ON THE EMBEDDING OF HOMEOMORPHISMS OF THE PLANE IN FLOWS

Thesis by

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Abstract

Homeomorphisms of the plane onto itself are studied, subject to the restriction that they should preserve the sense of orientation and have no fixed points. The author tries to determine which mappings in this general class can be embedded in one-parameter subgroups of the full homeomorphism group of the plane. Such subgroups are called flows.

By a theorem of Brouwer, $T^n p \to \infty$ as $n \to \pm \infty$ for any point p in the plane, if T is in the general class being studied. As a consequence, it is shown that if T is embedded in a flow then $\sum_{-\infty}^{\infty} T^n A$ is a proper subset of the plane for any compact set A. The author suspects that this property might be shared by all homeomorphisms in the general class.

It is found that for an arbitrary T there exists a natural partition of the plane into a collection of "fundamental regions", with the property that the restriction of T to any fundamental region must be embedded in a flow within that region whenever T over the whole plane is embedded in a flow. An example is given of a homeomorphism which, for this very reason, cannot be embedded in a flow over the whole plane E^2.

The author proves that if T satisfies the above condition that $\sum T^n A \neq E^2$ and if T has exactly one fundamental region, that being

E^2 itself, then T can be embedded in a flow, and indeed is equivalent to a translation.

Finally, it is shown by an example that even if the restrictions of T to its fundamental regions are all equivalent to translations, it might still be impossible to create a flow for T over all of E^2.

0. INTRODUCTION

The purpose of this dissertation is to investigate homeomorphisms of the Euclidean plane E^2 onto itself, with the object of finding which of these homeomorphisms are members of one-parameter subgroups of the full homeomorphism group of the plane. Such a one-parameter subgroup, or flow, is a collection of homeomorphisms $\{T^{\rho}p : -\infty < \rho < +\infty\}$ with T^{0} = the identity homeomorphism I, $T^{\alpha}(T^{\beta}p) = T^{\alpha+\beta}p$, and with the point $T^{\rho}p \in E^2$ jointly continuous in the real number ρ and the point p. We say that the homeomorphism T is "embedded in the flow $\{T^{\rho}\}$", if $T = T^{\rho}$ when $\rho = 1$.

Only those homeomorphisms which have no fixed points are considered here, although fragmentary results have been obtained in the general case.

1. A FEW GENERAL PROPERTIES OF PLANE HOMEOMORPHISMS

It will be shown later that if the homeomorphism T is embedded in a flow then T must preserve the sense of orientation. Thus we restrict ourselves at the outset to those homeomorphisms of the plane onto itself which have no fixed points and which preserve the sense of orientation.

The notation C^n, where $C \subset E^2$, will designate $T^n C$, the n-th iterate of the set C under the transformation T.

Section I rests on the following theorem of Brouwer (1):

<u>Theorem 1</u>: (Brouwer) Let T be an orientation-preserving homeomorphism of the plane onto itself without fixed points. Let E be a curve segment in the plane with endpoints p and $p' = T(p)$. Let $F = E - p'$ be the half-open segment obtained from E by deletion of p'. Then $F \cdot F' = 0$ implies that $F \cdot F^n = 0$ for all $n \neq 0$ and that the set $\sum_{-\infty}^{+\infty} F^n$ is a curve without self-intersections.

Brouwer's result has the following consequence:

<u>Theorem 2</u>: Let T be an orientation-preserving homeomorphism of the plane onto itself without fixed points. Let C be a connected compact set which is disjoint from C', its image under T. Then $C \cdot C^n = 0$ for all $n \neq 0$.

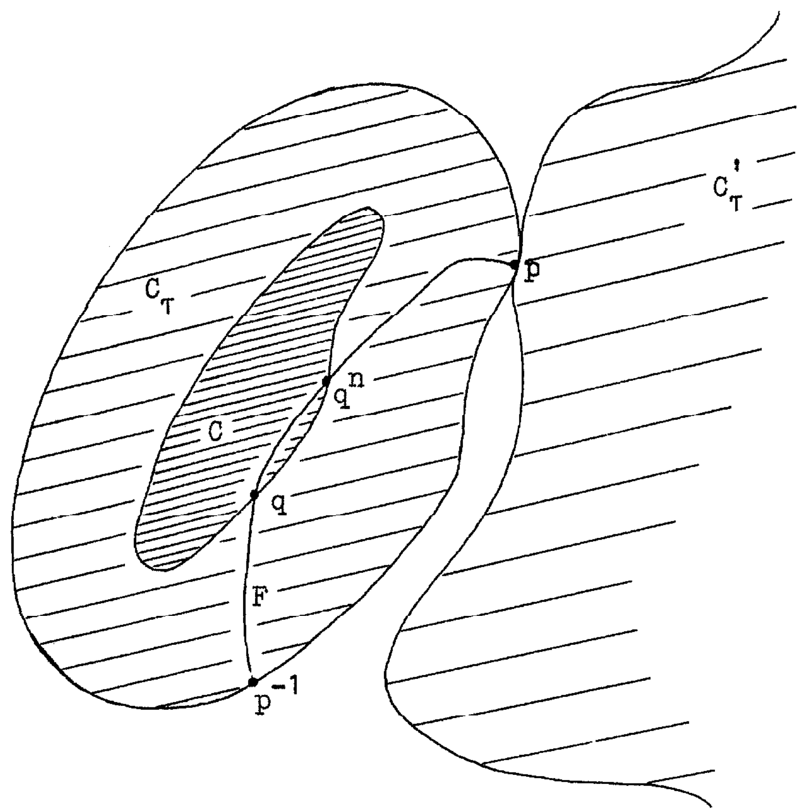
Proof: Set $C_\mu = \sum_{q \in C} D(q,\mu)$, where $D(q,\mu)$ is the open disc of radius μ centered at q. C_μ is just the set of all points whose distance to C is less than μ.

C_μ is an open set; because C is connected, C_μ is arcwise connected. Furthermore, any boundary point p of C_μ must be a boundary point of a disc centered in C and having radius μ. Consequently p can be connected to C by a straight line segment contained in C_μ, and we have shown that all boundary points of C_μ are accessible from within C_μ.

By compactness and continuity, C_μ is disjoint from $C'_\mu = T(C_\mu)$ for sufficiently small μ. Let $\tau = \text{l.u.b.}\ \{\mu : C_\mu \cdot C'_\mu = 0\}$. Then $C_\tau \cdot C'_\tau = 0$ while $\partial C_\tau \circ \partial C'_\tau \neq 0$. Let $p \in \partial C_\tau \cdot \partial C'_\tau$ be in the intersection of the boundaries of C_τ and C'_τ. Then the preimage p^{-1} of p is a boundary point of C_τ because p is a boundary point of C'_τ.

If $C \cdot C^n \neq 0$ for some $n \neq 0$, then C contains both q and q^n for some $q \in C$. We draw a simple curve segment F, as shown in Figure 1; F runs from p^{-1} to q to q^n to p, and is contained in C_τ except for its endpoints p and p'. It is possible to construct such an F precisely because C_τ is a connected open set all of whose boundary points are accessible.

Figure 1



The half-open segment F - p is contained in the set $C_T + p^{-1}$, and T(F-p) is contained in the set $C_T' + p$. Because $C_T + p^{-1}$ is disjoint from $C_T' + p$, we can conclude that F - p is disjoint from T(F-p), its image under T. Brouwer's theorem now tells us that F - p must be disjoint from all its images T^n(F-p), but this is impossible since F - p contains the points q and q^n. Therefore C cannot contain both q and q^n, and C must be disjoint from all its images C^n, Q.E.D.

A corollary of Theorem 2 which will be important for our purposes is the following:

Corollary 1: (Brouwer) Let T be an orientation-preserving homeomorphism of the plane onto itself without fixed points. Then for any point p of the plane, $p^n \to \infty$ as $n \to \pm \infty$.

Proof: If this were not the case, then the sequence $\{p^n\}$ would have an accumulation point $q \in E^2$. Since $q \neq q'$, there is a small closed disc B, centered at q, which does not meet its image B'. But B contains infinitely many points from the sequence $\{p^n\}$, hence in particular it contains two points p^n, p^{n+m} for some n, m where $m \neq 0$. Then B would meet B^m, which contradicts Theorem 2.

By taking C to be a single point, we obtain

Corollary 2: Let T be an orientation-preserving homeomorphism of the plane onto itself. If T has a periodic point, i.e. $T^n p = p$ for some point p, then T has a fixed point.

Thus the class of transformations we are studying is invariant under the operation of forming iterates.

The following corollary is not needed for an understanding of the rest of the paper, but is included for its own interest.

Corollary 3: (Montgomery) (2) Suppose that U is an orientation-preserving, measure-preserving homeomorphism of the open unit disc D onto itself. Then U has a fixed point in the interior of the disc.

<u>Proof</u>: Suppose, if possible, that U has no fixed points. Then there is a small closed disc B in D which is disjoint from its image UB. By Theorem 2, all the iterates of B must be disjoint; but because U preserves area, the set $\sum_{-\infty}^{+\infty} U^n B$ must have infinite area. Hence U has a fixed point.

2. SOME NECESSARY CONDITIONS FOR T TO BE EMBEDDED IN A FLOW

Suppose now that T is a homeomorphism of the plane without fixed points, and suppose that it is embedded in the flow $\{T^\rho\}$. We observe the following properties of $\{T^\rho\}$:

1) All the transformations T^ρ must preserve the sense of orientation.

<u>Proof</u>: T^α preserves orientation if $T^{\alpha/n}$ does; but $T^{\alpha/n}$ approximates $T^0 = I$ as $n \to \infty$. Hence $T^{\alpha/n}$ preserves orientation because I does.

2) $T^\rho(p) \to \infty$ as $\rho \to \pm\infty$ for any $p \in E^2$. In particular, each homeomorphism T^ρ has no fixed points for $\rho \neq 0$.

<u>Proof</u>: Otherwise, we would have $T^{n_k+\alpha_k} p \in B$, where B is a bounded set, $0 \le \alpha_k < 1$, and $\{n_k\}$ is an unbounded sequence of integers. Then we would have $T^{n_k} p \in \sum_{-1 \le \alpha \le 0} T^\alpha B$, which is itself a bounded set

by continuity and compactness. Hence p^{n_k} is a bounded sequence, contradicting Corollary 1 of Section I, which says that $T^n p \to \infty$.

The curve $\{T^\rho p : -\infty < \rho < +\infty\}$, we call the flow line F_p passing through p. It is easy to see that if F_p and F_q are two flow lines from the flow $\{T^\rho\}$, then either $F_p = F_q$ or $F_p \cdot F_q = 0$.

Observation 2) implies that if F_p is a flow line then $F_p + \{\infty\}$ can be viewed as a Jordan curve on the sphere obtained by adjoining the point at infinity. By the Jordan curve theorem, the complement of F_p in the plane consists of two disjoint connected open sets D_1 and D_2. If $q \in D_1$, then $F_q \subset D_1$, because F_q cannot cross over into D_2 without meeting F_p. Therefore each residual domain D_i is left invariant by each homeomorphism T^ρ of the flow.

A given transformation T might be embedded in many flows; our observations thus far in Section 2 have described the flows in which T is embedded without saying anything new about T itself. Our next theorem, however, will imply that if T has no fixed points and is embedded in a flow, then, for any compact set A in E^2, $\sum_{-\infty}^{+\infty} T^n A$ is disjoint from some nonvoid unbounded connected set.

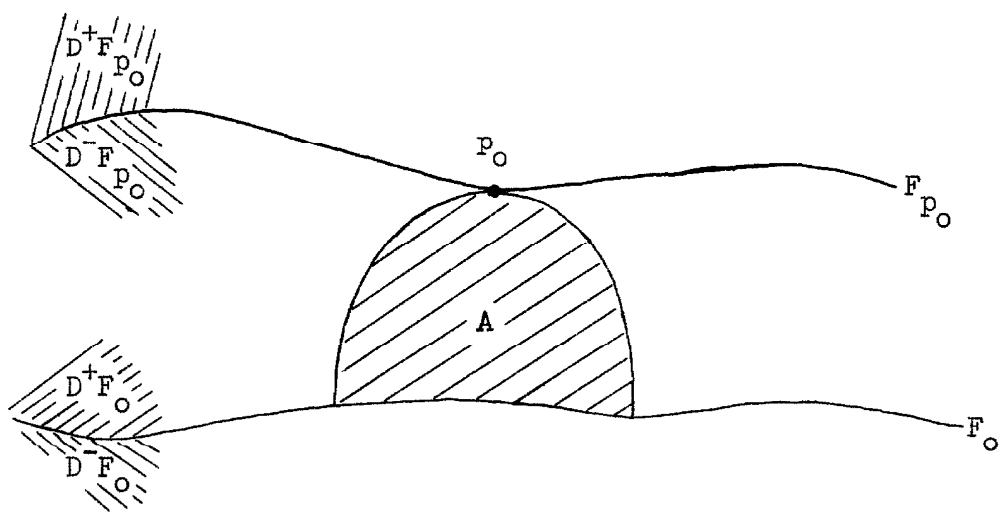
<u>Theorem 3</u>: Let $T \in \{T^\rho\}$; let F_o be a flow line from $\{T^\rho\}$. Suppose that a compact set A is contained in $\overline{D^+F_o} = F_o + D^+F_o$, the closure of one of the residual domains of F_o. Then a flow line F_{p_o} can be found in D^+F_o for which $F_{p_o} \cdot A \neq 0$ and yet $D^+F_{p_o} \cdot A = 0$, where $D^+F_{p_o}$

is that residual domain of F_{p_o} which does not meet F_o.

<u>Proof</u>: For any $q \in A$ let D^-F_q be that residual domain of F_q which meets F_o. For p and $q \in A$ we say that $F_p \leq F_q$ if and only if $D^-F_p \subset D^-F_q$. The relation $\leq$ is a partial ordering: since $F_p \leq F_q$ and $F_q \leq F_p$ imply that $D^-F_p = D^-F_q$ we can conclude that $F_p = F_q$, where we have used that part of the Jordan curve theorem which asserts that a Jordan curve is the boundary of each of its residual domains.

The proof will be complete when we have found a point $p_o \in A$ whose flow line is maximal in the partial ordering. The situation will then be as in Figure 2.

Figure 2



Let $\{F_\alpha\}$ be a totally ordered set of flow lines. Then $\{A \cdot \overline{D^+F_\alpha}\}$ is a nested collection of nonvoid compact sets, and hence a point $q \in A$ can be found for which $q \notin D^-F_\alpha$ for all α. Then F_q is disjoint from D^-F_α, and D^-F_α is contained in a residual domain of F_q, in particular the one that contains F_0. Therefore $F_q \geq F_\alpha$ for all α, and we have shown that every totally ordered collection has an upper bound.

By Zorn's lemma, let F_{p_0} be a maximal element from our partially ordered collection of flow lines. Then $A \cdot D^+F_{p_0} = 0$, or else we could find a flow line surpassing F_{p_0} in the partial ordering. F_{p_0} meets A itself because $p_0 \in A$, and we have proved the theorem.

<u>Remark</u>: Even if part of A were to lie in D^-F_0, $D^+F_{p_0}$ would still be an unbounded connected set not meeting any iterate of A. The writer suspects that T must have this property even if T is not embedded in a flow, i.e., that if T is an orientation-preserving homeomorphism without fixed points of the plane onto itself, or of a closed half-plane onto itself, then $\sum_{-\infty}^{+\infty} T^n A$ for a compact set A fails to meet some unbounded connected set in the plane, resp. half-plane.

3. THE FUNDAMENTAL REGIONS OF A HOMEOMORPHISM

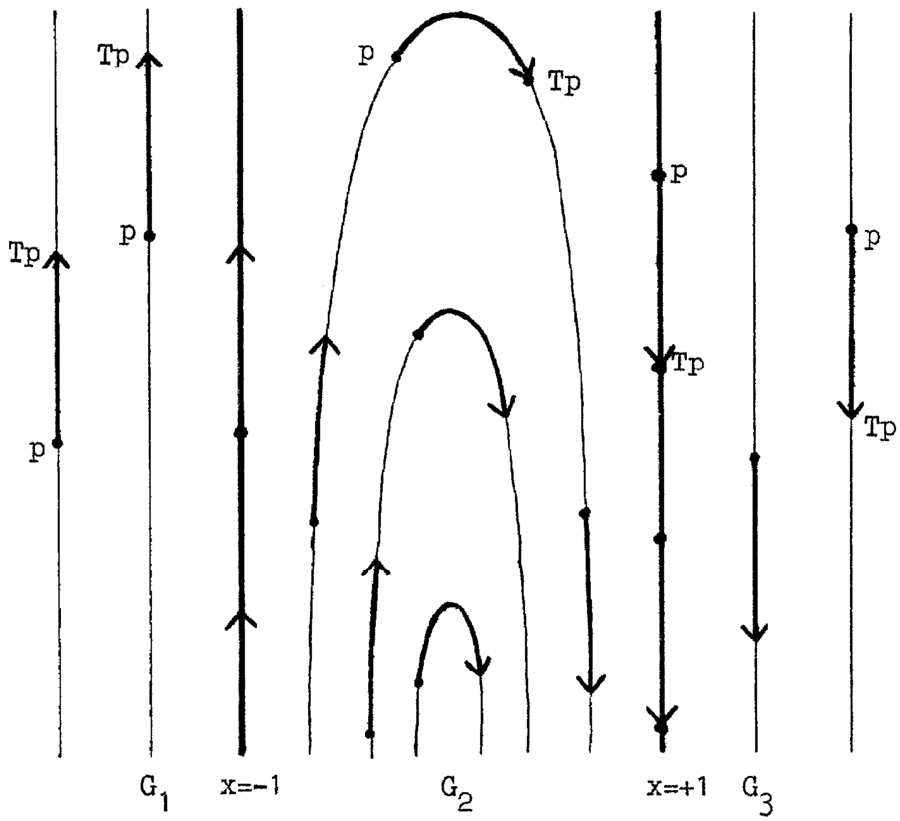
We have seen that, if $T \in \{T^\rho\}$, then every point $p \in E^2$ can be connected to its image p' by a curve $C = \{T^\rho p : 0 \leq \rho \leq 1\}$, where C has the property that $C^n \to \infty$ as $n \to \pm \infty$. (We say in general that the sequence of sets $\{E_n : -\infty < n < +\infty\}$ "diverges" if any compact set A meets E_n for only finitely many n.) Thus, when we are studying a homeomorphism T which is not known to be embedded in a flow, it is natural to ask which pairs of points p,q can be connected by a curve whose iterates diverge. We frame this idea in the form of an equivalence relation: for p and q $\in$ E, we say that $p \sim q$ if and only if p and q are endpoints of some curve segment whose iterates diverge.

The verification that $\sim$ is an equivalence relation is elementary, except for reflexivity, where one must apply the fact that $p^n \to \infty$ as $n \to \pm \infty$.

Thus the plane is partitioned into a collection of equivalence classes; we call these equivalence classes "the fundamental regions of T." By definition, each fundamental region is an arcwise connected set.

Example 1: Consider the homeomorphism defined in Figure 3. Each point p lies on a straight line or a curve; we define Tp to be a point lying on the same line or curve and separated from p by an arclength of 1.

Figure 3



T is obviously one-to-one, bicontinuous, onto, preserves orientation, and has no fixed points.

This particular transformation has three fundamental regions: $G_1 = \{(x,y) : x \leq -1\}$, $G_2 = \{(x,y) : -1 < x < +1\}$, and $G_3 = \{(x,y) : +1 \leq x\}$. It is easy to verify that any two points of G_i are equivalent under the relation $\sim$. But no point of G_1 can be equivalent under $\sim$ to a point of G_2, by the following reasoning: the iterates of any compact set can be confined between two vertical lines; any curve meeting G_1 and G_2 must possess two points which approach infinity in the opposite y-directions under positive iteration of the curve; therefore each of the positive iterates of the curve must cross the x-axis between the same two vertical lines. Hence the iterates of our curve cannot diverge. In like manner, no point of G_2 is equivalent to a point of G_3, by arguing on the negative iterates of T.

Concerning the fundamental regions G_i of a transformation T there is a certain dichotomy relation: each G_i is either invariant under T, $TG_i = G_i$, or else is disjoint from its image, $TG_i \cdot G_i = 0$. To see this, note that if $q \in TG_i \cdot G_i$ then $p \in G_i$ implies that $p \sim q$ and $p' \sim q$, hence $p \sim p'$ for all $p \in G$ and $TG_i = G_i$.

Of course, if T is embedded in a flow, then the first alternative must hold for all the fundamental regions of T. Indeed, every fundamental region G_i must be invariant under every homeomorphism T^{ρ} of any flow in which T is embedded. In particular, the boundaries and interiors of all fundamental regions must be preserved by any flow in which T is embedded. This implies that if $p \in \text{int } G_i$ then p can be connected to $p' \in \text{int } G_i$ by an arc contained in int G_i. The same statement holds for ∂G_i, the boundary of the fundamental region G_i.

Therefore, if T is embedded in a flow, then each fundamental region G_i of T satisfies

1) G_i is invariant under T,

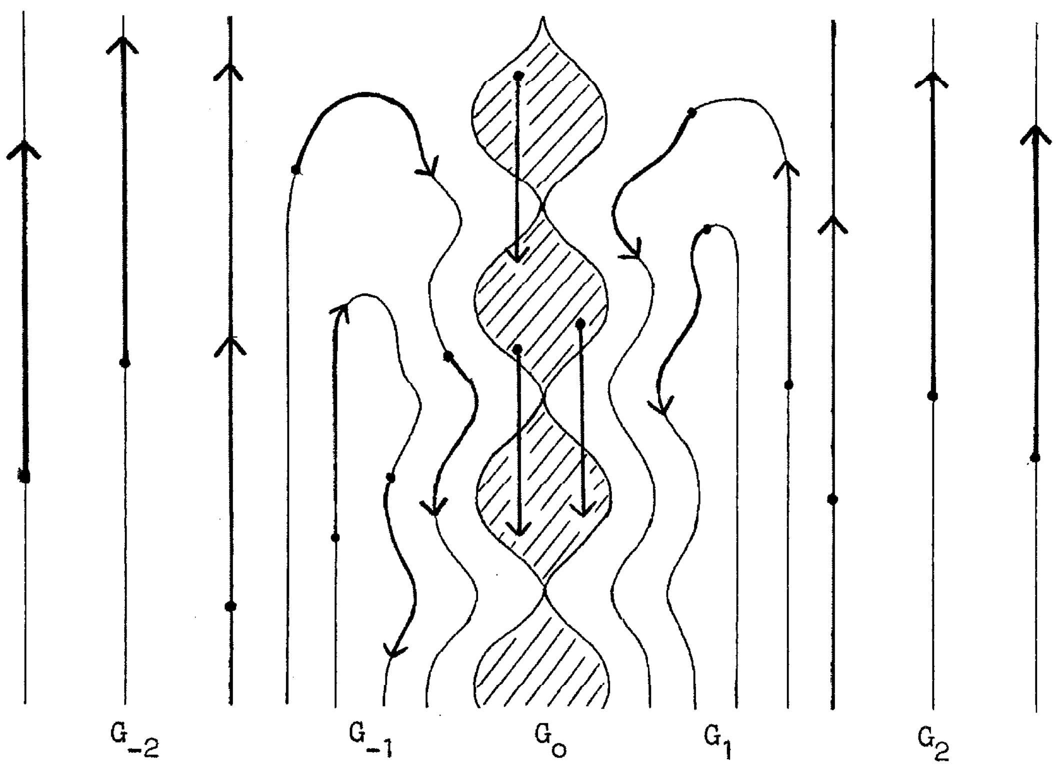
2) the connected components of int G_i are each invariant under T, and

3) the arcwise connected components of ∂G_i are each invariant under T.

It is immediate that if some fundamental region of some homeomorphism fails to satisfy one of the conditions 1)-3), then that homeomorphism cannot possibly be embedded in a flow. Indeed, examples can be given of homeomorphisms which violate 1), 2), or 3).

Example 2: We construct a homeomorphism T which violates condition 2) and hence cannot be embedded in a flow.

Figure 4



If p is on a line or a curve, Tp is separated from p by two units of arclength; if p is in the crosshatched region G then Tp is one unit south of p.

The fundamental regions of T are just G_{-2}, G_{-1}, G_0, G_1, and G_2, by the same technique as in the previous example. But int G_0 has infinitely many connected components, none preserved by T.

4. SUFFICIENT CONDITIONS FOR T TO BE EQUIVALENT TO A TRANSLATION, AND THUS BE EMBEDDED IN A FLOW

The simplest example of an orientation-preserving homeomorphism without fixed points is the translation which takes the point (x,y) into the point (x + 1,y). We call this translation L, and we note that L is embedded in the flow $\{L^\rho\}$, where $L^\rho(x,y) = (x + \rho, y)$.

We would like to determine those mappings whose structure is the same as the structure of L. Thus the homeomorphism T is "equivalent to a translation" if $T = ULU^{-1}$ where U is some homeomorphism of the plane onto itself. Naturally, T must then be embedded in the flow $\{T^\rho\} = \{UL^\rho U^{-1}\}$.

What happens when we compute the fundamental regions of T, where T is equivalent to a translation? We find that T has only one fundamental region, that being the entire plane E^2.

Let us agree that a "half-plane" in E^2 is one of the two residual domains of a simple curve which is a Jordan curve when viewed in the sphere; if L leaves a closed half-plane invariant, L has the property described in the Remark at the end of Section 2, namely, that for any compact set A in the half-plane there exists an unbounded connected set B in the half-plane which does not meet any iterate of A. This property is shared by any homeomorphism T which is equivalent to L.

We intend to show that these two conditions are also sufficient.

<u>Theorem 4</u>: Suppose that T is an orientation-preserving homeomorphism, without fixed points, of the plane onto itself. Suppose that

1) T has exactly one fundamental region, and
2) if T leaves a closed half-plane E invariant, then the iterates of any compact set $A \subset E$ fail to meet some unbounded connected set $B \subset E$.

Then T is equivalent to a translation.

<u>Note</u>: We do not know whether condition 2) is necessary.

Theorem 4 is proved by reducing it to the following theorem of Sperner:

<u>Theorem 5</u>: (Sperner) Let T be an orientation-preserving homeomorphism, without fixed points, of the plane onto itself. If the iterates of A diverge, whenever A is compact, then T is equivalent to a translation.

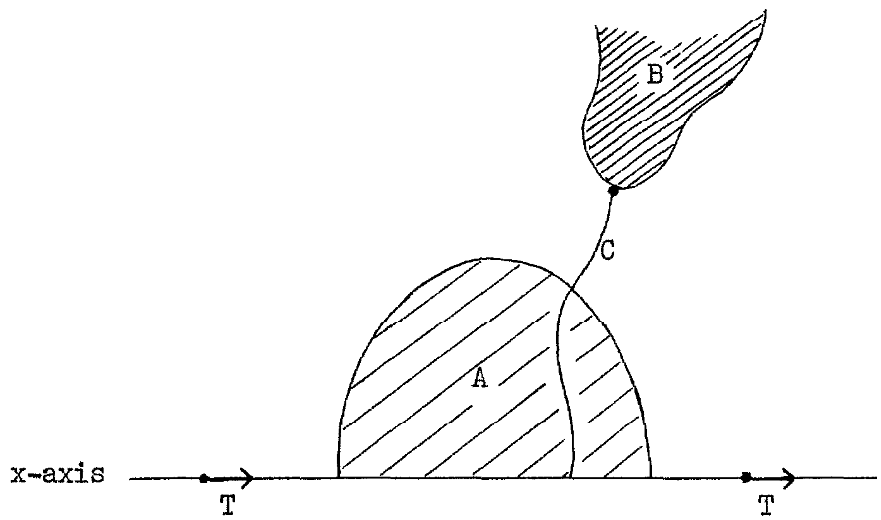
<u>Reduction of Theorem 4 to Theorem 5</u>: Let T satisfy the hypotheses of Theorem 4. Then $T^n A \to \infty$ as $n \to \pm \infty$ for any compact set A.

<u>Proof</u>: Choose $p \in E^2$, and let p be connected to p' by a curve C for which $\{C^n\}$ diverges; C may be chosen to be simple. Consider the subsegments of C whose endpoints correspond under T. For example, C itself is such a segment. Let F be a minimal subsegment in this class. Then $F \cdot F'$ consists only of the one endpoint of F, for otherwise a smaller subsegment could be found inside F whose endpoints correspond under T. By Theorem 1 (Brouwer) and by the fact that $\{F^n\}$ diverges, we see that $\sum_{-\infty}^{+\infty} F^n$ is a Jordan curve on the sphere and is left invariant by T. Let us represent $\sum F$ in our figures as the x-axis, and let us suppose without loss of generality that $T(x,0) = (x + 1,0)$.

Because T preserves orientation, the two half-planes are each invariant under T.

To show that the iterates of any compact set must diverge, it is sufficient to show that $A \cdot A^n$ is nonvoid for at most finitely many n if A is compact. Our constructions can be carried out equally well in both half-planes, so we content ourselves with showing that $A \cdot A^n = 0$ for sufficiently large n if A is any closed half-disc in the upper half-plane. The situation is depicted in Figure 5.

Figure 5

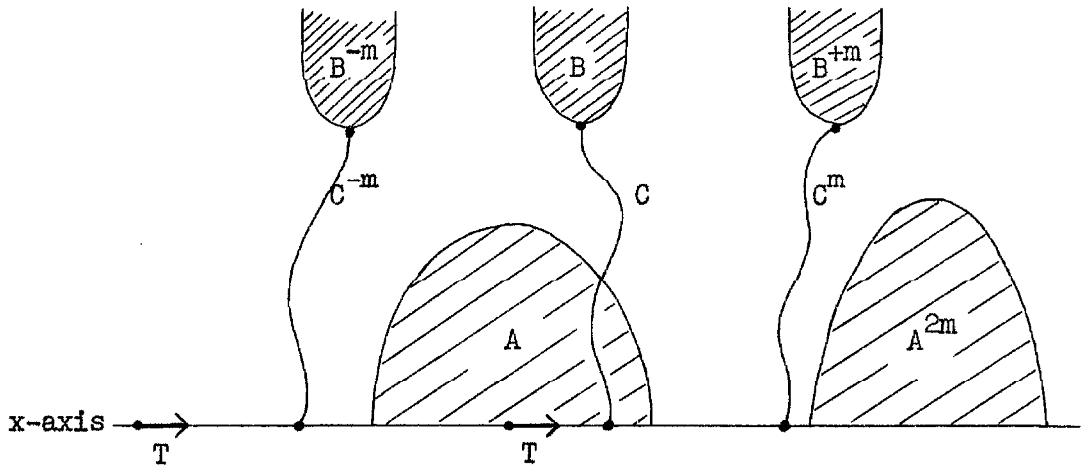


First we apply hypothesis 2) and find an unbounded connected set B which does not meet any iterate of A. Then by 1) we connect a point of B to a point on the x-axis by a curve whose iterates diverge. Let C denote a subsegment of this curve which is contained in the upper half-plane and which still connects B to the x-axis.

Let N be a positive integer large enough so that $m > N$ implies that $C^{-m} \cdot A = 0$, $C^{m} \cdot A = 0$, C^{-m} meets the x-axis to the left of A, and C^{m} meets the x-axis to the right of A. We will argue that $A \cdot A^{2m} = 0$.

The situation is as implied by Figure 6.

Figure 6



$B^m + C^m$ is disjoint from A because B^n never meets A for any n, and because m is large enough to ensure that $C^m \cdot A = 0$.

Likewise, $B^m + C^m$ is disjoint from A^{2m} because $B^{-m} + C^{-m}$ is disjoint from A. By our requirements of left and right, C^m meets the x-axis between A and A^{2m}.

If $A \cdot A^{2m}$ were nonvoid, a simple curve segment S could be drawn in $A + A^{2m}$ which would meet the x-axis only in A and in A^{2m}. A Jordan curve would then be formed by S with the appropriate segment of the x-axis. But $B^m + C^m$ is an unbounded connected set having points in the interior domain of this Jordan curve at the same time as it fails to intersect the Jordan curve itself, and we have a contradiction. Q.E.D.

Proof of Sperner's Theorem:

(The following proof follows a different line from that given by Sperner in (3), and Kerékjárto in (4).)

If p and $q \in E^2$, we say that p is congruent to q, written $p \equiv q(T)$, if $p^n = q$ for some n. Let I be the collection of all congruence classes.

Define the function $P : E^2 \to I$ to be the identification map. Then I is a topological space if we take the open sets in I to be those sets whose inverse images under P are open sets in E^2 (i.e., the identification topology). We call I the "identification space of the homeomorphism T."

Any point $\bar{p} \in I$ is contained in a neighborhood homeomorphic to a disc in the plane, for we need only choose a representative p from the congruence class $\bar{p}$, and let p be contained in a small disc D_p in the plane which is disjoint from all its iterates under T. Then every $\bar{q} \in I$ has at most one representative in D_p, and the restriction of P to D is a bicontinuous one-to-one mapping onto a neighborhood of $\bar{p}$.

Because $A^n \to \infty$ for compact sets, I is a Hausdorff space. For let $\bar{p}$ and $\bar{q}$ be two distinct points in I, and let p and q be representatives in E^2; we have $p \not\equiv q(T)$. Now p and q are contained in some large open disc D for which $D \cdot D^n = 0$ when $|n| \geq N$. Let $p \in Q_p \subset D$, where Q_p is a disc small enough so that q is not in the closure of $\sum_{-N}^{+N} Q_p^n$. Then q is not in the closure of $\sum_{-\infty}^{+\infty} Q_p^n$, because for $|n| \geq N$ we have $Q_p^n \subset D^n$ which does not meet D, in which q is located. Hence q is contained in some disc Q_q where Q_q meets no iterate of Q_p. Therefore $P(Q_p)$ and $P(Q_q)$ are disjoint neighborhoods of $\bar{p}$ and $\bar{q}$ in I.

We have shown that I, the identification space of T, is a Hausdorff space and is locally homeomorphic to E^2. These observations, together with the obvious facts that I is connected and has a countable base, imply that I is a two-dimensional manifold.

In showing that I is locally homeomorphic to E we actually showed that each point of I is contained in a neighborhood $\bar{N}$ for which $P^{-1}(\bar{N})$ is a disjoint countable collection of homeomorphic images of N. Therefore P is the projection mapping of a covering of I by E^2, and because E^2 is simply connected it is the universal covering manifold of I.

Now it will appear that the properties of T are reflected in corresponding properties of I. For example, if we had assumed that T were n times continuously differentiable, then I would be a C^n manifold. We actually have assumed that T preserves orientation; therefore I is an orientable manifold. To see this, let $\bar{J}$ be a sufficiently small Jordan curve in I. Then $\bar{J}$ has infinitely many disjoint preimages $\{J^n\}$ in E^2 under P. As a point travels around a curve J^n in the clockwise sense, each of the iterates under T of this point must travel around each of the other curves J^m in the same clockwise manner, because T preserves the sense of orientation. Thus, a given sense of orientation in $\bar{J}$ gives rise to one and the same orientation, say clockwise, for each of its preimages in E^2.

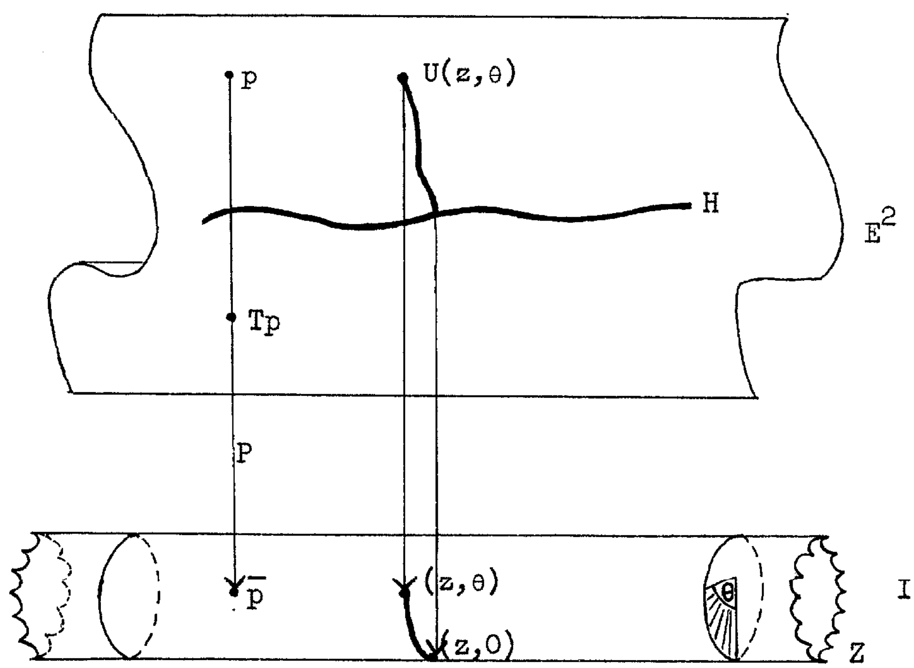
Let the curve $\bar{J}$ be moved through I, and let it return to its starting place. Then the preimages of $\bar{J}$ move from place to place in E^2, but they always preserve their clockwise orientations. When $\bar{J}$ comes back to its starting place, the preimages of $\bar{J}$ return, perhaps in a permuted order, to their starting places. But each preimage had the same orientation at every step of the way, and so $\bar{J}$ must have the same orientation at the end of its journey as it had at the beginning. Therefore a consistent orientation can be defined on I, and I is an orientable manifold.

We would like to know the fundamental group $\pi_1(I)$. This group is isomorphic to the group of covering transformations of the universal covering of I. But E^2 is the universal covering of I, and the covering transformations of E^2 are just the iterated powers of T. Therefore the fundamental group of I is infinite cyclic with one generator.

We have proved that I is an oriented two-manifold whose fundamental group is cyclic. It is proved in the Appendix that these facts are sufficient to guarantee that I is an infinite cylinder.

Let z and θ be coordinates on the cylinder, where (z,θ) and $(z,\theta + 1)$ refer to the same point of the cylinder. Let Z be a straight line along the cylinder, and let the angular coordinate be measured from Z, as indicated in Figure 7.

Figure 7



Let $H \subset E^2$ be one of the preimages of Z under the projection P. To every ordered pair of numbers (z,θ) we assign a point $U(z,\theta) \in E^2$ by the following procedure: $\{(z,t\theta) : 0 \leq t \leq 1\}$ is a curve wrapping around I, and it can be lifted into E^2 in such a manner that the end-point $(z,0)$ is lifted into H; we define $U(z,\theta)$ to be the other end-point of the lifted curve, the one that corresponds to (z,θ). One can verify that $U(z,\theta)$ is a homeomorphism from the z,θ-plane onto E^2.

Any curve $\{(z,\theta + t) : 0 \leq t \leq 1\}$ in I is a generator for the fundamental group of I. Therefore the endpoints of a lifting of this curve must correspond under a covering transformation which is a generator of the group of covering transformations. Hence the endpoints of a lifting must correspond under T itself.

Let us compare $U(z,\theta)$ and $U(z,\theta + 1)$. They are the endpoints of a lifting of $\{(z,\theta + t) : 0 \leq t \leq 1\}$, therefore they correspond under T. Thus we have $TU(z,\theta) = U(z,\theta + 1)$ or $T^{-1}U(z,\theta) = U(z,\theta + 1)$; by continuity a particular one of these two alternatives, say the first, must hold everywhere. But then $U^{-1}TU(z,\theta) = L(z,\theta)$, and we have shown that T is equivalent to a translation, the conclusion of Sperner's theorem.

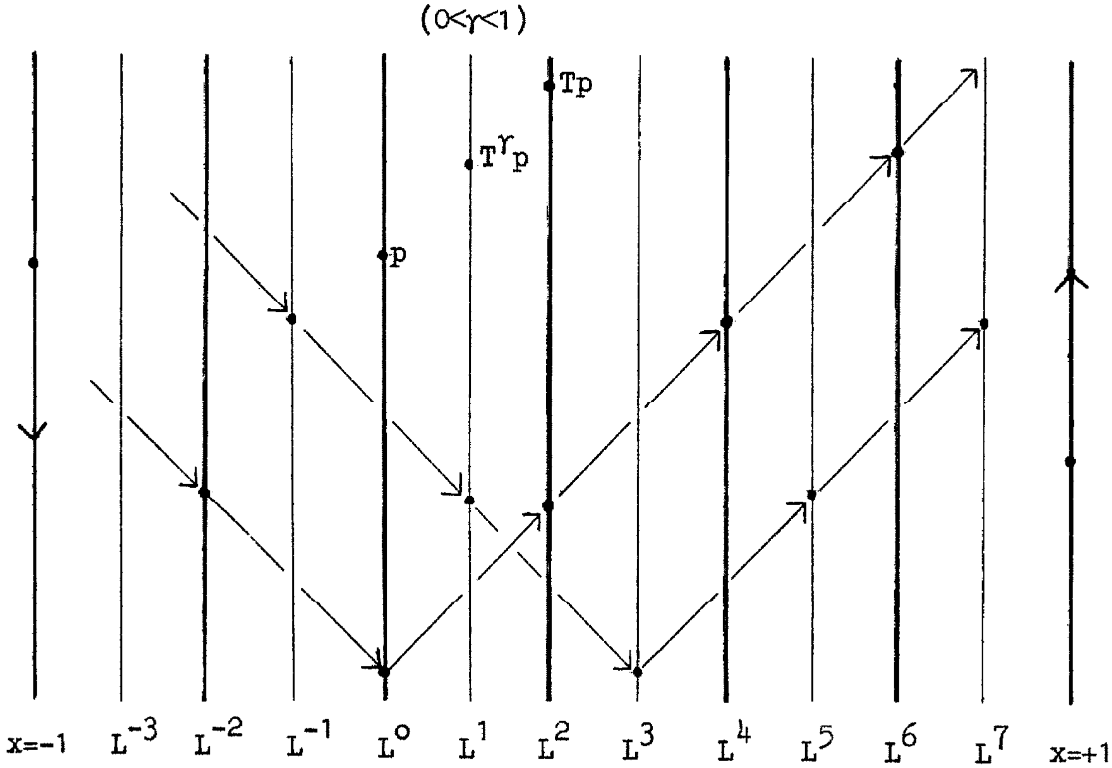
5. CONCLUSION

One might try to infer from the translation theorem and the results on fundamental regions that to embed a homeomorphism T in a flow, one need only consider the restrictions of T to each of its fundamental regions and then show that T is in fact equivalent to a translation in each of its fundamental regions. One would then try to synthesize these translations over subsets of E^2 into a flow over all of E^2.

Our concluding example shows that this synthesis may be impossible.

Example 3: T is defined by reference to Figure 8.

Figure 8



$$L^k = \left\{(x,y) \mid x = \frac{k}{1 + |k|}\right\}$$

The transformation T maps L^k isometrically onto L^{k+2} for all integers k. The amount of displacement in the y-direction is as suggested in the figure: L^o, L^2, L^4,..., as well as L^3, L^5, L^7,..., are displaced one unit northward in their passage from L^k to L^{k+2}. All other L^k are displaced one unit to the south.

For $x \leq -1$ we set $T(x,y) = (x,y - 1)$ and for $x \geq +1$ we set $T(x,y) = (x,y + 1)$. It only remains to define T between the lines L^k. Let this be done in such a manner that T over the whole plane is one-to-one, bicontinuous, orientation-preserving, and without fixed points.

One sees that T has the three fundamental regions $x \leq -1$, $-1 < x < +1$, and $+1 \leq x$. We will show that T cannot be embedded in a flow, despite the fact that the restrictions of T to its three fundamental regions are equivalent to translations.

For suppose, if possible, that T were embedded in a flow $\{T^\rho\}$. Then the lines $x = \pm 1$, being the boundaries of fundamental regions, are left invariant by the flow; but this can only mean that these lines are actually flowlines for $\{T^\rho\}$. We know that $T(-1,y)$ is south of $(-1,y)$; therefore $T^\gamma(-1,y)$ is south of $(-1,y)$ for all positive γ. Likewise, $T^\gamma(+1,y)$ is north of $(+1,y)$.

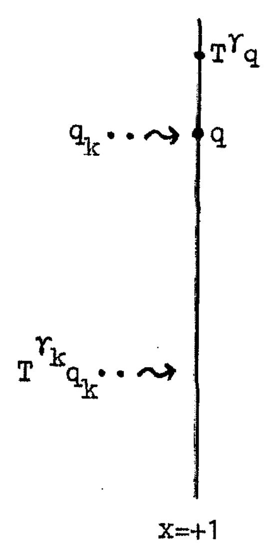
We will show that $\{T^\rho\}$ in $-1 < x < 1$ cannot possibly be defined so as to agree with what $\{T^\rho\}$ must do on both the two lines $x = \pm 1$ (although a flow can be made to agree with one line or the other). For let $p \in L^o$. For some γ in the range $0 < \gamma < 1$ we have $T^\gamma p \in L^1$, because the flowline from p to Tp has to cross L^1 in going from $p \in L^o$ to $Tp \in L^2$.

If $T^\gamma p$ is more than one unit north of p, then $T^{n+\gamma}p$ is always more than one unit north of $T^n p$ as $n \to -\infty$. Hence there are points near $x = -1$ which are moved at least one unit north by T^γ; this opposes the behavior of T^γ for points actually on the line, which is to move south. Thus we say that the point p has an irregularity near $x = -1$.

If $T^{\gamma}p$ is less than one unit north of p, then we see by the figure that $T^{1+\gamma}p$ is more than one unit south of Tp. As $n \to +\infty$, $T^{n+\gamma}p$ is always more than one unit south of $T^n p$, and we have an irregularity near $x = +1$.

Therefore each point on L^o has an associated irregularity on one side or the other. Let $\{p_k\}$ be a sequence of points in L^o tending to infinity in the -y direction, where each p_k has an irregularity on the same side, say near $x = +1$. Then for each p_k there is a γ_k, $0 < \gamma_k < 1$, for which $T^{\gamma_k}(p_k^n)$ is at least one unit south of p_k^n as $n \to +\infty$ (we have written $p_k^n = T^n p_k$). Then, as in Figure 9, there is a sequence of points $\{q_k\}$ tending to a limit point q on $x = +1$, where each q_k is some positive iterate of some point p_k.

Figure 9



But $T^{\gamma}q_n$ is at least one unit south of q_n for some γ in the range $0 < \gamma < 1$, while $T^{\gamma}q$ is north of q for all positive γ. Thus the continuity has been violated, and we must conclude that T cannot be embedded in a flow.

APPENDIX

Let M^∞ be a 2-manifold which

a) is orientable,

b) is noncompact,

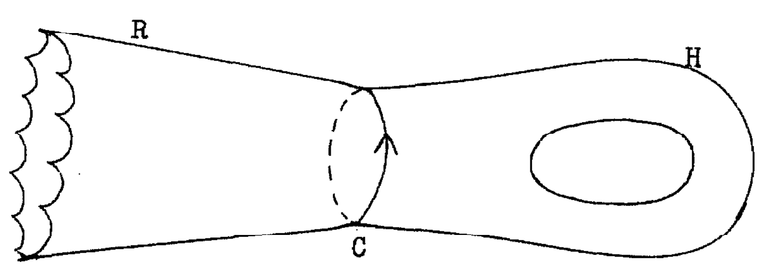
c) has no boundaries,

d) has for its fundamental group the infinite cyclic group with one generator.

Then M^∞ is an infinite cylinder.

Proof: By a result of Radó (5), M^∞ can be triangulated. Any finite submanifold K is a sphere with crosscaps, handles, and holes, by the surface classification theory. But no crosscaps can occur because M^∞ is orientable. If a handle H were to appear, then M^∞ would have the appearance of Figure 10, where $M^\infty = R + H$, and $R \cdot H = C$.

Figure 10



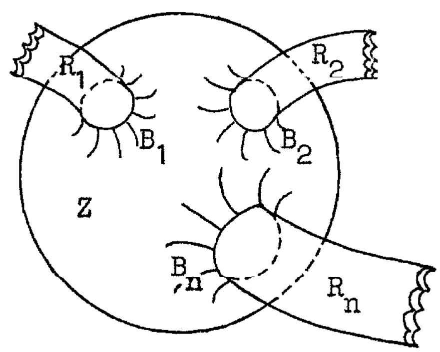
By van Kampen's theorem (6),

$$\pi_1(M^\infty) = \frac{\{a,b\} * \pi_1(R)}{aba^{-1}b^{-1} = r_1}, \quad r_1 \in \pi_1(R).$$

But this group is homomorphic to $\frac{\{a,b\}}{ab=ba}$ and thus cannot be infinite cyclic with one generator; we have proved that no handles can occur.

A finite submanifold K must therefore be a sphere with holes. A chain of adjacent triangles not in K can meet only one boundary of K, or else a handle would be formed. Therefore M^∞ has the form of Figure 11.

Figure 11



We have $M^\infty = Z + R_1 + R_2 + \cdots + R_n$ where $R_i \cdot R_k = 0$ and $Z \cdot R_k = B_k$.

If B_k is of finite order m in $\pi_1(R_k)$ then B_k^m is homotopic to a constant in R_k, and hence the boundary of an integer 2-chain C_k. C_k must have the same value up to sign for each triangle in R_k, hence R_k is a finite submanifold with one boundary and no crosscaps or handles, and R_k is a cell.

Among the R_k there can be at most two which are not cells. For suppose that R_1, R_2, and R_3 are not cells; then B_1, B_2, and B_3 are of infinite order in $\pi_1(R_1)$, $\pi_1(R_2)$, and $\pi_1(R_3)$. By van Kampen's theorem,

$$\pi_1(M^\infty) = \frac{\pi_1(R_1) * \pi_1(R_2) * \cdots * \pi_1(R_n)}{B_1 B_2 \ldots B_n = 1} .$$

$\pi_1(M^\infty)$ is homomorphic to the group in which all the groups $\pi_1(R_4)\ldots$ have been set equal to the identity. Then the subgroup generated by B_1, B_2, and B_3 has the single relation $B_1B_2B_3 = 1$. But this group is homomorphic to $\dfrac{\{B_1, B_2\}}{B_1B_2=B_2B_1}$. Hence $\pi_1(M^\infty)$ could not be cyclic with one generator, and at most two of the R_k are not cells.

If K contains a generator of $\pi_1(M^\infty)$ then exactly two of the R_k are not cells, for otherwise the generator would be homotopic to the identity.

Let K be a finite submanifold containing a generator of $\pi_1(M^\infty)$. Let M^1 be obtained from K by adjunction of all those R_k that are cells. We have shown that M^1 is a sphere with two holes, i.e., a compact cylinder. Let K^2 be obtained from M^1 by adjoining those triangles of M^∞ which meet the boundaries of M^1. Both boundaries of M^1 are in the interior of K^2 because M^∞ has no boundaries. M^2 is obtained from K^2 just as M^1 was from K^1 : all holes except two are filled in by cells, and M^2 is a compact cylinder.

M^∞ can be written as a nested union of compact cylinders $M^2 \subset M^3 \subset M^4 \subset \ldots$. The cylinder M^n is homeomorphic to that part of the z,θ-cylinder which lies between $z = -n$ and $z = +n$, by the mapping U_n, say. The $\{U_n\}$ can be defined so as to agree on the intersections of their domains, and the complete set $\{U_n\}$ gives a homeomorphism of M^∞ onto the z,θ-cylinder for $-\infty < z < +\infty$.

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