THEOREMS CONCERNING THE REDUCTION OF DRAG FOR SUPERSONIC AIRCRAFT

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ABSTRACT

In this report the problem of drag reduction or minimization for aircraft in supersonic flight is investigated within the framework of linearized theory of inviscid flow for fixed flight Mach number.

General theorems applicable to complete aircraft configurations are developed. These theorems state defining properties of distributions of thickness and normal force restricted to a particular region of the aircraft configurations that minimize the drag of the complete aircraft under the condition that the distribution of thickness and normal force on the remainder of the aircraft is specified in advance. These optimum distributions are further required to satisfy some of the various types of constraints that are commonly specified for aircraft.

The problem of finding the optimum distribution of thickness, lift and sideforce on a slender body of revolution is also studied under the assumption that the body carries no total lift or sideforce and can be represented by placing the distributions along the body axis. The case for which the Mach envelope of the body does not include all of the remaining portion of the aircraft configuration, upon which the thickness and normal force are specified, is solved. This solution covers the previously known case for which the Mach envelope of the body includes the entire aircraft.

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I. INTRODUCTION

1. Statement of Problem.

A crucial problem in the design of supersonic aircraft is that of achieving a relatively low drag, consistent with other design requirements of the aircraft. In recent years much theoretical research has been devoted to this problem. Most of this research has been done within the frame work of linearized theory of inviscid flow, and has dealt with specialized problems such as that of finding the minimum drag of a wing whose planform and lift are prescribed or the minimum drag of a fuselage whose length and volume are prescribed (see examples below).

The present report is an extension and generalization of previous research. Our aim is to develop general theorems applicable to complete aircraft configurations rather than to isolated components such as wings and fuselages. It will be assumed that certain characteristics of an aircraft are specified in advance. Examples of such specified characteristics are:

a. The aircraft configuration. By an aircraft configuration is meant a definite spatial arrangement of wings of fixed planform and bodies of fixed length.

b. The flight Mach number.

c. A set of constraints. Constraints are, for example: total lift, total fuselage volume, and maximum wing thickness at the mid-chord as a function of span.

d. The geometry, or the lift distribution, of a part of the airplane may be completely specified.

The problem is then to find an optimum geometrical shape of the aircraft, i.e. a shape which makes the drag a minimum.

We shall retain the assumptions of linearized theory of inviscid flow*. The aircraft may then be represented by a distribution of singularities (volume elements, lifting elements, etc., see below). A distribution of singularities that gives the minimum drag, consistent with the specified characteristics of the aircraft, will be called an optimum distribution. The geometry of the aircraft is then easily computed from the distribution of singularities; hence, if an optimum distribution of singularities has been found, the problem of finding the optimum geometry may be considered solved.

This report is concerned mainly with general theorems regarding optimum distributions. Examples will be given to show how ~~explicit~~ these theorems may be used to determine optimum distributions explicitly in special cases.

2. Assumptions of Linearized Theory.

Steady, frictionless, isentropic, supersonic flow past the aircraft is assumed. The velocity, density, and pressure are denoted by $\vec{q}$, p, and ρ, respectively. These quantities at upstream infinity (in fact, everywhere up stream of the region of influence of the aircraft) are denoted $\vec{q}_0 = U\vec{i}$, p_0, and ρ_0.

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* When the inviscid drag is known, one may make crude estimates of the viscous drag based on the concept of wetted area. Such estimates are useful, say, in comparing wings of different planforms. This problem will, however, not be considered here.

Perturbation quantities

$$\vec{q}' = \vec{q} - \vec{q}_o = \vec{q} - U\vec{i} = (u,v,w)$$
$$p' = p - p_o \qquad (2.1)$$
$$\rho' = \rho - \rho_o$$

are introduced.

The slopes of the surfaces of wings and bodies are assumed to be sufficiently small so that the perturbation quantities are well represented by linearized theory. A potential $\phi(x,y,z)$ is defined such that

$$\vec{q} = (u,v,w) = \text{grad } \phi \qquad (2.2)$$

where ϕ satisfies the wave equation

$$-\beta^2 \phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \qquad (2.3)$$

The perturbation pressure as given by linearized theory is

$$p' = -\rho_o Uu \qquad (2.4)$$

Boundary conditions are applied upon mean surfaces in the case of wings and in the case of bodies upon cylindrical surfaces or on a mean line near the body axis. In all cases the mean surfaces, cylinders and lines upon which boundary conditions are applied, are assumed to be parallel to the free stream direction.

3. Some Typical Problems of Drag Minimization.

The theorems to be derived here are of quite general nature. To aid in the understanding of the concepts that underlie these theorems, examples of more specialized problems which already have been studied extensively (see references below) are given.

1. Given a thin plane wing of fixed planform, total lift L, and freestream Mach number M, find the twist and camber that will cause this lift to be distributed so as to produce minimum drag.
2. Given the fixed planform and Mach number of problem (1) and the total wing volume V find the thickness distribution that produces minimum drag at zero lift.
3. Given a body of fixed length c, total volume V and Mach number M, find the distribution of cross-sectional area $t(x)$ that produces minimum drag.

The first two problems concern wings that are treated as planar systems. The general problem of planar wings of fixed planform has been studied in references (1) to (6). Numerous other studies have been made for particular planform shapes.

The mean surface of the wing is taken to be a plane, e.g. the x,y-plane. If this is done then it can be shown that drag due to lift can be separated from drag due to thickness, i.e. there is no interaction. This fact is not true in general for non-planar systems.

The twist and camber is described by a function $\alpha(x,y)$ defined as the angle that the middle surface of the wing makes with the free stream direction. $\alpha(x,y)$ is referred to as the angle of attack distribution and, within linearized theory, bears the following relation to the upwash

$$\alpha(x,y) = -\frac{w(x,y)}{U} \tag{3.1}$$

The wing thickness, defined as the difference in z-coordinate of the upper and lower wing surface, is described by the function $t(x,y)$. α and t are defined over the region S in the x,y-plane referred to as the wing planform, as shown in Fig. 3-1

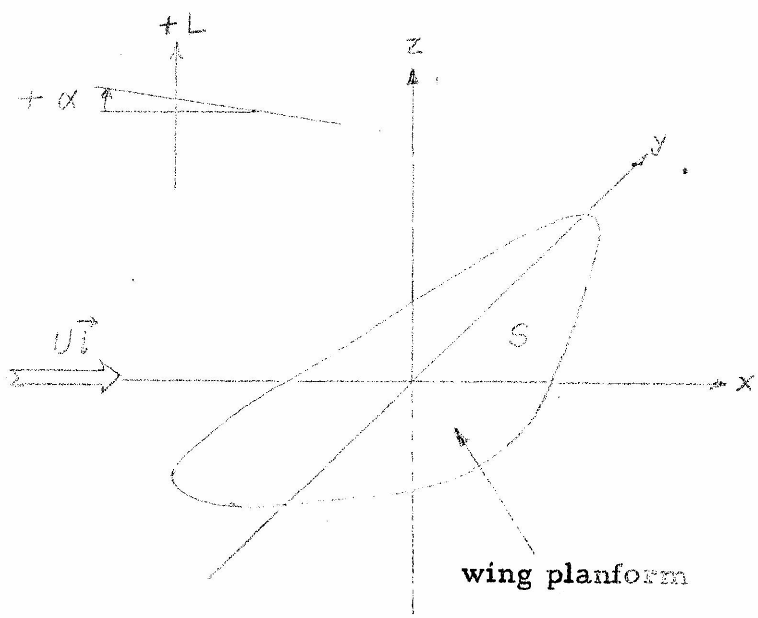


Fig. 3-1.

The local lift distribution, $\ell(x,y)$, is related to $\alpha(x,y)$ by the formulas of linearized theory. Similarly the linearized perturbation pressure $p' = -\rho_0 Uu$ is computed by linear theory from $t(x,y)$.

The total or net lift and wing volume are given by

$$L = \iint_S \ell(x,y)\, dx\, dy \tag{3.2}$$

$$V = \iint_S t(x,y)\, dx\, dy \tag{3.3}$$

The drags due to lift and thickness, as has been mentioned, are separable for the planar wing and are given by the formulas

$$D(\text{due to lift}) = \iint_S \alpha \, \ell \, dx \, dy = -\iint_S \frac{w}{U} \, \ell \, dx \, dy \qquad (3.4)$$

$$D(\text{due to thickness}) = \iint_S p' \frac{\partial t}{\partial x} \, dx \, dy$$

$$= -\iint_S \rho_o \, Uu \frac{\partial t}{\partial x} \, dx \, dy \qquad (3.5)$$

The total inviscid drag is thus given by the sum of equations 3.4 and 3.5.

Because of the linearity of the formulas superposition can be used. Thus, for example, if α_1 and α_2 are two angles-of-attack distributions, and if ℓ_1 and ℓ_2 are the corresponding lift distributions, then

$$\alpha = \alpha_1 + \alpha_2$$

implies

$$\ell = \ell_1 + \ell_2$$

and vice versa. The lift and drag are given by

$$L = \iint_S \ell_1 \, dx \, dy + \iint_S \ell_2 \, dx \, dy = L_1 + L_2$$

$$D = \iint_S (\alpha_1 + \alpha_2)(\ell_1 + \ell_2) \, dx \, dy$$

$$= \iint_S \alpha_1 \ell_1 \, dx \, dy + \iint_S (\alpha_1 \ell_2 + \alpha_2 \ell_1) \, dx \, dy +$$

$$+ \iint_S \alpha_2 \ell_2 \, dx \, dy$$

$$= D_1 + D_{12} + D_2 \, .$$

The middle term in the drag expression is called the interference drag and half of its value is symbolically represented by

$$(\ell_1, \ell_2) = \frac{1}{2} D_{12} = \frac{1}{2} \iint_S (\alpha_1 \ell_2 + \alpha_2 \ell_1) \, dx \, dy$$

$$= - \frac{1}{2} \iint_S \left[\frac{w_1}{U} \ell_2 + \frac{w_2}{U} \ell_1\right] dx \, dy \qquad (3.7)$$

The utility of this notation becomes apparent when several distributions are considered simultaneously. It is readily seen that

$$(\ell_1, \ell_2) = (\ell_2, \ell_1)$$

$$D_1 = (\ell_1, \ell_1), \quad D_2 = (\ell_2, \ell_2) \qquad (3.8)$$

$$D = (\ell_1 + \ell_2, \ell_1 + \ell_2) = (\ell_1, \ell_1) + 2(\ell_1, \ell_2) + (\ell_2, \ell_2)$$

In reference 3, the concept of orthogonality was introduced. Two lift distributions ℓ_1 and ℓ_2 were defined to be orthogonal if their interference drag is zero, i.e.

$$(\ell_1, \ell_2) = 0 \; .$$

The drag of two orthogonal distributions is then given simply by the sum of their individual drags, i.e.

$$D = D_1 + D_2 \; .$$

Similar relations exist between thickness distributions on the wing. The interference drag between two thickness distributions t_1 and t_2 is denoted by $2(t_1, t_2)$, where

$$(t_1, t_2) = - \frac{1}{2} \iint_S \rho_o U \left[u_1 \frac{\partial t_1}{\partial x} + u_2 \frac{\partial t_1}{\partial x}\right] dx \, dy \quad (3.9)$$

In references (1) and (2), R.T. Jones developed criteria that optimum distributions must satisfy for planar wings of fixed planform. These criteria are generalized in the theorems of this report so that they can be applied to a complete aircraft as well as to isolated wings or bodies. This will be done in Chapter III.

The criteria of references (1) and (2) were found useful in formulating integral equations for optimum distributions. These equations, however, do not lend themselves easily to mathematical analysis and hence have been solved in the exact sense for only a very limited number of planform shapes. Among these are the elliptical planforms discussed by Jones (ref. 2). It will be shown in Chapter IV how, in the particular case of singularities distributed along a streamwise line, an integral equation formulated from the general theorems of Chapter III can be solved to obtain an optimum distribution.

Integral methods have been developed (references 3, 4, 5, 6) to obtain approximate solutions that have drag values very near the minimum. The methods have been applied to planar wings of fixed planform for the lifting case and readily extend to cover the thickness case. These methods essentially consist of finding optimum linear combinations of a finite set of distributions for which the interference drag between individual distributions in the set can be calculated. No doubt these methods can be extended further with the aid of the theorems of Chapter III to cover the case of a complete aircraft.

II. PRELIMINARY DISCUSSION AND FORMULAS

4. Basic Singularities Used in the Linearized Representation of an Aircraft.

Certain basic solutions of the wave equation are utilized in the linearized description of the flow about an aircraft. The potential representing a source in supersonic flow is such a basic solution of the wave equation. Other basic solutions can be built up by applying certain limiting procedures to a superposition of sources. These resulting solutions are the doublet, horseshoe vortex, line vortex, etc.

Some of the basic solutions are described briefly in this section. For derivations and more detailed discussion see references (7) and (8).

Source.

A basic solution of equation 2.3 is the potential of a supersonic source, i.e.

$$\phi_s = \begin{cases} \dfrac{-1}{2\pi\left\{(x-\xi)^2-\beta^2[(y-\eta)^2+(z-\zeta)^2]\right\}^{1/2}} & \text{for } x-\xi \geq [(y-\eta)^2+(z-\zeta)^2]^{1/2} \\ 0 & \text{elsewhere} \end{cases} \tag{4.1}$$

It can be shown (ref. 7) that equation 4.1 represents a unit volume flow from the point (ξ, η, ζ).

Doublet.

The three-dimensional doublet or dipole, is a second basic solution of the wave equation and is obtained by allowing a source and sink of equal magnitude to approach each other, while the product of the source strength and the distance between the source and sink remains constant. The axis of the doublet is defined as the vector extending from the center of the source to the center of the sink. The doublet is thus a vector quantity and the above derivation is equivalent to taking the directional derivative of the source potential in the direction of the axis of the doublet. Thus

$$\begin{aligned} \phi_D &= (\text{grad } \phi_S) \cdot (a\vec{i} + b\vec{j} + c\vec{k}) \\ &= \frac{ax - \beta^2(by + cz)}{2\pi(x^2 - \beta^2 r^2)^{3/2}} \qquad x \geq \beta r = \beta\,[y^2 + z^2]^{1/2} \end{aligned} \tag{4.2}$$

is the potential of a unit doublet at the origin with the direction cosines of its axis given by (a,b,c).

Horseshoe vortex.

The potential of a third basic singularity, called the horseshoe vortex, is the potential due to a semi-infinite streamwise line of doublets of constant strength with their axes normal to the free stream and all pointing in the same direction, say $b\vec{j} + c\vec{k}$. Equivalently this potential can be obtained by differentiating the potential due to a semi-infinite streamwise line of sources in the direction $b\vec{j} + c\vec{k}$. Thus

$$\phi_{HSV} = [\text{grad} \int_0^{x-\beta r} \frac{d}{2\pi[(x-\xi)^2-\beta^2 r^2]^{1/2}}] \cdot (b\vec{j} + c\vec{k})$$

$$= \frac{x(by + cz)}{2\pi r^2[x^2-\beta^2 r^2]^{1/2}} \qquad x \geq \beta r \qquad (4.3)$$

represents a horseshoe vortex of unit strength ($\Gamma = 1$) located at the origin. The associated force is normal to the x-axis and has magnitude $\rho_o U$ and direction $b\vec{j} + c\vec{k}$.

If equation 4.3 is divided by $\rho_o U$, the potential for a unit "force element" is obtained. Thus if $b = 0$, $c = 1$, then

$$\phi_L = \frac{xz}{2\pi \rho_o U r^2[x^2-\beta^2 r^2]^{1/2}} \qquad x \geq \beta r \qquad (4.4)$$

is the potential of a unit <u>lifting element</u> located at the origin. Similarly, the potential of a unit <u>side force element</u> is

$$\phi_{SF} = \frac{xy}{2\pi \rho_o U r^2[x^2-\beta^2 r^2]^{1/2}} \qquad x \geq \beta r \qquad (4.5)$$

<u>Volume element</u>.

Another useful singularity is the "volume element". The potential of a unit volume element is the free stream velocity U times the potential of a unit dipole, with axis in the free stream direction (source upstream from the sink). To show this, consider the original construction of the dipole. Let

σ = volume of fluid per unit time flowing from source and absorbed by the sink, and

d = distance between source and sink.

Then the length of time that it takes for the fluid to go from the source to the sink is $\frac{d}{U}$ since the fluid travels at a velocity U. Therefore the volume occupied by the fluid is

$$\text{Vol} = \frac{\sigma d}{U} = \frac{\text{strength of dipole}}{U}$$

Thus the potential of a unit volume element is U times the potential of a unit dipole, that is

$$\phi_V = \frac{Ux}{2\pi(x^2-\beta^2 r^2)^{3/2}} \qquad x \geq \beta r \qquad (4.6)$$

5. Distributions of Singularities Representing Real Aircraft.

In treating problems of a more general nature than those given as examples in Section 3, it is convenient to consider an aircraft as a distribution of singularities.

The flow field around the wing of fixed planform described in Section 3 can be calculated by distributing lift and volume element singularities over a region S (the wing planform) in the x,y-plane. If the density of the lift and volume element distributions is set equal to the local lift and thickness intensities $\ell(x,y)$ and $t(x,y)$ on the wing, then the wing is represented in the mathematical sense by the singularity distributions.

Similarly, one may choose a non-planar surface S parallel to free stream direction. If suitable singularities (representing thickness and forces normal to S) are distributed over S one obtains a mathematical representation of flow past a non-planar wing having S as its mean surface.

The flow about a slender body of revolution can be described by placing a distribution of volume elements along the axis (a free streamline) of the body. The density of this linear distribution of volume elements is equated to the cross-sectional area $t(x)$ of the body. If the nose of the body is at the origin then the potential is given by

$$\begin{aligned} \phi &= \frac{U}{2\pi}\int_{o}^{x-\beta r} \frac{t(\xi)\,(x-\xi)\,d\xi}{[(x-\xi)^2-\beta^2 r^2]^{3/2}} \\ &= \frac{U}{2\pi}\,\frac{t(\xi)}{[(x-\xi)^2-\beta^2 r^2]^{1/2}}\Bigg|_{o}^{x-\beta r} \\ &\qquad - \frac{V}{2\pi}\int_{o}^{x-\beta r} \frac{t'(\xi)d\xi}{[(x-\xi)^2-\beta^2 r^2]^{1/2}} \end{aligned} \tag{5.1}$$

where the prime stands for differentiation in the x-direction.

If the body is smoothly faired then the infinite part of equation 5.1, corresponding to surface roughness (ref. 8), can be discarded. The remaining finite part

$$\phi = \frac{-U}{2\pi}\int_{o}^{x-\beta r} \frac{t'(\xi)\,d\xi}{[(x-\xi)^2-\beta^2 r^2]^{1/2}} \tag{5.2}$$

is the potential resulting from a lineal source distribution of strength $Ut'(x)$.

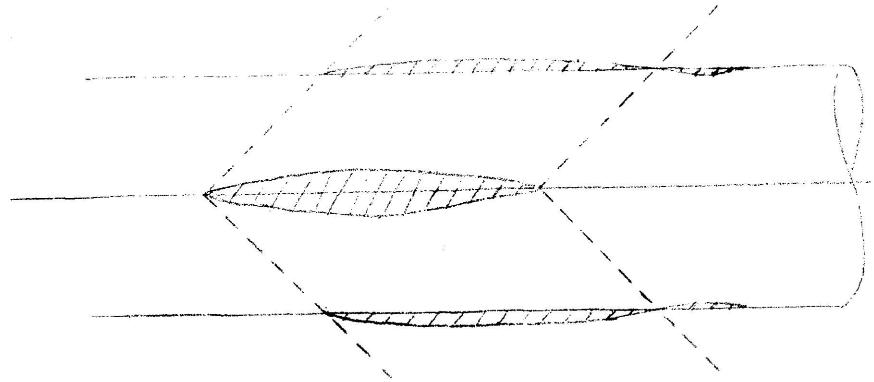
The potential due to a cylinder extending from $-\infty$ to $+\infty$ is zero since its cross-sectional area is a constant. If the body begins or ends in an infinite cylinder and slender body theory is used, then the length of the volume element distribution $t(x)$ will be semi-infinite, while the length of the source distribution $Ut'(x)$ will extend over a finite interval. In this case the value of t will be defined at the beginning and at the end of the interval over which $t'(x)$ is given. The following convention will be adopted: if $t'(x)$ is defined over an interval $a \leq x \leq b$ then

$$t(x) = \int_a^x t'(x)\,dx$$

unless otherwise stated.

A distribution of volume elements on the axis can also represent a "bump" on a cylinder, and "exact" linear theory used to compute the shape of the bump (ref. 8). It turns out (ref. 9) that the volume in the bump is just equal to the equivalent closed slender body, i.e. the sum of the volume elements (see fig. 5.1). The shape of the bump approaches the shape of the slender body as the radius of the cylinder approaches zero.

Negative volume elements can be interpreted as depressions on a cylinder and can be used in a superposition procedure as long as the resulting body or wing is real.



Volume in bump = Volume in slender body

Fig. 5-1.

6. Idealized Aircraft.

In formulating and proving general theorems and results it has been found convenient not to restrict oneself to singularity distributions representing conventional aircraft. Thus, instead of restricting the regions over which singularities are distributed to lines and surfaces, one may define the singularities in spatial regions. The resulting configuration of singularities is called an "idealized aircraft".

Low drag idealized aircraft may sometimes be approximated and sometimes realized in the exact sense by arrangements of real wings and bodies and hence may point the way to obtaining a practical low drag configuration. Of course, viscous drag must be taken into account in any final evaluation of the resulting (possibly unconventional) aircraft.

Furthermore, if a real aircraft can be enclosed in a three-dimensional region for which an optimum distribution of lift, sideforce, and volume element singularities can be found having the same total lift and volume as the real aircraft, then a general principle of minimum drag states that the minimum inviscid drag of the singularity distribution is less than any inviscid drag value that the actual aircraft can obtain. Thus for a given real configuration one may be able to compute useful lower bounds for the drag.

Finally, in formulating general theorems that can be applied to any aircraft, we do not know where the singularities might lie. Therefore we just state that they are defined in a certain region R which may be a three, two, or one-dimensional region depending on the particular application of the theorems.

For the rest of this report $t(x,y,z)$, $\ell(x,y,z)$ and $s(x,y,z)$ will be referred to simply as thickness, lift and sideforce distributions defined in a region R and shall be interpreted to mean singularity distributions defined in R. t will be said to be defined in R if R is the region containing all of the equivalent source singularities, $U \frac{\partial t}{\partial x}$. A distribution formed by the superposition of t, ℓ, and s will be denoted by

$$A(x,y,z) = t(x,y,z) + \ell(x,y,z) + s(x,y,z) \tag{6.1}$$

The "distant flow field" about a real aircraft can sometimes be represented by spatial singularity distributions.

This is done by means of "equivalent singularity distributions" some of which are discussed in reference 8.

Equivalent singularity distributions are defined as follows: two singularity distributions A_1 and A_2 are equivalent if and only if their respective flows are identical everywhere outside of some surface enclosing A_1 and A_2. This surface is usually the Mach envelope of a region within which A_1 and A_2 are defined. If A_1 and A_2 are equivalent, then the combined distribution $A_1 - A_2$ has zero net lift, zero net sideforce and zero drag. While the distant flow field produced by A_1 or A_2 is the same in the region outside of the enclosing surface, it may be quite different inside.

Thus, while the linearized flow field near the surfaces of a real aircraft is described by placing the singularity distributions on mean surfaces and mean lines, the distant flow field may be represented by an equivalent idealized aircraft.

7. Drag Formulas.

The non-viscous drag of wings and bodies as given by linearized theory may be obtained from two different points of view. First the drag can be evaluated by integrating the local pressure times frontal area over the wing and body surfaces. Second, the drag can be evaluated from momentum considerations involving the flow field produced at a great distance from the aircraft. It can be shown (ref. 8) that the drag as given by linearized theory is a quadratic expression in the linearized perturbation quantities.

Let u, v, and w be the perturbation velocities computed by linearized theory from the distribution $A(x,y,z)$ defined in a region R. The drag evaluated by the first method is

$$D = - \iiint_R [\rho_0 U u \frac{\partial t}{\partial x} + \frac{w}{U} \ell + \frac{v}{U} s] \, dx \, dy \, dz \qquad (7.1)$$

Each term has the following intuitive interpretation: the first term is the linearized pressure $-\rho_0 U u$ acting on the increments of frontal area $dt = \frac{\partial t}{\partial x} dx$; the second term is the downwash $(-w)$ times the strength of the local strength of the bound vortex $\Gamma = \frac{\ell}{\rho_0 U}$ times the free stream density ρ_0; the third term has a similar interpretation. Equation 7.1 is the generalization of equations 3.4 and 3.5, except that here the interference between lift, thickness and sideforce is accounted for.

Consider two distributions A_1 defined in R_1 and A_2 defined in R_2. The drag produced by the combined distribution $A_1 + A_2$ consists of the drag D_1 of A_1 plus the drag D_2 of A_2 plus an interference term D_{12} where $\frac{1}{2} D_{12}$ is denoted by the symbol (A_1, A_2). D_{12} is given by

$$D_{12} = 2(A_1, A_2) = - \iiint_{R_1} [\rho_0 U u_2 \frac{\partial t_1}{\partial x} + \frac{w_2}{U} \ell_1 + \frac{v_2}{U} s_1] \, dx \, dy \, dz$$

$$- \iiint_{R_2} [\ _0 U u_1 \frac{\partial t_2}{\partial x} + \frac{w_1}{U} \ell_2 + \frac{v_1}{U} s_2] \, dx \, dy \, dz \qquad (7.2)$$

It is evident from equation 7.2 that the interference drag is symmetric and bilinear, i.e.

$$(A_1, A_2) = (A_2, A_1) \tag{7.3a}$$

$$(A_1+A_2, A_3) = (A_1, A_3) + (A_2, A_3) = (A_3, A_1+A_2) \tag{7.3b}$$

The drag of A_1 or A_2 is then given by

$$\begin{aligned} D_1 &= (A_1, A_1) = D(A_1), \\ D_2 &= (A_2, A_2) = D(A_2) \end{aligned} \tag{7.3c}$$

Also (A_1, A_2) is communtative with scalars. Thus if <u>a</u> and <u>b</u> are any two scalars then

$$\begin{aligned} \text{Drag of } (aA_1+bA_2) &= D(aA_1+bA_2) = (aA_1+bA_2,\ aA_1+bA_2) \\ &= a^2(A_1, A_1) + 2ab(A_1, A_2) + b^2(A_2, A_2) \end{aligned} \tag{7.3d}$$

This symbolic notation is a direct generalization of equations 3.7 and 3.9. The relations 7.3a, b, c correspond to equations 3.8. (A_1, A_2) written out in full is

$$(A_1, A_2) = \left\{ \begin{array}{l} (t_1, t_2) + (t_1, \ _2) + (t_1, s_2) + \\ (\ _1, t_2) + (\ _1, \ _2) + (\ _1, s_2) + \\ (s_1, t_2) + (s_1, \ _2) + (s_1, s_2) \end{array} \right\}$$

The off-diagonal terms in this sum represent interference drag between the different types of distribution.

The method of Hayes (ref. 8,10) is an example of the second method of drag evaluation. Here, a large circular cylinder is employed as part of a control surface that surrounds the singularity distribution. The axis of the cylinder is chosen to be parallel to the free stream direction and to pass through or near the region R. The radius is made to become very large compared to the dimensions of R. Finally the cylindrical control surface is closed by a plane normal to the flow direction placed at a distance down stream from R that is much greater than the radius of the cylinder.

Wave drag is computed from the rate at which momentum in the free stream direction is carried across the cylindrical portion of the control surface, while vortex drag is computed from the rate at which momentum in the free stream direction is carried across the plane portion of the control surface.

Derivation of the method can be found in Hayes' original report (ref. 10) or in ref. (8). Only Hayes' construction and the formula for computing wave drag are given here.

The axis of the cylinder is taken as the x-axis and the polar angle $\theta = \tan^{-1} \frac{z}{y}$ is introduced. The region R is enclosed in a double Mach cone region R' shown in Fig. 7.1. The vertices of R' lie on the x-axis at the distances a and b as measured from some suitable origin.

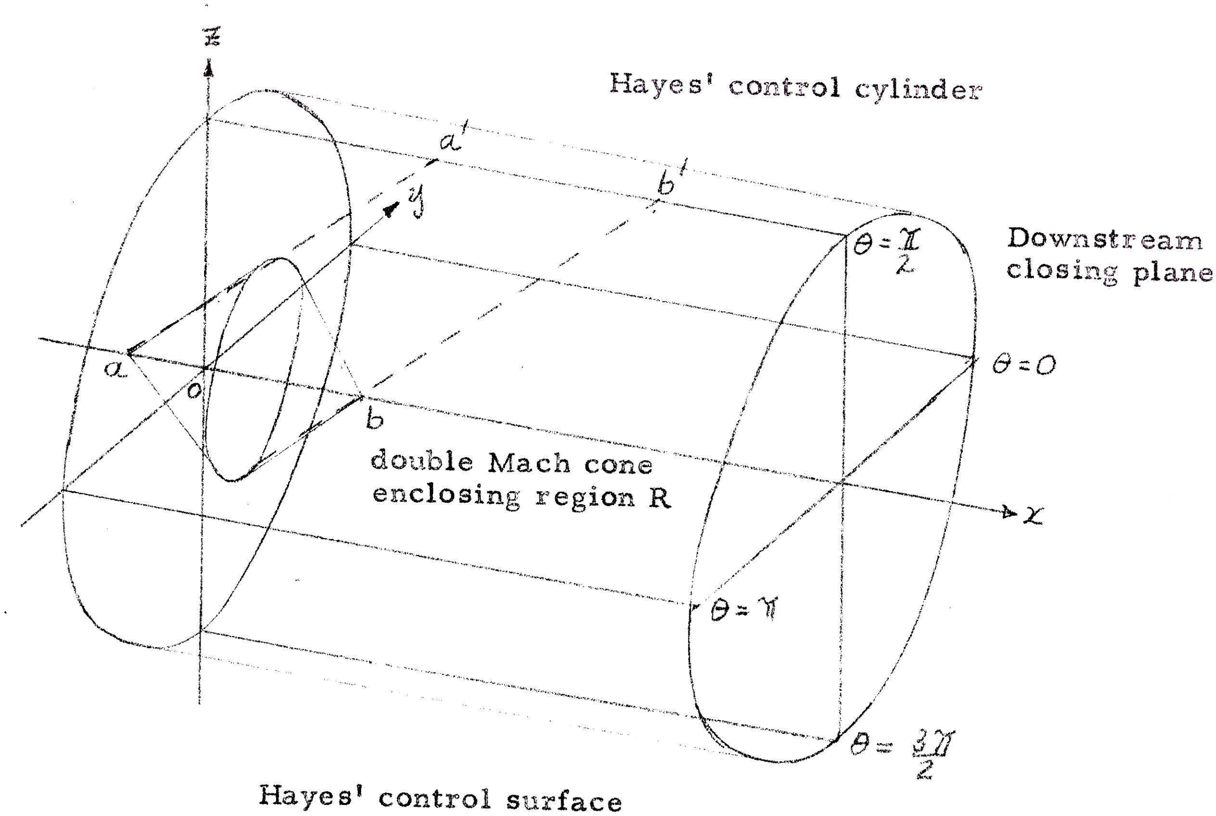
The equation

$$z \sin \theta + y \cos \theta = \frac{1}{\beta} (x - \xi) \qquad (7.4)$$

defines a two-parameter family of Mach planes for which ξ is the x-intercept. For fixed θ a set of parallel Mach planes is obtained. All of the singularities in the distribution $A(x,y,z)$ are displaced, in this set of parallel Mach planes, to the x-axis, where concentrated lineal distributions are formed

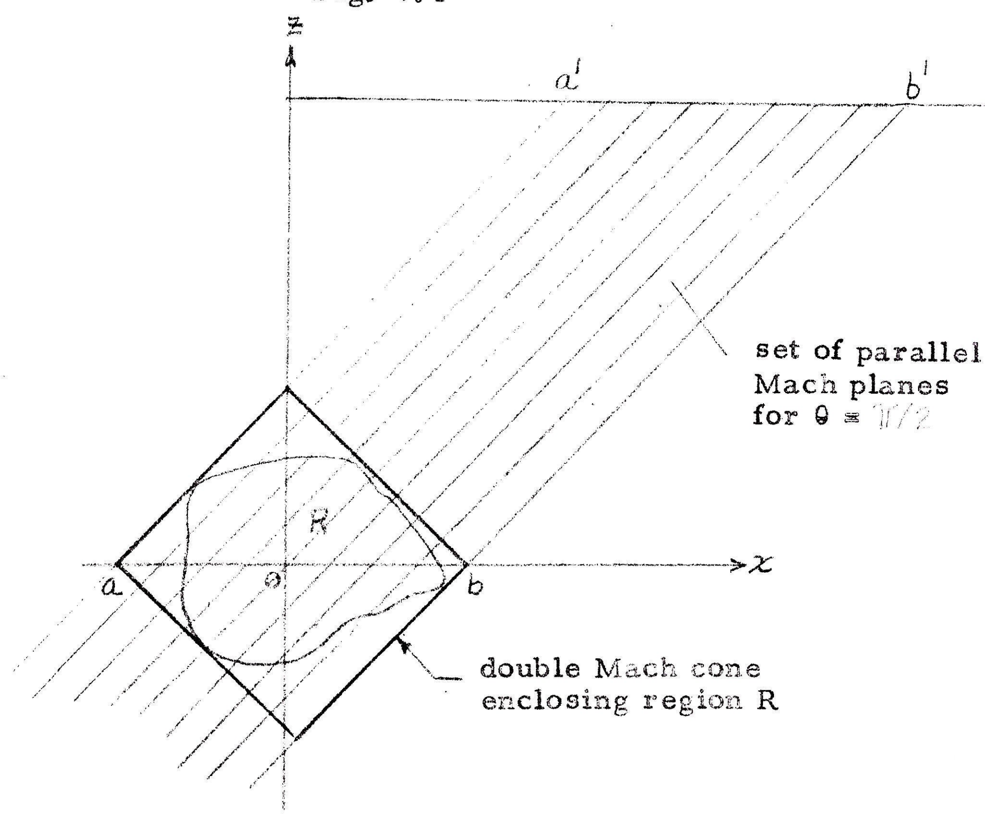
$$A(\xi, \theta) = t(\xi, \theta) + \ell(\xi, \theta) + s(\xi, \theta)$$

These lineal distributions are different in general for different values of θ.



z
Hayes' control cylinder
a'
b'
y
Downstream
closing plane
$\theta = \frac{\pi}{2}$
$\theta = 0$
a
o
b
double Mach cone
enclosing region R
x
$\theta = \pi$
$\theta = \frac{3\pi}{2}$
Hayes' control surface

Fig. 7.1



z
a'
b'
set of parallel
Mach planes
for $\theta = \pi/2$
R
x
a
o
b
double Mach cone
enclosing region R

Fig. 7.2

In computing the wave drag by Hayes' formula, the equivalent lineal source distribution for the angle θ is needed. This is given by

$$h(\xi,\theta) = Ut'(\xi,\theta) - \frac{\beta}{\rho_0 U}\left[\ell(\xi,\theta)\sin\theta + s(\xi,\theta)\cos\theta\right] \tag{7.5}$$

The contribution to the wave drag from the angle θ on Hayes' cylinder is given by Hayes' formula.

$$\frac{dD}{d\theta} = \frac{-\rho_0}{8\pi^2}\int_a^b h'(x,\theta)\,dx\int_a^b h'(\xi,\theta)\log|x-\xi|\,d\xi \tag{7.6}$$

The total wave drag is then given by

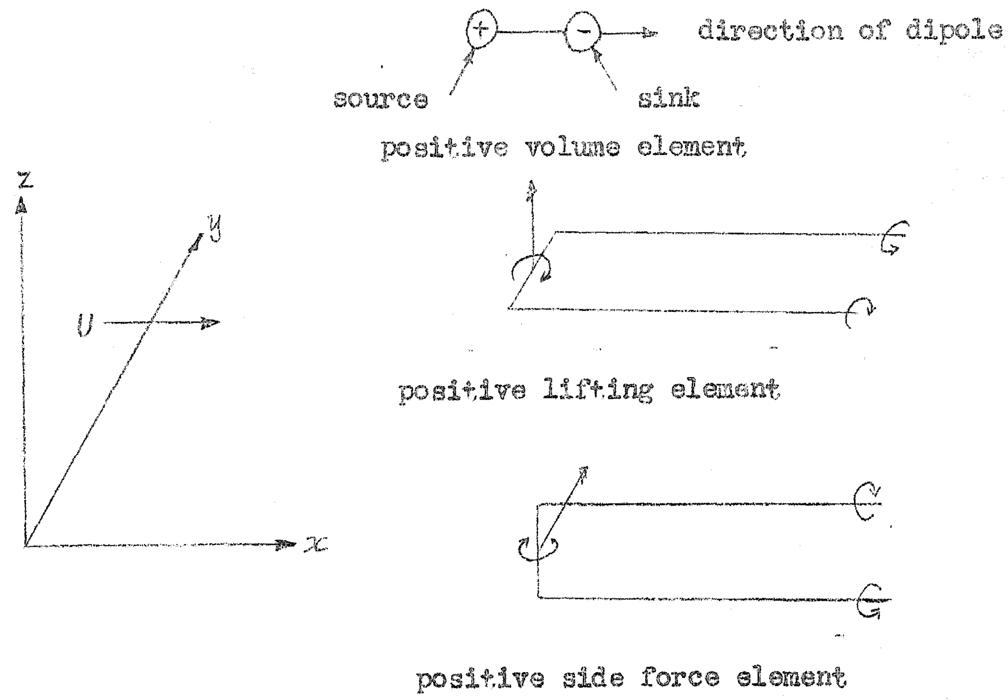
$$D_{wave} = \frac{-\rho_0}{8\pi^2}\int_\theta^{2\pi}\frac{dD}{d\theta}\,d\theta \tag{7.7}$$

8. General Reverse Flow Theorem and the Expression for Interference Drag in Terms of Combined Flow Velocities.

In the reversed flow field, produced by a distribution of thickness, lift and sideforce in the reversed flow, the free stream velocity vector is $q_o = -U\vec{i}$ and this is directed opposite to the free stream velocity vector in the forward flow $q_o = U\vec{i}$. The coordinate system and sign conventions used in expressing positive lift, sideforce, and velocity components in the forward flow are retained in the reversed flow. Drag, however, is the aerodynamic force in the appropriate free stream for either case.

Fig. 8-1 shows a positive volume element, lifting element, and sideforce element in forward and reverse flow. Note that in the positive volume element the source is upstream from the sink, and that vortices trail downstream from the bound vortex in either

FORWARD FLOW



REVERSED FLOW

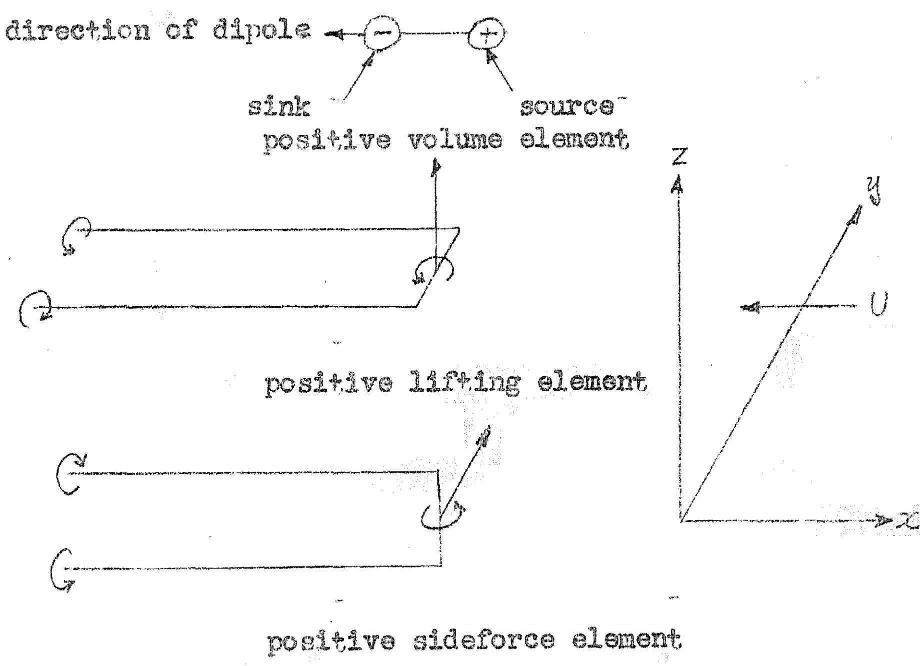


Fig. 8-1.

flow. Also note that the direction of vorticity in the bound vortex representing force elements is changed in the reversed flow in order to maintain the same positive direction of lift and side force as in the forward flow. The region of space influenced by the singularities is, in either case, the downstream Mach cone originating from the singularities.

Distributions and perturbation velocities considered in reversed flow will be denoted with the subscript r, while in the forward flow they will be denoted with the subscript f.

Consider the forward and reversed flow produced by the same distribution $A(x,y,z)$ of thickness, lift and sideforce, i.e.

$$A_f = A_r$$

The equivalent source distribution of the thickness t in A is $U \frac{\partial t}{\partial x}$ in the forward flow and $-U \frac{\partial t}{\partial x}$ in the reversed flow. Thus in the representation of thickness by sources and sinks, sources are replaced by sinks and sinks by sources when the flow is reversed on the same thickness distribution. This can be seen in the accompanying two-dimensional figure.

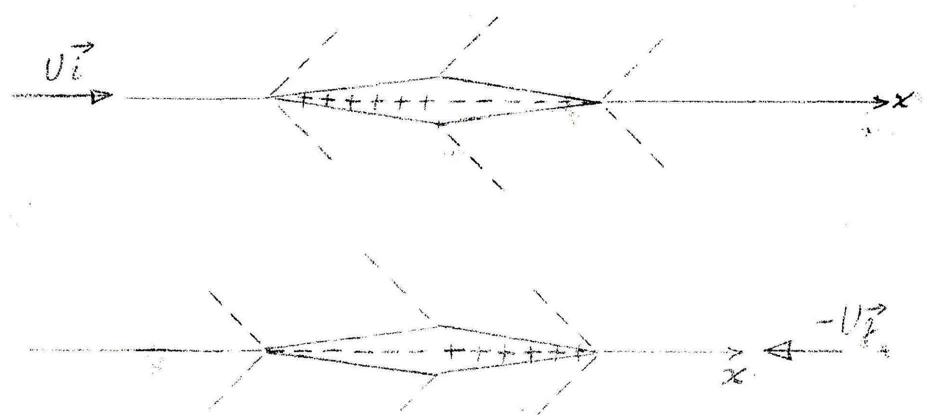


Fig. 8-2.

Let q_f and q_r be the respective velocity fields produced by the distribution A in the forward and reversed flow. The combined flow field is then defined as the average of $\vec{q}_f$ and $\vec{q}_r$. Quatntites in the combined flow will be denoted with a bar.

$$\bar{q} = \frac{1}{2}(q_f + q_r)$$

$$= \frac{1}{2}(Ui + q_f') + (-Ui + q_r')$$

$$= \frac{1}{2}(q_f' + q_r') = \bar{q}'$$

or

$$\bar{u} = \frac{1}{2}(u_f + u_r), \qquad \bar{v} = \frac{1}{2}(v_f + v_r), \qquad \bar{w} = \frac{1}{2}(w_f + w_r) \quad (8.1)$$

and in terms of the perturbation velocity potentials

$$\bar{q} = \text{grad}\, \bar{\phi}, \qquad \bar{\phi} = \frac{1}{2}(\phi_f + \phi_r), \qquad (8.2)$$

<u>General Reciprocity Theorem (ref. 11).</u>

If A_{1f} defined in R_1 and A_{2r} defined in R_2 are two distributions in forward and reversed flows respectively, and if u_{2f}, v_{2f}, w_{2f} are the perturbation velocities produced by A_{2f}, and u_{2r}, v_{2r}, w_{2r} are the perturbation velocities produced by A_{2r}, then

$$\underset{R_2}{}\left[\,{}_oUu_{1f}\,\frac{t_{2r}}{x} + \frac{w_{1f}}{U}\;{}_{2r} + \frac{v_{1f}}{U}\,s_{2r}\right] dx\,dy\,dz$$

$$= \underset{R_1}{}\left[\,{}_oUu_{2r}\,\frac{t_{1f}}{x} + \frac{w_{2r}}{U}\;{}_{1f} + \frac{v_{2r}}{U}\,s_{1f}\right] dx\,dy\,dz$$

(8.3)

With the aid of this theorem, the interference drag given by equation 7.2 can be expressed as either an integral over R_1 or an integral over R_2. Consider the first integral in equation 7.2. Let $A_1 = A_{1R}$ and $A_2 = A_{2F}$. Then by equation 8.3 one obtains

$$\iiint_{R_1} [\rho_o U u_2 \frac{\partial t_1}{\partial x} + \frac{w_2}{U} \ell_1 + \frac{v_2}{U} s_1]\, dx\, dy\, dz$$

$$= \iiint_{R_1} [\rho_o U u_{2f} \frac{\partial t_{1r}}{\partial x} + \frac{w_{2f}}{U} \ell_{1r} + \frac{v_{2f}}{U} s_{1r}]\, dx\, dy\, dz$$

$$= \iiint_{R_2} [\rho_o U u_{1r} \frac{\partial t_{2f}}{\partial x} + \frac{w_{1r}}{U} \ell_{2f} + \frac{v_{1r}}{U} s_{2f}]\, dx\, dy\, dz$$

$$= \iiint_{R_2} [\rho_o U u_{1r} \frac{\partial t_2}{\partial x} + \frac{w_{1r}}{U} \ell_2 + \frac{v_{1r}}{U} s_2]\, dx\, dy\, dz$$

Let A_1 be denoted by A_{1F} in the second integral of equation 7.2, then $u_1 = u_{1f}$, $w_1 = w_{1f}$, $v_1 = v_{1f}$. If equation 7.2 is divided by two one obtains

$$(A_1, A_2) = -\iiint_{R_2} [\rho_o U \frac{1}{2}(u_{1r} + u_{1f}) \frac{\partial t_2}{\partial x} + \frac{(w_{1r} + w_{1f})}{2U} \ell_2$$

$$+ \frac{(v_{1r} + v_{1f})}{2U} s_2]\, dx\, dy\, dz$$

In terms of the combined flow velocities

$$(A_1, A_2) = -\iiint_{R_2} [\rho_o U \bar{u}_1 \frac{\partial t_2}{\partial x} + \frac{\bar{w}_1}{U} \ell_2 + \frac{\bar{v}_1}{U} s_2]\, dx\, dy\, dz \qquad (8.4a)$$

$$= (A_2, A_1) = -\iiint_{R_1} [\ U \bar{u}_2 \frac{\partial t_1}{\partial x} + \frac{\bar{w}_2}{U} \ell_1 + \frac{\bar{v}_2}{U} s_1]\, dx\, dy\, dz \qquad (8.4b)$$

Equation 8.4b follows from equation 8.4a from the symmetry property of (A_1,A_2). The validity of Either R_1 or R_2, or both, are surface regions or portions of a curve, in which case the integrals become surface or line integrals.

III. GENERAL THEOREMS CONCERNING OPTIMUM DISTRIBUTIONS.

9. Formulation of the General Problem.

The three minimum drag problems given as examples in Section 3 can be formulated in terms of singularity distributions. In the first problem the distribution of lifting elements $\ell(x,y)$ defined on S that produces minimum drag for fixed total lift L is sought. The twist and camber is then determined from $\ell(x,y)$ by linearized theory. In the second problem the optimum volume element distribution $t(x,y)$, defined on S, is sought for fixed total volume V. In the third problem the optimum distribution of volume elements $t(x)$, defined on the body axis, is sought for fixed total volume.

These problems have three things in common: first, they all deal with singularity distributions defined in a fixed region; second, the constraints are scalar quantities that are linear functions of the intensities of the singularity distributions; third, only one type of singularity distribution is considered at a time, i.e. lift or thickness.

Problems of this type have the following formulation. Let A stand for either a lift, thickness, or sideforce distribution. Let R be the region in which A is defined. A linear operator W(A) is defined over R so as to produce a scalar quantity, e.g. total lift, total volume, etc. Linearity implies that if a is any scalar that multiplies every singularity in A, then

$$W(a\ A) = a\ W(A)\ .$$

$W(A)$ is then taken as the constraint and the problem is formulated:

Find the distribution A in R that makes the drag of A a mimimum for a fixed value of $W(A)$.

This problem can be generalized. Two distributions A_1 and A_2 are considered. The distributions A_1 is defined in a region R_1 and is held fixed while a distribution A_2 defined in a region R_2 is sought such that the drag of the combined distribution $A_1 + A_2$ is minimized under the condition that $W(A_2)$ is fixed. Stated formally:

Given a fixed region R_2 and a fixed distribution A_1 defined in R_1 find the distribution A_2 in R_2 that makes the drag of the combined distribution $A_1 + A_2$ a minimum for a fixed value of $W(A_2)$.

This general formulation is more useful when drag reduction for a complete aircraft is considered. Below are given a few of the reasons why this is so.

First, ease of fabrication and structural requirements may make it impossible to make drag reduction adjustments on thickness in some regions of aircraft. These regions may then be taken as the region R_1, and the regions where drag saving adjustments can be made as the region R_2.

Second, if the constraints are complicated in a region it may be advantageous from the mathematical point of view to break the region up into sub-regions where the constraints are simple. The region also may be complicated by re-entrant boundaries, etc. In this case the region is broken up into sub-regions with simple boundaries.

Third, when interference between different types of distributions is considered, it may be advantageous to make adjustment on one type of distribution in the region at a time, e.g. to adjust the thickness distribution while keeping the lift distribution fixed. In this case R_1 and R_2 may be the same region or one a sub-region of the other.

This generalized problem will be studied in the sections that follow.

10. The Fundamental Theorem of Drag Reduction.

In seeking the distributions A_2 that reduce the drag it may not be practical to find the absolute optimum distribution within the class of all possible distributions. In this event one may seek the relative optimum in a restricted class of distributions for which numerical computations can be made.

Examples of restricted classes of distributions are: conical distributions of lift on a delta wing; thickness distributions on a straight wing that are independent of span; all linear combinations of a finite set of distributions; all thickness distributions that vanish at the trailing edge of a wing, etc. A restricted class of distributions, C, will be defined as a set of distributions which is closed under linear combinations, i.e. if A' and A'' are any two members of C, then $\epsilon_1 A' + \epsilon_2 A''$ is a member of C where ϵ_1 and ϵ_2 are any two real scalars.

Let A_2 defined in R_2 be the relative optimum distribution in the class C for fixed $W(A_2)$ and for fixed A_1 in R_1. Since A_2 is optimum, it follows that the drag

$D(A_1 + A_2)$ cannot be improved by adding any distribution A_2' in C for which $W(A_2')$ is zero. Let ϵ be any scalar, then A_2' is in C, has zero W, and

$$D(A_1 + A_2 + \epsilon A_2') = (A_1 + A_2 + \epsilon A_2', A_1 + A_2 + \epsilon A_2')$$

$$= (A_1 + A_2, A_1 + A_2) + 2\epsilon(A_1 + A_2, A_2') + \epsilon^2(A_2', A_2')$$

$$\geq (A_1 + A_2, A_1 + A_2) = D(A_1 + A_2) = D_{opt.} \qquad (a)$$

The scalar ϵ in (a) can have any sign and magnitude. The inequality in (a) implies that

$$(A_1 + A_2, A_2') = 0 \qquad (10.1)$$

If equation 10.1 were not true one could always choose an sufficiently small such that

$$2\epsilon(A_1 + A_2, A_2') < 0 \qquad (b)$$

and

$$\epsilon^2(A_2', A_2') < |2\epsilon(A_1 + A_2, A_2')| \qquad (c)$$

Equations (b) and (c) and the inequality (a) are then contradictory. This proves equation 10.1.

Now let A_2 be an arbitrary distribution defined in R_2 and in the class C. The distribution

$$A_2' = \widetilde{A}_2 - \frac{W(\widetilde{A}_2)}{W(A_2)} A_2$$

has zero W and is in C. Hence, upon substitution into equation 10.1, one obtains

$$(A_1 + A_2, \widetilde{A}_2) = \frac{W(\widetilde{A}_2)}{W(A_2)} (A_1 + A_2, A_2) \qquad (10.2)$$

It has been proved that equation 10.2 is a necessary condition that A_2 must satisfy in order to be optimum for fixed W. If equation 10.2 holds for all $\tilde{A}_2$ in the class C then equation 10.2 is also sufficient.

Thus the necessary and sufficient condition that A_2 be an optimum in C for fixed W and A_1 is that equation 10.2 be satisfied for all $\tilde{A}_2$ in C.

11. General theorems concerning Optimum Thickness Distributions.

In formulating the theorems of this section certain restrictions are placed upon the regions where the optimum thickness distributions are defined.

Consider an arbitrary thickness distribution $t(x,y,z)$ in a region R. It shall be assumed that any infinite line $y = \text{constant}$, $z = \text{constant}$ has at most one connected line segment inside R. The upstream point where the line intersects the boundary of R will be called a leading point and its x-coordinate denoted by

$$x_L(y,z)$$

The downstream point of intersection will be called a trailing point and its x-coordinate denoted by

$$x_T(y,z)$$

The collection of all leading points will be called the leading boundary of R and the collection of all trailing points will be called the trailing boundary of R. In addition R may have cylindrical boundaries that are parallel to the x-axis. Unless otherwise stated it is assumed that $t(x,y,z)$ vanishes on the leading boundary of R, i.e.

$$t(x_L, y, z) = 0 \ .$$

Two quantities of interest in aircraft design that are linear functions of the singularities representing thickness are the total volume* V and the base area F. These are defined as follows:

$$V = V(t) = \iiint_R t(x,y,z)\ dx\ dy\ dz \tag{11.1}$$

$$F = F(t) = \iiint_R \frac{\partial t(x,y,z)}{\partial x}\ dx\ dy\ dz \tag{11.2}$$

Two fundamental theorems will now be proved concerning the necessary and sufficient condition that optimum thickness distributions must satisfy when either fixed volume or fixed base area is taken as the constraint.

Let W be the total volume and t_2 be an optimum for fixed V within the class of thickness distributions, defined in R_2, that vanish on the leading and trailing boundaries of R_2. If $\tilde{t}_2$ is an arbitrary member of this class then

$$\tilde{t}_2(x_L, y, z) = \tilde{t}_2(x_T, y, z) = 0 \tag{a}$$

and equations 10.2 and 11.1 yield

$$(A_1 + t_2, \tilde{t}_2) = \frac{(A_1 + t_2, t_2)}{V(t_2)} \iiint_{R_2} \tilde{t}_2\ dx\ dy\ dz \tag{b}$$

Now let $\bar{u}_1$ and $\bar{u}_2$ be combined flow perturbation velocities

---

* Note that the volume of t has no connection with the volume of the region R within which t is defined. $t(x,y,z)$ is a distribution of volume element singularities in a linearized representation of thickness. For example, if t is a distribution of volume element singularities on the mean surface of a wing, then the region of definition is two-dimensional and has zero volume, but the volume of the wing is given by the sum of the volume elements (cf. sections 5 and 6).

computed from A_1 and t_2 respectively. The left hand side of equation (b) may be expressed according to equation 8.4a.

$$-\iiint_{R_2} \rho_o U(\bar{u}_1+\bar{u}_2)\frac{\partial \tilde{t}_2}{\partial x}\, dx\, dy\, dz = \frac{(A_1+t_2, t_2)}{V(t_2)}\iiint_{R_2} \tilde{t}_2\, dx\, dy\, dz$$

Integration by parts and equation (a) then yield

$$\iiint_{R_2} \rho_o U \frac{\partial(\bar{u}_1+\bar{u}_2)}{\partial x}\tilde{t}_2\, dx\, dy\, dz = \frac{(A_1+t_2, t_2)}{V(t_2)}\iiint_{R_2} \tilde{t}_2\, dx\, dy\, dz \qquad (c)$$

Since (c) is valid for all $\tilde{t}_2$ it follows that

$$\frac{\partial(\bar{u}_1+\bar{u}_2)}{\partial x} = \frac{(A_1+t_2, t_2)}{\rho_o U V(t_2)} \quad \text{in } R_2 \qquad (11.3)$$

If A_1 is interpreted as the distribution of all singularities other than t_2, including lift and sidefore in R_2, then $\bar{u}_1 + \bar{u}_2$ is the x-component of the combined flow perturbation velocity produced by all singularity distributions in the flow field and $\bar{u}_1 + \bar{u}_2$ is simply denoted by $\bar{u}$. The term (A_1+t_2, t_2) can be expressed with the aid of equation 8.4a; equation 11.3 can then be expressed

$$\frac{\partial \bar{u}}{\partial x} = \frac{\partial(\bar{u}_1+\bar{u}_2)}{\partial x} = \frac{-\iiint_{R_2} \rho_o U\bar{u}\frac{\partial t_2}{\partial x}\, dx\, dy\, dz}{\rho_o U V(t_2)} \quad \text{in } R_2 \qquad (11.4)$$

The numerator in the right hand side of equation 11.4 can be interpreted as the drag produced by <u>all</u> combined flow pressures $(-\rho_o\ Uu)$ acting on the optimum thickness distribution t_2. These results are summarized in the first theorem.

<u>Theorem 1.</u> <u>The necessary and sufficient condition that</u> t_2 <u>is a thickness distribution in</u> R_2 <u>that minimizes</u> $D(A_1+t_2)$, <u>within the class of thickness distributions which vanish on the leading and trailing boundaries of</u> R_2, <u>for fixed</u> $V(t_2)$ <u>and fixed</u> A_1, <u>is that the gradient in the x-direction of</u> $(\bar{u}_1+\bar{u}_2)$ <u>be a constant in the region</u> R_2. <u>The value of this constant is given by equation 11.3. Further, if</u> A_1 <u>is the distribution of all thickness lift and sideforce excluding</u> t_2, <u>then the x-component of the gradient of</u> $\bar{u}$ <u>is constant in</u> R_2 <u>and has the value given by equation 11.4.</u>

Let W be the total base area and t_2 be an optimum in the class of thickness distributions in R_2 that vanish on leading boundaries of R_2. If $\tilde{t}_2$ is an arbitrary distribution in this class then equations 10.2, 11.2, and 8.7a yield

$$-\iiint_{R_2} \rho_o U(\bar{u}_1+\bar{u}_2) \frac{\partial \tilde{t}_2}{\partial x}\, dx\, dy\, dz = \frac{(A_1+t_2, t_2)}{F(t_2)} \iiint_{R_2} \frac{\partial \tilde{t}}{\partial x}\, dx\, dy\, dz \tag{d}$$

Since (d) is valid for <u>all</u> $\tilde{t}_2$ in the class it follows that

$$(\bar{u}_1+\bar{u}_2) = \frac{(A_1 + t_2, t_2)}{\rho_o U F(t_2)} \tag{11.5}$$

If A_1 includes all thickness lift and side force distributions other than t_2, then equation 11.7 has the expression

$$\bar{u} = (\bar{u}_1+\bar{u}_2) = \frac{-\iiint_{R_2} U\bar{u} \frac{\partial t}{\partial x}\, dx\, dy\, dz}{-\rho_o U F(t_2)} \tag{11.6}$$

The right hand side of equation 11.6 has an interpretation similar to that of equation 11.4. These results are summarized in the second theorem.

Theorem 2. The necessary and sufficient condition that t_2 is a thickness distribution in R_2 that minimizes $D(A_1+t_2)$, within the class of thickness distributions which vanish on the leading boundary of R_2, for fixed base area $F(t_2)$ and fixed A_1, is that $(\bar{u}_1+\bar{u}_2)$ be a constant in R_2. The value of this constant is given by equation 11.5. Further, if A_1 is the distribution of all thickness, lift, and sideforce excluding t_2, then $\bar{u}$ is constant in R_2 and has the value given by equation 11.6.

Theorems 1 and 2 will be used to prove three new theorems concerning optimum thickness distribution.

Volume and base area may also be prescribed as functions of y and z, e.g. the area of the airfoil sections of a wing prescribed as a function of wing span. Thus

$$V(y,z) = \int_{x_L}^{x_T} t(x,y,z)\,dx \qquad (11.7)$$

$$F(y,z) = \int_{x_L}^{x_T} \frac{\partial t(x,y,z)}{\partial x}\,dx = t(x_T,y,z) \qquad (11.8)$$

The total volume and base area are then given by

$$V = \iint_{\text{all } y,z \text{ in } R} V(y,z)\,dy\,dz$$

$$F = \iint_{\text{all } y,z \text{ in } R} F(y,z)\,dy\,dz = \iint_{\text{all } y,z \text{ in } R} t(x_T,y,z)\,dy\,dz$$

Consider the case where $V(y,z)$ is specified for the region R_2. Let t_2 be a thickness distribution that vanishes on the leading and trailing boundaries of R_2 and distributes $V(x,y)$ in the x-direction over R_2 for fixed A_1 such that $D(A_1+t_2)$

is minimized. Consider a cylinder with axis parallel to the x-axis and with cross-section dx dy. Let the coordinates of the center of this cylinder be located at $y = y_o$, $z = z_o$ such that y_o and z_o are in R_2 for some x. Denote the elementary region cut out of R_2 by this cylinder by

$$R_2(y_o,z_o) = \text{intersection of cylinder with } R_2$$

Denote the part of t_2 that lies in $R_2(y_o,z_o)$ by t_2', and the remainder, t_2-t_1, by t_2''. Let $\bar{u}_2'$ be produced by t_2', and $\bar{u}_2''$ produced by t_2''. Then

$$t_2 = t_2' + t_2'' \qquad (e)$$

$$\bar{u}_2 = \bar{u}_2' + \bar{u}_2'' \qquad (f)$$

By definition, t_2 is that distribution which distributes $V(y,z)$ optimally, and hence $D(A_1+t_2)$ cannot be reduced further by modifying t_2' in $R_2(y_o,z_o)$ for fixed

$$V(t') = V(y_o,z_o)\,dy\,dz \qquad (g)$$

and for fixed t_2'' and A_1. Furthermore t_2' vanishes on leading and trailing boundaries of $R(y_o,z_o)$. Therefore t_2' is optimum in $R(y_o,z_o)$ under the conditions of theorem 1. Equation 11.3 gives

$$\frac{\partial(\bar{u}_1+\bar{u}_2''+\bar{u}_2')}{\partial x} = \frac{(A_1+t_2'',t_2')}{\rho_o UV(t_2')} \quad \text{in } R(y_o,z_o) \qquad (h)$$

Equations (e), (f), (g), (h) and 8.3 yield

$$\frac{\partial(\bar{u}_1+\bar{u}_2)}{\partial x} = - \frac{\left[\int_{x_L}^{x_T} \rho_o U(\bar{u}_1+\bar{u}_2) \frac{\partial t'}{\partial x}\, dx\right]_{y_o, z_o} dy\, dz}{\rho_o U(y_o, z_o)\, dy\, dz}$$

$$\text{in} \quad R_2(y_o, z_o) \qquad \text{(i)}$$

Since y_o, z_o are arbitrary, equation (i) must hold throughout R_2. Removal of the subscripts on y_o and z_o and cancellation of the differential element $dy\, dz$ in equation (i) gives

$$\frac{\partial(\bar{u}_1+\bar{u}_2)}{\partial x} = \frac{-\int_{x_L}^{x_T} \rho_o U(\bar{u}_1+\bar{u}_2) \frac{\partial t_2}{\partial x}\, dx}{\rho_o UV(y,z)} \quad \text{in} \quad R_2 \qquad (11.7)$$

The right hand side of equation 11.7 is a function of y and z only and the numerator can be interpreted as the y,z-distribution of drag produced by the combined flow pressure $-\rho_o U(\bar{u}_1+\bar{u}_2)$ acting on the optimum distribution t_2. These results are summarized in theorem 3.

Theorem 3. The necessary and sufficient condition that t_2 is a thickness distribution that distributes $V(y,z)$ in the x-direction over R_2 so as to minimize $D(A_1+t_2)$ within the class of thickness distributions that vanish on the leading and trailing boundaries of R_2, for fixed A_1, is that the x-component of the gradient be a function of y and z only in R_2. The value of this function is given by equation 11.7.

A similar theorem can be derived for the case that $F(y,z)$ is specified. The proof of this theorem follows from theorem 2 and is similar to the proof of theorem 3.

<u>Theorem 4.</u> <u>The necessary and sufficient condition that t_2 is a thickness distribution in R_2, which vanishes on the leading boundary of R_2 and has the value $F(y,z)$ on the trailing boundary of R_2, that makes $D(A_1+t_2)$ a minimum for fixed A_1, is that $\bar{u}_1+\bar{u}_2$ be a function of y and z only in R_2. The value of this function is given by</u>

$$\bar{u}_1 + \bar{u}_2 = \frac{\int_{x_L}^{x_T} \rho_o U(\bar{u}_1+\bar{u}_2) \frac{\partial t_2}{\partial x} dx}{\rho_o U F_I(y,z)} \quad \text{in } R_2 \tag{11.8}$$

The integral on the right hand side of equation 11.8 has the same interpretation as the integral in equation 11.7. In both theorems 3 and 4 we can put $\bar{u} = \bar{u}_1+\bar{u}_2$ if all of the singularities are included in the distribution $A_1 + t_2$.

In many cases it is desirable to specify the value of the thickness as a function of y, z on some intermediate surface $x_I(y,z)$ and require the thickness to vanish on the leading and trailing boundaries of R_2. Thus x_I is defined such that for all y, z in R_2

$$x_L(y,z) \leq x_I(y,z) \leq x_T(y,z)$$

and one seeks the distribution t_2 that makes $D(A_1+t_2)$ minimum under the constraints

$$\begin{aligned} t_2(x_L,y,z) &= 0 \\ t_2(x_I,y,z) &= F_I(y,z) \qquad \text{(j)} \\ t_2(x_T,y,z) &= 0 \end{aligned}$$

This case is of practical importance in the design of thin wings for which volume is not the consideration, but in order to fulfill structural requirements, the distribution of thickness is prescribed along the span at some fraction of wing chord. The line along which the thickness is prescribed might correspond to the position of a tapered wing spar. The prescribed thickness would then be the depth of the spar plus the skin thickness. In this case F_I corresponds to frontal area.

Let t_2 be an optimum distribution in R_2 for fixed A_1 and for the constraints (j). Divide R_2 into two regions, R_L and R_T, such that $x = x_I$ forms the trailing boundary of R_L and the leading boundary of R_T. The optimum distribution can be decomposed into two distributions, t_L and t_T, such that

$$t_L = \begin{cases} t_2 & \text{in } R_L \\ F_I(y,z) & \text{in } R_T \end{cases} \tag{k}$$

$$t_T = \begin{cases} 0 & \text{in } R_L \\ t_2 - F_I(y,z) & \text{in } R_T \end{cases} \tag{l}$$

Then

$$t_2 = t_L + t_T \qquad \text{by (k) and (l)}$$

$$\bar{u}_2 = \bar{u}_L + \bar{u}_T$$

$$\left.\begin{aligned} t_L(x_L,y,z) &= 0 \\ t_L(x_I,y,z) &= F_I(y,z) \end{aligned}\right\} \text{ by (j) and (k)} \qquad \text{(m)}$$

$$\left.\begin{aligned} t_T(x_I,y,z) &= 0 \\ t_T(x_T,y,z) &= - F_I(y,z) \end{aligned}\right\} \text{ by (j) and (l)} \qquad \text{(n)}$$

Since t_2 is optimum, no improvement can be made by modifying t_L in R_L for fixed A_1 and t_T under the constraint (m). Thus t_L is optimum and theorem 4 can be applied to R_L yielding the result

$$\bar{u}_1+\bar{u}_T+u_L=\bar{u}_1+\bar{u}_2=\frac{\int_{x_L}^{x_I} \rho_o U(\bar{u}_1+\bar{u}_2)\frac{\partial t_2}{\partial x}\,dx}{\rho_o\, UF_I(y,z)} \quad \text{in } R_L \qquad (11.9a)$$

By the same argument and the constraints (n)

$$\bar{u}_1+\bar{u}_2 = -\frac{\int_{x_I}^{x_T} \rho_o U(\bar{u}_1+\bar{u}_2)\frac{\partial t_2}{\partial x}\,dx}{\rho_o\, UF_I(y,z)} \quad \text{in } R_T \qquad (11.9b)$$

The right hand sides of equations 11.9 a,b are functions of y and z only. However they are in general <u>different</u> functions in R_L and R_T respectively.

Theresults can be summarized in the following theorem.

<u>Theorem 5. The necessary and sufficient condition that</u> t_2 <u>is a thickness distribution in</u> R_2, <u>which vanishes on the leading and trailing boundaries of</u> R_2 <u>and has the prescribed</u>

value $F_I(y,z)$ on the intermediate surface $x_I(y,z)$, that makes $D(A_1+t_2)$ a minimum for fixed A_1, is that $\bar{u}_1+\bar{u}_2$ be a function y and z only in R_L and R_T respectively. These functions are, in general, different, and their values are given by equations 11.9a and 11.9b respectively.

12. General Theorems Concerning Optimum Normal Force Distributions.

This section deals with optimum distributions of normal force singularities used in the linearized description of the flow about wings.

As cited in section 7, the normal force elements are distributed on mean surfaces which are portions of free stream surfaces. Consider a point (x,y,z) on a mean surface. Denote the slope of the surface in the y,z-plane by $m(x,y,z)$, viz.

$$\frac{dz}{dy} = m(x,y,z) \tag{12.1}$$

For most mean surfaces $m(x,y,z)$ will be independent of x if real wings are represented. However, to cover cases where one surface contains different segments of the same free streamline, the x-variation of m will be retained. For example, for a particular y, z one may have

$$m(x,y,z) = \begin{cases} m_1(y,z) & \text{for} \quad a \leq x \leq b<c \\ \\ m_2(y,z) & \text{for} \quad c \leq x \leq d \end{cases}$$

At the point (x,y,z) on a mean surface, the lift and sideforce bear the relation

$$s(x,y,z) = - m(x,y,z)\,\ell(x,y,z) \tag{12.2}$$

so that the resulting force is normal to the mean surface. Thus if the lift is positive, the sideforce is negative provided m is positive, and positive if m is negative.

In formulating the theorems of this section, lift and sideforce singularities are not confined necessarily to mean surfaces but are generalized to spatial distributions defined in a region R. Thus $m(x,y,z)$ is a single valued function defined in R, and s and ℓ bear the relation 12.2. This is done in order to simplify the proofs, and the theorems are valid if R is taken as the mean surface of wings.

Two quantities of interest, which are linear functions of normal force distribution defined in R, are the total lift L and the total moment M. These are defined

$$L = L(\ell) = \iiint_R \ell(x,y,z)\, dx\, dy\, dz \tag{12.3}$$

$$M = M(\ell) = \iiint_R \ell(x,y,z)\,(x_o - x)\, dx\, dy\, dz \tag{12.4}$$

where x_o refers to the plane $x = x_o$ about which the moment is taken.

The general problem of this section is to find the necessary and sufficient conditions that a distribution of normal force $A_2 = \ell_2 + s_2$ in R_2 minimizes $D(A_1+A_2)$ under the relation 12.2 for fixed A_1, and $L(\ell_2)$ and for zero moment, $M(\ell_2)=0$. Thus not only is the total lift prescribed, but also the x-coordinate of the center of lift is prescribed.

The problem can be resolved into two problems. First, an optimum distribution A_{2L} in the restricted class C_1 of all lift and sideforce distributions satisfying 12.2 in R_2 is considered for fixed lift $L(\ell_{2L})$. A_{2L} must satisfy the integral form of the criterion given by equation 10.2, viz.

$$(A_1+A_{2L},\tilde{A}_2) = \frac{L(\tilde{\ell}_2)}{L(\ell_{2L})}(A_1+A_{2L},A_{2L}) \tag{a}$$

for all $\tilde{A}_2$ in the class C_1.

Second, A_{2L} is considered fixed along with A_1 and an optimum distribution A_{2M} for fixed moment $M(\ell_{2M})$ in the restricted class C_2 of all lift and sideforce distributions that satisfy equation 12.2 in R_2 and have zero net lift is considered. A_{2M} must satisfy the integrated criterion

$$(A_1+A_{2L}+A_{2M},A_2') = \frac{M(\ell_2')}{M(\ell_{2M})}(A_1+A_{2L}+A_{2M},A_{2M}) \tag{b}$$

for all A_2' in the class C_2

$$L(\ell_2') = 0, \qquad L(\ell_{2M}) = 0 \tag{c}$$

Equations (c) and 10.1 give

$$(A_1+A_{2L},A_2') = 0 \tag{d}$$

$$(A_1+A_{2L},A_{2M}) = 0 \tag{e}$$

Equation (b) can then be simplified with the aid of the orthogonality relations (d) and (e).

$$(A_{2M},A_2') = \frac{M(\ell_2')}{M(\ell_{2M})}(A_{2M},A_{2M}) \tag{f}$$

An arbitrary distribution A_2' of the class C_2 can be formed by considering an arbitrary distribution A_2 of the class C_1 and using the relation

$$A_2' = \tilde{A}_2 - \frac{L(\tilde{\ell}_2)}{L(\ell_{2L})} A_{2L} \qquad \text{(h)}$$

Then equations (f) and (h) yield

$$(A_{2M},\tilde{A}_2) = \frac{L(\tilde{\ell}_2)}{L(\ell_{2L})}(A_{2M},A_{2L}) + \left[\frac{M(\tilde{\ell}_2)}{M(\ell_{2M})} - \frac{L(\tilde{\ell}_2)}{L(\ell_{2L})}\frac{M(\ell_{2L})}{M(\ell_{2M})}\right](A_{2M},A_{2M}) \qquad \text{(i)}$$

Finally, we require

$$M(\ell_{2M}) = - M(\ell_{2L}) \qquad \text{(j)}$$

Equations (a), (i), and (j) combine to give

$$(A_1+A_{2L}+A_{2M},\tilde{A}_2) = \frac{(A_1+A_{2L}+A_{2M},A_{2L}+A_{2M})}{L(\ell_{2L})} L(\tilde{\ell}_2) + \frac{(A_{2M},A_{2M})}{M(\ell_{2M})} M(\tilde{\ell}_2) \qquad \text{(k)}$$

With the aid of equations (e) and (c), equation (k) can be expressed

$$(A_1+A_{2L}+A_{2M},\tilde{A}_2 - \frac{M(\tilde{\ell}_2)}{M(\ell_{2M})}A_{2M})$$
$$= \frac{(A_1+A_{2L}+A_{2M},A_{2L}+A_{2M})}{L(\ell_{2L}+\ell_{2M})} L(\tilde{\ell}_2 - \frac{M(\tilde{\ell}_2)}{M(\ell_{2M})}\ell_{2M}) \qquad \text{(l)}$$

The distribution $\tilde{A}_2 - \frac{M(\tilde{\ell}_2)}{M(\ell_{2M})} A_{2M}$ is an arbitrary member of a restricted class C_3 of all lift and sideforce distributions in

R_2 that satisfy equation 12.2 for which the x-coordinate of their centers of lift are all at $x = x_o$. Thus according to equation 10.2 the distribution

$$A_2 = A_{2L} + A_{2M} \qquad \text{(m)}$$

is an optimum in C_3 for fixed A_1 and $L(\ell_2)$.

The left hand sides of equations (a), (i) and (k) may be expressed according to equation 8.4a and 12.2, and the quantities $L(\tilde{\ell}_2)$ and $M(\tilde{\ell}_2)$ on the right hand sides, according to equations 12.3 and 12.4. Equation (a) then becomes

$$-\iiint_{R_2} \frac{1}{U} [(\bar{w}_1+\bar{w}_{2L}) - m(\bar{v}_1+\bar{v}_{2L})] \tilde{\ell}_2 \, dx\, dy\, dz$$
$$= \frac{(A_1+A_{2L}, A_{2L})}{L(\ell_{2L})} \iiint_{R_2} \tilde{\ell}_2 \, dx\, dy\, dz \qquad \text{(n)}$$

where $\bar{w}_1, \bar{v}_1$ is produced by A_1 and $\bar{w}_{2L}$, $\bar{v}_{2L}$ is produced by A_{2L}.*

Since $\tilde{A}_2$ is an arbitrary member of the class C_1 of <u>all</u> lift and sideforce distributions satisfying the relation 12.2, $\tilde{\ell}_2$ can be chosen arbitrarily. Thus for equation (n) to hold for all $\tilde{\ell}_2$,

$$(\bar{w}_1+\bar{w}_{2L}) - m(x,y,z)\,(\bar{v}_1+\bar{v}_{2L}) = -\frac{(A_1+A_{2L}, A_{2L})}{L(\ell_{2L})} \quad \text{in } R_2 \qquad (12.5)$$

Equation 12.5 is expressed in the following theorem.

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* Note that w_{2L} is produced by both ℓ_{2L} and s_{2L} in A_{2L}. Similarly remarks apply to v_{2L}.

Theorem 6. Let C_1 be the class of all lift and sideforce distributions that satisfy the relation 12.2 in R_2. Let A_{2L} be a lift and sideforce distribution in R_2 that minimizes $D(A_1+A_2)$ in the class C_1 for fixed A_1 and fixed total lift $L(\ell_{2L})$. The necessary and sufficient condition that A_{2L} is such an optimum is that $(\bar{w}_1+\bar{w}_{2L}) - m(\bar{v}_1+\bar{v}_{2L})$ be a constant in R_2. The value of this constant is given by equation 12.5.

Equations (i) and (k) can be expressed in forms similar to equation (n) and the following two theorems deduced.

Theorem 7. Let C_2 be the subclass of C_1 formed by taking all the distributions in C_1 that have zero net lift. Let A_{2L} be the optimum of theorem 6. Let A_{2M} be a lift and sideforce distribution that minimizes $D(A_1+A_{2L}+A_2)$ within the class C_2 for fixed A_1+A_{2L} and fixed total moment $M(\ell_2) = M(\ell_{2M})$. Then the optimum distribution A_{2M} is orthogonal to A_1+A_{2L} and must satisfy the necessary and sufficient condition that $\bar{w}_{2M} - m\bar{v}_{2M}$ be a linear function of x in R_2. This linear function is given by

$$\bar{w}_{2M} - m\bar{v}_{2M} = - \frac{(A_{2M},A_{2M})}{M(\ell_{2M})}\, x - \frac{(A_{2L},A_{2M})}{L(\ell_{2L})}\, \frac{M(\ell_{2L})}{M(\ell_{2M})}$$

$$- \frac{(A_{2M},A_{2M})}{M(\ell_{2M})} \left[\frac{M(\ell_{2L})}{L(\ell_{2L})} + x_o \right] \quad \text{in } R_2 \qquad (12.6)$$

Theorem 8. Let C_3 be the subclass of C_1 formed by taking all the distributions in C_1 that have zero net moment. The distribution that minimizes $D(A_1+A_2)$ in the class C_3 for fixed $L(\ell_2)$ and fixed A_1 is $A_2 = A_{2L}+A_{2M}$. A_{2L} is the optimum in theorem 6 and A_{2M} is the optimum in theorem 7, wherein we require that $L(\ell_{2L}) = L(\ell_2)$ and $M(\ell_{2M}) = -M(\ell_{2L})$, respectively. For A_2 to be this optimum, it is necessary and sufficient that $(\bar{w}_1+\bar{w}_2) - m(\bar{v}_1+\bar{v}_2)$ be a linear function of x. This linear function is given by

$$(\bar{w}_1+\bar{w}_2) - m(\bar{v}_1+\bar{v}_2) = \frac{(A_{2M},A_{2M})}{M(\ell_{2L})}(x_o-x) - \frac{(A_1+A_2,A_2)}{L(\ell_{2L})} \text{ in } R_2 \quad (12.7)$$

13. Discussions of Theorems.

The theorems of sections 11 and 12 state the defining conditions for optimum distributions in the presence of a fixed distribution. These defining conditions can be used to set up integral equations which in theory can be solved for A_2 in terms of the known quantities. For example, take Theorem 2. The condition that defines the optimum distribution t_2 is given by equation 11.5, which can be written,

$$\bar{u}_2 = -\bar{u}_1 - \frac{(A_1+t_2,t_2)}{UF(t_2)} \text{ in } R_2 \quad (a)$$

If the distribution A_1 is known then $\bar{u}_1$ can in principle be calculated utilizing the basic singularities of section 4. Similarly $\bar{u}_2$ can be expressed in terms of t_2. If $K(P,Q)$ is the x-component of the combined flow perturbation velocity at the point $P = (x,y,z)$ produced by a unit volume element located at $Q = (\xi,\eta,\zeta)$ then

$$\bar{u}_2(x,y,z) = \iiint\limits_{Q \text{ in } R_2} K(P,Q)\ t_2(Q)\ dQ \qquad (b)$$

where $dQ = d\xi\ d\eta\ d\zeta$. With the aid of equation (b) the integral equation (a) can be written

$$\iiint\limits_{Q \text{ in } R_2} K(P,Q)\ t_2(Q)\ dQ = -\bar{u}_1(P) - \frac{(A_1 + t_2, t_2)}{UF(t_2)}$$

$$\text{for all } P \text{ in } R_2 \qquad (13.1)$$

Since the integral equation is linear t_2 can be split into two distributions t_3 and t_4

$$t_2 = t_3 + Kt_4 \qquad (c)$$

such that

$$\iiint\limits_{Q \text{ in } R_2} K(P,Q)\ t_3(Q)\ dQ = -\bar{u}_1(P), \quad P \text{ in } R_2$$

$$\iiint\limits_{Q \text{ in } R_2} K(P,Q)\ t_4(Q)\ dQ = 1, \quad P \text{ in } R_2 \qquad (d)$$

The constant K is determined such that

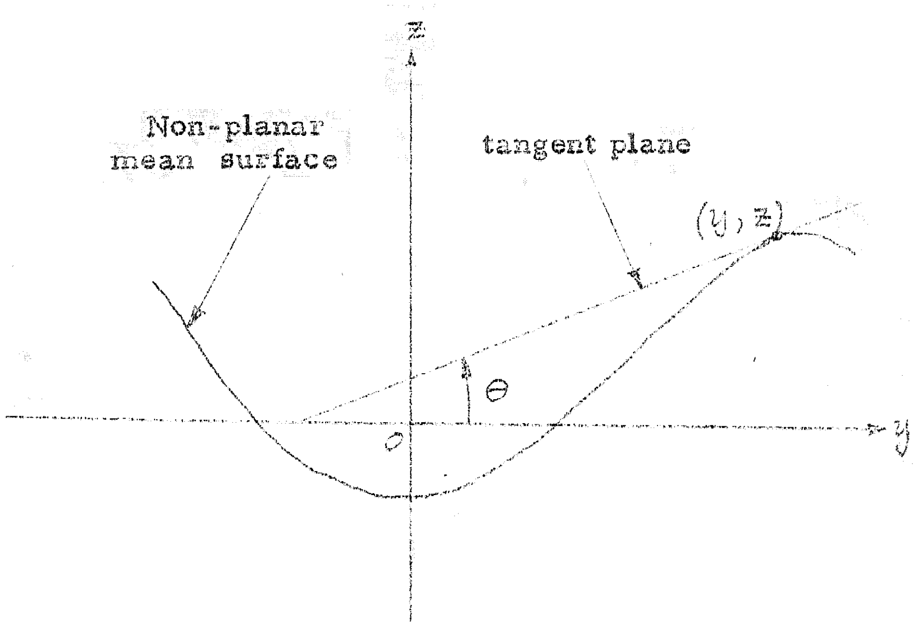
$$F(t_2) = F(t_3) + KF(t_4) \qquad (e)$$

Then t_2 is given by

$$t_2 = t_3 + \frac{F(t_2) - F(t_3)}{F(t_4)}\ t_4 \qquad (f)$$

In Chapter IV it will be shown how one may obtain explicit solutions of the integral equations if the optimum distribution A_2 is a distribution concentrated on a line parallel to the x-axis, i.e. in a lineal region R_2.

The theorems of section 12 have a special interpretation if the region R_2 is a portion S_2 of a free stream surface and if m is interpreted as the slope in the y, z-plane of this surface. Let $\theta(y,z)$ be the angle measured from the y-axis between the tangent plane of S_2 at the point y,z in the y,z-plane (see figure below).



Thus m is given by

$$m = m(x,y) = \tan \theta. \qquad (13.2)$$

If equations 12.5, 12.6, and 12.7 are multiplied by cos Θ then the resulting left hand sides are velocities normal to S_2 in the combined flow. The tangential velocity in the combined does not enter into the condition that A_2 be optimum. For example, theorem 6 states that the necessary and sufficient condition that the normal force A_2 be optimally distributed over S_2 for fixed A_1 and fixed lift is that the normal component of the combined flow velocity produced by A_1 and A_2 be a constant times $\cos\Theta(y,z)$, viz.

$$(\bar{w}_1+\bar{w}_{2L})\cos\Theta - (\bar{v}_1+\bar{v}_{2L})\sin\Theta = -U\,\frac{(A_1+A_{2L},A_{2L})}{L(\ell_{2L})}\cos\Theta \quad \text{on } S_2 \tag{13.4}$$

It is easily seen that at points where S_2 has a vertical tangent ($\Theta = \frac{\pi}{2}$) plane the normal velocity in the combined flow is $\bar{v}_1+\bar{v}_{2L}$ and must vanish if A_{2L} is optimum while $\bar{w}_1+\bar{w}_{2L}$ is not specified.

IV. OPTIMUM LINEAL DISTRIBUTIONS

14. Expression for Interference Drag Involving a Lineal Distribution.

Let $A_1(x,y,z) = t_1(x,y,z) + \ell_1(x,y,z) + s_1(x,y,z)$ be a distribution that has finite drag and is defined in a region R_1 which may be one, two, or three-dimensional. Let $\bar{u}_1(x)$, $\bar{v}_1(x)$ and $\bar{w}_1(x)$ be the combined flow perturbation velocities produced on the x-axis by A_1. Let $h_1(x,\theta)$ be the equivalent lineal source distribution on the x-axis as viewed from the angle θ on Hayes' control cylinder (cf. section 7). $h_1(x,\theta)$ is given by equation 7.5, i.e.

$$h_1(x,\theta) = Ut_1'(x,\theta) - \frac{\beta}{\rho_o U}\left[\ell_1(x,\theta)\sin\theta + s_1(x,\theta)\cos\theta\right] \quad \text{(a)}$$

Choose an interval $a < x < b$ large enough to include $h_1(x,\theta)$ for all θ. For the wave drag of A_1 to be finite it is necessary that $h_1(x,\theta)$ be differentiable in the interval $a < x < b$ and vanish at the end points, i.e.

$$h_1(a,\theta) = h_1(b,\theta) = 0 \quad \text{(b)}$$

for all θ (Ref. 8). (There may be a few θ-values for which this condition is not satisfied. However at these points, $dD_1/d\theta$ as defined by equation 7.6 must be integrable.)

Now consider a lineal distribution $A_2(x)=t_2(x)+\ell_2(x)+s_2(x)$ that has finite drag and is defined over an interval $-c < x < c$. For finite wave drag, t_2', ℓ_2, and s_2 must be differentiable in the interval $-c < x < c$ and vanish at $x = \pm c$, i.e.

$$t_2'(c) = t_2'(-c) = 0$$

$$\ell_2(c) = \ell_2(-c) = 0 \qquad (14.1)$$

$$s_2(x) = s_2(-c) = 0$$

Furthermore, for finite vortex drag the net lift and side force of A_2 must vanish (ref. 12), i.e.

$$\int_{-c}^{c} \ell_2(x)\,dx = 0, \quad \int_{-c}^{c} s_2(x)\,dx = 0 \qquad (14.2)$$

The interference drag between A_1 and A_2 thus appears only as wave drag interference and can be calculated either by the method of Hayes or by equation 8.4a.

The interference drag will be calculated by Hayes' method first. The equivalent lineal source distribution of A_2 is simply

$$h_2(x,\theta) = Ut_2'(x) - \frac{\beta}{\rho_o U}\left[\ell_2(x,\theta)\sin\theta + s_2(x,\theta)\cos\theta\right] \quad (c)$$

Equations (a) and (c) together with Hayes' formula 7.6 and 7.7 give

$$(A_1,A_2) = \frac{-\rho_o}{8\pi^2}\int_o^{2\pi} d\theta \int_a^b h_1'(x,\theta)\,dx \int_{-c}^{c} h_2'(\xi,\theta)\log|x-\xi|\,d\xi \qquad (14.3)$$

The inner integral in equation 14.3 can be integrated by parts noting that $h_2(x,\theta)$ is continuous over the interval $-c < x < c$. With the aid of equations 14.1 one obtains

$$\int_{-c}^{c} h_2'(\xi,\theta)\log|x-\xi|\,d\xi = -\int_{-c}^{c} \frac{h_2(\xi,\theta)}{x-\xi}\,d\xi \qquad (d)$$

Substituting equation d into equation 14.1 gives, after interchanging the bound variables and the order of integration,

$$(A_1,A_2) = \frac{\rho_o}{8\pi^2}\int_0^{2\pi} d\Theta \int_{-c}^{c} h_2(x,\Theta)\ dx \int_a^b \frac{h_1'(\xi,\Theta)}{x-\xi}\ d\xi \qquad \text{(e)}$$

Substituting the expression for $h_2(x,\Theta)$ given by equation c and performing the Θ integration first*, one obtains

$$\begin{aligned}(A_1,A_2) = -\int_{-c}^{c}\Bigg\{ & \rho_o U\left[\frac{1}{4\pi}\int_a^b \frac{1}{\xi - x}\left(\frac{1}{2\pi}\int_0^{2\pi} h_1'(\xi,\Theta)\,d\Theta\right)d\xi\right]t_2'(x) \\ & + \frac{1}{U}\left[\frac{-\beta}{4\pi}\int_a^b \frac{1}{\xi - x}\left(\frac{1}{2\pi}\int_0^{2\pi} h_1'(\xi,\Theta)\ \sin\Theta\ d\Theta\right)d\xi\right]\ell_2(x) \\ & + \frac{1}{U}\left[\frac{-\beta}{4\pi}\int_a^b \frac{1}{\xi - x}\left(\frac{1}{2\pi}\int_0^{2\pi} h_1'(\xi,\Theta)\ \cos\Theta\ d\Theta\right)d\xi\right]s_2(x)\Bigg\}\,dx\end{aligned}$$

(14.3)

If the interference drag is calculated by means of equation 8.4a one obtains the expression

$$(A_1,A_2) = -\int_{-c}^{c}\left[\rho_o U\ \bar{u}_1(x)\ t_2'(x) + \frac{\bar{w}_1(x)}{U}\ \ell_2(x) + \frac{\bar{v}_1(x)}{U}\ s_2(x)\right]dx$$

(14.4)

Comparison of equations 14.3 and 14.4 gives the following result, where $\langle f(\Theta)\rangle$ stands for the mean value of $f(\Theta)$, i.e.

$$\langle f(\Theta)\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\Theta)\ d\Theta\ ,$$

and where the order of Θ-integration and ξ-differentiation has been changed,

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* All of the requirements for changes in the orders of the integrations and differentiation, have been found to be satisfied if one deals only with distributions A_1 and A_2 that have finite wave drag.

$$\bar{u}_1(x) = \frac{1}{4\pi}\int_a^b \frac{\langle h_1(\xi,\theta)\rangle'}{\xi - x}\, d\xi \tag{14.5a}$$

$$\bar{w}_1(x) = \frac{-1}{4\pi}\int_a^b \frac{\langle h_1(\xi,\theta)\sin\theta\rangle'}{\xi - x}\, d\xi + \lambda \tag{14.5b}$$

$$\bar{v}_1(x) = \frac{-1}{4\pi}\int_a^b \frac{\langle h_1(\xi,\theta)\cos\theta\rangle'}{\xi - x}\, d\xi + \sigma \tag{14.5c}$$

Since the interval $-c < x < c$ and the distribution $A_2(x)$ can be chosen arbitrarily within the class of lineal distributions with finite drag, i.e. equations (14.2) apply; equations 14.5 a,b,c are valid for all x in the infinite interval. λ and σ are not determined in the comparison and they do not affect the calculation of interference drag as long as A_2 is restricted to the class of lineal distributions with finite drag. Equations 14.5b and 14.5c are exact if λ and σ are equated to the value of $\bar{w}_1$ and $\bar{v}_1$ in the Trefftz plane at $y = 0$, $z = 0$.

A lineal distribution $\bar{A}_1(x)=\bar{t}_1(x)+\ell_1(x)+\bar{s}_1(x)$ will now be found that produces exactly the same interference drag with any lineal distribution $A_2(x)$ as $A_1(x,y,z)$ does, i.e.

$$(A_1,A_2) = (\bar{A}_1,A_2) \quad \text{for all} \quad A_2(x) \tag{f}$$

Such a lineal distribution will be referred to as a "pseudo lineal distribution". The equivalent lineal source distribution of $\bar{A}_1(x)$ as viewed from the angle θ is

$$\bar{h}_1(x,\theta) = U\bar{t}_1'(x) - \frac{\beta}{\rho_o U}\left[\ell_1(x)\sin\theta + \bar{s}_1(x)\cos\theta\right] \tag{14.6}$$

It is easy to see from equations 14.5 a,b,c and 14.4 that for $\bar{A}_1(x)$ to be a pseudo lineal distribution of $A_1(x,y,z)$ it is sufficient that

$$\left\langle \bar{h}_1(x,\theta)\right\rangle = U\bar{t}_1'(x) = \left\langle h_1(x,\theta)\right\rangle \tag{14.7a}$$

$$\left\langle \bar{h}_1(x,\theta)\sin\theta\right\rangle = -\frac{\beta}{2\rho_o U}\,\bar{\ell}_1(x) = \left\langle h_1(x,\theta)\sin\theta\right\rangle \tag{14.7b}$$

$$\left\langle \bar{h}_1(x,\theta)\cos\theta\right\rangle = -\frac{\beta}{2\rho_o U}\,\bar{s}_1(x) = \left\langle h_1(x,\theta)\cos\theta\right\rangle \tag{14.7c}$$

Furthermore, from the differentiability condition on h_1 and equation(b), $\bar{t}_1'$, $\bar{\ell}_1$, and $\bar{s}_1$ satisfy the differentiability condition and

$$\bar{t}_1'(a) = \bar{t}_1'(b) = \ell_1(a) = \ell_1(b) = \bar{s}_1(a) = \bar{s}_1(b) = 0 \tag{14.8}$$

Hence the wave drag of $\bar{A}_1(x)$ is finite. One further requirement shall be made which fixes the pseudo lineal distribution A_1, viz.

$$\bar{t}_1(a) = 0 \tag{14.9}$$

The interference drag between A_1 and A_2 can thus be computed from 14.4, viz.

$$(A_1,A_2) = -\int_{-c}^{c}\left[\rho_o U\,\bar{u}_1\,t_2' + \frac{\bar{w}_1}{U}\,\ell_2 + \frac{\bar{v}_1}{U}\,s_2\right]dx \tag{14.10}$$

where $\bar{u}_1$, $\bar{w}_1$, and $\bar{v}_1$ are calculated directly from the distribution $A_1(x,y,z)$ or from equations similar to equations 14.5, viz.

$$\bar{u}_1 = \frac{U}{4\pi} \int_a^b \frac{\bar{t}_1''(\xi)}{\xi - x} d\xi \tag{14.11a}$$

$$\bar{w}_1 = \frac{\beta^2 U}{4\pi q} \int_a^b \frac{\bar{\ell}_1'(\xi)}{\xi - x} d\xi \tag{14.11b}$$

$$\bar{v}_1 = \frac{\beta^2 U}{4\pi q} \int_a^b \frac{\bar{s}_1'(\xi)}{\xi - x} d\xi \tag{14.11c}$$

obtained by using the pseudo lineal distribution $\bar{A}_1(x)$.

Equation (f) states that

$$(A_1 - \bar{A}_1, A_2) = 0 \quad \text{for all} \quad A_2(x) \quad \text{with finite wave plus vortex drag} \tag{14.12}$$

Similarly it can be shown that

$$(A_1 - \bar{A}_1, A_2)_{wave} = 0 \quad \text{for all} \quad A_2(x) \quad \text{with finite wave drag} \tag{14.13}$$

Thus if one puts $A_2 = \bar{A}_1$ in equation (14.13) one obtains, with its aid

$$D_{wave}(A_1) = D_{wave}(A_1 - \bar{A}_1 + \bar{A}_1) = D_{wave}(A_1 - \bar{A}_1) + D_{wave}(\bar{A}_1) \tag{14.14}$$

Net base areas, lift, and sideforce of the distribution A_1 are given (as in equations 11.2 and 12.3) by

$$F_1 = F(A_1) = \iiint_{R_1} \frac{\partial t_1}{\partial x} dx\, dy\, dz = \int_a^b t_1'(x,\theta)\, dx \tag{14.15a}$$

$$L_1 = L(A_1) = \iiint_{R_1} \ell_1\, dx\, dy\, dz = \int_a^b \ell_1(x,\theta)\, dx \tag{14.15b}$$

$$S_1 = S(A_1) = \iiint_{R_1} s_1\, dx\, dy\, dz = \int_a^b s_1(x,\theta)\, dx \tag{14.15c}$$

The equalities on the right hand sides of equations 14.15 follow from the method of construction, given in section 7, of the lineal projections $t_1^*(x,\theta)$, $\ell_1(x,\theta)$ and $s_1(x,\theta)$. From equations 14.7, 14.15 and equation (a) and the orthogonality between 1, $\sin\theta$ and $\cos\theta$, one obtains

$$F(\bar{t}_1) = \frac{1}{U}\int_a^b \left\langle h_1(x,\theta)\right\rangle \, dx = \frac{1}{U}\left\langle \int_a^b h_1(x,\theta)\, dx\right\rangle$$

$$= \frac{1}{U}\left\langle UF_1 - \frac{\beta}{\rho_o U}\left[L_1 \sin\theta + S_1 \cos\theta\right]\right\rangle = F_1 \qquad (14.16a)$$

and similarly

$$L(\bar{\ell}_1) = -\frac{2\rho_o U}{\beta}\left\langle UF_1 - \frac{\beta}{\rho_o U}\left[L_1 \sin\theta + S_1 \cos\theta\right]\right\rangle \sin\theta$$

$$= L_1 \qquad (14.16b)$$

$$S(\bar{s}_1) = -\frac{2\rho_o U}{\beta}\left\langle UF_1 - \frac{\beta}{\rho_o U}\left[L_1 \sin\theta + S_1 \cos\theta\right]\right\rangle \cos\theta$$

$$= S_1 \qquad (14.16c)$$

Unless L_1 and S_1 are zero, the equivalent pseudo distribution $\bar{A}_1(x)$ has infinite vortex drag.

15. Optimum Lineal Distributions in the Presence of a Fixed Distribution.

This section is devoted to the study of the problem of finding optimum lineal distributions of thickness, lift and side-force for an interval $-c \leq x \leq c$ on the x-axis such that the drag of the combined distribution A_1+A_2 is minimized. $A_1(x,y,z)$ is a given fixed distribution that may represent wings and bodies off the axis. A_2 is required to satisfy a set of constraints such as fixed total volume or fixed base area. In addition, A_2 must satisfy the conditions

$$L(\ell_2) = 0, \qquad S(\ell_2) = 0 \tag{15.1}$$

in order to have finite vortex drag (cf. section 14). According to Monk's stagger theorem and equations 15.1 only the wave drag of A_1+A_2 is minimized. Thus the pseudo lineal distribution $\bar{A}_1(x)$ will be used to replace $A_1(x,y,z)$ in the formulas. The drag of A_1+A_2 is then given by

$$D(A_1+A_2) = D(A_1) + 2(A_1+A_2,A_2) - (A_2,A_2) \tag{15.2}$$

As is indicated in the example of section 13, the optimum distribution A_2 can be split up into two distributions

$$A_2 = A_3 + A_4 \tag{15.3}$$

The first distribution A_3 is chosen such that the wave drag of A_1+A_3 is minimized irrespective of total volume, total base area, or equations 15.1 or other constraints which may be imposed on the distribution A_2. The necessary and sufficient condition that A_4 be this optimum is that

$$(A_1+A_3,\tilde{A})_{wave} = (A_1+A_3,\tilde{A})_{wave} = 0 \tag{15.4}$$

hold for all lineal distributions $\tilde{A}_3$ that have finite wave drag and are confined to the interval $-c \leq x \leq c$. The proof of equation 15.4 is similar to that used in deriving equation 10.1.

If equation 15.4 is expressed in the form of equation 14.9 one obtains

$$-\int_{-c}^{c} [\rho_o U(\bar{u}_1+\bar{u}_3)\ \tilde{t}_3' + \frac{\bar{w}_1+\bar{w}_2}{U}\tilde{\ell}_3 + \frac{\bar{v}_1+\bar{v}_3}{U}\tilde{s}_3]\ dx = 0 \qquad (15.5)$$

Since $\tilde{t}_3'$, $\tilde{\ell}_3$, and $\tilde{s}_3$ are arbitrary, one obtains

$$\bar{u}_1+\bar{u}_3 = 0, \qquad \bar{w}_1+\bar{w}_2 = 0, \qquad \bar{v}_1+\bar{v}_3 = 0$$

where the combined flow velocities are calculated with the aid of equations 14.11 a,b,c. One obtains, after cancelling out the constant factor,

$$\int_{-c}^{c} \frac{t_3''(\xi)}{\xi - x}\, d\xi = -\int_{a}^{b} \frac{\bar{t}_1''(\xi)}{\xi - x}\, d\xi \qquad (15.6a)$$

$$\int_{-c}^{c} \frac{{}_3'(\xi)}{\xi - x}\, d\xi = -\int_{a}^{b} \frac{\bar{\ell}_1'(\xi)}{\xi - x}\, d\xi \qquad (15.6b)$$

$$\int_{-c}^{c} \frac{s_3'(\xi)}{\xi - x}\, d\xi = -\int_{a}^{b} \frac{\bar{s}_1'(\xi)}{\xi - x}\, d\xi \qquad (15.6c)$$

for all x in the interval $-c \leq x \leq c$. For finite wave drag of the distributions A_3 and $\bar{A}_1$, it is required that

$$t_3'(\pm c) = \ell_3(\pm c) = s_3(\pm c) = 0 \qquad (15.6d)$$

$$\bar{t}_1'(a) = \bar{t}_1'(b) = \bar{\ell}_1(a) = \bar{\ell}_1(b) = \bar{s}_1(a) = \bar{s}_1(b) = 0 \qquad (15.6e)$$

In addition we will require

$$t_3(-c) = 0 \quad \text{and} \quad \bar{t}_1'(a) = 0 \qquad (15.6f)$$

These are the integral equations which must be solved for the distribution A_3.

The middle term of equation 15.1 can be expressed with the aid of equation 15.4 where we let $\tilde{A} = A_2$. Thus

$$(\bar{A}_1+A_2,A_2) = (\bar{A}_1+A_3+A_4,A_2)_{wave} = (A_4,A_2)_{wave}$$

$$= (A_4,A_3)_{wave} + (A_4,A_4)_{wave}$$

$$D(A_1+A_2) = D(A_1) + 2(A_4,A_3)_{wave} + 2(A_4,A_4)_{wave}$$

$$- (A_3,A_3)_{wave} - 2(A_3,A_4)_{wave} - (A_4,A_4)_{wave}$$

$$= D(A_1) - D_{wave}(A_3) + D_{wave}(A_4) \qquad (15.7)$$

The second distribution A_4 is chosen such that the quantity $D_{wave}(A_4)$ is minimized under the conditions:

$$L(\ell_4) = - L(\ell_3), \qquad S(s_4) = - S(s_3) \qquad (15.8)$$

derived from equations 15.1. In addition, t_4 must be such that the constraints on the combined thickness distribution $t_2 = t_3+t_4$ are satisfied. For example, if $F(t_2)$ is specified then the constraint for t_4 is

$$F(t_4) = F(t_2) - F(t_3) \qquad (15.9)$$

If A_2 is to have fixed volume and zero base area then

$$\left.\begin{aligned} F_4(t_4) &= - F(t_3) \\ V_4(t_4) &= V(t_2) - V(t_3) \end{aligned}\right\} \qquad (15.10)$$

are the constraints that A_4 must satisfy.

The problem of finding A_4 such that $D_{wave}(A_4)$ is minimized under the conditions 15.8 and the constraints given by equations 15.9 or 15.10 has been extensively studied in the references (see references (2), (8), (13), (14), for example).

A_4 can be decomposed into four basic orthogonal optimum distributions, viz.

Karman ogive distribution (ref. 13):

$$\left.\begin{aligned} t_{co} &= \left\{ \frac{x[c^2-x^2]^{1/2}}{c^2} + \sin^{-1}\frac{x}{c} + \frac{\pi}{2} \right\} F \\ F(t_{co}) &= F, \quad V(t_{co}) = \pi Fc \\ D(t_{co}) &= \frac{q}{\pi c^2} F^2 \end{aligned}\right\} \qquad (15.11)$$

Sears Haack distribution (refs. 14,15):

$$\left.\begin{aligned} t_{SH} &= \frac{8V}{3 \ c^4}(c^2 - x^2)^{3/2} \\ F(t_{SH}) &= 0, \qquad V(t_{SH}) = V \\ D(t_{SH}) &= \frac{8V}{\pi c^4} V^2 \end{aligned}\right\} \qquad (15.12)$$

Elliptic lift distribution (ref. 2):

$$\left.\begin{aligned} \ell_E &= \frac{2L}{\pi c^2}(c^2 - x^2)^{1/2} \\ D_{wave}(\ell_E) &= \frac{\beta^2 L^2}{2\pi qc^2} \end{aligned}\right\} \qquad (15.13)$$

Elliptic sideforce distribution:

$$\left.\begin{aligned} s_E &= \frac{2S}{\pi c^2}(c^2 - x^2)^{1/2} \\ D_{wave}(s_E) &= \frac{\beta^2 S^2}{2\pi qc^2} \end{aligned}\right\} \qquad (15.14)$$

By choosing $L = -L(\ell_3)$, $S = -S(s_3)$ in equations 15.13 and 15.14, respectively, conditions 15.8 are satisfied. The constraint 15.9 is satisfied by choosing $F = F(t_4) = F(t_2) - F(t_4)$ in equation 15.11. Similarly, the constraints 15.10 are satisfied by choosing $F = F(t_4) = -F(t_3)$ in equation 15.11 and $V = V(t_2) - V(t_3) + \pi c\, F(t_3)$ in equation 15.12

The remainder of this section is devoted to the calculation of $A_3(x)$, $L(\ell_3)$, $S(\sigma_3)$, $F(t_3)$ and $V(t_3)$. To include the most general case, (a) and (b) in equations 15.6 a,b,c are chosen such that

$$a \leq -c \leq x \leq c \leq b$$

Consider the integral equations 15.6 a,b,c. Since they are all of the same form only the first one will be discussed in detail* viz.

$$\int_{-c}^{c} \frac{t_3''(\xi)}{\xi - x}\, d\xi = -\int_a^b \frac{\bar{t}_1''(\xi)}{\xi - x}\, d\xi \qquad (15.15a)$$

where by equations 15.6 d,e,f, t_3 and $\bar{t}_1$ must satisfy the end conditions

$$t_3'(\pm c) = 0, \qquad t_3(-c) = 0 \qquad (15.15b)$$

$$\bar{t}_1(a) = \bar{t}_1'(b) = \bar{t}_1(a) = 0 \qquad (15.15c)$$

It will be shown that the solution of 15.15 a which satisfies the end conditions 15.15b

$$t''(x) = -\bar{t}_1''(x) + \frac{1}{\pi(c^2-x^2)^{1/2}} \left\{ \int_a^{-c} - \int_c^b \right\} \frac{\bar{t}_1''(\eta)(\eta^2-c^2)^{1/2}}{\eta - x}\, d\eta \qquad (15.16)$$

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* Equation 15.15a is similar to the airfoil equations and is fully discussed in ref. 16.

The following integrals are needed to verify that equation 15.16 is a solution of equation 15.15 and for use in obtaining integrals of $t_3''(x)$.

$$G(x,\eta) = \int_{-c}^{x} \frac{(\eta^2-c^2)^{1/2}}{(c^2-\xi^2)^{1/2}} \frac{d\xi}{\eta-\xi} = \sin^{-1}\frac{\eta x-c^2}{c|\eta-x|} + \begin{Bmatrix} -\frac{\pi}{2} \text{ for } \eta \leq -c \\ +\frac{\pi}{2} \text{ for } \eta \leq c \end{Bmatrix} \qquad \text{(a)}$$

where the principal value of $\sin^{-1}$ is assumed, i.e.

$$-\frac{\pi}{2} \leq \sin^{-1} \leq \frac{\pi}{2},$$

and where $|x| \leq c$. $G(x,\eta)$ has the following properties

$$\frac{\partial G(x,\eta)}{\partial \eta} = \frac{(c^2-x^2)^{1/2}}{(\eta^2-c^2)^{1/2}} \frac{1}{\eta - x} \qquad \text{(b)}$$

$$\left.\begin{aligned} &G(x,c) = 0, \qquad G(x,-c) = -\pi \\ &G(-c,\eta) = 0, \; G(c,\eta) = \begin{cases} -\pi & \text{for } \eta \leq c \\ 0 & \text{for } |\eta| < c \\ + & \text{for } \quad \geq c \end{cases} \end{aligned}\right\} \qquad \text{(c)}$$

The integral of $G(x,\eta)$ from $x = -c$ to $+c$ for $|\eta| > c$ is also needed.

$$\int_{-c}^{c} G(x,\eta)\,dx = c[G(c,\eta) + G(-c,\eta)] - \int_{-c}^{c} x \frac{\partial G}{\partial x}\,dx$$

$$= c\,G(c,\eta) - \int_{-c}^{c} \frac{(\eta^2-c^2)^{1/2}}{(c^2-x^2)^{1/2}} \frac{\eta-(\eta-x)}{\eta - x}\,dx$$

$$= c\,G(c,\eta) - \eta\,G(c,\eta) + \pi(\eta^2-c^2)^{1/2}$$

$$= (c-\eta)\,G(c,\eta) + \pi(\eta^2-c^2)^{1/2} \qquad \text{(d)}$$

Another integral which is needed is derived from equation (c)

$$\int_{-c}^{c} \frac{(\eta^2-c^2)^{1/2}}{(c^2-\xi^2)^{1/2}} \frac{d\xi}{(\xi-x)(\eta-\xi)}$$

$$= \frac{1}{\eta-x} \int_{-c}^{c} \frac{(\eta^2-c^2)^{1/2}}{(c^2-\xi^2)^{1/2}} \left(\frac{1}{\eta-\xi} - \frac{1}{x-\xi}\right) d\xi$$

$$= \frac{1}{\eta-x} \left[G(c,\eta) - G(c,x)\right] = \frac{1}{\eta-x} \begin{Bmatrix} -\pi & \text{for} & -c \\ +\pi & \text{for} & \_\ c \end{Bmatrix}, \quad |x| < c \tag{e}$$

We have, upon substituting t_3'' from equation 15.16 into equation 15.15a,

$$\int_{-c}^{c} \frac{t_3''(\xi)}{\xi - x} d\xi = -\int_{-c}^{c} \frac{\bar{t}_1''(\xi)}{\xi - x} d\xi$$

$$+ \frac{1}{\pi} \int_{-c}^{c} \frac{1}{(c^2-\xi^2)^{1/2}} \frac{d\xi}{\xi-x} \left\{ \int_{a}^{-c} - \int_{c}^{b} \right\} \frac{\bar{t}_1''(\eta)(\eta^2-c^2)^{1/2}}{\eta-\xi} d\eta \tag{f}$$

Interchanging the order of η and ξ -integration, noting equation (e), and replacing ξ by η in the first integral on the right hand side of equation (f), gives

$$\int_{-c}^{c} \frac{t_3''(\xi)}{\xi - x} d\xi = -\int_{-c}^{c} \frac{\bar{t}_1''(\eta)}{\eta - x} d\eta + \left\{ -\int_{a}^{-c} - \int_{a}^{b} \right\} \frac{\bar{t}_1''(\eta)}{\eta - x} d\eta$$

$$= -\int_{a}^{b} \frac{\bar{t}_1''(\eta)}{\eta - x} d\eta \tag{g}$$

as required. Thus $t_3(x)$ given by equation 15.16 is a solution of the integral equation 15.15a.

The integral of t_3 from $-c$ to x is given by

$$t_3'(x) = \int_{-c}^{x} t_3''(\xi)\, d\xi = -\int_{-c}^{x} \bar{t}_1''(\xi)\, d\xi + \frac{1}{\pi}\left\{\int_a^{-c} - \int_c^b\right\} G(x,\eta)\, \bar{t}_1''(\eta)\, d\eta \quad \text{(h)}$$

where the order of ξ and η -integrations has been interchanged and equation (a) has been used. Integration of the second integral on the right hand side of (h) by parts, yields

$$\begin{aligned} t_3'(x) &= -\bar{t}_1'(x) + \bar{t}_1'(-c) \\ &+ \frac{1}{\pi}\left[\bar{t}_1(-c)\, G(x,-c) - \bar{t}_1'(a)\, G(x,a) - \bar{t}_1'(b)\, G(x,b) + \bar{t}_1'(c)\, G(x,c)\right] \\ &- \frac{1}{\pi}\left\{\int_a^{-c} - \int_c^b\right\} \frac{\partial G(x,\eta)}{\partial \eta}\, \bar{t}_1'(\eta)\, d\eta \end{aligned} \quad \text{(i)}$$

Equations (b) and (c) and the end conditions 15.15c then give, finally,

$$t_3'(x) = -\bar{t}_1'(x) - \frac{(c^2-x^2)^{1/2}}{\pi}\left\{\int_a^{-c} - \int_c^b\right\} \frac{\bar{t}_1'(\eta)}{(\eta^2-c^2)^{1/2}}\, \frac{d\eta}{\eta - x} \quad (15.18)$$

An alternative form of equation (15.18) is obtained with the aid of the identity

$$\frac{(c^2-x^2)^{1/2}}{(\eta^2-x^2)^{1/2}} = -\frac{(\eta^2-c^2)}{(c^2-x^2)^{1/2}} + \frac{(\eta+x)\,(\eta-x)}{(c^2-x^2)^{1/2}\,(\eta^2-c^2)^{1/2}} \quad \text{(j)}$$

$$\begin{aligned} t_3'(x) = &\left[-\bar{t}_1'(x) + \frac{1}{\pi(c^2-x^2)^{1/2}}\left\{\int_a^{-c} - \int_c^b\right\} \frac{\bar{t}_1'(\eta)\,(\eta^2-c^2)}{\eta - x}\, d\eta\right] \\ &- \frac{1}{\pi(c^2-x^2)}\left\{\int_a^{-c} - \int_c^b\right\} \frac{\bar{t}_1'(\eta)\,(x+\eta)}{(\eta^2-c^2)^{1/2}}\, d\eta \end{aligned} \quad (15.19)$$

To show that t_3' vanishes at the end of the interval we let $x = c$ in equation (h) and obtain, with the aid of (c),

$$t_3'(c) = \int_{-c}^{c} t_3''(\xi)\, d\xi = -\int_a^b \bar{t}_1''(\eta)\, d\eta = -\bar{t}_1'(b) + \bar{t}_1'(a) \qquad \text{(k)}$$

The right hand side of equation (k) vanishes by equation 15.15c. Therefore $t_3'(c) = 0$ as required.

Further integrations of equation 15.19 yields

$$\begin{aligned} t_3(x) = {} & -\bar{t}_1(x) + \bar{t}_1(-c) + \frac{1}{\pi}\left\{\int_a^{-c} - \int_c^b\right\} G(x,\eta)\, \bar{t}_1'(\eta)\, d\eta \\ & + \frac{(c^2-x^2)^{1/2}}{\pi}\left\{\int_a^{-c} - \int_c^b\right\} \frac{\bar{t}_1'(\eta)}{(\eta^2-c^2)^{1/2}}\, d\eta \\ & - \frac{1}{\pi}\left[\sin^{-1}\frac{x}{c} + \frac{\pi}{2}\right]\left\{\int_a^{c} - \int_c^b\right\} \frac{\bar{t}_1'(\eta)\eta}{(\eta^2-c^2)^{1/2}}\, d\eta \end{aligned} \qquad (15.20)$$

$$\begin{aligned} F(t_3) = t_3(c) & = -\int_a^b \bar{t}_1'(\eta)\, d\eta - \left\{\int_a^{c} - \int_c^b\right\} \frac{\bar{t}_1'(\eta)}{(\eta^2-c^2)^{1/2}}\, d\eta \\ & = -F_1 - \left\{\int_a^{c} - \int_c^b\right\} \frac{\bar{t}_1'(\eta)}{(\eta^2-c^2)^{1/2}}\, d\eta \end{aligned} \qquad (15.21)$$

where F_1 is the base area of the distribution $A_1(x,y,z)$ (see equations 14.17a). Equation 15.20 has an alternative form similar to equation 15.18, viz.

$$\begin{aligned} t_3(x) = {} & \left[-\bar{t}_1(x) - \frac{(c^2-x^2)^{1/2}}{\pi}\left\{\int_a^{-c} - \int_c^b\right\} \frac{\bar{t}_1(\eta)}{(\eta^2-c^2)^{1/2}} \frac{d\eta}{\eta - x}\right] \\ & - \frac{1}{\pi} F_1 G(x,b) + \frac{(c^2-x^2)^{1/2}}{\pi}\left\{\int_a^{c} - \int_c^b\right\} \frac{\bar{t}_1'(\eta)}{(\eta^2-c^2)^{1/2}}\, d\eta \\ & - \frac{1}{\pi}\left[\sin^{-1}\frac{x}{c} + \frac{\pi}{2}\right]\left\{\int_a^{c} - \int_c^b\right\} \frac{\bar{t}_1'(\eta)\eta}{(\eta^2-c^2)^{1/2}}\, d\eta \end{aligned} \qquad (15.22)$$

Integration of equation 15.22 from $-c$ to c, with the aid of equations (d) and (c), yields the volume of t_3, viz.

$$V(t_3) = -V(\bar{t}_1) - \left\{ \int_a^{-c} - \int_c^b \right\} \frac{\bar{t}_1(\eta)\eta}{(\eta^2-c^2)^{1/2}} d\eta$$

$$- F_1[(b^2-c^2)^{1/2} -(b-c)]$$

$$+ \frac{c^2}{2} \left\{ \int_a^{-c} - \int_c^b \right\} \frac{\bar{t}_1'(\eta)}{(\eta^2-c^2)^{1/2}} d\eta - c \left\{ \int_a^{-c} - \int_c^b \right\} \frac{\bar{t}_1'(\eta)\eta}{(\eta^2-c^2)^{1/2}} d\eta$$

(15.23)

Solutions for ℓ_3 and s_3 are obtained by putting $\bar{\ell}_1$ or $\bar{s}_1$ in place of $\bar{t}_1'$ in equation 15.18, viz.

$$\ell_3(x) = -\bar{\ell}_1(x) - \frac{(c^2-x^2)^{1/2}}{\pi} \left\{ \int_a^{-c} - \int_c^b \right\} \frac{\bar{\ell}_1(\eta)}{(\eta^2-c^2)^{1/2}} \frac{d\eta}{\eta - x} \qquad (15.24)$$

$$s_3(x) = -\bar{s}_1(x) - \frac{(c^2-x^2)^{1/2}}{\pi} \left\{ \int_a^{-c} - \int_c^b \right\} \frac{\bar{s}_1(\eta)}{(\eta^2-c^2)^{1/2}} \frac{d\eta}{\eta - x} \qquad (15.25)$$

Similarly $L(t_3)$ and $S(t_3)$ are obtained with the aid of equations 14.17b and 14.17c, viz.

$$L(t_3) = -L_1 - \left\{ \int_a^{c} - \int_c^b \right\} \frac{\bar{\ell}_1(\eta)\eta}{(\eta^2-c^2)^{1/2}} d\eta \qquad (15.26)$$

$$S(t_3) = -S_1 - \left\{ \int_a^{c} - \int_c^b \right\} \frac{\bar{s}_1(\eta)\eta}{(\eta^2-c^2)^{1/2}} d\eta \qquad (15.27)$$

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