THE EMERGENCE AND PROPAGATION OF A PHASE BOUNDARY IN AN ELASTIC BAR

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I dedicate this thesis to my parents.

ABSTRACT

This dissertation is concerned with the dynamical analysis of an elastic bar whose stress-strain relation is not monotonic. Sufficiently large applied loads then require the strain to jump from one ascending branch of the stress-strain curve to another such branch. For a special class of these materials, a nonlinear initial-boundary value problem in one-dimensional elasticity is considered for a semi-infinite bar whose end is subjected to either a monotonically increasing prescribed traction or a monotonically increasing prescribed displacement. If the stress at the end of the bar exceeds the value of the stress at any local maximum of the stress-strain curve a strain discontinuity or "phase boundary" emerges at the end of the bar and subsequently propagates into the interior. For classically smooth solutions away from the phase boundary, the problem is reducible to a pair of differential-delay equations for two unknown functions of a single variable. The first of these two functions gives the location of the phase boundary, while the second characterizes the dynamical fields in the high-strain phase of the material. In these equations the former function occurs in the argument of the latter, so that the delays in the functional equations are unknown. A short-time analysis of this system provides an asymptotic description of the emergence and initial propagation of the phase boundary. For large-times, a different analysis indicates that the phase boundary velocity approaches a constant which depends on material properties and on the ultimate level reached by the applied load as well. Higher order corrections depend on the detailed way in which the load is applied. Estimates for the various dynamical field quantities are given and a priori conditions are deduced which determine whether the phase boundary eventually becomes the leading disturbance.

TABLE OF CONTENTS

INTRODUCTION

The phenomenon of phase changes has lately received increased attention in continuum mechanics. In the setting of one-dimensional finite elasticity, for example, the modelling of phase changes involves a strain energy density which is a non-convex function of the displacement gradient. This leads to a stress-strain relation which is not monotonic. Consequently the values of strain where the stress response changes monotonicity partition the strain axis into distinct intervals. We associate each such strain interval with a distinct material phase. Under certain loadings of a body composed of such a material, the possibility exists that surfaces emerge across which the strain is discontinous and which separate regions of different phase. The present study is motivated by questions raised by the recent literature involving solids with such multiple phases. These issues are addressed in detail in the following section.

In subsequent sections we formulate an initial-boundary-value problem associated with the dynamics of a homogeneous, elastic bar whose strain energy density is such a non-convex function of the displacement gradient; the objective is to illuminate the manner in which the phase boundaries are generated and propagated in response to given loading conditions. In order to avoid the consideration of reflected waves, we consider a semi-infinite bar which we take to be initially undeformed and at rest. Two problems are treated: in one, the traction at the end of the bar is prescribed as a function of time, corresponding to a load-controlled (or "soft") dynamical testing device; in the second — or "hard" — dynamical problem, the displacement history of the end of the bar is specified.

Conditions are then put on the strain energy density and on the loadings so as to focus attention on the issue of phase boundary propagation with a minimum of technical complication. We show how certain loadings require the emergence of a phase boundary at the end of the bar which subsequently propagates into the interior.

For a subclass of the materials considered, the location of the phase boundary is governed by a pair of differential-delay equations, where the delay is unknown. We first present a short-time analysis of this system in order to describe the emergence and initial propagation of the phase boundary. We then study this system at large times and show that — for the problems considered — the phase boundary propagates at a velocity which lies between the sound-speeds of the "low-strain" and "high-strain" phases of the material. Here the phase boundary may or may not eventually become the leading disturbance to travel down the bar. In either case the speed of propagation of the phase boundary approaches a constant which depends on the ultimate level reached by the applied load, as well as on the material.

All of the analysis described above is carried out in detail for the "soft" device. A summary of results for both the "hard" and "soft" device is presented in the concluding section. There we also describe the asymptotic nature of the associated dynamical fields for both short and large times, as well as discuss other pertinent aspects of the problem.

1. Background

For the one-dimensional homogeneous elastic bar in which the strain energy density is not a convex function of the strain, the stress will not be a monotone function of strain. Ericksen [1] has investigated equilibrium configurations of bars whose stress-strain relation is of the type depicted in Fig. 1, in which the abscissa, ε, is the strain and the ordinate, $\sigma = W'$, is the stress. When the bar is free of body forces, equilibrium configurations are states of uniform, i.e. spatially constant, stress throughout the bar. The non-monotonicity of the stress-strain relation leads to an interval $(\sigma_\alpha, \sigma_\beta)$ of possible values of equilibrium stress for which, at any location in the bar, there are three possible values of strain. Note that in the interval $(\sigma_\alpha, \sigma_\beta)$ there occurs a unique stress level, called the Maxwell stress and labeled σ_γ in Fig. 1, with the geometric property that the two closed loops in the stress-strain curve determined by the Maxwell stress have equal area. In using a traditional equilibrium criterion according to which the total energy in the bar is an extremum, we distinguish between the following equilibrium configurations. An equilibrium configuration is termed _stable_, _neutrally stable_, or _unstable_ _with respect to some other configuration satisfying the loading conditions_, if the total energy of the equilibrium configuration is respectively less than, equal to, or greater than the total energy of the competing configuration. If an equilibrium configuration is stable with respect to _all_ such competing configurations we say it is _absolutely stable_; if it is stable or neutrally stable with respect to all such competing configurations, we say it is _stable_. We call the equilibrium configuration _metastable_ if it is not stable, but if all competing configurations with respect to which the equilibrium configuration is unstable have strain fields that are not uniformly close to the strain field of the equilibrium configuration. Hence a metastable equilibrium configuration is either stable or neutrally stable with respect to all configurations whose strain fields differ only infinitesimally from the strain field of the equilibrium configuration everywhere in the bar. Finally if a configuration is neither stable nor metastable we say it is unstable. Consequently the type of stability (stable, metastable, unstable) depends on the size of the class of configurations with which the equilibrium configuration successfully competes as a minimizer of the total energy. Stable configurations are strong relative minimizers of the energy functional in that competitors need only satisfy the loading condition, so that any configuration is an eligible competitor provided it is in the function space in which the solution is sought. On the other hand, metastable configurations are merely weak relative minimizers of the energy functional, since the number of competitors is reduced by requiring in addition that the strain fields in the competitors be uniformly close to the strain field of the minimizer. Let $\gamma_1 < \gamma_2 < \gamma_3$ be the three Maxwell strains as depicted in Fig. 1. For a given equilibrium stress level σ, it is shown in [1] that if $\sigma < \sigma_\gamma$ $(\sigma > \sigma_\gamma)$ the unique configuration with $\varepsilon < \gamma_1$ $(\varepsilon > \gamma_3)$ everywhere in the bar is absolutely stable. If σ lies in the interval $[\sigma_\alpha, \sigma_\beta]$ and $\sigma < \sigma_\gamma (\sigma > \sigma_\gamma)$ there is one possible strain $\varepsilon_1 < \gamma_1 (\varepsilon_1 > \gamma_3)$ and another ε_2 with $\alpha_2 \le \varepsilon_2 < \gamma_3$ $(\gamma_1 \le \varepsilon_2 < \beta_1)$. It is shown that a configuration of the bar whose strain field is piecewise uniform and takes only the two values of ε_1 and ε_2 is metastable. If σ lies in the interval $(\sigma_\alpha, \sigma_\beta)$ and $\sigma < \sigma_\gamma (\sigma > \sigma_\gamma)$, there is a third possible strain ε_3 with $\beta_1 < \varepsilon_3 < \alpha_2$; any configuration whose strain field takes the value ε_3 at any point is unstable. If $\sigma = \sigma_\gamma$, there are no absolutely stable configurations, but any configuration with piecewise uniform strain taking only the values γ_1 and γ_3 is stable. If $\sigma = \sigma_\gamma$ and $\varepsilon = \gamma_2$ anywhere in the bar, the configuration is unstable. A particle P of the bar which was originally at x will be said to be in the low-strain (or first) phase if the strain $\varepsilon(x)$ corresponds to a point lying below the first ascending branch of the stress-strain curve (Fig. 1), so that $0 \le \varepsilon(x) \le \beta_1$. If $\beta_1 < \varepsilon(x) < \alpha_2$, P is said to be in the unstable phase, while P is in the high-strain (or second) phase if $\varepsilon(x) \ge \alpha_2$.

Consider a bar with one end fixed while the other end is loaded monotonically and quasistatically. The resulting equilibrium states will involve uniform stress at each instant. As long as the load level is below the Maxwell stress one would expect the bar to equilibrate in the absolutely stable first phase state with a uniform strain lying below the first ascending branch. However, once the load exceeds the Maxwell stress σ_γ, the configuration in which all particles are in the low-strain phase is only metastable, and so one would expect to eventually observe strains at the stabler value lying below the second ascending branch. Transitional equilibrium configurations would then be characterized by a partitioning of the bar into a number of co-existent phases of alternately higher and lower strains. As the load is increased still further, the regions of high strain would grow at the expense of the regions of low strain, leading eventually to the absolutely stable state in which the bar is everywhere in the high-strain phase. Finally, if one then conducts an analogous program of unloading, one would expect to observe portions of the bar in the low-strain phase at some load below the Maxwell stress. Hence one anticipates a hysteresis loop. The time at which these phase transistions occur would depend on the actual physical experiment where a certain unavoidable noise or disturbance level prevents achieving perfectly smooth loadings and unloadings. If the disturbance level is high, one expects that the bar would everywhere jump to the stabler phase at the Maxwell stress. However if it is low, one expects to be able to preserve the metastable one-phase states almost until that phase will no longer sustain the given equilibrium stress. Consequently the size of a measured hysteresis loop would be an indication of the disturbance level in the experiment.

As yet we have not excluded the possibility that different locations in the bar could change phase at different equilibrium stress levels. For a homogeneous bar in the envisioned loading experiment, one would assume that all transitions from low to high strain would occur at the same stress. Hence, in equilibrium, co-existence phases would occur only at some particular value of stress between σ_α and σ_β. The choice of such a transition stress under loading would then be a statement about the disturbance level of the system and amounts to choosing a particular inverse, $\varepsilon = (W')^{-1}(\sigma)$, to the stress response $\sigma = W'(\varepsilon)$. Ericksen [1] showed that when this transition occurs at the Maxwell stress, an arbitrary number and placement of alternating regions of high-and low-strain phases at this stress is consistent with either a specified displacement (hard) end condition or a specified load (soft) end condition. However, for the hard condition , the total extent of each phase present is uniquely determined from the condition which specifies the total length of the deformed bar.

An analogous situation occurs in the bending of an initially straight, homogeneous, inextensible elastica with a non-convex bending energy density [2]. In order to facilitate a discussion of the similarities in the bar and elastica problems, we shall introduce nomenclature from classical Gibbsian equilibrium thermodynamics. We identify a conjugate pair of dependent field variables for each system under consideration in such a way that the variable which appears in the argument of the energy density will be called the <u>extensive variable</u>, while the first derivative of the energy density with respect to this variable will be called the <u>intensive variable</u> [3]. For the bar and elastica problems, the extensive variables are respectively the strain and the curvature, while the intensive variables are respectively the stress and the moment. For energy densities of the type studied by Ericksen [1], Fig. 1 applies, provided the abscissa is identified with the extensive variable and the ordinate is identified with the intensive variable. Again we say a location in the body is in one of three distinct phases depending on whether the extensive variable is in the interval lying below the first ascending, descending, or second ascending branch of Fig. 1. In the body, the extensive variable may be discontinuous across certain <u>shock surfaces</u>. When the discontinuity separates values of the extensive variable which lie below distinct branches of the curve of Fig. 1, such a shock surface is simultaneously a phase boundary. In all cases, global equilibrium requires the continuity of the intensive variable.

Fosdick and James [2] examine two problems for an elastica for which Fig. 1 now describes the moment-curvature relation. Equilibrated bending moments are prescribed in one problem, while the other is one of prescribed slope difference between the ends of the elastica. Each problem is one of "pure bending" in that equilibrium configurations are those with uniform moment throughout the elastica. By minimizing the total energy subject to the given end conditions, we distinguish the same stability types introduced previously.[1] Once again if the equilibrium value of the intensive variable — in this case the moment — is not equal to σ_γ, that unique equilibrium configuration in which the extensive variable — in this case the curvature — is either everywhere less than γ_1 or everywhere greater than γ_3 is absolutely stable. If the intensive variable is equal to σ_γ, there are no absolutely stable configurations; but equilibrium configurations in which the extensive variable is limited to either γ_1 or γ_3 are stable. This case includes co-existent phases in which the deformed elastica consists of smoothly connected circular arcs whose curvatures alternate between γ_1 and γ_3. When the intensive variable is not equal to σ_γ, such coexistent phases between the two ascending branches of Fig. 1 are only metastable. In every case, if at any point in the body the value of the extensive variable lies below the descending branch, the system is unstable. For stable equilibrium configurations, co-existent phases occur only at the Maxwell moment σ_γ and the number of alternating regions of different phase, as well as their location, is arbitrary. Once again, for the <u>hard</u> boundary condition — prescribed

[1] Fosdick and James in [2] use a different terminology. Our stable and metastable equilibrium configurations correspond respectively to their Eulerian and weak Eulerian states.

terminal slope difference — the total extent of the elastica in each phase is fixed. No such restriction applies to the problem with the soft boundary condition.

For the problems reviewed above, equilibrium demands that the intensive variable be spatially constant. This leads to highly non-unique stable equilibrium configurations when that constant is the Maxwell value σ_γ. In more complicated problems, the intensive variable need not be independent of position. This eliminates much of the arbitrariness in the stable equilibrium configurations with co-existent phases. For example, bars where each particle has the same type of stress response as in Fig. 1., but with material and geometrical inhomogeneities, admit the possibility that the transition stress may vary throughout the bar. In addition if body forces are present, equilibrium configurations need not be configurations of spatially constant stress. Consequently the stress in the bar will coincide with the transition stress only at certain locations. As shown by James [4], this permits certain conclusions to be drawn about the number and location of equilibrium phase boundaries.

Another example is provided by an elastica with one end fixed and subject to an axial compressive force at the other end. Here buckling becomes a further potential source of non-uniqueness. For nonlinear moment-curvature relations, the number of available stable buckled configurations will in general increase at certain values of the applied load. For each such buckled state, the associated finite deformation provides a variable moment-arm for the applied load, so that equilibrium configurations will not be those with a uniform moment field. James [5] has studied buckled configurations when the moment-curvature relation is as in Fig. 1. He has shown that when the applied load is sufficiently large, candidates for stable "first mode" buckled configurations involve a region of high-curvature phase at the fixed end of the elastica, while the remaining portion is in the low-curvature phase. The location of the phase boundary is precisely determined, being further from the fixed end for larger applied loads.

A three-dimensional example is a problem of the twisting of an incompressible homogeneous elastic tube with a non-convex energy density, as studied by Abeyaratne [6]. By seeking radially symmetric solutions, Abeyaratne reduces the problem to a one-dimensional one. Here the intensive variable is the shear stress, which is not in general constant for equilibrium configurations. It is found that minimizers of the total stored energy[1] correspond to a unique configuration for all applied twisting angles. Some of these solutions are smooth, while others include a single circular phase boundary.

Knowles and Sternberg [7] have examined the relationship between the loss of ellipticity of the governing equations and the emergence of equilibrium shocks in the two-dimensional problem of plane finite elastostatics. It is shown that a necessary condition for the existence of a straight shock separating two homogeneous plane deformations is that the equilibrium equations must suffer a loss of strong ellipticity at some

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[1] Abeyaratne restricts competitors to be symmetric and also places an L_2 norm restriction on the closeness of the shear between minimizer and competitors. This restriction is weak enough not to unduly limit the class of competitors and, as in [2,4,5] where there are no norm restrictions, the jump in the intensive variable must occur at the Maxwell value.

homogeneous deformation which is a convex linear combination of the two plane homogeneous deformations separated by the shock. This fits in nicely with the one-dimensional problems reviewed above, where it was found that in stable and metastable equilibrium configurations the values of the extensive variable on opposite sides of a phase boundary are separated by an unstable branch of the constitutive relation.

The convexity of the relevant energy density plays a central role with respect to the question of the existence of equilibrium solutions which minimize the energy functional. For the elastic bar, Ball [8] demonstrates that if all specified displacement problems are to have continuously differentiable minimizers of the energy functional, then the corresponding strain energy density must necessarily be convex. When it is not, certain problems of specified displacement will have minimizers only if they admit discontinuous strains. Conversely, convexity of the strain energy is also a sufficient condition for the existence of minimizers.

Gurtin and Temam [9] have considered the minimization problem for finite elastostatic anti-plane shear. Here the extensive variable — the shearing stress — and the intensive variable — the shearing strain — are related as in Fig. 1. They consider an auxiliary minimization problem governed by a strain energy density which is the lower convex envelope of the original strain energy density. Then the "convexified problem" is governed by a relation between shearing stress and shearing strain in which the Maxwell line replaces that portion of the original curve which lies above the metastable and unstable intervals of shearing strain,

$\gamma_1 < \varepsilon < \gamma_3$. Unlike the original problem, the convexified problem is guaranteed to have a minimizer. The minimum value of the convexified energy functional is the infimum of the energy functional of the original problem evaluated at all configurations which satisfy the boundary conditions. Conversely if minimizers of the original problem exist, they will be minimizers of the convexified problem.

For three-dimensional finite elastostatics, Abeyaratne [10] has shown that the equilibrium shock conditions are given naturally by the Weierstrass-Erdmann corner conditions when piecewise smooth functions are admitted into contention as possible minimizers of the energy functional. The traction continuity condition arises from considering variations of the displacement components. If one then considers variations of the independent variables, the spatial coordinates, another vectorial shock condition is obtained. Continuity of the tangential displacement derivatives across the shock guarantees that all but one component of this shock condition is satisfied automatically. One is then left with a supplemental shock condition requiring an energy-like quantity to be continuous across the discontinuity surface. For the equilibrium bar problem of Ericksen [1], this condition is satisfied for a material with shocks if and only if the constant equilibrium stress in the bar is the Maxwell stress.

Knowles and Sternberg [11, 12] and Abeyaratne [13] have examined certain anti-plane shear crack problems with a non-convex energy density and have found equilibrium solutions with shocks issuing from the crack tips [11,12] or from points on the crack surface [13] and terminating in the interior of the body. These solutions do not satisfy the supplemental shock condition. If, however, one considers a one-parameter family of equilibrium states corresponding to a quasistatic loading, it can be shown that across the shocks appearing in the corresponding quasistatic solutions of [11,12,13] the jump in the supplemental quantity is of a particular sign (with respect to the direction with which the quasistatic shock is moving). Thus although the supplemental shock condition is not satisfied, a supplemental shock inequality is maintained. Moreover this inequality assures the dissipative character of the shock in the sense that it precludes quasistatic shocks which allow the body to store elastic energy faster than the rate at which work is being done. This dissipativity inequality was originally proposed as an admissibility criterion for elastic shocks by Knowles and Sternberg [7] and Knowles [14]. It is analogous to the entropy inequality for gas-dynamical shocks.

For the quasistatically loaded bar, this dissipativity inequality allows the emergence of shocks only if the stress equals or exceeds the Maxwell stress. Similarly Fosdick and James [2] find that the metastable equilibrium configurations for the pure bending problem of the elastica satisfy an analogue of the dissipation inequality across the curvature discontinuities. It thus appears that metastable quasistatic processes are associated with dissipative and hysteretic phenomena, while absolutely stable quasistatic processes are dissipation-free.

Of course, the quasistatic processes envisioned above are not in reality dynamic processes at all, but rather one-parameter families of equilibrium states. Although it is logical to identify the history parameter with time, such an approach neglects the effect of inertia and the consequent wave propagation phenomena associated with dynamical systems. Studying the emergence and evolution of different phases in a fully dynamic theory would seem to be prudent both for experimental reasons and to better understand the approach to an equilibrium consisting of co-existent phases. The latter would presumably help to resolve some of the questions unanswered by the static theory, in particular the ultimate location of equilibrium phase boundaries.

The following brief discussion of the dynamic behavior of systems with non-convex equilibrium energy densities will be limited to one-dimensional problems for bars. Even then there is a wide variety of dynamic problems which may be related to any one particular quasistatic problem, since non-equilibrium processes can be governed by an infinity of constitutive assumptions, all of which reduce in an equilibrium setting to the same purely elastic constitutive relation.

Determining the ultimate location of phase boundaries makes visco-elastic theories particularly appropriate, since the inherent dissipation of a viscoelastic dynamical process can lead to an asymptotic static state for large time. Dafermos [15] has studied an initial-boundary value problem for a special class of viscoelastic bars for which the constitutive law in the static case is consistent with a non-monotonic equilibrium stress-strain relation. He considers an initially deformed viscoelastic bar which is released with some initial velocity and is subsequently free from body forces and end loads. It is shown that if the viscosity is bounded above zero the bar will asymptotically approach some stress-free equilibrium state. Especially significant in light of our review of the static theory is that the asymptotic deformation gradient will in general be discontinuous, possibly even unbounded. Moreover, it is not completely apparent which of the various possible equilibrium configurations is ultimately approached.

The motion of a viscoelastic bar is typically governed by a partial differential equation of order three or more, in which case the well-developed mathematical techniques for second-order partial differential equations are unavailable. However the dynamics of a purely elastic bar are governed by a second-order quasilinear partial differential equation of mixed type, being hyperbolic when the strain lies below an ascending branch of the stress-strain curve, and elliptic when the strain lies below a descending branch. As naturally occurring initial and boundary conditions for dynamical problems lead to well-posed problems for hyperbolic partial differential equations but ill posed ones for elliptic equations, the dynamic theory itself provides impetus to seek solutions of the dynamic problem which have strains confined to the statically stable and metastable phases. With such a restriction the governing equation is equivalent to a hyperbolic system of two first-order ordinary differential equations and so the theory of Riemann invariants and characteristic curves in the $x-t$ plane is at one's disposal. Space-time curves across which the strain jumps from one branch to the other are the propagating phase boundaries.

To appreciate the specific dynamical phenomena associated with <u>non-monotonic</u> stress-strain relations it is helpful to first consider the simpler theory for the monotonic case. A good account of this theory may be found in Courant and Friedrichs [16]. It is well known [17] that solutions for hyperbolic systems with arbitrarily smooth initial data are not guaranteed to be globally smooth for all time; the elastic bar is no exception. The type of singularities encountered in this setting are associated with the intersection of characteristic curves at some finite distance from the curve of initial data. At such points the second derivatives of the displacement become unbounded. The remedy is to seek weak solutions to the governing equations by introducing shock curves across which the first derivatives of the displacement are discontinuous, but their jumps satisfy two shock conditions which are necessary for displacement continuity and momentum balance. As there may be many such weak solutions, an "entropy condition", due to Lax [18], may be introduced to select solutions of physical significance. This has the effect of confining the shock speed to the interval between the different acoustic speeds on each side of the shock.

It must be emphasized that these types of shocks are to be distinguished from phase boundaries, since the latter are not even admitted into a bar theory with a monotone stress-strain relation. For the non-monotonic theory both types of shocks become possible. By a conventional shock we shall mean a shock which arises from the intersection of characteristic curves associated with strains lying below a single branch of the stress-strain relation. Shock curves which separate distinct phases are referred to as phase boundaries. The term "shock" without an obvious referent will be used in discussing features common to both types. For example, as weak solutions require the same relations to hold across all curves with discontinuous strain, the same "shock" conditions apply in both cases.

The central role of monotonicity and convexity conditions in general hyperbolic systems of equations has been explored by Lax [18]. By applying the methods used in [18] to the equations for an elastic bar with a stress-strain relation as in Fig. 1, James [19] has demonstrated the local existence of classically smooth fields about an assigned moving phase boundary. James also treats two specific initial value problems. In the first, a value of strain is chosen from each of the two intervals lying below ascending branches of Fig. 1. One value is specified along the positive x axis and the other along the negative x axis. Hence the bar initially contains two phases. Two appropriately selected velocities complete the specified initial data. For this problem, two one-paramteter families of solutions are found, each of which contains a single constant-velocity phase boundary emerging from the point of initial strain discontinuity. The second initial value problem treated in [19] involves a constant value of strain associated with one of the ascending branches, and zero velocity, specified initially all along the x axis. In addition to the obvious static solution in which the initial stable or metastable state persists for all time, James establishes a two-parameter family of solutions which contain both a constant velocity phase boundary moving to the right and to the left. The region between the phase boundaries experiences a constant strain lying below the other ascending branch of the stress-strain curve. James points out that this second problem is naturally associated with necking phenomena, in that phase boundaries may spontaneously arise in the interior of the bar. It would however seem that achieving the initial condition of spatially constant non-zero strain would itself require a quasistatic process. Also the constant speed of the two phase boundaries depends on the doubly infinite spatial domain, so that subsequent boundary effects — such as those due to a loading device or due to waves reflected from the ends of the bar—are not taken into account.

James also addressed the question of admissibility of solutions to the dynamic problem by alternately considering static, viscous and entropy rate criteria. Particularly interesting is the "viscosity" criterion, according to which an inequality to be satisfied across shocks is deduced by considering solutions which are limits of viscoelastic solutions as the viscocity vanishes. For a phase boundary in which the traction discontinuity is small—so that the phase boundary is moving slowly and separates a region of statically stable phase from a region of statically metastable phase—the viscoelastic criterion requires that the phase boundary shall move in the direction that converts statically metastable phase to statically stable phase. Hence when the traction discontinuity is small, both the viscoelastic criterion of James and the dissipation criterion of Knowles allow and exclude the same phenomena. Indeed as the traction discontinuity vanishes, the phase boundary ceases its motion and the distinction between the two criteria vanishes.[1] For the first initial value problem considered by James, the viscoelastic criterion allows only one of the two families of solutions found. For the second problem, the occurrence of the double phase boundaries is possible only if the constant initial strain is a statically metastable solution.

[1]Compare (5.39) of [19] and (4.19) of [14].

2. Governing Equations for the Dynamically Loaded Elastic Bar. Characteristic Theory when a Single Phase is Present.

A. Formulation of the Problem

We consider a homogeneous, semi-infinite elastic bar which in the reference configuration occupies the interval $-\infty < x \le 0$. Motions are described by a mapping

$$y(x,t) = x + u(x,t) \ , \tag{2.1}$$

where $y(x,t)$ is the coordinate at time t of a particle which is at x in the reference configuration; $u(x,t)$ is the displacement. We assume the reference configuration to be undeformed.

In the fully three dimensional theory of elasticity the strain energy density $\hat{W}$ is a function of the deformation gradient tensor $\underset{\sim}{F}$. When the above deformation is expressed in a Cartesian reference frame $\underline{X}$ with the unit vector $\underset{\sim}{e}_1$ aligned with the bar, $\underset{\sim}{F}$ is described by means of its matrix $\underset{\sim}{F}^{\underline{X}}$:

$$\underset{\sim}{F}^{\underline{X}} = \begin{bmatrix} 1+u' & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ . \tag{2.2}$$

A prime will in general denote differentiation with respect to the argument and specifically with respect to x if there are multiple independent variables. The Piola stress tensor $\underset{\sim}{\sigma}$ is given by

$$\underset{\sim}{\sigma} = \frac{\partial}{\partial \underset{\sim}{F}} \hat{W}(\underset{\sim}{F}) \ . \tag{2.3}$$

For the longitudinal deformation (2.1) of an elastic bar, the strain energy density becomes a function of the displacement gradient—or strain—which we denote by

$$\varepsilon(x,t) = u'(x,t), \tag{2.4}$$

so that

$$\hat{W}(\underset{\sim}{F}) = W(\varepsilon). \tag{2.5}$$

It is assumed that for this type of deformation $W(\varepsilon)$ is such that the only nonzero stress is the normal component along the axis of the bar, which shall be denoted by σ and according to (2.3) is given by

$$\sigma(\varepsilon) = W'(\varepsilon)\ . \tag{2.6}$$

The longitudinal velocity in the bar is

$$v(x,t) = \dot{y}(x,t) = \dot{u}(x,t)\ , \tag{2.7}$$

where the superposed dot denotes time differentation.

With no body forces the only nontrivial equation of motion is that associated with the balance of momentum in the x direction:

$$\rho\ddot{u} = \frac{\partial}{\partial x} W'(u') = W''(u')u''\ , \tag{2.8}$$

where ρ, a positive constant, is the density of the undeformed bar. For a bar that is initially undisturbed and at rest,

$$u(x,0) = 0,\ \ v(x,0) = 0,\ \ -\infty < x \leq 0\ . \tag{2.9}$$

We shall consider two different types of loading device. In the hard device, the bar is subject to a controlled end displacement $u_0(t)$, so that

$$u(0,t) = u_0(t) \ , \quad t \geq 0 \ . \tag{2.10}$$

The other, a soft device, subjects the bar to an end load $\sigma_0(t)$. Here the appropriate boundary condition is

$$W'(u'(0,t)) = \sigma_0(t) \ , \quad t \geq 0 . \tag{2.11}$$

We shall seek solutions $u(x,t)$ which are twice continuously differentiable for $x \leq 0$, $t \geq 0$ except on at most a finite number of smooth shock curves, $x = s_i(t)$. Where u is twice continuously differentiable, we require that (2.8) be satisfied. Across a shock curve — $x = s(t)$, say — we shall require that the displacement shall remain continuous and that a global balance of momentum be maintained. This will hold provided the following shock conditions — sometimes called Rankine-Hugoniot conditions — are satisfied [20]:

$$\begin{aligned} \dot{s}[u'] + [\dot{u}] &= 0 \ , \\ \rho\dot{s}[\dot{u}] + [W'(u')] &= 0 \ , \end{aligned} \tag{2.12}$$

where [ ] denotes the jump across the shock (e.g. $[u'] \equiv u'(s(t)^+, t) - u'(s(t)^-, t)$).

B. <u>Comments on the Formulation</u>

Many of the equilibrium studies cited in the previous section [1,2,4,5,10] were formulated in a variational setting. It should be mentioned that this problem too has a variational formulation; one seeks extremals to the appropriate action functional, namely the time integral of the Lagrangian of the system. This yields (2.8) as the Euler-Lagrange equation. Moreover, the Weierstrass-Erdmann corner condition which arises from varying the strain and velocity lead to the shock condition which expresses the balance of momentum $(2.12)_2$. Variations with respect to x and t yield two more corner conditions which can be reduced to $(2.12)_1$, and an extra condition which requires that an energy-like quantity be conserved across a discontinuity curve. For static solutions this extra condition reduces to the no-dissipation condition derived in [10].

It can also be shown that (2.12) are the discontinuity conditions associated with the canonical weak formulation of this problem [19]. Whitham [21] points out that different weak formulations may yield the same partial differential equations in regions where the fields are sufficiently smooth, but may lead to different shock conditions. Realizing this, it is not difficult to discover that smooth solutions to

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\dot{u}^2 + W(u')\right) = \frac{\partial}{\partial x}\left(\dot{u}W'(u')\right) \tag{2.13}$$

are solutions to (2.8) where $\dot{u} \neq 0$. If u is not differentiable, the canonical weak formulation of (2.13) yields $(2.12)_1$ and the extra energy-like shock condition, but does not yield $(2.12)_2$. Hence we defer to the physics of the problem — in this case the balance of momentum — to require (2.12) as the only shock conditions which solutions must satisfy.

C. Smooth Solutions with One Phase Present

We shall consider materials with a non-convex strain energy density W and, in particular, we assume that:

$$\left.\begin{array}{l} \text{(i) } W \text{ is twice continuously differentiable on } [0,\infty), \\ \text{(ii) } W'(0) = 0, \quad W'(\varepsilon) > 0 \text{ for } \varepsilon > 0 , \end{array}\right\} \tag{2.14}$$

$$\left.\begin{array}{l} \text{(iii) there exists } \beta_1, \alpha_2 \text{ with } \beta_1 < \alpha_2 \text{ and} \\ \quad W''(\beta_1) = W''(\alpha_2) = 0 , \\ \quad W''(\varepsilon) > 0 \text{ for } 0 \le \varepsilon < \beta_1 , \ \varepsilon > \alpha_2 , \\ \quad W''(\varepsilon) < 0 \text{ for } \beta_1 < \varepsilon < \alpha_2 ; \end{array}\right\} \tag{2.15}$$

and

$$\text{(iv) } \lim_{\varepsilon\to\infty} W'(\varepsilon) = \infty , \tag{2.16}$$

whence there exists $\beta_2 > \alpha_2$ with

$$W'(\beta_2) = W'(\beta_1) = \sigma_\beta . \tag{2.17}$$

This leads to a stress-strain relation of the type discussed in the preceding section and depicted in Fig. 1.

When ε lies in the interval $0 \le \varepsilon < \beta_1$ or $\varepsilon > \alpha_2$, the governing equation (2.8) is hyperbolic; it is elliptic when $\beta_1 < \varepsilon < \alpha_2$. Consequently if we assume that the loading conditions (2.10) or (2.11) lead to a first phase strain field with $0 \le \varepsilon < \beta_1$ for all $x \le 0$ and $t \ge 0$, then any smooth solution to (2.8), (2.9) and (2.10) or (2.11) can be found by the theory of Riemann invariants and characteristic curves. A full account of this theory may be found in [16]. Define the local acoustic speed by

$$c(\varepsilon) = \sqrt{\frac{W''(\varepsilon)}{\rho}} \quad , \quad 0 \le \varepsilon \le \beta_1 \quad , \tag{2.18}$$

and let

$$\phi(\varepsilon) = \int_0^{\varepsilon} c(s)ds \quad , \quad 0 \le \varepsilon \le \beta_1 \quad . \tag{2.19}$$

From (2.18) it follows that $\phi(\varepsilon)$ is strictly increasing on $0 \le \varepsilon < \beta_1$. Let $\Phi(z)$ be the inverse of ϕ defined on $0 \le z < \phi(\beta_1)$. Then by (2.18) and (2.19), Φ is differentiable with

$$\Phi'(z) = (\frac{1}{\rho} W''(\Phi(z)))^{-\frac{1}{2}} \quad . \tag{2.20}$$

We note that if $W \in C^N([0,\beta_1))$, then $c \in C^{N-2}([0,\beta_1))$, $\phi \in C^{N-1}([0,\beta_1))$, and $\Phi \in C^{N-1}([0,\phi(\beta_1)))$. One finds that (2.8) is equivalent to

$$\begin{aligned} \frac{dv}{dt} - c(\varepsilon)\frac{d\varepsilon}{dt} &= 0 \quad \text{on} \quad \frac{dx}{dt} = c(\varepsilon) \; , \\ \frac{dv}{dt} + c(\varepsilon)\frac{d\varepsilon}{dt} &= 0 \quad \text{on} \quad \frac{dx}{dt} = -c(\varepsilon) \; . \end{aligned} \tag{2.21}$$

These equations in turn imply the existence of two families of characteristic curves. In the first family $(2.21)_1$, each member-curve has positive slope in the $x-t$ plane. We parametrize this family by the variable C_+. The second family $(2.21)_2$, whose member-curves have negative slope, we parameterize by C_-. It now follows from integrating (2.21) that there exist functions K^+ and K^- such that

$$\left.\begin{aligned} v - \phi(\varepsilon) &= K^{+}(C_{+}) \text{ on curves of constant } C_{+} \text{ given by } \frac{dx}{dt} = c(\varepsilon), \\ v + \phi(\varepsilon) &= K^{-}(C_{-}) \text{ on curves of constant } C_{-} \text{ given by } \frac{dx}{dt} = -c(\varepsilon). \end{aligned}\right\} \quad (2.22)$$

Since $\frac{dx}{dt} > 0$ on each curve C_{+}, all curves C_{+} originate on $t = 0$, $x \leq 0$ and terminate on $x = 0$, $t \geq 0$. On the other hand curves C_{-} originate on either $t = 0$, $x \leq 0$ or $x = 0$, $t \geq 0$ and propagate into $x < 0$, $t > 0$. Evaluating K^{+} on $t = 0$, $x \leq 0$ one infers from (2.4), (2.9), (2.19) and $(2.22)_1$ that $K^{+} \equiv 0$, whence

$$v = \phi(\varepsilon) . \quad (2.23)$$

Then on each curve C_{-}, (2.22) and (2.23) yield

$$v = \frac{1}{2} K^{-}(C_{-}) \quad , \quad \varepsilon = \Phi(\frac{1}{2} K^{-}(C_{-})) . \quad (2.24)$$

Moreover $(2.22)_2$ and $(2.24)_2$ show that each curve C_{-} is a straight ray.

If we further <u>assume</u> that a unique ray C_{-} passes through each point (x,t), $x \geq 0$, $t \leq 0$, then there is a simple geometrical construction of the solution. Consider first the soft device governed by (2.11). The first assumption $0 \leqslant \varepsilon < \beta_1$, together with (2.15), implies that the end-load induces a unique end-strain $\varepsilon(0,t) = \varepsilon_0(t)$ such that

$$W'(\varepsilon_0(t)) = \sigma_0(t) \quad , \quad t \geq 0 \quad , \quad (2.25)$$

and

$$0 \leq \varepsilon_0(t) < \beta_1 \quad . \quad (2.26)$$

By (2.15) and (2.26) it is possible to solve (2.25) for $\varepsilon_0(t)$ if and only if

$$0 \leq \sigma_0(t) < W'(\beta_1) = \sigma_\beta \ , \quad t \geq 0 . \tag{2.27}$$

Consider the ray C_- originating at $x = 0$ and $t = \tau$. This ray, which we denote by $C_-(\tau)$, has slope $\frac{dx}{dt} = -c(\varepsilon_0(\tau))$. Consequently a particular (x,t) lies on $C_-(\tau)$ if and only if

$$x = -c(\varepsilon_0(\tau))(t - \tau). \tag{2.28}$$

Since ε and v are constant on each ray C_-, it follows that when (2.28) holds, $\varepsilon = \varepsilon_0(\tau)$. Also by (2.23) it follows that $v = \phi(\varepsilon_0(\tau))$.

Now consider a particular (x,t) which lies on a ray C_- originating on $t = 0$, $x \leq 0$, so that

$$x < -c(\varepsilon_0(0))t. \tag{2.29}$$

The constancy of ε and v on C_- rays together with the initial condition (2.9) then indicates that $\varepsilon = 0$ and $v = 0$. Physically, the rays C_- propagate information about the end loading with speed $c(\varepsilon)$. Consequently (2.29) characterizes the region of the bar in front of the leading disturbance.

Summarizing these results, we have:

$$\left.\begin{aligned} &\text{(i)}\ \varepsilon = 0, \quad v = 0 \quad \text{for} \quad x < -c(\varepsilon_0(0))t \\ &\text{(ii)}\ \varepsilon = \varepsilon_0(\tau(x,t)), \quad v = \phi(\varepsilon_0(\tau(x,t)) \quad \text{for} \quad x \geq -c(\varepsilon_0(0))t \end{aligned}\right\} \tag{2.30}$$

where $\tau(x,t)$ is given implicitly by (2.28). We shall refer to (2.30) as the formal solution for a material in the first phase loaded by a soft device. The geometry of the rays C_- is depicted in Fig. 2.

Recall that in arriving at (2.30) we made two assumptions, the first being that $0 \le \varepsilon < \beta_1$ throughout the bar for all time, and the second that a unique ray C_- passes through each (x,t) in $x \leqslant 0$, $t \geqslant 0$. We now examine what these assumptions entail. According to (2.30), the bar will be everywhere in the first phase with $\varepsilon < \beta_1$ if the same is true of the induced end-strain $\varepsilon_0(t)$. Hence (2.27) is both a necessary and sufficient condition for our first assumption. We note that if (2.27) is not satisfied, then by (2.11) and (2.15) the end load $\sigma_0(t)$ cannot be sustained by an end-strain less than the value β_1. Consequently,the formal representation (2.30) would then itself be meaningless since $c(\varepsilon)$ and $\phi(\varepsilon)$ are only defined for $\varepsilon \le \beta_1$. Indeed an attempt to extend the definitions (2.18) and (2.19) into the statically unstable strain interval $\beta_1 < \varepsilon < \alpha_2$, results in $c(\varepsilon)$ and $\phi(\varepsilon)$ becoming imaginary. Most of this paper will be concerned with the consequences of smoothly loading the bar in such a way that eventually (2.27) no longer holds. In this endeavor it will be convenient to have (2.30) available to us when $\varepsilon < \beta_1$; consequently it is essential that we also examine our second assumption.

Denote the disturbed and tranquil regions of the bar by $\mathcal{R}_D$ and $\mathcal{R}_T$ where

$$\left.\begin{aligned} \mathcal{R}_D &= \{(x,t) \mid t \ge 0, \quad -c(\varepsilon_0(0))t \le x \le 0\}, \\ \mathcal{R}_T &= \{(x,t) \mid t \ge 0, \quad x < -c(\varepsilon_0(0))t\}. \end{aligned}\right\} \tag{2.31}$$

In $\mathcal{R}_T$, all the characteristic rays C_- originate on $x < 0$, $t=0$, so that by (2.9), (2.4) and $(2.21)_2$ they all have slope $\frac{dx}{dt} = -c(0)$. This guarantees that the rays C_- form an unambiguous cover for $\mathcal{R}_T$; in other words they span $\mathcal{R}_T$ and do not intersect each other.

In $\mathcal{R}_D$, the rays C_- will also form an unambiguous cover, provided (2.28) is uniquely invertible for $\tau(x,t)$ whenever $(x,t) \in \mathcal{R}_D$. From the construction of the rays C_- as depicted in Fig. 2., this will occur provided $c(\varepsilon_0(\tau))$ is a continuous and monotonically decreasing function of τ for $\tau \geq 0$. If $\sigma_0(t)$ is continuous or differentiable, it follows from (2.25), (2.4) and (2.14) that $\varepsilon_0(t)$ has the same degree of smoothness. When $\sigma_0(t)$ is differentiable

$$\dot{\varepsilon}_0(t) = \dot{\sigma}_0(t)/W''(\varepsilon_0(t)). \tag{2.32}$$

Suppose then that

$$\left.\begin{aligned} &\sigma_0(t) \in C^1([0,\infty)) , \\ &W(\varepsilon) \in C^3([0,\beta_1)) . \end{aligned}\right\} \tag{2.33}$$

It then follows that $c(\varepsilon_0(\tau))$ is monotonically decreasing provided

$$\frac{d}{dt} c(\varepsilon_0(t)) = \frac{1}{2} \rho^{-\frac{1}{2}} (W''(\varepsilon_0(t)))^{-\frac{3}{2}} W'''(\varepsilon_0(t)) \dot{\sigma}_0(t) \leq 0 , \; t \geq 0 . \tag{2.34}$$

By (2.15) this is equivalent to

$$W'''(\varepsilon_0(t)) \dot{\sigma}_0(t) \leq 0 , \quad t \geq 0 . \tag{2.35}$$

Since the bar is loaded from an undeformed rest configuration, it is natural to consider

$$\dot{\sigma}_0(t) \geq 0 , \quad t \geq 0 , \tag{2.36}$$

whereupon (2.35) becomes

$$W''''(\varepsilon_0(t)) \leq 0 \ , \quad t \geq 0 \ . \tag{2.37}$$

This in turn will be true if

$$W''''(\varepsilon) \leq 0 \ , \quad 0 \leq \varepsilon < \beta_1 \ . \tag{2.38}$$

Materials for which (2.38) holds are said to be <u>non-hardening in the first phase</u>. We have thus shown that (2.33), (2.36) and (2.38) are sufficient to guarantee that $c(\varepsilon_0(\tau))$ is a monotonically decreasing function of τ, so that (2.28) is uniquely invertible on $\mathcal{R}_D$.

Finally, the rays C_- form an unambiguous cover for all $x \leq 0$, $t \geq 0$, provided

$$\mathcal{R}_D \cup \mathcal{R}_T = \mathcal{R} \equiv \{(x,t) \mid x \leq 0, \ t \geq 0\}, \ \mathcal{R}_D \cap \mathcal{R}_T = \phi \quad \text{(the null set)}. \tag{2.39}$$

By (2.31) and the remarks immediately following it, (2.39) will hold provided $c(\varepsilon_0(0)) = c(0)$, which in turn will hold if

$$\varepsilon_0(0) = \sigma_0(0) = 0 \quad . \tag{2.40}$$

Physically, (2.40) is a condition that the initial loading be nonimpulsive. In summary, conditions sufficient to ensure that a unique ray C_- passes through each $(x,t) \in \mathcal{R}$ when $0 \leq \varepsilon < \beta_1$ is that (2.33), (2.36). (2.38) and (2.40) are satisfied. We shall refer to these conditions as those for smooth monotonic loading of a bar whose material is non-hardening in the first phase.

D. Discussion

For completeness we shall briefly discuss the phenomena associated with the violation of any one of conditions (2.33), (2.36), (2.38) or (2.40). This is instructive not only for its own sake, but will also lead to a better understanding of the dynamic phase changes associated with the violation of (2.27), which we will soon consider.

We have seen that a unique ray C_- passes through any point (x,t) in $\mathcal{R}_T$. Simply put, the importance of (2.33), (2.36) and (2.38) lies solely in that they are sufficient to ensure that $c(\varepsilon_0(\tau))$ is a continuous and monotonically decreasing function of τ, which by (2.28) guarantees that a unique ray C_- passes through each point (x,t) in $\mathcal{R}_D$. It is then (2.40) which provides that any point (x,t) in $\mathcal{R}$ is either in $\mathcal{R}_D$ or $\mathcal{R}_T$, but not both.

Condition (2.33) ensures the differentiability of $c(\varepsilon_0(\tau))$ and so reduces the investigation of the monotonicity condition to an examination of the signs of derivatives. Consequently one can sacrifice (2.33) provided $c(\varepsilon_0(\tau))$ remains monotonically decreasing. Of course discarding (2.33) would affect the differentiability of the solution (2.30).

We cannot, however, weaken (2.33) too much since then the continuity of $c(\varepsilon_0(\tau))$ is threatened. The continuity of $c(\varepsilon_0(\tau))$ can be attained by securing the continuity of the constituent functions c and ε_0. This in turn is guaranteed by requiring

$$\left.\begin{aligned} &\sigma_0(t) \in \mathcal{C}([0,\infty)) , \\ &W(\varepsilon) \in \mathcal{C}^2([0,\beta_1)) , \end{aligned}\right\} \qquad (2.41)$$

the second of which was an original requirement on the energy density (see (2.14)). Violations of either of (2.41) are not necessarily physically unreasonable; discontinuities in $\sigma_0(t)$ are associated with impulsive loading, while isolated failures of $(2.41)_2$ involve piecewise differentiable stress-strain relations.

Let us suppose that $c(\varepsilon_0(\tau))$ is monotonically decreasing, but that (2.41) fails, so that $c(\varepsilon_0(\tau))$ is discontinuous for some $t = t_d$, say. Then the characteristic rays abruptly change slope at $x = 0$, $t = t_d$, so that the region between the rays parametrized by $C_-(t_d^-)$ and $C_-(t_d^+)$ consists of points (x,t) for which (2.28) cannot be solved for $\tau(x,t)$. Hence we must extend the partial solution (2.30) to the wedge-shaped region enclosed between $C_-(t_d^-)$ and $C_-(t_d^+)$. Consider first the case where $W''(\varepsilon)$ has an isolated discontinuity at $\varepsilon = \varepsilon_d$. Then the characteristic rays abruptly change slope at $\tau = t_d$ where $\sigma_0(t_d) = W'(\varepsilon_d)$. On both $C_-(t_d^-)$ and $C_-(t_d^+)$, (2.30) implies that $\varepsilon = \varepsilon_0(t_d)$ and $v = \phi(\varepsilon_0(t_d))$. Hence the solution is completed by defining $\varepsilon = \varepsilon_0(t_d)$ and $v = \phi(\varepsilon_0(t_d))$ in the region enclosed between $C_-(t_d^-)$ and $C_-(t_d^+)$. It may be mentioned that this solution for a kinked stress-strain curve can be generated as the limit of solutions with smooth stress-strain curves whose slope rapidly changes over a small strain interval, Fig. 3.

Now we consider an isolated discontinuity in $\sigma_0(t)$ at $t = t_d$. This prompts a discontinuity in $\varepsilon_0(t)$ at $t = t_d$. Consequently ε and v have different constant values on $C_-(t_d^-)$ and $C_-(t_d^+)$. To extend (2.30) into the region enclosed between these rays one may consider end-tractions which are continuous but have large derivatives on some finite time interval. By computing the solution via (2.30) and taking the limit as the interval-length vanishes, it may be verified that the traction-discontinuity leads to the well-known centered simple waves [16], Fig. 4. Similarly a failure of (2.40) also gives rise to a centered simple wave. In these traction-impulse constructions, a subtle but important point is that if $W'(\varepsilon)$ is linear between $\varepsilon=\varepsilon(t_d^-)$ and $\varepsilon=\varepsilon(t_d^+)$ — i.e., in (2.38), $W'''(\varepsilon)=0$ for $\varepsilon_0(t_d^-)\le\varepsilon\le\varepsilon_0(t_d^+)$ — then all the characteristic rays in the centered simple wave have the same slope. Accordingly the simple wave degenerates to the single characteristic $C_-(t_d)$, across which v and ε are discontinuous. Courant and Friedrichs [16] remark[1] that such a degenerate simple wave "...does not deserve the name 'shock'...". Nevertheless, it may be verified that the shock conditions (2.12) are satisfied across such a degenerate simple wave. It is, however, a particularly well-behaved shock in that, as the limit of classically smooth solutions, it is the limit of solutions for which (2.23) is satisfied. Accordingly (2.23) remains satisfied on both sides of such a shock, so preserving straight line characteristics throughout $\mathcal{R}_D$, each of which still propagates a constant value of v and ε. We shall see that such a fortuitous situation is not maintained in either the conventional shocks we now consider, nor in the propagating phase boundary.

We have so far examined conditions in which the constructed solution (2.30) leads to regions of $\mathcal{R}$ devoid of any rays C_-. A contrary state of affairs occurs when $c(\varepsilon_0(t))$ is increasing on some interval in t. Then (2.28) has multiple solutions in some region of $\mathcal{R}_D$. This can occur if either $c(\varepsilon)$ is increasing on some interval or if $\sigma_0(t)$

[1] p. 242.

is decreasing on some interval. When (2.33) holds, these eventualities are associated with a violation of (2.38) and (2.36), respectively. In either case, the result is that the rays C_- may intersect each other. Not surprisingly, in those regions of $\mathcal{R}_D$ where (2.28) has multiple solutions, the remedy is to discard all but one such solution. Presumably this can be accomplished by introducing within $\mathcal{R}$ a finite number of conventional shock curves $x = s_i(t)$ separating regions with classically smooth solutions to (2.8). Across these curves ε and v have jump discontinuities restricted by (2.12).

We note that in the $x - t$ plane, the first conventional shock $x = s_1(t)$ so generated must lie to the right of the leftmost envelope of intersecting characteristics C_-. The solution (2.30) will continue to hold for $x < s_1(t)$ provided $\tau(x,t)$ is taken to be the minimum positive root of (2.28). In $x > s_1(t)$ the argument leading to (2.23) is no longer valid since we cannot integrate (2.21) across $s_1(t)$ without using (2.12) to match integration constants. Suppose we introduce N shock curves. Then all we can conclude from (2.21) is that there must exist $2N$ functions, $K_2^+, K_3^+ \ldots K_{N+1}^+, K_2^-, \ldots K_{N+1}^-$, such that if

$$s_{i-1}(t) < x < s_i(t), \quad i = 1, \ldots, N; \text{ or } s_i(t) < x \leq 0, \ i = N, \tag{2.42a}$$

then C_+ is constant on curves obeying

$$\frac{dx}{dt} = c(\varepsilon), \quad v - \phi(\varepsilon) = K_i^+(C_+), \tag{2.42b}$$

and C_- is constant on curves obeying

$$\frac{dx}{dt} = -c(\varepsilon), \quad v + \phi(\varepsilon) = K_i^-(C_-) \ . \tag{2.42c}$$

It must be remembered that the number and location of these conventional shocks $x = s_i(t)$ are part of the unknowns in the problem. After the introduction of these shock curves, any point $(x,t) \in \mathcal{R}$ should lie on the intersection of unique C_- and C_+ characteristic curves. Moreover, as the shocks were introduced to eliminate multiple characteristics C_- through certain points in $\mathcal{R}$, we expect that the shocks themselves should not give rise to additional characteristics C_-. In other words all curves C_- should intersect either $t = 0$ or $x = 0$. Indeed the requirement that all curves C_- should originate on initial or boundary data is a special case of an admissibility—or entropy—condition formulated by Lax for more general hyperbolic systems [18].

So far we have dealt exclusively with the soft device governed by the traction boundary condition (2.11). The results for the hard device governed by the kinematic boundary condition (2.10) are also quite similar for the case where $\varepsilon(x,t) < \beta_1$. For convenience we define

$$v_0(t) = \dot{u}_0(t) \quad , \quad t \geq t_0 \quad . \tag{2.43}$$

Then smooth solutions to (2.8), (2.9) and (2.10) are given by

$$\varepsilon = 0, \ v = 0 \quad \text{for} \quad x < -c(\Phi(v_0(0)))t \quad , \tag{2.44}$$

which is the tranquil region in front of the leading disturbance. While

$$v = v_0(\tau(x,t)) \quad , \quad \varepsilon = \Phi\big(v_0(\tau(x,t))\big) \quad , \tag{2.45}$$

in the disturbed region

$$x \geq -c\big(\Phi(v_o(0))\big)t \quad . \tag{2.46}$$

Here $\tau(x,t)$ is given implicitly by

$$x = -c\big(\Phi(v_o(\tau))\big)(t-\tau) \quad . \tag{2.47}$$

The stipulation that $\varepsilon(x,t)$ is less than the value β_1 will hold provided

$$v_o(t) < \phi(\beta_1) \quad , \quad t \geq 0 \quad , \tag{2.48}$$

while (2.47) will have a unique solution in the disturbed region provided both

$$v_o(0) = 0 \quad , \tag{2.49}$$

and $c\big(\Phi(v_o(\tau))\big)$ is a continuous and monotonically decreasing function of τ. If we consider materials which are nonhardening in the first phase — so that (2.38) holds — then a condition sufficient to ensure the above monotonicity condition is that $v_o(t)$ be differentiable and

$$\dot{v}_o(t) \geq 0 \quad . \tag{2.50}$$

When (2.38), (2.49) or (2.50) are violated, one encounters difficulties similar to those of the specified-load problem. In these instances analogous remedies are invoked.

Finally, we remark that the consideration of compressive end-loadings from an initially undisturbed rest state would require the stress response to be known for negative strains. If

$$W''(\varepsilon) > 0 \quad , \quad W'''(\varepsilon) \leq 0 \quad \text{for} \quad \varepsilon < 0 \ , \tag{2.51}$$

the associated dynamical fields are smooth provided there is no unloading, in contrast to the behavior encountered in the analagous piston problem of gas dynamics [16].

3. Characteristic Theory when Two Phases are Present.

So far we have only considered loadings which permit the material to remain in the first phase throughout the bar for all time. We now consider the consequences of end conditions that lead to $\varepsilon > \beta_1$, so that solutions are not restricted to the initial hyperbolic regime of strain $0 \le \varepsilon < \beta_1$.

A. Formulation of the Two-Phase Problem for the Soft Loading Device.

We now return to the traction problem governed by (2.8), (2.9), (2.11) and (2.12). As in the situations leading to conventional shocks, phase boundaries arise even with arbitrarily smooth data. To focus attention on the specific phenomenon of phase changes, we shall assume

$$W(\varepsilon) \in C^{\infty}([0,\infty)), \quad \sigma_0(t) \in C^{\infty}([0,\infty)). \tag{3.1}$$

As in the previous section we further assume that (2.36), (2.38) and (2.40) hold. For convenience we collect these together below:

$$\begin{aligned} &\text{i)}\ \sigma_0(0) = 0, \quad \dot{\sigma}_0(t) \ge 0 \quad t \ge 0, \\ &\text{ii)}\ W'''(\varepsilon) \le 0, \quad 0 \le \varepsilon < \beta_1 . \end{aligned} \tag{3.2}$$

We have already shown that if, in addition, (2.27) holds, then there is a classically smooth solution (2.30) with $\varepsilon < \beta_1$ throughout the bar for all time. Thus we only need consider the case where $\sigma_0(t)$ attains or exceeds the value $W'(\beta_1) = \sigma_\beta$. Let t_β be the time for which (2.27) is first violated so that $t = t_\beta$ is the minimum value of t satisfying

$$\sigma_0(t) = \sigma_\beta . \tag{3.3}$$

Now (2.15) still guarantees the existence of a unique function $\varepsilon_0(t)$ defined on the interval $0 < t < t_\beta$ such that

$$W'(\varepsilon_0(t)) = \sigma_0(t) \ , \tag{3.4}$$

and

$$0 \le \varepsilon_0(t) < \beta_1 \ . \tag{3.5}$$

If we examine the previous solution (2.30) with $\varepsilon_0(t)$ defined through (3.4) and (3.5), we find from (2.18), (2.15) and (3.3) that $c\Big(\varepsilon_0(\tau(x,t))\Big)$ approaches zero as $\tau(x,t)$ approaches the value t_β. As a result, the rays C_- in the $x-t$ plane become infinitely steep as their t-intercept approaches t_β. Moreover the set of rays C_- form a nonintersecting cover of $\mathcal{R}\setminus\{(0,t)\,|\,t \ge t_\beta\}$; see Fig. 5. Thus if we append the restriction $\tau < t_\beta$ to (2.30), we obtain a smooth solution everywhere except for points (x,t) with $x = 0$ and $t \ge t_\beta$.

It is natural to ask whether the set of such points with $x = 0$ and $t \ge t_\beta$ could form some sort of stationary "end-shock". For this to be the case $\dot{s}(t) = 0$, so that $(2.12)_2$ and (2.30) would now imply

$$W'(\varepsilon(0,t)) = W'\Big(\varepsilon_0(\tau(0^-,t))\Big) = \sigma_\beta \ , \quad t \ge t_\beta \ . \tag{3.6}$$

This, with (2.11), necessitates that (3.3) must hold for all times t greater than the value t_β. One also sees that if $\sigma_0(t)$ is maintained at the value σ_β for some interval, say $t_\beta \le t \le t_K$, then we may extend (2.30) <u>smoothly</u> onto $x = 0$ and $t_\beta \le t \le t_K$ by $\varepsilon(0,t) = \beta_1$ and $v(0,t) = \phi(\beta_1)$ for $t_\beta \le t \le t_K$. We are, however, faced with the problem of not being able to construct a solution by this method for the set of points $x = 0$ and $t > t_K$. Consequently such stationary end-shocks are precluded if $\sigma_0(t)$ exceeds the value σ_β. For simplicity we shall suppose that $t = t_\beta$ is the only root of (3.3). Then it follows from (2.11), (2.15) and $(3.2)_1$ that

$$\varepsilon(0,t) > \beta_2 \ , \qquad t > t_\beta \ . \tag{3.7}$$

Now (3.4) and (3.5) ensure that $\varepsilon(0,t) < \beta_1$ is consistent with the loading condition (2.11) for $t < t_\beta$. The relations (2.15), (2.17) and (3.3) show that $\varepsilon(0,t_\beta)$ may take on either the value β_1 or the value β_2. Hence it is natural to seek solutions where $\varepsilon(0,t)$ jumps from β_1 to β_2 at $t = t_\beta$. Once one concludes that the induced end-strain <u>must</u> be discontinuous, it is clear that a jump from the first to the second ascending branches of the stress-strain curve could occur at the end of the bar for any t in the interval $[t_\alpha, t_\beta]$, where (see Fig. 1),

$$\sigma_0(t_\alpha) = \sigma_\alpha < \sigma_\beta = \sigma_0(t_\beta) \ . \tag{3.8}$$

In the face of this nonuniqueness, we opt to seek solutions for which the end-strain lies below the first ascending branch for as long as feasible, as there is no mathematical need to introduce a second phase until the loss of hyperbolicity is imminent at $t = t_\beta$. Consequently we shall define a <u>particular inverse</u> for $W'(\varepsilon)$, denoted by $T(\cdot)$, such that

$$\left.\begin{aligned} &T:[0,\infty) \to [0,\beta_1] \cup (\beta_2,\infty) \,, \\ &T(W'(\varepsilon)) = \varepsilon \quad \text{for} \quad 0 \le \varepsilon \le \beta_1 \,, \quad \varepsilon > \beta_2 \,. \end{aligned}\right\} \qquad (3.9)$$

By (3.1), T is differentiable to all orders except at σ_β where T is discontinuous (Fig. 6). With this choice of inverse, the induced end-strain is given by

$$\varepsilon_0(t) \equiv T(\sigma_0(t)) \,, \quad t \ge 0 \,. \qquad (3.10)$$

We note that $\varepsilon_0(t)$ is discontinuous at $t = t_\beta$. For all other $t \ge 0$, $\varepsilon_0(t)$ is differentiable to all orders, with the first derivative again given by (2.32). Moreover, since $W''(\beta_1) = 0$ and $W''(\beta_2) > 0$ it follows that the right hand limit of all derivatives of $\varepsilon_0(t)$ exist at $t = t_\beta$, whereas these derivatives become unbounded as t approaches t_β from the left.

Since a stationary end-shock is incompatible with an end-load greater than the value σ_β, it follows that the high-strain phase will not be confined to the end of the bar, but that a phase boundary will emerge at the end of the bar at time t_β and subsequently travel into the interior. In front of this (unknown) phase boundary, which we denote by $x = s(t)$, the strain and velocity fields are still given by (2.30), while for $s(t) < x \le 0$ we seek solutions with $\varepsilon > \beta_2$.

Note that for our particular choice of inverse to the stress response, the phase boundary is <u>not</u> genearated <u>at</u> the Maxwell stress σ_γ, which in the equilibrium case is the stress where the configuration loses absolute stability. Instead the phase boundary emerges when the end-load reaches σ_β, which in the equilibrium setting is the stress at which the configuration loses metastability. Consequently we are assuming a type of "superstraining" analogous to the superheating a liquid undergoes when heated above its boiling point. This would seem reasonable for a smoothly applied end-loading; in this case we regard the phase-boundary as emerging in response to the incipient loss of hyperbolicity.

We now extend the definitions (2.18) and (2.19) to the high-strain phase by writing

$$c(\varepsilon) = \sqrt{\frac{W''(\varepsilon)}{\rho}} \quad , \quad 0 \le \varepsilon \le \beta_1 \ , \quad \varepsilon \ge \alpha_2 \ , \tag{3.11}$$

and

$$\phi(\varepsilon) = \begin{cases} \int_0^{\varepsilon} c(s)ds \ , & 0 \le \varepsilon \le \beta_1 \ , \\ \phi(\beta_1) + \int_{\alpha_2}^{\varepsilon} c(s)ds \ , & \varepsilon \ge \alpha_2 \ . \end{cases} \tag{3.12}$$

Once again $\phi(\varepsilon)$ is monotonic on its domain of definition, so that we can extend the previously defined inverse Φ to the range of ϕ as given by (3.12). $\Phi(z)$ is continuous on its domain $0 \le z < \lim_{\varepsilon \to \infty} \phi(\varepsilon)$, and by (3.1) is differentiable to all orders except at $z = \phi(\beta_1)$ $(= \phi(\alpha_2))$. At $z = \phi(\beta_1)$, the one-sided derivatives of $\Phi(z)$ exist from the right but are unbounded from the left. $\Phi'(z)$ and the higher derivatives may still be found from (2.20)

Denote by $\mathcal{R}_1$ and $\mathcal{R}_2$ the regions in the low- and high-strain phases respectively, so that

$$\left.\begin{aligned} \mathcal{R}_1 &= \{(x,t)\,|\,(x \le 0,\ t \le t_\beta) \cup (x < s(t),\ t > t_\beta)\}\ , \\ \mathcal{R}_2 &= \{(x,t)\,|\,s(t) < x \le 0\ ,\ t > t_\beta\}\ . \end{aligned}\right\} \quad (3.13)$$

We shall refer to $\mathcal{R}_1$ as the region in front of the phase boundary and $\mathcal{R}_2$ as the region behind the phase boundary.

In $\mathcal{R}_1$, the dynamical fields are still given by (2.30) with the restriction that $\tau < t_\beta$. In crossing the phase boundary, we require that the shock conditions (2.12) be satisfied. In $\mathcal{R}_2$, the extension of c and ϕ into the second hyperbolic regime of strain allows us to adopt a scheme like that discussed in the preceding section for finding solutions behind conventional shocks.

In $\mathcal{R}_2$ we shall parametrize the positively-sloped characteristic curves by the time at which they intersect $x = 0$. This is accomplished by letting $\tau^+(x,t)$ be the time at which the positively-sloped characteristic curve passing through the point (x,t) in $\mathcal{R}_2$ terminates on $x = 0$. Notice that from this definition follows $\tau^+(0,t) = t$. Similarly let $\tau^-(x,t)$ parametrize the negatively-sloped characteristic curve in $\mathcal{R}_2$ which originates on $x = 0$ at $t = \tau$. Under the extended definition of $c(\varepsilon)$, (2.21) continues to hold in $\mathcal{R}_2$. As before, these equations may be integrated to yield the Riemann invariants K_2^+ and K_2^-, where

$$v(x,t) - \phi(\varepsilon(x,t)) = K_2^+(\tau^+(x,t)), \qquad (3.14a)$$

on curves parametrized by $\tau^+(x,t)$, for which

$$\frac{dx}{dt} = c(\varepsilon(x,t))\ ,\quad \tau^+(0,t) = t\ . \qquad (3.14b)$$

Similarly,

$$v(x,t) + \phi(\varepsilon(x,t)) = K_2^-(\tau^-(x,t)) \ , \tag{3.14c}$$

on curves parametrized by $\tau^-(x,t)$, for which

$$\frac{dx}{dt} = -c(\varepsilon(x,t)) \quad , \quad \tau^-(0,t) = t \ . \tag{3.14d}$$

Here

$$\tau^+ : \mathcal{R}_2 \to [t_\beta, \infty) \ , \quad \tau^- : \mathcal{R}_2 \to [t_\beta, \infty) \ , \tag{3.14e}$$

$$K_2^+ : [t_\beta, \infty) \to \mathbb{R} \ , \quad K_2^- : [t_\beta, \infty) \to \mathbb{R} \ , \tag{3.14f}$$

constitute unknowns to be determined. There is a final unknown, namely the phase boundary location $s(t)$. We have seen that

$$s(t) \leq 0 \quad \text{for} \quad t \geq t_\beta \ , \qquad s(t_\beta) = 0 \ . \tag{3.15}$$

We shall seek $s(t) \in C^1([t_\beta, \infty)) \cap C^\infty((t_\beta, \infty))$. The shock conditions (2.12) become

$$[\varepsilon(s(t),t)]\dot{s}(t) + [v(s(t),t)] = 0, \ [v(s(t),t)]\dot{s}(t) + \frac{1}{\rho}[W'(\varepsilon(s(t),t))] = 0$$

$$\text{for} \quad t \geq t_\beta \ . \tag{3.16}$$

Here $\varepsilon(s(t)^-,t)$ and $v(s(t)^-,t)$ are given by (2.30) with $\tau(s(t),t) < t_\beta$. We must also satisfy the boundary condition

$$\varepsilon(0,t) = \varepsilon_0(t) \quad , \quad t \geq t_\beta \ . \tag{3.17}$$

For (x,t) in $\mathcal{R}_2$, (3.14) shows that the velocity and strain are given by

$$v(x,t) = \frac{1}{2}K_2^-(\tau^-(x,t)) + \frac{1}{2}K_2^+(\tau^+(x,t)),$$
$$\varepsilon(x,t) = \Phi\left(\frac{1}{2}K_2^-(\tau^-(x,t)) - \frac{1}{2}K_2^+(\tau^+(x,t))\right) \quad . \tag{3.18}$$

We may use (3.14) and (3.17) to eliminate K_2^- as follows:

$$2\phi(\varepsilon_0(t)) = 2\phi(\varepsilon(0,t)) = K_2^-(\tau^-(0,t)) - K_2^+(\tau^+(0,t)) = K_2^-(t) - K_2^+(t), \tag{3.19}$$

so that

$$K_2^-(t) = K_2^+(t) + 2\phi(\varepsilon_0(t)) \quad . \tag{3.20}$$

Thus (3.18) becomes

$$v(x,t) = \frac{1}{2}K_2^+(\tau^+(x,t)) + \frac{1}{2}K_2^+(\tau^-(x,t)) + \phi(\varepsilon_0(\tau^-(x,t))),$$
$$\varepsilon(x,t) = \Phi\left(-\frac{1}{2} K_2^+(\tau^+(x,t)) + \frac{1}{2} K_2^+(\tau^-(x,t)) + \phi(\varepsilon_0(\tau^-(x,t)))\right). \tag{3.21}$$

Collecting our results so far, we may summarize the problem as follows:

When $W(\varepsilon)$ and $\sigma_0(t)$ are subject to (2.14), (2.15), (2.16), (2.17), (3.1), (3.2), and with $\varepsilon_0(t)$, $c(\varepsilon)$, $\phi(\varepsilon)$, $\Phi(z)$ defined through (3.10), (3.11), (3.12), we seek four functions $K_2^+(z)$, $\tau^+(x,t)$, $\tau^-(x,t)$ and $s(t)$, such that:

(i) $s(t) \in C^1([t_\beta,\infty)) \cap C^\infty((t_\beta,\infty))$, $s(t) \le 0$, $s(t_\beta) = 0$ and $s(t)$ partitions $\mathcal{R}$ into $\mathcal{R}_1$ and $\mathcal{R}_2$ as defined by (3.13);

(ii) In $\mathcal{R}_1$, $\varepsilon(x,t)$ and $v(x,t)$ are given by (2.30) with the restriction $\tau < t_\beta$;

(iii) In $\mathcal{R}_2$, $\varepsilon(x,t)$ and $v(x,t)$ are given by (3.21);

(iv) $\tau^+ : \mathbb{R}_2 \to [t_\beta, \infty)$, $\tau^+(0,t) = t$ and τ^+ is constant on curves for which $\frac{dx}{dt} = c(\varepsilon(x,t))$;

(v) $\tau^- : \mathbb{R}_2 \to [t_\beta, \infty)$, $\tau^-(0,t) = t$ and τ^- is constant on curves for which $\frac{dx}{dt} = -c(\varepsilon(x,t))$;

(vi) The jump conditions (3.16) are satisfied.

(3.22)

Solutions to (3.22) give a single phase boundary preceding a region of high-strain material phase and propagating into a region of low-strain material phase. In both of these regions the strain and velocity fields are classically smooth solutions to (2.8). It should be emphasized that the existence of such a smooth solution behind the phase boundary is by no means guaranteed. In this regard recall that the material condition $W'''(\varepsilon) \leqslant 0$ for $0 \leqslant \varepsilon \leqslant \beta_1$, is necessary for the existence of smooth fields in front of the phase boundary, whereas we have not as yet imposed a comparable conditon on $W'''(\varepsilon)$ for $\varepsilon \geq \alpha_2$.

B. Statement of the Two-Phase Problem for the Hard Loading Device.

Before investigating (3.22) further we shall give a statement of the corresponding free boundary problem for the hard device governed by (2.10). Here we again define $c(\varepsilon)$, $\phi(\varepsilon)$ and $\Phi(z)$ by (3.11) and (3.12). If we consider a non-impulsive, smooth loading of a bar of material that is non-hardening in the first phase, we are assured that (2.38), (2.49) and (2.50) hold. It is then possible for the bar to remain everywhere in the first phase until time t_β , where $t = t_\beta$ is the largest root of

$$v_o(t) = \phi(\beta_1) \ . \tag{3.23}$$

We shall only consider the case in which (3.23) has a unique root, rather than an interval of roots. For $t > t_\beta$ near the end $x = 0$ it is necessary that the material be in the high-strain phase. If $x = s(t)$ denotes the resulting propagating phase boundary, we seek three additional functions $K_2^+(z)$, $\tau^+(x,t)$, and $\tau^-(x,t)$ such that:

(i) $s(t) \in C^1([t_\beta,\infty)) \cap C^\infty((t_\beta,\infty))$, $s(t) \le 0$, $s(t_\beta) = 0$ and $s(t)$ partitions $\mathcal{R}$ into $\mathcal{R}_1$ and $\mathcal{R}_2$ as defined by (3.13);

(ii) In $\mathcal{R}_1$, $\varepsilon(x,t)$ and $v(x,t)$ are given by (2.44), (2.45), (2.46) and (2.47) with the restriction $\tau < t_\beta$;

(iii) In $\mathcal{R}_2$, $\varepsilon(x,t)$ and $v(x,t)$ are given by

$$v(x,t) = \frac{1}{2}K_2^+(\tau^+(x,t)) - \frac{1}{2}K_2^+(\tau^-(x,t)) + v_0(\tau^-(x,t)),$$

$$\varepsilon(x,t) = \Phi\left(-\frac{1}{2}K_2^+(\tau^+(x,t)) - \frac{1}{2}K_2^+(\tau^-(x,t)) + v_0(\tau^-(x,t))\right);$$

(iv) $\tau^+ : \mathcal{R}_2 \to [t_\beta,\infty)$, $\tau^+(0,t) = t$ and τ^+ is constant on curves for which $\frac{dx}{dt} = c(\varepsilon(x,t))$;

(v) $\tau^- : \mathcal{R}_2 \to [t_\beta,\infty)$, $\tau^-(0,t) = t$ and τ^- is constant on curves for which $\frac{dx}{dt} = -c(\varepsilon(x,t))$;

(vi) The jump conditions (3.16) are satisfied.

(3.24)

4. A Special Class of Materials.

The application of characteristic theory leading to the formulation (3.22) requires that a unique member of both families of characteristic curves should pass through each point (x,t) in $\mathcal{R}$. For portions of the bar in the low-strain phase, we were able to use the initial conditions (2.9) to show that the Riemann invariant associated with the positively-sloped family of characteristic curves vanishes identically (see the discussion preceding (2.23)). However, it is necessary to stipulate that $W'''(\varepsilon) \le 0$ for $0 \le \varepsilon < \beta_1$ in order to guarantee that the negatively-sloped family of characteristic curves (here rays) forms an unambiguous cover of $\mathcal{R}_1$. We now inquire as to what requirement (if any) on $W(\varepsilon)$ and its derivatives in $\varepsilon > \alpha_2$ will guarantee that each of the two families of characteristic curves will unambiguously cover $\mathcal{R}_2$. Should these covering properties fail, τ^+ or τ^- would either be undefined or multi-valued at certain locations in $\mathcal{R}_2$, and the problem governed by (3.22) would not be well-posed. For example, multi-valuedness of τ^+ or τ^- would suggest the presence of conventional shocks in the high-strain phase portion of the bar. Unfortunately we do not know of a-priori conditions that are both necessary and sufficient for $\tau^+(x,t)$ and $\tau^-(x,t)$ to be both defined and single-valued for all $(x,t) \in \mathcal{R}_2$. However, in what follows we give a condition on the material sufficient to ensure the existence and single-valuedness of τ^+ and τ^- throughtout $\mathcal{R}_2$.

We shall henceforth consider only materials with the property that there exists a strain δ, $\alpha_2 < \delta < \beta_2$, such that

$$W'''(\varepsilon) = 0 \quad \text{for} \quad \varepsilon \ge \delta \, . \tag{4.1}$$

Thus there are constants $E > 0$ and D such that

$$W'(\varepsilon) = E\varepsilon + D \quad \text{for} \quad \varepsilon \geq \delta \quad . \tag{4.2}$$

In other words, we shall deal only with energy densities for which the stress is a linear function of strain for values of strain greater than δ, where δ lies below the second ascending branch, but is less than β_2. A diagram of such a stress-strain curve satisfying (2.14), (2.15), (2.16), (2.17), (2.38), (4.1) and (4.2) is found in Fig. 7.

A. Reduction of the Problem to a Pair of Differential-Delay Equations.

From (4.2), (3.11) and(3.12) it then follows that for such a material

$$c(\varepsilon) = \sqrt{\frac{E}{\rho}} \quad , \qquad \varepsilon \geq \delta \ , \tag{4.3}$$

$$\phi(\varepsilon) = \phi(\delta) + \sqrt{\frac{E}{\rho}}\,(\varepsilon - \delta), \quad \varepsilon \geq \delta \ , \tag{4.4}$$

and

$$\Phi(z) = \sqrt{\frac{\rho}{E}}\, z - \sqrt{\frac{\rho}{E}}\, \phi(\delta) + \delta, \quad z \geq \phi(\delta) \ . \tag{4.5}$$

Turning our attention to the hard-device problem in the form (3.22), we have by (4.2), (3.9) and (3.10) that

$$\varepsilon_0(t) = \frac{1}{E}(\sigma_0(t) - D) > \beta_2 \quad \text{for} \quad t > t_\beta \ . \tag{4.6}$$

We seek solutions which are in the linear range corresponding to the high-strain phase, i.e. $\varepsilon > \delta$. The constant acoustic speed (4.3) in $\mathcal{R}_2$ then allow us to integrate $(3.22)_{iv}$ and $(3.22)_v$ and solve for $\tau^+(x,t)$ and $\tau^-(x,t)$. The result is

$$\left.\begin{aligned} \tau^{+}(x,t) &= t - \sqrt{\frac{\rho}{E}}\,x \quad , \\ \tau^{-}(x,t) &= t + \sqrt{\frac{\rho}{E}}\,x \quad . \end{aligned}\right\} \tag{4.7}$$

Thus for $(x,t) \in \Re_2$, (3.21), (4.4), (4.5) and (4.7) yield

$$\left.\begin{aligned} v(x,t) &= \frac{1}{2}K_2^{+}(t-\sqrt{\frac{\rho}{E}}x) + \frac{1}{2}K_2^{+}(t+\sqrt{\frac{\rho}{E}}x) + \sqrt{\frac{E}{\rho}}\varepsilon_0(t+\sqrt{\frac{\rho}{E}}x) - \sqrt{\frac{E}{\rho}}\delta + \phi(\delta), \\ \varepsilon(x,t) &= -\frac{1}{2}\sqrt{\frac{\rho}{E}}K_2^{+}(t-\sqrt{\frac{\rho}{E}}x) + \frac{1}{2}K_2^{+}(t+\sqrt{\frac{\rho}{E}}x) + \varepsilon_0(t+\sqrt{\frac{\rho}{E}}x). \end{aligned}\right\} \tag{4.8}$$

It is convenient to define the new function f in terms of K_2^{+} by

$$f(z) = \frac{1}{2}\sqrt{\frac{\rho}{E}}K_2^{+}(-\sqrt{\frac{\rho}{E}}z) - \frac{1}{2}\delta + \frac{1}{2}\sqrt{\frac{\rho}{E}}\phi(\delta), \tag{4.9}$$

whereupon (4.8) becomes

$$\begin{aligned} v(x,t) &= \sqrt{\frac{E}{\rho}}f(x-\sqrt{\frac{E}{\rho}}t) + \sqrt{\frac{E}{\rho}}f(-x-\sqrt{\frac{E}{\rho}}t) + \sqrt{\frac{E}{\rho}}\varepsilon_0(t+\sqrt{\frac{\rho}{E}}x), \\ \varepsilon(x,t) &= -f(x-\sqrt{\frac{E}{\rho}}t) + f(-x-\sqrt{\frac{E}{\rho}}t) + \varepsilon_0(t+\sqrt{\frac{\rho}{E}}x) . \end{aligned} \tag{4.10}$$

That $\varepsilon(x,t)$ and $v(x,t)$ are each linear combinations of functions of the variables $x + \sqrt{\frac{E}{\rho}}t$ and $x - \sqrt{\frac{E}{\rho}}t$ is a consequence of the fact that, for the linear portion of the stress-strain relation, (2.8) becomes

$$\frac{\partial^2 u}{\partial t^2} - \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2} = 0 . \tag{4.11}$$

Differentiating (4.11) one shows that ε and v also satisfy (4.11). Consequently we have simply retrieved a form of D'Alemberts solution to the classical wave equation. Note also that in $\Re_2$ <u>both</u> families of characteristic curves are straight rays with the respective slopes $\sqrt{\frac{E}{\rho}}$ and $-\sqrt{\frac{E}{\rho}}$. Here, however, neither family of characteristic rays propagate constant values of ε or v.

Before we describe how the results of this analysis simplify the formulation (3.22), we make a few observations. From the jump conditions (3.16) we see that the square of the phase boundary velocity is given by

$$\dot{s}^2(t) = \frac{1}{\rho}\frac{[W'(\varepsilon(s(t),t))]}{[\varepsilon(s(t),t)]} \quad , \tag{4.12}$$

and so for a normalized density ($\rho = 1$) is given by the <u>slope of the secant line</u> to the stress-strain curve between $\varepsilon(s(t)^+,t)$ and $\varepsilon(s(t)^-,t)$. On the other hand, by (3.11), the square of the acoustic speed at any location is

$$c^2(\varepsilon(x,t)) = \frac{1}{\rho} W''(\varepsilon(x,t)), \tag{4.13}$$

and so, for a normalized density, $c^2(\varepsilon)$ is given by the <u>slope</u> of the stress-strain curve at $\varepsilon(x,t)$.

In the problem we are considering, the stress-strain curve for a strain associated with $\mathcal{R}_2$ has slope E, which is necessarily greater than the slope of a secant line projected back to a point on the first branch when $\varepsilon(s(t)^+,t)$ is near β_2 and $\varepsilon(s(t)^-,t)$ is near β_1. Hence (4.12) and (4.13) imply that for some $\Delta > 0$,

$$|\dot{s}(t)| < c(\varepsilon(s(t)^+,t) = \sqrt{\frac{E}{\rho}}\,, \qquad t_\beta \le t \le t_\beta + \Delta \quad . \tag{4.14}$$

For the above time interval the phase boundary is subsonic with respect to the material behind it. In this investigation we shall restrict attention to phase boundaries for which this is true for all time. Hence we impose the condition

$$|\dot{s}(t)| < c(\varepsilon(s(t)^{+},t) = \sqrt{\frac{E}{\rho}}\,, \quad t \geq t_{\beta} \quad , \tag{4.15}$$

an important consequence of this restriction being the lower bound

$$-\sqrt{\frac{E}{\rho}} < \dot{s}(t) \quad , \quad t \geq t_{\beta} \; . \tag{4.16}$$

Integrating (4.16) and using the conditions (3.15), we arrive at

$$s(t) - \sqrt{\frac{E}{\rho}}\,t \leq -s(t) - \sqrt{\frac{E}{\rho}}\,t \leq -\sqrt{\frac{E}{\rho}}\,t_{\beta} < 0 \quad . \tag{4.17}$$

By $(3.13)_2$ we further have

$$s(t) - \sqrt{\frac{E}{\rho}}\,t \leq x - \sqrt{\frac{E}{\rho}}\,t \leq -x - \sqrt{\frac{E}{\rho}}\,t \leq -s(t) - \sqrt{\frac{E}{\rho}}\,t \leq -\sqrt{\frac{E}{\rho}}\,t_{\beta} \quad \text{for} \quad (x,t) \in \mathcal{R}_2 \; . \tag{4.18}$$

Hence $f(z)$ need only be defined on $z \leq -\sqrt{\frac{E}{\rho}}\,t_{\beta}$, which we shall henceforth take to be the domain of definiton for f. This could have been predicted from (4.9) and (3.14f). Notice also that the bound (4.16) assures that, for increasing t, $s(t)$ intersects members of the negatively-sloped family of characteristic rays in $\mathcal{R}_2$ which arise progressively later on $x = 0$, $t \geq t_{\beta}$.

We can incorporate the initial conditions (2.9) into the resulting end-strain by setting $\varepsilon_0(t)$ equal to zero for $t < 0$. Then by (3.10) and (4.6)

$$\varepsilon_0(t) = \begin{cases} 0 & t \leq 0 \quad , \\ T(\sigma_0(t)) & 0 \leq t \leq t_{\beta}, \\ \frac{1}{E}\,(\sigma_0(t) - D) & t > t_{\beta} \quad . \end{cases} \tag{4.19}$$

Note that $\varepsilon_o(t)$ is continuous everywhere except at $t = t_\beta$ and continuously differentiable to all orders except at $t = 0$ and $t = t_\beta$. We also define $\sigma_o(t) = 0$ for $t < 0$ so that (3.10) holds for all time. Finally we define $\tau(x,t) = 0$ for $(x,t) \in \mathcal{R}_T$. These extensions allow us to use the same formulation in both the disturbed and undisturbed regions of the bar.

By means of (4.10) the problem formulated in (3.22) is reduced to finding functions s(t) and f(z) for which

(i) $s(t) \in C^1([t_\beta,\infty)) \cap C^\infty((t_\beta,\infty))$, $\dot{s}(t) \leq 0$ and

$$s(t_\beta,) = 0; \tag{4.20a}$$

(ii) $$f(z): (-\infty, -\sqrt{\tfrac{E}{\rho}}t_\beta] \to \mathbb{R}\ ; \tag{4.20b}$$

(iii) $$[\varepsilon(s(t),t)]\dot{s}(t) + [v(s(t),t)] = 0 \quad \text{for} \quad t \geq t_\beta, \tag{4.20c}$$

and

$$[v(s(t),t)]\dot{s}(t) + \frac{1}{\rho}[W'(\varepsilon(s(t),t))] = 0 \text{ for } t \geq t_\beta, \tag{4.20d}$$

where

$$[v] \equiv [v(s(t),t)] =$$

$$\sqrt{\frac{E}{\rho}}f(s(t) - \sqrt{\frac{E}{\rho}}t) + \sqrt{\frac{E}{\rho}}f(-s(t) - \sqrt{\frac{E}{\rho}}t) + \sqrt{\frac{E}{\rho}}\,\varepsilon_o(t + \sqrt{\frac{\rho}{E}}s(t)) - \phi(\varepsilon_o(\tau)), \tag{4.20e}$$

$$[\varepsilon] \equiv [\varepsilon(s(t),t)] =$$

$$-f(s(t) - \sqrt{\frac{E}{\rho}}t) + f(-s(t) - \sqrt{\frac{E}{\rho}}t) + \varepsilon_o(t + \sqrt{\frac{\rho}{E}}s(t)) - \varepsilon_o(\tau), \tag{4.20f}$$

$$[W'] \equiv [W'(\varepsilon(s(t),t))] =$$

$$-Ef(s(t)-\sqrt{\frac{E}{\rho}}t)+Ef(-s(t)-\sqrt{\frac{E}{\rho}}t)+E\varepsilon_0(t+\sqrt{\frac{\rho}{E}}s(t))+D-W'(\varepsilon_0(\tau)), \quad (4.20g)$$

and $\tau=\tau(s(t),t)$ is the unique root less than t_β of

$$s(t) = -c(\varepsilon_0(\tau))(t-\tau) \quad . \qquad (4.20h)$$

In summary, for an induced end-strain given by (3.10) the problem formulated in (4.20) completely characterizes the phase boundary and dynamical fields associated with the smooth monotonic loading (in a soft device) of a bar of material that is non-hardening in the first phase and satisfies (4.2). The system (4.20) consists of two differential-delay equations for the unknown functions $f(z)$ and $s(t)$. Notice that $s(t)$ appears in the retarded argument of f so that the amount of delay constitutes part of the solution.

It will be convenient to consider the following linear combinations of (4.20c) and (4.20d):

$$\sqrt{\frac{E}{\rho}}\dot{s}\,[\varepsilon]+(\sqrt{\frac{E}{\rho}}+\dot{s})[v]+\frac{1}{\rho}[W']=0 \quad , \qquad (4.21a)$$

$$\sqrt{\frac{E}{\rho}}\,\dot{s}[\varepsilon]+(\sqrt{\frac{E}{\rho}}-\dot{s})[v]-\frac{1}{\rho}[W']=0 \quad . \qquad (4.21b)$$

Upon substitution from (4.20e,f,g) into (4.21), we arrive at

$$2\sqrt{\frac{E}{\rho}}(\dot{s}(t)+\sqrt{\frac{E}{\rho}})f(-s(t)-\sqrt{\frac{E}{\rho}}t)+\Gamma_1(\dot{s}(t),s(t),t)=0\ , \qquad (4.22a)$$

and

$$2\sqrt{\frac{E}{\rho}}(-\dot{s}(t)+\sqrt{\frac{E}{\rho}})f(s(t)-\sqrt{\frac{E}{\rho}}t)+\Gamma_2(\dot{s}(t),s(t),t)=0 \qquad t\geq t_\beta \tag{4.22b}$$

where

$$\Gamma_1(\dot{s},s,t)=\frac{2}{\sqrt{\rho E}}\ (\dot{s}+\sqrt{\frac{E}{\rho}})\sigma_0(t+\sqrt{\frac{\rho}{E}}s)-\frac{2D}{\sqrt{\rho E}}(\dot{s}+\frac{1}{2}\sqrt{\frac{E}{\rho}})$$

$$-\sqrt{\frac{E}{\rho}}\dot{s}\varepsilon_0(\tau(s,t))-(\dot{s}+\sqrt{\frac{E}{\rho}})\phi(\varepsilon_0(\tau(s,t)))-\frac{1}{\rho}\sigma_0(\tau(s,t)) \quad , \tag{4.23a}$$

and

$$\Gamma_2(\dot{s},s,t)=\frac{-D}{\rho}-\sqrt{\frac{E}{\rho}}\ \dot{s}\varepsilon_0(\tau(s,t))+(\dot{s}-\sqrt{\frac{E}{\rho}})\phi(\varepsilon_0(\tau(s,t)))$$

$$+\frac{1}{\rho}\ \sigma_0(\tau(s,t)). \tag{4.23b}$$

We shall use (4.22) and (4.23) in the forthcoming asymptotic analysis.

B. An Alternative Formulation Involving an Integral Equation.

It is also possible through (4.22) and (4.23) to eliminate f by recasting the problem as an integral equation for $s(t)$.

Define

$$Y(t) = \int_{t_\beta}^{t} f(-\sqrt{\frac{E}{\rho}}\ s)ds \quad \text{for} \quad t\geq t_\beta \quad . \tag{4.24}$$

Then (4.22a) can be written as

$$2\frac{E}{\rho}\frac{d}{dt}Y(\sqrt{\frac{\rho}{E}}\ s(t)+t)+\Gamma_1(\dot{s}(t),s(t),t) = 0 \quad \text{for} \quad t\geq t_\beta \ . \tag{4.25}$$

Integrating (4.25), using $Y(t_\beta)=0$ and $s(t_\beta)=0$, we find

$$2\frac{E}{\rho}\gamma(\sqrt{\frac{\rho}{E}}s(t)+t)+\int_{t_\beta}^{t}\Gamma_1(\dot{s}(s),s(s),s)ds=0 \quad \text{for} \quad t\geq t_\beta \quad . \tag{4.26}$$

Similarly (4.22b) leads to

$$2\frac{E}{\rho}\gamma(-\sqrt{\frac{\rho}{E}}s(t)+t)+\int_{t_\beta}^{t}\Gamma_2(\dot{s}(s),s(s),s)ds=0 \quad \text{for} \quad t\geq t_\beta \ . \tag{4.27}$$

By (4.15), both $-\sqrt{\frac{\rho}{E}}s(t)+t$ and $\sqrt{\frac{\rho}{E}}s(t)+t$ are continuous and monotonically increasing. Thus for any $t\geq t_\beta$, there exist times t_a and t_b such that

$$t=\sqrt{\frac{\rho}{E}}s(t_a)+t_a \quad \text{and} \quad t=-\sqrt{\frac{\rho}{E}}s(t_b)+t_b \ . \tag{4.28}$$

Note that

$$t_a\geq t\geq t_b\geq t_\beta^- \ . \tag{4.29}$$

Evaluating (4.26) at t_a and (4.27) at t_b we may eliminate $\gamma(t)$ to obtain

$$\int_{t_\beta}^{t_a}\Gamma_1(\dot{s}(s),s(s),s)ds-\int_{t\beta}^{t_b}\Gamma_2(\dot{s}(s),s(s),s)ds=0 \ . \tag{4.30}$$

To simplify (4.30) we note that by (4.28) and (4.23a), the first term in the integral on the left is

$$\frac{2}{\sqrt{\rho E}}\int_{t_\beta}^{t_a}(\dot{s}(s)+\sqrt{\frac{E}{\rho}})\sigma_o(s+\sqrt{\frac{\rho}{E}}s(s))ds=\frac{2}{\rho}\int_{t_\beta}^{t}\sigma_o(s)ds \ . \tag{4.31}$$

With the help of (4.23), (4.29) and (4.31), (4.30) yields the following equation for $s(t)$:

$$\frac{1}{\sqrt{\rho E}}\int_{t_\beta}^{t}\sigma_o(s)ds=\int_{t_\beta}^{t_b}\Gamma_3(\dot{s}(s),s(s),s)ds+\int_{t_b}^{t_a}\Gamma_4(\dot{s}(s),s(s),s)ds,$$

$$t\geq t_\beta. \tag{4.32a}$$

Here

$$\Gamma_3(\dot{s},s,t)=\frac{D}{E}\dot{s}+\sqrt{\frac{\rho}{E}}\dot{s}\phi(\varepsilon_o(\tau(s,t)))+\frac{1}{\sqrt{\rho E}}\sigma_o(\tau(s,t)) , \tag{4.32b}$$

$$\Gamma_4(\dot{s},s,t)=\frac{D}{E}(\dot{s}+\frac{1}{2}\sqrt{\frac{E}{\rho}})+\frac{1}{2}\dot{s}\,\varepsilon_o(\tau(s,t))+\frac{1}{2}\sqrt{\frac{\rho}{E}}(\sqrt{\frac{E}{\rho}}+\dot{s})\phi(\varepsilon_o(\tau(s,t)))$$

$$+\frac{1}{2\sqrt{\rho E}}\sigma_o(\tau(s,t)) , \tag{4.32c}$$

and $t_a=t_a(t)$ and $t_b=t_b(t)$ are the roots of

$$\sqrt{\frac{\rho}{E}}s(t_a)+t_a=t=-\sqrt{\frac{\rho}{E}}s(t_b)+t_b . \tag{4.32d}$$

Notice that we can evaluate the left-hand side of (4.32a) as well as the first terms in the integrals on the right-hand side. For an induced end-strain given by (3.10), system (4.32) also completely characterizes the phase boundary and dynamical fields associated with the smooth monotonic loading (in a soft device) of a bar of material that is non-hardening in the first phase and satisfies (4.2). Of the two formulations (4.20) and (4.32), we shall consider only (4.20) in the asymptotic analysis carried out in the next two sections. We shall, however, comment further on (4.32) in the final section.

5. Emergence of the Phase Boundary.

We now study the behavior of solutions of the system (4.20) near $t = t_\beta$, at which time the phase boundary first appears. Since $s(t_\beta) = 0$, we have

$$s(t_\beta) - \sqrt{\frac{E}{\rho}}t_\beta = - s(t_\beta) - \sqrt{\frac{E}{\rho}}\, t_\beta = -\sqrt{\frac{E}{\rho}}\, t_\beta \ , \tag{5.1}$$

so that at the initial time t_β, the two arguments of f involved in (4.20) are the same. Differential-delay equations whose arguments coalesce initially are called singular, and in general they permit the specification of only the initial values of the unknown functions; see [22]. In our case, the initial value of s is given: $s(t_\beta) = 0$. Moreover, we may let $t \to t_\beta^+$ in (4.20) to determine $f(-\sqrt{\frac{E}{\rho}}\, t_\beta)$. Recalling that $\varepsilon_o(t)$ is discontinuous at $t = t_\beta$, we have

$$\lim_{t \to t_\beta^+} \varepsilon_o(t + \sqrt{\frac{\rho}{E}} s(t)) = \varepsilon_o(t_\beta^+) = \beta_2 \ , \tag{5.2}$$

$$\lim_{t \to t_\beta^+} \varepsilon_o(\tau(s(t),t)) = \varepsilon_o(t_\beta^-) = \beta_1 \ , \tag{5.3}$$

so that (4.20e)-(4.20g), when evaluated at $t = t_\beta$, become

$$[v]_{t_\beta} = 2\sqrt{\frac{E}{\rho}} f(-\sqrt{\frac{E}{\rho}} t_\beta) + \sqrt{\frac{E}{\rho}} \beta_2 - \phi(\beta_1) \ , \tag{5.4}$$

$$[\varepsilon]_{t_\beta} = \beta_2 - \beta_1 \ , \tag{5.5}$$

$$[W']_{t_\beta} = 0 \ . \tag{5.6}$$

We may then infer from (4.20c) and (4.20d) that

$$(\beta_2 - \beta_1)\dot{s}(t_\beta) + 2\sqrt{\frac{E}{\rho}}f(-\sqrt{\frac{E}{\rho}}t_\beta) + \sqrt{\frac{E}{\rho}}\beta_2 - \phi(\beta_1) = 0 \quad , \tag{5.7}$$

$$\left(2\sqrt{\frac{E}{\rho}}f(-\sqrt{\frac{E}{\rho}}t_\beta) + \sqrt{\frac{E}{\rho}}\beta_2 - \phi(\beta_1)\right)\dot{s}(t_\beta) = 0 \quad . \tag{5.8}$$

These equations imply that

$$\dot{s}(t_\beta) = 0 \quad , \tag{5.9}$$

$$f(-\sqrt{\frac{E}{\rho}}t_\beta) = \frac{1}{2}\left(\sqrt{\frac{\rho}{E}}\,\phi(\beta_1) - \beta_2\right). \tag{5.10}$$

We observe that, from (5.4) and (5.10), $[v]_{t_\beta} = 0$, so that at time t_β the jump conditions (4.20c) and (4.20d) reduce to the corresponding conditions for the equilibrium problem as given by Ericksen [1].

Although there are local existence and uniqueness theorems for <u>nonsingular</u> differential-delay systems and for singular systems with <u>known</u> delays [22], these do not apply to (4.20). We shall therefore take for granted the existence of a solution to our system. Moreover, we shall assume that, near $t = t_\beta$, the unknowns s and f are asymptotically characterized by

$$s(t) \sim \sum_{k=0}^{\infty} s_k (t - t_\beta)^{m_k} \quad , \tag{5.11}$$

$$f(z) \sim f_0 + \sum_{k=1}^{\infty} f_k(-z - \sqrt{\frac{E}{\rho}}\,t_\beta)^{n_k} \quad , \tag{5.12}$$

where the constants s_k, m_k f_k and n_k, which are to be determined, must be such that

$$m_0 > 1, \quad n_1 > 0, \quad s_0 < 0, \quad f_0 = \frac{1}{2}\left(\sqrt{\frac{\rho}{E}}\,\phi(\beta_1) - \beta_2\right), \qquad (5.13)^1$$

$$m_j > m_k, \quad n_j > n_k \quad \text{for} \quad j > k \ . \qquad (5.14)$$

We shall determine m_0, n_1, s_0 and f_1 explicitly, so that the dominant terms for $t \to t_\beta$ in (5.11) and (5.12) will be known.

For simplicity, we restrict attention to loadings for which

$$\dot{\sigma}_0(t_\beta) > 0 \ , \qquad (5.15)$$

and materials such that

$$W''''(\beta_1) < 0 \ ; \qquad (5.16)$$

the latter of these conditions ensures that the stress-strain curve is locally parabolic (rather than, say, quartic) near $\varepsilon = \beta_1$. We have

$$\sigma_0(t) = \sigma_\beta + \dot{\sigma}_0(t_\beta)(t - t_\beta) + O[(t-t_\beta)^2] \quad \text{as} \quad t \to t_\beta \ . \qquad (5.17)$$

Then by (3.9) and (4.19), the discontinuous function $\varepsilon_0(t)$ is given near $t = t_\beta$ by

$$\varepsilon_0(t) = \begin{cases} \beta_1 - \left[\dfrac{-2\dot{\sigma}_0(t_\beta)}{W''''(\beta_1)}\right]^{\frac{1}{2}} (t_\beta - t)^{\frac{1}{2}} + o[(t_\beta - t)^{\frac{1}{2}}] & \text{as} \quad t \to t_\beta^- \ , \\ \beta_2 + \dfrac{\dot{\sigma}_0(t_\beta)}{E}(t - t_\beta) + o(t - t_\beta) & \text{as} \quad t \to t_\beta^+ \ . \end{cases} \qquad (5.18)$$

---

[1]See (5.10)

For use in (4.20e), we require the appropriate approximation to $\phi(\varepsilon)$ for $\varepsilon \to \beta_1^-$. From (2.18) and (2.19),

$$\phi(\varepsilon) = \phi(\beta_1) - \frac{1}{\sqrt{\rho}} \int_{\varepsilon}^{\beta_1} \sqrt{W''(s)}\, ds \quad , \tag{5.19}$$

so that making use of (2.15) and (5.16) we obtain

$$\phi(\varepsilon) = \phi(\beta_1) - \frac{2}{3}\sqrt{-W'''(\beta_1)/\rho}\, (\beta_1 - \varepsilon)^{\frac{3}{2}} + o[(\beta_1 - \varepsilon)^{\frac{3}{2}}] \quad \text{as} \quad \varepsilon \to \beta_1^- \quad . \tag{5.20}$$

Since $\tau(s(t),t)$ occurs as an argument in (4.20), we require its behavior for $t \to t_\beta^-$. To this end, one first observes that, by (4.20h), (2.15), (5.16) and $(5.18)_1$, near $t = t_\beta^+$,

$$s(t) \sim -K_1 (t_\beta - \tau)^{\frac{1}{4}} (t - \tau) \quad , \tag{5.21}$$

where the constant K_1 is given by

$$K_1 = \left[\frac{-2W'''(\beta_1)\dot{\sigma}_o(t_\beta)}{\rho^2} \right]^{\frac{1}{4}} > 0 \quad . \tag{5.22}$$

A detailed analysis, making use of (5.11), (5.21) and (5.22), shows that the dominant behavior of $\tau(s(t),t)$ depends on the value of the unknown exponent m_o in (5.11) as follows: as $t \to t_\beta^+$,

$$\tau(s(t),t) = \begin{cases} t_\beta - (\frac{-s_o}{K_1})^{5/4}(t - t_\beta)^{4m_o/5} + o[(t - t_\beta)^{4m_o/5}] & \text{if} \quad m_o < 5/4 , \\ t_\beta - \lambda(t - t_\beta) + o(t - t_\beta) & \text{if} \quad m_o = 5/4 \\ t_\beta - (\frac{-s_o}{K_1})^4 (t - t_\beta)^{4m_o - 4} + o[(t - t_\beta)^{4m_o - 4}] & \text{if} \quad m_o > 5/4 . \end{cases} \tag{5.23}$$

Here the constant λ is the unique positive root of

$$\lambda^{5/4} + \lambda^{1/4} + \frac{s_o}{K_1} = 0 \quad . \tag{5.24}$$

For convenience, we define constants $q = q(m_o)$ and $K_2 = K_2(m_o, s_o)$ by

$$q = \begin{cases} 4m_o/5 & \text{for} \quad m_o < 5/4 , \\ 1 & \text{for} \quad m_o = 5/4 , \\ 4m_o - 4 & \text{for} \quad m_o > 5/4 , \end{cases} \tag{5.25}$$

and

$$K_2 = \begin{cases} (-s_o/K_1)^{5/4} & \text{for} \quad m_o < 5/4 , \\ \lambda & \text{for} \quad m_o = 5/4 , \\ (-s_o/K_1)^4 & \text{for} \quad m_o > 5/4 . \end{cases} \tag{5.26}$$

Then (5.23) can be written as

$$t_\beta - \tau = K_2(t - t_\beta)^q + o[(t - t_\beta)^q] \quad \text{as} \quad t \to t_\beta^+ \quad . \tag{5.27}$$

Since by (5.13) we have $m_o > 1$, it follows that

$$q > 4/5 \ . \tag{5.28}$$

We are now able to determine the behavior near $t = t_\beta$ of the functions $W'(\varepsilon_o(\tau))$, $\varepsilon_o(\tau)$, $\phi(\varepsilon_o(\tau))$ and $\varepsilon_o(t + \sqrt{\frac{\rho}{E}}\, s(t))$ which occur in (4.20):

$$W'(\varepsilon_o(\tau)) = \sigma_o(\tau) = \sigma_\beta - \dot{\sigma}_o(t_\beta) K_2 (t - t_\beta)^q + o[(t - t\ )^q], \tag{5.29}$$

$$\varepsilon_o(\tau) = 1 - [-2\dot{\sigma}_o(t_\beta) K_2 / W'''(\beta_1)]^{\frac{1}{2}} (t - t_\beta)^{q/2} + o[(t - t_\beta)^{q/2}] \ , \tag{5.30}$$

$$\phi(\varepsilon_o(\tau)) = \phi(\beta_1) - 2/3(-W'''(\beta_1))^{-\frac{1}{4}} \rho^{-\frac{1}{2}} (2\dot{\sigma}_o(t_\beta) K_2)^{\frac{3}{4}} (t - t_\beta)^{3q/4}$$

$$+ \ o[(t - t_\beta)^{3q/4}] \ , \tag{5.31}$$

$$\varepsilon_o(t + \sqrt{\frac{\rho}{E}}\, s(t)) = \beta_2 + \frac{\dot{\sigma}_o(t_\beta)}{E} (t - t_\beta) + \sqrt{\frac{\rho}{E^3}}\, s_o \dot{\sigma}_o(t_\beta)(t - t_\beta)^{m_o}$$

$$+ \ o[(t - t_\beta)^{m_o}] + O[(t - t_\beta)^2] \ . \tag{5.32}$$

Furthermore, (5.11), (5.12) together with the binomial expansion then yield

$$f(s(t) - \sqrt{\frac{E}{\rho}}\, t) + f(-s(t) - \sqrt{\frac{E}{\rho}}\, t) = 2f_o + 2f_1 (\sqrt{\frac{E}{\rho}})^{n_1} (t - t_\beta)^{n_1}$$

$$+ \ 2f_2 (\sqrt{\frac{E}{\rho}})^{n_2} (t - t_\beta)^{n_2} + o\left[(t - t_\beta)^{\min\{n_1 + m_o - 1,\, n_2\}}\right] \ , \tag{5.33}$$

$$-f(s(t)-\sqrt{\frac{E}{\rho}}t)+f(-s(t)-\sqrt{\frac{E}{\rho}}t)=2f_1n_1s_o(\sqrt{\frac{E}{\rho}})^{n_1-1}(t-t_\beta)^{n_1+m_o-1}$$

$$+2f_1n_1s_1(\sqrt{\frac{E}{\rho}})^{n_1-1}(t-t_\beta)^{n_1+m_1-1}+2f_2n_2s_o(\sqrt{\frac{E}{\rho}})^{n_2-1}(t-t_\beta)^{m_o+n_2-1}$$

$$+o\left[(t-t_\beta)^{\min\{n_1+m_1-1,2m_o+n_1-2,m_o+n_2-1\}}\right]$$

$$+O\left[(t-t_\beta)^{\min\{3m_o+n_1-3,m_1+n_2-1,m_o+n_3-1\}}\right] . \quad (5.34)$$

Equations (5.33) and 5.34) make it possible to determine the behavior near $t=t_\beta$ of the various physical quantities which suffer jumps across the phase boundary. Thus from (4.20e),(5.33), (5.32) and (5.31)

$$[v]=2(\sqrt{\frac{E}{\rho}})^{n_1+1}f_1(t-t_\beta)^{n_1}+2(\sqrt{\frac{E}{\rho}})^{n_2+1}f_2(t-t_\beta)^{n_2}$$

$$+\frac{\dot{\sigma}_o(t_\beta)}{\sqrt{\rho E}}(t-t_\beta)+\frac{2}{3}(-W'''(\beta_1))^{\frac{1}{4}}\rho^{-\frac{1}{2}}(2\dot{\sigma}_o(t_\beta)K_2)^{\frac{3}{4}}(t-t_\beta)^{3q/4}$$

$$+o\left[(t-t_\beta)^{\min\{n_1+m_o-1,n_2,1,3q/4\}}\right] , \quad (5.35)$$

while (4.20f), (5.34), (5.32). (5.30) and (5.11) give

$$[\varepsilon]\dot{s}=(\beta_2-\beta_1)m_os_o(t-t_\beta)^{m_o-1}+(\beta_2-\beta_1)m_1s_o(t-t_\beta)^{m_1-1}$$

$$+m_os_o\left(-2\dot{\sigma}_o(t_\beta)K_2/W'''(\beta_1)\right)^{\frac{1}{2}}(t-t_\beta)^{q/2+m_o-1}$$

$$+O\left[(t-t_\beta)^{\min\{n_1+2m_o-2,m_o\}}\right]+o\left[(t-t_\beta)^{\min\{n_2+m_o-1,\ q/2+m_o-1,m_1-1\}}\right] . \quad (5.36)$$

From (4.20g),(5.34), (5.32) and (5.29) we have

$$\frac{1}{\rho}[W'] = 2f_1n_1s_0\left(\sqrt{\frac{E}{\rho}}\right)^{n_1+1}(t-t_\beta)^{n_1+m_0-1} + 2f_1n_1s_1\left(\sqrt{\frac{E}{\rho}}\right)^{n_1+1}(t-t_\beta)^{n_1+m_1-1}$$

$$+ 2f_2n_2s_0\left(\sqrt{\frac{E}{\rho}}\right)^{n_2+1}(t-t_\beta)^{m_0+n_2-1} + \frac{\dot{\sigma}_0(t_\beta)}{\rho}(t-t_\beta) + \frac{s_0\dot{\sigma}_0(t_\beta)}{\sqrt{\rho E}}(t-t_\beta)^{m_0}$$

$$-\frac{\dot{\sigma}_0(t_\beta)K_2}{\rho}(t-t_\beta)^q + o\left[(t-t_\beta)^{\min\{n_1+m_1-1,\,2m_0+n_1-2,\,m_0+n_2-1,\,m_0,\,q\}}\right]$$

$$+ O\left[(t-t_\beta)^{\min\{3m_0+n_1-3,\,m_1+n_2-1,\,m_0+n_3-1\}}\right] . \qquad (5.37)$$

Finally (5.35) and (5.11) give

$$[v]\dot{s} = 2m_0s_0\left(\sqrt{\frac{E}{\rho}}\right)^{n_1+1}f_1(t-t_\beta)^{n_1+m_0-1} + 2m_1s_1\left(\sqrt{\frac{E}{\rho}}\right)^{n_1+1}f_1(t-t_\beta)^{n_1+m_1-1}$$

$$+2m_0s_0\left(\sqrt{\frac{E}{\rho}}\right)^{n_2+1}f_2(t-t_\beta)^{n_2+m_0-1} + \frac{m_0s_0\dot{\sigma}_0(t_\beta)}{\sqrt{\rho E}}(t-t_\beta)^{m_0}$$

$$+\frac{2}{3}m_0s_0(-W'''(\beta_1))^{\frac{1}{4}}\rho^{-\frac{1}{2}}(2\dot{\sigma}_0(t_\beta)K_2)^{\frac{3}{4}}(t-t_\beta)^{\frac{3q}{4}+m_0-1}$$

$$+o\left[(t-t_\beta)^{\min\{n_1+m_1-1,\,n_2+m_0-1,\,n_1+2m_0-2,\,m_0,\,\frac{3q}{4}+m_0-1\}}\right] . \qquad (5.38)$$

It is now necessary to enter the two shock conditions (4.20c) and (4.20d) and investigate the possible dominant balances among the various terms as $t \to t_\beta$. This process makes use of the asymptotic results given above and requires a detailed analysis, too laborious to be included here, of various possible cases. One is thus led to the values

$$m_o = 3/2 \quad , \quad n_1 = 1/2 \tag{5.39}$$

for the leading exponents and to the formulas

$$s_o = -\left[\frac{\dot{\sigma}_o(t_\beta)}{3\rho(\beta_2-\beta_1)}\right]^{\frac{1}{2}} , \quad f_1 = \frac{1}{4}\left(\frac{E}{\rho}\right)^{-\frac{3}{4}}\left[\frac{3(\beta_2-\beta_1)\dot{\sigma}_o(t_\beta)}{\rho}\right]^{\frac{1}{2}} > 0 \ . \tag{5.40}$$

Thus the dominant terms in the expansions (5.11) and (5.12) are determined:

$$s(t) \sim s_o(t - t_\beta)^{\frac{3}{2}} \quad , \tag{5.41}$$

$$f(z) \sim f_o + f_1\left(-z-\sqrt{\frac{E}{\rho}}\, t_\beta\right)^{\frac{1}{2}} \quad , \tag{5.42}$$

where s_o and f_1 are given by (5.40). Although we have not pursued in full detail the higher order corrections to (5.41) and (5.42), we conjucture that the values

$$m_k = \frac{k+3}{2} \quad , \quad n_k = \frac{k}{2} \tag{5.43}$$

for the exponents in (5.11) and (5.12) provide for a consistent expansion scheme.

6. Large-Time Behavior of the Phase Boundary

We now undertake an asymptotic analysis to determine the large-time behavior of the propagating phase boundary. The loading condition $\dot{\sigma}_0(t) \geq 0$ suggests that we seek solutions such that

$$\ddot{s}(t) < 0 \quad \text{for} \quad t > t_\beta \quad . \tag{6.1}$$

The validity of (6.1) in some neighborhood of time t_β follows from the results of the previous section. Although we conjecture that (6.1) follows from the governing equations, either (4.20) or (4.32), we have been unable to confirm this. We shall show, however, that assumption (6.1) leads to physically reasonable behavior of the solution to our problem.

From (4.15), (6.1), and the short-time analysis we have

$$-\sqrt{\frac{E}{\rho}} < \dot{s}(t) < 0 \ , \quad t > t_\beta \quad . \tag{6.2}$$

Moreover, since $\dot{s}(t)$ is monotonic decreasing and bounded below, it follows that it approaches some limit as $t \to \infty$. Thus we define the asymptotic phase boundary speed α by

$$\alpha = -\lim_{t \to \infty} \dot{s}(t) \tag{6.3}$$

hence

$$0 < \alpha \leq \sqrt{\frac{E}{\rho}} \quad . \tag{6.4}$$

The acoustic speed $c(x,t)$ is constant on each negatively-sloped characteristic ray in $\mathcal{R}_1$; moreover, $c(0,t_\beta^-) = 0$. Thus (6.1) guarantees that, with increasing time, the phase boundary will intersect characteristic rays in $\mathcal{R}_1$ which arise on $x = 0$ at progressively earlier times; see Fig. 8. This geometric state of affairs is described by the inequalities:

$$\frac{d}{dt}\,\tau(s(t),t) < 0\ , \qquad t > t_\beta \quad , \tag{6.5}$$

and

$$c\left(\varepsilon_0(\tau(s(t),t))\right) < -\dot{s}(t)\ ,\ t > t_\beta \quad . \tag{6.6}$$

The validity and equivalence of (6.5) and (6.6) also follow from (6.1), (4.20h) and the short-time analysis of the previous section. From (2.15), (2.18), (2.19), (2.32), (2.36), and (6.5) we infer

$$\frac{d}{dt}\,\varepsilon_0(\tau(s(t),t)) \leq 0 \quad \text{and} \quad \frac{d}{dt}\,\phi\left(\varepsilon_0(\tau(s(t),t))\right) \leq 0\ , \tag{6.7}$$

so that $\varepsilon(s(t)^-,t)$ and $v(s(t)^-,t)$ are decreasing with time. We have already observed that the first inequality in (6.2) guarantees that, with increasing time, the phase boundary will intersect members of the negatively-sloped family of characteristic rays in $\mathcal{R}_2$ which issue progressively later on $x = 0$.[1] This alone does not allow us to draw any conclusion regarding the monotonicity of $\varepsilon(s(t)^+,t)$ and $v(s(t)^+,t)$. We will, however, be able to establish such a monotonicity result once we determine some of the properties of the unknown function $f(z)$. Meanwhile, we consolidate (6.2) and (6.6) into

$$-\sqrt{\frac{E}{\rho}} < \dot{s}(t) < -c\left(\varepsilon_0(\tau(s(t),t))\right) \quad , \quad t > t_\beta\ . \tag{6.8}$$

Thus the speed of the phase boundary is confined to the interval between the different acoustic speeds on each side of this boundary. We note that (6.8) is precisely the entropy condition given by Lax [18].

It is apparent that as $t \to \infty$ one of the following two alternatives must occur.

---

[1] See the discussion following (4.18).

Alternative I: The phase boundary in the x-t plane does not cross the initial characteristic $x = -c(0)t$ and so is confined to $\mathcal{R}_D$. For this to occur it is necessary that

$$-c(0)t \leq s(t) \qquad \text{for} \qquad t > t_\beta \quad . \tag{6.9}$$

Since by (6.3), s(t) is asymptotically linear, (6.9) will hold if and only if

$$\alpha \leq c(0) = \sqrt{\frac{W''(0)}{\rho}} \quad . \tag{6.10}$$

Thus for large-time, the curve $x = s(t)$ will be asymptotically parallel to a limiting characteristic C_- ray in $\mathcal{R}_1$. We shall let τ_∞ denote the time at which this limiting characteristic arises on $x = 0$. Hence,

$$\tau_\infty = \lim_{t\to\infty} \tau(s(t),t) \quad , \tag{6.11}$$

and

$$\alpha = c(\varepsilon_0(\tau_\infty)) \quad . \tag{6.12}$$

Alternative II: The phase boundary in the x-t plane crosses the initial characteristic at some time t_L which satisfies the equation

$$s(t_L) = -c(0)t_L \quad . \tag{6.13}$$

Then

$$s(t) < -c(0)t \qquad \text{for} \qquad t > t_L \quad , \tag{6.14}$$

so that after time t_L the phase boundary is the leading disturbance. From (6.14) it follows that this alternative will occur if and only if

$$\alpha > c(0) = \sqrt{\frac{W''(0)}{\rho}} \quad . \tag{6.15}$$

It is clear from (6.10) and (6.15) that whether or not the phase boundary becomes the leading disturbance depends on the asymptotic phase boundary speed α. Thus α determines which of the foregoing alternatives occurs.

To ascertain the large-time behavior of the phase boundary, we shall study the large-time behavior of solutions to the pair of differential-delay equations (4.22). A major difficulty in any such asymptotic scheme stems from the lack of an explicit formula for $\tau(s(t),t)$ when the phase boundary is in $\mathcal{R}_D$. We faced the same difficulty in connection with the short-time behavior where an asymptotic analysis led to (5.23). For large time, another approximation scheme immediately suggests itself. In this case we wish to consider $t \gg t_\beta > \tau(s(t),t)$, so that from (4.20h),

$$c(\varepsilon_0(\tau)) = \frac{-s(t)}{t-\tau} \sim \frac{-s(t)}{t} \quad . \tag{6.16}$$

Now with (3.2), the discussion in section 2 shows that $c(\varepsilon_0(\tau))$ is continuous and monotonically decreasing as a function of τ. If $c(\varepsilon_0(\tau))$ is strictly decreasing, then it is invertible. By (2.34), this invertibility is assured if both

$$\dot{\sigma}_0(t) > 0 \quad , \qquad 0 \le t \le t_\beta \tag{6.17}$$

and

$$W'''(\varepsilon) < 0 \quad , \qquad 0 \le \varepsilon \le \beta_1 \quad . \tag{6.18}$$

Note that (6.18) tightens assumption (2.38) by requiring that the material is not only "non-hardening in the first phase" but also genuinely nonlinear or "softening in the first phase." In the remaining analysis we shall assume that (6.17) and (6.18) hold. Thus we may define an inverse Ω for $c(\varepsilon_0(\cdot))$, where

$$\left.\begin{aligned} &\Omega: [0,c(0)] \to [0,t_\beta] \quad , \\ &z = c\big(\varepsilon_0(\Omega(z))\big) \quad , \qquad 0 \le z \le c(0) \quad , \\ \text{and}& \\ &t = \Omega(c(\varepsilon_0(t)) \quad , \qquad 0 \le t \le t_\beta \quad . \end{aligned}\right\} \quad (6.19)$$

By (6.16) it now follows that

$$\tau(s(t),t) \simeq \Omega\left(\frac{-s(t)}{t}\right) \qquad \text{for} \qquad t >> t_\beta \quad , \tag{6.20}$$

provided

$$0 \le \frac{-s(t)}{t} \le c(0) \quad . \tag{6.21}$$

If the phase boundary becomes the leading disturbance, then Alternative II holds and $s(t) \in \mathcal{R}_T$ for $t > t_L$; consequently, $\tau(s(t),t) = 0$ for $t > t_L$. Thus a large-time asymptotic study of (4.22) requires an explicit approximation for $\tau(s(t),t)$ only for Alternative I, the alternative where the phase boundary does <u>not</u> become the leading disturbance. Then (6.1), (6.3), and (6.10) yield

$$0 \le -\dot{s}(t) \le c(0) \quad , \qquad t > t_\beta \quad . \tag{6.22}$$

Since $s(t)$ is asymptotically linear, (6.22) implies (6.21) for large times. Thus when Alternative I holds we may appeal to (6.20) in the asymptotic analysis. We note that for such an analysis, Ω need only be defined on some one-sided neighborhood of α, say $(\alpha-\delta,\alpha]$. Hence we could weaken (6.17) and (6.18) to

$$\dot{\sigma}_0(\tau_\infty) > 0 \qquad \text{and} \qquad W'''(\varepsilon_0(\tau_\infty)) < 0 \quad . \tag{6.23}$$

However, as τ_∞ is not known in advance, it is convenient to retain (6.17) and (6.18).

When Alternative I holds, (6.20) furnishes the leading term in an asymptotic expansion of $\tau(s(t),t)$ for large t. This can be seen by considering the following iteration scheme suggested by (4.20h):

$$\tau_{n+1} = \Omega\left(\frac{-s(t)}{t-\tau_n}\right) \quad , \qquad \tau_0 = 0 \quad , \tag{6.24}$$

for which $\tau(s(t),t)$ is a fixed point and (6.20) is the first iterate. Equation (6.24) generates a formal series solution to (4.20h) of the form

$$\tau(s(t),t) = \sum_{k=0}^{\infty} A_k\left(\frac{-s(t)}{t}\right) t^{-k} \quad , \tag{6.25}$$

where $A_k(z)$ is a finite sum of products of z, $\Omega(z)$, $\frac{d}{dz}\Omega(z)\cdots\frac{d^k\Omega(z)}{dz^k}$. The first three A_k's are given by

$$\left.\begin{aligned} A_0(z) &= \Omega(z) \quad , \\ A_1(z) &= z\Omega(z)\,\frac{d\Omega(z)}{dz} \quad , \\ \text{and} \qquad A_2(z) &= z^2\Omega(z)\left(\frac{d\Omega(z)}{dz}\right)^2 - z(\Omega(z))^2\,\frac{d\Omega(z)}{dz} \\ &\quad + \frac{1}{2}\,z^2(\Omega(z))^2\,\frac{d^2\Omega(z)}{dz^2} \quad . \end{aligned}\right\} \tag{6.26}$$

Differentiation of the second of (6.19) yields

$$\frac{d}{dz}\,\Omega(z) = \frac{1}{\dot{c}(\varepsilon_0(\Omega(z)))\;\dot{\varepsilon}_0(\Omega(z))} \quad . \tag{6.27}$$

It then follows from (2.18) and (2.32) that (6.17) and (6.18) are precisely the conditions necessary for $d\Omega/dz$ to be finite. It similarly follows that all the $A_k(z)$ are finite, although not necessarily uniformly bounded, so that (6.25) is indeed an asymptotic series.

From (6.17) and (6.18) it follows that $\varepsilon_0(t)$ and $c(\varepsilon)$ are invertible in the first phase; their inverses, denoted by ε_0^{-1} and c^{-1}, are then defined on $0 \leq \varepsilon \leq \beta_1$ and $0 \leq c \leq c(0)$, respectively. We then have

$$\Omega(z) = \varepsilon_0^{-1}(c^{-1}(z)) \quad , \tag{6.28}$$

and hence arrive at the following asymptotic formulas for Alternative I:

$$\tau(s(t),t) = \varepsilon_0^{-1}(c^{-1}(\tfrac{-s(t)}{t})) + O(\tfrac{1}{t}) \quad , \tag{6.29}$$

$$\varepsilon(s(t)^-,t) = \varepsilon_0(\tau(s(t),t)) \sim c^{-1}(\tfrac{-s(t)}{t}) \quad , \tag{6.30}$$

$$W'(\varepsilon(s(t)^-,t)) = W'(\varepsilon_0(\tau(s(t),t))) \sim W'(c^{-1}(\tfrac{-s(t)}{t})) \quad , \tag{6.31}$$

$$v(s(t)^-,t) = \phi(\varepsilon_0(\tau(s(t),t))) \sim \phi(c^{-1}(\tfrac{-s(t)}{t})) \quad . \tag{6.32}$$

Entering (4.22) and (4.23) with (6.29)-(6.32), we arrive at the <u>asymptotic governing equations when the phase boundary does not become the leading disturbance</u> (Alternative I):

$$\sqrt{\tfrac{E}{\rho}}\,(\dot{s}(t)+\sqrt{\tfrac{E}{\rho}})\; f(-s(t)-\sqrt{\tfrac{E}{\rho}}t) + \frac{1}{\sqrt{\rho E}}\,(\dot{s}(t)+\sqrt{\tfrac{E}{\rho}})\;\sigma_0(t+\sqrt{\tfrac{\rho}{E}}\,s(t)) -$$

$$- \frac{D}{\sqrt{\rho E}} (\dot{s}(t) + \frac{1}{2}\sqrt{\frac{E}{\rho}}) - \frac{1}{2}\sqrt{\frac{E}{\rho}}\, \dot{s}(t)\; c^{-1}(\frac{-s(t)}{t}) - \frac{1}{2}\,(\dot{s}(t) + \sqrt{\frac{E}{\rho}})\phi(c^{-1}(\frac{-s(t)}{t}))$$
$$- \frac{1}{2\rho} W'(c^{-1}(\frac{-s(t)}{t})) = 0 \quad , \tag{6.33a}$$

$$\sqrt{\frac{E}{\rho}}\,(-\dot{s}(t) + \sqrt{\frac{E}{\rho}})\; f(s(t) - \sqrt{\frac{E}{\rho}}\,t) - \frac{D}{2\rho} - \frac{1}{2}\sqrt{\frac{E}{\rho}}\,\dot{s}(t)\; c^{-1}(\frac{-s(t)}{t})$$
$$+ \frac{1}{2}\,(\dot{s}(t) - \sqrt{\frac{E}{\rho}})\,\phi(c^{-1}(\frac{-s(t)}{t})) + \frac{1}{2\rho} W'(c^{-1}(\frac{-s(t)}{t})) = 0 \quad . \tag{6.33b}$$

Conversely entering (4.22) and (4.23) with $\varepsilon_0(\tau(s(t),t)) = 0$, we arrive at the <u>exact governing equations when the phase boundary becomes the leading disturbance</u> and $t > t_L$ (Alternative II):

$$\sqrt{\frac{E}{\rho}}\,(\dot{s}(t) + \sqrt{\frac{E}{\rho}})\; f(-s(t) - \sqrt{\frac{E}{\rho}}\,t) + \frac{1}{\sqrt{\rho E}}\,(\dot{s}(t) + \sqrt{\frac{E}{\rho}})\; \sigma_0(t + \sqrt{\frac{\rho}{E}}\, s(t))$$
$$- \frac{D}{\sqrt{\rho E}}\,(\dot{s}(t) + \frac{1}{2}\sqrt{\frac{E}{\rho}}) = 0 \quad , \tag{6.34a}$$

and

$$\sqrt{\frac{E}{\rho}}\,(-\dot{s}(t) + \sqrt{\frac{E}{\rho}})\; f(s(t) - \sqrt{\frac{E}{\rho}}\,t) - \frac{1}{2}\,\frac{D}{\rho} = 0 \quad . \tag{6.34b}$$

In proceeding with the analysis of (6.33), (6.34) we shall treat four separate cases:

(I)	$W''(0) \geq E$,	$\sigma_0(t) \to +\infty$,	(6.35)
(II)	$W''(0) \geq E$,	$\sigma_0(t) \to \sigma_\infty < +\infty$,	(6.36)
(III)	$W''(0) < E$,	$\sigma_0(t) \to \sigma_\infty < +\infty$,	(6.37)
(IV)	$W''(0) < E$,	$\sigma_0(t) \to +\infty$.	(6.38)

Since the end condition (6.17) implies that $\lim_{t \to \infty} \sigma_0(t) \leq +\infty$, the four cases listed above are exhaustive. Although it seems likely that $W''(0) \geq E$ for real materials, we shall treat cases (III) and (IV) as well for the sake of completeness. We note from (6.4) and (6.15) that the condition

$$W''(0) < E \tag{6.39}$$

is necessary for the phase boundary to become the leading disturbance.

We have so far remained silent on the behavior of $f(z)$, the remaining unknown in the differential-delay equations. We first show that $f(z)$ is monotonically decreasing on its interval of definition $(-\infty,-\sqrt{\frac{E}{\rho}}\, t_\beta]$. For this purpose, as well as for the forthcoming analysis, we will need the following lemma.

Lemma: If $W(\varepsilon)$ satisfies (2.14), (2.15), (2.16), (2.17), (4.2), and (6.18), then

$$W'(c^{-1}(\xi)) - E\,c^{-1}(\xi) - D > 0 \quad , \tag{6.40}$$

where

$$0 \leq \xi \leq \min\{\sqrt{\frac{W''(0)}{\rho}}\,,\sqrt{\frac{E}{\rho}}\} \ . \tag{6.41}$$

We note that $\min\{\sqrt{\frac{W''(0)}{\rho}},\sqrt{\frac{E}{\rho}}\}$ is $\sqrt{\frac{W''(0)}{\rho}}$ in cases III, IV; it is $\sqrt{\frac{E}{\rho}}$ in Cases I, II.

Proof of Lemma:

Let

$$H(\xi) = W'(c^{-1}(\xi)) - E\,c^{-1}(\xi) - D \quad , \tag{6.42}$$

then since $c^{-1}(0) = \beta_1$ it follows that

$$H(0) = W'(\beta_1) - E\beta_1 - D > W'(\beta_1) - E\beta_2 - D = 0 \ . \tag{6.43}$$

Since $H(0) > 0$, we need only demonstrate that $H(\xi)$ is monotonically increasing. Writing

$$H'(\xi) = (W''(c^{-1}(\xi)) - E)\, \frac{d}{d\xi}\, c^{-1}(\xi) \ , \tag{6.44}$$

we examine each factor of $H'(\xi)$ individually. Now

$$\frac{d}{d\xi}\, c^{-1}(\xi) = \frac{1}{\frac{d}{d\varepsilon}\, c(\varepsilon)\Big|_{\varepsilon = c^{-1}(\xi)}} = \frac{\sqrt{\rho W''(c^{-1}(\xi))}}{W'''(c^{-1}(\xi))} \le 0 \ . \tag{6.45}$$

As for the first factor of (6.44),

$$W''\left(c^{-1}\left(\min\left\{\sqrt{\frac{W''(0)}{\rho}}, \sqrt{\frac{E}{\rho}}\right\}\right)\right) - E = \begin{cases} W''(0) - E \leqslant 0 & \text{for} \quad W''(0) \leqslant E \\ E - E = 0 & \text{for} \quad W''(0) \geqslant E \end{cases} \tag{6.46}$$

and

$$\frac{d}{d\xi}\left(W''(c^{-1}(\xi)) - E\right) = W'''(c^{-1}(\xi))\, \frac{d}{d\xi}\, c^{-1}(\xi) \ge 0 \ . \tag{6.47}$$

Now (6.46) and (6.47) give

$$W''(c^{-1}(\xi)) - E \le 0 \ . \tag{6.48}$$

Hence we obtain the monotonicity result

$$H'(\xi) \ge 0 \ , \tag{6.49}$$

which completes the proof of the Lemma.

<u>Corollary</u>: *For $W(\varepsilon)$ as in the above Lemma and ε^* defined by*

$$\varepsilon^* = \begin{cases} 0 & \text{for } W''(0) \leq E \ , \\ c^{-1}\left(\sqrt{\frac{E}{\rho}}\right) & \text{for } W''(0) \geq E \ ; \end{cases} \tag{6.50}$$

$$W'(\varepsilon) - E\varepsilon - D > 0 \quad , \tag{6.51}$$

when

$$\varepsilon^* \leq \varepsilon \leq \beta_1 \quad . \tag{6.52}$$

We now show that $f'(z) \leq 0$. Differentiating (4.22b) and using (4.23b) we find after some algebra that

$$f'\left(s(t) - \sqrt{\frac{E}{\rho}}\,t\right) \equiv \frac{d}{dz} f(z)\Big|_{z = s(t) - \sqrt{\frac{E}{\rho}}\,t}$$

$$= \frac{1}{2\sqrt{\frac{E}{\rho}}\left(\sqrt{\frac{E}{\rho}} - \dot{s}(t)\right)^2} \left\{ \frac{\ddot{s}(t)}{\rho\left(\sqrt{\frac{E}{\rho}} - \dot{s}(t)\right)} \Big[W'(\varepsilon_0(\tau(s(t),t)) - E\varepsilon_0(\tau(s(t),t)) - D \Big] - \left[\left(\sqrt{\frac{E}{\rho}} - c(\varepsilon_0(\tau(s(t),t)))\right)\left(\dot{s}(t) + c(\varepsilon_0(\tau(s(t),t)))\right) \right] \times \frac{d}{dt}\varepsilon_0(\tau(s(t),t)) \right\} \quad . \tag{6.53}$$

Later on we shall also require $f'\left(-s(t) - \sqrt{\frac{E}{\rho}}\,t\right)$, which we display at this juncture. From (4.22a), (4.23a), we obtain after elementary manipulations,

$$f'(-s(t)-\sqrt{\tfrac{E}{\rho}}t) \equiv \frac{d}{dz} f(z)\Big|_{z=-s(t)-\sqrt{\frac{E}{\rho}}t}$$

$$= \frac{1}{2\sqrt{\frac{E}{\rho}}(\sqrt{\frac{E}{\rho}}+\dot{s}(t))^2}\left\{\frac{\ddot{s}(t)}{\rho(\sqrt{\frac{E}{\rho}}+\dot{s}(t))}\left[W'(\varepsilon_0(\tau(s(t),t))) - E\varepsilon_0(\tau(s(t),t)) - D\right]\right.$$

$$-\left[\left(\sqrt{\tfrac{E}{\rho}}+c(\varepsilon_0(\tau(s(t),t)))\right)\left(\dot{s}(t)+c(\varepsilon_0(\tau(s(t),t)))\right)\right]\frac{d}{dt}\varepsilon_0(\tau(s(t),t))$$

$$\left. + 2(\dot{s}(t)+\sqrt{\tfrac{E}{\rho}})^2\ \dot{\varepsilon}_0(t+\sqrt{\tfrac{\rho}{E}}\,s(t))\right\} \quad . \tag{6.54}$$

Now recall that

$$\beta_1 \geq \varepsilon_0(\tau(s(t),t)) \geq 0 \quad . \tag{6.55}$$

In addition, if $W''(0) \geq E$, the phase boundary cannot become the leading disturbance; accordingly, (6.7), (6.30), (6.3), (6.45), and (6.4) yield

$$\beta_1 \geq \varepsilon_0(\tau(s(t),t)) \geq c^{-1}(\alpha) \geq c^{-1}(\sqrt{\tfrac{E}{\rho}}) \quad \text{when } W''(0) \geq E. \tag{6.56}$$

Consequently $\varepsilon_0(\tau(s(t),t))$ always occurs in the interval given by (6.52), so that the Corollary yields

$$W'(\varepsilon_0(\tau(s(t),t))) - E\varepsilon_0(\tau(s(t),t)) - D > 0 \quad . \tag{6.57}$$

Examining the second bracketed quantity in (6.53) we have with the aid of (6.8) that

$$\left[\left(\sqrt{\tfrac{E}{\rho}} - c(\varepsilon_0(\tau(s(t),t)))\right)\left(\dot{s}(t)+c(\varepsilon_0(\tau(s(t),t)))\right)\right] < 0, \quad t > t_\beta . \tag{6.58}$$

Thus applying (6.1), (6.8), (6.57), (6.58), and (6.7) to (6.53), we conclude that

$$f'(s(t) - \sqrt{\frac{E}{\rho}}\,t) < 0 \quad , \qquad t > t_\beta \quad . \tag{6.59}$$

Moreover, $\dot{s}(t) \leq 0$ and $s(t_\beta) = 0$ guarantees that $t \mapsto s(t) - \sqrt{\frac{E}{\rho}}\,t$ is a continuous, one-to-one mapping of the interval $(-\infty, t_\beta]$ onto $(-\infty, -\sqrt{\frac{E}{\rho}}\,t_\beta]$. This allows us to conclude from (6.59) that

$$f'(z) < 0 \quad \text{for} \quad -\infty < z < -\sqrt{\frac{E}{\rho}}\,t_\beta \quad , \tag{6.60}$$

thus establishing the monotonicity of f. It is interesting to note that it is not obvious from (6.54) that $f'(-s(t) - \sqrt{\frac{E}{\rho}}\,t) < 0$, although of course (6.60) provides this result as well.

Having shown that f is monotonically decreasing, we now demonstrate that f(z) is bounded. From (5.10) we already have the minimum value for f. To show that f is bounded above, we appeal to (4.22b), which yields

$$f(z) \leq \max_{t \geq t_\beta} \frac{-\Gamma_2(\dot{s}(t), s(t), t)}{2\sqrt{\frac{E}{\rho}}\,(-\dot{s}(t) + \sqrt{\frac{E}{\rho}})} \leq \frac{\frac{D}{\rho} + 2\sqrt{\frac{E}{\rho}}\,\phi(\beta_1)}{2\,\frac{E}{\rho}} \quad . \tag{6.61}$$

In view of (6.60), (6.61), $\lim_{z \to -\infty} f(z)$ exists, and so we may define

$$f_{-\infty} \equiv \lim_{z \to -\infty} f(z) \quad . \tag{6.62}$$

Combining (5.10) and (6.62),

$$\frac{1}{2}\,(\sqrt{\frac{\rho}{E}}\,\phi(\beta_1) - \beta_2) = f(-\sqrt{\frac{E}{\rho}}\,t_\beta) \leq f(z) \leq f_{-\infty} \quad . \tag{6.63}$$

The preceding discussion concerning the behavior of s(t) for large positive t and of f(z) for large negative z leads us to seek asymptotic solutions of (6.33) or (6.34) in the form

$$s(t) = -\alpha t + q(t) \quad , \tag{6.64}$$

$$f(z) = f_{-\infty} + f_1(z) \quad , \tag{6.65}$$

where

$$q(t) = o(t) \ , \quad \dot{q}(t) = o(1) \quad \text{as} \quad t \to \infty \ , \tag{6.66}$$

$$f_1(z) = o(1) \qquad \text{as} \quad z \to -\infty \quad . \tag{6.67}$$

We shall give formulas determining α and $f_{-\infty}$, and shall seek more specific information about $q(t)$ and $f_1(z)$.

In the ensuing analysis of (6.33) or (6.34) we are faced with determining the asymptotic behavior of $f(-s(t) - \sqrt{\frac{E}{\rho}}t)$, $f(s(t) - \sqrt{\frac{E}{\rho}}\, t)$ and $\sigma_0(t + \sqrt{\frac{\rho}{E}}\, s(t))$ as $t \to \infty$. Here we shall use the fact that since f and σ_0 are monotonic, their asymptotic behavior as their argument approaches some value (finite or infinite) is simply given by the function evaluated at its asymptotic argument.[1]

Case I: $W''(0) \geq E$, $\sigma_0(t) \to \infty$

Here (6.33) necessarily applies and substitution of (6.64) into (6.33a) yields the asymptotic equation:

$$\sqrt{\frac{E}{\rho}}\,(-\alpha + \sqrt{\frac{E}{\rho}})\; f(-s(t) - \sqrt{\frac{E}{\rho}}t) + \frac{1}{\sqrt{\rho E}}(-\alpha + \sqrt{\frac{E}{\rho}})\sigma_0((1 - \alpha\sqrt{\frac{\rho}{E}})t)$$

$$- \frac{D}{\sqrt{\rho E}}\,(-\alpha + \frac{1}{2}\sqrt{\frac{E}{\rho}}) + \frac{1}{2}\sqrt{\frac{E}{\rho}}\alpha c^{-1}(\alpha) - \frac{1}{2}(-\alpha + \sqrt{\frac{E}{\rho}})\;\phi(c^{-1}(\alpha)) - \frac{1}{2}\rho W'(c^{-1}(\alpha)) = 0 \tag{6.68}$$

The last four terms are bounded for all α with $0 < \alpha \leq \sqrt{\frac{E}{\rho}}$, as is the first term, because of (6.63). However, if $\alpha \neq \sqrt{\frac{E}{\rho}}$, the second term is

[1] This procedure was tacitly employed in obtaining (6.30)-(6.32).

unbounded. Thus we conclude that

$$\alpha = \sqrt{\frac{E}{\rho}} \quad . \tag{6.69}$$

When (6.69) holds, (4.17) requires that

$$q(t) \geq \sqrt{\frac{E}{\rho}}\, t_\beta \quad . \tag{6.70}$$

On account of (6.69), equation (6.33a) to dominant order yields

$$\sqrt{\frac{E}{\rho}}\, \dot{q}(t)\ f(-q(t)) + \frac{1}{\sqrt{\rho E}}\, \dot{q}(t)\ \sigma_0\left(\sqrt{\frac{\rho}{E}}\ q(t)\right) + \frac{1}{2}\frac{D}{\rho}$$

$$+ \frac{1}{2}\frac{E}{\rho}\, c^{-1}\left(\sqrt{\frac{E}{\rho}}\right) - \frac{1}{2\rho}\, W'\left(c^{-1}\left(\sqrt{\frac{E}{\rho}}\right)\right) = 0 \quad , \tag{6.71}$$

while (6.33b) to dominant order gives

$$2\,\frac{E}{\rho}\, f\left(-2\sqrt{\frac{E}{\rho}}\,t\right) - \frac{1}{2}\frac{D}{\rho} + \frac{1}{2}\frac{E}{\rho}\, c^{-1}\left(\sqrt{\frac{E}{\rho}}\right) -$$

$$- \sqrt{\frac{E}{\rho}}\,\phi\left(c^{-1}\left(\sqrt{\frac{E}{\rho}}\right)\right) + \frac{1}{2\rho}\, W'\left(c^{-1}\left(\sqrt{\frac{E}{\rho}}\right)\right) = 0 \quad . \tag{6.72}$$

From (6.72) and (6.65) we obtain

$$f_{-\infty} = -\frac{1}{4E}\left(W'\left(c^{-1}\left(\sqrt{\frac{E}{\rho}}\right)\right) + E\, c^{-1}\left(\sqrt{\frac{E}{\rho}}\right) - D\right) + \frac{1}{2}\sqrt{\frac{\rho}{E}}\,\phi\left(c^{-1}\left(\sqrt{\frac{E}{\rho}}\right)\right) . \tag{6.73}$$

Turning our attention to (6.71) we note that the last three terms are $O(1)$ and by the Lemma do not sum to zero. Since $\dot{q}(t) = o(1)$ and f is bounded, it follows that to effect the dominant balance, $\sigma_0\left(\sqrt{\frac{\rho}{E}}\,q(t)\right)$ must be unbounded as $t \to \infty$. This requires that

$$\lim_{t\to\infty} q(t) = +\infty \quad , \tag{6.74}$$ [1]

[1] Hence (6.70) is satisfied for large-time.

and that the leading behavior of $q(t)$ is supplied by the solution of

$$\dot{q}(t)\sigma_o\left(\sqrt{\tfrac{\rho}{E}}\, q(t)\right) = \frac{1}{2}\sqrt{\frac{E}{\rho}}\left[W'\left(c^{-1}\left(\sqrt{\tfrac{E}{\rho}}\right)\right) - Ec^{-1}\left(\sqrt{\tfrac{E}{\rho}}\right) - D\right] > 0 , \qquad (6.75)$$

the inequality being a consequence of the Lemma. Integration of the above expression yields

$$\int^q \sigma_o\left(\sqrt{\tfrac{\rho}{E}}\, s\right)ds \sim \frac{1}{2}\sqrt{\frac{E}{\rho}}\left[W'\left(c^{-1}\left(\sqrt{\tfrac{E}{\rho}}\right)\right) - Ec^{-1}\left(\sqrt{\tfrac{E}{\rho}}\right) - D\right]t \quad . \qquad (6.76)$$

Let us consider as examples power-law and exponential loadings:

Example 1: Asymptotic power-law loading

Let

$$\sigma_o(t) = k_1 t^n + o(t^n) , \qquad k_1 > 0, \quad 0 < n < \infty . \qquad (6.77)$$

Upon substitution into (6.76) this yields

$$\frac{1}{2}\sqrt{\frac{E}{\rho}}\left[W'\left(c^{-1}\left(\sqrt{\tfrac{E}{\rho}}\right)\right) - Ec^{-1}\left(\sqrt{\tfrac{E}{\rho}}\right) - D\right]t$$

$$\sim \int^q k_1\left(\sqrt{\tfrac{\rho}{E}}\, s\right)^n ds \sim k_1\left(\frac{\rho}{E}\right)^{n/2} \frac{q^{n+1}}{n+1} , \qquad (6.78)$$

so that

$$q(t) \sim \sqrt{\frac{E}{\rho}}\left(\frac{n+1}{2k_1}\left(W'\left(c^{-1}\left(\sqrt{\tfrac{E}{\rho}}\right)\right) - Ec^{-1}\left(\sqrt{\tfrac{E}{\rho}}\right) - D\right)\right)^{\frac{1}{n+1}} t^{\frac{1}{n+1}} . \qquad (6.79)$$

Example 2: Asymptotic exponential loading

Let

$$\sigma_o(t) = k_2\, e^{k_3 t} + o(e^{k_3 t}) , \quad k_2 > 0, \quad k_3 > 0 . \qquad (6.80)$$

Substitution into (6.76) now yields

$$\frac{1}{2}\sqrt{\frac{E}{\rho}}[W'(c^{-1}(\sqrt{\frac{E}{\rho}})) - Ec^{-1}(\sqrt{\frac{E}{\rho}}) - D]t$$

$$\sim \int^{q} k_2 e^{\sqrt{\frac{\rho}{E}}k_3 s} ds \sim \frac{k_2}{k_3}\sqrt{\frac{E}{\rho}}\ e^{\sqrt{\frac{\rho}{E}}k_3 q} \ , \tag{6.81}$$

whereupon

$$q(t) \sim \frac{1}{k_3}\sqrt{\frac{E}{\rho}}\ \ell n\ t \quad . \tag{6.82}$$

From these examples, we see that the faster $\sigma_0(t)$ tends to infinity, the smaller is the dominant behavior of q(t). Hence the phase boundary approaches its asymptotic speed $\sqrt{\frac{E}{\rho}}$ faster for faster loading rates.

Since the final round of the above analysis involved a dominant balance among the O(1) terms in (4.22), further corrections to f(z) and q(t) will involve consideration of higher order terms from (6.30)-(6.32). Because this requires considering the $O(\frac{1}{t})$ term in (6.25), it appears that higher order analysis would become an order of magnitude more tedious.

Case II: $W''(0) \geq E$, $\sigma_0(t) \to \sigma_\infty < \infty$

Again the phase boundary cannot become the leading disturbance so that (6.33) apply. We can dismiss the possibility $\alpha = \sqrt{\frac{E}{\rho}}$, since in that case the first two summands in (6.33a) are o(1) and we are left with the O(1) terms which must satisfy

$$\frac{1}{2\rho}\,(D + Ec^{-1}(\sqrt{\frac{E}{\rho}}) - W\,(c^{-1}(\sqrt{\frac{E}{\rho}}))) = 0 \ , \tag{6.83}$$

contradicting the Lemma. Hence we conclude that

$$0 < \alpha < \sqrt{\frac{E}{\rho}} \quad . \tag{6.84}$$

Consequently, not only is

$$f(s(t) - \sqrt{\frac{E}{\rho}}t) \sim f_{-\infty} \qquad \text{as} \quad t \to \infty , \tag{6.85}$$

but also

$$f(-s(t) - \sqrt{\frac{E}{\rho}}t) \sim f((\alpha - \sqrt{\frac{E}{\rho}})\, t) \sim f_{-\infty} \qquad \text{as} \quad t \to \infty . \tag{6.86}$$

Entering with (6.85), (6.86), and (6.64) into (6.33), we are lead to the asymptotic equations:

$$\sqrt{\frac{E}{\rho}}\,(-\alpha + \sqrt{\frac{E}{\rho}})\; f_{-\infty} + \frac{1}{\sqrt{\rho E}}\,(-\alpha + \sqrt{\frac{E}{\rho}})\sigma_{\infty} - \frac{D}{\sqrt{\rho E}}(-\alpha + \frac{1}{2}\sqrt{\frac{E}{\rho}})$$

$$+ \frac{1}{2}\,\alpha\sqrt{\frac{E}{\rho}}\; c^{-1}(\alpha) - \frac{1}{2}(-\alpha + \sqrt{\frac{E}{\rho}})\phi(c^{-1}(\alpha)) - \frac{1}{2\rho}\, W'(c^{-1}(\alpha)) = 0, \tag{6.87}$$

and

$$\sqrt{\frac{E}{\rho}}\,(\alpha + \sqrt{\frac{E}{\rho}})\; f_{-\infty} - \frac{D}{2\rho} + \frac{1}{2}\sqrt{\frac{E}{\rho}}\;\alpha c^{-1}(\alpha)$$

$$- \frac{1}{2}(\alpha + \sqrt{\frac{E}{\rho}})\; \phi(c^{-1}(\alpha)) + \frac{1}{2\rho}\, W'(c^{-1}(\alpha)) = 0 \; . \tag{6.88}$$

Eliminating $f_{-\infty}$ between (6.87) and (6.88) we arrive at an equation for α:

$$G(\alpha) \equiv (\alpha^2 - \frac{E}{\rho})\,\frac{\sigma_{\infty}}{E} + \frac{1}{\rho}\, W'(c^{-1}(\alpha)) - \alpha^2 c^{-1}(\alpha) - \frac{\alpha^2 D}{E} = 0 \; . \tag{6.89}$$

Any root of $G(\alpha)=0$ in the interval $0 < \alpha < \sqrt{\frac{E}{\rho}}$ furnishes a candidate for the asymptotic phase boundary speed. We now show that there is a unique root α of (6.89) in the above interval. To this end note that

$$G(0) = -\frac{1}{\rho}\sigma_\infty + \frac{1}{\rho} W(c^{-1}(0)) = -\frac{1}{\rho}(\sigma_\infty - \sigma_0(t_\beta)) < 0 \quad , \tag{6.90}$$

and

$$G(\sqrt{\frac{E}{\rho}}) = \frac{1}{\rho}(W'(c^{-1}(\sqrt{\frac{E}{\rho}})) - E\, c^{-1}(\sqrt{\frac{E}{\rho}}) - D) > 0 \ , \tag{6.91}$$

where the inequality in (6.91) follows from the Lemma. Thus we are guaranteed at least one root in the desired interval. To establish the uniqueness of this root, we demonstrate that $G(\alpha)$ is strictly increasing on the interval $0 < \alpha < \sqrt{\frac{E}{\rho}}$. From (6.89),

$$\frac{dG}{d\alpha} = \frac{2\alpha}{E}(\sigma_\infty - Ec^{-1}(\alpha) - D) + (\frac{1}{\rho} W''(c^{-1}(\alpha)) - \alpha^2)\frac{d}{d\alpha} c^{-1}(\alpha) \ . \tag{6.92}$$

Whereas (2.18) gives

$$(\frac{1}{\rho} W''(c^{-1}(\alpha)) - \alpha^2) = 0 \quad , \tag{6.93}$$

which implies that (6.92) may be rewritten as

$$\frac{dG}{d\alpha} = \frac{2\alpha}{E}([\sigma_\infty - \sigma_\beta] + [\sigma_\beta - W'(c^{-1}(\alpha))] + [W'(c^{-1}(\alpha)) - Ec^{-1}(\alpha) - D]) \ . \tag{6.94}$$

Since each of the bracketed quantities in (6.94) is positive on $0 < \alpha < \sqrt{\frac{E}{\rho}}$, it follows that

$$\frac{dG}{d\alpha} > 0 \qquad \text{for} \qquad 0 < \alpha < \sqrt{\frac{E}{\rho}} \ . \tag{6.95}$$

Consequently (6.89) furnishes a unique value for the asymptotic phase boundary speed in the interval $0 < \alpha < \sqrt{\frac{E}{\rho}}$. One may now find $f_{-\infty}$ from either (6.87) or (6.88).

From (6.89) it follows that the larger σ_∞, the larger is the asymptotic phase boundary speed. Moreover, as the last three terms of (6.89) are bounded, the unique root approaches $\sqrt{\frac{E}{\rho}}$ as σ_∞ tends to infinity. This, of course, is to be expected from the results for Case I.

In contrast to Case I, we have obtained here, from the $O(1)$ terms in (6.33), only the leading asymptotic terms for s(t) and f(z) which are α and $f_{-\infty}$. Hence the first corrections to both s(t) and f(z) will undoubtedly require consideration of the $O(\frac{1}{t})$ term in (6.25).

We now consider materials which admit the possibility that the propagating phase boundary becomes the leading disturbance, that is, materials for which

$$W''(0) < E \quad , \qquad c(0) = \sqrt{\frac{W''(0)}{\rho}} < \sqrt{\frac{E}{\rho}} \; . \tag{6.96}$$

It then follows from (2.14), (2.15), (2.17), (4.2), (6.18), and an application of the mean value theorem that

$$D < 0 \quad . \tag{6.97}$$ [1]

This inequality is also apparent from the graph of an appropriate stress-strain curve that obeys (6.96).

Since the phase boundary will <u>not</u> become the leading disturbance if $\alpha \leq c(0) = \sqrt{\frac{W''(0)}{\rho}}$ (see (6.10)), (6.33) applies when

$$0 < \alpha \leq c(0) \quad . \tag{6.98}$$

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[1] $D = W'(\beta_2) - E\beta_2 < W'(\beta_1) - E\beta_1 = W'(0) + W''(0)\beta_1 + \frac{1}{2} W'''(\bar{\varepsilon})\beta_1^2 - E\beta_1$

$= (W''(0) - E)\beta_1 + \frac{1}{2} W'''(\bar{\varepsilon})\beta_1^2 < (W''(0) - E)\,\beta_1 < 0$, where $0 < \bar{\varepsilon} < \beta_1$.

Conversely, the phase boundary becomes the leading disturbance if $\alpha > c(0)$, so that (6.34) applies when

$$c(0) < \alpha \leq \sqrt{\frac{E}{\rho}} \quad . \tag{6.99}$$

<u>Case III: $W''(0) < E$, $\sigma_0(t) \to \sigma_\infty < \infty$</u>

Let us suppose, prior to the analysis, that the phase boundary does not become the leading disturbance. Then we can once again appeal to the analysis which yielded (6.89), provided we also satisfy (6.98). The roots of $G(\alpha) = 0$, with α satisfying (6.98) furnish candidates for the asymptotic phase boundary speed. Once again,

$$G(0) < 0 \quad , \quad \frac{dG}{d\alpha} > 0 \qquad \text{for} \qquad 0 < \alpha < c(0). \tag{6.100}$$

Hence (6.89) has a unique root in the interval given by (6.98) if and only if

$$G(c(0)) = (c(0)^2 - \frac{E}{\rho}) \frac{\sigma_\infty}{E} - \frac{c(0)^2 D}{E} \geq 0 \quad . \tag{6.101}$$

Therefore, a necessary condition for the phase boundary not to become the leading disturbance is that

$$\sigma_\infty \leq \frac{-D\, c(0)^2}{\frac{E}{\rho} - c(0)^2} \quad . \tag{6.102}$$

Now let us suppose first that the phase boundary becomes the leading disturbance. Again we denote the time at which this occurs by t_L. Then (6.34) applies when $t > t_L$. Here the asymptotic phase boundary speed must lie in the interval given by (6.99). As before we dismiss the possibility that $\alpha = \sqrt{\frac{E}{\rho}}$ since then the asymptotic equation resulting from (6.34) requires that

$$\frac{1}{2}\frac{D}{\rho} = 0 \quad , \tag{6.103}$$

contradicting (6.97). Hence, $c(0) < \alpha < \sqrt{\frac{E}{\rho}}$, so that we again have (6.85) and (6.86). The asymptotic equations provided by (6.34) are

$$\sqrt{\frac{E}{\rho}}\,(-\alpha + \sqrt{\frac{E}{\rho}})\, f_{-\infty} + \frac{1}{\sqrt{\rho E}}\,(-\alpha + \sqrt{\frac{E}{\rho}})\sigma_{\infty} - \frac{D}{\sqrt{\rho E}}\,(-\alpha + \frac{1}{2}\sqrt{\frac{E}{\rho}}) = 0, \tag{6.104}$$

and

$$\sqrt{\frac{E}{\rho}}\,(\alpha + \sqrt{\frac{E}{\rho}})\, f_{-\infty} - \frac{1}{2}\frac{D}{\rho} = 0 \quad . \tag{6.105}$$

Elimination of $f_{-\infty}$ between (6.104) and (6.105) leads to the equation

$$(-\alpha^2 + \frac{E}{\rho})\,\sigma_{\infty} + \alpha^2 D = 0 \quad , \tag{6.106}$$

which has the single positive root

$$\alpha = \frac{\sqrt{\frac{E}{\rho}}}{(1 - \frac{D}{\sigma_{\infty}})^{1/2}} \quad . \tag{6.107}$$

By (6.97), $\alpha < \sqrt{\frac{E}{\rho}}$. Finally, the first inequality in (6.99) requires

$$\frac{\sqrt{\frac{E}{\rho}}}{(1 - \frac{D}{\sigma_{\infty}})^{1/2}} > c(0) \quad , \tag{6.108}$$

or equivalently,

$$\sigma_{\infty} > \frac{-D\ c(0)^2}{\frac{E}{\rho} - c(0)^2} \quad . \tag{6.109}$$

Inequalities (6.102) and (6.109) reveal the circumstances under which the phase boundary becomes the leading disturbance. If the ultimate end load σ_{∞} is less than or equal to a cut-off load

$$\sigma_L \equiv \frac{-D\ c(0)^2}{\frac{E}{\rho} - c(0)^2} \quad , \tag{6.110}$$

the phase boundary is confined to the disturbed region and the asymptotic phase boundary speed is given by the unique root of $G(\alpha) = 0$ in the interval $0 < \alpha \leq c(0)$. Conversely, if $\sigma_{\infty} > \sigma_L$, the phase boundary becomes the leading disturbance and the asymptotic phase boundary speed is given by (6.107). We note that an application of the mean-value theorem (in a manner similar to that which led to (6.97)) yields

$$\sigma_L > \sigma_\beta = \sigma_o(t_\beta) \quad . \tag{6.111}$$

If we define α as a function of σ_{∞} via (6.89) when (6.102) holds, and via (6.107) when (6.109) holds, it will follow that α is a continuous and increasing function on $\sigma_\beta < \sigma_{\infty} < \infty$. As σ_{∞} approaches σ_β, one has $\alpha \to 0$. When $\sigma_{\infty} = \sigma_L$, it follows that $\alpha = c(0)$, so that in the x-t plane the phase boundary is confined to the disturbed region, but is asymptotically parallel to the leading characteristic curve $x = -c(0)t$. Finally as $\sigma_{\infty} \to \infty$, α approaches $\sqrt{\frac{E}{\rho}}$.

Having obtained α, we find $f_{-\infty}$ when $\sigma_{\infty} \leq \sigma_L$ from either (6.87) or (6.88). Here, as in Case II, further corrections to $s(t)$ and $f(z)$

will again require consideration of the $O(\frac{1}{t})$ term in (6.25).

When $\sigma_\infty > \sigma_L$ we obtain $f_{-\infty}$ from either (6.104) or (6.105). The result is

$$f_{-\infty} = \frac{D}{2\sqrt{\rho E}} \frac{1}{(\alpha + \sqrt{\frac{E}{\rho}})} = \frac{D}{2E} \frac{(1 - \frac{D}{\sigma_\infty})^{1/2}}{1 + (1 - \frac{D}{\sigma_\infty})^{1/2}} < 0 . \qquad (6.112)$$

Asymptotic corrections to <u>all</u> orders for f(z) and s(t) are found from (6.34). Since the analysis is not difficult and the result is interesting, we proceed to obtain some of these corrections. Let

$$\sigma_0(t) = \sigma_\infty + \sigma_1(t) , \quad \dot{\sigma}_1(t) > 0 , \quad \sigma_1(t) = o(1) \quad \text{as} \quad t \to \infty . \qquad (6.113)$$

Substituting from (6.113), (6.64), (6.65) into (6.34) and cancelling the previously balanced $O(1)$-terms eventually yields the asymptotic equations:

$$(f_{-\infty} + \frac{\sigma_\infty}{E} - \frac{D}{E}) \dot{q}(t) + (-\alpha + \sqrt{\frac{E}{\rho}}) f_1((\alpha - \sqrt{\frac{E}{\rho}}) t)$$
$$+ (-\alpha + \sqrt{\frac{E}{\rho}}) \frac{1}{E} \sigma_1((1 - \sqrt{\frac{\rho}{E}} \alpha) t) = 0 , \qquad (6.114)$$

$$-\dot{q}(t) f_{-\infty} + (\alpha + \sqrt{\frac{E}{\rho}}) f_1((-\alpha - \sqrt{\frac{E}{\rho}}) t) = 0 . \qquad (6.115)$$

Eliminating $\dot{q}(t)$ between (6.114) and (6.115), one has

$$(f_{-\infty} + \frac{\sigma_\infty}{E} - \frac{D}{E})(\alpha + \sqrt{\frac{E}{\rho}}) f_1((-\alpha - \sqrt{\frac{E}{\rho}}) t)$$
$$+ f_{-\infty} (-\alpha + \sqrt{\frac{E}{\rho}}) f_1((\alpha - \sqrt{\frac{E}{\rho}}) t)$$
$$+ \frac{f_{-\infty}}{E} (-\alpha + \sqrt{\frac{E}{\rho}}) \sigma_1((1 - \sqrt{\frac{\rho}{E}} \alpha) t) = 0 , \qquad (6.116)$$

which may be used to determine the asymptotic behavior of $f_1(z)$. From (6.115) the asymptotic behavior of $\dot{q}(t)$ is then given by

$$\dot{q}(t) \sim \frac{(\alpha + \sqrt{\frac{E}{\rho}})}{f_{-\infty}} f_1((-\alpha - \sqrt{\frac{E}{\rho}})t) \quad . \tag{6.117}$$

We are also now in a position to determine the asymptotic behavior of $\int_{t^*}^{t} f_1((-\alpha - \sqrt{\frac{E}{\rho}})s)ds$, where t^* is some fixed time greater than t_L. If this integral is unbounded as $t \to \infty$, it follows upon integration of (6.117) that the contribution due to the lower limit of integration t^* is subdominant. In this case we may further conclude that

$$q(t) \sim \frac{(\alpha + \sqrt{\frac{E}{\rho}})}{f_{-\infty}} \int_{t^*}^{t} f_1((-\alpha - \sqrt{\frac{E}{\rho}})s)ds \quad . \tag{6.118}$$

Consider as an <u>example</u>,

$$\sigma_0(t) = \sigma_\infty + \sigma_1 t^{-n} + o(t^{-n}) \ , \quad \sigma_1 < 0 \ , \ n > 0 \ . \tag{6.119}$$

Then (6.116) suggests the asymptotic form

$$f_1(z) \sim \overline{f}_1(-z)^{-n} \qquad \text{as} \qquad z \to -\infty \quad , \tag{6.120}$$

where $\overline{f}_1$ is a constant to be determined. Substitution from (6.119) and (6.120) into (6.116) yields

$$\overline{f}_1 = \frac{\frac{-f_{-\infty}}{\sqrt{\rho E}}(1 - \sqrt{\frac{\rho}{E}}\alpha)^{-n+1} \quad \sigma_1}{f_{-\infty}((\alpha + \sqrt{\frac{E}{\rho}})^{-n+1} + (-\alpha + \sqrt{\frac{E}{\rho}})^{-n+1}) + \frac{\sigma_\infty - D}{E}(\alpha + \sqrt{\frac{E}{\rho}})^{-n+1}} < 0 \ , \tag{6.121}$$

where the inequality follows from $\sigma_1 < 0$, $\alpha < \sqrt{\frac{E}{\rho}}$ and (6.112). Further from (6.117),

$$\dot{q}(t) \sim \frac{\bar{f}_1}{f_{-\infty}} \left(\alpha + \sqrt{\frac{E}{\rho}}\right)^{-n+1} t^{-n} > 0 \quad , \tag{6.122}$$

so that as $t \to \infty$,

$$q(t) \sim \begin{cases} \dfrac{\bar{f}_1 \left(\alpha + \sqrt{\frac{E}{\rho}}\right)^{-n+1}}{f_{-\infty}(-n+1)} \, t^{-n+1} & \text{for } 0 < n < 1 \, , \\ \dfrac{\bar{f}_1}{f_{-\infty}} \, \ell n \, t & \text{for } n = 1 \, , \\ \text{constant} & \text{for } n > 1 \, . \end{cases} \tag{6.123}$$

Case IV: $W''(0) < E$, $\sigma_0(t) \to \infty$

Since in this case $\sigma_0(t)$ eventually exceeds σ_L, the results of the analysis for Case III allow us to conclude that the phase boundary will become the leading disturbance at some time, again denoted by t_L. For $t > t_L$, the governing equations are once more given exactly by (6.34). An argument like that given in Case I yields

$$\alpha = \sqrt{\frac{E}{\rho}} \quad , \tag{6.124}$$

This result could have been anticipated by formally letting $\sigma_\infty \to \infty$ in (6.107). We also find

$$f_{-\infty} = \frac{D}{4E} < 0 \quad . \tag{6.125}$$

As in Case I, it follows that

$$\lim_{t \to \infty} q(t) = +\infty , \tag{6.126}$$[1]

$$\int^{q} \sigma_0\left(\sqrt{\frac{\rho}{E}}\, s\right) ds \sim -\frac{1}{2}\sqrt{\frac{E}{\rho}}\, Dt > 0 \quad . \tag{6.127}$$[2]

Let us again consider the asymptotic power-law loading of Example 1, whence

$$\sigma_0(t) = k_1 t^n + o(t^n) \ , \quad k_1 > 0 \ , \quad 0 < n < \infty .$$

Then (6.127) and (6.77) yield

$$q(t) \sim \left(-\frac{n+1}{2k_1} D\right)^{\frac{1}{n+1}} \sqrt{\frac{E}{\rho}}\, t^{\frac{1}{n+1}} \ . \tag{6.128}$$

The simple form of (6.34) allows the computation of the next corrections for $s(t)$ and $f(z)$, provided we know the next term in an asymptotic series for $\sigma_0(t)$. Consider the following extension of (6.77):

$$\sigma_0(t) = k_1 t^n + k_2 t^m + o(t^m) \ , \quad k_1 > 0 \ , \quad n > 0 \ , \quad m < n \ . \tag{6.129}$$

An asymptotic analysis of (6.34) then reveals that

$$s(t) \sim -\sqrt{\frac{E}{\rho}}\, t + \left(-\frac{n+1}{2k_1} D\right)^{\frac{1}{n+1}} \sqrt{\frac{E}{\rho}}\, t^{\frac{1}{n+1}} + \bar{q}_1\, t^P \quad \text{as} \quad t \to \infty , \tag{6.130}$$

$$f(z) \sim \frac{D}{4E} + \bar{f}_1 (-z)^{\frac{-n}{n+1}} \qquad \text{as} \quad z \to -\infty \ , \tag{6.131}$$

where

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[1] See the discussion leading to (6.74).

[2] See (6.76).

$$P = \begin{cases} \dfrac{1-n+m}{n+1} & \text{for} \quad m \geq 0 \, , \\ \\ \dfrac{1-n}{n+1} & \text{for} \quad m \leq 0 \, , \end{cases} \qquad (6.132)$$

and $\overline{f}_1$, $\overline{q}_1$ are computable constants.

7. Summary and Conclusions

In this final section we shall describe the strain, stress and velocity fields in the bar, with particular reference to their asymptotic behavior for times both very near t_β and much greater than t_β. Also we present a summary of our results for the bar loaded in a hard device. We close with a discussion of other pertinent aspects of the problems we have considered.

A. The Strains and Velocities on the Phase Boundary

We have already seen that

$$\frac{d}{dt}\varepsilon(s(t)^-,t) \leq 0, \quad \frac{d}{dt} v(s(t)^-,t) \leq 0, \quad t \geq t_\beta, \qquad (7.1)^1$$

hence the strain, stress and velocity immediately ahead of the phase boundary are monotonically decreasing with time. Immediately behind the phase boundary we have from (4.10) that

$$\frac{d}{dt} v(s(t)^+,t) = \sqrt{\frac{E}{\rho}}\; f'(s(t) - \sqrt{\frac{E}{\rho}}\, t)(\dot{s}(t) - \sqrt{\frac{E}{\rho}}) + \sqrt{\frac{E}{\rho}}\, f'(-s(t) - \sqrt{\frac{E}{\rho}}\, t)(-\dot{s}(t) - \sqrt{\frac{E}{\rho}}) + \dot{\varepsilon}_0(t + \sqrt{\frac{\rho}{E}}\, s(t))(\sqrt{\frac{E}{\rho}} + \dot{s}(t)), \qquad (7.2)$$

and

$$\frac{d}{dt}\varepsilon(s(t)^+,t) = -\, f'(s(t) - \sqrt{\frac{E}{\rho}}\, t)(\dot{s}(t) - \sqrt{\frac{E}{\rho}}) + f\,(-s(t) - \sqrt{\frac{E}{\rho}}\, t) \times (-\dot{s}(t) - \sqrt{\frac{E}{\rho}}) + \dot{\varepsilon}_0(t + \sqrt{\frac{\rho}{E}}\, s(t))(1 + \sqrt{\frac{\rho}{E}}\, \dot{s}(t)). \qquad (7.3)$$

From (2.32), (6.2), (6.17), (6.60) and (7.2) it follows that

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[1] See (6.7).

$$\frac{d}{dt} v(s(t)^{+},t) > 0 \quad , \quad t > t_{\beta}. \tag{7.4}$$

To obtain a corresponding monotonicity result for $\varepsilon(s(t)^{+},t)$ we must go beyond (7.3) since there the first term works against us. Substituting from (6.53) and (6.54) into (7.3) we find after some algebra that

$$\frac{d}{dt} \varepsilon(s(t)^{+},t) = \left[\frac{c(\varepsilon_0(\tau(s(t),t)))^2 - \dot{s}(t)^2}{\frac{E}{\rho} - \dot{s}(t)^2}\right] \frac{d}{dt} \varepsilon_0(\tau(s(t),t))$$
$$+ \frac{\ddot{s}(t)}{2\sqrt{\rho E}} \left[\frac{1}{(\dot{s}(t) - \sqrt{\frac{E}{\rho}})^2} - \frac{1}{(\dot{s}(t) + \sqrt{\frac{E}{\rho}})^2}\right] (W'(\varepsilon_0(\tau(s(t),t))) -$$
$$E\varepsilon_0(\tau(s(t),t)) - D). \tag{7.5}$$

Hence (6.1), (6.7) (6.8), (6.57) and (7.5) yield

$$\frac{d}{dt} \varepsilon(s(t)^{+},t) > 0 \; , \quad t > t_{\beta} \; . \tag{7.6}$$

Thus the strain, stress and velocity immediately behind the phase boundary are monotonically increasing with time.

B. Summary of Results for Small Time

From (5.40) -(5.42), the small-time asymptotic results for s(t) and f(z) are summarized as follows:

$$s(t) = -\left(\frac{\dot{\sigma}_0(t_\beta)}{3\rho(\beta_2 - \beta_1)}\right)^{1/2} (t - t_\beta)^{3/2} + o\left[(t - t_\beta)^{3/2}\right] \; , \tag{7.7}$$

$$f(z) = \frac{1}{2}\left(\sqrt{\frac{\rho}{E}}\,\phi(\beta_1) - \beta_2\right) + \frac{1}{4}\left(\frac{E}{\rho}\right)^{-3/4}\left(\frac{3(\beta_2 - \beta_1)\,\dot{\sigma}_0(t_\beta)}{\rho}\right)^{1/2}$$
$$(-z - \sqrt{\frac{E}{\rho}}\,t_\beta)^{1/2} + o\left[(-z - \sqrt{\frac{E}{\rho}}\,t_\beta)^{1/2}\right] . \tag{7.8}$$

An appeal to (4.10), (4.2), (7.7) and (7.8) determines the strain, stress and velocity immediately behind the phase boundary:

$$\varepsilon(s(t)^+,t) = \beta_2 + \frac{3}{4}\frac{\dot{\sigma}_o(t_o)}{E}(t - t_\beta) + o[(t - t_\beta)], \qquad (7.9)$$

$$\sigma(s(t)^+,t) = \sigma_\beta + \frac{3}{4}\dot{\sigma}_o(t_\beta)(t - t_\beta) + o[(t - t_\beta)], \qquad (7.10)$$

$$v(s(t)^+,t) = \phi(\beta_1) + \frac{1}{2}\left(\frac{3(\beta_2-\beta_1)\dot{\sigma}_o(t_\beta)}{\rho}\right)^{1/2}(t-t_\beta)^{1/2} + o[(t-t_\beta)^{1/2}]. \qquad (7.11)$$

As required, these three field quantities are smoothly increasing with time. In particular we note that although the quantity $\beta_2-\beta_1$ — which goes to zero with the length of the interval for the unstable material phase — appears in the denominator of s_o, it does not occur in the denominators of any of the expansion coefficients appearing in (7.9)-(7.11).

For comparison, we display the leading terms in the series expansions for the prescribed load and the resulting strain and velocity at the end of the bar:

$$\varepsilon(0,t) = \beta_2 + \frac{\dot{\sigma}_o(t_\beta)}{E}(t - t_\beta) + o[(t - t_\beta)^2], \qquad (7.12)$$

$$\sigma(0,t) = \sigma_\beta + \dot{\sigma}_o(t_\beta)(t - t_\beta) + o[(t-t_\beta)^2], \qquad (7.13)$$

$$v(0,t) = \phi(\beta_1) + \frac{1}{2}\;\frac{3(\beta_2-\beta_1)\dot{\sigma}_o(t_\beta)}{\rho}^{1/2}(t- t_\beta)^{1/2} + o[(t-t_\beta)^{1/2}]. \qquad (7.14)$$

Between the end of the bar and the phase boundary, where

$$s(t) < x \leq 0\ , \qquad (7.15)$$

we have that

$$\beta_2 + \frac{3}{4}\frac{\dot{\sigma}_o(t_\beta)}{E}(t-t_\beta) + o[(t-t_\beta)] \le \varepsilon(x,t) \le \beta_2 + \frac{\dot{\sigma}_o(t_\beta)}{E}(t-t_\beta) + o[(t-t_\beta)], \tag{7.16}$$

$$\sigma_\beta + \frac{3}{4}\dot{\sigma}_o(t_\beta)(t-t_\beta) + o[(t-t_\beta)] \le \sigma(x,t) \le \sigma_\beta + \dot{\sigma}_o(t_\beta)(t-t_\beta) + o[(t-t_\beta)], \tag{7.17}$$

$$v(x,t) = \phi(\beta_1) + \frac{1}{2}\left(\frac{3(\beta_2-\beta_1)\dot{\sigma}_o(t_\beta)}{\rho}\right)^{1/2}(t-t_\beta)^{1/2} + o[(t-t_\beta)^{1/2}]. \tag{7.18}$$

We note that the proposed expansion scheme (5.11), (5.12) and (5.43) would suggest that the "o" error bounds appearing in (7.7) -(7.11), (7.16)-(7.18) may be tightened to "O" of the next larger half-integer power.

We turn now to a determination of the various field quantities immediately ahead of the phase boundary. Here from (5.22), (5.23), (5.39) and (5.40):

$$\tau(s(t),t) = t_\beta - \left(\frac{-\dot{\sigma}_o(t_\beta)}{18(\beta_2-\beta_1)^2 W'''(\beta_1)}\right)(t-t_\beta)^2 + o[(t-t_\beta)^2]. \tag{7.19}$$

Furthermore (2.30), (5.25), (5.26), (5.29)-(5.31), (5.39) and (5.40) yield:

$$\varepsilon(s(t)^-,t) = \beta_1 - \left(\frac{-\dot{\sigma}_o(t_\beta)}{3(\beta_2-\beta_1)W'''(\beta_1)}\right)(t-t_\beta) + o[(t-t_\beta)], \tag{7.20}$$

$$\sigma(s(t)^-,t) = \sigma_\beta - \left(\frac{-\dot{\sigma}_o(t_\beta)^2}{18(\beta_2-\beta_1)^2 W'''(\beta_1)}\right)(t-t_\beta)^2 + o[(t-t_\beta)^2], \tag{7.21}$$

$$v(s(t)^-,t) = \phi(\beta_1) - \left(\frac{-2\,\dot{\sigma}_o(t_\beta)^{3/2}}{3^{5/2}\rho^{1/2}(\beta_2-\beta_1)^{3/2}W'''(\beta_1)}\right)(t-t_\beta)^{3/2} + o[(t-t_\beta)^{3/2}]. \tag{7.22}$$

As required, the above field quantities are smoothly decreasing with time. We also note the presence of $\beta_2-\beta_1$ in the denominators of the coefficients of the terms giving the leading order time dependence. The final quantity which we shall examine is the acoustic speed immediately ahead of the phase boundary. From (4.20h), (7.7) and (7.19) it follows that

$$c(\varepsilon(s(t)^-,t)) = \left(\frac{\dot{\sigma}_0(t_\beta)}{3\rho(\beta_2-\beta_1)}\right)^{1/2}(t-t_\beta)^{1/2} + o[(t-t_\beta)^{1/2}]. \qquad (7.23)$$

We note that (7.7) and (7.23) confirm (6.6) for small-time. The determination of the exact order of the remainder terms appearing in (7.20)-(7.23) is complicated by the multiple nesting of functional arguments occurring in the various formulae, as well as by the need to determine the next order correction to (7.19).

C. Summary of Results for Large Time

We have treated separately the four cases (6.35)-(6.38) and have found that

$$s(t) = -\alpha t + q(t) \qquad \text{as } t \to \infty, \qquad (7.24)$$

$$f(z) = f_{-\infty} + f_1(z) \qquad \text{as } z \to -\infty, \qquad (7.25)$$

where

$$0 < \alpha \leq \sqrt{\frac{E}{\rho}} \qquad (7.26)$$

and

$$q(t) = o(t), \quad \dot{q}(t) = o(1) \qquad \text{as } t \to \infty, \qquad (7.27)$$

$$f_1(z) = o(1) \qquad \text{as } z \to -\infty. \qquad (7.28)$$

We first considered the cases given by (6.35) and (6.36), for which

$$W''(0) \geq E. \qquad (7.29)$$

For these cases the phase boundary cannot become the leading disturbance. When the applied load approaches a finite limit σ_∞ the asymptotic phase boundary speed α is given by the unique root of (6.89) occurring within the interval $0<\alpha<\sqrt{\frac{E}{\rho}}$. It follows that α is an increasing function of σ_∞ and approaches $\sqrt{\frac{E}{\rho}}$ as σ_∞ tends to infinity. The constant $f_{-\infty}$ may then be found from either (6.87) or (6.88). When the applied load is unbounded, $\alpha = \sqrt{\frac{E}{\rho}}$ and $\lim_{t\to\infty} q(t) = \infty$. In particular the dominant behavior of q(t) is given implicitly by (6.76), while $f_{-\infty}$ is given by (6.73).

For the cases given by (6.37) and (6.38), we have

$$W''(0) < E. \tag{7.30}$$

For these cases, if the applied load remains below the value σ_L, given by (6.110), the phase boundary does not become the leading disturbance. Here α is the unique root of (6.89) occurring within the interval $(0,c(0)]$, while $f_{-\infty}$ is again found from either (6.87) or (6.88). If the applied load exceeds σ_L the phase boundary eventually becomes the leading disturbance. In particular if the applied load approaches a limit $\sigma_\infty > \sigma_L$, α is given by (6.107) and $f_{-\infty}$ by (6.112). If the applied load is unbounded, $\alpha = \sqrt{\frac{E}{\rho}}$ and $f_{-\infty} = D/4E$. In this latter circumstance q(t) is also unbounded, the dominant behavior being given implicitly by (6.127).

We have already noted that ahead of the phase boundary, $\varepsilon(s(t)^-,t)$, $\sigma(s(t)^-,t)$ and $v(s(t)^-,t)$ are monotonically decreasing with time. Of course if the phase boundary becomes the leading disturbance, say again at $t = t_L$, these quantities are all zero for $t > t_L$. Otherwise the phase boundary is confined to the disturbed region and is asymptotically parallel to the characteristic ray C_- in R_1 which issues from $x = 0$ at time $t = \tau_\infty$, where

$$\tau_\infty = \Omega(\alpha) \ . \qquad (7.31)^1$$

Then (6.30)-(6.32) yield the asymptotic results:

$$\varepsilon(s(t)^-,t) \sim c^{-1}(\alpha) \quad \text{as } t \to \infty, \qquad (7.32)$$

$$\sigma(s(t)^-,t) \sim W'(c^{-1}(\alpha)) \text{ as } t \to \infty, \qquad (7.33)$$

$$v(s(t)^-,t) \sim \phi(c^{-1}(\alpha)) \quad \text{as } t \to \infty. \qquad (7.34)$$

Consider the fixed particle P initially at x. Since $s(t_\beta) = 0$, it follows from (5.9) and (6.1) that for any given $x < 0$ the equation $x = s(t)$ has a unique root $t = t_x$ with $t_x > t_\beta$. Consequently the particle P finds itself in the high-strain phase for times greater than t_x. For fixed x we have

$$\lim_{t\to\infty} f(x - \sqrt{\tfrac{E}{\rho}}\, t\ ) = f_{-\infty}\ , \qquad \lim_{t\to\infty} f(-x - \sqrt{\tfrac{E}{\rho}}\, t\ ) = f_{-\infty}, \qquad (7.35)$$

which, with (4.6) and (4.10), yield the following large-time asymptotic results for the stress, strain and velocity at the particle P initially at x:

$$\sigma(x,t) \sim \sigma_0(t + \sqrt{\tfrac{\rho}{E}}\, x) \qquad \text{as } t \to \infty, \qquad (7.36)$$

$$\varepsilon(x,t) \sim \varepsilon_0(t + \sqrt{\tfrac{\rho}{E}}\, x) = [\sigma_0(t + \sqrt{\tfrac{\rho}{E}}\, x) - D]/E \quad \text{as } t \to \infty, \qquad (7.37)$$

$$v(x,t) \sim [\sigma_0(t + \sqrt{\tfrac{\rho}{E}}\, x) - D]/\sqrt{\rho E} + 2\sqrt{\tfrac{E}{\rho}}\, f_{-\infty} \quad \text{as } t \to \infty. \qquad (7.38)$$

It is interesting to compare (7.36)-(7.38) to the stress, strain and velocity fields occurring in a companion problem involving a linearly elastic bar, for which the stress is related to the strain via

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[1] See (6.11), (6.12) and (6.28).

$$\sigma = E\varepsilon + D_2, \qquad \varepsilon \geq 0. \tag{7.39}$$

In this companion problem we shall denote the strain, stress and velocity fields by $\hat{\varepsilon}(x,t)$, $\hat{\sigma}(x,t)$ and $\hat{v}(x,t)$. The response of such a linearly elastic bar subject to the prescribed end-loading

$$\hat{\sigma}(0,t) = \sigma_0(t), \tag{7.40}$$

in the disturbed region $x \geqslant -\sqrt{\frac{E}{\rho}}\, t$ is given by

$$\hat{\sigma}(x,t) = \sigma_0(t + \frac{\rho}{E} x) , \tag{7.41}$$

$$\hat{\varepsilon}(x,t) = [\sigma_0(t + \sqrt{\frac{\rho}{E}}\, x) - D_2] \,/\, E , \tag{7.42}$$

$$\hat{v}(x,t) = [\sigma_0(t + \sqrt{\frac{\rho}{E}}\, x) - D_2] \,/ \sqrt{\rho E} . \tag{7.43}$$

Two cases of interest in comparing (7.36)-(7.38) to (7.41)-(7.43) are $D_2 = 0$ — in which case the linearly elastic bar when stress-free is unstrained — and $D_2 = D$, in which instance (7.39) is the linear extrapolation back through $\varepsilon = 0$ of the final linear portion of the stress-strain curve occurring in the original problem. Recall that D can be either positive or negative[1]. According to (6.63),

$$f_{-\infty} \geq f(z) \geq f(-\sqrt{\frac{E}{\rho}}\, t_\beta) = \frac{1}{2}\left(\int_0^{\beta_1} \sqrt{\frac{W''(s)}{E}}\, ds - \beta_2\right), \tag{7.44}$$

so that $f(-\sqrt{\frac{E}{\rho}}\, t_\beta)$ may be either positive or negative depending on the value of β_2 and on the detailed behavior of $W(\varepsilon)$ within the interval $(0,\beta_1)$. Similarly $f_{-\infty}$ may be either positive or negative[2].

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[1] As we have seen $D < 0$ when $W''(0) < E$; see (6.97).

[2] For example when $W''(0) < E$, it follows that $f(-\sqrt{\frac{E}{\rho}}\, t_\beta) < 0$. However if $W''(\varepsilon) > 9E$ for $0 \leq \varepsilon \leq \frac{1}{2}\beta_1$, and $\beta_2 < \frac{3}{2}\beta_1$, it follows that $f(-\sqrt{\frac{E}{\rho}}\, t_\beta) > 0$. In this case (7.44) provides $f_{-\infty} > 0$. Conversely, (6.112) furnishes an example with $f_{-\infty} < 0$.

Consequently, apart from the monotonicity of f, there is little we can say concerning its range without stipulating further conditions on the stress-strain relation.

One case of interest is that of a material with a small nonlinearity. Suppose

$$|W''(\varepsilon) - E| \le \delta_2 \quad \text{for} \quad 0 \le \varepsilon < \beta_1 - \delta_1, \text{ and that } \beta_2 = \beta_1 + \delta_1, \tag{7.45}$$

where

$$\beta_1 >> \delta_1 > 0 \quad \text{and} \quad E >> \delta_2 > 0. \tag{7.46}$$ [1]

Here the material is "almost linear" except in a δ_1-neighborhood about $\varepsilon = \beta_1$. For such a material

$$\lim_{\delta_1, \delta_2 \to 0} f(-\sqrt{\frac{E}{\rho}}\, t_\beta) = \lim_{\delta_1, \delta_2 \to 0} \frac{1}{2} \left(\int_0^{\beta_1} \sqrt{\frac{W''(s)}{E}}\, ds - \beta_2 \right) = 0, \tag{7.47}$$

and

$$\lim_{\delta_1, \delta_2 \to 0} D = \lim_{\delta_1, \delta_2 \to 0} (W'(\beta_1) - E\beta_2) = 0 \, . \tag{7.48}$$

We now show that in all of the cases set down in (6.35)-(6.38),

$$\lim_{\delta_1, \delta_2 \to 0} f_{-\infty} = 0 \, . \tag{7.49}$$

When the phase boundary becomes the leading disturbance, (7.49) follows from (7.48) and either (6.112) or (6.125). When the phase boundary is confined to the disturbed region, we examine either (6.73) or (6.88). First note that for $\varepsilon < \beta_1$

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[1] If it were not for (6.18), we would take $\delta_2 = 0$.

$$\lim_{\delta_1,\delta_2\to 0} W'(\varepsilon) = E\varepsilon \quad , \tag{7.50}$$

$$\lim_{\delta_1,\delta_2\to 0} \phi(\varepsilon) = \lim_{\delta_1,\delta_2\to 0} \int_0^{\varepsilon}\sqrt{\frac{W''(s)}{\rho}}\, ds = \int_0^{\varepsilon}\sqrt{\frac{E}{\rho}}\, ds = \sqrt{\frac{E}{\rho}}\, \varepsilon. \tag{7.51}$$

When (6.88) applies, we have

$$\lim_{\delta_1,\delta_2\to 0} f_{-\infty} = \frac{1}{\sqrt{\frac{E}{\rho}}(\alpha+\sqrt{\frac{E}{\rho}})} \lim_{\delta_1,\delta_2\to 0}\Big[\frac{D}{2\rho} - \frac{1}{2}\sqrt{\frac{E}{\rho}}\alpha\, c^{-1}(\alpha) + \frac{1}{2}(\alpha+\sqrt{\frac{E}{\rho}})\phi(c^{-1}(\alpha))$$

$$- \frac{1}{2\rho} W'(c^{-1}(\alpha))\Big]$$

$$= \frac{1}{\sqrt{\frac{E}{\rho}}(\alpha+\sqrt{\frac{E}{\rho}})}\Big[\lim_{\delta_1,\delta_2\to 0}(\frac{D}{2\rho}) + \frac{1}{2}\alpha \lim_{\delta_1,\delta_2\to 0}\Big(-\sqrt{\frac{E}{\rho}}c^{-1}(\alpha) + \phi(c^{-1}(\alpha))\Big)$$

$$+ \frac{1}{2}\lim_{\delta_1,\delta_2\to 0}\Big(\sqrt{\frac{E}{\rho}}\phi(c^{-1}(\alpha)) - \frac{1}{\rho} W'(c^{-1}(\alpha))\Big)\Big] = 0\ . \tag{7.52}$$

Similarly (7.48), (7.50) and (7.51) yield (7.49) when (6.73) applies. This completes the verification of (7.49) in all cases. Finally from (7.44),(7.47) and (7.49) we have

$$\lim_{\delta_1,\delta_2\to 0} f(z) = 0\ . \tag{7.53}$$

It is immediate from (7.45) that the strain, stress and velocity ahead of the phase boundary will, as δ_1 and δ_2 approach zero, approach the corresponding values occurring in a material with the stress-strain relation

$$\sigma = E\varepsilon \qquad \varepsilon \geq 0\ . \tag{7.54}$$

What (7.53) guarantees is that the same is true <u>behind</u> the phase boundary[1]. Thus it follows that the jumps in strain, stress and velocity across the phase boundary will tend to zero with δ_1 and δ_2. In other words the solution to the nonlinear problem approaches the "shock-free" solution to the linear problem as the nonlinearity becomes small[2].

D. Summary of Results for Hard Loading

So far we have only considered "soft" loading conditions for the materials introduced in the fourth section. For hard-loading, we have formulated in (3.24) the free boundary problem governing the phase boundary for materials unrestricted by assumptions as to the curvature of the stress-strain relation in the high-strain material phase. As one would expect, if we confine our attention to the subclass of materials for which (4.2) also holds, we are able to integrate (3.24) partially whereupon we arrive at a pair of differential-delay equations formally resembling (4.22) and (4.23). We may then recast the problem as an integral equation formally resembling (4.32).

Proceeding as in the analysis of the soft device, we arrive at qualitatively similar results. The major difference between hard and soft loading occurs in the small-time asymptotic results. We restrict attention to the generic case: $W'''(\beta_1) < 0$ and $\dot{v}_0(t_\beta) > 0$. Here $t = t_\beta$ is the unique root of (3.23). Upon completion of an analysis parallel to that of the fifth section, one is led to the following asymptotic results:

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[1] Formally set f=0 in (4.10).

[2] This is reassuring, since naively letting $\beta_2 \to \beta_1$, in (7.7) and (7.20)-(7.22) would have led us astray.

$$s(t) = \frac{-\dot{v}_o(t_\beta)}{2(\beta_2-\beta_1)}(t - t_\beta)^2 + o[(t - t_\beta)^3] , \tag{7.55}$$

$$\varepsilon(s(t)^+,t) = \beta_2 + \frac{\rho\dot{v}_o^2(t_\beta)}{E(\beta_2-\beta_1)}(t - t_\beta)^2 + o[(t - t_\beta)^2] , \tag{7.56}$$

$$\sigma(s(t)^+,t) = \sigma_\beta + \frac{\rho\dot{v}_o^2(t_\beta)}{\beta_2-\beta_1}(t - t_\beta)^2 + o[(t - t_\beta)^2] , \tag{7.57}$$

$$v(s(t)^+,t) = v_o(t_\beta) + \dot{v}_o(t_\beta)(t - t_\beta) + O[(t - t_\beta)^2] . \tag{7.58}$$

Also, in analogy with (7.16) and (7.18), here we have for

$$x < s(t) \leq 0 \tag{7.59}$$

that

$$\beta_2 + \frac{\rho\dot{v}_o^2(t_\beta)}{E(\beta_2-\beta_1)}(t - t_\beta)^2 + o[(t-t_\beta)^2] \leq \varepsilon(x,t) \leq \beta_2 + \frac{3\rho\dot{v}_o^2(t_\beta)}{2E(\beta_2-\beta_1)}(t-t_\beta)^2 + o[(t - t_\beta)^2] \tag{7.60}$$

and

$$v(x,t,) = v_o(t) + o[(t-t_\beta)^2] . \tag{7.61}$$

Moreover, in contrast to the soft loading case, indications are that here the phase boundary location $s(t)$ — as well as the strain, stress and velocity fields immediately on either side of the phase boundary — can apparently be expanded in integer powers of $t-t_\beta$.

Turning to the large-time analysis we again represent $\dot{s}(t)$ in the form (7.24) subject to (7.26) and (7.27). As with soft-loading, the phase boundary cannot become the leading disturbance for materials with $W''(0) \geq E$. In this case, if the end velocity is unbounded, $\alpha = \sqrt{\frac{E}{\rho}}$ and $q(t)$ is unbounded, while if $v_o(t)$ approaches a limit v_∞ we find α as the unique root within the interval $(0,\sqrt{\frac{E}{\rho}})$ of the equation

$$(\alpha^2 - \frac{E}{\rho})[v_\infty - \phi(c^{-1}(\alpha))] + \frac{\alpha}{\rho}[W'(c^{-1}(\alpha)) - Ec^{-1}(\alpha) - D] = 0. \quad (7.62)$$

For materials with $W''(0) < E$ the phase boundary becomes the leading disturbance if and only if the increasing velocity at the end of the bar eventually exceeds the value $v = v_L$, where

$$v_L = \frac{-Dc(0)}{\rho(\frac{E}{\rho} - c(0)^2)} \quad . \quad (7.63)$$

If $v_0(t)$ is unbounded, $\alpha = \sqrt{\frac{E}{\rho}}$ and $q(t)$ is unbounded; otherwise α is found from (7.62) when $v_\infty \leq v_L$, while it is given by

$$\alpha = \frac{D}{2\rho v_\infty} + \sqrt{(\frac{D}{2\rho v_\infty})^2 + \frac{E}{\rho}} \quad (7.64)$$

when $v_\infty > v_L$. In all cases it is possible to obtain higher order corrections through methods analogous to those employed in Section 6.

E. Further Remarks.

The integral equation formulation for this problem remains relatively unexplored. It can be shown that for a certain type of piecewise-linear stress-strain relation, the integrals can be evaluated yielding a scalar equation relating the quantities t, t_a and t_b. Such an equation could perhaps permit the determination of s(t). It should be noted that if s(t) is known, it is a simple matter to determine u(x,t) in $\mathcal{R}_2$ by a procedure we now briefly outline. Since (4.15) ensures that the curve x = s(t) is never parallel to the characteristic rays in $\mathcal{R}_2$, the phase boundary is an acceptable curve on which to specify Cauchy data for the classical wave equation (4.11) which governs u(x,t) in $\mathcal{R}_2$. The proper

Cauchy data to specify on the phase boundary may be found from $\varepsilon(s(t)^+,t)$ and $v(s(t)^+,t)$, which in turn are given in terms of the known functions $\varepsilon(s(t)^-,t)$, $v(s(t)^-,t)$ and $\dot{s}(t)$ by means of the jump conditions (2.12). Then, as is well known, an application of Green's theorem in the plane will yield $u(x,t)$ in $\mathcal{R}_2$. It is essential to locate the curve $x = s(t)$ properly in order to guarantee that this solution to the classical wave equation will satisfy the condition prescribed on $x = 0$.

If this procedure is employed with $s(t)$ as yet unknown, it gives a functional representation for $u(x,t)$ behind the phase boundary in terms of $s(t)$ and $\dot{s}(t)$. For the problem of <u>hard</u> loading, equating the prescribed value of $u(0,t)$ to that provided by this functional representation furnishes an elegant alternative derivation of the integral equation. However, since the <u>soft</u> boundary condition amounts to prescribing the normal derivative of $u(x,t)$ on $x = 0$, this method will not supply the integral equation for the case of soft loading.

One notable aspect of our analysis is the inequality (6.8) which assures that the phase boundary is subsonic with respect to the material in the high-strain phase behind it, but supersonic with respect to the material in the low-strain phase ahead of it. According to James [19] certain uniaxial tensile experiments on a bar in a dead-loading (soft) device indicate that phase boundaries appear which separate approximately homogeneously deformed phases. These experiments also indicate that the phase boundary travels more slowly than the sound speed of either phase. That the phase boundary we describe is supersonic with respect to the material ahead of it is equivalent to the geometrical requirement that the leading side of the phase boundary intersect characteristic rays which arise either at the end of the bar $x = 0$ for $t < t_\beta$, or on the line of initial data $x < 0$, $t = 0$. From the discussion following (3.5) we recall that the vanishing of the sound speed at $x = 0$, $t = t_\beta^-$ ensures that these characteristic rays form an unambiguous cover of $\mathbb{R}\setminus\{(0,t)|t \geq t_\beta\}$. Consequently any phase boundary emerging at time t_β must at least initially be supersonic with respect to the material ahead of it[1]. Moreover, when (6.1) holds the curvature of the phase boundary in the x-t plane assures that the leading side of the phase boundary continues to intersect new members of this covering set of characteristic rays, thus ensuring the supersonic condition (6.6).

For a phase boundary to advance subsonically into a material in the low-strain phase, characteristic curves must arise on the leading side of the phase boundary. Such a violation of the entropy condition given by Lax would seem possible if the phase boundary were to emerge on $x = 0$ before the sound speed of the material vanishes, i.e. before the loss of hyperbolicity at time t_β. Suppose the phase boundary emerges on $x = 0$ at time $t_e < t_\beta$. Then a sufficient condition for the phase boundary to be initially subsonic with respect to both material phases is that it emerge with zero velocity. In this circumstance any point in the region of the x-t plane between the characteristic ray originating at $x = 0$, $t = t_e$ and the phase boundary would lie on a negatively-sloped characteristic ray which originates on the phase boundary. Consequently the solution representation (2.30) would apply only up to the characteristic ray originating at $x = 0$, $t = t_e$. The obvious candidate for a solution in the region $-c(\varepsilon_0(t_e))(t-t_e) < x < s(t)$ would be the extension

[1] As in confirmed by (7.7) and (7.23).

of the strain and velocity fields as constants from $x = -c(\varepsilon_0)(t_e))(t-t_e)$.

The time at which the phase boundary emerges is determined by the particular inverse to the stress response function $\sigma = W'(\varepsilon)$ which operates at the end of the bar. In our study we have assumed an inverse which results in a change of phase only when the current phase can no longer support the applied load. A consequence of this assumption is that the sound speed of the original phase vanishes where the phase boundary emerges. The vanishing of the sound speed is a phenomenon not without precedent in continuum mechanics. Consider the isentropic one-dimensional non-steady flow engendered by withdrawing a piston from a tube containing an ideal polytropic gas initially at rest. If the final piston velocity exceeds the escape speed, the sound speed is zero at the leading edge of the gas as it moves into the newly created volume. Although this piston problem has many formal similarities to our problem of the elastic bar, there is no appropriate analogue of a second phase. Here the region between the piston and the leading edge of the gas is simply a vacuum or zone of cavitation. A relevant solid mechanical analogue of the piston problem would seem to be the monotonic loading of an elastic bar whose stress response curve is initially increasing with strain and then decreasing <u>without a final increasing portion</u>, for then, just as speeds above the escape speed cannot be achieved by the gas, equilibrium stresses above a certain level cannot be sustained by the elastic solid[1].

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[1] For a discussion of such a stress response in (incompressible) elastic solids and the corresponding analogy with compressible fluids, reference may be made to Section 4. of [23].

As discussed in the third section, it is possible to introduce a second phase emerging at the end of the bar at any time in the interval $[t_\alpha, t_\beta]$[1] . Besides our choice of $t = t_\beta$, another plausible candidate is the time t_M when the applied load reaches the Maxwell stress. Then $\sigma_0(t_M) = \sigma_\gamma$ and we expect to observe $\sigma = \sigma_\gamma$, $\varepsilon = \gamma_1$, and $v = \phi(\gamma_1)$ in the region of the x-t plane between the "Maxwell characteristic" and the phase boundary. Although we have yet to investigate such a problem, we point out that since the fields in front of the phase boundary would, at least initially, be homogeneous, such a study has the analytical advantage of eliminating the need to invert (4.20h).

Besides the Maxwell characteristic, there is another canonical characteristic. We observe that, at the end of the bar, the particles are initially subsonic since $v(0,0) = 0 < \sqrt{\frac{W''(0)}{\rho}} = c(0,0)$; at time t_β, however, they are supersonic since $v(0,t_\beta) = \phi(\beta_1) > 0 = c(0,t_\beta)$. Hence there is a "transonic characteristic" on which the particles are moving to the right at the acoustic speed. We can locate this characteristic by solving for the associated "transonic strain". The latter is the unique root within the interval $(0,\beta_1)$ of the equation

$$c(\varepsilon) - \phi(\varepsilon) = \frac{1}{\sqrt{\rho}} \left(\sqrt{W''}(\varepsilon) - \int_0^\varepsilon \sqrt{W''(s)\ ds} \right) = 0. \qquad (7.65)$$

In contrast to the familiar situation of two-dimensional isentropic steady compressible fluid flow, the governing equations for the dynamics of an elastic bar do not change their type as the continuum passes from subsonic to supersonic. It should be noted from (7.65) that the transonic

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[1] See the discussion surrounding eqn. (3.8).

strain is completely determined by the first ascending branch of the stress-strain curve. On the other hand, the Maxwell strain γ_1 for the material in the low-strain phase depends on the global stress-strain relation via the equal area rule. Consequently the transonic strain may be either greater than, equal to, or less than γ_1. In other words, whether the particles become supersonic before or after they reach the Maxwell stress depends on the particular stress-strain curve under consideration.

Another interesting result of our analysis is that the velocity of the phase boundary approaches the sound speed of the high-strain phase $\sqrt{\frac{E}{\rho}}$ as the applied load tends to infinity. This bears a resemblance to results in other free-boundary problems in mechanics. Burridge and Keller [24] treat a semi-infinite one-dimensional linear-elastic adhesive-tape which is initially completely attached to a plane surface and subject to an in-plane tensile force at its end. Peeling is initiated by suddenly applying a sufficiently large normal force to the end of the tape. The free-boundary is the point separating the portion of the tape still adhering to the surface from that part which has already peeled. If the pair of forces at the end of the tape is maintained, the free-boundary will move at a constant velocity which is a monotonically increasing function of the magnitude of the applied normal force. Moreover the value of this function approaches the fixed sound speed of the elastic adhesive-tape as the magnitude of the applied normal force tends to infinity. The analogy, however, cannot be carried too far, since the tape problem incorporates a condition which characterizes the breaking of the adhesive bond between the tape and the surface, while the problem of the elastic two-phase bar incorporates no analogous failure condition external to elasticity theory. The adhesive failure condition, which applies only at the free-boundary, is dissipative in the sense that it lowers the free-boundary speed[1]. It would seem that a corresponding effect on the phase boundary speed might occur in the problem treated here if a surface energy were associated with the phase boundary. As James [5] has observed, the effects of such surface energies have so far been ignored.

Finally we note that in this study we have totally neglected unloading, a process which we would expect to promote conventional shocks; see [16]. In addition we have dealt with a very specialized class of materials that not only made possible the integration of some of the governing equations, but also precluded conventional shocks behind the phase boundary. For more general materials, even determining the smoothness of the fields behind the phase boundary would perhaps demand mathematical techniques entirely different from those employed in this study; see, for example, [25].

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[1] See Eq. 2.7 of [24].

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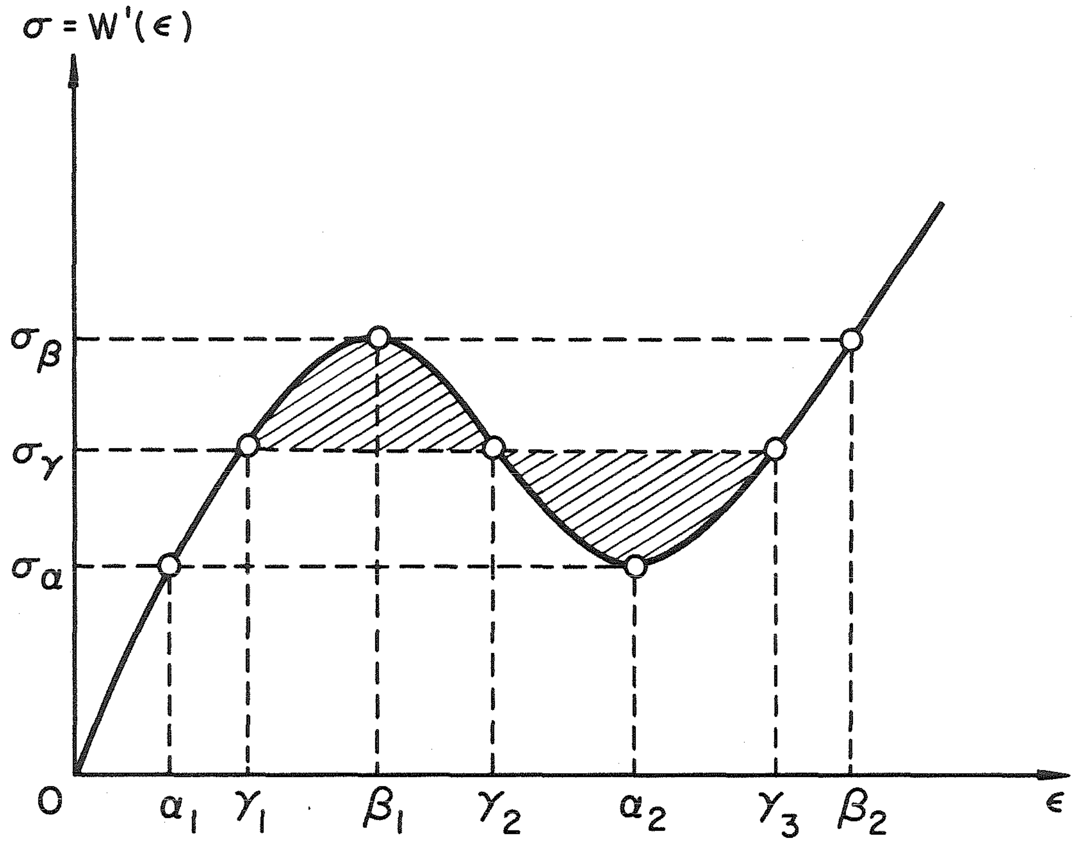
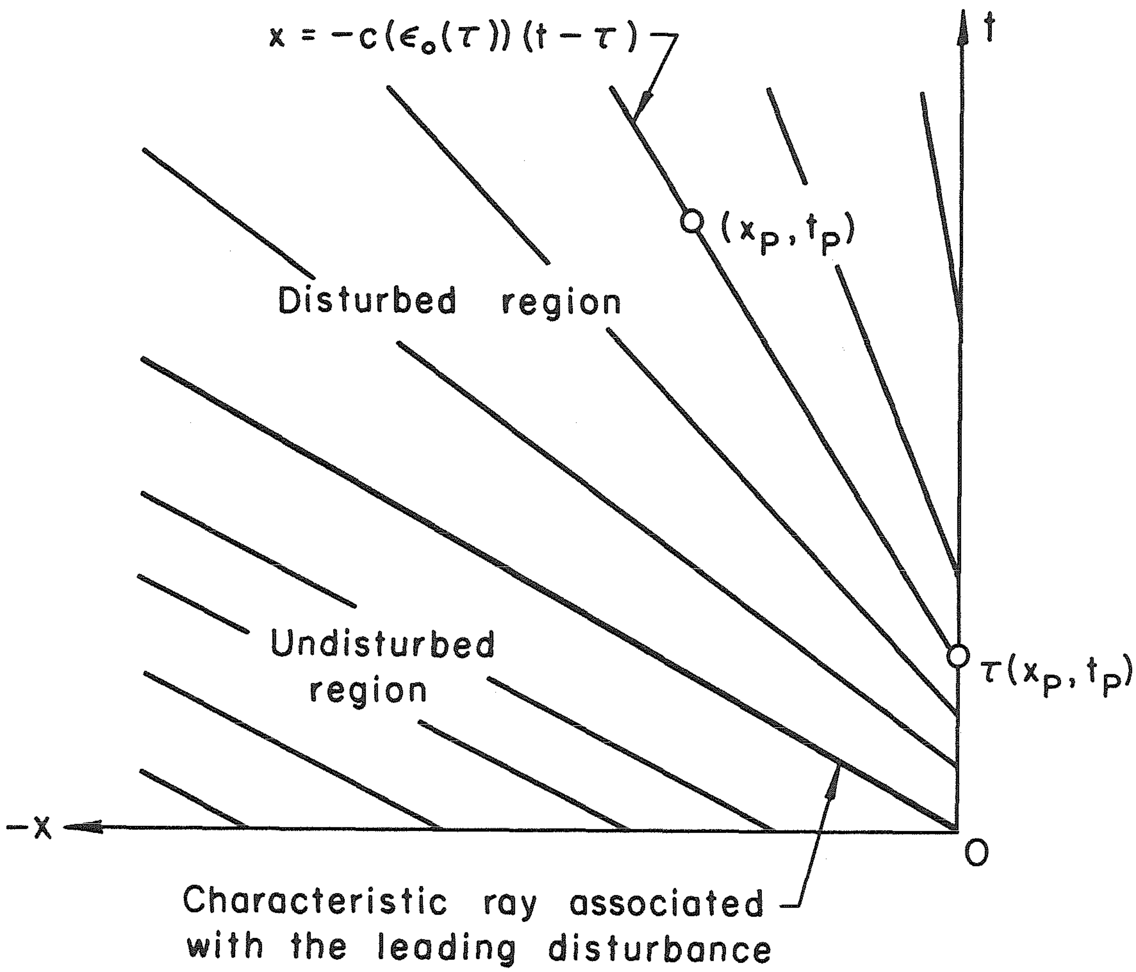


Figure 1. Stress-strain curve for an elastic material which admits multiple phases.



Undisturbed Region

$$R_T = \left\{(x,t) \mid t \geq 0,\ x < -c(\epsilon_o(0))\, t\right\}$$

$$\frac{dx}{dt} = -c(\epsilon_o(0))$$

Disturbed Region

$$R_D = \left\{(x,t) \mid t \geq 0,\ -c(\epsilon_o(0))\, t \leq x \leq 0\right\}$$

$$\frac{dx}{dt} = -c(\epsilon_o(\tau(x,t)))$$

Figure 2. Geometry of the characteristic rays C_- for an elastic single-phase bar.

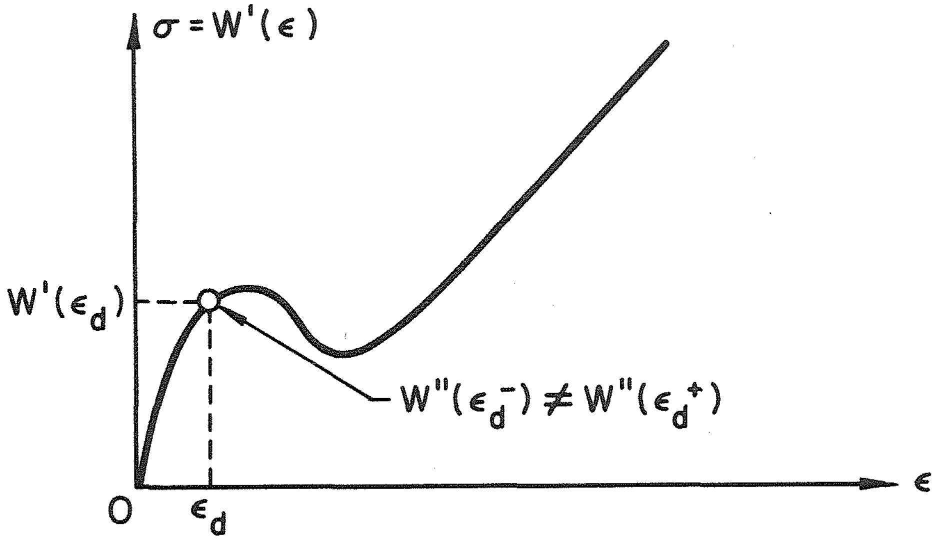


Figure 3a. Stress-strain curve with an isolated discontinuity in $W''(\varepsilon)$ at $\varepsilon = \varepsilon_d$.

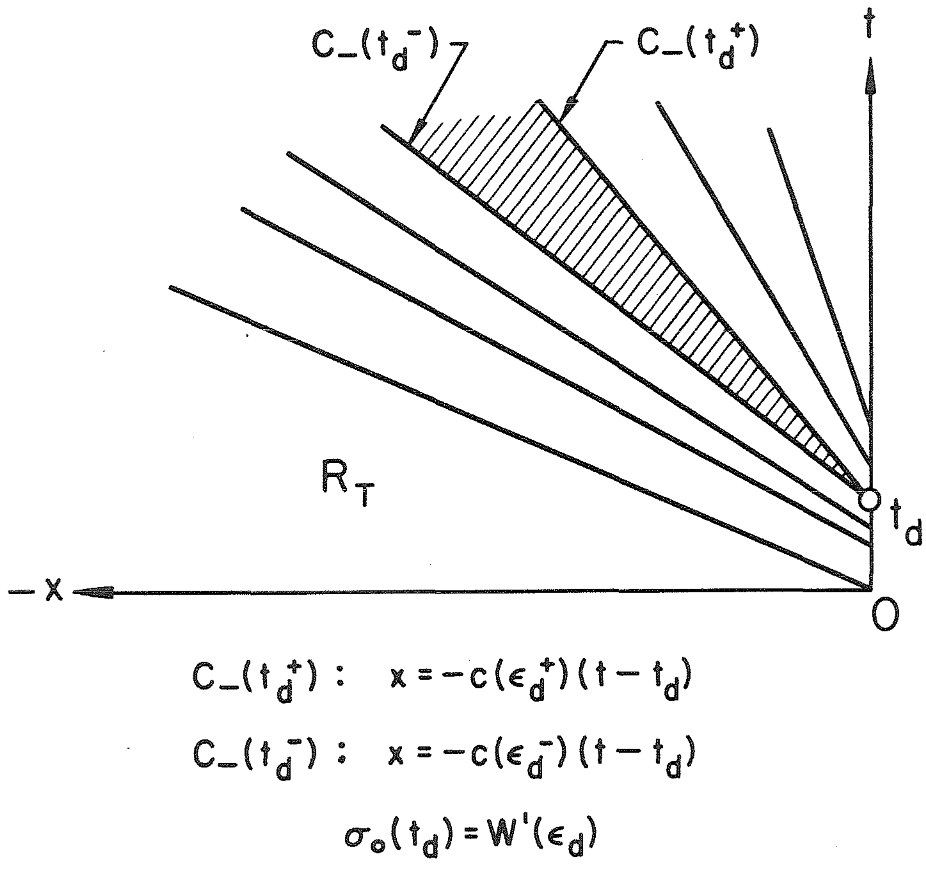


Figure 3b. Region between the characteristic rays associated with strain ε_d which involves constant strain and velocity.

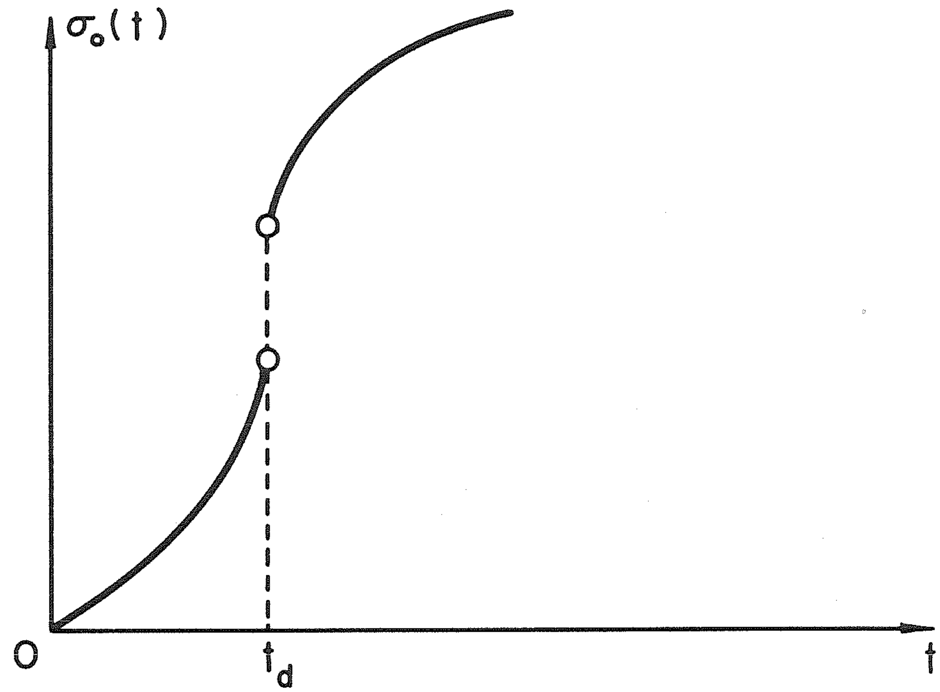
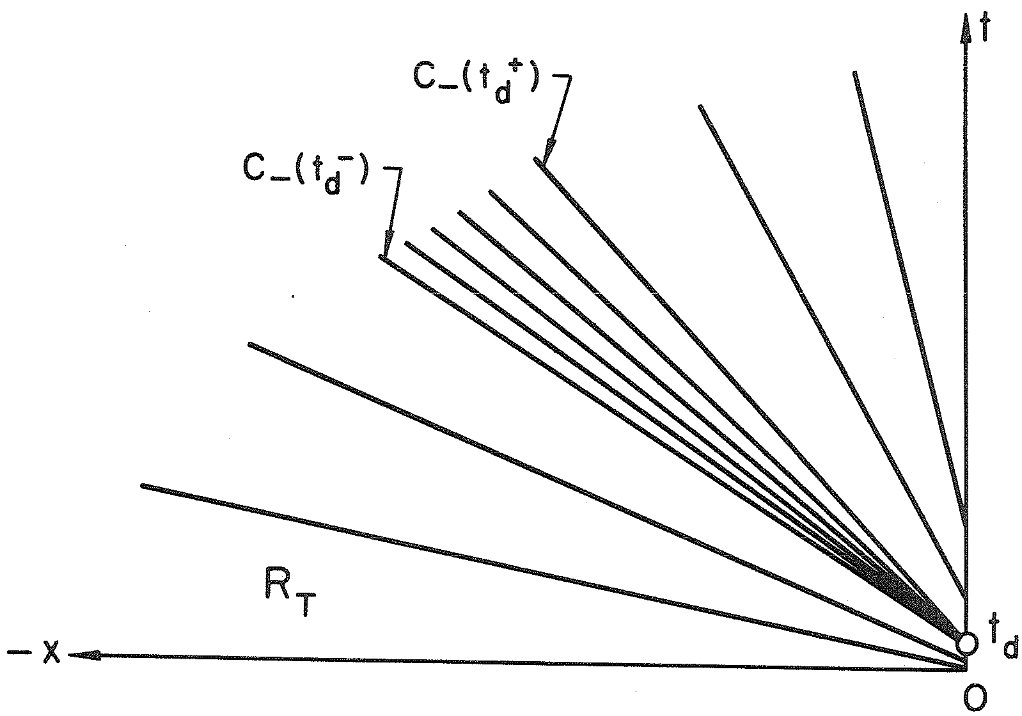


Figure 4a. Traction impulse delivered at time t_d.



$$C_-(t_d^+)\ :\quad x = -c(\epsilon_o(t_d^+))(t - t_d)$$

$$C_-(t_d^-)\ :\quad x = -c(\epsilon_o(t_d^-))(t - t_d)$$

Figure 4b. Centered simple wave associated with the traction impulse.

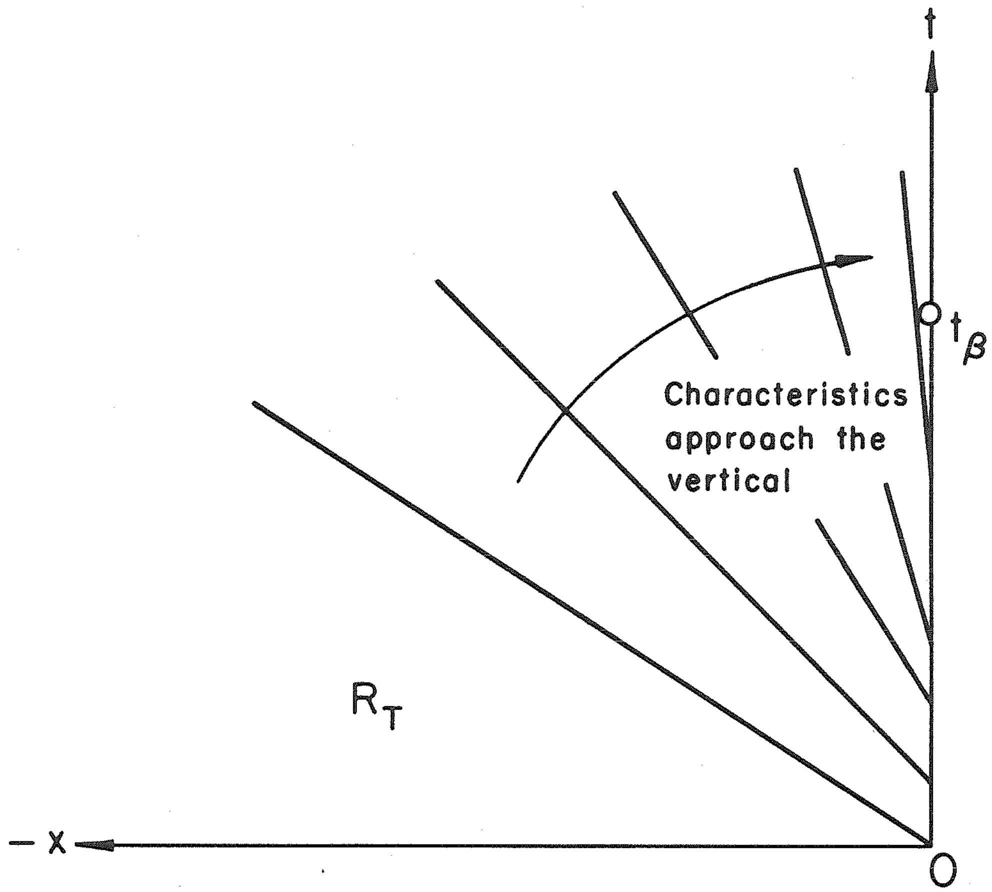


Figure 5. The smooth monotone loading past $\sigma = \sigma_\beta$ $(=\sigma_0(t_\beta))$ of a bar whose material is non-hardening in the first phase is associated with a pattern of characteristic rays C_- which provide an unambiguous covering of the region $\mathcal{R}\setminus\{(0,t)\,|\,t \geq t_\beta\}$.

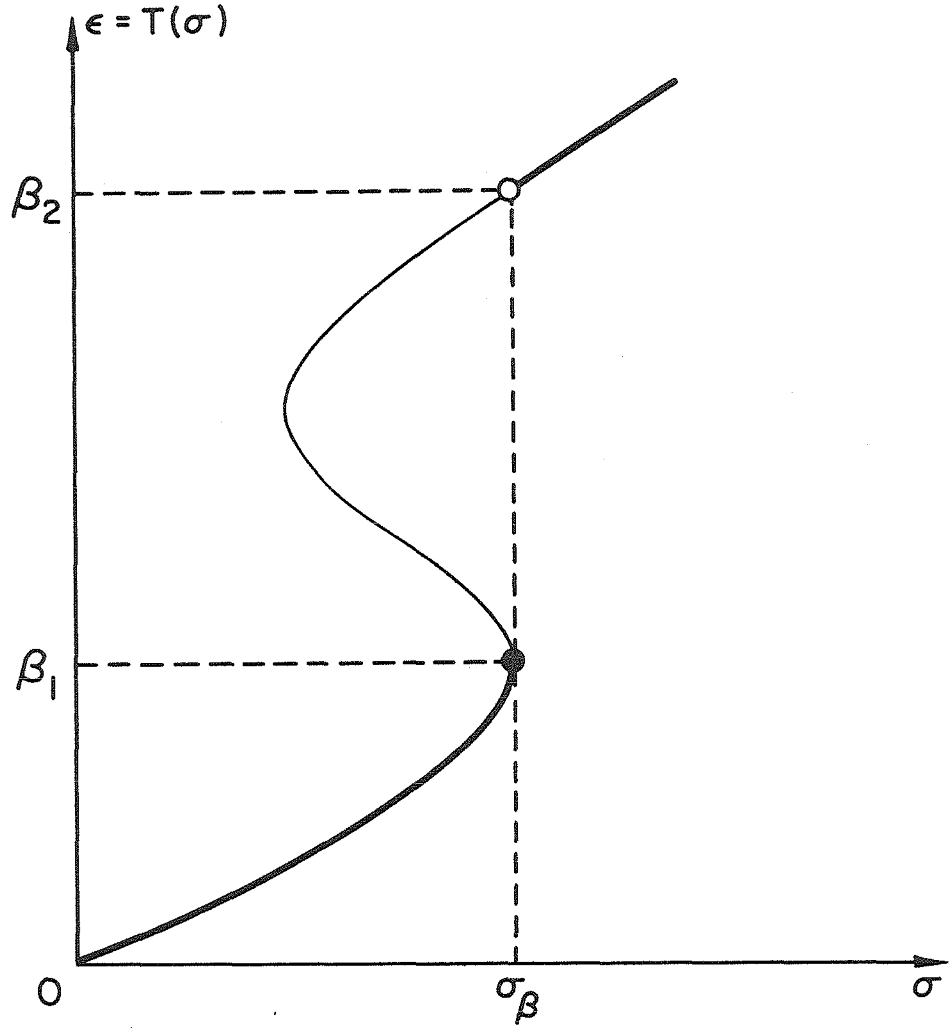


Figure 6. The particular inverse to the stress response curve of Figure 1 which results in the material assuming the high-strain phase only when the low-strain phase can no longer support the applied load.

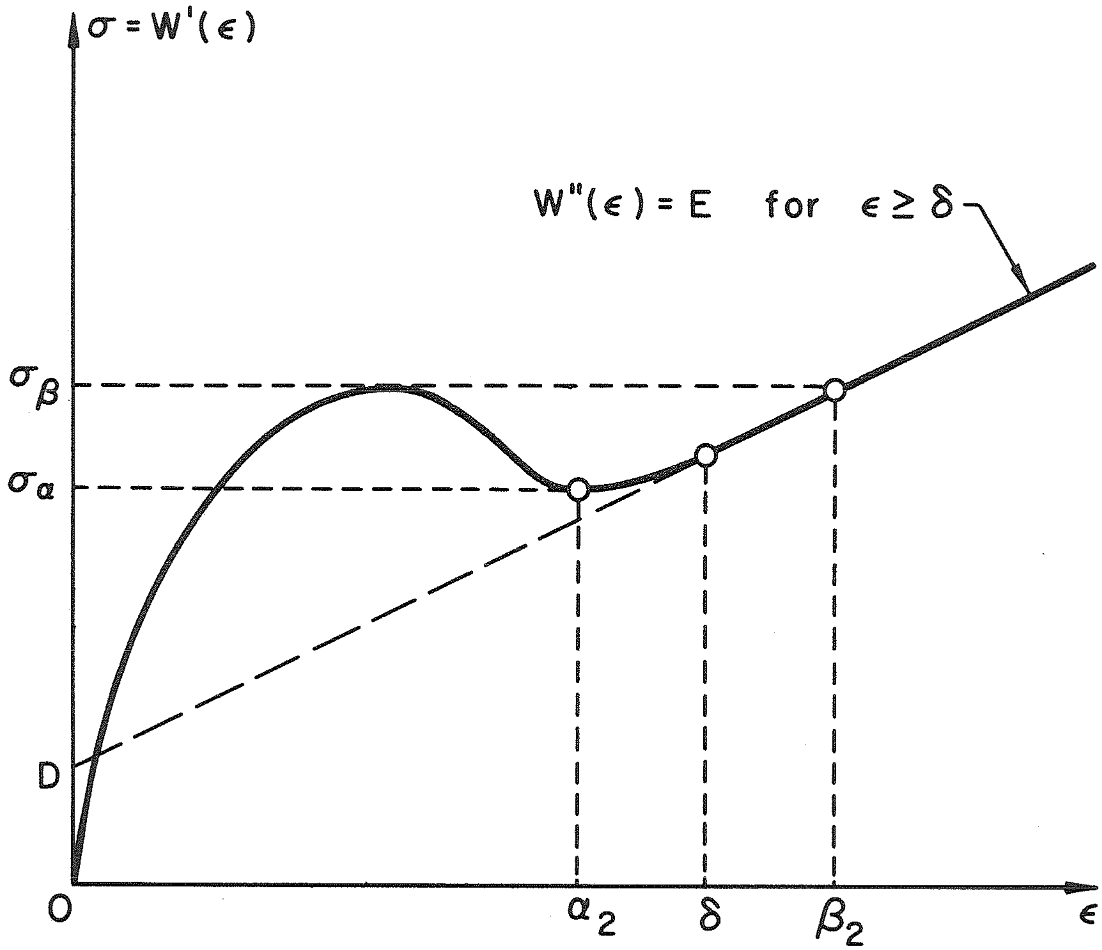


Figure 7. Representative stress-strain curve for the family of materials introduced in Section 4.

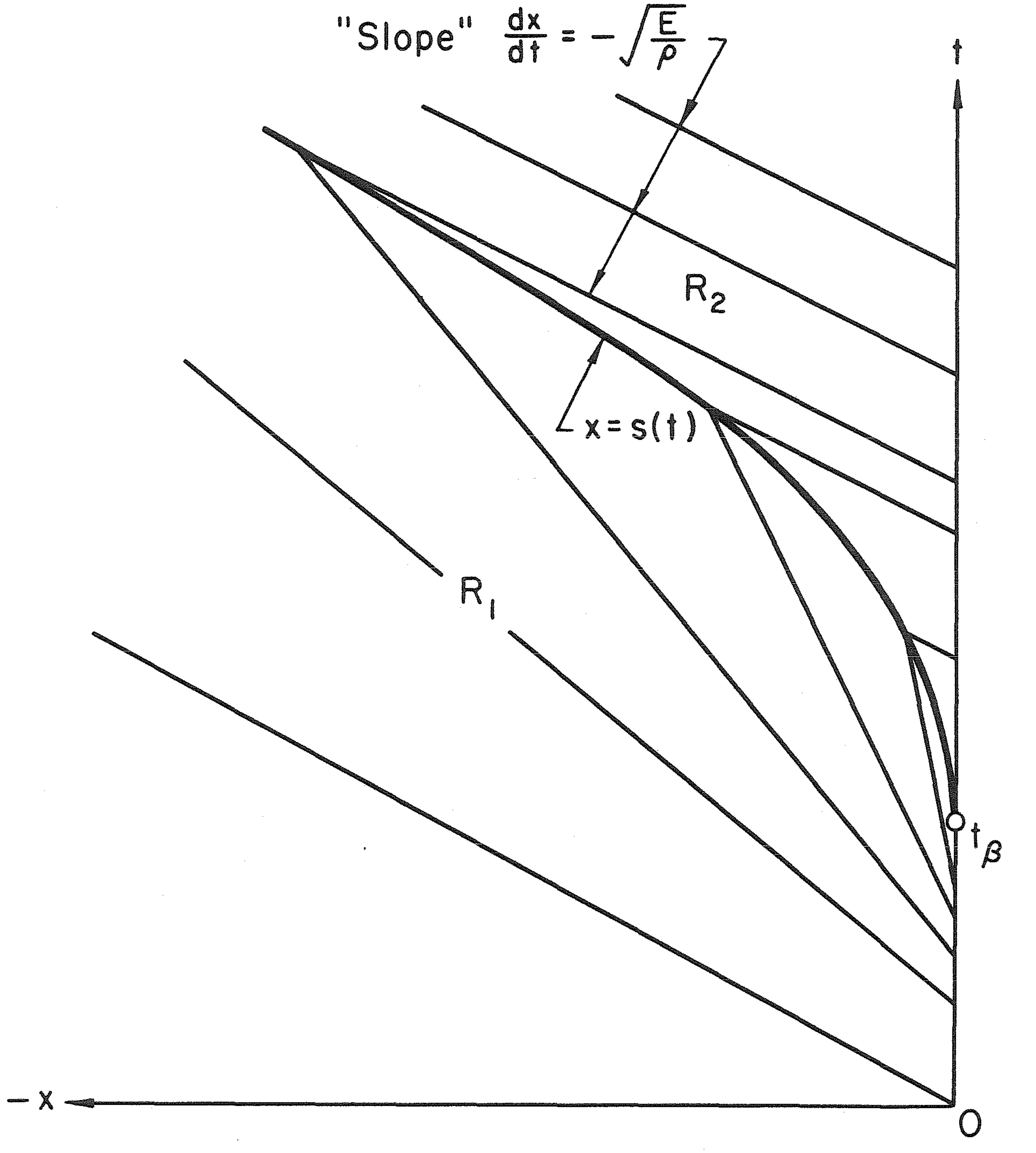


Figure 8. Geometry of the characteristic rays C_- for the material whose stress response curve is depicted in Figure 7.