delta) \times [T_0, \infty)$

$$f_{x,T}(y) = i(A(x,y,T) + \tfrac{1}{2}\langle\Omega(y-b),(y-b)\rangle - (A(x,\bar{y},T) + \tfrac{1}{2}\langle\Omega(\bar{y}-b),(\bar{y}-b)\rangle)) \text{ for}$$

$y \in B(b,\delta)$, where $\bar{y} = \bar{y}(x,T)$ is the unique minimum point of $y \mapsto A(x,y,T) + \frac{1}{2}\langle\Omega(y-b),(y-b)\rangle$ and is the only point where $f_{x,T}$ is real and stationary.

Now using the method of stationary phase we get

$$I_{0,0}(x,T,\lambda) = \int_{|y-b|\leq\frac{\delta}{2}} e^{-\lambda(Ax,y,T)+\frac{1}{2}\langle\Omega(y-b),(y-b)\rangle} e^{i\lambda f_{x,T}(y)} a_0^T(x,y)\, dy$$

$$= e^{-\lambda(A(x,\bar{y},T)+\frac{1}{2}\langle\Omega(\bar{y}-b),(\bar{y}-b)\rangle)}\Bigg\{\left(\det\left[\frac{[\lambda f''_{x,T}(\bar{y})]}{2\pi i}\right]\right)^{-\frac{1}{2}}$$

$$\left[(L_0 a_0^T(x,y))(\bar{y}) + \frac{(L_1\ a_0^T(x,y))(\bar{y})}{\lambda} + \cdots + \frac{(L_{k-1}\ a_0^T(x,y))(\bar{y})}{\lambda^{(k-1)}}\right] +$$

$$+\ 0\ \frac{\left(\sum_{|\alpha|\leq 2k}\ \sup_{(x,y,T)\in B(x_0,\delta)\times B(b,\frac{\delta}{2})\times[T_0,\infty)} |\partial_y^\alpha a_0^T(x,y)|\right)}{\lambda^k}\Bigg\}$$

for any integer $k \geq 1$ and we take

(4.26) $\quad k \in [\frac{d+4}{2}, \frac{d+4}{2} + 1)$. By (4.25), $|y-\bar{y}/|f'_{x,T}(y)| \leq c_7 < \infty$

and (4.28) $\frac{\partial^{|\alpha|}}{\partial y^{\alpha}} A(x,y,T) = 0(1)$ uniformly for $|\alpha| \leq 3k + 1$

and $(x,y,T) \in B(x_0, \delta) \times B(b,\delta) \times [T_0, \infty)$ and thus the

$0(\cdot)$ is uniform for $(x,T) \in B(x_0, \delta) \times [T_0, \infty)$.

By (4.19)

$$\left(\det\left[\frac{\lambda f''_{x,T}(\bar{y})}{2\pi i}\right]\right)^{-\frac{1}{2}} = \left(\det\left[\frac{\lambda(2\Omega+0(e^{-\delta_1 T}))}{2\pi}\right]\right)^{-\frac{1}{2}} = \lambda^{-\frac{d}{2}}(\det\Omega)^{-\frac{1}{2}}\pi^{\frac{d}{2}}(1+0(e^{-\delta_1 T}))$$

uniformly for $(x,T) \in B(x, \delta) \times [T_0, \infty)$.

By (4.17)

$$L_0(a_0^T(x,y))(\bar{y}) = a_0^T(x,\bar{y}) = a_0(x)(1 + 0(e^{-\delta_1 T})),$$

$(L_1 a_0^T(x,y))(\bar{y})$ is a differential operator of order two acting on $a_0^T(x, \cdot)$ at $\bar{y}$ with coefficients that are rational homogeneous functions of degree -1 in $f''_{x,T}(\bar{y}) = 2\Omega + 0(e^{-\delta_1 T})$, $f^{(3)}_{x,T}(\bar{y}) = A^{(3)}_{yyy}(x,y,T)|_{y=\bar{y}}$ and $f^{(4)}_{x,T}(\bar{y}) = A^{(4)}_{yyyy}(x,y,T)|_{y=\bar{y}}$. By (4.17) $\frac{\partial^{|\alpha|}}{\partial y^{\beta}} a_0^T(x,y)|_{y=\bar{y}} = C_{0,\beta}\, a_0(x)$ where $C_{0,\beta}$ are constants and $a_0(x)$ is C^{∞} on $B(x_0, \delta)$ by Theorem A and by (4.19) $\frac{\partial^{|\beta|}}{\partial y^{\beta}} A(x,y,T)|_{y=\bar{y}} = C_{4,\beta} + 0(e^{-\delta_1 T})$ where $c_{4,\beta}$ is constant. Hence $L_1(a_0^T(x,\cdot)(\bar{y}) = a_{0,1}(x) + 0(e^{-\delta_1 T})$ uniformly for $(x,T) \in B(x_0, \delta) \times [T_0, \infty)$ where $a_{0,1}(x)$ is in $C^\infty(B(x_0, \delta))$.

Since we have uniform bound for any fixed number of derivatives of $a_0^T(x,\cdot)$ and $A(x,\cdot,T)$ in (4.16) and (4.18) and the form of each L_j, $j = 2, \ldots, k$ where $k \in [\frac{d+4}{2}, \frac{d+4}{2} + 1)$ we have, by above

(4.27) $\quad I_{0,0}(x,T,\lambda) =$

$$\left\{\left[\lambda^{-\frac{d}{2}}(\det \Omega)^{-\frac{1}{2}}\pi^{\frac{d}{2}}\left\{a_0(x)(1+0(e^{-\delta_1 T}))+\frac{a_{0,1}(x)(1+0(e^{\delta_1 T}))}{\lambda}+\cdots+0(\frac{1}{\lambda^{(k-1)}})\right\}\right]\right.$$

$$\left. + 0(\lambda^{-(\frac{d+4}{2})})\right\} \exp(-\lambda(A(x,\bar{y},T) + \tfrac{1}{2}\langle\Omega(\bar{y}-b),(\bar{y}-b)\rangle)) =$$

$$= \lambda^{\frac{d}{2}}(\det \Omega)^{-\frac{1}{2}}\pi^{\frac{d}{2}}\, a_0(x)(1 + 0(e^{-\delta_1 T}))\, e^{-\lambda(\rho(x,b)+0(e^{-2\omega_1 T}))}$$

$$\cdot\left(1 + \frac{\tilde{a}_{0,1}(x)}{\lambda} + 0(\frac{1}{\lambda^2})\right)$$

uniformly for $(x,T) \in B(x_0, \delta) \times [T_0, \infty)$ where a_0 is as in Theorem A and $\tilde{a}_{0,1} = \frac{a_{0,1}}{a_0} \in C^\infty(B(x_0, \delta))$. Above we used $a_0(x) \geq$ const. > 0 by Theorem A, $A(x,\bar{y},T) = \rho(x,b) + 0(e^{-2\omega_1 T})$ by (4.20) and $\bar{y} = \bar{y}(x,T) = b + 0(e^{-\omega_1 T})$ by (4.13). All estimates being uniform in those x and T's.

In a similar way using also (4.21): For each $\alpha \in \mathbb{N}_0^d$, $\frac{\partial^{|\alpha|}}{\partial y\alpha} a_1^T(x,y)| = 0(T)$ uniformly for $(x,y,T) \in B(x_0 , \delta) \times B(b,\delta) \times [T_0 , \infty)$ and (4.22) $a_1^T(x,\bar{y}) = c_5 \cdot T + a_1(x) + 0(Te^{-\delta_1 T})$ uniformly, we get by estimating

$$(L_j a_0^T(x,y\ )\ a_1^T(x,y))(\bar{y}\ ) = 0(T) \text{ for } j = 1, ..., k - 1$$

and

$$(\sum_{|\alpha|\leq 2k} \sup_{(x,T,y)} |\partial_y^\alpha(a_0^T(x,y)\ a_1^T(x,y))|) = 0(T)$$

uniformly for $(x,T) \in B(x_0 , \delta) \times [T_0 , \infty)$, with k as in (4.26) above.

$$(4.28) \qquad I_{1,0}(x,y,T) = \int_{|y-b|\leq\frac{\delta}{2}} e^{-\lambda(A(x,y,T)+\frac{1}{2}\langle\Omega(y-b),(y-b)\rangle)}$$

$$a_0^T(x,y)\ a_1^T(x,y)\ dy = = \lambda^{-\frac{d}{2}}(\det\Omega)^{-\frac{1}{2}}\ \pi^{\frac{d}{2}}\ a_0(x)(1 + 0(e^{-\delta_1 T}))$$

$$(c_5 \cdot T + a_{1,0}(x) + 0(Te^{-\delta_1 T}))e^{-\lambda(\rho(x,b)+0(e^{-2\omega_1 T}))}(1 + 0(\tfrac{T}{\lambda}))$$

uniformly for $(x,T) \in B(x_0 , \delta) \times [T_0 , \infty)$.

§4.6. <u>The terms with the polynomials</u>.

If $h_{x,T}(y) \in C^\infty(B(b,\delta))$ for $(x,T) \in B(x_0 , \delta) \times [T_0 , \infty]$ and $P_\alpha(y-b) = (y-b)^\alpha$ where $\alpha = (\alpha_1 ,..., \alpha_d) \in \mathbb{N}_0^d$ is multi index with $|\alpha| = \sum_{i=1}^{d} |\alpha_i| = n$ then if we take $k \in [\frac{n+d+6}{2} , \frac{n+d+8}{2})$ we have

$$(4.29) \quad \lambda^{\frac{n}{2}} \int_{|y-b|<\frac{\delta}{2}} P_\alpha(y-b)\, h_{x,T}(y)\, e^{-\lambda(A(x,y,T)+\frac{1}{2}\langle\Omega(y-b),(y-b)\rangle)}\, dy$$

$$= \lambda^{\frac{n}{2}}[\lambda^{-\frac{d}{4}}(\det \Omega)^{-\frac{1}{2}}\, \pi^{\frac{d}{2}}(1 + 0(e^{-\delta_1 T}))(L_0(h_{x,T}\, P_\alpha))(\overline{y})$$

$$+ \frac{L_1(\ )}{\lambda} + \cdots + \frac{L_{k-1}(h_{x,T}P_\alpha)(\overline{y})}{\lambda^{(k-1)}}) +$$

$$0\, \frac{\left(\sum_{|\beta|\leq 2k} \sup_{(x,y,T)} |\partial_y^\beta h_{x,T}(y)| \right)}{\lambda^k}\,]\, \exp(-\lambda(\rho(x,b) + 0(e^{-2\omega_1 T})))$$

uniformly for $(x,T) \in B(x_0\, , \delta) \times [T_0\, , \infty)$ and for $\lambda > 0$ (as in the proof of (4.27), this uniformity follows from (4.18) and (4.25)).

Since L_j is a differential operator of order 2j and $\overline{y} = \overline{y}(x,T) = b + 0(e^{-\omega_1 T})$ uniformly for $(x,T) \in B(x_0\, , \delta) \times [T_0\, , \infty)$ we have $L_j(h_{x,T}(y)\ P_\alpha(y))(\overline{y}) = \widetilde{0}(e^{-\omega_1(n-2j)T})$ for integers $j \leq \frac{n}{2}$ and $L_j(h_{x,T}(y)\ P_\alpha(y))(\overline{y}) = \widetilde{0}(1)$ for $j \geq \frac{n}{2}$ where $\sim$ is to remind us on the dependence of $h_{x,T}(y)$ and its y-derivatives of order $\leq 2j$.

If n is odd we write (4.29) as

$$(4.30)\qquad \lambda^{\frac{n}{2}} \int_{|y-b|\le\frac{\delta}{2}} P_\alpha(y-b)\, h_{x,T}(y)\, e^{-\lambda(A(x,y,T)+\frac{1}{2}\langle\Omega(y-b),(y-b)\rangle)}\, dy$$

$$= \lambda^{\frac{n}{2}}[\lambda^{-\frac{d}{2}}(\det\Omega)^{-\frac{1}{2}}\pi^{\frac{d}{2}}(1+0(e^{-\delta_1 T}))$$

$$\{(\widetilde{0}(e^{-\omega_1 nT}) + \widetilde{0}(\frac{e^{-\omega_1(n-2)T}}{\lambda}) + \cdots + \widetilde{0}(\frac{e^{-\omega_1 T}}{\lambda^{\frac{(n-1)}{2}}}) +$$

$$+ \frac{L_{\frac{n+1}{2}}(h_{x,T}(y)P_\alpha(y))(\bar{y})}{\lambda^{(\frac{n+1}{2})}} + \widetilde{0}(\frac{1}{\lambda^{(\frac{n+3}{2})}}) + \cdots + \widetilde{0}(\frac{1}{\lambda^{\frac{(n+d+4)}{2}}})\}$$

$$+ \widetilde{0}(\frac{1}{\lambda^{\frac{(n+d+6)}{2}}})]\exp(-\lambda(\rho(x,b)+0(e^{-2\omega_1 T}))) =$$

$$= \lambda^{-\frac{d}{2}}(\det\Omega)^{-\frac{1}{2}}\pi^{\frac{d}{2}}(1+0(e^{-\delta_1 T}))$$

$$\left(\frac{(L_{\frac{n+1}{2}}(h_{x,T}(y)P_\alpha(y))(\bar{y})}{\lambda^{\frac{1}{2}}} + \widetilde{0}(\frac{1}{\lambda^{\frac{3}{2}}}) + \widetilde{0}(e^{-\omega_1 T}\lambda^{\frac{1}{2}})\right)e^{-\lambda(\rho(x,b)+0(e^{-2\omega_1 T}))}$$

uniformly for $(x,T,\lambda) \in B(x_0, \delta) \times [T_0, \infty) \times (0,\infty)$ when $e^{-\omega_1 T}\lambda^{\frac{1}{2}} \le$ some constant and

$$(4.31)\qquad \widetilde{0}(\cdot) = 0(\cdot \sum_{|\beta|\le n+d+8} \sup_{(x,y,T)} |\partial_y^\beta h_{x,T}(y)|).$$

Similar if $|\alpha| = n$ is even we have for $P_\alpha(y-b) = (y-b)^\alpha$

(4.32) $$\lambda^{\frac{n}{2}} \int_{|y-b|\leq\frac{\delta}{2}} P_\alpha(y-b)\, h_{x,T}(y)\, e^{-\lambda(A(x,y,T)+\frac{1}{2}\langle\Omega(y-b),(y-b)\rangle)}\, dy$$

$$= \lambda^{-\frac{d}{2}} (\det \Omega)^{-\frac{1}{2}} \pi^{\frac{d}{2}} (1 + 0(e^{-\delta_1 T}))(L_{\frac{n}{2}}(h_{x,T}(y)P_\alpha(y))(\bar{y})$$

$$+ \tilde{0}(\tfrac{1}{\lambda}) + \tilde{0}(e^{-\omega_1 T}\lambda^{\frac{1}{2}})) \text{ uniformly for } (x,T,\lambda) \in B(x_0, \delta)$$

$$\times [T_0, \infty) \times (0,\infty) \text{ as long as } e^{-\omega_1 T}\lambda^{\frac{1}{2}} \leq \text{some constant and}$$

with $\tilde{0}(\cdot)$ as in (4.31).

Write each

$$P_{i,b}(y) = \sum_{k=1}^{m_i} \Big(\sum_{|\alpha|=2k} c_\alpha^{(i)} (y-b)^\alpha + \sum_{|\alpha|=2k+1} c_\alpha^{(i)} (y-b)^\alpha \Big)$$

$$= \sum_{k=1}^{m_i} \Big(\sum_{|\alpha|=2k} c_\alpha^{(i)} P_\alpha(y-b) + \sum_{|\alpha|=2k+1} c_\alpha^{(i)} P_\alpha(y-b) \Big)$$

then with $h_{x,T}(y) = a_0^T(x,y)$, and hence $\tilde{0}(\cdot) = 0(\cdot)$ by (4.16), we have by (4.30) and (4.32)

$$(4.33) \qquad I_{0,i}(x,T,\lambda) = \int_{|y-b|\le\frac{\delta}{2}} e^{-\lambda(A(x,y,T)+\frac{1}{2}\langle\Omega(y-b),(y-b)\rangle)}$$

$$a_0^T(x,y)\ P_{i,b}(\lambda^{\frac{1}{2}}(y-b))\ dy =$$

$$= \Bigg(\sum_{k=1}^{m_i} \Big(\sum_{|\alpha|=2k} c_\alpha^{(i)} \lambda^{-\frac{d}{2}} (\det \Omega)^{-\frac{1}{2}} \pi^{\frac{d}{2}} (1 + 0(e^{-\delta_1 T}))$$

$$(L_k(a_0^T(x,y)\ P\alpha(y))(\bar{y}) + 0(\tfrac{1}{\lambda}) + 0(e^{-\omega_1 T}\ \lambda^{\frac{1}{2}}\})$$

$$+ \sum_{|\alpha|=2k-1}^{m_i} \Big\{ C_\alpha^{(i)}\ \lambda^{-\frac{d}{2}} (\det \Omega)^{-\frac{1}{2}} \pi^{\frac{d}{2}} (1 + 0(e^{-\delta_1}))$$

$$\frac{(L_k(a_0^T(x,y)\ P\alpha(y))(\bar{y})}{\lambda^{\frac{1}{2}}} + 0(\frac{1}{\lambda^{\frac{3}{2}}}) + 0(e^{-\omega_1 T}\ \lambda^{\frac{1}{2}}))\Big\}\Bigg)$$

$$\exp(-\lambda(\rho(x,b) + 0(e^{-2\omega_1 T})))$$ with uniformity as in (4.30).

Note the first contribution is from the even part and the second is from the odd part of the polynomial.

If $2i = |\alpha|$ then $L_i(a_0^T(x,\cdot)\ P\alpha(\cdot))(\bar{y}) =$ constant $a_0(x) + 0(e^{-\delta_1 T})$ uniformly for those (x,T)'s, since L_i is a differential operator of order 2i acting on $a_0^T(x,\cdot)\ P\alpha(\cdot)$ at $y = \bar{y}$, with coefficients, being rational functions of y-derivatives of $A(x,y,T)$ or order $2,3, \ldots, 2i + 2$ at $y = \bar{y}$, of the form (constant $+ 0(e^{-\delta_1 T}))$.

Unless all derivatives act on P_α we get $0(e^{-\omega_1 T}) = 0(|\bar{y} - b|)$.

When all derivatives act on P_α, we get

$$a_0^T(x,\bar{y})(\text{constant} + 0(e^{-\delta_1 T})) = (a_0(x) + 0(e^{-\delta_1 T}))$$

$$(\text{constant} + 0(e^{-\delta_1 T})) = \text{constant}\ (a_0(x) + 0(e^{-\delta_1 T})).$$

Similarly, if $2i - 1 = |\alpha|$

$$(L_i(a_0^T(x,y)\ P_\alpha(y))\ (\bar{y}) = \text{constant}\ a_0(x) + 0(e^{-\delta_1 T}),$$

since by (4.17) $\dfrac{\partial^{|\beta|}}{\partial y^\beta} a_0^T(x,y)|_{y=\bar{y}} = c_{0,\beta}\ a_0(x) + 0(e^{-\delta_1 T})$ uniformly $|\beta| \leq 2$

and we get $0(e^{-\omega_1 T})$ unless one derivative acts on a_0 and differentiation of order $2i -$ 1 on P_α. In that case we get $\dfrac{\partial}{\partial y_i} a_0^T(x,y)|_{y=\bar{y}}$ (constant $+ 0(e^{-\delta_1 T})) =$ constant $(a_0(x) + 0(e^{-\delta_1 T}))$.

Recall $P_{1,b}$ is odd and we get by above from (4.33)

$$(4.34) \qquad I_{0,1} = \sum_{k=1,}^{m_1} \sum_{|\alpha|=2k-1} (c_\alpha^{(i)} \lambda^{-\frac{d}{2}} (\det \Omega)^{-\frac{1}{2}} \pi^{\frac{d}{2}} (1 + 0(e^{-\delta_1 T}))$$

$$\frac{((c_k\, a_0(x) + 0(e^{-\delta_1 T}))}{\lambda^{\frac{1}{2}}} + 0(\lambda^{-\frac{3}{2}}) + 0(e^{-\omega_1 T} \lambda^{\frac{1}{2}}))\}$$

$$\exp(-\lambda(\rho(x,b) + 0(e^{-2\omega_1 T})) = \lambda^{-\frac{d}{2}} (\det \Omega)^{-\frac{1}{2}} \pi^{\frac{d}{2}} (1 + 0(e^{-\delta_1 T}))$$

$$(\frac{(a_{0,1}(x) + 0(e^{-\delta_1 T}))}{\lambda^{\frac{1}{2}}} + 0(\lambda^{-\frac{3}{2}}) + 0(e^{-\omega_1 T} \lambda^{\frac{1}{2}}))$$

$$\exp(-\lambda(\rho(x,b) + 0(e^{-2\omega_1 T})))$$

uniformly for $(x,T) \in B(x_0, \delta) \times [T_0, \infty)$ when $\lambda^{\frac{1}{2}} e^{-\omega_1 T} \leq$ some constant where

$a_{0,1}(x) =$ constant $a_0(x) \in C^\infty(B(x_0, \delta))$. Similarly

$$(4.35) \qquad I_{0,2}(x,T,\lambda) = \lambda^{-\frac{d}{2}} (\det \Omega)^{-\frac{1}{2}} \pi^{\frac{d}{2}} (1 + 0(e^{-\delta_1 T}))$$

$$\{a_{0,2}(x) + 0(e^{-\delta_1 T}) + 0(\lambda^{-\frac{1}{2}}) + 0(e^{-\omega_1 T} \lambda^{\frac{1}{2}})\} \exp(-\lambda(\rho(x,b)$$

$$+ 0(e^{-2\omega_1 T})))$$

with same uniformity as in (4.34) where $a_{0,2}(x) = \text{const.}\ a_0(x) \in C^\infty(B(x_0, \delta)$.

Adding (4.27), (4.34) and (4.35) gives

$$(4.36)\quad \sum_{i=0}^{2} \frac{I_{0,i}(x,T,\lambda)}{\lambda^{\frac{i}{2}}} = \lambda^{-\frac{d}{2}}(\det \Omega)^{-\frac{1}{2}}\pi^{\frac{d}{2}}(1 + 0(e^{-\delta_1 T}))$$

$$\exp(-\lambda(\rho(x,b)+0(e^{-2\omega_1 T}))$$

$$[a_0(x)(1 + \frac{\tilde{a}_{0,1}(x)}{\lambda} + 0(\frac{1}{\lambda^2})) + \frac{1}{\lambda^{\frac{1}{2}}}$$

$$(\frac{(a_{0,1}(x)+0(e^{-\delta_1 T}))}{\lambda^{\frac{1}{2}}} + 0(\lambda^{-\frac{3}{2}}) + 0(e^{-\omega_1 T}\lambda^{\frac{1}{2}}))$$

$$+ \frac{1}{\lambda}(a_{0,2}(x) + 0(e^{-\delta_1 T}) + 0(\lambda^{-\frac{1}{2}}) + 0(e^{-\omega_1 T}\lambda^{\frac{1}{2}}))] =$$

$$= \lambda^{-\frac{d}{2}}(\det \Omega)^{-\frac{1}{2}}\pi^{\frac{d}{2}}(1 + 0(e^{-\delta_1 T}))\, a_0(x)(1 + \frac{\tilde{\tilde{a}}_{0,1}(x)}{\lambda} + 0(\lambda^{-\frac{3}{2}}))$$

$\exp(-\lambda(\rho(x,b) + 0(e^{-2\omega_1 T})))$ uniformly for $(x,T,\lambda) \in B(x_0, \delta)$

$\times [T_0, \infty) \times (0,\infty)$ when $e^{-\omega_1 T} \cdot \lambda^{\frac{1}{2}} \leq$ a constant and where

$\tilde{\tilde{a}}_{1,0}(x) \in C^\infty(B(x_0, \delta))$.

Now take $h_{x,T}(y) = a_0^T(x,y)\, a_1^T(x,y)$.

By (4.21) $\frac{\partial^\alpha}{\partial y^\alpha} a_1^T(x,y) = 0(T)$ uniformly and so $\tilde{0}(\cdot)$ in (4.31) is 0(T).

From (4.3) and (4.31) we get

$$(4.37) \qquad I_{1,i}(x,T,\lambda) = \int_{|y-b| \le \frac{\delta}{2}} e^{-\lambda(A(x,y,T)+\frac{1}{2}<\Omega(y-b),(y-b)>)} a_0^T(x,y)$$

$$a_1^T(x,y)\, P_{i,b}(\lambda^{\frac{1}{2}}(y-b))\, dy$$

$$= \Big(\sum_{k=1}^{m_i} \Big(\sum_{\alpha:|\alpha|=2k} (c_\alpha^i\, \lambda^{-\frac{d}{2}} (\det \Omega)^{-\frac{1}{2}}\, \pi^{\frac{d}{2}}(1 + 0(e^{-\delta,T}))$$

$$(L_k(a_0^T(x,y)a_1^T(x,y)P_\alpha(\lambda)(\bar{y}) + 0(\tfrac{T}{\lambda}) + 0(Te^{-\omega_1 T}\lambda^{\frac{1}{2}}))$$

$$+ \sum_{|\alpha|=2k-1} (c_\alpha^i\, \lambda^{-\frac{d}{2}}(\det \Omega)^{-\frac{1}{2}}\, \pi^{\frac{d}{2}}(1 + 0(e^{-\delta_1 T}))$$

$$\frac{(L_k(a_0^T(x,y)a_1^T(x,y)P_\alpha(y))(\bar{y})}{\lambda^{\frac{1}{2}}}$$

$$+ 0(\tfrac{T}{\lambda^{\frac{3}{2}}}) + 0(Te^{-\omega_1 T}\lambda^{\frac{1}{2}})) \exp(-\lambda(\rho(x,b) + 0(e^{-2\omega_1 T})))$$

for $(x,T,\lambda) \in B(x_0, \delta) \times [T_0, \infty) \times (0,\infty)$ with uniformity when $\lambda^{\frac{1}{2}} e^{-\omega_1 T} \le$ a constant.

When i = 1 we have the contribution from the even part in (4.37) vanishing since $P_{1,b}$ is odd and hence

$$(4.38) \qquad I_{1,1}(x,T,\lambda) = \lambda^{-\frac{d}{2}}(1 + 0(e^{-\delta_1 T}))(0(\frac{T}{\lambda^{\frac{1}{2}}}) + 0(Te^{-\omega_1 T}\lambda^{\frac{1}{2}}))$$

$$\exp(-\lambda(\rho(x,b) + 0(e^{-2\omega_1 T}))) \text{ with uniformity as in (4.36).}$$

Generally we have

$$I_{1,i}(x,T,\lambda) = \lambda^{-\frac{d}{2}}(1 + 0(e^{-\delta_1 T}))(0(T) + 0(Te^{-\omega_1 T}\lambda^{\frac{1}{2}}))$$

$\exp(-\lambda(\rho(x,b) + 0(e^{-2\omega_1 T}))$ with the usual uniformity, which we use with i = 2 together with (4.37) and (4.38) and get

$$(4.39) \qquad \sum_{i=0}^{2} I_{1,i}(x,T,\lambda)/\lambda^{(1+\frac{i}{2})} = \lambda^{-\frac{d}{2}}(\det \Omega)^{-\frac{1}{2}}\pi^{\frac{d}{2}} a_0(x)(1 + 0(e^{-\delta_1 T}))$$

$$[\frac{1}{\lambda}(c_5 \cdot T + a_{1,0}(x) + 0(Te^{-\delta_1 T}))(1 + 0(\frac{T}{\lambda}))$$

$$+ \frac{1}{\lambda^{\frac{3}{2}}}(0(\frac{T}{\lambda^{\frac{1}{2}}}) + 0(Te^{-\omega_1 T}\lambda^{\frac{1}{2}}))$$

$$+ \frac{1}{\lambda^2}(0(T) + 0(Te^{-\omega_1 T}\lambda^{\frac{1}{2}}))]\exp(-\lambda(\rho(x,b) + 0(e^{-2\omega_1 T})))$$

$$= \lambda^{-\frac{d}{2}}(\det \Omega)^{-\frac{1}{2}}\pi^{\frac{d}{2}} a_0(x)\,(1 + 0(e^{-\delta_1 T}))$$

$$\left[\frac{c_5 T + a_{1,0}(x) + 0(Te^{-\delta_1 T}))}{\lambda} + 0(\frac{T^2}{\lambda^2}) + 0(\frac{Te}{\lambda}^{-2\omega_1 T})\right]$$

$\exp(-\lambda(\rho(x,b) + 0(e^{-2\omega_1 T})))$ uniformly for

$(x,T,\lambda) \in B(x_0\ , \delta) \in B(x_0\ , \delta) \times [T_0\ , \infty) \times (0,\infty)$

with $e^{-\omega_1 T}\ \lambda^{\frac{1}{2}} \leq$ constant where $a_{1,0} \in C^\infty(B(x_0\ , \delta))$.

Now we finally worry about

§4.7. The Remainder.

By (4.36)

$$\sum_{i=0}^{2} I_{0,i}(x,T,\lambda)/\lambda^{\frac{i}{2}} = 0(\lambda^{-\frac{d}{2}} \exp(-\lambda(\rho(x,b) + 0(e^{-2\omega_1 T})))$$

with the usual uniformity and by (4.9)

$$(4.40)\qquad R_2(x,T,\lambda) = \lambda^{\frac{3d}{4}}\ e^{-Te_0}\ \beta(\lambda)(0(\frac{T^4}{\lambda^2}) + 0(\frac{T^2}{\lambda} \exp(-\lambda\ c_1\ \frac{\delta^2}{T^4}))$$

$$(\sum_{i=1}^{2} I_{0,i}(x,T,\lambda)/\lambda^{\frac{i}{2}})$$

$$= 0(\lambda^{\frac{d}{4}}\ \beta(\lambda)e^{-Te_0}(0(\frac{T^4}{\lambda^2}) + (\frac{T^2}{\lambda} \exp(-\lambda\ c_1\ \frac{\delta^2}{T^4}))$$

$$\exp(-\lambda(\rho(x,b) + 0(e^{-2\omega_1 T})))).$$

By (4.8)

$$R_1(x,T,\lambda) = \beta(\lambda) \int_{B(b,\frac{\delta}{2})} e^{-T\frac{H(\lambda)}{\lambda}}(x,y)\,\tilde{r}_{2,b}(y,\lambda)\,dy$$

and by (1.35) $e^{-T\frac{H(\lambda)}{\lambda}}(x,y) = 0(\lambda^{\frac{d}{2}} e^{-Te_0} e^{-\lambda A(x,y,T)})$ uniformly for $(x,y,T) \in$

$B(x_0 , \delta)$

$\times B(b,\delta) \times [T_0 , \infty)$ when $T^4 \leq \lambda$ and $T, \lambda \to \infty$.

We use the Cauchy-Schwartz inequality

$$(4.41) \qquad R_1(x,T,\lambda) \leq \beta(\lambda) \| e^{-T\frac{H(\lambda)}{\lambda}}(x, \cdot)\|_2 \, \|\tilde{r}_{2,b}(\cdot ,\lambda)\|_2$$

with $\| \cdot \|_2 =$ the $L^2(B(b,\frac{\delta}{2}))$-norm.

By (4.4) $\|\tilde{r}_{2,b}(\cdot ,\lambda)\|_2 = 0(\lambda^{-\frac{3}{2}})$. Now write as in (4.15) $A(x,y,T) = A(x,\bar{\bar{y}},T) -$

$i\tilde{f}_{x,T}(y)$.

$$\|e^{-T\frac{H(\lambda)}{\lambda}}(x,\cdot)\|_2 = 0\Big(\lambda^{\frac{d}{2}} e^{-Te_0}\Big(\int_{B(b,\frac{\delta}{2})} e^{-2\lambda A(x,y,T)}dy\Big)^{\frac{1}{2}}\Big)$$

$$= 0\Big(\lambda^{\frac{d}{2}} e^{-Te_0} e^{-\lambda A(x,\bar{\bar{y}},T)}\Big(\int e^{2i\lambda \tilde{f}_{x,T}(y)}dy\Big)^{\frac{1}{2}}\Big)$$

$$= \lambda^{\frac{d}{2}} e^{-Te_0} e^{-\lambda A(x,\bar{\bar{y}},T)} 0(\lambda^{-\frac{d}{4}})$$

uniformly for $(x,T) \in B(x_0, \delta_0) \times [T_0, \infty)$, by (4.25), (4.18), (4.19) and Hörmander's stationary phase theorem.

Now (4.20) of lemma 7 and (4.41) imply

$$\text{(4.42)} \qquad R_1(x,T,\lambda) = 0(e^{-Te_0} \lambda^{\frac{d}{4}} \lambda^{-\frac{3}{2}} \exp(-\lambda(\rho(x,b) + 0(e^{-2\omega_1 T})))$$

uniformly for $(x,T,\lambda) \in B(x_0, \delta) \times [T_0, \infty) \times (0,\infty)$ when $T^4 \leq \lambda$.

§4.8. Cancellations and taking T = constant $\log \lambda$ completes the proof.

From (4.5), (4.36), (4.39), (4.40), and (4.42) we have

$$\text{(4.42)} \qquad \int_{|y-b|\leq\frac{\delta}{2}} e^{-T\frac{H(\lambda)}{\lambda}}(x,y)\, \Omega_0(y,\lambda)\, dy = \lambda^{\frac{d}{4}} \beta(\lambda)\, f_b(x)\, e^{-\lambda\rho(x,b)}$$

$$(1 + 0(\lambda e^{-2\omega_1 T}))(1 + 0(e^{-\delta_1 T}))\, e^{-Te_0}$$

$$[1 + \frac{(c_5 \cdot T + f_{1,b}(x) + 0(Te^{-\delta_1 T}))}{\lambda} + 0(\frac{T^4}{\lambda^2}) + 0(\lambda^{-\frac{3}{2}})$$

$+ \; 0(\frac{T^2}{\lambda} \exp(-\lambda \frac{c_4}{T^4}))]$ uniformly for $x \in B(x_0, \delta)$ as $T, \lambda \to \infty$ and

$T^4 \leq \lambda \leq e^{\omega_1 T}$, where $f_b(x) = \pi^{-\frac{d}{4}}\left(\det\left(\sqrt{V''(b)}\,\right)\right)^{\frac{1}{4}} a_0(x)$ and a_0

as in Theorem A.

Here $f_{1,b}(x) = a_{1,0}(x) + \tilde{\tilde{a}}_{0,1}(x) \in C^{\infty}(B(x_0, \delta))$ with $a_{1,0}(x)$ from (4.38) and $\tilde{\tilde{a}}_{0,1}$ from (4.36), $e_0 = \frac{\mathrm{trace}\sqrt{V''(b)}}{2} = \frac{\mathrm{trace}\ \Omega}{2}$, $\omega_1 = \omega_1(b) = \sqrt{\text{smallest eigenvalue of } V''(b)}$ and $0 < \delta_1 \leq \frac{\omega_1}{2}$, $c_5 = c_5(b)$, $c_1 = c_1(b) > 0$ are constants and with $\beta(\lambda)$ as in assumption $(*)$.

We get similar contribution from $\{y : |y-a| \leq \frac{\delta}{2}\}$ and a small one from $\mathbb{R}^d \backslash (B(a,\frac{\delta}{2}) \cup B(b,\frac{\delta}{2}))$ by (4.3), which adds up to

$$(4.43) \qquad \Omega_0(x,\lambda) = e^{T(e_0 + \frac{e_0^{(1)}}{\lambda} + 0(\frac{1}{\lambda^2}))} e^{-Te_0} \{\lambda^{\frac{d}{4}} \alpha(\lambda) f_a(x) e^{-\lambda\rho(x,a)}$$

$$(1 + 0(\lambda e^{-2\omega_1(a)T}))(1 + 0(e^{-\delta_1(a)T}))$$

$$[1 + \frac{(c_5(a)T + f_{1,a}(x) + 0(Te^{-\delta_1(a)T}))}{\lambda} +$$

$$+ 0(\frac{1}{\lambda^{\frac{3}{2}}}) + 0(\frac{T^4}{\lambda^2}) + 0(\frac{T^2}{\lambda} \exp(-\lambda \frac{c_4(a)}{T^4}))]$$

$$+ \lambda^{\frac{d}{4}} \beta(\lambda)\, f_b(x)\, e^{-\lambda\rho(x,b)} (1+ 0(\lambda e^{-2\omega_1(b)T}))(1+0(\lambda e^{-\delta_1(b)T}))$$

$$[1 + \frac{(c_5(b)T+f_{1,b}(x)+0(Te^{-\delta_1 T}))}{\lambda} + 0(\lambda^{-\frac{3}{2}}) + 0(\frac{T^4}{\lambda^2})$$

$$+ 0(\frac{T^2}{\lambda} \exp(-\lambda \frac{c_4(b)}{T^4}))]\}$$ uniformly for $x \in B(x_0, \delta)$ when $T, \lambda \to \infty$

with $T^4 \leq \lambda \leq \min(e^{\omega_1(a)T}, e^{\omega_1(b)T})$.

Take $$\delta_0 = \min\{\delta_1(a), \delta_1(a), \delta_1(b), \omega_1(a), \omega_1(b)\} > 0$$

and write $e^{T(e_0 + \frac{e_0^{(1)}}{\lambda} 0(\frac{1}{\lambda^2}))} = e^{Te_0}(1 + e_0^{(1)} \cdot \frac{T}{\lambda} + 0(\frac{T^2}{\lambda^2}))$ as $T, \lambda \to \infty$, $\frac{T}{\lambda} \to 0$.

If $\lambda \leq e^{\delta_0 T}$ then $(1+ 0(\lambda e^{-2\omega_1(c)T})) = (1 + 0(e^{-\delta_0 T})$ and if $T^8 \leq \lambda$ then $\frac{T^4}{\lambda^2} =$

$0(\frac{1}{\lambda^{\frac{3}{2}}})$ and $\frac{T^2}{\lambda} e^{-\lambda\frac{c_1\delta^2}{T^4}} = 0(e^{-\lambda^{\frac{1}{2}}})$ and we write (4.43) as

$$(4.44)\qquad \Omega_0(x,\lambda) = \lambda^{\frac{d}{4}}\{\alpha(\lambda)\, f_a(x)\, e^{-\lambda\rho(x,b)}\,(1 + 0(e^{-\delta_0 T}))$$

$$\left[1 + \frac{(c_1(a) + e_0^{(1)})T + f_{1,b}(x) + 0(Te^{-\delta_0 T}))}{\lambda} + 0(\lambda^{-\frac{3}{2}})\right]$$

$$+ \lambda^{\frac{d}{4}}\,\beta(\lambda)\, f_b(x)\, e^{-\lambda\rho(x,b)}\,(1+0(e^{-\delta_0 T}))$$

$$\left[1 + \frac{(c_1(b) + e_0^{(1)})T + f_{1,b}(x) + 0(Te^{-\delta_0 T}))}{\lambda} + 0(\lambda^{-\frac{3}{2}})\right]$$

uniformly for $x \in B(x_0\,,\delta)$ when $T, \lambda \to \infty$ and $T^8 \leq \lambda \leq e^{\delta_0 T}$.

To show $c_1(a) + e_0^{(1)} = 0$ take $\overline{x} \in B(x_0\,,\delta)$ with $\rho(\overline{x},a) < \rho(\overline{x},b)$. Put $T = \frac{8}{\delta_0}\log\lambda$ in the R.H.S. of (4.44) and then $T = \frac{4}{\delta_0}\log\lambda$ in the R.H.S. and subtract. That gives

$$0 = \Omega_0(\overline{x},\lambda) - \Omega_0(\overline{x},\lambda) = \lambda^{\frac{d}{4}}\,\alpha(\lambda)\, f_a(\overline{x})\, e^{-\lambda\rho(\overline{x},a)}\,(1 + 0(\lambda^{-4}))$$

$$\left(\frac{(c_1(a) + e_0^{(1)})\,(\frac{8}{\delta_0}\log\lambda - \frac{4}{\delta_0}\log\lambda) + 0(\log\frac{\lambda}{\lambda^4})}{\lambda} + 0(\lambda^{-\frac{3}{2}})\right)$$

the prefactors are not zero and we get

$$(c_1(a) + e_0^{(1)}) \frac{\log\lambda}{\lambda} = 0(\lambda^{-\frac{3}{2}}) \text{ as } \lambda \to \infty$$

which implies $c_1(a) + e_0^{(1)} = 0$. By taking $\approx{x} \in B(x_0, \delta)$ with $\rho(\approx{x},b) < \rho(\approx{x},a)$ we get $c_1(b) + e_0^{(1)} = 0$. Now we go back to (4.44) with $T = \frac{4}{\delta_0} \log \lambda$ and use $c_1(a) + e_0^{(1)} = 0 = c_1(b) + e_0^{(1)}$ and we get

$$\Omega_0(x,\lambda) = \lambda^{\frac{d}{4}}\alpha(\lambda)\, f_a(x)\, e^{-\lambda\rho(x,a)}(1 + 0(\lambda^{-4})) \tag{4.45}$$

$$\left[1 + \frac{(f_{1,a}(x) + 0(\lambda^{-4} \log \lambda))}{\lambda} + 0(\lambda^{-\frac{3}{2}})\right]$$

$$+ \beta(\lambda)\, f_b(x)\, e^{-\lambda\rho(x,b)}\, (1 + 0(\lambda^{-4}))$$

$$\left[1 + \frac{(f_{1,b}(x) + 0(\lambda^{-4} \log \lambda))}{\lambda} + 0(\lambda^{-\frac{3}{2}})\right]$$

$$= \lambda^{\frac{d}{4}}\{\alpha(\lambda)\, f_a(x)\, e^{-\lambda\rho(x,a)}\, (1 + \frac{f_{1,a}(x)}{\lambda} + 0(\lambda^{-\frac{3}{2}}))$$

$$+ \beta(\lambda)\, f_b(x)\, e^{-\lambda\rho(x,b)}\, (1 + \frac{f_{1,b}(x)}{\lambda} + 0(\lambda^{-\frac{3}{2}}))\}$$

uniformly for $x \in B(x_0, \delta)$ as $\lambda \to \infty$.

5. The Feynman-Kac Formula and Large Deviations.

§5.1. A use of the Cameron-Martin formula

§5.2. Large deviations, Varadhan's theorem and Schilder's theorem.

5. The Feynman-Kac formula and large deviations.

§5.1 A use of the Cameron-Martin formula.

References for this section are Simon [2] and [5], poor Barry!

Given $x,y \in \mathbf{R}^d$ and $t > 0$, let $C_{x,y}([0,t])$ be the Banach space of continuous functions $Z:[o,t] \to \mathbf{R}^d$ with $Z(0) = x$ and $Z(t) = y$ and with the supremum norm

$$(5.1) \qquad \|Z\|_{L^\infty[o,t]} = \sup_{0 \leq s \leq t} \left(\sum_{i=1}^{d} Z_i^2(s) \right)^{\frac{1}{2}}$$

Let $P_{x,y:t}$ be the Gaussian measure on $C_{xy}([o,t])$ with mean $m(s) = E_{x,y;t}(Z(s)) = (1 - s)\,x + sy$ if $0 \leq s \leq t$ and covariance.

$$(5.2) \qquad E_{x,y;t}((Z_i(s) - m_i(s)\,(Z_j(u) - m_j(u)) = \delta_{ij}\, s(1 - u/t)$$

if $0 \leq s \leq u \leq t$ and for $i,j \in \{1, \ ,d\}$.

When $x = 0 = y$, $\{Z,(s)\}_{0 \leq s \leq t}$ is called the Brownian bridge or tied down Brownian motion. We will denote $E_{00;t}(\,\cdot\,)$ by $E^t_Z(\,\cdot\,)$. The conditional Wiener measure is given by

$$\mu_{0,x,y,t}(\,\cdot\,) = (2\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{2t}\right) P_{x,y;t}(\,\cdot\,)$$

If $V \geq 0$ and continuous and $H = -\frac{1}{2}\,\Delta + V$ the Feynman-Kac formula says

$$(5.3) \qquad e^{-tH}(x,y) = \int \exp\left(-\int_0^t V(\omega(s))\,ds\right) d\mu_{0,x,y;t}\,(\omega)$$ which can be written as

$$(5.4)\qquad e^{-tH}(x,y) = (2\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{2t}\right)$$

$$E_Z^{\frac{1}{2}}\left(\exp\left(-\int_0^t V\left(\left(1-\tfrac{s}{t}\right)x + \tfrac{s}{t}y + \sqrt{t}\, Z\left(\tfrac{s}{t}\right)\right) ds\right)\right).$$

In the double well case, we are staring at, we take $t = T/\lambda$ and $H = H(\lambda)$ and we have

$$(5.5)\qquad e^{-T\frac{H(\lambda)}{\lambda}}(x,y) = \int \exp\left(-\lambda^2 \int_0^{\frac{T}{\lambda}} V(\omega(s))\, ds\right) d\mu_{0,x,y;T/\lambda}(\omega)$$

$$= \left(\frac{\lambda}{2\pi T}\right)^{\frac{d}{2}} \exp\left(-\frac{\lambda|x-y|^2}{2T}\right) E_Z^{\frac{1}{2}}\left(\exp\left(-\int_0^{\frac{T}{\lambda}} V\left(\left(1-\tfrac{s\lambda}{T}\right)x + \tfrac{s\lambda}{T}y + \sqrt{T/\lambda}\, Z\left(\tfrac{s\lambda}{T}\right)\right) ds\right)\right).$$

Writing formally $\int d\mu_{0xy:t}(\omega) =$ "$\int$" $d^\infty\omega \exp\left(-\frac{1}{2}\int_0^t \dot\omega^2(s)\, ds\right)$ $\delta(\omega(0)-x)\, \delta(\omega(t)-y)$ gives $e^{-tH(\lambda)}(x,y) =$ "$\int$" $d^\infty \omega\delta(\omega(0)-x)\, \delta(\omega(t)-y)$ $\exp\left(-\frac{1}{2}\int_0^t \dot\omega^2(s)\, ds - \lambda^2 \int_0^t V(\omega(s))\, ds\right)$.

With $t = \frac{T}{\lambda}$ and $\gamma(s) = \omega(s/\lambda)$ we have

$$e^{-T\frac{H(\lambda)}{\lambda}}(x,y) = \text{“}\int\text{”}\, d^\infty \gamma\delta(\gamma(0)-x)\, \delta(\gamma(T)-y) \exp\left(-\lambda\int_0^T \left(\tfrac{1}{2}\dot\gamma^2 + V(\gamma)\right) ds\right)$$

which reminds on (1.17) $\lim_{\lambda\to\infty} \frac{\log e^{-\frac{TH(\lambda)}{\lambda}}(x,y)}{\lambda} = -A(x,y,T)$ u.o.c. in $\mathbb{R}^d \times \mathbb{R}^d \times 0,\infty)$, where $A(x,y,T) = \inf\left\{\int_0^T \left(\frac{1}{2}\dot\gamma^2 + V(\gamma)\right)dt : \gamma(0) = x,\ \gamma(T) = y\right\}$.

(1.17) was proven using large deviations that we say a few words about below. Now we prepare for going beyond the leading order in (1.17), when there is a unique minimal action path $g_T^{x,y}$ and the second variation is positive definite. We want to estimate as a small error term the contribution in (5.5) from the paths in the complement of some neighborhood of $g_T^{x,y}(\cdot\ \lambda)$. Then we make Taylor expansions of the integrand around $g_T^{x,y}$ in this neighborhood (see Schilder [1]) to get finer asymptotics of the contribution from this neighborhood.

Our first step is to use Cameron-Martin formula (Cameron-Martin [1], Freidlin [1]) that implies: If $f \in C_{00}([0,t])$ and $\int_0^t \dot{f}^2(s)\, ds < \infty$ then $\mu_{0,0,0;t}\ (f + Z)$ is absolutely continuous with respect to $\mu_{0,0,0;t}\ (Z)$ and the Radon-Nikodym derivative is given by

$$(5.6) \qquad \exp(-\tfrac{1}{2} \int_0^t (\dot{f}(s))^2\, ds) \exp(-\int_0^t \dot{f}(s)\, dZ(s))$$

where $\int_0^t \dot{f}(s)\, dZ(s)$ is Itô stochastic integral, to obtain

Sublemma 1.

If g_T^{xy} is a miminal action path and E_Z^T denotes expectation w.r.t. Brownian bridge on [0,T] then (See also Davies and Truman [1])

$$(5.7) \qquad e^{-T\frac{H(\lambda)}{\lambda}}(x,y) = \int \exp(-\lambda^2 \int_0^{\frac{T}{\lambda}} (\omega(s))\, d\mu_{0,x:y,\frac{T}{\lambda}}(\omega) =$$

$$= \left(\frac{\lambda}{2\pi T}\right)^{\frac{d}{2}} \exp(-\lambda A(x,y,T))\, E_Z^T(\exp(-\int_0^T (V(g_T^{xy}(t) + \lambda^{-\frac{1}{2}} Z(t)) -$$

$$- V(g_T^{xy}(t)) - \lambda^{-\frac{1}{2}} \nabla V(g_T^{xy})) \cdot Z(t))\, dt)) \text{ and if } r > 0 \text{ then}$$

(5.8) $$F(x,y,T,\lambda,r) := \int \exp(-\lambda^2 \int_0^{\frac{T}{\lambda}} V(\omega(s))\, ds)$$

$$\chi(\omega \in C_{x,y}([0,\tfrac{T}{\lambda}]) : \|\omega - g_T^{xy}(\cdot\ \lambda)\|_\infty \leq r)$$

$$d\mu_{0,x,y;\frac{T}{\lambda}}(\omega) = \left(\frac{\lambda}{2\pi T}\right)^{\frac{d}{2}} \exp(-\lambda A(x,y,T))\, E_Z^T\, (\exp(-\int_0^T (V(g_T^{xy}(t) +$$

$$+ \lambda^{-\frac{1}{2}} Z(t)) - V(g_T^{xy}(t)) - \lambda^{-\frac{1}{2}} \nabla V(g_T^{xy}) \cdot Z(t)))dt))$$

$$\chi(Z \in C_{00}([0,T])\ (\|Z\|_\infty \leq r\lambda^{\frac{1}{2}}))\ .$$

Proof. We prove (5.8) then (5.7) follows by taking $r \to \infty$. If $\{\omega(s)\}_{0 \leq s \leq t}$ is the conditional Wiener process on $[0,t]$ starting at x and ending at y and $\{\alpha(s)\}_{0 \leq s \leq 1}$ is the Brownian bridge on $[0,1]$ then (see Simon [5], chapter 2) $\omega(s) \doteq (1 - \frac{s}{t})\, x + \frac{s}{t}\, y + \sqrt{t}\ \alpha(\frac{s}{t})$ and $\alpha(\frac{s}{t}) \doteq t^{-\frac{1}{2}}\, \alpha(s)$ where $\doteq$ means the distribution on both sides are the same.

(5.9) Therefore $$F(x,y,T,\lambda,r) = \int \exp\,(-\lambda^2 \int_0^{\frac{T}{\lambda}} V(\omega(s))\, ds)$$

$$\chi(\omega \in C_{x,y}([0,\tfrac{T}{\lambda}]) : |\omega(s) - g_T^{xy}(s \cdot \lambda)| \leq r)$$

$$d\mu_{0,x,y;\frac{T}{\lambda}}(\omega) = \left(\frac{\lambda}{2\pi T}\right)^{\frac{d}{2}} \exp(-\frac{\lambda|x-t|^2}{2T})$$

$$E_\alpha^T[\exp(-\lambda \int_0^T V((1-\tfrac{t}{T})\, x + \tfrac{t}{T} y + \lambda^{-\frac{1}{2}}\, \alpha(t))\, dt)$$

$$\chi(z \in C_{00}([0,T\}) : |(1-\tfrac{t}{T})\, x + \tfrac{t}{T} y + \lambda^{-\frac{1}{2}}\, \alpha\,(t) - g_T^{xy}(t)| \leq r$$

for $t \in \{0,T])]$.

Now $\alpha \to \alpha + f =: Z$ where $f(t) = -\lambda^{\frac{1}{2}}(g_T^{xy}(t) - (1-\frac{t}{T})\, x + \frac{t}{T} y)$

(so $(1-\frac{t}{T})\, x + \frac{t}{T} y + \lambda^{-\frac{1}{2}} \alpha(t) = g_T^{xy}(t) + \lambda^{-\frac{1}{2}} Z(t))$ is a bijection $C_{00}([0,T]) \to C_{00}([0,T])$, $\int_0^T |\dot{f}|^2\, dt < \infty$ and so from (5.6) we have the Radon-Nikodym derivative is given by $\frac{d\mu_T(\alpha)}{d\mu_T(Z)} = \exp(-\int_0^T \langle \dot{f}(t),\, dZ(t)\rangle - \frac{1}{2}\int_0^T |\dot{f}(t)|^2\, dt) = \exp(-\lambda^{\frac{1}{2}} \int_0^T < \nabla V(g_T^{x,y}(t)) \cdot Z(t) > dt) \exp(--\frac{\lambda}{2}\int_0^T (\dot{g}_T^{xy})^2\, dt) \exp(\lambda\frac{(x-y)^2}{2T}))$ where we integrate by parts the ItÔ-integral since $\dot{f}$ is absolutely continuous (Nelson [1]) and used $\ddot{g}_T^{x,y}(t) = \nabla V(g_T^{x,y})$ and where $d\mu_T(Z) = (2\pi T)^{\frac{d}{2}}\, d\mu_{0,0,0;T}(Z)$ is the Brownian bridge measure.

By (5.9)

$$F(x,y,T,\lambda,r) = \left(\frac{\lambda}{2\pi T}\right)^{\frac{d}{2}} \exp(-\frac{\lambda}{2} \int_0^T (\dot{g}_T^{xy}(t)^2\, dt)$$

$$E_Z^T(\exp(-\lambda\int_0^T (V(g_T^{xy} + \lambda^{-\frac{1}{2}} Z) - \lambda^{-\frac{1}{2}} V'(g_T^{xy}) Z)\, dt)$$

$$\chi[Z \in C_{00}([0,T]) : \|Z\|_{L^\infty[0,T]} \leq r\lambda^{\frac{1}{2}}]) = \left(\frac{\lambda}{2\pi T}\right)^{d/2} \exp\,(-\,\lambda\, A(x,yT))$$

$$E_Z^T(\,\exp\,(-\lambda\int_0^T (V(g_T^{xy}(t) + \lambda^{-\frac{1}{2}} Z(t)) - V(g_T^{xy}(t)) - \lambda^{-\frac{1}{2}}$$

$$V'(g_T^{x,y}(t)) \cdot Z(t))\, dt)\, \chi\,(Z : \|Z\|_{L^\infty[0,T]} \leq r\, \lambda^{\frac{1}{2}})].$$

which finishes the proof.

To use this (see lemmas 8, 9 and 10 below) we need:

§5.2 Large deviations - Varadhan's Theorem and Schilder's Theorem.

References for this section are Varadhan [1], [2]. Stroock [1] and Jain [1]. Let X be a separable complete metric space, $\mathfrak{B}$ = the Borel $\sigma-$ field over X and P_ε be a family of probability measures.

We have in mind $X = C_{00}([0,T])$ and with $\varepsilon = \frac{1}{\lambda}$, $P_\varepsilon(A) = \mu_T(\lambda^{\frac{1}{2}}A)$ where

5.10) $\mu_T(\cdot) = (2\pi T)^{\frac{d}{2}} \mu_{0,0,0;T}(\cdot)$ is the Brownian bridge measure,

so formally $P_\varepsilon(A) = \text{“}\int_A\text{”} \exp\left(-\frac{1}{2\varepsilon}\int_0^T \dot{Z}^2(s)\,ds\right) d^\infty Z$

and we want results like Laplace method (Erdelyi [1]), when

$$X = \mathbb{R} \text{ and } P_\varepsilon(A) = \int_A \frac{e^{-\frac{x^2}{2\varepsilon}}}{(2\pi\varepsilon)^{\frac{1}{2}}}\,dx .$$

(5.11) Definition (Varadhan [1]).

We say $\{P_\varepsilon\}$ obeys the large deviation principle with rate function I( ) if there exists a function $I(\cdot)$ from X into $[\mathbb{R}]$ such that

(i) $0 \leq I(x) \leq \infty$ for all $x \in X$

(ii) $I(\cdot)$ is lower-semicontinuous

(iii) For each $\ell < \infty$ the set $\{x : I(x) \leq \ell\}$ is compact subset in X

(iv) For each closed set $C \subseteq X$ $\overline{\lim_{\varepsilon \to 0}}\, \varepsilon \log P_\varepsilon(C) \leq -\inf_{x\in C} I(x)$

(V) For each open set $G \subset X$ $\underline{\lim_{\varepsilon \to 0}}\, \varepsilon \log P_\varepsilon(G) \geq -\inf_{x\in G} I(x)$.

Then we have

Varadhan's Theorem (Varadhan [2], [3]). Let P_ε satisfy the large deviation principle with a rate function $I(\cdot)$. Then for any $F : X \to \mathbf{R}$ bounded and continuous we have

(5.12) (i) For any closed set C in X

$$\overline{\lim_{\varepsilon \to 0}}\ \varepsilon \log \int_C \exp(-\frac{F(x)}{\varepsilon})\, dP_\varepsilon(x) \le -\inf_{x \in C} (F(x) + I(x))$$

(ii) For every open set G in X

$$(5.13) \quad \underline{\lim_{\varepsilon \to 0}}\ \varepsilon \log \int_G \exp(-\frac{F(x)}{\varepsilon})\, dP_\varepsilon(x) \ge -\inf_{x \in G} (F(x) + I(x))$$

In particular if G = X = C then

$$(5.14) \quad \lim_{\varepsilon \to 0} \varepsilon \log \int_X \exp(-\frac{F(x)}{\varepsilon})\, dP_\varepsilon(x) = -\inf_{x \in X} (F(x) + I(x)).$$

Moreover

(5.15) If $F : X \to \mathbf{R} \cup \{\infty\}$ is lower semicontinuous and bounded from below then (5.12) holds.

Now we let $X = C([0,\infty), \mathbf{R}^d)$ with the uniform convergence at compacts, and M the Borel field over X. The d-dimensional Brownian-motion $\{b_j(s)\}_{s \ge 0, 1 \le j \le d}$ that are mean zero Gaussian random variables with covariance $E(b_j(t)\, b_k(s)) = \delta_{jk} \min(t,s)$ give a probability measure P on (X,M). Let $P_\varepsilon(A) := P(\varepsilon^{-\frac{1}{2}} A)$.

$$X_T := C_0([o,T], \mathbb{R}^d) = \{f \in C([0,T], \mathbb{R}^d) : f(0) = 0\}$$

and

$$I_T(f) = \frac{1}{2} \int_0^T |\dot f|^2(t)\, dt \text{ for } f \in X_T.$$

Then:

Theorem (Schilder). The $P_\varepsilon|_{X_T}$'s satisfy the large deviation principle with rate function I_T.

Schilder [1] showed if $F : C([0,T]) \to \mathbf{R}$ is continuous and $F(Z) + \frac{1}{2}\int_0^T |\dot{Z}|^2(t)$ dt has a unique minimum at some Z_0 with some additonal hypotheses among which was: There are $\varepsilon > 0$ and $\delta > 0$ such that $E(\exp\frac{[-(1-\varepsilon)D^2F(Z_0+\eta)Z^2]}{2}) \leq$ constant $< \infty$ if $\|\eta\|_{L^\infty\{0,T]} \leq \delta$. (the second variation positive definite) then

$$E(\exp(-\varepsilon^{-1}F(\varepsilon^{\frac{1}{2}}Z)) = (\Gamma_0 + \Gamma_1\varepsilon^{\frac{1}{2}} + \Gamma_2\varepsilon + \cdots + \Gamma_n\,\varepsilon^{\frac{n}{2}} + 0(\varepsilon^{\frac{(n+1)}{2}}))$$

$$\exp\Big(-\varepsilon^{-1}(F(Z_0) + \tfrac{1}{2}\int_0^T |\dot{Z}|^2\,(t)\,dt)\Big)$$

For further results see for instance: Azencott [1], Chevet [1], Donsker-Varadhan [1] Ellis-Rosen [1], [2] and [3], Freidlin-Wentzell [1] Pincus [1], Simon [5] and Stroock [1].

6. The Proof of Theorem A.

§6.1. Some lemmas and the idea of the proof.

We make the same assumptions as in lemmas 2-7 on x_0 and b:

(1.33) $V''(b) = \Omega^2$ where $\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_d)$ with $0 < \omega_1 \leq \omega_2 \leq \cdots \leq \omega_d$.

(2.5) $\rho(x_0, b) < \rho(x_0, a) + \rho(a,b)$.

(2.6) There is a unique Agmon geodesic $g^{x_0,b}$ minimizing

$$\rho(x_0, b) = \inf\{\int_0^\infty (\tfrac{1}{2}\dot{\gamma}^2 + V(\gamma))dt \mid \gamma(0) = x_0, \gamma(\infty) = b\}$$

and

(2.7) The second variation of $g^{x_0,b}$ is positive definite.

Lemma 8. With δ_0, T_0 as in lemma 2, r_0 in lemma 4, $a_0^T(x,y)$ and $a_1^T(x,y)$ as described in Theorem A, we have

1. For some $k_5 > 0$

(6.1)
$$F(x,y,T,\lambda,\tfrac{r}{T}) = \int \exp(-\lambda^2 \int_0^{\frac{T}{\lambda}} V(\omega(s))ds)$$

$$\chi(\{\omega \in C_{x,y}([0,\tfrac{T}{\lambda}]) : \|\omega - g_T^{x,y}(\cdot\,\lambda)\|_\infty \leq \tfrac{r}{T}\})d\mu_{0,x,y,\frac{T}{\lambda}}(\omega)$$

$$= \exp(-\lambda A(x,y,T))(\tfrac{\lambda}{2\pi})^{\frac{d}{2}} b_0^T(x,y)\{1 + \frac{a_1^T(x,y)}{\lambda}$$

$$+ 0(\frac{T^4}{\lambda^2}) + 0(\frac{T^2}{\lambda} \exp(\frac{-\lambda k_5 r^2}{T^4}))\}$$ uniformly for

$$(x,y,T,r,\lambda) \in (B(x_0, \delta_0) \cup B(b,\delta_0)) \times B(b,\delta_0) \times$$

$$\times [T_0, \infty) \times (0,r_0] \times (0,\infty)$$

where

$$(6.2) \qquad b_0^T(x,y) = 2^{\frac{d}{2}} \left(\prod_{i=1}^{d} \omega_i \right)^{\frac{1}{2}} a_0^T(x,y)\, e^{-Te_0} \quad \text{and} \quad e_0 = \frac{\operatorname{trace} \sqrt{V''(b)}}{2} \ .$$

2. If $x = w_0 \in B(x_0 , \delta_0) \cup B(b,\delta_0)$ $z = w_{m+1} \in B(b,\delta_0)$ and $T_1 \in [T_0 , \infty)$ then

$$\inf_{w_1,\dots,w_m \in \mathbb{R}^d} \{ \sum_{i=0}^{m} A(w_i,w_{i+1} ,\dots, T_1)\} = A(x,z,(m+1)T_1)$$

and is attained if and only if

$$w_i = g_{(m+1)T_1}^{x,z}(iT_1) \text{ for } i \in \{1,\dots,m\}.$$

Moreover

$$(6.3) \quad \int_{(B(b,\delta_0))^m} \left(\prod_{i=0}^{m} \left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}} \cdot b_0^{T_1}(w_i , w_{i+1}) \right.$$

$$\exp(-\lambda A(w_i , w_{i+1} , T_1))\, dw ,\dots, dw_m$$

$$= \left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}} b_0^{(m+1)T_1} (x,z) \exp(-\lambda A(x,z,(m+1)T_1)\ (1 + 0(\tfrac{1}{\lambda}))^m$$

where $0(\frac{1}{\lambda})$ is uniform for $(x,z,T_1,\lambda,m) \in (B(x_0,\delta_0) \cup B(b,\delta_0))$

$\times\ B(b,\delta_0) \times [T_0 , \infty) \times (0,\infty) \times \mathbb{N}$.

(A Proof in Chapter 7.)

Observe that (6.1) gives the lower bound in (1.35) of Theorem A and part 2 says the leading order of the approximation in part 1 satisfies the semigroup property.

Lemma 9. There are positive constants k_6 and λ_0 such that

$$(6.4) \qquad \exp(-T\frac{H(\lambda)}{\lambda})(x,y) = F(x,y,T,\lambda,r) + 0(\lambda^{\frac{d}{2}} e^{-\lambda k_6 r^2}) \exp(-\lambda A(x,y,T))$$

$$= F(x,y,T,\lambda,r)(1 + 0(e^{-\lambda k_6 r^2})) \quad \text{uniformly for}$$

$$(x,y,T,r,\lambda) \in (B(x_0,\delta_0) \cup B(b,\delta)) \times B(b,\delta_0) \times$$

$$\times [T_0 , 2T_0] \times [0,r_0] \times [\lambda_0 , \infty).$$

(A Proof in Chapter 8.)

Note that (6.1) and (6.4) yield (see (6.19) below)

$$(6.5) \qquad \exp(-T\frac{H(\lambda)}{\lambda})(x,y) =$$

$$(\frac{\lambda}{2\pi})^{\frac{d}{2}} b_0^T(x,y)\{1 + \frac{a_1^T(x,y)}{\lambda} + 0(\frac{1}{\lambda^2})\}\exp(-\lambda A(x,y,T))$$

uniformly for $(x,y,T) \in B(x_0 , \delta_0) \times B(b,\delta_0) \times [T_0 , 2T_0]$

as $\lambda \to \infty$ (see also Davies and Truman [1])

Lemma 10. With $|V'(x)| = 0(e^{A|x|^2})$ for statement about derivatives, there exist $\beta \in (0,1)$, k_7 , k_8 , and λ_0 positive such that

(i) If $x_0 \in K = \{x : \rho(x,a) = \rho(x,b) = \frac{\rho(a,b)}{2}\}$

$x \in B(x_0, \delta_0)$, $w \notin B(a,\delta_0) \cup B(b,\delta_0)$, $T \in [T_0, \infty)$,

$T_1 \in [T_0, 2T_0]$, $\lambda \in [\lambda_0, \infty)$ and $i \in \{1, ..., d\}$ then

(6.6)

$$\left.\begin{array}{l} \exp(-T_1 \frac{H(\lambda)}{\lambda})(x,w) \\ \\ |\frac{1}{\lambda}\frac{\partial}{\partial x_i} e^{-T_1\frac{H(\lambda)}{\lambda}}(x,w)| \end{array}\right\} \leq \left\{\begin{array}{l} k_7 \exp(-\lambda(\min_{c\in\{a,b\}}\{\sup_{y\in B(c,\beta\delta_0)}\{\rho(x,c)+\rho(c,y)\}+k_8\}) \\ \\ k_7 \exp(-\lambda(A(x,y,T)+k_8)) \ \forall y\in B(b,\beta\delta_0) \cup B(a,\beta\delta_0) \end{array}\right.$$

(ii) If $x \in B(b,\delta_0)$, $y \in B(b,\beta\delta_0)$, $w \notin B(b,\delta_0)$

$T \in [T_0, \infty)$, $T_1 \in [T_0, 2T_0]$, $\lambda \in [\lambda_0, \infty)$ and $i \in \{1,...,d\}$ then

$$(6.7)\quad \left.\begin{array}{l} \exp(-T_1 \frac{H(\lambda)}{\lambda})(x,w) \\ |\frac{1}{\lambda}\frac{\partial}{\partial x_i} e^{-T_1\frac{H(\lambda)}{\lambda}}(x,w)| \end{array}\right\} \leq \left\{\begin{array}{l} k_7 \exp(-\lambda(\sup_{y\in B(b,\beta\delta_0)}\{\rho(x,b)+\rho(b,y)\}+k_8)) \\ k_7 \exp(-\lambda(A(x,y,T)+k_8)) \end{array}\right.$$

and similar when $(x,y) \in B(a,\delta_0) \times B(a,\beta\delta_0)$

(iii) With everything as in lemma 8.1 and

$|V'(x)| = 0(e^{A|x|^2})$ for some $A < \infty$ then *uniformly for*

$(x,y,T) \in B(x_0, \delta) \times B(b,\delta_0) \times [T_0, 2T_0]$ and $i \in \{1,...,d\}$

$$(6.8)\qquad -\frac{1}{\lambda}\frac{\partial}{\partial x_i} e^{-T\frac{H(\lambda)}{\lambda}}(x,y) = \left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}}$$

$$\left\{\frac{\partial A(x,y,T)}{\partial x_i} b_0^T(x,y) + \frac{\left(\frac{\partial A(x,y,T)}{\partial x_i} b_0^T(x,y)\, a_1^T(x,y) - \frac{\partial b_0^T(x,y)}{\partial x_i}\right)}{\lambda} + 0\left(\frac{1}{\lambda^2}\right)\right\}, \text{ as } \lambda \to \infty$$

A Proof in Chapter 10.

Now we want to get the upper bound in (1.35) by showing the main contribution in

$$(6.9)\qquad e^{-T\frac{H(\lambda)}{\lambda}}(x,y) = \int \exp\left(-\lambda^2 \int_0^{\frac{T}{\lambda}} V(\omega(s))ds\right) d\mu_{0,x,y,\frac{T}{\lambda}}(\omega)$$

is from $\{\omega : |\omega(s) - g_T^{x,y}(s\lambda)| \leq \frac{r_0}{T}\}$ and use lemma 8.1. If $T \in [2T_0 , \infty)$ we write $T = (n + 1)\, T_1$ with $T_1 \in [T_0 , 2T_0)$ and for $(x,y) \in B(x_0 , \delta) \times B(b,\delta)$ where $\delta \in (0,\beta\delta_0]$ we write $w_0 := x$, $w_{n+1} := y$ and

$$(6.10)\qquad e^{-T\frac{H(\lambda)}{\lambda}}(x,y) = \int_{(\mathbb{R}^d)^n} \prod_{i=0}^{n} e^{-T_1\frac{H(\lambda)}{(\lambda)}}(w_i , w_{i+1})\, dw_1 , ..., dw_n .$$

First we show the main contribution in (6.10) is from $(B(b,\delta_0))^n$. In the final formula for the splitting this step puts the contributions from paths going between a and b more than once, among them multi-instantons (see Coleman [1] and Zinn-Justin [1]), into error terms. The second step is to show that $w_i \in B(g_T^{x,y}(iT_1), \frac{\text{constant}}{T})$ is what really matters. In step 3 we bound

$$\int_{w_i \in B(g_T^{x,y}(iT_1),\frac{\text{constant}}{T})} \prod_{i=0}^{n} e^{-T_1\frac{H(\lambda)}{\lambda}}(w_i , w_{i+1})\, dw_1 , ..., dw_n$$

from above by

$$\int \exp(-\lambda^2 \int_0^{\frac{T}{\lambda}} V(\omega(s)ds)\ \chi(\{\omega \in C_{x,y}([0, \tfrac{T}{\lambda}])$$

$$\|\omega - g_T^{x,y}(\cdot\ \lambda)\|_{L^\infty[0,\frac{T}{\lambda}]} \le \frac{r_0}{T}\})\ d\mu_{0,x,y,\frac{T}{\lambda}}(\omega)\ (1 + \text{“something small”})$$

which we know by lemma (8.1) and finishes the proof of the upper bound in (1.35).

To get (1.36) we differentiate (6.10) under the integral sign

$$(6.11)\qquad \frac{\partial}{\partial x_i} e^{-T\frac{H(\lambda)}{\lambda}}(x,y) = \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i} e^{-T_1\frac{H(\lambda)}{\lambda}}(x,w_1)\, e^{-nT_1\frac{H(\lambda)}{\lambda}}(w_1\,,y)\,dw_1$$

and we only need the analysis of $\frac{\partial}{\partial x_i} e^{-T_1\frac{H(\lambda)}{\lambda}}(x,w_1)$ in lemma 10, (9.3) of Proposition 2 that says

$$(6.12)\qquad \overline{\lim_{\lambda\to\infty}} \frac{\log\left|\frac{\partial}{\partial x_i} e^{-T\frac{H(\lambda)}{\lambda}}(x,y)\right|}{\lambda} \le -A(x,y,T)$$

u.o.c. in $\mathbb{R}^d \times \mathbb{R}^d \times (0,\infty)$

and $T \in [T_0\,,\infty)$-analysis of $e^{-T\frac{H(\lambda)}{\lambda}}(w_1\,,y)$

that we have then finished.

Now with k_8 as in lemma 10 we pick $\delta \in (0,\beta\delta_0]$ such that

$$(6.13) \qquad \sup_{x\in B(x_0,\delta)} |\rho(x,a)-\rho(x,b)| \le \frac{k_8}{2}$$

$$\sup_{y\in B(b,\delta)} \{\rho(y,b)\} \le \sup_{y\in B(a,\beta\delta_0)} \{\rho(y,a)\}$$

and we do the first step of section 6.2.

§6.2. Going from $(\mathbb{R}^d)^n$ to $(B(b,\delta_0))^n$ in (6.10)

In this section we will prove that if $\delta \in (0,\beta\delta_0]$ satisfies (6.13) and $T_1 \in [T_0, 2T_0)$ then uniformly for $x = w_0 \in B(x_0, \delta)$, $y = w_{n+1} \in B(b,\delta)$, $T = (n+1)\,T_1 \in [2T_0, \infty)$ and $\lambda \in [\lambda_0, \infty)$ we have

$$(6.14) \qquad \exp(-T\frac{H(\lambda)}{\lambda})(x,y) = \int_{(B(b,\delta_0))^n} \prod_{i=0}^{n} e^{-T_1\frac{H(\lambda)}{\lambda}} (w_i, w_{i+1})\, dw_1 \dots dw_n$$

$$+ 0(\exp(-\lambda(\rho(x,b) + \sup_{y\in B(b,\delta)} \{\rho(b,y)\} + \frac{k_8}{2}))(1 + 0(\frac{T}{\lambda})) \text{ and}$$

$$\int_{(B(b,\delta_0))^n} \prod_{i=0}^{n} e^{-T_1\frac{H(\lambda)}{\lambda}} (w_i, w_{i+1})\, dw_1 \dots dw_n =$$

$$= (\frac{\lambda}{2\pi})^{\frac{d}{2}}\, b_0^T (x,y) \exp(-\lambda A(x,y,T))\, (1 + 0(\frac{T}{\lambda})).$$

<u>Proof</u>. For $x \in B(x_0, \delta)$, $y \in B(b,\beta\delta_0)$, $T = (n+1)\,T_1$ with $T_1 \in [T_0, 2T_0)$ we write

$$(6.15)\qquad e^{-T\frac{H(\lambda)}{\lambda}}(x,y) = \int \prod_{i=0}^{n} e^{-T_1\frac{H(\lambda)}{\lambda}}(w_i, w_{i+1})\, dw_1 \dots dw_n$$

$$= \left(\int_{B(a,\delta_0)} + \int_{B(b,\delta_0)} + \int_{\mathbb{R}^d \setminus (B(a,\delta_0) \cup B(b,\delta_0))} \right)$$

$$e^{-T\frac{H(\lambda)}{\lambda}}(x,w_1)\; e^{-nT_1\frac{H(\lambda)}{\lambda}}(w_i, y)\, dw_1$$

$=: I_1 + III_1 + V_1$ which defines I_1, III_1, and V_1 (in the same order).

For $j = 2, \dots, n$ put

$$(6.16)\qquad I_j = \int_{w_k \in B(a,\delta_0) \text{ for } k=1,\dots,j} \prod_{i=0}^{n} e^{-T_1\frac{H(\lambda)}{\lambda}}(w_i, w_{i+1})\, dw_1 \dots dw_n$$

and

$$(6.17)\qquad II_j = \int_{\substack{w_k \in B(a,\delta_0) \text{ for } k=1,\dots,j-1 \\ w_j \notin B(a,\delta_0)}} \prod_{i=0}^{n} e^{-T_1\frac{H(\lambda)}{\lambda}}(w_i, w_{i+1})\, dw_1 \dots dw_n$$

For $\ell \in \{1, ..., n-1\}$

$$I_\ell = \int_{w_k \in B(b,\delta_0),\ k=1,...,\ell} \prod_{i=0}^{n} e^{-T_1 \frac{H(\lambda)}{\lambda}} (w_i , w_{i+1})\, dw_1 \ldots dw_n$$

$$= \int_{w_k \in B(b,\delta_0),\ k=1,...,\ell+1} \prod_{i=0}^{n} e^{-T_1 \frac{H(\lambda)}{\lambda}} (w_i , w_{i+1})\, dw_1 \ldots dw_n$$

$$+ \int_{\substack{w_k \in B(a,\delta_0) \quad \text{for } k=1,...,\ell \\ w_{\ell+1} \notin B(a,\delta_0)}} \prod_{i=0}^{n} e^{-T_1 \frac{H(\lambda)}{\lambda}} (w_i , w_{i+1})\, dw_1 \ldots dw_n$$

$$= I_{\ell+1} + II_{\ell+1}$$

and so

$$\text{(6.18)} \qquad I_1 = I_2 + II_2 = (I_3 + II_3) + II_2 = I_n + \sum_{j=2}^{n} II_j \,.$$

Before continuing we note by (6.4) of lemma 9

$$\exp(-T_1 \frac{H(\lambda)}{\lambda}) (u,v) = F(u,v,T_1,\lambda,\frac{r_0}{T_1}) + 0(\lambda^{\frac{d}{2}} e^{-\lambda A(x,y,T_1)} e^{-\lambda k_6 r_0^2})$$

uniformly for $(u,v,T_1,\lambda) \in (B(x_0 , \delta_0) \cup B(b,\delta_0)) \times B(b,\delta) \times [T_0 , 2T_0] \times [\lambda_0 , \infty)$.

Hence by lemma 8.1

(6.19) $$\exp(-T_1\frac{H(\lambda)}{\lambda})\,(u,v) = (\frac{\lambda}{2\pi})^{\frac{d}{2}}\, b_0^{T_1}\,(u,v)\,\exp(-\lambda A(u,v,T_1))$$

$$\{1 + \frac{a_1^{T_1}}{\lambda} + 0\Big(\frac{T_1^4}{\lambda^2}\Big) + 0\Big(\frac{T_1^2}{\lambda}\,\exp(-\,\frac{\lambda k_1 r_0^3}{2T_1^4})\Big)\}$$

$$(1 + 0(\lambda^{\frac{d}{2}}\, e^{-\lambda k_6 r_0^2})) =$$

$$= ((\frac{\lambda}{2\pi})^{\frac{d}{2}}\; b_0^{T_1}(u,v)\,\exp(-\lambda A(u,v,T_1))\,(1 + \frac{a_1^{T_1}(u,v)}{\lambda} + 0(\frac{1}{\lambda^2}))$$

$$= ((\frac{\lambda}{2\pi})^{\frac{d}{2}}\; b_0^{T_1}(u,v)\,\exp(-\lambda A(u,v,T_1))\,(1 + 0(\frac{1}{\lambda}))$$

uniformly for $(u\,,\,v\,,\,T_1\;\lambda) \in (B(x_0\,,\,\delta_0) \cup B(b,\delta_0)) \times B(b,\delta_0)$

$\times\,[T_0\,,\,2T_0] \times [\lambda_0\,,\,\infty)$

and similarly we have for

$$(u\,,\,v,\,T_1\,,\lambda) \in (B(x_0\,,\,\delta_0) \cup B(a,\delta_0)) \times B(a,\delta_0) \times [T_0\,,\,2T_0] \times [\lambda_0\,,\,\infty).$$

For $j \geq 2$ we write

$$(6.20)\quad II_j = \int_{B(a,\delta_0)} \left(\int_{\substack{w_2,\dots w_{j-1} \in B(a,\delta_0) \\ w_j \notin B(a,\delta_0)}} \prod_{i=0}^{n} e^{-T_1 \frac{H(\lambda)}{\lambda}} (w_i\,,\, w_{i+1})\, dw_1\,,\, dw_n \right)$$

$$= \int_{B(a,\delta_0)} e^{-T\frac{H(\lambda)}{\lambda}} (x,w_1)\, \widetilde{II}_j(w_1\,,\, y)\, dw_1\,.$$

By the semigroup property and $e^{-t\frac{H(\lambda)}{\lambda}}(x,y) \le (\frac{\lambda}{2\pi t})^{\frac{d}{2}} \exp(-\frac{\lambda|x-y|^2}{2t})$ if $t > 0$ we have

$$\widetilde{II}_j(w_1\,,\, y) = \int_{\substack{w_2,\dots w_{j-1} \in B(a,\delta_0) \\ w_j \notin B(a,\delta_0) \\ w_{j+1},\dots w_n \in \mathbb{R}^d}} \left(\prod_{i=0}^{j-1} e^{-T_1 \frac{H(\lambda)}{\lambda}} (w_i\,,\, w_{i+1}) \right)$$

$$\left(\prod_{i=j}^{n} e^{-T_1 \frac{H(\lambda)}{\lambda}} (w_i\,,\, w_{i+1}) \right) dw_1\,,\, \dots,\, dw_n$$

$$= \int_{\substack{w_2,\dots,w_{j-1} \in B(a,\delta_0) \\ w_j \notin B(a,\delta_0)}} \prod_{i=1}^{j-1} e^{-T_1 \frac{H(\lambda)}{\lambda}} (w_i\,,\, w_{i+1})$$

$$e^{-(n-j)T_1 \frac{H(\lambda)}{\lambda}} (w_j\,,\, y)\, dw_2\,,\, \dots,\, dw_j$$

$$\leq \Big(\sup_{w_j \notin B(a,\delta_0)} \Big(\int_{w_1,\dots,w_{j-1} \in B(a,\delta_0)} \prod_{i=1}^{j-1} e^{-T_1 \frac{H(\lambda)}{\lambda}} (w_i, w_{i+1}) dw_2, \dots, dw_{j-1} \Big)\Big)$$

$$\int e^{-(n-j)T_1 \frac{H(\lambda)}{\lambda}} (w_j, y)\, dw_j$$

$$\leq \sup_{w_j \notin B(a,\delta_0)} \Big(\int_{w_1,\dots,w_{j-1} \in B(a,\delta_0)} \prod_{i=1}^{j-1} e^{-T_1 \frac{H(\lambda)}{\lambda}} (w_i, w_{i+1})\, dw_2, \dots, dw_{j-1} \Big)$$

where if $j = 2$ we mean $\sup_{w_2 \notin B(a,\delta_0)} (e^{-T_1 \frac{H(\lambda)}{\lambda}} (w_1, w_2))$ which by (6.7) is less than k_5 $\exp(-\lambda(A(w_1, z, S) + k_6))$ for any $z \in B(a, \beta\delta_0)$ and $S \in [T_0, \infty)$ so we have

(6.21) $\quad \tilde{\Pi}_2(w_1, y) \leq k_5 \exp(-\lambda(A(w_1, z, S) + k_6))$ for any such z and S.

If $j \geq 3$ then by (6.19), (6.3) and (6.7)

(6.22) $$\tilde{\Pi}_j(w_1, y) \leq \sup_{w_j \notin B(a,\delta_0)} \Big(\int_{w_2,\dots,} \cdots \int_{w_{j-1} \in B(a,\delta_0)}$$

$$\Big\{ \prod_{i=1}^{j-2} ((\tfrac{\lambda}{2\pi})^{\frac{d}{2}} b_0^{T_1}(w_i, w_{i+1})\, e^{-\lambda A(w_i, w_{i+1}, T_1)} (1 + 0(\tfrac{1}{\lambda}))$$

$$e^{-T_1 \frac{H(\lambda)}{\lambda}} (w_{j-1}, w_j)\Big\}\, dw_1, \dots, dw_{j-1}$$

$$\leq \int_{w_{j-1}\in B(a,\delta_0)} (\frac{\lambda}{2\pi})^{\frac{d}{2}} b_0^{(j-2)T_1}(w_1, w_{j-1})\, e^{-\lambda A(w_1,w_{j-1},(j-2)T_1)}$$

$$k_7 \exp(-\lambda(A(w_{j-1}, z, S) + k_8))\, dw_{j-1}(1 + 0(\tfrac{1}{\lambda}))^{(j-2)}$$

for any $z \in B(a,\beta\delta_0)$ and $s \in [T_0, \infty)$.

If $x \in B(x_0,\delta_0) \cup B(a,\delta_0), y \in B(a,\delta_0)$ (and similar for b) and $T,S \in [T_0,\infty)$ then

$\min\limits_{w\in B(a,\delta_0)} \{A(x,w,T) + A(w,y,S)\} = A(x,y,T+S)$ and is attained at the unique point

$\overline{w} = \overline{w}(x,y,T,S) = g^{x,y}_{T+S}(T) \in B(a, \frac{\delta_0}{2})$, by lemma 2.

Now we will use Hörmander's stationary phase theorem.

To get uniformity, as in (4.11), we want

$$(6.23)\qquad \frac{|w-\overline{w}|}{|A'_w(x,w,T)+A'_w(w,y,S)|} \leq \text{constant} < \infty \text{ for all}$$

$$(x,y,T,s) \in (B(x_0,\delta_0) \cup B(a,\delta_0) \times B(a,\delta_0) \times [T_0, \infty) \times [T_0, \infty).$$

That follows from

$A'_w(x,w,T) + A'_w(w,y,S) =$

$$= (A'_w(x,w,T) + A'_w(w,y,S)) - (A'_w(x,w,T) + A'_w(w,y,S))|_{w=\overline{w}}$$

$$= \int_0^1 (A''_{ww}(x,\overline{w} + t(w-\overline{w}),T) + A''_{ww}(\overline{w} + t(w-\overline{w}),y,S))(w - \overline{w})\, dt$$

$$= \int_0^1 (2\Omega + 0(e^{-\delta_1 T} + e^{-\delta_1 S} + |\overline{w} + t(w-\overline{w}) - a|))(w - \overline{w})\, dt\, .$$

In the last step we used (4.18).

With k = integer part of $(\frac{d}{2}+2) =: [\frac{d}{2}+2]$ we get from (4.10)

$$(6.24) \quad \int_{B(a,\delta_0)} (\tfrac{\lambda}{2\pi})^{\frac{d}{2}} b_0^T(u,w)\, e^{-\lambda A(x,w,T)}\, e^{-\lambda A(w,y,S)}\, dw$$

$$= 0(e^{-Te_0(a)}\, e^{-\lambda A(x,y,T+S)} (1 + 0(\tfrac{1}{\lambda}))$$

uniformly for $(x,y,T,S,\lambda) \cup (B(x_0, \delta_0)) \times [T_0, \infty) \times [T_0, \infty) \times (0,\infty)$.

Since, by a statement similar to (6.2) near a, instead of near b, we have

$$|\partial_y^\alpha b_0^T(x,y)| = e^{-Te_0(a)} 0(|\partial_y^\alpha a_0^T(x,y)|)$$

where $e_0(a) = \text{trace}\sqrt{V''(a)}$.

The uniformity in (6.24) follows from (6.23) together with (4.18) and (4.16) that say $|\partial_y^\alpha A(x,y,T)|$ for $|\alpha| \leq 3[\frac{d}{2}+2]+1$ and $|\partial_y^\beta a_0^T(x,y)|$ for $|\beta| \leq 2[\frac{d}{2}+2]$ are uniformly bounded on

$$(B(x_0, \delta_0) \cup B(a,\delta_0)) \times B(a,\delta_0) \times [T_0, \infty).$$

So (6.21), (6.22), and (6.24) gives

If $j \geq 2$ then

(6.25) $\widetilde{II}_j(w_1\,,y) = 0(e^{-(j-2)T_1 e_0(a)} \exp(-\lambda[A(w_1\,,z\,,(j-2)T_1 + S) + k_8]))$

$(1 + 0(\frac{1}{\lambda}))^{(j-1)}$ uniformly for any $z \in B(a,\beta\delta_0)$ and $S \in [T_0\,,\infty)$.

Put (6.19) and (6.24) into (6.20) and use (6.23) and you get:

If $j \in \{2, ..., n\}$ then $II_j = 0(e^{-(j-1)T_1 e_0(a)}\, e^{-\lambda\{A(x,z,(j-1)T_1)+ S)+k_8\}}(1 + 0(\frac{1}{\lambda}))^j)$.

Taking $S \to \infty$ (use Proposition 1) and then inf over $z \in B(a,\beta\delta_0)$ gives

(6.26) $II_j = 0(e^{-jT_1 e_0(a)} \exp(-\lambda\{\rho(x,a) + \sup_{y\in B(a,\beta\delta_0)} (\rho(y,a))+k_8\}))(1 + 0(\frac{1}{\lambda}))^j.$

Using $j \leq n = 0(T)$ in (6.26) we get

(6.27) $\sum_{j=2}^{n} II_j = 0(\exp(-\lambda\{\rho(x,a) + \sup_{y\in B(a,\beta\delta_0)} (\rho(y,a)) + k_8\})(1 + 0(\frac{T}{\lambda})))$

uniformly for $(x,y,T,\lambda) \in B(x_0,\delta_0) \times B(b,\beta\delta_0) \times [T_0\,,\infty) \times [\lambda_0\,,\infty)$.

In

$$I_n = \int_{w_k \in B(a,\delta_0),\ k=1,...,n} \prod_{i=0}^{n} e^{-T_1\frac{H(\lambda)}{\lambda}}(w_i\,,w_{i+1})\, dw_1\,, ..., dw_n$$

we use (6.19) for $e^{-T_1\frac{H(\lambda)}{\lambda}}(w_i , w_{i-1})$ for i = 0, 1, ..., n − 1 and (1.17) for i = n,

which implies

$$e^{-T_1\frac{H(\lambda)}{\lambda}}(w_n , y) \leq C_\varepsilon \exp(-\lambda[A(w_n , y , T_1) - \varepsilon])$$

uniformly on $\overline{B(a,\delta_0)} \times \overline{B(b,\beta\delta_0)} \times [T,2T_0]$, and we have, since

$$(1 + 0(\tfrac{1}{\lambda}))^n = 1 + 0(\tfrac{T}{\lambda})$$

$$I_n \leqq \int_{(B(a,\delta_0))^n} \prod_{i=0}^{n-1}\Big((\tfrac{\lambda}{2\pi})^{\frac{d}{2}} b_0^{T_1}(w_i , w_{i+1})\, e^{-\lambda A(w_i,w_{i+1},T_1)}\Big)$$

$$C_\varepsilon\, e^{-\lambda[A(w_n,y,T_1)-\varepsilon]}\, dw_1 , ..., dw_n\, (1 + 0(\tfrac{T}{\lambda})) .$$

By (6.3) and (6.2) we get

$$I_n = 0(C_\varepsilon\, e^{-nT_1e_0(a)}\, \lambda^{\frac{d}{2}} \int_{B(a,\delta_0)}$$

$$e^{-\lambda[A(x , w_n , nT_1) + A(w_n , y , T_1)-\varepsilon]}\, dw_n\, (1 + 0(\tfrac{T}{\lambda}))^2)$$

since $A(w_n , y , T_1) \geq \rho(w_n , y)$ and

$$A(x , w_n , nT_1) = \rho(x,a) + \rho(a,w_n) + 0(e^{-\omega_1(a)nT_1})$$

uniformly for $(x,w_n) \in \overline{B(x_0 , \delta)} \times \overline{B(a,\delta_0)}$ we get

$$I_n = 0\left(C_\varepsilon e^{-Te_1(a)} \lambda^{\frac{d}{2}} e^{-\lambda[\inf_{w\in B(a,\delta_0)} \{\rho(x,a)+\rho(a,w)+\rho(w,y)\}+0(e^{-\omega_1(a)T})-\varepsilon]} \right)$$

Since y is in $B(b,\beta\delta_0)$, I_n is smaller than the R.H.S. of (6.27) and so by (6.18)

$$(6.28) \qquad I_1 = 0\left(\exp(-\lambda[\rho(x,a) + \sup_{y\in B(a,\beta\delta_0)} [\rho(y,a)] + k_8\right) (1 + 0(\tfrac{T}{\lambda}))$$

uniformly for $(x,y,T,\lambda) \in B(x_0 , \delta_0) \times B(b,\beta\delta_0) \times [2T_0 , \infty) \times [\lambda_0 , \infty)$.

Next write

$$III_1 = III_2 + IV_2 = \cdots = III_n + \sum_{j=2}^{n} IV_j$$

where II_j is defined by replacing a by b, the definition of I_j , and in the same way we get IV_j from II_j .

We can replace a by b in the estimates for II_j and copy estimates for IV_j . We get from (6.27)

$$(6.29) \qquad III_1 = \int_{(B(b,\delta_0))^n} (\prod_{i=0}^{n} e^{-T_1\frac{H(\lambda)}{\lambda}} (w_i , w_{i=1}))\, dw_1 \dots dw_n$$

$$+ 0(e^{-\lambda[\rho(x,b)+ \sup_{y\in b(b,\beta\delta_0)} (\rho(b,y))+k_8]} (1 + 0(\tfrac{T}{\lambda}))$$

where by (6.19) and (6.3)

$$(6.30)\qquad \int_{(B(b,\delta_0))^n} \Big(\prod_{i=0}^{n} e^{-T_1\frac{H(\lambda)}{\lambda}}(w_i\,,\,w_{i+1})\Big)\,dw_1\,,\,\ldots,\,dw_n =$$

$$= \left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}} b_0^T(x,y)\exp(-\lambda A(x,y,T))(1+0(\tfrac{T}{\lambda}))\ \text{uniformly for}$$

$$(x,y,T,\lambda) \in B(x_0\,,\,\delta_0) \times B(b,\beta\delta_0) \times [2T_0\,,\,\infty) \times [\lambda_0\,,\,\infty).$$

Finally by (6.6) of lemma 10.1

$$(6.31)\qquad V_1 = \int_{\mathbb{R}^d\setminus(B(a,\delta_0)\cup B(b,\delta_0))} e^{-T_1\frac{H(\lambda)}{\lambda}}(x,w_1)\, e^{-nT_1\frac{H(\lambda)}{\lambda}}(w_1\,,\,y)\,dw_1$$

$$\leq \Big(\sup_{w_1\notin B(a,\delta_0)\cup B(b,\delta_0)} e^{-T_1\frac{H(\lambda)}{\lambda}}(x,w_1)\Big)\Big(\int e^{-nT_1\frac{H(\lambda)}{\lambda}}(w_1,y)\,dw_1\Big)$$

$$\leq k_7\exp(-\lambda(\min_{c\in\{a,b\}}[\sup_{y\in B(c,\beta\delta_0)}\{\rho(x,c)+\rho(c,y)\}]+k_8))$$

and so

$$e^{-T\frac{H(\lambda)}{\lambda}}(x,y) = I_1 + III_1 + V_1 =$$

$$= 0(e^{-\lambda[\rho(x,a)+\sup_{y\in B(a,\beta\delta_0)}(\rho(a,y))+k_8]}(1+0(\tfrac{T}{\lambda})))$$

$$+ \int_{(B(b,\delta_0))^n}\Big(\prod_{i=0}^{n} e^{-T_1\frac{H(\lambda)}{\lambda}}(w_i,w_{i+1})\Big)\,dw_1\,,\,\ldots,\,dw_n +$$

$$+ 0(e^{-\lambda[\rho(x,b)+\sup_{y\in B(b,\beta\delta_0)}(\rho(b,y))+k_8]}(1 + 0(\tfrac{T}{\lambda})))$$

$$+ 0(\exp(-\lambda(\min_{c\in\{a,b\}}[\sup_{y\in B(c,\beta\delta_0)}\{\rho(x,c) + \rho(c,y)\}] + k_8))$$

$$= \int_{(B(b,\delta_0))^n} (\prod_{i=0}^{n} e^{-T_1\frac{H(\lambda)}{\lambda}}(w_i , w_{i+1}))\, dw_1 , \ldots, dw_n$$

$$+ 0(\exp(-\lambda[\rho(x,b) + \sup_{y\in B(b,\delta)}\{\rho(y,b)\} + \tfrac{k_8}{2}]))(1 + 0(\tfrac{T}{\lambda}))$$

uniformly for $(x,y,T,\lambda) \in B(x_0 , \delta) \times B(b,\delta) \times [2T_0 , \infty] \times [\lambda_0 , \infty]$.

Here we used (6.15) in the first step, (6.28), (6.29) and (6.31) in the second and (6.13) in the third. (6.14) follows now by (6.30).

§6.3 <u>The main contribution in (6.10) is from</u> $\times_{i=0}^{n} B(g_T^{x,y}(iT_1, \tfrac{r}{T})$.

We want to go beyond the $0(\frac{T}{\lambda})$-term in the leading term of (6.14) given by

$$\int_{(B(b_1\delta_0))^n} \left(\prod_{i=0}^{n} e^{-T_1 H\frac{(\lambda)}{\lambda}}(w_i, w_{i+1})\right) dw_1 \ldots dw_n = \tag{6.32}$$

$$\left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}} b_0^T(x,y) \exp(-\lambda A(x,y,T) (1 + 0(\tfrac{T}{\lambda}))$$

As a step towards reducing the problem to (6.1) in Lemma 8.1 we have:

With k_1 as (2.33) we have

(6.33)
$$\int_{(B(b_1\delta_0))^n} \prod_{i=0}^{n} e^{-T_1 \frac{H(\lambda)}{\lambda}(w_i, w_{i+1})} dw_1 \dots dw_n$$

$$= \int_{w_i \in B(g_T^{xy}(iT_1), \frac{r}{T})} \prod_{i=0}^{n} e^{-T\frac{H(\lambda)}{\lambda}(w_i, w_{i+1})} dw_1 \dots dw_n$$

$$+ 0((1 + 0(\tfrac{T}{\lambda}))\, T\lambda^d\, e^{-Te_0(b)} \exp(-\lambda(A(x,y,T) + \frac{k_1 T^2}{T^2})))$$

uniformly for $(x,y,T,\lambda,r) \in B(x_0,\delta) \times B(b,\delta) \times [2T_0, \infty) \times [\lambda_0, \infty) \times (0, r_0]$.

Remarks 1. If we put (6.19) into (6.32) we get $\sum_{i=0}^{n} A(w_i, w_{i+1}, T_1)$ in the exponent

with $w_0 = x$ and $w_{n+1} = y$. By the uniqueness of g_T^{xy},

$$\min_{w_1 \cdots w_n} \left\{\sum_{i=0}^{n} A(w_i, w_{i+1}, T_1)\right\} = A(x,y,T)$$

is attained if and only if $w_i = g_T^{xy}(iT_1)$ for $i \in \{1, \dots, n\}$.

2. Using the uniformity in (6.33) and taking $r \downarrow 0$ we get

$$\int_{(B(b_1\delta_0))^n} \prod_{i=0}^{n} e^{-T_1 \frac{H(\lambda)}{\lambda}(w_1, w_{i+1})} dw_1 \dots dw_n =$$

$$= 0((1 + 0(\tfrac{T}{\lambda}))\, T\lambda^d e^{-Te_0(b)} \exp(-\lambda\, A(x,y,T))$$

which $T \cdot \lambda^{\frac{d}{2}}$ times the behaviour in (6.32).

Proof of (6.33):

Put

$$\text{(6.34)} \qquad VI_0 = \int_{(B(b_1\delta_0))^n} \prod_{i=0}^{n} e^{-T_1\frac{H(\lambda)}{\lambda}} (w_i\,, w_{i+1})\, dw_1 \ldots dw_n$$

for $r \in (0, r_0]$ and for $j \in \{1, \ldots, n\}$

$$\text{(6.35)} \qquad VI_j = \int_{\substack{w_k \in (g_T^{xy}(kT_1), \frac{r}{T}) \quad k=1,\ldots,j \\ w_{j+1},\ldots,w_n \in B(b,\delta_0)}} \prod_{i=0}^{n} e^{-T\frac{H(\lambda)}{\lambda}} (w_i\,, w_{i+1})\, dw_1 \ldots dw_n$$

and

$$\text{(6.36)} \qquad VII_j = \int_{\substack{w_k \in B(g_T^{xy}(kT_1), \frac{r}{T}) \ k=1,\ldots,j-1 \\ w_j \in B(b,\delta_0)\setminus B(g_T^{xy}(jT_1), \frac{r}{T}) \\ w_k \in B(b,\delta_0) \quad k=j+1,\ldots,n}} \prod_{i=0}^{n} e^{-T_1\frac{H(\lambda)}{\lambda}} (w_i, w_{i+1})\, dw_1\, dw_n.$$

We see $VI_j = VI_{j+1} + VII_{j+1}$ for $j = 0, n-1$ and so

$$\text{(6.37)} \qquad VI_0 = VI_n + \sum_{j=1}^{n} VII_j.$$

By (6.30) and (6.3) we get from (6.36):

$$(6.38)\qquad VII_j \le \int_{\substack{w_k\in B(b,\delta_0)\text{ for }k\in\{1,\dots,n\}\setminus\{j\}\\ w_j\in B(b,\delta_0)\setminus B(g_T^{xy}(jT_1),\frac{r}{T})}} \prod_{i=0}^{n} e^{-T_1\frac{H(\lambda)}{\lambda}}(w_i, w_{i+1})dw_1\cdots dw_n$$

$$= \int_{w_j\in B(b,\delta_0)\setminus B(g_T^{xy}(jT_1),\frac{r}{T})} \left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}} b_0^{jT_1}(x,w_j)e^{-\lambda A(x,w_j,jT_1)}(1+0(\tfrac{1}{\lambda}))^{2j}$$

$$\left[\left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}} b_0^{(n+1-j)T_1}(w_j,y))e^{-\lambda A(w_j,y,(n+1-j)T_1)}(1+0(\tfrac{1}{\lambda}))^{2(n+1-j)}\right] dw_j\,.$$

Now we use (2.33) that says:

$$\int_0^T\left(\tfrac{1}{2}\dot\gamma^2 + V(\gamma)\right)dt \ge A(x,y,T) + k_1\min(\delta_0^2, \|\gamma - g_T^{xy}\|^2_{L^\infty[0,T]}$$

for all γ such that $\gamma - g_T^{xy} \in D_0(0,T)$, uniformly for $(x,y,T) \in B(x_0,\delta_0) \times B(b,\delta_0) \times [T_0, \infty)$. Hence

$$\inf_{w_j\in B(b,c_0)\setminus B(g_T^{xy}(jT_1),\frac{r}{T})} \{A(x,w_j,jT_1) + A(w_j,y,(n+1)T_1)\}$$

$$\ge \inf_{\gamma:\|\gamma-g_T^{xy}\|_{L^\infty[0,T]}\ge\frac{r}{T}} \left\{\int_0^T\left(\tfrac{1}{2}\dot\gamma^2 + V(\gamma)\right)dt\right\} \ge A(x,y,T) + k_1\frac{r^2}{T^2}.$$

By (6.2)

$$\sup_{\substack{y,w_j\in B(b,\delta_0)\\ x\in B(x_0,\delta_0)}} (b_0^{jT_1}(x,w_j)\, b^{(n+u-j)T_1}(w_j,y)) = 0\,(e^{-(n+1)Te_0(b)})$$

and since $(1 + 0(\frac{1}{\lambda}))^{2(n+1)} = 1 + 0(\frac{T}{\lambda})$

$$VII_j = 0(\lambda^d e^{-Te_0} e^{-\lambda(A(x,y,T)+\frac{k_1 r_2}{T^2})} (1 + 0(\frac{T}{\lambda}))$$

uniformly for

$$(x,y,T,\lambda,r,j) \in B(x_0,\delta) \times B(b,\delta) \times [T_0, \infty) \times [\lambda_0, \infty) \times (0, r_0] \times \{1, ..., n\}$$

Now (6.33) follows from (6.37) and definitions (6.34) and (6.35).

§6.4 The conclusion of the proof of (1.35)

We start by; If

(6.39) $$r = \frac{r_0}{(1+k_2)}$$

then

(6.40) $$\int_{w_j \in B(g_T^{xy}(jT_1),\frac{r}{T}) \text{ for } j=1,2,...,n} e^{-T_1 \frac{H(\lambda)}{\lambda}} (w_i,w_{i+1}))dw_1 \ldots dw_n$$

$$= \left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}} b_0^T(x,y)\{1 + \frac{a_1^T(xy)}{\lambda} + (\frac{T^4}{\lambda^2}) + 0(T \exp(-\frac{\lambda k_6 r^2}{T^2}))$$

$$+ 0\left(\frac{T^2}{\lambda} \exp(-\frac{\lambda k_5 r^2}{T^4})\right)\} \exp(-\lambda A(x,y,T))$$

uniformly for $(x,y,T,\lambda) \in B(x_0,\delta) \times B(b,\delta) \times [2T_0,\infty) \times [\lambda_0, \infty)$.

Proof. Put $v_j = g_T^{xy}(jT_1)$ for $j \in \{0,1,\dots,n,n+1\}$ then $v_0 = x = w_0$ and $v_{n+1} = y = w_{n+1}$ and $g_{T_1}^{v_j,v_{j+1}}(t) = g_T^{xy}(t+jT_1)\Big|_{[0,T_1]}$ by the uniqueness in lemma 2.2. If $w_j \in B(g_T^{xy}(jT_1),\frac{r}{T})$ for $j = 1, \dots, n$ then $g_{T_1}^{w_jw_{j+1}}$ is again unique and by (2.38) $|g_{T_1}^{w_j,w_{j+1}}(t)-g_{T_1}^{v_j,v_{j+1}}| \le k_2(|w_j - v_j|e^{-\omega_1 t} + |w_{j+1} - v_{j+1}|e^{-\omega_1|T_1-t|}) \le 2k_2\frac{r}{T}$, uniformly. Hence if

$$|\alpha_j(s) - g_{T_1}^{w_jw_{j+1}}(s\lambda)| \le \tfrac{t}{T} \text{ for } s \in [0, \tfrac{T}{\lambda}]$$

then

$$|\alpha_j(s) - g_T^{xy}(jT_1 + s\lambda)| = |\alpha_j(s) - g_{T_1}^{v_1v_2}(s\lambda)|$$

$$\le |\alpha_j(s) - g_{T_1}^{w_jw_{j+1}}(s\lambda)| + |g_{T_1}^{w_jw_{j+1}}(s\lambda) - g_{T_1}^{v_jv_{j+1}}(s\lambda)|$$

$$\le \tfrac{r}{T} + \frac{2k_2r}{T} = (1 + 2k_2)\,\tfrac{r}{T} = \tfrac{r_0}{T}, \text{ by (6.39).}$$

So we have a string of (n + 1)-sausages inside a long and fat one:

$$(6.41) \qquad A := \bigcup_{\substack{w_i \in B(g_T^{xy}(iT_1),\frac{r}{T})\\ i=1,\dots,n}} \{\alpha \in C_{x,y}([0,\tfrac{T}{\lambda}]) : \alpha(j\tfrac{T_1}{\lambda}) = w_j \text{ for } j = \{0,1,\dots,n+1\}$$

$$\text{and } |\alpha(j\tfrac{T_1}{\lambda} + s) - g_{T_1}^{w_jw_{j+1}}(s\lambda)| \le \tfrac{r}{T} \text{ on } [0,\tfrac{T_1}{\lambda}] = [0,\tfrac{T}{(n+1)\lambda}] \text{ for } j \in \{0, \dots, n\}\}$$

$$\subseteq \{\alpha \in C_{x,y}([0, \tfrac{t}{\lambda}]) : |\alpha(s) - g_T^{xy}(s\lambda)| \le \tfrac{r_0}{T}\} =: B \quad \text{for r given in (6.39).}$$

By definition

$$F(x,y,T,\lambda,\epsilon) = \int \exp(-\lambda^2 \int_0^{\frac{T}{\lambda}} V(\alpha(s))\, ds) \times (\{\alpha \in C_{x,y}([0, \tfrac{T}{\lambda}]):$$

$$\|\alpha - g_T^{xy}(\cdot\lambda)\|_{L^\infty[0,\frac{T}{\lambda}]} \le \varepsilon\})\, d\mu_{0,x,y,\frac{T}{\lambda}}(\alpha)$$

Since $\mu_{0,u,v,t}$ is the restriction of the Wiener-measure to $C_{u,v}([0,t])$ we get:

$$(6.42) \qquad \int_{w_k \in B(g_T^{xy}(kT_1,\frac{r}{T}))k=1,\dots,n} \prod_{i=0}^{n} F(w_i, w_{i+1}, T_1, \lambda, \tfrac{r}{T})\, dw_1, \dots, dw_n$$

$$= \int_{\substack{w_k \in B(g_T^{xy}(kT_1),\frac{r}{T})\\ k=1,\dots,n}} \Big(\prod_{i=0}^{n} [\int \exp(-\lambda^2 \int_0^{\frac{T_1}{\lambda}} V(\alpha(s))ds)\, \chi(\{\alpha \in C_{w_i w_{i+1}}([0,\tfrac{T_1}{\lambda}]):$$

$$|(\alpha(s) - g_{T_1}^{w_i w_{i+1}}(s\lambda)| \le \tfrac{r}{T}\})d\mu_{0,w_i,w_{i+1},\frac{T_1}{\lambda}}(\alpha) dw_1,\dots,dw_n$$

$$= \int_A \exp(-\lambda^2 \int_0^{\frac{T}{\lambda}} V(\alpha(s)ds)d\mu_{0,x,y:\frac{T}{\lambda}}(\alpha)$$

$$\le \int_B \exp(-\lambda^2 \int_0^{\frac{T}{\lambda}} V(\alpha(s)ds)d\mu_{0,x,y:\frac{T}{\lambda}}(\alpha) = F(x,y,T,\lambda, \tfrac{r_0}{T})$$

that we know how to expand, by (6.1).

Recall Lemma 9 that says

$$\exp(-T_1 \frac{H(\lambda)}{\lambda})\,(w_i\, ,\, w_{i+1}) = F(w_i\, ,\, w_{i+1},\, T_1, \lambda, \tfrac{r}{T})\,(1 + 0(\exp(-\lambda \frac{k_6 r^2}{T^2})))$$

uniformly. Using $(n + 1) = 0(T)$, (6.41) and (6.1) with r as in (6.39), we conclude:

$$(6.43) \qquad \int_{w_k \in B(g_T^{xy}(kT_1), \frac{r}{T}) \text{ for } k=1,\ldots,n} \Big(\prod_{i=0}^{n} e^{-T_1 \frac{H(\lambda)}{\lambda}} (w_i, w_{i+1}) \Big) dw_1, \ldots, dw_n$$

$$\int_{w_k \in B(g_T^{xy}(kT_1), \frac{r}{T}) \text{ for } k=1,\ldots,n} \Big(\prod_{i=0}^{n} F(w_i, w_{i+1}, T_1, \lambda, \tfrac{r}{T}) \Big) dw_1, \ldots dw_n$$

$$(1 + 0(T \exp \frac{-\lambda k_6 r^2}{T^2}))) \leq F(x,y,T,\lambda,\tfrac{r_0}{T}) \; (1 + 0(T \exp(\frac{-\lambda k_6 r^2}{T^2})))$$

$$= \exp(-\lambda A(x,y,T)) \left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}} b_0^T(x,y) \left(1 + \frac{a_1^T(x,y)}{\lambda} + 0\,(\frac{T^4}{\lambda^2})\right.$$

$$\left. + \, 0\Big(\frac{T^2}{\lambda} \exp(\frac{-\lambda k_5 r^2}{T^4})\Big) + 0(T \exp(\frac{-\lambda k_6 r^2}{T^2}))\right)$$

uniformly which proves (6.40).

Now we get:

$$(6.44) \qquad e^{-T\frac{H(\lambda)}{\lambda}}(x,y) \overset{(6.10)}{=} \int_{(\mathbb{R}^d)^n} \Big(\prod_{i=0}^{n} e^{-T_1 \frac{H(\lambda)}{\lambda}} (w_i, w_{i+1}) \Big) dw_1, \ldots, dw_n$$

$$\overset{(6.14)}{=} \int_{(B(b,\delta_0))^n} \Big(\prod_{i=0}^{n} e^{-T_1 \frac{H(\lambda)}{\lambda}} (w_i, w_{i+1}) \Big) +$$

$$0(\exp(-\lambda(\rho(x,b) + \sup_{y \in B(b,\delta)} \{\rho(b,y)\} + \frac{k_8}{2}))) \; (1 + 0(\frac{T}{\lambda}))$$

$$\overset{(6.33)}{=} \int_{\overset{n}{\underset{i=0}{X}}(B(g_T^{xy}(iT_1),\frac{r}{T}))} \left(\prod_{i=0}^{n} e^{-T_1\frac{H(\lambda)}{\lambda}}(w_i,w_{i+1})\right)dw_1, \ldots, dw_n$$

$$+ 0((1 + 0(\frac{T}{\lambda}))\, T\lambda^d e^{-T\ell_0(b)} \exp(-\lambda(A(x,y,T) + \frac{k_1r^2}{T^2})))$$

$$+ 0(\exp[-\lambda(\rho(x,b) + \sup_{y\in B(b,\delta)} \{\rho(b,y)\} + \frac{k_8}{4})])\ (1 + 0(\frac{T}{\lambda}))$$

$$\overset{(6.40)}{=} \left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}} b_0^T(xy)\ \{1 + \frac{a_1^T(x,y)}{\lambda} + 0(\frac{T^4}{\lambda^2})$$

$$+ 0\left(\frac{T^2}{\lambda} \exp(-\lambda \frac{k_5r^2}{T^4})\right) + 0(T \exp\left(-\lambda \frac{k_6r^2}{T^2}\right))$$

$$+ 0(T\lambda^{\frac{d}{2}} \exp\left(-\lambda\frac{k_1r^2}{T^2}\right))\} \exp(\alpha-\lambda A(x,y,T))$$

$$+ 0(\exp[-\lambda(\rho(x,b) + \sup_{y\in B(b,\delta)} \{\rho(b,y)\} + \frac{k_8}{2})])$$

$$= \left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}} b_0^T(x,y)\ \{1 + \frac{a_1^T(x,y)}{\lambda} + 0(\frac{T^4}{\lambda^2}) + 0(\frac{T^2}{\lambda} \exp(-\lambda \frac{k_5r^2}{T^4}))\}$$

$$\exp(-\lambda A(x,y,T))$$

Uniformly $(x,y) \in B(x_0,\delta) \times B(b, \delta)$ as $T, \lambda \to \infty$ and $T^4 \leq \lambda$. Where we used Proposition 1 in the last step.

§6.5. Proof of (1.36)

Similar to (6.15) we write

$$-\frac{1}{\lambda}\frac{\partial}{\partial x_i} e^{-T\frac{H(\lambda)}{\lambda}}(x,y) =$$

$$\int -\frac{1}{\lambda}\frac{\partial}{\partial x_i} e^{-T_0\frac{H(\lambda)}{\lambda}}(x,w_1)\Big(\prod_{i=1}^{n} e^{-T_1\frac{H(\lambda)}{\lambda}} w_i,w_{i+1})\Big)\, dw_1,\ldots, dw_n$$

when $T = T_0 + nT_1 \in [2T_0, \infty)$. The reason for this "$T_0$" is to make the large T dependence in formula (6.48) below simpler (T_0 is fixed but T_1 depends on T).

When we obtained (6.14) from (6.15) we only used the estimates in (6.6), (6.7) and the upperbound in (1.17). We have (6.6) and (6.7) again for $\left|\frac{1}{\lambda}\frac{\partial}{\partial x_i} e^{-T\frac{H(\lambda)}{\lambda}}(x,y)\right|$ and we replace the upperbound in (1.17) by (9.3) below that says

$$\overline{\lim_{\lambda\to\infty}} \frac{\log\left|\frac{\partial}{\partial x_i} e^{-T\frac{H(\lambda)}{\lambda}}(x,y)\right|}{\lambda} \leq -A(x,y,T)$$

uniformly on compacts in $\mathbb{R}^d \times \mathbb{R}^d \times (0, \infty)$.

So similar to (6.14) we have

$$(6.45) \quad -\frac{1}{\lambda}\frac{\partial}{\partial x_i} e^{-T\frac{H(\lambda)}{\lambda}}(x,y) = \int_{(B(b,\delta_0))^n} -\frac{1}{\lambda}\frac{\partial}{\partial x_i} e^{-T_0\frac{H(\lambda)}{\lambda}}(x,w_1)$$

$$\Big(\prod_{i=1}^{n} e^{-T_1\frac{H(\lambda)}{\lambda}}(w_i, w_{i+1})\Big)\, dw_1,\ldots,dw_n +$$

$$+ 0(\exp\{-\lambda(\rho(x,b) + \sup_{y\in B(b,\delta)} (\rho(b,y)) + \frac{k_8}{2})\} (1 + 0(\frac{T}{\lambda}))$$

uniformly for $x \in B(x_0, \delta)$, $y = w_{n+1} \in B(b, \delta)$, $T = T_0 + nT_1$ where $T_1 \in [T_0, 2T_0)$ and $n \in \mathbb{N}$, when δ satisfies (6.13).

For $w_1 \in B(b,\delta_0)$ and $y = w_{n+1} \in B(b,\delta) \subseteq B(b,\beta\delta_0)$ we obtain an expansion of $\int(\prod_{i=1}^{n} e^{-T_1\frac{H(\lambda)}{\lambda}}(w_i,w_{i+1}))dw_2, \ldots, dw_n$ as in (6.43) which along with (6.8) yield

(6.46) $$\int_{(B(b,\delta_0))^n} -\frac{1}{\lambda}\frac{\partial}{\partial x_i} e^{-T_0\frac{H(\lambda)}{\lambda}}(x,w_1) \Big(\prod_{i=1}^{n} e^{-T_1\frac{H(\lambda)}{\lambda}}(w_i,w_{i+1})\Big)dw_1,\ldots,dw_n$$

$$= \int_{B(b,\delta_0)} \left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}} \Bigg\{\frac{\partial A(x,w_1,T_0)}{\partial x_i} b_0^{T_0}(x,w_1) +$$

$$\frac{\Big(\frac{\partial A}{\partial x_i}(x,w_1,T_0)\, b_0^{T_0}(x,w_1)\, a_1^{T_0}(x,w_1) - \frac{\partial b_0^{T_0}}{\partial x_i}(x,w_1)\Big)}{\lambda} + 0(\frac{1}{\lambda^2})\Bigg\}$$

$$\exp(-\lambda A(x,w_1,T_0)) \left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}} b_0^{T-T_0}(w_1,y)$$

$$\Big(1 + \frac{a_1^{(T-T_0)}(w_1,y)}{\lambda} + e(T,\lambda)\Big) \exp(-\lambda A(w_1,y,T-T_0))$$

dw_1 uniformly where

$$e(T,\lambda) = 0\left(\frac{T^4}{\lambda^2}\right) + 0\Big(\frac{T^2}{\lambda} \exp\Big(-\frac{\lambda k_5 r^2}{T^4}\Big)\Big).$$

By the uniqueness of g_T^{xy} in Lemma 2, $\bar{w} := g_T^{xy}(T_0)$ is the only point where the inf in $\inf_{w_1} \{A(x,w_1,T_0) + A(w_1,y,T-T_1)\} = A(x,y,T)$ and $g_{T_0}^{x,\bar{w}} = g_T^{xy}\big|_{[0,T_0]}$. Hence by (2.26)

$$\frac{\partial A(x,\bar{w},T_0)}{\partial x_i} = -(\dot{g}_{T_0}^{x,\bar{w}}(0))_i = -(\dot{g}_{T_0}^{x,\bar{y}}(0))_i = \frac{\partial A(x,y,T)}{\partial x_i}.$$

By sublemma 2 $b_0^T(x,y) = 2^{\frac{d}{2}}\left(\det \sqrt{V''(b)}\right)^{-\frac{1}{2}} e^{-\frac{T}{2}\sum_{i=1}\omega_i} a_0^T(x,y)$. The bounds on the derivatives of $a_0^T(x,y)$, $a_1^T(x,y)$ and $A(x,y,T)$, $A''_{yy}(x,y,T) \geq$ constant > 0, (4.25) in Lemma 7 and Hörmander's stationary phase theorem imply:

$$(6.47) \quad \int_{B(b,\delta_0)} \left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}} \left\{\frac{\partial A(x,w,T_0)}{\partial x_i} b_0^{T_0}(x,w) + \frac{\left(\frac{\partial}{\partial x_i} A(x,w,T_0)\, b_0^{T_0}(x,w)\, a_1^{T_0}(x,w) - \frac{\partial b^{T_0}(x,w)}{\partial x_i}\right)}{\lambda}\right\} \exp(-\lambda A(x,w,T_0))$$

$$\left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}} b_0^{T-T_0}(w,y)\left(1 + \frac{a_1^{T-T_0}(w,y)}{\lambda}\right) \exp(-\lambda A(w,y,T-T_0)))$$

$$dw = \left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}} \left(\frac{\partial A(x,y,T)}{\partial x_i} b_0^T(x,y) + \frac{c_i^T(x,y)}{\lambda} + 0(\frac{T}{\lambda^2})\right) \exp(-\lambda A(x,y,T))$$

uniformly for $(x,y,T,\lambda) \in B(y_0,\delta_0) \times B(b,\delta) \times [2T_0,\infty) \times (0,\infty)$,

where

(6.48) $$c_i^T(x,y) = \frac{\partial A(x,y,T)}{\partial x_i} b_0^T(x,y)\,(a_1^{T_0}(x,\bar{w}) + a_1^{T-T_0}(\bar{w},y)$$

$$+ (\det((A''_{ww}(x,w,T_0) + A''_{ww}\,(w,y,T-T_0))\Big|_{w-\bar{w}}))^{-\frac{1}{2}}$$

$$[L_1\Big(\frac{\partial A(x,w,T_0)}{\partial x_i} b_0^{T_0}(x,w) b_0^{T-T_0}(w,y)\Big) - \frac{\partial}{\partial x_i} b_0^{T_0}(x,w) b_0^{T-T_0}(w,y)]\Big|_{w=\bar{w}}.$$

$$=: 2^{\frac{d}{2}}(\det \sqrt{V''(b)})^{-\frac{1}{2}} e^{-\frac{T}{2}\sum_{i=1}^{d}\omega_i} a_{1,i}^T(x,y).$$

Now (1.36) follows from (6.45), (6.46), and (6.47). The estimates in (1.37), (1.38), (1.39), and (1.41) are included in Lemma 7, in (7.8) below and Lemma 3.2.

(1.40) follows from

$$\frac{\partial A(x,y,T)}{\partial x_i} - \frac{\partial \rho(x,b)}{\partial x_i} = -(\dot{g}_T^{xy}(0) - \dot{g}^{x,b}(0))_i$$

$$= -(\dot{g}_T^{xy}(\tfrac{T}{2}) - \dot{g}^{x,b}(\tfrac{T}{2}))_i + \int_0^{\frac{T}{2}} (\ddot{g}_T^{xy}(s) - \ddot{g}^{xb}(s))_i\,ds = 0(e^{-\omega_1\frac{T}{2}})$$

$$+\int_0^{\frac{T}{2}} (V'(g_T^{xy}(s)) - V'(g^{x,b}(s))_i\,ds = 0(e^{-\omega_1\frac{T}{2}})$$

by (2.26), (2.20), (2.14), (2.31), and (2.42).

§6.6. A Proof of Theorem A.1.

We only do (1.34) for $e^{-T\frac{H(\lambda)}{\lambda}}(x,y)$. The proof for $\left|\frac{1}{\lambda}\frac{\partial}{\partial x_i} e^{-T\frac{H(\lambda)}{\lambda}}(x,y)\right|$ then follows since the estimates (6.6), (6.7) and upperbound in (1.17) of $e^{-T_1\frac{H(\lambda)}{\lambda}}(x,w_1)$, that we use below, are also valid for $\left|\frac{\partial}{\partial x_i} e^{-T_1\frac{H(\lambda)}{\lambda}}(x,w_1)\right|$. (See (6.6), (6.7) and (6.12).).

Let $\bar{\delta} \in (0, \delta]$. For $x \in B(x_0, \delta)$ $y \notin B(a, \bar{\delta}) \cup B(b, \bar{\delta})$ and $T = (n + 1)T_1$ with

$T_1 \in [T_0, 2T_0]$ we write (6.15):

$$e^{-T\frac{H(\lambda)}{\lambda}}(x,y) = \int \left(\prod_{i=0}^{n} e^{-T\frac{H(\lambda)}{\lambda}}(w_i,w_{i+1})\right) dw_1,\ldots,dw_n$$

$$= \left(\int_{B(a,\delta_0)} + \int_{B(b,\delta_0)} + \int_{\mathbb{R}^d\setminus(B(a,\delta_0))\cup B(a,\beta\delta_0)}\right) e^{-T_1\frac{H(\lambda)}{\lambda}}(x,w_1) e^{-nT_1\frac{H(\lambda)}{\lambda}}(w_1,y)dw_1 =$$

$I_1 + III_1 + V_1$ which defines I_1 II_1 and V_1. As before in (6.18) we write $I_1 = I_n + \sum_{j=2}^{n} II_j$

where

$$II_j = \int_{\substack{w_k \in B(a,\delta_0) \text{ for } k=1,\ldots,j-1 \\ w_j \notin B(a,\delta_0)}} \left(\prod_{i=0}^{n} e^{-T_1\frac{H(\lambda)}{\lambda}}(w_i, w_{i+1})\right) dw_1,\ldots dw_n$$

and

$$I_n = \int_{w_k \in B(a,\delta_0) k=1,\ldots,n} \prod_{i=0}^{n} e^{-T_1 \frac{H(\lambda)}{\lambda}} (w_i, w_{i+1})) \, dw_1,\ldots dw_n.$$

Also $III_1 = III_n + \sum_{j=2}^{n} IV_j$ with

$$III_n = \int_{w_k \in B(b,\delta_0) \text{ for } k=1,\ldots,n} \Big(\prod_{i=0}^{n} e^{-T_1 \frac{H(\lambda)}{\lambda}} (w_i, w_{i+1}) \Big) \, dw_1,\ldots dw_n$$

and

$$IV_j = \int_{\substack{w_k \in B(b,\delta_0) \text{ for } k=1,\ldots,j \\ w_j \notin B(b,\delta_0)}} \Big(\prod_{i=0}^{n} e^{-T_1 \frac{H(\lambda)}{\lambda}} (w_i, w_{i+1}) \Big) \, dw_1,\ldots dw_n$$

We estimate II_j, IV_j and V_1 as before and we get:

$$\sum_{j=2}^{n} II_j = 0(\exp(-\lambda[\rho(x,a) + \sup_{z \in B(a,\beta\delta_0)} \{\rho(z,a)\} + k_8])) \, (1 + 0(\tfrac{T}{\lambda}))) \tag{6.49}$$

$$\sum_{j=2}^{n} IV_j = 0(\exp(-\lambda[\rho(x,b) + \sup_{z \in B(b,\beta\delta_0)} \{\rho(z,b)\} + k_8])) \, (1 + 0(\tfrac{T}{\lambda}))) \tag{6.50}$$

and

$$V_1 \le k_7 \exp(-\lambda(\min_{c \in \{a;b\}} [\sup_{z \in B(c,\beta\delta_0)} \{\rho(x,c) + \rho(c,z)\} + k_8]))$$

If $y \notin B(a, \delta_0)$ then using (6.19) lemma 8.2 and (6.6) gives

$$I_n \le k_7 \int_{w_k \in B(a,\delta_0)} \Big(\prod_{i=0}^{n-1} \Big(\Big(\frac{\lambda}{2\pi}\Big)^{\frac{d}{2}} b_0^{T_1}(w_i, w_{i+1}) \, e^{-\lambda A(w_i, w_{i+1}, T_1)} (1 + 0(\tfrac{1}{\lambda})) \Big) \Big)$$

$$e^{-\lambda(A(w_n,z,s)+k_8)} \, dw_1,\ldots,dw_n$$

$$= k_7 \int_{w_n \in B(a,\delta_0)} \left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}} b_0^{nT_1}(x_1 w_n) e^{-\lambda A(x,w_n,nT_1)} e^{-\lambda(A(w_n,z,s)+k_8)} dw_n (1+0(\tfrac{T}{\lambda}))$$

$$= 0(e^{-Te_0(a)} e^{-\lambda A(x,z,nT_1+s)})(1 + 0(\tfrac{T}{\lambda}))$$

for any $z \in B(a, \beta\delta_0)$ and $s \geq T_0$. Take $s \to \infty$ and inf over z and you get:

(6.51) $\quad I_n = 0(\text{the R.H.S. of (6.49)})$, if $y \notin B(a, \delta_0)$.

If $y \in B(a,\delta_0)\backslash B(a,\beta\delta_0)$ we expand:

$$e^{-T_1\frac{H(\lambda)}{\lambda}}(w_n,y) = \left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}} b_0^{T_1}(w_n,y)\, e^{-\lambda A(w_n,y,T_1)}(1 + 0(\tfrac{1}{\lambda}))$$

uniformly for $(w_n,y,T_1) \in B(a,\delta_0) \times B(b,\delta_0) \times [T_0, 2T_0]$ and then

$$I_n = 0(e^{-T_0 e_0(a)} e^{-\lambda A(x,y,T)}(1 + 0(\tfrac{T}{\lambda})))$$

Since (see (1.38) and (1.39))

$$A(x,y,T) = \rho(x,a) + \rho(a,y) + 0(e^{-\omega_1(a)T})$$

and $\rho(a,y) = \frac{1}{2}\sum_{i=1}^{d} \omega_i(a)(y_i - b_i)^2 + 0(|y-b|^3)$ in coordinates such $V''(a)$ is given by a diagonal matrix:

$$V''(a) = \Omega^2 \text{ where}$$

$$\Omega = \text{diag}(\omega_1(a),\ldots,\omega_d(a)) \text{ with } 0 < \omega_1(a) \leq \cdots \leq \omega_d(a)$$

we have

$$I_n = 0(e^{-Te_0(a)} e^{-\lambda(\rho(x,a)+\gamma\bar{\delta}^2)} (1 + 0(\tfrac{T}{\lambda}))$$

uniformly for $x \in B(x_0,\delta)$ $y \in B(b,\delta_0)\backslash B(b,\bar{\delta})$ as $T,\lambda \to \infty$.

Now we get by (6.49)

$$I_n = 0(e^{-Te_0(a)} e^{-\lambda(\rho(x,a)+\gamma\bar{\delta}^2)})$$

uniformly for $x \in B(x_0,\delta)$ $y \notin B(a,\bar{\delta})$ as $T,\lambda \to \infty$ and $\frac{T}{\lambda} \to 0$.

Similarly, we get

$$III_n = 0(e^{-Te_0(b)} e^{-\lambda(\rho(x,b)+\gamma\bar{\delta}^2)})$$

uniformly for $x \in B(x_0,\delta)$ $y \notin B(b,\bar{\delta})$ as $T, \lambda \to \infty$ and $\frac{T}{\lambda} \to \infty$. Collecting terms we get:

$$e^{-T\frac{H(\lambda)}{\lambda}}(x,y) = 0\ (\min\{e^{-Te_0(a)} e^{-\lambda\rho(x,a)}, e^{-Te_0(b)} e^{-\lambda\rho(x,b)}\} e^{-\lambda\gamma\bar{\delta}^2})$$

uniformly for $x \in B(x_0,\delta)$ $y \notin B(a, \bar{\delta}) \cup B(b, \bar{\delta})$ as $T, \lambda \to \infty$ and $\frac{T}{\lambda} \to 0$.

7. A Proof of Lemma 8.

§7.1. Asymptotics for a Gaussian path integral.

Our goal is to prove the asymptotics of F(x,y,T,λ,r) in (6.1).

By sublemma 1 in Chapter 5

$$(7.1)\qquad F(x,y,T,\lambda,\tfrac{r}{T}) = \left(\frac{\lambda}{2\pi T}\right)^{\frac{d}{2}} \exp\left(-\lambda A(x,y,T)\right)$$

$$E_Z^T[\exp(-\lambda \int_0^T (V(g_T^{xy}(t) + \lambda^{-\frac{1}{2}} z(t)) - V(g_T^{xy}(t)) - \lambda^{-\frac{1}{2}} V'(g_T^{xy}(t))Z(t))dt)$$

$$\chi(Z \in C_{00}[0,T] : \lambda^{-\frac{1}{2}} \|Z\| \le \tfrac{r}{T})] \text{ for } r \in [0, \delta_0].$$

Now Taylor's formula implies the exponent at the integrand is given by

$$(7.2)\qquad \frac{1}{2!}\int_0^1 \langle V''(g_T^{xy}(t) + S\lambda^{-\frac{1}{2}} Z(t)\, Z(t), Z(t)\rangle\, ds$$

$$= \frac{1}{2!}\langle V''(g_T^{xy}(t) + \lambda^{-\frac{1}{2}} \eta(t))\, Z(t) \cdot Z(t)\rangle$$

where $|\eta(t)| \le |Z(t)|$ for all $t \in [0,T]$. When $\lambda^{-\frac{1}{2}} \|Z\|_{L^\infty[0,T]} \le \frac{r}{T}$ each matrix element at $V''(g_T^{xy} + \lambda^{-\frac{1}{2}}\eta) - V''(g_T^{xy})$ is $0(\frac{r}{T})$ uniformly. Hence the expression in (7.2) can be estimated from below by $\frac{1}{2!}\langle (V''(g_T^{xy}) + W_T^r)\, Z \cdot Z\rangle$ where

(7.3) W_T^r is a symmetric continuous matrix function on $[0 \cdot T]$ with $\int_0^T |W_T^r(t)|dt = 0(r)$ uniformly.

By Lemma 4 there is a $r_0 \in (0, \delta_0]$ such that the Dirichlet's operator

$$(7.4) \qquad \left(\frac{-d^2}{dt^2} + V''(g_T^{xy}(t)) + W_T^r(t)\right) > 0 \text{ on } D_0(0,T)$$

for all $(x,y,T,r) \in B(x_0 , \delta_0) \times B(b,\delta_0) \times [T_0 , \infty) \times [0,r_0]$.

We will now evaluate the Gaussian integrals

$$E_z^T(\exp(-\tfrac{1}{2}\int_0^T \langle (V''(g_T^{xy}) + W_T^r) Z,Z\rangle\, dt)).$$

Recall we assume (1.34) $V''(b) = \Omega^2$.

Sublemma 2. If $(x,y,T,r) \in B(x_0 , \delta_0) \times B(b,\delta_0) \times [T_0 , \infty) \times [0,r_0]$ so that (7.4) holds, then

$$(7.5) \qquad T^{-\frac{d}{2}} E_z^T (\exp(-\tfrac{1}{2}\int_0^T \langle V''(g_T^{xy}) + W_T^r) Z,Z\rangle\, dt)) = b_r^T(x,y) \text{ where}$$

$$(7.6) \qquad b_r^T(x,y) = \det(C_r^{x,y,T}(0)))^{-\frac{1}{2}} \text{ and } C(t) = C_r^{x,y,T}(t)$$

is the matrix solution of

$$(7.7) \qquad \begin{cases} \ddot{C}(t) = (V''(g_T^{xy}(t)) + W_T^r(t))\, C(t) \\ \text{with } C(T) = 0 \text{ and } \dot{C}(T) = -I \end{cases}$$

Moreover

$$(7.8) \qquad b_r^T(x,y) = 2^{\frac{d}{2}} (\det \Omega)^{\frac{1}{2}} e^{-\frac{T}{2}\operatorname{trace}\Omega}\, a_r^T(x, y) \text{ and } a_r^T(xy) =$$

$$= (\det X^{x,b}(0))^{-\frac{1}{2}} (1 + 0(e^{-\delta_1 T} + |y - b| + r)) =$$

$$= a_0^T(x,y)(1 + 0(r)) \text{ uniformly.}$$

Here X^{xb} and δ_1 are as in Lemma 2 so det $(X^{x,b}(0)) \geq$ constant > 0 on $B(x_0, \delta_0)$.

2. If $\bar{z} = g_T^{xy}(t_0)$ for $t_0 \in [0,T]$ then

$$(7.9) \quad b_0^T(x,y) = b_0^{t_0}(x,\bar{z})\left(\det[A''_{zz}(x,z,t_0) + A''_{zz}(z,y,T-t_0)]\Big|_{z=\bar{z}}\right) b_0^{T-t_0}(\bar{z},y)$$

Remark. We obtain (7.5) by reducing to a similar problem for the Brownian motion, which is solved in Truman [1]. See also Montroll [1] who proves (7.5) in one dimension.

(7.9) necessarily holds if our expansions of the heat kernel satisfies the semigroup property. Similar enters for short time asymptotics of diffusion processes on a Riemannian manifold, see Molchanov [1] proposition 10.4 and A. Bellaiche [1] proposition on 8.12.

Proof of Sublemma 2.1. Let E(·) denote the expectation with respect to Brownian motion then (Truman [1], section 3):

If

$$(7.10) \quad \ddot{A}(t) = (V''(g_T^{xy}) + W_T^r)\, A(t) \text{ on } [0,T] \text{ with } A(T) = I \text{ and } \dot{A}(T) = 0$$

has a nonsingular matrix solution and for $b \in C_0[0, T]$

$$(7.11) \quad m(b) = \exp\left(-\tfrac{1}{2}\int_0^T \langle (V''(g_T^{xy}) + W_T^r)\, b,b\rangle dt\right)$$

then

$$(7.12) \quad E(m(b)) = (\det A(0))^{-\frac{1}{2}}.$$

With $X = X_r^{xyT}$ and $Y = Y_r^{xyT}$ as in (2.45) of Lemma 4, Chapter 4.

(7.13) $\quad A(t) := (Y(t) - X(t)(\dot{X}(t))^{-1}\dot{Y}(T))\,(Y(T) - X(T)(\dot{X}(T))^{-1}\dot{Y}(T))^{-1}$

is the unique solution of (7.10) and one can see from (2.22) and (2.46) that it is nonsingular.

Write

(7.14) $\quad E(m(b)) = \int_{\mathbb{R}^d} E(m(b)|b(T) = c)\,dc$

and let B(t) be the unique (recall (7.4))) solution of

(7.15) $\quad \ddot{B}(t) = (V''(g_T^{xy}) + W_T^r)\,B(t)$ on [0,T] with $B(0) = 0$ and $B(T) = I$.

For $c \in \mathbb{R}^d$ let $f_c(t) = B(t)c$ then $f_c(0) = 0, f_c(T) = c$, $\dot{f}_c \in L^2$ and $b \mapsto b + f_c$ gives a bijection $C_{0,0}\,[0,T] \to C_{0,c}\,[0,T]$. The Radon-Nikodym is given by (see (5.6))

$$J(b) := \exp(-\tfrac{1}{2}\int_0^T |\dot{f}_c(t)|^2\,dt)$$

$$\exp(\int_0^T \dot{f}_c(t)\,db(t))$$

Using $E(\,\cdot\,|b(T) = c) = (2\pi T)^{-\frac{d}{2}}\,E_z^T(\;)$, (7.12), (7.14) and (7.15) gives

$$(\det A(0))^{-\frac{1}{2}} = E(m(b)) = \int_{\mathbb{R}^d} E(m(b)|b(T) = c)\,dc$$

$$= \int E(J(b)\,M(f_c + b)\mid b(T) = 0)\,dc$$

$$= (2\pi T)^{-\frac{d}{2}} \int \exp(-\tfrac{1}{2}\langle \dot{B}(T)\,c,c\rangle)$$

$$E_z^T(\exp(-\tfrac{1}{2}\int_0^T \langle (V''(g_T^{xy}) + W_T^r)\,z,z\rangle))\,dc$$

$$= T^{-\frac{d}{2}} (\det \dot{B}(T))^{-\frac{1}{2}}$$

$$E_z^T \left(\exp\left(-\tfrac{1}{2} \int_0^T \langle (V''(g_T^{xy}) + W_T^r)\, z,z\rangle \, dt\right)\right)$$

or

(7.16) The L.H.S. of (7.5) $= (\det [A(0)(\dot{B}(T))^{-1}])^{-\frac{1}{2}}$.

From (7.13) and (7.15) one can show

$$\det [A(0)(\dot{B}(T))^{-1}] = \det C(0)$$

where

$$C(t) := C_r^{x,y,T}(t) = -(Y(t) - X(t)X^{-1}(T)Y(T))\,(\dot{Y}(T) - \dot{X}(T)X^{-1}(T)Y(T))^{-1}$$

which satisfies (7.7). To show (7.8) one uses (2.22), (2.23), (2.46), (2.47) and (2.18) but we leave out the details.

<u>Proof of Sublemma 2.2.</u>

If $\bar{z} = g_T^{x,y}(t_0)$ then $g_{t_0}^{x,\bar{z}}(t) = g_T^{x,y}(t)|_{[0,t_0]}$ and $g_{T-t_0}^{\bar{z},y}(t) = g_T^{x,y}(t + t_0)$.

By part 1, $b_0^{t_0}(x,\bar{z}) = (\det C_1(0))^{-\frac{1}{2}}$ where $\ddot{C}_1 = V''(g_T^{x,y})C_1$ $\quad C_1(t_0) = 0$ and $\dot{C}_1(t_0) = -I$. So with $X = X^{x,y,T}$ and $Y = Y^{x,y,T}$ $\quad C_1(0) = -(Y(0) - X(0)X^{-1}(t_0)\, Y(t_0))\,(\dot{Y}(t_0) - \dot{X}(t_0)\,(X(t_0))^{-1}\, Y(t_0))^{-1}$. Similarly

$$b_0^{T-t_0}(\bar{z},y) = (\det C_2(0))^{-\frac{1}{2}}$$

where
$$C_2(0) = -(Y(t_0) - X(t_0)\,X^{-1}(T)\,Y(T))$$
$$(\dot{Y}(T) - \dot{X}(T)\,X^{-1}(T)\,Y(T))^{-1}$$

From (2.26) in Lemma 2 we see
$$A''_{zz}(x,z,t_0)|_{z=\bar{z}}\ (\dot{Y}(t_0)-\dot{X}(t_0)(X(0)^{-1}Y(0))\ (Y(t_0)-X(t_0)\ (X(0))^{-1}Y(0))^{-1}$$

and
$$A''_{zz}(z,y,T-T_0)|_{z=\bar{z}} = -(\dot{X}(t_0)-\dot{Y}(t_0)(Y(T))^{-1}X(T))\ (X(t_0)-Y(t_0)(Y(T))^{-1}X(T))^{-1}\ .$$

The R.H.S. of (7.9) $= (\det N)^{-\frac{1}{2}}$ where
$$N = (Y(0)-X(0)X^{-1}(t_0)Y(t_0))(\dot{Y}(t_0)-\dot{X}(t_0)(X(t_0))^{-1}Y(t_0))^{-1}$$
$$[(\dot{Y}(t_0)-\dot{X}(t_0)(X(0)^{-1}Y(0))\ (Y(t_0)-X(t_0)(X(0))^{-1}Y(0))^{-1}$$
$$-(\dot{X}(t_0)-\dot{Y}(t_0)(Y(T))^{-1}X(T))\ (X(t_0)-Y(t_0)(Y(T))^{-1}X(T))^{-1}]$$
$$(Y(t_0)-X(t_0)(X(T))^{-1}Y(T))\ (\dot{Y}(T)-\dot{X}(T)(X(T))^{-1}Y(T))^{-1}$$

and we need to show
$$\det N = \det[-(Y(0)-X(0)X^{-1}(T)Y(T))(\dot{Y}(T)-\dot{X}(T)X^{-1}(T)\ Y(T))^{-1}].$$

Actually $N = -(Y(0)-X(0)X^{-1}(T)Y(T))\ (\dot{Y}(T)-\dot{X}(T)X^{-1}(T)Y(T))^{-1}$ which can be proved by straightforward calculations.

§7.2 More convenient path space measures.

By (7.4) $\left(-\frac{d^2}{dt^2} + V''(g_T^{xy}) + W_T^r\right) > 0$ on $D_0(0,T)$ for all $(x,y,T,r) \in B(x_0, \delta_0) \times B(b,\delta_0) \times [T_0, \infty) \times [0,r_0]$ and recall we studied it's Green's matrix $G_r^{x,y,T}$ in Lemma 5 in Chapter 2.

Now we have

Sublemma 3. If $(x,y,T,r) \in B(x_0, \delta_0) \times B(b,\delta_0) \times [T_0, \infty) \times [0,r_0]$ then

$$(7.17) \quad E_z^T(z_i(s)\, z_j(t) \exp(-\tfrac{1}{2} \int_0^T \langle (V''(g_T^{xy}) + W_T^r)\, z,z\rangle\, du)) =$$

$$= \left(G_r^{x,y,T}(s,t)\right)_{i,j} E_z^T(\exp(-\tfrac{1}{2} \int_0^T \langle (V''(g_T^{xy}) + W_T^r)\, z,z\rangle\, du))$$

hence by sublemma 2

$$(7.18) \quad T^{-\frac{d}{2}} E_z^T\, [z_i(s)\, z_j(t)\, (\exp(-\tfrac{1}{2} \int_0^T \langle (V''(g_T^{xy}) + W_T^r)\, z,z\rangle\, du)]$$

$$= b_r^T(x,y) \left(G_r^{x,y,T}(s,t)\right)_{i,j} \text{ for } s,\, t \in [0,T] \text{ and } i,\, j \in \{1,\dots,d\}.$$

(7.19) Remark. To evaluate integrals with more general polynomials in z than in (7.8), e.g., for $d = 1$

$$I := \frac{E_z^T}{T^{\frac{1}{2}}} \left(\left(\int_0^T V^{(3)}\, (g_T^{xy}(t)) z^3(t) dt\right)^2 \exp(-\tfrac{1}{2}\int_0^T V''(g_T^{xy}) + W_T^r)\, z^2(a)\, du)\right)$$

one can use Lemma 20.4 in Simon [5] that says: Let $X_1, \dots, X_{2k}$ be jointly Gaussian random variables. Then

$$\langle X_1 \cdots X_{2k}\rangle = \sum_{\text{pairings}} \langle X_{i_1}, X_{j_1}\rangle \cdots \langle X_{i_k}, X_{j_k}\rangle$$

where $\sum_{\text{pairings}}$ denotes the sum over all $(2k)!/2^k\, k!$ ways of breaking $(1, \ldots, 2k)$ into k pairs. So we get with $G = G_r^{x,y,T}$ (see also Mizrahi [1])

$$I = b_r^T(x,y) \int_0^T \int_0^T V^{(3)}(g_T^{xy}(t))\, V^{(3)}(g_T^{xy}(s))$$

$(9G(t,t)\, G(t,s)\, G(s,s) + 6G^3(t,s))\, dt\, ds$. Since $G(t,s) = 0(e^{-\omega_1|t-s|})$ uniformly the integrand is $0(e^{-\omega_1|t-s|})$ and hence $I = 0(b_r^T(x,y)\, T)$

Generally if $\alpha_1, \ldots, \alpha_n \in N_0^d$ and $|\alpha_1| + \cdots + |\alpha_n| = 2k$ we get

(7.20) $$T^{-\frac{d}{2}} E_z^T \left[\int_0^T \cdots \int_0^T z_{(t_1)}^{\alpha_1} \cdots z_{(t_n)}^{\alpha_n}\, dt_1 \cdots \exp\left(-\tfrac{1}{2} \int_0^T \langle V''(g_T^{xy}) + W_T^r\, z,z\rangle\, du\right)\right]$$

$$= \sum_{\text{finite}} b_r^T(x,y) \int_0^T \cdots \int_0^T G(\ ) \cdots G(\ ) dt_1 \cdots dt_n = 0(b_r^T(x,y) T^n)$$

unformly for $(x,y,T,r) \in B(x_0, \delta_0) \times B(b,\delta_0) \times [T_0, \infty) \times [0,r_0]$.

We base a proof of sublemma 3 on the following

<u>Proposition</u> (Ellis and Rosen [1], Lemma 4.4).

Suppose that A_1 is a symmetric strictly positive trace class operator on a real separable Hilbert space H_1 and Λ is a symmetric operator in $B(H_1)$ such that $A_1^{-1} + \Lambda > 0$ on $D(A_1^{-1})$. Then $\Delta_1 := \det(I + A_1\, \Lambda) = \det(I + \sqrt{A_1}\, \Lambda\, \sqrt{A_1})$ is well-defined and $\Delta_1 > 0$.

Also $B_1 := (A_1^{-1} + \Lambda)^{-1}$ exists and is given by $B_1 = \sqrt{A_1}\ (I + \sqrt{A_1}\ \Lambda \sqrt{A_1})^{-1} \sqrt{A_1}$. B_1 is the covariance operator of a mean zero Gaussian measure P_{B_1} on H_1 and

(7.21) $$dP_{B_1}(y) = \sqrt{\Delta_1}\ \exp(-\tfrac{1}{2}\langle \Lambda y,y\rangle)\ dP_{A_1}(y)$$

Remark. Here $B(H_1)$ is the space of bounded linear operators and $<\cdot, \cdot>$ is the inner product on H_1.

A Proof of Sublemma 3 (see also Davies and Truman [1]).

Let μ_T be the Brownian bridge measure on $C_{00}([0,T], \mathbb{R}^d)$ as in (5.10). Then

(7.22) $$\int z_i(s)\ z_j(t)\ d\mu_T(z) = \delta_{ij}\ s(1 - \tfrac{t}{T}) =: (\rho(s,t))_{i,j}$$

if $0 \leq s \leq t \leq T$ and $i, j \in \{1, \ldots, d\}$.

Define $A_1 : L^2([0,T],\mathbb{R}^d,dt) \to L^2([0,T],\mathbb{R}^d,dt)$ by $A_1\ f(s) = \int_0^T \rho(s,t)\ f(t)\,dt$. Then $A_1 = \left(-\frac{d^2}{dt^2}\right.$ on $[0,T]$ with Dirichlet's boundary conditions$)^{-1}$ is symmetric, strictly positive and trace class. Now we use an idea from Stroock [1] and Chevet [1], for instance, to embed C_{00} into L^2 and work with the induced measure on L^2.

So let $\Phi : C_{00} \to L^2$ be the natural embedding, which is continuous. Then A_1 is the covariance operator for the induced measure $P_{A_1} := \mu_T \circ \Phi^{-1}$, by (7.22).

Define $\Lambda : L^2 \to L^2$ as the multiplication operator $f \mapsto (V''(g_T^{xy}) + W_T^r)f$ for each (x,y,T,r). Then Λ is bounded and symmetric.

With $H_1 = L^2([0,T])$ we get, using (7.4), $B_1 = B_1^{x,y,T,r} := \Big(\Big(-\frac{d^2}{dt^2} + V''(g_T^{xy}) + W_T^r\Big)$ on [0,T] with Dirichlet's boundary conditions$\Big)^{-1}$ is the covariance operator of a mean zero Gaussian measure P_{B_1} on L^2 such that

$$dP_{B_1}(\alpha) = \sqrt{\Delta_1}\ \exp(-\tfrac{1}{2}\int_0^T \langle (V''(g_T^{xy}) + W_T^r)\ \alpha,\alpha\rangle\ du)\ dP_{A_1}(\alpha).$$

Since

$$1 = P_{B_1}(L^2) = \sqrt{\Delta_1}\ \int_{L^2} \exp(\ \cdot\ )\ dP_{A_1}(\alpha) =$$

$$= \sqrt{\Delta_1}\ \int_{C_{00}([0,T])} \exp(\ )d\mu_T(z)$$

$$= \sqrt{\Delta_1}\ \ E_z^T(\exp(\ ))$$

we have

$$\sqrt{\Delta_1} = (E_z^T(\exp(-\tfrac{1}{2}\int_0^T \langle (V''(g_T^{xy}) + W_T^r)\ z,z\rangle\ du)))^{-1} \tag{7.23}$$

With $\alpha = \Phi(z)$

$$E_z^T(z_i(s)\ z_j(t)\ \exp(-\tfrac{1}{2}\int_0^T \langle (V''(g_T^{xy}) + W_T^r)\ z,z\rangle\ du))$$

$$= \int_{L^2} \alpha_i(s)\alpha_j(t)\ \exp(-\tfrac{1}{2}\int_0^T \langle (V''(g_T^{xy}) + W_T^r)\ \alpha,\alpha\rangle\ du))\ dP_{A_1}(\alpha)$$

$$= \Big(\sqrt{\Delta_1}\Big)^{-1} \int_{L^2} \alpha_i(s)\ \alpha_j(t)\ dP_{B_1}(\alpha)$$

$$= \Big(\sqrt{\Delta_1}\Big)^{-1} \Big(G_r^{x,y,T}(s,t)\Big)_{i,j}$$

which along with (7.23) gives (7.17).

§7.3 Keeping track of the large T dependence.

We will do the usual Taylor expansions (see Schilder [1] and Davies and Truman [1]) paying attention to large T's.

We write (7.1) with r replaced by $\frac{r}{T}$ as

$$F(x,y,T,\lambda,\tfrac{r}{T}) = \left(\frac{\lambda}{2\pi T}\right)^{\frac{d}{2}} \exp(-\lambda A(x,y,T)\ (F_1(x,y,T,\lambda,\tfrac{r}{T}) + F_2(x,y,T,\lambda,\tfrac{r}{T}))$$

where

$$F_i(x,y,T,\lambda,\tfrac{r}{T}) = E_z^T(\exp(-\lambda \int_0^T (V(g_T^{xy} + \lambda^{-\frac{1}{2}} z) - V(g_T^{xy}) - \lambda^{\frac{1}{2}} V'(g_T^{xy})z)dt)\chi_i(z))$$

where

$$\chi_i(z) = \{z \in C_{00}[0,T]: \lambda^{-\frac{1}{2}}\|z\|_\infty \le \tfrac{r}{T} \text{ and}$$

$$(-1)^i \int_0^T (V(g_T^{xy} + \lambda^{-\frac{1}{2}} z) - V(g_T^{xy}) - \lambda^{-\frac{1}{2}} V'(g_T^{xy}) \cdot z - \frac{\lambda^{-1}}{2} \langle V''(g_T^{xy})\, z,z\rangle)\, dt \ge 0\}$$

Then we expand the integrand in F_1 and F_2, using

$$\exp(x) = \sum_{i=0}^{n-1} \frac{x^i}{i!} + R_n(x)$$

where

$$R_n(x) \le \begin{cases} \frac{x^n}{n!} \exp(x) \text{ if } x \ge 0 \\ \frac{|x|^n}{n!} \text{ if } x < 0, \end{cases}$$

collect the terms and we get

$$(7.24) \quad F(x,y,T,\lambda,\tfrac{r}{T}) = \lambda^{\frac{d}{2}} \exp(-\lambda A(x,y,T))\ (I_0 + I_1 + I_2 + I_3 + J_1 + J_2)$$

where

(7.25) $$I_k = \frac{1}{k!}(2\pi T)^{-\frac{d}{2}} E_z^T$$

$$\left([-\lambda \int_0^T (V(g_T^{xy} + \lambda^{-\frac{1}{2}} z) - V(g_T^{xy}) - \lambda^{-\frac{1}{2}} V'(g_T^{xy})\, z - \frac{\lambda^{-1}}{2!} \langle V''(g_T^{xy})\, z,z\rangle)\, dt\right]^k$$

$$\exp(-\tfrac{1}{2}\int_0^T \langle V''(g_T^{xy})\, z,z\rangle\, dt)\, \chi(z : \lambda^{-\frac{1}{2}} \|z\|_\infty \le \tfrac{r}{T}))$$

for $k \in \{0,1,2,3\}$, $$J_1 = 0((2\pi T)^{-\frac{d}{2}} E_z^T$$

$$\left(|-\lambda\int_0^T (V(g_T^{xy} + \lambda^{-\frac{1}{2}} z) - V(g_T^{xy}) - \lambda^{-\frac{1}{2}} V'(g_T^{xy}) \cdot z - \frac{\lambda^{-1}}{2} \langle V''(g_T^{xy}) z,z\rangle\right) dt|^4$$

$$\exp[-\lambda \int_0^T (V(g_T^{xy} + \lambda^{-\frac{1}{2}} z) - V(g_T^{xy}) - \lambda^{-\frac{1}{2}} V'(g_T^{xy}) \cdot z)dt]\chi_1(z))$$

and $$J_2 = 0((2\pi T)^{-\frac{d}{2}} E_z^T$$

$$(|-\lambda\int_0^T (V(g_T^{xy} + \lambda^{-\frac{1}{2}} z) - V(g_T^{xy}) - \lambda^{-\frac{1}{2}} V'(g_T^{xy}) \cdot z - \frac{\lambda^{-1}}{2} \langle V''(g_T^{xy})\, z,z\rangle\Big)\, dt|^4$$

$$\exp(-\tfrac{1}{2}\int_0^T \langle V''(g_T^{xy})\, z;z\rangle\, dt)\, \chi_2(z)) .$$

For the error terms we use

$$\lambda(V(g_T^{xy} + \lambda^{-\frac{1}{2}} z) - V(g_T^{xy}) - \lambda^{-\frac{1}{2}} V'(g_T^{xy})z - \frac{\lambda^{-1}}{2} \langle V''(g_T^{xy})\, z,z\rangle) =$$

$$= 0\Big(\lambda^{-\frac{1}{2}} \sum_{|\alpha|=3} |z^\alpha| = 0\Big(\lambda^{-\frac{1}{2}} \sum_{i=1}^{d}\Big(1 + z_i^6(t)\Big)\Big)$$

uniformly for $\lambda^{-\frac{1}{2}} \|z\|_\infty \leq \frac{r}{T}$. Therefore we have a crude estimate

$$J_1 = 0((2\pi T)^{-\frac{d}{2}} E_z^T[(\int_0^T \sum_{i=1}^{d} (1 + z_i^6(t)\, dt)^4$$

$$\exp(-\lambda \int_0^T \langle (V''(g_T^{xy}) + w_T^r)\, z,z\rangle\, dt] \cdot \lambda^{-2} =$$

$$= 0(b_r^T(x,y)\, \frac{T^4}{\lambda^2}) = 0\Big(b_0^T(x,y)\, (1 + 0(r))\, \frac{T^4}{\lambda^2}\Big) = 0\Big(b_0^T(x,y)\, \frac{T^4}{\lambda^2}\Big)$$

where we used (7.20) and sublemma 2. Similar $J_2 = 0\Big(b_0^T(x,y)\, \frac{T^4}{\lambda^2}\Big)$ uniformly.

Using (7.5), (7.18), (7.20) Hölder's inequality and (see Simon [2] or Stroock [1])

$$(7.26) \qquad (2\pi T)^{-\frac{d}{2}} E_z^T(\chi(z : \|z\|_\infty > \delta)) = 0\Big(\exp\Big(-\frac{\delta^2 k(d)}{T}\Big)\Big)$$

one can show there exists $\epsilon_0 \geq 0$ such that

$$(7.27) \qquad I_0 = (2\pi)^{\frac{d}{2}}\, b_0^T(x,y)\, (1 + 0(\exp(-\frac{\lambda r^2 \varepsilon_0 k(d)}{2T^4})) ,$$

$$I_1 = -\frac{\lambda^{-1}}{4!}(2\pi T)^{-\frac{d}{2}} E_z^T [\int_0^T V^{(4)}(g_T^{xy}; z,z,z,z)\, dt \exp(-\tfrac{1}{2}\int_0^T \langle V''(g_T^{xy})\, z,z\rangle\, dt)]$$

$$+ b_0^T(x,y)\, [0(\frac{T}{\lambda^2}) + 0(\frac{T}{\lambda} \exp(-\frac{\lambda r^2 \varepsilon_0 k(d)}{2T^4}))]$$

$$I_2 = \frac{1}{\lambda}\, \frac{1}{(2!)(3!)^2}\, E_z^T[(\int_0^T V^{(3)}(g_T^{xy}; z,z,z)\, dt)^2 \exp(-\tfrac{1}{2} \int_0^T \langle V''(g_T^{xy})\, z,z\rangle\, dt)]$$

$$+ b_0^T(x,y)\, (0(\frac{T^2}{\lambda} \exp(-\frac{\lambda r^2 \varepsilon_0 k(d)}{2T^4}) + 0\, (\frac{T^2}{\lambda^2}))$$

and $I_3 = 0\Big(b_0^T(x,y)\, \frac{T^3}{\lambda^2}\Big)$.

Here $$V^{(k)}(x;y,\ldots,y) = \sum_{|\alpha|=k} k!\,\frac{\partial^{|\alpha|}V(x)}{\partial x^\alpha}\,\frac{y^\alpha}{\alpha!}\ .$$

We will give the details for I_1 but first we note by (7.24) and the estimates for I_1, I_2, I_3, T_1 and T_2 above we have

$$F(x,y,T,\lambda,\tfrac{r}{T}) = \left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}} b_0^T(x,y)\,[1 + \frac{a_1^T(x,y)}{T} + 0\left(\frac{T^4}{\lambda^2}\right)$$
$$= 0\left(\frac{T^2}{\lambda}\exp\left(-\frac{\lambda r^2\varepsilon_0 k(d)}{2T^4}\right)\right)]$$

where

$$(7.28)\qquad b_0^T(x,y)\,a_1^T(x,y) =$$
$$= T^{-\frac{d}{2}} E_z^T\left(\left[\frac{1}{2!}\,\frac{1}{(3!)^2}\left(\int_0^T V^{(3)}(g_T^{xy};z,z,z)\,dt\right)^2 - \frac{1}{4!}\int_0^T V^{(4)}(g_T^{xy};z,z,z,z)\,dt\right]\right.$$
$$\left.\exp\left(-\frac{1}{2}\int_0^T \langle V''(g_T^{xy})\,z,z\rangle\,dt\right)\right)$$

Here $b_0^T(x,y)$ is as in sublemma 2 and $a_1^T(x,y) = 0(T)$ follows from Remark (7.19) but we refer to the proof of lemma 7 in Chapter 11.

So now the proof of the asymptotics of I_1 in (7.27).

By Taylor's theorem (Hörmander [1]) and (7.25)

$$(7.29)\quad I_1 = (2\pi T)^{-\frac{d}{2}} E_z^T\Big([\lambda \int_0^T (\frac{V^{(3)}}{3!} (g_T^{xy}; \lambda^{-\frac{1}{2}} z, \lambda^{-\frac{1}{2}} z, \lambda^{-\frac{1}{2}} z)$$

$$+ \frac{V^{(4)}}{4!} (g_T^{xy}(t); \lambda^{-\frac{1}{2}} z, \ldots, \lambda^{-\frac{1}{2}} z) + \frac{V^{(5)}}{5!} (g_T^{xy}(t); \lambda^{-\frac{1}{2}} z, \ldots, \lambda^{-\frac{1}{2}} z)$$

$$+ \int_0^1 \frac{V^{(6)}}{5!} (g_T^{xy} + s\lambda^{-\frac{1}{2}} z; \lambda^{-\frac{1}{2}} z, \ldots, \lambda^{-\frac{1}{2}} z)(1-s)^5\, ds\Big)\, dt]$$

$$\exp(-\tfrac{1}{2} \int_0^T \langle V''(g_T^{xy})\, z,z\rangle\, dt)\, \chi(z : \frac{\|z\|_\infty}{\lambda^{\frac{1}{2}}} \le \tfrac{r}{T})) =: I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4}$$

which defines $I_{1,1} \cdots I_{1,4}$ (in the same order).

By symmetry

$$(7.30)\quad I_{1,1} = 0 = I_{1,3}\,.$$

Since $\lambda^{-\frac{1}{2}} \|z\|_\infty \le \frac{r}{T}$, derivatives of V are uniformly bounded on compacts and $|z^\alpha(t)| \le \sum_{i=1}^d z_i^6(t)$ if $|\alpha| = 6$ we get

$$(7.31)\quad I_{1.4} = 0(\lambda^{-2} \sum_{i=1}^d \int_0^T T^{-\frac{d}{2}} E_z^T(z_i^6(t) \exp(-\tfrac{1}{2} \int_0^T \langle V''(g_T^{xy})\, z,z\rangle\, dt)$$

$$= 0(\lambda^{-2} \sum_{i=1}^d \int_0^T b_0^T(x,y)\, G_{i,i}^3(t,t)\, dt) = b_0^T(x,y)\, 0\Big(\frac{T}{\lambda^2}\Big) \text{ uniformly.}$$

(7.27) now follows from

$$(7.32) \qquad \lambda^{-1} T^{-\frac{d}{2}} E_z^T \left(\int_0^T V^{(4)}(g_T^{xy} ; z,z,z,z)\, dt \exp(-\tfrac{1}{2} \int_0^T \right.$$

$$\langle V''(g_T^{xy})\, z,z\rangle\, du)\ \chi(z : \lambda^{-\frac{1}{2}} \|z\|_\infty > \tfrac{r}{T}))$$

$$= 0(\lambda^{-1} \sum_{i=1}^{d} \int_0^T T^{-\frac{d}{2}} E_z^T [z_i^4(t) \exp(-\tfrac{1}{2} \int_0^T \langle V''(g_T^{xy})\, z,z\rangle\, du)\ \chi(z : \lambda^{-\frac{1}{2}} \|z\| > \tfrac{r}{T})]$$

$$= 0(\lambda^{-1} \sum_{i=1}^{d} \int_0^T [T^{-\frac{d}{2}} E_z^T ((z_i^4(t))^{(\frac{T+\varepsilon_0}{T})}$$

$$\exp(-\tfrac{1}{2}\Big(\frac{T+\varepsilon_0}{T}\Big) \int_0^T \langle V''(g_T^{xy})\, z,z\rangle\, du)]^{(\frac{T}{T+\varepsilon_0})}$$

$$\left[T^{-\frac{d}{2}} E_z^T (z : \lambda^{-\frac{1}{2}} \|z\| > \tfrac{r}{T})\right]^{\frac{\varepsilon_0}{T+\varepsilon_0}}$$

$$= 0(\lambda^{-1} \sum_{i=1}^{d} \int_0^T (b_{\varepsilon_0}^T(x,y)\, (1 + (G_{\varepsilon_0}^{x,y,T}(t,t)_{i,i}^4))^{(\frac{T}{T+\varepsilon_0})} \quad dt$$

$$(\exp \frac{(-\lambda r^2 k(d))}{T^3})^{\frac{\varepsilon_0}{T+\varepsilon_0}} = 0\ (\tfrac{T}{\lambda} \exp\Big(\frac{-\lambda r^2 k(d)\varepsilon_0}{2T^4}\Big)\Big)$$

where we used

$$|V^{(4)}(g_T^{xy} ; z,z,z,z)| = 0\ (\sum_{i=1}^{d} z_i^4(t))$$

Hölder's inequality with $P = \dfrac{T+\varepsilon_0}{T}$ and $q = \dfrac{T+\varepsilon_0}{\varepsilon_0}$ where without loss of much generality

$$\frac{T+\varepsilon_0}{T} \le 2 \text{ for } T \in [T_0\,,\infty) \text{ so } (z_i^4(t))^{(\frac{T+\varepsilon_0}{T})} \le 1 + z_i^8(t),$$

sublemma 2 with $W_T^{\varepsilon_0}(t) = \frac{\varepsilon_0}{T} V''(g_T^{xy}(t))$, (7.20) and (7.26).

(7.29)-(7.32) imply (7.27) which completes the proof.

§7.4 <u>A sketch of a proof of Lemma 8.2</u>.

Equation (6.3) of Lemma 8.2 follows from: If $(x,y,T,S) \in (B(x_0\,,\delta_0 \cup B(b,\delta_0)) \times B(b,\delta_0) \times [T_0\,,\infty) \times [T_0\,,\infty)$ then $\inf_z\{A(x,z,T) + A(z,y,S)\} = A(x,y,T+S)$ is attained at the unique point $\bar{z} = g_{T+S}^{xy}(T) \in B(b, \frac{\delta_0}{2})$ (uniqueness by Lemma 2 Chapter 2 and the estimate by (2.39)) and

$$\int_{B(b,\delta_0)} \left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}} b_0^T(x,z) \exp(-\lambda A(x,z,T)) \left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}} b_0^S(z,y) \exp(-\lambda A(x,z,S))\, dz$$

$$= \left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}} b_0^{T+S}(x,y) \exp(-\lambda A(x,y,T+S))\,(1 + 0(\tfrac{1}{\lambda}))$$

uniformly for

$$(x,y,T,S,\lambda) \in (B(x_0\,,\delta_0) \cup B(b,\delta_0)) \times B(b,\delta_0) \times [T_0\,,\infty) \times [T_0\,,\infty) \times (0\,,\infty).$$

A Proof uses Hörmander's stationary phase theorem in §4.3 with k = integer part of $(\frac{d}{2} + 2)$. The argument of the important uniformity is the same as in (6.24), that we refer to.

The prefactor being $\left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}}$ follows from (4.10) and Sublemma 2.2.

8. A Proof of Lemma 9.

It suffices, by lemma 8.1, to show

$$(8.1)\qquad G(x,y,T,\lambda,r) := e^{-T\frac{H(\lambda)}{\lambda}}(x,y) - F(x,y,T,\lambda,r) =$$

$$= 0(\lambda^{\frac{d}{2}} \exp(-\lambda A(x,y,T)) \exp(-\lambda k_6\, r))$$

for some $k_6 > 0$, uniformly.

By sublemma 1, chapter 5,

$$(8.2)\qquad G(x,y,T,\lambda,r) = \left(\frac{\lambda}{2\pi T}\right)^{\frac{d}{2}} \exp\,(-\lambda A(x,y,T))$$

$$E_z^T(\exp(-\lambda \int_0^T (V(g_T^{xy} + \lambda^{-\frac{1}{2}} z) - V(g_T^{xy})$$

$$-\lambda^{-\frac{1}{2}} V'(g_T^{xy})Z)dt)\ \chi(z : r \le \lambda^{-\frac{1}{2}}\|z\|))\ .$$

For small $r_0 > 0$ and large R_0 , yet to be chosen, we write

$$\chi(z : r < \lambda^{-\frac{1}{2}}\|z\|) \le \chi(z : r \le \lambda^{-\frac{1}{2}}\|z\|_\infty \le r_0)$$

$$+\ \chi(z : r_0 \le \lambda^{-\frac{1}{2}}\|z\| \le R_0)$$

$$+\ \chi(z : R_0 \le \lambda^{-\frac{1}{2}}\|z\|) \text{ where } \chi(z : A) = \begin{cases} 1 \text{ if } z \in A \\ 0 \text{ if } z \notin A \end{cases}.$$

Accordingly we write

$$(8.3)\qquad G(x,y,T,\lambda,r) \le G_1(x,y,T,\lambda,r) + G_2(x,y,T,\lambda) + G_3(x,y,T,\lambda).$$

Estimating G_2 uniformly is the main problem now since we know how to handle G_1 from chapter 7 and we estimate G_3 using (7.26) that says

$$(8.4) \qquad (2\pi T)^{-\frac{d}{2}} E_z^T(Z : \|z\|_{L^\infty[0,T]} > \delta) = 0(\exp(-\frac{\delta^2 k(d)}{T}))$$

(recall $T \in [T_0, 2T_0]$.).

More precisely, we pick R_0 such that

$$R_0 \geq 2 \sup|g_T^{x,y}(t) - ((1 - \tfrac{t}{T})\, x + \tfrac{t}{T}\, y)|$$

$$(x,y,T,t) \in B(x_0, \delta_0) \times B(b,\delta_0) \times [T_0, 2T_0] \times [0,T]$$

and

$$\frac{R_0^2\, k(d)}{8T_0} > \sup_{(x,y,T)} (A(x,y,T)) + 100\, \delta_0^2$$

and we get

$$(8.5) \qquad G_3(x,y,T,\lambda) = 0(\exp(-\lambda(\sup(A(x,y,T)) + 100\, \delta_0^2))$$

uniformly, but we leave out the details.

Now fix $r_0 \in (0,\delta_0]$ such that

$$(8.6) \qquad \int_0^T (\dot\eta^2 + (1 + r_0)\langle V''(\gamma)\eta,\eta\rangle)dt \geq \frac{k_1}{2} \|\eta\|^2_{L^\infty[0,T]}$$

for all $\eta \in D_0(0,T)$, γ with $\|\gamma - g_T^{xy}\|_\infty \leq r_0$

and $(x,y,T) \in B(x_0, \delta_0) \times B(b,\delta_0) \times [T_0, 2T_0]$.

Then using the mean value theorem to estimate the exponent in this integrand, Holder's inequality with $p = (1 + r_0)$ and (8.4) (similar to (7.32)) one can show

(8.7) $G_1(x,y,T,\lambda,r) = 0(\lambda^{\frac{d}{2}} \exp(-\lambda A(x,y,T)) \exp(-\frac{\lambda k(d) r_0}{2T_0(1+r_0)}))$ uniformly for $(x,y,T,\lambda,r) \in \overline{B(x_0, \delta_0} \times \overline{B(b,\delta_0)} \times [T_0, 2T_0] \times [1,\infty) \times (0,r_0]$ (after taking slightly smaller δ_0).

Finally we look at G_2.

Put

(8.8) $L := \{z \in C_{0,0}([0,T]) : r_0 \le \|z\|_\infty \le R_0\}$

and set

(8.9) $f(z) = f^{x,y,T}(z) = \int_0^T (V((g_T^{x,y}) + z) - V(g_T^{x,y}) - V'(g_T^{xy})z)\, dt$ near L and extend it to all of $C_{0,0}([0,T])$ so that it is bounded and continuous into $\mathbf{R}$.

Also set

(8.10) $H(x,y,T,\lambda) = E_z^T(\exp(-\lambda f(\lambda^{-\frac{1}{2}} z)) : z \in L)$

and $\mathcal{A}(\gamma) := \int_0^T (\frac{1}{2}\dot{\gamma}^2 + V(\gamma))\, dt$ if $\gamma \in C_{x,y}([0,T])$ and $\dot{\gamma} \in L^2([0,T])$.

Then $G_2(x,y,T,\lambda) = (\frac{\lambda}{2\pi T})^{\frac{d}{2}} \exp(-\lambda A(x,y,T))\, H(x,y,T,\lambda)$

and we will show there exist $\varepsilon_0 > 0$ and λ_0 such that

(8.11) $H(x,y,T,\lambda) \leq \frac{1}{\varepsilon_0} e^{-\lambda\varepsilon_0}$ if $\lambda \geq \lambda_0$

for all $(x,y,T) \in \overline{B(x_0, \delta_0)} \times \overline{B(b,\delta_0)} \times [T_0, 2T_0]$.

That will complete the proof of (8.1).

With $\varepsilon = \frac{1}{\lambda}$ and $P^T_\varepsilon(A) = \mu_T(\lambda^{\frac{1}{2}} A)$ (μ_T is the measure corresponding to $E^T_z(\cdot)$) we have according to Schilder and Varadhan (see §5.2)

$$\overline{\lim_{\lambda \to \infty}} \frac{\log H(x,y,T,\lambda)}{\lambda} =$$

$$= \overline{\lim_{\varepsilon \to 0}} \varepsilon \log \int_L \exp(-\varepsilon^{-1}(f(z)) dP^T_\varepsilon(z) \leq - \inf_{z \in L}(f(z) + I(z))$$

where

$$I(z) = \frac{1}{2} \int_0^T |\dot{z}|^2 dt.$$

By the Euler-Lagrange equations

$$f(z) + I(z) = \mathcal{A}(g^{xy}_T + z) - \mathcal{A}(g^{xy}_T)$$

and therefore by (2.33)

$$\inf_{z \in L}(f(z) + I(z)) \geq k_1 \min(r_0^2, \delta_0^2) = k_1 r_0^2$$

and we have:

For each $(x,y,T) \in \overline{B(x_0, \delta_0)} \times \overline{B(b,\delta_0)} \times [T_0, 2T_0]$ there are constants $C(x,y,T)$ and $\lambda_0(x,y,T)$ such that

(8.12) $H(x,y,T,\lambda) \leq C(x,y,T) e^{-\lambda\frac{k_1 r_0^2}{2}}$ if $\lambda \geq \lambda_0(x,y,T)$.

To prove uniformity for $(x,y,T)(x_1,y_1,T_1) \in \overline{B(x_0,\delta_0)} \times \overline{B(b,\delta_0)} \times [T_0, 2T_0]$, (so $\frac{1}{2} \le \frac{T}{T_1} \le 2$) we denote

$$g(t) = g_T^{x,y}(t\,\tfrac{T}{T_1}),\ g_1(t) = g_{T_1}^{x_1y_1}(t),\ \|\cdot\|_\infty = \|\cdot\|_{L^\infty([0,T_1])}$$

and we write $|H(x_1, y_1, T_1, \lambda) - H(x,y,T,\lambda)| =$

$$= |E_z^{T_1}(\exp(-\lambda \int_0^{T_1} (V(g_1 + \lambda^{-\frac{1}{2}} z) - V(g_1) - \lambda^{-\frac{1}{2}} V'(g_1)z]dt)$$

$$\chi(z : r_0 \le \lambda^{-\frac{1}{2}}\|z\|_\infty \le R_0)dt -$$

$$- E_z^{T_1}(\exp(-\lambda \tfrac{T}{T_1} \int_0^{T_1} [V(g + (\lambda\tfrac{T}{T_1})^{-\frac{1}{2}} z) - V(g) - (\lambda\tfrac{T}{T_1})^{-\frac{1}{2}} V'(g)\, z]\, dt)$$

$$\chi(z : r_0 \le (\lambda\tfrac{T}{T_1})^{-\frac{1}{2}}\|z\|_\infty \le R_0)) \le H(x_1,y_1,T_1,\lambda)(1 + \sup_{r_0 \le \lambda^{-\frac{1}{2}}\|z\|_\infty \le R_0}$$

$$[\exp(-\lambda \int_0^{T_1} ([\tfrac{T}{T_1} V(g(t) + \lambda^{-\frac{1}{2}}(\tfrac{T}{T_1})^{-\frac{1}{2}} z(t)) - V(g_1(t) + \lambda^{-\frac{1}{2}} z(t))]$$

$$- [\tfrac{T}{T_1} V(g(t)) - V(g_1(t))] - (\lambda^{-\frac{1}{2}}) [((\tfrac{T}{T_1})^{\frac{1}{2}} V'(g(t)) - V'(g_1(t)))\, z(t)])\, dt)]$$

$$+ E_z^T[\exp(-\lambda \int_0^T (V(g_T^{xy} + \lambda^{-\frac{1}{2}} z) - V(g_T^{xy}) - \lambda^{-\frac{1}{2}} V'(g_T^{xy})\, z)\, dt)$$

$$(\chi(z : 2^{-\frac{1}{2}} r_0 \le \lambda^{-\frac{1}{2}}\|z\|_\infty \le r_0) + \chi(z : R_0 \le \lambda^{-\frac{1}{2}}\|z\|_\infty))]$$

where we used $z(at) \doteq a^{\frac{1}{2}} z(t)$ (see Simon [5], chapter 2) and

$$\{z : r_0 \le (\tfrac{\lambda T}{T_1})^{-\frac{1}{2}} \|z\|_\infty \le R_0\} \le \{z : 2^{-\frac{1}{2}} r_0 \le \lambda^{-\frac{1}{2}} \|z\|_\infty \le r_0\}$$

$$\cup \{z : r_0 \le \lambda^{-\frac{1}{2}} \|z\|_\infty \le R_0\}\} \cup \{z : R_0 \le \lambda^{-\frac{1}{2}} \|z\|_\infty \} .$$

By (2.37) $\|g_T^{xy}(\cdot \tfrac{T}{T_1}) - g_{T_1}^{x_1 y_1}\|_{L^\infty[0,T_1]} \to 0$ as $(x,y,T) \to (x_1 , y_1 , T_1)$.

Hence for (x,y,T) near (x_1 , y_1 , T_1)

$$|H(x_1 , y_1 , T_1 , \lambda) - H(x,y,T,\lambda)| \le H(x_1 , y_1 , T_1 , \lambda)(1 + 0(e^{\lambda \varepsilon}))$$

$$+ 0(\exp(- \frac{\lambda r_0^3 k(d)}{4T_0(1+r_0)})) + 0(\exp(-\lambda\, 100\, \delta_0^2))$$

where we used also (8.7), (8.5) and the definition of G_3 . Hence we can use the same exponential bound in a neighborhood of (x_1 , y_1 , T_1) and so (8.11) follows from (8.12) by compactness.

9. Differentiating the Heat Kernel.

We want an estimate of $|\frac{\partial}{\partial x_i} e^{-T\frac{H(\lambda)}{\lambda}}(x,y)|$ similar to $e^{-T\frac{H(\lambda)}{\lambda}}(x,y) \le (\frac{\lambda}{2\pi T})^{\frac{d}{2}}$ $\exp(-\frac{\lambda|x-y|^2}{2T})$ when $|x|$ and T are bounded and a replacement for the upperbound in (see (1.17)) $\lim_{\lambda\to\infty} \frac{\log e^{-\frac{H(\lambda)}{\lambda}}(x,y)}{\lambda} = -A(x,y,T)$ u.o.c as $\lambda \to \infty$.

Proposition 2. Let V be a double well with $|V'(x)| = 0(e^{A|x|^2})$ and $i \in \{1, ..., d\}$.

Then

1. With k(d) as in (7.26)

$$(9.1) \qquad -\frac{1}{\lambda}\frac{\partial}{\partial x_i} e^{-T\frac{H(\lambda)}{\lambda}} = (\frac{\lambda}{2\pi T})^{\frac{d}{2}} \exp(-\frac{\lambda|x-y|^2}{2T})$$

$$E_z^T[\{\frac{(x_i-y_i)}{T} + \int_0^T \frac{\partial V}{\partial x_i}((1-\tfrac{t}{T})x + \tfrac{t}{T}y + \lambda^{-\frac{1}{2}}z(t))(1-\tfrac{t}{T})\,dt\}$$

$$\exp(-\lambda\int_0^T V((1-\tfrac{t}{T})x + \tfrac{t}{T}y + \lambda^{-\frac{1}{2}}z(t))\,dt)]$$

whenever $(x,y,T,\lambda) \in \mathbb{R}^d \times \mathbb{R}^d \times (0,\infty) \times [\frac{16AT}{k(d)}, \infty)$ and obeys

$$(9.2) \qquad |-\frac{1}{\lambda}\frac{\partial}{\partial x_i} e^{-T\frac{H(\lambda)}{\lambda}}(x,y)| =$$

$$= 0(\lambda^{\frac{d}{2}} \exp(-\frac{\lambda|x-y|^2}{2T})\{\frac{|x_i-y_i|}{T} + T^{\frac{3}{2}}\exp(4A(|x|^2+|y|^2))\})\text{ uniformly.}$$

2.

$$(9.3) \qquad \overline{\lim_{\lambda\to\infty}} \frac{\log\left|\frac{\partial}{\partial x_i} e^{-T\frac{H(\lambda)}{\lambda}}(x,y)\right|}{\lambda} \le -A(x,y,T) \text{ u.o.c. in } \mathbb{R}^d \times \mathbb{R}^d \times (0,\infty).$$

Remark. Let $|x| \le \frac{R}{2}$, $|y| \ge R$ and $T \le T_1$ for some R and T_1 . Then

$$(9.4) \qquad \begin{bmatrix} e^{-T\frac{H(\lambda)}{\lambda}}(x,y) \\ \left|-\frac{1}{\lambda}\frac{\partial}{\partial x_i} e^{-T\frac{H(\lambda)}{\lambda}}(x,y)\right| \end{bmatrix} \le 0(\exp(-\frac{\lambda|y|^2}{8T_1}))$$

uniformly for $\lambda \ge$ some λ_0 , since in (9.2)

$$\lambda^{\frac{d}{2}} \exp(-\frac{\lambda|x-y|^2}{2T}) \exp(4A(|x|^2 + |y|^2))$$

$$= 0(\lambda^{\frac{d}{2}} \exp[-\frac{\lambda|y|^2}{4T}(1 - \frac{32AT}{\lambda})])$$

$$= 0(\lambda^{\frac{d}{2}} \exp[-\frac{\lambda|y|^2}{4T_1}(1 - \frac{32AT_1}{\lambda})])$$

Remark (9.5). When $|V'(x)| = 0(e^{A|x|^2})$ one can use Proposition 2 to prove the second part of (1.20). Namely, for each n $\overline{\lim_{\lambda\to\infty}} \frac{\log|\nabla\Omega_n(x,\lambda)|}{\lambda} \le -\min\{\rho(x,a), \rho(x,b)\}$ u.o.c in $\mathbb{R}^d$.

Write $|\nabla\Omega_n(x,\lambda)| \leq e^{T\frac{E_n(\lambda)}{\lambda}} \int |\nabla_x e^{-T\frac{H(\lambda)}{\lambda}}(x,y)||\Omega_n(y,\lambda)|\, dy$ for x in a compact which is inside $B(0\ ,\frac{R}{2})$ for some large R. When $|y| \geq R$ use (9.4) and (1.21), that says $|\Omega_n(y,\lambda)| = 0(e^{-\lambda c_2|y|})$ for $|y|$ and λ large.

For $|y| \leq R$ use (9.3) an, (1.19) that says

$$\lim_{T\to\infty} A(x,y,T) = \min\{\rho(x,a) + \rho(a,y),\ \rho(x,b) + \rho(b,y)\} \text{ u.o.c.}$$

and the first part of (1.20), i.e., $\overline{\lim_{\lambda\to\infty}} \frac{\log|\Omega_n(x,\lambda)|}{\lambda} \leq -\min\{\rho(x,a),\ \rho(x,b)\}$ u.o.c.

Finally we recall by (1.2) $\frac{E_n(\lambda)}{\lambda} = 0(1)$ as $\lambda \to \infty$, so fixing a large T and taking $\lambda \to \infty$ gives the result.

Proof of Proposition 2.1.

Write the Feynman-Kac formula (5.5) as

$$\text{(9.5)} \qquad e^{-T\frac{H(\lambda)}{\lambda}}(x,y) = \left(\frac{\lambda}{2\pi T}\right)^{\frac{d}{2}} \exp\left(-\frac{\lambda|x-y|^2}{2}\right) E_z^T(k(x;z))$$

with $k(x;z) = k(x,y,T,\lambda;z) = \exp(-\lambda\int_0^T V((1 - \frac{t}{T})\, x + \frac{t}{T}\, y + \lambda^{-\frac{1}{2}} z(t))\, dt)$

whenever $z \in C_{0,0}([0,T])$.

To differentiate we write

$$(9.6)\qquad \frac{k(x+\varepsilon e_i;z)-k(x;z)}{\varepsilon} = \frac{\int_0^{\varepsilon} \frac{d}{ds}\, k(x+se_i;z)ds}{\varepsilon}$$

$$= -\frac{\lambda}{\varepsilon}\int_0^{\varepsilon}\Big\{\int_0^T \frac{\partial V}{\partial x_i}((1-\tfrac{t}{T})(x+se_i)+\tfrac{t}{T}\,y+\lambda^{-\frac{1}{2}}z(t))$$

$$(1-\tfrac{t}{T})dt)\; k(x+se_i;z)\; ds$$

$$\to -\lambda(\int_0^T \frac{\partial V}{\partial x_i}((1-\tfrac{t}{T})\,x+\tfrac{t}{T}\,y+\lambda^{-\frac{1}{2}}z(t))(1-\tfrac{t}{T})dt)\; k(x;z)$$

$$=: -\lambda \ell_i(x;z)\; k(x;z) \text{ as } \varepsilon \to 0$$

for each $z \in C_{0,0}([0,T])$.

By hypothesis $|V'(x)| = 0(e^{A|x|^2})$ so

$$(9.7)\qquad |\frac{\partial V}{\partial x_i}((1-\tfrac{t}{T})(x+se_i)+\tfrac{t}{T}\,y+\lambda^{-\frac{1}{2}}z(t)| =$$

$$0\Big(e^{4A((|x|+\varepsilon)^2+|y|^2)}\; e^{\frac{2A}{\lambda}\|z\|^2_{L^\infty[0,T]}}\Big)$$

By (7.26) with $\|\cdot\|_\infty = \|\cdot\|_{L^\infty[0,T]})$ we have

$$(2\pi T)^{-\frac{d}{2}}\; E_z^T(z : \|z\|_\infty \geq \delta) = 0(\exp(-\frac{\delta^2 k(d)}{T}))$$

and we get

$$(9.8) \qquad (2\pi T)^{-\frac{d}{2}} E_z^T(\exp(\frac{2A}{\lambda} \|z\|_\infty^2))$$

$$\leq \sum_{n=0}^{\infty} \Big(\sup_{n \leq \|z\|_\infty \leq n+1} [\exp(\frac{2A}{\lambda}\|z\|_\infty^2)]\Big)(2\pi T)^{-\frac{d}{2}} E_z^T(z : n \leq \|z\|_\infty \leq n + 1)$$

$$= 0(\sum_0^\infty \exp(\frac{2A}{\lambda}(n + 1)^2 \exp(- \frac{n^2 k(d)}{T}))$$

$$= 0(\exp(\frac{2A}{\lambda}) + \sum_1^\infty \exp(- \frac{k(d)n^2}{T} (1 - \frac{4AT(1+\frac{1}{n})^2}{k(d)\lambda}))$$

$$= 0(\exp(\frac{2A}{\lambda}) + \sum_1^\infty \exp(- \frac{k(d)n^2}{T})$$

$$= 0(\exp(\frac{2A}{\lambda}) + T^{\frac{1}{2}}) \text{ when } \lambda \in [\frac{16AT}{k(d)}, \infty).$$

Now (9.6), (9.7) and the dominated convergence theorem give

$$(9.9) \qquad \frac{E_z^T(k(x + \varepsilon e_i , z)) - E_z^T(k(x:z))}{\varepsilon} \to -\lambda E_z^T(\ell_i(x;z)k(x;z)) \text{ as } \varepsilon \to 0$$

whenever $(x,y,T,\lambda) \in \mathbb{R}^d \times \mathbb{R}^d \times (0,\infty) \times [\frac{16AT}{k(d)}, \infty)$.

(9.9) and (9.5) imply (9.1) and together with (9.7) and (9.8) they imply (9.2).

Proof of Proposition 2.2.

Recall (9.5)

$$e^{-T\frac{H(\lambda)}{\lambda}}(x,y) = (\frac{\lambda}{2\pi T})^{\frac{d}{2}} \exp(-\frac{\lambda|x-y|^2}{2T})\, E_z^T(k(x,y,T,\lambda;z))$$

and (9.1) can be written as

$$(9.10) \quad -\frac{1}{\lambda}\frac{\partial}{\partial x_i} e^{-T\frac{H(\lambda)}{\lambda}}(x,y) = (\frac{\lambda}{2\pi T})^{\frac{d}{2}} \exp(-\frac{\lambda|x-y|^2}{2T})$$

$$\Big(\frac{(x_i-y_i)}{T} E_z^T(k(x,y,T,\lambda;z)\Big) + E_z^T(\ell_i(x,y,T,\lambda;z)\, k(x,y,T,\lambda;z)).$$

By (1.17) (see Simon [2])

$$(9.11) \quad \lim_{\lambda\to\infty} \frac{\log e^{-T\frac{H(\lambda)}{\lambda}}(x,y)}{\lambda} = -A(x,y,T) \text{ u.o.c.}$$

and we only need to look at the second term.

Let $L \subseteq \mathbb{R}^d \times \mathbb{R}^d \times (0,\infty)$ be a compact set and

$$I(x,y,T,\lambda,R) = (\frac{\lambda}{2\pi T})^{\frac{d}{2}} \exp(-\frac{\lambda|x-y|^2}{2T})\, E_z^T(|\ell_i(z)|\, k(x),\, \chi(z : \|z\|_{L^\infty[0,T]} \le R\lambda^{\frac{1}{2}}))$$

and

$$II = II(x,y,T,\lambda,R) =$$

$$= (\frac{\lambda}{2\pi T})^{\frac{d}{2}} \exp(-\frac{\lambda|x-y|^2}{2T})\, E_z^T(|\ell_i(z)|k(z)\chi(z : \|z\|_{L^\infty[0,T]} \ge R\lambda^{\frac{1}{2}})).$$

Then the second term in (9.10) is $0(I + II)$.

By Cauchy-Schwartz (9.6), (9.7), (9.8), and (7.26)

$$II(x,y,T,\lambda R) = 0(\lambda^{\frac{d}{2}}(E_z^T(\exp(\frac{4A}{\lambda}\|z\|^2_{L^\infty[0,T]})))^{\frac{1}{2}}$$

$$(E_z^T(z : \|z\|^2_{L^\infty[0,T]} \geq R\lambda^{\frac{1}{2}})))^{\frac{1}{2}} = 0(\lambda^{\frac{d}{2}} \exp(-\frac{\lambda R^2 k(d)}{2T_{max}}))$$

uniformly for $(x,y,T) \in L$ and $R > 0$ where $T_{max} = \max_{(x,y,T)\in L}\{T\}$.

Now fix R_0 such that $\frac{R_0 k(d)}{2T_{max}} \geq \sup_L \{A(x,y,T)\} + 1$ and then $II(x,y,T,\lambda,R_0)$ is out of the story.

So now we consider $I(x,y,T,\lambda,R_0)$ since $|x|$, $|y|$ and T are all bounded for

$$(x,y,T) \in L,\ I(x,y,T,\lambda,R_0) = 0(T^{\frac{3}{2}} e^{4A(|x|^2+|y|^2)} e^{2AR_0^2} (\frac{\lambda}{2\pi T})^{\frac{d}{2}} \exp(-\frac{\lambda|x-y|^2}{2T})$$

$$E_z^T(k(z)\ \chi(z : \|z\|^2_{L^\infty[0,T]} \leq \lambda^{\frac{1}{2}} R_0) = 0(e^{-T\frac{H(\lambda)}{\lambda}}(x,y))$$

and (9.3) follows from (9.11).

10. A Proof of lemma 10

10. Proof of Lemma 10.

We will use some basic rough estimates on the minimal action the Agmon distance and their paths that we state in sublemma 4 below. These estimates will also be used to prove Proposition 1.

§10.1. Preliminaries (sublemma 4).

From the definition of a double well follows there are constants C_0, C_1, $C_2 > 0$ such that

(10.1) $$C_1^2 \leq V''(x) \leq C_2^2 \text{ if } x \in B(a,C_0) \cup B(b,C_0)$$

(10.2) $$C_1^2|x-c|^2 \leq V'(x)\,(x-c) \leq C_2^2\,|x-c|^2$$
if $x \in B(c, C_0)$ for $c \in \{a, b\}$ and

(10.3) $$V(x) \geq \frac{C_1\,C_0^2}{2} \text{ if } x \notin B(a,C_0) \cup B(b,C_0)$$

We will assume $B(a,C_0)$ and $B(b,C_0)$ are far apart compared to the "diameter" of $B(a,C_0)$ and $B(b,C_0)$ in the Agmon metric, more precisely we picked C_0 such that

(10.4) $$\inf_{(x,y)\in B(a,C_0)\times B(b,C_0)}\{\rho(x,y)\} < \max\{2\sup_{x\in B(a,C_0)}\{\rho(x,a)\},2\sup_{x\in B(b,C_0)}\{\rho(x,b)\}\}$$

We notice if $\gamma(t) \in B(c,C_0)$ for $t \in [0,T]$ and γ satisfies the Euler-Largrage equations $\ddot{\gamma}(t) = \nabla V(\gamma(t))$ then $\alpha(t) := |\gamma(t)-C|^2$ satisfies $\ddot{\alpha}(t) = 2|\dot{\gamma}(t)|^2 + 2\langle\ddot{\gamma}(t),(\gamma(t)-C)\rangle \geq 2\langle\nabla V(\gamma(t)), (\gamma(t)-C)\rangle \geq 2C_1^2|\gamma(t)-C|^2$. This makes easy to get rough estimates for minimal action paths and Agmon gedesics using differential inequalities (Protter-Weinberger [1]).

<u>Sublemma 4</u>.

(1) If $|x-C| \leq \frac{C_0}{\left(\frac{1}{2}+\frac{C_2}{C_1}\right)}\frac{1}{2}$ then $|x-C|\, e^{-C_2 t} \leq |g^{x,C}(t)-C|$

$\leq |x-C|\, e^{-Ct}$ and $\frac{C_1{}^2}{2C_2}\,|x-C|^2 \leq \rho(x,C) \leq \frac{C_2{}^2}{2C_1}\,|x-C|^2$

(2) If $0 \leq t_1 \leq t_2 \leq T$, $0 \leq \alpha \leq 1$ and $|g_T^{xy}(t) - C| \leq \alpha\, C_0$

for $i \in \{1, 2\}$ and $|g_T^{xy}(t_i) - C| \leq \alpha\, C_0$ for $t \in [t_1, t_2]$

then $|g_T^{xy}(t) - C|^2 \leq |g_T^{xy}(t_1) - C|^2 \frac{\sinh C_1(t_2\text{-}t)}{\sinh C_1(t_2\text{-}t_1)} +$

$|g_T^{xy}(t_2) - C|^2 \frac{\sinh C_1(t\text{-}t_1)}{\sin h\, C_1(t_2\text{-}t_1)} \leq \alpha^2 C_0{}^2$

(3) If $|x - C| \leq C_0$ then $A(x, C, T) \leq \frac{C_2}{2}\,|x - C|\,\frac{\cosh C_2 T}{\sinh C_2 T}$

(4) If $0 \leq \alpha \leq \beta \leq 1$ then

$$\inf_{\substack{|x-C|=\alpha C_0\\ |y-C|=\beta C_0\\ T>0}} A(x, y, T) \geq \frac{C_1 C_0^2}{4}\,(\beta^2 - \alpha^2)$$

The proof of this sublemma is in Chapter 12.

Now we prove (6.6) and (6.7) of lemma 10 will only use rough estimates valid for both $e^{-T\frac{H(\lambda)}{\lambda}}(x, y)$ and $|-\frac{1}{\lambda}\frac{\partial}{\partial x_i} e^{\frac{-TH(\lambda)}{\lambda}}(x, y)|$. Namely

$$(10.5) \qquad \begin{bmatrix} \overline{\lim}_{\lambda\to\infty} \dfrac{\log e^{-T\frac{H(\lambda)}{\lambda}}(x,y)}{\lambda} \\ \overline{\lim}_{\lambda\to\infty} \log |-\frac{1}{\lambda}\frac{\partial}{\partial x_i} e^{-T\frac{H(\lambda)}{\lambda}}(x,y)| \end{bmatrix} \leq -A(x,y,T)$$

uniformly on compacts (see(1.17) and (9.3)) and: If $|x| \leq \frac{R}{2}$, $|y| \geq R$ and $T \in [T_0, 2T_0]$

the for $i \in \{1,...,d\}$

$$(10.6) \qquad \begin{bmatrix} \exp\left(-T\frac{H(\lambda)}{\lambda}\right)(x,y) \\ |-\frac{1}{\lambda}\frac{\partial}{\partial xi} e^{-T\frac{H(\lambda)}{\lambda}}(x,y)| \end{bmatrix} = 0\left(\exp\left(-\frac{\lambda|y|^2}{16T_0}\right)\right)$$

uniformly for $\lambda \geq \lambda_0$ (λ_0 maybe getting larger) (see (9.10)). Finally we notice one can take δ_0 smaller and T_0 larger in lemmas 2-9 without a loss of generality.

§10.2. Proof of (6.6 and (6.7) of lemma 10.

We will assume

$$(10.7) \qquad \delta_0 \in \left(0, C_0(\tfrac{1}{2} + \tfrac{C_2}{C_1})^{-\frac{1}{2}}\right)$$

and put

(10.8) $\beta := \frac{1}{128} \frac{C_1^2}{C_2^2}$

Recall Proposition 1 says $A(x,y,T) = \min\{\rho(x,a) + \rho(a,y), \rho(x,b) + \rho(b,y)\} + 0(e^{-\alpha T})$ uniformly an compacts in $\mathbf{R}^d \times \mathbf{R}^d$ as $T \to \infty$, so we may assume (taking T_0 larger)

$$(10.9) \quad \left[\begin{array}{l} |A(x,y,T) - (\rho(x,C) + \rho(C,y))| \leq \frac{C_1 \delta_0{}^2}{32} \\ \text{if } (x,y) \in (\overline{B(x_0,\delta_0)} \times \overline{B(C, \delta_0)}) \cup (\overline{B(C, C_0)} \times \overline{B(C,C_0)}) \end{array} \right]$$

for $C \in \{a,b\}$ and $T \geq T_0$

and

(10.10)

$$T_0 \frac{C_1 \beta^2 \delta_0^2}{2} \geq \begin{array}{c} \sup\limits_{C\in\{a,b\}} \sup\limits_{(x,y)\in(B(x_0,\delta_0)\times B(C,C_0))\cup(B(C,C_0)\times B(C,C_0))} \{\rho(x,c)+\rho(c,y)\} + 100\, C_1\, C_0^2 \\ \\ \sup\limits_{C\in\{a,b\}} \sup\limits_{(x,y)\in(B(C,C_0))^2} \{\rho(x,c)+\rho(c,y)\} + 100\, C_1\, C_0^2 \end{array}$$

By (10.1) and (10.2)

(10.11) $V(x) \geq C_1 \frac{\beta^2 \delta_0{}^2}{2}$ if $x \notin B(a,\beta\delta_0 \cup B(b,\beta\delta_0)$

and for $T_1 \in [T_0, 2T_0]$ $x \in B(x_0,\delta_0)$

and $w \notin B(a,\delta_0) \cup B(b,\delta_0)$ we have:

(10.12) $A(x,w,T_1) \geq T_0 \frac{C_1 \beta^2 \delta_0{}^2}{2}$ if no minimal action path g_T^{xw} enters

$B(a,\beta\delta_0) \cup B(b,\beta\delta_0)$ and $A(x,w,T_1) \geq \inf_{\substack{(z_1,z_2) \in (B(a,\beta\delta_0))^2 \cup (B(b,\beta\delta_0))^2 \\ s_1,s_2}} \{A(x,z_1,S_1) + A(z_2,w,S_2)]$

otherwise.

In the first case we only keep the potential term of the action and use (10.10) and (10.11).

In the second case we only keep the contribution until the first time that a path enters $B(a,\beta\delta_0) \cup B(b,\beta\delta_0)$ and from the last time it leaves $B(a,\beta\delta_0) \cup B(b,\beta\delta_0)$. If $(z_1,z_2) \in B(b,\beta\delta_0) \cup B(b,\beta\delta_0)$ then $A(x,z_1,S_1) + A(z_2,w,S_2) \geq \rho(x,z_1) + \rho(z_2,w) \geq \rho(x,b) - \rho(z_1,b) + \rho(b,w) - \rho(b_1,z_2)$ since $\rho(x,y) = \inf_{T>0} A(x,y,T)$ (Carmona-Simon [1]) and the triangle inequality $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$. By sublemma 4.1 $\rho(z_i, b) \leq \frac{1}{2} \frac{C_2^2}{C_1} |z_i - b|^2 \leq \frac{1}{2} \frac{C_2^2}{C_1} \beta^2 \delta_0{}^2$ and since $w \notin B(a,\delta_0) \cup B(b,\delta_0)$ implies $\rho(b,w) \geq \inf_{|z-b|=\delta_0} \rho(b,z) \geq \frac{C_1\delta_0^2}{4}$ by sublemma 4.4. We obtain by above

$$
\begin{aligned}
A(x,z_1,S_1) + A(z_2,w,S_2) &\geq \\
&\geq \rho(x,z_1) + \rho(z_2,w) \\
&\geq \rho(x,b) + \frac{C_1\delta_0^2}{4} - \frac{C_2^2}{C_1} \beta^2 \delta_0{}^2 \\
&\geq \rho(x,b) + \frac{31}{128} C_1 \delta_0{}^2
\end{aligned}
$$

by the choice of β in (10.8).

If $y \in B(b,\beta\delta_0)$ then by sublemma 4.1

$$\rho(b,y) \le \frac{C_2^2}{2C_1} |y-b|^2$$

$$\le \frac{C_2^2}{2C_1} \beta^2\delta_0^2 \le \frac{1}{256} C_1\delta_0^2$$

i.e.,

$$0 \ge \rho(b,y) - \frac{1}{256} C_1\delta_0^2$$

and we get

$$A(x,z_1,S_1) + A(z_2,w,S_2) \ge \rho(x,b) + \rho(b,y) + \frac{61}{256} C_1\delta_0^2$$

for all $y \in B(b,\beta\delta_0)$.

By (10.9) and (10.10) we get

$$(10.13) \quad A(x,w,T_1) \ge \begin{bmatrix} \min_{c\in\{a,b\}} \{\rho(x,c)+\rho(c,y)\} + \frac{53}{256} C_1\delta_0^2 \\ A(x,y,T) + \frac{53}{256} c_1\delta_0^2 \end{bmatrix}$$

for all $x \in B(x_0 , \delta_0)$, $T_1 \in [T_0 , 2T_0]$, $w \notin B(b,\delta_0) \cup B(a,\delta_0)$,

$y \in B(b,\beta\delta_0)$ and $T \ge T_0$.

Looking at $(z_1, z_2) \in B(a,\beta\delta_0) \times B(a,\beta\delta_0)$ and $y \in B(a,\beta\delta_0)$ we get a similiar statement where a and b are interchanged and hence:

$$A(x,w,T_1) \ge \begin{bmatrix} \min_{c\in\{a,b\}} \{\rho(x,c)+\rho(c,y)\} + \frac{53}{256} C_1\delta_0^2 \\ A(x,y,T) + \frac{53}{256} C_1\delta_0^2 \end{bmatrix}$$

for all $x \in B(x_0,\delta_0)$, $T_1 \in [T_0,2T_0]$, $w \notin B(a,\delta_0) \cup B(b,\delta_0)$, $y \in B\ (a,\beta\delta_0)$ and $T \in [T_0, \infty)$.

Now we consider large $|w|$'s. Pick large R so that $B(x_0, \delta_0)$, $B(a,C_0)$ and $B(b,C_0) \subseteq B(0,\frac{R}{2})$. Bearing in mind (10.6) we make

$$\frac{R^2}{16T_0} \geq \begin{cases} \{\rho(x,c)+\rho(c,y)\} + 100\, C_1 C_0^2 \\ A(x,y,T) + 100\, C_1 C_0^2 \end{cases}$$

$\forall (x,y,T) \in B(x_0 , \delta_0) \times B(C,C_0) \times [T_0 , \infty)$ and $C \in \{a,b\}$. So we have the wanted

estimates in (6.6) if $(x,w,T_1,\lambda) \in B(x_0 , \delta_0) \times (\mathbb{R}^d - B(0,R)) \times [T_0 , 2T_0] \times [\lambda_0 , \infty)$. For this R, we use (10.5).

With $\varepsilon = \frac{1}{250} C_1 \delta_0^2$ there exist $k_7 = C_\varepsilon$ and $\lambda_0 = \lambda_0(\varepsilon)$ (still larger λ_0!) such that uniformly for $(x,y,T) \in B(x_0 , \delta_0) \times B(0,R) \times [T_0 , 2T_0]$ we have

$$\left. \begin{array}{l} e^{-T_1 H\frac{(\lambda)}{\lambda}}(x,w) \\ |\frac{1}{\lambda}\frac{\partial}{\partial x_i} e^{-T_1 H\frac{(\lambda)}{\lambda}}(x,w)| \end{array} \right\} \leq C_\varepsilon \exp(-\lambda(A(x,w,T_1) - \varepsilon))$$

$$= k_7 \exp(-\lambda(A(x,w,T_1) - \frac{C_1 \delta_0{}^2}{256}))$$

which, by above, gives (6.6) of Lemma 10, with $k_8 = \frac{52}{256} C_1 \delta_0^2$.

Proof of (6.7) is similiar.

We get uniform estimates of $A(x,w,T_1)$ for $x \in B(b,\delta_0)$ $w \notin B(b,\delta_0)$ (not just on compacts in w) and $T_1 \in [T_0 , 2T_0]$ by writing $A(x,w,T_1) \geq (T_0 \frac{C_1 \beta^2 \delta_0^2}{2})$ if no minimal action path $g_{T_1}^{xy}$ enters $B(a,\beta\delta_0) \cup B(b,\beta\delta_0)$,

$$A(x,w,T_1) \geq \inf_{\substack{z_1 \in B(b,\delta_0) \\ z_2 \in B(a,\beta\delta_0)}} \{\rho(z_1,z_2)\}$$

if some $g_{T_1}^{x,w}$ enters $B(a,\delta_0)$ and

$$A(x,w,T_1) \geq \inf_{\substack{z_1,z_2 \in B(b,\beta\delta_0) \\ W \notin B(b,\delta_0) \\ s_1,s_2>0}} \{A(x,z_1,s_1) + A(z_2,w,s_2))$$

if some $g_{T_1}^{xy}$ enters $B(b,\beta\delta_0)$.

This first lower bound is large since we made it large in (10.10), the second is large since the wells are far apart and β is small. Using how β was chosen we see the third one is bigger than

$$\begin{bmatrix} \rho(x,b)+\rho(b,y) \\ A(x,y,T) \end{bmatrix} + \frac{53}{256} C_1 \delta_0^2$$

for all $y \in B\ (b,\beta\delta_0)$ and $T \in [T_0, \infty)$, as in (10.13).

Now we can finish the proof as above using (10.6) first and then (10.5).

§10.3. Sketch of a proof of lemma 10.3.

Recall, sublemma 1 says:

$$\text{(10.14)}\quad e^{-T\frac{H(\lambda)}{\lambda}}(x,y) = \left(\frac{\lambda}{2\pi T}\right)^{\frac{d}{2}} \exp(-\lambda A(x,y,T))$$

$$E_z^T \left(\exp(-\lambda \int_0^T (V(g_T^{x,y}(t) + \lambda^{-\frac{1}{2}} z(t) - V(g_T^{x,y}(t)) - \lambda^{-\frac{1}{2}} V'(g_T^{x,y})\, z(t))\, dt\right),$$

lemma 2 says $g_T^{x,y}$ is unique for $(x,y,T) \in B(x_0,\delta_0) \times B(b,\delta_0) \times B(b,\delta_0) \times [T_0,2T_0)$ and is differentiable with respect to x.

For $(x,y,T,\lambda) \in B(x_0, \delta_0) \times B(b,\delta_0) \times [T_0, 2T_0] \times [1, \infty]$ and $z \in C_{00}([0,T])$ set

$$(10.15)\quad f(x;z) = f(x,y,T,\lambda;z) := \exp(-\lambda \int_0^T (V(g_T^{x,y}(t) + \lambda^{-\frac{1}{2}} z(t) - V(g_T^{x,y}(t)) - \lambda^{-\frac{1}{2}} V'(g_T^{x,y}(t))\, z(t))\, dt) \;.$$

Then

$$(10.16)\quad \frac{f(x+he_i;z)-f(x,z)}{h} = \int_0^h \frac{\frac{d}{ds} f(x+se_i:z)}{h} ds$$

$$= -\frac{\lambda}{h} \int_0^h \left\{ \int_0^T \sum_{j=1}^d \left(\frac{\partial V}{\partial u_j} (g_T^{x+se_i,y} (t) + \lambda^{-\frac{1}{2}} z(t)) - \frac{\partial V}{\partial u_j} (g_T^{x+se_i,y} (t)\right) - \lambda^{-\frac{1}{2}} \sum_{k=1}^d \frac{\partial^2 V}{\partial u_k \partial u_j} (g_T^{x+se_i,y} (t) z_k(t)) \left(\frac{\partial}{\partial x_i} g_T^{x+se_i,y} (t)\right)_j dt \right\}$$

$$f(x+se_i;z)\, ds \rightarrow -\lambda \int_0^T \Big(\sum_{j=1}^d \Big(\frac{\partial V}{\partial u_j} (g_T^{x,y}(t)+\lambda^{-\frac{1}{2}} z(t)) - \frac{\partial V}{\partial u_j}(g_T^{x,y}(t)) - \lambda^{-\frac{1}{2}} \sum_{k=1}^d \frac{\partial^2 V}{\partial u_j \partial u_k} (g_T^{x,y}(t))\, z_k(t)\Big) \frac{\partial (g_T^{x,y}(t))_j}{\partial x_i} \Big)\, dt\, f(x;z)$$

$$=: -\lambda\, h_i(x;z)\, f(x;z) \cdot$$

Since $|\nabla V(x)| = 0(e^{A|x|^2})$ implies $|\nabla V(g_T^{x,y} + \lambda^{-\frac{1}{2}} z)| = 0\Big(e^{2A(\|g_T^{x,y}\|_\infty^2 + \lambda^{-1} \|z\|_\infty^2)}\Big)$

the integrand in

$$(10.17)\quad E_z^T \frac{(f(x+he_i;z)-E_z^T(f(x;z))}{h} = E_z^T\Big(\frac{(f(x+he_i;z)-f(x:z)}{h}\Big)$$

is uniformly bounded by an integrable functional of the form

$$0(e^{2\frac{A\|z\|_\infty^2}{\lambda}} e^{B\lambda^{\frac{1}{2}}\|z\|_\infty}) \text{ for } \lambda \text{ large.}$$

The dominated convergences theorem and (10.16) implies the expression in (10.17) tends to $-\lambda E_z^T(h_i(x;z)\, f(x;z))$ as $h \to 0$.

By (10.14) we have

$$(10.18)\quad -\frac{1}{\lambda}\frac{\partial}{\partial x_i} e^{-T\frac{H(\lambda)}{\lambda}}(x,y) = \Big(\frac{\partial A(x,y,T)}{\partial x_i}\Big) e^{-T\frac{H(\lambda)}{\lambda}}(x,y) +$$

$$+\Big(\frac{\lambda}{2\pi T}\Big)^{\frac{d}{2}} E_z^T\Big(\Big(\int_0^T \sum_{j=1}^d \Big(\frac{\partial V}{\partial u_j}(g_T^{x,y} + \lambda^{-\frac{1}{2}} z) - \frac{\partial V(g_T^{x,y})}{\partial u_j} -$$

$$-\lambda^{-\frac{1}{2}} \sum_{k=1}^d \frac{\partial^2 V(g_T^{x,y})}{\partial u_j \partial u_k} z_k\Big)\Big(\frac{\partial (g_T^{x,y})j}{\partial x_j} dt\Big) \exp\Big(-\lambda \int_0^T (V(g_T^{x,y}(t) + \lambda^{-\frac{1}{2}} z(t)) -$$

$$- V(g_T^{x,y}(t)) - \lambda^{-\frac{1}{2}} V'(g_T^{x,y}(t))z(t))\, dt\Big)\Big)$$

$$\exp\,(-\lambda A(X,Y,T)) =: \frac{\partial A(x,y,T)}{\partial x_i} e^{-T\frac{H(\lambda)}{\lambda}}(xy) + I(x,y,T,\lambda)$$

and we need to expand $I(x,y,T,\lambda)$. As before we divide the path space $C_{00}([0,T])$ into 3 pieces for $0 < r < R$:

$$C_{00}([0,T]) = \{Z : \lambda^{-\frac{1}{2}}\|z\|_\infty < r\} \cup \{z : r \le \lambda^{-\frac{1}{2}}\|z\|_\infty \le R\} \cup (z : \lambda^{-\frac{1}{2}}\|z\|_\infty > R\}$$

and accordingly

(10.19) $\quad I(x,y,T,\lambda) = II_r(x,y,T,\lambda) + III_{r,R}(x,y,T,\lambda) + IV_R(x,y,T,\lambda).$

Since $T \in [T_0, 2T_0]$, which is bounded, we estimate

$$IV_R\Big(x,y,T,\lambda\Big) = 0\,(e^{\frac{2A\|z\|_\infty^2}{\lambda}}\, e^{\lambda^{\frac{1}{2}} B\|z\|_\infty}\, \chi(z{:}\|z\| > R\lambda^{\frac{1}{2}})\Big)$$

$$= 0\left(\sum_{n=[R]}^{\infty} e^{\frac{2A(n+1)^2}{\lambda}}\, e^{\lambda^{\frac{1}{2}}(n+1)B}\, e^{-\lambda\frac{n^2k(d)}{2T_0}}\right)$$

where [R] is the integer part of R.

Here we used (7.26)

$$(2\pi T)^{\frac{d}{2}} E_z^T(z{:}\|z\| > r) = 0\,(\exp\left(-\frac{r^2\,k(d)}{T}\right) = 0(\exp\left(-\frac{r^2k(d)}{2T_0}\right)),$$

and for the characteristic function we have

$$\chi\{z{:}\|z\| > k) \leq \sum_{n=k}^{\infty} \chi(z{:}\|z\| > n)$$

and a uniform estimate for the integrand.

For R large compared to B we get

(10.20) $$IV_R(x,y,T,\lambda) = 0(e^{-\frac{\lambda R^2 k,d)}{4T0}}) \text{ as } \lambda \to \infty$$ and we fix R_0 such that

$$IV_{R_0}(x,y,T,\lambda) = 0\left(e^{-\lambda[\sup_{(x,y,T)} A\{x,y,T\}+1]}\right) \text{ as } \lambda \to \infty.$$

Next, using (2.49) we fix r_0 such that $-\frac{d^2}{dt^2} + V''(\gamma) > 0$ on $D_{00}(0,T)$ if $\|\gamma - g_T^{xy}\| \le \frac{r_0}{2}$ for all $(x,y,T) \in B(x_0, \delta_0) \times B(b,\delta_0) \times [T_0, 2T_0]$. Then we make a Taylor expansion of the integrand in $II_{r_0}(x,y,T,\lambda)$ similiar as in the proof of lemma 8.1. Bounded T makes it easier and we get:

(10.21) $$II_{r_0}(x,y,T,\lambda) = \left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}} [\frac{1}{\lambda}\frac{1}{2!}$$

$$E_z^T\left(\int_0^T V^{(3)}\left(g_T^{xy}; z, z, \frac{\partial g_T^{x,y}}{\partial x_i}\right) dt\right.$$

$$\left.\exp\left(-\frac{1}{2}\int_0^T \langle V''(g_T^{xy})\, z, z\rangle\, du\right)\right) + 0(\frac{1}{\lambda^2})]$$

$$\exp(-\lambda A(x,y,T)).$$

Finally by the bound on $|V'(x)|$

(10.22) $III_{r_0,r_0}(x,y,T,\lambda) =$

$$= 0(e^{2AR_0^2}(\frac{\lambda}{2\pi T})^{\frac{d}{2}} E_z^T\Big(\exp\Big(-\int_0^T (V(g_T^{xy}+\lambda^{-\frac{1}{2}} z) - V(g_T^{xy}) - \lambda^{-\frac{1}{2}} V'(g_T^{xy})\, z)dt\Big)$$

$$\chi(z : r_0 \le \lambda^{-\frac{1}{2}}\|z\|_\infty \le R_0)\Big) \exp(-\lambda A(x,y,T)) = 0(\, e^{-\lambda(A(x,y,T)+\varepsilon_0)})$$

uniformly for $(x,y,T,\lambda) \in B(x_0, \delta_0) \times B(b,\delta_0) \times [T_0, 2T_0] \times [\lambda_0, \infty)$

according to (8.11) in the proof of lemma 9.

Now (10.19)-(10.22) give

(10.23) $I(x,y,T,\lambda) = II_{r_0}(x,y,T,\lambda) + 0\,(e^{-\lambda(A(x,y,T)+\varepsilon_0)}) =$

$$= \Big(\frac{\lambda}{2\pi T}\Big)^{\frac{d}{2}} [\tfrac{1}{\lambda}(-\frac{\partial}{\partial x_i})\, E_z^T(\exp(-\frac{1}{2!}\int_0^T \langle V''(g_T^{xy})z,z\rangle\, dt)$$

$$+ 0(\frac{1}{\lambda^2})]\, \exp(-\lambda A(x,y,T))$$

$$= \Big(\frac{\lambda}{2\pi}\Big)^{\frac{d}{2}} (\frac{-\frac{\partial b_0^T(x,y)}{\partial x_i}}{\lambda} + 0(\frac{1}{\lambda^2}))\, \exp(-\lambda A(x,y,T))$$

where we used the definition of $b_0^T(x,y)$ in sublemma 2.

Now (10.18), (10.23) and (6.5) give (6.8).

11. A Proof of Lemma 7.

§11.1. Lemma 7.1

§11.2. Lemma 7.2

§11.3. Lemma 7.3

§11.4. A sketch of a proof of lemma 7.4 and 7.5

11. A proof of lemma 7.

§11.1. Lemma 7.1.

We first want to show (4.17): If $\bar{y} = b + 0(e^{-\omega_1 T})$ uniformly on $B(x_0, \delta_0) \times [T_0, \infty)$, as in lemma 6, then for $\beta \in \mathbb{N}_0^d$ with $|\beta| \leq 2$.

(11.1) $$\frac{\partial^{|\beta|}}{\partial y^\beta} a_0^T(x,y)\,|_{y=\bar{y}} = C_{0,\beta}\, a_0(x) + 0\,(e^{-\delta_1 T})$$

unformily on $B(x_0, \delta_0) \times [T_0, \infty)$.

Here $C_{0\beta}$ are constants and $a_0(x) = (\det X^{x,b}(0))^{-\frac{1}{2}}$ and $X^{x,b}$ as is in lemma 2. For $\beta = 0$, (11.1) follow from (7.8) that says

(11.2) $$a_0^T(x,y) = a_0(x)\,(1 + 0(e^{-\delta_1 T} + |y-b|))$$

uniformly on $B(x_0, \delta_0) \times B(b,\delta_0) \times (T_0, \infty)$.

Now recall (see sublemma 2)

(11.3) $$b_0^T(x,y) = 2^{\frac{d}{2}}\,(\Pi\omega_i)^{\frac{1}{2}}\, e^{-\frac{T}{2}\sum_{i=1}^{d}\omega_i}\, a_0^T(x,y)$$

and so it suffices to show

(11.4) $$\frac{\partial^{|\beta|}}{\partial y^\beta} b_0^T(x,y)_{y=\bar{y}} = (C_{0\beta} + 0(e^{-\delta_1 T}))\, b_0^T(x,y)|_{y=\bar{y}} \quad \text{for } |\beta| \leq 2.$$

To do that we differentiate

$$(11.5)\qquad b_0^T(x,y) = T^{-\frac{d}{2}} E_z^T(\exp(-\tfrac{1}{2}\int_0^T \langle V''(g_T^{xy})z,z\rangle\, dt)) \text{ from sublemma 2}$$

and use

$$(11.6)\qquad T^{-\frac{d}{2}} E_z^T[z_i(s)\, z_j(t) \exp(-\tfrac{1}{2}\int_0^T \langle V''(g_T^{xy})z,z\rangle\, du)]$$

$$= b_0^T(x,y)\, (G^{x,y,T}(s,t))_{ij} \text{ from sublemma 3.}$$

We get

$$(11.7)\qquad \frac{\partial}{\partial y_i} b_0^T(x,y) = T^{-\frac{d}{2}} E_z^T\, [(-\tfrac{1}{2}\int_0^T \sum_{k,n,m} \frac{\partial^3 v(g_T^{xy}(t))}{\partial u_k \partial u_n \partial u_m} z_n(t)\, z_m(t)$$

$$\frac{\partial (g_T^{xy})_k}{\partial y_i} dt) \exp(-\tfrac{1}{2}\int_0^T \langle V''(g_T^{xy})z,z\rangle\, du)]$$

$$= (-\tfrac{1}{2}\int_0^T \sum_{k,n,m} \frac{\partial^3 V(g_T^{xy}(t))}{\partial u_k \partial u_n \partial u_m} G_{n,m}^{x,y,T}(t,t) \frac{\partial (g_T^{xy})_k}{\partial y_i} dt)\, b_0^T(x,y)$$

and

$$(11.8)\qquad \frac{\partial^2}{\partial y_j \partial y_i} b_0^T(x,y) = [\frac{\partial}{\partial y_j}(-\tfrac{1}{2}\int_0^T \sum \frac{\partial^3 V(g_T^{xy}(t))}{\partial u_k \partial u_n \partial u_m} G_{n,m}^{x,y,T}(t,t) \frac{\partial (g_T^{xy}(t))_k}{\partial y_i} dt)$$

$$+ \tfrac{1}{4}(\int_0^T \sum \frac{\partial^3 V(g_T^{xy}(t))}{\partial u_k \partial u_n \partial u_m} G_{n,m}^{x,y,T}(t,t) \frac{\partial (g_T^{x,y}(t))_k}{\partial y_i} dt)$$

$$(\int_0^T \sum \frac{\partial^3 V(g_T^{xy}(s))}{\partial u_k \partial u_n \partial u_m} G_{n,m}^{x,y,T}(s,s) \frac{\partial (g_T^{xy}(s))_k}{\partial y_i} ds)]\, b_0^T(x,y)$$

and we need to show

$$I_1 := \int_0^T \frac{\partial^3 V(g_T^{xy}(t))}{\partial u_k \partial u_n \partial u_m} (G^{x,y,T}(t,t))_{n,m} \frac{\partial (g_T^{xy}(t))_k}{\partial y_i}\Big|_{y=\bar{y}} \, dt$$

$$I_2 := \int_0^T \frac{\partial^4 V(g_T^{xy}(t))}{\partial u_\ell \partial u_k \partial u_n \partial u_m} (G^{x,y,T}(t,t))_{n,m} \frac{\partial (g_T^{xy}(t))_\ell}{\partial y_j} \frac{\partial (g_T^{xy}(t))_k}{\partial y_i}\Big|_{y=\bar{y}} \, dt$$

$$I_3 := \int_0^T \frac{\partial^3 V(g_T^{xy}(t))}{\partial u_k \partial u_n \partial u_m} \frac{\partial G_{n,m}^{x,y,T}(t,t)}{\partial y_j} \frac{\partial (g_T^{xy}(t))_k}{\partial y_i}\Big|_{y=\bar{y}} \, dt$$

$$I_4 := \int_0^T \frac{\partial^3 V(g_T^{xy}(t))}{\partial u_k \partial u_n \partial u_m} G_{n,m}^{x,y,T}(t,t) \frac{\partial^2 (g_T^{xy}(t))_k}{\partial y_j \partial y_i}\Big|_{y=\bar{y}} \, dt$$

are all of the form "a constant + $0(e^{-\delta_1 T})$" uniformly for $(x,T) \in B(x_0, \delta_0) \times (T_0, \infty)$.

Take $t = T - v$, by (2.34)

$$\text{(11.10)} \quad \frac{\partial^3 V(g_T^{xy}(T-v))}{\partial u_k \partial u_n \partial u_m}\Big|_{y=\bar{y}}, \ \frac{\partial^4 V(g_T^{xy}(T-v))}{\partial u_\ell \partial u_k \partial u_n \partial u_m}\Big|_{y=\bar{y}} = \text{constant} + 0(e^{-\omega_1(T-v)})$$

By lemma 5.2

$$\text{(11.11)} \quad G^{x,y,T}(T-v,T-v)_{y=\bar{y}} = (I - e^{-2\Omega v})(2\Omega)^{-1} + 0(e^{-\delta_1(T-v)}) + 0(e^{-\delta_1 T})$$

and

$$\text{(11.12)} \quad \frac{\partial G^{xy,T}(T-v,T-v)}{\partial y_j}\Big|_{y=\bar{y}} = B(e_j,v)\,(2\Omega)^{-1} - (A(e_j,0) - A(e_j,v) - B(e_j,0))\,(2\Omega)^{-1} e^{-2\Omega v} + 0(e^{-\delta_1(T-v)})$$

where $A(e_j,v)$ and $B(e_j,v)$ are diagonal matrices in

$C_b^\infty([0,\infty),\mathbb{R}^{d^2})$, for each $j \in \{1,\dots,d\}$.

By (2.26), (2.22) and $0 < \delta_1 \le \frac{\omega_1}{2}$ we have

$$(11.13) \quad \frac{\partial(g_T^{xy}(T-v))_k}{\partial y_i}\Big|_{y=\bar{y}} = \Big[(I + 0(e^{-\delta_1(T-v)}))\, e^{-\Omega v}\, (I + 0(e^{-\delta_1 T}))\Big]_{k,i}$$

$$= \delta_{k,i}\, e^{-\omega_i v} + 0(e^{-\delta_1(T+v)})$$

Using also (2.56) we get

$$(11.14) \quad \frac{\partial^2(g_T^{xy}(T-v))_k}{\partial y_j \partial y_i}\Big|_{y=\bar{y}} = (B(e_j,v) + 0(e^{-\delta_1(T-v)}))$$

$$e^{-\Omega v}\,(I + 0(e^{-\delta_1 T})) - (I + 0(e^{-\delta_1(T-v)}))\, e^{-\Omega v}(B(e_j,0)$$

$$+ 0(e^{-\delta_1 T}))]_{k,i} = (B(e_j,v) - B(e_j,0))_{k,i}\, e^{-\omega_i v} + 0(e^{-\delta_1(T+v)}).$$

The estimates in (11.7)-(11.11) are all uniform for $(x,T,v) \in B(x_0,\delta_0) \times (T_0,\infty) \times [0,T]$ and imply I_1, I_2, I_3 and I_4 are of the form constant $+ 0(e^{-\delta_1 T})$ uniformly. For example:

$$(11.16)\quad I_4 = \int_0^T \frac{\partial^3 V(g_T^{xy}(T-v))}{\partial u_k \partial u_n \partial u_m} (G^{x,y,T}(T-v,\, T-v))_{n,m} \frac{\partial^2 (g_T^{xy}(T-v))_k}{\partial y_j \partial y_i}\Big|_{y=\bar{y}}\, dt$$

$$= \int_0^T (\text{constant} + 0(e^{-\delta_1(T-v)}))$$

$$[(1 - e^{-2\omega_n v})\, \delta_{n,m} + 0(e^{-\delta_1(T-v)}) + 0(e^{-\delta,T})]$$

$$[(B(e_j,v) - B(e_j,0))_{k,i}\, e^{-\omega_i v} + 0(e^{-\delta_1(T+v)})]\, dv$$

$$= \int_0^T (\text{constant } (1-e^{-2\omega_n V})\, \delta_{n,m}(B(e_j,v) - B(e_j 0))_{k,i}\, e^{-\omega_i v}\, dv$$

$$+ \int_0^T 0(e^{-\delta_1(T+v)})\, dv$$

$$= \text{a new constant} + 0(e^{-\delta_1 T}).$$

In the last step we used $(B(e_j,v))_{k,i}$ is bounded and $\delta_1 \le \frac{\omega_1}{2} < \omega_i$. To prove

(4.16) $\frac{\partial^{|\alpha|}}{\partial y^\alpha} a_0^T(x,y) = 0(1)$ uniformly for $(x,y,T) \in B(x_0,\, \delta_0) \times B(b,\delta_0) \times [T_0,\infty)$ it

suffices (see (11.3)) to show:

(11.17) For each $\alpha \in \mathbb{N}_0^d\ \frac{\partial^{|\alpha|}}{\partial y \alpha} b_0^T(x,y) = 0(1)\ b_0^T(x,y)$

uniformly on $B(x_0 \delta_0) \times B(b_0\ \delta_0) \times (T_0,\, \infty)$

For $|\alpha| = 0$ (4.16) follows from sublemma 2. For $|\alpha| \geq 1$ we see from (11.7), (11.8) and differentiating (11.8) that $\frac{\partial^{|\alpha|}}{\partial y\alpha} b_0^T(x,y)$ is $b_0^T(x,y)$ times sum of products of terms that are bounded by

$$I_5 := 0\left(\int_0^T \left|\frac{\partial^{|\alpha_1|)}}{\partial y\alpha_1} V(g_T^{xy}(t))\right| \left|\frac{\partial^{|\alpha_2|}}{\partial y\alpha_2} G^{x,y,T}(t,t)\right| \left|\frac{\partial^{|\alpha_3|}}{\partial y\alpha_3}(g_T^{xy}(t)\right| \cdots \left|\frac{\partial^{\alpha_n}}{\partial y\alpha_n}(g_T^{xy}(t))\right| dt\right)$$

where $n \geq 3$ and $|\alpha_3|, \ldots, |\alpha_n|$ are ≥ 1. Using lemma 2 and lemma 4

$$\frac{\partial^{|\alpha_1|}}{\partial y^{\alpha_1}} V(g_T^{xy}(t)| = 0(1), \quad \frac{\partial^{|\alpha_2|}}{\partial y\alpha 2} G^{xy,T}(t,t) = 0(1)$$

and
$$\frac{\partial^{|\alpha_i|}}{\partial y\alpha_i}(g_T^{xy}(t) = 0(e^{-\omega_1(T-t)})$$

so
$$I_5 = 0\,(\int_0^T e^{-\omega_1(T-t)}) \, dt) = 0(1).$$

Uniformly for $(x,y,T) \in B(x_0, \delta_0) \times B(b,\delta_0) \times [T_0, \infty)$ which implies (11.17).

§11.2. Lemma 7.2.

If $(x,y,T) \in B(x_0, \delta_0) \times B(b,\delta_0) \times [T_0, \infty)$ then by (2.26)

$$\text{(11.18)} \quad \frac{\partial A(x,y,T)}{\partial y_i} = (\dot{g}_T^{xy}(T)_i, \text{ where by (2.34)}$$

(11.19) $|\dot{g}^{xy}_{T}(t)| = 0(e^{-\omega_1 t} + \min(|y-b|, \delta_0)\, e^{-\omega_1(T-t)})$ uniformly,

and

(11.20) $A''_{yy}(x,y,T) = (\dot{Y}(T) - \dot{X}(T)\,(X(0)^{-1}Y(0))\,(Y(T) - X(T)\,(X(0))^{-1}Y(0))^{-1}.$

Here $X = X^{x,y,T}$ and $Y = Y^{x,y,T}$ satisfy (see (2.22)).

$$X^{x,y,T}(t) = (I + 0\,(e^{-\delta_1 t} + |y-b|))\, e^{-\Omega t} = -\dot{X}^{x,y,T}(t)\,\Omega^{-1}$$

and

(11.21) $Y^{x,y,T}(t) = (I + 0(e^{-\delta_1 t} + |y-b|))\, e^{\Omega t} = \dot{Y}^{x,y,T}(t)\,\Omega^{-1}$

uniformly for $(x,y,T,t) \in B(x_0, \delta_0) \times B(b,\delta_0) \times [T_0, \infty) \times [0,T]$

Now (11.21) into (11.20) implies

(11.22) $A''_{yy}(x,y,T) = \left(\Omega + 0(e^{-\delta_1 T} + |y-b|)\right) e^{\Omega T}\, ((I + 0(e^{-\delta_1 T}$

$+ |y-b|))\, e^{\Omega T}\big)^{-1} = \Omega + 0(e^{-\delta_1 T} + |y-b|)$

uniformly for (x,y,T)'s. If $y = \overline{y}$ or $y = \overline{\overline{y}}$,

that are $b + 0(e^{-\omega_1, T})$ uniformly for $(x,T) \in B(x_0, \delta_0)$,

then $A''_{yy}(x,y,T) = \Omega + 0(e^{-\delta_1 T})$

uniformly for $(x, T) \in B(x_0, \delta_0) \times (T_0, \infty)$.

Differentiating $A''_{yy}(x,y,T)$ in (11.20) with respect to y, we get a sum of terms of the form

$$I_6^{x,T} = \pm\left[\frac{\partial^{|\alpha_1|}}{\partial y^{\alpha_2}}\left(\dot{Y}(T)-\dot{X}(T)(X(0))^{-1}Y(0)\right)\right]\left(Y(T)-X(T)X^{-1}(0)Y(0)\right)^{-1}$$

$$\left[\frac{\partial^{|\alpha_2|}}{\partial y^{\alpha_2}}\left(Y(T)-X(T)X^{-1}(0)Y(0)\right)\right]\left(Y(T)-X(T)X^{-1}(0)Y(0)\right)^{-1}\cdots$$

$$\left[\frac{\partial^{|\alpha_n|}}{\partial y^{\alpha_n}}\left(Y(T)-X(T)X^{-1}(0)Y(0)\right)\right]\left(Y(T)-X(T)X^{-1}(0)Y(0)\right)^{-1}.$$

By (2.24) $\frac{\partial^{|\alpha_1|}}{\partial y^{\alpha_1}}\left(\dot{Y}(T)-\dot{X}(T)(X(0))^{-1}Y(0)\right)$ and $\frac{\partial^{|\alpha_i|}}{\partial y^{\alpha_i}}\left(Y(T)-X(T)X^{-1}(0)Y(0)\right)$ are

$0(1)e^{\Omega T}$ and by (2.22) $\left(Y(T)-X(T)X^{-1}(0)Y(0)\right)^{-1} = e^{-\Omega T}\left(I+0(e^{-\delta_1 T})+|y-b|\right)$

and, by above, we get $I_6 = 0(1)$ and hence $\frac{\partial^{|\alpha|}}{\partial y^{\alpha}} A(x,y,T) = 0(1)$ uniformly for $(x,y,T) \in B(x_0, \delta_0) \times B(b, \delta_0) \times [T_0, \infty)$ for each $\alpha \in N_0^d$, with $|\alpha| \geq 3$.

When $y = \overline{y} = b + 0\,(e^{-\omega_1 T})$

$$\frac{\partial^{|\alpha_1|}}{\partial y^{\alpha_1}} \left(\dot{Y}(T) - \dot{X}(T)(X(0))^{-1} Y(0)\right)\Big|_{y=\overline{y}} =$$

$$= \left(B(\alpha_1, 0) + 0(e^{-\delta_1 T})\right) e^{\Omega T}\, \Omega.$$

$$\frac{\partial^{|\alpha_i|}}{\partial y^{\alpha_i}} \left(Y(T) - X(T)(X(0))^{-1} Y(0)\right)\Big|_{y=\overline{y}} =$$

$$= \left[B(\alpha_i, 0) + 0(e^{-\delta_1 T})\right] e^{\Omega T}$$

by (2.55) and (2.56) and

$$\left(Y(T) - X(T)X^{-1}(0)Y(0)\right)^{-1}\Big|_{y=\overline{y}} = e^{-\Omega T}(I + 0(e^{-\delta_1 T})).$$

Hence each

$$I_6\big|_{y=\overline{y}} = \pm\, B(\alpha_1, 0)\, \Omega B(\alpha_2, 0) \cdots B(\alpha_n, 0) + 0(e^{-\delta_1 T})$$

and therefore

$$\frac{\partial^{|\beta|}}{\partial y^{\beta}} A(x,y,T)\big|_{y=\overline{y}} = \text{constant} + 0(e^{-\delta_1 T)})$$

uniformly for $(x,T) \in B(x_0, \delta_0) \times [T_0, \infty)$, for each $\beta \in \mathbb{N}_0^d$ with $|\beta| \geq 3$.

To prove (4.20) we write, for $\hat{y} \in \{\bar{y}, \bar{\bar{y}}\}$, $A(x,\hat{y},T) = \rho(x,b) + A(x,b,T) - \rho(x,b) + A(x,\hat{y},T) - A(x,b,T)$. By Proposition 1 (2.26 and (2.34) $\rho(x,b) - A(x,b,T) =$

$$= \lim_{s\to\infty}(A(x,b,s) - A(x,b,T)$$

$$= \lim_{S\to\infty}\left(\int_T^S \frac{\partial A(x,b,T')}{\partial T'}\, dT'\right)$$

$$= \lim_{S\to\infty}\left(\int_T^S \tfrac{1}{2}(\dot{g}_{T'}^{xb}(T'))^2\, dT'\right)$$

$$= \lim_{S\to\infty}\left(\int_T^S 0(e^{-2\omega_1 T'})\, dT' = 0(e^{-2\omega_1 T})\right.$$

uniformly for $(x,T) \in B(x_0, \delta_0) \times (T_0, \infty)$.

By Taylor's theorem, (11.18, (11.19) and (11.22) we get

$$A(x,\hat{y},T) - A(x,b,T) = \langle A_y'(x,y,T), (y-b)\rangle|_{y=\hat{y}}$$

$$+ \tfrac{1}{2}\int_0^1 \langle A_{yy}''(x,b+s(\hat{y}-b),T)(\hat{y}-b),(\hat{y}-b)\rangle\, ds$$

$$= 0((e^{-\omega_1 T} + |\hat{y}-b|)|\hat{y}-b|) +$$

$$+ \tfrac{1}{2}\int_0^1 \langle(\Omega + 0(e^{-\delta_1 T}))(\hat{y}-b),(\hat{y}-b)\rangle\, ds$$

$$= 0(e^{-2\omega_1 T}) \text{ uniformly for } (x, T) \in B(x_0, \delta_0) \times (T_0, \infty).$$

This completes the proof of (4.20) and of lemma 7.2.

§11.3. Lemma 7.3.

By (7.50), (7,20) and Simon [5] lemma 20.4 $a_1^T(x,y)$ is a finite sum of constants times terms of the following types:

Type 1: $I = I^{x,y,T} = \int_0^T \partial^\alpha V(g(t))\, G_{j_1j_2}(t,t)\, G_{j_3j_4}(t,t)\, dt$ where $\alpha \in \mathbb{N}^d$, $|\alpha| = 4$,

Type 2: $II = II^{x,y,T} = \int_0^T \int_0^T \partial^\alpha V(g(t))\, \partial^\beta V(g(s))\, G_{j_1j_2}(t,s)\, G_{j_3j_4}(t,s)\, G_{j_5j_6}(t,s)\, dt\, ds$

where $|\alpha| = 3 = |\beta|$ and

Type 3: $III = III^{x,y,T} = \int_0^T \int_0^T \partial^\alpha V(g(t)\, \partial^\beta V(g(s))\, G_{j_1j_2}(t,t)\, G_{j_3j_4}(t,s)\, G_{j_5j_6}(s,s)\, ds\, dt$

where $|\alpha| = 3 = |\beta|$.

Here $g = g_T^{xy}$ and $G = G^{x,y,T}$ for $(x,y,T) \in B(x_0, \delta_0) \times B(b, \delta_0) \times [T_0, \infty)$.

See also Mizrahi [1], and Davies and Truman [1].

We want to show

(11.23) For each $\gamma \in \mathbb{N}_0^d$ $\frac{\partial^{|\gamma|}}{\partial y^\gamma} a_1^T(x,y) = 0(T)$

uniformly on $B(x_0, \delta_0) \times B(b, \delta_0) \times [T_0, \infty)$

and

(11.24) If $\bar{y} = \bar{y}(x,T) = b + 0(e^{-\omega_1 T})$

uniformly for $(x,T) \in B(x_0, \delta_0) \times (T_0, \infty)$

then $a_1^T(x,y)|_{y=\bar{y}} = (\text{constant})\, T + f(x) + 0(T\, e^{-\delta_1 T})$

uniformly on $B(x_0, \delta_0), \times [T_0, \infty)$ where $f \in C^\infty(B(x_0, \delta_0), \mathbb{R})$.

It suffices to show (11.23) and (11.24) are valid for a term of each of the 3 types.

To see that (11.23) holds for $I^{x,y,T}$, $II^{x,y,T}$, and $III^{x,y,T}$ we differentiate under the integral signs and use (2.24), (2.26), and (2.53) to show the integrand is 0(1) uniformly for $I^{x,y,T}$ and $0(e^{-\omega_1(t-s)})$ uniformly for the others.

To prove (11.24) for each type we use

(11.25) $G^{x\bar{y},T}(t,s) = G_1^{xb}(t,s) + G_2^T(s,t) + R^{x,T}(s,t)$

from lemma 5.2 and the estimates in (2.58) - (2.60) together with

(11.26) $g_T^{x\bar{y}}(t) = g^{x,b}(t) + 0(e^{-\omega_1 T}\, e^{-\omega_1(T-t)})$ by (2.42).

We will only do the calculations for type 3, say. The others are similiar.

Using (11.26) and $\delta_1 \le \frac{\omega_1}{2} < \omega_1$ we get:

$$(11.27)\quad III^{x\bar{y},T} = \int_0^T \partial^\alpha V(g^{xb}(t)\Big(G^{x\bar{y},T}(t,t)_{j_1j_2}\Big\{(\int_0^T + \int_0^T)\partial^\beta V(g^{xb}(s)\Big)$$

$$G^{x\bar{y},T}(t,s)_{j_3j_4}\left(G^{x\bar{y},T}(s,s)\right)_{j_5j_6} ds\Big\}\, dt + 0\left(e^{-\delta,T}\right)$$

$$= \int_0^T \partial^\alpha V(g^{xb}(t)\left(G^{x\bar{y},T}(t,t)\right)_{j_1j_2}\Big\{(\int_0^T + \int_0^T)\ \partial^\beta V\Big(g^{xb}(s)\Big)$$

$$\left(G_1^{xb}(t,s) + G_2^T(t,s) + R^{x,T}(t,s)\right)_{j_3j_4}$$

$$\left(G_1^{xb}(s,s) + G_2^T(s,s) + R^{x,T}(s,s)\right)_{j_5j_6} ds\Big\}\ dt + 0(e^{-\delta,T}) =$$

$$= \int_0^T \partial^\alpha V(g^{xb}(t)\Big(G^{x,T}(t,t)\Big)_{j_1j_2}\Big\{\sum_{i=1}^{7} I_i^{x,T}(t) + 0(\int_0^T |R^{x,T}(s,s)|ds)\Big\}dt + 0(e^{-\delta_1 T})$$

$$= \int_0^T \partial^\alpha V(g^{xb}(t))\Big(G^{x,T}(t,t)\Big)_{j_1j_2}\Big\{\sum_{i=1}^{7} I_i^{x\bar{y}T}(t)\Big\}\ dt + 0\ (Te^{-\delta_1 T})$$

uniformly for $(x,T) \in B(x_0, \delta_0) \times [T_0, \infty)$.

Where

$$I_1^{x,T}(t) = \int_0^T \partial^\beta V(g^{xb}(s))\Big(G_1^{x,b}\ (t,s)\Big)_{j_3j_4}\left(G_1^{x,b}(s,s)\right)_{j_5j_6} ds$$

$$I_2^{x,T}(t) = \int_0^t \partial^\beta V(g^{xb}(s))\Big(G_1^{x,b}(t,s)\Big)_{j_3j_4}\left(G_2^T\ (s,s)\right)_{j_5j_6} ds$$

$$I_3^{x,T}(t) = \int_t^T \partial^\beta V\Big(g^{xb}(s)\Big)\Big(G_1^{x,b}(t,s)\Big)_{j_3j_4}\left(G_2^T\ (s,s)\right)_{j_5j_6} ds$$

$$I_4^{x,T}(t) = \int_0^t \partial^\beta V(g^{xb}(s))\left(G_2^T(t,s)\right)_{j_3j_4}\left(G_1^{x,b}(s,s)\right)_{j_5j_6} ds$$

$$I_5^{x,T}(t) = \int_t^T \partial^\beta V(g^{xb}(s))\left(G_2^T(t,s)\right)_{j_3j_4}\left(G_1^{x,b}(s,s)\right)_{j_5j_6} ds$$

$$I_6^{x,T}(t) = \int_0^t \partial^\beta V\left(g^{xb}(s)\right)\left(G_2^T(t,s)\right)_{j_3j_4}\left(G_2^T(s,s)\right)_{j_5j_6} ds$$

and

$$I_7^{x,T}(t) = \int_t^T \partial^\beta V(g^{xb}(s))\left(G_2^T(t,s)\right)_{j_3j_4}\left(G_2^T(s,s)\right)_{j_5j_6} ds$$

Now we just have to look at each one of them.

By (2.58) and (2.14) we have: For $\alpha \in \mathbb{N}_0^{\,d}$

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha}\left(G_1^{x,b}(t,s) - e^{-\Omega|t-s|}(2\Omega)^{-1}\right) = 0\left(e^{-\delta_1 \min\{t,s\}}e^{-\omega_1|t-s|}\right) \tag{11.28}$$

and

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha}\left(g^{x,b}(t) - b\right) = 0\left(e^{-\omega_1 t}\right) \quad \text{uniformly.} \tag{11.29}$$

Hence

$$I_1^{x,T}(t) = \int_0^T \partial^\beta V(g^{xb}(s))\left(G_1^{x,b}(t,s)\right)_{j_3j_4}\left(G_1^{x,b}(s,s)\right)_{j_5j_6} ds$$

$$= \int_0^T \partial^\beta V(b)\,\frac{e^{-\omega_{j_3}|t-s|}}{4\,\omega_{j_3}\omega_{j_5}}\,\delta_{j_3j_4}\,\delta_{j_5j_6}\,ds +$$

$$+ \int_0^T \partial^\beta V(b) \left[\left(G_1^{x,b}(t,s)\right)_{j_3j_4} \left(G_1^{x,b}(s,s)\right)_{j_5j_6} - \frac{e^{-\omega_{j_3}|t-s|}}{4\,\omega_{j_3}\omega_{j_5}} \right] ds$$

$$+ \int_0^T (\partial^\beta V(g^{x,b}(s)) - \partial^\beta V(b)) \left(G_1^{x,b}(t,s)\right)_{j_3j_4} \left(G_1^{x,b}(s,s)\right)_{j_5j_6} ds$$

$$= \frac{\partial^\beta V(b)}{2\,\omega_{j_3}\omega_{j_5}} \left(1 - \tfrac{1}{2}\left(e^{-\omega_{j_3}t} + e^{-\omega_{j_3}(T-t)}\right)\right)$$

$+ f(x,t) + e^{x,T}(t)$ where

$$(11.30) \quad f(x, t) = \int_0^\infty \partial^\beta V(b) \left[\left(G_1^{x,b}(t,s)\right)_{j_3j_4} \left(G_1^{x,b}(s,s)\right)_{j_5j_6} - \frac{e^{-\omega_{j_3}|t-s|}}{4\,\omega_{j_3}\omega_{j_5}} \right] ds$$

$$+ \int_0^T (\partial^\beta V(g^{x,b}(s)) - \partial^\beta V(b)) \left(G_1^{x,b}(t,s)\right)_{j_3j_4} \left(G_1^{x,b}(s,s)\right)_{j_5j_6} ds$$

and by (1128) and (11.29):

$$(11.31) \quad e^{x,T}(t) = \int_T^\infty \partial^\beta V(b) \left[\left(G_1^{x,b}(t,s)\right)_{j_3j_4} \left(G_1^{x,b}(s,s)\right)_{j_5j_6} - \frac{e^{-\omega_{j_3}|t-s|}}{4\,\omega_{j_3}\omega_{j_5}} \right] ds$$

$$+ \int_T^\infty (\partial^\beta V(g^{x,b}(s)) - \partial^\beta V(b)) \left(G_1^{x,b}(t,s)\right)_{j_3j_4} \left(G_1^{x,b}(s,s)\right)_{j_5j_6} ds =$$

$$= \int_T^\infty 0(e^{-\delta_1 s}\, e^{-\omega_1|t-s|} + e^{-\delta_1 \min\{s,t\}}\, e^{-\omega_1|t-s|})\, ds$$

$$+ \int_T^\infty 0(e^{-\omega_1 s})ds = 0(e^{-\delta_1 T})$$

uniformly.

Differentiating (11.30) under the integral sign and using (11.28) and (11.29) gives:

For each $\gamma \in \mathbb{N}_0^d$

(11.32) $\frac{\partial^{|\gamma|}}{\partial x^\gamma} f(x,t) = 0(e^{-\delta_1 t})$ uniformly for $(x,t) \in B(x_0, \delta_0) \times [0, \infty)$.

Now we state what one gets for the others $I_i^{x,T}$'s using (11.28), (11.29) and

$$G_2^T(t,s) = -\, e^{-\Omega(2T-s-t)}\, (2\Omega)^{-1}.$$

If $\qquad C = \frac{\partial^\beta V(b)}{2\, \omega_{j_3}\, \omega_{j_5}}\, \delta_{j_3 j_4}\, \delta_{j_5 j_6}$

then

$$I_2^{xT}(t) = \frac{C}{\omega_{j_3} + 2\omega_{j_5}}\, e^{-2\omega_{j_5}(T-t)} + 0(e^{-\omega_1(T-t)}\, e^{-\delta_1 T})$$

$$I_3^{x,T}(t) = C \begin{bmatrix} e^{-2\omega_{j_5}(T-t)} (T-t) \text{ if } \omega_{j_3} = 2\omega_{j_5} \\ \dfrac{e^{-\omega_{j_5}(T-t)} - e^{-2\omega_{j_5}(T-t)}}{(2\omega_{j_5} - \omega_{j_3})} \text{ if } \omega_{j_3} \neq 2\omega_{j_5} \end{bmatrix}$$

$$+ 0(e^{-\omega_1(T-t)} e^{-\delta_1 T})$$

$$I_4^{x,T}(t) + I_5^{x,T}(t) = \frac{C}{2\omega_{j_3}} + 0(e^{-\delta_1 T})$$

and

$$I_6^{x,T}(t) + I_7^{x,T}(t) = \frac{C}{2\omega_{j_3}} + 0(e^{-\omega_1 T})$$

Adding up gives:

$$\sum_{i=1}^{7} I_i^{x,T}(t) = d_1 + d_2 e^{-\omega_{j_3} t} + d_3 e^{-\omega_{j_3}(T-t)} \tag{11.33}$$

$$+ (d_4 + d_5 (T-t)) e^{-\omega_{j_3}(T-t)} + f(x,t) + 0 (e^{-\delta_1 T})$$

for some constants $d_1 \dots d_5$ with $d_1 = \frac{5}{8} \dfrac{\partial^\beta V(b)}{2\, \omega_{j_3}^2 \omega_{j_5}}$ and f satisfying (11.32).

Now we put (11.33) into (11.27) and we get:

$$III^{x,\bar{y},T} = \int_0^T (\partial^\alpha V(b) + \partial^\alpha V(g^{xb}(t)) - \partial^\alpha V(b))(G_1^{x,b}(t,t) + G_2^T(t,t) +$$

$$+ \mathbb{R}^{x,T}(t,t))_{j_1 j_2} (d_1 + d_2 e^{-\omega_{j_3} t} + d_3 e^{-\omega_{j_3}(T-t)} + (d_4 + d_5(T-t)$$

$$e^{-\omega_{j_5}(T-t)} + f(x,t) + 0(e^{-\delta_1 T})) dt + 0(e^{-\delta_1 T})$$

where $(G_1^{x,b}(t,t))_{j_1 j_2} = \dfrac{\delta_{j_1 j_2}}{2\omega_{j_1}} + 0(e^{-\delta_1 T}) + 0(e^{-\delta_1 T})$

$$(G_2^T(t,t))_{j_1 j_2} = \frac{e^{-2\omega_{j_1}(T-t)}}{2\omega_{j_1}} \delta_{j_1 j_2} \text{ and } |R^{x,T}(t,t)| = 0(e^{-\delta_1 T})$$

by (11.28), (11.29) and (2.60).

Hence

$$III^{x,\bar{y},T} = \left(\frac{5}{16} \frac{\partial^\alpha V(b)\, \partial V^\beta(b)}{\omega_{j_1} \omega_{j_3}^2 \omega_{j_5}} \delta_{j_1 j_2} \delta_{j_3 j_4} \delta_{j_5 j_6}\right) T$$

$$+ \text{constant} + \int_0^T (\partial^\alpha V(g^{x,b}(t)) - \partial^\alpha V(b))\, d_1 +$$

$$\partial^\alpha (g^{x,b}(t))\, (f(x,t) + d_2 e^{-\omega_{j_3} t})\, dt + 0(Te^{-\delta_1 T})$$

which implies (11.24) holds by differentiating under the integral sign and apply (11.29) and (11.32).

§11.4. A sketch of a proof a lemma 7.4 and 7.5.

Recall that in the definition of a_{ij} in (6.48) $L_1(\cdot)$ is a differential operator of order 2.

Hence (4.23) follows from the results above and by showing $\frac{\partial^{|\alpha|+|\beta|}}{\partial y^{\alpha} \partial x^{\beta}} b_0^T(x,y) = 0(1)\, b_0^T(x,y)$ which can be done by differentiating in (11.5) under the integral sign.

To prove (4.24) one uses $\bar{W} = g^{x,b}(T_0) + 0(Te^{-\omega_1 T})$ uniformly by (2.42).

Hence, for instance, $\frac{\partial^{|\beta|} b_0^{T_0}(x, W)}{\partial W^{\beta}} = \frac{\partial^{|\beta|}}{\partial W^{\beta}} b_0^T(x,g^{x,b}(T_0)) + 0(e^{-\omega_1 T})$

since all W – derivatives of order $|\beta| + 1$ are uniformly bounded. (4.25) is proven in the same way as (6.23).

12. The Proofs of Sublemma 4 and Proposition 1.

§12.1. Sublemma 4.4

§12.2. Sublemma 4.1.

§12.3. Sublemmas 4.2 and 4.3.

§12.4. The upperbound in Proposition 1.

§12.5. The lowerbound in Proposition 2.

§12.1. Proof of sublemma 4.4.

We need to prove: If $0 \le \alpha < \beta \le 1$ and $c \in \{a,b\}$ then

$$(12.1) \qquad \inf_{\substack{|x-c|=\alpha c_0 \\ |x-c|=\beta c_0 \\ T>0}} A(x,y,T) \ge \frac{c_1 c_0^2}{4}(\beta^2 - \alpha^2)$$

where c_1 and c_0 are such that (10.1)-(10.3) hold.

Recall

$$A(x,y,T) = \inf\{\int_0^T (\tfrac{1}{2}\dot\gamma^2 + V(\gamma))dt : \gamma(0) = x,\ \gamma(T) = y\}$$

and note we only need to consider paths γ in (12.1) with $\alpha c_0 \le |\gamma(t) - c| \le \beta c_0$ for $t \in [0,T]$ since $\gamma|_{[t_1,t_2]}$, where $t_1 := \max\{t \in [0,T] : |\gamma(t) - c| = \alpha c_0\}$, and $t_2 := \min\{t \in [t_1, T] : |\gamma(t) - c| = \beta c_0\}$, has less action than γ.

From (10.1) and Taylor's formula we get $V(x) \ge \frac{c_1^2}{2}(x - c)^2$ if $|x - c| \le c_0$. For this harmonic oscillator we solve the Euler-Lagrange equations $\ddot\gamma = c_1^2(\gamma - c)$. The minimal action paths are given by: $\gamma_T^{xy}(t) := c + (x - c)\dfrac{\sinh c_1(T-t)}{\sinh c_1 T} + (y - c)\dfrac{\sinh c_1 t}{\sinh c_1 T}$. Hence the R.H.S. if (12.1) is greater than

$$\inf_{\substack{|x-c|=\alpha c_0\\|x-c|=\beta c_0\\T>0}}\left\{\frac{c_1}{2}\left(\frac{[(x-c)^2+(y-c)^2]\cosh c_1T-2\langle(x-c),(y-c)\rangle}{\sinh c_1T}\right)\right\}\geq$$

$$\geq \inf_{T>0}\left\{\frac{c_1c_0^2\alpha^2}{2}\left(\frac{(1+\frac{\beta^2}{\alpha^2})\cosh c_1T-2\frac{\beta}{\alpha}}{\sinh c_1T}\right)\right\}$$

which attains its infimum value when $\cosh c_1T = \frac{1}{2}(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}) > 1$ and leads to (12.1).

§12.2. The Proof of Sublemma 4.1.

We recall the statement: If $|x - c| < \dfrac{c_0}{(\frac{1}{2}+\frac{c_2}{c_1})}$ then

(12.2) $|x - c|\, e^{-c_2t} \leq |g^{x,c}(t) - c| \leq |x - c|\, e^{-c_1t}$

and

(12.3) $\dfrac{c_1^2}{2c_2}|x - c|^2 \leq \rho(x,c) \leq \dfrac{c_2^2}{2c_1}|x - c|^2$.

First we show that

(12.4) $|g^{x,c}(t) - c| \leq c_0$ if $t \geq 0$ and x as above and then we use (10.2),

$c_1|x - c|^2 \leq \langle\nabla V(x),(x-c)\rangle \leq c_2|x - c|^2$ for $|x - c| \leq c_0$,

to prove (12.2) and (12.3).

Taking

$$\gamma(t) = (x - c)\, e^{-c_2t} + c \text{ for } t \geq 0$$

as a trial path for

$$\rho(x,c) = \inf\{\int_0^\infty (\tfrac{1}{2}\dot\gamma^2 + V(\gamma))dt \mid \gamma(0) = x,\ \gamma(\infty) = c\}.$$

We get, by (10.1),

$$(12.5)\quad \int_0^\infty (\tfrac{1}{2}\dot\gamma^2 + V(\gamma))dt \le \int_0^\infty (\tfrac{1}{2}\dot\gamma^2 + \tfrac{c_2^2}{2}(\gamma-b)^2)dt =$$

$$= \frac{c_2}{2}|x-c|^2 < \frac{c_2 c_0^2}{2(\frac{1}{2}+\frac{c_2}{c_1})}.$$

While going from $\{x : |x-c| = \frac{c_0}{(\frac{1}{2}+\frac{c_2}{c_1})^{\frac{1}{2}}}\}$ to $\{x : |x-c| = c_0\}$ and then to c makes

$$\rho(x,c) = \int_0^\infty (\tfrac{1}{2}(\dot g^{x,c}(t)^2 + V(g^{x,c}(t)))dt \ge \frac{c_1c_0^2}{4}\left(1 - \frac{1}{(\frac{1}{2}+\frac{c_2}{c_1})}\right) + \frac{c_1c_0^2}{4} = \frac{c_2c_0^2}{2(\frac{1}{2}+\frac{c_2}{c_1})}$$

and so by (12.5) we get (12.4).

Now

$$(12.6)\quad \tfrac{1}{2}(\dot g^{x,c}(t))^2 - V(g^{x,c}(t)) =$$

$$= \lim_{T\to\infty} (\tfrac{1}{2}(\dot g^{x,c}(t))^2 - V(g^{x,c}(t)) - \tfrac{1}{2}(\dot g^{x,c}(T))^2 - V(g^{x,c}(T)))$$

$$= \lim_{T\to\infty} \int_t^T \langle(\ddot g^{x,c}(t) - \nabla V(g^{x,c}(t))),\ \dot g^{x,c}(t)\rangle dt = 0$$

and so when we put $f(t) = |g^{x,c}(t) - c|^2$ we get

$$\ddot{f}(t) = 2(\dot{g}^{x,c})^2 + 2\langle \ddot{g}^{x,c},(g^{x,c}-c)\rangle = 4V(g^{x,c}) + 2\langle \nabla V(g^{x,c}),(g^{x,c}-c)\rangle$$

By (10.1) and (10.2)

(12.7) $4c_1^2\, f(t) \leq \ddot{f}(t) \leq 4c_2^2\, f(t).$

For $n \geq 1$ put $f_n(t) = f(t)|_{[0,n]}$,

$$h_n(t) = f(0)\,\frac{\sinh 2c_1(n-t)}{\sinh 2c_1 n} + f(n)\,\frac{\sinh 2c_1 t}{\sinh 2c_1 n}$$

and

$$k_n(t) = f(0)\,\frac{\sinh 2c_2(n-t)}{\sinh 2c_2 n} + f(n)\,\frac{\sinh 2c_2 t}{\sinh 2c_2 n}$$

Here h_n and k_n solve the comparison equations

$$\ddot{h}_n(t) = 4c_1^2\, h_n(t),\ h_n(0) = f_n(0),\ h_n(n) = f_n(n)$$

and

$$\ddot{k}_n(t) = 4c_2^2 k_n(t),\ k_n(0) = f_n(0),\ k_n(n) = f_n(n).$$

By Protter-Weinberger [1]

$$k(t) \leq f_n(t) \leq h_n(t) \text{ if } t \in [0,n],$$

hence

$$|x-c|^2\, e^{-2c_2 t} = f(0)\, e^{-2c_2 t} = \lim_{n\to\infty} k_n(t)$$

$$\leq \lim_n f_n(t) = f(t) \leq \lim_{n\to\infty} h_n(t) = |x-c|^2\, e^{-2c_1 t}$$

which gives (12.2).

Using $\frac{1}{2}\, c_1^2|x-c|^2 \leq V(x) \leq \frac{1}{2}\, c_2^2|x-c|^2$ if $|x-c| \leq c_0$, (12.6) and (12.2) we get

$$\rho(x,c) = 2\int_0^{\infty} V(g^{x,c}(t))\,dt \le \int_0^{\infty} c_2^2|x-c|^2\, e^{-2c_1 t}\,dt \le \frac{c_2^2|x-c|^2}{2c_1}$$

and

$$\rho(x,c) \ge \int_0^{\infty} c_1^2|x-c|^2\, e^{-2c_2 t}\,dt = \frac{c_1^2|x-c|^2}{2c_2}$$

which finishes the proof.

§12.3. <u>The Proof of Sublemma 4.2 and 4.3</u>.

Put $\quad f_T(t) := |g_T^{x,y}(t + t_1) - b|^2$ for $0 \le t \le t_2 - t_1$.

Then

$$\ddot{f}_T(t) = 2(\dot{g}_T^{x,y})^2 + 2\langle \ddot{g}_T^{x,y}(t),(\ddot{g}_T^{x,y}(t)-c)\rangle$$

$$\ge 2\langle \nabla V(g_T^{x,y}(t)),(g_T^{x,y}(t)-c)\rangle$$

$$\ge 2c_1^2|g_T^{x,y}(t) - c|^2 \ge c_1^2\, f_T(t)$$

where used the assumption $|g_T^{x,y}(t) - c| \le c_0$ if $t_1 \le t \le t_2$, (10.2) and the Euler-Lagrange equations. Therefore (Protter and Weinberger [1])

$$f_T(t) \le f_T(0)\,\frac{\sinh c_1(t_2-t_1-t)}{\sinh c_1(t_2-t_1)} + f_T(t_2 - t_1)\,\frac{\sinh c_1 t}{\sinh c_1(t_2-t_1)}$$

if $t \in [0,t_2-t_1]$ and so $|g_T^{x,y}(t) - c|^2 = f_T(t)$

$$\le |g_T^{x,y}(t_1) - c|^2\,\frac{\sinh c_1(t_2-t)}{\sinh c_1(t_2-t_1)} + |g_T^{x,y}(t_2) - c|^2\,\frac{\sinh c_1(t-t_1)}{\sinh c_1(t_2-t_1)} \le$$

$$\leq \max_{i\in\{1,2\}}\left\{|g_T^{x,y}(t_i)-c|^2\right\}\left(\frac{\sinh c_1(t_2-t)+\sinh c_1(t-t_1)}{\sinh c_1(t_2-t_1)}\right)$$

$$\leq \max_{i\in\{1,2\}}\left\{|g_T^{x,y}(t_i)-c|^2\right\} \text{ for all } t\in[t_1\,,t_2]$$

and sublemma (4.2) follows.

To prove sublemma (4.3) take $\gamma_T(t) = (x-c)\,\dfrac{\sinh c_2(T-t)}{\sinh c_2 T} + c$ as a trial path for $A(x,c,T)$ and use $V(x) \leq \frac{c_2^2}{2}|x-c|^2$ if $|x-c| \leq c_0$.

§12.4. <u>The Upperbound in Proposition 1</u>.

We prove the upperbound in 3 steps.

First we consider x and $y = b$ with $\rho(x,b) < \rho(x,a) + \rho(a,b)$ then $x = a$ and $y = b$ and finally the general case.

<u>Step 1</u>. If $\rho(x_0\,,b) < \rho(x_0\,,a) + \rho(a,b)$ then there exist $\delta(x_0) > 0$ such that $A(x,b,T) \leq \rho(x,b) + 0(e^{-2c_1T})$ uniformly for $x \in B(x_0\,,\delta)$ as $T \to \infty$.

To prove step 1, pick $\delta > 0$ such that

(12.8) $\quad \rho(x,b) < \rho(x,a) + \rho(a,b)$ for all $x \in \overline{B(x_0,\delta)}$.

<u>Claim</u>. There exists $\delta_2 > 0$ s.t.

(12.9) $\quad \left(\bigcup_{x\in B(x_0,\delta)} \{g^{x,b}(t),\, t\in[0,\infty]\}\right) \cap B(a,\delta_2) = \emptyset$.

Otherwise, we have $x_n \in \overline{B(x_0,\delta)}$, $x_n \to \bar{x}$ and $y_n := g^{x_n,b}(t_n) \to a$ as $n \to \infty$.

This implies (proof below)

(12.10) $\rho(\bar{x},b) \geq \rho(\bar{x},a) + \rho(a,b)$

which contradicts (12.8) and hence proves (12.9) by contradiction.

We write $g_n(t) := g^{x_n,b}(t)$ and to prove (12.10) we notice

(12.11) $\underline{\lim}_{n\to\infty} t_n > 0$ since

$$|g_n(t_n) - x_n|^2 = |g_n(t_n) - g_n(0)|^2 \leq t_n \int_0^{t_n} \dot{g}_n^2(u)\, du$$

$$\leq (\sup_n\{\rho(x_n,b)\})\, t_n \ , \quad x_n \to \bar{x} \neq a \text{ and } g_n(t_n) \to a \text{ as } n \to \infty .$$

By lemma 4, the triangle inequality and (12.11)

(12.12) $$\int_0^{T_n} (\tfrac{1}{2}\dot{g}_n^2 + V(g_n))dt \geq A(x_n , y_n , t_n)$$

$$\geq A(x_n , a, 2t_n))dt - A(y_n , a, t_n)$$

$$\geq \rho(x_n , a) - \frac{c_2|y_n - a|^2}{2} \frac{\cosh c_2 t_n}{\sinh c_2\, t_n} \to \rho(\bar{x},a)$$

as $n \to \infty$, since $x_n \to \bar{x}$ and $y_n \to a$ as $n \to \infty$. Now pick $s_n \to \infty$ as $n \to \infty$ such that $s_n - t_n \to \infty$ and with $z_n := g_n(s_n) \to b$ as $n \to \infty$. Then

$$(12.13)\quad \int_{t_n}^{\infty}(\tfrac{1}{2}\dot{g}_n^2 + V(g_n))dt \geq \int_{t_n}^{s_n}(\tfrac{1}{2}\dot{g}_n^2 + V(g_n))dt$$

$$\geq A(y_n , z_n , s_n - t_n) \geq A(a,b,3(s_n - t_n))$$

$$- A(a , y_n , s_n - t_n) - A(z_n , b , s_n - t_n)$$

$$\geq \rho(a,b) - \frac{c_2|a-y_n|^2}{2}\frac{\cosh c_2(s_n-t_n)}{\sinh c_2(s_n-t_n)} - \frac{c_2|z_n-b|^2}{2}\frac{\cosh c_2(s_n-t_n)}{\sinh c_2(s_n-t_n)}$$

$$\to \rho(a,b) \text{ as } n \to \infty .$$

Now (12.12) and (12.13) imply (12.10) and by above we have (12.9). Pick $\delta_2 > 0$ with $\delta_2 \leq \frac{c_0}{(\frac{1}{2}+\frac{c_2}{c_1})^{\frac{1}{2}}}$ so that (12.9) holds and $\delta_3 > 0$ such that $V(x) \leq \delta_3$ implies $x \in B(a,\delta_2) \cup B(b,\delta_2)$. If $x \in \overline{B(x_0,\delta_0)}$ then by above $V(g^{x,b}(t_0)) \leq \delta_3$ implies $|g^{x,b}(t_0) - b| \leq \delta_2 \leq \frac{c_0}{(\frac{1}{2}+\frac{c_2}{c_1})^{\frac{1}{2}}}$ and by sublemma 4.1

$$|g^{x,b}(t) - b| \leq \frac{c_0}{(\frac{1}{2}+\frac{c_2}{c_1})^{\frac{1}{2}}} e^{-c_1(t-t_0)} \text{ if } t \geq t_0 .$$

Since

$$\delta_3|\{t : V(g^{x,b}(t)) \geq \delta_3\}| \leq \int_0^{\infty}(\tfrac{1}{2}(\dot{g}^{x,b}(t))^2 + V(g^{x,b}))\, dt \leq \max_{x\in\overline{B(x_0,\delta)}}\rho(x,b)$$

we get

$$(12.2)\qquad |g^{x,b}(t)-b| \leq \frac{c_0}{(\frac{1}{2}+\frac{c_2}{c_1})^{\frac{1}{2}}} e^{-c_1(t-T_0)} \text{ for all } x \in \overline{B(x_0,\delta)}$$

$$\text{if } t \geq T_0 := \frac{\max_{\overline{b(x_0,\delta)}}(\rho(x,b))}{\delta_3} .$$

Now set, for $T \geq T_0 + 1$,

$$\gamma_T^{x,b}(t) := \begin{bmatrix} g^{x,b}(t) \text{ if } t \in [0,T-1] \\ (b-g^{x,b}(T-1))(t-T+1) + g^{x,b}(T-1) \text{ if } t \in (T-1,T] \end{bmatrix}$$

Then $$A(x,b,T) \leq \int_0^T (\tfrac{1}{2}\dot\gamma_T^2 + V(\gamma_T))\, dt =$$

$$= \rho(x,b) - \int_{T-1}^{\infty} (\tfrac{1}{2}(\dot g^{x,b}(t))^2 + V(g^{x,b}(t)))\, dt + \int_{T-1}^{T} (\tfrac{1}{2}\dot\gamma_T^2 + V(\gamma_T))dt$$

$$\leq \rho(x,b) + \int_{T-1}^{T} (\tfrac{1}{2}|g^{x,b}(T-1)-b|^2 + \tfrac{c_2^2}{2}|g^{x,b}(T-T)^2)dt$$

$$\leq \rho(x,b) + (\frac{1+c_2^2}{2}) \frac{c_0^2}{(\frac{1}{2}+\frac{c_2}{c_1})} e^{-2c_1(T-1-T_0)}$$

for all $T \geq T_0 + 1$ and $x \in \overline{B(x_0,\delta_1)}$ which completes the proof of step 1.

<u>Step</u> <u>2</u>. If K_1 is compact then (and similar when b is replaced by a).

(12.13) $\quad A(x,b,T) \leq \rho(x,b) + 0(e^{-\frac{2c_1T}{3}})$ uniformly for $x \in K_1$ as $T \to \infty$.

<u>Proof</u>. Given $x \in K_1$. If $\rho(x,b) < \rho(x,a) + \rho(a,b)$ then (12.13) holds near c by step 1.

If $\rho(x,b) = \rho(x,a) + \rho(a,b)$ then for some x_0 with $\rho(x_0, a) = \frac{1}{2}\rho(a,b) = \rho(x_0, b)$,

$\rho(x,b) = \rho(x,a) + \rho(a,x_0) + \rho(x_0, b)$ and (12.13) holds near x by applying step 1 to

$$A(x,b,T) \le A(x,a,\tfrac{T}{3}) + A(a,x_0,\tfrac{T}{3}) + A(x_0,b,\tfrac{T}{3}) .$$

Uniformity on compacts follows also.

Step 3. Given a compact $K_1 \times K_2 \subseteq \mathbf{R}^d \times \mathbf{R}^d$. By the triangle inequality and step 2 we have: Uniformly for $x \in K_1$ and $y \in K_2$

$$A(x,y,T) \le \min_{c\in\{a,b\}} \{A(x,c,\tfrac{T}{2}) + A(c,y,\tfrac{T}{2})\}$$

$$= \min_{c\in\{a,b\}} \{\rho\{x,c) + 0(e^{-\frac{c_1T}{3}}) + \rho(c,y) + 0(e^{-\frac{c_1T}{3}})\}$$

$$= \min_{c\in\{a,b\}} \{\rho\{x,c) + \rho(c,y)\} + 0(e^{-\frac{c_1T}{3}})\}$$

as $T \to \infty$.

§12.5. Proof of the Lower Bound in Proposition 1.

Let $K_1 \times K_2$ be a compact subset of $\mathbb{R}^d \times \mathbb{R}^d$. Put

$$E = \sup_{(x,y,T)\in K_1\times K_2\times[1,\infty)} \{A(x,y,T)\} \tag{12.14}$$

and pick $\delta_4 > 0$ such that

(12.15) $V(x) \le \delta_4$ implies $x \in B(a,\frac{c_0}{2}) \cup B(b,\frac{c_0}{2})$, where c_0 is as in (10.1).

We reduce to proving

(12.16) There is a T_0 such that for each $T \in [T_0, \infty)$ there exists a time interval $[t_T, t_T + S_T] \subseteq [0,T]$

with $S_T \geq \dfrac{T - \frac{E}{\delta_4}}{\frac{16E}{3c_1c_0^2+1}} =: c_3 T - c_4$ and so that $g_T^{x,y}(t) \in B(a, \frac{c_0}{2}) \cup B(b, \frac{c_0}{2})$

for $(x,y,T,t) \in K_1 \times K_2 \times [T_0, \infty] \times [t_T, t_T + S_T]$.

Assuming (12.16) we now prove the lowerbound.

By sublemma 4.2

(12.17) $$|g_T^{x,y}(t) - c|^2 \leq |g_T^{x,y}(t_T) - c|^2 \frac{\sinh c_1(t_T + S_T - t)}{\sinh c_1 S_T}$$

$$+ \; |g_T^{x,y}(t_T + S_T) - c|^2 \frac{\sinh c_1(t - t_T - t)}{\sinh c_1 S_T} \leq$$

$$\leq \frac{c_0^2}{4} \left(\frac{\sinh c_1(t_T + S_T - t) + \sinh c_1(t - t_T)}{\sinh c_1 S_T}\right) \leq \frac{c_0^2}{4}$$

if $t \in [t_T, t_T + S_T]$ where $c \in \{a,b\}$.

Put $R_T = t_T + \frac{1}{2} S_T$ then by (12.16)

(12.18) $$R_T \, , T - R_T \geq \tfrac{1}{2}(c_3 T - c_4)$$

and by (12.15)

$$(12.19) \qquad \left.\begin{array}{l} |g_T^{x,y}(R_T)-c|^2 \\ |g_T^{x,y}(T-R_T)-c|^2 \end{array}\right\} = 0(e^{-\frac{c_1 S_T}{2}}) = 0(e^{-\frac{c_1 c_3}{4}T})$$

uniformly for those (x,y,T)'s.

Now using the triangle inequality $A(x,y,T) = A(x,g_T^{x,y}(t),t) + A(g_T^{x,y}(t),y,T-t)$

if $t \in (0,T)$, $\rho(x,y) = \inf_{T>0} \{A[x,y,T]\}$, sublemma 4.3, (12.18) and (12.19) gives

$$\begin{aligned} A(x,y,T) &= A(x,g_T^{x,y}(R_T),R_T) + A(g_T^{x,y},(R_T),y,T-R_T) \\ &\geq A(x,c,T) - A(g_T^{x,y}(R_T),c,T-R_T) + \\ &+ A(c,y,T) - A(g_T^{x,y}(R_T),c,R_T) \geq \\ &\geq \rho(x,c) + \rho(c,y) - \frac{c_2}{2}|g_T^{x,y}(R_T)-c|^2 \frac{\cosh(c_2 R_T)}{\sinh(c_2 T)} \\ &- \frac{c_2}{2}|g_T^{x,y}(T-R_T)-c|^2 \frac{\cosh(c_2(T-R_T))}{\sinh(c_2(T-R_T))} \\ &= \rho(x,c) + \rho(c,y) + 0(e^{-\frac{c_1 c_3 T}{4}}) \end{aligned}$$

uniformly for $(x,y,T) \in K_1 \times K_2 \times [T_0,\infty)$. This gives the lowerbound in Proposition 1.

We prove (12.16) by estimating the total time $g_T^{x,y}(t)$ is inside $B(a,\frac{c_0}{2}) \cup \overline{B(a,\frac{c_0}{2})}$

from below and the number of times $g_T^{x,y}(t)$ is there from above.

With E and δ_4 as in (12.14) and (12.15)

$$|\{t \in [0,T] : g_T^{x,y} \in \overline{B(a, \tfrac{c_0}{2})} \cup \overline{B(a, \tfrac{c_0}{2})}\}| \geq \tag{12.20}$$

$$|\{t \in [0,T] : V(g_T^{x,y}(t)) \leq \delta_4\}| =$$

$$T - |\{t \in [0,T] : V(g_T^{x,y}(t)) > \delta_4\}| \geq T - \frac{E}{\delta_4}$$

if $(x,y,T) \in K_1 \times K_2 \times [1,\infty]$. The last inequality is valid since

$$\delta_4 |\{t \in [0,T] : V(g_T^{x,y}(t)) \leq \delta_4\}| \leq \int_0^T V(g_T^{x,y}(t)\, dt$$

$$\leq \sup_{(x,y,T) \in K_1 \times K_2 \times [T,\infty]} \{A(x,y,T)\} = E.$$

If $c \in \{a,b\}$ then by sublemma 4.2 $g_T^{x,y}$ never leaves $\{x : |x-c| \leq \frac{c_0}{2}\}$ and comes back to that set without leaving $\{x : |x-c| \leq c_0\}$. Doing so contributes at least $c_1 \frac{c_0^2}{4} (1 - \frac{1}{4}) = \frac{3}{16} c_1 c_0^2$ to A(x,y,T) by sublemma (4.4). Hence, the number of times $g_T^{x,y}$ is inside $B(a,\frac{c_0}{2}) \cup B(b, \frac{c_0}{2}) \leq$ (number of times $g_T^{x,y}$ can go from $\{x : |x-c| = \frac{c_0}{2}\}$ to $\{x : |x-c| = c_0\}$ for $c \in \{a,b\}$) $+ 1 \leq \frac{16E}{3c_1 c_1^2} + 1$.

Now write

$$\{t \in [0,T] : g_T^{x,y}(t) \in \overline{B(a,\frac{c_0}{2})} \cup \overline{B(a,\frac{c_0}{2})}\} = \bigcup_{i=1}^{N} [t_i \, , t_i + s_i]$$

where $$s_i > 0 \text{ and } N = N(x,y,T) \leq \frac{16E}{3c_1c_0^2} + 1$$

then by (12.20)

$$T - \frac{E}{\delta_4} \leq \sum_{i-1}^{N} s_i \leq N(x,y,T) \max_i \{s_i(x,y,T)\}$$

which implies (12.16).

13. Some Basic Properties of Agmon Geodesics and Minimal Action Paths.

§13.1. Infimum is attained and so on (sublemma 5).

§13.2. Proof of lemma 1.

§13.3. On the uniqueness and nondegeneracy of $g^{x,b}$ (sublemma 6).

§13.4. There is a unique and nondegenerate minimum of the action for large T's (sublemma 7).

13. Some Basic Properties of Agmon Geodesics and Minimal Action Paths.

§13.1. Infimum is attained and so on (sublemma 5).

If $T > 0$ let $J = [0,T]$, $[-T,T]$, $[0,\infty)$ or $(-\infty,\infty)$ and for $\gamma \in C(J)$ define

$$(13.1) \quad \mathcal{A}(\gamma) := \int_J (\tfrac{1}{2}\dot{\gamma}^2 + V(\gamma))\, dt$$

which we will only write when $\mathcal{A}(\gamma) < \infty$.

We start by showing infimum in

$$(13.2) \quad \rho(x_0, b) = \inf\{\int_0^\infty (\tfrac{1}{2}\dot{\gamma}^2 + V(\gamma)dt \mid \gamma(0) = x_0, \gamma(\infty) = b\}$$

is attained when

$$(13.3) \quad \rho(x_0, b) < \rho(x_0, a) + \rho(a,b).$$

Sublemma 5.

1. Assume (13.3). If $\gamma_n \in C([0,T_n])$ for $T_n \in (0,\infty]$ where $T_n \to \infty$, $\gamma_n(0) \to x_0$, $\gamma_n(T_n) \to b$, or $\gamma_n(t) \to b$ as $t \to \infty$ if $T_n = \infty$, and $\mathcal{A}(\gamma_n) \to \rho(x_0, b)$ as $n \to \infty$ then there exists $g^{x_0,b} \in C^\infty([0,\infty))$ minimizing (13.2) and a subsequence γ_{n_k} such that $\|\gamma_{n_k} - g^{x_0,b}\|_{L^\infty[0,T_{n_k}]} \to 0$ as $k \to \infty$. Hence by continuity of $\rho(x,b)$: If $g^{x_0,b}$ is unique and for x near x_0, $g^{x,b}$ is some minimizing path for $\rho(x,b)$ then

$$\|g^{x,b} - g^{x_0,b}\|_{L^\infty([0,\infty))} \to 0 \text{ as } x \to x_0 .$$

2. Assume

$$\rho(x_0\,, b) + \rho(b,y_0) < \rho(x_0\,, a) + \rho(a,y_0). \tag{13.4}$$

If $\gamma_n \in C([0,T_n])$ where $T_n < \infty$ $T_n \to \infty$, $\gamma_n(0) \to x_0$, $\gamma_n(T_n) \to y_0$ and $\mathcal{A}(\gamma_n) \to \rho(x_0\,, b) + \rho(b,y_0)$ as $n \to \infty$ then there are $g^{x_0,b}$ and $g^{y_0,b}$ in $C^\infty([0,\infty))$ minimizing $\rho(x_0\,, b)$ and $\rho(y_0\,, b)$ in (13.2) and a subsequence γ_{n_k} such that $\|g^{x_0,b} - \gamma_{n_k}\|_{L^\infty[0,\frac{T_{n_k}}{2}]} \to 0$ and $\|g^{y_0,b} - \gamma_{n_k}(T_{n_k} - \cdot\,)\|_{L^\infty[0,\frac{T_{n_k}}{2}]} \to 0$ as $k \to \infty$.

Hence by Proposition 1: If $g^{x_0,b}$ and $g^{x_0,b}$ are unique and for (x,y,T) near $(x_0\,, y_0\,,\infty)$ $g_T^{x,y}$ is a minimal action path then

$$\| g_T^{x,y} - g^{x_0,b} \|_{L^\infty[0,\frac{T}{2}]} \to 0$$

and

$$\|g_T^{x,y}(T - \cdot\,) - g^{y_0,b}\|_{L^\infty[0,\frac{T}{2}]} \to 0$$

as

$$(x,y,T) \to (x_0\,, y_0\,, \infty).$$

3. Let $g^{x,b}$ be a minimizing path for (13.2), where x satisfies (13.3), and $x_0 = g^{x,b}(t_0)$ for some $t_0 > 0$. Then there is a unique $g^{x_0,b}(t)$ given by $g^{x_0,b}(t) = g^{x,b}(t + t_0)|_{[0,\infty)}$. Moreover its second variation is positive definite.

Remark. The last part of sublemma 5.3 implies (see (2.9)): There is no nonzero solution of the Jacobi's equations.

(13.5) $\ddot{f}(t) = V''(g^{x,b}(t))\, f(t)$ on $[0,\infty)$ vanishing twice on $[0,\infty)$.

Since every solution of (13.5) that vanishes at ∞, decays exponentially, this can by (13.8) below, be strengthened to: No nonzero solution of (13.5) vanishes twice on $[0,\infty]$. So we remind on Jacobi's condition (see Hestenes [1], p. 124) that says: An endpoint and a point between endpoints of a geodesic, without corners, are not conjugate.

Proof of Sublemma 5.1.

We can assume γ_n is defined on $[0,\infty)$, by extending it to be $(\gamma_n(T_n) - b)\, e^{-(t-T_n)} + b$ on (T_n, ∞), if $T_n < \infty$.

Going to a subsequence we may assume $\dot{\gamma}_n \to \dot{\gamma}$ weakly in $L^2(0,\infty)$, since a bounded set in a reflexive Banach space is weakly sequentially compact (Berger [1], p. 31) and then $\int_0^\infty \dot{\gamma}^2\, dt \le \underline{\lim}_n \int_0^\infty \dot{\gamma}_n^2\, dt$ (also Berger).

Now Ascoli's Theorem (Royden [1]) implies there is a subsequence (call it also) γ_n such that $\gamma_n \to \gamma$ uniformly on compacts if we show $\{\gamma_n\}_{n \in N}$ is an equicontinuous family and pointwise bounded.

Equicontinuity follows from

$$\sup_{u \in [s,t]} |f(u) - f(s)|^2 \leq (t - s) \int_s^t |\dot{f}(v)|^2 \, dv, \; \gamma_n(0) \to x_0$$

and $\sup_n \mathcal{A}(\gamma_n) < \infty$.

Pointwise bound follows from

$$\begin{aligned} \infty > \sup_n \mathcal{A}(\gamma_n) &\geq \mathcal{A}(\gamma_n|_{[0,t]}) \\ &\geq \rho(\gamma_n(t), \gamma_n(0)) \geq \rho(\gamma_n(t), b) - \rho(b, \gamma_n(0)) \\ &\geq \sqrt{\frac{\delta}{2}} \, |\gamma_n(t)| - \text{constant} \end{aligned}$$

where we used: There exist $\delta > 0$ and $R > 0$ such that $\rho(x,b) \geq \sqrt{\frac{\delta}{2}} \, |x|$ if $|x| \geq R$ (see beginning of §13.2).

Then $\gamma(0) = \lim_n \gamma_n(0) = x_0$ and for any $L > 0$ by the Fatou lemma and above

$$\begin{aligned} \int_0^L (\tfrac{1}{2} \dot{\gamma}^2 + V(\gamma) \, dt &\leq \underline{\lim}_n \int_0^L (\tfrac{1}{2} \dot{\gamma}_n^2 + V(\gamma_n))dt \\ &\leq \underline{\lim}_n \int_0^\infty (\tfrac{1}{2} \dot{\gamma}_n^2 + V(\gamma_n))dt = \rho(x_0 \, , b), \end{aligned}$$

for any $L > 0$. Therefore $\int_0^\infty (\frac{1}{2} \dot{\gamma}^2 + V(\gamma)) \, dt \leq \rho(x_0 \, , b)$ so $\gamma(t) \to a$ or $\gamma(t) \to b$

for any $L > 0$. Therefore $\int_0^\infty (\frac{1}{2}\dot{\gamma}^2 + V(\gamma))\,dt \le \rho(x_0, b)$ so $\gamma(t) \to a$ or $\gamma(t) \to b$ as $t \to \infty$. Recall $\gamma_n(T_n) \to b$ as $n \to \infty$ and so if $\gamma(t) \to a$ as $t \to \infty$ there is a sequence $0 < S_n < T_n$ such that $\gamma_n(S_n) \to a$ as $n \to \infty$ and we get

$$\rho(x_0, b) = \lim_n \mathcal{A}(\gamma_n) = \underline{\lim}_n \int_0^{S_n} (\ ) + \underline{\lim} \int_{S_n}^\infty (\ ) \ge \rho(x_0, a) + \rho(a,b),$$

which contradicts assumption (13.3). Hence $\gamma(t) \to b$, γ is a minimizing path of (13.2) and we write $\gamma = g^{x_0,b}$.

Now we want to show $\|\gamma_n - g^{x_0,b}\|_{L^\infty(0,\infty)} \to 0$ as $n \to \infty$ for this subsequence, which we do by help from sublemma 4.

Given a small $\varepsilon > 0$ pick $T > 0$ such that $|g^{x_0,b}(t) - b| < \varepsilon$ if $t \ge T$ and pick M such that $T_n \ge T$, $\|\gamma_n - g^{x_0,b}\|_{L^\infty[0,T]} < \varepsilon$, $\mathcal{A}(\gamma_n) \le \rho(x_0, b) + \varepsilon^2$ and $|\rho(\gamma_n(0),b) - \rho(x_0, b)| \le \varepsilon^2$ if $n \ge M$.

Then for $n \ge M$

$$\begin{aligned}\int_0^T (\tfrac{1}{2}\dot{\gamma}_n^2 + V(\gamma_n))dt &\ge \rho(\gamma_n(0),\gamma_n(T)) \\ &\ge \inf_{\substack{|y-b|\le\varepsilon \\ n\ge M}} (\rho(\gamma_n(0),b) - \rho(y,b)) \\ &\ge \rho(x_0, b) - (1 + \frac{c_2}{2c_1^2})\,\varepsilon^2\end{aligned}$$

by sublemma 4.1. By above

$$\rho(x_0\,,\,b) + \varepsilon^2 \geq \mathcal{A}(\gamma_n) = (\int_0^T + \int_T^\infty)$$

$$\geq \rho(x_0\,,\,b) - (1 + \frac{c_2}{2c_1})\,\varepsilon^2$$

or

$$\int_T^\infty (\tfrac{1}{2}\,\dot\gamma_n^2 + V(\gamma_n))\,dt \leq (2 + \frac{c_2}{2c_1})\,\varepsilon^2$$

and

$$|\gamma_n(T) - b| \leq \varepsilon\,.$$

By sublemma 4.4 this implies

$$\sup_{n\geq m}\ \sup_{t\in[T,\infty)} |\gamma_n(t) - b| \leq \sqrt{1+\frac{4}{c_1}(2+\frac{c_2^2}{2c_1})\,\varepsilon}\ .$$

Assumption (13.3) makes the minimizing paths $g^{x_0,b}$ for (13.2) stay outside a neighborhood of a and by sublemma 4.1 stay away from b on any bounded subinterval of $[0,\infty)$ and by Hilbert's differentiability theorem (see Hestenes [1]), $g^{x_0,b} \in C^\infty([0,\infty))$ as function of t.

Proof of Sublemma 5.2.

As in the beginning of the proof of sublemma 5.1 we get $\gamma_1\,,\,\gamma_2 \in C(0,\infty)$ and a subsequence (call it also) γ_n with

$$\gamma_n|_{[0,\frac{T_n}{2}]} \to \gamma_1\,,\ \gamma_n(T_n - \cdot\,)|_{[0,\,\frac{T_n}{2}]} \to \gamma_2$$

uniformly on compacts in $(0,\infty)$, $\int_0^\infty \dot\gamma_1^2\, dt \le \underline{\lim}_n \int_0^{\frac{T_n}{2}} \dot\gamma_n^2\, dt$

and $$\int_0^\infty \dot\gamma_2^2\, dt \le \underline{\lim}_n \int_0^{\frac{T_n}{2}} \dot\gamma_n^2(T-t)\, dt \text{ as } n \to \infty .$$

Using Fatou's lemma we get

$$\int_0^\infty (\tfrac{1}{2}\dot\gamma_1^2 + V(\gamma_1))\, dt \le \rho(x_0, b) + \rho(b, y_0)$$

and

$$\int_0^\infty (\tfrac{1}{2}\dot\gamma_2^2 + V(\gamma_2))\, dt \le \rho(x_0, b) + \rho(b, y_0) \cdot$$

Therefore

$$\gamma_1(t) \to c_1 \in \{a,b\} \text{ and } \gamma_2(t) \to c_2 \in \{a,b\}$$

as $t \to \infty$, implying there is a subsequence $u_\ell \to \infty$ as $\ell \to \infty$ and sequences u_n and v_n with $u_{n_\ell} \le \frac{T_{n_\ell}}{2} \le v_{n_\ell}$ such that $\gamma_{n_\ell}(u_{n_\ell}) \to c_1$ and $\gamma_{n_\ell}(v_{n_\ell}) \to c_2$ as $\ell \to \infty$.

Hence

$$\rho(x_0, b) + \rho(b, y_0) =$$

$$= \lim_{\ell\to\infty} \int_0^{T_{n_\ell}} (\tfrac{1}{2}\dot\gamma_\ell^{2n} + V(\gamma_{n_\ell}))\, dt \ge \underline{\lim}_\ell \int_0^{T_{n_\ell}} (\,) + \underline{\lim}_\ell \int_0^{T_{n_\ell}} (\,) \ge$$

$$\ge \underline{\lim}_\ell \int_0^{u_{n_\ell}} (\,) + \underline{\lim}_\ell \int_{u_{n_\ell}}^{v_{n_\ell}} (\,) + \underline{\lim}_\ell \int_0^{T_{n_\ell}} (\,) \ge$$

$$\ge \rho(x_0, c_1) + \rho(c_1, c_2) + \rho(c_2, y_0)$$

and we conclude from (13.4) that $c_1 = c_2 = b$.

Now $\gamma_1(0) = x_0$, $\gamma_2(0) = y_0$ and $\gamma_1(\infty) = b = \gamma_2(\infty)$ implies

$$\rho(x_0 , b) \le \int_0^{\infty}(\tfrac{1}{2}\,\dot\gamma_1^2 + V(\gamma_1))\,dt$$

and

$$\rho(y_0 , b) \le \int_0^{\infty}(\tfrac{1}{2}\dot\gamma_2^2 + V(\gamma_2))\,dt.$$

We also have for any $L > 0$

$$\int_0^{L}(\tfrac{1}{2}\,\dot\gamma_n^2 + V(\gamma_n))\,dt + \int_0^{L}(\tfrac{1}{2}\,\dot\gamma_2^2 + V(\gamma_2))\,dt$$

$$\le \underline{\lim_{n}} \int_0^{L}(\tfrac{1}{2}\,\dot\gamma_n^2 + V(\gamma_n))\,dt + \underline{\lim_{n}} \int_0^{L}(\tfrac{1}{2}(\dot\gamma_n^2(T-t))^2 + V(\gamma_n(T-t))\,dt$$

$$\le \underline{\lim_{n}} \int_0^{T_n}(\tfrac{1}{2}\,\dot\gamma_n^2 + V(\gamma_n)) = \rho(x_0 , b) + \rho(b,y_0).$$

Hence

$$\int_0^{\infty}(\tfrac{1}{2}\,\dot\gamma_1^2 + V(\gamma_1))\,dt = \rho(x_0 , b)$$

and

$$\int_0^{\infty}(\tfrac{1}{2}\,\dot\gamma_2^2 + V(\gamma_2))\,dt = \rho(y_0 , b)$$

so

$$\gamma_1 = g^{x_0,b} \text{ and } \gamma_2 = g^{y_0,b} .$$

Since $$\rho(x_0 , b) + \rho(b,y_0) < \rho(x_0 , a) + \rho(a,y_0)$$

implies $$\rho(x_0 , b) < \rho(x_0 , a) + \rho(a,b)$$

and $$\rho(y_0 , b) < \rho(y_0 , a) + \rho(a,b)$$

we get, by sublemma 5.1, a new subsequence γ_{n_k} from the subsequence γ_n above such that

$$\|g^{x_0,b} - \gamma_{n_k}\|_{L^\infty[0,\frac{T_{n_k}}{2}]} \to 0$$

and

$$\|g^{x_0,b} - \gamma_{n_k}(T - \cdot)\|_{L^\infty[0,\frac{T_{n_k}}{2}]} \to 0 \text{ as } k \to \infty .$$

$g^{x,b}(t) := g^{x_0,b}(t + t_0)|_{[0,\infty)}$ minimizes (13.2) with $x_0 = x$ and to prove the uniqueness one can use an argument similar to one given in Kobayashi [1], p. 99.

Now assume the second variation of $g^{x,b}$ is not positive definite (and we will get contradiction to the second variation of $g^{x_0,b}$ is non-negative). Then by (2.9) there are $t_0 \le t_1 < t_2 < \infty$ and a solution $f \neq 0$ of $\ddot{f}(t) = V''(g^{x_0,b}(t))\, f(t)$ on (t_1, t_2) with $f(t_1) = 0 = f(t_2)$.

Then $\dot{f}(t_1) \neq 0$ and so there is some $h \in C_0^\infty((0,\infty),\mathbb{R}^d)$ with $\langle \dot{f}(t_1), h(t_1)\rangle < 0$ but $h(t_2) = 0$. Now for small $\delta > 0$ put $\varphi_\delta(t) = f(t) + \delta h(t)$ where f is extended to be zero outside (t_1 , t_2).

Then

$$0 \le \int_0^\infty (|\dot{\varphi}_\delta|^2 + \langle V''(g^{x_0,b})\varphi_\delta , \varphi_\delta\rangle)\, dt$$

$$= \int_{t_1}^{t_2} (|\dot{f}|^2 + \langle V''(g^{x_0,b})\, f,f\rangle)\, dt$$

$$+ 2\delta \int_{t_1}^{t_2} (\langle \dot{f},\dot{h}\rangle + \langle V''(g^{x_0,b})f,h\rangle)\, dt$$

$$+ \delta^2 \int_0^\infty (|\dot{h}|^2 + \langle V''(g^{x_0,b})h,h\rangle)\, dt$$

$$= 2\delta \, \langle \dot{f}(t_1),h(t_1)\rangle + \delta^2 \int_0^\infty (|\dot{h}|^2 + \langle V''(g^{x_0,b})\, h,h\rangle)\, dt$$

which is strictly negative for δ small enough.

§13.2. Proof of Lemma 1.

First we show $K = \{x : \rho(x,a) = \rho(x,b) = \frac{\rho(a,b)}{2}\}$ is compact. By continuity of $x \mapsto \rho(x,a)$ and $x \mapsto \rho(x,b)$ K is closed and it is bounded since $\rho(x,a), \rho(x,b) \to \infty$ as $|x| \to \infty$. To prove the last statement pick $\delta > 0$ and $R > 0$ such that $V(y) \geq \delta$ if $|y| \geq R$. When

$$|x| \geq 2R,\ \rho(x,a), \rho(x,b) \geq \inf_{|y|=R} \rho(x,y) = \inf_{|y|=R} \{\inf_{\gamma} [\int_0^1 \sqrt{2V(\gamma)}\, |\dot\gamma| ds : \gamma(0) = x,$$

$$\gamma(1) = y\} \geq \inf_{|y|=R} \{\inf_{\gamma} \{\sqrt{2\delta} \int_0^1 |\dot\gamma(s)| ds : \gamma(0) = x,$$

$$\gamma(1) = y\} \geq \sqrt{2\delta}(|x|-|y|) \geq \sqrt{\tfrac{\delta}{2}}\, |x|.$$

If $x_0 \in K$ then using sublemma 5.1 we see the problem

$$\rho(a,b) = \inf\{\int_{-\infty}^{\infty} (\tfrac{1}{2}\dot\gamma^2 + V(\gamma))dt \,|\, \gamma(-\infty) = a,\ \gamma(\infty) = b,\ \gamma(0) = x_0\} \tag{13.6}$$

has a minimizing path g_{x_0}.

Pick $t_0 \in (0,\infty)$ and put $x = g_{x_0}(-t_0)$. Then $g^{x,b}(t) := g_{x_0}(t - t_0)|_{[0,\infty]}$ minimizes $\rho(x,b)$ in (1.3.2). Since $x_0 = g^{x,b}(t_0)$ and $\rho(x,b) < \rho(a,b) < \rho(x,a) + \rho(a,b)$ we get from sublemma 5.3 $g^{x_0,b}(t) = g^{x,b}(t + t_0)|_{[0,\infty)}$ is the unique minimizing for $\rho(x_0\,, b)$ in (13.2) and has a positive definite second variation. Similarly $g_{x_0}|_{(-\infty,0]}\,(-t) = g^{x_0,a}(t)$ is the unique minimizing path for $\rho(x_0\,, a)$ and has positive definite second variation. Therefore this g_{x_0} is the unique minimizing path for (13.6).

Now we apply lemma 2 and we get unique $g^{x,a}$ and $g^{x,b}$ for x near x_0 and their second variations are strictly positive. Moreover $\rho(x,a)$ and $\rho(x,b)$ are C^∞ near x_0 and $\rho'(x,c) = -\dot{g}^{x,c}(0)$. By above $g^{x_0,b} = g_{x_0}|_{[0,\infty]}$ and $g^{x_0,a} = g_{x_0}|_{(-\infty,0)}$ so $\rho'(x,a) - \rho'(x,b)|_{x=x_0} = 2\dot{g}_{x_0}(0)$ which is not zero since $\frac{1}{2}\dot{g}_{x_0}^2(0) = V(x_0) \neq 0$.

The continuity of $\rho'(x,a) - \rho'(x,b)$, implies it's nonzero near x_0 and lemma 1 follows, since K is compact.

§13.3. <u>On Uniqueness and Nondegeneracy of $g^{x,b}$ (Sublemma 6)</u>.

We will show that if (13.3) holds and

(13.7) there is a unique $g^{x,b}$ minimizing (13.2) and the second variation of $g^{x,b}$ is positive definite holds at $x = x_0$

then (13.7) holds for x near x_0.

<u>Sublemma 6</u>. Assume (13.3) and (13.7) for $x = x_0$.

Then there are $\delta_0 > 0$ and $k > 0$ such that if $x \in B(x_0, \delta_0)$ then there is a unique $g^{x,b}$ and if $\|\gamma - g^{x,b}\|_{L^\infty(0,\infty)} \le \delta_0$ then

(13.8) $$\int_0^\infty (\dot{\eta}^2 + \langle V''(\gamma)\eta,\eta\rangle)dt \ge k\|\eta\|^2_{L^\infty[0,\infty)}$$

for all $\eta \in D_0(0,\infty) := \{\eta \in AC([0,\infty),\mathbb{R}^d)$

$\eta(0) = 0$ and $\eta, \dot{\eta} \in L^2([0,\infty),\mathbb{R}^d)\}$.

Proof. By sublemma 5.1 $\|g^{x,b} - g^{x_0,b}\|_{L^\infty[0,\infty)} \to 0$ as $x \to x_0$ so it suffices to prove

(13.9) There exist $\delta_1 > 0$ and $k_1 > 0$ such that $\|\gamma - g^{x_0,b}\|_{L^\infty_{[0,\infty)}} \leq \delta_1$ implies $\int_0^\infty (\dot\eta^2 + \langle V''(\gamma)\eta_1\eta\rangle)\, dt \geq k_1 \|\eta\|^2_{L^\infty[0,\infty)}$ for all $\eta \in D_0(0,\infty)$.

To see (13.9) implies there is a unique $g^{x,b}$ for x near x_0, we assume γ_1 and γ_2 are two minimizing path for $\rho(x,b)$ in (13.2). By sublemma 5.1 they are both close to $g^{x_0,b}$ for x close to x_0. Hence, $\gamma_1 = \gamma_2 + \eta$ for $\eta \in D_0(0,\infty)$ and we get by (13.9), using γ_1 and γ_2 satisfy the Euler-Lagrange equation $\ddot\gamma = \nabla V(\gamma)$:

$$0 = \mathcal{A}(\gamma_1) - \mathcal{A}(\gamma_2) = \mathcal{A}(\gamma_2 + \eta) - \mathcal{A}(\gamma_2)$$

$$= \int_0^\infty [\frac{(\dot\gamma_2 + \dot\eta)^2}{2} - \frac{\dot\gamma_2^2}{2} + V(\gamma_2 + \eta) - V(\gamma_2))dt$$

$$= \int_0^\infty (\frac{\dot\eta^2}{2} - \ddot\gamma_2 \cdot \eta + V(\gamma_2 + \eta) - V(\gamma_2))\, dt$$

$$= \int_0^\infty (\frac{\dot\eta^2}{2} + V(\gamma_2 + \eta) - V(\gamma_2) - \nabla V(\gamma_2) \cdot \eta)\, dt$$

$$= \frac{1}{2} \int_0^\infty (\frac{\dot\eta^2}{2} + \langle V''(\gamma_2 + \mathfrak{Z})\eta,\eta\rangle)\, dt$$

where $\|\mathfrak{Z}\|_{L^\infty[0,\infty)} \leq \|\eta\|_{L^\infty[0,\infty)} = \|\gamma_1 - \gamma_2\|_{L^\infty[0,\infty)}$. Hence $\gamma_2 + \mathfrak{Z}$ is close to $g^{x_0,b}$ and by (13.9)

$$0 = \mathcal{A}(\gamma_1) - \mathcal{A}(\gamma_2) \geq \tfrac{1}{2}k_1\|\eta\|^2_{L^\infty(0,\infty)} = \tfrac{1}{2}k_1\|\gamma_1 - \gamma_2\|^2_{L^\infty[0,\infty)}$$

and we have $\gamma_1 = \gamma_2$.

To prove (13.9) we split $[0,\infty)$ into two pieces $[0,t_0 + 1]$ and $[t_0 + 1, \infty)$, for $t_0 \geq 0$ to be chosen. On $[0, t_0 + 1]$ we use the positivity of the second variation of $g^{x_0,b}$ and on $[t_0 + 1, \infty)$ we use $V''(b) > 0$.

Recall (10.1), there are c_0 , c_1 and $c_2 > 0$ such that if $c \in \{a,b\}$ and $|x-c| \leq c_0$ then $c_1^2 \leq V''(x) \leq c_2^2$. Pick $t_0 \geq 0$ such that $|g^{x_0,b}(t) - b| \leq \frac{c_0}{2}$ if $t \geq t_0$. Then

(13.10) there exist $\delta_2 > 0$ and $k_2 > 0$

such that $\|\gamma - g^{x_0,b}\|_{L^\infty[0,t_0+1]} \leq \delta_2$

implies $\int_0^{t_0+1} (\dot\eta^2 + \langle V''(\gamma)\eta_1\eta\rangle)\, dt \geq k_2\|\eta\|^2_{L^\infty[0,t_0+1]}$

$\forall \eta \in D_0(0,\infty)$

and

(13.11) If $\|\gamma - g^{x_0,b}\|_{L^\infty[t_0+1,\infty)} \leq \frac{c_0}{2}$ then

$\int_{t_0+1}^{\infty} (\dot\eta^2 + \langle V''(\gamma)\eta_1\nu\rangle)\, dt \geq \frac{c_1}{2}\|\eta\|^2_{L^\infty[t_0+1,\infty]}$

$\forall \eta \in D_0(0,\infty)$

Now (13.10) and (13.11) imply (13.9) with $k_1 = \min\{k_2 , \frac{c_1}{2}\}$ and $\delta_1 = \min\{\delta_2 , \frac{c_0}{2}\}$.

To prove (13.10) set

$$k_3 := \inf_{\substack{\eta \in D_0(0,\infty) \\ \int_0^{t_0+1} |\dot\eta|^2 dt = 1}} \{\int_0^{t_0+1} (\dot\eta^2 + \langle V''(g^{x_0,b})\eta,\eta\rangle) dt\}$$

Then $k_3 > 0$; proof by contradiction. Assume $k_3 = 0$. Then there exist η_n with $\|\dot\eta_n\|_{L^2[0,t_0+1]} = 1$ and

$$\int_0^{t_0+1} (\dot\eta_n^2 + \langle V''(g^{x_0,b})\eta_n,\eta_n\rangle)\, dt \to 0 \text{ as } n \to \infty .$$

By Alaoglu's Theorem, a bounded set in a reflexive Banach space is weakly sequentially compact (see Berger [1]) and we may assume $\dot\eta_n \overset{\text{weakly}}{\to} \dot\eta \in L^2[0,t_0+1]$. Since $|\eta_n(t) - \eta_n(s)|^2 \le (t-s) \int_s^t \dot\eta_n^2(u)\, du \le (t-s)$ if $t - s > 0$ and for all n and $\eta_n(0) = 0$ we may assume by the Ascoli's theorem (Royden [1]) that $\eta_n \to \eta$ in $C[0,t_0+1]$. If we show $\eta = 0$ we get contradiction, by above, since

$$(\int_0^{t_0+1} \langle V''(g^{x_0,b})\eta_n,\eta_n\rangle dt| = 0(\|\eta_n\|^2_{L^\infty[0,t_0+1]}) \to 0 \text{ as } n \to \infty .$$

We have a small problem in using the positivity of the second variation of $g^{x_0,b}$ since we only know $\eta_n|_{[0,t_0+1]} \in D(0,t_0+1) := \{\eta \in AC([0,t_0+1],\mathbb{R}^d), \dot\eta \in L^2$ and $\eta(0) = 0\}$ and it is not necessarily in $D_0(0,t_0+1)$ (we don't have $\eta_n(t_0+1) = 0$).

Define $D(0,t_0+1)\, f \ni \mapsto \tilde f \in D_0(0,t_0+1)$ by

$$\tilde f(t) = \begin{bmatrix} f(t) \ \text{ if } 0 \le t \le t_0 \\ f(t)(t_0+1-t) \text{ if } t_0 < t \le t_0+1 \end{bmatrix}$$

If we show $\tilde\eta = 0$ then $\eta = 0$ and we are done.

On $[t_0 , t_0 + 1]$ we have

$$\dot{\tilde f}^2 \le 2\dot f^2(t_0+1-t)^2 + 2f^2 \text{ and } c_1^2 \le V''(g^{x_0,b}(t)) \le c_2^2 ,$$

hence $\int_{t_0}^{t_0+1} (\dot{\tilde f}^2 + \langle V''(g^{x_0,b})\tilde f,\tilde f\rangle)\, dt$

$$\le (2 + \frac{(2+c_2^2)}{c_1^2})) \int_{t_0}^{t_0+1} (\dot f^2 + \langle V''(g^{x_0,b})f,f\rangle)\, dt .$$

Therefore, by uniform convergence of η_n and weak L^2-convergence of $\dot\eta_n$, we have

$$\int_0^{t_0+1} (\dot{\tilde\eta}^2 + \langle V''(g^{x_0,b})\, \tilde\eta,\tilde\eta\rangle)\, dt \le$$

$$\le \varliminf_n \int_0^{t_0+1} (|\dot{\tilde\eta}_n|^2 + \langle V''(g^{x_0,b})\, \tilde\eta_n , \tilde\eta_n\rangle)\, dt$$

$$\le (2 + (\frac{2+c_2^2}{c_1^2}))\, \varliminf_n \int_0^{t_0+1} (\dot\eta_n^2 + \langle V''(g^{x_0,b})\, \eta_n , \eta_n\rangle)\, dt = 0$$

By the positivity of the second variation of $g^{x_0,b}$, $\tilde\eta = 0$ and we conclude $k_3 > 0$.

Therefore

$$(13.12)\qquad \int_0^{t_0+1} (\dot\eta^2 + \langle V''(g^{x_0,b})\,\eta,\eta\rangle)\,dt \ge k_3\|\eta\|^2_{L^\infty[0,t_0+1]}$$

for all $\eta \in D_0(0,\infty)$ where $k_3 > 0$.

Having in the mind $|\eta(t)|^2 \le t \int_0^t |\dot\eta^2(u)|\,du$ if $\eta(0) = 0$, hence

$$\|\eta(t)\|^2_{L^\infty[0,t_0+1]} \le (t_0+1)\|\dot\eta\|^2_{L^\infty[0,t_0+1]} \text{ and}$$

$$\|\eta(t)\|^2_{L^2[0,t_0+1]} \le \frac{(t_0+1)^2}{2}\|\dot\eta\|^2_{L^\infty[0,t_0+1]},$$

we pick $\delta_1 > 0$ such that $\|\gamma - g^{x_0,b}\|_{L^\infty[0,\infty)} \le \delta_1$

implies $V''(\gamma(t)) \ge V''(g^{x_0,b}(t)) - \dfrac{k_3}{(t_0+1)^2}$

for $t \in [0,\infty)$

(V is uniformly continuous on compacts).

For such γ's

$$\int_0^{t_0+1} (\dot\eta^2 + \langle V''(\gamma)\,\eta,\eta\rangle)\,dt \ge$$

$$\ge k_3\|\dot\eta\|^2_{L^2[0,t_0+1]} - \frac{k_3}{(t_0+1)^2}\|\eta\|^2_{L^2[0,t_0+1]}$$

$$\ge \frac{k_3}{2}\|\dot\eta\|^2_{L^2[0,t_0+1]} \ge \frac{k_3}{(t_0+1)^2}\|\eta\|^2_{L^\infty[0,t_0+1]},$$

which completes the proof of (13.10).

Now we prove (13.11). From the choice of t_0 and since

$$\|\gamma - g^{x_0,b}\|_{L^\infty[t_0+1,\infty]} \le \frac{c_0}{2}$$

we have,

$$(13.13)\qquad \int_{t_0+1}^{\infty} (\dot\eta^2 + \langle V''(\gamma)\,\eta,\eta\rangle)\,dt \ge \int_{t_0+1}^{\infty} (\dot\eta^2 + c_1^2\,\eta^2)\,dt.$$

$|\eta(t)|$ is continuous and tends to 0 at ∞. Pick t_1 and t_2 with $t_0 \le t_1 < \infty$,

$$|\eta(t_1)| = \|\eta\|_{L^\infty[t_0+1,\infty)} ,$$

$$|\eta(t_2)| = \tfrac{1}{2}\|\eta\|_{L^\infty[t_0+1,\infty)}$$

and

$$|\eta(t_1)| \ge |\eta(t)| \ge |\eta(t_2)| \text{ for } t \in [t_1 , t_2].$$

Then by (13.13)

$$\int_{t_0+1}^{\infty} (\dot\eta^2 + \langle V''(\gamma)\,\eta,\eta\rangle)\,dt \ge \int_{t_1}^{t_2} (\dot\eta^2 + c_1\,\eta^2)\,dt \ge$$

$$\ge \frac{\||\eta(t_2)|-|\eta(t_2)|\|^2}{(t_2 - t_1)} + c_1^2 \inf_{t_1\le t\le t_2} |\eta(t)|(t_2-t_1)$$

$$\ge \frac{1}{4}\,\|\eta\|^2_{L^\infty[t_0+1,\infty)} \left(\frac{1}{(t_2-t_1)} + c_1^2(t_2-t_1)\right)$$

$$\ge \frac{c_1}{2}\,\|\eta\|^2_{L^\infty[t_0+1,\infty)} .$$

§13.4. <u>There is a unique and nondegenerate minimum of the action for large T's.</u>

The results stated below imply lemmas 3.1, 3.3 and a part of lemma 2.2.

<u>Sublemma 7</u>. Assume (13.4)

$$\rho(x_0\ ,\ b) + \rho(b,y_0) < \rho(x_0\ ,\ a) + \rho(a,y_0)$$

and

there are unique $g^{x_0,b}$ and $g^{y_0,b}$, each one having positive definite second variation.

Then there are positive constants δ_0, T_0 and k such that if

$$(x,y,T) \in B(x_0\ ,\ \delta_0) \times B(y_0\ ,\ \delta_0) \times [T_0\ ,\ \infty)$$

then

1. There is a unique minimal action path $g_T^{x,y}$

and if $\|\gamma - g_T^{x,y}\|_{L^\infty[t_0+1,\infty)} \leq \delta_0$

then $\int_0^T (\dot{\eta}^2 + \langle V''(\gamma)\ \eta,\ \eta\rangle)\ dt \geq k\|\eta\|^2_{L^\infty[0,T]}$

$\forall \eta \in D_0(0,T) = \{\eta \in AC([0,T],\mathbb{R}^d) : \dot{\eta} \in L^2[0,T]\ \eta(0) = 0 = \eta(T)\}.$

2. $\mathcal{A}(g_T^{x,y} + \eta) - \mathcal{A}(g_T^{x,y}) \geq k \min\{\delta_0^2\ ,\ \|\eta\|^2_{L^\infty[0,T]}\}$ for all $\eta \in D_0(0,T)$.

3. If $(x_n\ ,\ y_n\ ,\ T_n) \to (x,y,T)$ as $n \to \infty$ then

$$\|g_{T_n}^{x_n,y_n}(\ \cdot\ \tfrac{T_n}{T}) - g_T^{x,y}\|_{L^\infty[0,T]} \to 0 \text{ as } n \to \infty\ .$$

Proof of Sublemma 7.1.

Pick $t_0 > 0$ such that $|g^{x_0,b}(t) - b| \le \frac{c_0}{6}$ and $|g^{y_0,b}(t) - b| \le \frac{c_0}{6}$ if $t \ge t_0$ where c_0 is as in (10.1)-(10.3).

By (13.10) there are $\delta_4 > 0$ and $k_4 > 0$ such that If

(13.14) $\quad \|\gamma - g^{x_0,b}\|_{L^\infty[t_0+1]} \le \delta_4$ or $\|\gamma - g^{y_0,b}\|_{L^\infty[t_0+1]} \le \delta_4$

then $\int_0^{(t_0+1)} (\dot\eta^2 + \langle V''(\gamma)\,\eta,\eta\rangle)\,dt \ge k_4\|\eta\|^2_{L^\infty[t_0+1]}$

for all $\eta \in D(0, t_0 + 1) = \{\eta \in AC([0, t_0 + 1], \mathbb{R}^d) :$

$\dot\eta \in L^2[0, t_0 + 1]$ and $\eta(0) = 0\}$.

By sublemma 5.2 there exists $\delta_5 > 0$ and $T_0 \ge 4(t_0 + 1)$ such that

(13.15) $\quad$ If $|x - x_0| \le \delta_5$, $|y - y_0| \le \delta_5$ and $T \ge T_0$

then $\|g_T^{x,y} - g^{x_0,b}\|_{L^\infty[0,\frac{T}{2}]} \le \frac{1}{6}\min(\delta_4, c_0)$

and $\|g_T^{x,y}(T - \cdot) - g^{x_0,b}\|_{L^\infty[0,\frac{T}{2}]} \le \frac{1}{6}\min(\delta_4, \frac{c}{0})$.

Now we prove that if

(13.16) $\quad |x - x_0| \le \delta_5$, $|y - y_0| \le \delta_5$ and $T \ge T_0$

then $\|\gamma - g_T^{x,y}\|_{L^\infty[0-T]} \le \frac{2}{3}\min(\delta_4, c_0)$,

for some minimal action path $g_T^{x,y}$, implies

$$\int_0^T (\dot\eta^2 + \langle V''(\gamma)\,\eta,\eta\rangle)\,dt \ge \min(k_4\,,\frac{c_1}{2}\,,\frac{c_1^2}{2})\|\eta\|^2_{L^\infty[0,T]}$$

for all $\eta \in D_0(0,T)$.

(13.14) and (13.15) imply

$$(13.17)\qquad \int_0^{(t_0+1)} (\dot\eta^2 + \langle V''(\gamma)\,\eta,\eta\rangle)\,dt \ge k_4\|\eta\|^2_{L^\infty[0,t_0+1]}$$

$$\text{and } \int_{T-(t_0+1)}^{T} (\dot\eta^2 + \langle V''(\gamma)\,\eta,\,\eta\rangle)\,dt \ge k_4\|\eta\|^2_{L^\infty[T-(t_0+1),T]}$$

for all $\eta \in D_0(0,T)$ and γ's like in (13.16) as can be checked using the triangle inequality.

On $[t_0+1,T-(t_0+1)]$ we have $V''(\gamma) \ge c_1^2$ since $\|\gamma-b\|_{L^\infty[\ell+1,T-(\ell+1)]} \le c_0$.

There we split into two cases depending on whether

$$|\eta(t)| \ge \frac{1}{2}\|\eta\|_{L^\infty[t_0+1,T-(t_0+1)]}$$

for all $t \in [t_0 + 1, T-(t_0+1)]$ or not. In the first case we get

$$(13.18)\qquad (\int_{t_0+1}^{T-(t_0+1)} (\dot\eta^2 + \langle V''(\gamma)\,\eta,\,\eta\rangle)\,dt \ge c_1^2 \int_{t_0+1}^{T-(t_0+1)} \eta^2(t)\,dt$$

$$\ge \frac{c_1^2}{4}\|\eta\|^2_{L^\infty[t_0+1,T-(t_0+1)]}(T_0 - 2(t_0+1))$$

$$\ge \frac{2(t_0+1)}{4}\, c_1^2\|\eta\|^2_{L^\infty[t_0+1,T-(t_0+1)]} \ge \frac{c_1^2}{2}\|\eta\|^2_{L^\infty[t_0+1,T-(t_0+1)]}\,.$$

In the second case do as in the proof of (13.11) and get

(13.19) $$\int_{t_0+1}^{T-(t_0+1)} (\dot{\eta}^2 + \langle V''(\gamma)\,\eta,\,\eta\rangle)\,dt \geq \frac{c_1}{2}\|\eta\|^2_{L^\infty[t_0+1,T-(t_0+1)]}\,.$$

Now write $$\int_0^T = \int_0^{T_0+1} + \int_{t_0+1}^{T-(t_0+1)} + \int_{T-(t_0+1)}^T$$

and (13.16) follows from (13.17)-(13.19).

(13.15) implies that if $(x,y,T) \in B(x_0\,,\delta_5) \times B(y_0\,,\delta_5) \times [T_0\,,\infty]$ then the difference of any two minimal action paths between x and y in time T, has uniform norm less than $\frac{1}{3}\min\{\delta_4\,,c_0\}$. Hence the uniqueness of $g_T^{x,y}$ follows from

(13.20) If $(x,y,T) \in B(x_0\,,\delta_5) \times B(y_0\,,\delta_5) \times [T_0\,,\infty]$

and $g_T^{x,y}$ is some minimal action path, then

$\eta \in D_0(0,T)$, $\|\eta\|_{L^\infty[0,T]} \leq \frac{1}{3}\min(\delta_4\,,c_0)$

implies $\mathcal{A}(g_T^{x,y} + \eta) - \mathcal{A}(g_T^{x,y}) \geq k_5\|\eta\|^2_{L^\infty[0,T]}$

where $k_5 = \frac{1}{2}\min\{k_4\,,\frac{c_1}{2}\,,\frac{c_1^2}{2}\}$.

To prove (13.20) we integrate by parts, use $\ddot{g}_T^{x,y} = \nabla V(g_T^{x,y})$ and we get

$\mathcal{A}(g_T^{x,y} + \eta) - \mathcal{A}(g_T^{x,y}) =$

$$= \int_0^T (\tfrac{1}{2}\dot{\eta}^2 + V(g_T^{x,y} + \eta) - V(g_T^{x,y}) - \nabla V(g_T^{x,y})\eta)\,dt$$

$$= \frac{1}{2}\int_0^T (\dot{\eta}^2 + \langle V''(g_T^{x,y} + \mathfrak{Z})\,\eta,\,\eta\rangle\,dt$$

where $\|\mathfrak{Z}\|_{L^\infty[0,T]} \leq \|\eta\|_{L^\infty[0,T]}$.

Now (13.20) follows from (13.16) and sublemma 7.1 from (13.16) and the uniqueness of $g_T^{x,y}$.

Proof of Sublemma 7.3. We prove a slightly stronger statement that we use in proving sublemma 7.2.

(13.21) If $\gamma_n \in C([0,T_n])$, $T_n \to T$, $\gamma_n(0) \to x$ and $\gamma_n(T) \to y$,

where $(x,y,T) \in B(x_0, \delta_5) \times B(y_0, \delta_5) \times [T_0, \infty)$,

and $\mathcal{A}(\gamma_n) = \int_0^{T_n} (\frac{1}{2}\dot\gamma_n^2 + V(\gamma_n))\, dt \to A(x,y,T)$

then $\sup_{0 \le t \le T} |\gamma_n(\frac{tT_n}{T}) - g_T^{x,y}(t)| \to 0$ as $n \to \infty$.

If (13.21) is false then we have for a subsequence, as in the proof of (13.10),

$$\sup_{0 \le t \le T} |\gamma_n(\tfrac{tT_n}{T}) - g_T^{x,y}(t)| \ge \varepsilon_0 > 0, \ \gamma_n(\tfrac{tT_n}{T}) \to \gamma(t)$$

in C[0,T] (by Ascoli) and $\frac{d}{dt}\gamma_n(\frac{tT_n}{T}) \to \dot\gamma(t)$ weakly in $L^2[0,T]$ (by the Alaoglu theorem).

Then $\gamma(0) = x$, $\gamma(T) = y$, $\|\gamma - g_T^{x,y}\|_{L^\infty[0,T]} \ge \varepsilon_0$ but

$$\int_0^T (\tfrac{1}{2}\dot\gamma^2 + V(\gamma))\, dt \le \underline{\lim}_n \int_0^T ((\tfrac{d}{dt}\gamma_n(\tfrac{tT_n}{T}))^2 + V(\gamma_n(\tfrac{tT_n}{T})))\, dt$$

$$= \underline{\lim}_n \frac{T}{T_n} \int_0^{T_n} (\tfrac{1}{2}(\tfrac{T_n}{T})^2\, \dot\gamma_n^2(u) + V(\gamma_n(u))\, dt$$

$$= A(x,y,T)$$

which contradicts the uniqueness of $g_T^{x,y}$ for (x,y,T)'s as in (13.20) and completes the proof of (13.21).

<u>Proof of Sublemma 7.2</u>. By (13.20) it suffices to show that if

(13.22) $\delta = \frac{1}{3}\min(\delta_4 , c_0)$ then

$$\inf_{(x,y,T)\in B(x_0,\frac{\delta_5}{2})\times B(y_0,\frac{\delta_5}{2})\times[T_0,\infty]} \{\mathcal{A}(g_T^{x,y} + \eta_T) - \mathcal{A}(g_T^{x,y})\} > 0$$

$\eta_T \in D_0(0,T)$ and $\|\eta_T\| \geq \delta$.

Proof by contradiction: If not we have

$$(x_n , y_n , T_n) \to (x,y,T) \in B(x_0 , \delta_5) \times B(y_0 , \delta_5) \times [T_0 ,\infty)$$

$$\mathcal{A}(g_{T_n}^{x_n,y_n} + \eta_{T_n}) = \mathcal{A}(g_{T_n}^{x_n,y_n}) \to 0 \text{ as } n \to \infty$$

with $$\|\eta_{T_n}\|_{L^\infty[0,T_n]} \geq \delta > 0 .$$

If $T < \infty$ we get a contradiction using (13.21). If $T = \infty$ sublemma 5.2 gives contradiction, since $g^{x_0,b}$ and $g^{y_0,b}$ are unique.

14. Some Estimates (Sublemma 8).

Let C_0 , C_1 and C_2 be as in (10.1) and put

$$(14.1)\quad C_3 = \frac{C_0}{(\frac{1}{2}+\frac{c_2}{c_2})^{\frac{1}{2}}} .$$

Then we have

Sublemma 8.

1. Uniformly for $x \in B(b,c_3)$ and $i \in \{1, \dots, d\}$

$$(14.2)\quad (g^{x,b}(t)-b)_i = (x-b)_i\, e^{-\omega_i t} + 0(|x-b|^2\, e^{-\min\{\omega_i, 2\omega_1\}t})$$

$$= -\omega_i^{-1}(\dot{g}^{x,b}(t))_i$$

and hence

$$(14.3)\quad \rho(x,b) = \tfrac{1}{2}\langle\Omega(x-b),(x-b)\rangle + 0(|x-b|^3) \text{ uniformly.}$$

2. For any compact $K_1 \subseteq \{x \;\; \rho(x,b) < \rho(x,a) + \rho(a,b)\}$ there is a constant D_1 such that

$$(14.4)\quad |\dot{g}^{xb}(t)|,\ |g^{xb}(t) - b| \le D_1\, e^{-\omega_i t} \text{ uniformly for } (x,t) \in K_1 \times [0,\infty)$$

and

$$(14.5)\quad |g^{x_1 b}(t) - g^{x_2 b}(t)| \le D_1 \| g^{x_1 b} - g^{x_2 b} \|_{L^\infty(0,\infty)}\, e^{-\omega_1 \frac{t}{2}}$$

uniformly for $(x_1 , x_2 , t) \in K_1^2 \times [0,\infty)$.

3. For any compact $K_1 \subseteq \{(x,y) \in \mathbf{R}^d \times \mathbf{R}^d$

$$\rho(x,b) + \rho(b,y) < \rho(x,a) + \rho(a,y)\}$$

there are T_0 and D_2 such that uniformly for

$$(x,y,T,t) \in K_2 \times [T_0 , \infty) \times [0,T]$$

$$(14.6) \quad |\dot{g}_T^{x,y}(t)|, |g_T^{x,y}(t) - b| \le D_2(e^{-\omega_1 t} + \min\{|y-b|,c_3\}e^{-\omega_1(T-t)}).$$

Hence

$$(14.7) \quad E_T(x,y) = \tfrac{1}{2}(\dot{g}_T^{x,y}(t))^2 - V(g_T^{x,y}(t)) = 0(e^{-\omega_1 T})$$

and

$$(14.8) \quad A(x,y,T) = \rho(x,b) + \rho(b,y) + 0(e^{-\omega_1 T}) \text{ uniformly for those } (x,y,T)\text{'s.}$$

4. Let K_1 be as in part 2 then there are $\delta_0 > 0$ and T_0 such that

$$(14.9) \quad \dot{g}_T^{x,y}(T) = \Omega(y - b)(1 + 0(e^{-2\omega_1 T})) + 0(e^{-\omega_1 T} + |y-b|^2)$$

uniformly for $(x,y,T) \in K_1 \times B(b,\delta_0) \times [T_0 , \infty)$.

5. Assume $\rho(x_0 , b) < \rho(x_0 , a) + \rho(a,b)$, there is only one $g^{x_0,b}$ and it has positive definite second variation. Then there are $\delta_0 > 0$ and $T_0 < \infty$ such that (recall sublemma 6 and 7 say there are unique $g^{x,b}$ and $g_T^{x,y}$)

$$(14.10) \quad \begin{bmatrix} |g_T^{x,y}(t) - g_T^{x,b}(t)|^2 \\ |g_T^{x,y}(t) - g^{x,b}(t)|^2 \end{bmatrix} = 0(e^{-\omega_1 T}\, e^{-c_1 t} + |y-b|^2\, e^{-c_1(T-t)})$$

$$\text{if } t \in [0,T]$$

$$(14.11)\qquad \begin{bmatrix} |g_T^{x,y}(t) - g^{x,b}(t)|^2 \\ |g_T^{x,y}(T-t) - g^{y,b}(t)|^2 \end{bmatrix} = 0(e^{-c_1 t} + e^{-c_1(\frac{T}{2}-t)})\, e^{-\omega_1 T}$$

$$\text{if } t \in [0,\tfrac{T}{2}]$$

and

$$(14.12)\qquad |g_T^{x_1 y_1}(t) - g_T^{x_2 y_2}(t)|^2 = 0\Big(\|g_T^{x_1 y_1} - g_T^{x_2 y_2}\|^2_{L^\infty[0,T]}\, e^{-c_1 t} + |y_1 - y_2|^2\, e^{-c_1(T-t)}\Big) \text{ if } t \in [0,T]$$

everything uniform on $B(x_0\, ,\, \delta_0) \times B(b,\delta_0) \times [T_0\, ,\, \infty)$.

Proof of Sublemma 8.1. Write

$$V'(x) = \begin{bmatrix} \omega_1^2\,(x-b)_1 \\ \vdots \\ \omega_d^2\,(x-b)_d \end{bmatrix} + R(x)$$

where $0 < \omega_1 \leq \omega_2 \leq \cdots \leq \omega_d$ and

$$(14.13)\qquad R(x) \leq D_0\, |x - b|^2 \text{ for } |x - b| \leq c_0\, .$$

With $g = g^{x,b}$ we set

$$(14.14)\qquad g = f + r$$

where

$$(14.15)\qquad f_i(t) = (x - b)_i\, e^{-\omega_i t} + b_i$$

i.e.,

(14.15) $f_i(t) = (x - b)_i\, e^{-\omega_i t} + b_i$

i.e., $\ddot{f}_i(t) = \omega_i^2(f_i - b_i)$, $f(0) = x$ and $f(\infty) = b$

so $\ddot{r}(t) = \omega_i^2\, r_i(t) + R_i(g(t))$, $r_i(0) = 0$ and $r_i(\infty) = 0$.

Therefore

(14.16) $$r_i(t) = -\frac{e^{-\omega_i t}}{\omega_i} \int_0^t \sinh(\omega_i s) R_i(g(s))ds - \frac{\sinh \omega_i t}{\omega_i} \int_t^\infty e^{-\omega_i s} R_i(g(s))ds$$

and

(14.17) $$\dot{r}_i(t) = e^{-\omega_i t} \int_0^t \sinh(\omega_i s)\, R_i(g(s))\, ds -$$

$$- \cosh \omega_i t \int_t^\infty e^{-\omega_i s}\, R_i(g(s))\, ds$$

Recall sublemma 4.1 says

(14.18) $|g^{x,b}(t) - b| \le |x - b|\, e^{-c_1 t}$ if $|x - b| \le c_3$.

Without loss of generality we assume

$nc_1 \notin \{\omega_1, ..., \omega_d\}$ for $n = 1, 2, ...$ (take smaller c_1 if needed).

If $c_1 < 2c_2 < \cdots < 2^{n_0+1} c_1$ and we put

$$q(n) = \sum_{i=1}^{d} \frac{1}{(\omega_i^2 - (2^{n+1} c_1)^2)}$$

for $n = 0, 1, ..., n_0$ and for $r > 0$, $\beta_0(r) = 1$ and $\beta_{j+1}(r) = 1 + 2D_0\, \beta_j^2(r)\, q(j)\, r$, we get:

(14.19) If $0 \leq j \leq n_0$ and $|x - b| \leq c_3$ then

$$|g^{x,b}(t) - b| \leq |x - b|\ \beta_j(|x-b|)\ e^{-2^j c_1 t} \text{ implies}$$

$$|g^{x,b}(t) - b| \leq |x - b|\ \beta_j(|x-b|)\ e^{-\min(\omega_1, 2^{j+1} c_1)t} .$$

For $j = 0$ this follows from (14.14).

For the induction step, we assume $|g^{x,b}(t) - b| \leq \beta|x - b|\ e^{-\alpha t}$ for $t \geq 0$

and $0 < \alpha < \omega_1$ and $2\alpha \neq \omega_1$, for $i = 1, \ldots, d$. We obtain by (14.9) and (14.12)

$$|r_i(t)| \leq \frac{e^{-\omega_i t}}{\omega_i} \int_0^t e^{\omega_i s} |R_i(g(s))|\ ds + \frac{e^{\omega_i t}}{\omega_i} \int_t^\infty e^{-\omega_i s} |R_i(g(s))|\ ds$$

$$\leq \frac{D_0 \beta^2 |x-b|^2}{\omega_i} \left\{ \frac{e^{-2\alpha t} - e^{-\omega_i t}}{|\omega_i - 2\alpha|} + \frac{e^{-2\alpha t}}{(\omega_i + 2\alpha)} \right\}$$

$$\leq \frac{2 D_0 \beta^2}{|\omega_i^2 - (2\alpha)^2|}\ e^{-\min\{\omega_i, 2\alpha\}t} .$$

Hence

$$|g_i(t) - b| \leq |f_i(t) - b_i| + |r_i(t)| \leq$$

$$|x_i - b_i| e^{-\omega_i t} + \frac{2 D_0 \beta^2}{|\omega_i^2 - (2\alpha)^2|} |x - b|^2\ e^{-\min(\omega_i, 2\alpha)t}$$

which proves the induction step and therefore implies (14.19) with $j = n_0$

(14.20) $$|g^{x,b}(t) - b| \leq |x - b|\ \beta_{n_0+1}(|x - b|)\ e^{-\omega_1 t}$$

whenever $|x - b| \leq c_3$ and $t \geq 0$.

Now (14.20) and (14.13) put into (14.16) and (14.17) imply (14.2). (14.3) follows by putting (14.2) into

$$\rho(x,b) = \int_0^\infty (\tfrac{1}{2}(\dot g^{x,b}(t))^2 + V(\dot g^{x,b}(t))\, dt = \int_0^\infty (\dot g^{x,b}(t))^2\, dt.$$

Proof of Sublemma 8.2.

Let $x_0 \in \{x : \rho(x,b) < \rho(x,a) + \rho(a,b)\}$ and pick $\delta > 0$ such that

$$\overline{B(x_0,\delta)} \subseteq \{x : \rho(x,b) < \rho(x,a) + \rho(a,b)\}.$$

By compactness it suffices to prove (14.4) for $x \in B(x_0 , \delta)$.

By (12.9) there exist $\varepsilon > 0$ such that

$$\bigcup_{x \in B(x_0,\delta)} \{g^{x,b}(t) : t \in [0,\infty]\} \cap B(a,\varepsilon) = \phi. \tag{14.21}$$

Once $g^{x,b}$ is inside $B(b,c_3)$ it doesn't leave $B(b,c_3)$, by (14.14) and we bound from above the total time the $g^{x,b}$'s spend outside $B(b,c_3)$ by

$$\rho(x,b) \geq \int_{\{t:|g^{x,b}(t)-b|\geq\varepsilon\}} V(g^{x,b}(t))\, dt \geq \frac{c_1\varepsilon^2}{2} |\{t : |g^{x,b} - b| \geq \varepsilon\}| \,.$$

Therefore there is a $t_0 < \infty$ so that $|g^{x,b}(t_0)-b| \leq c_3$ for all $x \in \overline{B(x_0,\delta)}$ and (14.2) implies (14.4) for $x \in \overline{B(x_0,\delta)}$.

To prove (14.5) we pick $\varepsilon > 0$ such that $|x - b| \leq \varepsilon_0$ implies $V''(x) \geq \frac{\omega_1^2}{2}$ and T_1 such that by (14.4) $|g^{x,b}(t) - b| \leq D_1 e^{-\omega_1 t} \leq \frac{\varepsilon_0}{3}$ for $x \in K_1$ and $t \geq T_1$.

Now we get

$$\ddot{g}^{x_1b}(t) - \ddot{g}^{x_1b}(t) = V'(g^{x_1b}(t)) - V'(g^{x_1b}(t)) \tag{14.22}$$

$$= \int_0^1 V''(g^{x_2b}(t) + s(g^{x_1b}(t) - g^{x_2b}(t)))\,(g^{x_1b}(t) - g^{x_2b}(t))\,ds$$

$$= V''(g^{x_2b} + \mathcal{Z}(t))\,(g^{x_1b}(t) - g^{x_2b}(t))$$

where

$$|g^{x_2b}(t) + \mathcal{Z}(t) - b| \leq |g^{x_2b}(t) - b| + |g^{x_2b}(t) - g^{x_1b}(t)|$$

$$\leq \frac{\varepsilon_0}{3} + |g^{x_2b}(t) - b| + |g^{x_1b}(t) - b| \leq \varepsilon_0$$

if $\quad t \geq T_1$ and x_1 , $x_2 \in K_1$.

Hence $\quad \dfrac{d^2}{dt^2}|g^{x_1b}(t) - g^{x_2b}(t)|^2$

$$= 2\langle V''(g^{x_2b}(t) + \mathcal{Z}(t))(g^{x_1b}(t) - g^{x_2b}(t)), (g^{x_1b} - g^{x_2b})\rangle$$

$$\geq \omega_1^2 |g^{x_1b}(t) - g^{x_2b}(t)|^2 \text{ if } t \geq T_1 .$$

As in the proof of sublemma 4.1 we get

$$|g^{x_1b}(t) - g^{x_2b}(t)|^2 \leq |g^{x_1b}(T_1) - g^{x_2b}(T_1)|^2\, e^{-\omega_1(t-T_1)}$$

$$\leq e^{-\omega_1 T_1} \|g^{x_1b} - g^{x_2b}\|^2_{L^\infty[0,\infty)}\, e^{-\omega_1 t}$$

if $t \geq T_1$ which gives (14.5).

Proof of Sublemma 8.3 and 8.4.

It suffices, by compactness, to show (14.6) holds near any (x_0 , y_0) with

$$\rho(x_0 , b) + \rho(b,y_0) < \rho(x_0 , a) + \rho(a,y_0).$$

Given such (x_0 , y_0) we pick $\delta_1 > 0$ such that

(14.23) $\rho(x,b) + \rho(b,y) < \rho(x,a) + \rho(a,y)$ for all $x \in \overline{B(x_0,\delta_1)}$.

Now we claim there exist $\delta_2 > 0$ and $T_1 > 0$ such that

$$(14.24) \quad \left[\bigcup_{T \geq T_1} \Big(\bigcup_{\substack{x \in B(x_0,\delta_1) \\ y \in B(y_0,\delta_1)}} \{g_T^{x,y}(t) : t \in [0,T]\} \Big) \right] \cap B(a,\delta_2) = \phi.$$

We prove the claim by contradiction. Otherwise there exist $T_n \to \infty$, $0 \leq t_n \leq T_n$, $x_n \to \bar{x} \in \overline{B(x_0,\delta_1)}$, $y_n \to \bar{y} \in \overline{B(y_0,\delta_1)}$ and $g_{T_n}^{x_n,y_n}(t_n) \to a$ as $n \to \infty$.

Then by Proposition 1

$$\rho(\bar{x},b) + \rho(b,\bar{y}) = \min_{c \in \{a,b\}} \{\rho(\bar{x},c) + \rho(c,\bar{y})\}$$

$$= \lim_{n \to \infty} A(x_n , y_n , T_n) \geq \underline{\lim_{n}} \int_0^{t_n} (\tfrac{1}{2}(\dot{g}_{T_n}^{x_n,y_n})^2 + V(g_{T_n}^{x_n,y_n}))\, dt$$

$$+ \underline{\lim_{n}} \int_{t_n}^{T_n} (\tfrac{1}{2}(\dot{g}_{T_n}^{x_n,y_n})^2 + V(g_{T_n}^{x_n,y_n}))\, dt$$

$\geq \rho(\bar{x},a) + \rho(a,\bar{y})$ which contradicts (14.23).

Put

(14.25) $\delta_3 = \min\left\{\delta_2 , \frac{c_0}{(1+\frac{4c_2}{c_2})^{\frac{1}{2}}}\right\}$ and pick $\delta_4 > 0$ such that

$V(x) \leq \delta_4$ implies $x \in B(a,\delta_3) \cup B(b,\delta_3)$ since $\delta_3 \leq \delta_2$

and (14.24) we have

$$|\{t \in [0,T] : |g_T^{x,y}(t) - b| \leq \delta_3\}| \geq$$

$$|\{t \in [0,T] : V(g_T^{x,y}(t)) \leq \delta_4\}| \geq$$

$$= T - |\{t \in [0,T] : V(_T^{x,y}(t)) > \delta_4^2\}|$$

$$\geq T - \frac{2}{\delta_4} \max_{\substack{x\in B(x_0,\delta_1) \\ y\in B(y_0,\delta_1)}} \{\rho(x,b) + \rho(b,y)\} =: T - t_0$$

(defining t_0) for all

$x \in B(x_0 , \delta_1)$, $y \in B(y_0 , \delta_1)$ and $T \geq$ some $T_2 \geq T_1$.

Here we used

$$\delta_4^2 |t \in [0,T] : V(g_T^{x,y}(t)) \geq \delta_4\}| \geq A(x,y,T)$$

and Proposition 1 that says

$$A(x,y,T) \to \min_{c\in(a,b)} \{\rho(x,c) + \rho(c,y)\}$$

$$= \rho(x,b) + \rho(b,y) \leq \max\{\rho(x,b) + \rho(b,y)\}$$

uniformly on $\overline{B(x_0,\delta_1)} \times \overline{B(x_0,\delta_1)}$ as $T \to \infty$.

For $T \geq T_2$ put $t_1 := \min_{0 \leq t \leq T} \{t : |g_T^{x,y}(t) - b| \leq \delta_3\}$

and $$t_2 := \begin{bmatrix} T & \text{if } |y-b| \leq \delta_3 \\ \max\{t \in [0,T] : (g_T^{x,y}(t) - b| = \delta_3\} & \text{if } |y-b| > \delta_3 \end{bmatrix}$$

Then $t_2 - t_1 \geq T - t_0$ and we now show

(14.26) $|g_T^{x,y}(t) - b| \leq c_0$, $\forall t \in [t_1 , t_2]$ and then we use sublemma 4.2.

The path

$$\gamma_T(t) = \begin{bmatrix} g_T^{x,y}(t) \text{ if } 0 \leq t \leq t_1 \text{ or } t_2 \leq t \leq T \\ b + (g_T^{x,y}(t_1) - b) \dfrac{\sinh c_2(t_2 - t)}{\sinh c_2(t_2 - t_1)} + (g_T^{x,y}(t_2) - b) \dfrac{\sinh c_2(t - t_1)}{\sinh c_2(t_2 - t_1)} \end{bmatrix}$$

if $t_1 < t < t_2$

shows $$\int_{t_1}^{t_2} (\tfrac{1}{2}(g^{x,y}(t))^2 + V(g_T^{x,y}(t)) \, dt \leq$$

$$\leq \frac{c_2}{2}\left(\frac{[|g_T^{x,y}(t_1) - b|^2 + |g_T^{x,y}(t_2) - b|^2] \cosh(c_2(t_2 - t_1)) - 2\langle (g_T^{x,y}(t_1) - b), (g_T^{x,y}(t_2) - b)\rangle}{\sinh(c_2(t_2 - t_1))}\right)$$

$$\leq c_2 \, \delta_3^2 \left(\frac{\cos h(c_2(t_2 - t_1)) + 1}{\sin(c_2(t_2 - t_1))}\right) \leq 2c_2 \, \delta_3^2$$

if $T \geq$ some $T_3 \geq T_2$, $|x - x_0| \leq \delta_1$ and

$|y - y_0| \leq \delta_1$ (since $t_2 - t_1 \geq T - t_0$).

On the other hand leaving $\{x : |x-b| \le c_0\}$ in $[t_1 , t_2]$ would make

$\int_{t_1}^{t_2} (\frac{1}{2}(\dot{g}_T^{x,y}(t))^2 + V(g_T^{x,y}(t))\, dt > 2(\frac{c_1 c_0^2}{4}(1 - \frac{\delta_3^2}{c_0^2}))$ by sublemma 4.4. Then by above

$2c_2\, \delta_3^2 > \frac{c_1 c_0^2}{2}(1 - \frac{\delta_3^2}{c_0^2}))$ or $(1 + 4\frac{c_2}{c_1})\, \delta_0^2 > c_0^2$ which contradicts (and explains!) (14.25).

Therefore we have (14.26) and sublemma 4.2 implies

$$|g_T^{x,y}(t) - b|^2 \le \delta_3^2 \frac{\sin h(c_1(t_2-t))}{\sin h(c_1(t_2-t_1))} + \min(\delta_3^2 , |y-b|^2) \frac{\sin h(c_1(t-t_1))}{\sin h(c_1(t_2-t_1))} .$$

We have uniform bounds on $|g_T^{x,y}(t) - b|$ on $[0,t_1]$ and $[t_2 , T]$ and then since $t_2 - t_1 \ge T - t_0$ where t_0 is independent of T we obtain

$$|g_T^{x,y}(t)-b|^2 \le 0(e^{-c_1 t} + \min(\delta_3^2 , |y-b|^2)e^{-c_1(T-t)}) \tag{14.27}$$

uniformly for $|x - x_0| \le \delta_1$, $|y - y_0| \le \delta_1$ and $T \ge T_3$.

Now we want to improve this and get (14.6). As before we write

$$V'(x) = \begin{bmatrix} \omega_1^2\, (x-b)_1 \\ \vdots \quad\quad \vdots \\ \omega_d^2\, (x-b)_d \end{bmatrix} + R(x)$$

where $R(x) \le D_0|x - b|^2$ if $|x - b| \le c_0$

and $0 < \omega_1 < \omega_2 < \cdots < \omega_d$.

On $[t_1, t_2]$ put $g_T^{x,y}(t) = f(t) + r(t)$ where

$$f_i(t) := b_i + (g_T^{x,y}(t_1)-b)_i \frac{\sinh \omega_i(t_2-t)}{\sinh \omega_i(t_2-t_1)} + (g_T^{x,y}(t_2)-b)_i \frac{\sinh \omega_i(t-t_1)}{\sinh \omega_i(t_2-t_1)}$$

satisfies $\ddot{f}_i = \omega_i^2(f_i - b_i)$

$$f(t_1) = g_T^{x,y}(t_1) \text{ and } f(t_2) = g_T^{x,y}(t_2) .$$

Therefore r_i satisfies

$$\ddot{r}_i(t) - \omega_i^2 r_i(t) = R_i(g_T^{x,y}(t))$$

$$r_i(t_1) = 0 = r_i(t_2)$$

i.e.,

$$(14.28) \qquad r_i(t) = -\frac{\sinh \omega_i(t_2-t)}{\omega_i \sinh \omega_i(t_2-t_1)} \int_{t_1}^{t} \sinh \omega_i(s-t_1)\, R_i(g_T^{x,y}(s))\, ds$$

$$- \frac{\sinh \omega_i(t-t_1)}{\omega_i \sinh \omega_i(t_2-t_1)} \int_{t}^{t_2} \sinh \omega_i(t_2-s)\, R_i(g_T^{x,y}(s))\, ds$$

and

$$(14.29) \qquad \dot{r}_i(t) = \frac{\cosh \omega_i(t_2-t)}{\sinh \omega_i(t_2-t_1)} \int_{t_1}^{t} \sinh \omega_i(s-t_1)\, R_i(g_T^{x,y}(s))\, ds$$

$$- \frac{\cosh \omega_i(t-t_1)}{\sinh \omega_i(t_2-t_1)} \int_{t}^{t_2} \sinh \omega_i(t_2-s)\, R_i(g_T^{x,y}(s))\, ds$$

since $t_1 \le t_0$ and $T - t_2 \le t_0$ where t_0 is independent of T and

$$R_i(g_T^{x,y}(t)) = 0(e^{-c_1 t} + \gamma^2 e^{-c_1(T-t)})$$

uniformly if $\gamma = \min(\delta_3 , |y-b|^2)$, we obtain

$$|\dot r_i(t)|,\ |r_i(t)| = 0(e^{-\omega_1 t}\int_0^t e^{\omega_i s}(e^{-c_1 s} + \gamma^2\, e^{-c_1(T-s)})\, ds$$

$$+\, e^{-\omega_i(T-t)}\int_t^T e^{\omega_i(T-s)}(e^{-c_1 s} + \gamma^2\, e^{-c_1(T-s)}) ds)$$

$$= 0(e^{-c_1 t} + \gamma^2\, e^{-c_1(T-t)})\ .$$

Since $|\dot f(t)|,\ |f(t) - b| \le 0(e^{-\omega_1 t} + \gamma e^{-\omega_1(T-t)})$ we get

$$|\dot g_T^{x,y}(t)|,\ |g_T^{x,y}(t) - \ | \le 0(e^{-c_1 t} + \gamma e^{-c_1(T-t)})\ .$$

Using this estimate in (14.24) and (14.28) yields $|\dot r_i(t)|, |r_i(t)| = 0(e^{-\min(\omega_1, 2c_1)(T-t)})$.

Assuming $nc_1 \notin \{\omega_1\ , \ldots, \omega_d\}$ for $n = 1, \ldots, 2, \ldots$ we get by induction

$$|\dot r_i(t)|,\ |r_i(t)| = 0(e^{-\omega_1 t} + \gamma^2 e^{-\omega_1(T-t)})$$

and hence

$$|\dot g_T^{x,y}(t)|,\ |g_T^{x,y}(t) - b| = 0(e^{-\omega_1 t} + \gamma e^{-\omega_1(T-t)})$$

uniformly. This completes the proof of (14.6).

If $K_2 = K_1 \times \{y : |y-b| \le \delta_3\}$ as in part 4, then $t_2 = T$ and we get

$$(\dot g_T^{x,y}(T))_i = \dot f_i(T) + \dot r_i(T)$$

$$= -\omega_i(g_T^{x,y}(t_1)-b)_i\left(\frac{1}{\sinh\ \omega_i(T-t_1)}\right) + \omega_i(y - b)_i\, \frac{\cosh\ \omega_i(T-t_1)}{\sinh\ \omega_i(T-t_1)}$$

$$+\ 0\ (e^{-\omega_1 T} + |y - b|^2)$$

$$= \omega_i(y - b)_i\ (1 + 0(e^{-2\omega_i T})) + 0(e^{-\omega_i T} + |y-b|^2)$$

uniformly in $K_2 \times [T_3\ , \infty)$ which proves (14.9).

By the Euler-Lagrange equations $\frac{1}{2}(\dot{g}_T^{x,y}(t))^2 - V(g_T^{x,y}(t))$ is independent of $t \in [0,T]$ and (14.7) follows from (14.6).

To prove (14.8)

$$A(x,y,T) = \rho(x,b) + \rho(b,y) + 0(e^{-\omega_1 T})$$

uniformly we use

(14.30) $\quad A(x,y,T) = A(x,g_T^{x,y}(\frac{T}{2}),\frac{T}{2}) + A(g_T^{x,y}(\frac{T}{2}),y,\frac{T}{2})$ for the lowerbound

and

(14.31) $\quad A(x,y,T) \leq A(x,b,\frac{T}{2}) + A(b,y,\frac{T}{2})$ for the upperbound.

If $\rho(x,b) + \rho(b,y) < \rho(x,a) + \rho(a,y)$ then $\rho(x,b) < \rho(x,a) + \rho(a,b)$ and so estimate (14.4) holds for $g^{x,b}(t)$. Taking

$$\gamma_T(t) = \begin{bmatrix} g^{x,b}(t) \text{ for } t \in [0,T-1] \\ (b-g^{x,b}(T-t))(t-T+1) + g^{x,b}(T-1) \end{bmatrix}$$
$$\text{for } t \in (T-1,T)$$

as a trial path for $A(x,b,T)$ gives

(14.32) $\quad A(x,b,T) \leq \rho(x,b) \times 0(e^{-2\omega_1 T})$ uniformly.

For the same reason $A(b,y,T) \leq \rho(b,y) + 0(e^{-2\omega_1 T})$, and by (14.31) we get the upperbound in (14.8)

$$A(x,y,T) \leq \rho(x,b) + \rho(b,y) + 0(e^{-\omega_1 T})$$

uniformly.

The triangle inequality, sublemma 4 and (14.6) give the lowerbound by (14.26)

$$A(x,y,T) = A(g_T^{x,y}(\tfrac{T}{2}),\tfrac{T}{2}) + A(g_T^{x,y}(\tfrac{T}{2}),y,\tfrac{T}{2})$$

$$\geq A(x,b,T) - A(g_T^{x,y}(\tfrac{T}{2}),b,\tfrac{T}{2})$$

$$+ \quad A(b,y,T) - A(b,g_T^{x,y}(\tfrac{T}{2}),\tfrac{T}{2}) \geq$$

$$\geq \rho(x,b) + \rho(b,y) - 2A(g_T^{x,y}(\tfrac{T}{2}),b,\tfrac{T}{2})$$

$$\geq \rho(x,b) + \rho(b,y) - c_2|g_T^{x,y}(\tfrac{T}{2})-b|^2 \frac{\cosh(c_2\frac{T}{2})}{\sinh(c_2\frac{T}{2})}$$

$$\geq \rho(x,b) + \rho(b,y) + 0(e^{-\omega_1 T})$$

uniformly.

<u>Proof</u> <u>of</u> <u>Sublemma</u> <u>8.5</u>. We reduce to proving that if t_0 is such that

(14.43) $\quad |g^{x_0 b}(t)-b| \leq \frac{c_0}{2}$ for $t \geq t_0$

then

(14.44) $\quad$ If $g = g_T^{x_1 y_1}$ or $g = g^{x,b}$ then

$$\frac{d^2}{dt^2}|g_T^{x,y}(t)-g(t)|^2 \geq c_1^2|g_T^{x,y}(t)-g(t)|^2$$

if $t \in [t_0 \, , T]$ and if $g = g_T^{x,b}$ or $g = g^{x,b}$ and $s \geq 2(t_0 + 1)$

then

(14.45) $\quad \|g_T^{x,y} - g\|^2_{L^\infty[0,s]} = 0(|\dot g_T^{x,y}(s)-\dot g(s)| \; |g_T^{x,y}(s)-g(s)|)$

uniformly near $(x_0 \, , b \, , \infty)$.

By the differential inequality in (14.44) for $g = g_T^{x,y}$ and the endpoint conditions

$$|g_T^{x,y}(t_0) - g_T^{x_1y_1}(t_0)|^2 \leq \|g_T^{x,y} - g_T^{x_1y_1}\|_{L^\infty[0,T]},$$

$$|g_T^{x,y}(T) - g^{x_1y_1}(T)|^2 = |y - y_1|^2_T$$

we get (see Protter and Weinberger [1]).

$$(14.46) \qquad |g_T^{x,y}(t) - g(t)|^2 \leq |g_T^{x,y}(t_0) - g(t_0)|^2 \frac{\sin h\, c_1(T-t)}{\sin h\, c_1(T-t_0)}$$

$$+ |g_T^{x,y}(T) - g(T)|^2 \frac{\sin h\, c_1(t-t_0)}{\sin h\, c_1(T-t_0)} =$$

$$= 0(\|g_T^{x,y} - g_T^{x,y}\|^2_{L^\infty[0,T]} e^{-c_1 t} + |y - y_1|^2 e^{-c_1(T-t)})$$

uniformly, which proves (14.12).

If we take $S = \frac{T}{2}$ and $g = g^{x,b}$ or $g = g_T^{x,y}$ in (14.44) we get

$$|g_T^{x,y}(t_0) - g(t_0)|^2 \leq \|g_T^{x,y} - g\|^2_{L^\infty[0,\frac{T}{2}]} = 0(e^{-\omega_1 T})$$

uniformly by (14.4) and (14.6).

Also in both cases

$$|g_T^{x,y}(T) - g(T)|^2 = 0(|y-b|^2 + e^{-2\omega_1 T})$$

and now by (14.44) we have the first part of (14.46) which implies (recall $c_1 \le \omega_1$)

$$|g_T^{x,y}(t) - g(t)|^2 \le 0(e^{-\omega_1 T})\frac{\sinh c_1(T-t)}{\sinh c_1(T-t_0)}$$

$$+ 0(|y-b|^2 + e^{-2\omega_1 T})\frac{\sinh c_1(t-t_0)}{\sinh c_1(T-t_0)}$$

$$= 0(e^{-\omega_1 T}\, e^{-c_1 t} + |y-b|^2\, e^{-c_1(T-t)})$$

and we have proven (14.10).

The first part of (14.11) follows from the differential inequality in (14.44) applied to $[t_0 , \frac{T}{2}]$ and the bound $\|g_T^{x,y} - g^{x,b}\|_{L^\infty[0,\frac{T}{2}]} = 0(e^{-\omega_1 T})$ mentioned above.

The second part can be proven by an analog of (14.45) on $[T - \bar{S}, T]$.

Next we will prove (14.44). Using sublemma 5 in chapter 13 we choose δ_0 and T_0 such that $g_T^{x_1 y_1}(t), g^{x,b}(t) \in B(b,\frac{c_0}{2})$ for all

$$(x_1 , y_1 , T , t) \in B(x_0 , \delta_0) \times B(b,\delta_0) \times [T_0 , \infty) \times [t_0 , T]$$

where t_0 is from (14.43).

If $g = g_T^{x_1 y_1}$ or $g = g^{x,b}$ then

$$(14.47)\qquad \frac{d^2}{dt^2}|g_T^{x,y}(t) - g(t)|^2 = 2|\dot g_T^{x,y} - \dot g|^2$$

$$+ 2\langle(\ddot g_T^{x,y} - \ddot g),(g_T^{x,y} - g)\rangle =$$

$$= 2|\dot g_T^{x,y} - \dot g|^2 + 2\langle(V'(g_T^{x,y}) - V'(g)),(g_T^{x,y} - g)\rangle$$

$$= 2|\dot g_T^{x,y} - \dot g|^2 + 2\langle\int_0^1 V''(g+u(g_T^{x,y}-g))(g_T^{x,y}-g),(g_T^{x,y}-g)\rangle\, du$$

$$= 2|\dot g_T^{x,y} - \dot g|^2 + 2\langle V''(g+\mathcal{Z})(g_T^{x,y}-g),(g_T^{x,y}-g)\rangle\,.$$

When $t \in [t_0 , T]$, $g(t) + \mathcal{Z}(t) \in B(b,c_0)$ and we get by (10.1) and (14.47)

$$\frac{d^2}{dt^2}|g_T^{x,y} - g|^2 \geq 2c_1^2|g_T^{x,y} - g|^2$$

$$\geq c_1^2|g_T^{x,y} - g|^2$$

which is (14.44).

(14.47) also implies for $s \geq 2(t_0 + 1)$

$$\int_{t_0+1}^{s} \frac{d^2}{dt^2}|g_T^{x,y}(t) - g(t)|^2\, dt =$$

$$\leq 2\int_{t_0+1}^{s} (|\dot g_T^{x,y} - \dot g|^2 + c_1^2|\, g_T^{x,y} - g|^2)\, dt$$

$$\leq \min(c_1 , c_1^2)\|g_T^{x,y} - g\|^2_{L^\infty[t_0+1,s]}$$

as in the proof of (13.16).

If $g = g_T^{x,y}$ or $g = g^{x,b}$ then $\int_0^{t_0+1} \frac{d^2}{dt^2}|g_T^{x,y} - g|^2\, dt$

$$= \int_0^{t_0+1} (2|\dot{g}_T^{x,y} - \dot{g}|^2 + 2\langle V''(g + \mathcal{Z})(g_T^{x,y} - g),(g_T^{x,y} - g)\rangle\, dt$$

$$\geq k_1 \|g_T^{x,y} - g\|^2_{L^\infty[0,t_0+1]}$$

after taking smaller $\delta_0 > 0$ and larger $T_0 < \infty$ if needed. By the above we have:

If $s \geq 2(t_0 + 1)$ then

$$2\langle(\dot{g}_T^{x,y}(s) - \dot{g}(s)),(g_T^{x,y}(s) - g(s))\rangle$$

$$= \int_0^s \frac{d^2}{dt^2} |g_T^{x,y} - g|^2\, dt$$

$$= 2\int_0^s (|\dot{g}_T^{x,y} - \dot{g}|^2 + \langle V''(g+\mathcal{Z})(g_T^{x,y} - g),(g_T^{x,y} - g)\rangle\, dt$$

$$\geq \min(k_1 , c_1 , c_1^2)\|g_T^{x,y} - g\|^2_{L^\infty[0,s]} \text{ for } g = g_T^{x,y} \text{ or } g^{x,b}$$

which completes the proof.

15. On the Solutions of the Jacobi Equations.

§15.1. Asymptotics.

§15.2. Estimates and derivatives of minimal action paths and Agmon geodesics.

§15.1. Asymptotics.

We assume (1.33) $V''(b) = \Omega^2$ where $\Omega = \mathrm{diag}(\omega_1, \dots, \omega_d)$ with $0 < \omega_1 \le \omega_2 \le \omega_d$ and we put

$$(15.1)\quad \delta_1 = \min_{\{j:\omega_{j-1} \neq \omega_j\}} \{ \tfrac{c_1}{2}, \omega_j - \omega_{j-1} \} > 0 .$$

Sublemma 9. Assume $\rho(x_0, b) < \rho(x_0, a) + \rho(a,b)$ there is a unique $g^{x_0,b}$ and it has positive definite second variation. Then there exist δ_0 and T_0 positive such that

1. For all $x \in B(x_0, \delta_0)$

(15.2) $\ddot{Z}(t) = V''(g^{x,b}(t))\, Z(t)$ on $[0,\infty)$ has matrix solutions $X^{x,b}$ and $Y^{x,b}(t)$

such that uniformly for $x \in B(x_0, \delta_0)$

$$(15.3)\quad X^{x,b}(t) = (I + 0(e^{-\delta_1 t}))e^{-\Omega t} = -\dot{X}^{x,b}(t)\, \Omega^{-1} \text{ and}$$

$$(15.4)\quad Y^{x,b}(t) = (I + 0(e^{-\delta_1 t}))\, e^{\Omega t} = \dot{Y}^{x,b}(t)\, \Omega^{-1} \text{ as } t \to \infty$$

where $X^{x,b}(t)$ is nonsingular by (13.8).

Moreover

$$(15.5)\quad |(X^{x,b}(t) - X^{x_0 b}(t))e^{\Omega t}| = 0(e^{-\delta_1 t} \|g^{x,b} - g^{x_0 b}\|_\infty) =$$

$$= |(\dot{X}^{x,b}(t) - \dot{X}^{x_0 b}(t))e^{\Omega t}|$$

and

$$(15.6)\quad |(Y^{x,b}(t) - Y^{x_0 b}(t))e^{\Omega t}| = 0(e^{-\delta_1 t} \|g^{x,b} - g^{x_0 b}\|_\infty) =$$

$$= |(\dot{Y}^{x,b}(t) - \dot{Y}^{x_0 b}(t))e^{\Omega t}|).$$

If x_0 then $X^{b,b}(t) = e^{-t\Omega}$ and $Y^{b,b}(t) = e^{\Omega t}$ and

(15.7) $$|X^{y,b}\, e^{\Omega t} - I| = 0(e^{-\delta_1 t}|y-b|) = |\dot{X}^{y,b}(t)\, e^{\Omega t} + \Omega|$$

and

(15.8) $$|Y^{x,b}(t)\, e^{-\Omega t} - I| = 0(e^{-\delta_1 t}|y-b|) = |\dot{Y}^{x,b}(t)\, e^{\Omega t} - \Omega|$$

uniformly for $y \in B(b,\delta_0)$ as $t \to \infty$.

2. If $W(t)$ for $t \in (0,\infty)$ are $(d \times d)$-matrices that are integrable with $\int_0^\infty |W(t)|dt < \infty$ then

(15.9) $\ddot{Z}(t) = (V''(g^{x,b}(t)) + W(t))\, Z(t)$ on $[0,\infty)$ has solutions $X_W^{x,b}$ and $Y_W^{x,b}$ satisfying

(15.10) $$X_W^{x,b}(t) - X^{x,b}(t) = 0(\int_0^t e^{-\delta_1(t-s)}|W(s)|ds + \int_t^\infty |W(s)|ds)e^{-\Omega t}$$
$$= \dot{X}_W^{x,b}(t) - \dot{X}^{x,b}(t)$$

and

(15.11) $$Y_W^{x,b}(t) - Y^{x,b}(t) = 0(\int_0^t e^{-\delta_1(t-s)}|W(s)|ds + \int_t^\infty |W(s)|ds)e^{\Omega t}$$
$$= \dot{Y}_W^{x,b}(t) - \dot{Y}^{x,b}(t)$$ uniformly for $x \in B(x_0\, ,\, \delta_0)$.

3. The system

(15.12) $\ddot{Z}(t) = V''g_T^{x,y}(t))\, Z(t)$ on $[0,T]$ has matrix solutions $X^{x,y,T}$ and $Y^{x,y,T}$

such that

(15.13) $(X^{x,y,T}(t)-X^{x,b}(t))\, e^{\Omega t} = 0(e^{-\omega_1 \frac{T}{2}} + |y-b|)$

$= (\dot{X}^{x,y,T}(t)-\dot{X}^{x,b}(t))\, e^{\Omega t}$

and

(15.14) $(Y^{x,y,T}(t) - Y^{x,b}(t))\, e^{-\Omega t} = 0(e^{-\omega_1 \frac{T}{2}} + |y-b|)$

$= (\dot{Y}^{x,y,T}(t)-\dot{Y}^{x,b}(t))\, e^{-\Omega t}$.

4. For $(r,T) \in [0,r_1] \times [T_0, \infty)$ let $W_T^r(t)$ be continuous symmetric $(d \times d)$-matrices on $[0,T]$ with

(15.15) $\int_0^T |W_T^r(t)|\, dt = 0(r)$ as $r \downarrow 0$ uniformly for $T \in [T_0, \infty)$.

Then there exists a r_0 such that $(-\frac{d^2}{dt^2} + V''(g_T^{x,y}(t)) + W_T^r(t)) > 0$ on $D_0(0,T)$ (see (2.30) and (2.31)) for all $(x,y,T,r) \in B(x_0, \delta_0) \times B(b,\delta_0) \times [T_0, \infty) \times [0,r_0]$.

Moreover

(15.16) $\ddot{Z} = (V''(g_T^{x,y}(t) \times W_T^r(t))Z(t)$ on $[0,T]$ has matrix solutions $X_r^{x,y,T}$ and $Y_r^{x,y,T}$ satisfying

(15.17) $X_r^{x,y,T}(t) - X^{x,y,T}(t) =$

$= 0(\int_0^t e^{-\delta_1(t-s)} |W_T^r(s)| ds + \int_t^T |W_T^r(s)| ds) e^{-\Omega t} =$

$= \dot{X}_r^{x,y,T}(t) - \dot{X}^{x,y,T}(t)$

and

(15.18) $Y_r^{x,y,T}(t) - Y^{x,y,T}(t) =$

$$0(\int_0^t e^{-\delta_1(t-s)}|W_T^r(s)| + \int_t^T |W_T^r(s)|ds)e^{\Omega t} =$$

$$= \dot{Y}_r^{x,y,T}(t) - \dot{Y}^{x,y,T}(t).$$

Proof. To prove (15.3) we write (15.2) as

$$\ddot{Z}(t) = V''(g^{x,b}(t))\, Z = (\Omega^2 + V''(g^{x,b}(t)) - \Omega^2)\, Z(t)$$

as a first order system

(15.19) $\dot{\varphi}(t) = (\Lambda + R^{x,b}(t)\, \varphi(t)$

where $\Lambda = \mathrm{diag}(\lambda_1 , \lambda_2 , ..., \lambda_{2d})$ with $\lambda_{2i-1} = \omega_i$

and $\lambda_{2i} = -\omega_i$ for $i \in \{1,...,d\}$ and $R^{x,b}(t)$

satisfies $|R^{x,b}(t)| = 0(|V''(g^{x,b}(t)) - V''(b)|)$

$= 0(|g^{x,b}(t) - b|) = 0(e^{-\omega_1 t})$,

by (14.4).

Then we use successive approximations as described in exercise 29 of chapter 3 in Coddington-Levinson [1].

Notice φ, Λ and $R^{x,b}$ in (15.19) are (2d x 2d)-matrices. We get (15.19) from (15.2) by defining a (2d x 2d)-matrix through

$$(15.21)\qquad \begin{bmatrix} \mathfrak{Z}_{2i-1,2j-1} = Z_{ij}\,,\ \mathfrak{Z}_{2i,2j-1} = 0 \\ \mathfrak{Z}_{2i-1,2j} = 0 \text{ and } \mathfrak{Z}_{2i,2j} = \dot{Z}_{ij} \end{bmatrix}$$

Then set

$$(15.22)\qquad \varphi = C\mathfrak{Z} \text{ where}$$

$$C = \mathrm{diag}\left(\begin{bmatrix} \omega_1 & 1 \\ -\omega_1 & 1 \end{bmatrix}, \ldots, \begin{bmatrix} \omega_d & 1 \\ -\omega_d & 1 \end{bmatrix} \right)$$

For $j \in \{1,\ldots,d\}$ be $P_{1,j}$ and $P_{2,j}$ be projection matrices such that

$$(15.23)\qquad P_{1,j} + P_{2,j} = I \quad \text{and}$$

$$V_{1,j}(t) := e^{\Lambda t}\, P_{1,j} \text{ contains } e^{\lambda_k t} \text{ with } \lambda_k < \lambda_j \text{ and}$$

$$V_{2,j}(t) := e^{\Lambda t}\, P_{2,j} \text{ contains } e^{\Lambda_k t} \text{ with } \lambda_k \geq \lambda_j\,.$$

Then there are constants $k_{1,j}$ and $k_{2,j}$ such that $k_{1,d} + k_{2,d} = 2d$ and

$$(15.24)\qquad |V_{1,j}(t)| \leq k_{1,j}\, e^{(\lambda_j - \delta_i)t} \qquad \text{if } t \geq 0$$

$$|V_{2,j}(t)| \leq k_{2,j}\, e^{\lambda_j t} \qquad \text{if } t \leq 0$$

(recall (15.1)) where $|A| = \sum_{1 \leq i,j \leq n} |a_{i,j}|$ for an $(n \times n)$ $-$ matrix A.

Pick t_0 such that

(15.25) $(2d)\int_{t_0}^{\infty}|R^{x,b}(t)|\,dt < \frac{1}{2}$ for all $x \in B(x_0\,,\,\delta_0)$

where δ_0 is as in sublemma 8.5 in chapter 14.

With $\varphi_{j,-1}^{x,b}(t) = 0$, $\varphi_{j,0}^{x,b}(t) = e^{\lambda_j t} f_j$ where $f_j = (\delta_{i,j})_{i=1}^{2,d}$ and

$$\varphi_{j,\ell+1}^{x,b}(t) = e^{\lambda_j t} f_j + \int_{t_0}^{t} V_{1,j}(t-s)\, R^{x,b}(s)\, \varphi_{j,\ell}(s)\, ds \tag{15.26}$$

$$-\int_{t}^{\infty} V_{2,j}(t-s)\, R^{x,b}(s)\, \varphi_{j,\ell}(s)\, ds$$

we get by induction, using (15.24) and (15.25),

$$|\varphi_{j,\ell+1}^{x,b} - \varphi_{j,\ell}^{x,b}(t)| \le \frac{e^{\lambda_j t}}{2^{\ell+1}} \text{ and therefore} \tag{15.27}$$

$$\varphi^{x,b}{}_{j,\ell}(t) = \sum_{k=0}^{\ell} (\varphi^{x,b}{}_{j,k}(t) - \varphi^{x,b}{}_{j,k-1}(t)) \to \varphi_j^{x,b}(t)$$

uniformly on compacts as $\ell \to \infty$.

By (15.27) $|\varphi_j^{x,b}(t)| \le 2e^{\lambda_j t}$ and then by (15.26) it satisfies

$$\varphi_j^{x,b}(t) = e^{\lambda_j t} f_j + \int_{t_0}^{t} V_{1,j}(t-s)\, R^{x,b}(s)\, \varphi_j^{x,b}(s) - \tag{15.28}$$

$$- \int_{t}^{\infty} V_{2,j}(t-s)\, R^{x,b}(s)\, \varphi_j^{x,b}(s)\, ds$$

and hence it is a solution of (15.19).

Using (15.20), (15.24) and $\delta_1 \le \frac{\omega_1}{2}$ one can show

$$|\varphi_j^{x,b}(t) - e^{\lambda_j t} f_j| =$$

$$= 0(\int_{t_0}^{t} e^{(\lambda_j-\delta_1)(t-s)} e^{-\omega_1 s} ds + \int_t^\infty e^{-\lambda_j(t-s)} e^{-\omega_1 s} ds)$$

$$= 0(e^{(\lambda_j-\delta_1)t}) \text{ uniformly.}$$

Extend the $\varphi_j^{x,b}(t)$ as a solution of (15.19) to all of $(0,\infty)$ and form a matrix solution $\Phi^{x,b}(t)$ whose j^{th}-column is $\varphi_j^{x,b}$. Then

(15.29) $\Phi^{x,b}(t) = (I + 0(e^{-\delta_1 t}))\, e^{\Lambda t}$ uniformly as $t \to \infty$ and $\Phi^{x,b}(t)$

is a fundamental solution of (15.19) on $(0,\infty)$,
since by Liouville-formula (Hartman [1])

(15.30) $\det \Phi^{x,b}(s) = \det \Phi^{x,b}(t) \exp(\int_t^s \text{trace}(\Lambda + R^{x,b}(s))ds)$
and $\det \Phi^{x,b}(t) \neq 0$ for t large.

To make it easier to control the x-dependence of the solutions we don't use these φ's but put

(15.31) $V_{2,j}^{x,b}(t,s) = \Phi^{x,b}(t), P_{1,j}(\Phi^{x,b}(s))^{-1}$ and

$V_{2,j}^{x,b}(t,s) = \Phi^{x,b}(t), P_{2,j}(\Phi^{x,b}(s))^{-1}$

where $P_{1,j}$ and $P_{2,j}$ are from (15.23). Then by (15.29)

$$(15.32)\qquad |V_{1j}^{x,b}(t,s)| = 0(e^{(\lambda_j-\delta_1)(t-s)}) \quad \text{if } s,t \geq 0,\ t-s \geq 0$$

$$|V_{2j}^{x,b}(t,s)| = 0(e^{\lambda_j(t-s)}) \qquad \text{if } s,t \geq 0,\ t-s \leq 0$$

again uniformly for $x \in B(x_0, \delta_0)$.

Now we define $\psi_j^{x_0,b}(t) = \varphi_j^{x_0,b}(t)$ and

$$(15.33)\qquad \psi_j^{x,b}(t) = \psi_j^{x_0,b}(t) + \int_0^t V_{1j}^{x,b}(t,s)(R^{x,b}(s) - R^{x_0,b}(s))\,\psi_j^{x_0,b}(s)\,ds$$

$$-\int_t^{\infty} V_{2j}^{x,b}(t,s)(R^{x,b}(s) - R^{x_0,b}(s))\,\psi_j^{x_0,b}(s)\,ds$$

which solves (15.19) and satisfies

$$(15.34)\qquad \psi_j^{x,b}(t) = e^{\lambda_j t} f_j + 0(e^{(\lambda_j-\delta_1)t}).$$

Going back to the second order system (15.2) $\psi_{2j}^{x,b}(t)$ give a solution $X_j^{x,b}(t)$ (recall $\lambda_{2j} = -\omega_j$) satisfying $X_j^{x,b}(t) = e^{-\omega_j t}(e_j + 0(e^{-\delta_1 t})) = -\frac{\dot{X}^{x,b}(t)}{\omega_j}(t)$ where $e_j = (\delta_{ij})_{i=1}^{d}$ and $X_{2j-1}^{x,b}(t)$ gives solutions $Y_j^{x,b}(t)$ of (15.2) satisfying $Y_j^{x,b}(t) = e^{\omega_j t}(e_j + 0(e^{-\delta_1 t})) = \frac{\dot{Y}^{x,b}(t)}{\omega_j}$.

Hence the matrix $X^{x,b}(t)$ whose j^{th} column is $X_j^{x,b}(t)$ is a solution of (15.2) satisfying (15.3). Similarly we get a matrix function $Y^{x,b}(t)$ from the $Y_j^{x,b}$ is that satisfies (15.2) and (15.4).

Using (14.5) that says

$$|g^{x,b}(t) - g^{x_0,b}(t)| = 0(\|g^{x,b} - g^{x_0,b}\|_\infty\, e^{-\omega_1 \frac{t}{2}})$$

in (15.33) gives

$$|\psi_j^{x,b}(t) - \psi_j^{x_0,b}(t)| = 0(e^{(\lambda_j - \delta_1)t} \|g^{x_0,b} - g^{x,b}\|_\infty)$$

which implies (15.5) and (15.6).

If $x_0 = b$ then $R^{x_0,b} = 0$ so $\psi_j^{b,b}(t) = \varphi_j^{x_0,b}(t) = e^{\lambda_j t}$ and hence $X^{bb}(t) = e^{-\Omega t}$ and $Y^{b,b}(t) = e^{\Omega t}$.

In that case $\|g^{y,b}(t) - b\|_\infty = 0(|y-b|)$ and (15.7) follows from (15.5) and (15.8) from (15.6).

<u>Proof of Sublemma 9.2</u>.

We follow the proof of part 1 above and write (15.9) as a first order system

(15.35) $\quad \dot\varphi(t) = (\Lambda + R^{x,b}(t) + \tilde W(t))\, \varphi(t)$ with Λ and $R^{x,b}$ from (15.19)

and

(15.36) $\quad \tilde W(t) = 0(|W(t)|)$.

Successive approximations give solutions $\varphi_{j,w}^{x,b}$ of (15.35) such that

(15.37) $$e^{-\lambda_j t} |\varphi_{j,w}^{x,b}(t) - e^{\lambda_j t} f_j|$$

$$= 0(e^{-\delta_1 t} \int_{t_0}^{t} e^{\delta_1 s}(|R^{x,b}(s)| + |\tilde W(s)|)ds + \int_t^\infty (|R^{x,b}(s)| + |\tilde W(s)|)ds)$$

uniformly for $(x,t) \in B(x_0, \delta_0) \times [t_0, \infty)$, for some $t_0 \geq 0$.

Now extend $\varphi_{j,w}^{x,b}$ as a solution of (15.35) on all of $(0,\infty)$ and let it be the j^{th}-column of a matrix $\Phi_w^{x,b}(t)$ on $(0,\infty)$.

According to (15.37)

(15.38) $\Phi_w^{x,b}(t) = (I + 0(1))\, e^{\Lambda t}$ uniformly,

thus by an analog of (15.30) $\Phi_w^{x,b}(t)$

is a fundamental solution of (15.35) on $(0,\infty)$.

If P_{1j} and P_{2j} are as in (15.23) and we put $V_{1j,w}^{x,b}(t,s) := \Phi_w^{x,b}(t)\, P_{1j}(\Phi_w^{x,b}(s))^{-1}$ and $V_{2j,w}^{x,b}(t,s) := \Phi_w^{x,b}(t)\, P_{2j}(\Phi_w^{x,b}(s))^{-1}$ then, by (15.38)

(15.39) $|V_{1j,w}^{x,b}(t,s)| = 0(e^{(\lambda_j-\delta_1)(t-s)})$ if $t,s \geq 0$ and $t - s \geq 0$

$|V_{2j,w}^{x,b}(t,s)| = 0(e^{\lambda_j(t-s)})$ if $t,s \geq 0$ and $t - s \leq 0$

uniformly for $x \in B(x_0, \delta_0)$.

With $\psi_j^{x,b}(t)$ from (15.32) we set

(15.40) $$\psi_{j,w}^{x,b}(t) = \psi_j^{x,b}(t) + \int_0^t V_{1j,w}^{x,b}(t,s)\, \tilde{W}(s)\, \psi_j^{x,b}(s)$$

$$-\int_t^\infty V_{1j,w}^{x,b}(t,s)\, \tilde{W}(s)\, \psi_j^{x,b}(s)\, ds$$

which satisfies (15.35) and using (15.39), (15.34) and (15.36) we get

$$(15.41)\qquad |\psi^{x,b}_{j,w}(t) - \psi^{x,b}_{j}(t)| = 0(\int_0^t e^{(\lambda_j-\delta_1)(t-s)} |\tilde{W}(s)| e^{\lambda_j s} ds$$

$$+ \int_t^{\infty} e^{\lambda_j(t-s)} |\tilde{W}(s)| e^{\lambda_j s} ds) =$$

$$= 0(\int_0^t e^{-\delta_1(t-s)} |W(s)| ds + \int_t^{\infty} |W(s)| ds) e^{\lambda_j t} .$$

Now $\psi^{x,b}_{2j,w}(t)$ in (15.39) gives a solution $X^{x,b}_{j,w}(t)$ of (15.9) that by (15.40) obeys (recall $\lambda_{2j} = -\omega_j$)

$$X^{x,b}_{j,w}(t) - X^{x,b}_{j}(t) = 0(\int_0^t e^{-\delta_1(t-s)} |W(s)| ds + \int_0^t |W(s)| ds) e^{-\omega_j t}$$

$$= \dot{X}^{x,b}_{j,w}(t) - \dot{X}^{x,b}_{j}(t)$$

where $X^{x,b}_{j}(t)$ are as in the proof of part 1 and (15.10) follows if we let $X^{x,b}_{j,w}(t)$ be the j^{th}-column of $X^{x,b}_{w}(t)$.

Similarly the $\psi^{x,b}_{2j-1,w}$'s give a matrix solution $Y^{x,b}_{w}(t)$ of (15.9) satisfying (15.11).

<u>Proof of Sublemma 9.3</u>.

We will follow the proof of sublemma 9.1 closely. Write (15.12) as a first order system

$$(15.42)\qquad \dot{\varphi}^{x,y,T}(t) = (\Lambda + R^{x,y,T}(t))\, \varphi^{x,y,T}(t) \text{ on } [0,T]$$

where Λ is as in (15.19) and

(15.43) $|R^{x,y,T}(t)| = 0(|V''(g_T^{x,y}(t)) - V''(b)|) = 0(|g_T^{x,y}(t) - b|)$

$= (e^{-\omega_1 t} + |y-b|\, e^{-\omega_1(T-t)})$

uniformly for

$(x,y,T,t) \in b(x_0 , \delta_0) \times B(b,\delta_0) \times [T_0 ,\infty) \times [0,T]$, by (14.6).

As in the proof of part 1 we get, using successive approximations, $\Phi^{x,y,T}(t,s)$

of

(15.42) such that if $P_{1,j}$ and $P_{2,j}$ are as in (15.23) then

$$V_{1,j}^{x,y,T}(t,s) := \Phi^{x,y,T}(t)\, P_{1,j}(\Phi^{x,y,T}(s))^{-1}$$

and

$$V_{2,j}^{x,y,T}(t,s) := \Phi^{x,y,T}(t)\, P_{2,j}(\Phi^{x,y,T}(s))^{-1}$$

satisfy

(15.44) $|V_{1,j}^{x,y,T}(t,s)| = 0(e^{(\lambda_j-\delta_1)(t-s)})$ if $T \geq t,\ s \geq 0$

and $t - s \geq 0$ and

$|V_{2,j}^{x,y,T}(t,s)| = 0(e^{\lambda_j(t-s)})$ if $T \geq t,\ s \geq 0$ and $t - s < 0$

uniformly for $(x,y,T) \in B(x_0 , \delta_0) \times B(b,\delta_0) \times [T_0 , \infty)$.

For $(x,y,T,t) \in B(x_0 , \delta_0)\times B(b,\delta_0) \times [T_0 , \infty) \times [0,T]$ put

$$(15.45)\qquad \psi_j^{x,y,T}(t) = \psi_j^{x,b}(t) + \int_0^t V_{1j}^{x,y,T}(t,s)(R^{x,y,T}(s)-R^{x,b}(s))\,\psi_j^{x,b}(s)\,ds$$

$$-\int_t^T V_{2j}^{x,y,T}(t,s)(R^{x,y,T}(s) - R^{x,b}(s))\,\psi_j^{x,b}(s)\,ds$$

with $\psi_j^{x,b}(t)$ as in (15.33), which solves (15.42).

On $[0,\frac{T}{2}]$ we estimate

$$|R^{x,y,T}(s) - R^{x,b}(s)| = 0(|g_T^{x,y}(s) - g^{x,b}(s)|)$$

$$= 0(e^{-\frac{c_1 s}{2}} + e^{-c_1(\frac{T}{2}-s)})\,e^{-\omega_1\frac{T}{2}}$$

uniformly by (14.11) and on $[\frac{T}{2}, T]$ we get

$$|R^{x,y,T}(s) - R^{x,b}(s)| = 0(|g_T^{x,y}(s) - g^{x,b}(s)|)$$

$$= 0(|g_T^{x,y}(s) - b| + |g^{x,b}(s) - b|)$$

$$= 0(e^{-\omega_1 s} + |y-b|\,e^{-\omega_1(T-s)})$$

uniformly by (14.4) and (14.6).

Those estimates and (15.44) put into (15.45) give

$$(15.46)\qquad \psi_j^{x,y,T}(t) = \psi_j^{x,b}(t) + 0(e^{-\omega_1\frac{T}{2}} + |y-b|)\,e^{\lambda_j t}$$

which implies (15.13) and (15.14).

The Outlines Of A Proof Of Sublemma 9.5.

The positivity of $(-\frac{d^2}{dt^2} + V''(g_T^{x,y}) + W_T^r)$ on $D_0(0,T)$ follows from the positivity of $(-\frac{d^2}{dt^2} + V''(g_T^{x,y}))$ (see sublemma 7.1 in chapter 13) and the estimate (15.15).

We write (15.16) as a first order system

(15.47) $\dot{\varphi} = (\Lambda + R^{x,y,T} + \tilde{W}_T^r)\varphi$ and we get, using successive approximations, a fundamental solution $\Phi(t) = \Phi^{x,y,T,r}(t)$ so that

$V_{1,j}^{x,y,T,r}(t,s) := \Phi(t), P_{1,j}(\Phi(s))^{-1}$ and

$V_{2,j}^{x,y,T,r}(s) := \Phi(t), P_{2,j}(\Phi(s))^{-1}$

satisfy the estimate in (15.44)

With $\psi_j^{x,y,T}(t)$ as in (15.45) we set

$$\psi_j^{x,y,T}(t) = \psi_j^{x,y,T,r}(t) + \int_0^t V_{1,j}^{x,y,T,r}(t,s)\, \tilde{W}_T^r(s)\, \psi_j^{x,y,T}(s)\, ds$$
$$- \int_t^T V_{1,j}^{x,y,T,r}(t,s)\, \tilde{W}_T^r(s)\, \psi_j^{x,y,T}(s)\, ds$$

which solves (15.47) and we get

$$|\psi_j^{x,y,T,r}(t) - \psi_j^{x,y,T}(t)| = 0(\int_0^t e^{-\delta_1(t-s)} |W_T^r(s)|\, ds + \int_t^T |W_t^r(s)| ds)\, e^{\lambda_j t}$$

which implies (15.17) and (15.18).

§15.2. Estimates And Derivatives Of Minimal Action Paths And Agmon Goedesics.

Sublemma 10. Under the assumptions of sublemma 9 and with δ_0 and T_0 as there, we have

1. $g_T^{x,y}$ and $g^{x,b}$ are differentiable w.r.t. x and y for

$$(x,y,T) \in B(x_0,\delta_0) \times B(b,\delta_0) \times [T_0,\infty).$$

With $X = X^{x,y,T}$ and $Y = Y^{x,y,T}$ as in (15.13)

$$(15.48)\quad (\tfrac{d}{dt})^j \tfrac{\partial}{\partial x_i} g_T^{x,y}(t) = \tfrac{\partial}{\partial x_i}(\tfrac{d}{dt})^j g_T^{x,y}(t) =$$

$$(\tfrac{d}{dt})^j(X(t)-Y(t)(Y(T))^{-1}X(T))(X(0)-Y(0)(Y(T))^{-1}X(T))^{-1}\, e_i$$

and

$$(15.49)\quad (\tfrac{d}{dt})^j \tfrac{\partial}{\partial y_i} g_T^{x,y}(t) = \tfrac{\partial}{\partial y_i}(\tfrac{d}{dt})^j g_T^{x,y}(t) =$$

$$(\tfrac{d}{dt})^j(Y(t)-X(t)(X(0))^{-1}Y(0))(Y(T)-X(T)(X(0))^{-1}Y(0))^{-1}\, e_i$$

and with $X^{x,b}$ as in (15.3).

$$(15.50)\quad (\tfrac{d}{dt})^j \tfrac{\partial}{\partial x_i} g_T^{x,y}(t) = \tfrac{\partial}{\partial x_i}(\tfrac{d}{dt})^j g_T^{x,y}(t) = (\tfrac{d}{dt})^j X^{x,b}(t)(X^{x,b}(0))^{-1}\, e_i$$

for $i \in \{1,\ldots,d\}$ and $j \in \{0,1\}$.

2. Moreover

$$(15.51)\quad |g_T^{x_1,y_1}(t) - g_T^{x_2,y_2}(t)|,\ |\dot g_T^{x_1,y_1}(t) - \dot g_T^{x_2,y_2}(t)|$$

$$= 0(|x_1-x_2|\, e^{-\omega_1 t} + |y_1-y_2|\, e^{-\omega_1(T-t)})$$

(15.52) $$|g^{x_1,b}(t) - g^{x_2,b}(t)|, |\dot{g}^{x_1,b}(t) - \dot{g}^{x_2,b}(t)| = 0(|x_1 - x_2| e^{-\omega_1 t})$$

and

(15.53) $$|g_T^{x,b}(t) - g^{x,b}(t)|, |\dot{g}_T^{x,b}(t) - \dot{g}^{x,b}(t)| = 0(e^{-\omega_1 T} e^{-\omega_1 (T-t)})$$

uniformly for $x, x_1, x_2 \in B(x_0, \delta_0)$, $y, y_1, y_2 \in B(b, \delta_0)$ and $T \in [T_0, \infty)$.

A Proof Of Sublemma 10.2.

We observe that (15.52) follows from (15.51) and (15.53) by writing

$$g^{x_1,b}(t) - g^{x_2,b}(t) = (g^{x_1,b}(t) - g_T^{x_1 b}(t))$$

$$+ (g_T^{x_1,b}(t) - g_T^{x_2,b}(t)) + (g_T^{x_2,b}(t) - g^{x_2,b}(t))$$

(similar for t-derivatives) and taking $T \to \infty$. Also (15.53) follows from (15.51) since $g^{x,b}(t)|_{[0,T]} = g_T^{x,\tilde{y}}(t)$ where $\tilde{y} = g^{x,b}(T) = b + 0(e^{-\omega_1 T})$ uniformly by (14.4).

To prove

$$|g_T^{x_1,y_1}(t) - g_T^{x_2,y_2}(t| = 0(|x_1 - x_2| e^{-\omega_1 t} + |y_1 - y_2| e^{-\omega_1 (T-t)})$$

in (15.51) it suffices to show

(15.54) $$|g_T^{x_1,y}(t) - g_T^{x_2,y}(t)| = 0(|x_1 - x_2| e^{-\omega_1 t})$$

and

$$(15.55)\qquad |g_T^{x,y_1}(t) - g_T^{x,y_2}(t)| = 0(|y_1 - y_2|\, e^{-\omega_1(T-t)})$$

uniformly, as we see from writing

$$g_T^{x_1,y_1} - g_T^{x_2,y_2} = (g_T^{x_1,y_1} - g_T^{x_2,y_1}) + (g_T^{x_2,y_1} - g_T^{x_2,y_2}).$$

Now we show (15.54), (15.55) is proven in a similar way. Write

$$h(t) = g_T^{x_1,y}(t) - g_T^{x_2,y} \text{ then } h(0) = x_1 - x_2\,,\ h(T) = 0$$

$$\text{and } \ddot h = V'(g_T^{x_1,y}) - V'(g_T^{x,y}(t))$$

$$= \int_0^1 V''(g_T^{x_2,y} + s(g_T^{x_1,y} - g_T^{x_2,y}))\,(g_T^{x_2,y} - g_T^{x_1,y})\, ds =$$

$$(V''(g_T^{x_0,y}) + W)h \text{ where}$$

$$\int_0^T |W(t)|\, dt = 0(\int_0^T \int_0^1 |sg_T^{x_1,y} + (1-s)\, g_T^{x_2,y} - g_T^{x_2,y}|\, ds\, dt)$$

$$\overset{(14.12)}{=} 0(\int_0^T \int_0^1 s\|g_T^{x_1,y} - g_T^{x_0,y}\|_{L^\infty[0,T]}\, e^{-c_1 t} +$$

$$+ (1-s)\|g_T^{x_2,y} - g_T^{x_0,y}\|_{L^\infty[0,T]}\, ds\, dt) = 0(f(\delta)),$$

if we define

$$f(\delta) = \sup_{x\in B(x_0,\delta)} (\|g_T^{x,y} - g_T^{x_0,y}\|_{L^\infty[0,T]}) \cdot$$

By sublemma 5.2, $f(\delta) \to 0$ as $\delta \downarrow 0$, and by sublemma 9.4, $\ddot Z = (V''(g_T^{x_0,y}) + W)Z$ on $[0,T]$ has matrix solutions X and Y such that

(15.56) $\quad X(t) - X^{x_0,y,T}(t) = 0(f(\delta))\, e^{-\Omega t} = \dot{X}(t) - \dot{X}^{x_0,y,T}(t)$

and

(15.57) $\quad Y(t) - Y^{x_0,y,T}(t) = 0(f(\delta))\, e^{\Omega t} = \dot{Y}(t) - \dot{Y}^{x_0,y,T}(t)$

and by taking δ_0 smaller $X(0)$ is invertible for all x_1 and x_2 in $B(x_0\,,\,\delta_0)$.

Now (15.54) follows from

$$g_T^{x_1,y}(t) - g_T^{x_2,y}(t) = h(t) = (X(t) - Y(t)(Y(T))^{-1}\, X(T))$$

$$(X(0) - Y(0)(Y(T))^{-1}\, X(T))^{-1}\,(x_1 - x_2)$$

(15.56), (15.57) and the estimates in lemma 2.

The second part of (15.51), $|\dot{g}_T^{x_1,y}(t) - \dot{g}_T^{x_2,y}(t) = 0(|x_1 - x_2|\, e^{-\omega_1 t})$ follows from $\dot{g}_T^{x_1,y}(t) - \dot{g}_T^{x_2,y}(t) = \int_0^1 \frac{d}{ds}\, \dot{g}_T^{(1-s)x_1+sx_2,y}(t)\, ds$ and (15.48) that we prove next.

Proof of Sublemma 10.1.

For small ε we are interested in

$$f_\varepsilon(t) := (g_T^{x+\varepsilon e_i,y}(t) - g_T^{x,y}(t))\, \varepsilon^{-1}\,.$$

We observe that $\ddot{f}_\varepsilon(t) = \int_0^1 V''(g_T^{x+s\varepsilon e_i,y}(t))\, ds\, f_\varepsilon(t) = (V''(g_T^{x,y}(t)) + W_\varepsilon^T(t))$ $f_\varepsilon(t)$, $f_\varepsilon(0) = 0$ and $f_\varepsilon(T) = e_i$ where $|W_\varepsilon^T(t)| = 0(\varepsilon e^{-\omega_1 t})$ by (15.54).

By sublemma (9.4)

$$f_\varepsilon(t) = (Y_\varepsilon(t) - X_\varepsilon(t)(X_\varepsilon(0))^{-1} Y_\varepsilon(0)) \tag{15.58}$$

$$(Y_\varepsilon(T) - X_\varepsilon(T)(X_\varepsilon(0)^{-1} Y_\varepsilon(0))^{-1} e_i$$

where

$$X_\varepsilon(t) - X^{x,y,T}(t) = 0(\varepsilon)\, e^{-\Omega t} = \dot{X}_\varepsilon(t) - \dot{X}^{x,y,T}(t).$$

and

$$Y_\varepsilon(t) - Y^{x,y,T}(t) = 0(\varepsilon)\, e^{\Omega t} = \dot{Y}_\varepsilon(t) - \dot{Y}^{x,y,T}(t)$$

Now we take ε to zero in (15.58) and we get (15.48) for $g_T^{x,y}$. To obtain (15.48) for $\dot{g}_T^{x,y}$ we differentiate (15.58) w.r.t. t and then take $\varepsilon \to 0$.

16. Proofs and Sketches of Proofs of Lemmas 2 to 6.

§16.1. A sketch of a proof of lemma 2.

§16.2. A proof of lemma 3.

§16.3. A sketch of a proof of lemma 4.

§16.4. A sketch of a proof of lemma 5.

§16.5. A proof of lemma 6.

16. Proofs and Sketches of Proofs of Lemmas 2 to 6.

§16.1. A Sketch of a Proof of Lemma 2 .

Proof of Lemma 2.1. Sublemma 6 in section 13.3 says there is a unique $g^{x,b}$ and it has positive definite second variation for $x \in B(x_0 , \delta_0)$ for some $\delta_0 > 0$ (which will be getting smaller as the proof goes on). (2.14) with $|\alpha| = 0$ is (14.4) in sublemma 8.2 (2.15) is (14.2) and (2.16) is (15.3) and (15.4) is sublemma 9.

(2.18) that says det $X^{x,b}(t) \geq$ constant $e^{-t \sum_{i=t}^{d} \omega_i} > 0$ follows from $X^{x,b}(t) = (I + 0(e^{-\delta_1 t}))\, e^{-\Omega t}$ (see (2.16)) and $X^{x,b}(t)$ is nonsingular by (13.8) of sublemma 6.

More precisely, we can find t_0 such that

$$\det X^{x,b}(t) = (1 + 0(e^{-\delta_1 t}))\, e^{-t \sum_{i=1}^{d} \omega_i} > \tfrac{1}{2}\, e^{-t\sum_{i=1}^{d} \omega_i}$$

if $(x,t) \in \overline{B(x_0,\delta_0)} \times [t_0 , \infty)$ (slightly smallr δ_0). Then by compactness of $\overline{B(x_0,\delta_0)} \times [0,t_0]$ and continuity of $\det(X^{x,b}(t))$ (its nonzero and positive for large t)

$$\min_{\overline{B(x_0,d_0)}\times[0,t_0]} \det(X^{x,b}(t)) > 0$$

so (2.18) follows.

(2.19) is (15.50) of sublemma 10 and (2.20) follows from (2.19), (2.14) with $|\alpha| = 0$ and $\rho(x,b) = \int_0^{\infty} (\tfrac{1}{2}(\dot{g}^{x,b}(t))^2 + V(g^{x,b}(t)))\, dt$.

(2.14) for $|\alpha| > 0$ follows from (2.19) and (2.17) and so to complete the proof it suffices to prove (2.17).

We use the proof of lemma 9.1 and we let $\varphi_{j,\ell}^{x,b}(t)$ be as in (15.26),

(16.1) $\quad \varphi_j^{x,b}(t) = \sum_{k=0}^{\infty} (\varphi_{j,k}^{x,b}(t) - \varphi_{j,k-1}^{x,b}(t))$ as in (15.28),

$V_{1,j}^{x,b}(t,s)$ and $V_{2,j}^{x,b}(t,s)$ as in (15.31) and as in (15.38)

(16.2) $$\psi_j^{x,b}(t) = \psi_j^{x_0,b}(t) + \int_0^t V_{1,j}^{x,b}(t,s)(R^{x,b}(s) - R^{x_0,b}(s))\, \psi_j^{x_0,b}(s)\, ds$$

$$- \int_t^{\infty} V_{2,j}^{x,b}(t,s)(R^{x,b}(s) - R^{x_0,b}(s))\psi_j^{x_0,b}(s)\, ds$$

for $x \in B(x_0, \delta_0)$ and $t \in (0,\infty)$.

(2.17) follows if we show that for each $\alpha \in N_0^d$ with $|\alpha| > 1 \frac{\partial^{|\alpha|}}{\partial x^\alpha} \psi_j^{x,b}(t) = 0(e^{(\lambda_j - \delta_1)t})$ uniformly on $B(x_0, \frac{\delta_0}{2}) \times (0,\infty)$, which can be done by induction. Let $m = 2, 3, 4, \ldots$ and $\alpha, \beta, \gamma \in N_0^d$ and let I_m, II_m and III_m denote the following statements. If $0 \le |\alpha| \le m-2$, $0 \le |\beta| \le m-1$ and $1 \le |\gamma| \le m-1$ then

$$I_m: \quad \Big|\Big(\frac{\partial^{|\alpha|} \varphi_{j,\ell+1}^{x+\varepsilon e_j,b}}{\partial x^\alpha}(t) - \frac{\partial^{|\alpha|} \varphi_{j,\ell+1}^{x,b}}{\partial x^\alpha}(t)\Big)$$

$$-\Big(\frac{\partial^{|\alpha|} \varphi_{j,\ell}^{x+\varepsilon e_j,b}}{\partial x^\alpha}(t) - \frac{\partial^{|\alpha|} \varphi_{j,\ell}^{x,b}}{\partial x^\alpha}(t)\Big)\Big| \le D_{m-1,\ell+1}\, e^{(\lambda_j - \delta_1)t}\, \varepsilon$$

uniformly for $(x,t) \in B(x_0, (1 - (\frac{1}{4} + \cdots + \frac{1}{2^m}))\delta_0)$ where $D_{m-1,\ell+1} \le \frac{E_{m-1} \cdots E_1 (\ell+1)^{m-1}}{2^\ell}$, for $\ell = 0, 1, 2, \ldots$, E_i is independent of ℓ and t_0 as in (15.25).

$$\text{II}_m: \frac{\partial^{|\beta|}}{\partial x^\beta} V_{1,j}^{x,b}(t,s) = 0(e^{(\lambda_j-\delta_1)(t-s)}) \text{ if } t, s \geq 0, t - s \geq 0,$$

$$\text{and } \frac{\partial^{|\beta|}}{\partial x^\beta} V_{2,j}^{x,b}(t,s) = 0(e^{\lambda_j(t-s)}) \text{ if } t, s \geq 0, t - s \leq 0$$

uniformly $x \in B(x_0 , (1 - (\frac{1}{4} +\cdots+ \frac{1}{2^m})\delta_0))$.

Finally

$$\text{III}_m: \frac{\partial^{|\gamma|}}{\partial x^\gamma} \psi_j^{x,b}(t) = 0(e^{(\lambda_j-\delta_1)t})$$

uniformly on $B(x_0 , (1 - (\frac{1}{4} +\cdots+ \frac{1}{2^m})\delta_0)) \times (0,\infty)$.

We leave out the proof but we point out that I_m implies we can interchange differentiation and summation in (16.1) and get $\frac{\partial^\beta}{\partial x^\beta} \varphi_j^{x,b}(t) = (e^{(\lambda_j-\delta_1)t})$ which is used in the proof of II_m . Notice also $\frac{\partial^\beta}{\partial x^\beta} R^{x,b}(t) = 0(|\frac{\partial^\beta}{\partial x^\beta} (g^{x,b}(t)-b)|)$

$$= \begin{bmatrix} 0|g^{x,b}(t)-b| \quad \text{if } |\beta| = 1 \\ 0(\sum\limits_{|\alpha|=|\beta|-1} | \frac{\partial^\beta}{\partial x^\beta} (X^{x,b}(t)(X^{x,b}(0))^{-1})|) \text{ if } |\beta| \geq 1 \end{bmatrix}$$

and II_m gives by induction we can differentiate under the integral sign in (6.2).

A sketch of a proof of lemma 2.2.

By sublemma 7.1 in section 13.3 there is a unique $g_T^{x,y}$ for $(x,y,T) \in B(x_0, \delta_0) \times B(b,\delta_0) \times [T_0, \infty)$ and it has positive definite second variation. (2.25) follows from (15.13) and (15.14), (2.22) from (2.25) and (2.16) since we assume $\delta_1 \le \frac{\omega_1}{2}$.

(2.23) follows from (2.25) and (2.18). (2.26) except $\frac{\partial A(x,y,T)}{\partial T} = \frac{1}{2}(\dot{g}_T^{x,y}(T))^2 + V(y)$ follows from (15.48) and (15.49) in sublemma 10.

(2.26) follows by showing $g_T^{x,y}$ is differentiable w.r.t. T and differentiate $A(x,y,T) = \int_0^T (\frac{1}{2}(\dot{g}_T^{x,y})^2 + V(g_T^{x,y}))\, dt$ and use $g_T^{x,y}(T) = y$ for all T, $g_T^{x,y}(0) = x$ for all T and the Euler-Lagrange equations.

To prove (2.26) one can show that $\psi_j^{x,y,T}(t)$ in (15.45) in the proof of sublemma 9.3 obeys $\frac{\partial^{|\alpha|+|\beta|}}{\partial x^\alpha \partial y^\beta} \psi_j^{x,y,T}(t) = 0(1)\, e^{\lambda_j t}$ by an induction similar to that one sketched for $\psi_j^{x,b}(t)$ above.

Now finally (2.27) follows from (2.24) and (2.26).

§16.2. A Proof of Lemma 3.

Lemma 3.1 and 3.3 is contained in sublemma 7, section 13.4. Lemma 3.2 is contained in sublemma 8.1 and 8.3, section 14.1. Lemma 3.4 except (2.41) follows from sublemma 10.2 in chapter 15. (2.41 follows by setting $\alpha(t) = |g_T^{x_1,y_1}(t) - g_T^{x_2,y_2}(t)|^2$ then $\ddot{\alpha} = 2\dot{\alpha}^2 + 2\langle\ddot{\alpha},\alpha\rangle \ge c_1^2\, \alpha$, by use of Euler-Lagrange equations and the mean value theorem. Now (Protter-Weinberger [1]) this implies $\alpha(t) \le |x_1 - x_2|^2 \frac{\sinh c_1(T-t)}{\sinh c_1 T} + |y_1-y_2|^2 \frac{\sinh c_1 t}{\sinh c_1 T}$ so $\alpha(t) \le \max(|x_1 - x_2|^2, |y_1-y_2|^2)$.

§16.3. A Sketch of a Proof of Lemma 4.

Lemma 4.1 and the positivity in (2.49) of lemma 4.2 follows from sublemma 9.4. That $G_r^{x,y,T}(t,s)$ defined in lemma 4 is a Green's matrix for $(-\frac{d^2}{dt^2} + V''(g_T^{x,y}) + W_T^r)$ on [0,T] with Dirichlet's boundary condition can be easily checked but see also Heimes [1]. Notice $Z(t) := \dot{Y}(t) - \dot{X}(t)\, X^{-1}(t)\, Y(t)$ satisfies $\dot{Z}(t) = -\dot{X}(t)\, X^{-1}(t)\, Z(t)$ and is nonsingular at $t = T$, by the asymptotics in (2.46) and (2.22) and so it is always nonsingular by the Liouville formula (Hartman [1]).

One can obtain the estimates in (2.51) and (2.52) by inserting (2.46) and (2.22) into (2.50). The estimates in (2.53) follows from (2.50) and (2.24).

§16.4. A Sketch of a Proof of Lemma 5.

(2.55) and (2.56) for $|\alpha| = 1$ are obtained by going though the successive approximations to obtain (recall (15.44) and (2.42))

$$V_{1,j}^{x,\bar{y},T}(T-V,T-u) = (I + 0(e^{-\delta_1(T-v)}))\, V_{1,j}(u-v)(I + 0(e^{-\delta_1(T-u)}))$$

and similar for $V_{2,j}^{x,\bar{y},T}(T-v,T-u)$. Then by differentiating (15.45) and by showing

$$\psi_j^{x,\bar{y},T}(T-u) = e^{\lambda_j(T-u)}(f_j + 0(e^{-\delta_1(T-u)}))$$

$$\frac{\partial R^{x,y,T}}{\partial y_i}(T-u)|_{y=\bar{y}} = \frac{\partial R^{y,b}}{\partial y_i}(u)|_{y=b} + 0(e^{-\delta_1(T-u)}e^{-\omega_1 u})$$

and

$$R^{x,\bar{y},T}(T-u) = R^{x,b}(T-u) + 0(e^{-\omega_1 T}(e^{-\omega_1 u} + e^{-\omega_1(T-u)}))$$

we get

$$\frac{\partial \psi^{x,\bar{y},T}}{\partial y_i}(T-v) = [\int_v^\infty V_{1,j}(u-v)\,\frac{\partial R^{y,b}}{\partial y_i}(u)|_{y=b}\, e^{-\lambda_j u}\, du$$

$$-\int_0^v V_{2,j}(u-v)\,\frac{\partial R^{y,b}}{\partial y_i}(u)|_{y=b}\, e^{-\lambda_j u}\, du]\, e^{\lambda_j T} f_j$$

$$+ 0(e^{(\lambda_j-\delta_1)(T-v)})$$

$$=: (q(v)f_j + 0(e^{-\delta_1(T-v)}))\, e^{-\lambda_j(T-v)}$$

whereby (15.24) and

$$\frac{\partial R^{y,b}}{\partial y_i}(u) = 0(|\frac{\partial}{\partial y_i}g_{x,b}(u)|) \overset{(2.14)}{=} 0(e^{-\omega_1 u}), \text{ we have}$$

$$q(v) = 0(e^{-\lambda_j v}(\int_v^\infty e^{(\lambda_j-\delta_1)(u-v)}\, e^{-\omega_1 u}\, e^{-\lambda_j u}\, du$$

$$+ \int_0^v e^{\lambda_j(u-v)}\, e^{-\omega_1 u}\, e^{-\lambda_j u}\, du) = 0(1)$$

which gives (2.55) and (2.56) with $|\alpha| = 1$. (2.55) and (2.56), for $|\alpha| > 1$, are proven similarly by induction.

The first part of lemma (5.2) follows from the formula for $G^{x,y,T}(t,s)$ in (2.51) and the estimates in lemma 2. The second part is then obtained using (2.55) and (2.56).

§16.5. Proof of Lemma 6.

By (2.26) and (14.9)

$$A'_y(x,y,T) = \dot{g}_T^{x,y}(T) = \Omega(y - b)(1 + 0(e^{-2\omega_1 T})) + 0(e^{-\omega_1 T} + |y-b|^2)$$

uniformly for $(x,y,T) \in B(x_0, \delta_0) \times B(b,\delta_0) \times [T_0, \infty)$. Hence $A'_y(x,y,T) = 0$ implies $y - b = (e^{-\omega_1 T})$ uniformly. Now (4.18) that says

$$A''_{yy}(x,y,T) = \Omega + 0(e^{-\delta_1 T} + |y-b|)$$

uniformly implies $\overline{B(b,\delta)} \ni y \mapsto A(x,y,T)$ attains its absolute minimum at a unique point $\bar{\bar{y}}$ satisfying $\bar{\bar{y}} = b + 0(e^{-\omega_1 T})$ uniformly for $(x,T) \in B(x_0, \delta) \times [T_0, \infty)$ some $\delta \in (0,\delta_0]$ (taking T_0 larger, if needed). The proof of the statement for $B(b,\delta) \ni y \mapsto A(x,y,T) + \frac{1}{2}\langle\Omega(y-b), (y-b)\rangle$ is similar.

References

S. Agmon [1], Lectures on exponential decay of solutions of second order elliptic equations: Bounds on eigenfunctions of N-Body Schrödinger operators, MathematicalNotes 29. Princeton Univ. Press, Princeton, N.J. 1982.

R. Azencott [1], Grandes dérviations et applications. Lecture Notes in Math., no. 774, Springer, Berlin-New York, 1980.

A. Bellaiche [1], Propriétés Extrémales Des Géodésiques Société Mathématique de France, Astérisque 84-85 (1981), Chapitre 7, p. 83–130.

C. Bellaiche [1], Le Comportement Asymptotique De P(t, x, y), Quand t → 0 (Points Eloignes). Société Mathématique de France Astérisque 84-85 (1981) Chapitre 9, p. 151–187.

M. Berger [1], Nonlinearity and Functional Analysis, Academic Press, New York, 1977.

S. Breen [1], Large order perturbation theory for the anharmonic oscillator: UBC preprint.

R. H. Cameron and W. T. Martin [1], Transformation of Wiener integrals under translations. Ann. Math. 45 (1944), 386–396.

R. Carmona and B. Simon [1], Pointwise Bounds on Eigenfunctions and Wave Packets in N-Body Quantum Systems V. Lower bounds and path integrals, Comm. Math. Phys., 80, 59-98 (1981).

S. Chevet [1], Gaussian Measures and Large Deviations. Probability in Banach Spaces, IV, 30-46, Lecture Notes in Math., 990, Springer Berlin-New York, 1983.

E. A. Coddington and N. Levinson [1] The Theory of Ordinary Differential Equations. Robert E. Krieger Publishing Company, Malabar, Florida, 1984.

S. Coleman [1], The Whys of Subnuclear Physics. International School of Subnuclear Physics, Erice 1977. Edited by Antonio Zichichi.

J. M. Combes, P. Duclos and R. Seiler [1], Krein's Formula and One-Dimensional Multiple-Well, Journal of Functional Analysis, 52, 257-301, 1983.

J. M. Combes, P. Duclos and R. Seiler [2], Convergent Expansions for Tunneling, Comm. Math. Phys., 92, 229-245, 1983.

I. Davies and A. Truman [1], Laplace asymptotic expansions of conditional Wiener integrals and generalized Mehler kernel formulas, J. Math. Phys. 23(11), 1982, 2059-2070.

R. S. Ellis and J. S. Rosen [1], Asymptotic analysis of Gaussian integrals, I. Isolated minimum points, Trans. Amer. Math. Soc. 273 (1982), no. 2, 447–481.

R. S. Ellis and J. S. Rosen [2], Laplace's method for Gaussian integrals with an application to statistical mechanics, Ann. Prob. 10 (1982), no. 1, 47–66.

R. S. Ellis and J. S. Rosen [3], Asymptotic analysis of Gaussian integrals. II. Manifold of minimum points, Comm. Math. Phys. 82 (1981/82), no. 2, 153–181.

A. Erdelyi [1], Asymptotic Expansions, Dover, New York, 1956.

M. Freidlin [1], Functional Integration and Partial Differential Equations, Annals of Mathematics Studies Number 109, Princeton Univ. Press, Princeton, New Jersey, 1985.

M. I. Freidlin and A. D. Wentzell [1], Random Perturbations of Dynamical Systems, Grundlehren der Mathematischen Wissenshaffer 260, Springer, Berlin-New York, 1984.

I. M. Gelfand and S. V. Fomin [1], Calculus of Variations, revised English edition, Prentice-Hall: Englewood Cliffs, N.J., 1963.

P. Hartman [1], Ordinary differential equations, Wiley, New York, 1963.

E. M. Harrell [1], Double Wells, Comm. Math. Phys. 75, 239-261, (1980).

E. M. Harrell [2], On the Rate of Asymptotic Eigenvalue Degeneracy, Comm. Math. Phys. 60, 73–95, (1978).

K. A. Heimes [1], Green's functions for linear second order systems, SIAM J. Math. Anal., Vol. 9, No. 1, (1978), 207-214.

B. Helffer and J. Sjöstrand [1], Multiple wells in the semi-classical limit I, Comm. Partial Differential Equations, 9 (4), 1984, 337–408.

B. Helffer and J. Sjöstrand [2], Puits multiples en limite semiclassique, II, Intersection moleculaire, symétries, perturbation, Ann. Inst. H. Poincaré.

B. Helffer and J. Sjöstrand [3], Multiple wells in the semiclassical limit III, Interaction through non-resonant wells, Math. Nachr. 124, (1985), 263–313.

M. R. Hestenes [1], Calculus of variations and optimal control theory, Wiley, New York, 1966.

L. Hörmander [1]: The Analysis of Linear Partial Differential Operators I, Springer, Berlin-New York, 1983.

N. C. Jain, An Introduction to Large Derivatives, Probability in Banach Spaces V, 273–296, Lecture Notes in Math., 1153, Springer, Berlin-New York, 1985.

G. Jona-Lasinio, F. Martinelli and E. Scoppola [1], New Approach to the Semiclassical Limit of Quantum Mechanics I. Multiple Tunnelings in One-dimension, Comm. Math. Phys., 80, (1981) 223–254.

G. Jona-Lasinio, F. Martinelli and E. Scoppola [2], The Semiclassical Limit of Quantum Mechanics: A Qualitative Theory via Stochastic Mechanics, Physics Reports 77, no. 3, (1981), 313–327.

M. Kac and C. Thompson, Phase transitions and eigenvalue degeneracy of a one-dimensional anharmonic oscillator, Stnd. Appl. Math. 48, (1969), 257–264.

S. Kobayashi [1]: On conjugate and cut loci, in MAA Studies in Mathematics, Vol. 4: Studies in Global Geometry and Analysis, S. S. Chern, editor.

L. D. Landau and E. M. Lipschitz Quantum Mechanics, non-relativistic theory, Pergamon Press, 1977.

L. Lithner [1], A theorem of Phragmon Lindelof type for second order elliptic operators, Ark. Mat. 5 (1964), 281–285.

M. M. Mizrahi [1], The semiclassical expansion of the anharmonic-oscillator propagator, J. Math. Phys. 20(5), 1975, 844–855.

S. Molchanov [1], Diffusion Processes and Riemannian Geometry, Russian Math. Surveys 30:1 (1975), 1–63.

E. W. Montroll [1], Markoff Chains, Wiener Integrals, and Quantum Theory, Comm. Pure Appl. Math., Vol. 5 (1952), 415–453.

E. Nelson [1], Dynamical Theories of Brownian Motion, Princeton, Univ. Press, Princeton, N.J., 1967.

F.W.J. Olver [1], Asymptotics and Special Functions, Academic Press, New York, 1974.

M. Pincus [1], Gaussian processes and Hammerstein integral equations, Trans. Amer. Math. Soc. 134 (1968), 193–216.

M. H. Protter and H. F. Weinberger [1], Maximum Principles in Differential Equations, Prentice-Hall, 1967.

M. Reed and B. Simon [1], Methods of Modern Mathematical Physics IV, Analysis of Operators, Academic Press, New York, 1978.

M. Reed and B. Simon [2], Methods of Modern Mathematical Physics II, Fourier Analysis, Self-adjointness, Academic Press, New York, 1975.

H. L. Royden [1], Real Analysis, second edition, The Macmillan Company, New York, 1968.

M. Schilder [1], Some asymptotic formulas for Wiener integrals, Trans. Amer. Math. Soc. 125 (1966), 63-85.

B. Simon [1], Semiclassical analysis of low lying eigenvalues I. Nondegenerate minima: Asymptotic Expansions. Ann. Inst. Henri Poincaré, Physique théorique, Vol. XXXVIII, no. 3, 1983, p. 295-307.

B. Simon [2], Semiclassical analysis of low lying eigenvalues II. Tunneling. Annals of Mathematics, Second Series, 1984, 120, no. 1, 89–118.

B. Simon [3], Instantons, Double Wells and Large Deviations, Bull. Am. Math. Soc. Vol. 8, Number 2, 1983.

B. Simon [4], Semiclassical analysis of low lying eigenvalues IV. The flea on the elephant. Journal of Functional Analysis, 1985, 63, no. 1, 123–136.

B. Simon [5], Functional Integration and Quantum Physics, Academic Press, New York, 1979.

B. Simon [6], Schrödinger Semigroups, Bull, Amer. Math. Soc. 7, (1982), 447–526.

J. Sjöstrand [1], Multiple Wells and Tunneling, Proceedings of Symposia in Pure Mathematics, Vol. 43, (1985).

D. W. Stroock [1], An Introduction to the Theory of Large Deviations, Springer-Verlag, New York, 1984.

A. Truman [1], Classical mechanics, the diffusion (heat) equation, and the Schrödinger equation, J. Math. Phys. 18, no. 12, 1977, 2308–2315.

S.R.S. Varadhan [1], Large Deviations and Applications. CBMS-NSF Regional Conference Series in Applied Mathematics 46, SIAM, 1984.

S.R.S. Varadhan [2], Lectures on Diffusion Processes and Partial Differential Equations, Tata Institute of Fundamental Research, Bombay, 1980.

S.R.S. Varadhan [3], Diffusion processes in a small time interval. Comm. Pure Appl. Math., 20 (1967), 659-685.

J. Zinn-Justin [1], Multi-Instanton Contributions in Quantum Mechanics, Nuclear Physics B192 (1981), 125-140.