LINEAR PROGRAMMING METHODS FOR THE NUMERICAL SOLUTION OF PARABOLIC EQUATIONS BACKWARDS IN TIME

Thesis by

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In Partial Fulfillment of the Requirements

for the degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1983

(submitted May 18, 1983)

ACKNOWLEDGMENTS

I gratefully acknowledge support during my stay at Caltech in the form of teaching assistantships, summer support from the Kaplun Memorial Fund, and a Charles Lee Powell Fellowship. I am deeply indebted to my advisor, Professor Joel Franklin, whose skill and expertise in ill-posed problems made this thesis possible. His patience and advice have helped me many times. I would also like to thank all of the students and faculty who have helped make my stay here more interesting and enjoyable.

ABSTRACT

This thesis investigates linear programming methods for the numerical solution of parabolic equations backwards in time. These problems are ill-posed. Hence an approximate numerical solution for such problems can only be obtained if additional constraints (called a regularization) are imposed on the solution in order to guarantee its stability under small perturbations. Previous authors have implemented regularizations on the backward heat equation which used (linear or nonlinear) least squares, or linear programming. These regularizations use the exact form of the kernel for the heat equation, however, and so are not generalizable to problems with an unknown kernel or unknown eigenfunction expansion. Furthermore, the least squares methods can not easily handle the nonnegativity constraint that a positive temperature, for example, must have.

In the first part of this thesis, linear regularizations which can be used to solve any linear parabolic equation on a finite domain backwards in time are introduced. It is then shown how a numerical approximation to the solution of the regularized problem can be obtained by using linear programming and any stable and consistent difference method (such as Crank-Nicholson). The convergence of these algorithms is shown to be a direct consequence of the Lax equivalence theorem. The stability, accuracy, and results of actual numerical experiments using this linear programming method are analyzed.

The second part of this thesis shows how these regularizations can be used on weakly nonlinear equations. This is done by introducing a successive approximation method, and solving a linear program at each step in the iteration. The stability, accuracy, and results of numerical experiments for this algorithm are also examined.

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I. INTRODUCTION AND REGULARIZATION OF ILL-POSED PROBLEMS

I.1. Introduction.

No model is perfect; any mathematical description of a physical system must ignore some features of the system. If the wrong features are ignored, the model will be bad. This situation often occurs when the description of the system is ill-posed.

Consider the linear problem

$$Ku=f, \tag{1.1}$$

where K is a bounded linear operator mapping a Banach space X into a Banach space Y, f is known, and u is to be determined. This problem is defined to be well posed if it has a unique solution which depends continuously on f; otherwise it is improperly posed (or ill-posed). The restriction that u depend continuously on f prevents a small change in f from creating a large change in u. This is important, for if (1.1) represents a model of a physical situation, then a small measurement error in f should cause only a small error in the calculated solution u.

There are three ways in which (1.1) can be ill-posed. First, there could be more than one solution, which occurs when $Kv=0$ for some $v\neq0$. Second, (1.1) might have no solution at all, which occurs when f is not in the range of K. Third, a unique solution to (1.1) might exist which does not depend continuously on the 'data' f. This last case occurs when K has an unbounded inverse. These three cases are related, as the following theorem shows.

<u>Theorem 1.1</u>: If the domain of K is X, if the range of K is dense in Y, and if K is invertible, then the range of K is all of Y if and only if K^{-1} is

bounded.

Proof: This theorem is an immediate consequence of the closed graph theorem. See [33], page 209.

If an ill-posed problem comes from a model of a physical system, certain key features of the system are being ignored in the model. In order to make the model well posed, more features of the system must be specified.

Example 1 (Backward heat equation): Suppose that the temperature distribution on a thin rod is measured at time t=1, and the temperature is sought for some t'<1. Heat conduction is a diffusive process, and so like many diffusive processes that occur in chemistry and physics, this phenomenon can be modeled by a parabolic partial differential equation. The equation often has variable coefficients, but for this process they are constant. Here the problem is to find u(x,t) satisfying

$$
\begin{aligned}
&u_t = u_{xx} \\
(1.2)\qquad &u(0,t)=0 \\
&u(\pi,t)=0 \\
&u(x,1)=f(x) \qquad\qquad (0\le x\le \pi,\ 0\le t\le 1).
\end{aligned}
$$

This problem is ill-posed, however. A small perturbation of sin(nx)/n in f(x) creates a large perturbation of $e^{n^2}\sin(nx)/n$ in u(x,0).

Certain key features of the experiment are ignored in this description. One of them is the data error in f(x). An additional constraint on the solution is also needed. This constraint should reflect a known feature of the temperature distribution. For example, suppose f(x) is measured to within a tolerance of .01, and that it is known (either from observations or from the system itself) that at time t=0 the temperature u satisfies $0\le u\le 1$.

With this, (1.2) becomes

$$
(1.2') \qquad
\begin{aligned}
&u_t = u_{xx} \\
&|u(x,1)-f(x)| \leq .01, \\
&u(0,t)=u(\pi,t)=0 \\
&0 \leq u(x,0) \leq 1.
\end{aligned}
$$

Later it will be shown that if u_1 and u_2 are any two solutions to (1.2'), then

$$\|u_1(x,t)-u_2(x,t)\|_2 \leq 2(.01)^t.$$

Therefore, the solutions to (1.2') do not have the instability that solutions to (1.2) have.

Example 2 (Numerical differentiation): Suppose that an experimenter measures a function f(x) for $0 \leq x \leq 1$ and seeks its derivative. Setting

$$(1.3) \qquad g(x)=f'(x)$$

will not work here, since f(x) is a measured quantity and might not even be differentiable. Furthermore, a small error of $n^{-\frac{1}{2}}x^n$ in f(x) will cause a large error of $n^{\frac{1}{2}}x^n$ in g(x). Problem (1.3) is therefore ill-posed.

Equation (1.3) requires information about the data error in f(x) as well as an additional condition on g to remove this instability. On the other hand,

$$
(1.3') \qquad
\begin{aligned}
&|f(x)-f_0(x)| \leq \varepsilon \\
&\int_0^1 g(x)\,dx = f_0(x)-f(0) \\
&\|g(x)\|_2 \leq M_0 \\
&\|g'(x)\|_2 \leq M_1
\end{aligned}
$$

has both of these. If a selection rule is used to specify which of all g(x) that satisfy (1.3') is the approximate solution to (1.3), then (1.3') becomes well posed. Franklin [16] and Cullum [10] have studied this problem using a selection method due to Tikhonov [35,36].

For the above examples, the original description of the experiment resulted in an ill-posed problem. A more thorough description of the experiment was needed to make the problem well posed.

The solution at t=0 to the ill-posed backward heat equation (1.2) solves the Fredholm integral equation of the first kind

$$\int_0^{\pi} k(x,s)u(s,0)ds=f(x), \tag{1.4}$$

where

$$k(x,s)=[2/\pi]\sum_{n=1}^{\infty} e^{-n^2}\sin(nx)\sin(ns). \tag{1.5}$$

From (1.5) it can be seen that all derivatives of k(x,s) exist and are continuous. Therefore, if u(x,0) is continuous, all derivatives of f(x) must exist and be continuous.

Finding the solution to any Fredholm integral equation of the first kind is ill-posed. This is easily seen, for if k(x,s) is any bounded, measurable function, then

$$\int_0^{\pi} k(x,s)\sin(ns)ds \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{1.6}$$

This shows that an arbitrarily small change in f(x) can lead to an O(1) change of sin(nx) in u(x,0). Another example of an ill-posed problem that can be written as an integral equation is the problem of finding u(x,1) from

(1.7)
$$\begin{aligned} &u_{xx}+u_{yy}=0 \\ &u(0,y)=0 \\ &u(\pi,y)=0 \\ &u(x,0)=f(x) \\ &u_y(x,0)=0 \qquad (0\le x\le\pi,\ 0\le y\le 1). \end{aligned}$$

Here a small change of [sin(nx)]/n in f will produce a large change of 2[cosh(n)sin(nx)]/n in u(x,1). In this case, the solution to (1.7) at y=1 solves the Fredholm integral equation

$$\int_0^{\pi} k(x,s)u(s,1)ds=f(x),$$

where

$$k(x,s)=\frac{1}{\pi}\sum_{n=1}^{\infty}\sin(nx)\sin(ns)/[\cosh(n)].$$

Computers produce roundoff errors. However, even a small error in the data can lead to a large error in the solution of an ill-posed problem, as these examples show. Hence without the imposition of additional constraints on the solution, numerical methods to solve these problems will be unstable. The assumptions which remove this instability are often called a regularization. For example, consider the inequalities

(1.8)
$$\begin{aligned} &\|Kv-f\|_p\le \varepsilon \qquad (p\ge 1), \\ &\|Bv\|_q\le M \qquad (q>1), \end{aligned}$$

where B is a bounded linear operator with bounded inverse, and $\varepsilon \ll M$. These inequalities might arise in heat conduction problems, for example, if u is known only approximately (to within measurement error) at t=1, and is known to be bounded at t=0. Instead of looking for the unique solution to (1.1), we now try to find any solution to (1.8) for small enough ε. We then hope that the solution to these inequalities is, in some sense, 'close' to u. If it is, then (1.8) regularizes the ill-posed problem. These inequalities will regularize the backward heat equation (1.2) for $0<t\leq 1$, as will be shown later. Cannon [4] and Douglas [11] used this regularization with $p=q=\infty$ to calculate approximate solutions to the problems of analytic continuation and the backward heat equation, both problems in which $k(x,s)$ is known exactly. Others, such as Miller [23,24], have used least squares algorithms to solve (1.8) when $p=q=2$ and the kernel $k(x,s)$ is known. Not all ill-posed final-boundary value problems have known kernels, however. For example, the problem of finding $u(x,t)$ for $t<1$ from

$$\begin{aligned} &r(x,t)u_t=(p(x,t)u_x)_x \\ &u(x,1)=g(x) \\ &u(0,t)=0 \\ &u(\pi,t)=0 \end{aligned} \tag{1.9}$$

is also ill-posed if $r,p>0$. The kernel for this problem can not in general be determined, however.

The inequalities (1.8) will not in general regularize the Fredholm equation (1.4). One regularization that will work for these problems is due to Tikhonov [35,36]. In it, (1.8) is replaced with the variational problem of finding u such that

$$\|Ku-f\|_2^2+\varepsilon\sum_{m=0}^{n}\|r_m(x)u^{(m)}\|_2^2=\text{minimum} \qquad (n\geq 1) \tag{1.10}$$

for small $\varepsilon>0$. Tikhonov showed that if the integral equation (1.4) has a continuous solution $v(x)$, then the solutions to (1.10) will converge uniformly to $v(x)$ as $\varepsilon\to 0$. Franklin [15], however, has shown that convergence of Tikhonov's method can be arbitrarily slow. Because of this, there is little computational advantage in using Tikhonov's method instead of the regularization (1.8) for many ill-posed partial differential equations.

In this thesis, regularizations to (1.4) of the form

$$\sum_{m=0}^{n} \|u^{(m)}\|_\infty \le M \qquad (n \ge 0),$$

(1.11)

$$\|Ku-f\|_q \le \varepsilon \qquad (q=1 \text{ or } \infty)$$

are used to obtain numerical solutions to parabolic equations such as (1.9) backwards in time. It should be emphasized that the choice of n, M, and q depend on the additional information known about the solution u. Without this information, there is no particular reason to choose any n, M, or q. For backward parabolic equations, this regularization takes the form

$$\sum_{m=0}^{n} \left\| \frac{\partial^m u(x,0)}{\partial x^m} \right\|_\infty \le M,$$

$$\|u(x,1)-f(x)\|_q \le \varepsilon.$$

As in Tikhonov's method, solutions to these inequalities will converge uniformly to the solution of (1.3) as $\varepsilon \to 0$ if this solution obeys certain smoothness conditions.

Now for these backward problems,

$$u(x,1)=Ku(x,0)$$

by definition. Given u(x,0), however, u(x,1) can be approximately determined by using any stable and consistent difference method, such as Crank-Nicolson. In this case, we have

$$\begin{pmatrix} u(x_1,1) \\ \vdots \\ u(x_N,1) \end{pmatrix} = A \begin{pmatrix} u(x_1,0) \\ \vdots \\ u(x_N,0) \end{pmatrix}$$

for some matrix A. Here A can be thought of as an approximation to the kernel K. Substituting this and the appropriate centered differences into (1.11') results in a system of linear inequalities, and these inequalities can be solved by linear programming. A priori and a posteriori error estimates will be derived for this procedure, and it will be shown that the error in the discretized problem as well as the analytical problem is logarithmically convex. This method will also be compared with other methods, such as Tikhonov's method, and it will be shown how to obtain a good initial guess of the solution in order to reduce the computation necessary to solve the linear programming problem.

This algorithm will also be applied to solve weakly nonlinear parabolic equations backwards in time. For example, consider the problem of finding $u(x,t)$ $(t<1)$ from

$$(1.12)\qquad \begin{aligned} &r(x,t)u_t=(p(x,t)u_x)_x+cF(u)\\ &u(0,t)=0\\ &u(\pi,t)=0\\ &u(x,1)=f(x) \qquad (0\leq x\leq\pi,\ 0\leq t\leq 1),\end{aligned}$$

where F is continuous and $0<c<<1$. A successive approximation method can be formulated to find an approximate solution to this problem. Let $u^{(0)}(x,t)$ be an approximate solution to (1.12) when $c=0$. This function can be approximated by using the regularization (1.11). For $n>0$, let $u^{(n)}$ solve

$$r(x,t)u_t^{(n)}=(p(x,t)u_x^{(n)})_x+cF(u^{(n-1)}),$$

together with appropriate regularized boundary conditions at $x=0$, $x=\pi$, and $t=1$. Each $u^{(n)}$ solves a linear problem, and so it can be approximately determined with the aid of regularization (1.11). The error in this algorithm can often be made arbitrarily small if ε is chosen small enough and n is chosen large enough. Error analysis and sufficient conditions for convergence will be derived in this thesis, and these theoretical conditions will be verified with numerical experiments.

This thesis is divided into four parts. This section and the next section, which make up the first part, cover background material on ill-posed problems, functional analysis, and regularization techniques for ill-posed problems.

The second part of this thesis covers the description and analysis of the linear programming methods proposed here for linear backward parabolic equations. Sections II.1 and II.2 are devoted to studying the logarithmic convexity of solutions for the cases of time-independent coefficients and time-dependent coefficients. In section II.1 it is shown that if u(x,t) solves a nonsingular parabolic equation with time-independent coefficients on a finite domain, then

(1.13) $$\|u(x,t)\|_2 \leq \|u(x,0)\|_2^{1-t} \, \|u(x,1)\|_2^{t}.$$

There are analogous (although more complicated) results for the case of time-dependent coefficients. These problems have been studied before, for example by Payne [26], Carasso [9], Franklin [16], and others. Hence some of the material in these sections is not new; that which is not is clearly indicated. The results obtained in these two sections can be used to study the error in using regularizations such as (1.8) for backward parabolic problems. In particular, if u and v both solve (1.11), then

$$\|u(x,t)-v(x,t)\|_2 \leq 2\varepsilon^{t} M^{1-t},$$

from (1.13). In section II.3 the details of the numerical method for approximately solving backward parabolic equations using the regularization (1.11) are given. The resulting system of linear inequalities takes the form (for m=0, $q=\infty$)

$$-M \leq u(x_i, 0) \leq M$$

$$-\varepsilon \leq \sum_{j=1}^{n} a_{ij} u(x_j, 0) - f(x_i) \leq \varepsilon \qquad (i=1,\ldots,n),$$

and for a given ε and M this system can be solved with linear programming. Sometimes, however, the bound M at t=0 and the error ε at t=1 might not be known precisely. In section II.4 methods are given for solving parabolic equations backwards in time when either ε or M is unknown. These regularizations require finding a solution to (1.11) which minimizes either ε, M, or some linear combination of them. This can be done to guarantee that the solution is accurate at t=1 and that it does not blow up at t=0. Linear programming is ideally suited for extremal problems such as these. There are four cases to consider; each of the constants ε and M can be either known or unknown. Miller [24] showed that only one of these cases can be solved by using regular least squares. All four cases can be solved by using linear programming, however. Four types of linear programming problems are described, and the equivalence of the errors for these problems is proven. In section II.5 the stability and accuracy of the linear programming solutions is analyzed, and it is shown here how the stability of the difference scheme affects the stability of the linear programming solution. It is also proven here that the linear programming solution converges to the exact solution of the ill-posed problem if and only if the matrix $A=(a_{ij})$ is obtained from a stable and consistent difference scheme. Finally, in II.6

the results of numerical experiments are given.

In the third part of this thesis the results of part II are extended to weakly nonlinear parabolic equations. Logarithmic convexity results and estimates of the regularization error for these problems are given in III.1. Here it is proven that if F is Lipschitz continuous with Lipschitz constant L, and if u and v both solve

$$u_t=u_{xx}+cF(u)$$
$$\|u(x,0)\|_2\leq M \qquad (1.14)$$
$$\|u(x,1)-f(x)\|_2\leq \varepsilon \quad (0\leq t\leq 1,\ 0\leq x\leq \pi),$$

then

$$\|u(x,t)-v(x,t)\|_2\leq 2\sqrt{\pi}\,\varepsilon^t M^{1-t}/(1-cL)$$

if $cL<1$. This shows that the bound on the solution at t=0 will only restrict the growth of the error if c is small enough. This section also gives an example of how the Lipschitz constant L can be estimated from the bound on u at t=0. In section III.2 the successive approximation method used to solve these problems is described. The numerical error for this method is analyzed in section III.3. Sufficient conditions for the approximation method to converge are given here. The results of numerical experiments for weakly nonlinear problems are given in III.4.

Chapter IV, the conclusion, summarizes the results of this thesis. This chapter also mentions how linear programming methods similar to those used here can be used on other ill-posed problems, such as the Cauchy problem for (linear and weakly nonlinear) elliptic equations, and the problem of numerical differentiation with inexact data.

Appendix A provides background information on linear programming and

Chebyshev approximation, subjects frequently mentioned in this text. This appendix also contains an explanation of why, in this author's opinion, linear programming methods are better than least squares methods for solving parabolic equations backwards in time. There is a table of symbols after the appendix which defines some of the notation used in the text.

I.2. Regularization.

Regularization will be defined in this section, and it will be shown how some results from functional analysis can be used to prove regularization theorems for ill-posed problems. Several examples will demonstrate how these regularizations can be applied to the backward heat equation.

Let X and Y be Banach spaces, and let K be a bounded linear operator with an unbounded inverse mapping X into Y. Let U be any closed set in X. The operator K^{-1} is said to be (weakly) regularized on U if

$$(2.1) \qquad Ku_i \longrightarrow Ku \quad \text{as } i \longrightarrow \infty \quad (u, u_i \in U)$$

implies that

$$u_i \longrightarrow u \quad \text{(weakly)} \quad \text{as } i \longrightarrow \infty .$$

Equivalently, we say that the problem Ku=f is (weakly) regularized on U.

<u>Lemma 2.1</u>: Let K be as above. Then K^{-1} is regularized on compact sets in X.

<u>Proof</u>: Let U be compact, and let u, u_i satisfy (2.1). By the Bolzano-Weierstrass theorem, $\{u_i\}$ has a convergent subsequence $\{u_{i_j}\}$. Let

$$v = \lim_{i_j \to \infty} u_{i_j} .$$

Then

$$Kv = K \lim_{i_j \to \infty} u_{i_j} = Ku$$

and so

$$u = v,$$

since K is invertible.

The following theorem is well known.

Theorem 2.1: Bounded, closed sets in a reflexive Banach space are weakly compact.

Proof: See [33], pp. 177.

Corollary 2.1: Let X be a reflexive Banach space, and let K be as above. Then K^{-1} is weakly regularized on U if U is bounded and closed.

Proof: Apply Theorem 2.1 to Lemma 2.1.

Corollary 2.2: Let $X=L_2[a,b]$, let $Y=C[a,b]$, and let U be a uniformly bounded family of functions in X with uniformly bounded derivatives in $L_2[a,b]$. Then K^{-1} is regularized on U.

Proof: Let u, $\{u_i\}$ be elements of U that satisfy (2.1). Since

$$||u_i'||_2 \leq M$$

for some M>0, $\{u_i'\}$ has a weakly convergent subsequence in $L_2[a,b]$ by Theorem 2.1. Let $\{u_{i_j}'\}$ be any weakly convergent subsequence of $\{u_i'\}$, and let v(x) be its weak limit. Also let

$$w(x)=\int_a^x v(s)ds + u(a).$$

Then for fixed x ($a \leq x \leq b$),

$$Kw(x)=K[\int_a^x v(s)ds + u(a)]$$

$$=K \int_a^x (\operatorname{Lim}_{i_j \to \infty} u'_{i_j}(s)ds) + u(a)$$

$$=K \operatorname{Lim}_{i_j \to \infty} u_{i_j}(x).$$

These last two equalities were obtained from the fact that ($\int \cdot\, ds$) is a linear functional and from the definition of weak limit. But this implies that

$$Ku = \lim_{i_j \to \infty} Ku_{i_j} = Kw,$$

and so

$$u=w.$$

This proves the corollary.

Corollary 2.3: Let $X=C[a,b]$, and let U be a uniformly bounded, equicontinuous family of functions. Then K^{-1} is regularized on U.

Proof: Apply the Arzela-Ascoli theorem to Lemma 2.1.

Example 1 (Backward heat equation): Let $u(x,t)$ solve

$$u_t=u_{xx} \quad (0\leq x\leq \pi,\ t\geq 0),$$

$$u(0,t)=f_0(t) \quad (t\geq 0),$$

$$u(\pi,t)=f_1(t) \quad (t\geq 0)$$

$$u(x,1)=g(x) \quad (0\leq x\leq \pi),$$

$$\|u(x,0)\|_q\leq M$$

and let $v(x,t;\varepsilon)$ satisfy

(2.2a) $$v_t=v_{xx} \quad (0\leq x\leq \pi,\ t\geq 0),$$

(2.2b) $$v(0,t)=f_0(t) \quad (t\geq 0),$$

(2.2c) $$v(\pi,t)=f\ (t) \quad (t\geq 0),$$

(2.2d) $$\|v(x,1)-g(x)\|_p\leq \varepsilon\ ,$$

(2.2e) $$\|v(x,0)\|_q\leq M,$$

where $p\geq 1$ and $q>1$. We will show that

$$|v(x,t;\varepsilon)-u(x,t)| \to 0 \text{ as } \varepsilon \to 0 \quad (t>0,\ 0\leq x\leq \pi).$$

Let

$$w(x,t;\varepsilon)=v(x,t;\varepsilon)-u(x,t),$$

so that w solves

(2.3a) $$w_t=w_{xx} \quad (0\leq x\leq \pi,\ t\geq 0),$$

(2.3b) $w(0,t)=0 \qquad (t \geq 0)$,

(2.3c) $w(\pi,t)=0 \qquad (t \geq 0)$,

(2.3d) $\|w(x,1)\|_p \leq \varepsilon \qquad (0 \leq x \leq \pi)$,

(2.3e) $\|w(x,0)\|_q \leq 2M \qquad (0 \leq x \leq \pi)$.

The set of all functions in $L_q[a,b]$ that satisfy (2.3e) is weakly compact. For $\varepsilon_i \to 0$, let $w(x,t;\varepsilon_i)$ be solutions to (2.3), and let $z(x)$ be any weak limit of a subsequence of the w. Then for any continuous $K(x,s)$,

$$\int_0^{\pi} z(s)K(x,s)\,dx = \lim_{\varepsilon_i \to 0} \int_0^{\pi} w(s,0;\varepsilon_i)K(x,s)\,ds ,$$

by the definition of weak convergence. Therefore, if

$$K(x,s)=[2/\pi]\sum_{n=1}^{\infty}[\sin(nx)\sin(ns)\exp(-n^2)],$$

then

$$0= \lim_{\varepsilon_i \to 0} \|w(s,1;\varepsilon_i)\|_p = \lim_{\varepsilon_i \to 0} \left\| \int_0^{\pi} w(s,0;\varepsilon_i)K(x,s)\,ds \right\|_p$$

$$= \int_0^{\pi} z(s)K(x,s)\,ds .$$

Hence, by the uniqueness of solutions to the heat equation,

$$z(x)=0 \qquad (0 \leq x \leq \pi).$$

We have shown that $w(x,0;\varepsilon) \to 0$ weakly as $\varepsilon \to 0$. It follows therefore by the definition of weak convergence that for $t>0$,

$$w(x,t;\varepsilon)= \int_0^{\pi} w(s,0;\varepsilon)[2/\pi]\sum_{n=1}^{\infty}(\sin(nx)\sin(ns)\exp(-n^2 t))\,ds$$

$$\to 0 \text{ as } \varepsilon \to 0.$$

This was what we wanted to show.

<u>Example 2</u>: Note that the regularization (2.2d,e) is not enough to guarantee that $v(x,0;\varepsilon) \to u(x,0)$ as $\varepsilon \to 0$. Merely restricting the solution to lie in a bounded set is not enough here. However, if $v(x,t;\varepsilon)$ and $u(x,t)$ also satisfy

(2.2f) $\|u_x(x,0)\|_2, \|v_x(x,0)\|_2 \le M_1$,

then this is enough to insure that $\|u(x,0)-v(x,0;\varepsilon)\| \longrightarrow 0$ as $\varepsilon \longrightarrow 0$, by Corollary 2.2. Regularizations like this were first studied by Tikhonov [35,36].

<u>Example 3</u>: Uniform convergence at t=0 can be obtained if, instead of (2.2d,e), the regularization

(2.2g) $|v(x,1;\varepsilon)-g(x)| \le \varepsilon \qquad (0 \le x \le \pi)$,

(2.2h) $|v(x,0;\varepsilon)| \le M_0 \qquad (0 \le x \le \pi)$,

(2.2i) $|v_x(x,0)| \le M_1 \qquad (0 \le x \le \pi)$.

is used. This is a consequence of Corollary 3.3.

In this section several ways in which ill-posed problems can be regularized were illustrated, and some of these regularizations were applied to the backward heat equation. The question of which regularization to choose for a particular ill-posed problem depends upon the physical information available, as well as the numerical efficiency and accuracy of the algorithm used to solve the regularized problem. In the following sections the numerical efficiency, stability, and error for regularizations of backward parabolic problems which can be solved using linear programming will be analyzed.

II. THE LINEAR PROBLEM

II.1. Logarithmic Convexity for Problems with Time-independent Coefficients.

Several theorems are proven in this section on the logarithmic convexity of solutions to linear parabolic equations with time-independent coefficients. These convexity results can be used to obtain error estimates for the numerical solution of regularized parabolic equations backwards in time. Some of the results in this section have appeared in print before, either by Miller [25], or Payne [26].

A positive real function F is defined to be logarithmically convex on [a,b] if for $a<x<b$ $F(x)$ satisfies

$$F(x) \leq [F(a)]^{\frac{b-x}{b-a}} [F(b)]^{\frac{x-a}{b-a}}$$

The following is the most well known logarithmic convexity result.

Theorem 1.1 (Hadamard's three circle theorem): Suppose $f(z)$ is analytic and single-valued on the annulus $a<|z|<1$, continuous on the closure, and

$$|f(z)| \leq m = a^{c}, \qquad |z| = a,$$

$$|f(z)| \leq 1, \qquad |z| = 1.$$

Then for $a \leq r \leq 1$,

$$|f(z)| \leq m^{\frac{\log r}{\log a}} = r^{c} \qquad , |z| = r.$$

Furthermore, this bound is optimal when c is a positive integer, since the analytic function z^{c} assumes the bound.

Proof: See [33], pp. 270. If F is the maximum modulus of f,

$$F(r) = \sup_{|z|=r} |f(z)|,$$

then Theorem 1.1 says that F(r) is logarithmically convex.

Miller [24] has used this theorem to show that solutions of the heat equation are logarithmically convex. Here a different approach suggested by Franklin is used, and logarithmic convexity results are obtained for a more general class of problems.

Theorem 1.2: Let H be any Hilbert space of functions on [0,1], with norm $\|\ \|_2$, and let L be any symmetric, positive definite operator mapping H into itself. Furthermore, let u(x,t) be in H for all fixed $t \geq 0$, and let u(x,t) solve

(1.1) $$u_t = Lu \quad (t>0,\ 0 \leq x \leq 1).$$

Then

$$\|u(x,t)\|_2 \leq \|u(x,0)\|_2^{1-t} \|u(x,1)\|_2^{t}.$$

Proof: Let

$$F(t) = .5 \mathrm{Log} \langle u(x,t), u(x,t) \rangle = \mathrm{Log}\ \|u(x,t)\|_2^2.$$

To show that F is convex, we must show that

$$F''(t) \geq 0 \qquad (0<t<1).$$

But

$$F'(t) = \langle Lu,u \rangle / \langle u,u \rangle,$$

and

$$F''(t) = 2(\langle Lu,Lu \rangle \langle u,u \rangle - \langle Lu,u \rangle^2)/\langle u,u \rangle^2 \geq 0,$$

by the Cauchy-Schwartz inequality. This proves the theorem.

Theorem 1.3: Let u(x,t) solve (1.1), where L is a symmetric, positive definite operator defined on C[0,1]. Then

$$\|u(x,.5)\|_2^2 \leq \|u(x,0)\|_p \ \|u(x,1)\|_q$$

if $(1/p)+(1/q)=1$.

Proof: The solution to (1.1) is

$$u(x,t) = e^{tL}u(x,0),$$

and so

$$\langle u(x,.5),u(x,.5)\rangle = \langle e^{.5L}u(x,0), e^{-.5L}u(x,1)\rangle$$
$$= \langle u(x,0),u(x,1)\rangle,$$

since L is Hermitian. The desired result now follows from Holder's inequality.

Theorems 1.2 and 1.3 have been used by Payne [26], Miller [25], and others to obtain L_2 error estimates for solutions to backward parabolic problems. Here we derive a new logarithmic convexity result that will enable us to obtain pointwise error estimates for these problems. We first need some lemmas.

Lemma 1.1: If

$$f(0)=f(\pi)=0,$$

and

$$\int_0^{\pi}(f(x))^2\,dx \le M_1^2,$$
$$\int_0^{\pi}(f'(x))^2\,dx \le M_2^2,$$

then

$$|f(x)|^2 \le 2M_1 M_2.$$

Proof: For $0 \le s \le \pi$, we have

$$[f(s)]^2 = \int_0^s [f(x)^2]'\,dx$$
$$= 2\int_0^s f(x)f'(x)\,dx \quad ,$$

and so

$$[f(s)]^2 \leq 2\left(\int_0^s [f(x)]^2 dx\right)^{1/2} \left(\int_0^s [f'(x)]^2 dx\right)^{1/2}$$

$$\leq 2M_1 M_2.$$

Lemma 1.2: Any u(x,t) continuous in $[0,\pi]\times[0,\infty]$ that satisfies

$$r(x)u_t=(p(x)u_x)_x-q(x)u \quad (0<x<\pi,\ t>0)$$

$$u(0,t)=0 \qquad (t\geq 0)$$

(1.2) $$u(\pi,t)=0 \qquad (t\geq 0)$$

$$\|u(x,0)\|_{2,r}\leq M$$

$$\|u(x,1)\|_{2,r}\leq \varepsilon$$

with $p,r>0$ and $q\geq 0$ also satisfies

$$\|u_x(x,1)\|_{2,p}\leq K\varepsilon[\text{Log}\ (M/\varepsilon)]^{1/2}$$

for some constant K depending only on the functions p, q, and r.

Proof: Any solution to (1.2) has the form

$$u(x,t)=\sum_{n=1}^{\infty} a_n X_n(x)\exp(-\lambda_n^2 t) \qquad (0\leq x\leq \pi,\ t\geq 0),$$

where $X_n(x)$, λ_n^2 solve the Sturm-Liouville problem

(1.3) $$(p(x)X_n'(x))'+(\lambda_n^2 r(x)-q(x))X_n=0,$$
$$X_n(0)=X_n(\pi)=0.$$

If the X_n are normalized by

(1.4) $$\int_0^\pi r(x)[X_n(x)]^2 dx=1,$$

then

$$a_n=\int_0^\pi r(x)X_n(x)u(x,0)dx.$$

Hence from

$$\|u(x,1)\|_{2,r} = \sum_{n=1}^{\infty} a_n^2 \exp(-2\lambda_n^2)$$

and

$$u_x(x,t) = \sum_{n=1}^{\infty} a_n X_n'(x)\exp(-\lambda_n^2 t) \quad (0\le x\le\pi,\ t>0),$$

it follows that

$$\|u_x(x,1)\|_{2,p} = \sum_{n=1}^{\infty} a_n^2\exp(-2\lambda_n^2)\int_0^{\pi} p(x)[X_n'(x)]^2dx \quad +$$

$$2\sum_{m<n} a_m a_n \exp(-\lambda_m^2-\lambda_n^2)\int_0^{\pi} p(x)X_m'(x)X_n'(x)dx$$

(1.5)
$$\le \sum_{n=1}^{\infty} a_n^2\exp(-2\lambda_n^2)\int_0^{\pi} p(x)(X_n'(x))^2dx \quad +$$

$$2\sum_{m<n} a_m a_n\exp(-\lambda_m^2-\lambda_n^2)\left(\int_0^{\pi} p(x)(X_m'(x))^2\,dx\right)^{1/2}\left(\int_0^{\pi} p(x)(X_n'(x))^2\,dx\right)^{1/2}$$

Using (1.3) and integration by parts, we see that

$$\int_0^{\pi}(X_n(x))^2[\lambda_n^2 r(x)-q(x)]dx = -\int_0^{\pi}[p(x)X_n'(x)]'X_n(x)dx$$

$$= \int_0^{\pi} p(x)[X_n'(x)]^2dx.$$

It is well known that $\lambda_n = O(n)$ (see [6] for example), and so using (1.4) we see that there are constants K_1 and K_2 such that

(1.6)
$$K_1 n^2 \le \int_0^{\pi} p(x)[X_n'(x)]^2dx \le K_2 n^2.$$

Therefore, from (1.5) we have that

$$\|u_x(x,1)\|_{2,p} \le K_2\sum_{n=1}^{\infty} n^2 a_n^2\exp(-2\lambda_n^2) + 2K_2\sum_{m<n} a_m a_n\exp(-\lambda_m^2-\lambda_n^2)$$

$$= K_2\left(\sum_{n=1}^{\infty} n a_n\exp(-\lambda_n^2)\right)^2.$$

For $\delta>0$, let $C(\delta)$ satisfy

(1.7)
$$C(\delta)\exp(\delta\lambda_n^2) \ge n \qquad (n=1,2,\dots).$$

It follows then that

$$\|u_x(x,1)\|^2_{2,\rho} \leq K_2[C(\delta)]^2 \sum_{n=1}^{\infty} a_n^2 \exp(-\lambda_n^2(2-2\delta))$$

(1.8) $$= K_2[C(\delta)]^2 \|u(x,1-\delta)\|^2_{2,r}$$

$$\leq K_2[C(\delta)]^2(\varepsilon^{1-\delta}M^{\delta})^2,$$

the last inequality being a consequence of Theorem 2. We now must choose $C(\delta)$.

Using $\lambda_n = O(n)$, from (1.7) we must have

$$C(\delta) \geq [n\exp(-\delta\lambda_n^2)]^2$$

$$\geq n^2\exp(-2\delta K_3 n^2)$$

for all positive integral values of n and for some positive constant K_3. Consider now the function

(1.9) $$F(x) = x^2\exp(-2\delta K_3 x^2).$$

The maximum of this function can be found from elementary calculus.

$$F'(x) = (2x-4\delta K_3 x^3)\exp(-2\delta K_3 x^2)$$

$$= 0 \qquad \text{when } 2\delta K_3 x^2 = 1.$$

Substituting this value into F gives

$$F(x;\delta) = 1/(2\delta K_3 e).$$

Let

$$C(\delta)^2 = F(x;\delta).$$

$[C(\delta)(M/\varepsilon)^{\delta}]$ becomes unbounded as $\delta \to 0$ and as $\delta \to \infty$. We now seek to choose $\delta > 0$ so that this quantity is minimized. This is accomplished by letting

$$\delta = 1/[2\mathrm{Log}(M/\varepsilon)],$$

so that

(1.10) $$[C(\delta)(M/\varepsilon)^{\delta}]^2 = [\mathrm{Log}\,(M/\varepsilon)]/K_3.$$

Combining (1.8) and (1.10) now gives

$$\|u_x(x,1)\|^2_{2,\rho} \le K_2 \varepsilon^2 [\text{Log } (M/\varepsilon)]/K_3.$$

This proves the lemma.

Theorem 1.4: There is a constant K depending only upon p, q, and r such that if u(x,t) satisfies the conditions of Lemma 1.2, then

$$|u(x,t)| \le K \varepsilon^t M^{1-t} \text{Log}(M/\varepsilon)^{1/4} .$$

Proof: From Theorem 1.2, we have

$$\|u(x,t)\|_2 \le C_1 \varepsilon^t M^{1-t} \quad (t \ge 0)$$

for some constant C_1, and from Lemma 1.2 we have

$$\|u_x(x,t)\|_2 \le C_2 \varepsilon^t M^{1-t} \text{Log}(M/\varepsilon)^{1/2} \quad (t>0)$$

for some constant C_2. Now apply Lemma 1.1.

More logarithmic convexity results for ill-posed problems can be found in [24].

II.2. Logarithmic Convexity for Problems with Time-dependent Coefficients.

Two logarithmic convexity theorems for solutions of time-dependent differential equations in a Hilbert space are proven in this section. One of these functional analytic results is due to Carasso [9]; the other is due to Agmon and Niremberg [1]. Here it is shown how these theorems can be applied to obtain error estimates for the solution of parabolic equations with time-dependent coefficients backwards in time.

Let H be a Hilbert space of functions on $[0,\pi]$, and let $L(t):H\longrightarrow H$ be a linear operator for all fixed t in (0,1). Here L(t) will be called differentiable at $t=t'$ if there exists an operator $L_{t'}$ such that

(2.1) $$\lim_{t\longrightarrow t'} \frac{L(t)-L(t')}{t-t'}\, u = L_{t'} u$$

for all u in H.

Consider now the problem

(2.2a) $$u_t=L(t)u \qquad (t\geq 0)$$

(2.2b) $$u(0)=u_0\,,$$

where L is a linear negative definite operator and is differentiable in the sense of (2.1) for all $t\in(0,1)$. The unique solution to this problem is given by

$$u(t)= e^{\int_0^t L(t)dt}\, u_0\,.$$

Let

$$F(t)=\text{Log}\,\langle u(t),u(t)\rangle.$$

Then

$$F'(t)=2\langle u(t),u'(t)\rangle/\langle u(t),u(t)\rangle$$
$$= 2\langle u,L(t)u\rangle/\langle u,u\rangle \leq 0,$$

and

$$F''(t)=2(\langle u,L(t)u\rangle/\langle u,u\rangle)'$$
$$= 4\ (\langle Lu,Lu\rangle\langle u,u\rangle-\langle u,Lu\rangle^{2})/\langle u,u\rangle^{2}$$
$$+2\ \langle u,L_t u\rangle/\langle u,u\rangle$$
$$\geq 2\langle u,L_t u\rangle/\langle u,u\rangle$$

by the Cauchy-Schwartz inequality.

An equivalent form of the following theorem was first stated by Carasso [9].

Theorem 2.1: Let u solve (2.2a), and let

$$M=\langle u(0),u(0)\rangle,$$
$$\varepsilon=\langle u(1),u(1)\rangle.$$

If u also satisfies

$$2\langle u,L_t u\rangle \geq C\langle u,u\rangle$$

for all $t\in(0,1)$, then if $C\geq 0$

$$\langle u(t),u(t)\rangle \leq \varepsilon^{t}M^{1-t},$$

and if $C<0$

$$\langle u(t),u(t)\rangle \leq e^{-Ct(1-t)}\varepsilon^{t}M^{1-t}.$$

Proof : Define

$$G(t)=\mathrm{Log}\langle u(t),u(t)\rangle-[t\,\mathrm{Log}\,\varepsilon + (1-t)\mathrm{Log}\ M], \tag{2.3}$$

so that

$$G(0)=G(1)=0,$$

and

$$G''(t)=F''(t)=(\mathrm{Log}\langle u(t),u(t)\rangle)'' \geq C.$$

If $C \geq 0$ then

$$G''(t)=F''(t)\geq 0 \qquad (0<t<1), \tag{2.4}$$

which implies that the maximum of G must occur at either t=0 or t=1. Hence

$$G(t)\leq 0 \qquad (0<t<1),$$

or equivalently

$$\langle u(t),u(t)\rangle \leq \varepsilon^{t} M^{1-t}.$$

Assume now that C<0. Let

$$G(s)= \sup_{0<t<1} G(t) \quad .$$

We can assume without loss of generality that $s\neq 0$ or 1, so that

$$G'(s)=0.$$

It follows from (2.4) that

$$-G'(t)=G'(s)-G'(t)\geq C(s-t) \geq C(1-t) \quad (1>s>t>0),$$

and so

$$G(t)-G(0) = G(t) \leq t[-C(1-t)] \quad (1>s>t>0).$$

Similarly, for 1>t>s>0 we have

$$G'(t)\geq C(t-s)\geq Ct \qquad (1>t>s>0),$$

so that

$$G(t)-G(1)=G(t)\leq -Ct(1-t) \quad (1>t>s>0).$$

Hence,

$$\exp(G(t)) \leq \exp[-Ct(1-t)] \qquad (0\leq t\leq 1). \tag{2.5}$$

The desired inequality now follows immediately from (2.3) and (2.5).

This theorem, which was first proven by Agmon and Niremberg [1], can be used to provide error estimates for the solution to regularized time-dependent parabolic equations backwards in time (although they never did so).

Theorem 2.2: If there are positive constants c and k such that

$$\langle L_t u,u\rangle \geq c\langle (L-kI)u,u\rangle \qquad (0<t<1),$$

then

$$e^{2k(w(t)-t)}\langle u(t),u(t)\rangle \leq \langle u(0),u(0)\rangle^{1-w(t)}\langle u(1),u(1)\rangle^{w(t)},$$

where

$$w(t)=(\exp(ct)-1)/(\exp(c)-1).$$

Proof: Let

(2.6) $$s=\exp(ct)$$

and let

$$F(s)=\mathrm{Log}\langle \exp(-kt)u,\exp(-kt)u\rangle = \mathrm{Log}\langle s^{-k/c}u(s),s^{-k/c}u(s)\rangle.$$

We will show that F is a convex function of s. Now

$$du/ds=(du/dt)(dt/ds)=u_t/(cs),$$

and so

$$F'(s) = \frac{2s^{-2k/c}\langle u,du/ds\rangle+(-2k/c)\langle u,u\rangle s^{-2k/c-1}}{s^{-2k/c}\langle u,u\rangle}$$

$$= [2/(cs)](\langle u,u_t\rangle-k\langle u,u\rangle)/\langle u,u\rangle$$

$$= [2/(cs)](\langle u,Lu\rangle-k\langle u,u\rangle)/\langle u,u\rangle.$$

Also, since

$$d(du/dt)/ds = d(Lu)/ds = (L(t)u)_t/(cs) = (L^2u+L_tu)/(cs)$$

we have that

$$F''(s) = -2k/(cs^2) - [2/(cs^2)]\langle u,Lu\rangle/\langle u,u\rangle$$

$$+[2/(cs)](\langle u,Lu\rangle/\langle u,u\rangle)$$

$$= [2/(cs^2)]\{-k\langle u,u\rangle^2-\langle u,Lu\rangle\langle u,u\rangle+(2/c)\langle Lu,Lu\rangle\langle u,u\rangle$$

$$-\langle u,Lu\rangle^2/c +\langle u,u\rangle\langle L_tu,u\rangle/c\}/\langle u,u\rangle^2.$$

Hence, using the Cauchy-Schwartz inequality gives

$$F''(s) \geq \frac{2}{(cs^2)\|u\|_2^2}\left(-k\|u\|_2^2 - \langle u, Lu\rangle + \|Lu\|_2^2/c + \langle L_t u, u\rangle/c\right)$$

$$\geq \frac{2}{(cs)^2\|u\|_2^2}\left(\langle L_t u, u\rangle - c\langle u, Lu\rangle - ck\langle u, u\rangle\right).$$

But this last quantity is nonnegative by hypothesis. Therefore, if

$$G(s)=F(s) - \left(\frac{e^c - s}{e^c - 1} F(s=1) + \frac{s-1}{e^c - 1} F(s=e^c)\right) \quad (2.7)$$

then the convexity of G implies that

$$G(s) \leq G(0) = G(1) = 0.$$

Substituting (2.6) back into (2.7) and raising exponents gives

$$e^{-2kt}\langle u(t), u(t)\rangle \leq \langle u(0), u(0)\rangle^{1-w(t)} \langle u(1), u(1)\rangle^{w(t)} e^{2Kw(t)},$$

or

$$\|u(t)\|_2 \leq e^{K(w(t)-t)} \|u(0)\|_2^{1-t} \quad \|u(1)\|^t \quad ,$$

where

$$w(t)=(\exp(ct)-1)/(\exp(c)-1). \quad (2.8)$$

This proves the theorem.

We now show how error estimates can be derived from these theorems.

<u>Example 1</u>: Let u_1 and u_2 both satisfy

$$u_t = (p(x,t)u_x)_x = L(t)u \quad (0<x<\pi,\ 0<t<1),$$

$$u(0,t) = u(\pi,t) = 0 \quad (0<t<1),$$

$$\|u(x,1)-f(x)\| \leq \varepsilon \quad (0<x<\pi),$$

$$\|u(x,0)\| \leq M \quad (0<x<\pi),$$

where p is positive and differentiable on $[0,\pi]\times[0,1]$. The operator L_t is given by

$$L_t u = (Lu)_t - Lu_t$$

$$= [p_t(x,t)u_x]_x.$$

Hence,

$$\langle u, L_t u\rangle = -\langle p_t(x,t)u_x, u_x\rangle.$$

If p is monotonically decreasing in t, then

$$\langle u, L_t u\rangle \geq 0,$$

and so the error estimate

$$\|u_1(t)-u_2(t)\|_2 \leq 2\varepsilon^t M^{1-t}.$$

can be obtained from Theorem 2.1.

Example 2: Let u_1 and u_2 both solve

$$u_t = u_{xx} - q(x,t)u \qquad (0<x<\pi,\ 0<t<1),$$

$$u(0,t)=u(\pi,t)=0 \qquad (0<t<1),$$

$$\|u(x,1)-f(x)\|_2 \leq \varepsilon,$$

$$\|u(x,0)\|_2 \leq M,$$

where $q(x,t)$ is nonnegative and differentiable on $[0,\pi]\times[0,1]$. Let

$$N = \sup_{\substack{0<x<\pi \\ 0<t<1}} |q_t(x,t)|.$$

Then

$$\langle u, L_t u\rangle = \langle -q_t u, u\rangle \geq -N\langle u,u\rangle,$$

and so Theorem 2.1 gives the estimate

$$\|u_1(t)-u_2(t)\|_2 \leq 2e^{\frac{Nt(1-t)}{2}}\varepsilon^t M^{1-t}.$$

Example 3: Let u_1 and u_2 both solve

$$u_t = (p(x,t)u_x)_x - q(x,t)u,$$

$$u(0,t)=u(\pi,t)=0 \qquad (0<t<1),$$

$$\|u(x,1)-f(x)\|_2 \leq \varepsilon,$$

$$\|u(x,0)\|_2 \leq M,$$

where p is positive and q is nonnegative on $[0,\pi]\times[0,1]$. Here

$$\langle u, L_t u\rangle = -\langle q_t u, u\rangle - \langle p_t u_x, u_x\rangle,$$

and

$$\langle u, Lu\rangle = -\langle qu, u\rangle - \langle pu_x, u_x\rangle.$$

Now if

$$c = \sup_{\substack{0<x<\pi \\ 0<t<1}} \left\{ p_t(x,t)/p(x,t),\ q_t(x,t)/q(x,t),\ 0 \right\},$$

then

$$\langle u, L_t u\rangle - c\langle u, Lu\rangle \geq 0,$$

and so applying Theorem 2.2 gives the error estimate

$$\| u_1(x,t) - u_2(x,t) \|_2 \leq 2\varepsilon^{w(t)} M^{1-w(t)},$$

where w(t) is defined by (2.8).

The logarithmic convexity theorems given in this section and in the previous section are useful in obtaining estimates of the regularization error when solving ill-posed parabolic equations backwards in time. A linear programming algorithm for solving these problems is described in the next section.

II.3. The Numerical Method.

In this section we shall show how to find an approximate solution to

$$
\text{(3.1)} \qquad
\begin{aligned}
&r(x,t)u_t=(p(x,t)u_x)_x-q(x,t)u,\\
&u(x,1)=g(x),\\
&u(0,t)=f_0(t),\\
&u(\pi,t)=f_1(t),
\end{aligned}
$$

in the region $(0,\pi)x(0,1)$ by utilizing a priori information about $u(x,t)$. In particular, it shall be shown how linear programming can be used to approximately solve (3.1). Here we assume that $r(x,t)$ and $p(x,t)$ are positive and continuous, and that $p_x(x,t)$ exists in this region. We also assume that

$$g(0)=f_0(1),\ g(\pi)=f_1(1).$$

Notice that (3.1) includes the backward heat equation (a classical ill-posed problem) as a special case.

First we show how (3.1) can be reduced to an ill-posed problem with homogeneous boundary conditions at $x=0$ and $x=\pi$.

Now

$$u(x,t)=v(x,t)+w(x,t),$$

where $v(x,t)$ solves

$$
\text{(3.2)} \qquad
\begin{aligned}
&r(x,t)v_t=(p(x,t)v_x)_x-q(x,t)v,\\
&v(x,0)=(x*f_1(0)+(\pi-x)*f_0(0))/\pi,\\
&v(0,t)=f_0(t),\\
&v(\pi,t)=f_1(t) \qquad (0<x<\pi,\ 0<t<1),
\end{aligned}
$$

and $w(x,t)$ solves

$$(3.3)\qquad \begin{aligned} &r(x,t)w_t=(p(x,t)w_x)_x-q(x,t)w,\\ &w(x,1)=g(x)-v(x,1),\\ &w(0,t)=0,\\ &w(\pi,t)=0 \qquad (0<x<\pi,\ 0<t<1). \end{aligned}$$

Problem (3.2) is an initial-boundary value problem for a parabolic equation, and it is well posed. Hence (3.2) can be solved numerically by any stable difference method, such as Crank-Nicolson. Problem (3.3), however, is a final-boundary value problem, and this problem is ill-posed. Since we can solve for $v(x,t)$ in $(0,\pi)\times(0,1)$, we have succeeded in reducing (3.1), which has general boundary conditions at $x=0$ and $x=\pi$, to (3.3), which has homogeneous boundary conditions.

Since (3.3) is ill-posed, a direct numerical solution of it can not be obtained without further assumptions about the solution $w(x,t)$. Any small truncation error in the evaluation of $g(x)$ may lead to an arbitrarily large error in calculating w.

Assume therefore that f_0, f_1, and g are not known precisely, but that

$$(3.4)\qquad \begin{aligned} &|f_0(t)-F_0(t)|\le \varepsilon'' && (0<t<1)\\ &|f_1(t)-F_1(t)|\le \varepsilon' && (0<t<1)\\ &|g(x)-G(x)|\le \varepsilon && (0<x<\pi) \end{aligned}$$

where F_0, F_1, and G are known, continuous functions in their domains, and

$$0<\ \varepsilon,\ \varepsilon',\ \varepsilon''<<1.$$

Assume also that there is a constant M such that

$$(3.5)\qquad |u(x,0)|\le M \qquad (0\le x\le \pi).$$

Assumption (3.4) might occur, for example, if f_0, f_1, and g are measured quantities in an experiment and are therefore subject to measurement errors. Assumption (3.5), on the other hand, can be interpreted as known information about the past temperature of a thin rod, for example. Under these assumptions, u(x,t) solves

$$
(3.6)\qquad
\begin{aligned}
&r(x,t)u_t=(p(x,t)u_x)_x-q(x,t)u,\\
&|u(x,1)-G(x)|\le \varepsilon,\\
&|u(0,t)-F_0(t)|\le \varepsilon',\\
&|u(\pi,t)-F_1(t)|\le \varepsilon'',\\
&|u(x,0)|\le M \qquad (0<x<\pi,\ 0<t<1).
\end{aligned}
$$

Let $u_0(x,t)$ solve (3.1) and satisfy (3.5), and let $u_\varepsilon(x,t)$ be any twice continuously differentiable solution to (3.6) in $(0,\pi)\times(0,1)$. Let

$$e(x,t)=u_0(x,t)-u_\varepsilon(x,t),$$

so that e(x,t) satisfies

$$
(3.7)\qquad
\begin{aligned}
&r(x,t)e_t=(p(x,t)e_x)_x-q(x,t)e,\\
&|e(x,1)|\le \varepsilon,\\
&|e(0,t)|\le \varepsilon',\\
&|e(\pi,t)|\le \varepsilon'',\\
&|e(x,0)|\le 2M.
\end{aligned}
$$

If r, p, and q are functions of x only, then we have the following lemma.

<u>Lemma 3.1</u>: If e(x,t) solves (3.7), then for $0<t\le 1$ we have

$$\int_0^\pi (e(x,t))^2\,dx\le 4\pi M^{2(1-t)}\varepsilon^{2t}$$

<u>Proof</u>: This result is a direct consequence of the logarithmic convexity of solutions of (3.7) (see section II.1). Notice that the above inequality implies that

$$\int_0^\pi (u_0(x,t)-u_\varepsilon(x,t))^2\,dx \longrightarrow 0 \text{ as } \varepsilon\longrightarrow 0 \quad (0<t\le 1).$$

Hence inequalities (3.4),(3.6) regularize (3.1) in $L_2(0,\pi)$ for $0<t\leq 1$.

We will now show how (3.6) can be solved by linear programming. First notice that the error estimate in Lemma 3.1 is independent of the errors ε' and ε'' at x=0 and $x=\pi$. Also, since the forward problem is well posed, the instability in the backward problem is not due to the boundary conditions at x=0 or $x=\pi$, but to the instability of u(x,t) to small perturbations in g(x). Therefore, instead of trying to find solutions to (3.6), we will find solutions to

$$
(3.8)\qquad
\begin{aligned}
&r(x,t)u_t=(p(x,t)u_x)_x-q(x,t)u\\
&|u(x,1)-G(x)|\leq \varepsilon\\
&u(0,t)=F_0(t)\\
&u(\pi,t)=F_1(t)\\
&|u(x,0)|\leq M.
\end{aligned}
$$

The generalization to other types of boundary conditions is readily apparent. As before, we can split the solution of this problem into two components, v(x,t) and w(x,t), where v(x,t) solves a well posed initial value problem, and w(x,t) solves

$$
(3.9)\qquad
\begin{aligned}
&r(x,t)w_t=(p(x,t)w_x)_x-q(x,t)w\\
&|w(x,1)-(G(x)-v(x,1))|\leq \varepsilon\\
&w(0,t)=0\\
&w(\pi,t)=0\\
&|w(x,0)+v(x,0)|\leq M.
\end{aligned}
$$

Let

$$h=\pi/N$$

for some positive integer N, and let

$$x_j=jh \qquad (j=0,1,\ldots,N).$$

Consider any stable and consistent difference approximation to the differential equation, such as

$$r(x_i,t_j+.5k)(u(x_i,t_j+k)-u(x_i,t_j))/k=D^+(p(x_i,t_j)D^-u(x_i,t_j)) \\ -q(x_i,t_j)u(x_i,t_j), \tag{3.10}$$

and choose

$$k=1/m$$

for some positive integer m. This explicit difference scheme is known to be stable and $O(k+h^2)$ if r=p=1, q=0, and k/h^2 is small enough.

Now (3.10) can be written as

$$\begin{pmatrix} u(x_1,t_j+k) \\ \vdots \\ u(x_{N-1},t_j+k) \end{pmatrix} = A \begin{pmatrix} u(x_1,t_j) \\ \vdots \\ u(x_{N-1},t_j) \end{pmatrix}$$

for some (N-1)x(N-1) matrix A. Hence,

$$\begin{pmatrix} u(x_1,1) \\ \vdots \\ u(x_{N-1},1) \end{pmatrix} = \prod_{j=0}^{m-1} A_j \begin{pmatrix} u(x_1,0) \\ \vdots \\ u(x_{N-1},0) \end{pmatrix} = A \begin{pmatrix} u(x_1,0) \\ \vdots \\ u(x_{N-1},0) \end{pmatrix} \tag{3.11}$$

Here A can be thought of as an approximate Green's function for the partial differential equation.

Substituting (3.11) into the inequalities of (3.9), we obtain

$$-\varepsilon \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \le A \begin{pmatrix} u(x_1,0) \\ \vdots \\ u(x_{N-1},0) \end{pmatrix} - \begin{pmatrix} G(x_1)-v(x_1,1) \\ \vdots \\ G(x_{N-1})-v(x_{N-1},1) \end{pmatrix} \le \varepsilon \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \tag{3.12a}$$

and

$$(3.12b)\quad -M\begin{pmatrix}1\\ \cdot\\ \cdot\\ \cdot\\ 1\end{pmatrix} \le \begin{pmatrix}u(x_1,0)\\ \cdot\\ \cdot\\ \cdot\\ u(x_{N-1},0)\end{pmatrix} + \begin{pmatrix}v(x_1,0)\\ \cdot\\ \cdot\\ \cdot\\ v(x_{N-1},0)\end{pmatrix} \le M\begin{pmatrix}1\\ \cdot\\ \cdot\\ \cdot\\ 1\end{pmatrix}.$$

This is a system of 4N-4 inequalities, and these inequalities can be solved by the Simplex algorithm of linear programming if $v(x,t)$ and A are known. Since $v(x,t)$ is the solution to a well posed initial boundary value problem, it can be approximated by any of several techniques. The matrix A can also be obtained easily.

For any integer j between 1 and N-1, let

$$z_i(x_j,0)=\delta_{ij}.$$

Using (3.10), we can calculate $z_i(x_j,1)$. Let

$$a_{ij}=z_i(x_j,1).$$

Clearly, a_{ij} is the (i,j) component of the matrix A. Note that if r, p, and q are functions of x only, and if $m=2^n$ for some integer n, then $A=(A_1)^m$ can be calculated quite efficiently. First, calculate A_1. Then set

$$(A_1)^{2^j} = [(A_1)^{2^{j-1}}]^2 \qquad (j=1,\ldots,n).$$

Linear programming problems involve finding extremal solutions to systems of linear inequalities. In other words, with linear programming one can find solutions to (3.12) which minimize (or maximize) some linear combination of the $u(x_i,0)$ and $v(x_i,0)$. Using this fact, one can solve (3.12) even when ε and/or M are unknown.

If M is known and ε is unknown, ε can be treated as a variable which will be minimized. This corresponds to Chebyshev approximation with inequality constraints (see Appendix A); the aim is to find the unknown $u(x_i,0)$ that satisfy the inequality constraints and most closely solve

(3.12a). Note that this linear program always has an optimal solution. Note also that if ε=0 in this optimal solution, then (3.12a) becomes an equality, and the solution at t=0 can be obtained by setting

$$u(x_j,1) = G(x_j)-v(x_j,1)$$

and using a stable forward difference scheme in the backward direction. This rarely happens in practice, however, unless M is taken to be very large. If the optimal solution has $\varepsilon>0$, as is usually the case, then this value of ε gives an indication of the accuracy of the data G(x). Of course, this value of ε will not be the true error in G(x); it will be a combination of the true error, the discretization error, and the error in calculating v(x,1). If v(x,t) and the matrix A are determined by an $O(h^2)$ difference method, for example, then

$$\varepsilon_{true} - Ch^2 \leq \varepsilon_{opt} \leq \varepsilon_{true} + Ch^2$$

for some constant C.

If ε is known and M is unknown in (3.12), then M can be treated as a variable which must be minimized using linear programming. This corresponds to finding any solution to (3.12a) which does not grow too quickly as t approaches zero. Similarly, if neither ε nor M is known, then both ε and M can be treated as variables with some linear combination ε+cM minimized. (The constant c here should be a very small positive number; the reasons for this are discussed in the next section.) A remarkable fact can be deduced from these last two methods. Since M need not be specified, no a priori assumption is really needed at t=0 (except the assumption that the solution is continuous there) to approximately solve the backward heat equation! One of the goals of the next section is to prove the validity of validity of this last statement. In other words, in the next section it will be proven that the methods described here actually work.

Other inequality constraints can be used with (3.12). For example, if the initial temperature profile sought has bounded derivatives, then inequalities of the form

(3.13) $$-M_2 h \leq [u(x_i,0)+v(x_i,0)] - [u(x_{i-1},0)+v(x_{i-1},0)] \leq M_2 h$$

are needed. Similarly, if the initial profile is everywhere nonnegative, then inequalities of the form

$$u(x_i,0) \geq -v(x_i,0)$$

are needed.

One well known principle of linear programming is that the computation necessary to solve a linear program can be significantly reduced if one already knows a solution close to the extremal solution (see Franklin [14]). This fact can be used to formulate an efficient algorithm for mesh refinement.

First, solve (3.12) with a fairly coarse mesh. Then, using interpolation on the coarse mesh, find the approximate solution on a finer mesh. Use this interpolated solution as the initial estimate of the solution to (3.12) on the finer mesh. This technique will lead to a significant decrease in the computation time necessary to solve the linear program on the finer mesh, since even when using a coarse mesh most distinguishing features of the true solution are found in the approximate solution. This technique was used by this author with considerable success, even when the coarse mesh had as few as six gridpoints.

The total error obtained in using the numerical methods described in this section is a combination of the regularization error (inherent in the regularized problem) and the discretization error (inherent in the numerical method). In the next section, it will be proven that the four algorithms proposed here are equivalent, and estimates of the regularization error will be obtained. It will be shown that these estimates approach zero as ϵ does, therefore proving that these regularizations work. The effects of bounding a derivative, such as in (3.13), will also be discussed. The stability of the numerical algorithms proposed here is analyzed in section II.5.

II.4. Regularization Error for the Numerical Method.

The previous section showed how discretization of the problem

$$
\begin{aligned}
& r(x,t)u_t=(p(x,t)u_x)_x-q(x,t)u \\
& u(0,t)=F_0(t) \\
& u(\pi,t)=F_1(t) \\
& |u(x,1)-G(x)|\leq \varepsilon \\
& |u(x,0)|\leq M \qquad (0<x<\pi,\ 0<t<1)
\end{aligned}
\tag{4.1}
$$

leads to the system of inequalities

$$
\begin{gathered}
-\varepsilon\begin{pmatrix}1\\ \cdot\\ \cdot\\ \cdot\\ 1\end{pmatrix} \leq A\begin{pmatrix}w(x_1,0)\\ \cdot\\ \cdot\\ \cdot\\ w(x_{N-1},0)\end{pmatrix}-\begin{pmatrix}G(x_1)-v(x_1,1)\\ \cdot\\ \cdot\\ \cdot\\ G(x_{N-1})-v(x_{N-1},1)\end{pmatrix}\leq \varepsilon\begin{pmatrix}1\\ \cdot\\ \cdot\\ \cdot\\ 1\end{pmatrix}, \\
\begin{pmatrix}-M\\ \cdot\\ \cdot\\ \cdot\\ -M\end{pmatrix}\leq\begin{pmatrix}w(x_1,0)+v(x_1,0)\\ \cdot\\ \cdot\\ \cdot\\ w(x_{N-1},0)+v(x_{N-1},0)\end{pmatrix}\leq\begin{pmatrix}M\\ \cdot\\ \cdot\\ \cdot\\ M\end{pmatrix}.
\end{gathered}
\tag{4.2}
$$

The analysis that follows was inspired by a paper from Keith Miller [23], in which he discusses how least squares can be applied to ill-posed problems. Here it is not assumed that either ε or M is known precisely, but it is assumed that there is an ε and an M such that u(x,t) satisfies (4.1). There are four cases, depending upon whether ε and M are known precisely or not.

Case 1: ε,M both known.

This is the simplest case of all. Here we attempt to find any u(x,t) that solves (4.1) for some specific ε and M. Discretization leads to the problem of finding any $w(x_1,0),\ldots,w(x_{N-1},0)$ that solve (4.2). The Simplex algorithm for linear programming will provide a solution to (4.2) if one exists, or report that a solution does not exist.

<u>Case 2</u>: ε known, M unknown.

Suppose the temperature of a thin rod of slowly varying thickness is measured at t=1 to within a certain tolerance ε, and the temperature at t=0 is known to be bounded, but the bound is not known precisely. In this case we might seek a solution to (4.1) which is within the measurement error at t=1, and which does not blow up at t=0. For example, we might seek the 'smallest' M such that (4.1) is satisfied for some u(x,t). After discretization, the problem becomes to minimize M subject to the linear inequalities (4.2). This problem can also be solved readily by linear programming.

<u>Case 3</u>: ε unknown, M known.

This case occurs, for example, if the temperature at t=1 is measured with an instrument of uncertain accuracy. Here (as in Chebyshev approximation) the aim is to find the $w(x_i,0)$ such that (4.2) is satisfied, and such that

$$\varepsilon = \max_{0<i<N} |(Aw)_i - G(x_i)|$$

is minimized.

<u>Case 4</u>: ε, M both unknown.

This method is applicable when the temperature in a rod is measured at t=1 and known to be bounded at t=0. Neither the accuracy of the measurement, nor the maximum temperature at t=0 is known, however. Once again, we seek a solution which closely matches the observations at t=1, and which is not too large at t=0. This can be accomplished if we try to find the unknown $w(x_i,0)$ such that (4.2) is satisfied, and such that

$$\|A\underset{\sim}{w}-\underset{\sim}{G}\|_\infty + c\|w\|_\infty$$

is minimized for some small $c>0$. The analysis which follows will show that a good choice for c is $c=O(\varepsilon/M)$.

Notice that all four of these cases can be handled by linear programming, whereas Miller [23] reported that only case 4 can be solved by regular least squares.

We shall now prove the equivalence of the errors for these four cases.

Let $u_2(x,t)$, $u_3(x,t)$, and $u_4(x,t)$ be approximate solutions to (4.1) obtained from methods 2, 3, and 4 (with $c=\varepsilon/M$), and define $W(\varepsilon,M)$ to be the set of all functions $u(x,t)$ that satisfy (4.1) and are continuous in $[0,\pi]\times[0,1]$. Let

$$R_1(\varepsilon,M;t) = \sup_{u_1,v_1\in W} \|u_1(x,t)-v_1(x,t)\|_\infty,$$

$$R_2(\varepsilon,M;t) = \sup_{u_1\in W} \|u_2(x,t;\varepsilon,M)-u_1(x,t)\|_\infty,$$

$$R_3(\varepsilon,M;t) = \sup_{u_1\in W} \|u_3(x,t;\varepsilon,M)-u_1(x,t)\|_\infty,$$

$$R_4(\varepsilon,M;t) = \sup_{u_1\in W} \|u_4(x,t;\varepsilon,M)-u_1(x,t)\|_\infty.$$

<u>Lemma 4.1</u>: $R_i(\varepsilon,M;t)$ is a monotone nondecreasing function of ε and M for $i=1,2,3,4$.

<u>Proof</u>: If $\varepsilon'\le\varepsilon$ and $M'\le M$, then $W(\varepsilon',M') \subseteq W(\varepsilon,M)$.

<u>Lemma 4.2</u>: If (4.1) has a solution u_0 with $\varepsilon=0$, then

$$R_1(3\varepsilon,3M;t) \le 4R_1(1.5\varepsilon,1.5M;t).$$

<u>Proof</u>: Let $u_1,v_1\in W(3\varepsilon,3M)$, and let

$$w(x,t) = (3u_0(x,t)+u_1(x,t))/4.$$

Then $w(x,t)\in W(\varepsilon,1.5M)\subseteq W(1.5\varepsilon,1.5M)$. Furthermore,

$$|w(x,t)-u_0(x,t)| = |u_1(x,t)-u_0(x,t)|/4$$

for all fixed $(x,t)\in[0,\pi]\times[0,1]$. This shows that

$$R_1(1.5\varepsilon,1.5M;t)\geq [R_1(3\varepsilon,3M;t)]/4,$$

which proves the result.

Lemma 4.3: $R_2(\varepsilon,M;t)\leq R_1(\varepsilon,M;t)\leq 2R_2(\varepsilon,M;t)$

Proof: $u_2(x,t;\varepsilon,M)\in W(\varepsilon,M)$, so the first inequality is immediately satisfied. On the other hand,

$$\sup_{u_1,v_1\in W} \|u_1(x,t)-v_1(x,t)\|_\infty \leq \sup_{u_1\in W} \|u_1(x,t)-u_2(x,t)\|_\infty$$

$$+ \sup_{v_1\in W} \|v_1(x,t)-u_2(x,t)\|_\infty$$

$$= 2R_2(\varepsilon,M;t).$$

Similarly, we have the following.

Lemma 4.4: $R_3(\varepsilon,M;t)\leq R_1(\varepsilon,M;t)\leq 2R_3(\varepsilon,M;t)$.

Lemma 4.5: $R_4(\varepsilon,M;t)\leq 4R_1(\varepsilon,M;t)\leq 8R_4(\varepsilon,M;t)$.

Proof: For all $u_1(x,t)$ in $W(\varepsilon,M)$,

$$\|u_1(x,1)-G(x)\|_\infty+(\varepsilon/M)\|u_1(x,0)\|_\infty\leq\varepsilon+(\varepsilon/M)M = 2\varepsilon$$

Hence

$$|u_4(x,1)-G(x)|\leq 2\varepsilon$$

and

$$|u_4(x,0)|\leq 2M.$$

This shows that $u_4(x,t)$ lies in $W(2\varepsilon,2M)$, so

$$\sup_{u_1\in W(\varepsilon,M)} \|u_1-u_4\|_\infty\leq \sup_{u_1,v_1\in W(2\varepsilon,2M)} \|u_1-v_1\|_\infty$$

$$\leq 4R_4(\varepsilon,M;t)$$

by Lemma 4.2. Also,

$$\sup_{u_0, v_0 \in W(\varepsilon, M)} \|u_0 - v_0\|_\infty \leq \sup_{u_0, v_0 \in W(\varepsilon, M)} \|u_0 - u_4\|_\infty + \|u_4 - v_0\|_\infty$$

$$= 2R_4(\varepsilon, M; t)$$

Combining these two results proves the lemma.

Lemmas 4.2-4.5 show that the regularization errors for any two methods are equivalent, that is, that their ratios are bounded by a constant.

Let us now consider the problem of finding a continuous $u(x,t)$ in $[0,\pi]\times[0,1]$ satisfying

$$r(x)u_t = (p(x)u_x)_x \qquad (r,p>0)$$

$$u(0,t)=0$$

(4.3) $$u(\pi,t)=0$$

$$\|u(x,1)\|_{2,r} \leq \varepsilon$$

$$\|u(x,0)\|_{2,r}, \|u_x(x,0)\|_{2,p} \leq 1.$$

Equation (4.3) is not an example of a problem that can be solved by linear programming, but the analysis that follows will show how little extra accuracy is gained when a bound is imposed on $u_x(x,0)$.

Let W' be the class of all continuous functions $u(x,t)$ which satisfy (4.3). Define the modulus of regularization for this problem to be

$$R(t,\varepsilon) = \sup_{u \in W'} \|u(x,t)\|_{2,r}.$$

The following theorem was first proven by Franklin [15] for the case when $p=r=1$.

<u>Theorem 4.1</u>: There is a constant C_1 such that

$$R(t,\varepsilon) \leq C_1 \left((\mathrm{Log}(1/\varepsilon))^{-\frac{1}{2}}\right)^{1-t} \varepsilon^t.$$

In particular, $R(0,s) \to 0$ as $s \to 0$. Furthermore, this bound is best possible, since if $r=p=1$ and $\varepsilon = \exp(-n^2)/n$,

$$|\exp(-n^2 t)\sin(nx)/n| \geq 2((\mathrm{Log}(1/\varepsilon))^{-1/2})^{1-t}\varepsilon^{t} .$$

Proof: The optimality of the bound is a straightforward exercise in algebra, and so will be omitted here. Note that due to the logarithmic convexity of solutions of parabolic problems (see section II.1), it is only necessary to show that

$$R(0,\varepsilon)<C_1(\mathrm{Log}(1/\varepsilon))^{-1/2}$$

for some positive C_1.

We know from Sturm-Liouville theory that any u(x,t) in W must have the form

$$(4.4) \qquad u(x,t)=\sum_{n=1}^{\infty} a_n X_n(x)e^{-\lambda_n^2 t},$$

where $X_n(x)$ and λ_n^2 are the eigenfunctions and eigenvalues for the problem

$$(4.5) \qquad \begin{aligned} &(p(x)X_n'(x))' + (\lambda_n^2 r(x))X_n=0 \qquad (0<x<\pi) \\ &X_n(0)=0 \\ &X_n(\pi)=0 \end{aligned}$$

and that

$$||u(x,0)||_{2,r} = \sum_{n=1}^{\infty} a_n^2 .$$

Furthermore, using (4.4) and Cauchy's inequality, we find that

$$\begin{aligned} ||u_x(x,0)||_{2,p} &= \int_0^{\pi} p(x)\Big(\sum_{i=1}^{\infty} a_i X_i'(x)\Big)^2 dx \\ (4.6) \qquad &= \sum_{i=1}^{\infty} a_i^2\Big[\int_0^{\pi} p(x)(X_i'(x))^2 dx\Big] + 2\sum_{m<n} a_m a_n \int_0^{\pi} p(x)X_m'(x)X_n'(x)dx. \end{aligned}$$

But

$$\begin{aligned} \int_0^{\pi} p(x)X_m'(x)X_n'(x)dx &= -\int_0^{\pi} [p(x)X_m'(x)]'X_n(x)dx \\ &= \lambda_m^2 \int_0^{\pi} r(x)X_m(x)X_n(x)dx=0, \end{aligned}$$

and we showed in section II.1 that there are positive constants c_1 and c_2 such that

$$c_1 n^2 \leq \int_0^{\pi} p(x)(X_n'(x))^2\, dx \leq c_2 n^2 .$$

Therefore,

$$c_1 \sum_{n=1}^{\infty} n^2 a_n^2 \leq \| u_x(x,0) \|_{2,p} \leq c_2 \sum_{n=1}^{\infty} n^2 a_n^2 .$$

But

$$\| u_x(x,0) \|_{2,p} \leq 1,$$

and so from (4.7) we obtain

$$|a_n| \leq (n^2 c_1)^{-1/2} .$$

On the other hand, from (4.4) and (4.3) we have

$$|a_n| \leq 1.$$

Also,

$$\| u(x,1) \|_{2,r}^2 = \sum_{n=1}^{\infty} a_n^2 e^{-2\lambda_n^2} \leq \varepsilon^2 ,$$

and so together with (4.3) we obtain

$$|a_n| \leq e^{\lambda_n^2} \varepsilon .$$

Now clearly, for fixed ε we have

$$\varepsilon e^{\lambda_n^2} \longrightarrow \infty,\ (n^2 c_1)^{-1/2} \longrightarrow 0 \text{ as } n \longrightarrow \infty .$$

Assume that ε is small enough so that

(4.8) $$\varepsilon e^{\lambda_n^2} < (n^2 c_1)^{-1/2}$$

for some positive integer n. Let $N(\varepsilon)$ be the greatest positive integer that satisfies (4.8). Hence

$$(N+1) e^{\lambda_{N+1}^2} \geq (\varepsilon^2 / c_1)^{-\frac{1}{2}} \geq N e^{\lambda_N^2} .$$

Taking logarithms,

$$\mathrm{Log}(N+1) + \lambda_{N+1}^2 \geq \mathrm{Log}(1/\varepsilon) + .5\mathrm{Log}(c_1) \geq \mathrm{Log}(N) + \lambda_N^2 .$$

This implies the existence of positive constants K_1 and K_2 such that

(4.9) $$K_1 N(\varepsilon)^2 \leq \mathrm{Log}(1/\varepsilon) \leq K_2 N(\varepsilon)^2.$$

Let

$$h_n = \begin{cases} \varepsilon e^{\lambda_n^2} & \text{if } n<N(\varepsilon) \\ (na_n)/N & \text{if } n \geq N(\varepsilon). \end{cases}$$

Now clearly

$$|h_n| \geq |a_n|,$$

and so

$$\sum_{n=1}^{\infty} |a_n|^2 \leq \sum_{n=1}^{\infty} |h_n|^2$$

$$= \varepsilon^2 \sum_{n<N} (e^{2\lambda_n^2}) + \sum_{n \geq N} (n^2 a_n^2)/(N^2)$$

$$= \varepsilon^2 \sum_{n<N} (e^{2\lambda_n^2}) + O(1/N^2)$$

$$\leq \frac{1}{C_1} \sum_{n<N} (e^{2\lambda_n^2})/(N^2 e^{2\lambda_N^2}) + O(1/N^2)$$

$$= O(1/N^2) + O(1/N^2)$$

But, from (4.9),

$$1/N^2 = O((\mathrm{Log}(1/\varepsilon))^{-1}).$$

This proves the result.

The above theorem shows that imposing a bound on $u_x(x,0)$ will regularize (4.1) at t=0. For example, if the inequalities

(4.10) $$|u_\varepsilon(x,0)| \leq M \ , \ |u(x,1)-f(x)| \leq \varepsilon \quad (0<x<\pi)$$

are used as the regularization, there is no guarantee that $u_\varepsilon(x,0) \to u_0(x,0)$ as $\varepsilon \to 0$. However, if the inequality

(4.11) $$|u_{\varepsilon x}(x,0)| \leq M_1$$

is also used, then Theorem 4.1 implies that

$$\|u_\varepsilon(x,0)-u_0(x,0)\|_2 \le C_1 (\mathrm{Log}(1/\varepsilon))^{-1/2} \to 0 \text{ as } \varepsilon \to 0. \tag{4.12}$$

This convergence is very slow, however. The error ε must be chosen smaller than exp(-100) in order to have $(\mathrm{Log}(1/\varepsilon))^{-1/2}<.1$. Therefore, since (4.10) regularizes the backward parabolic problem (4.1) for t>0, (4.11) should not be used if u(x,t) is only sought for some t>0.

The next section will show that solutions to the system of discrete inequalities (4.2) are stable under perturbations in g, and obey logarithmically convex error bounds when the matrix A is obtained from a consistent and stable numerical method.

II.5. Stability and Discretization Error.

In this section, some stability theorems about the approximate linear programming solution to the problem

(5.1a) $u_t = Lu = (p(x)u_x)_x - q(x)u \quad (0 \le x \le \pi,\ t \ge 0)$

(5.1b) $u(0,t) = f_0(t) \quad (t \ge 0)$

(5.1c) $u(\pi,t) = f_1(t) \quad (t \ge 0)$

(5.1d) $|u(x,0)| \le M \quad (0 \le x \le \pi)$

(5.1e) $|u(x,1) - g(x)| \le \epsilon \quad (0 \le x \le \pi)$

will be proven. Here, as before, it is assumed that $q(x)$, $p'(x)$, and $g(x)$ are continuous, and that $p(x)$ is positive and $q(x)$ is nonnegative on $[0,\pi]$. In particular, it will be shown that the numerical method described in section II.3 is convergent if the one-step difference approximation to (5.1a) is stable and consistent.

A difference scheme of the form

(5.2)
$$\begin{aligned} \underset{\sim}{v}^{(n+1)} &= (v(x_1,(n+1)k), \ldots, v(x_{N-1},(n+1)k)) \\ &= A(k,h)\underset{\sim}{v}^{(n)} \end{aligned}$$

is called stable for $0 \le t \le 1$ if there is an $s>0$ such that the infinite set of matrices $A(k,h)$ satisfies

$$\|A(k,h)^m\| \le C_1 \exp(cmk) \quad (0<k,h<s,\ 0 \le mk \le 1)$$

for some constants C_1, c. The difference equation (5.2) is consistent if $\|(A(k,h)-I)/k - L\| \to 0$ as $k,h \to 0$. The difference equation is said to be convergent if, for fixed $mk \le 1$, $\|A(k,h)^m u(x,0) - u(x,mk)\| \to 0$ as $k,h \to 0$ for all functions $u(x,t)$ that satisfy (5.1a,b,c).

The following lemma gives a simple example of stable difference schemes on $0 \leq t \leq 1$.

Lemma 5.1: Any consistent difference approximation to (5.1a) in $[0,\pi]x[0,1]$ which obeys the maximum principle

$$\max_{0<i<N} (A^n \underset{\sim}{v}^{(0)}) = \max_{\substack{0<i<N \\ 0<j}} \{f_0(jk), f_1(jk), v(x_i, 0)\} \qquad (n \geq 0),$$

is stable in $L_\infty[0,\pi]$.

Proof: Applying the maximum principle to both v and -v, we get that

$$|v(x_i, t_j)| \leq \sup_{\substack{0<x<\pi \\ 0<t<1}} \{|f_0(t)|, |f_1(t)|, |v(x,0)|\}$$

which is independent of k or h. This shows that $|v(x_i, t_j)|$ is bounded in $[0,\pi]x[0,1]$ independent of the fineness of the mesh, which proves the lemma.

The following theorem is very important; it will be used to prove that the algorithms proposed in the previous section for solving (5.1) are stable.

Theorem 5.1 (Lax Equivalence Theorem): Given a properly posed initial-value problem, and a consistent finite difference approximation to it, stability is the necessary and sufficient condition for convergence.

Proof: See [29], page 45.

Consider now the implicit difference scheme

(5.3a) $v(x_i, t_j+k) = v(x_i, t_j) + (k/h^2) D^+ p(x_i) D^- v(x_i, t_j+k) - kq(x_i) v(x_i, t_j+k)$

where

(5.3b) $h = (x_{i+1} - x_i) = \pi/N_1$

for some integer N_1, and

(5.3c) $k = (t_{j+1} - t_j) = 1/N_2$.

We will now obtain error estimates for the approximate linear programming solution of (5.1) using the difference scheme (5.3). The techniques used in these derivations will then be generalized to include stable and consistent Hermitian difference methods. Finally, it will be shown that the difference scheme need only be stable and consistent (not necessarily Hermitian) for the linear programming algorithm to be convergent. The following lemma shows that (5.3) is stable.

Lemma 5.2: Let $v(x_i,t_j)$ satisfy (5.3) in $[0,\pi]\times[0,1]$. Then the maximum of $v(x_i,t_j)$ occurs when either $x_i=0$, $x_i=\pi$, or $t_j=0$.

Proof: The difference scheme (5.3a) can be rewritten as

$$(1+r(p(x_i)+p(x_i+h))+kq(x_i))v(x_i,t_j+k) = v(x_i,t_j)+rp(x_i+h)v(x_i+h,t_j+k)+rp(x_i)v(x_i-h,t_j+k)\ , \tag{5.4}$$

where

$$r=k/h^2.$$

Since r and k are positive, (5.4) takes the form

$$v(x_i,t_j+k)=a(x_i)v(x_i,t_j)+b(x_i)v(x_i+h,t_j)+c(x_i)v(x_i-h,t_j)$$

where

$$a(x_i),\ b(x_i),\ c(x_i)>0,$$

and

$$a(x_i)+b(x_i)+c(x_i)\le 1.$$

Hence

$$v(x_i,t_j+k)\le\max\{v(x_i,t_j),v(x_i+h,t_j),v(x_i-h,t_j)\}.$$

This shows that the maximum of $v(x_i,t_j)$ can not occur if $0<x_i<\pi$ or $t_j>0$, which proves the lemma.

From (5.4), we have that

$$\begin{pmatrix} v(x_1,t_j) \\ \vdots \\ v(x_{N-1},t_j) \end{pmatrix} = A^{-1} \begin{pmatrix} v(x_1,t_j+k) \\ \vdots \\ v(x_{N-1},t_j+k) \end{pmatrix}$$

where

$$A^{-1} = \begin{pmatrix} 1+r(p(x_1)+p(x_2))+kq(x_1) & -rp(x_2) & & 0 \\ -rp(x_2) & \ddots & \ddots & \\ & \ddots & \ddots & -rp(x_{N-1}) \\ 0 & & -rp(x_{N-1}) & 1+r(p(x_{N-1})+p(x_N))+kq(x_{N-1}) \end{pmatrix}$$

This matrix is symmetric and diagonally dominant. Therefore, it has all real eigenvalues, and by applying Gerschgorin's circle theorem we see that all of the eigenvalues λ_i^{-1} of A^{-1} satisfy

$$|\lambda_i^{-1}|>1$$

and so the eigenvalues λ_i of the real symmetric matrix A satisfy

$$|\lambda_i|<1,$$

which implies that

(5.5) $$||A||_2<1.$$

Now let $u=(u_1,\ldots,u_{N-1})$ and $v=(v_1,\ldots,v_{N-1})$ be any vectors in $\mathbb{R}^{N-1}$, and define the inner product $<\ ,\ >$ by

$$<u,v>=h(u_1v_1+\ldots+u_{N-1}v_{N-1}).$$

Then

(5.6) $$<Au,Au>=<A^TAu,u> \leq <A^2u,A^2u>^{1/2}<u,u>^{1/2},$$

by the Cauchy-Schwartz inequality. Hence if

$$E(t_j)=<(u(x_1,t_j),\ldots,u(x_{N-1},t_j)),(u(x_1,t_j),\ldots,u(x_{N-1},t_j))>,$$

then from (5.6) it follows that $E(t_j)$ is logarithmically convex, or

$$E(t_j)\leq E(0)^{1-t_j}\,E(1)^{t_j}.$$

Let $u_0(x,t)$ be the solution to (5.1) with $\varepsilon=0$, and let $w(x_i,t_j)$ be determined by the difference scheme (5.4) with boundary conditions

$$w(x_i,0)=u_0(x_i,0) \qquad (i=1,\dots,N_1-1),$$

(5.7) $$w(0,t_j)=f_0(t_j) \qquad (j=0,\dots,N_2),$$

$$w(\pi,t_j)=f_1(t_j) \qquad (j=0,\dots,N_2).$$

Let

$$e(x_i,t_j)=w(x_i,t_j)-u_0(x_i,t_j).$$

Expanding $u_0(x_i,t_j)$ in a Taylor series gives

$$(e(x_i,t_j+k)-e(x_i,t_j))/k=D^+(p(x_i)D^-e(x_i,t_j+k))-q(x_i)e(x_i,t_j+k)$$
$$+O(k+h^2),$$

for sufficiently smooth $u(x,t)$. Hence,

$$\begin{pmatrix} e(x_1,t_j) \\ \cdot \\ \cdot \\ \cdot \\ e(x_{N_1-1},t_j) \end{pmatrix} = A^{-1}\begin{pmatrix} e(x_1,t_j+k) \\ \cdot \\ \cdot \\ \cdot \\ e(x_{N_1-1},t_j+k) \end{pmatrix} +O(k^2+kh^2).$$

Using (5.5), it follows that there is a constant C such that

(5.8) $$\begin{aligned} ||e(x_i,t_j+k)||_2 &\le ||A||_2||e(x_i,t_j)||_2+C(k^2+kh^2) \\ &\le ||e(x_i,t_j)||_2+Ck(k+h^2). \end{aligned}$$

But

$$e(x_i,0)=0 \;,$$

and so (5.8) implies that

$$||e(x_i,1)||_2\le C(k+h^2).$$

We are now ready to prove the following theorem.

<u>Theorem 5.2</u>: Let $u_0(x,t)$ solve (5.1) with $\varepsilon=0$, and let $v_\varepsilon(x_i,t_j)$ be the linear programming solution to (5.1) using the difference scheme (5.3). Then there is a constant C such that

$$||u_0(x_i,t_j)- v_\varepsilon(x_i,t_j)||_2\le\sqrt{\pi}(\varepsilon+C(k+h^2))^{t_j}(2M)^{1-t_j}+C(k+h^2)t.$$

<u>Proof</u>: We know from the above that

$$||A^{k}\underset{\sim}{u}_0(x_i,0)-u_0(x_i,1)||_2\le C(k+h^2)$$

for some constant C. We also have from (5.1) that at t=1

$$||v_\varepsilon(x_i,1)-u_o(x_i,1)||_2 \leq \sqrt{\pi}\,\varepsilon,$$

so that

$$||v_\varepsilon(x_i,1)-A^{1/k}\underset{\sim}{u}_o(x_i,0)||_2 \leq \sqrt{\pi}\,\varepsilon+C(k+h^2).$$

Furthermore, at t=0

$$||v_\varepsilon(x_i,0)-u_o(x_i,0)||_2 \leq 2\sqrt{\pi}M.$$

From the logarithmic convexity of the difference scheme (5.3a), this gives us

(5.9) $$||v_\varepsilon(x_i,t_j)-A^j\underset{\sim}{u}_o(x_i,0)||_2 \leq \sqrt{\pi}(\varepsilon+C(k+h^2))^{t_j}(2M)^{1-t_j}.$$

On the other hand, we already know that

(5.10) $$||A^j\underset{\sim}{u}_o(x_i,0)-u(x_i,t_j)||_2 \leq C(k+h^2)t.$$

The desired result now comes from (5.9), (5.10), and the triangle inequality.

Note that if k is chosen $O(\varepsilon)$ and h is chosen $O(\varepsilon^{1/2})$, then the truncation error will not dominate the total error at t=1. With these choices for k and h, the approximation error (5.10) becomes $O(\varepsilon)$, which is dominated by the logarithmic convexity error $O(\varepsilon^t)$ of (5.9) for $t \leq 1$.

The following theorem generalizes Theorem 5.2 to arbitrary one step Hermitian finite difference methods.

Theorem 5.3: Let

(5.11) $$\begin{pmatrix} u(x_1,t_j+k) \\ \cdot \\ \cdot \\ \cdot \\ u(x_{N_1-1},t_j+k) \end{pmatrix} = A(k,h)\begin{pmatrix} u(x_1,t_j) \\ \cdot \\ \cdot \\ \cdot \\ u(x_{N_1-1},t_j) \end{pmatrix}$$

represent any stable finite difference approximation to (5.1a) with truncation error $O(h^{N_3}+k^{N_4})$ for some integers N_3 and N_4, where A is an Hermitian matrix. Also let $v(x_i,t_j;\varepsilon)$ be any linear programming solution to (5.1), and let $u_0(x,t)$ solve (5.1) exactly with $\varepsilon=0$. Then there is a constant C such that

$$\|u_0(x_i,t_j)-v(x_i,t_j;\varepsilon)\|_2 \le \sqrt{\pi}\,(\varepsilon+C(k^{N_4}+h^{N_3}))^{1-t_j}(2M)^{t_j}+C(k^{N_4}+h^{N_3})t.$$

<u>Proof</u>: The logarithmic convexity of solutions of the matrix equation (5.11) is guaranteed since the matrix A is Hermitian (see (5.6)). Now let $w(x_i,t_j)$ be determined by the difference scheme (5.11) with boundary conditions

$$w(x_i,0) = u_0(x_i,0) \qquad (i=1,\ldots,N_1-1),$$

$$w(0,t_j) = f_0(t_j) \qquad (j=0,\ldots,N_2),$$

$$w(\pi,t_j) = f_1(t_j) \qquad (j=0,\ldots,N_2),$$

and let

$$e(x_i,t_j) = w(x_i,t_j) - u_0(x_i,t_j).$$

Then e satisfies

$$e(x_i,0) = 0,$$

$$\begin{pmatrix} e(x_1,t_j+k) \\ \cdot \\ \cdot \\ \cdot \\ e(x_{N_1-1},t_j+k) \end{pmatrix} = A \begin{pmatrix} e(x_1,t_j) \\ \cdot \\ \cdot \\ \cdot \\ e(x_{N_1-1},t_j) \end{pmatrix} + \underset{\sim}{b}_j,$$

where

$$\|b_j\|_2 \le C_1\, k(h^{N_3}+k^{N_4})$$

for some constant C_1. Hence

$$\begin{pmatrix} e(x_1,t_j+k) \\ \cdot \\ \cdot \\ \cdot \\ e(x_{N_1-1},t_j+k) \end{pmatrix} = \sum_{s=0}^{j} A^{j}\underset{\sim}{b}_{j-s},$$

and so

$$\|e(x_i,t_j+k)\|_2 \leq C_1 k(h^{N_3}+k^{N_4}) \sum_{s=0}^{j} \|A^s\|_2$$
$$\leq C_1 C_2 (j+1) k(h^{N_3}+k^{N_4})$$
$$= C_1 C_2 (t_j+k)(h^{N_3}+k^{N_4})$$

for some constant C_2, since the difference method is stable. The magnitude of the truncation error is therefore $O(t(h^{N_3}+k^{N_4}))$ at time t. The rest of the proof follows exactly as in Theorem 5.2.

The next theorem is the main result of this section.

Theorem 5.4: Consider the regularized ill-posed final value problem (5.1) with a consistent finite difference approximation to it. Stability of the finite difference approximation is the necessary and sufficient condition for convergence of the algorithm (as $\varepsilon,k,h \to 0$) to the solution of (5.1) with $\varepsilon=0$, whenever this solution exists.

Proof: Assume that the algorithm converges on $0<t\leq 1$ as $\varepsilon,k,h \to 0$. This means that for fixed $mk\leq 1$,

$$\|A(k,h)^m \underset{\sim}{u}_\varepsilon(x,0)-u_0(x,mk)\|_2 \to 0 \text{ as } \varepsilon,k,h \to 0$$

for all $u_\varepsilon(x,t)$ that satisfy (5.1). But

$$\|u_\varepsilon(x,mk)-u_0(x,mk)\|_2 \to 0 \text{ as } \varepsilon \to 0,$$

since (5.1) is regularized. Hence

$$\|A(k,h)^m u_\varepsilon(x,0)-u_\varepsilon(x,mk)\|_2 \to 0 \text{ as } k,h,\varepsilon \to 0,$$

and so taking the limit as $\varepsilon \to 0$ gives us

$$\|A(k,h)^m u_0(x,0)-u_0(x,mk)\|_2 \to 0 \text{ as } k,h \to 0.$$

But since u_0 is arbitrary, this shows that the difference scheme is convergent. Hence, by the Lax equivalence theorem, the difference scheme is stable.

Now suppose that the finite difference approximation is stable and consistent. Then the difference scheme is convergent, by the Lax equivalence theorem. In other words, for fixed $nk \leq 1$,

$$\|A(k,h)^n u(x,0) - u(x,nk)\|_2 \longrightarrow 0 \text{ as } h,k \longrightarrow 0$$

for all functions $u(x,t)$ which satisfy (5.1a,b,c). Let $u_\varepsilon(x_i,t_j)$ be the linear programming solution to (5.1) using this difference scheme, and let $v_\varepsilon(x_i,t_j)$ be determined by the difference scheme (5.3), together with the initial condition

$$v_\varepsilon(x_i,0) = u_\varepsilon(x_i,0).$$

For all fixed $\varepsilon > 0$,

(5.12) $$\|u_\varepsilon(x_i,t_j) - v_\varepsilon(x_i,t_j)\|_2 \longrightarrow 0 \text{ as } k,h \longrightarrow 0,$$

since the difference scheme determined by (5.3) and $A(k,h)$ are both convergent. Also, from convergence of the difference method,

$$\|u_\varepsilon(x_i,1) - u_0(x_i,1)\|_2 \leq \varepsilon + o(1),$$

and so

$$\|v_\varepsilon(x_i,1) - u_\varepsilon(x_i,1)\|_2 \leq \varepsilon + o(1) + C(k+h^2) = \varepsilon + o(1).$$

Therefore, as in the proofs to Theorems 5.2 and 5.3, it follows that

(5.13) $$\|u_0(x_i,t_j) - v_\varepsilon(x_i,t_j)\|_2 \leq \sqrt{\pi}(\varepsilon + o(1))^{t_j}(2M)^{1-t_j} + o(1).$$

The desired result now follows from (5.12), (5.13), and the triangle inequality.

Some results of numerical experimentation are reported in the next section.

II.6. Report on Numerical Experiments.

These calculations were done backwards in time on parabolic problems of the form

$$
(6.1)\qquad
\begin{aligned}
& r(x,t)u_t=(p(x,t)u_x)_x-q(x,t)u && (t>0, 0<x<\pi)\\
& u(0,t)=0 && (t\geq 0)\\
& u(\pi,t)=0 && (t\geq 0)\\
& u(x,1)=g(x) && (0<x<\pi).
\end{aligned}
$$

Here it was assumed that r and p are positive and that q is nonnegative in $(0,\pi)x(0,1)$.

For figures 1-3, r=p=1 and q=0, so that (6.1) becomes the heat equation. With

$$g(x)=\exp(-9)\sin(3x)+3\exp(-1.)\sin(x)$$

the regularization

$$
(6.2)\qquad
\begin{aligned}
& |u(x,1)-g(x)|\leq \epsilon = .005,\\
& 0\leq u(x,0)\leq M
\end{aligned}
$$

was used, where M was assumed unknown and minimized. Using the explicit, stable difference scheme

$$(6.3)\qquad r(x,t)[u(x,t+k)-u(x,t)]=kD^{+}[p(x,t)D^{-}u(x,t)]-q(x,t)u(x,t)$$

with $h=\pi/30$ and $k=1/900$, these figures show the solution and its approximation at t=.2, t=.1, and t=0. The approximation is quite good at t=.1, but is very bad at t=0, as the theory predicts.

For figures 4 and 5, g(x) was determined by a forward integration of 3sin(x) with r(x)=1+.1x, p(x)=1-.1x, and q(x)=0. The same regularization (6.2) and the same difference scheme (6.3) were used here.

Figures 6 and 7 demonstrate the ability of the algorithm to handle time-dependent coefficients. Here $r=1+.1t$, $p=1-.1t$, $q=0$, and $g(x)$ was again determined by a forward integration of $3\sin(x)$. There is excellent agreement at $t=.4$ and still good agreement at $t=.2$ between the approximate solution and the exact solution.

In figure 8, the effect that bounding a derivative has on the solution at $t=0$ can be seen. Here the parameters were

$$g(x)=[8/(e\pi)]\sin(x) + [8/(27\pi e^{9})]\sin(3x),$$

$$\varepsilon = .01,$$

$$M=6,$$

and

$$\|u(x,1)-g(x)\|_\infty+(\varepsilon/M)(\|u(x,0)\|_\infty+\|u_x(x,0)\|_\infty)$$

was minimized. Franklin [16] first showed that this method will converge logarithmically slowly toward the exact solution at $t=0$, and indeed the approximation is not too good here.

Finally, the ability to resolve higher order harmonics is seen in figures 9-11. For these graphs, Crank-Nicolson was used, together with the regularization (6.2) with $\varepsilon=.01$. For figure 9, $h=\pi/25$, $k=1/32$, and $t=.125$. For figures 10 and 11, $h=\pi/50$, and $k=1/64$. Notice the greater accuracy of the approximation in figure 10 to that in figure 9. Figure 11 is interesting because it shows that in some sense this method even picks up the periodicity of the solution at $t=0$. The following table compares the actual solution of this problem to the computed solution.

Time	$h=\pi/25,\ k=1/32$		$h=\pi/50,\ k=1/64$	
	$\frac{\|\|e\|\|_\infty}{\|\|u\|\|_\infty}$	$\frac{\|\|e\|\|_2}{\|\|u\|\|_2}$	$\frac{\|\|e\|\|_\infty}{\|\|u\|\|_\infty}$	$\frac{\|\|e\|\|_2}{\|\|u\|\|_2}$
.000	.512	.367	.587	.402 (.402)
.125	.075	.069	.033	.027 (.253)
.250	.043	.044	.018	.019 (.160)
.375	.036	.036	.017	.017 (.101)
.500	.030	.030	.016	.016 (.063)
.625	.025	.025	.014	.014 (.040)
.750	.020	.020	.013	.013 (.025)
.875	.016	.015	.012	.012 (.016)
1.00	.012	.011	.011	.010 (.010)

Table 1
Percent error as a function of time for the backward continuation of the heat equation. The parenthesized numbers in the right column represent the a posteriori error estimate ($\varepsilon^t M^{1-t}$) derived in the text.

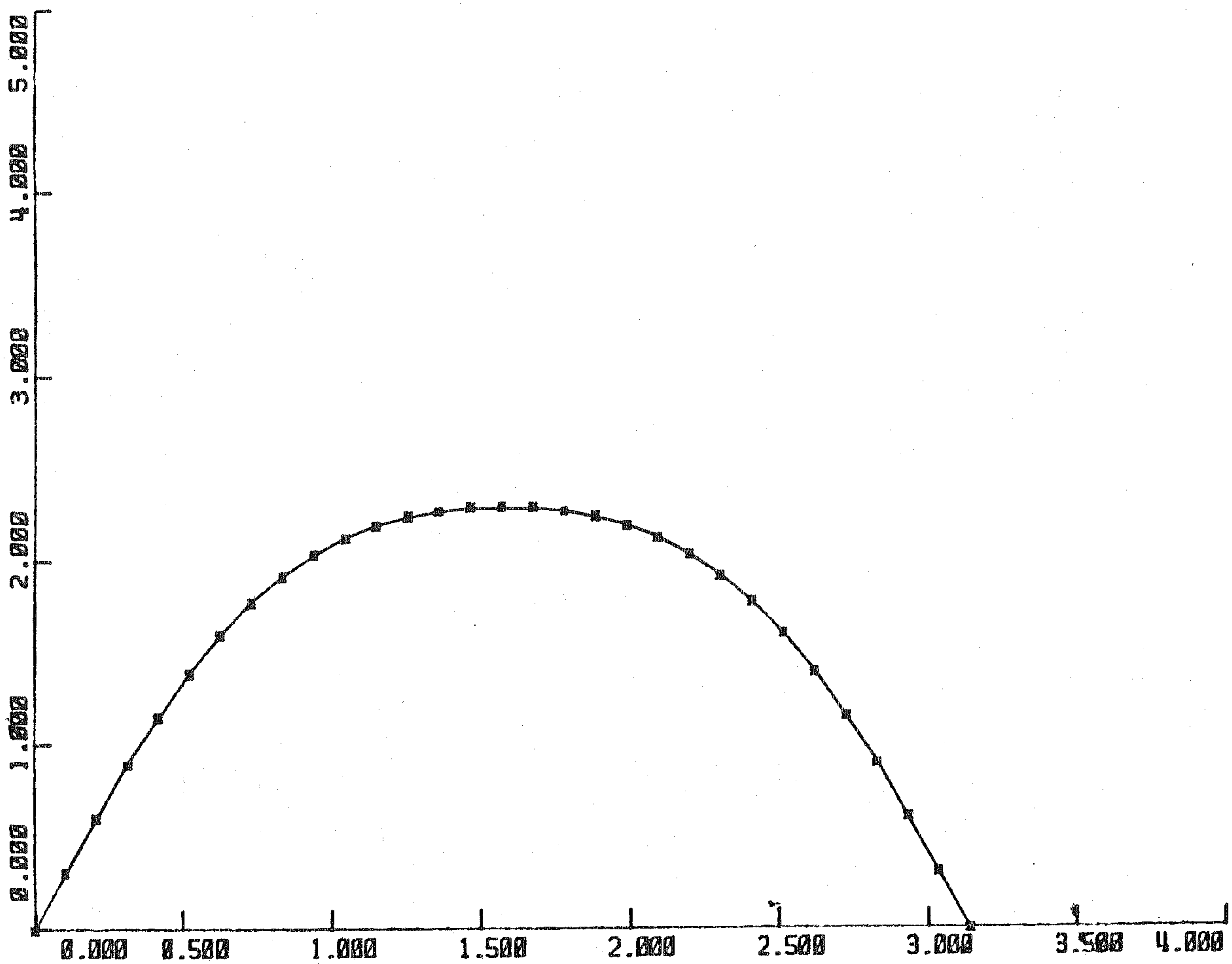


Figure 1: Exact and approximate solution to (6.1) and (6.2) with r=1, p=1, q=0, ϵ=.005. This graph is for t=1/5.

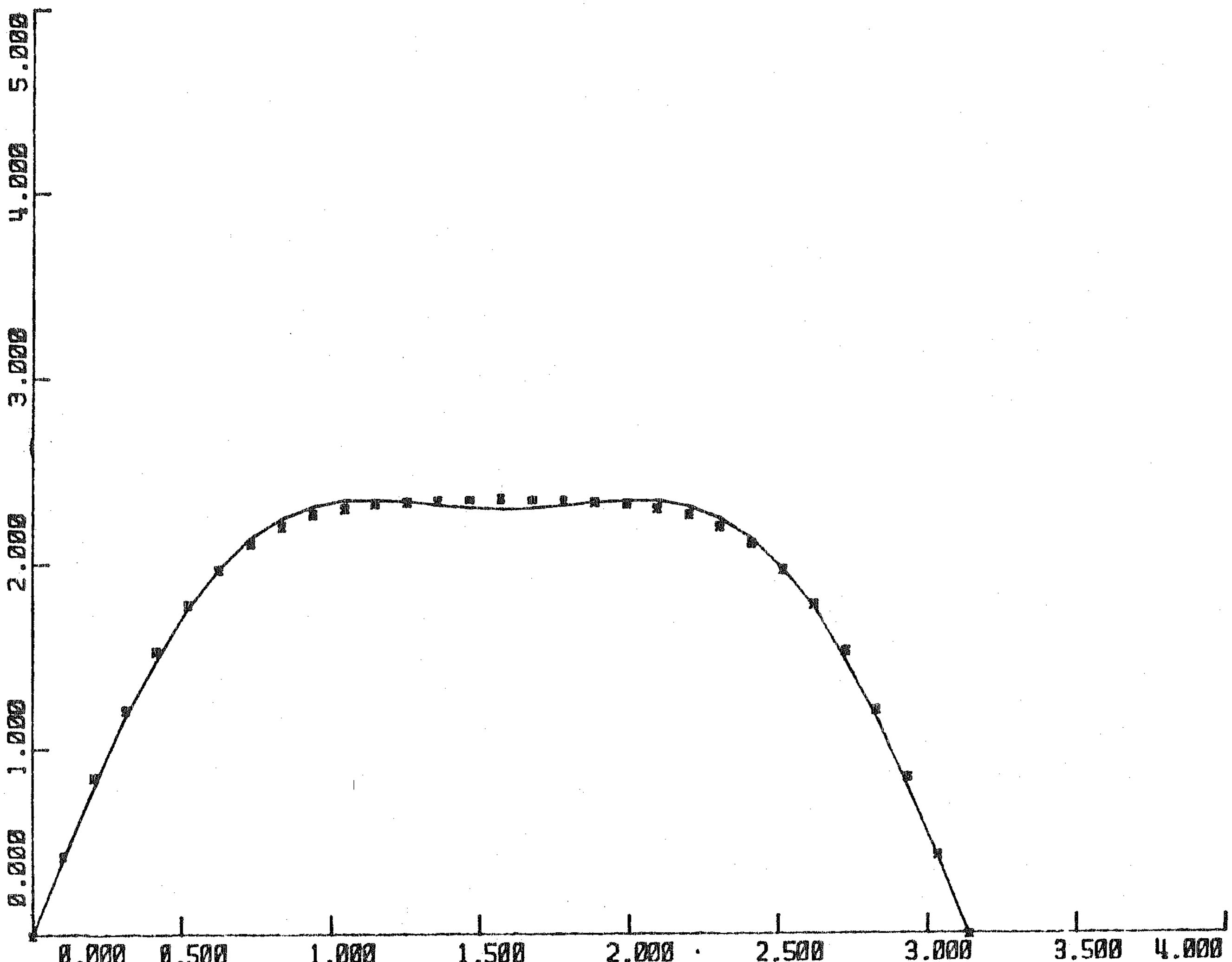


Figure 2: Same problem as Figure 1, with t=1/10.

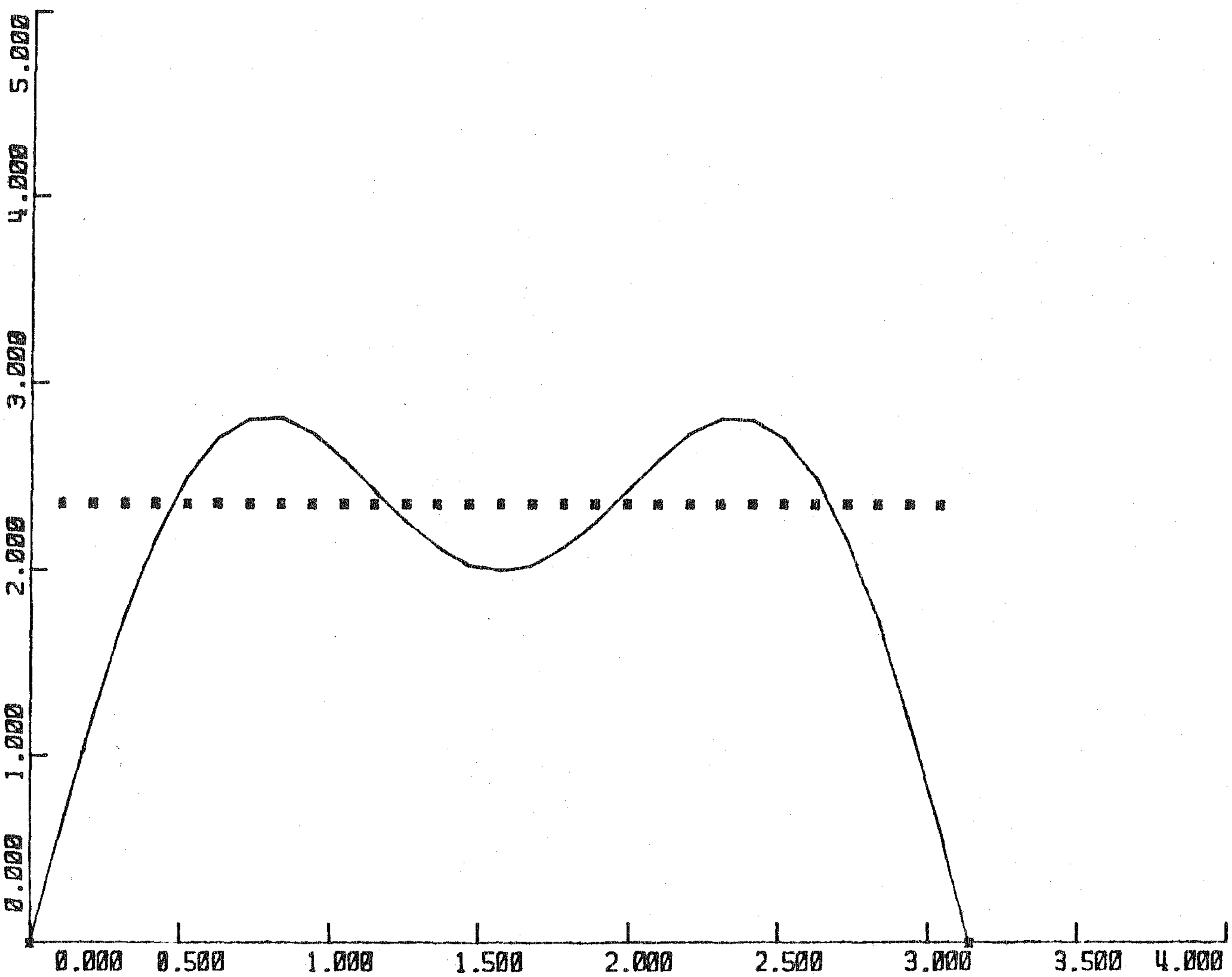
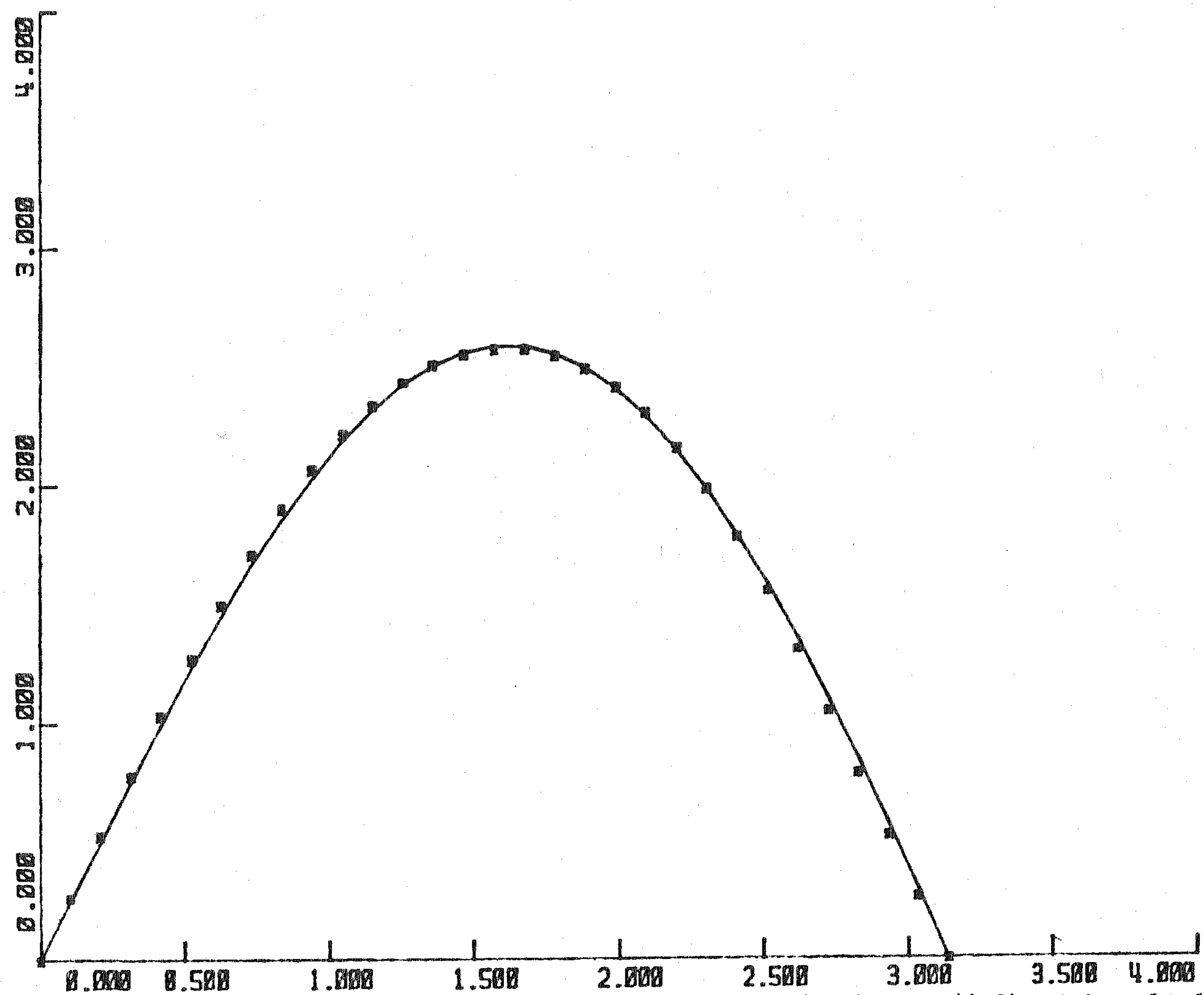


Figure 3: Same problem as Figure 1, with t=0.



<u>Figure 4</u>: Exact and approximate solution to (6.1) and (6.2) with r=1+.1x, p=1-.1x, q=0, t=2/5.

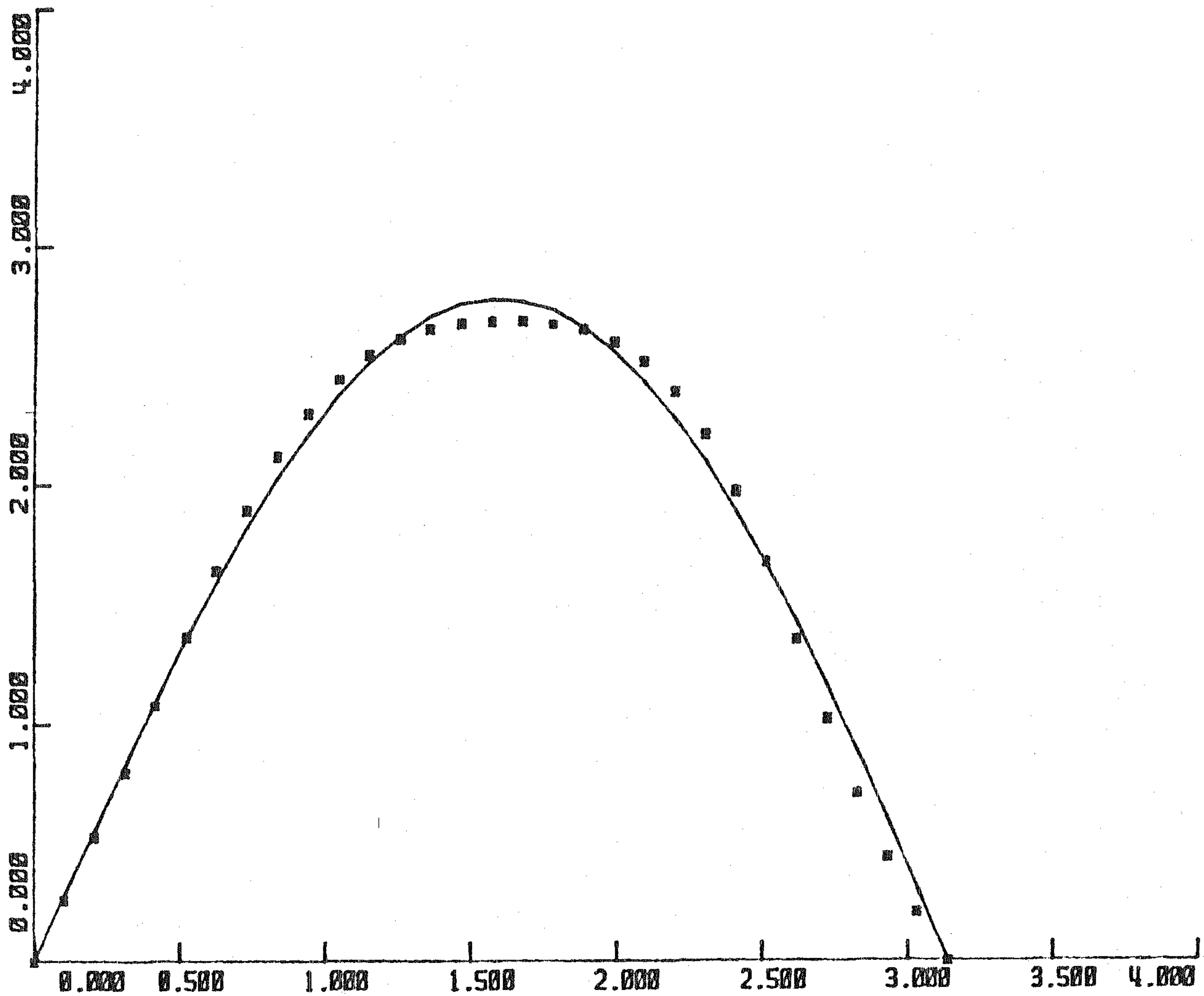
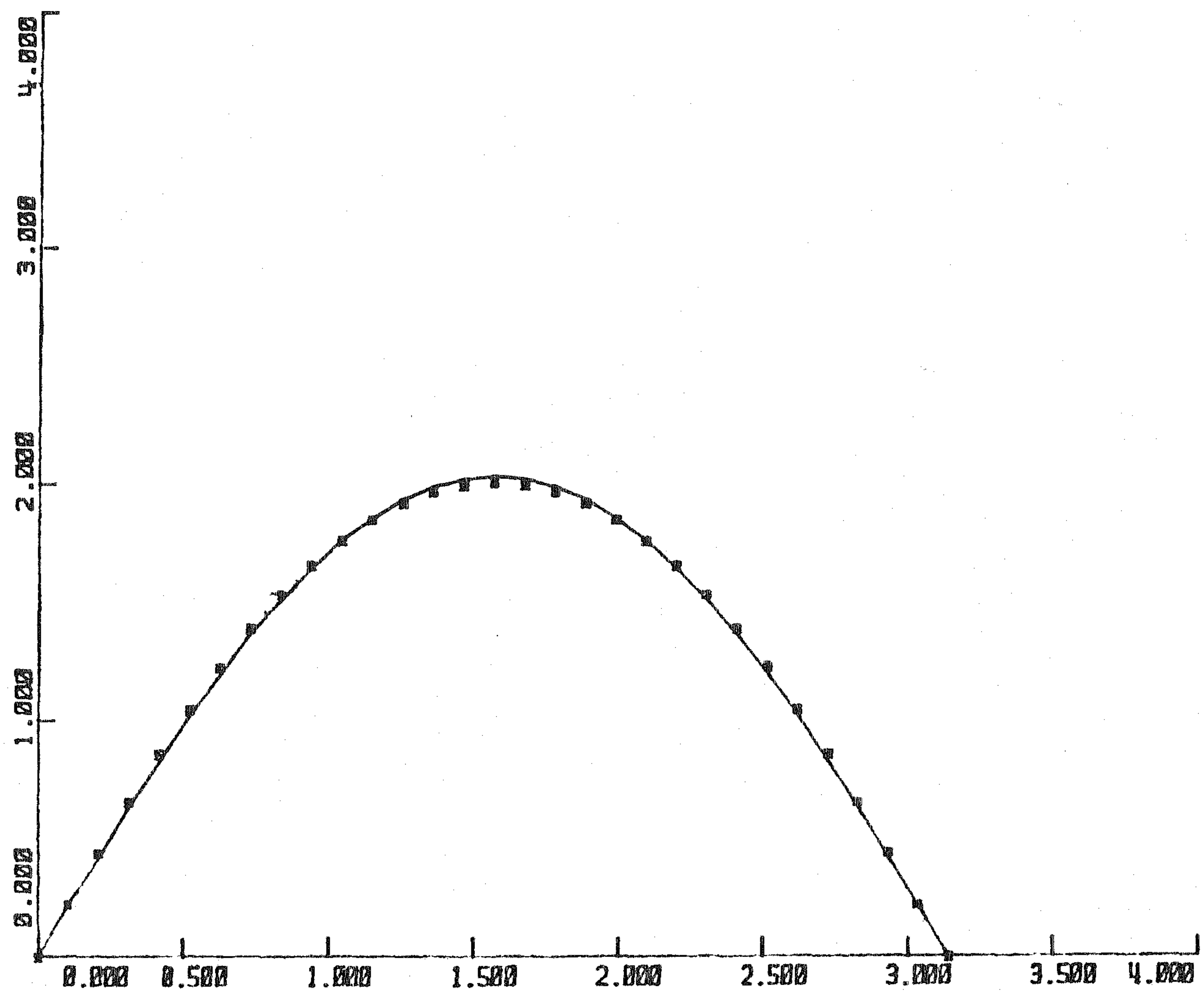


Figure 5: Same problem as Figure 4, with $t=1/5$.



<u>Figure 6</u>: Exact and approximate solution to (6.1) and (6.2) with r=1+.1t, p=1-.1t, q=0, ε=.005. This graph is for t=2/5.

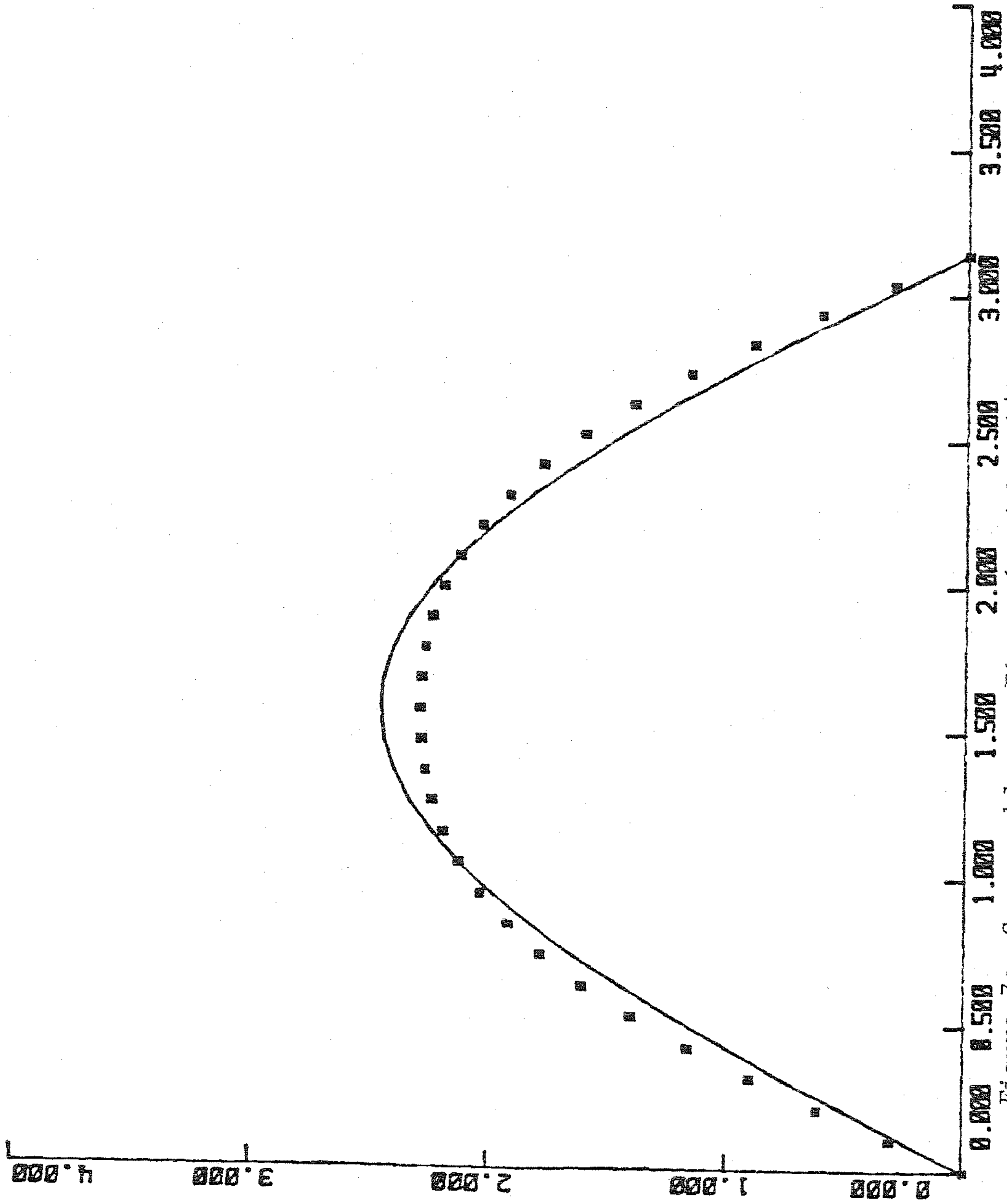


Figure 7: Same problem as Figure 6, with t=1/5.

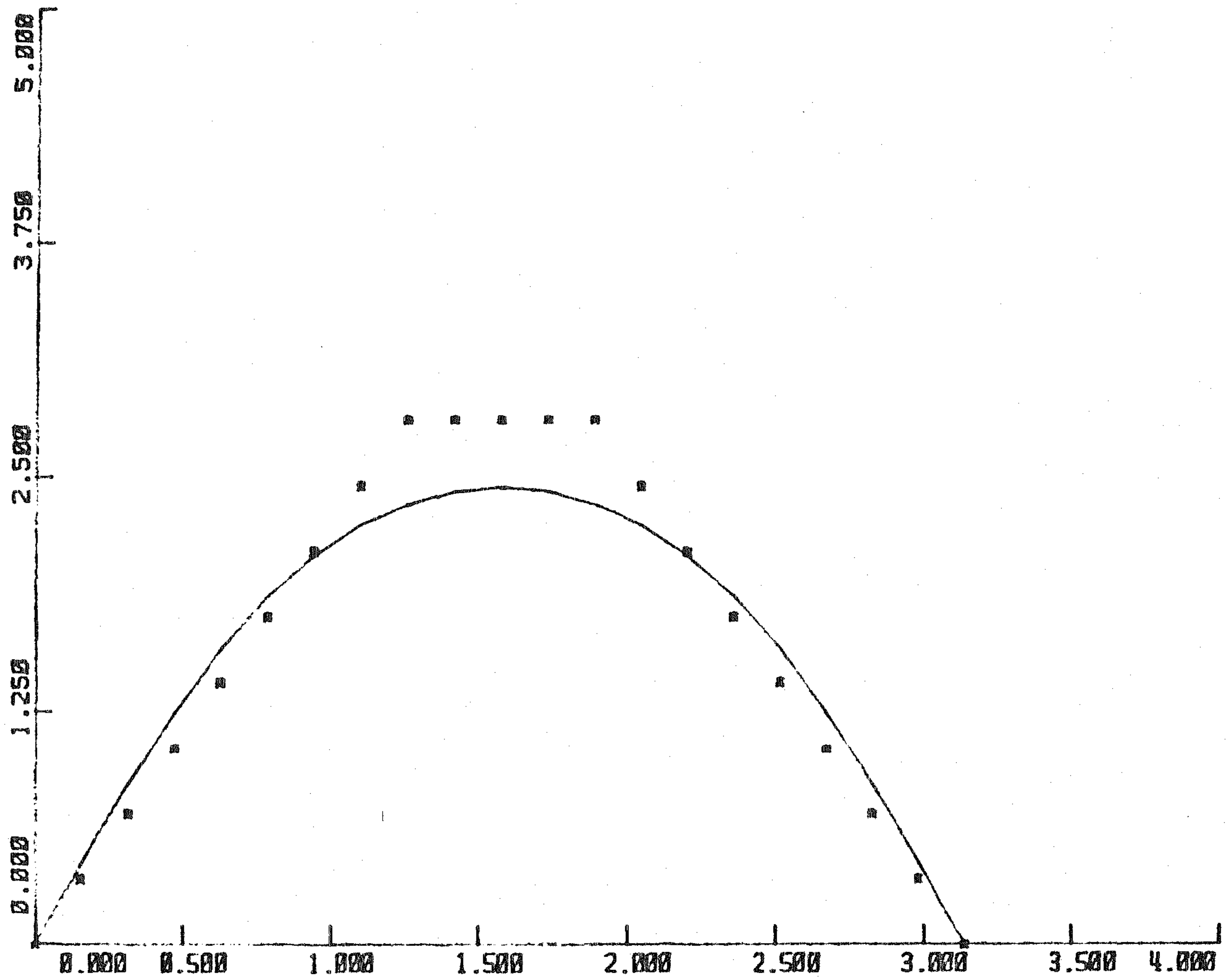


Figure 8: Exact and approximate solution to (6.1) and (6.2) with r=1, p=1, q=0, ϵ=.01. Here a bound on $u_x(x,0)$ is specified. This graph is for t=0.

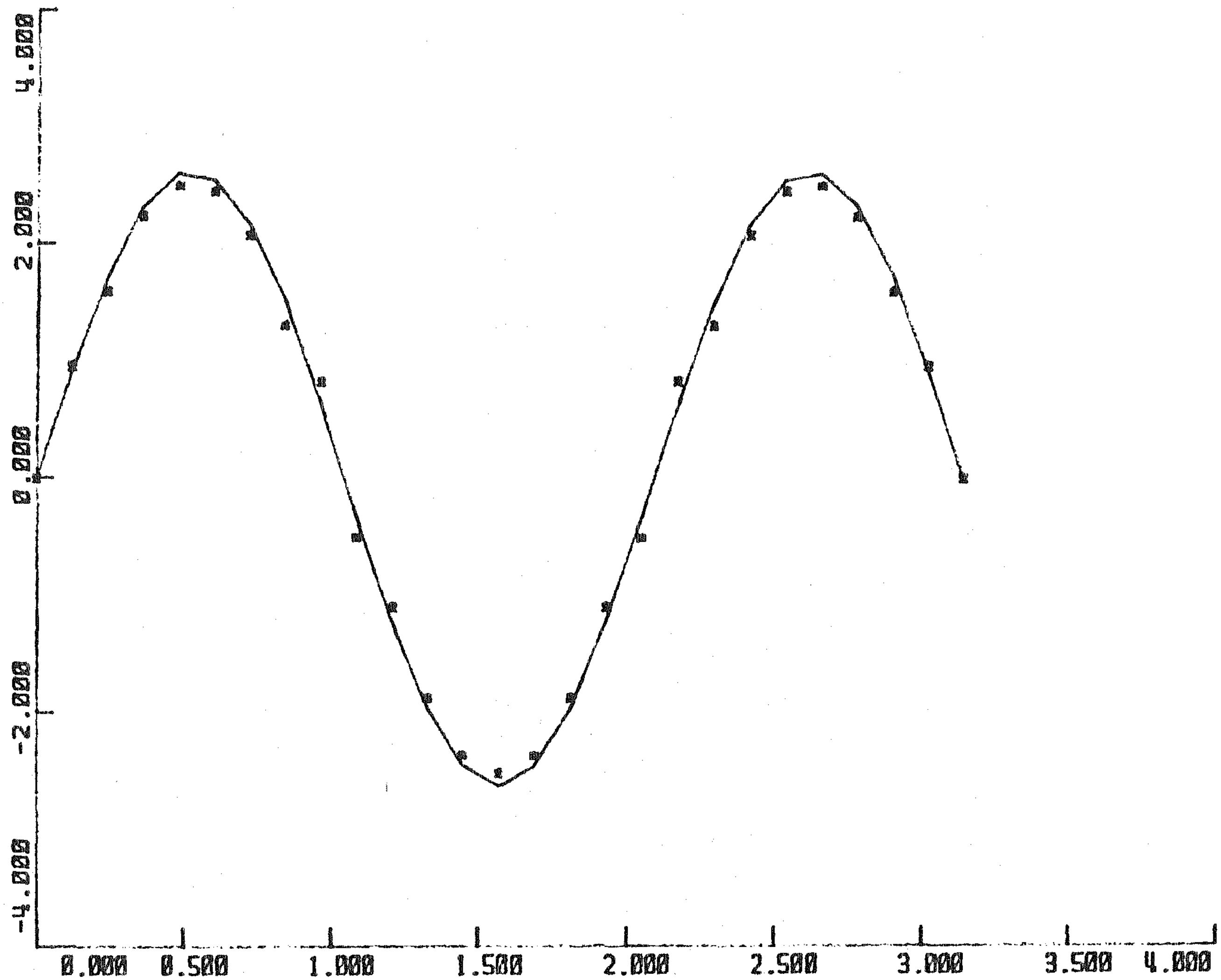


Figure 9: Exact and approximate solution to the heat equation at $t=1/8$ using Crank-Nicholson with $h=\pi/25$, $k=1/32$, $\epsilon=.01$. Compare with Figure 10.

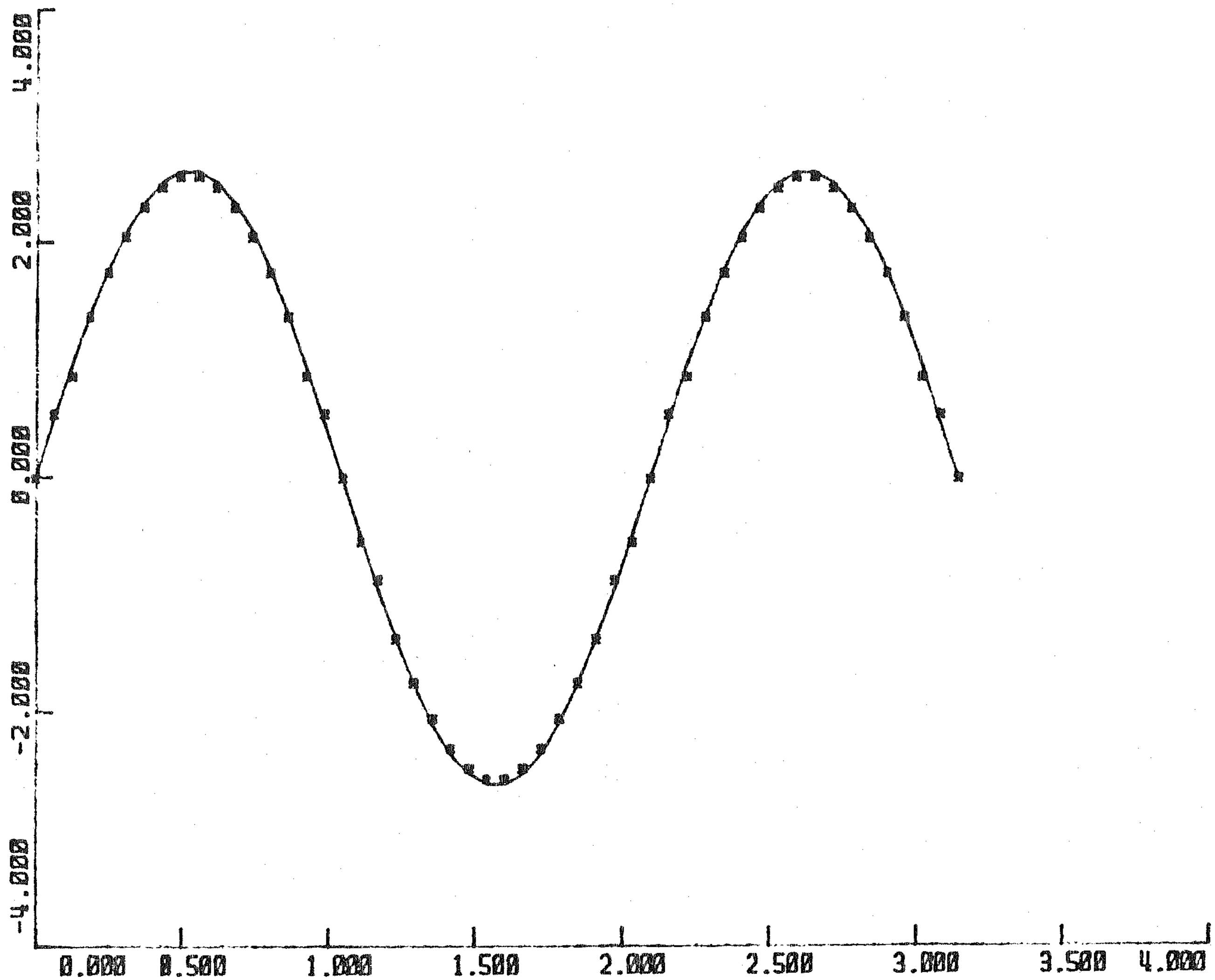


Figure 10: Exact and approximate solution to the heat equation at t=1/8 using Crank-Nicholson with $h=\pi/50$, $k=1/64$, and $\varepsilon=.01$.

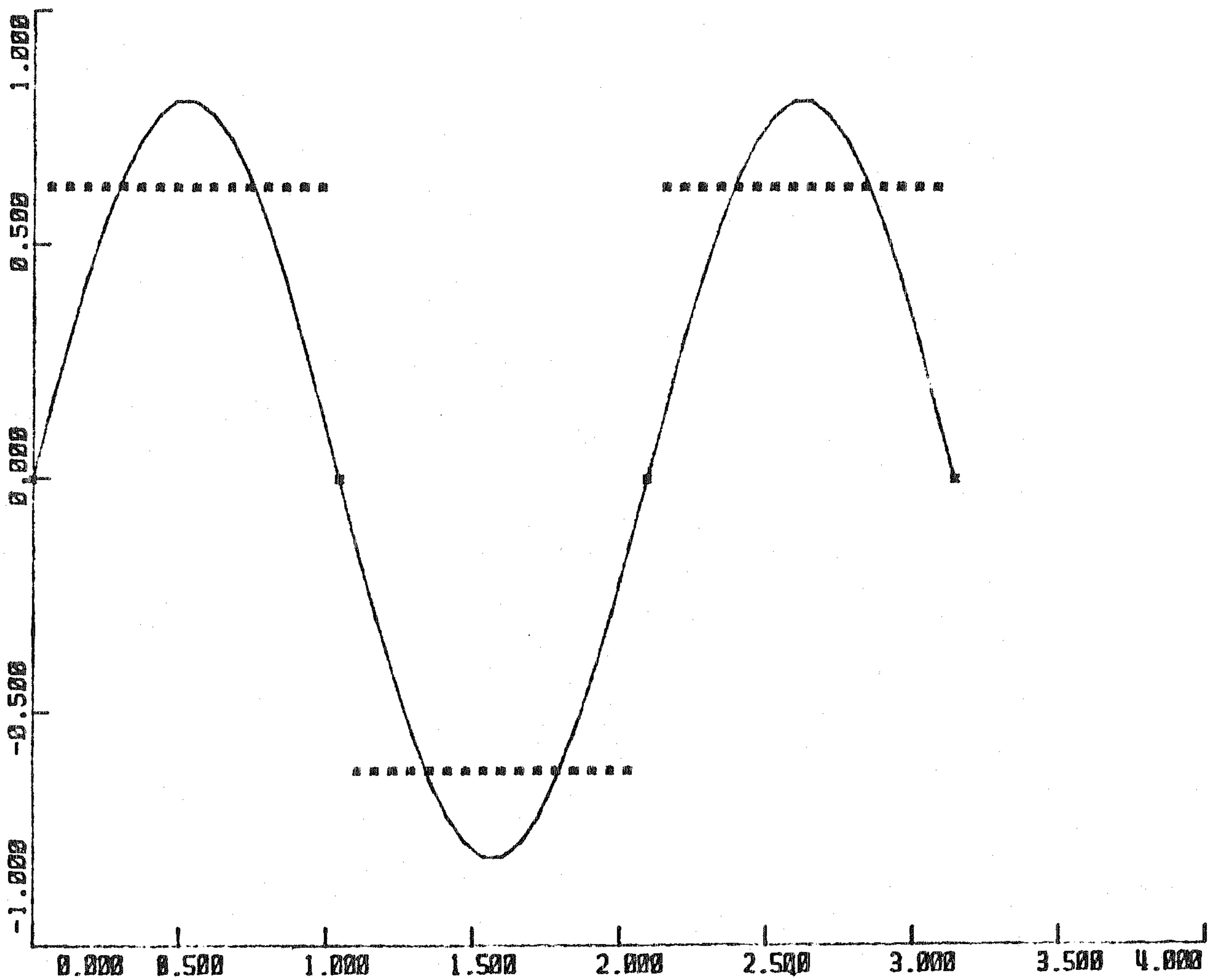


Figure 11: Same problem as Figure 10, with t=0.

III. THE WEAKLY NONLINEAR PROBLEM

III.1. Logarithmic Convexity Bounds.

This chapter shows how the linear programming methods described in the previous chapter can be used to obtain approximate solutions to weakly nonlinear backward parabolic problems. In this section logarithmic convexity bounds for solutions of the weakly nonlinear equation

$$(1.1)\qquad u_t = u_{xx}+cF(u) \qquad ((x,t)\in[0,\pi]\times[0,1]),$$

are derived, where $c<<1$, F is continuous, and $F(0)=0$. It will be shown that these problems, like the linear problems, require a good description of the data error and a restraint on $u(x,0)$ to remove any instability in them. Other sections describe the proposed numerical method, analyze the error in this method, and give the results of numerical experiments.

Let K be a self adjoint, linear negative definite operator mapping a dense subset of a Hilbert space into itself. (Here we associate Ku with u_{xx}). Then any solution to

$$(1.2)\qquad \begin{aligned} &u_t=Ku+cF(u)\\ &u(0)=u_0 \end{aligned}$$

can be found by the method of integrating factors to satisfy

$$(1.3)\qquad u(t)=e^{Kt}u_0+c\int_0^t e^{K(t-s)}F(u(s))ds.$$

Suppose now that F is Lipschitz continuous in u with Lipschitz constant L, and that $cL<1$. The following theorem proves that $\|u(t)\|_2<\infty$ for $0<t<1$.

<u>Theorem</u> <u>1.1</u>: If u solves (1.3), then

$$\sup_{0<t<t'} \|u(t)\|_2 \leq \|u_0\|_2/(1-cLt').$$

Proof: From (1.3) and the Lipschitz continuity of F,

$$\sup_{0<t<t'} \|u(t)\|_2 \leq \sup_{0<t<t'} \|e^{Kt}\|_2 \|u_0\|_2 + cL\int_0^{t'} \|e^{K(t-s)}u(s)\|_2 ds$$

$$\leq \|u_0\|_2 + cLt' \sup_{0<t<t'} \|u(t)\|_2 .$$

Hence, for sufficiently small c,

$$\sup_{0<t<t'} \|u(t)\|_2 \leq \|u_0\|_2/(1-cLt') < \infty$$

if $0\leq t'\leq 1$.

The above theorem shows that the auxiliary conditions

$$\|u(x,0)\|_2 \leq M,$$

(1.4)

$$\|u(x,1)-f(x)\|_2 \leq \varepsilon$$

can only regularize (1.2) if c is small enough. The next theorem bounds the error in solutions to (1.2) and (1.4).

Theorem 1.2: Let u and v be any two solutions to (1.2) and (1.4), let F be as above, and let

$$w=u-v.$$

Then if $cL<1$,

$$\|w(x,t)\|_2 \leq 2\sqrt{\pi}\,\varepsilon^t M^{1-t}/(1-cL).$$

Proof: The function $w(x,t)$ solves

$$w_t=Kw+c[F(u)-F(v)],$$

where

$$K=\frac{\partial^2}{\partial x^2}.$$

Hence, using the method of integrating factors, w is seen to satisfy

$$w(t)=e^{kt}w(0)+c\int_0^t e^{k(t-s)}[F(u(s))-F(v(s))]ds$$

(1.5)
$$\|w(0)\|_2\le 2M$$
$$\|w(1)\|_2\le 2\varepsilon .$$

But $e^{kt}w(0)$ solves the heat equation, so using the logarithmic convexity of solutions to the heat equation and the Lipschitz continuity of F gives

(1.6)
$$\|w(t)\|_2\le 2(\varepsilon+cL\int_0^1\|e^{k(1-s)}w(s)\|_2ds)^t(M)^{1-t}+cL\int_0^t\|e^{k(t-s)}w(s)\|_2ds.$$

But from (1.5), we get

$$\|e^{k(1-s)}w(s)\|_2=\|w(1)-c\int_0^1 e^{k(1-s)}[F(u(s))-F(v(s))]ds\|_2$$

(1.7)

$$\le\varepsilon+cL\int_0^1\|e^{k(1-s)}w(s)\|_2ds.$$

Successive applications of (1.7) to (1.6) now proves the theorem.

The above two theorems show that the inequalities

$$|u(x,0)|\le M$$
$$|u(x,1)-f(x)|\le\varepsilon$$

regularize the weakly nonlinear (1.1) for small enough c. As with linear ill-posed problems, a good description of the data error and an additional constraint on the solution u are needed here to solve this problem.

Note that the results in Theorems 1.1 and 1.2 continue to hold if (1.1) is replaced by

$$r(x)u_t=(p(x)u_x)_x+cF(x,t,u)$$

and $\|\;\|_2$ is replaced by $\|\;\|_{2,r}$. For the time-dependent coefficient case

$$r(x,t)u_t=(p(x,t)u_x)_x+cF(x,t,u)$$

there are results analogous to these theorems based upon the logarithmic convexity of time-dependent equations derived in section II.2.

Example: Consider the problem

(1.8a) $$u_t=u_{xx}+.01u^3$$

with initial-boundary conditions

(1.8b)
$$|u(x,0)| \leq 1,$$
$$u(0,t) = u(\pi,t) = 0.$$

The function $F(u)=u^3$ is not uniformly Lipschitz continuous on the real line. Nevertheless, the condition (1.8b) can be used to find a Lipschitz constant valid for all solutions to (1.8) on $[0,\pi]x[0,1]$.

Decompose u(x,t) into two components, v and w, where v(x,t) satisfies the homogeneous problem

$$v_t=v_{xx}$$
$$v(x,0)=u(x,0)$$
$$v(0,t) = v(\pi,t) = 0,$$

and w(x,t) is the solution to

(1.9)
$$w_t=w_{xx}+.01(v+w)^3$$
$$w(x,0)=0$$
$$w(0,t) = w(\pi,t) = 0.$$

The maximum principle for the heat equation gives

(1.10) $$|v(x,t)| \leq 1 \qquad ((x,t)\in[0,\pi]x[0,1]).$$

It remains to estimate w(x,t).

For fixed t, let

$$w(x_0,t) \doteq \sup_{0<x<\pi} w(x,t).$$

If x_0=0 or π, then $w(x,t)\leq 0$ for $0\leq x\leq\pi$. Otherwise,

(1.11) $$[w(x_0,t)]_t\leq .01(v+w(x_0,t))^3.$$

Using (1.9), (1.10), and (1.11), it is clear that $w(x_0,t)$ is bounded by solutions to

$$dz/dt = .01(1+z)^3 ,$$

$$z(0)=0.$$

The solution to this differential equation is given by

$$z(t) = (1-.02t)^{-1/2} -1$$

$$\leq .011 \qquad \text{for } 0 \leq t \leq 1.$$

Therefore,

$$w(x_0,t) = \sup_{0<x<\pi} w(x,t) \leq .011$$

in $[0,\pi]\times[0,1]$. A similar calculation shows that

$$\inf_{0<x<\pi} w(x,t) \geq -.011 .$$

Hence,

$$|u(x,t)| \leq |v(x,t)| + |w(x,t)| \leq 1.011,$$

and so

$$[F(u)]' = (u^3)' = 3u^2$$

$$\leq 3(1.011)^2$$

$$\leq 3.07.$$

Therefore, 3.07 is a Lipschitz constant for F(u) valid for all u(x,t) that solve (1.8). This example illustrates how a bound at t=0 can be used to find a Lipschitz constant valid for all u(x,t) that satisfy the prescribed bound.

In the next section it will be shown how the solutions to (1.1) and (1.2) can be approximated by using a successive iteration and linear programming.

III.2. The Numerical Method.

In this section an algorithm for computing numerical solutions to the weakly nonlinear equation

(2.1) $r(x,t)u_t = (p(x,t)u_x)_x + cF(x,t,u) \quad ((x,t)\in[0,\pi]\times[0,1])$

backwards in time is described. Here it is assumed that r and p are positive, that F is continuous, and that $F(x,t,0)=0$.

Let $u(x,t)$ solve (2.1), together with boundary conditions

$$u(x,1) = g(x),$$

(2.2) $$u(0,t)= f_0(t),$$

$$u(\pi,t)=f_1(t).$$

The results of the previous section show that $u(x,t)$ can be approximately determined (for small c) if, instead of (2.2), the regularized boundary conditions

$$|u(x,1)-g(x)|\leq \varepsilon,$$

$$u(0,t)=f_0(t) \ ,$$

(2.3) $$u(\pi,t)=f_1(t) \ ,$$

$$|u(x,0)|\leq M$$

are used for some ε and M. A successive approximation method using linear programming is used here to find solutions to (2.1) and (2.3).

Let $u^{(0)}$ solve (2.1) and (2.3) with c=0, and for n>0 let $u^{(n)}$ solve

(2.4) $$r(x,t)u_t^{(n)}=(p(x,t)u_x^{(n)})_x +cF(x,t,u^{(n-1)}).$$

Each $u^{(n)}$ is now the solution to a linear problem. This iteration can be stopped after a predetermined number of steps, or it can be stopped whenever $\|u^{(n)}-u^{(n-1)}\|$ gets small enough.

The iteration (2.4) is not the only successive approximation that can be used on the weakly nonlinear (2.1). For example, if $F=u^n$ for some n>1, and G=F/u, then

(2.4') $$r(x,t)u_t^{(n)} = (p(x,t)u_x^{(n)})_x + cu^{(n)}G(x,t,u^{(n-1)})$$

also results in a linear equation at each step. Iteration (2.4) is analyzed here, however, because of its simplicity.

Clearly

$$u^{(n)}(x,t)=v^{(n)}(x,t)+w^{(n)}(x,t),$$

where $v^{(n)}(x,t)$ solves

$$\begin{aligned} &r(x,t)v_t^{(n)} = (p(x,t)v_x^{(n)})_x+cF(x,t,u^{(n-1)}),\\ &v^{(n)}(x,0) = (x*f_1(0)+(\pi-x)*f_0(0))/\pi,\\ &v^{(n)}(0,t) = f_0(t),\\ &v^{(n)}(\pi,t)=f_1(t), \end{aligned} \tag{2.5}$$

and $w^{(n)}(x,t)$ solves

$$\begin{aligned} &r(x,t)w_t^{(n)} = (p(x,t)w_x^{(n)})_x,\\ &|w^{(n)}(x,1)+v^{(n)}(x,1)-g(x)| \le \varepsilon,\\ &w^{(n)}(0,t) = 0,\\ &w^{(n)}(\pi,t) = 0,\\ &|w^{(n)}(x,0)+v^{(n)}(x,0)| \le M. \end{aligned} \tag{2.6}$$

Problem (2.5) is an inhomogeneous initial-boundary value problem; numerical solutions to this well posed problem may be found with arbitrary accuracy by using any consistent and stable difference scheme with a fine enough mesh. Problem (2.6), however, is a regularized backward parabolic problem, and can be solved using linear programming by the methods of Chapter II. At each step any of four linear programming problems can be solved, corresponding to whether the bounds ε and M are known or unknown. Problem (2.4) can therefore be reduced from an inhomogeneous backward problem with inhomogeneous boudary conditions to a homogeneous problem with homogeneous boundary conditions.

Since problems (2.5) and (2.6) are being solved iteratively, this fact can sometimes be taken advantage of in order to reduce the computation and increase the accuracy of the numerical solutions. It has already been mentioned that the computation necessary to solve an extremal linear program can be significantly reduced if one starts from a solution which is, in some sense, 'close' to the extremal solution. Of the four cases analyzed in II.6 (corresponding to ε and M either known or unknown), three of these cases require solving an extremal linear program. These cases occur when either ε or M is unknown. For these cases, using $u^{(n-1)}(x,t)$ as an initial estimate of $u^{(n)}(x,t)$ has been found to significantly reduce the computation time necessary to solve (2.6).

The accuracy and convergence properties of this algorithm will be analyzed in the next section.

III.3. Error and Convergence of the Numerical Method.

The error and convergence properties of the successive approximation method described in the previous section are analyzed here. We start by considering the convergence properties of the related forward iteration

$$
\begin{aligned}
&u_t^{(0)}=u_{xx}^{(0)} \\
(3.1)\qquad &u_t^{(n)}=u_{xx}^{(n)}+cF(x,t,u^{(n-1)}) &&(n>0) \\
&u^{(n)}(x,0)=f(x) &&(n\geq 0) \\
&u^{(n)}(0,t)=u^{(n)}(\pi,t)=0 &&(n\geq 0).
\end{aligned}
$$

Theorem 3.1: Let F be Lipschitz continuous in u with Lipschitz constant L, and let u(x,t) be the continuous solution to

$$
\begin{aligned}
&u_t=u_{xx}+cF(x,t,u) \\
(3.2)\qquad &u(0,t)=u(\pi,t)=0 \\
&u(x,0)=f(x).
\end{aligned}
$$

Then the forward iteration (3.1) converges to u(x,t) whenever $cL<1$.

Proof: Let

$$e^{(n)}(x,t)=u^{(n)}(x,t)-u(x,t),$$

so that $e^{(n)}$ solves

$$
\begin{aligned}
&e_t^{(n)}=e_{xx}^{(n)}+c[F(x,t,u^{(n-1)})-F(x,t,u)] \\
&e^{(n)}(0,t)=e^{(n)}(\pi,t)=0 \\
&e^{(n)}(x,0)=0.
\end{aligned}
$$

Hence, using the Lipschitz continuity of F,

$$\sup_{0<t<1} \|e^{(n)}(x,t)\|_2 < cL \sup_{0<t<1} \|e^{(n-1)}(x,t)\|_2 .$$

It now follows from the contraction mapping theorem that $e^{(n)}(x,t) \rightarrow 0$ as $n \rightarrow \infty$. This proves the theorem.

Remember from section III.1 that the pair of inequalities

$$\|u(x,1)-g(x)\|_2 \leq \varepsilon$$

$$\|u(x,0)\|_2 \leq M$$

will only regularize (3.2) if $cL<1$. This means that the numerical method described in III.2 will only work when c is small. How does the error depend on c? This question will now be answered.

Let $u(x,t)$ solve

$$(3.3) \quad \begin{aligned} u_t &= u_{xx}+cF(x,t,u) \\ u(0,t) &= f_0(t) \\ u(\pi,t) &= f_1(t) \\ u(x,1) &= g(x) \qquad (0<x<\pi,\ 0<t<1), \end{aligned}$$

and let $u^{(n)}(x,t)$ be iterate n in the successive approximation to (3.3). Then if $n=0$, $u^{(0)}$ solves

$$(3.4a) \qquad u_t^{(0)} = u_{xx}^{(0)}$$

or, if $n>0$, u solves

$$(3.4b) \qquad u_t^{(n)} = u_{xx}^{(n)}+cF(x,t,u^{(n-1)})$$

together with boundary conditions

$$(3.4c) \quad \begin{aligned} u^{(n)}(0,t) &= f_0(t) \\ u^{(n)}(\pi,t) &= f_1(t) \\ |u^{(n)}(x,1)-g(x)| &\leq \varepsilon \\ |u^{(n)}(x,0)| &\leq M_0. \end{aligned}$$

Let the error at step n be given by

$$e^{(n)}(x,t) = u^{(n)}(x,t)-u(x,t).$$

This error will now be estimated.

When $n=0$, $e^{(0)}(x,t)$ solves

$$(3.5)\qquad \begin{aligned} &e^{(0)}_t = e^{(0)}_{xx} - cF(x,t,u)\\ &|e^{(0)}(x,1)| \le \varepsilon\\ &|e^{(0)}(x,0)| \le 2M\\ &e^{(0)}(0,t) = e^{(0)}(\pi,t) = 0, \end{aligned}$$

while when $n>0$,

$$(3.6)\qquad \begin{aligned} &e^{(n)}_t = e^{(n)}_{xx} + c[F(x,t,u^{(n-1)}) - F(x,t,u)]\\ &|e^{(n)}(x,1)| \le \varepsilon\\ &|e^{(n)}(x,0)| \le 2M\\ &e^{(n)}(0,t) = e^{(n)}(\pi,t) = 0. \end{aligned}$$

As usual, $e^{(n)}(x,t)$ can be split into two components, $e_1^{(n)}$ and $e_2^{(n)}$, where $e_1^{(n)}$ solves the homogeneous equation

$$(3.7)\qquad \begin{aligned} &e^{(n)}_{1\,t} = e^{(n)}_{1\,xx}\\ &e^{(n)}(x,0) = e_1^{(n)}(x,0)\\ &e_1^{(n)}(0,t) = e_1^{(n)}(\pi,t) = 0, \end{aligned}$$

and $e_2^{(n)}$ solves the inhomogeneous equation

$$(3.8)\qquad \begin{aligned} &e^{(n)}_{2\,t} = e^{(n)}_{2\,xx} + c[F(x,t,u^{(n-1)}) - F(x,t,u)]\\ &e^{(n)}(0,t) = e^{(n)}(\pi,t) = e^{(n)}(x,0) = 0 \end{aligned}$$

when $n \ge 1$. The quantities $e_1^{(n)}$ and $e_2^{(n)}$ will now be estimated.

Let

$$M_1 = \sup_{\substack{0<x<\pi\\ 0<t<1}} |u(x,t)| .$$

From (3.4a), it can be shown that $\|e_2^{(0)}\|_2$ is bounded above by solutions to

$$dy/dt = cLM_1 ,\ y(0) = 0,$$

and below by solutions to

$$dy/dt = -cLM_1 ,\ y(0) = 0.$$

Hence,

$$(3.9)\qquad \|e_2^{(0)}(x,t)\|_2 \le cLM_1 t.$$

On the other hand, (3.7) and (3.9) together imply that

$$|e_1^{(0)}(x,0)| \le 2M_0,$$

and that

$$\|e_1^{(0)}(x,1)\|_2 \le \|e_2^{(0)}(x,1)\|_2 + \|e^{(0)}(x,1)\|_2$$
$$\le \varepsilon\sqrt{\pi} + cLM_1 .$$

Therefore,

$$\|e_1^{(0)}(x,t)\|_2 \le \sqrt{\pi}(\varepsilon + cLM_1)^t (2M_0)^{1-t} \tag{3.10}$$

by the logarithmic convexity of solutions to the heat equation. Together, (3.9) and (3.10) show that the error in the first iterate can be bounded above by

$$\begin{aligned}\|e^{(0)}(x,t)\|_2 &\le \|e_1^{(0)}(x,t)\|_2 + \|e_2^{(0)}(x,t)\|_2 \\ &\le \sqrt{\pi}[(\varepsilon + cLM_1)^t (2M_0)^{1-t} + cLM_1 t].\end{aligned} \tag{3.11}$$

When n>0, however, it follows from (3.8) that

$$.5\langle e_2^{(n)}, e_2^{(n)}\rangle_t \le cL\langle e^{(n-1)}, e_2^{(n)}\rangle$$
$$\le cL\|e^{(n-1)}\|_2 \|e_2^{(n)}\|_2 .$$

Integrating this inequality gives

$$\|e_2^{(n)}(x,t)\| \le cL \int_0^t \|e^{(n-1)}(x,s)\|_2 ds. \tag{3.12}$$

Furthermore,

$$|e_1^{(n)}(x,0)| \le 2M_0$$

and

$$\|e_1^{(n)}(x,1)\|_2 \le \|e^{(n)}(x,1)\|_2 + \|e_2^{(n)}(x,1)\|_2$$

$$\le \varepsilon\sqrt{\pi} + cL\int_0^1 \|e^{(n-1)}(x,s)\|_2 ds.$$

Therefore, by the logarithmic convexity of solutions to the heat equation,

$$\|e_1^{(n)}(x,t)\|_2 \le \varepsilon\sqrt{\pi} + cL\int_0^1 \|e^{(n-1)}(x,s)\|_2 ds)^t (2M_0)^{1-t} . \tag{3.13}$$

Hence, using (3.12) and (3.13),

$$\|e^{(n)}(x,t)\|_2 \le (\varepsilon\sqrt{\pi} + cL\int_0^1 \|e^{(n-1)}(x,s)\|_2\, ds)^t (2M_0)^{1-t} \tag{3.14}$$

$$+cL\int_0^t \|e^{(n-1)}(x,s)\|_2\, ds.$$

An iterated inequality similar to this one is used to prove the following error estimate.

<u>Theorem</u> <u>3.2</u>: There is a constant A such that

$$\|e^{(n)}(x,t)\|_2 \le A(\varepsilon + (cL)^{n+1} M_1)^t (2M_0)^{1-t} + (2cL)^{n+1} M_1/2.$$

<u>Proof</u>: The error $e^{(n)}$ is given exactly in $L_2[0,\pi]$ by

$$e^{(n)}(x,t) = e^{Kt}[e^{(n)}(x,0)] + c\int_0^t e^{K(t-s)}[F(u^{(n-1)}(s)) - F(u(s))]ds,$$

where K is the operator $\frac{\partial^2}{\partial x^2}$ operating on twice differentiable functions that vanish at x=0 and x=π. Using this equation, we see that $e^{(n)}(x,t)$ must also satisfy the more restrictive inequality

$$\|e^{(n)}(x,t)\|_2 \le (\varepsilon\sqrt{\pi} + cL\int_0^1 \|e^{K(1-s)}e^{(n-1)}(x,s)\|_2\, ds)^t (2M_0)^{1-t} \tag{3.14'}$$

$$+cL\int_0^t \|e^{K(t-s)}e^{(n-1)}(x,s)\|_2\, ds$$

in addition to (3.14). But

$$e^{K(1-s)}e^{(n-1)}(x,s) = e^{(n-1)}(x,1) - c\int_s^1 e^{K(1-s)}[F(u^{(n-1)}(s)) - F(u(s))]ds.$$

Therefore, using this in (3.14') gives

$$\|e^{(n)}(x,t)\| \le (\varepsilon\sqrt{\pi} + cL[\varepsilon\sqrt{\pi} + cL\int_0^1 \|e^{K(1-s)}e^{(n-2)}(x,s)\|_2\, ds])^t (2M_0)^{1-t}$$

$$+cL(\|e^{(n-1)}(x,t)\| + cL\int_0^t \|e^{K(t-s)}e^{(n-2)}(x,s)\|_2\, ds).$$

Continuing in this manner, a simple induction proves that

$$\|e^{(n)}(x,t)\|_2 \leq [\frac{\varepsilon\sqrt{\pi}}{1-cL} + (cL)^{n+1}M_1]^t (2M_0)^{1-t} + (2cL)^{n+1}M_1/2$$

$$\leq A(\varepsilon+(cL)^{n+1}M_1)^t (2M_0)^{1-t} + (2cL)^{n+1}M_1/2$$

for $A=\frac{2\sqrt{\pi}}{1-cL}$. This proves the theorem.

Theorem 3.2 does not imply that the successive approximation algorithm will always converge. It does indicate, however, that the error at time t>0 can be made arbitrarily small (for small c) if ε is chosen small enough and if n is chosen large enough. In fact the algorithm should not always converge, for if any solution to (3.4) is sought, then the best that can be hoped for is that the difference between iterates satisfies

$$|u^{(n)}(x,1)-u^{(n-1)}(x,1)| \leq 2\varepsilon$$

and

$$|u^{(n)}(x,0)-u^{(n-1)}(x,0)| \leq 2M_0.$$

This is only one of the four cases analyzed in section II.6, though. Any solution will do for this case; particular extremal solutions are sought in the other three cases. Numerical experiments indicate that for these other cases the algorithm converges if cL<<1.

In the next section the results of some numerical experiments are given, and the error and convergence properties of these experiments are compared with the results obtained here and in section III.1.

III.4. Report on Numerical Experiments.

The calculations reported here were done on problems of the form

(4.1a) $$u_t = u_{xx} + cF(u)$$

with boundary conditions

$$u(0,t) = u(\pi,t) = 0,$$

(4.1b) $$|u(x,1)-g(x)| \leq \varepsilon,$$

$$|u(x,0)| \leq M.$$

Here M was assumed unknown and minimized, and ε was given to be .01.

A simple perturbation expansion of the problem

(4.2a) $$u_t = u_{xx} + cu^3$$

with boundary conditions

$$u(0,t) = u(\pi,t) = 0$$

(4.2b)

$$u(x,0) = \sin(x)$$

shows that

$$u(x,t) = \sin(x)\exp(1.-t) + c(.375)\sin(x)(\exp(3.-t)-\exp(3-3t))$$
$$+ c(\sin(3x)(\exp(3-9t)-\exp(3-3t))/24 + O(c^2).$$

Set

$$v(x,t) = \sin(x)\exp(1-t)+.01(.375\sin(x)(\exp(3-t)-\exp(3-3t)))$$
$$+.01(\sin(3x)(\exp(3-9t)-\exp(3-3t)))/24,$$

and let

$$g(x) = v(x,1) = 1.0240\sin(x) - .0004\sin(3x).$$

Table 1 shows the difference between the first few iterates obtained using our algorithm and v(x,t) given above. Note that the error for these iterates is considerably smaller than the logarithmic convexity estimate obtained in III.1. This error estimate goes to zero as ε and c do if

(4.3) $$c \sup_u |F'(u)| < 1.$$

For the problem given above, iteration 1 provided

$$\sup |u^{(1)}| = 2.165$$

and so

$$c \sup |F'(u^{(1)})| = .01(3)(2.165)^2 = .1406,$$

which is well within the tolerance of (4.3). Figures 1 and 2 show the difference between the solution calculated in iterate 6 and v(x,t) for t=.25 and t=.375.

Figure 3 was obtained from the problem

$$u_t = u_{xx} + cu^2,$$

$$u(0,t) = u(\pi,t) = 0, \tag{4.4}$$

$$u(x,0) = \exp(4)\sin(2x).$$

A simple perturbation expansion for this shows that

$$u(x,t) = \exp(4-4t)\sin(2x) + O(c),$$

and so g(x) was set to

$$g(x) = u(x,1) = \sin(2x).$$

For this problem, iteration 1 provided

$$\sup |u^{(1)}| = 41.467,$$

and so

$$c \sup |F'(u^{(1)})| = .01(2)(41.467) = .829.$$

The iterations converged very slowly for this problem. Several iterations were required to obtain two place accuracy between iterates at t=0. Figure 3 shows the result of the third iteration at t=0. Note the slight asymmetry about $x=\pi/2$. This is probably due to roundoff error; the true solution is symmetric about this point. The computed solution should not be expected to be accurate at t=0. It should be periodic, however (see II.8).

Finally, an experiment was made on (4.2a) using

$$u(x,0) = \exp(9)\sin(3x),$$

$$u(0,t) = u(\pi,t) = 0.$$

Clearly,

$$u(x,t) = \exp(9-9t)\sin(3x) + O(c).$$

The first iteration provided a periodic function of x satisfying

$$\sup |u^{(1)}| = 5471.$$

Hence, for this problem

$$c \sup |F'(u^{(1)})| = .01(3)(5471)^2 >> 1.$$

The iterations diverged for this problem.

	Iterate 1		Iterate 2		Iterate 3		
	$\|\|e\|\|_2$	$\|\|e\|\|_\infty$	$\|\|e\|\|_2$	$\|\|e\|\|_\infty$	$\|\|e\|\|_2$	$\|\|e\|\|_\infty$	
t=0	1.450	1.739	1.417	1.690	1.450	1.670	(1.450)
t=1/16	.6698	.6017	.6564	.5660	.6524	.5487	(1.065)
t=1/8	.3795	.3222	.3726	.2989	.3722	.3207	(.7807)
t=1/4	.1276	.1028	.1272	.1222	.1357	.1435	(.4051)
t=1/2	.01851	.02031	.02714	.02959	.04786	.04741	(.1218)

Table 1: Error at time t vs. iteration. The parenthesized numbers in the right column represent the a posteriori error estimate ($\epsilon^t M^{1-t}$) derived in section III.1.

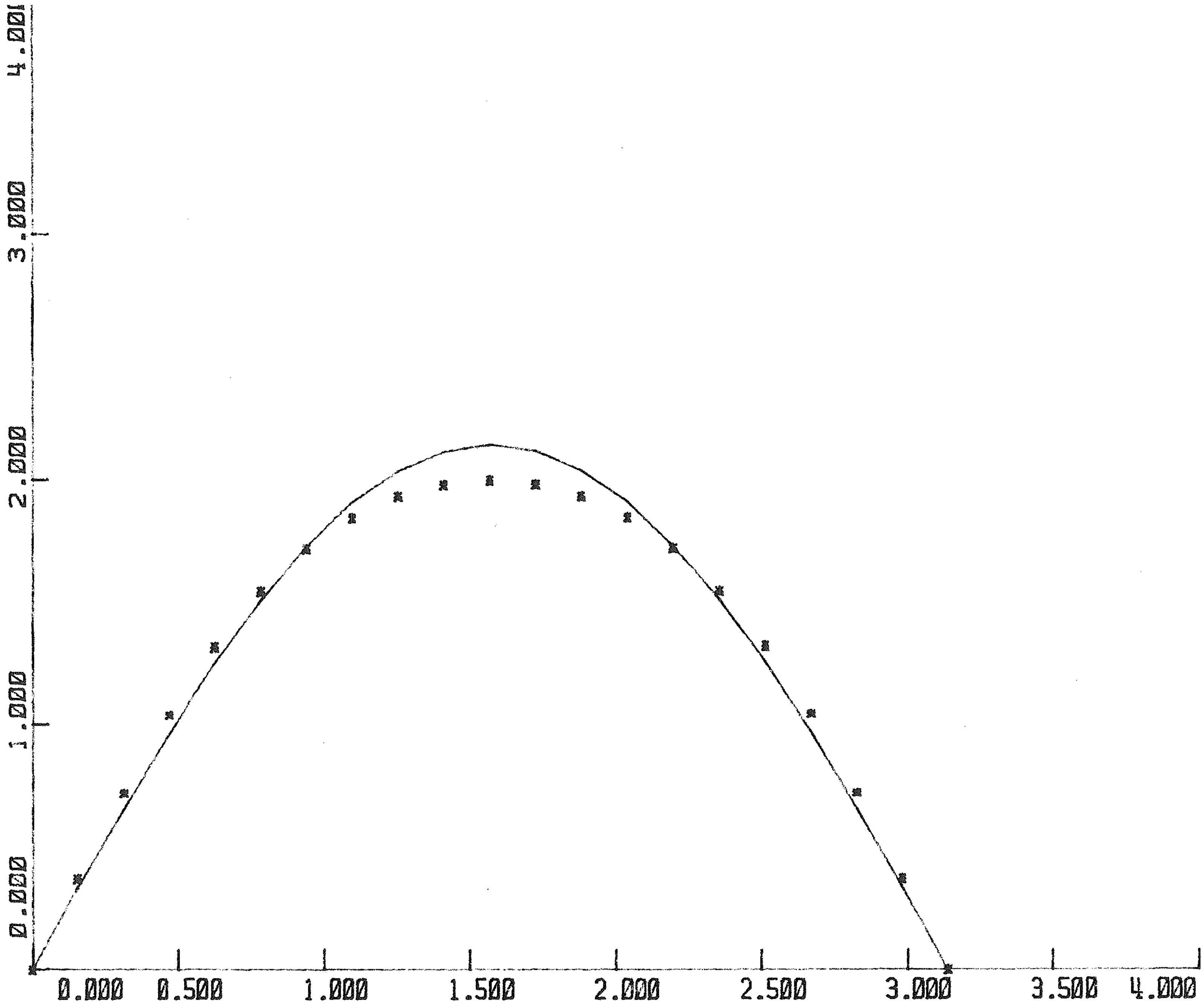


Figure 1: Asymptotic vs. numerical solution to (4.2a) at t=1/4.

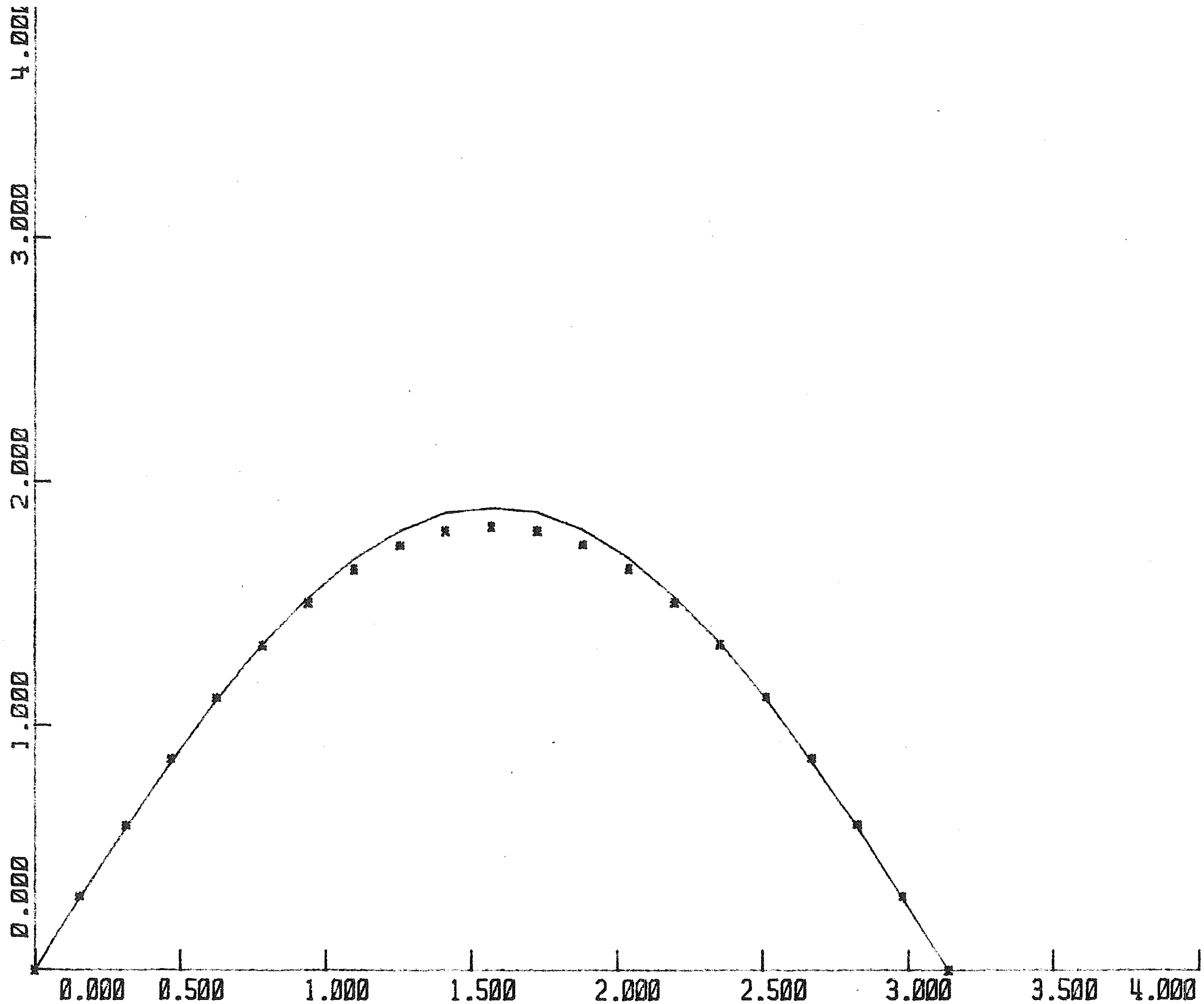


Figure 2: Asymptotic vs. numerical solution to (4.2a) at t=3/8.

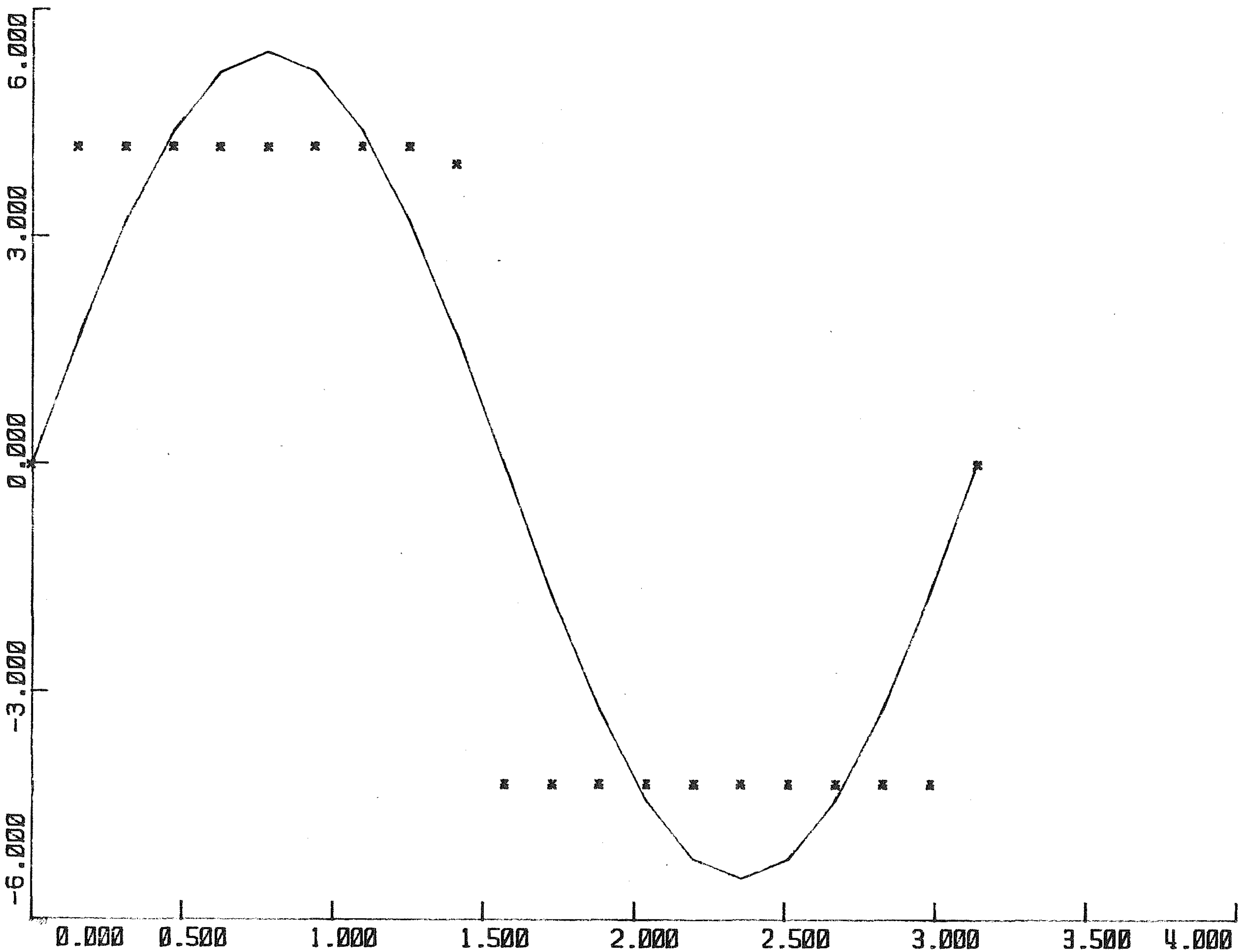


Figure 3: Asymptotic vs. numerical solution to (4.4) at t=0.

IV. CONCLUSION.

This thesis has shown how to approximately solve parabolic problems on a finite domain backwards in time by using linear programming. Here it is not necessary to assume that the Green's function or that the eigenvalues and eigenfunctions for the associated Sturm-Liouville problem are known explicitly.

Forward parabolic problems, such as the heat equation and the Fokker-Planck equation, can be used to model a variety of diffusive phenomena. Backward parabolic problems are ill-posed; a small change in the terminal data can lead to a large change in the solution sought. Since roundoff errors are inevitable in any numerical method, this instability is very undesirable. A technique to remove this instability is called a regularization. An important class of regularizations are those that restrict the search for a solution to a compact set.

For example, consider the problem of finding $u(x,0)$ from

$$\begin{aligned} &u_t = u_{xx} && (0<x<\pi, t>0), \\ &u(x,1) = g(x) && (0<x<\pi), \\ &u(0,t) = u(\pi,t) = 0 && (t>0). \end{aligned} \tag{1}$$

One regularization for this problem is

$$\begin{aligned} &u_t = u_{xx}, \\ &\|u(x,1)-g(x)\|_\infty \le \varepsilon, \\ &u(0,t) = u(\pi,t) = 0, \\ &\|u(x,0)\|_\infty \le M_0, \\ &\|u_x(x,0)\|_\infty \le M_1. \end{aligned} \tag{2}$$

Here problem (1) has been replaced by problem (2), which is in some sense 'near' to the original problem for ε small. The restrictions on $u(x,0)$ and u (x,0) imply that any solution to (2) must lie in a uniformly bounded, equicontinuous family of functions; this family is compact, by the Arzela-Ascoli theorem. On the other hand, the problem

(3)
$$\begin{aligned} &u_t = u_{xx},\\ &\|u(x,1)-g(x)\|_\infty \le \varepsilon,\\ &u(0,t) = u(\pi,t) = 0,\\ &\|u(x,0)\|_\infty \le M_0 \end{aligned}$$

does not regularize (1) at t=0. It does regularize (1) for $0<t\le 1$, however. This is proven in section I.2; the proof uses the fact that bounded sets are weakly compact in L_p for $1<p<\infty$.

A discrete approximation can be obtained for (2) or (3). Let

$$h = \pi/N_1,\ k=1/N_2$$

for some integers N_1 and N_2, and let

$$u_i = u(ih,0),$$
$$g_i = g(ih).$$

Then upon discretization, (3) becomes

(4)
$$\begin{aligned} &\|A^{N_2}\underset{\sim}{u}-g\|_\infty \le \varepsilon,\\ &\|u\|_\infty \le M_0, \end{aligned}$$

where A is an (N-1)x(N-1) matrix determined by the difference scheme used. Solutions to this system of inequalities can now be determined by linear programming.

The error involved in replacing (1) with (2) or (3) is called the regularization error, and it is logarithmically convex. That is, if u_ε solves (3) and u_0 solves (1), then

$$\|u_\varepsilon - u_0\|_2 \le \varepsilon^t (2M_0)^{1-t}.$$

This shows that the error in u_ε goes to zero as ε does for $0<t\le 1$. This

result also holds if the heat equation in (1) is replaced by the more general space-dependent problem

$$r(x)u_t = (p(x)u_x)_x - q(x)u.$$

On the other hand, if the equation considered is of the form

$$r(x,t)u_t = (p(x,t)u_x)_x$$

then the error satisfies

$$\|u_\varepsilon - u_0\|_2 \leq \varepsilon^{w(t)}(2M_0)^{1-w(t)}$$

where

$$w(t) = [\exp(ct)-1]/[\exp(c)-1]$$

and

$$c = \sup_{\substack{0<x<\pi \\ 0<t<1}} \{p_t/p\}.$$

It is not necessary to know precisely what either ε or M is in (2). Clearly, if both are known then any solution to (4) will suffice. This is not always the case, however. If a temperature distribution at t=1 is determined by measurement, then an unknown ε will correspond to uncertain accuracy in the measurements. An unknown M, on the other hand, will occur if the solution is only known to exist and be continuous at t=0. There are four cases to consider (each of ε and M may be either known or unknown). If M is known and ε is not, for example, then ε can be considered a variable which can be minimized using linear programming. This case is equivalent to Chebyshev approximation with inequality constraints (see Appendix A). The regularization error for each of these four cases is equivalent, as is shown in section II.4.

In section II.5, error estimates for the discretization (4) were derived, and it was shown that the algorithm is convergent to the true solution of (1) is the matrix A is obtained from any stable and consistent difference scheme (such as Crank-Nicolson). This important theorem was confirmed by the results given of numerical experiments of various linear parabolic problems backwards in time.

In chapter III, these results were extended to regularized, weakly nonlinear parabolic problems such as

$$u_t = u_{xx} + cF(x,t,u)$$

(5)
$$\|u(x,1)-g(x)\|_\infty \leq \varepsilon$$

$$u(0,t) = u(\pi,t) = 0$$

$$\|u(x,0)\|_\infty \leq M_0,$$

where $c \ll 1$. If F is Lipschitz continuous in u with Lipschitz constant L, it was shown in III.1 that

(6)
$$\|u_\varepsilon - u_0\|_2 \leq 2\sqrt{\pi}(\varepsilon)^t (M_0)^{1-t}/(1-cL)$$

if $cL<1$. Moreover, Lipschitz continuity is not really much of a restriction on F, for the constraint at t=0 can usually be used to restrict the range of u under consideration. For example, if $F=u^3$ and $M_0=1$, then $L \leq 3.07$, as was shown in III.1.

A successive approximation method for solving (5) can be obtained by letting $u^{(0)}(x,t)$ solve

$$u_t^{(0)} = u_{xx}^{(0)},$$

and for n>0 letting $u^{(n)}(x,t)$ solves

$$u_t^{(n)} = u_{xx}^{(n)} + cF(u^{(n-1)}),$$

It was shown in section III.3 that if u solves these equations together with boundary conditions

$$\|u^{(n)}(x,1)-g(x)\|_\infty \leq \varepsilon$$

and

$$\|u^{(n)}(x,0)\|_\infty \leq M_0 \qquad (n \geq 0),$$

then

$$||u_{\varepsilon}(x,t)-u_{o}(x,t)||_{2} \leq A(\varepsilon+(cL)^{n+1}M_{1})^{t}(2M_{o})^{1-t}$$
$$+ (2cL)^{n+1}M_{1}/2.$$

Each $u^{(n)}$ here solves a regularized linear equation, and the linear programming techniques of chapter II can be used to do this. This algorithm will not converge in all of the four cases described in section II.4, however. In the case that both ε and M are known, then the best that can be hoped for is that

$$|u^{(n)}(x,1)-u^{(n-1)}(x,1)| \leq 2\varepsilon$$

and

$$|u^{(n)}(x,0)-u^{(n-1)}(x,0)| \leq 2M_{o}.$$

The results of numerical experiments for weakly nonlinear problems are described in III.4. The experiments performed converge for $cL<1$ and diverge for $cL>1$.

Linear programming techniques similar to those proposed in this thesis can be applied to other ill-posed problems of the form

$$Ku=f,$$

where K is a bounded linear operator with an unbounded inverse. A regularization for these problems will take the form

(7)
$$||Ku-f||_{\infty} \leq \varepsilon,$$
$$||u||_{\infty} \leq M_{o},$$

and (possibly)

(8) $$||u'||_{\infty} \leq M_{1}.$$

After discretization, solutions to (7) can be found by using linear programming.

<u>Example 1</u> (Numerical differentiation): Suppose that

(9) $$Ku_0(x) = \int_0^x u_0(y)dy = f(x),$$

and assume that

$$f(1)=0.$$

The problem is to find $u_0(x)=f'(x)$. Divided differences are usually used to approximate u_0, but this technique will often not work well if f is a measured quantity and thus known only to within measurement error. In this case the measured function f might not even be differentiable. This problem has been analyzed before by Franklin [16] and Cullum [10]. Franklin showed that for this problem the error in the solution to the regularized problem (7,8) is $O(\varepsilon^{1/2})$.

Example 2 (Cauchy problem): Let u(x,y) solve

$$u_{xx}+u_{yy}=0,$$
$$u(x,0)=f(x),$$
$$u_y(x,0)=0,$$
$$u(0,y)=u(\pi,y)=0 \qquad (-\infty<x,y<\infty).$$

If

$$f(x)=[\sin(nx)]/n,$$

then

$$u(x,y)=[\sin(nx)\cosh(ny)]/n.$$

This problem is ill-posed; a small perturbation in f can lead to a large error in u. Solutions to this problem are (almost) logarithmically convex (see Miller [25]). Therefore, the error analysis for this problem is similar to that for the backward heat equation. Extensions of this problem can include variable coefficient problems, as well as weakly nonlinear problems. These extensions can be handled in a manner similar to the methods proposed in this thesis for extensions of the backward heat equation. An analysis of numerical methods for solving the Cauchy problem for elliptic equations would make a good research problem.

Example 3 (Harmonic continuation outside a circle): Let $f(r,\theta)$ be harmonic on $(r,\theta)\in[0,1]\times[0,2\pi)$, and let

$$f(r_1,\theta)=g(\theta) \qquad (0\leq\theta<2\pi)$$

for some $r_1<1$. The problem here is to find $f(1,\theta)$. This problem is ill-posed because an $O(r_1^n)$ perburtation of $r^n\sin(n\theta)$ in g can lead to an O(1) error in $f(1,\theta)$. Poisson's integral formula states that

$$f(r_1,\phi) = \frac{1}{2\pi}\int_0^{2\pi} f(1,\theta)\,\frac{(1-r_1^2)}{[1-2r_1\cos(\phi-\theta)+r_1^2]}\,d\theta .$$

The problem now is to invert this formula. Douglas [12] has shown how the similar problem of analytic continuation can be regularized and solved using linear programming. Extensions of this problem might include an analysis of what happens when the boundary $|z|=r_1$ is incompletely specified, or when the boundary is irregular.

There are many other linear ill-posed problems; these are only a few. More research is needed for a better understanding of these problems and techniques used to solve them, for as G. Anger stated in the 1979 preface to Inverse and Improperly Posed Problems in Differential Equations,

> "The practical importance of inverse and improperly posed problems is such that they may be considered among the pressing problems of current mathematical research. Many problems remain unsolved in the study of inverse problems."

Appendix A: Linear Programming and Chebyshev Approximation.

In linear programming one seeks the maximum (or minimum) of a linear function of several variables subject to linear constraints on the variables. The following are two examples of linear programs.

Example 1: Minimize $z=x_1+2x_2$ subject to

$$x_1, x_2 \geq 0$$

$$x_1+x_2=1.$$

The solution to this problem is given by

$$x_1=1,\ x_2=0,\ z=1.$$

Example 2: Maximize y subject to

$$y \leq 1$$

$$y \leq 2$$

The solution to this problem is clearly

$$y=1.$$

Example 2 is called the dual of Example 1, and the solutions of these two examples are closely related, as shall be seen.

In general, let A be an m by n matrix, and let $\underset{\sim}{b}$ and $\underset{\sim}{c}$ be vectors with m and n components respectively. Consider the problem of finding the vector $\underset{\sim}{x}$ that solves

$$A\underset{\sim}{x}=\underset{\sim}{b},\ \underset{\sim}{x} \geq 0, \qquad (A.1)$$

$$\langle \underset{\sim}{c}, \underset{\sim}{x} \rangle = \text{minimum}.$$

This is called the canonical minimum problem of linear programming, and Dantzig's simplex algorithm (described in Franklin [13]) provides a means for determining when a solution exists and computing the solution. The solution to this problem is closely related to the solution of the dual linear program

(A.2) $$\underset{\sim}{y}^T A \leq \underset{\sim}{c}^T$$
$$\langle \underset{\sim}{y}, \underset{\sim}{b} \rangle = \text{maximum},$$

as the following theorem shows.

Theorem A.1 (Equilibrium Theorem): The vector $\underset{\sim}{x}=(x_1,\ldots,x_n)^T$ is optimal for (A.1) if and only if there exists a vector $\underset{\sim}{y}=(y_1,\ldots,y_m)^T$ optimal for (A.2) satisfying

(A.3) $$\sum_{i=1}^{m} y_i a_{ij} = c_j \text{ if } x_j > 0.$$

Furthermore, if $\underset{\sim}{x}$ is optimal for (A.1) and $\underset{\sim}{y}$ is optimal for (A.2), then

(A.4) $$\langle \underset{\sim}{c}, \underset{\sim}{x} \rangle = \langle \underset{\sim}{b}, \underset{\sim}{y} \rangle.$$

This well known result can be found in [13].

A vector $\underset{\sim}{x}$ that satisfies the linear constraints in (A.1) is called a feasible solution. For any feasible $\underset{\sim}{x}$, define the set

(A.5) $$S(\underset{\sim}{x};A)=\{a^j : x_j > 0\},$$

where a^j denotes column j of the matrix A. If the vectors in $S(\underset{\sim}{x};A)$ are linearly independent, then $\underset{\sim}{x}$ is called a basic feasible solution. The following theorem guarantees the existence of basic feasible solutions.

Theorem A.2: If (A.1) has any feasible solution, it has a basic feasible soluion. If (A.1) has any optimal solution, it has a basic optimal solution.

A simple proof of this theorem can be found in [13].

Assume now that

$$\text{rank}(A)=m,$$

and that b is independent of any m−1 columns of A. If these conditions are met, then (A.1) is called nondegenerate. Suppose that x is a basic optimal solution to (A.1). Then since $\underset{\sim}{b}$ is independent of any m-1 columns of A, $\underset{\sim}{x}$ must have m non-zero components. From the equilibrium theorem, (A.3) gives m equations for the m unknown components of the dual solution $\underset{\sim}{y}$. Furthermore, since $\underset{\sim}{x}$ is basic, the vectors in $S(\underset{\sim}{x};A)$ are linearly independent. Hence, given $\underset{\sim}{x}$, the vector $\underset{\sim}{y}$ that solves the dual program (A.2) can be found by simply solving the m linear equations (A.3). This shows that for nondegenerate canonical minimum problems, the solution to the dual problem can be found directly from the solution to the primal problem.

Linear programming can be used to solve problems in Chebyshev approximation. Stiefel [32] was the first to show how this is done. Given an overdetermined and inconsistent system of linear equations

(A.6) $$a_{i1}x_1+a_{i2}x_2+\dots+a_{in}x_n=b_i \quad (i=1,\dots,m),$$

the Chebyshev approximation problem is to find an $x_1,\dots,x_n$ that approximately solve (A.6). In particular, if

$$z_i=|a_{i1}x_1+a_{i2}x_2+\dots+a_{in}x_n-b_i| \quad (i=1,\dots,m),$$

then the problem is to find

(A.7) $$z=\min_{(x_1,\dots,x_n)}\ \max_i z_i$$

This is a problem in linear programming. To see this, consider the problem of minimizing z subject to

(A.8) $$\begin{aligned} &a_{i1}x_1+a_{i2}x_2+\dots+a_{in}x_n-b_i\le z \\ &a_{i1}x_1+a_{i2}x_2+\dots+a_{in}x_n-b_i\ge -z \qquad (i=1,\dots,m), \\ &z\ge 0. \end{aligned}$$

It is easy to verify that solving this program will give us the Chebyshev approximants x_i, as well as the desired minimum z. This problem can be put into canonical minimum form so that it can be solved by the Simplex algorithm.

Set

$$t_i=(b_i-a_{i1}x_1-\dots-a_{in}x_n)+z,$$

$$w_i=z+(a_{i1}x_1+\dots+a_{in}x_n-b_i) \qquad (i=1,\dots,m).$$

Clearly

$$t_i,w_i\geq 0 \qquad (i=1,\dots,m).$$

Also choose

$$r_j,s_j\geq 0 \qquad (j=1,\dots,n)$$

so that

$$x_j=r_j-s_j \qquad (j=1,\dots,n).$$

Substituting these relationships into (A.8) yields the following problem: minimize z subject to

(A.9)
$$\begin{aligned} &a_{i1}(r_1-s_1)+\dots+a_{in}(r_n-s_n)-z+t_i=b_i,\\ &a_{i1}(r_1-s_1)+\dots+a_{in}(r_n-s_n)+z-w_i=b_i\\ &r_j,s_j,z,t_i,w_i\geq 0 \qquad (i=1,\dots,m;j=1,\dots,n). \end{aligned}$$

This is a canonical minimum program in 2n+2m+1 unknowns and 2m equations.

It can be more efficient numerically to write (A.8) in canonical maximum form, however. For this, the problem (A.8) can be written as follows:

maximize $-z$ subject to

$$a_{i1}x_1+\dots+a_{in}x_n-z\leq b_i$$

$$-a_{i1}x_1-\dots-a_{in}x_n-z\leq -b_i \qquad (i=1,\dots,m)$$

This is the dual of a canonical minimum problem, and has only (n+1) unknowns and 2m inequalities. The corresponding minimum problem, which can be derived from (A.8), is to minimize

$$\sum_{i=1}^{m} b_i(u_i - v_i) \quad \text{subject to}$$

$$\text{(A.10)} \qquad \begin{aligned} &\sum_{i=1}^{m} a_{ij}(u_i - v_i) = 0 \qquad (j=1,\dots,n), \\ &\sum_{i=1}^{m} (u_i + v_i) = 1, \\ &u_i, v_i \geq 0 \qquad (i=1,\dots,m). \end{aligned}$$

Since (A.10) involves fewer unknowns than (A.9), practice has shown that it is generally more efficient to solve (A.10) first, and then to solve the dual program (A.8) using the equilibrium theorem and (A.3).

Both linear programming and the method of least squares can be used to find approximate solutions of ill-posed problems. We conclude this appendix with a comparison of these two methods. The method of linear programming can be used to compute approximate solutions to (A.6) that satisfy

$$\|A\underset{\sim}{x} - \underset{\sim}{b}\|_1 = \text{minimum},$$

or

$$\|A\underset{\sim}{x} - \underset{\sim}{b}\|_\infty = \text{minimum}.$$

The method of least squares, on the other hand, gives the solution to

$$\|Ax - b\|_2 = \text{minimum}$$

as

$$\hat{x} = (A^T A)^{-1} A^T b.$$

Both methods allow us to find a 'best' approximation to $Ax=b$ in a suitable normed space. Is there any advantage in using linear programming instead of least squares? The answer is yes. With linear programming we are allowed to impose additional linear inequality constraints on the solution, such as

(A.11) $|x_i| \leq M$ $(i=1,\ldots,n)$,

(A.12) $x_i \geq 0$ $(i=1,\ldots,n)$,

or

(A.13) $\sum_{i=1}^{n} e_i x_i = f$.

Constraints such as (A.11) occur in regularized ill-posed problems, while constraints such as (A.12) often are required when the nonnegativity of a solution must be guaranteed, such as in heat conduction problems. Both linear programming and least squares can incorporate equality constraints such as (A.13). Neither (A.11) nor (A.12) can be handled by conventional least squares techniques, however. Therefore, in this author's opinion, it is better to use linear programming than least squares on linear backward heat diffusion problems.

SYMBOLS

$L_p[a,b]$	The Banach space of functions on $[a,b]$ with norm $\|\ \|_p$ $(1 \leq p < \infty)$.
$\|\ \|_p$	The norm for L_p; $\|f\| = \left(\int_a^b \|f\|^p dx\right)^{1/p}$.
$L_{p,r}[a,b]$	The Banach space of functions on $[a,b]$ with norm $\|\ \|_{p,r}$. Note $r(x)$ must be positive on $[a,b]$.
$\|\ \|_{p,r}$	$\|f\|_{p,r} = \left(\int_a^b \|f\|^p r(x) dx\right)^{1/p}$.
$C[a,b]$	The space of continuous functions on $[a,b]$ with norm $\|\ \|_\infty$.
$C^p[a,b]$	The space of functions continuously differentiable p times on $[a,b]$.
$\|\ \|_\infty$	$\|f\|_\infty = \sup_{a<x<b} \|f(x)\|$.
$\langle\ ,\ \rangle$	The inner product in a Hilbert space. For example, if $\underset{\sim}{c}=(c_1,\ldots,c_n)^T$, and $\underset{\sim}{x}=(x_1,\ldots,x_n)^T \in \mathbb{R}^n$, $\langle c,x\rangle = \sum_{i=1}^{n} c\ x$.
D^+	$D^+u(x)=[u(x+h)-u(x)]/h$. Note that h is assumed known here.
D^0	$D^0u(x)=[u(x+h)-u(x-h)]/(2h)$.
D^-	$D^-u(x)=[u(x)-u(x-h)]/h$.
δ_{ij}	$\delta_{ij}=1$ if $i=j$, 0 otherwise.

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