The Lyapunov Exponents for Schrödinger Operators and Jacobi Matrices with Slowly Oscillating Potentials

Thesis by

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Abstract

In the first part, we study the one-dimensional half-line Schrödinger operator

$$H_\nu = -\frac{d^2}{dx^2} + \cos(x^\nu) \qquad x \in [0, \infty) \tag{1}$$

with $0 < \nu < 1$. For each $\theta \in [0, \pi)$, let H_ν^θ denote the unique self-adjoint realization of H_ν on $L^2[0, \infty)$ with boundary condition at 0 given by $u(0)\cos\theta + u'(0)\sin\theta = 0$.

By studying the integrated density of states, we prove the existence of the Lyapunov exponent and the Thouless formula for (1). This yields an explicit formula for these Lyapunov exponents. By applying rank one perturbation theory, we also obtain some spectral consequences. Our main results are the following

Theorem. *Let $\gamma_0(E) = [\max(0, -E)]^{\frac{1}{2}}$ and $k_0(E) = \pi^{-1}[\max(0, E)]^{\frac{1}{2}}$. Then for all $E \notin R_\nu$, where R_ν is the resonance set for (1) which has both Lebesgue measure zero and Hausdorff dimension zero, we have*

$$\gamma(E) = \gamma_0(E) + \int_{-\infty}^{\infty} \ln|E - E'|d(k - k_0)(E')$$

where $\gamma(E)$ is the Lyapunov exponent for H_ν, and $k(E)$ is the integrated density of states for H_ν.

Theorem. *For all $E \notin R_\nu$, where R_ν is the resonance set for (1) which has both Lebesgue measure zero and Hausdorff dimension zero, the operator H_ν in (1) has Lyapunov behavior with the Lyapunov exponent given by*

$$\gamma(E) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [\max(0, \cos x - E)]^{\frac{1}{2}}\, dx. \tag{2}$$

Theorem. *For a.e. $\theta \in [0, \pi)$ (with respect to Lebesgue measure), H_ν^θ has dense pure point spectrum on $(-1, 1)$, and the eigenfunction of H_ν^θ to all eigenvalues $E \in (-1, 1)$ decay like $e^{-\gamma(E)x}$ at ∞ for almost every θ, where $\gamma(E)$ is the Lyapunov exponent for (1) which is given by (2).*

Theorem. *For $\theta \neq \frac{\pi}{2}$, the singular continuous part, $(d\mu_\theta)_{\mathrm{sc}}$, of the spectral measure $d\mu_\theta$ for H_ν^θ is supported on a Hausdorff dimension zero set.*

In the second part, we extend the above arguments to the Jacobi matrix on $L^2(\mathbb{Z}^+)$ which is a discrete analog of the Schrödinger operator (1). Let

$$(h(\nu,\lambda)u)(n) = u(n+1) + u(n-1) + \lambda\cos(n^\nu)u(n) \qquad n \in \mathbb{Z}^+ \tag{3}$$

with $|\lambda| < 2$ and $0 < \nu < 1$.

Similarly, by studying the integrated density of states for (3), we can prove the existence of the Lyapunov exponents and the Thouless formula for (3). Then, we can compute an explicit formula for these Lyapunov exponents. By applying rank one perturbation theory again, we can also obtain some interesting spectral consequences for $h(\nu,\lambda)$. We have the following theorems.

Theorem. *There exists a Lebesgue measure zero and Hausdorff dimension zero set $\overline{R}_\nu$, which we call the resonance set for (3). For all $E \notin \overline{R}_\nu$, $h(\nu,\lambda)$ has Lyapunov behavior with the Lyapunov exponent given by*

$$\gamma(E) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Re\, cosh^{-1}\left(\frac{E-\lambda\cos x}{2}\right) dx. \tag{4}$$

Theorem. *For almost all $|\lambda| < 2$ (with respect to Lebesgue measure), $h(\nu,\lambda)$ has dense pure point spectrum on $(-2-|\lambda|, -2+|\lambda|) \cup (2-|\lambda|, 2+|\lambda|)$, and the eigenvectors to all eigenvalues E decay like $e^{-\gamma(E)n}$ at infinity, where $\gamma(E)$ is the Lyapunov exponent for (3) which is given by (4).*

Theorem. *For $\lambda \neq 0$, $(d\mu_\lambda)_{sc}$, the singular continuous part of the spectral measure $d\mu_\lambda$ for $h(\nu,\lambda)$, is supported on a Hausdorff dimension zero set.*

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Chapter 1

Introduction

Our goal here is to prove Lyapunov behavior and compute Lyapunov exponents for the one-dimensional half-line Schrödinger operator

$$H_\nu = -\frac{d^2}{dx^2} + \cos(x^\nu) \qquad x \in [0, \infty) \tag{1.1}$$

on $L^2[0, \infty)$ and the one-dimension half-line Jacobi matrix which is a discrete analog of the Schrödinger operator (1.1)

$$(h(\nu, \lambda)u)(n) = u(n+1) + u(n-1) + \lambda \cos(n^\nu)u(n) \qquad n \in \mathbb{Z}^+ \tag{1.2}$$

on $\ell^2(\mathbb{Z}^+)$, where $0 < \nu < 1$ and $|\lambda| < 2$.

We first study the Schrödinger operator H_ν in detail. Then without many difficulties, we can extend the argument to the Jacobi matrix $h(\nu, \lambda)$.

It's clear that H_ν is regular at 0 and is limit point at infinity (for the definition of regular and limit point, see [26] or [17]). Therefore, for each $\theta \in [0, \pi)$, H_ν has a unique self-adjoint realization on $L^2[0, \infty)$ with boundary condition at 0 given by

$$u(0)\cos\theta + u'(0)\sin\theta = 0$$

which will be denoted by H_ν^θ.

In the spectral theory of Schrödinger operators, most work has concentrated on the potential $V(x)$, either $V(x) \to 0$ as $|x| \to \infty$ or $V(x)$ is periodic or almost periodic. Such models have been investigated particularly well. Comparatively new are the models with oscillating but not periodic nor almost periodic potentials. Due to recent discoveries of H. Behncke ([2]), W. Kirsch, S.A. Molchanov and L.A. Pastur ([13], [14]) and G. Stolz ([23], [24]), it is clear that some such models may yield very interesting spectrum. As two of his particular examples, Stolz has studied the spectral properties for (1.1) and (1.2) in his papers [24] and [23]. Let $\sigma(H)$, $\sigma_{ac}(H)$, $\sigma_{sing}(H)$, $\sigma_{sc}(H)$ and $\sigma_{pp}(H)$ denote the spectrum, absolutely continuous spectrum, singular spectrum, singular continuous spectrum and pure point spectrum resp. for H. Then from Stolz's papers, we know that

(1) For (1.1), we have $\sigma(H_\nu) = [-1,\infty)$, $\sigma_{ac}(H_\nu) = [1,\infty)$, and $\sigma_{sing}(H_\nu) = [-1,1]$;

(2) For (1.2), we have $\sigma(h(\nu,\lambda)) = [-2-|\lambda|, 2+|\lambda|]$, $\sigma_{ac}(h(\nu,\lambda)) = [-2+|\lambda|, 2-|\lambda|]$, and $\sigma_{sing}(h(\nu,\lambda)) = [-2-|\lambda|, -2+|\lambda|] \cup [2-|\lambda|, 2+|\lambda|]$.

We already see that these two models have some subtle and fascinating spectral properties, especially, for $E \in (-1,1)$ in the continuous case, and $E \in (-2-|\lambda|, -2+|\lambda|) \cup (2-|\lambda|, 2-|\lambda|)$ in the discere case. We will continue working on these models. In particular, we will prove Lyapunov behavior and compute Lyapunov exponents for these two models.

We know that the Lyapunov exponent is an important tool in the spectral theory for one-dimensional Schrödinger operators and Jacobi matrix operators with almost periodic or random potentials. In [20, 21], the rank one perturbation theory shows that Lyapunov behavior can also be used to study Schrödinger operators with deterministic potentials. However, for almost periodic or random potentials, we have the subadditive ergodic theorem to guarantee the existence of the Lyapunov exponent, but for deterministic potentials, it is often difficult to prove Lyapunov behavior. But at least there are two ways we can try. At the beginning, we tried to compute a Lyapunov exponent by directly constructing the solutions using the WKB approximation solutions. Since WKB solutions are singular at each classical turning point and there are infinite many turning points in our models, it's quite difficult to justify an approximation solution. We haven't succeeded in this approach. However, this heuristic argument does reveal the mystery of the beautiful Lyapunov exponent formulae. In the Schrödinger operator case, if we use the WKB solutions and the connection formula (see [9], [10]) around each turning point without justifying them, then we can easily obtain the Lyapunov exponent formula (3.22) and the resonance set in Simon's conjecture (see the remark after our definition of resonance set in section 3.1). In the Jacobi matrices case, in fact, physicists have already found the Lyapunov exponent formula for $h(\nu, \lambda)$ (see [5]) by using WKB approximation solutions without mathematically justifying them. Also, they haven't considered the resonance set in [5]. However, they do give a lot of numerical results in [5] to support the Lyapunov exponent formula. In any way, we need to find a mathematical proof for all these formulae. Our approach here is by first studying the integrated density of states and proving the Thouless formula.

In reference, the Thouless formula was only proved for random potentials and almost periodic potentials (see [1]). The reason for this is that to prove the Thouless formula, we first need to know the existence of the Lyapunov exponents. For random potentials and almost periodic potentials, the existence of the Lyapunov exponents is a standard consequence of the subadditive ergodic theorem (see [16]). In our case, to prove the Thouless formulae for both the Schrödinger operator (1.1) and the Jacobi matrix (1.2), we can closely follow the proof given in [1]. However, since there is no subadditive ergodic theorem that we can use, we will prove the existence of the Lyapunov exponent directly by using information on the integrated density of states.

We first note that the potential in (1.1) (the same is true for (1.2)) is slowly varying. It has the property of "locally constancy" for large x. In fact, we have

$$\left|\frac{dV}{dx}\right| = \left|\frac{\sin\sqrt{x}}{2\sqrt{x}}\right| \to 0, \qquad \text{for } x \to \infty.$$

This asymptotic property makes the Dirichlet-Neumann bracketing technique and

WKB technique work perfectly. When we restrict H_ν to any finite interval $[0, L]$ ($L > 0$), we can estimate the integrated density of states, $k^{(L)}(E)$, by using the Dirichlet-Neumann bracketing technique (or together with the WKB technique). Therefore, we know how $k^{(L)}(E)$ converges to $k(E)$, the integrated density of states for H_ν on $[0, \infty)$. We'll use this information and all the techniques given in [1] to prove the existence of Lyapunov exponents and the Thouless formula. In fact, our method works for much more general cases.

Since variations of boundary condition are rank one perturbations (see [20]), we can apply the rank one perturbation theory to our model. By doing this, we will obtain some interesting facts. First, we know that there are always a lot of energies E (in fact, it's a dense G_δ set) in some intervals for which there is no Lyapunov behavior (see [20] or [6]). Therefore, the existence of the Lyapunov exponent can only be proved for a.e. E in some intervals. There is no way to remove the resonance sets in Theorem 3.2, 3.3 and Theorem 4.2, 4.3. Second, del Rio et al. [7] prove that for a dense G_δ of θ, the spectrum of H_ν^θ in $[-1, 1]$ is purely singular continuous. Following our formulae for $\gamma(E)$, we show that for a.e. θ, the spectrum of H_ν^θ in $[-1, 1]$ is purely point spectrum. Similar statement is true in the Jacobi matrix case. Third, our formula for $\gamma(E)$ in Theorem 3.2, $E \in (-1, 1)$, which we prove of an explicitly given set of measure zero, it's strictly positive. It is known (see [8]) that since $[-1, 1] \subset \sigma(H_\nu)$, the complement of $\{E | \, \gamma(E) \text{ exists and is positive}\}$ is a dense G_δ in $[-1, 1]$. By our construction, this dense G_δ has Lebesgue measure zero; indeed, it has Hausdorff dimension zero. This give us some interesting information on the singular continuous part of the spectral measure.

We are unaware of any other explicit (non-random) Schrödinger operators with a computable positive Lyapunov exponent. The explicit formula for $\gamma(E)$ is quasi-classical.

Chapter 2

The Integrated Density of States for Schrödinger Operators

To prove the Thouless formula in the next chapter, we need to study the integrated density of states, $k(E)$, and the existence of the Lyapunov exponent. Also, as we mentioned in chapter 1, we need information how rapidly $k^{(\ell)}(E)$ converges to $k(E)$ to establish the existence of the Lyapunov exponent. So, in this chapter, we first study the main technical object, the integrated density of states for equation (1.1), in detailed. We will prove a formula for the integrated density states, and more important, we will estimate how fast $k^{(\ell)}(E)$ converges to $k(E)$. The rate of the convergence will determine how large of the resonance set we need. That's why we present section 3 where we give faster convergent of $k^{(\ell)}(E)$ to $k(E)$.

The basic idea to compute the integrated density of states uses the standard Dirichlet-Neumann bracketing technique. This technique says that if we break a domain into several subdomains and use constant potential on each subdomain to estimate the number of eigenvalues of the restricted operator, then by summing them together, we can get the number of eigenvalues of the operator on the whole domain. Since the potentials in our problem are slowly oscillating, i.e., the potentials have the property of locally constant. Dirichlet-Neumann bracketing technique works perfectly. In section 2, we will introduce the Dirichlet-Neumann bracketing technique and use it to estimate the $k^{(\ell)}(E)$ for the special case H_ν with $\nu = \frac{1}{2}$. Of course, for general H_ν with $0 < \nu < 1$, it can be handled in a similar way.

The idea and the approach in section 2 are simple, but the result is not very good, i.e., the convergence of $k^{(\ell)}(E)$ to $k(E)$ is quite slow. Although this result is enough to prove the Thouless formula in the next chapter, we will need to use a larger resonant set than the one we will give in the next chapter. Of course, if we don't care about the resonant set very much (in any way, it's a Lebesgue measure zero and Hausdorff dimension zero set!), then we may skip the more complicating analysis in section 3. However, the method we used in section 3 is quite interesting. Therefore, we still like to present it here.

The basic idea in section 3 is that we will combine the Dirichlet-Neumann bracketing technique with the basic WKB technique. By the results of oscillation theory for regular Sturm-Liouville operator, we know that the number of eigenvalues is equal to the number of zeros (plus or minus a constant) for a real solution. Due to the property of locally constant, WKB solutions away from turning points should be quite accurate in some sense. In fact, we show that it's accurate enough to estimate the number of zeros for an exact solution. However, the WKB approximation breaks down in the vicinity of a classical turning point. So, we need to break the nth potential well into several parts. In the part where the potential is far away from the turning points, we will construct an exact (real) solution by using WKB solutions. However, when the potential is near the turning points, we need to use the Dirichlet-Neumann bracketing technique again.

2.1 Notations and Definitions

In this section, we will give some definitions and notations which will be needed in the following sections and the next chapter. In the following, when we write H_ν or H for short, we always mean the Schrödinger operator given by (1.1).

For convenience, we introduce the following notations. Define

$$L = S_\nu(\ell) = (2\pi\ell)^{\frac{1}{\nu}}, \qquad \Omega_\ell = [S_\nu(\ell-1), S_\nu(\ell)], \qquad \text{for } \ell = 1, 2, \cdots.$$

Ω_ℓ is the lth potential well for potential $V(x) = \cos x^\nu$ $(0 < \nu < 1)$. Let $H_D(\Omega)$, (resp. $H_N(\Omega)$) denotes the self-adjoint operator $H_0 + V(x)$ on $L^2(\Omega)$ with Dirichlet (resp. Neumann) boundary conditions, where $H_0 = -\Delta$. When $\Omega = (0, L)$, we use $H_D(L)$, (resp. $H_N(L)$) to denote $H_D(\Omega)$, (resp. $H_N(\Omega)$). In this case, we use $H_{DN}(L)$, (resp. $H_{ND}(L)$) denotes the self-adjoint operator $H_0 + V(x)$ on $L^2(0, L)$ with Dirichlet (resp. Neumann) boundary condition at 0 and Neumann (resp. Dirichlet) boundary condition at L.

Definition. For any self-adjoint operator A, define

$$N(E, A) = \dim P_{(-\infty,E)}(A) = \sum_{E_k<E} 1$$

where $P_\Omega(A)$ is the spectral projection for the operator A, and $\{E_k\}$ are the eigenvalues of A with $E_1 \leq E_2 \leq E_3 \leq \cdots$.

Now, let $H_{bc}(S(\ell))$ be any self-adjoint realization of H_ν which is given by (1.1) on $L^2(0, S(\ell))$ with some given boundary conditions at 0 and $S(\ell)$. Let $N_{bc}(E, \ell) = N(E, H_{bc}(S(\ell)))$.

Definition. Let $N_{bc}(E, \ell)$ be as above. We define

$$k^{(\ell)}(E) = \frac{1}{S_\nu(\ell)} N_{bc}(E, \ell) \qquad \text{and} \quad k(E) = \lim_{\ell\to\infty} k^{(\ell)}(E) \ .$$

Then $k(E)$ is called the integrated density of states for (1.1).

We will show that in the above definition, the limit $k(E)$ exists and is independent of the choice of boundary conditions. Therefore, we may use $N(E, \Omega)$ to denote $N(E, H_{bc}(\Omega))$.

Finally, to develop the Dirichlet-Neumann bracketing technique, we need the following definition

Definition (see [18]). Let A and B be self-adjoint operators that are nonnegative where A is defined on a dense subset of a Hilbert space $\mathcal{H}$ and B is defined on a dense subset of a Hilbert subspace $\mathcal{H}_1 \subset \mathcal{H}$. We write $0 \leq A \leq B$ if and only if

(1) $Q(B) \subset Q(A)$, where $Q(A)$ and $Q(B)$ are the formal domain of A and B resp., and

(2) For any $\psi \in Q(B)$,

$$0 \leq (\psi, A\psi) \leq (\psi, B\psi).$$

2.2 The Integrated Density of States I: Dirichlet-Neumann Bracketing Technique

For simplicity, we only study the $\nu = \frac{1}{2}$ case in this section. Of course, the same argument works for the general case. Therefore, the H in this section is referred to the following operator

$$H = -\frac{d^2}{dx^2} + \cos(\sqrt{x}) \qquad x \in [0, \infty). \tag{2.1}$$

Also, for convenience, we use $S(\ell) = (2\pi\ell)^2$ instead of $S_{1/2}(\ell)$.

The goal in this section is to compute the integrated density of states for (2.1) using the standard Dirichlet-Neumann bracketing technique, which is based on the following two lemmas.

Lemma 2.1 (See [18]). *If $0 \leq A \leq B$, then*

$$\dim P_{[0,E]}(A) \geq \dim P_{[0,E]}(B), \qquad \textit{for all } E > 0.$$

Lemma 2.2 (see [18]). *Let $\Delta_D(\Omega)$ (resp. $\Delta_D(\Omega)$) be the free Laplance on $L^2(\Omega)$ with Dirichlet (resp. Neumann) boundary condition. Then we have*

(a) *If $\Omega \subset \Omega'$, then $0 \leq -\Delta_D(\Omega') \leq -\Delta_D(\Omega)$.*

(b) *For any Ω, $0 \leq -\Delta_N(\Omega) \leq -\Delta_D(\Omega)$.*

(c) *Let Ω_1, Ω_2 be disjoint open subsets of an open set Ω so that $\overline{(\Omega_1 \cup \Omega_2)}^{int} = \Omega$, and $\Omega \setminus \Omega_1 \cup \Omega_2$ has Lebesgue measure 0. Then*

$$0 \leq -\Delta_D(\Omega) \leq -\Delta_D(\Omega_1 \cup \Omega_2),$$

and

$$0 \leq -\Delta_N(\Omega_1 \cup \Omega_2) \leq -\Delta_N(\Omega).$$

Under our notations which are given in section 2.1 and by using these two lemmas, we have the following inequalities

$$\sum_{j=1}^{\ell} N(E, H_D(\Omega_j)) \leq N(E, H_D(L)) \leq N(E, H_N(L)) \leq \sum_{j=1}^{\ell} N(E, H_N(\Omega_j)). \quad (2.2)$$

Now, we will prove the existence of the integrated density of states in three steps.

(1). Eigenvalues of free Laplance.

It's easy to check that

(a). The eigenvalues of $-\Delta_D$ on $L^2(a,b)$ are

$$E_k = \left(\frac{k\pi}{b-a}\right)^2 \qquad k = 1, 2, 3, \cdots$$

with the corresponding eigenfunctions

$$\Phi_k(x) = \begin{cases} (\frac{b-a}{2})^{-\frac{1}{2}} \cos[\frac{k\pi}{b-a}(x - \frac{a+b}{2})], & \text{for } k = 1, 3, \cdots; \\ (\frac{b-a}{2})^{-\frac{1}{2}} \sin[\frac{k\pi}{b-a}(x - \frac{a+b}{2})], & \text{for } k = 2, 4, \cdots. \end{cases}$$

(b). The eigenvalues of $-\Delta_N$ on $L^2(a,b)$ are

$$E_k = \left(\frac{k\pi}{b-a}\right)^2 \qquad k = 0, 1, 2, 3, \cdots$$

with the corresponding eigenfunctions

$$\Psi_k(x) = \begin{cases} (\frac{b-a}{2})^{-\frac{1}{2}} \sin[\frac{k\pi}{b-a}(x - \frac{a+b}{2})], & \text{for } k = 1, 3, \cdots; \\ (\frac{b-a}{2})^{-\frac{1}{2}} \cos[\frac{k\pi}{b-a}(x - \frac{a+b}{2})], & \text{for } k = 2, 4, \cdots; \\ \frac{1}{2}\sqrt{2} & \text{for } k = 0. \end{cases}$$

Therefore, we have the following

Lemma 2.3. *If we let $N_D(E;a,b)$ (respectively, $N_N(E;a,b)$) denote the dimension of the spectral projection $P_{(-\infty,E]}$ for $-\Delta_D$ (respectively, $-\Delta_N$) on (a,b). Then for $E<0$, we have*

$$N_D(E;a,b) = N_N(E;a,b) = 0 \tag{2.3}$$

and for $E \geq 0$, we have

$$\left| N_D(E;a,b) - \frac{\sqrt{E}}{\pi}(b-a) \right| \leq 1, \tag{2.4}$$

$$\left| N_N(E;a,b) - \frac{\sqrt{E}}{\pi}(b-a) \right| \leq 1. \tag{2.5}$$

(2). Estimation of $N(E, H_D(\Omega_j))$ and $N(E, H_N(\Omega_j))$

Let a_k, b_k be as in the figure 1 such that $\cup_k [a_k, b_k] = \Omega_j$. Let $I_k^{(j)} = (a_k, b_k) \subset \Omega_j$, with $b_k - a_k = j^\alpha$ for some $0<\alpha<1$, and let

$$V_k^D = \sup\{V(x)\ |x \in [a_k, b_k]\} \qquad \text{and} \qquad V_k^N = \inf\{V(x)\ |x \in [a_k, b_k]\}.$$

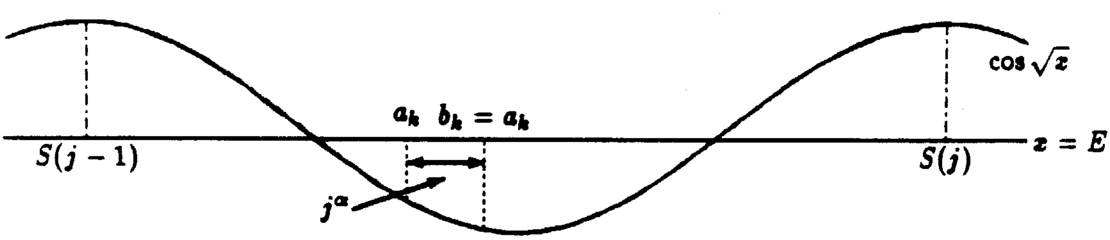

FIGURE 1: THE jTH POTENTIAL WELL, Ω_j

Define $B_D(I_k^{(j)}) = -\Delta_D(I_k^{(j)}) + V_k^D$ and $B_N(I_k^{(j)}) = -\Delta_N(I_k^{(j)}) + V_k^N$, then

$$0 \leq H_D(I_k^{(j)}) \equiv -\Delta_D(I_k^{(j)}) + V(x) \leq B_D(I_k^{(j)})$$

and

$$0 \leq B_N(I_k^{(j)}) \leq -\Delta_N(I_k^{(j)}) + V(x) \equiv H_N(I_k^{(j)}).$$

Thus, by lemma 2.1, we have

$$N(E, B_D(I_k^{(j)})) \leq N(E, H_D(I_k^{(j)})), \qquad N(E, H_N(I_k^{(j)})) \leq N(E, B_N(I_k^{(j)})).$$

Now, by lemma 2.2, we have

$$\begin{aligned} N(E, H_D(\Omega_j)) &\geq N(E, H_D(\cup I_k^{(j)})) \\ &= \sum_k N(E, H_D(I_k^{(j)})) \geq \sum_k N(E, B_D(I_k^{(j)})) \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} N(E, H_N(\Omega_j)) &\leq N(E, H_N(\cup I_k^{(j)})) \\ &= \sum_k N(E, H_N(I_k^{(j)})) \leq \sum_k N(E, B_N(I_k^{(j)})). \end{aligned} \tag{2.7}$$

Therefore, to estimate $N(E, H_D(\Omega_j))$ and $N(E, H_N(\Omega_j))$, we only need to estimate $N(E, B_N(I_k^{(j)}))$ and $N(E, B_D(I_k^{(j)}))$. But by (2.3) and (2.5),

$$\begin{aligned} N(E, B_N(I_k^{(j)})) &= N_N(E; a_k, b_k) \\ &= \begin{cases} \frac{\sqrt{E - V_k^N}}{\pi}(b_k - a_k) + C_0(k), & \text{if } E \geq V_k^N, \\ 0, & \text{if } E < V_k^N \end{cases} \end{aligned}$$

where $|C_0(k)| \leq 1$ for all k.

Thus, if we define $[f(x)]_+ = \max\{0, f(x)\}$, then we have

$$N(E, B_N(I_k^{(j)})) = \frac{[E - V_k^N]_+^{\frac{1}{2}}}{\pi}(b_k - a_k) + C_0(k, E). \tag{2.8}$$

where $C_0(k, E) = C_0(k)$ if $E \geq V_k^N$ and $C_0(k, E) = 0$ if $E < V_k^N$. Since $C_0(k) \leq 1$ for all k, we also have $C_0(k, E) \leq 1$ for all k and E.

Notice that we can write

$$[E - V_k^N]_+^{\frac{1}{2}}(b_k - a_k) = \int_{a_k}^{b_k} [E - V_k^N]_+^{\frac{1}{2}}\, dx.$$

Therefore,

$$[E - V_k^N]_+^{\frac{1}{2}}(b_k - a_k) - \int_{a_k}^{b_k} [E - V(x)]_+^{\frac{1}{2}}\, dx$$

$$= \int_{a_k}^{b_k} \{[E - V_k^N]_+^{\frac{1}{2}} - [E - V(x)]_+^{\frac{1}{2}}\}\, dx \stackrel{\text{def}}{=} \pi J. \tag{2.9}$$

Since

$$\left\{[E - V_k^N]_+^{\frac{1}{2}} - [E - V(x)]_+^{\frac{1}{2}}\right\}^2$$

$$\leq \left|[E - V_k^N]_+^{\frac{1}{2}} - [E - V(x)]_+^{\frac{1}{2}}\right| \left\{[E - V_k^N]_+^{\frac{1}{2}} + [E - V(x)]_+^{\frac{1}{2}}\right\}$$

$$\leq \frac{1}{2\sqrt{a_k}}(b_k - a_k) \qquad \text{for } x \in I_k^{(j)}.$$

By Schwarz inequality, we have,

$$|J| \leq \frac{1}{\pi}(b_k - a_k)^{\frac{1}{2}} \left[\int_{a_k}^{b_k} \{[E - V(x)]_+^{\frac{1}{2}} - [E - V_k^N]_+^{\frac{1}{2}}\}^2\, dx\right]^{\frac{1}{2}}$$

$$\leq \frac{1}{\sqrt{2}\pi} a_k^{-\frac{1}{4}} (b_k - a_k)^{\frac{3}{2}}$$

$$\leq C_1 j^{\frac{3}{2}\alpha - \frac{1}{2}} \tag{2.10}$$

where C_1 is a constant which is independent of j $(j > 1)$, k and α.

Therefore, by (2.8)–(2.10), we have

$$N(E, B_N(I_k^{(j)})) \leq \frac{1}{\pi} \int_{a_k}^{b_k} [E - V(x)]_+^{\frac{1}{2}}\, dx + C_1 j^{\frac{3}{2}\alpha - \frac{1}{2}} + 1.$$

By summing over k and using (2.7),

$$N(E, H_N(\Omega_j)) \leq \frac{1}{\pi} \int_{S(\ell-1)}^{S(\ell)} [E - V(x)]_+^{\frac{1}{2}}\, dx + C_2 j^{\frac{1}{2}(1+\alpha)} + C_3 j^{1-\alpha} \tag{2.11}$$

where $C_2 > 0$ and $C_3 > 0$ are constants.

Similarly, if we use (2.4) and (2.6) instead of (2.5)and (2.7), then we have

$$N(E, H_D(\Omega_j)) \geq \frac{1}{\pi} \int_{S(\ell-1)}^{S(\ell)} [E - V(x)]_+^{\frac{1}{2}}\, dx - C_2 j^{\frac{1}{2}(1+\alpha)} - C_3 j^{1-\alpha}. \tag{2.12}$$

(3). The existence of the integrated density of states

First, let's compute

$$k_D^{(\ell)}(E) \equiv \frac{1}{S(\ell)} N(E, H_D(L)), \qquad k_N^{(\ell)}(E) \equiv \frac{1}{S(\ell)} N(E, H_N(L)).$$

By summing over j in (2.11), (2.12) and using (2.2),

$$\frac{1}{\pi}\int_0^{S(\ell)} [E-V(x)]_+^{\frac{1}{2}}\, dx - C_2\ell^{\frac{1}{2}(3+\alpha)} - C_3\ell^{2-\alpha} \le N(E, H_D(L))$$

$$\le N(E, H_N(L)) \le \frac{1}{\pi}\int_0^{S(\ell)} [E-V(x)]_+^{\frac{1}{2}}\, dx + C_2\ell^{\frac{1}{2}(3+\alpha)} + C_3\ell^{2-\alpha}.$$

So, if we take $\alpha = 1/3$, then we have

$$\frac{1}{\pi}\int_0^{S(\ell)} [E-V(x)]_+^{\frac{1}{2}}\, dx - C\ell^{\frac{5}{3}} \le N(E, H_D(L))$$

$$\le N(E, H_N(L)) \le \frac{1}{\pi}\int_0^{S(\ell)} [E-V(x)]_+^{\frac{1}{2}}\, dx + C\ell^{\frac{5}{3}} \tag{2.13}$$

where $C = C_2 + C_3 > 0$ is a constant.

Also, we have the following estimation

$$\frac{1}{S(\ell)}\int_0^{S(\ell)} [E-V(x)]_+^{\frac{1}{2}}\, dx = \frac{1}{(2\pi\ell)^2}\int_0^{2\pi\ell} 2x[E-\cos x]_+^{\frac{1}{2}}\, dx$$

$$= \frac{1}{(2\pi\ell)^2}\sum_{k=1}^{\ell}\int_{2(k-1)\pi}^{2k\pi} 2x[E-\cos x]_+^{\frac{1}{2}}\, dx$$

$$= \frac{1}{(2\pi\ell)^2}\sum_{k=1}^{\ell}\int_{-2\pi}^{0} (2z+4k\pi)[E-\cos z]_+^{\frac{1}{2}}\, dz$$

$$= [I + II]$$

where

$$I = \frac{1}{(2\pi\ell)^2}\sum_{k=1}^{\ell}\int_{-2\pi}^{0} 2z[E-\cos z]_+^{\frac{1}{2}}\, dz, \qquad II = \frac{1}{(2\pi\ell)^2}\sum_{k=1}^{\ell}\int_{-2\pi}^{0} 4k\pi[E-\cos z]_+^{\frac{1}{2}}\, dz.$$

Since $\int_{-2\pi}^{0} 2z[E-\cos z]_+^{\frac{1}{2}}\,dz$, and $\int_{-2\pi}^{0}[E-\cos z]_+^{\frac{1}{2}}\,dz$ are bounded,

$$I = O\left(\frac{1}{\ell}\right), \qquad \text{and}$$

$$II = \frac{\ell+1}{2\pi\ell}\int_{-\pi}^{\pi}[E-\cos x]_+^{\frac{1}{2}}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}[E-\cos x]_+^{\frac{1}{2}}\,dx + O\left(\frac{1}{\ell}\right).$$

Thus, by the above estimations and (2.13), we have

$$\begin{aligned} \left|k_D^{(\ell)}(E) - \frac{1}{2\pi^2}\int_{-\pi}^{\pi}[E-\cos x]_+^{\frac{1}{2}}\,dx\right| &= O(\ell^{-\frac{1}{3}}), \\ \left|k_N^{(\ell)}(E) - \frac{1}{2\pi^2}\int_{-\pi}^{\pi}[E-\cos x]_+^{\frac{1}{2}}\,dx\right| &= O(\ell^{-\frac{1}{3}}). \end{aligned} \tag{2.14}$$

Since variations of boundary condition are rank one perturbations (see [20]), we have

$$|N(E, H_N(L)) - N(E, H_{\rm bc}(L))| \le 2 \tag{2.15}$$

where $N(E, H_{\rm bc}(L)$ is defined by any other self-adjoint boundary conditions.

Thus, by (2.14) and (2.15), we have proved the following theorem.

Theorem 2.1. *The integrated density of states for Schrödinger operator (2.1) exists, which is independent of the boundary conditions, and is given by*

$$k(E) = \frac{1}{2\pi^2}\int_{-\pi}^{\pi}[E-\cos x]_+^{\frac{1}{2}}\,dx.$$

Moreover, we have the following estimation

$$|k^{(\ell)}(E) - k(E)| = O(\ell^{-\frac{1}{3}}). \tag{2.16}$$

2.3 The Integrated Density of States II: WKB Technique

In this section, we will study the integrated density of states for equation (1.1) with general $0 < \gamma < 1$, especially, we will obtain faster convergent of $k^{(\ell)}(E)$ to $k(E)$.

First, we will need some standard results of the oscillation theory for regular Sturm-Liouville operator. These results can be found in the ordinary differential equations textbook [3, for example] or [26].

Lemma 2.3[26]. *Let $\tau u(x) = -u''(x)+q(x)u(x)$ be a regular Sturm-Liouville operator on (a,b), A be a self-adjoint realization of τ with separated boundary conditions at a and b given by*

$$u(a)\cos\alpha - u'(a)\sin\alpha = 0, \qquad u(b)\cos\beta - u'(b)\sin\beta = 0$$

with $\alpha \in [0,\pi)$ and $\beta \in (0,\pi]$. Then the eigenvalues of A are bounded from below. If the eigenvalues are arranged such that

$$E_0 < E_1 < E_2 < E_3 < \cdots < E_n \to \infty,$$

then the eigenfunction $u_n(x)$ corresponding to E_n has exactly n zeros in (a,b).

Lemma 2.4 [26]. *Let τ be a regular Sturm-Liouville operator, A be any self-adjoint realization with separated boundary conditions. If $(\tau - E)u(x) = 0$ has a real solution with n zeros in (a,b), then the number, $N(E)$, of eigenvalues of A in $(-\infty, E]$ satisfies*

$$n-1 \le N(E) \le n+2.$$

Remark. If A is any self-adjoint realization of τ, then we have the estimation (see [26]): $n-2 \le N(E) \le n+3$.

Now, we are ready to estimate $N(E, H_{bc}(\Omega_n))$, the number of eigenvalues of $H_{bc}(\Omega_n)$ in $(-\infty, E]$. We will use the notations introduced in section 2.1. In the following, we let $E \in (-1,1)$ be fixed. Let $p(x) = E - V(x)$, and a_n and b_n be the two turning points in the nth potential well, that is, a_n, $b_n \in \Omega_n$ such that $p(a_n) = p(b_n) = 0$. Let $d_n = C_d\,(2\pi n)^\alpha$ with $\alpha = \frac{3(1-\nu)}{5\nu}$, where C_d is a constant (independent of n) which will be determined later. Now, we can divide the nth potential well Ω_n into several pieces

$$P_1 = \{x|\, a_n + d_n \le x \le b_n - d_n\};$$

$$P_2 = \{x|\, a_n < x < a_n + d_n\} \cup \{x|\, b_n - d_n < x < b_n\};$$

$$P_3 = \{x|\, p(x) \le 0\} = \{x|\, S(n-1) \le x \le a_n\} \cup \{x|\, b_n \le x \le S(n)\}.$$

According to the decomposition of Ω_n, we divide the estimation of $N(E, H_(\Omega_n))$ into three steps.

(1). The control of $N(E, H_{bc}(P_1))$

First, we prove the following lemma

Lemma 2.5. *If $p(x) > 0$ on $(\alpha_n, \beta_n) \subseteq \Omega_n$, and the following integral is very small, say*

$$\int_{\alpha_n}^{\beta_n} p^{-\frac{1}{4}}(x) \big| [p^{-\frac{1}{4}}(x)]'' \big|\, dx < \frac{1}{100}. \tag{2.17}$$

Then there exists a real solution of

$$u''(x) + p(x)u(x) = 0 \qquad x \in (\alpha_n, \beta_n) \tag{2.18}$$

which vanishes K times on the interval (α_n, β_n), where

$$\left| K - \frac{1}{\pi} \int_{\alpha_n}^{\beta_n} p^{\frac{1}{2}}(x)\, dx \right| \le C_1$$

and C_1 is a constant (independent of n).

Proof. We will use the WKB approximation solutions. Let

$$\psi_\pm(x) = p^{-\frac{1}{4}}(x) e^{\pm i\eta(x)} \qquad \text{with } \eta(x) = \int_{\alpha_n}^{x} p^{\frac{1}{2}}(s)\, ds.$$

Then a simple computation shows that $\psi_\pm(x)$ satisfy the following differential equation of second order

$$\psi_\pm''(x) + p(x)\psi_\pm(x) = F(x)\psi_\pm(x)$$

where

$$F(x) = p^{\frac{1}{4}}(x)[p^{-\frac{1}{4}}(x)]''. \tag{2.19}$$

We only need to study the $\psi_+(x)$ case. For convenience, let $\psi(x) = \psi_+(x)$. Now, assume that $u_1(x) = \varepsilon_+(x)\psi(x)$ is an exact solution to (2.18) with $\varepsilon_+(\alpha_n) = 1$ and $\varepsilon'_+(\alpha_n) = 0$. Then $\varepsilon_+(x)$ satisfies

$$\begin{cases} \psi(x)\varepsilon''_+(x) + 2\psi'(x)\varepsilon'_+(x) + F(x)\psi(x)\varepsilon_+(x) = 0 \\ \varepsilon_+(\alpha_n) = 1, \quad \varepsilon'_+(\alpha_n) = 0. \end{cases} \tag{2.20}$$

Since the above equation is equivalent to

$$\begin{cases} (\psi^2(x)\varepsilon'_+(x))' = -F(x)\psi^2(x)\varepsilon_+(x) \\ \varepsilon_+(\alpha_n) = 1, \quad \varepsilon'_+(\alpha_n) = 0, \end{cases}$$

we can write the equation (2.20) as the following integral equation

$$\varepsilon_+(x) = 1 - \int_{\alpha_n}^{x} \psi^{-2}(t) \int_{\alpha_n}^{t} F(s)\psi^2(s)\varepsilon_+(s)\, ds\, dt,$$

that is,

$$\varepsilon_+(x) = 1 - \int_{\alpha_n}^{x} K(x,t)F(t)\varepsilon_+(t)\, dt \tag{2.21}$$

where

$$K(x,t) = \psi^2(t) \int_{t}^{x} \psi^{-2}(s)\, ds \qquad \text{for } t \le x. \tag{2.22}$$

Let $C[\alpha_n, \beta_n]$ denote all the continuous functions on $[\alpha_n, \beta_n]$. For $f \in C[\alpha_n, \beta_n]$, the norm of f is defined by $\|f\| = \max_{\alpha_n \le x \le \beta_n]} |f(x)|$. If we define $T : C[\alpha_n, \beta_n] \to C[\alpha_n, \beta_n]$ by

$$T(f)(x) = 1 - \int_{\alpha_n}^{x} K(x,t)F(t)f(t)\, dt,$$

then by (2.21), $\varepsilon_+(x)\psi(x)$ is an exact solution to (2.18) if and only if $\varepsilon_+(x)$ is a fixed point of the operator T.

(i) *The existence of the fixed point of* T. We have

$$|T(f)(x) - T(g)(x)| \le \|f - g\| \int_{\alpha_n}^{\beta_n} |K(x,t)F(t)|\, dt.$$

But by (2.22), we know that

$$K(x,t) = \psi^2(t) \int_t^x \psi^{-2}(s)\, ds = \frac{i}{2} p^{-\frac{1}{2}}(t) \left[e^{2i(\eta(t)-\eta(x))} - 1 \right].$$

So,

$$\left| K(x,t) \right| \leq p^{-\frac{1}{2}}(t) \qquad \text{for } \alpha_n \leq t \leq x \leq \beta_n.$$

Thus,

$$\left| K(x,t) F(t) \right| \leq p^{-\frac{1}{4}}(t) \left| [p^{-\frac{1}{4}}(t)]''(t) \right|.$$

Therefore, by the hypothesis of the lemma, the Lipschiz norm of T satisfies that $\|T\|_{\mathrm{Lip}} \leq \frac{1}{100}$. By the Lipschiz fixed point theorem, there exists a fixed point, $\varepsilon_+(x)$, of T.

(ii) *Estimation of* $\varepsilon_+(x)$ *and* $\varepsilon_+'(x)$ *on* (α_n, β_n). Since

$$\|T(1) - 1\| = \|(\varepsilon_+ - 1) - T(\varepsilon_+) + T(1)\| \geq (1 - \frac{1}{100}) \|\varepsilon_+ - 1\|$$

and

$$\|T(1) - 1\| = \max_{\alpha_n \leq x \leq \beta_n} \left| \int_{\alpha_n}^x K(x,t) F(t)\, dt \right| \leq \frac{1}{100},$$

we have $\|\varepsilon_+ - 1\| \leq \frac{1}{99}$, i.e.,

$$|\varepsilon_+(x) - 1| \leq \frac{1}{99} \qquad \text{for all } x \in [\alpha_n, \beta_n]. \tag{2.23}$$

Also, we have

$$\begin{aligned} |\varepsilon_+'(x)| &= \left| [T(\varepsilon_+)(x)]' \right| \\ &= \left| K(x,x) F(x) \varepsilon_+(x) + \int_{\alpha_n}^x \psi^2(t) \psi^{-2}(x) F(t) \varepsilon_+(t)\, dt \right| \\ &\leq \frac{1}{100} \|\varepsilon_+\| p^{\frac{1}{2}}(x) \\ &\leq \frac{1}{99} p^{\frac{1}{2}}(x) \qquad \text{for all } x \in [\alpha_n, \beta_n]\,. \end{aligned} \tag{2.24}$$

So, we have proved that $u_1(x) = \varepsilon_+(x)\psi_+(x)$ is an exact solution to equation (2.18), where $\varepsilon_+(x)$ satisfies (2.23) and (2.24). Now, since the coefficients in equation (2.18) are real, the complex conjugate of $u_1(x)$ is also a solution of (2.18). Thus, if we take $\varepsilon_-(x) = \overline{\varepsilon_+(x)}$, the complex conjugate, then $u_2(x) = \varepsilon_-(x)\psi_-(x)$ is also an exact solution to equation (2.18) with

$$|\varepsilon_-(x) - 1| \leq \frac{1}{99}, \qquad |\varepsilon'_-(x)| \leq \frac{1}{99}p^{\frac{1}{2}}(x) \qquad \text{for all } x \in [\alpha_n, \beta_n]. \tag{2.25}$$

(iii) *Finish the proof of the lemma.* Now, let $u(x) = \varepsilon_+(x)\psi_+(x) + \varepsilon_-(x)\psi_-(x)$, then $u(x)$ is a real solution to equation (2.18). We can write $u(x)$ as

$$\begin{aligned} u(x) &= 2p^{-\frac{1}{4}}(x)\{\cos\eta(x) + e^{i\eta(x)}(\varepsilon_+(x) - 1) + e^{-i\eta(x)}(\varepsilon_-(x) - 1)\} \\ &= 2p^{-\frac{1}{4}}(x)\{\cos\eta(x) + \delta(x)\} \end{aligned}$$

where $\delta(x) = e^{i\eta(x)}(\varepsilon_+(x) - 1) + e^{-i\eta(x)}(\varepsilon_-(x) - 1)$.

By the estimation (2.23)–(2.25), it's easy to see that $\delta(x)$ satisfies

$$|\delta(x)| \leq \frac{2}{99}, \qquad |\delta'(x)| \leq \frac{4}{99}p^{\frac{1}{2}}(x) \qquad \text{for all } x \in [\alpha_n, \beta_n]. \tag{2.26}$$

Since $p^{-\frac{1}{4}}(x) > 0$ on (α_n, β_n), the number of zeros of $u(x)$ is the same as the number of zeros of $v(x) \equiv \cos\eta(x) + \delta(x)$. Now, let

$$\cup I_i = \left\{ x \in (\alpha_n, \beta_n) \,\middle|\, |\cos\eta(x)| < \frac{2}{99} \right\}.$$

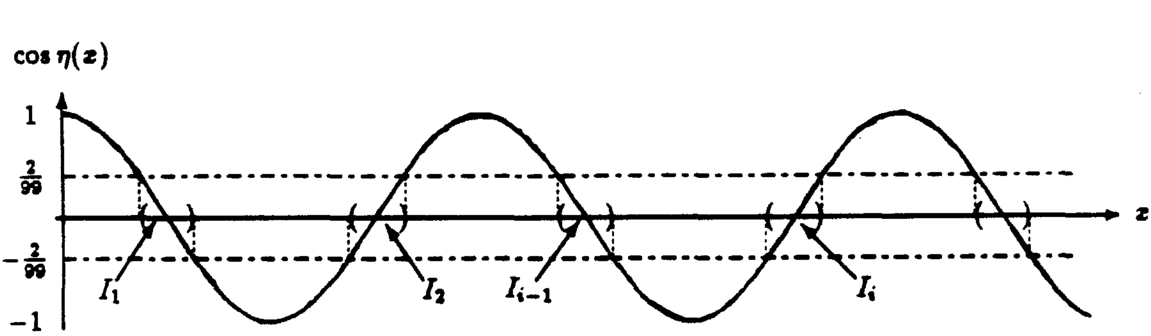

FIGURE 2: THE INTERVAL I_i AROUND THE ZERO OF $cos\eta(x)$

Then outside $\cup I_i$, $v(x)$ has the same signs as $\cos\eta(x)$. Inside I_i, as $\cos\eta(x)$ vanishes once, $v(x)$ vanishes at least once in each I_i. Also, by (2.26), we have

$$\begin{aligned} v'(x) &= p^{\frac{1}{2}}(x)\sin\eta(x) + \delta'(x) \\ &\geq \left[\sqrt{1-\left(\frac{2}{99}\right)^2} - \frac{4}{99}\right]p^{\frac{1}{2}}(x) \qquad (\text{for } x \in I_i) \\ &> 0. \end{aligned}$$

This means that $v(x)$ vanishes at most once in each I_i. Thus, in (α_n, β_n), $u(x)$ (or $v(x)$) vanishes as many times as $\cos\eta(x)+C$ with some constant $|C| \leq 2$. Therefore, K, the number of zeros of $u(x)$ in (α_n, β_n), satisfies

$$\left|K - \frac{1}{\pi}\int_{\alpha_n}^{\beta_n} p^{\frac{1}{2}}(x)\,dx\right| \leq 4. \quad \square$$

Now, let $\alpha_n = a_n + d_n$ and $\beta_n = b_n - d_n$. To apply the lemma 2.5, we only need to check the inequality (2.17). In fact, our d_n is chosen in such a way that (2.17) holds. For $p(x) = E - \cos(x^\nu)$ with $p(a_n) = p(b_n) = 0$, it's easy to show that for $x \in P_1$,

$$p(x) \geq p(b_n - d_n) \geq C_1' d_n b_n^{\nu-1} = C_1 d_n n^{-\frac{1-\nu}{\nu}} \tag{2.27}$$

and

$$\begin{aligned} p^{-\frac{1}{4}}(x)[p^{-\frac{1}{4}}(x)]'' &= -\frac{1}{4}p^{-\frac{3}{2}}(x)p''(x) + \frac{5}{16}p^{-\frac{5}{2}}(x)[p'(x)]^2 \\ &= -\frac{1}{4}\frac{p^{-\frac{3}{2}}(x)}{x^{2-2\nu}}\left[\frac{\nu(\nu-1)\sin(x^\nu)}{x^\nu} + \nu^2\cos(x^\nu)\right] \\ &\quad + \frac{5\nu^2}{16x^{2-2\nu}}p^{-\frac{5}{2}}(x)\sin^2(x^\nu). \end{aligned} \tag{2.28}$$

Therefore, by (2.27) and (2.28),

$$|p^{-\frac{1}{4}}(x)[p^{-\frac{1}{4}}(x)]''| \leq \frac{C_2'}{x^{2(1-\nu)}}p^{-\frac{5}{2}}(x) \leq C_2 d_n^{-\frac{5}{2}} n^{\frac{1-\nu}{2\nu}}$$

and

$$\int_{P_1} p^{-\frac{1}{4}}(x)\left| [p^{-\frac{1}{4}}(x)]'' \right| dx \le C_2 d_n^{-\frac{5}{2}} n^{\frac{1-\nu}{2\nu}} (b_n - a_n) \le C_2 d_n^{-\frac{5}{2}} n^{\frac{3(1-\nu)}{2\nu}}$$

$$= (2\pi)^{-\frac{3(1-\nu)}{2\nu}} C_2 C_d^{-\frac{5}{2}}.$$

Thus, if we choose C_d so large that

$$C_d \ge (100C_2)^{\frac{2}{5}} (2\pi)^{-\frac{3(1-\nu)}{5\nu}}, \tag{2.29}$$

then we have

$$\int_{P_1} p^{-\frac{1}{4}}(x)\left| [p^{-\frac{1}{4}}(x)]'' \right| dx \le \frac{1}{100}.$$

Therefore, by choosing C_d such that (2.29) holds, we have shown that (2.17) holds. So, by lemma 2.4 and lemma 2.5, the number, $N(E, H_{bc}(P_1))$, of eigenvalues of H in $(-\infty, E]$ satisfies

$$\left| N(E, H_{bc}(P_1)) - \frac{1}{\pi} \int_{P_1} p^{\frac{1}{2}}(x)\, dx \right| \le C$$

where C is a constant (independent of n).

(2). The control of $N(E, H_{bc}(P_2))$

On the interval P_2, we will again use the Dirichlet-Neumann bracketing technique. All of the computation is exactly like what we did in section 2. So, we will not repeat that computation. Like in section 2, we can break the P_2 into some small pieces with each having length $n^{\frac{1}{3}\frac{1-\nu}{\nu}}$. Then, we have

$$\left| N(E, H_{bc}(P_2)) - \frac{1}{\pi} \int_{P_2} p^{\frac{1}{2}}(x)\, dx \right| = O(n^{\frac{4}{15}\frac{1-\nu}{\nu}}).$$

(3). The control of $N(E, H_{bc}(P_3))$

On P_3, $p(x) < 0$. Therefore, by the argument in section 2, $N(E, H_{bc}(P_3)) < 2$.

Now, by step 1, 2 and 3, together with the Dirichlet-Neumann bracketing technique, we have proved that

$$\left|N(E, H_{bc}(\Omega_n)) - \frac{1}{\pi}\int_{\Omega_n} [E - V(x)]_+^{\frac{1}{2}}\, dx\right| = O(n^{\frac{4}{15}\frac{1-\nu}{\nu}}).$$

Therefore

$$\left|N(E, H_{bc}(L)) - \frac{1}{\pi}\int_0^{S_\nu(\ell)} [E - V(x)]_+^{\frac{1}{2}}\, dx\right| = O(\ell^{\frac{4}{15}\frac{1-\nu}{\nu}+1}). \tag{2.30}$$

Theorem 2.2. *For Schrödinger*

$$H_\nu = -\frac{d^2}{dx^2} + \cos(x^\nu) \qquad x \in [0, \infty)$$

with $0 < \nu < 1$, *the integrated density of states is given by*

$$k(E) = \frac{1}{2\pi^2}\int_{-\pi}^{\pi} [E - \cos x]_+^{\frac{1}{2}}\, dx.$$

Moreover, we have the following estimation

$$|k^{(\ell)}(E) - k(E)| = O(\ell^{-\kappa(\nu)}) \tag{2.31}$$

where

$$\kappa(\nu) = \min\left\{\frac{11(1-\nu)}{15\nu}, 1\right\}. \tag{2.32}$$

Proof. In fact, we already compute it in section 2.3. Here we only want to emphasize that for all $0 < \nu < 1$, we have the same formula for $k(E)$. Also, we need to give the explicit formula of the convergence for all $0 < \nu < 1$, i.e., to give the formula (2.31) and (2.32).

Recall that $S_\nu(\ell) = (2\pi\ell)^{\frac{1}{\nu}}$, and by the definition of integrated density of states,

$$k^{(\ell)}(E) = \frac{N(E, H_{bc}(L))}{S_\nu(\ell)}.$$

So, by the estimation (2.30), we have

$$k^{(\ell)}(E) = \frac{1}{\pi S_\nu(\ell)} \int_0^{S_\nu(\ell)} [E - \cos x^\nu]_+^{\frac{1}{2}}\, dx + O(S_\nu^{-1} \ell^{\frac{4}{15}\frac{1-\nu}{\nu}+1})$$

$$= \frac{1}{\pi\nu(2\pi\ell)^{\frac{1}{\nu}}} \int_0^{2\pi\ell} y^{\frac{1}{\nu}-1}\, [E - \cos y]_+^{\frac{1}{2}}\, dy + O(\ell^{-\frac{11}{15}\frac{1-\nu}{\nu}}) \qquad (y = x^\nu)$$

$$= \frac{1}{\pi\nu(2\pi\ell)^{\frac{1}{\nu}}} \sum_{k=1}^{\ell} \int_{-\pi}^{\pi} (z + 2k\pi)^{\frac{1}{\nu}-1}\, [E - \cos z]_+^{\frac{1}{2}}\, dz + O(\ell^{-\frac{11}{15}\frac{1-\nu}{\nu}})$$

$$= \frac{1}{\pi\nu(2\pi\ell)^{\frac{1}{\nu}}} \sum_{k=1}^{\ell} (2k\pi)^{\frac{1}{\nu}-1} \int_{-\pi}^{\pi} [E - \cos z]_+^{\frac{1}{2}}\, dz + O(\ell^{-1}) + O(\ell^{-\frac{11}{15}\frac{1-\nu}{\nu}})$$

$$= \frac{1}{2\pi^2} \int_{-\pi}^{\pi} [E - \cos x]_+^{\frac{1}{2}}\, dx + O(\ell^{-\kappa(\nu)})$$

where $\kappa(\nu)$ is defined by (2.32).

Therefore,

$$k(E) = \lim_{\ell\to\infty} k^{(\ell)}(E) = \frac{1}{2\pi^2} \int_{-\pi}^{\pi} [E - \cos x]_+^{\frac{1}{2}}\, dx$$

and

$$|k^{(\ell)}(E) - k(E)| = O(\ell^{-\kappa(\nu)}). \quad \square$$

Remark. When $\nu = \frac{1}{2}$, we have shown that $|k^{(\ell)}(E) - k(E)| = O(\ell^{-\frac{11}{15}})$, compare with the result of theorem 2.1 in section 2, the convergence is much faster here.

Chapter 3

The Thouless Formula and Lyapunov Exponent for Schrödinger Operators

Now, we begin to study the Lyapunov exponent by first proving the Thouless formula which relates the Lyapunov exponent to the integrated density of states for the Schrödinger operators H_ν. In [1], Thouless formula is proved for almost periodic potentials and random potentials. In their proof, they need to use the subadditive ergodic theorem to guarantee the existence of the Lyapunov exponent. To prove the Thouless formula in our case, we can closely follow the proof given in [1] for Schrödinger operators. However, we will prove the existence of the Lyapunov exponent by using information on how fast $k^\ell(E)$ converges to $k(E)$ which is given in Theorem 2.2.

In the section 1, we will introduce some definitions and notations that are needed in this chapter. Especially, we will define a resonant set and prove that it has both Lebesgue measure and Hausdorff dimension zero. In section 2, we will prove the Thouless formula for the Schrödinger operator (1.1). In section 3, we will compute an explicit formula of Lyapunov exponent by using the integrated density of states formula which is given in Chapter 2 and the Thouless formula.

3.1 Preliminary

First, we define the transfer matrix for the Schrödinger operator (1.1) as follows. Let $u(x,a,E)$, $v(x,a,E)$ $(a \geq 0, x \geq 0)$ solve the equation $-\phi'' + (V(x) - E)\phi = 0$ with the boundary conditions at a given by $u(a) = 0$, $u'(a) = 1$; $v(a) = 1$, $v'(a) = 0$. Then the transfer matrix is defined by

$$T_{a,x}(E) = \begin{pmatrix} v(x,a,E) & u(x,a,E) \\ \frac{\partial v(x,a,E)}{\partial x} & \frac{\partial u(x,a,E)}{\partial x} \end{pmatrix}. \tag{3.1}$$

In particular, when $a = 0$, we use $T_x(E)$ to denote $T_{0,x}(E)$.

Definition. For a given E, if $\gamma(E) = \lim_{x\to\infty} x^{-1} \ln \|T_x(E)\|$ exists, then we say that for the energy E, H has Lyapunov behavior, and $\gamma(E)$ is called the Lyapunov exponent for H.

Remark. In general, to define a Lyapunov exponent, we need to take the limit over $|x| \to \infty$. But in our case, we only consider the positive half line, as H_ν is defined on $L^2[0, \infty)$.

To give the Thouless formula, we first need to define the resonant set. In chapter 2 section 1, we have defined the operators $H_D(L)$, $H_N(L)$, $H_{DN}(L)$ and $H_{ND}(L)$. Now, let $\{E_k(\ell, D)\}$, $\{E_k(\ell, N)\}$, $\{E_k(\ell, DN)\}$, and $\{E_k(\ell, ND)\}$ be the corresponding eigenvalues. Also, recall that in theorem 2.2, we defined a function $\kappa(\nu)$ for each $\nu \in (0,1)$ by $\kappa(\nu) = \min\{\frac{11(1-\nu)}{15\nu}, 1\}$.

Definition. For each given $\nu \in (0,1)$, let $\epsilon_\nu > 0$ be a fixed number such that ϵ_ν is much small than $\kappa(\nu)$, where $\kappa(\nu)$ is defined by (2.16). Then the resonance set, R_ν, for the operator H_ν is defined by

$$R_\nu = R_D \cup R_N \cup R_{DN} \cup R_{ND} \tag{3.2}$$

where

$$R_D = \bigcup_{d=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \bigcup_{k} \{\, E \in [-d, d] \mid |E - E_k(n, D)| < \exp(-n^{\kappa(\nu)-\epsilon_\nu}) \,\}. \tag{3.3}$$

Similarly, we can define R_N, R_{DN} and R_{ND} by replacing $\{E_k(\ell, D)\}$ in (3.3) by $\{E_k(\ell, N)\}$,$\{E_k(\ell, DN)\}$ and $\{E_k(\ell, ND)\}$ resp.

Remark. In Simon's conjecture, instead of (3.2) and (3.3), the resonant set in $[-1,1]$ is defined by

$$R_\nu = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \bigcup_{k} \{E \in [-1,1] \mid |E - E_k^{(n)}| < \exp(-n^{\frac{1-\nu}{2\nu}})\}$$

where $\{E_k^{(n)}\}$ are the eigenvalues of $H_\nu = H_0 + V(x)$ on the nth potential well, $[(2n\pi - 2\pi)^2, (2n\pi)^2]$, with the Dirichlet boundary conditions. We believe that this is the reasonable definition of the resonant set. However, in our proof of Thouless formula, we need to use the resonant set defined by (3.2) and (3.3) which is larger than that in Simon's conjecture. Also, it should be possible that we can reduce our resonant set to that in Simon's conjecture, but we will not study this here.

Next, we show that the resonant set R_ν defined by (3.2) and (3.3) has both Lebesgue measure and Hausdorff dimension zero. So, first, we introduce the general notations of Hausdorff α-dimensional measure and Hausdorff dimension.

Let $M \subset \mathbb{R}^n$ be fixed. and let $\mathcal{A}(M,\rho)$ be the collection of all open cover of M with diameter less than or equal to ρ, i.e.,

$$\mathcal{A}(M,\rho) = \left\{ \{E_i\} \middle| M \subseteq \bigcup_{i=1}^{\infty} E_i, \quad E_i \ (i = 1, 2, \cdots) \text{ are open sets with } \operatorname{diam} E_i \leq \rho \right\}$$

where

$$\operatorname{diam} E_i = \sup_{x,y \in E_i} \{\|x - y\|\}.$$

Then we define

$$\mathcal{H}_\alpha(M,\rho) = \inf_{\{E_i\} \in \mathcal{A}(M,\rho)} \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} E_i)^\alpha \right\}.$$

Definition. $\mathcal{H}_\alpha(M) = \lim_{\rho \to 0} \mathcal{H}_\alpha(M,\rho)$ is called the Hausdorff α-dimensional measure of M.

It's clear that if $\rho_1 \geq \rho_2$, then $\mathcal{H}_\alpha(M,\rho_1) \leq \mathcal{H}_\alpha(M,\rho_2)$. So, the limit in the above definition exists (may be infinite). Also, we can show that if $\mathcal{H}_\alpha(M) \leq \infty$, than for all $\alpha < \alpha'$, $\mathcal{H}'_\alpha(M) = 0$. Therefore, we can define

Definition. For $M \subseteq \mathbb{R}^n$, $\dim_H M = \inf\{\alpha | \mathcal{H}_\alpha(M) = 0\}$ is called the Hausdorff dimension of M.

Lemma 3.1. *If* $\sum_{n=1}^{\infty} |A_n| < \infty$, *then we have*

$$\left|\limsup_{n\to\infty} A_n\right| = \left|\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} A_n\right| = 0$$

where $|\cdot|$ *denotes the Lebesgue measure.*

Proof. Let $B_m = \cup_{n=m}^{\infty} A_n$, then $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$. Thus,

$$\left|\limsup_{n\to\infty} A_n\right| = \lim_{m\to\infty} |B_m| \leq \lim_{m\to\infty} \sum_{n=m}^{\infty} |A_n| = 0. \quad \square$$

Lemma 3.2 [27]. *If* $\dim_H A_n = d$ *for each* A_n *in a countable collection* $\{A_n\}$, *then* $\dim_H(\bigcup_n A_n) = d$.

Theorem 3.1. *Let* R_ν *be the resonant set for* H *which is defined by (3.2) and (3.3). Then*

$$|R_\nu| = \dim_H R_\nu = 0.$$

Proof. For any fixed $d > 0$, let

$$S_d = \bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\bigcup_{k}\{E \in [-d,d] \mid |E - E_k(n,D)| < \exp(-n^{\kappa(\nu)-\epsilon_\nu})\}$$

where ϵ_ν is given in the definition of R_ν. Then by lemma 3.2, it's enough to prove that $|S_d| = \dim_H S_d = 0$.

Let $A_{n,k} = \{E \in [-d,d] \mid |E - E_k(n,D)| < e^{-n^{\kappa(\nu)-\epsilon_\nu}}\}$, and $B_m = \bigcup_{n=m}^{\infty}\bigcup_k A_{n,k}$. Then $\bigcap_{m=M}^{\infty} B_m = S_d$ for any $M \geq 1$.

From chapter 2, we know that the number of the eigenvalues $\{E_k(n,D)\}$ in $[-d,d]$ is at most, say, $Cn^{\frac{1}{\nu}}$ (C is a constant). Therefore,

$$\sum_{n=1}^{\infty}\left|\bigcup_k A_{n,k}\right| \leq \sum_{n=1}^{\infty} Cn^{\frac{1}{\nu}} \exp(-n^{1-\epsilon_\nu}) < \infty.$$

By lemma 3.1, $|S_d| = \left|\limsup_{n\to\infty}\bigcup_k A_{n,k}\right| = 0$. Thus, $|R_\nu| = 0$.

To prove that $\dim_H S_d = 0$, by definition, it's equivalent to prove that $\forall \alpha > 0$ and $\forall \rho > 0$, we have $\mathcal{H}_\alpha(S_d, \rho) = 0$. Now, let $\alpha > 0$ and $\rho > 0$ be given and fixed. It's enough to prove that $\forall \varepsilon > 0$, there exists an open cover of S_d, $\{E_i\}$, with diameter $< \rho$ such that

$$\sum_{i=1}^{\infty} (\text{diam} E_i)^\alpha < \varepsilon.$$

For the given $\rho > 0$, $\exists N_1 > 1$ such that

$$\text{diam} A_{n,k} = \exp(-n^{\kappa(\nu)-\epsilon_\nu}) < \rho \qquad \text{when } n \geq N_1. \tag{3.4}$$

Since $\sum_k \text{diam}(A_{n,k})^\alpha \leq C n^{\frac{1}{\nu}} \exp(-n^{\kappa(\nu)-\epsilon_\nu})$, for the given $\alpha > 0$ and $\varepsilon > 0$, $\exists N_2 > 1$ such that

$$\sum_{n=N_2}^{\infty} \sum_k \text{diam}(A_{n,k})^\alpha < \varepsilon.$$

Now, let $N = \max\{N_1, N_2\}$. As $\bigcup_{n=N}^{\infty} \bigcup_{k=1} A_{n,k} = B_N \supseteq S_d$, and by (3.4), $\{A_{n,k} \,|\, n \geq N, \text{and all possible } k\}$ is an open cover of S_d with diameter less than ρ. Thus, we have proved that $\mathcal{H}_\alpha(S_d, \rho) = 0$ for any $\rho > 0$ and $\alpha > 0$. So, $\dim_H S_d = 0$. Therefore, $\dim_H R_\nu = 0$. $\square$

3.2 The Thouless Formula

Our goal in this section is to prove the following:

Theorem 3.2 (Thouless formula). *Let H_ν be the Schrödinger operator given by (1.1). Let $\gamma_0(E) = [\max(0, -E)]^{\frac{1}{2}}$ and $k_0(E) = \pi^{-1}[\max(0, E)]^{\frac{1}{2}}$. Then for any $E \notin R_\nu$, where R_ν is defined by (3.2) and (3.3), we have*

$$\gamma(E) = \gamma_0(E) + \int_{-\infty}^{\infty} \ln|E - E'| \, d(k - k_0)(E') \tag{3.5}$$

where $\gamma(E)$ is the Lyapunov exponent for H_ν, and $k(E)$ is the integrated density of states for H_ν.

To prove this theorem, we will divide it into several steps. The first 3 steps are already given in [1]. So, we will not give the detailed proof for these results.

Step 1. *Control of* $\ell^{-1}\ln u_0(x,E)$. Let $u_0(x,E) = \sin(x\sqrt{E})/\sqrt{E}$. Then we have

Lemma 3.3 [1]. *For a.e. E,*

$$\lim_{\ell\to\infty} \ell^{-1}\ln|u_0(\ell,E)| = \gamma_0(E), \tag{3.6}$$

the limit being though the integers.

Proof. For $E<0$, this is easy. For $E>0$, we note that for a.e. E, we have that

$$|\ell\sqrt{E}+\pi j| \geq c|\ell|^{-m} \tag{3.7}$$

for suitable c, m. Then $u_0(\ell\sqrt{E}) \geq c'|\ell|^{-m}$, so the limit in question is 0. □

Step 2. *Finite eigenvalue estimates.* By general principals, we have

Lemma 3.4 [1]. *Let $E_k(\ell)$ be the eigenvalue of H_ν on $L^2[0,S_\nu(\ell)]$ with vanishing boundary conditions. $E_k^{(0)}(\ell) = (\pi k/S_\nu(\ell))^2$ be the corresponding eigenvalue of H_0. Then*

$$|E_k(\ell) - E_k^{(0)}| \leq \|V\|_\infty = 1. \tag{3.8}$$

Step 3. *Product formula in finite volume.* Recall that $S_\nu(\ell) = (2\pi\ell)^{\frac{1}{\nu}}$ and $\{E_k(\ell)\}$ (resp. $\{E_k^{(0)}(\ell)\}$) are eigenvalues of H (resp. H_0) on $L^2[0,S_\nu(\ell)]$. We claim that

Lemma 3.5 [1]. *For a fixed ℓ, we have that*

$$\frac{u(S_\nu(\ell),E)}{u_0(S_\nu(\ell),E)} = \prod_{k=1}^{\infty}\left[\frac{E-E_k(\ell)}{E-E_k^{(0)}(\ell)}\right]. \tag{3.9}$$

Lemma (3.4) implies the absolute convergence of the production in (3.9) as well as that in $\prod_{k=1}^{\infty} E_k^{(0)}(\ell)/E_k(\ell)$. Standard integral equation techniques show that for ℓ fixed, (a) $\lim_{E\to-\infty} u/u_0 = 1$. (b) $|u_0(S_\nu(\ell), E)| \le C_1(\ell)\exp(C_2(\ell)\sqrt{|E|})$ for complex E. Thus, by the Hadamard product formulas:

$$u(S_\nu(\ell), E) = c\prod_{k=1}^{\infty}(1 - E_k^{-1}E),$$

$$u_0(S_\nu(\ell), E) = d\prod_{k=1}^{\infty}\left[1 - (E_k^{(0)})^{-1}E\right].$$

From this and (3.8) (to justify some interchanges of product), we obtain (3.9) up to an overall constant

$$cd^{-1}\prod_{k=1}^{\infty} E_k^{(0)}/E_k$$

in front. Since $u/u_0 \to 1$ as $E \to -\infty$, we see that this constant must be 1.

Step 4. *Control of the limit at infinity.* From [1], we have the following limit

$$\lim_{M\to\infty}\left[\int_{k(E')\le M} \ln|E - E'|\,dk(E') - \int_{k_0(E')\le M} \ln|E - E'|\,dk_0(E')\right]$$
$$= \int_{-\infty}^{\infty} \ln|E - E'|\,d(k - k_0)(E'). \tag{3.10}$$

Step 5. *Control of $\frac{1}{S_\nu(\ell)}\ln|u(S_\nu(\ell), E)|$ when $\ell \to \infty$.* This is the key step to prove the Thouless formula. First, we will prove the following lemma

Lemma 3.6. *For all $E \notin R_D$, we have*

$$\lim_{\ell\to\infty}\frac{1}{S_\nu(\ell)}\ln\prod_{k=1}^{\infty}\left|\frac{E - E_k(\ell)}{E - E_k^{(0)}(\ell)}\right| = \int_{-\infty}^{\infty}\ln|E - E'|d(k - k_0)(E'). \tag{3.11}$$

Proof. For a given $E \notin R_D$, without loss of generality, we can also suppose that $E \notin R_D^{(0)}$, where $R_D^{(0)}$ is the corresponding resonance set for H_0 with Dirichlet boundary condition. From now on, we always suppose that E is fixed and $E \notin R_D \cup R_D^{(0)}$.

For each fixed E, we can choose $M(\ell)$ such that $M(\ell) \to \infty$ as $\ell \to \infty$ and $a_i(\ell) > E + 1$ $(i = 0, 1)$, where

$$a_0(\ell) = \sup\{E' \mid k_0^{(\ell)}(E') \le M(\ell)\}, \qquad a_1(\ell) = \sup\{E' \mid k^{(\ell)}(E') \le M(\ell)\}.$$

For convenience, we define

$$f_\ell(E) = \frac{1}{S_\nu(\ell)} \ln \prod_{k=1}^{\infty} \left| \frac{E - E_k(\ell)}{E - E_k^{(0)}(\ell)} \right|, \qquad f(E) = \int_{-\infty}^{\infty} \ln|E - E'|\, d(k - k_0)(E').$$

Then we have

$$\begin{aligned} |f_\ell(E) - f(E)| =& \Bigg| \frac{1}{S_\nu(\ell)} \ln \prod_{k \le M(\ell) S_\nu(\ell)} \left| [E - E_k(\ell)]/[E - E_k^{(0)}(\ell)] \right| \\ &+ \frac{1}{S_\nu(\ell)} \ln \prod_{k > M(\ell) S_\nu(\ell)} \left| [E - E_k(\ell)]/[E - E_k^{(0)}(\ell)] \right| - f(E) \Bigg| \\ \le& \Bigg| \int_{-\infty}^{a_1(\ell)} \ln|E - E'|\, d(k^{(\ell)} - k)(E') \\ &- \int_{-\infty}^{a_0(\ell)} \ln|E - E'|\, d(k_0^{(\ell)} - k_0)(E') \Bigg| \\ &+ \left| \int_{a_1(\ell)}^{\infty} \ln|E - E'|\, dk(E') - \int_{a_0(\ell)}^{\infty} \ln|E - E'|\, dk_0(E') \right| \\ &+ \left| \frac{1}{S_\nu(\ell)} \ln \prod_{k > M(\ell) S_\nu(\ell)} \left| [E - E_k(\ell)]/[E - E_k^{(0)}(\ell)] \right| \right|. \end{aligned} \tag{3.12}$$

By (3.10), we have

$$\lim_{\ell \to \infty} \left| \int_{a_1(\ell)}^{\infty} \ln|E - E'|\, dk(E') - \int_{a_0(\ell)}^{\infty} \ln|E - E'|\, dk_0(E') \right| = 0. \tag{3.13}$$

Since $E_k^{(0)}(\ell) = \left(\pi k / S_\nu(\ell)\right)^2$, by using lemma 3.4, we have

$$\begin{aligned} \ln \prod_{k > M(\ell) S_\nu(\ell)} \left| \frac{E - E_k(\ell)}{E - E_k^{(0)}(\ell)} \right| &= \sum_{k > M(\ell) S_\nu(\ell)} \ln \left| 1 + \frac{E_k(\ell) - E_k^{(0)}}{E_k^{(0)}(\ell) - E} \right| \\ &\le \sum_{k > M(\ell) S_\nu(\ell)} S_\nu^2(\ell) / [\pi^2 k^2 - S_\nu^2(\ell) E] \\ &\le S_\nu(\ell) \int_{M(\ell)}^{\infty} \frac{dx}{\pi^2 x^2 - E}. \end{aligned}$$

Therefore,

$$\left| \frac{1}{S_\nu(\ell)} \ln \prod_{k > M(\ell) S_\nu(\ell)} \left| [E - E_k(\ell)]/[E - E_k^{(0)}(\ell)] \right| \right| = O\left(\frac{1}{M(\ell)} \right). \tag{3.14}$$

So, it remains to estimate

$$J_\ell \equiv \left| \int_{-\infty}^{a_1(\ell)} \ln|E-E'|\, d(k^{(\ell)}-k)(E') - \int_{-\infty}^{a_0(\ell)} \ln|E-E'|\, d(k_0^{(\ell)}-k_0)(E') \right|.$$

We define

$$I_\ell(E) = \big[E-\delta_\ell, E+\delta_\ell\big], \qquad \delta_\ell = \frac{1}{3}\exp(-\ell^{\kappa(\nu)-\epsilon_\nu}) \tag{3.15}$$

where $\kappa(\nu)$ is defined by (2.16) and ϵ_ν is given in definition of the resonance set.

Since $E \notin R_D \cup R_D^{(0)}$, there are no eigenvalues of $H_D(L)$ and $H_{0D}(L)$ on the interval $I_\ell(E)$ which is defined by (3.15). Thus, $k^{(\ell)}(E)$, $k_0^{(\ell)}(E)$ are constant on the interval $I_\ell(E)$. Also, we notice that

$$\left| \int_{I_\ell(E)} \ln|E-E'|\, dk(E') \right| \le C_E[|I_\ell(E)|]^{\frac{1}{2}} \tag{3.16}$$

where C_E is a constant for a given E. So, we have

$$\begin{aligned} J_\ell = \Bigg| &\int_{I_\ell(E)} \ln|E-E'|\, d(k^{(\ell)}-k)(E') - \int_{I_\ell(E)} \ln|E-E'|\, d(k_0^{(\ell)}-k_0)(E') \\ &- \int_{(-\infty,a_0(\ell)]\setminus I_\ell(E)} \ln|E-E'|\, d(k_0^{(\ell)}-k_0)(E') \\ &+ \int_{(-\infty,a_1(\ell)]\setminus I_\ell(E)} \ln|E-E'|\, d(k^{(\ell)}-k)(E') \Bigg| \\ \le &\left| \int_{(-\infty,a_1(\ell)]\setminus I_\ell(E)} \ln|E-E'|\, d(k^{(\ell)}-k)(E') \right| + \left| \int_{I_\ell(E)} \ln|E-E'|\, dk(E') \right| \\ &+ \left| \int_{(-\infty,a_0(\ell)]\setminus I_\ell(E)} \ln|E-E'|\, d(k_0^{(\ell)}-k_0)(E') \right| \\ &+ \left| \int_{I_\ell(E)} \ln|E-E'|\, dk_0(E') \right|. \end{aligned} \tag{3.17}$$

By (3.16), we know that

$$\lim_{\ell\to\infty} \int_{I_\ell(E)} \ln|E-E'|\, dk(E') = 0. \tag{3.18}$$

Similarly,

$$\lim_{\ell\to\infty} \int_{I_\ell(E)} \ln|E-E'|\, dk_0(E') = 0. \tag{3.19}$$

Using integration by parts, we have

$$\begin{aligned}
&\left| \int_{(-\infty,a_1(\ell)]\setminus I_\ell(E)} \ln|E-E'|\, d(k^{(\ell)}-k)(E') \right| \\
&\quad \le (k^{(\ell)}-k)(a_1(\ell)) \ln|E-a_1(\ell)| + \{(k^{(\ell)}-k)(E+\delta_\ell) \\
&\qquad - (k^{(\ell)}-k)(E-\delta_\ell)\} \ln\delta_\ell + \left| \int_{(-\infty,a_1(\ell)]\setminus I_\ell(E)} \frac{(k^{(\ell)}-k)(E')}{E'-E}\, dE' \right|.
\end{aligned}$$

By theorem 2.1 and (3.15), we know that

$$\lim_{\ell\to\infty} (k^{(\ell)}-k)(a_1(\ell)) \ln|E-a_1(\ell)| = 0$$

$$\lim_{\ell\to\infty} \{(k^{(\ell)}-k)(E+\delta_\ell) - (k^{(\ell)}-k)(E-\delta_\ell)\} \ln\delta_\ell = 0$$

and

$$\begin{aligned}
\left| \int_{(-\infty,a_1(\ell)]\setminus I_\ell(E)} \frac{(k^{(\ell)}-k)(E')}{E'-E}\, dE' \right| &\le C_1 \ell^{-\kappa(\nu)} \left| \int_{(-\infty,a_1(\ell)]\setminus I_\ell(E)} \frac{1}{E'-E}\, dE' \right| \\
&\le \ell^{-\kappa(\nu)} \{C_2 \ln\delta_\ell + C_3 \ln|a_1(\ell)-E|\} \\
&\to 0 \qquad \text{as } \ell\to\infty.
\end{aligned}$$

Thus,

$$\lim_{\ell\to\infty} \left| \int_{(-\infty,a_1(\ell)]\setminus I_\ell(E)} \ln|E-E'|\, d(k^{(\ell)}-k)(E') \right| = 0. \tag{3.20}$$

Similarly,

$$\lim_{\ell\to\infty} \left| \int_{(-\infty,a_0(\ell)]\setminus I_\ell(E)} \ln|E-E'|\, d(k_0^{(\ell)}-k_0)(E') \right|. \tag{3.21}$$

So, by (3.17)–(3.21),

$$\lim_{\ell\to\infty} J_\ell = 0. \tag{3.22}$$

Now, by (3.12)–(3.14) and (3.22), we have proved that $\lim_{\ell\to\infty} |f_\ell(E) - f(E)| = 0$. Therefore, Lemma 3.6 is proved. □

Now, by combining the results of Lemma 3.5 and Lemma 3.6, we have proved the following result.

For all $E \notin R_D$, we have that

$$\lim_{\ell\to\infty} \frac{1}{S_\nu(\ell)} \ln\left| \frac{u(S_\nu(\ell),E)}{u_0(S_\nu(\ell),E)} \right| = \int_{-\infty}^{\infty} \ln|E-E'| d(k-k_0)(E').$$

By using Lemma 3.3, we obtain the following control on the limit

Lemma 3.7. *For all $E \notin R_D$, we have*

$$\lim_{\ell\to\infty} \frac{1}{S_\nu(\ell)} \ln |u(S_\nu(\ell), E)| = \gamma_0(E) + \int_{-\infty}^{\infty} \ln |E - E'|\, d(k - k_0)(E').$$

Step 6. *Completion of the proof for Thouless formula.* Similar to the proof of Lemma 3.7, by using different boundary conditions, we can obtain control of $\frac{1}{S_\nu(\ell)} \ln\big|v(S_\nu(\ell), E)\big|$, $\frac{1}{S_\nu(\ell)} \ln\big|\frac{\partial u(S_\nu(\ell),E)}{\partial x}\big|$ and $\frac{1}{S_\nu(\ell)} \ln\big|\frac{\partial v(S_\nu(\ell),E)}{\partial x}\big|$. Therefore, we obtain the control of $\frac{1}{S(\ell)} \ln \|T_{S(\ell)}(E)\|$, namely

Lemma 3.8. *For $E \notin R_\nu$, where R_ν is the resonance set defined by (3.2) and (3.3), then*

$$\lim_{\ell\to\infty} \frac{1}{S(\ell)} \ln \|T_{S(\ell)}(E)\| = \gamma_0(E) + \int_{-\infty}^{\infty} \ln |E - E'|\, d(k - k_0)(E')$$

where $\|\cdot\|$ denotes the matrix norm, and $T_x(E)$ is defined by (3.1).

Now, Theorem 3.2 follows from Lemma 3.8 and the definition of the Lyapunov exponent.

3.3 The Lyapunov Exponent Formula

In this section, we want to compute an explicit formula for the Lyapunov exponent by using the Thouless formula and the formula for integrated density of states. First, (3.5) asserts that $\pi k + i\gamma$ is the boundary value of an analytic function in the upper half plane. Let $F(z) = \pi k(z) + i\gamma(z)$ for $\operatorname{Im} z \geq 0$, and define $\tilde{F}(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sqrt{z - \cos x}\, dx$ with branch cut from -1 to $+\infty$ along the real axis. Then $\tilde{F}(z)$ is analytic for $\operatorname{Im} z > 0$ and by Theorem 2.1, $\operatorname{Re} \tilde{F}(z) \to \pi k(E)$ as $z \to E$ ($\operatorname{Im} z > 0$, $E \in \mathbb{R}$). Therefore,

$$\gamma(E) = \lim_{\operatorname{Im} z>0, z\to E} \operatorname{Im} \tilde{F}(z) + C$$

where C is a real constant. That is,

$$\gamma(E) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [\cos x - E]_+^{\frac{1}{2}}\, dx + C.$$

Notice that for $E > 1$, $\gamma(E) = 0$ and the integral in the right-hand side is also zero, so $C = 0$. Therefore, we have

Theorem 3.3. *For all $E \notin R_\nu$, where R_ν is defined by (3.2) and (3.3), the operator H_ν in (1.1) has Lyapunov behavior with the Lyapunov exponent given by*

$$\gamma(E) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [\cos x - E]_+^{\frac{1}{2}}\, dx \tag{3.23}$$

where $[f(x)]_+ = \max\{0, f(x)\}$.

Remarks. 1. In fact, there is no mystery for this beautiful Lyapunov exponent formula if we use the WKB (see [9],[10]) heuristic argument. However, it's not easy to justify the WKB solutions.

2. Note that while R_ν is ν-dependent, the right-hand side of (3.23) is ν-independent!

3.4 Some Spectral Properties

As we have already proved that for a.e. $E \in [-1, 1]$, H_ν has positive Lyapunov exponent, by simply applying the Kotani's argument (see [15]) or the general theory of rank one perturbations we can get dense pure points spectrum on $(-1, 1)$ for almost boundary conditions. Also, we can show that the eigenfunctions are exponentially decaying. The result on pure point spectrum is an unpublished result by Kirsch and Stolz which is stated in [14] by Kirsch, Molchanov and Pastur, and the result on exponentially decaying is proved by Stolz in [25]. Now, we can give an explicit decaying rate of eigenfunctions. A key fact needed is that the θ-mean of the Weyl-Titchmarsh spectral measures $\rho_\theta(E)$ of H_ν is absolutely continuous (see [20, 21]), that is,

$$\int_0^{\pi} [d\rho_\theta(E)]\, d\theta = dE \tag{3.24}$$

in the sense that if $f \in L^1(\mathbb{R}, dE)$, then $f \in L^1(\mathbb{R}, d\rho_\theta)$ for a.e. $\theta \in [0, \pi)$, $\int f(E)\, d\rho_\theta(E) \in L^1(\mathbb{R}, d\theta$ and

$$\int_0^\pi \left(\int f(E) d\rho_\theta(E) \right) d\theta = \int f(E)\, dE.$$

Another fact we will need is the following Osceledec lemma

Lemma 3.9 [16, 19]. *Let $T(x)$ be matrices of $SL(2, \mathbb{R})$ for every $x \in \mathbb{R}$ satisfying*

$$\lim_{x \to \infty} \frac{1}{x} \ln \|T(x)\| = \gamma < \infty \qquad \textit{and} \tag{3.25}$$

$$\lim_{x \to \infty} \sup_{0 \le \delta \le 1} \frac{1}{x} \ln \|T(x+\delta)T_{-1}(x)\| = 0. \tag{3.26}$$

Then there exists a one-dimensional subspace V of $\mathbb{R}$ so that

$$\lim_{x \to \infty} \frac{1}{x} \ln \|T(x)u\| = -\gamma \qquad \textit{if } u \in V,$$

$$\lim_{x \to \infty} \frac{1}{x} \ln \|T(x)u\| = \gamma \qquad \textit{if } u \notin V.$$

In our case, $T(x) = T_x(E)$ which is defined by (3.1). So, for each $E \notin R_\nu$, the condition (3.25) holds with $\gamma = \gamma(E)$ by theorem 3.3 and condition (3.26) always valids because $|V(x)| \le 1$.

Theorem 3.4. *Let H_ν^θ be the operator H_ν given by (1.1) with the θ boundary condition at 0, $u(0)\cos\theta + u'(0)\sin\theta = 0$. Then for a.e. $\theta \in [0, \pi)$ (with respect to Lebesgue measure), H_ν^θ has dense pure point spectrum on $(-1, 1)$, and the eigenfunctions of H_ν^θ to all eigenvalues $E \in (-1, 1)$ decay like $e^{-\gamma(E)x}$ at ∞ for almost every θ, where $\gamma(E)$ is given by (3.23).*

Proof. The first part is already contained in [15]. Also, it follows from the Simon Wolff criterion (see the remark at the end of this section). So, we will not show it here. Now, we want to show the exponential decay part. By theorem 3.3, and lemma 3.9, we know that for $E \notin R_\nu$, $H_\nu u = Eu$ has a solution decaying like $e^{-\gamma(E)x}$ at ∞. By (3.24) and theorem 3.1,

$$\int_0^\pi \rho_\theta(R_\nu)\, d\theta = |R_\nu| = 0.$$

So, $\rho_\theta(R_\nu) = 0$ for a.e. $\theta \in [0,\pi)$. Also, by first part of the theorem, ρ_θ is pure point measure on $(-1,1)$ for a.e. θ. Thus, if $u(x)$ is an eigenfunction of H_ν^θ with eigenvalues $E \in (-1,1)$, then $u(x) \in L^2(0,\infty)$, and $u(x)$ satisfies equation (1.1). Thus, (1.1) has an L^2-solution and a solution decaying like $e^{-\gamma(E)x}$ at ∞. In view of their Wronskian or using the limit point properties of H_ν at $+\infty$, these two solutions should coincide except a constant factor. This shows that the eigenfunctions are decaying like $e^{-\gamma(E)x}$ at ∞. □

Before we present another application of our theorem 3.1 and 3.3, we first introduce some results from general rank one perturbation theory. All these results can be found in [20, 21].

Let A be a positive self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Let

$$A_\alpha = A + \alpha B, \qquad B = (\phi, \cdot)\phi.$$

By the spectral theorem, $\mathcal{H}$ is unitary equivalent to $L^2(\mathbb{R}, d\mu_0)$ in such a way that A is multiplication by x and $\phi = 1$. Here $d\mu_0$ is the spectral measure of ϕ for A. Also, we use $d\mu_\alpha$ to denote the spectral measure of ϕ for A_α. Now, we define

$$F_\alpha(z) \equiv \int \frac{d\mu_\alpha(x)}{x - z}$$

and

$$G(x) \equiv \int \frac{d\mu_0(x)}{(x-z)^2}, \qquad B(x) \equiv [G(x)]^{-1}.$$

Then from [20, 21], we have

Lemma 3.10 [20, 21]. *For an open interval (a,b). The following are equivalent:*

(a) For a.e. α, A_α has only point spectrum in (a,b).

(b) For a.e. x in (a,b), $B(x)>0$.

Definition. For a given Borel measure η on $\mathbb{R}$, we say that η is supported on A if $\eta(\mathbb{R}\setminus A)=0$.

Lemma 3.11 [20, 21]. *Let $(d\mu_\alpha)_{\text{sc}}$ be the singular continuous part of the spectral measure $d\mu_\alpha$. Then for $\alpha\neq 0$, $(d\mu_\alpha)_{\text{sc}}$ is supported on*

$$S_\alpha \equiv \{x\in\mathbb{R}\,|\,F_0(x+i0)=-\alpha^{-1};\; G(x)=\infty\}.$$

Recall that H_ν^θ is the operator H_ν given by (1.1) with the boundary condition at 0 given by

$$u(0)\cos\theta+u'(0)\sin\theta=0 \qquad \theta\in[0,\pi).$$

So, $\theta=\frac{\pi}{2}$ is the Neumann boundary condition. Let A be the $\theta=\frac{\pi}{2}$ operator, $\delta(x)$ be the Dirac delta function and define

$$A_\alpha = A+\alpha\delta(x).$$

In [20], Simon has showed that $H_\nu^\theta = A_{-\cot(\theta)}$. So, variations of boundary condition are precisely rank one perturbation. Also, we have

Lemma 3.12 [20]. *Let A be as above. Then $G(E)<\infty$ if and only if both of the following hold:*

(a) E is not an eigenvalues of A;

(b) Equation $H_\nu(x)=Eu$ has a solution which is L^2 at $+\infty$.

Now, our second result based on theorem 3.1 and 3.3 is a simply consequence of the above lemmas.

Theorem 3.5. *Let H_ν^θ be the operator H_ν given by (1.1) with the boundary condition at 0 given by $u(0)\cos\theta + u'(0)\sin\theta = 0$ for $\theta \in [0,\pi)$. Then for $\theta \neq \frac{\pi}{2}$, the singular continuous part, $(d\mu_\theta)_{\mathrm{sc}}$, of the spectral measure $d\mu_\theta$ for H_ν^θ is supported on a Hausdorff dimension zero set.*

Proof. We know that there is no singular spectrum outside $[-1,1]$, and for $E \in [-1,1] \setminus R_\nu$ (R_ν is defined by (3.2) and (3.3)), there exists positive Lyapunov exponent. So, by lemma 3.9, for such E, $H_\nu(x) = Eu$ has an L^2-solution (in fact, it's exponentially decaying solution) at $+\infty$. By lemma 3.11 and lemma 3.12, we know that $(d\mu_\theta)_{\mathrm{sc}}$ is supported on

$$R_\nu \cup \{E \in (-1,1) \,|\, E \text{ is an eigenvalue of } H_\nu^{\pi/2}\}.$$

But the second part is a countable set, therefore has $(d\mu_\theta)_{\mathrm{sc}}$ zero measure. Thus, $(d\mu_\theta)_{\mathrm{sc}}$ is supported on R_ν and theorem 3.1 says that $\dim_H R_\nu = 0$. □

Remark. By lemma 3.12 and the above proof, we have showed that for all $E \in ([-1,1] \setminus R_\nu) \cap \{E \,|\, E \text{ is not an eigenvalue of } H_\nu^{\pi/2}\}$, we have $G(E) < \infty$ (or equivalently, $B(E) > 0$). But the above set has full Lebesgue measure. Therefore, by Simon Wolff criterion (lemma 3.10), we know that for a.e. $\theta \in [0,\pi)$, H_ν^θ has only point spectrum. This gives a proof to the first part of theorem 3.4.

Chapter 4

The Jacobi Matrix Operators

In this chapter, we will study the Jacobi matrix operators which are discrete analogs of Schrödinger operators (1.1). Let

$$(h_0 u)(n) = u(n+1) + u(n-1)$$

and

$$(h(\nu,\lambda)u)(n) = (h_0 u)(n) + \lambda \cos(n^\nu) u(n) \qquad n \in \mathbb{Z}^+ \tag{4.1}$$

with $|\lambda| < 2$ and $0 < \nu < 1$. In the case that we don't need to specify the dependence of ν and λ, we will write h instead of $h(\nu, \lambda)$ for short.

The method and argument in this chapter are complete analogs of the Schrödinger operator case. First, we study the integrated density of states for $h(\nu, \lambda)$, especially we need to study how $k^{(l)}(E)$ convergent to $k(E)$. By using this information, we can prove the existence of the Lyapunov exponent and the Thouless formula. Then we give the explicit formula of the Lyapunov exponent for $h(\nu, \lambda)$. After we have shown that there exists positive Lyapunov exponent in some interval (for a.e. E), we can use the rank one perturbation theory again and get some spectral properties for $h(\nu, \lambda)$.

4.1 The Integrated Density of States

In this chapter, we will use the same notations as the previous chapter, but with a little different meaning. For example, we will use $S_\nu(l)$ to denote the integral part of of $(2\pi l)^{\frac{1}{\nu}}$, and use Ω_n to denote all integers which lie in the nth potential well, $[S_\nu(n-1), S_\nu(n)]$.

Let $h^{(l)}$ denote the restriction of operator $h(\nu,\lambda)$ on $\ell^2([0,S_\nu(l)])$ with vanishing boundary conditions, and let $N^{(l)}(E)$ denote the number of eigenvalues of $h^{(l)}$ on $(-\infty,E]$.

Definition. Let

$$k^{(l)}(E)=\frac{1}{S_\nu(l)+1}N^{(l)}(E),\qquad \text{and } k(E)=\lim_{l\to\infty}k^{(l)}(E).$$

Then $k(E)$ is called the integrated density of states for (4.1).

For more precise and mathematical definition of integrated density of states and its relation with the above one, see [1].

First, let's provide the explicit eigenvalues and eigenvectors of h_0. Let

$$(h_0u)(n)=Eu(n) \tag{4.2}$$

on $\ell^2(n_1,n_2)$ with boundary condition given by $u(n_1)=u(n_2)=0$. We know that the general solution to equation (4.2) is $u(n)=C_1z_1^n+C_2z_2^n$, where

$$z_{1,2}=\frac{1}{2}(E\pm\sqrt{E^2-4}). \tag{4.3}$$

So, there are no eigenvalues on $\{E\,|\,|E|\geq 2\}$. Now, let $E<2$, then the $z_{1,2}$ which are defined by (4.3) can be written as $z_{1,2}=e^{\pm i\theta}$ with $2\cos\theta=E$, $2\sin\theta=\sqrt{4-E^2}$ and $\theta\in(0,\pi)$. Thus, the eigenvalues in $(-2,2)$ are

$$E=2\cos\frac{(2k+1)\pi}{n_2-n_1},\qquad \text{for } k=0,1,\dots,\left[\frac{n_2-n_1-1}{2}\right],$$

$$E=2\cos\frac{2k\pi}{n_2-n_1},\qquad \text{for } k=1,2,\dots,\left[\frac{n_2-n_1}{2}\right]$$

where the $[\cdot]$ denotes the integral part, with the corresponding eigenfunctions given by

$$u(n)=\cos\left[\frac{(2k+1)\pi}{n_2-n_1}\Big(n-\frac{n_2+n_1}{2}\Big)\right],\qquad \text{for } k=0,1,\dots,\left[\frac{n_2-n_1-2}{2}\right],$$

$$u(n)=\cos\left[\frac{2k\pi}{n_2-n_1}\Big(n-\frac{n_2+n_1}{2}\Big)\right],\qquad \text{for } k=1,2,\dots,\left[\frac{n_2-n_1-1}{2}\right].$$

Therefore, $N_0(E)$, the number of eigenvalues on $(-\infty, E]$, is given by

$$N_0(E) = \begin{cases} 0, & \text{if } E \le -2; \\ \frac{n_2-n_1}{\pi} \arccos\left(-\frac{E}{2}\right) + \delta(E), & \text{if } |E| < 2; \\ n_2 - n_1 - 1, & \text{if } E \ge 2 \end{cases} \tag{4.4}$$

where $|\delta(E)| < 2$.

If we notice that

$$\operatorname{Im}\cosh^{-1} x = \begin{cases} 0, & \text{if } x \ge 1; \\ \arccos x, & \text{if } |x| < 1; \\ \pi, & \text{if } x \le -1, \end{cases}$$

then we can rewrite (4.4) as

$$\left| N_0(E) - \frac{n_2 - n_1}{\pi} \operatorname{Im}\cosh^{-1}\left(-\frac{E}{2}\right)\right| < 2. \tag{4.5}$$

Now, we want to estimate $N(E, h(\Omega_n))$, where $h(\Omega_n)$ is the Jacobi operator (4.1) restricted on the lattices which lie in Ω_n with vanishing boundary condition. First, we divide Ω_n into some small intervals $\{I_k^{(n)}\}$ with $|I_k^{(n)}| = n^\alpha$, where α will be determined later. We can define $h(I_k^{(n)})$ in a similar way as $h(\Omega_n)$. Let $I_k^{(n)} = (a_k, b_k)$,

$$V(x) = \lambda \cos x^\nu, \qquad V_k = \sup\{V(x) \mid x \in [a_k, b_k]\}$$

and define $B(I_k^{(n)}) = h_0 + V_k$. Then by (4.5), we have

$$\left| N(E, B(I_k^{(n)})) - \frac{b_k - a_k}{\pi} \operatorname{Im}\cosh^{-1}\left(\frac{V_k - E}{2}\right)\right| \le C_1$$

where C_1 is a constant.

Notice that

$$\frac{b_k - a_k}{\pi} \operatorname{Im}\cosh^{-1}\left(\frac{V_k - E}{2}\right) = \frac{1}{\pi}\int_{a_k}^{b_k} \operatorname{Im}\cosh^{-1}\left(\frac{V_k - E}{2}\right) dx$$

and

$$\left|\text{Im}\cosh^{-1}\left(\frac{V(x)-E}{2}\right) - \text{Im}\cosh^{-1}\left(\frac{V_k-E}{2}\right)\right| \leq C_2 n^{\frac{1}{2}\alpha - \frac{1}{2}\frac{1-\nu}{\nu}}$$

for $x \in I_k^{(n)}$ and $|I_k^{(n)}| = n^\alpha$.

Then we have

$$\left|\frac{1}{\pi}\int_{a_k}^{b_k} \text{Im}\cosh^{-1}\left(\frac{V(x)-E}{2}\right)dx - \frac{b_k-a_k}{\pi}\text{Im}\cosh^{-1}\left(\frac{V_k-E}{2}\right)\right| \leq C_2$$

if we take $\alpha = \frac{1}{3}\frac{1-\nu}{\nu}$. Thus, like in section 2.3, by using the Dirichlet-Neumann bracketing technique, we have

$$\left|N(E, h(I_k^{(n)})) - \frac{1}{\pi}\int_{a_k}^{b_k} \text{Im}\cosh^{-1}\left(\frac{V(x)-E}{2}\right)dx\right| \leq C_3,$$

by summing over all $I_k^{(n)}$ in Ω_n, we have

$$\left|N(E, h(\Omega_n)) - \frac{1}{\pi}\int_{\Omega_n} \text{Im}\cosh^{-1}\left(\frac{V(x)-E}{2}\right)dx\right| \leq C_4\frac{|\Omega_n|}{|I_k^{(n)}|} \leq C_5 n^{\frac{2}{3}\frac{1-\nu}{\nu}},$$

by summing over n from 1 to l, we have

$$\left|N(E, h^{(l)}) - \frac{1}{\pi}\int_0^{S_\nu(l)} \text{Im}\cosh^{-1}\left(\frac{V(x)-E}{2}\right)dx\right| \leq C_6 l^{\frac{2}{3}\frac{1-\nu}{\nu}+1} \tag{4.6}$$

where all C_i are some constants.

Theorem 4.1. *The integrated density of states of (4.1) is given by*

$$k(E) = \frac{1}{2\pi^2}\int_{-\pi}^{\pi} \mathit{Im}\,\mathit{cosh}^{-1}\left(\frac{\lambda\cos x - E}{2}\right)dx. \tag{4.7}$$

Moreover, we have

$$|k^{(l)}(E) - k(E)| = O(l^{-\frac{1}{3}\frac{1-\nu}{\nu}}). \tag{4.8}$$

Proof. By using 4.6, the proof is similar to the proof of theorem 2.2. So, we will not give the detailed computation.

$$k^{(l)}(E) = \frac{1}{(S_\nu(l)+1)\pi}\left[\int_0^{S_\nu(l)} \text{Im}\cosh^{-1}\left(\frac{\lambda\cos x^\nu - E}{2}\right)dx + O(l^{\frac{2}{3}\frac{1-\nu}{\nu}+1})\right]$$

$$= \frac{1}{2\pi^2}\int_{-\pi}^{\pi} \text{Im}\cosh^{-1}\left(\frac{\lambda\cos x - E}{2}\right)dx + O(l^{-\frac{1}{3}\frac{1-\nu}{\nu}}).$$

Therefore,

$$k(E) = \lim_{l\to\infty} k^{(l)}(E) = \frac{1}{2\pi^2}\int_{-\pi}^{\pi} \operatorname{Im}\cosh^{-1}\left(\frac{\lambda\cos x - E}{2}\right) dx$$

and

$$|k^{(l)}(E) - k(E)| = O(l^{-\frac{1}{3}\frac{1-\nu}{\nu}}). \quad \Box$$

4.2 The Thouless Formula

For each fixed E, we consider the difference equation of second order

$$u(n+1) + u(n-1) + (V(n) - E)u(n) = 0 \tag{4.9}$$

where $V(n) = \lambda \cos n^\nu$ with $|\lambda| < 2$ and $\nu \in (0,1)$.

For convenience, let $u_n = u(n)$, $V_n = V(n)$ and define

$$\bar{u}(n) = \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix}, \qquad A_n(E) = \begin{pmatrix} E - V_n & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.10}$$

Then a function $u(n)$ satisfies (4.9) if and only if it satisfies

$$\bar{u}(n+1) = A_{n+1}\bar{u}(n).$$

Let

$$T_{a,n}(E) = A_n(E)A_{n-1}(E)\dots A_{a+1}(E) \qquad (a, n \in \mathbb{Z}, \text{ and } a, n \geq 0), \tag{4.11}$$

then $\bar{u}(n) = T_{a,n}(E)\bar{u}(a)$ defines a solution of (4.9) with the initial condition $\bar{u}(a) = (u_{a+1}\, u_a)^t$. In particular, when $a = 0$, we use $T_n(E)$ to denote $T_{0,n}(E)$.

Definition. For a given E, if the limit $\gamma(E) = \lim_{l\to\infty} l^{-1} \ln \|T_l(E)\|$ exist, then we say that $h(\nu, \lambda)$ which is given by (4.1) has Lyapunov behavior at E, and $\gamma(E)$ is called the Lyapunov exponent.

By (4.10) and (4.11), $T_n(E)$ can be defined inductively,

$$T_1(E) = A_1(E), \qquad T_n(E) = \begin{pmatrix} E - V_n & -1 \\ 1 & 0 \end{pmatrix} T_{n-1}(E) \qquad (n \geq 2).$$

Thus, by induction argument, it's easy to show that

$$T_n(E) = \begin{pmatrix} P_n(E) & Q_{n-1}(E) \\ P_{n-1}(E) & Q_{n-2}(E) \end{pmatrix} \qquad (n \geq 2)$$

where P_k and Q_k are polynomials of the degree k in the form $P_k(E) = E^k + \cdots$; and $Q_k(E) = -E^k + \cdots$.

Notice that

$$u(n+1) = P_n(E)u(1) + Q_{n-1}(E)u(0). \tag{4.12}$$

So, it's easy to see that $P_n(E) = 0$ if and only if $h(\nu, \lambda)hu = Eu$ has a solution with $u(0) = u(n+1) = 0$; and $Q_n(E) = 0$ if and only if $h(\nu, \lambda)u = Eu$ has a solution with $u(1) = u(n+2) = 0$. Therefore,

$$P_n(E) = \prod_{k=1}^{n} (E - E_k^{(n)}),$$
$$Q_n(E) = \prod_{k=1}^{n} (E - \tilde{E}_k^{(n)})$$

where $E_k^{(n)}$ (resp. $\tilde{E}_k^{(n)}$) are the eigenvalues of operator $h(\nu, \lambda)$ with boundary condition $u(0) = u(n+1) = 0$ (resp. u(1) = u(n+2) =0).

Recall that $S_\nu(n)$ is the integral part of $(2\pi n)^{\frac{1}{\nu}}$. So, we have just described the $E_k^{(S_\nu(n))}$ and $\tilde{E}_k^{(S_\nu(n))}$. Now, we can define

Definition. The resonance set, R_ν, for $h(\nu, \lambda)$ is defined by $R_\nu = R_\nu^{(1)} \cup R_\nu^{(2)}$, where

$$\begin{aligned} R_\nu^{(1)} &= \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \bigcup_{k} \{E \in [-4, 4] \mid |E - E_k^{(S_\nu(n))}| < \exp(-n^{\frac{1}{4}\frac{1-\nu}{\nu}})\}, \\ R_\nu^{(2)} &= \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \bigcup_{k} \{E \in [-4, 4] \mid |E - \tilde{E}_k^{(S_\nu(n))}| < \exp(-n^{\frac{1}{4}\frac{1-\nu}{\nu}})\}. \end{aligned} \tag{4.13}$$

Lemma 4.1. *For each $\nu \in (0,1)$, we have $|R_\nu| = \dim R_\nu = 0$.*

Proof. See the proof of theorem 3.1 in chapter 3. □

Like in the Schrödinger operator case, we will prove that the Lyapunov exponent exists outside the resonance set. First, we will show that the Thouless formula is true outside the resonance set.

From our definition of $k^{(l)}(E)$, we know that

$$dk^{(l)}(E) = \frac{1}{S_\nu(l)} \sum_i \delta(E - E_i^{(S_\nu(l))})\, dE.$$

Define

$$f_l(E) \stackrel{\text{def}}{=} \frac{1}{S_\nu(l)} \ln |P_{S_\nu(l)}(E)| = \int \ln |E - E'| dk^{(l)}(E').$$

Lemma 4.2. *Let $f(E) = \int \ln |E - E'|\, dk(E')$. Then there exists a subsequence $\{l_i\}$ with $l_i = i^{n_0}$ (n_0 is a fixed constant) such that for each $E \notin R_\nu$, we have*

$$f_{l_i}(E) \to f(E) \qquad \text{as } i \to \infty.$$

Proof. Let $E \notin R_\nu$ be fixed It's enough to that

$$\sum_{i=1}^{\infty} |f_{l_{i+1}}(E) - f_{l_i}(E)| < \infty. \tag{4.14}$$

Let

$$I_n(E) = \left[E - \frac{1}{3}\exp(-n^{\frac{1}{4}}), E + \frac{1}{3}\exp(-n^{\frac{1}{4}})\right].$$

Notice that $dk^{(l)}(E)$ (for all l) have compact support, say, $K = [-2 - |\gamma|, 2 + |\gamma|]$, and as$E \notin R_\nu$, $k^{(l_{i+1})}(E)$, $k^{(l_i)}(E)$ are constant on the interval $I_{l_{i+1}}(E)$. Thus,

$$
\begin{aligned}
f_{l_{i+1}}(E) - f_{l_i}(E) &= \int_{K\setminus I_{l_{i+1}}(E)} \ln|E-E'| d(k^{(l_{i+1})} - k^{(l_i)})(E') \\
&= \ln \delta_{i+1}\left[k^{(l_{i+1})}(E+\delta_{i+1}) - k^{(l_{i+1})}(E-\delta_{i+1})\right] \\
&\quad - \ln \delta_{i+1}\left[k^{(l_i)}(E+\delta_{i+1}) - k^{(l_i)}(E-\delta_{i+1})\right] \\
&\quad + \int_{K\setminus I_{l_{i+1}}(E)} \frac{(k^{(l_{i+1})} - k^{(l_i)})(E')}{E-E'}\, dE' \\
&\stackrel{\text{def}}{=} J_1 + J_2 + J_3
\end{aligned}
$$

where $\delta_{i+1} = \frac{1}{3}\exp(-l_{i+1}^{\frac{1}{4}\frac{1-\nu}{\nu}})$.

By theorem 4.1, it's easy to see that

$$
J_1 = O(l_{i+1}^{-\frac{1}{12}\frac{1-\nu}{\nu}}), \qquad \text{and } J_2 = O(l_{i+1}^{\frac{1}{4}\frac{1-\nu}{\nu}} l_i^{-\frac{1}{3}\frac{1-\nu}{\nu}}). \tag{4.15}
$$

As $(k^{(l_{i+1})} - k^{(l_i)})(E') = O(l_i^{-\frac{1}{3}\frac{1-\nu}{\nu}})$,

$$
|J_3| \le O(l_i^{-\frac{1}{3}\frac{1-\nu}{\nu}}) \int_{K\setminus I_{l_{i+1}}(E)} \frac{1}{|E-E'|}\, dE' = O(l_{i+1}^{\frac{1}{4}\frac{1-\nu}{\nu}} l_i^{-\frac{1}{3}\frac{1-\nu}{\nu}}). \tag{4.16}
$$

Therefore, by (4.15) and (4.16),

$$
|f_{l_{i+1}}(E) - f_{l_i}(E)| = O(l_{i+1}^{-\frac{1}{12}\frac{1-\nu}{\nu}}) + O(l_{i+1}^{\frac{1}{4}\frac{1-\nu}{\nu}} l_i^{-\frac{1}{3}\frac{1-\nu}{\nu}}).
$$

Thus, for example, if we take $l_i = i^{\frac{36\nu}{1-\nu}}$, then we have

$$
|f_{l_{i+1}}(E) - f_{l_i}(E)| = O(i^{-3}).
$$

Therefore, (4.14) follows. □

Now, since the above argument also works if we use $\tilde{E}_k^{(l)}$ instead of $E_k^{(l)}$. Therefore, by lemma 4.2, we have in fact proved that

Lemma 4.2′. *Let* $f(E) = \int \ln|E - E'|\,dk(E')$ *and* $l_i = i^{\frac{36\nu}{1-\nu}}$. *Then for each* $E \notin R_\nu$*, we have*

$$\lim_{i\to\infty} \frac{1}{S_\nu(l_i)} \ln\|T_{l_i}(E)\| = f(E). \tag{4.17}$$

Lemma 4.3. *Let* $T_n(E)$ *be defined by (4.10) and (4.11). Then for each fixed* E*, there exists a constant* $c > 0$ *such that*

$$e^{-c|m|} \le \frac{\|T_{a,n+m}(E)\|}{\|T_{a,n}(E)\|} \le e^{c|m|} \qquad (a \in \mathbb{Z}, a \ge 0). \tag{4.18}$$

Proof. By definition, $T_{a,n}(E) = A_n(E)A_{n-1}(E)\cdots A_{a+1}(E)$. Notice that for all $k \ge 0$, $A_k^{-1}(E)$ exists. It is easy to show that

$$\begin{aligned}\|A_{n+m}^{-1}(E)A_{n+m-1}^{-1}(E)\cdots A_{n+1}^{-1}\|^{-1} &\le \frac{\|T_{a,n+m}(E)\|}{\|T_{a,n}(E)\|}\\ &\le \|A_{n+m}(E)A_{n+m-1}(E)\cdots A_{n+1}\|.\end{aligned}$$

Now, since V_n is bounded, both $\|A_k(E)\|$ and $\|A_k^{-1}(E)\|$ are bounded for all $k \ge 0$. That is, there exists a constant $c > 0$, such that $\|A_k(E)\| \le e^c$ and $\|A_k^{-1}\| \le e^c$ for all $k \ge 0$. Thus, (4.18) follows. □

Now, we are ready to give one of our main results in this section.

Theorem 4.2 (Thouless formula). *Let* $h(\nu,\lambda)$ *be the Jacobi matrix operator defined by (4.1) and* R_ν *be the resonance set defined by (4.13). Then for* $E \notin R_\nu$*, we have*

$$\gamma(E) = \int \ln|E - E'|\,dk(E') \tag{4.19}$$

where $\gamma(E)$ *is the Lyapunov exponent and* $k(E)$ *is the integrated density of states for* $h(\nu,\lambda)$.

Proof. Let's fix $E \notin R_\nu$. By lemma 4.2′, we have a subsequence $l_i = i^{\frac{36\nu}{1-\nu}}$ such that (4.17) is true.

Now, let $n_i = S_\nu(l_i) = [(2\pi)^{\frac{1}{\nu}} i^{\frac{36}{1-\nu}}]$. Again, the $[\cdot]$ means the integral part. For any m with $n_i \le m < n_{i+1}$, by lemma 4.3, we have

$$e^{-c(n_{i+1}-n_i)} \le e^{-c(m-n_i)} \le \frac{\|T_m(E)\|}{\|T_{n_i}(E)\|} \le e^{c(n_{i+1}-n_i)}.$$

Therefore,

$$\left|\frac{\ln \|T_m(E)\|}{m} - \frac{\ln \|T_{n_i}\|}{n_i}\frac{n_i}{m}\right| \le c\frac{n_{i+1}-n_i}{n_i}.$$

Notice that by our choice of n_i, we have

$$\frac{n_{i+1}-n_i}{n_i} \to 0, \qquad \frac{n_i}{m} \to 1 \qquad \text{as } i \to \infty.$$

Therefore,

$$\lim_{m\to\infty} \frac{\ln \|T_m(E)\|}{m} = \lim_{i\to\infty} \frac{\ln \|T_{n_i}\|}{n_i}.$$

Together with lemma 4.2′, we have proved the Thouless formula. □

4.3 The Lyapunov Exponent

Using the explicit formula for integrated density of states and the Thouless formula, we are ready to derive the explicit formula for the Lyapunov exponent. As we will need to compute some integrals, we first quote some results from integral tables.

Lemma 4.4. *We have the following integral formulae*

$$\int_0^\pi \ln(a + b\cos x)\, dx = \pi \ln \frac{a+\sqrt{a^2-b^2}}{2} \qquad (a \ge |b| > 0), \tag{4.20}$$

$$\int_0^{\frac{\pi}{2}} \ln |a^2 - \sin^2 x| dx = \int_0^{\frac{\pi}{2}} \ln |a^2 - \cos^2 x| dx = -\pi \ln 2 \quad (a^2 \le 1). \tag{4.21}$$

Proof. Check from integral tables. For example, see [11, pp527-528]. □

Theorem 4.3. *Let $h(\nu,\lambda)$ be the Jacobi matrix operator defined by (4.1) and R_ν be the resonance set defined by (4.13). Then for $E \notin R_\nu$, $h(\nu,\lambda)$ has Lyapunov behavior with Lyapunov exponent given by*

$$\gamma(E) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Re\, cosh^{-1}\left(\frac{E-\lambda\cos x}{2}\right) dx. \tag{4.22}$$

Proof. Let $f(x) \stackrel{\text{def}}{=} \lambda\cos x$. First, by the formula for integrated density of states, (4.7), we know that

$$dk(E) = \begin{cases} \frac{dE}{2\pi^2}\int_{-\pi}^{\pi} \mathrm{Re}\left[1-\left(\frac{f(x)-E}{2}\right)^2\right]^{-\frac{1}{2}} dx, & \text{for } |f(x)-E| \le 2, \\ 0, & \text{for } |f(x)-E| > 2. \end{cases}$$

Therefore, by Thouless formula,

$$\begin{aligned} \gamma(E) &= \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-2+f(x)}^{2+f(x)} \ln|E-E'| \left[1-\left(\frac{f(x)-E}{2}\right)^2\right]^{-\frac{1}{2}} dE'dx \\ &= \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-1}^{1} \ln|E-f(x)+2y|\, d\arccos y\, dx \qquad (E' = f(x)-2y) \\ &= \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{0}^{\pi} \ln|E-f(x)+2\cos\theta|\, d\theta\, dx \qquad (\theta = \arccos y). \end{aligned} \tag{4.23}$$

Now, for each fixed $x \in [-\pi,\pi]$, we need to compute the integral

$$\int_{-\pi}^{\pi}\int_{0}^{\pi} \ln|E-f(x)+2\cos\theta|\, d\theta \stackrel{\text{def}}{=} K(x). \tag{4.24}$$

For a fixed x, let's denote $a = E - f(x)$.

(1) If $|a| \ge 2$, then by the integral formula (4.20), we have

$$K(x) = \int_0^{\pi} \ln|a-2\cos x|\, dx = \pi\ln\frac{|a|+\sqrt{a^2-4}}{2}.$$

(2) If $|a| < 2$, then we will formula (4.21).

$$K(x) = \int_0^\pi \ln 2\, d\theta + \int_0^{\frac{\pi}{2}} \ln\left|\frac{a}{2} - \cos\theta\right| d\theta + \int_{\frac{\pi}{2}}^{\pi} \ln\left|\frac{a}{2} - \cos\theta\right| d\theta$$

$$= \pi \ln 2 + \int_0^{\frac{\pi}{2}} \ln\left|\frac{a}{2} - \cos\theta\right| d\theta + \int_0^{\frac{\pi}{2}} \ln\left|\frac{a}{2} + \cos\theta\right| d\theta$$

$$= \pi \ln 2 + \int_0^{\frac{\pi}{2}} \ln\left|\frac{a^2}{4} - \cos^2\theta\right| d\theta$$

$$= \pi \ln 2 - \pi \ln 2 = 0.$$

Therefore,

$$K(x) = \begin{cases} \pi \ln \frac{|E-\lambda\cos x| + \sqrt{(E-\lambda\cos x)^2 - 4}}{2}, & \text{for } |E - \lambda\cos x| \geq 2 \\ 0, & \text{for } |E - \lambda\cos x| < 2 \end{cases}$$

$$= \pi \mathrm{Re} \cosh^{-1}\left(\frac{E - \lambda\cos x}{2}\right). \tag{4.25}$$

Combine (4.23)-(4.25), we have proved

$$\gamma(E) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{Re}\cosh^{-1}\left(\frac{E - \lambda\cos x}{2}\right) dx. \quad \Box$$

4.4 Some Spectral Consequences

The Jacobi matrix operators are discrete analogs of Schrödinger operators. Essentially, the same argument for Schrödinger operators is also valid for Jacobi matrices. So, we will not give the detailed proof in this section.

Recall that the Jacobi matrix operator we discussed is

$$h(\nu, \lambda) = h_0 + V(n), \qquad V(n) = \lambda \cos(n^\nu) \tag{4.26}$$

with $0 < \nu < 1$ and $|\lambda| < 2$.

Let δ_0 be the vector in ℓ^2 which is given by

$$\delta_0(n) = \delta_{n0}.$$

Let P be the projection on δ_0: $P = (\delta_0, \cdot)\delta_0$. Fix V and let A be the Jacobi matrix associated to $V - \delta_{\cdot 0}(V(0))$. Then

$$h(\nu, \lambda) = A + V(0)P.$$

So, the variations of V are precisely rank one perturbations. Therefore, by applying the general rank one perturbation theory (see [20, 21]) and using the various known facts (lemma 3.9, lemma 4.1, theorem 4.3 and etc.), we can show that

Theorem 4.4. *Let $h(\nu, \lambda)$ be the Jacobi matrix operator given by (4.26). Then for almost all $|\lambda| < 2$ (with respect to Lebesgue measure), $h(\nu, \lambda)$ has dense pure point spectrum on $(-2 - |\lambda|, -2 + |\lambda|)$ and $(2 - |\lambda|, 2 + |\lambda|)$, and the eigenfunctions to all the eigenvalues decay like $e^{-\gamma(E)n}$ at ∞, where $\gamma(E)$ is given (4.22).*

Theorem 4.5. *Let $h(\nu, \lambda)$ be the Jacobi matrix operator given by (4.26). Then $(d\mu)_{sc}$, the singular continuous part of the spectral measure $d\mu$, is supported on a Hausdorff dimension zero set.*

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