Evolution Equations
and Semigroups of Operators
with the Disjoint Support Property

Thesis by

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Abstract

Let X_1, X_2 be locally compact Hausdorff spaces, E_1, E_2 Banach spaces.

Theorem. *T is an operator in $L\big(C_0(X_1, E_1), C_0(X_2, E_2)\big)$ with the disjoint support property if and only if $\exists Y$ open, $Y \subset X_2$ $\exists Q \in C_b\big(Y, L_s(E_1, E_2)\big)$ $\exists \varphi \in C(Y, X_1)$ such that :*

(1) $\forall y \in Y \;\; Q(y) \neq 0.$

(2) $\forall \varepsilon > 0 \; \forall u \in E_1 \; \forall K$ *compact,* $K \subset X_1 \; \exists F$ *compact,* $F \subset Y$ *with the following property:*

$$y \in Y \setminus F, \quad \varphi(y) \in K \qquad \Rightarrow \qquad \|Q(y)u\| < \varepsilon.$$

(3) $\forall f \in C_0(X_1, E_1)$

$$(Tf)(y) = \begin{cases} Q(y)f(\varphi(y)), & \text{if } y \in Y, \\ 0, & \text{if } y \in X_2 \setminus Y. \end{cases}$$

Let X be a locally compact Hausdorff space, E a Banach space.

Theorem. *$\{T(t)\}_{t\in\mathbf{R}}$ is a C_0-group on $C_0(X, E)$ with the disjoint support property if and only if $\exists \varphi$ a continuous flow, $\exists Q$ a continuous cocycle of φ such that $\forall t \in \mathbb{R}$ $\forall x \in X \; \forall f \in C_0(X, E) \; (T(t)f)(x) = Q_t(x)f(\varphi_t(x))$.*

There is a corresponding result about C_0-semigroups on $C_0(X, E)$ with the disjoint support property, where semiflows and semicocycles play the roles of flows and cocycles respectively.

Suppose $-\infty \leq a < b \leq +\infty$, X is either (a, b) or $[a, b]$, where by $[-\infty, b]$ we mean $(-\infty, b]$, and by $[a, +\infty]$ we mean $[a, +\infty)$.

Theorem. *Let $\{T(t)\}_{t\in\mathbf{R}}$ be a C_0-group on $C_0(X)$ with the disjoint support property. Then $\exists U \subset X$, U is the union of pairwise disjoint intervals (a_i, b_i), $i \in I$,*

where I is either finite or countable and $\exists\psi\colon U \to \mathbb{R}$ such that $\forall i \in I$ $\psi_i = \psi|_{(a_i,b_i)}\colon (a_i, b_i) \to \mathbb{R}$ is a homeomorphism and the corresponding group dual

$$C_0(X)^{\odot} = M(X \setminus U) \oplus L^1(U, d\psi).$$

The above theorem generalizes the well-known result of A. Plessner that if $f\colon \mathbb{R} \to \mathbb{C}$ and $\mathrm{Var}_{\mathbf{R}}[f] < +\infty$, then f is absolutely continuous if and only if $\mathrm{Var}_{\mathbf{R}}[f(\cdot + t) - f(\cdot)] \to 0$ as $t \to 0$.

The following theorem generalizes the result of N. Wiener and R. C. Young about the behavior of measures on $\mathbb{R}$ under translation.

Theorem. *Let $\{T(t)\}_{t\in\mathbf{R}}$ be a C_0-group on $C_0(X)$ with the disjoint support property. Then $\forall\mu \in M(X)$*

$$\limsup_{t\to 0} \|T^*(t)\mu - \mu\| \geq 2\|\mu_d\|,$$

where μ_d is the component of μ in $C_0(X)^{\odot d}$. Moreover, if $\limsup_{t\to 0} \|T(t)\| = 1$, then the last inequality becomes an equality.

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Chapter 0

Introduction

Let E be a Banach space, $s \in \mathbb{R}$, $\{A(t)\}_{t \geq s}$ a one parameter family of linear operators with domains $D(A(t))$ consisting of linear subspaces of E.

Definition. An *evolution equation* is a differential equation

$$\dot{u}(t) = A(t)u(t), \quad t \geq s,$$

where $u(t)$, $t \geq s$ is an E-valued function.

If $s = 0$ and $A(t) = A$ is constant, the evolution equation

$$\dot{u}(t) = Au(t), \quad t \geq 0$$

is called *autonomous.*

In many cases (see [Pa, Ch. 4]) the solution of an autonomous evolution equation with initial value $u(0) = u_0$ is given by $u(t) = T(t)u_0$, where $\{T(t)\}_{t \geq 0}$ is a C_0-semigroup with infinitesimal generator A.

Definition. A family $\{T(t)\}_{t \geq 0}$, where $\forall t \geq 0\ T(t) \in L(E)$ is called a C_0*-semigroup* if

(1) $\forall t, s \geq 0\ T(t+s) = T(t)T(s)$.

(2) $T(0) = I$.

(3) $\forall u \in E\ T(t)u \to u$ as $t \downarrow 0$.

The *infinitesimal generator* A of a C_0-semigroup is defined by

$$Au = \lim_{t \downarrow 0} \frac{T(t)u - u}{t}$$

for all $u \in D(A)$, where $D(A)$ is the set of all $u \in E$ for which this limit exists.

$D(A)$ is a norm-dense linear subspace of E, and A is a closed linear operator (see [Pa, Ch.1]).

It is shown in [Pa, Ch. 5] that the solution of a non-autonomous evolution equation with initial values $u(s) = u_s$, $s \in \mathbb{R}$ is often given by $u(t,s) = U(t,s)u_s$, where $\{U(t,s)\}_{t \geq s}$ is an evolution family with the property that

$$\frac{\partial U(t,s)}{\partial t} = A(t)U(t,s), \qquad \frac{\partial U(t,s)}{\partial s} = -U(t,s)A(s).$$

Definition. A two-parameter family $\{U(t,s)\}_{t \geq s}$, where $\forall t \geq s$ $U(t,s) \in L(E)$ is called an *evolution family* if

(1) $\forall t \geq r \geq s$ $U(t,r)U(r,s) = U(t,s)$.
(2) $\forall s \in \mathbb{R}$ $U(s,s) = I$.
(3) The map $(t,s) \to U(t,s)$ is strongly continuous for $t \geq s$.

In [Ra] René Rau showed that the study of evolution families can be reduced to the study of semigroups by defining

$$(T(t)f)(x) = U(x, x-t)f(x-t), \quad x \in \mathbb{R},$$

where $t \geq 0$, $x \in \mathbb{R}$, $f \in C_0(\mathbb{R}, E)$. This semigroup has an important property: $\forall t \geq 0$ $T(t)$ has the *disjoint support property*, i.e. it maps functions with disjoint support to functions with disjoint support. We study operators with the disjoint support property in great detail in Chapters 1 and 2.

If X is a compact Hausdorff space, E is a Banach space, then operators with the disjoint support property acting on $C(X, E)$ are classified in [JR]. In Chapter 1 we extend this result to locally compact Hausdorff spaces.

In Chapter 2 we study semigroups and groups of operators with the disjoint support property. For the scalar case ($E = \mathbb{C}$) they were classified in [Na, B-II.3]. We extend these results to the case when E is an arbitrary Banach space.

Operators with the disjoint support property are closely related to disjointness preserving operators on Banach lattices. Roughly speaking, a Banach lattice is a Banach space with two lattice operations $\vee$ and $\wedge$ defined on it. Also every element u of a Banach lattice has an absolute value defined by $|u| = u \vee (-u)$. For an introduction to the theory of Banach lattices, we refer to [AB] and [LZ].

Many function spaces are Banach lattices where $\vee$ and $\wedge$ are defined by

$$(f \vee g)(x) = \max\{f(x), g(x)\}, \qquad (f \wedge g)(x) = \min\{f(x), g(x)\}.$$

Definition. In a Banach lattice two elements u and v are called *disjoint* or *orthogonal* (in symbols, $u \perp v$) if $|u| \wedge |v| = 0$.

Let E_1, E_2 be Banach lattices, $T\colon E_1 \to E_2$ a linear operator.

Definition. T is called *positive* if $\forall u \geq 0\ Tu \geq 0$.

T is called a *lattice homomorphism* if it preserves the lattice operations.

T is called *disjointness preserving* if $\forall u, v \in E_1\ (u \perp v \Rightarrow Tu \perp Tv)$.

It can be shown that T is a lattice homomorphism if and only if it is positive and disjointness preserving. Since every invertible positive operator whose inverse is also positive is a lattice homomorphism (see [AB, 7.3]), every positive group on a Banach lattice is a group of lattice homomorphisms. If $E_i = C_0(X_i)$, $i = 1, 2$, then an operator $T \in L(E_1, E_2)$ has the disjoint support property if and only if it is disjointness preserving. Therefore, every positive group on $C_0(X)$ is a group with the disjoint support property. Positive groups are studied in [dP], [Na], [vN] and other sources. We deal with groups with the disjoint support property on $C_0(a, b)$ in Chapter 3.

Given a C_0-semigroup $\{T(t)\}_{t\geq 0}$, the adjoint operators $\{T^*(t)\}_{t\geq 0}$ also form a semigroup of operators, i.e. satisfy (1) and (2) in the definition of a C_0-semigroup. However, this semigroup is not, in general, strongly continuous, i.e. does not satisfy

(3). [HP] introduced the *sun-dual* $E^{\odot}$ which is the subspace of E^* on which the adjoint semigroup is strongly continuous.

It follows from [Pl] that $C_0(\mathbb{R})^{\odot}$ with respect to the translation group is $L^1(\mathbb{R}, dx)$. In Chapter 3 we generalize this result for an arbitrary C_0-group with the disjoint support property on $C_0(a, b)$ and find the sun-duals for such groups.

[dP] showed that for any positive group on a Banach lattice E whose dual E^* has an order continuous norm its sun-dual is a *projection band*, i.e.

$$E^* = E^{\odot} \oplus E^{\odot d},$$

where $E^{\odot d} = \{ u^* \in E^* : \forall v^* \in E^{\odot}\ u^* \perp v^* \}$ is the *disjoint complement* of $E^{\odot}$. For instance, for the translation group $C_0(\mathbb{R})^{\odot d}$ is the projection band of singular measures.

[WY] proved that $\forall \mu \in M(\mathbb{R})$

$$\limsup_{t \to 0} \|\mu_t - \mu\| = 2\|\mu_d\|,$$

where $\forall F \subset \mathbb{R}$, F Borel $\mu_t(F) = \mu(F - t)$, μ_d is the component of μ in $C_0(\mathbb{R})^{\odot d}$. This result was later generalized in [dP] for positive groups. In Chapter 3 we also obtain Wiener-Young type theorem for groups with the disjoint support property on $C_0(a, b)$.

Chapter 1

Operators on $C_0(X, E)$ with the disjoint support property

Let X be a locally compact Hausdorff space, E a Banach space.

Definition. A function $f\colon X \to E$ is called *vanishing at infinity* if $\forall \varepsilon > 0\ \exists K \subset X$, K compact such that $\forall x \in X \setminus K\ \|f(x)\| < \varepsilon$.

We denote the space of all continuous functions $f\colon X \to E$ vanishing at infinity by $C_0(X, E)$, the space of all continuous functions $f\colon X \to E$ with compact support by $C_c(X, E)$, the space of all regular E^*-valued Borel measures on X with finite variation by $M(X, E^*)$. For the theory of vector-valued measures and integration with respect to these measures we refer to [Di].

Lemma 1. *Suppose $U \subset X$, U is open, $\mu \in M(X, E^*)$ is such that $\|\mu\|(U) \neq 0$, where $\|\mu\|$ is the variation of μ. Then $\exists f \in C_c(X, E)$ with $\operatorname{supp} f \subset U$ such that $\int\limits_X \langle f, d\mu\rangle \neq 0$.*

Proof. Since $\|\mu\|(U) \neq 0$, $\exists A$ Borel, $A \subset U$, such that $\mu(A) \neq 0$. The regularity of μ implies now that $\exists K$ compact, $K \subset A$ such that $\mu(K) \neq 0$. Therefore, $\exists u \in E_1$, $\|u\| = 1$ such that $\langle u, \mu(K)\rangle \neq 0$. Then $\varepsilon = |\langle u, \mu(K)\rangle| > 0$.

Since $\|\mu\|$ is a positive regular Borel measure on K, $\exists W$ open with compact closure such that $K \subset W \subset \overline{W} \subset U$, $\|\mu\|(W) < \|\mu\|(K) + \varepsilon$. Using Urysohn's Lemma, we can find $g \in C_c(X)$ such that $g = 1$ on K, $\operatorname{supp} g \subset W$ and $\|g\| = 1$. Define $f = g \otimes u$. Then

$$\int\limits_X \langle f, d\mu\rangle = \int\limits_K \langle f, d\mu\rangle + \int\limits_{W\setminus K} \langle f, d\mu\rangle.$$

Observe that

$$\Big|\int\limits_K \langle f, d\mu\rangle\Big| = \varepsilon, \qquad \Big|\int\limits_{W\setminus K} \langle f, d\mu\rangle\Big| \leq \|\mu\|(W \setminus K)\ \|f\| < \varepsilon.$$

It follows that $|\int\limits_X \langle f, d\mu\rangle| > 0$. ■

The following theorem generalizes the well-known result of I. Singer ([Si]) about the representation of the linear functionals on the space of vector-valued continuous functions on a compact space.

Theorem 2. $C_0(X,E)^* = M(X,E^*)$.

Proof. Let $\hat{X}$ be the one-point compactification of X. We can identify $C_0(X,E)$ with the subspace of functions $f \in C(\hat{X},E)$ such that $f(\infty) = 0$. Singer's theorem cited above implies that $C(\hat{X},E)^* = M(\hat{X},E^*)$. Then

$$C_0(X,E)^* = M(\hat{X},E^*)/C_0(X,E)^\perp$$

(see for example [Ru1, 4.9]).

Next we will show that $C_0(X,E)^\perp = \{\, \delta_\infty \otimes u^* : u^* \in E^* \,\}$, where δ is the Dirac measure. The $\supset$ inclusion is trivial. To prove $\subset$, suppose $0 \neq \mu \in C_0(X,E)^\perp$. Then its variation $\|\mu\|$ is a positive Borel regular measure on $\hat{X}$. Such measures always have nonempty supports (see [HR, 11.25]). We want to show that $\operatorname{supp}\|\mu\| = \{\infty\}$. Suppose it is not true, i.e. $\exists x \in X \cap \operatorname{supp}\|\mu\|$. Then for any open $U \subset X$ such that $x \in U$ we have $\|\mu\|(U) \neq 0$. Applying Lemma 1, we will obtain a function $f \in C_c(X,E)$ such that $\int\limits_X \langle f, d\mu\rangle \neq 0$. This contradicts the fact that μ annihilates $C(X,E)$. Therefore, the inclusion $\subset$ is also proved.

The statement of the theorem follows now from

$$M(\hat{X},E^*) = M(X,E^*) \oplus M(\{\infty\},E^*).$$ ■

Definition. A measure $\mu \in M(X,E^*)$ is said to have *the disjoint support property* if $\forall f_1, f_2 \in C_0(X,E)$

$$\|f_1(\cdot)\| \wedge \|f_2(\cdot)\| = 0 \qquad \Rightarrow \qquad \Big|\int\limits_X \langle f_1, d\mu\rangle\Big| \wedge \Big|\int\limits_X \langle f_2, d\mu\rangle\Big| = 0.$$

Proposition 3. *A measure $\mu \in M(X, E^*)$ has the disjoint support property if and only if $\exists x \in X$, $\exists u^* \in E^*$ such that $\mu = \delta_x \otimes u^*$.*

Proof. NECESSITY. If $\mu = 0$, then it is obvious. Let $\mu \neq 0$. We want to show that $\operatorname{supp} \|\mu\|$ consists of a single point. Suppose $\exists x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $x_1, x_2 \in \operatorname{supp} \|\mu\|$. Then $\exists U_1, U_2$ open disjoint such that $x_i \in U_i$ and $\|\mu\|(U_i) \neq 0$, $i = 1, 2$. Apply Lemma 1 to construct $f_1, f_2 \in C_c(X, E)$ such that $\operatorname{supp} f_i \subset U_i$, $i = 1, 2$ and $\int_X \langle f_1, d\mu \rangle \neq 0$, $\int_X \langle f_2, d\mu \rangle \neq 0$. Clearly, $\|f_1(\cdot)\| \wedge \|f_2(\cdot)\| = 0$ but $|\int_X \langle f_1, d\mu \rangle| \wedge |\int_X \langle f_2, d\mu \rangle| \neq 0$. Therefore, $\exists x \in X$ such that $\operatorname{supp} \|\mu\| = \{x\}$. Let $u^* = \mu(\{x\})$. Then $\mu = \delta_x \otimes u^*$.

SUFFICIENCY. Obvious. ■

Let X_1, X_2 be locally compact Hausdorff spaces, E_1, E_2 Banach spaces.

Definition. An operator $T \in L\big(C_0(X_1, E_1), C_0(X_2, E_2)\big)$ is said to have *the disjoint support property* if $\forall f_1, f_2 \in C_0(X_1, E_1)$

$$\|f_1(\cdot)\| \wedge \|f_2(\cdot)\| = 0 \qquad \Rightarrow \qquad \|(Tf_1)(\cdot)\| \wedge \|(Tf_2)(\cdot)\| = 0.$$

Let $L_s(E_1, E_2)$ be the space of all bounded linear operators from E_1 to E_2 with the strong operator topology.

Theorem 4. *T is an operator in $L\big(C_0(X_1, E_1), C_0(X_2, E_2)\big)$ with the disjoint support property if and only if $\exists Y$ open, $Y \subset X_2$ $\exists Q \in C_b\big(Y, L_s(E_1, E_2)\big)$ $\exists \varphi \in C(Y, X_1)$ such that :*

(1) *$\forall y \in Y$ $Q(y) \neq 0$.*

(2) *$\forall \varepsilon > 0$ $\forall u \in E_1$ $\forall K$ compact, $K \subset X_1$ $\exists F$ compact, $F \subset Y$ with the following property:*

$$y \in Y \setminus F, \quad \varphi(y) \in K \qquad \Rightarrow \qquad \|Q(y)u\| < \varepsilon.$$

(3) $\forall f \in C_0(X_1, E_1)$

$$(Tf)(y) = \begin{cases} Q(y)f(\varphi(y)), & \text{if } y \in Y, \\ 0, & \text{if } y \in X_2 \setminus Y. \end{cases}$$

Proof. NECESSITY. Let $N = \{ y \in X_2 : \forall f \in C_0(X_1, E_1)\ (Tf)(y) = 0 \}$. Since $\forall f \in C_0(X_1, E_1)$ its null set is closed, and N is the intersection of all such null sets, it follows that N is a closed subset of X_2. Therefore, $Y = X_2 \setminus N$ is open.

Fix $y \in Y$, $v^* \in E_2^*$. Then $\mu = T^*(\delta_y \otimes v^*) \in M(X_1, E_1^*)$. Suppose also that

$$\|f_1(\cdot)\| \wedge \|f_2(\cdot)\| = 0$$

for some $f_1, f_2 \in C_0(X_1, E_1)$. Then

$$\int_{X_1} \langle f_i, d\mu \rangle = \int_{X_2} \big\langle Tf_i, d(\delta_y \otimes v^*) \big\rangle = \big\langle (Tf_i)(y), v^* \big\rangle, \quad i = 1, 2.$$

Since T has the disjoint support property,

$$\|(Tf_1)(y)\| \wedge \|(Tf_2)(y)\| = 0,$$

which implies that

$$\big|\big\langle (Tf_1)(y), v^* \big\rangle\big| \wedge \big|\big\langle (Tf_2)(y), v^* \big\rangle\big| = 0.$$

It follows that

$$\Big| \int_{X_1} \langle f_1, d\mu \rangle \Big| \wedge \Big| \int_{X_1} \langle f_2, d\mu \rangle \Big| = 0,$$

whence μ is a measure with the disjoint support property. Applying Proposition 3, we will get that

$$T^*(\delta_y \otimes v^*) = \delta_x \otimes u^*,$$

where $u^* = u^*(y, v^*) \in E_1^*$, $x = x(y, v^*) \in X_1$.

Let $W_y = \{ v^* \in E_2^* : u^*(y, v^*) \neq 0 \}$. Then $\forall y \in Y\ W_y \neq \varnothing$. To see this, suppose that $W_y = \varnothing$ for some $y \in Y$. It implies that $\forall v^* \in E_2\ T^*(\delta_y \otimes v^*) = 0$

which means that $\forall f \in C_0(X_1, E_1)$ $\forall v^* \in E_2$ $\int\limits_{X_2} \langle Tf, d(\delta_y \otimes v^*)\rangle = 0$. It follows that $\langle (Tf_i)(y), v^*\rangle = 0$. Therefore, $\forall f \in C_0(X_1, E_1)$ $(Tf)(y) = 0$. This contradicts the fact that $y \in Y$.

The next step is to show that $x(y, v^*)$ does not depend on v^* whenever $v^* \in W_y$. To see this, suppose $\exists v_1^*, v_2^* \in W_y$, $v_1^* \neq v_2^*$ such that $x_1 \neq x_2$, where $x_i = x(y, v_i^*)$, $i = 1, 2$. Let $u_i^* = u^*(y, v_i^*)$, $i = 1, 2$. Then $\exists u_1, u_2 \in E_1$ such that $\langle u_i, u_i^*\rangle \neq 0$, $i = 1, 2$. By Urysohn's lemma $\exists g_1, g_2 \in C_c(X_1)$ with disjoint supports such that $g_i(x_i) = 1$, $i = 1, 2$. Let $f_i = g_i \otimes u_i$, $i = 1, 2$. It follows that

$$\|f_1(\cdot)\| \wedge \|f_2(\cdot)\| = 0.$$

Since T has the disjoint support property, this implies that

$$\|(Tf_1)(y)\| \wedge \|(Tf_2)(y)\| = 0.$$

On the other hand,

$$\langle (Tf_i)(y), v_i^*\rangle = \langle u_i, u_i^*\rangle \neq 0, \quad i = 1, 2,$$

whence $(Tf_1)(y) \neq 0$ and $(Tf_2)(y) \neq 0$. Contradiction.

If $v^* \notin W_y$, then $u^*(y, v^*) = 0$ and $x = x(y, v^*)$ is not uniquely defined. Therefore, for such v^* we can define $x(y, v^*) = x(y, v_0^*)$, where v_0^* is any vector in W_y. Hence, $\varphi(y) = x(y, v^*)$ does not depend on v^*. We conclude that $\forall f \in C_0(X_1, E_1)$ $\forall v^* \in E_2^*$ $\forall y \in Y$

$$\langle (Tf)(y), v^*\rangle = \langle Tf, \delta_y \otimes v^*\rangle = \langle f, T^*(\delta_y \otimes v^*)\rangle = \langle f, \delta_x \otimes u^*\rangle = \langle f(\varphi(y)), u^*(y, v^*)\rangle.$$

Thus, the following formula holds:

$$(*) \qquad \langle (Tf)(y), v^*\rangle = \langle f(\varphi(y)), u^*(y, v^*)\rangle.$$

For each $y \in Y$, $u \in E_1$ define $Q(y)u = (Tf)(y)$, where $f \in C_0(X_1, E_1)$ is any function such that $f(\varphi(y)) = u$. We have to prove that $Q(y)u$ is well defined.

Suppose we have two functions f_1 and f_2 as above. Let $f = f_1 - f_2$. It follows from $(*)$ that $\forall v^* \in E_2^*\ \langle (Tf)(y), v^* \rangle = 0$, whence $(Tf_1)(y) = (Tf_2)(y)$. Thus, $Q(y)u$ is well defined and $\forall f \in C_0(X_1, E_1)\ \forall y \in Y\ Q(y)f(\varphi(y)) = (Tf)(y)$. This proves (3).

Also observe that $\forall u \in E_1\ \|Q(y)u\| \leq \|T\|\,\|f\|$. Since we can always choose f such that $\|f\| \doteq \|u\|$, $\|Q(y)u\| \leq \|T\|\,\|u\|$. Thus, $\forall y \in Y\ Q(y) \in L(E_1, E_2)$.

Suppose $Q(y) = 0$ for some $y \in Y$. Then it follows from (3) that $\forall f \in C_0(X_1, E_1)$ $(Tf)(y) = 0$. That contradicts the fact that $y \in Y$. This proves (1).

Our next step is to establish the continuity properties of the functions φ and Q. We will start with φ.

Suppose a net $y_\alpha \to y$ in Y but $\varphi(y_\alpha)$ does not converge to $\varphi(y)$. It means that there exists a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ $\exists U$ open, $\varphi(y) \in U \subset X_1$ such that $\{\varphi(y_\beta)\} \cap U = \varnothing$. By Urysohn's lemma $\exists g \in C_c(X_1)$ such that $\operatorname{supp} g \subset U$ and $g(\varphi(y)) = 1$. Also, since $Q(y) \neq 0$, $\exists u \in E_1$ such that $u \notin ker(Q(y))$. Let $f = g \otimes u$. Then $Tf \in C(X_1, E_1)$. Observe that since $\{y_\beta\}$ is a subnet of $\{y_\alpha\}$,

$$Q(y_\beta)f(\varphi(y_\beta)) \to Q(y)f(\varphi(y)).$$

However, $Q(y_\beta)f(\varphi(y_\beta)) = 0$, $Q(y)f(\varphi(y)) = Q(y)u \neq 0$. Contradiction. Thus, $\varphi \in C(Y, X_1)$.

Now we turn our attention to Q. We have already seen that Q is bounded, namely $\forall y \in Y\ \|Q(y)\| \leq \|T\|$. To prove continuity, suppose again that a net $y_\alpha \to y$ in Y. We claim that $Q(y_\alpha) \to Q(y)$ in $L_s(E_1, E_2)$. Let U be a neighborhood of $\varphi(y)$ with compact closure. Since φ is continuous, we can assume, without loss of generality, that $\{\varphi(y_\alpha)\} \subset U$. By Urysohn's lemma $\exists g \in C_c(X_1)$ such that $g|_{\overline{U}} = 1$. Let $u \in E_1$, $f = g \otimes u$. Then

$$Q(y_\alpha)f(\varphi(y_\alpha)) \to Q(y)f(\varphi(y))$$

implies that $Q(y_\alpha)u \to Q(y)u$. This proves $Q \in C_b\big(Y, L_s(E_1, E_2)\big)$.

Finally, we must establish (2). Suppose, $\varepsilon > 0$, $u \in E_1$, $K \subset X_1$, K is compact. By Urysohn's lemma $\exists g \in C_c(X_1)$ such that $g|_K = 1$. Let $f = u \otimes g$, $F = \{y \in X_2 : \|(Tf)(y)\| \geq \varepsilon\}$. Then, since Tf vanishes at infinity, F is a compact subset of Y. If $y \in Y \setminus F$ and $\varphi(y) \in K$, then since $f(\varphi(y)) = u$, $\|Q(y)u\| < \varepsilon$.

SUFFICIENCY. Let $f \in C_0(X_1, E_1)$. From the continuity properties of the functions φ and Q it immediately follows that Tf is continuous at each point of Y. Our objective now is to prove that Tf is continuous on $X_2 \setminus Y$ and that it vanishes at infinity.

To this end, let $\varepsilon > 0$ and suppose that $M > 0$ is such that $\forall y \in Y$ $\|Q(y)\| \leq M$. Since f vanishes at infinity, $\exists K$ compact, $K \subset X_1$ such that $\|f(x)\| < \frac{\varepsilon}{M}$ $\forall x \notin K$. Observe that $f(K)$ is compact in E_1. Let $\{u_1, u_2, \ldots, u_n\}$ be an $\frac{\varepsilon}{2M}$-net for $f(K)$. Applying (2), we will obtain $F_1, F_2, \ldots, F_n$ compact, $F_i \subset Y$ such that

$$y \in Y \setminus F_i, \quad \varphi(y) \in K \qquad \Rightarrow \qquad \|Q(y)u_i\| < \frac{\varepsilon}{2}, \qquad i = 1, \ldots, n.$$

Let $F = \bigcup_{i=1}^{n} F_i$. If $y \notin F$, then there are three possibilities:

1. $y \notin Y$. In this case $(Tf)(y) = 0$.
2. $y \in Y$, $\varphi(y) \notin K$. In this case $\|Q(y)f(\varphi(y))\| < M\frac{\varepsilon}{M} = \varepsilon$.
3. $y \in Y$, $\varphi(y) \in K$. In this case $\|f(\varphi(y)) - u_i\| < \frac{\varepsilon}{2M}$ for some i, $1 \leq i \leq n$ which implies that

$$\|Q(y)f(\varphi(y))\| \leq \|Q(y)u_i\| + \|Q(y)[f(\varphi(y)) - u_i]\| < \frac{\varepsilon}{2} + M\frac{\varepsilon}{2M} = \varepsilon.$$

Therefore, $\forall y \notin F$ $\|(Tf)(y)\| < \varepsilon$, which proves that Tf vanishes at infinity. Also, $X_2 \setminus F$ is an open neighborhood for any $y \notin Y$. Since for such y $(Tf)(y) = 0$, it means that Tf is continuous at y.

Finally, the boundedness of Q implies that T is a bounded operator, and it follows from (3) that T has the disjoint support property. ■

As the following example shows, if T has the disjoint support property, then in general Q and φ cannot be extended to functions continuous on X_2.

Example 5. Suppose $X_1 = X_2 = \mathbb{R}$, $E_1 = E_2 = \mathbb{C}$. Define T as follows: $\forall f \in C_0(\mathbb{R})$

$$(Tf)(x) = \begin{cases} (\operatorname{sign} x)\, f(\log|x|), & \text{if } x \neq 0, \\ 0, & \text{else.} \end{cases}$$

Let K be a compact subset of $\mathbb{R}$. Then $K \subset [a,b]$ for some $a, b \in \mathbb{R}$. Let $F = [-e^b, -e^a] \cup [e^a, e^b]$. Clearly, F is a compact subset of $\mathbb{R} \setminus \{0\}$ and $\{\, x \in \mathbb{R} \setminus \{0\} : x \notin F,\ \log|x| \in K \,\} = \varnothing$. Therefore, by Theorem 4, T is an operator with the disjoint support property, however neither $\operatorname{sign} x$ nor $\log|x|$ can be extended to a function continuous on $\mathbb{R}$.

Corollary 6. *T is an operator in $L\big(C_0(X_1, E_1), C_0(X_2, E_2)\big)$ such that it is invertible and both T and T^{-1} have the disjoint support property if and only if the following conditions are satisfied:*

(1) *there exists a homeomorphism $\varphi\colon X_2 \to X_1$.*

(2) $\exists Q \in C_b\big(X_2, L_s(E_1, E_2)\big)$ $\exists R \in C_b\big(X_1, L_s(E_2, E_1)\big)$ *such that* $\forall y \in X_2$ $Q(y)$ *is invertible and* $\forall x \in X_1$ $R(x) = Q(\varphi^{-1}(x))^{-1}$.

(3) $\forall f \in C_0(X_1, E_1)$ $\forall y \in X_2$ $(Tf)(y) = Q(y) f(\varphi(y))$.

In this case $\forall g \in C_0(X_2, E_2)$ $\forall x \in X_1$ $(T^{-1}g)(x) = R(x) g(\varphi^{-1}(x))$.

Proof. NECESSITY. Since $\exists T^{-1}$, $\{\, y \in X_2 : \forall f \in C_0(X_1, E_1)\ (Tf)(y) = 0 \,\} = \varnothing$, $\{\, x \in X_1 : \forall g \in C_0(X_2, E_2)\ (T^{-1}g)(x) = 0 \,\} = \varnothing$. Then by Theorem 4 $\exists \varphi \in C(X_2, X_1)$ $\exists \psi \in C(X_1, X_2)$ $\exists Q \in C_b\big(X_2, L_s(E_1, E_2)\big)$ $\exists R \in C_b\big(X_1, L_s(E_2, E_1)\big)$ such that $\forall f \in C_0(X_1, E_1)$ $\forall g \in C_0(X_2, E_2)$ $\forall x \in X_1$ $\forall y \in X_2$

$$(**) \qquad \begin{aligned} Q(y) R(\varphi(y)) g((\psi \circ \varphi)(y)) &= g(y), \\ R(x) Q(\psi(x)) f((\varphi \circ \psi)(x)) &= f(x). \end{aligned}$$

$\forall u_1 \in E_1$ $\forall u_2 \in E_2$ $\forall x \in X_1$ $\forall y \in X_2$ we can always find $f \in C_0(X_1, E_1)$, $g \in C_0(X_2, E_2)$ such that $f(x) = f((\varphi \circ \psi)(x)) = u_1$, $g(y) = g((\psi \circ \varphi)(y)) = u_2$.

Then it follows from (**) that $Q(y)R(\varphi(y))u_1 = u_1$, $R(x)Q(\psi(x))u_2 = u_2$. Thus,

$$(***)\qquad \begin{aligned} Q(y)R(\varphi(y)) &= I_{E_1},\\ R(x)Q(\psi(x)) &= I_{E_2}. \end{aligned}$$

Now (**) and (***) combined imply that $\forall x \in X_1\ \forall y \in X_2\ \forall f \in C_0(X_1, E_1)$ $\forall g \in C_0(X_2, E_2)$ $g((\psi \circ \varphi)(y)) = g(y)$, $f((\varphi \circ \psi)(x)) = f(x)$. Since the functions from $C_0(X_i, E_i)$ separate points of X_i, $i = 1, 2$, it follows that $(\psi \circ \varphi)(y) = y$, $(\varphi \circ \psi)(x) = x$ whence $\psi = \varphi^{-1}$. Let $y = \varphi(x)$. (***) now implies that

$$Q(\psi(x))R(x) = I_{E_1},$$

$$R(x)Q(\psi(x)) = I_{E_2}.$$

We conclude that $R(x) = Q(\psi(x))^{-1}$.

SUFFICIENCY. Define the operator T as in (3). To prove that T has the disjoint support property, we need to verify condition (2) of Theorem 4. To this end, let K be a compact subset of X_1, $F = \psi(K)$. Then $(X_2 \setminus F) \cap \{\, y \in X_2 : \varphi(y) \in K \,\} = \varnothing$. It follows that condition (2) of Theorem 4 is satisfied, hence T has the disjoint support property.

$\forall g \in C_0(X_2, E_2)\ \forall x \in X_1$ define the operator S by $(Sg)(x) = R(x)g(\psi(x))$. Using a similar reasoning to the above one, we can show that S also has the disjoint support property. Now it is not difficult to see that $\forall f \in C_0(X_1, E_1)\ \forall g \in C_0(X_2, E_2)$ $TSg = g$, $STf = f$. This implies that $S = T^{-1}$. ∎

Remark. If $E_1 = E_2 = \mathbb{C}$, then operators with the disjoint support property are disjointness preserving, and vice versa (for the theory of disjointness preserving operators see [AB] and [MN]). Therefore, in this case the condition that T^{-1} has the disjoint support property in Corollary 6 is redundant since by [MN, Cor.3.1.21] the inverse of a disjointness preserving operator, when it exists, is also a disjointness preserving operator. In general, however, this condition is not redundant as the following example shows.

Example 7. Let $X_1 = \{0\}$, $X_2 = \{1, 2\}$, $E_1 = E_2 = l^2$. Then X_1 and X_2 with the discrete topology are compact Hausdorff spaces, $C_0(X_1, E_1) = l^2$, $C_0(X_2, E_2) = l^2 \times l^2$. For any $f \in l^2$ define $T\colon C_0(X_1, E_1) \to C_0(X_2, E_2)$ by

$$Tf = (\{f_1, f_3, f_5, \dots\}, \{f_2, f_4, f_6, \dots\}).$$

Clearly, T has the disjoint support property. For any $(g, h) \in l^2 \times l^2$ define $S\colon C_0(X_2, E_2) \to C_0(X_1, E_1)$ by

$$S(g, h) = \{g_1, h_1, g_2, h_2, g_2, h_3, \dots\}.$$

It follows that $S = T^{-1}$ but S does not have the disjoint support property since $\forall f \in l^2$, $f \neq 0$ $\|(f, 0)(\cdot)\| \wedge \|(0, f)(\cdot)\| = 0$ but $\|(S(f, 0))(\cdot)\| \wedge \|(S(0, f))(\cdot)\| \neq 0$.

Suppose that X_1 and X_2 are compact Hausdorff.

Corollary 8. *T is an operator in $L(C(X_1, E_1), C(X_2, E_2))$ with the disjoint support property if and only if $\exists Q \in C(X_2, L_s(E_1, E_2))$ $\exists \varphi \in C(Y, X_1)$, where $Y = \{y \in X_2 : Q(y) \neq 0\}$ such that condition (3) of Theorem 4 is satisfied.*

Proof. NECESSITY. Apply Theorem 4 and let Y be as in this theorem. Define $Q(y) = 0$ $\forall y \notin Y$. Then $Y = \{y \in X_2 : Q(y) \neq 0\}$ and $\forall u_1 \in E_1$ $\forall y \in X_2$ $Q(y)u_1 = (T(\mathbb{1}_{X_1} \otimes u_1))(y)$, where $\mathbb{1}_{X_1}$ is a constant 1-function defined on X_1. Therefore, $Q(y) \in C(X_2, L_s(E_1, E_2))$.

SUFFICIENCY. Define operator T as in (3) of Theorem 4. To prove that T has the disjoint support property, we need to verify condition (2) of Theorem 4. Suppose K be a compact subset of X_1, $F = \varphi^{-1}(K)$. Since K is closed, φ is continuous, F is a closed subset of Y. Then F is compact because X_2 is. Finally, $(X_2 \setminus F) \cap \{y \in X_2 : \varphi(y) \in K\} = \varnothing$ implies that condition (2) of Theorem 4 is satisfied. ∎

Remark. Corollary 8 was first proved by [JR] using a slightly different approach.

Corollary 8 shows that whenever X_1 and X_2 are compact, Q is continuous on X_2 rather than just on Y as in the general case. However, it is still impossible in general to extend φ to a function continuous on X_2 as the following example shows.

Example 9. Suppose $X_1 = X_2 = [-1, 1]$, $E_1 = E_2 = \mathbb{C}$. Define T as follows: $\forall f \in C[0, 1]$

$$(Tf)(x) = \begin{cases} xf(\operatorname{sign} x), & \text{if } x \neq 0, \\ 0, & \text{else.} \end{cases}$$

Corollary 8 now implies that T has the disjoint support property but we cannot extend $\operatorname{sign} x$ to a function continuous on $[-1, 1]$.

Remark. Corollary 8 shows that the description of operators with the disjoint support property is nicer when both X_1 and X_2 are compact. So in the case where X_1 and X_2 are not compact, one might be tempted to try to extend an operator $T \in L\big(C_0(X_1, E_1), C_0(X_2, E_2)\big)$ with the disjoint support property to an operator $\hat{T} \in L\big(C(\widehat{X_1}, E_1), C(\widehat{X_2}, E_2)\big)$ such that $\hat{T}$ also has the disjoint support property, where $\widehat{X_1}$ and $\widehat{X_2}$ are compactifications of X_1 and X_2 respectively such that any $f \in C_0(X_i, E_i)$ can be extended to $\hat{f} \in C(\widehat{X_i}, E_i)$, $i = 1, 2$. The Alexandroff compactification is an example of such a compactification. When E_1, E_2 are finite-dimensional, the Stone-Čech compactification gives another example. Such an extension of an operator T, however, does not always exist. To see this, we need the following lemma.

Lemma 10. *Let X be a locally compact Hausdorff space, $\hat{X}$ a compactification of X such that any $f \in C_0(X)$ can be extended to $\hat{f} \in C(\hat{X})$. Then such an extension is unique for any $f \in C_c(X)$.*

Proof. Let $\operatorname{supp} f \subset K$, K compact, $\hat{x} \in \hat{X} \setminus X$. Since $\hat{X}$ is a compactification of X, $\exists h \colon X \to \hat{X}$ such that h is a homeomorphism of X onto $h(X)$ and such that $h(X)$ is dense in $\hat{X}$. Therefore, $\exists x_\alpha$ a net such that $\{x_\alpha\} \subset X$ and $x_\alpha \to \hat{x}$ in $\hat{X}$.

Suppose $\exists x_\beta$ a subnet of x_α such that $\{x_\beta\} \subset K$. Since K is compact in X, $\exists x_\gamma$ a subnet of x_β such that $x_\gamma \to x$ in X for some x in K. h is a continuous map, so $h(x_\gamma) \to h(x)$ in $\hat{X}$. This contradicts $x_\alpha \to \hat{x}$.

Therefore, the net x_α is eventually in $X \setminus K$. Let $\hat{f}$ be a continuous extension of f on $\hat{X}$. Then eventually $\hat{f}(x_\alpha) = 0$ and since $\hat{f} \in C(\hat{X})$, $\hat{f}(\hat{x}) = 0$. Thus, $\hat{f}|_{\hat{X}\setminus X} = 0$. We conclude that the extension is unique. ■

We proceed with our reasoning now. Consider the operator T from Example 5. Suppose we can extend it to $\hat{T} \in C(\widehat{\mathbb{R}})$. Then by Corollary 8, $\exists \hat{Q} \in C(\widehat{\mathbb{R}})$ $\exists \hat{\varphi} \in C(\hat{Y}, \widehat{\mathbb{R}})$, where $\hat{Y} = \{\, \hat{y} \in \widehat{\mathbb{R}} : \hat{Q}(\hat{y}) \neq 0 \,\}$ such that $\forall \hat{f} \in C(\widehat{\mathbb{R}})$

$$(\hat{T}\hat{f})(\hat{y}) = \begin{cases} \hat{Q}(\hat{y})\hat{f}(\hat{\varphi}(\hat{y})), & \text{if } \hat{y} \in \hat{Y}, \\ 0, & \text{else.} \end{cases}$$

Let $y \in \mathbb{R} \setminus \{0\}$. Clearly, $y \in \hat{Y}$. Suppose $\hat{\varphi}(y) \neq \log|y|$. Then by Lemma 10 $\exists f \in C_c(\mathbb{R})$ such that $f(\log|y|) = 1$ and $\hat{f}(\hat{\varphi}(y)) = 0$, where $\hat{f}$ is a continuous extension of f. Therefore,

$$\operatorname{sign} y = (Tf)(y) = (\hat{T}\hat{f})(y) = 0.$$

Contradiction with $y \neq 0$. Thus, $\hat{\varphi}(y) = \log|y|$. As a consequence, we conclude that $\forall y \in \mathbb{R} \setminus \{0\}$ $\hat{Q}(y) = \operatorname{sign}(y)$. This is clearly impossible since $\hat{Q}$ is a continuous function and sign is not. Contradiction.

Another approach would be to try to extend T to $\hat{T} \in L\big(C(\widehat{X_1}, E_1), C(\hat{Y}, E_2)\big)$, where $\hat{Y}$ is a compactification of Y. Whether it is possible or not is still an open question, however, it is clear that the Alexandroff compactification will not work since if $f \in C_0(X_2, E_2)$, then $f|_Y$ might not be in $C_0(Y, E_2)$.

Suppose X is a locally compact Hausdorff space and E is a Banach space.

Definition. An operator $T \in L\big(C_0(X, E)\big)$ is called *local* if $\forall f_1, f_2 \in C_0(X, E)$

$$\|f_1(\cdot)\| \wedge \|f_2(\cdot)\| = 0 \qquad \Rightarrow \qquad \|(Tf_1)(\cdot)\| \wedge \|f_2(\cdot)\| = 0.$$

Clearly, every local operator has the disjoint support property. Therefore, we can apply Theorem 4 to get the following theorem characterizing local operators.

Theorem 11. *T is a local operator in $L(C_0(X,E))$ iff $\exists Q \in C_b(X, L_s(E))$ such that $\forall f \in C_0(X,E)$ $\forall x \in X$*

$$(Tf)(x) = Q(x)f(x).$$

Proof. NECESSITY. Again let Y be as in Theorem 4, and define $Q(y) = 0$ $\forall y \notin Y$. Then $Y = \{x \in X : Q(x) \neq 0\}$. We have to show that $\forall x \in Y$ $\varphi(x) = x$. Suppose this is not true, i.e. $\exists x \in Y$ such that $\varphi(x) \neq x$. Using Urysohn's lemma, we can construct $g_1, g_2 \in C_c(X)$ such that $\operatorname{supp} g_1 \cap \operatorname{supp} g_2 = \varnothing$ and $g_1(x) = 0$, $g_1(\varphi(x)) = 1$, $g_2(x) = 1$, $g_2(\varphi(x)) = 0$. Since $Q(x) \neq 0$, $\exists u \neq 0$, $u \notin ker(Q(x))$. Let $f_i = g_i \otimes u$, $i = 1, 2$. Then $\|f_1(\cdot)\| \wedge \|f_2(\cdot)\| = 0$. However,

$$(Tf_1)(x) = Q(x)f_1(\varphi(x)) = Q(x)u \neq 0,$$
$$f_2(x) = u \neq 0.$$

This contradicts the fact that $\|(Tf_1)(\cdot)\| \wedge \|f_2(\cdot)\| = 0$.

The only thing which remains to be proved is that Q is continuous outside Y. Now that we know that $\varphi = id$, this can be done in the same manner as we proved the continuity of Q in Theorem 4.

SUFFICIENCY. Obvious. ■

Proposition 12.

(1) *Local operators form a closed subalgebra of $L_s(C_0(X,E))$.*

(2) *If T is local and invertible, than T^{-1} is also local.*

Proof. (1). The only nontrivial part is to show that local operators form a set closed in $L_s(C_0(X,E))$. Suppose that $\forall f \in C_0(X,E)$ $T_n f \to Tf$, where $\forall n \in \mathbb{N}$ T_n

is local. Let $f_1, f_2 \in C_0(X,E)$ be such that $\|f_1(\cdot)\| \wedge \|f_2(\cdot)\| = 0$. Since $\forall n \in \mathbb{N}$ T_n is local, $\|(T_n f_1)(\cdot)\| \wedge \|f_2(\cdot)\| = 0$. Then since by [AB, 11.1] the operation $\wedge$ is continuous, $\|(Tf_1)(\cdot)\| \wedge \|f_2(\cdot)\| = 0$.

(2). By Theorem 10 $\exists Q \in C_b(X, L_s(E))$ such that $\forall f \in C_0(X,E)$ $\forall x \in X$

$$(Tf)(x) = Q(x)f(x).$$

Let $S = T^{-1}$, $x \in X$, $u \in E$. Then $\exists f \in C_0(X,E)$ such that $f(x) = u$. Let $g = Sf$. Then

$$u = f(x) = (Tg)(x) = Q(x)g(x).$$

Therefore, $Q(x)$ is onto.

Suppose that $Q(x)u = 0$ for some u, $\|u\| = 1$. Since $Q \in C_b(X, L_s(E))$, $\forall n \in \mathbb{N}$ $\exists U_n$, U_n is a neighborhood of x such that $\forall y \in U_n$ $\|Q(y)u\| < \frac{1}{n}$. By Urysohn's Lemma, $\exists g_n \in C_c(X,E)$ such that $\|g_n\| = 1$ and $\operatorname{supp} g_n \subset U_n$. Let $f_n = g_n \otimes u$, $h_n = Tf_n$. Then $\forall y \in X$

$$\|h_n(y)\| = \|(Tf_n)(y)\| = \|g_n(y)Q(y)u\| < \frac{1}{n}.$$

Thus, $\forall n \in \mathbb{N}$ $\|h_n\| < \frac{1}{n}$ and $\|Sh_n\| = \|f_n\| = 1$ contradicting the boundedness of S. Hence, we may conclude that $Q(x)$ is 1-1.

We have proved that $\forall x \in X$ $Q(x)$ is a bijection. Let $R(x) = Q(x)^{-1}$. Then by the Open Mapping Theorem $\forall x \in X$ $R(x) \in L(E)$. Therefore, $\forall f \in C_0(X,E)$ $\forall x \in X$

$$(Sf)(x) = R(x)^{-1}f(x).$$

Using the argument similar to the one at the end of the necessity part of the proof of Theorem 4, we conclude that $R \in C_b(X, L_s(E))$. Thus, by Theorem 11, the operator S is local. ∎

Suppose that X is again a locally compact Hausdorff space but E is now a Banach lattice.

Theorem 13. *T is an orthomorphism on $C_0(X,E)$ iff $\exists Q \in C_b(X, L_s(E))$ such that $\forall x \in X$ $Q(x) \in Orth(E)$ and $\forall f \in C_0(X,E)$ $\forall x \in X$*

$$(Tf)(x) = Q(x)f(x).$$

Proof. NECESSITY. For each $x \in Y$, $u \in E$ define $Q(x)u = (Tf)(x)$, where f is any $C_0(X,E)$ function such that $f(x) = u$. We have to prove that $Q(x)u$ is well defined. Suppose we have two functions f_1 and f_2 as above. Let $f = f_1 - f_2$. [AB, 15.5] implies that $|T| \leq \|T\|I$ which means that

$$0 \leq (|T||f|)(x) \leq \|T\||f|(x) = 0.$$

We conclude that $(|T||f|)(x) = 0$. Since $|Tf| = |T||f|$ (see [AB, 8.6]), it follows that $(Tf)(x) = 0$. Thus, $(Tf_1)(x) = (Tf_2)(x)$. Therefore, $Q(x)$ is well defined and $\forall f \in C_0(X,E)$ $\forall x \in X$ $(Tf)(x) = Q(x)f(x)$. It is also easy to see that $\forall x \in X$ $\|Q(x)\| \leq \|T\|$. In the same manner as we proved the continuity of Q in Theorem 4, we can prove that Q is continuous at any point $x \in X$. Thus, $Q \in C_b(X, L_s(E))$.

Finally we have to show that $\forall x \in X$ $Q(x) \in Orth(E)$. Let $u_1 \perp u_2$ for some $u_1, u_2 \in E$. Choose $g \in C_0(X)$ such that $g(x) = 1$. Let $f_i = g \otimes u_i$, $i = 1, 2$. Then $f_1 \perp f_2$. Since T is an orthomorphism, $Tf_1 \perp f_2$ which implies that $(Tf_1)(x) \perp f_2(x)$ whence $Q(x)u_1 \perp u_2$. This proves $Q(x)$ is an orthomorphism on E.

SUFFICIENCY. Obvious. ■

Chapter 2

C_0-semigroups with the disjoint support property

Let X be a locally compact Hausdorff space, $\{ Y_t \subset X : t \geq 0 \}$ a collection of subsets of X, $\Pi = \{ (t,x) : t \geq 0, x \in Y_t \}$.

Definition. $\{Y_t\}$ is called *a collection of decreasing open sets* if

(1) $Y_0 = X$.

(2) $\forall t, s$ such that $0 \leq t \leq s$, $Y_s \subset Y_t$.

(3) Π is open in $[0, +\infty) \times X$.

Lemma 1. *$\{Y_t\}$ is a collection of decreasing open sets if and only if*

(a) *$\forall t \geq 0$ Y_t is open.*

(b) *If we define $I_x = \{ t \geq 0 : x \in Y_t \}$, $x \in X$, then $\forall x \in X$ $\exists a$ $0 < a \leq +\infty$ such that $I_x = [0, a)$.*

Proof. NECESSITY. (a) immediately follows from (3).

If $\forall t \geq 0$ $x \in Y_t$, then $I_x = [0, +\infty)$. Suppose this is not the case. Then (1) guarantees that $I_x \neq \varnothing$, (3) that I_x is open in $[0, +\infty)$, and (2) that I_x is connected. We conclude that (b) is true.

SUFFICIENCY. (1) and (2) follows immediately from (b).

Suppose $(t, x) \in \Pi$. (b) implies that $I_x = [0, a)$, where $t < a \leq +\infty$. Let $\alpha = t + 1$, if $a = +\infty$ and $\alpha = \frac{t+a}{2}$, otherwise. Then $\alpha \in I_x$. (a) implies that Y_α is open, therefore, $\exists U$ open neighborhood of x such that $U \subset Y_\alpha$. It follows from (b) that $\forall s$ $0 \leq s < \alpha$ $\forall y \in U$ $y \in Y_s$. We conclude that $[0, \alpha) \times U \subset \Pi$. Since $[0, \alpha) \times U$ is open in $[0, +\infty) \times X$ and $(t, x) \in [0, \alpha) \times U$, (3) is true. ∎

Let $\{Y_t\}$ be a collection of decreasing open sets.

Definition. A mapping $\varphi\colon \Pi \to X$ is called *a partial semiflow* if

(1) $\forall t \geq 0\ \varphi_t \in C(Y_t, X)$.

(2) $\varphi_0 = id_X$.

(3) If $x \in Y_{t+s}$ for some $t, s \geq 0$, then $\varphi_t(x) \in Y_s$ and $\varphi_s(\varphi_t(x)) = \varphi_{t+s}(x)$.

Definition. A partial semiflow φ is called *continuous* if $\varphi \in C(\Pi, X)$.

Example 2. Let $X = (0, +\infty)$, $Y_t = (\sqrt{t}, +\infty)$. $\forall x > \sqrt{t}$ define $\varphi_t(x) = \sqrt{x^2 - t}$. It follows that $I_x = [0, x^2)$ and $\{Y_t\}$ is a collection of decreasing open sets by Lemma 1. Straightforward calculation now shows that φ is a continuous partial semiflow.

Let φ be a partial semiflow, E a Banach space.

Definition. A mapping $Q\colon \Pi \to L_s(E)$ is called *a partial semicocycle of* φ if

(1) $\forall t \geq 0\ Q_t \in C_b(Y_t, L_s(E))$.

(2) $\forall (t, x) \in \Pi\ Q_t(x) \neq 0$.

(3) $\forall x \in X\ Q_0(x) = I$.

(4) If $x \in Y_{t+s}$ for some $t, s \geq 0$, then $Q_{t+s}(x) = Q_t(x)Q_s(\varphi_t(x))$.

(5) If $t, s \geq 0$, $x \in Y_t$, $\varphi_t(x) \in Y_s$, then either $x \in Y_{t+s}$ or $Q_t(x)Q_s(\varphi_t(x)) = 0$.

Definition. A partial semicocycle Q is called *continuous* if $Q \in C(\Pi, L_s(E))$.

Example 3. Let X, Y_t and φ be as in Example 2. Let also $E = \mathbb{C}$. $\forall x \in Y_t$ define $Q_t(x) = e^{x - \sqrt{x^2 - t}}$. Observe that $\forall x \geq \sqrt{t}$

$$x - \sqrt{x^2 - t} = \frac{t}{x + \sqrt{x^2 - t}} \leq \frac{t}{x} \leq \sqrt{t}.$$

Therefore $\forall t \geq 0\ Q_t \in C_b(Y_t)$. It is now easy to verify that Q is a continuous partial semicocycle of φ.

Definition. A C_0-semigroup $\{T(t)\}_{t \geq 0}$ on $C_0(X, E)$ is said to have *the disjoint support property* if $\forall t \geq 0\ T(t)$ has the disjoint support property.

Theorem 4. *$\{T(t)\}_{t\geq 0}$ is a C_0-semigroup on $C_0(X,E)$ with the disjoint support property if and only if $\exists\{Y_t\}$ a collection of decreasing open sets, $\exists\varphi$ a continuous partial semiflow, $\exists Q$ a continuous partial semicocycle of φ such that*

(1) *$\forall t \geq 0$ Q_t and φ_t satisfy condition (2) of Theorem 1.4.*

(2) *$\exists\delta > 0$ $\exists M \geq 0$ such that $\forall t$ $0 \leq t < \delta$ $\forall x \in Y_t$ $\|Q_t(x)\| \leq M$.*

(3) *$\forall t \geq 0$ $\forall f \in C_0(X,E)$*

$$(T(t)f)(x) = \begin{cases} Q_t(x)f(\varphi_t(x)), & \textit{if } x \in Y_t, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. NECESSITY. Theorem 1.4 implies that $\exists\{Y_t\}$ a collection of open sets $\exists\varphi$: $\Pi \to X$ $\exists Q\colon \Pi \to L_s(E)$ such that $\forall t \geq 0$ $\varphi_t \in C(Y_t, X)$, $Q_t \in C_b(Y_t, L_s(E))$, $\forall(t,x) \in \Pi$ $Q_t(x) \neq 0$ and such that (3) is satisfied.

Our first step is to show that Y_t is a collection of decreasing open sets.

The semigroup property $T(0) = I$ implies that $\{\, x \in X : \forall f \in C_0(X,E)$ $(T(0)f)(x) = 0 \,\} = \varnothing$. It follows that $Y_0 = X$.

Let $0 \leq s \leq t$, $x \in Y_t$. Suppose $x \notin Y_s$. Then $\forall f \in C_0(X,E)$ $(T(s)f)(x) = 0$. In particular, $\forall f \in C_0(X,E)$ $(T(t)f)(x) = (T(s)T(t-s)f)(x) = 0$. Since $x \in Y_t$, it follows from (3) that $\forall f \in C_0(X,E)$ $(T(t)f)(x) = Q_t(x)f(\varphi(x))$. $\forall u \in E$ we can always find an $f \in C_0(X,E)$ such that $f(\varphi(x)) = u$. Thus, $Q_t(x) = 0$. Contradiction.

Suppose $(x,t) \in \Pi$ and U is an open neighborhood of $\varphi(x)$. Since $Q_t(x) \neq 0$, $\exists u \in E$ such that $\|Q_t(x)u\| = 1$. By Urysohn's lemma $\exists f \in C_c(X,E)$ such that $\operatorname{supp} f \subset U$ and $f(\varphi_t(x)) = u$. The strong continuity of $\{T(t)\}_{t\geq 0}$ implies that $\exists\varepsilon > 0$ such that $J \subset I_x$ and $\forall s \in J$ $\|T(s)f - T(t)f\| < \frac{1}{2}$, where $J = [0,\varepsilon)$, if $t = 0$ and $J = (t-\varepsilon, t+\varepsilon)$, if $t > 0$. Let $V = \{\, y \in X : \|(T(t)f)(y)\| > \frac{1}{2} \,\}$. Then V is an open neighborhood of x. It follows that $\forall s \in J$ $\forall y \in V$ $(T(s)f)(y) \neq 0$. (3) now implies that $\forall s \in J$ $\forall y \in V$ $y \in Y_s$. Thus, $J \times V \subset \Pi$. Therefore, since

$J \times V$ is open in $[0, +\infty) \times X$, Π is open in $[0, +\infty) \times X$ and we conclude that Y_t is a collection of decreasing open sets.

Note also that $\forall s \in J\ \forall y \in V\ Q_s(y)f(\varphi_s(y)) \neq 0$, whence $f(\varphi_s(y)) \neq 0$, whence $\varphi_s(y) \in U$. We conclude that $\varphi \in C(\Pi, X)$.

We turn our attention to Q now. Suppose $(t, x) \in \Pi$, $u \in E$. Since Π is open in $[0, +\infty) \times X$, $\exists K_x$ a compact neighborhood of x, $\exists K_t$ a compact neighborhood of t, such that $K_t \times K_x \subset \Pi$. The continuity of φ implies that if $K = \varphi(K_t \times K_x)$, then K is compact in X. By Urysohn's lemma $\exists g \in C_c(X)$ such that $g|_K = 1$. Let $f = g \otimes u$, $\varepsilon > 0$ arbitrary. Since $T(t)f \in C_0(X, E)$, $\exists U$ open, $x \in U \subset K_x$ such that $\forall y \in U\ \|(T(t)f)(y) - (T(t)f)(x)\| < \frac{\varepsilon}{2}$. The strong continuity of $\{T(t)\}_{t \geq 0}$ implies that $\exists J$ open in $[0, +\infty)$, $t \in J \subset K_t$ such that $\forall s \in J\ \|T(s)f - T(t)f\| < \frac{\varepsilon}{2}$. Therefore, $\forall y \in U\ \forall s \in J$

$$
\begin{aligned}
&\|(T(s)f)(y) - (T(t)f)(x)\| \\
\leq &\|(T(s)f)(y) - (T(t)f)(y)\| + \|(T(t)f)(y) - (T(t)f)(x)\| \\
< &\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

Since $f|_K = u$, it means that $\forall (s, y) \in J \times U\ \|Q_s(y)u - Q_t(x)u\| < \varepsilon$. Thus, $Q \in C(\Pi, L_s(E))$.

The next step is to prove that φ and Q are partial semiflow and partial semicocycle respectively. $\forall u \in E\ \forall x \in X\ \exists f \in C_0(X, E)$ such that $f(\varphi_0(x)) = u$. The semigroup property $T(0) = I$ implies that $\forall x \in X\ \forall u \in E\ Q_0(x)u = u$. In other words, $\forall x \in X\ Q_0(x) = I$. It follows that $\forall f \in C_0(X, E)\ \forall x \in X\ f(\varphi_0(x)) = f(x)$. Since continuous functions on a locally compact Hausdorff space separate the points of the space, $\forall x \in X\ \varphi_0(x) = x$. In other words, $\varphi_0 = id_X$.

Let $t, s \geq 0$, $x \in Y_{t+s}$. Suppose that $\varphi_t(x) \notin Y_s$. Then $\forall f \in C_0(X, E)$ $(T(s)f)(\varphi_t(x)) = 0$. The semigroup property $T(t + s) = T(t)T(s)$ combined with

(3) now imply that $\forall f \in C_0(X,E)$

$$Q_{t+s}(x)f(\varphi_{t+s}(x)) = Q_t(x)(T(s)f)(\varphi_t(x)) = 0.$$

Fix $u \in E$. By choosing an $f \in C_0(X,E)$ such that $f(\varphi_{t+s}(x)) = u$, we conclude that $\forall u \in E$ $Q_{t+s}(x)u = 0$. Therefore, $Q_{t+s}(x) = 0$. Contradiction. Thus, $\varphi_t(x) \in Y_s$.

Assume now that $t, s \geq 0$, $x \in Y_t$, $\varphi_t(x) \in Y_s$, and $x \notin Y_{t+s}$. Then $\forall f \in C_0(X,E)$

$$Q_t(x)Q_s(\varphi_t(x))f(\varphi_s(\varphi_t(x))) = (T(t)T(s)f)(x) = (T(t+s)f)(x) = 0.$$

Fix $u \in E$. By choosing an $f \in C_0(X,E)$ such that $f(\varphi_s(\varphi_t(x))) = u$, we conclude that $\forall u \in E$ $Q_t(x)Q_s(\varphi_t(x))u = 0$.

Suppose again that $t, s \geq 0$, $x \in Y_{t+s}$. It follows that $\forall f \in C_0(X,E)$

$$(*) \qquad Q_{t+s}(x)f(\varphi_{t+s}(x)) = Q_t(x)Q_s(\varphi_t(x))f(\varphi_s(\varphi_t(x))).$$

By choosing an $f \in C_0(X,E)$ such that $f(\varphi_{t+s}(x)) = f(\varphi_s(\varphi_t(x))) = u$, where $u \in E$ is fixed, we conclude that

$$(**) \qquad Q_{t+s}(x) = Q_t(x)Q_s(\varphi_t(x)).$$

If we could show that φ is a partial semiflow, this would prove that Q is a partial semicocycle of φ.

Now we must show φ is a partial semiflow. If not, then $\exists t, s \geq 0$ $\exists x \in Y_{t+s}$ such that $\varphi_{t+s}(x) \neq \varphi_s(\varphi_t(x))$. Since $x \in Y_{t+s}$, $Q_{t+s}(x) \neq 0$, therefore, $\exists u \in E$ such that $Q_{t+s}(x)u \neq 0$. Hence, it is possible to find $g \in C_0(X)$ such that $g(\varphi_{t+s}(x)) = 0$ and $g(\varphi_s(\varphi_t(x))) = 1$. Let $f = g \otimes u$. Then it follows from $(*)$ and $(**)$ that

$$0 = Q_{t+s}(x)f(\varphi_{t+s}(x)) = Q_{t+s}(x)f(\varphi_s(\varphi_t(x))) = Q_{t+s}(x)u \neq 0.$$

Contradiction.

Finally, we have to prove (2). Since $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup and therefore locally bounded, it follows from $\|T(t)\| = \sup\{\, \|Q_t(x)\| : x \in Y_t \,\}$.

SUFFICIENCY. Theorem 1.4 implies that $\forall t \geq 0$ $T(t)$ has the disjoint support property. Also it is clear from property (2) of a partial semiflow and property (3) of a partial semicocycle that $T(0) = I$.

Let $t, s \geq 0, x \in X, f \in C_0(X, E)$. Then there are four possibilities:

1. $x \in Y_{t+s}$. In this case $(T(t+s)f)(x) = (T(t)T(s))f(x)$ follows from property (3) of a partial semiflow and property (4) of a partial semicocycle.

2. $x \notin Y_{t+s}$, $x \in Y_t$, $\varphi_t(x) \in Y_s$. In this case $(T(t+s)f)(x) = 0 = (T(t)T(s)f)(x)$ follows from property (5) of a partial semicocycle.

3. $x \notin Y_{t+s}$, $x \in Y_t$, $\varphi_t(x) \notin Y_s$. In this case

$$(T(t)T(s)f)(x) = Q_t(x)(T(s)f)(\varphi_t(x)) = 0 = (T(t+s)f)(x).$$

4. $x \notin Y_{t+s}$, $x \notin Y_t$. In this case $\forall g \in C_0(X, E)$ $(T(t)g)(x) = 0$. In particular, $(T(t)T(s)f)(x) = 0 = (T(t+s)f)(x)$.

Therefore, $\{T(t)\}_{t\geq 0}$ is a semigroup. We have to show now that it is a strongly continuous one. Let $f \in C_0(X, E)$, $\mu \in M(X, E^*)$, $t_n \downarrow 0$. Suppose δ is as in (2). Without loss of generality we may assume that $\{t_n\} \subset [0, \delta)$. Fix $x \in X$ and let $J = I_x \cap [0, \delta)$. Since φ and Q are continuous and $t_n \in J$ for n large enough, it follows that $Q_{t_n}(x)f(\varphi_{t_n}(x)) \to f(x)$. In other words, $(T(t_n)f)(x) \to f(x)$. Also, by (2), $\forall t \in [0, \delta)$ $\|T(t)\| \leq M$. The Dominated Convergence Theorem for vector-valued measures (see [Di, Th. II.8.3]) now implies that

$$\int_X \langle T(t_n)f, d\mu\rangle \to \int_X \langle f, d\mu\rangle.$$

We conclude that $\{T(t)\}_{t\geq 0}$ is a weakly continuous semigroup, and by [Da, 1.23] it is also a strongly continuous one. ■

Corollary 5. *If* $\{T(t)\}_{t\geq 0}$ *is a* C_0*-semigroup on* $C_0(X)$ *with the disjoint support property,* $t, s \geq 0$*, then* $x \in Y_{t+s}$ *if and only if* $x \in Y_t$ *and* $\varphi_t(x) \in Y_s$*, where* Y_t *and* φ *are as in Theorem 4.*

Proof. Necessity follows from the definition of a partial semiflow. To prove the sufficiency, suppose $x \in Y_t$, $\varphi_t(x) \in Y_s$ but $x \notin Y_{t+s}$. Following the proof of Theorem 4, we conclude that $Q_t(x)Q_s(\varphi_t(x)) = 0$. By definition of a partial semicocycle $\forall (t,x) \in \Pi$ $Q_t(x) \neq 0$. Therefore, since Q is complex-valued, $Q_t(x)Q_s(\varphi_t(x)) \neq 0$. Contradiction. ■

Remark. It is still an open problem whether the conclusion of Corollary 5 is true in general. If the answer is positive, we can modify the definition of partial semiflow and semicocycle in the following way. In the definition of a partial semiflow we can change (3) to (3′):

(3′) If $t, s \geq 0$, then $x \in Y_{t+s}$ if and only if $x \in Y_t$ and $\varphi_t(x) \in Y_s$.

In this case $\varphi_{t+s}(x) = \varphi_t(x)(\varphi_s(x))$ and in the definition of a partial semicocycle we can get rid of (5) entirely.

As the following theorem shows, we can get rid of condition (2) in Theorem 4 under the assumption that X is a compact Hausdorff space. Whether condition (2) in Theorem 4 is superfluous or not in general is an open problem.

Let X be a compact Hausdorff space, E a Banach space.

Theorem 6. $\{T(t)\}_{t\geq 0}$ *is a* C_0*-semigroup on* $C(X,E)$ *with the disjoint support property if and only if* $\exists Q \in C(X \times [0,+\infty), L_s(E))$ *such that* $Y_t = \{x \in X : Q_t(x) \neq 0\}$, $t \geq 0$ *form a collection of decreasing open sets,* $\exists \varphi$ *a continuous partial semiflow such that* $Q|_\Pi$ *is a continuous partial semicocycle of* φ *and* $\forall t \geq 0$ $\forall f \in C_0(X,E)$

$$(T(t)f)(x) = \begin{cases} Q_t(x)f(\varphi_t(x)), & \textit{if } x \in Y_t, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. NECESSITY. Let Q be as in Theorem 4. Extend it to $X \times [0, +\infty)$ by defining it to be zero outside Π. The only thing that needs to be proved is that $Q \in C(X \times [0, +\infty), L_s(E))$. Let $x \in X$, $t \geq 0$, $u \in E$, $\varepsilon > 0$. Since $\{T(t)\}_{t \geq 0}$ is a C_0-semigroup, $\exists J$ open neighborhood of t in $[0, +\infty)$ such that $\forall s \in J$

$$\|T(t)(\mathbb{1} \otimes u) - T(s)(\mathbb{1} \otimes u)\| < \frac{\varepsilon}{2}.$$

Also since $T(t)(\mathbb{1} \otimes u) \in C(X, E)$ $\exists U$ open neighborhood of x in X such that $\forall y \in U$

$$\|(T(t)(\mathbb{1} \otimes u))(x) - (T(t)(\mathbb{1} \otimes u))(y)\| < \frac{\varepsilon}{2}.$$

Therefore, $\forall s \in J$ $\forall y \in U$

$$\begin{aligned}
&\|(T(s)(\mathbb{1} \otimes u))(y) - (T(t)(\mathbb{1} \otimes u))(x)\| \\
\leq &\|(T(s)(\mathbb{1} \otimes u))(y) - (T(t)(\mathbb{1} \otimes u))(y)\| \\
+ &\|(T(t)(\mathbb{1} \otimes u))(y) - (T(t)(\mathbb{1} \otimes u))(x)\| \\
< &\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

It is easy to see that $\forall t \geq 0$ $\forall x \in X$ $(T(t)(\mathbb{1} \otimes u))(x) = Q_t(x)u$. Thus, $\forall (s, y) \in J \times U$ $\|Q_t(x)u - Q_s(y)u\| < \varepsilon$. Hence, $Q \in C(X \times [0, +\infty), L_s(E))$.

SUFFICIENCY. Corollary 1.8 implies that $\forall t \geq 0$ $T(t)$ has the disjoint support property.

Since $Q \in C(X \times [0, 1], L_s(E))$ and $X \times [0, 1]$ is compact, $\forall u \in E$ $\exists M > 0$ such that $\forall t \in [0, 1]$ $\forall x \in X$ $\|Q_t(x)u\| < M$. Therefore, by the Uniform Boundedness Principle, $\exists C > 0$ such that $\forall t \in [0, 1]$ $\forall x \in X$ $\|Q_t(x)\| < C$.

Now we can proceed as in the sufficiency part of Theorem 4. ∎

We turn our attention to C_0 groups with the disjoint support property. Let X be a locally compact Hausdorff space, E a Banach space.

Definition. A mapping $\varphi\colon \mathbb{R}\times X \to X$ is called *a flow* if

(1) $\varphi_0 = id_X$.

(2) $\forall t,s \in \mathbb{R}\ \forall x \in X\ \varphi_s(\varphi_t(x)) = \varphi_{t+s}(x)$.

A flow φ is called *continuous* if $\varphi \in C(\mathbb{R}\times X, X)$.

Definition. A mapping $Q\colon \mathbb{R}\times X \to L_s(E)$ is called *a cocycle of* φ if

(1) $\forall t \in \mathbb{R}\ Q_t \in C_b(X, L_s(E))$.

(2) $\forall x \in X\ Q_0(x) = I$.

(3) $\forall t,s \in \mathbb{R}\ \forall x \in X\ Q_{t+s}(x) = Q_t(x)Q_s(\varphi_t(x))$.

A cocycle Q is called *continuous* if $Q \in C(\mathbb{R}\times X, L_s(E))$.

Lemma 7. *If φ is a flow and Q is a cocycle of φ, then $\forall t \in \mathbb{R}$ φ_t and Q_t are invertible and $\forall x \in X$ $\varphi_t^{-1}(x) = \varphi_{-t}(x)$, $Q_t^{-1}(x) = Q_{-t}(\varphi_t(x))$.*

Proof. It follows from the definition of a flow that $\forall t \in \mathbb{R}\ \forall x \in X$

$$\varphi_{-t}(\varphi_t(x)) = \varphi_t(\varphi_{-t}(x)) = \varphi_0(x) = x.$$

It follows from the definition of a cocycle that $\forall t \in \mathbb{R}\ \forall x \in X$

$$Q_t(x)Q_{-t}(\varphi_t(x)) = Q_0(x) = I,$$

$$Q_{-t}(\varphi_t(x))Q_t(x) = Q_{-t}(\varphi_t(x))Q_t(\varphi_{-t}(\varphi_t(x)) = Q_0(\varphi(x)) = I. \qquad \blacksquare$$

Definition. A C_0-group $\{T(t)\}_{t\in\mathbf{R}}$ on $C_0(X,E)$ is said to have *the disjoint support property* if $\forall t \in \mathbb{R}\ T(t)$ has the disjoint support property.

Theorem 8. *$\{T(t)\}_{t\in\mathbf{R}}$ is a C_0-group on $C_0(X,E)$ with the disjoint support property if and only if $\exists\varphi$ a continuous flow, $\exists Q$ a continuous cocycle of φ such that $\forall t \in \mathbb{R}\ \forall x \in X\ \forall f \in C_0(X,E)\ (T(t)f)(x) = Q_t(x)f(\varphi_t(x))$.*

Proof. NECESSITY. This part of the proof can be obtained by mimicking the necessity part of the proof of Theorem 4. Since $\forall t \in \mathbb{R}$ $T(t)$ is invertible, $\{\, x \in X : \forall f \in$

$C_0(X,E)$ $(T(t)f)(x) = 0\} = \varnothing$. Thus, $\forall t \in \mathbb{R}$ $Y_t = X$. This considerably simplifies the necessity part of the proof of Theorem 4.

SUFFICIENCY. Let $t \in \mathbb{R}$. Corollary 1.6 guarantees that $T(t)$ is invertible and both $T(t)$ and $T(t)^{-1}$ have the disjoint support property. The fact that $T(t)$ is a group follows easily from the definitions of a flow and a cocycle.

Observe that $\forall t \in \mathbb{R}$ $\|T(t)\| = \sup\{\, \|Q_t(x)u\| : x \in X,\ u \in E,\ \|u\| \le 1\,\}$ and $\forall u \in E$, $\|u\| \le 1$ $\|Q_t(x)u\| \in C(\mathbb{R})$. Since the supremum of any collection of lower semicontinuous functions is also a lower semicontinuous function (see [Ru2, 2.8.c]), $\forall t \in \mathbb{R}$ $\|T(t)\|$ is a lower semicontinuous function. In particular, it is measurable. Also, since $\{T(t)\}_{t\in\mathbf{R}}$ is a group, $\log\|T(t)\|$ is a subadditive function. Thus, by [HP, 7.4.1], $\|T(t)\|$ is bounded on compact intervals of $\mathbb{R}$. Hence, condition (2) of Theorem 4 is satisfied and we can complete the proof as in the sufficiency part of the proof of Theorem 4. ■

Now we are going to extend the notion of locality to unbounded operators.

Definition. An operator A on $C_0(X,E)$ with the domain $D(A)$ is called *local* if $\forall f_1 \in D(A)$ $\forall f_2 \in C_0(X,E)$

$$\|f_1(\cdot)\| \wedge \|f_2(\cdot)\| = 0 \qquad \Rightarrow \qquad \|(Af_1)(\cdot)\| \wedge \|f_2(\cdot)\| = 0.$$

Theorem 9. *If $\{T(t)\}_{t\ge 0}$ is a C_0-semigroup on $C_0(X,E)$ with the disjoint support property, A its infinitesimal generator, then A is local.*

Proof. Let $t \ge 0$, $x \in X$, $f_1 \in D(A)$, $f_2 \in C_0(X,E)$, $\|f_1(\cdot)\| \wedge \|f_2(\cdot)\| = 0$. Then

$$\begin{aligned}
&\frac{\|(T(t)f_1)(x) - f_1(x)\|}{t} \wedge \|f_2(x)\| \\
\le&\frac{\|(T(t)f_1)(x)\|}{t} \wedge \|f_2(x)\| + \frac{\|f_1(x)\|}{t} \wedge \|f_2(x)\| \\
=&\frac{\|(T(t)f_1)(x)\|}{t} \wedge \|f_2(x)\| \\
\le&\frac{\|(T(t)f_1)(x)\|}{t} \wedge \|(T(t)f_2)(x) - f_2(x)\| + \frac{\|(T(t)f_1)(x)\|}{t} \wedge \|(T(t)f_2)(x)\|
\end{aligned}$$

$$= \frac{\|(T(t)f_1)(x)\|}{t} \wedge \|(T(t)f_2)(x) - f_2(x)\|$$
$$\leq \|(T(t)f_2)(x) - f_2(x)\|.$$

C_0-continuity of $\{T(t)\}_{t\geq 0}$ implies that $\lim_{t\downarrow 0} \|(T(t)f_2)(x) - f_2(x)\| = 0$. Since $\wedge$ is a continuous opèration (see [AB, 11.1]), it follows that $\|(Af_1)(x)\| \wedge \|f_2(x)\| = 0$. ■

Definition. A C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $C_0(X, E)$ is called *local* if $\forall t \geq 0$ $T(t)$ is local.

Corollary 10. *A uniformly continuous semigroup on $C_0(X, E)$ with the disjoint support property is local.*

Proof. Let $\{T(t)\}_{t\geq 0}$ be a uniformly continuous C_0-semigroup on $C_0(X, E)$ with the disjoint support property, A its infinitesimal generator. Then by Theorem 9 A is local. Also since $\{T(t)\}_{t\geq 0}$ is uniformly continuous, A is bounded (see for example [Pa, 1.1.4]). Thus, by Theorem 1.11, $\exists Q \in C_b(X, L_s(E))$ such that $\forall f \in C_0(X, E)$ $\forall x \in X$ $(Af)(x) = Q(x)f(x)$. Again using [Pa, 1.1.4]), it follows that $\forall t \geq 0$ $T(t) = e^{tA}$. In other words, $\forall f \in C_0(X, E)$ $\forall x \in X$ $(T(t)f)(x) = (e^{tQ(x)}f)(x)$, i.e. $T(t)$ is local. ■

Proposition 11. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on $C_0(X, E)$, A its infinitesimal generator. Then the following are equivalent:*

(1) *$\{T(t)\}_{t\geq 0}$ is local.*

(2) *$R(\lambda, A)$ is local for some $\lambda \in \rho(A)$.*

(3) *$\forall \lambda \in \rho(A)$ $R(\lambda, A)$ is local.*

Proof. (1)⇒(2). Let ω_0 be the growth bound of $\{T(t)\}_{t\geq 0}$. Then [Pa, 1.5.4] implies that if $\lambda > \omega_0$, then $\forall f \in C_0(X, E)$

$$R(\lambda, A)f = \int_0^{+\infty} e^{-\lambda t} T(t) f dt.$$

By Proposition 1.12(1) local operators form a closed subalgebra of $L_s(C_0(X,E))$ and since $\forall t \geq 0$ $T(t)$ is local, it follows that $R(\lambda, A)$ is local.

(2)⇒(3). Let $\mu \in \rho(A)$. The resolvent equation ([Yo, Th.VIII.2.2])

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A)$$

implies that

$$R(\mu, A) = (I + (\mu - \lambda)R(\lambda, A))^{-1}R(\lambda, A).$$

By Proposition 1.12(2) $(I + (\mu - \lambda)R(\lambda, A))^{-1}$ is local, therefore $R(\mu, A)$ is local as well.

(3)⇒(1). [Pa, 1.8.3] implies that $\forall t \geq 0$ $\forall f \in C_0(X,E)$

$$T(t)f = \lim_{n \to +\infty} \left[\frac{n}{t}R(\frac{n}{t}, A)\right]^n f.$$

Again using the fact that local operators form a closed subalgebra of $L_s(C_0(X,E))$, we conclude that $\forall t \geq 0$ $T(t)$ is local. ■

Chapter 3

$C_0(a,b)^{\odot}$ with respect to C_0-groups with the disjoint support property

Throughout this chapter $-\infty \le a < b \le +\infty$, X is either (a,b) or $[a,b]$, where by $[-\infty, b]$ we mean $(-\infty, b]$, and by $[a, +\infty]$ we mean $[a, +\infty)$.

Let $\{T(t)\}_{t\in\mathbf{R}}$ be a C_0-group on a Banach space E.

Definition. The *group dual of E with respect to $\{T(t)\}_{t\in\mathbf{R}}$*, denoted $E^{\odot}$ and pronounced E-sun is defined in the following way:

$$E^{\odot} = \{\, u^* \in E^* : \lim_{t\to 0} \|T^*(t)u^* - u^*\| = 0 \,\}.$$

[vN] is an excellent source of information about the semigroup and group duals of Banach spaces and related subjects.

Let $\{T(t)\}_{t\in\mathbf{R}}$ be a C_0-group on $C_0(X)$ with the disjoint support property. It follows from Theorem 2.8 that $\exists\varphi\colon \mathbb{R}\times X \to X$ a continuous flow $\exists q\colon \mathbb{R} \to C_b(X)$ a continuous cocycle of φ such that $\forall t \in \mathbb{R}$ $\forall f \in C_0(X)$ $\forall x \in X$ $(T(t)f)(x) = q_t(x)f(\varphi_t(x))$.

In [Na, B-II.3.21] W. Arendt characterized all continuous flows on X. For the sake of completeness we list this result here providing a somewhat more detailed proof than the original one.

Theorem 1. *φ is a continuous flow on X if and only if the following conditions are satisfied:*

(1) *$\exists U \subset X$, U is the union of pairwise disjoint intervals (a_i, b_i), $i \in I$, where I is either finite or countable.*

(2) *$\exists\psi\colon U \to \mathbb{R}$ such that $\forall i \in I$ $\psi_i = \psi|_{(a_i,b_i)}\colon (a_i, b_i) \to \mathbb{R}$ is a homeomorphism.*

(3) $\forall t \in \mathbb{R}$

$$\varphi_t(x) = \begin{cases} \psi_i^{-1}(\psi_i(x)+t), & \textit{if } x \in (a_i, b_i), \\ x, & \textit{if } x \notin U. \end{cases}$$

Proof. SUFFICIENCY First we will establish that φ is a flow on $\mathbb{R}$. Let $x, s, t \in \mathbb{R}$. Then either $x \in U$ or $x \notin U$. If $x \notin U$, then $\varphi_t(x) = \varphi_s(x) = \varphi_{t+s}(x) = x$ and therefore $\varphi_t(\varphi_s(x)) = \varphi_{t+s}(x)$. If $x \in (a_i, b_i)$ for some $i \in I$, then

$$\varphi_t(\varphi_s(x)) = \psi_i^{-1}\left(\psi_i\left(\psi_i^{-1}(\psi_i(x)+s)\right)+t\right) = \psi_i^{-1}(\psi_i(x)+t+s) = \varphi_{t+s}(x).$$

It is also clear that $\forall x \in \mathbb{R}$ $\varphi_0(x) = x$.

The next step is to prove the continuity of φ. It is fairly clear that if $x \in U$ or $x \notin \bar{U}$, then $\forall t \in \mathbb{R}$ $\varphi_t(x)$ is continuous at (t, x). Suppose $x \in \bar{U} \setminus U$, $t \in \mathbb{R}$. Since $\varphi_t(x) = x$, we have to prove that

$$\varphi_s(y) \to x \text{ as } y \downarrow x,\ s \to t \qquad \text{and} \qquad \varphi_s(y) \to x \text{ as } y \uparrow x,\ s \to t.$$

We will prove only the first fact of the above two. The second one can be proved in the same manner.

Let $\varepsilon > 0$. Then there are two possibilities: either $(x, x+\varepsilon) \subset U$ or $(x, x+\varepsilon) \not\subset U$. Let us consider the second case first. Then $\exists \delta$ $0 < \delta < \varepsilon$ such that $x + \delta \notin U$. Let $x < y < x + \delta$. If $y \notin U$, then $\forall s \in \mathbb{R}$ $\varphi_s(y) = y$ and therefore $x < \varphi_s(y) < x + \varepsilon$. If, on the other hand, $y \in (a_i, b_i)$ for some $i \in I$, then

$$x \le a_i < y < b_i \le x + \delta < x + \varepsilon.$$

Note that $\forall s \in \mathbb{R}$ $\varphi_s(y) \in (a_i, b_i)$ and therefore $x < \varphi_s(y) < x + \varepsilon$.

Assume now $(x, x+\varepsilon) \subset U$. Then $\exists i \in I$ such that $a_i = x < x + \varepsilon \le b_i$. Since ψ_i is a homeomorphism, without losing generality we may assume that it is a decreasing function. Let

$$\delta = \psi_i^{-1}(\psi_i(x+\varepsilon) + |t| + 1) - x > 0.$$

Thus, $\psi_i(x+\delta) = \psi_i(x+\varepsilon) + |t| + 1$. Therefore, $\forall y \in (x, x+\delta)$, $\forall |s| < |t| + 1$

$$\psi_i(y) + s > \psi_i(x+\delta) - |t| - 1 = \psi_i(x+\varepsilon).$$

It follows that $x < \varphi_s(y) < x+\varepsilon$. This concludes the proof that $\varphi_s(y) \to x$ as $y \downarrow x$, $s \to t$.

Necessity. First we will prove that $\forall t \in \mathbb{R}$ φ_t is a strictly increasing function. Assume this is not true. Then $\exists t \in \mathbb{R}$ $\exists x, y \in X$, $x < y$ such that $\varphi_t(x) \geq \varphi_t(y)$. Since φ is a continuous function and since $\varphi_0 = id_X$, $\exists s \in (0, t]$ such that $\varphi_s(x) = \varphi_s(y)$. Contradiction with the fact that φ_s is a homeomorphism.

Let $K = \{x \in X : \forall t \in \mathbb{R}\ \varphi_t(x) = x\}$. Clearly K is a closed set. Also if $X = [a, b]$, it is easy to see that $a, b \in K$. Therefore, $U = X \setminus K$ is open in $\mathbb{R}$ and thus is a union of of pairwise disjoint intervals (a_i, b_i), $i \in I$, where I is either finite or countable.

Let $i \in I$, $x \in (a_i, b_i)$, $\beta(t) = \varphi_t(x)$. We claim that β is an injection. Suppose this is false, i.e. $\exists r, s \in \mathbb{R}$ such that $\varphi_s(x) = \varphi_r(x)$ which means that $\varphi_{s-r}(x) = x$. Scaling by t, if necessary, we may assume that $\varphi_1(x) = x$. Let $y = \varphi_{1/2}(x)$. It follows from the definition of a flow that $\varphi_{1/2}(y) = x$. Since as we showed above $\varphi_{1/2}$ is a strictly increasing function, we conclude that $x = y$.

Using a similar argument, we can show that $\forall n \in \mathbb{N}$ $\varphi_{1/2^n}(x) = x$ and thus $\forall m \in \mathbb{Z}$ $\varphi_{m/2^n}(x) = x$. Since numbers $\{m/2^n\}_{n\in\mathbb{N},\ m\in\mathbb{Z}}$ are dense in $\mathbb{R}$ and since φ is a continuous flow, it follows that $\forall t \in \mathbb{R}$ $\varphi_t(x) = x$, i.e. $x \in K$. Contradiction with $x \in (a_i, b_i)$. Thus, β is an injection.

Our next claim is that β maps $\mathbb{R}$ onto (a_i, b_i). Suppose $\varphi_t(x) \notin (a_i, b_i)$. Then $\exists s \in (0, t]$ such that $y = \varphi_s(x) \in K$. From the definition of K it follows that $\varphi_{-s}(y) = y$ and thus $x = y \in K$. Contradiction with $x \in (a_i, b_i)$. We conclude that the image of β is contained in (a_i, b_i). Thus, $\beta\colon \mathbb{R} \to (c, d)$ is a homeomorphism and $(c, d) \subset (a_i, b_i)$. Without loss of generality we may assume that β is an increasing function. Therefore, $\beta(t) \to d$ as $t \to +\infty$. Let $s \in \mathbb{R}$. Then

$$\varphi_s(d) = \varphi_s(\lim_{t\to+\infty} \varphi_t(x)) = \lim_{t\to+\infty} \varphi_s(\varphi_t(x)) = \lim_{t\to+\infty} \varphi_{t+s}(x) = d.$$

Hence, $d \in K$ and thus $d = b_i$. Analogously $c = a_i$. This establishes the claim.

Finally, let $\psi_i = \beta^{-1}$, $y \in (a_i, b_i)$, $s = \psi_i(y)$, i.e. $y = \varphi_s(x)$. Then

$$\varphi_t(y) = \varphi_t(\varphi_s(x)) = \varphi_{t+s}(x) = \psi_i^{-1}(s+t) = \psi_i^{-1}(\psi_i(y)+t). \qquad \blacksquare$$

Remark. If φ is a continuous flow, then $\forall n \in \mathbb{Z}\ \varphi_n = \varphi_1^n$, i.e. φ_n is the nth iterant of the function φ_1. Theorem 1 shows that the functions $\{\varphi_t\}_{t\in\mathbf{R}}$ are the *continuous iterants of* φ_1 *in the sense of Abel*, i.e. there exists a function ψ such that φ_1 satisfies the *Abel equation* $\psi(\varphi_1(x)) - \psi(x) = 1$ (see [Ku, Ch.VII]), and consequently $\forall t \in \mathbb{R}$ $\forall x \in X\ \psi(\varphi_t(x)) - \psi(x) = t$.

Suppose μ is a nonnegative Borel measure on X, $t \in \mathbb{R}$. $\forall F \subset X$, F Borel define $\mu_t(F) = \mu(\varphi_{-t}(F))$. By [DS, III.10.8] μ_t is a nonnegative Borel measure on X and $\forall F \subset X$, F Borel $\forall f \in C_0(X)$

$$(*) \qquad \int_F f\, d\mu_t = \int_{\varphi_{-t}(F)} f \circ \varphi_t\, d\mu.$$

It can be also easily seen that if $\mu \in M(X)$, then so is μ_t and the above equality holds as well.

Suppose $\mu \in M(X)$, $t \in \mathbb{R}$. $\forall F \subset X$, F Borel define

$$\mu_t'(F) = \int_F q_t \circ \varphi_{-t}\, d\mu_t.$$

Since $q_t \in C_b(X)$, [DS, III.10.4] implies that $\mu_t' \in M(X)$ and $\forall F \subset X$, F Borel $\forall f \in C_0(X)$

$$(**) \qquad \int_F f\, d\mu_t' = \int_F q_t \circ \varphi_{-t} \cdot f\, d\mu_t = \int_{\varphi_{-t}(F)} q_t \cdot f \circ \varphi_t\, d\mu.$$

Lemma 2. $\forall t \in \mathbb{R}\ \forall \mu \in M(X)\ \mu_t' = T^*(t)\mu$.

Proof. It follows from above that $\forall t \in \mathbb{R}\ \mu_t' \in M(X)$ and $\forall f \in C_0(X)$

$$\int_X T(t) f\, d\mu = \int_X q_t \cdot f \circ \varphi_t\, d\mu = \int_X f\, d\mu_t'. \qquad \blacksquare$$

Lemma 3. *If $\mu \in M(X)$ and $\mu|_U = 0$, then $\mu \in C_0(X)^{\odot}$ with respect to $\{T(t)\}_{t\in\mathbf{R}}$.*

Proof. Suppose $t \in \mathbb{R}$, $F \subset X$ is Borel. Then

$$\mu_t'(F) = \int_X q_t \chi_{\varphi_{-t}(F)}\, d\mu.$$

Since $\mu|_U = 0$, $\mu(\varphi_{-t}(F) \cap U) = 0$ and therefore

$$\mu_t'(F) = \int_X q_t \chi_{\varphi_{-t}(F)\setminus U}\, d\mu.$$

Since φ_{-t} is a bijection and since $\varphi_{-t}(U) = U$, $\varphi_{-t}(F) \setminus U = \varphi_{-t}(F \setminus U)$. Also since $\varphi_{-t}|_{X\setminus U} = id_{X\setminus U}$, $\varphi_{-t}(F \setminus U) = F \setminus U$. We obtain:

$$\mu_t'(F) = \int_X q_t \chi_{F\setminus U}\, d\mu = \int_X q_t \chi_F\, d\mu$$

since $\mu(F \cap U) = 0$. Thus,

$$\mu_t'(F) - \mu(F) = \int_X (q_t - 1)\chi_F\, d\mu.$$

Let $\mathcal{F}$ be the set of all partitions $\{F_j\}$ of X. Then $\forall t \in \mathbb{R}$

$$\begin{aligned} \|\mu_t' - \mu\| &= \sup_{\{F_j\}\in\mathcal{F}} \sum_{j=1}^{\infty} |(\mu_t' - \mu)(F_j)| \\ &\le \sup_{\{F_j\}\in\mathcal{F}} \sum_{j=1}^{\infty} \int_X |q_t - 1| \chi_{F_j}\, d\mu = \int_X |q_t - 1|\, d\mu. \end{aligned}$$

Since $\{T(t)\}_{t\in\mathbf{R}}$ is a C_0-group, it is locally bounded. It is not very difficult to see that $\|T(t)\| = \|q_t\|$. Hence, $\exists D > 0$ such that $\forall |t| \le 1\ \|q_t\| \le D$ and therefore $\|q_t - 1\| \le D + 1$. Since by the definition of a cocycle $q_0 = 1$ and since $(D+1)\chi_X \in L^1(X, \mu)$, it follows from the Dominated Convergence Theorem that

$$\int_X |q_t - 1| \, d\mu \to 0 \quad \text{as} \quad t \to 0.$$

Hence, $\|\mu_t' - \mu\| \to 0$ as $t \to 0$. Since by Lemma 2 $\mu_t' = T^*(t)\mu$, it follows that $\mu \in C_0(X)^{\odot}$. ∎

Lemma 4. *Suppose $i \in I$, $\alpha \in L^1((a_i, b_i), d\psi)$. $\forall F$ Borel, $F \subset X$ define*

$$\mu(F) = \int_{a_i}^{b_i} \alpha \chi_F \, d\psi,$$

where ψ is as in Theorem 2. Then $\mu \in C_0(X)^{\odot}$ with respect to $\{T(t)\}_{t \in \mathbf{R}}$.

Proof. $d\psi$ is either a nonnegative or nonpositive measure on (a_i, b_i). Without loss of generality we may assume it is nonnegative. Suppose $(c, d) \subset (a_i, b_i)$. Since ψ_i is a continuous function, $d\psi(c, d) = \psi(d) - \psi(c)$. Then $\forall t \in \mathbb{R}$

$$\begin{aligned}(d\psi)_t(c,d) &= d\psi(\varphi_{-t}(c), \varphi_{-t}(d)) = \psi(\varphi_{-t}(d)) - \psi(\varphi_{-t}(c)) \\ &= \psi(d) - t - \psi(c) + t = \psi(d) - \psi(c) = d\psi(c,d).\end{aligned}$$

It follows that $\forall G \subset (a_i, b_i)$, G open $(d\psi)_t(G) = d\psi(G)$. [Ru2, 2.18] implies that both $d\psi$ and $(d\psi)_t$ are regular. Thus, $\forall F \subset (a_i, b_i)$, F Borel $(d\psi)_t(F) = d\psi(F)$ which means that $\forall t \in \mathbb{R}$ $(d\psi)_t = d\psi$ on (a_i, b_i).

Let $F \subset X$ be Borel, $t \in \mathbb{R}$. Then by $(*)$ and $(**)$

$$\begin{aligned}(\mu_t' - \mu)(F) &= \int_{a_i}^{b_i} q_t \alpha \chi_{\varphi_{-t}(F)} \, d\psi - \int_{a_i}^{b_i} \alpha \chi_F \, d\psi \\ &= \int_{a_i}^{b_i} q_t \circ \varphi_{-t} \cdot \alpha \circ \varphi_{-t} \cdot \chi_{\varphi_{-t}(F)} \circ \varphi_{-t} \, d\psi_t - \int_{a_i}^{b_i} \alpha \chi_F \, d\psi \\ &= \int_{a_i}^{b_i} (q_t \circ \varphi_{-t} \cdot \alpha \circ \varphi_{-t} - \alpha) \chi_F \, d\psi\end{aligned}$$

since $(d\psi)_t = d\psi$ on (a_i, b_i), $\chi_{\varphi_{-t}(F)} \circ \varphi_{-t} = \chi_F$. Using the argument similar to the one in the proof of Lemma 3, we conclude that $\forall t \in \mathbb{R}$

$$\|\mu_t' - \mu\| \leq \int_{a_i}^{b_i} |q_t \circ \varphi_{-t} \cdot \alpha \circ \varphi_{-t} - \alpha| \, d\psi.$$

We have shown in the proof of Lemma 3 that $\exists D > 1$ such that $\forall |t| \leq 1$ $\|q_t\| \leq D$. Let $\varepsilon > 0$. Since $\alpha \in L^1((a_i, b_i), d\psi)$, $\exists g \in C_c(a_i, b_i)$ such that

$$\int_{a_i}^{b_i} |\alpha - g| \, d\psi < \frac{\varepsilon}{3D} < \frac{\varepsilon}{3}.$$

Since $(d\psi)_t = d\psi$ on (a_i, b_i), it follows from $(*)$ that $\forall |t| \leq 1$

$$\int_{a_i}^{b_i} |q_t \circ \varphi_{-t} \cdot \alpha \circ \varphi_{-t} - q_t \circ \varphi_{-t} \cdot g \circ \varphi_{-t}| \, d\psi = \int_{a_i}^{b_i} |q_t \alpha - q_t g| \, d\psi \leq D \frac{\varepsilon}{3D} = \frac{\varepsilon}{3}.$$

Let $K = \operatorname{supp} g$, $K' = \varphi(K \times [-1, 1]) \subset (a_i, b_i)$. Since φ is a continuous flow, K' is compact. Suppose $x \in (a_i, b_i) \setminus K'$. It means that $\forall y \in K$ $\forall |t| \leq 1$ $x \neq \varphi_{-t}(y)$ which implies that $\varphi_t(x) \neq y$. Hence, $\varphi_t(x) \notin K$ and $g(\varphi_t(x)) = 0$. It follows that $\forall |t| \leq 1$ $\operatorname{supp}(g \circ \varphi_t - g) \subset K'$. Therefore, $\forall |t| \leq 1$

$$|q_t \circ \varphi_{-t} \cdot g \circ \varphi_{-t} - g| \leq \|g\|(D + 1)\chi_{K'}.$$

Since $\chi_{K'} \in L^1((a_i, b_i), d\psi)$, the Dominated Convergence Theorem implies that $\exists \delta > 0$ such that $\forall |t| < \delta$

$$\int_{a_i}^{b_i} |q_t \circ \varphi_{-t} \cdot g \circ \varphi_{-t} - g| \, d\psi < \frac{\varepsilon}{3}.$$

Hence,

$$\int_{a_i}^{b_i} |q_t \circ \varphi_{-t} \cdot \alpha \circ \varphi_{-t} - \alpha| \, d\psi$$

$$\leq \int_{a_i}^{b_i} |q_t \circ \varphi_{-t} \cdot \alpha \circ \varphi_{-t} - q_t \circ \varphi_{-t} \cdot g \circ \varphi_{-t}| \, d\psi$$
$$+ \int_{a_i}^{b_i} |q_t \circ \varphi_{-t} \cdot g \circ \varphi_{-t} - g| \, d\psi + \int_{a_i}^{b_i} |\alpha - g| \, d\psi$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Combining all the results, we obtain that $\forall |t| < \delta \ \|\mu_t' - \mu\| < \varepsilon$. Thus, $\lim_{t\to 0} \mu_t' = \mu$ and $\mu \in C_0(X)^{\odot}$ with respect to $\{T(t)\}_{t\in\mathbf{R}}$. ■

Lemma 5. $C_0(X)^{\odot} \subset M(X \setminus U) \oplus L^1(U, d\psi)$.

Proof. Suppose $\mu \in M(X)$. Then

$$\mu = \nu + \sum_{i\in I} \mu_i,$$

where $\nu|_U = 0$ and $\forall i \in I \ \mu_i|_{(X)\setminus(a_i,b_i)} = 0$. Let A be the infinitesimal generator of $\{T(t)\}_{t\in\mathbf{R}}$, ω_0 the growth bound of $\{T(t)\}_{t\in\mathbf{R}}$, $\lambda > \omega_0$. Since by [vN, 1.3.1] $C_0(X)^{\odot} = \overline{D(A^*)}$, it suffices to prove that $R(\lambda, A^*)\nu \in M(X \setminus U)$ and $\forall i \in I$ $R(\lambda, A^*)\mu_i \in L^1(U, d\psi)$.

Let $f \in C_0(X)$. Then it follows from [Pa, 1.5.4] that $\forall x \in \mathbb{R}$

$$(R(\lambda, A)f)(x) = \int_0^{+\infty} e^{-\lambda t} q_t(x) f(\varphi_t(x)) \, dt.$$

Hence, since $\nu|_U = 0$ and $\varphi_t|_{X\setminus U} = id_{X\setminus U}$

$$\langle R(\lambda, A^*)\nu, f\rangle = \langle \nu, R(\lambda, A)f\rangle$$
$$= \int_X d\nu(x) \int_0^{+\infty} e^{-\lambda t} q_t(x) f(x) \, dt = \int_X f(x) H_\lambda(x) \, d\nu(x),$$

where

$$H_\lambda(x) = \int_0^{+\infty} h_\lambda(t, x) \, dt, \qquad h_\lambda(t, x) = e^{-\lambda t} q_t(x).$$

Let $\varepsilon = (\lambda - \omega_0)/2$. We have mentioned in the proof of Lemma 3 that $\forall t \in \mathbb{R}$ $\|T(t)\| = \|q_t\|$. Then $\exists B > 0$ such that $\forall t \in \mathbb{R}$ $\|q_t\| \le Be^{t(\omega_0+\varepsilon)}$, and $\forall x \in X$

$$|H_\lambda(x)| \le \int_0^{+\infty} |h_\lambda(t,x)|\,dt \le B\int_0^{+\infty} e^{-\varepsilon t}\,dt = \frac{B}{\varepsilon}.$$

Thus, if $\forall F \subset X$, F Borel we define

$$\xi(F) = \int_F H_\lambda(x)\,d\nu(x),$$

then by [DS, III.10.4] $\xi \in M(X)$ and $\xi = R(\lambda, A^*)\nu$. Also since $\nu|_U = 0$, $\xi|_U = 0$ as well.

Let $i \in I$. Without loss of generality we may assume that ψ is nondecreasing on (a_i, b_i). Then

$$\langle R(\lambda, A^*)\mu_i, f\rangle = \langle \mu_i, R(\lambda, A)f\rangle = \int_{a_i}^{b_i} d\mu_i(x) \int_0^{+\infty} e^{-\lambda t} q_t(x) f(\psi_i^{-1}(\psi(x)+t))\,dt.$$

Suppose $t = \psi(s) - \psi(x)$. Then by [DS, III.10.8]

$$\int_{a_i}^{b_i} d\mu_i(x) \int_0^{+\infty} e^{-\lambda t} q_t(x) f(\psi_i^{-1}(\psi(x)+t))\,dt$$
$$= \int_{a_i}^{b_i} d\mu_i(x) \int_x^{b_i} h_\lambda(\psi(s)-\psi(x), x) f(s)\,d\psi(s).$$

Applying Fubini's Theorem, we will get:

$$\int_{a_i}^{b_i} d\mu_i(x) \int_x^{b_i} h_\lambda(\psi(s)-\psi(x), x) f(s)\,d\psi(s)$$
$$= \int_{a_i}^{b_i} f(s)\,d\psi(s) \int_{a_i}^{s} h_\lambda(\psi(s)-\psi(x), x)\,d\mu_i(x)$$
$$= \int_{a_i}^{b_i} F_i(s) f(s)\,d\psi(s),$$

where

$$F_i(s) = \int_{a_i}^{s} h_\lambda(\psi(s) - \psi(x), x)\, d\mu_i(x).$$

We need to show that $F_i \in L^1((a_i, b_i), d\psi)$. Again using Fubini's Theorem and [DS, III.10.8], we will obtain:

$$\int_{a_i}^{b_i} |F_i(s)|\, d\psi(s) \le \int_{a_i}^{b_i} d\psi(s) \int_{a_i}^{s} |h_\lambda(\psi(s) - \psi(x), x)|\, d|\mu_i|(x)$$

$$= \int_{a_i}^{b_i} d|\mu_i|(x) \int_{x}^{b_i} |h_\lambda(\psi(s) - \psi(x), x)|\, d\psi(s) = \int_{a_i}^{b_i} d|\mu_i|(x) \int_{0}^{+\infty} |h_\lambda(t, x)|\, dt$$

$$\le \frac{B}{\varepsilon} |\mu_i|(a_i, b_i) < +\infty$$

since $\forall x \in X \ \int_0^{+\infty} |h_\lambda(t, x)|\, dt \le B/\varepsilon$.

Hence, $\forall i \in I \ R(\lambda, A^*)\mu_i = \nu_i$, where $\forall F \subset \mathbb{R}$, F Borel

$$\nu_i(F) = \int_{a_i}^{b_i} F_i \chi_F\, d\psi,$$

which means that ν_i can be associated with a function from $L^1((a_i, b_i), d\psi)$. ■

Theorem 6. *Let $\{T(t)\}_{t\in\mathbf{R}}$ be a C_0-group on $C_0(X)$ with the disjoint support property. Then $\exists U \subset X$, U is the union of pairwise disjoint intervals (a_i, b_i), $i \in I$, where I is either finite or countable and $\exists \psi \colon U \to \mathbb{R}$ such that $\forall i \in I$ $\psi_i = \psi|_{(a_i,b_i)} \colon (a_i, b_i) \to \mathbb{R}$ is a homeomorphism and the corresponding group dual $C_0(X)^\odot = M(X \setminus U) \oplus L^1(U, d\psi)$.*

Proof. Follows from Lemmas 3, 4 and 5. ■

Remark. If $U = \varnothing$, then $\forall t \in \mathbb{R}\ \forall f \in C_0(X)\ \forall x \in X\ (T(t)f)(x) = q_t(x)f(x)$, i.e. $T(t)$ is local, and as the result shows in that case $\{T^*(t)\}_{t\in\mathbf{R}}$ is a C_0-group.

Remark. The above theorem generalizes the well-known result of A. Plessner ([Pl]) that if $f\colon \mathbb{R} \to \mathbb{C}$ and $\mathrm{Var}_{\mathbf{R}}[f] < +\infty$, then f is absolutely continuous if and only if $\mathrm{Var}_{\mathbf{R}}[f(\cdot + t) - f(\cdot)] \to 0$ as $t \to 0$.

The following theorem generalizes the result of N. Wiener and R. C. Young ([WY]) about the behavior of measures on $\mathbb{R}$ under translation.

Theorem 7. *Let $\{T(t)\}_{t\in\mathbf{R}}$ be a C_0-group on $C_0(X)$ with the disjoint support property. Then $\forall \mu \in M(X)$*

$$\limsup_{t\to 0} \|T^*(t)\mu - \mu\| \geq 2\|\mu_d\|,$$

where μ_d is the component of μ in $C_0(X)^{\odot d}$. Moreover, if $\limsup_{t\to 0}\|T(t)\| = 1$, then the last inequality becomes an equality.

Proof. First we will prove that

$$\limsup_{t\to 0} \|T^*(t)\mu - \mu\| = \limsup_{t\to 0} \|T^*(t)\mu_d - \mu_d\|.$$

Suppose that φ and q are the flow and the cocycle of $\{T(t)\}_{t\in\mathbf{R}}$. Then it is not difficult to see that $|T(t)|$ also has the disjoint support property with the flow φ and cocycle $|q|$. Thus, by Theorem 6 both groups $\{T(t)\}_{t\in\mathbf{R}}$ and $\{|T(t)|\}_{t\in\mathbf{R}}$ have the same $C_0(X)^{\odot}$.

Observe that $\forall t \in \mathbb{R}$ $\{T(t)\}_{t\in\mathbf{R}}$ is a positive disjointness preserving group. Since $M(X)$, being an AL-space, has an order continuous norm (see [AB, p. 187]), it follows from [vN, Th. 8.1.6] that $C_0(X)^{\odot}$ is a projection band in $M(X)$. Thus,

$$M(X) = C_0(X)^{\odot} \oplus C_0(X)^{\odot d}.$$

Suppose $\mu \in M(X)$. Then there exist unique $\mu_0 \in C_0(X)^{\odot}$, $\mu_d \in C_0(X)^{\odot d}$ such that $\mu = \mu_0 + \mu_d$. We claim that $\forall t \in \mathbb{R}$ $C_0(X)^{\odot}$ and $C_0(X)^{\odot d}$ are invariant under $T^*(t)$.

Since $T^*(s)\mu_0 \to \mu_0$ as $t \to 0$, it follows that $T^*(t)T^*(s)\mu_0 \to T^*(t)\mu_0$ as $t \to 0$. Hence, $T^*(s)T^*(t)\mu_0 \to T^*(t)\mu_0$ as $t \to 0$ and $T^*(t)\mu_0 \in C_0(X)^{\odot}$. Since $T(t)$ is disjointness preserving, so is $T^*(t)$ (see [MN, 3.1.21]). Therefore, since $\forall t \in \mathbb{R}$

$\forall \nu \in C_0(X)^{\odot}$ $T^*(-t)\nu \perp \mu_d$, $\nu \perp T^*(t)\mu_d$ and $T^*(t)\mu_d \in C_0(X)^{\odot d}$. Thus, the claim is established.

It follows that $\forall t \in \mathbb{R}$

$$(T^*(t)\mu_0 - \mu_0) \perp (T^*(t)\mu_d - \mu_d).$$

Since $M(X)$ is an AL-space, we conclude that

$$\begin{aligned}
&\limsup_{t\to 0} \|T^*(t)\mu - \mu\| \\
&= \limsup_{t\to 0} \|(T^*(t)\mu_0 - \mu_0) + (T^*(t)\mu_d - \mu_d)\| \\
&= \lim_{t\to 0} \|T^*(t)\mu_0 - \mu_0\| + \limsup_{t\to 0} \|T^*(t)\mu_d - \mu_d\| \\
&= \limsup_{t\to 0} \|T^*(t)\mu_d - \mu_d\|.
\end{aligned}$$

Next step is to prove that

$$\limsup_{t\to 0} \|T^*(t)\mu_d - \mu_d\| = \limsup_{t\to 0} \|T^*(t)\mu_d\| + \|\mu_d\|.$$

Let m be the Lebesgue measure on $\mathbb{R}$. Since $\{|T(t)|\}_{t\in\mathbf{R}}$ is a positive C_0-group, it follows from [dP, 2.3] that $\forall\mu$ in $C_0(X)^{\odot d}$ $|T(t)|^*\mu \perp \mu$ m-a.e. on $\mathbb{R}$. It follows from [MN, 3.1.21] that both $T^*(t)$ and $|T(t)|^*$ are disjointness preserving, and therefore by [AB, 8.6] $|T^*(t)\mu| = |T^*(t)||\mu|$ and $|T(t)|^*|\mu| = ||T(t)|^*\mu|$. Since by [MN, 3.1.21] $|T^*(t)| = |T(t)|^*$, we obtain:

$$|T^*(t)\mu| = |T^*(t)||\mu| = |T(t)|^*|\mu| = ||T(t)|^*\mu|.$$

Therefore, $|T(t)^*\mu| \wedge |\mu| = ||T(t)|^*\mu| \wedge |\mu| = 0$ which implies that $T(t)^*\mu \perp \mu$ m-a.e. on $\mathbb{R}$. Again using the fact that $M(X)$ is an AL-space, we conclude that

$$\limsup_{t\to 0} \|T^*(t)\mu_d - \mu_d\| = \limsup_{t\to 0} \|T^*(t)\mu_d\| + \|\mu_d\|.$$

The only thing left to show that

$$\limsup_{t\to 0} \|T^*(t)\mu - \mu\| \geq 2\|\mu_d\|$$

is to prove that $\limsup_{t\to 0}\|T^*(t)\mu_d\| \geq \|\mu_d\|$. Let $\varepsilon > 0$. Then $\exists f \in C_0(X)$ such that

$$\|\mu_d\| - \varepsilon < |\langle \mu_d, f\rangle|.$$

Since $\{T(t)\}_{t\in\mathbf{R}}$ is a C_0-group,

$$|\langle T^*(t)\mu_d, f\rangle| = |\langle \mu_d, T(t)f\rangle| \to |\langle \mu_d, f\rangle| \quad \text{as} \quad t \to 0.$$

Thus, $\exists \delta > 0$ such that $\forall |t| < \delta$

$$\|T^*(t)\mu_d\| \geq |\langle T^*(t)\mu_d, f\rangle| > |\langle \mu_d, f\rangle| - \varepsilon > \|\mu_d\| - 2\varepsilon.$$

It follows that $\limsup_{t\to 0}\|T^*(t)\mu_d\| \geq \|\mu_d\|$.

Finally if $\limsup_{t\to 0}\|T(t)\| = 1$, then the inequality becomes an equality since

$$\limsup_{t\to 0}\|T^*(t)\mu_d\| \leq \limsup_{t\to 0}\|T^*(t)\|\,\|\mu_d\| = \|\mu_d\|. \qquad \blacksquare$$

Corollary 8. *Let $\{T(t)\}_{t\in\mathbf{R}}$ be a C_0-group on $C_0(X)$ with the disjoint support property such that one of the following two conditions is satisfied:*

(1) *$\{T(t)\}_{t\in\mathbf{R}}$ is a contraction group.*

(2) *X is compact.*

Then $\forall \mu \in M(X)$

$$\limsup_{t\to 0}\|T^*(t)\mu - \mu\| = 2\|\mu_d\|,$$

where μ_d is the component of μ in $C_0(X)^{\odot d}$.

Proof. The only thing we need to prove is that $\limsup_{t\to 0}\|T(t)\| = 1$.

Case (1). In the proof of Theorem 7 we showed that $\limsup_{t\to 0}\|T(t)\| \geq 1$. However, since $\{T(t)\}_{t\in\mathbf{R}}$ is a contraction group, $\limsup_{t\to 0}\|T(t)\| \leq 1$. Thus, $\limsup_{t\to 0}\|T(t)\| = 1$.

Case (2). Since $[-1,1]\times X$ is compact, $|q|$ is uniformly continuous on $[-1,1]\times X$. Thus, $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\forall |t| < \delta\ \forall x \in X\ ||q_t(x)| - 1| < \varepsilon$. In other words,

$$1 - \varepsilon < |q_t(x)| < 1 + \varepsilon.$$

It follows that

$$1-\varepsilon \leq \|q_t\| \leq 1+\varepsilon$$

and therefore $\lim_{t\to 0}\|q_t\| = 1$. It is not difficult to see that $\forall t \in \mathbb{R}\ \|T(t)\| = \|q_t\|$. Hence, $\lim_{t\to 0}\|T(t)\| = 1$. ■

Remark. Let $\{T(t)\}_{t\in\mathbf{R}}$ be a C_0-group on $C_0(X)$ with the disjoint support property such that $\sup_{t\in\mathbf{R}}\|T(t)\| = \sup_{t\in\mathbf{R}}\|q_t\| = M < +\infty$. $\forall f \in C_0(X)$ define $|||f||| = \sup_{t\in\mathbf{R}}\|T(t)f\|$. Then (see [Pa, Th. 1.5.2]) $\forall t \in \mathbb{R}\ \forall f \in C_0(X)\ \|f\| \leq |||f||| \leq M\|f\|$ and $|||T(t)f||| = |||f|||$. Hence, by Theorem 7

$$\limsup_{t\to 0} |||T^*(t)\mu - \mu||| = 2|||\mu_d|||,$$

where μ_d is the component of μ in $C_0(X)^{\odot d}$.

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