Broadband Wireless Broadcast Channels: Throughput, Performance, and PAPR Reduction

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To

my parents

and

Eram

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Abstract

The ever-growing demand for higher rates and better quality of service in cellular systems has attracted many researchers to study techniques to boost the capacity and improve the performance of cellular systems. The main candidates to increase the capacity are to use multiple antennas or to increase the bandwidth. This thesis attempts to solve a few challenges regarding scheduling schemes in the downlink of cellular networks, and the implementation of modulation schemes suited for wideband channels.

Downlink scheduling in cellular systems is known to be a bottleneck for future broadband wireless communications. Information theoretic results on broadcast channels provide the limits for the maximum achievable rates for each receiver and transmission schemes to achieve them. It turns out that the sum-rate[1] capacity of a multi-antenna broadcast channel heavily depends on the availability of channel state information (CSI) at the transmitter. Unfortunately, the dirty paper coding (DPC) scheme which achieves the capacity region is extremely computationally intensive especially in multiuser context. Furthermore, relying on the assumption that full CSI is available from all the n users may not be feasible in practice.

In the first part of the thesis, we obtain the scaling law of the sum-rate capacity for large n and for a homogeneous fading MIMO (multiple input multiple output) broadcast channel, and then propose a simple scheme that only requires little (partial) CSI and yet achieves the same scaling law. Another important issue in downlink schedul-

[1]sum-rate (or throughput) refers to the sum of the transmission rates to all users.

ing is to maintain fairness among users with different distances to the transmitter. Interestingly, we prove that our scheduling scheme becomes fair provided that the number of transmit antennas is large enough. We further analyze the impact of using a throughput optimal scheduling on the delay in sending information to the users. Finally, we look into the problem of differentiated rate scheduling in which different users demand for different sets of rates. We obtain explicit scheduling schemes to achieve the rate constraints.

In the second part of the thesis, we focus on orthogonal frequency division multiplexing (OFDM), which is the most promising technique for broadband wireless channels (mainly due to its simplicity of channel equalization even in a severe multipath fading environment). The main disadvantage of this modulation, however, is its high peak to mean envelope power ratio (PMEPR). This is due to the fact that the OFDM signal consists of many (say n) harmonically related subcarriers which may, in the worst-case, add up constructively and lead to large peaks (of order n) in the signal.

Despite this worst-case performance, we show that when each subcarrier is chosen from some given constellation, the PMEPR behaves like $\log n$ almost surely, for large n. This implies that there exist almost full-rate codes with a PMEPR of $\log n$ for large n. We further prove that there exist codes with rate not vanishing to zero such that the PMEPR is less than a constant (independent of n). We also construct high rate codes with a guaranteed PMEPR of $4 \log n$. Simulation results show that in a system with 128 subcarriers and using 16QAM, the PMEPR of a multicarrier signal can be reduced from 13.5 to 3.4 which is within 1.6dB of the PMEPR of a single carrier system.

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Chapter 1

Introduction

1.1 Capacity Increase in Wireless Systems

The wireless industry has been confronted with an ever-growing demand for higher rates and better quality of service. The down-link scheduling in cellular systems is known to be the bottleneck for future broadband wireless communications. From the early 1980s when cellular systems were first deployed, there has been a tremendous increase in the number of subscribers, a huge improvement in the quality of service, and also in the availability of a variety of services other than voice. The success in cellular systems also motivated the use of wireless systems for many other applications such as local area networks (LAN) and bluetooth, which is a short-range wireless connection protocol.

The main property of a wireless channel is its random time-varying behavior due to the mobility of users and other objects, as well as obstacles in the environment. More specifically, the channel to a given user might have poor conditions at some times and favorable conditions at some other times. This is the so-called "fading" behavior of the channel. In many situations, multiple copies of the transmitted signal may be received with different delays and different strengths. This is called "multipath fading" and can severely deteriorate the performance when the transmitted signals have shorter duration (e.g., broadband transmission). Also, since the subscribers are mobile, it is generally assumed that the processing capability as well as the average transmit power are limited resources for the handheld devices. Therefore, any communication scheme that aims for higher transmission rates through a wireless channel has to deal with fading channels, limited average power, and low complexity receivers.

In order to improve the capacity of a cellular system with an average power constraint, one can either decrease the cell size, increase the bandwidth, or add more antennas at the transmitter and/or receivers. Increasing the number of cells (or decreasing the cell size) has been the most traditional method for increasing the capacity and was studied in the 1970s at Bell Labs. However this method has some drawbacks. First of all, this would increase the number of handshakes as users move between the cells more often. Secondly, there is a cost associated with deploying new base stations as it requires buying land and expanding the backbone. Finally, it is quite clear that this method can not be progress indefinitely! Therefore, this thesis deals with a few challenges associated with the last two options to increase capacity, namely, adding more antennas or increasing the bandwidth.

1.1.1 Use of Multiple Antennas

Multiple-antenna communications systems have generated a great deal of interest since they are capable of considerably increasing the capacity of a wireless link. In fact, it was known for a long time that, if perfect channel state information (CSI) were available at the transmitter and receiver, then they could jointly diagonalize the channel, thereby creating as many parallel channels as the minimum of the number of transmit/receive antennas and thus increase the capacity of the channel by this same factor. More surprisingly, it was later shown that the same capacity scaling is true if the channel is not known at the transmitter [18, 51] and even if it is not known at the receiver [19, 20] (provided the coherence interval of the channel is not too short).

While these are all true for point-to-point communications links, there has only

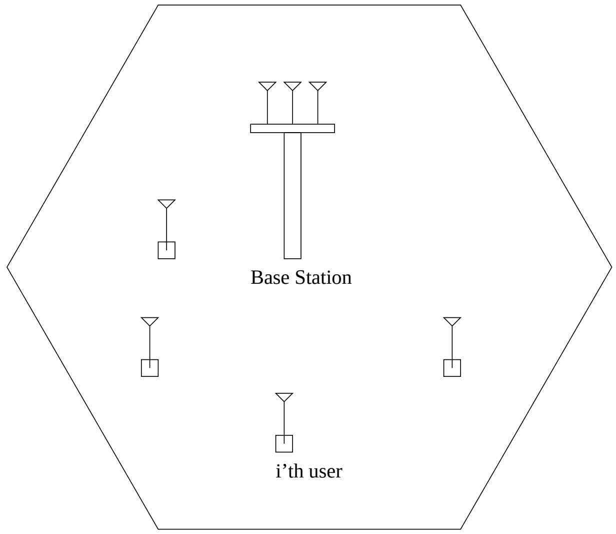

Figure 1.1: A cellular system with multiple antennas in the transmitter (base station)

been recent interest in the role of multiple antenna systems in a multi-user network environment, and especially in broadcast and multi-access scenarios. In fact, broadcast and multiple access channels refer to the downlink and uplink of cellular system (as shown in Fig. 1.1). Therefore, information theoretic results on broadcast channels provide the limits for the maximum achievable rates for each receiver (i.e., the capacity region) and transmission schemes to achieve the region. For example, the sum-rate capacity is a point on the boundary of the region that shows the maximum total rate (or throughput) that can be conveyed by the transmitter to all the users.

In multi-antenna broadcast channels, unlike point-to-point multi-antenna channels, the multi-user capacity depends heavily on whether the transmitter knows the channel coefficients to each user [35, 9, 53]. For instance, in a Gaussian broadcast channel with M transmit antennas and n single antenna users, the sum rate capacity scales like $M \log\log n$ for large n if perfect channel state information (CSI) is available at the transmitter, yet only logarithmically with M if it is not. It is also known that the sum-rate capacity can be achieved using dirty paper coding (DPC). Unfortunately, implementation of DPC, especially in the multi-user context, is extremely computationally expensive. Furthermore, DPC relies on the assumption that full CSI is available from all the n users which may not be feasible in practice. These issues raise the following question:

- Considering that in a typical cellular system, the number of users is much larger than the number of transmit antennas, do there exist schemes that lead to near-sum-rate capacity with less complexity and less feedback?

This question is mainly dealt with in Chapters 2 and 3 in which we first obtain the scaling law of the sum-rate capacity when n is much larger than M, and then we propose a simple scheme with partial CSI that achieves the same scaling law for the sum-rate as obtained with DPC.

1.1.2 Performance Issues

Many applications in wireless systems such as video-on-demand not only require higher rates but also are sensitive to delay [79, 83]. In fact, resource allocation in wireless systems aims for two conflicting goals, firstly providing quality of service such as delay and fairness to users, and secondly maximizing the throughput of the system [80]. This conflict is due to the fact that capacity-optimal schemes exploit the time-varying nature of the channel by allocating the power to the user that has the "best" channel conditions, and therefore, there is no consideration for the delay and fairness in these types of transmission [28].

On the other hand, in future cellular systems, users might request for different applications such as voice, internet, or video-on-demand. This implies that the base station has to provide differentiated services to different users yet at the same time, maximize the throughput. Therefore, the best operating point on the capacity region will no longer be the sum-rate capacity point. While achieving the sum-rate capacity point has been studied before, it is not easy to generalize the results to achieve other points on the capacity region.

Therefore, this thesis studies the performance of capacity-optimal scheduling schemes in terms of fairness among users and delay in giving service to different users in chapter 3 and 5. Also, in chapter 4, we obtain explicit transmission schemes that provide differentiated rates to users while maximizing the throughput of the systems.

1.1.3 Increasing the Bandwidth

Increasing the bandwidth is one the candidates to increase the capacity of cellular systems. The main challenge in systems with large bandwidth is to alleviate the frequency selectivity of the channel which implies that different frequency components of the transmit signal will be affected differently by the channel. This will require the receiver to equalize the channel which can be computationally intense [31].

Orthogonal frequency devision multiplexing (OFDM) is one of the most appealing techniques for wideband channels as it significantly simplifies the equalization [44]. This is done by breaking the wideband transmit signal into many narrowband subchannels. As each subchannel has a much smaller bandwidth, the channel roughly behaves as a constant in each coherence interval; this constant is of course different from one subchannel to the other. Each subchannel can be then equalized easily by just dividing the signal by the channel constant. In fact, OFDM has been proposed for several high speed wireless standards such as IEEE802.11a,g, digital audio/video broadcasting (DAB/DVB) [31].

The main disadvantage of this modulation, however, is its high peak-to-mean-envelope-power-ratio (PMEPR) [43, 44]. This is due to the fact that the OFDM signal consists of many, say n, harmonically related subcarriers which may add up constructively and lead to large peaks in the signal. In fact the worst case PMEPR is of the order of n. This high PMEPR will significantly hamper the power efficiency of the power amplifier in the transmitter front end, which in return reduces the battery life time.

A whole host of methods have been proposed for PMEPR reduction, such as coding, selective mapping, reserved carriers, and clipping [37, 36, 43, 38, 39]. Most of these methods are heuristic and do not provide much analytical insight into the problem. The main goal Chapters 6,7, and 8 has been to provide theoretical limits for PMEPR reduction using coding. One of the fundamental questions in this area has been to obtain the distribution of the PMEPR when each subcarrier is chosen from some given constellation.

We can, in fact, obtain the asymptotic behavior of distribution of PMEPR for a large number of subcarriers, n, and prove that for almost all the practical constellations, the PMEPR behaves like $\log n$ almost surely [4]. This implies that there exist almost full-rate codes with the PMEPR of $\log n$ for large n. This result raises two interesting questions:

- Can we *construct* high rate codes with a *guaranteed* PMEPR of $\log n$?
- Do there exist codes with rate not vanishing to zero such that the PMEPR is less than $\log n$ (say constant)?

We can, in fact, answer both of the aforementioned questions. In particular, for a q-ary constellation, we design a code with rate $1 - \log_q 2$, PMEPR of $4 \log n$ for any n, and decoding/encoding of order n. Furthermore, we can prove that the answer to the second answer is affirmative. For large n, we show that there exist exponentially many codewords (in n) with a PMEPR bounded by a constant independent of n. These two results are shown by balancing the PMEPR of each codeword using the sign of each subcarrier. Simulation results confirm that a significant improvement in PMEPR can be achieved using our coding scheme.

1.2 Contributions of the Thesis

In this section, we review the contents of each chapter and mention the main contributions. The thesis is organized in such a way that different chapters can be more or less read independently.

It should be also mentioned that most of our analytical results are asymptotic which implies that one (or more) parameter in the problem is assumed to grow to infinity. This is for the following reasons. First of all, it is the asymptotic regime which is pften of practical interests; for instance, in cellular systems, the number of users is assumed to be large or in a multicarrier system the number of subcarriers is assumed to be large. Secondly, the analysis is often not tractable in the general case. Finally, the asymptotic results usually give a lot of insight into the problem which can not be otherwise obtained. Therefore, throughout the thesis, we normally focus on situations where some parameters in the system gets large while other parameters are fixed.

In Chapter 2, broadcast channels are defined and known capacity results are presented under differing assumptions regarding the amount of channel state information (CSI) at the transmitter. In the case where full CSI is available in the transmitter, it has been shown that the sum-rate capacity in single transmit antenna systems can be achieved by transmitting only to the receiver with the most favorable channel condition. This scheduling is called "opportunistic." For the multi-antenna broadcast channel, however, the sum-rate capacity is achieved using dirty paper coding (DPC). In order to gain a better insight into the behavior of the sum-rate with the number of users, n, and number of transmit antennas, M, we obtain the scaling laws of the sum-rate for multi-antenna channels when n is large. It turns out that opportunistic transmission can incur a big loss compared to that of DPC. Furthermore, in the case where the transmitter has full CSI, the sum-rate capacity does grow linearly with M, yet only logarithmically with M when the transmitter has no CSI. Therefore the need for some sort of CSI is a necessity if we would like to boost the capacity by a factor of M.

In Chapter 3, we propose a simple scheme that only requires little feedback (partial CSI) and achieves the same scaling law for the sum-rate as what we obtained using DPC when n is large. This is motivated by the fact that in systems with large n, obtaining full CSI from all users may not be feasible. Since lack of CSI does not lead to multi-user gains, it is therefore of interest to investigate transmission schemes that employ only partial CSI. Our scheme is one that constructs M random beams and that transmits information to the users with the highest signal-to-noise-plus-interference ratios (SINRs), which can be made available to the transmitter with very little feedback. For fixed M and n increasing, the throughput of our scheme scales as $M \log\log nN$, where N is the number of receive antennas of each user. This is precisely the same scaling obtained with perfect CSI using dirty paper coding. We furthermore show that linear increase in throughput with M can be obtained provided that M does not not grow faster than $\log n$. We also study the fairness of our scheduling in a heterogeneous network and show that, when M is large enough, the system becomes interference-dominated and the probability of transmitting to any user converges to $\frac{1}{n}$, irrespective of its path-loss. In fact, using $M = \alpha \log n$ transmit antennas emerges as a desirable operating point, both in terms of providing linear scaling of the throughput with M as well as in guaranteeing fairness.

In Chapter 4, we consider the downlink of a wireless cellular system where users have different rate demands; therefore, our main interest is no longer to achieve the sum-rate capacity of broadcast channel. In particular, we assume n homogenous users are divided into M groups, each group of which requires the same rate, and where the ratio of the groups' rates are given. The transmitter would like to maximize the throughput (sum of the rates to all users) while maintaining the rational rate constraints. In general, this problem appears to be computationally intractable since the ergodic capacity region is described as the convex hull of (an infinite) set of rates. To illustrate this, we first consider systems where $n = 2$ and $n = 3$ and where each user requires a different rate. We focus on the achievable region by using weighted opportunistic scheduling (WO)—a generalization of opportunistic scheduling—in which we transmit to only the user that has the largest "weighted" signal to noise ratio (SNR). It turns out that determining the explicit relationship between the appropriate weights of the schedule and the desired ratio of the rates is analytically intractable even for the case of $n = 3$. For this reason, and also because most practical systems have many users, we focus on the asymptotic regime of large n where explicit results can be found. In particular, we propose three scheduling schemes to provide the rational rate constraints namely, the aforementioned WO, time division opportunistic (TO), and superposition coding (SC). In TO, each group has its own time slot in which the transmitter chooses the user with the best SNR from the corresponding group. Superposition coding is the scheme that achieves the information-theoretic capacity region. For each scheduling we give an explicit scheme to guarantee the rational rate constraints. We also analyze the throughput loss due to the rate constraints for all three different schemes. In particular, we show that the throughput loss compared to the maximum throughput (i.e., the sum rate capacity without any rate constraints) tends to zero for large n. Thus, there is not much of a penalty in providing different levels of service to different users. We also analyze the convergence rate of all the schemes and provide simulations supporting the theoretical analysis.

In Chapter 5, we look into the delay analysis for the opportunistic transmission in broadcast channels. We consider a single-antenna broadcast block fading channel with n users where the transmission is packet-based and all the users are backlogged. We define the delay as the minimum number of channel uses that guarantees *all* n users successfully receive m packets. This is a more stringent notion of delay than average delay and is the worst case delay among the users. A delay optimal scheduling scheme, such as round-robin, achieves the delay of mn. For the opportunistic scheduling (which is throughput optimal) where the transmitter sends a packet to the user with the best channel conditions at each channel use, we derive the mean and variance of the delay for any m and n. For large n and in a homogeneous network, it is proved that the expected delay in receiving one packet by all the receivers scales as $n \log n$, as opposed to n for the round-robin scheduling. We also show that when m grows faster than $(\log n)^r$, for some $r > 1$, then the delay scales as mn. This roughly determines the time-scale required for the system to behave fairly in a homogeneous network. We then propose a scheme to significantly reduce the delay at the expense of a small throughput hit. We further look into the advantage of multiple transmit antennas on the delay. For a system with M antennas in the transmitter where at each channel use packets are sent to M different users, we obtain the expected delay in receiving one packet by all the users.

In the following sections, we turn our attention to the peak to average power reduction of OFDM systems. In Chapter 6, we introduce OFDM signals and define the peak to mean envelope power ratio (PMEPR). We derive lower and upper probability bounds for the PMEPR distribution when the number of subcarriers n is large. Even though the worst case PMEPR is of the order of n, the main result is that the PMEPR of a random codeword $C = (c_1, \ldots, c_n)$ is $\log n$ with probability one asymptotically, for the following three general cases: (i) c_i is chosen independently and identically from a complex QAM constellation in which the real and imaginary part of c_i each has i.i.d. and even distribution (*not necessarily uniform*), (ii) c_i is chosen from a PSK constellation where the distribution over the constellation points is invariant under $\pi/2$ rotation, (iii) C is chosen uniformly from a complex sphere of dimension n. Furthermore, we use these bounds to obtain a Varsharmov-Gilbert (VG) style bound for the achievable rate and minimum Hamming distance of codes with entries chosen from QAM/PSK constellations and with PMEPR of less than $\log n$ for sufficiently large n. It is proved that asymptotically, the VG bound remains the same for codes with PMEPR of less than $\log n$.

The results in Chapter 6 motivates the question of how much reduction beyond $\log n$ can one asymptotically achieve with coding, and at what price in terms of the rate loss. In Chapter 7, by optimally choosing the sign of each subcarrier, we prove the existence of q-ary codes of *constant* PMEPR for sufficiently large n, and with a rate loss of at most $\log_q 2$. We also obtain a Varsharmov-Gilbert type upper bound on the rate of a code given its minimum Hamming distance with constant PMEPR, for large n.

Since ours is an existence result, we also study the problem of designing signs for PMEPR reduction in Chapter 8. We investigate practical schemes to search over the sign vector with linear or polynomial complexity in n while providing a worst case guarantee on the resulting PMEPR. In particular, we first propose two algorithms one based on derandomization and the other one base on a greedy approach. Both of these algorithms have order n complexity in the transmitter and no additional complexity in the receiver, while guaranteeing that the worst case PMEPR is less than $4 \log n$. For symmetric q-ary constellations, this algorithm constructs a code with rate $1 - \log_q 2$ and with PMEPR of $c \log n$ with simple encoding and decoding. In fact, simulation results show a much better performance than the upper bound; for example in a system with 128 subcarriers and using 16QAM constellation, the PMEPR has been reduced from 13.5 to 3.4 which is within 1.6dB of the PMEPR of single carrier system. Finally we propose a modification of both algorithms using sphere decoding that lead to much better PMEPR at the cost cubic computational complexity at the transmitter. At the end of this Chapter, we also introduce another PMEPR reduction method using constellation modification by adjusting the amplitude and signs of each subcarrier.

Finally in Chapter 9, we discuss a few interesting open problems that have been brought up by the research undertaken in this thesis.

Chapter 2

Basics of Broadcast Channels

2.1 Introduction

A broadcast channel consists of one transmitter, many receivers (say n), and a transmission medium that is shared by all the receivers. This channel was first introduced by Cover in [22] in the 1970s and since then has attracted a lot of interest in the research community. Here R_i refers to the rate for the i'th receiver (or user). A rate n-tuple $(R_1, \ldots, R_n)$, is said to be achievable for the broadcast channel if reliable communication at these rates for each user is possible. The capacity region of a broadcast channel is the closure of the set of achievable rates.

Clearly, the downlink scheduling in cellular systems can be modeled as a broadcast channel in which the channels to the users are randomly changing over time. Therefore, information theoretic results on broadcast channels provide the limits for the maximum achievable rates for each receiver in the downlink of cellular systems. This further motivates the study of the capacity region and transmission schemes (i.e., scheduling) in broadcast channels.

The capacity region for single antenna broadcast channels was first studied in the 1970s by several authors (see for example [22, 11] and the references therein). It has been shown that the single antenna broadcast channel falls into the category of degraded channels and its capacity region is known to be achieved by superposition coding [11]. Here degradedness (roughly) means that the broadcast channel is equivalent to the cascade of n channels where the output of each one corresponds to one of the receivers. Superposition coding is a method to superimpose high rate and low rate information to different users in one codeword. In this coding the best user will be able to decode the information to all n users. This will be further explained in Section 2.2.

Recently, there has been a large amount of interest in the area of multi-antenna broadcast channels; more specifically, the capacity region of MIMO (Multiple Input Multiple Output) Gaussian broadcast channels in which the additive noise is modeled as a white Gaussian noise. This recent interest is mainly due to an impressive capacity increase in a point-to-point communication link using multiple transmit and receive antennas. Unfortunately, MIMO broadcast channels in general do not fall into the category of degraded channels, and therefore, the problem cannot be solved with the known information-theoretic techniques. In the case of *Gaussian* MIMO broadcast channels and when *the channel state information is known in both transmitter and receivers (so called full CSI assumption)*, there has been recent progress on the capacity region in the past three years. It is shown by several authors [9, 10, 12, 54] that the sum-rate capacity (which is the point on the boundary of the capacity region that maximizes $\sum_{i=1}^{n} R_i$) is achieved by dirty paper coding (DPC). In fact DPC was first introduced by Costa in 1985 to obtain the capacity of point-to-point channels with known interference at the transmitter and receiver. The main result of Costa is that if the interference is known non-causally, the capacity is the same as that of a channel with no interference and can be achieved by DPC. The main idea behind DPC is to presubtract the interference at the transmitter while maintaining the average power constraint. This technique is then applied by Caire and Shamai [9] to a broadcast channel with two users to obtain the sum-rate capacity and then it is generalized to more than two users in [10, 12, 54]. Very recently, Weingarten et al. [53] proved that the capacity region can be achieved by dirty paper coding.

While the above results suggest that sum-rate capacity increases linearly in the number of transmit antennas, they all rely on the assumption that the channel is known perfectly at the transmitter. In many applications, however, it is not reasonable to assume that all the channel coefficient to every user can be made available to the transmitter. This is especially true if the number of transmit antennas M and/or the number of users n is large (or if the users are mobile and are moving rapidly). Therefore, it would be worthwhile analyzing the sum-rate (or throughput) of the broadcast channel under different amounts and types of channel state information (CSI) at the transmitter or the receivers.

For point-to-point multiple antenna links, it is known that the capacity scales linearly with the minimum of the number of transmit and receive antennas, no matter whether CSI is available in either the transmitter or the receivers [51, 18, 19, 20]. It also should be mentioned that, although CSI does not affect the capacity that much, it can greatly simplify the decoding/encoding in the system.

The main goal of this chapter is to look into the capacity of broadcast channels under two extreme cases, namely, when the transmitter has full CSI or no CSI. Our main focus would be on the sum-rate capacity of the system as it refers to the maximum throughput in the downlink scheduling. It also should be mentioned that the sum-rate capacity can be written as a $\log\det$ convex optimization problem with n variables and can be therefore evaluated numerically. However, little analytical insight is gained from the expression. In order to better understand the behavior of the sum-rate capacity, we investigate the scaling laws of the sum-rate capacity of Gaussian MIMO broadcast channels with many users n when the transmitter has M antennas and each receiver is equipped with N antennas. Previously, in [14, 16], asymptotic results for the sum rate of DPC and beamforming have been derived when n and M have the same growth rate. Furthermore, in [15], the asymptotics of the sum-rate for DPC is derived for large signal to noise ratios and large M when the other parameters of the system are fixed. However, motivated by a cellular system with large number of users (say 100) and having $M \leq 5$, we consider a different region in which n is large and M is either fixed or growing to infinity with much less pace, i.e., logarithmically with n.

Here is the outline of this chapter. Section 2.2 introduces our channel model and notation. In Section 2.3, we review the existing results on the capacity of single and multi-antenna broadcast channels under differing CSI assumptions. In Section 2.4, we obtain the scaling laws of the sum-rate for the MIMO broadcast channel using different scheduling schemes and Section 2.5 concludes the chapter.

2.2 Channel Model and Notation

A broadcast channel is defined as a channel with one transmitter and many receivers where the transmitter or the receivers may be equipped with multiple antennas as shown in Fig. 1.1. As we are interested in a cellular system, throughout the thesis, we assume the information intended for different users is independent.

We consider a Gaussian broadcast channel with n users, a transmitter with M antennas and receivers equipped with N antennas. Therefore we may write the received vector at the i'th receiver at the t'th channel use as

$$Y_i(t) = H_i(t)S(t) + N_i(t), \qquad i = 1, \ldots, n, \tag{2.1}$$

where $H_i(t)$ is an $N \times M$ matrix that represents the channel, $S(t)$ $(M \times 1)$ is the transmit symbol, and $N_i(t)$ $(N \times 1)$ is the additive noise. Both $H_i(t)$'s and $N_i(t)$'s have independently and identically distributed (i.i.d.) complex Gaussian entries with zero mean and unit variance, $CN(0, 1)$. $N_i(t)$ here randomly and independently changes to another value at $t + 1$'th channel use.

In a wireless channel, we need to model the environment between the transmitter and the receivers as well. Here, we consider a block fading narrowband model for the channel, which implies that the channel, e.g. $H_i(t)$, remains constant for T channel uses where T is the coherence interval of the channel. The channel then randomly changes to another value (see [41] for example and reference therein). Here T denotes the coherence time of the channel.

Moreover, the expected value of the total transmit power is assumed to be P, i.e., $E\{S^*S\} = P$. Therefore the received signal to noise ratio (SNR) of the i'th user will be $E\{|H_iS|^2\} = P$. Throughout the thesis it is assumed that the users are homogeneous, i.e., they have the same SNR. In this thesis, we only consider a short-term average transmit power constraint which implies that the transmitter has to use the power P at each channel use. This is mainly motivated by the fact that the base station is subject to a short term power constraint which should be satisfied for each fading state.

Assuming that the channel changes over time in the stationary and ergodic manner, throughout the thesis, we focus on the ergodic rates and the capacity region refers to the ergodic capacity region. Therefore, computation of the rates involves averaging over the channel realization. We may then drop the time index from (2.1) for the sake of brevity.

Throughout the thesis we extensively use notation for orders as follows: $f(x) = O(g(x))$ means that there exists a non-negative constant α such that $\lim_{x\to\infty} |\frac{f(x)}{g(x)}| \leq \alpha$. Similarly $f(x) = o(g(x))$ denotes that the limit goes to zero. Finally, $f(x) = \Theta(g(x))$ implies that there exist positive constants α and β such that $\beta \leq \lim_{x\to\infty} |\frac{f(x)}{g(x)}| \leq \alpha$.

2.3 Earlier Results on the Capacity of Broadcast Channels

In this section, we review the earlier results on the capacity of broadcast channel under differing CSI assumptions. We start with single antenna systems and then we present the results for the multi-antenna broadcast channel.

2.3.1 Single Antenna Broadcast Channel

As mentioned, the broadcast channel was first introduced by Cover [22]. It is shown that a single antenna broadcast channel falls into the category of degraded channels. The capacity region for degraded broadcast channel has been shown to be achieved by superposition coding [11].

To further clarify the definitions, we consider a broadcast channel with 2 users. Similar results are valid for channels with more than 2 users. The broadcast channel consists of an alphabet $\mathcal{X}$ and two output alphabets $\mathcal{Y}_1$ and $\mathcal{Y}_2$ and a probability transition function $p(y_1, y_2|x)$. A code $((2^{KR_1}, 2^{KR_2}), K)$ for a broadcast channel consists of an encoder

$$X : (\{1, 2, \ldots, 2^{KR_1}\} \times \{1, 2, \ldots, 2^{KR_2}\}) \to \mathcal{X}^K$$

and two decoders corresponding to each user as

$$\mathcal{Y}_1^K \to \{1, 2, \ldots, 2^{KR_1}\}$$
$$\mathcal{Y}_3^K \to \{1, 2, \ldots, 2^{KR_2}\}$$

We define the average probability of error as the probability P_e^K that the decoded message is not equal to the transmitted message. A rate pair (R_1, R_2) is then said to be achievable for the broadcast channel if there exists a sequence of $((2^{KR_1}, 2^{KR_2}), K)$ codes such that P_e^K tends to zero as K becomes large. The capacity region is the closure of the set of achievable rates. It should be mentioned that the capacity region depends only on the conditional of the marginal distributions of $p(y_1|x)$ and $p(y_2|x)$.

A broadcast channel is called physically degraded if $p(y_1, y_2|x) = p(y_1|x)p(y_2|y_1)$. Since the capacity region only depends on the marginal, we can generalize the idea of physically degraded channels and define stochastically degraded channels that have the same capacity region. A broadcast channel is call stochastically degraded if its conditional marginal distribution is the same as that of a physically degraded one, i.e., if there exists a distribution $p'(y_2|y_1)$ such that

$$p(y_2|x) = \sum_{y_1} p(y_1|x)p^{'}(y_2|y_1)$$

A physically degraded channel means that the channel between X and Y_2 can be represented as the cascade of the channel between X and Y_1 (denoted by $p(Y_1|X)$) and another channel between Y_1 and Y_2 (denoted by $p(Y_2|Y_1)$). This definition can be easily generalized to a channel with more than two users.

The capacity region can be represented as a convex hull of the closure of all (R_1, R_2) satisfying

$$\begin{aligned} R_2 &\leq I(U; Y_2), \\ R_1 &\leq I(X; Y_1|U) \end{aligned} \tag{2.2}$$

for some joint distribution $p(u)p(x|u)p(y, z|x)$ where $I(\cdot;\cdot)$ denotes the mutual information and the auxiliary random variable U has cardinality bounded by the cardinality of $\mathcal{X}, \mathcal{Y}_1$ and $\mathcal{Y}_2$. Here the auxiliary random variable U will serve as a cloud center that can be distinguished by both receivers $\mathcal{Y}_2$ and $\mathcal{Y}_1$. Each cloud consists of

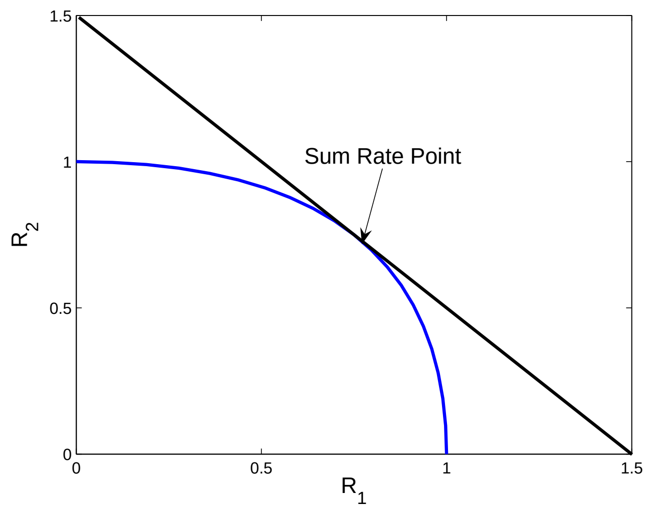

Figure 2.1: Ergodic capacity region for a homogeneous broadcast channel with two users

2^{KR_1} codewords $\mathcal{X}^K$ distinguishable by the receiver Y_1. The worst receiver can only see the clouds, while the better receiver can see the individual codewords within the clouds. Therefore the best receiver can decode the information to the worst receiver. This is in fact the main idea behind superimposing the information intended to both receivers in one codeword.

While the previous results hold for a general broadcast channel, in this thesis we are interested in a time-varying Gaussian broadcast channel defined in Section 2.2. The ergodic capacity of a gaussian broadcast channel can be also achieved by superposition coding. Fig. 2.1 shows the ergodic capacity region for the two-user homogeneous broadcast channel. Clearly the maximum throughput or the sum-rate of the broadcast channel refers to the point on the boundary of the region that maximizes $R_1 + R_2$ as shown in Fig. 2.1.

In a fading broadcast channel with short-term average power constraint, it has been proved that to achieve the sum-rate capacity, the transmitter has to assign all the power to the receiver with the best channel condition [28]. This is intuitively obvious since if the transmitter assigns some portion of the power to a receiver with less capacity, it is clear that the sum of the rates will be less than the case where the transmitter assigns all its power to the user with the largest capacity. We call this opportunistic time-sharing method of transmission to only the best user as the *opportunitic scheduling* for brevity [28, 57].

It is worth mentioning that the single antenna broadcast channel is degraded even when channel state information is not available at either the receivers or the transmitter. For the multi-antenna broadcast channel, however, the channel is not degraded when full CSI is available at the transmitter and the receivers. Intuitively, the reason is that the channels are matrices and it is not clear how the transmitter can compare the channels of different receivers with each other. In the next subsection, we review the results for the capacity region of a multi-antenna broadcast channel for differing amount of CSI at the transmitter and the receivers.

2.3.2 MIMO Broadcast with no CSI at the Transmitter

Assuming that the transmitter or the receivers are equipped with multiple antennas, the MIMO broadcast channel is still degraded if the transmitter has no CSI [9]. This result is valid for both cases where the receivers either have full CSI or have no CSI.

Knowing that the channels are degraded, we can state that the capacity region of the MIMO broadcast channel defined is with superposition coding [11] when the transmitter has no CSI, irrespective of whether the receivers have full CSI or not.

Knowing the capacity region, we investigate the behavior of the sum-rate capacity with respect to the number of users and the number of receive/trasnmit antennas. Suppose $\mathcal{C}_{BC}$ is the capacity region of the channel, hence any n-tuple rate vector $(R_1, \ldots, R_n)$ belonging to the set $\mathcal{C}_{BC}$ is achievable. Since the MIMO BC channel is degraded, the sum-rate capacity can be bounded by the individual capacity[1] of the

[1]By individual capacity C_i, we mean the capacity of the point-to-point link between the trans-

user with the largest number of receive antennas [11],

$$\max_{(R_1,\ldots,R_n)\in\mathcal{C}_{BC}} \sum_{i=1}^{n} R_i \leq \max_{1\leq i\leq n} C_i, \tag{2.3}$$

where C_i is the individual capacity of the i'th user.

It is known that with full CSI in the receiver and for large signal to noise ratios (SNR), the capacity of the point-to-point communication link between the transmitter and the first receiver, C_1, scales like [18],

$$C_1 = \min(M, N) \log P + O(1), \tag{2.4}$$

where we used the fact that the total average transmit power is P.

For the case with no CSI in the receivers, it is also shown that for large signal-to-noise ratios [19, 20],

$$C_1 = \min(M, N, \lfloor T/2 \rfloor) \left(1 - \frac{\min(M, r_1, \lfloor T/2 \rfloor)}{T}\right) \log P + O(1), \tag{2.5}$$

where T is the coherence interval of the channel.

Assuming $M > N$, Eqs. (2.4), (2.5) together with (2.3) imply that the sum-rate capacity with no CSI at the transmitter does not depend on M for large SNRs no matter whether the receivers have full CSI or not. Furthermore, the sum-rate capacity does not depend on the number of users n, which is to be expected, since we have no information about the users at the transmitter and the transmitter cannot exploit the multiuser nature of the system.

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mitter and the i'th receiver.

2.3.3 MIMO Broadcast with Full CSI at the Transmitter

As mentioned, when full CSI is available at the transmitter, the broadcast channel is not degraded as it is not clear how to compare channel matrices with each other. Therefore the problem becomes more complicated and is still open in the general case. However, for the Gaussian broadcast channel, it has been shown shown by several authors [9, 10, 12] that the sum-rate capacity is achieved by so called *dirty paper coding* (DPC). In this scheme, since the interference is known non-causally, the transmitter presubtracts the interference. The main challenge however is to satisfy the average transmit power constraint. This can be done by dirty paper coding, which was first proposed by M. Costa [9]. Only recently, it has been proved that the capacity region of multi-antenna Gaussian broadcast channels in fact can be achieved by dirty paper coding [53]. It should be mentioned that the dirty paper coding scheme, especially in the multi-user context, is extremely computationally intensive [55]. This is mainly due to the fact that the implementation of DPC requires a vector quantization of large dimensions which has exponential complexity. Having mentioned that, suboptimal schemes such as channel inversion or Tomlinson-Harashima precoding [54, 55, 56] have been proposed that give relatively close performance to the optimal schemes.

Here we consider the system model in (2.1), for a channel with n receivers each one equipped with N antennas and a base station with M antennas. Given a set of $N \times N$ positive semi-definite matrices $P_i \geq 0$ which satisfy the power constraint $tr(\sum_{i=1}^{n} P_i \leq P$, and a permutation function π on the set $\{1, \ldots, n\}$, the following rates are achievable in the MIMO Gaussian broadcast channel using DPC [53]

$$R_k^{DP}(\pi, P_1, \ldots, P_n) = \frac{1}{2} \log \frac{|H_k(\sum_{i=1}^{k} P_{\pi(i)})H_k^* + I|}{|H_k(\sum_{i=1}^{k-1} P_{\pi(i)})H_k^* + I|} \tag{2.6}$$

for all $k = 1, \ldots, n$. The DPC region is given by the convex hull of all the achievable rates as

$$\mathcal{R}^{DP} = \mathcal{C}\left(\bigcup_{\pi}\bigcup_{P_1,\ldots,P_n} R_k^{DP}(\pi, P_1, \ldots, P_n)\right) \tag{2.7}$$

In [53], it is further proved that the capacity region of the MIMO Gaussian broadcast channel is equivalent to the DPC rate region defined in (2.7).

In fact, the expression for the sum-rate capacity can be simplified and written as,

$$R_{DP} = E\left\{\max_{\{P_1\geq 0,\ldots,P_n\geq 0, tr(\sum P_i)\leq P\}} \log\det\left(1+\sum_{i=1}^{n} H_i^* P_i H_i\right)\right\}, \tag{2.8}$$

where H_i are $N \times M$ channel matrices with i.i.d. $CN(0,1)$ distributions, and P is the average power constraint. Since the $\log\det$ function is concave and the function inside the $\log\det$ is a linear in P_i's, the maximization inside the expectation in (2.21) is a concave optimization problem and can be solve in polynomial time. However, as the number of users increases, numerically evaluating R_{DP} becomes cumbersome. Furthermore, the expression in (2.21) gives little inside into the behavior of the sum-rate as a function of the number of users and number of transmit/receive antennas. Therefore, in the next sections, we look into the scaling behavior of the sum-rate capacity for broadcast channels. Along with dirty paper coding, we also analyze the scaling law of the sum-rate when the transmitter uses the opportunistic scheduling to the user with the maximum instantaneous capacity. This is a very simple scheduling while achieving the sum-rate capacity for a single antenna broadcast channel.

2.4 Scaling Laws of Sum-Rate in MIMO Systems

As mentioned in previous section, numerically evaluating the sum-rate capacity requires a lot of computation and lead to little inside specially for a large number of users. As in cellular systems, the number of users is much larger than the number of antennas, in this section, we investigate the scaling laws of the sum-rate capacity of

Gaussian MIMO broadcast channels with many users n using opportunitic transmission and DPC when the transmitter has M antennas and each receiver is equipped with N antennas. Previously, in [14, 16], asymptotic results for the sum-rate of DPC and beamforming have been derived when n and M have the same growth rate. Furthermore, in [15], the asymptotics of the throughput for DPC and opportunistic time-sharing are derived for large signal to noise ratios and large M when the other parameters of the system are fixed. However, motivated by a cellular system with a large number of users (say 100) and having $M \leq 5$, we consider a different region in which n is large and M is either fixed or growing to infinity with much less pace, i.e., logarithmically with n.

2.4.1 Scaling Laws of Opportunistic Time-Sharing

In a single antenna broadcast system with full CSI at the transmitter, the sum-rate capacity can be achieved by using time-sharing and sending to the user with the largest simultaneous capacity. However, for a multi-antenna broadcast system with full CSI at the transmitter, this is not the case. In this section, we derive the scaling laws for the sum-rate of multi-antenna broadcast channels using time-sharing to the strongest user, e.g., opportunistic transmission, for a large number of users.

It is clear that, by only sending to the strongest user, the sum-rate (denoted by $E\{R_{ts}\}$) can be written as [15, 13]

$$E\{R_{ts}\} = E\left\{\max_{i=1,\ldots,n} C(H_i, P)\right\} = E\left\{\max_{i=1,\ldots,n} \max_{P_i \geq 0, tr(P_i) \leq P} \log\det\left(I + H_i P_i H_i^*\right)\right\}, \tag{2.9}$$

where $C(H_i, P)$ is the capacity of the link between the transmitter and the i'th receiver with the channel matrix H_i, and P_i $(M \times M)$ is the optimal covariance matrix of the transmitted signal. Lemma 2.1 considers the case where M and N are fixed and n grows to infinity. Lemma 2.2 derives the result for the case that M is also growing to infinity but logarithmically with n.

Lemma 2.1 *For M, N, and P fixed, we have*

$$\lim_{n\to\infty} \frac{E\{R_{ts}\}}{\min(M,N)\log\log n} = 1. \tag{2.10}$$

Proof: First of all we assume $M \geq N$, the case where $N > M$ can be analyzed in the same way. Using the inequality $\det(A) \leq \left(\frac{tr(A)}{N}\right)^N$ where A is an $N \times N$ matrix, we can bound $C(H_i, P)$ as

$$C(H_i, P) \leq N \log\left(1 + \frac{1}{N} tr(H_i P_i H_i^*)\right). \tag{2.11}$$

Defining $H_i = [h_1^i \ \ldots \ h_M^i]$ (where h_j^i $(N \times 1)$ is the j'th column of H_i), we can use the inequality

$$tr(H_i P_i H_i^*) \leq \max_{1\leq j\leq M} h_j^{i*} h_j^i tr(P_i). \tag{2.12}$$

We can also find a lower bound by assigning equal power to N transmit antennas instead of M. Clearly, this leads to

$$C(H_i, P) \geq N \log\left(1 + \frac{P}{N}\lambda_{\min}(H_i^{'} H_i^{'*})\right), \tag{2.13}$$

where $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of its argument and the matrix $H_i^{'}$ $(N \times N)$ is a truncated version of H_i $(N \times M)$ by omitting $M - N$ columns of H_i. Using (6.45) and (6.46), we may write the expected sum-rate of the opportunistic scheme as

$$E\left\{N\log\left(1+\frac{P}{N^2}\max_{1\le i\le n}\lambda_{\min}(H_i^{'}H_i^{'*})\right)\right\}\le E\{R_{ts}\}\le$$
$$E\left\{\max_{i=1,\ldots,n}N\log\left(1+\frac{P}{N}\max_{1\le j\le M}h_j^{i*}h_j^i\right)\right\}. \tag{2.14}$$

It is worth noting that $h_j^{i*}h_j^i$ has a $\chi^2(2N)$ distribution. In [66], it is shown that $N\lambda_{\min}(H_i^{'}H_i^{'*})$ is exponentially distributed ([66], Theorem 5.5, p. 62). Therefore, using the results in extreme value theory (see Appendix 10.1), it can be shown that

$$\Pr\left\{\log nM+2(N-2)\log\log nM+O(\log\log\log n)\le\max_{1\le i\le nM}\kappa_i^{'}\right.$$
$$\left.\le\log nM+2N\log\log nM+O(\log\log\log n)\right\}=1-O\left(\frac{1}{\log^2 n}\right), \tag{2.15}$$

where $\kappa_i^{'}$ are i.i.d. and have $\chi^2(2N)$ distribution.

Noting that $N\lambda_{\min}(H_i^{'}H_i^{'*})$ has $\chi^2(2)$ distribution, we can similarly prove that

$$\Pr\left\{\log n-2\log\log n\le\max_{1\le i\le n}N\lambda_{\min}(H_i^{'}H_i^{'*})\le\log n+2\log\log n\right\}=1-O\left(\frac{1}{\log^2 n}\right). \tag{2.16}$$

We also know that the sum-rate capacity is certainly bounded by the capacity of a MIMO single user link with M transmit and nN receiver antennas (cooperation allowed between users), i.e., C is of the order of $M\log n$ [27, 52]. Defining $A=\log nM+2N\log\log nM+O(\log\log\log n)$, we can now derive an upper bound for

$E\{R_{ts}\}$ as

$$\begin{aligned} E\{R_{ts}\} &\leq E\left\{R_{ts}|\max_{1\leq i\leq n}\kappa_i^{'} \leq A\right\}\Pr\left\{\max_{1\leq i\leq n}\kappa_i^{'} \leq A\right\} \\ &\quad +O\left(M\log(nN)\Pr\{\max_{1\leq i\leq n}\kappa_i^{'} \geq A\}\right) \\ &= N\log\left(1+\frac{P}{N}\log n\right)+O(\log\log\log n). \end{aligned} \tag{2.17}$$

Similarly, a lower bound can be written as

$$\begin{aligned} E\{R_{ts}\} &\geq E\left\{R_{ts}|\log n \leq \max N\lambda_{\min}(H_iH_i^*) \leq \log n + 2\log\log n\right\} \times \\ &\quad \Pr\left\{\log n \leq \max N\lambda_{\min}(H_iH_i^*) \leq \log n + 2\log\log n\right\} \\ &= N\log\left(1+\frac{P}{N^2}\log n\right)\left(1-\frac{1}{\log n}\right). \end{aligned} \tag{2.18}$$

Eqs. (2.17) and (2.18) complete the proof and lead to (2.10). □

In practice, the number of users n is large; however, it is not infinite. Therefore, it would be of interest to analyze the sum-rate when M is also growing to infinity but with a much lower pace than n, say as $\log n$. For example, in a cellular network with 100 users and four or five transmit antennas, it is clear that M is in the order of $\log n$. The next lemma analyzes the sum-rate of opportunistic time-sharing when M is of the order of $\log n$.

Lemma 2.2 *For $M = \beta_1 \log n$ where β_1 is a constant independent of n and for N and P fixed, we have*

$$\lim_{n\to\infty}\frac{E\{R_{ts}\}}{N\log\log n} = 1. \tag{2.19}$$

Proof: The proof is along the same line as the proof of Lemma 2.1. Clearly, assuming $M \geq N$, Eq. (2.14) holds for any M, N and P. Furthermore, the derivation of the upper and lower bounds in Lemma 2.1 was based on the distribution of $\max_{1 \le i \le nM} \kappa_i^{'}$ where $\kappa_i^{'}$'s either have $\chi^2(2N)$ or $\chi^2(2)$ distributions for any M and N, respectively. As N is assumed fixed, the bounds both still hold and therefore $E\{R_{ts}\}$ is growing like $N \log\log n$. □

2.4.2 Scaling Laws for Dirty Paper Coding

In [27], assuming a transmitter with M antennas, single antenna receivers and total average transmit power of M, it is proved that the sum-rate capacity of DPC scales like $M \log\log n$ for large values of n and when M is fixed. In this section, we first generalize this result to the case of having multiple antenna users, i.e., $N \ge 1$, and when the average total transmit power is fixed. Again, we further look into the scaling laws of the sum-rate when M is also going to infinity logarithmically with n, i.e., with a much lower pace than n.

In the following lemma, we show that when M is fixed the sum-rate scales like $M \log\log nN$ as n grows to infinity and for any N no matter whether N grows to infinity or not.

Lemma 2.3 *For M and P fixed and any N, we have*

$$\lim_{n \to \infty} \frac{E\{R_{DPC}\}}{M \log\log nN} = 1. \tag{2.20}$$

Proof: The sum-rate in MIMO BC channel has been recently addressed by several authors [9, 10, 12]. Using the duality between the broadcast channel and MAC, the sum-rate of MIMO BC, $E\{R_{DPC}\}$ is equal to [10, 12],

$$E\{R_{DPC}\} = E\left\{ \max_{\{P_1 \ge 0, \ldots, P_n \ge 0, \sum_{i=1}^{n} tr(P_i) \le P\}} \log\det\left(I + \sum_{i=1}^{n} H_i^* P_i H_i \right) \right\}, \tag{2.21}$$

where H_i are $N \times M$ channel matrices with i.i.d. $CN(0,1)$ distributions, P_i $(N \times N)$ is the optimal power scheduling, and P is the total transmit power.

Using the inequality $\det(A) \leq \left(\frac{tr(A)}{M}\right)^M$ where A is an $M \times M$ matrix, we can write (2.21) as

$$E\{R_{DPC}\} \leq ME\left\{\max_{\{P_1,\ldots,P_n,\sum_{i=1}^n tr(P_i)\leq P\}} \log\left(1+\frac{\sum tr(H_i^* P_i H_i)}{M}\right)\right\}. \qquad (2.22)$$

Denoting the matrix $H_i^* = \begin{bmatrix} g_1^{i*} & \ldots & g_N^{i*} \end{bmatrix}$ (where g_j^i $(1 \times M)$ is the j'th row of H_i), we can state the following inequality,

$$tr(H_i^* P_i H_i) \leq \max_{1\leq j\leq N} g_j^i g_j^{i*} tr(P_i). \qquad (2.23)$$

Using (2.23) and (2.22), we obtain

$$\begin{aligned} E\{R_{DPC}\} &\leq ME\left\{\max_{\{P_1,\ldots,P_n,\sum tr(P_i)\leq P\}} \log\left(1+\frac{\sum_{i=1}^n \max_{1\leq j\leq N} g_j^i g_j^{i*} tr(P_i)}{M}\right)\right\} \\ &\leq ME\left\{\max_{\{P_1,\ldots,P_n,\sum tr(P_i)\leq P\}} \log\left(1+\frac{\max_{1\leq k\leq n}\max_{1\leq j\leq N} g_j^k g_j^{k*} \sum_{i=1}^n tr(P_i)}{M}\right)\right\} \\ &= ME\left\{\log\left(1+\frac{P}{M}\max_{1\leq i\leq nN}\kappa_i\right)\right\}. \end{aligned} \qquad (2.24)$$

where κ_i's are i.i.d. random variables with $\chi^2(2M)$ distribution. Eq. (2.15) states that with high probability the maximum of nN i.i.d. random variables with $\chi^2(2M)$ distribution behaves like $\log nN + O(\log\log n)$ (see Appendix 10.1 and also Eq. (2.16)). Therefore, similar to the argument in (2.17), we may write

$$E\{R_{DPC}\} \leq M\log(1+P\log nN) + O(\log\log\log n). \qquad (2.25)$$

To prove that $M\log\log nN$ is achievable, we use the scheme proposed in [27] with partial side information that achieves $M \log\log nN$ when M is fixed. It is worth noting that in [27], the average transmit power was M ($P = M$), however, since M is fixed, it is easy to see that changing the average total transmit power from M to P (another constant) does not affect the scaling law of the sum-rate. Therefore,

$$E\{R_{DPC}\} \geq M \log\log nN + O(\log\log\log n). \tag{2.26}$$

Eq. (2.25) and (2.26) complete the proof of the lemma. □

The next lemma considers a different region in which M is also logarithmically increasing with n.

Lemma 2.4 *For $M = \beta \log n$ and fixed N, P and β, we have*

$$\lim_{n\to\infty} \frac{E\{R_{DPC}\}}{M} = \gamma. \tag{2.27}$$

where γ is a constant independent of n. Furthermore, we can bound γ by $\gamma \leq \log(1+\alpha)$ where α is the unique solution to $\alpha - \beta\log\alpha = 1 + \beta - \beta\log\beta$.

Proof: As we stated in the proof of Lemma 2.1 (i.e., Eq. (2.24)), we can write the following upper bound for the the sum-rate capacity for any n and M,

$$E\{R_{DPC}\} \leq ME\left\{\log\left(1 + \frac{P}{M}\max_{1\leq i\leq nN} \kappa_i\right)\right\}, \tag{2.28}$$

where κ_i's are i.i.d. $\chi^2(2M)$ random variables. The only difference here is that M is also a function of n and is going to infinity. In Appendix 2.6.1, we prove that

$$\Pr\left\{\max_{1\leq i\leq nN} \kappa_i \leq \log n + O(\log\log n)\right\} = 1 - O\left(\frac{1}{\log n}\right), \tag{2.29}$$

the upper bound in the lemma follows by using the same technique as in the proof of Lemma 2.3.

In order to find a lower bound, we may use any suboptimal scheduling and show that its sum-rate is bigger than αM where α is a constant independent of n. This is in fact done in [27] in which a random beamforming method is proposed that achieves the linear scaling of sum-rate with M even when M grows logarithmically with M. This will be more described in Chapter 3. This completes the proof of the lemma. □

2.5 How Much Does CSI Affect the Sum-Rate Capacity?

In this Chapter, we reviewed the results on the capacity region of broadcast channels. It is shown that the amount of channel state information significantly changes the capacity, and in particular, the sum-rate capacity. For instance, in a Gaussian broadcast channel with M transmit antennas and n single antenna users, the sum rate capacity scales like $M \log \log n$ for large n if perfect channel state information (CSI) is available at the transmitter, yet only logarithmically with M if it is not.

In systems with large n, obtaining full CSI from all users may not be feasible. Since lack of CSI does not lead to multi-user gains, it is therefore of interest to investigate transmission schemes that require less complexity and less feedback. We considered the opportunistic scheduling which a simple scheduling that achieves the sum-rate capacity of the single antenna broadcast channel. The scaling laws of DPC and opportunistic scheduling are compared for the case of large number of users and when the transmit antennas are fixed or growing logarithmically with n. Lemma 2.1 and 8.1 imply that for M and N fixed and when $M \geq N$, we have

$$\lim_{n\to\infty} \frac{R_{DP}}{R_{ts}} = \lim_{n\to\infty} \frac{M \log \log n}{N \log \log n} = \frac{M}{N}. \tag{2.30}$$

In fact, Eq. (2.30) proves the conjecture of [15] for a large number of users and when the number of transmit antennas is fixed.

For the case where the number of receive antennas is one, it is clear from (2.30) that opportunistic scheduling leads to scaling law of only $\log\log n$ as opposed to $M\log\log n$ achieved by DPC. This further motivates the investigation of simple transmission schemes that not only require less computation (compared to DPC) but also relaxes the assumption of having full CSI in the transmitter. This is indeed the main goal of the next Chapter in which a transmission scheme is proposed that requires little CSI and achieves a sum-rate that has the same scaling law as that of dirty paper coding.

2.6 Appendix

2.6.1 Proof of Eq. (2.29)

In this appendix, we investigate the behavior of the maximum of n i.i.d. random variable κ_i for $i = 1, \ldots, n$ with $\chi^2(2M)$ distribution where $M = \beta \log n$. Clearly the cumulative distribution function of κ_i can be written as

$$F(x) = \Pr\{\kappa_i \leq x\} = 1 - e^{-x}\sum_{m=0}^{M-1}\frac{x^m}{m!} = 1 - \frac{\Gamma(M,x)}{\Gamma(M)}. \tag{2.31}$$

In order to find the behavior of the maximum of κ_i's, we have to compute $F^n(x)$. Following the technique in [27], we initially solve the the following equality,

$$F(x_l) = 1 - \frac{\Gamma(M,x_l)}{\Gamma(M)} = 1 - \frac{(\log n)^3}{n}. \tag{2.32}$$

It is worth noting that both arguments of the incomplete gamma function in Eq. (5.87) are going to infinity. The asymptotic expansion of the incomplete gamma function has been studied by Tricomi [4, 8] and it is shown that

$$\Gamma(M,x) = \frac{e^{-x}x^M}{x-M+1}\left\{1 - \frac{M-1}{(x-M+1)^2} + \frac{2(M-1)}{(x-M+1)^3} + O\left(\frac{(M-1)^2}{(x-M+1)^4}\right)\right\} \tag{2.33}$$

as the modulus of $\sqrt{M}/(x-M)$ tends to zero. We can also write the asymptotic expansion of the gamma function in [63] as

$$\log\Gamma(M) = M\log M - M - \frac{1}{2}\log M + O(1). \tag{2.34}$$

Using the asymptotic expansions, we can solve (5.87) to get $x_l = \alpha\log n - \frac{5}{2}\log\log n + o(\log\log n)$ where α satisfies

$$\alpha - \beta\log\alpha = 1 + \beta - \beta\log\beta. \tag{2.35}$$

Therefore the probability that the maximum of κ_i's is less than x_l can be written as

$$\Pr\left\{\max_{1\le i\le n}\kappa_i \le x_l\right\} = (F(x_l))^n = \left(1 - \frac{(\log n)^3}{n}\right)^n = O\left(e^{-(\log n)^3}\right). \tag{2.36}$$

We can also similarly find x_u such that $F(x_u) = 1 - \frac{1}{n\log n}$ as $x_u = \alpha\log n + \frac{3}{2}\log\log n + o(\log\log n)$. Therefore,

$$\Pr\left\{\max_{1\le i\le n}\kappa_i \le x_u\right\} = (F(x_u))^n = \left(1 - \frac{1}{n\log n}\right)^n = 1 - O(\frac{1}{\log n}). \tag{2.37}$$

Eqs. (2.36) and (2.37) can be combined to get

$$\Pr\left\{\alpha\log n - \frac{5}{2}\log\log n + o(\log\log n) \le \max_{1\le i\le n}\kappa_i \le \alpha\log n + \frac{3}{2}\log\log n + o(\log\log n)\right\} = 1 - O\left(\frac{1}{\log n}\right).$$

This completes the derivation of Eq. (2.29). □

Chapter 3

Multi-Antenna Broadcast Channels

3.1 Introduction

Multiple-antenna communications systems have generated a great deal of interest since they are capable of considerably increasing the capacity of a wireless link. In fact, it was known for a long time that, if perfect channel state information were available at the transmitter and receiver, then they could jointly diagonalize the channel, thereby creating as many parallel channels as the minimum of the number of transmit/receive antennas and thus increase the capacity of the channel by this same factor. More surprisingly, it was later shown that the same capacity scaling is true if the channel is not known at the transmitter [18, 51] and even if it is not known at the receiver [19, 20] (provided the coherence interval of the channel is not too short).

As mentioned in Chapter 2, while these are all true for point-to-point communications links, there has only been recent interest in the role of multiple-antenna systems in a multiuser network environment, and especially in broadcast and multi-access scenarios. There has been a line of work studying scheduling algorithms in multiple-input multiple-output (MIMO) broadcast channels [52] with the main result being that, due to channel-hardening in MIMO systems, many of the multiuser gains disappear. There has been another line of work studying the sum-rate capacity, and in fact the capacity region, of MIMO broadcast channels [9, 10, 12]. It has been shown that the sum-rate capacity is achieved by dirty paper coding and, moreover recently, it has been shown that dirty paper coding in fact achieves the capacity region of the Gaussian MIMO broadcast channel [53].

While the above results suggest that capacity increases linearly in the number of transmit antennas, they all rely on the assumption that the channel is known perfectly at the transmitter. Moreover, the dirty paper coding scheme, especially in the multiuser context, is extremely computationally intensive (although suboptimal schemes such as channel inversion or Tomlinson-Harashima precoding [54, 55, 56] give relatively close performance to the optimal schemes). One may speculate whether, as in the point-to-point case, it is possible to get the same gains without having channel knowledge at the transmitter. Unfortunately, it is not too difficult to convince oneself that if no channel knowledge is available at the transmitter, then sum-rate capacity (maximum throughput) scales only logarithmically in the number of transmit antennas. In fact, in this case increasing the number of transmit antennas yields no gains since the same performance can be obtained with a single transmit antenna operating at higher power.

In many applications, however, it is not reasonable to assume that all the channel coefficients to every user can be made available to the transmitter. This is especially true if the number of transmit antennas M and/or the number of users n is large (or if the users are mobile and are moving rapidly). Since perfect channel state information may be impractical, yet no channel state information is useless, it is very important to devise and study transmission schemes that require only partial channel state information at the transmitter. This is the main goal of this chapter.

The scheme we propose is one that constructs M random orthonormal beams and transmits to users with the highest signal-to-noise-plus-interference ratios (SINRs). In this sense, it is in the same spirit as the work of [57] where the transmission of random beams is also proposed.

However, our scheme differs in several key respects. First, we send multiple beams (in fact, M of them) whereas [57] sends only a single beam. Second, whereas the main concern in [57] is to improve the proportional fairness of the system (by giving different users more of a chance to be the best user) our scheme aims at capturing as much of the broadcast channel capacity as possible. Fairness[1] is achieved in our system as a convenient by-product.

We should remark that our scheme requires far less feedback than one that provides full channel state information. To have full channel knowledge at the receiver, each user must feedback M complex numbers (its channel gains) to the transmitter. Here each user need only feed back one real number (its best SINR) and the corresponding index, which is an integer number. In fact, it turns out that only users who have favorable SINRs need to do so, which can considerably reduce the amount of feedback required.

Based on asymptotic analysis, we show that, for fixed M and n increasing, our proposed scheme achieves a throughput[2] of $M \log\log nN$, where N is the number of receive antennas of each user. Happily, this is the same as the scaling law of the sum-rate capacity when perfect channel state information is available, as shown in Chapter 2 [58], and so, asymptotically, our scheme does not suffer a loss in this regime. One may ask, how fast may M grow to guarantee a linear scaling of the throughput R with M? We show that the answer is $M = O(\log n)$; more precisely, if $\frac{M}{\log n} \to \alpha$ then $\frac{R}{M} \to \alpha'$, whereas if $\frac{M}{\log n} \to \infty$ then $\frac{R}{M} \to 0$.

In schemes (such as ours) that exploit multiuser diversity there is often tension between increasing capacity (by transmitting to the strongest users) and fairness. The reason is that the strongest users (here meaning the users closest to the base sta-

[1] In this thesis, by fairness we mean that the probability of choosing users with different signal to noise ratios is equal.

[2] In this thesis, throughput refers to the achievable sum average rate by the scheduling scheme.

tion) may dominate the network. Fortunately, we show that in our scheme, provided the number of transmit antennas is large enough, the system becomes interference dominated and so, although close users receive strong signal they also receive strong interference. Therefore it can be shown that, for large enough M and in a heterogeneous network, the probability of any user having the highest SINR converges to $\frac{1}{n}$, irrespective of how strong their signal strength is. A more careful study of this issue reveals that the choice of $M = \alpha \log n$ transmit antennas is a desirable operating point, both in terms of providing linear scaling of the throughput with M as well as in guaranteeing fairness.

The remainder of this chapter is organized as follows: Section 3.2 describes the formulation of the problem. Our proposed scheduling algorithm is introduced in Section 3.3. In Section 3.4, the asymptotic analysis of the throughput of our scheme is presented for the case where the number of users is increasing, M (number of transmit antennas) is fixed, and each user has single receive antenna ($N = 1$). Section 3.5 considers the case where M is allowed to grow to infinity as well. In Section 3.6, different scenarios for $N > 1$ are considered and the asymptotic behavior of their throughput is obtained. Fairness of our scheduling when the users have different signal to noise ratios is considered in Section 3.7. Section 3.8 presents the simulation result for the throughput and fairness of our proposed scheduling. Finally Section 3.9 concludes the chapter.

3.2 Problem Formulation

We consider a multi-antenna Gaussian broadcast channel with n receivers equipped with N antennas and a transmitter with M antennas. We consider the block fading model for the channel described by a propagation matrix that is constant during the coherence interval of T. Since in a typical cellular system, the number of users is much larger than the number of transmit antennas, and also the number of antennas in the base station (or the transmitter) is greater than the number of antennas in the receiver, we often assume $n \gg M$ and $N \leq M$ throughout the chapter.

Let $S(t)$ be the $M \times 1$ vector of the transmit symbols at time slot t, and let $Y_i(t)$ be the $N \times 1$ vector of the received signal at the i'th receiver related by

$$Y_i(t) = \sqrt{\rho_i} H_i S(t) + W_i, \qquad i = 1, \ldots, n, \tag{3.1}$$

where H_i is an $N \times M$ complex channel matrix, known perfectly to the receiver, W_i is a $N \times 1$ additive noise, and the entries of H_i and W_i are i.i.d. complex Gaussian with zero mean and variance one, $CN(0,1)$. Moreover, the total transmit power is assumed to be M, i.e., $E\{S^*S\} = M$, in other words, the transmit power per antenna is one[3]. Therefore the received signal to noise ratio (SNR) of the i'th user will be $E\{\rho_i |H_i S|^2\} = P = M\rho_i$; however to simplify the notation we refer to ρ_i as the SNR of the i'the user.

To analyze the throughput of the system we consider a homogeneous network in which all the users have the same SNR, i.e., $\rho_i = \rho$ for $i = 1, \ldots, n$. However, in the last part of the chapter, we look into the fairness issue when the network is heterogeneous in which the users have different SNRs.

3.3 Scheduling Based on Random Beamforming

The capacity of point-to-point multi-antenna systems has been investigated with different assumption for the channel state information (CSI) whether the receiver or the transmitter knows the channel or not. As it is shown in [18, 51] if the receiver

[3]This is in contrast to the convention used in single link MIMO channels where the total transmit power, $E\{S^*S\} = 1$, is fixed. There this is done to make a fair comparison with a single antenna channel operating at the same transmit power. Here, however, since we will be transmitting to M different users, we would like to make a comparison to M independent single antenna links each operating at unit power. Hence our normalization will be $E\{S^*S\} = M$.

knows the channel, the capacity scales like $\min(M,N)\log\rho$ no matter whether the transmitter knows the channel or not. Indeed, it is shown in [19, 20] that when the receiver does not know CSI, the capacity scales like $\min(M,N)(1-\frac{\min(M,N)}{T})\log\rho$ where T is the coherence interval of the channel.

While the full CSI in the transmitter does not seem to be beneficial in the point-to-point communication, the knowledge of the channel is crucial in broadcast channels [52, 21]. For the case with the full CSI available at both the transmitter and the receivers, it is shown that the sum rate capacity of the Gaussian broadcast channel can be achieved by using dirty paper coding [9, 10, 12]. More precisely, for the case where $N=1$, the sum rate capacity, R_{DP}, can be written as

$$R_{DP} = E\left\{\max_{\{P_1,\ldots,P_n,\sum P_i = M\rho\}} \log\det\left(1+\sum_{i=1}^{n} H_i^* P_i H_i\right)\right\}, \tag{3.2}$$

where H_i is $1\times M$ channel matrix and $M\rho$ is the total average power. In the previous chapter, the following lemma is proved (see also [102]):

Lemma 3.1 *Suppose both the transmitter and receivers know the channel perfectly in a Gaussian broadcast channel with n single antenna receivers with average transmit power of $M\rho$, and the transmitter has M antennas. Let also M and ρ be fixed, then for sufficiently large n, the sum rate capacity scales like $M\log\log n$.*

Therefore when the transmitter and receivers have full CSI, the sum-rate capacity scales linearly with M. On the other hand, having full CSI in both sides requires a lot of feedback and practically it is unrealistic. This motivates the question of how much partial side information is needed in the transmitter that provides us a linear scaling of the throughput with M and reduces the amount of feedback [59, 52, 21].

In order to exploit having multiple antennas in the transmitter without having full CSI in the transmitter, we propose a scheme that constructs M random beams and transmits to the users with the highest signal to interference plus noise ratios

(SINRs). For simplicity, we assume $N = 1$ and we choose M random orthonormal vectors ϕ_m ($M \times 1$) for $m = 1, \dots, M$ where ϕ_i's are generated according to an isotropic distribution [20]. Then at time slot t, the m'th vector is multiplied by the m'th transmit symbol $s_m(t)$, so that the transmitted signal is

$$S(t) = \sum_{m=1}^{M} \phi_m(t) s_m(t), \qquad t = 1, \dots, T. \tag{3.3}$$

Following our earlier assumption and using the independence of s_i's, the average transmit power per antenna is one, equivalently $E\{|s_i|^2\} = 1$, and henceforth the total transmit power is $E\{S^* S\} = M$. After T channel uses, we independently choose another set of orthonormal vectors $\{\phi_m\}$, and so on. We assume s_m's are letters from codewords of a Gaussian capacity-achieving codebook. We further assume that the coding is performed across several blocks.

From now on, for simplicity, we drop the time index from $S_i(t)$ and $Y_i(t)$, and therefore, the received signal at the i'th receiver is

$$Y_i = \sum_{m=1}^{M} \sqrt{\rho} H_i \phi_m s_m + W_i, \qquad i = 1, \dots, n. \tag{3.4}$$

We assume that the i'th receiver knows $H_i \phi_m$ for $m = 1, \dots, M$ (this can be readily arranged by training). Therefore, the i'th receiver can compute the following M SINRs by assuming that s_m is the desired signal and the other s_i's are interference as follows,

$$\mathrm{SINR}_{i,m} = \frac{|H_i \phi_m|^2}{1/\rho + \sum_{k \neq m} |H_i \phi_k|^2}, \qquad m = 1, \dots, M. \tag{3.5}$$

Note that on average the SINRs behave like[4] $\mathrm{SINR}_{i,m} \approx \frac{1}{1/\rho + (M-1)} \approx \frac{1}{M-1}$. Thus if

[4] This can be made more precise, however for the sake of brevity we just mention a sketchy argument.

we randomly assign beams to users, the throughput will be

$$\begin{aligned} R = E\left\{\sum_{i=1}^{M} \log\left(1+\text{SINR}_{i,m}\right)\right\} &= ME\log(1+\text{SINR}_{i,m}) \\ &\leq M\log(1+E\{\text{SINR}_{i,m}\}) \\ &\approx M\log(1+\frac{1}{M-1}) \\ &< \frac{M}{M-1} \approx 1. \end{aligned} \tag{3.6}$$

Thus, even though we are sending M different signals, we do not get M-fold increase in the throughput. Therefore, the side information in the transmitter is crucial to exploit the multiuser diversity.

Suppose now each receiver feeds back its maximum SINR, i.e., $\max_{1\leq m\leq M}\text{SINR}_{i,m}$, along with the index m in which the SINR is maximized. Therefore, in the transmitter, instead of randomly assigning each beam to one of the users, the transmitter assigns s_m to the user with the highest corresponding SINR, i.e., $\max_{1\leq i\leq n}\text{SINR}_{i,m}$. So if we do the above scheduling, the throughput can be written roughly as

$$R \approx E\left\{\sum_{m=1}^{M} \log\left(1+\max_{1\leq i\leq n}\text{SINR}_{i,m}\right)\right\} = ME\left\{\log\left(1+\max_{1\leq i\leq n}\text{SINR}_{i,m}\right)\right\}, \tag{3.7}$$

where we used "$\approx$" instead of "$=$" since there is a small probability that user i may be the strongest user for more than one signal s_m. In Section 3.4, we shall see that this is very unlikely as n increases, and so the above approximation approaches equality.

It is important to note that compared to (3.6), we have a maximization over i inside the logarithm. Thus, we need to study the distribution of $\max_{1\leq i\leq n}\text{SINR}_{i,m}$, which, as we shall see, has a huge effect on the end result. We also remark that, as we shall see in Section 3.4, it is not even necessary for all users to send back their strongest SINR, which considerably reduces the required feedback.

3.4 Throughput Analysis: $N = 1$, M is fixed

In this section we obtain lower and upper bounds for the throughput when M is fixed, $N = 1$, and n is going to infinity. Using M random beams and sending to the users with the highest SINRs, we can bound the throughput R_{MBF}, as

$$R_{MBF} \leq E\left\{\sum_{m=1}^{M} \log\left(1 + \max_{i=1,\ldots,n} \text{SINR}_{i,m}\right)\right\}, \tag{3.8}$$

where this is an upper bound since we ignored the probability that user i be the maximum SINR user twice (if this is the case, the transmitter has to choose another user with SINR less than the maximum SINR, which therefore decreases the throughput). On the other hand, the following lemma states a lower bound for the throughput as well.

Lemma 3.2 *Let R_{MBF} be the throughput of the random beamforming scheduling. Then,*

$$R_{MBF} \geq M\left(1 - \{\Pr\{\text{SINR}_{i,1} \leq 1\}\}^n\right) E\left\{\log\left(1 + \max_{i=1,\ldots,n} \text{SINR}_{i,1}\right) | \max_{i=1,\ldots,n} \text{SINR}_{i,1} \geq 1\right\}. \tag{3.9}$$

Proof: First of all we make the following observation: For any $r \in \{1, \ldots, M\}$, conditioning on the fact that $\text{SINR}_{j,r} = \max_{1 \leq i \leq n} \text{SINR}_{i,r} \geq 1$, then $\text{SINR}_{j,r}$ has to be the maximum over $m = 1, \ldots, M$ as well, i.e., $\text{SINR}_{j,r} = \max_{m=1,\ldots,M} \text{SINR}_{j,m}$. This can be easily proved as follows; assuming $\text{SINR}_{j,r} > 1$, we have

$$|H_j \phi_r|^2 \geq 1/\rho + \sum_{k \neq r} |H_j \phi_k|^2 \geq |H_j \phi_m|^2 \qquad m = 1, \ldots, M.$$

Now we can write $\mathrm{SINR}_{j,m}$ and $m \neq r$ as

$$\mathrm{SINR}_{j,m} = \frac{|H_j\phi_m|^2}{1/\rho + \sum_{k\neq m}|H_j\phi_k|^2} < \frac{|H_j\phi_m|^2}{|H_j\phi_r|^2} \leq 1 \tag{3.10}$$

and hence $\mathrm{SINR}_{i,r}$ is the maximum over $m = 1\ldots, M$ as well, i.e., $\mathrm{SINR}_{\mathrm{i,r}} = \max_{m=1,\ldots,M} \mathrm{SINR}_{i,m}$. Therefore, it is impossible for a user to be the maximum SINR for two signals conditional on the fact that $\max_{1\leq i\leq n} \mathrm{SINR}_{i,r} \geq 1$. Thus the throughput can be bounded as

$$\begin{aligned} R_{MBF} \geq \quad & \sum_{m=1}^{M} \Pr\left\{\max_{i=1,\ldots,n} \mathrm{SINR}_{i,m} \geq 1\right\} \times \\ & E\left\{\log\left(1 + \max_{i=1,\ldots,n} \mathrm{SINR}_{i,m}\right) | \max_{i=1,\ldots,n} \mathrm{SINR}_{i,m} \geq 1\right\}. \end{aligned} \tag{3.11}$$

Since $\Phi = (\phi_1 \ldots \phi_M)$ is a unitary matrix, so $H_i\Phi$ is a vector with i.i.d. $CN(0,1)$ entries. This implies that $|H_i\phi_m|^2$ are i.i.d. over m (and also over i) with $\chi^2(2)$ distribution. Therefore $\mathrm{SINR}_{i,m}$ for $i = 1, \ldots, n$, are i.i.d. but not independent over $m = 1, \ldots, M$. Thus

$$\Pr\left\{\max_{i=1,\ldots,n} \mathrm{SINR}_{i,m} \leq 1\right\} = \left(\Pr\left\{\mathrm{SINR}_{i,1} \leq 1\right\}\right)^n. \tag{3.12}$$

Substituting (3.12) in (3.11) completes the proof. □

As we shall show later, the lower and upper bounds for the throughput become tight for sufficiently large n and when $\lim_{n\to\infty} \frac{M}{\log n} = 0$. In this case, conditioning on $\max_{1\leq i\leq M} \mathrm{SINR}_{i,m} \geq 1$ in Lemma 3.2 can be replaced by $\max_{1\leq i\leq M} \mathrm{SINR}_{i,m} \geq \eta$ where η is a constant independent of n and the bounds remain tight. This implies that the receiver is only required to feed back its maximum SINR if it is greater than η along with the index m corresponding to the signal. Therefore the amount of feedback here will be $n\Pr\{\max_{1\leq m\leq M} \mathrm{SINR}_{i,m} \geq \eta\}$ real numbers, and M integers (at most). However, in the case with full CSI in the transmitter, the amount of feedback is $2nM$ real numbers which is roughly $2M$ times bigger than what we need in our scheme. Furthermore the complexity of our scheme is much less than the proposed schemes to implement dirty paper coding with full CSI using nested lattices or trellis precoding [55, 54].

In order to evaluate the lower and upper bounds, we have to obtain the distribution of $\mathrm{SINR}_{i,m}$. As mentioned earlier, $|H_i\phi_m|^2$'s are i.i.d. over m (and also over i) with $\chi^2(2)$ distribution. Thus,

$$\mathrm{SINR}_{i,m} = \frac{|H_i\phi_m|^2}{1/\rho + \sum_{k\neq m}|H_i\phi_k|^2} = \frac{z}{1/\rho + y}, \tag{3.13}$$

where z has $\chi^2(2)$, and y has $\chi^2(2M-2)$ distributions (denoted by $f_Y(\cdot)$). Conditioning on y, the probability distribution function (PDF) of $\mathrm{SINR}_{i,m}$, $f_s(x)$, can be written as

$$\begin{aligned} f_s(x) &= \int_0^\infty f_{s|Y}(x|y) f_Y(y)dy \\ &= \int_0^\infty (1/\rho + y)e^{-(1/\rho+y)x} \times \frac{y^{M-2}e^{-y}}{(M-2)!}dy \\ &= \frac{e^{-x/\rho}}{(1+x)^M}\left(\frac{1}{\rho}(1+x) + M - 1\right). \end{aligned} \tag{3.14}$$

We can also calculate the cumulative distribution function (CDF) of $\mathrm{SINR}_{i,m}$, $F_s(x)$, as

$$F_s(x) = \int_0^x \frac{e^{-x/\rho}}{(1+x)^M}\left(\frac{1}{\rho}(1+x) + M - 1\right)dx = 1 - \frac{e^{-x/\rho}}{(1+x)^{M-1}}, \qquad x \geq 0. \tag{3.15}$$

Since $\mathrm{SINR}_{i,m}$ for $i = 1, \ldots, n$, are i.i.d. random variables, the CDF of $\max\limits_{1\leq i\leq n}\mathrm{SINR}_{i,m}$ for $m = 1\ldots, M$ is $(F_s(x))^n$. Using the obtained CDF we can now evaluate the throughput of our proposed randomly chosen beamforming technique:

Lemma 3.3 *For any ρ, M, and n, the throughput of the randomly chosen beamform-*

ing satisfies

$$M\int_1^{\infty}\log(1+x)nf_s(x)F_s^{n-1}(x)dx \le R_{MBF} \le M\int_0^{\infty}\log(1+x)nf_s(x)F_s^{n-1}(x)dx, \tag{3.16}$$

where $f_s(x)$ and $F_s(x)$ are as defined in (3.14) and (3.15), respectively.

Proof: The upper bound clearly follows from (3.8) by substituting the distribution of the maximum SINR in (3.8). To prove the lower bound, we can write the conditional distribution of $\max_{1\le i\le n}\mathrm{SINR}_{i,1}$ given that $\max_{1\le i\le n}\mathrm{SINR}_{i,1} \ge 1$ as

$$\Pr\{\max_{i=1,\dots,n}\mathrm{SINR}_{i,1} < a | \max_{i=1,\dots,n}\mathrm{SINR}_{i,1} > 1\} = \begin{cases} \frac{F_s^n(a)-F_s^n(1)}{1-F_s^n(1)} & a \ge 1, \\ 0 & a \le 1. \end{cases} \tag{3.17}$$

Now taking the derivative of the CDF, and substituting the PDF in (3.9), we can derive the lower bound as stated in (3.16). □

Lemma 3.3 can be used to evaluate the throughput for any n, ρ, and M. However in many systems ρ and M are fixed, but n (the number of users) is large. It is therefore useful to investigate this regime. In what follows, we will focus on the scaling laws of the throughput for large n.

In fact the asymptotic behavior of the distribution of the maximum of n i.i.d. random variables has been extensively studied in the literature [60, 61, 62]. In Appendix 10.1 we review results that we need in this chapter. Corollary 10.3 in Appendix 10.1 can be used to state the following result.

Lemma 3.4 *Let* $\mathrm{SINR}_{i,m}$ $i = 1, \dots, n$ *be* n *i.i.d. random variables with distribution* *function $f_s(x)$ as in (3.14). Then, for M and ρ fixed and n sufficiently large,*

$$\Pr\left\{\rho\log n-\rho M\log\log n+O(\log\log\log n)\leq \max_{1\leq i\leq n}\mathrm{SINR}_{i,m}\leq \right.$$
$$\left.\rho\log n-\rho(M-2)\log\log n+O(\log\log\log n)\right\}\geq 1-O\left(\frac{1}{\log n}\right) \quad (3.18)$$

In particular,

$$\Pr\left\{\left|\frac{\max_{1\leq i\leq n}\mathrm{SINR}_{i,m}}{\rho\log n}-1\right|\leq O\left(\frac{\log\log n}{\log n}\right)\right\}\geq 1-O\left(\frac{1}{\log n}\right). \quad (3.19)$$

Remark 3.1: Lemma 3.4 shows that when M is fixed and n increases, the maximum SINR behaves like $\rho\log n+O(\log\log n)$. On the other hand, from the expression for the SINR defined in (3.5), it is clear that the numerator is a $\chi^2(2)$ random variable and the interference terms constitute a $\chi^2(2M-2)$ random variable. It is well-known that (see Example 1 in Appendix 10.1) the maximum of n i.i.d. $\chi^2(2)$ behaves like $\log n$ for large n. One may then suspect that $\max_{i=1,\ldots,n}\mathrm{SINR}_{i,m}$ should behave like $\frac{\log n}{\frac{1}{\rho}+M-1}$, arguing that when the numerator takes on its maximum the denominator takes on its average value. What is interesting about Lemma 3.4 is that this heuristic argument is not true. It turns out that $\max_{i=1,\ldots,n}\mathrm{SINR}_{i,m}$ is achieved when the numerator behaves as $\log n$ *and* the interference terms are arbitrarily small, this yielding the behavior $\rho\log n$.

Proof: We use Corollary 10.3 in Appendix 10.1 to find the asymptotic distribution of the maximum of n i.i.d. random variable $\mathrm{SINR}_{i,m}$ for $i=1,\ldots,n$. The growth function $g_s(x)$ for $x \geq 0$, here, is

$$g_s(x) = \frac{1 - F_s(x)}{f_s(x)} = \frac{\frac{e^{-x\rho}}{(1+x)^{M-1}}}{\frac{e^{-x/\rho}}{(1+x)^M}\left(\frac{1}{\rho}(1+x) + M - 1\right)} = \frac{1+x}{(1+x)/\rho + M - 1} = \rho - \frac{\rho(M-1)}{(1+x)/\rho + M - 1}. \tag{3.20}$$

Clearly $\lim_{x\to\infty} g_s(x) = \rho > 0$ so that the first condition of Corollary 10.3 is met. To verify the second condition we need to find u_n defined via the equation $1 - F_s(u_n) = 1/n$. Thus

$$1 - F_s(u_n) = \frac{e^{-u_n/\rho}}{(1+u_n)^{M-1}} = \frac{1}{n} \Longrightarrow \frac{u_n}{\rho} + (M-1)\log(1+u_n) = \log n. \tag{3.21}$$

Eq. (3.21) implies that $u_n = \rho \log n - \rho(M-1)\log\log n + O(\log\log\log n)$ for large n and fixed M. Taking derivatives, it is straightforward to verify that $g^{(m)}(u_n) = O(\frac{1}{(1+u_n)^{m+1}})$ for large n. Corollary 10.3 therefore applies, and so

$$\Pr\{\rho \log\log n \leq \max_{1\leq i\leq n} \text{SINR}_{i,m} - u_n \leq \rho\log\log n\} \geq 1 - O(\frac{1}{\log n}). \tag{3.22}$$

The theorem follows by substituting the value of u_n in (3.22). □

We can now state the following theorem to prove the asymptotic linear scaling of the throughput with M when M is fixed.

Theorem 3.5 *Let M and ρ be fixed and $N = 1$. Then*

$$\lim_{n\to\infty} \frac{R_{MBF}}{M\log\log n} = 1. \tag{3.23}$$

Proof: We derive upper and lower bounds for R_{MBF} when n is sufficiently large. For large n, using the upper bound in (3.8) and Eq. (3.19), we may write

$$\begin{aligned}
R_{MBF} &\leq E\left\{\sum_{m=1}^{M} \log\left(1+\max_{i=1,\ldots,n} \mathrm{SINR}_{i,m}\right)\right\} \\
&\leq M \Pr\left\{\max_{i=1,\ldots,n} \mathrm{SINR}_{i,m} \leq u_n + \rho\log\log n\right\} \log\left(1+u_n+\rho\log\log n\right) \\
&\quad + M \Pr\left\{\max_{i=1,\ldots,n} \mathrm{SINR}_{i,m} \geq u_n + \rho\log\log n\right\} \log\left(1+\rho n\right) \\
&\leq M\log(1+u_n+\rho\log\log n) + MO(1/\log n)\log(1+\rho n) \\
&\leq M\log(1+u_n+\rho\log\log n) + O(1),
\end{aligned}$$

where $u_n = \rho\log n - \rho(M-1)\log\log n + O(\log\log\log n)$ as derived in Lemma 3.4 and we used the fact that the throughput is bounded by $M\log(1+\rho n)$ (the capacity of a MIMO point to point system with M transmit and n receive antennas). In order to find a lower bound, We can use the lower bound in (3.9) and Lemma 3.4 to write

$$\begin{aligned}
R_{MBF} &\geq \Pr\left\{\max_{i=1,\ldots,n} \mathrm{SINR}_{i,m} \geq 1\right\} \times \\
&\qquad E\left\{\sum_{m=1}^{M} \log\left(1+\max_{i=1,\ldots,n} \mathrm{SINR}_{i,m}\right) \Big| \max_{i=1,\ldots,n} \mathrm{SINR}_{i,m} \geq 1\right\} \\
&\geq \Pr\left\{\max_{i=1,\ldots,n} \mathrm{SINR}_{i,m} \geq 1\right\} \times \\
&\quad \frac{M\log(1+u_n-\rho\log\log n)\Pr\left\{\max_{i=1,\ldots,n} \mathrm{SINR}_{i,m} \geq u_n-\rho\log\log n\right\}}{\Pr\left\{\max_{i=1,\ldots,n} \mathrm{SINR}_{i,m} \geq 1\right\}} \\
&= M\log(u_n-\rho\log\log n)\left(1-O\left(\frac{1}{\log n}\right)\right),
\end{aligned} \tag{3.24}$$

where we used the definition of the conditional probability. The corollary follows by substituting the value of u_n and observing that both the lower and upper bounds converge to $M\log\log n + O(\log\log\log n)$. □

Remark 3.2: Using Eq. (3.18) it is not hard to obtain the next order term in R_{MBF} as follows:

$$R_{MBF} \geq M \log(\rho \log n - \rho M \log\log n) + o(\log\log\log n). \tag{3.25}$$

Theorem 3.5 states that for fixed M as n grows to infinity the throughput scales like $M \log\log n$. Interestingly in Lemma 3.1, we showed that $M \log\log n$ is in fact the best sum-rate capacity that can be achieved with full knowledge of the channel using dirty paper coding [12, 10, 9]. Therefore, as far as the scaling law of the throughput is concerned, we are not losing anything in terms of the throughput provided that M is fixed. This in fact raises the question of how far we can increase M and still maintain the linearly scaling of throughput with M. This question will be answered in the next section.

3.5 Linear Scaling of Throughput with M

In this section, we consider the case where the number of transmit antennas M is allowed to grow to infinity. We would like to see how fast can M grow to retain the linear scaling of throughput with M. Similar to the previous section, we assume each receiver has a single antenna and the total average transmit power is M, i.e., the average transmit power per antennas is one.

Since M is also going to infinity, the results in Appendix 10.1 do not apply. Therefore, we need to directly analyze the asymptotic behavior of the maximum SINR when both n and M grow to infinity. Of course, the asymptotics will depend on the growth rate of M relative to n.

In what follows, we first show that if $\lim_{n\to\infty} \frac{M}{\log n}$ is a constant, the throughput of our scheme still exhibits linear growth, i.e., $\lim_{n\to\infty} \frac{R_{MBF}}{M}$ is a constant independent of n.

Furthermore, we show that if M grows faster than $\log n$, i.e., $\lim_{n\to\infty} \frac{M}{\log n} = \infty$, then the ratio of the throughput to M tends to zero. Therefore throughput will linearly scale with M provided that M does not grow faster than $\log n$.

Theorem 3.6 *Suppose the transmitter has M antennas, each receiver is equipped with a single antenna, and that we use random beamforming to users with the highest SINR's. Then, if $M = \frac{\log n + 3\log\log n - c/\rho}{\log(1+c)} + 1$, where c is a positive constant. Then,*

$$\Pr\left\{c - \alpha\frac{\log\log n}{\log n} \leq \max_{1\leq i\leq n} \mathrm{SINR}_{i,m} \leq c\right\} \geq 1 - O\left(\frac{1}{\log^3 n}\right), \tag{3.26}$$

where $\alpha = 7(1+c)\log(1+c)$. Consequently,

$$\lim_{n\to\infty} \frac{R_{MBF}}{M\log(1+c)} = 1. \tag{3.27}$$

Proof: First of all, note that

$$1 - F_s(c) = \frac{e^{-c/\rho}}{(1+c)^{M-1}} = e^{-c/\rho + (M-1)\log(1+c)}. \tag{3.28}$$

Inserting the value of M in (3.28) yields, $1 - F_s(c) = \frac{1}{n\log^3 n}$. Therefore,

$$\begin{aligned}
\Pr\{\max_{1\leq i\leq n} \mathrm{SINR}_{i,m} \leq c\} = \{F_s(c)\}^n &= (1 - \frac{1}{n\log^3 n})^n \\
&= e^{n\log\left(1 - \frac{1}{n\log^3 n}\right)} \\
&= e^{-1/\log^3 n + O(\frac{1}{n\log^6 n})} \\
&= 1 - O(1/\log^3 n), \qquad (3.29)
\end{aligned}$$

where we used the fact that $\log(1-x) = -x + O(x^2)$ for small x. Similarly setting

$c' = c - \frac{\alpha \log\log n}{\log n}$ we have

$$\begin{aligned}
1 - F_s(c') &= e^{-c/\rho + \frac{\alpha \log\log n}{\rho \log n} + (M-1)\log(1+c) - (M-1)\log(1 - \frac{\alpha \log\log n}{(1+c)\log n})} \\
&= e^{-c/\rho + (M-1)\log(1+c)} \times e^{\frac{\alpha \log\log n}{\rho \log n}} \times e^{-(M-1)\log(1-\frac{\alpha \log\log n}{(1+c)\log n})} \\
&= \frac{e^{\frac{\alpha \log\log n}{(1+c)\log(1+c)} + O(\log\log n/\log n)}}{n \log^3 n} \\
&= \frac{\log^4 n \times (1 + O(\log\log n/\log n))}{n} = \frac{\log^4 n + o(\log^4 n)}{n},
\end{aligned}$$

where in the third step we used $\alpha = 7(1+c)\log(1+c)$ and in the forth step we used the identity $e^x = 1 + O(x)$ for small x. We can now state that

$$\begin{aligned}
\Pr\{\max_{1\le i\le n} \text{SINR}_{i,m} \le c - \frac{\alpha \log\log n}{\log n}\} &= (F_s(c'))^n \\
&= \left(1 - \frac{\log^4 n + o(\log^4 n)}{n}\right)^n \\
&= e^{-\log^4 n + o(\log^4 n)} = O\left(\frac{1}{n^4}\right), \quad (3.30)
\end{aligned}$$

where in the last step we have been very conservative. Now, using (3.30) and (3.29), we get

$$\begin{aligned}
\Pr\left\{c - \alpha\frac{\log\log n}{\log n} \le \max_{1\le i\le n} \text{SINR}_{i,m} \le c\right\} &\ge \Pr\left\{\max_{1\le i\le n} \text{SINR}_{i,m} \le c\right\} - 1 \\
&\quad + \Pr\left\{c - \alpha\frac{\log\log n}{\log n} \le \max_{1\le i\le n} \text{SINR}_{i,m}\right\} \\
&= 1 - O(\frac{1}{\log^3 n}). \quad (3.31)
\end{aligned}$$

In order to find bounds on the throughput, we use a similar argument as in the proof of Theorem 3.5 to show that

$$\begin{aligned}
R_{MBF} &\leq E\left\{\sum_{m=1}^{M} \log\left(1+\max_{i=1,\ldots,n} \mathrm{SINR}_{i,m}\right)\right\} \\
&= \Pr\{\max_{i=1,\ldots,n} \mathrm{SINR}_{i,m} \leq c\} \times \\
&\quad E\left\{\sum_{m=1}^{M} \log\left(1+\max_{i=1,\ldots,n} \mathrm{SINR}_{i,m}\right) | \max_{i=1,\ldots,n} \mathrm{SINR}_{i,m} \leq c\right\} \\
&\quad + \Pr\{\max_{i=1,\ldots,n} \mathrm{SINR}_{i,m} \geq c\} \times \\
&\quad E\left\{\sum_{m=1}^{M} \log\left(1+\max_{i=1,\ldots,n} \mathrm{SINR}_{i,m}\right) | \max_{i=1,\ldots,n} \mathrm{SINR}_{i,m} \geq c\right\} \\
&\leq M\log(1+c) + O\left(\frac{M\log n}{\log^3 n}\right) \\
&= M\log(1+c) + O\left(\frac{1}{\log n}\right), \qquad (3.32)
\end{aligned}$$

where we used the fact the sum-rate is bounded by $M\log(1+\rho n)$ and $M = O(\log n)$. In order to derive a lower bound for the throughput, for $c > 1$ we can use Lemma 3.2 and the fact that the maximum SINR is almost surely equal to $c > 1$, to obtain a lower bound as

$$R_{MBF} \geq M\log\left(1+c-\alpha\frac{\log\log n}{\log n}\right)\left(1-O\left(\frac{1}{n^4}\right)\right) \qquad (3.33)$$

for $c > 1$. Clearly, for $c < 1$, the lower bound in Lemma 3.2 is not tight. Therefore, in order to find a lower bound, we define the event A as the event that for all m, $c-\epsilon \leq \max_{1\leq i\leq n} \mathrm{SINR}_{i,m} \leq c$ where $\epsilon = \alpha\frac{\log\log n}{\log n}$. We also define the event B as the event that each user i can at most be the maximum for one signal s_m.

Therefore the throughput can be written as

$$\begin{aligned} R_{MBF} = E\{rate\} &= E\{rate|A \cap B\}\Pr\{A \cap B\} + E\{rate|A' \cup B'\}(1 - \Pr\{A \cap B\}) \\ &\geq E\{rate|A \cap B\}\Pr\{A \cap B\} \\ &\geq M \log\left(1 + c - \alpha\frac{\log\log n}{\log n}\right)\Pr\{B|A\}\Pr\{A\}, \end{aligned} \tag{3.34}$$

where in the last inequality we used the fact that given the events A and B, the transmission rate corresponding to the signal s_m is greater than $\log\left(1 + c - \alpha\frac{\log\log n}{\log n}\right)$. Now we can use the union bound to find a lower bound for $\Pr\{A\}$ as

$$\begin{aligned} \Pr\{A\} &= \Pr\left\{\forall m,\ \max_{1 \leq i \leq n} \mathrm{SINR}_{i,m} \in [c - \epsilon,\ c]\right\} \\ &\geq 1 - \Pr\left\{\exists m,\ \max_{1 \leq i \leq n} \mathrm{SINR}_{i,m} \notin [c - \epsilon,\ c]\right\} \\ &\geq 1 - M\Pr\left\{\max_{1 \leq i \leq n} \mathrm{SINR}_{i,m} \notin [c - \epsilon,\ c]\right\} \\ &= 1 - O\left(\frac{M}{\log^3 n}\right), \end{aligned} \tag{3.35}$$

where we used (3.31) in the last inequality. In Appendix 3.10.1, it is shown that

$$\Pr\{B|A\} \geq 1 - O\left(\frac{(\log\log n)^2}{\sqrt{\log n}}\right). \tag{3.36}$$

Therefore, inserting (3.36) and (3.35) into (3.34), we get

$$R_{MBF} \geq M\log\left(1 + c - \alpha\frac{\log\log n}{\log n}\right)\left(1 - O\left(\frac{(\log\log n)^2}{\sqrt{\log n}}\right)\right)\left(1 - O\left(\frac{\log n}{\log^3 n}\right)\right). \tag{3.37}$$

Theorem 3.6 follows using (3.37) and (3.33) for $c \leq 1$. $\square$

Theorem 3.6 shows that when M grows like $\log n$, the throughput still scales linearly with M. In the next theorem, we show that increasing M at a rate faster than $\log n$ results in sublinear scaling of the throughput with M. It is also worth noting that the sum rate capacity of the broadcast channel with full CSI also scales linearly with M, i.e., $\lim_{n\to\infty} \frac{R_{DP}}{M} = \alpha$ where α is a constant independent of n [58]. Therefore, in this regime, up to a constant multiplicative factor, the scaling law of the throughput of our scheme is still the same as that of dirty paper coding.

Remark 3.3: Similar to Remark 3.1, from the SINR expression one may expect that $\max_{i=1,\ldots,n} \text{SINR}_{i,m}$ behaves as $\frac{\log n}{M} = \frac{\log n}{\log n / \log(1+c)} = \log(1+c)$. The argument is that the maximum is achieved when the numerator behaves as $\log n$ and the denominator behaves as the mean of $\chi^2(2M-2)$ (here it is not reasonable to assume that the numerator can be $\log n$ and the interference terms arbitrarily small, since we have $O(\log n)$ interference terms). However careful analysis of Theorem 3.6 shows that this heuristic is false: $\max_{i=1,\ldots,n} \text{SINR}_{i,m}$ is achieved when the numerator behaves as $\log n$ and the interference terms as $\frac{\log n}{c} < \frac{\log n}{\log(1+c)}$.

Theorem 3.7 *Consider the setting of Theorem 3.6. If M grows faster than* $\log n$*, i.e.,* $\lim_{n\to\infty} \frac{M}{\log n} = \infty$*, then* $\lim_{n\to\infty} \frac{R_{MBF}}{M} = 0$.

Proof: Let u_n be a positive sequence such that $\lim_{n\to\infty} u_n = 0$. For such a u_n, let $M = \frac{2\log n}{\log(1+u_n)} + 1$, clearly $\lim_{n\to\infty} \frac{M}{\log n} = \infty$ if and only if $\lim_{n\to\infty} u_n = 0$. With the choice of u_n, we have

$$1 - F_s(u_n) = e^{-u_n/\rho - (M-1)\log(1+u_n)} = \frac{e^{-u_n/\rho}}{n^2} \tag{3.38}$$

and therefore,

$$\begin{aligned} \Pr\{\max_{1\le i\le n} \text{SINR}_{i,m} \le u_n\} = \{F_s(u_n)\}^n &= \left(1 - \frac{e^{-u_n/\rho}}{n^2}\right)^n \\ &\ge \left(1 - \frac{1}{n^2}\right)^n \\ &= 1 - O\left(\frac{1}{n}\right). \end{aligned}$$

Using a similar argument as in the proof of Theorem 3.6, we can therefore bound the throughput as

$$R_{MBF} \leq M \log(1 + u_n) + O\left(\frac{M \log n}{n}\right). \tag{3.39}$$

Eq. (3.39) implies that $\lim_{n\to\infty} \frac{R_{MBF}}{M} = 0$ since $\lim_{n\to\infty} u_n = 0$. □

It is worth mentioning that with full CSI at the transmitter, the sum-rate capacity linearly scales with M even when M is of the order of n [14]. This can be seen by a simple zero forcing beamforming scheme that creates M parallel channels as long as the channel matrix is full rank [56]. Our scheme has access only to partial CSI, and can therefore only guarantee a linear scaling in M, provided that M does not grow faster than $\log n$.

3.6 Throughput Analysis: $N > 1$, M is fixed

In the previous sections, we focused on the case where each receiver is equipped with only one antenna. When the users have multiple receive antennas, the sum-rate capacity of DPC scales as $M \log\log nN$ [58]. Insofar as our scheme is concerned, there are three distinct possibilities:

1. *Treating each receive antenna as an independent user.* In this case, we effectively have nN single antenna receivers. Therefore, each receiver should feed back N times the amount of information since each user has N independent antennas and therefore it has N maximum SINRs corresponding to each receive antenna. The transmitter then assigns s_m for $m = 1, \ldots, M$ to the antenna of that user with the highest SINR, i.e., $\max_{1\leq i\leq nN} \text{SINR}_{i,m}$. Since we have nN i.i.d. SINRs, the maximization will be over nN i.i.d. random variables instead of n ones which was the case for $N = 1$.

2. *Assigning at most one beam to each user.* In this case, the SINR can be written as

$$\mathrm{SINR}_{i,m} = \frac{\phi_m^* H_i^* H_i \phi_m}{1/\rho + \sum_{k\neq m} \phi_k^* H_i^* H_i \phi_k}, \qquad m = 1, \ldots, M, \tag{3.40}$$

   where as in (3.1), H_i is the $N \times M$ channel matrix for the i'th user. Again we send the symbol s_m to the user corresponding to $\max_{1\leq i\leq n} \mathrm{SINR}_{i,m}$. Note that in this case each user just feeds back its maximum SINR and the corresponding index m in which it is maximum.

3. *Assigning multiple beams to each user.* For simplicity, let us assume $K = M/N$ is an integer.[5] In this case, we either assign N beams to a user or no beams at all. Therefore to find the best user, instead of feeding back SINRs, each receiver has to feed back its capacity, computed as

$$C_{i,m} = \log\left\{\det\left\{I + \Phi_m^* H_j^* H_j \Phi_m \left(\frac{1}{\rho} I + \sum_{k\neq m} \Phi_k^* H_j^* H_j \Phi_k\right)^{-1}\right\}\right\}, \tag{3.41}$$

   for $m = 1, \ldots, M/N$ where the Φ_k's $(k = 1, \ldots, K)$ are $M \times N$ random orthonormal matrices chosen according to an isotropic distributions. In other words, $\Phi = (\Phi_1 \ \ldots \ \Phi_K)$ is an $M \times M$ unitary matrix.

As mentioned, the first case is effectively the same as having nN users with single receive antennas. The second case is a generalization of the case with $N = 1$, and it turns out the analysis of this is very similar to that of the case with $N = 1$. On the other hand, the last case is quite different from the previous two cases and requires more effort to be analyzed. In terms of the amount of feedback, clearly the first case requires N times more feedback than the second and third cases.

[5] The more general case can be handled in a straightforward fashion, but will not be done here for the sake of brevity.

Case 1:

Here, since we consider all the receive antennas as separate users (no cooperation among receivers), we have nN users with single receive antennas. In this case, the formulation of the problem is the same as that in Section 3.4 with the only difference being that n is replaced by nN. Therefore, we can state the following limit result as a simple consequence of Theorem 3.5,

$$\lim_{n\to\infty} \frac{R_{MBF}}{M \log\log nN} = 1, \tag{3.42}$$

when M is fixed and for any N.

Remark 3.4: In fact, it has been recently shown in [58] that when M is fixed, n is large, and for any N, the sum rate capacity scales like $M \log\log nN$ in the presence of full CSI in the transmitter using dirty paper coding. Therefore, treating each antenna as an independent user does give the right scaling law for the throughput.

Case 2:

Here we send at most one symbol per user. Therefore each user has to feed back its maximum SINR calculated as in (3.40), where H_i is the $N \times M$ channel matrix for the i'th user. Similar to Section 3.4, using the orthogonality of ϕ_m's, we first write the SINR as $\text{SINR}_{i,m} = \frac{z}{1/\rho+y}$ where z has $\chi^2(2N)$ and y has $\chi^2(2N(M-1))$ distributions that are independent.[6] Therefore, we may write the PDF of the $\text{SINR}_{i,m}$ (denoted by f_{s_N}) as

$$\begin{aligned} f_{s_N}(x) &= \frac{x^{2N-2}e^{-x/\rho}}{(2N-1)!(NM-N-1)!}\int_0^\infty (u+1/\rho)^{2N-1}u^{NM-N-1}e^{-u(1+x)}du \\ &= \frac{x^{2N-2}e^{-x/\rho}}{(2N-1)!(NM-N-1)!}\sum_{i=0}^{2N-1}\binom{2N-1}{i}\frac{1}{\rho^{2N-i-1}}\times\frac{(N(M-1)+i-1)!}{(1+x)^{N(M-1)+i}}. \end{aligned} \tag{3.43}$$

[6] The reason is that Φ is a unitary matrix and H_i is a matrix of i.i.d. $CN(0,1)$.

The above $f_{s_N}(x)$ can be used to evaluate exactly the throughput using Lemma 3.2. The asymptotic analysis can be done similarly to that of Section 3.4, although the analysis becomes more cumbersome.

Theorem 3.8 *Let* $\max_{1\le i\le n} \mathrm{SINR}_{i,m}$ *where* $\mathrm{SINR}_{i,m}$ *for* $i = 1, \ldots, n$ *be* n *i.i.d. random variables defined in (3.40) and let* M *and* N *be fixed numbers. Then for sufficiently large* n*,*

$$\Pr\Big\{\rho\log n - \rho(N(M+1)-2)\log\log n + O(\log\log\log n) \le \max_{1\le i\le n}\mathrm{SINR}_{i,m}$$
$$-O(\log\log\log n) \le \rho\log n - \rho(N(M-3)+1)\log\log n\Big\} \ge 1 - O(\frac{1}{\log n}) \quad (3.44)$$

and therefore,

$$\lim_{n\to\infty}\frac{R_{MBF}}{M\log\log n} = 1. \quad (3.45)$$

Proof: We use Corollary 10.3 in Appendix 10.1 to prove the first part of the theorem. We first check whether the growth function has a positive constant limit or not. Using L'Hopital's rule and (3.43), we get

$$\lim_{x\to\infty} g_{s_N}(x) = \lim_{x\to\infty}\frac{-f_{s_N}(x)}{f'_{s_N}(x)} = \rho > 0. \quad (3.46)$$

So the first condition in Corollary 10.3 is met. Furthermore, taking the integral of $f_{s_N}(x)$ in (3.43), it is quite straightforward to show that $g_{s_N}^{(m)}(x) = O(1/x^m)$ for large x.

To verify the last condition, we need to find u_n defined as the solution of $1 - F_{s_N}(u_n) = \frac{1}{n}$. Since solving the equation is involved, we can find upper and lower bounds for u_n, i.e., $u_n^l \leq u_n \leq u_n^u$, by first deriving lower and upper bounds for $f_{s_N}(x)$:

$$\binom{NM+N-2}{2N-1}\frac{e^{-x/\rho}}{(1+x)^{N(M+1)-3}} \leq f_{s_N}(x) \leq \frac{\sum_{i=0}^{2N-1}\binom{2N-1}{i}\frac{(N(M-1)+i-1)!}{\rho^{2N-i-1}}}{(2N-1)!(NM-N-1)!}\frac{e^{-x/\rho}}{(1+x)^{N(M-3)+2}} \tag{3.47}$$

for $x > 1$. The lower bound follows by replacing $u + 1/\rho$ by u in the integral of (3.43), which then becomes an exponential integral. The upper bound can be also derived by using $x < x+1$ and $1 < x+1$ in the expansion in (3.43).

Replacing $f_{s_N}(x)$ by its lower bound allows us to compute u_n^l via

$$\int_{u_n^l}^{\infty} \frac{\gamma_1 e^{-x/\rho}}{(1+x)^{N(M+1)-3}} dx = \frac{1}{n}, \tag{3.48}$$

where $\gamma_1 = \binom{NM+N-2}{2N-1}$. Using the identity $\int_a^{\infty} \frac{e^{-bx}}{(1+x)^M} dx = e^b b^{M-1}\Gamma(-M+1, b(a+1))$ and the asymptotic expansion $\Gamma(-M+1, x) = x^M e^{-x}(1 + O(1/x))$ for large $|x|$ as in [63], we obtain

$$\gamma_1'\Gamma(-M+1, \frac{1}{\rho}(u_n^l+1)) = \frac{1}{n} \Longrightarrow u_n^l = \rho \log n - \rho(N(M+1)-3) \log\log n + O(\log\log\log n), \tag{3.49}$$

where γ_1' only depends on N, M, ρ, and does not depend on n.

We can similarly find the upper bound as $u_n^u = \rho \log n + \rho(N(M-3)+1) \log\log n + O(\log\log\log n)$. Using Corollary 10.3 and the bounds for u_n, we can use the same argument as in the proof of Theorem 3.5 (i.e., Eq. (3.24)), to prove the first part of the theorem. The second part of the Theorem follows by using the same argument as in the proof of theorem 3.5 (i.e., Eq. (3.24)) and Eq. (3.44). □

Remark 3.5: Using (3.44) it follows that the next order term in R_{MBF} is

$$R_{MBF} \leq M \log(\rho \log n - \rho(N(M-3)-1) \log\log n) + o(\log\log\log n). \tag{3.50}$$

Note that using (3.25), the expansion for Case 1 is

$$R_{MBF} \geq M \log(\rho \log nN - \rho M \log\log nN) + o(\log\log\log n),$$

which implies that the scheme of Case 2 is worse than that of Case 1 (it is even worse than using $N = 1$ receive antenna, which can be explained by the channel hardening that occurs for $N > 1$).

Case 3:

Here each user feeds back its largest capacity $C_{i,m}$ defined in (3.41), and so we need to analyze the (asymptotic) distribution of $\max_{i=1\dots,n} C_{i,m}$ to find an upper bound for the throughput. While, in principle, this can be done, the algebra is extremely tedious. It turns out that an upper bound for the throughput can be derived if we replace $C_{i,m}$ by the simple upper bound

$$C_{i,m} \leq C_{i,m}^u = \log\det\left(I + \rho \Phi_m^* H_i^* H_i \Phi_m\right). \tag{3.51}$$

The analysis of $C_{i,m}^u$ is easier because the eigenvalues of $\Phi_m^* H_i^* H_i \Phi_m$ are readily characterized (via Wishart distribution [66]) than those appearing in $C_{i,m}$. Since $H_i \Phi$ has the same distribution as H_i, the $N \times N$ matrices $H_i \Phi_m$ consists of i.i.d. $CN(0,1)$ and are also independent over i and m. Therefore we need to study

$$C_i^u = \log\det(I + \rho G_i^* G_i), \qquad i = 1, \dots, n, \tag{3.52}$$

where G_i is $N \times N$ and has i.i.d. $CN(0,1)$ entries. Consequently the throughput will be $R_{MBF} \leq \frac{M}{N} E\{\max_{1\leq i \leq n}(C_1^u, \dots, C_n^u)\}$ since we have M/N random beams. Now letting $\lambda_1^i, \dots, \lambda_N^i$ be the eigenvalues of the matrix $G_i^* G_i$, we can state the following inequality for R_{MBF}:

$$R_{MBF} \leq ME\left\{\max_{1\leq i\leq n} \log\left(1+\frac{\rho}{N}\sum_{k=1}^{N}\lambda_k^i\right)\right\}, \tag{3.53}$$

where we used the inequality $\det(I+G_i^*G_i) \leq \left(1+\frac{tr(G_i^*G_i)}{N}\right)^N$. The next theorem presents an asymptotic result for the throughput of Case 3.

Theorem 3.9 *Let M and N be fixed and n increasing, then the throughput of Case 3 is bounded by*

$$R_{MBF} \leq M\log\left(1+\frac{\rho}{N}\log n + O(\log\log n)\right). \tag{3.54}$$

Proof: In order to evaluate the upper bound in (3.53), we may use the fact that $tr\{G^*G\}$ has $\chi^2(2N^2)$ distribution. It is in fact shown in Example 1 of Appendix 10.1 that for large n, the maximum of n $\chi^2(2N^2)$ random variables satisfies

$$\Pr\left\{\log n + (N^2-2)\log\log n + O(\log\log\log n) \leq \max_{1\leq i\leq n}\sum_{k=1}^{N}\lambda_k^i < \log n + N^2\log\log n + O(\log\log\log n)\right\} \geq 1 - O\left(\frac{1}{\log n}\right).$$

Therefore we can use the above result, (3.53), and using the same argument as in Theorem 3.5 (i.e., (3.24)) to show that

$$R_{MBF} \leq M\log\left(1+\frac{\rho}{N}\log n + O(\log\log n)\right) + O(1). \tag{3.55}$$

□

Again when M and N are fixed, the throughput achieved by Case 3 has the leading order term of $M\log\log n$. The only effect is observed in the lower order terms (the $\log\log\log n$ term), and therefore, we conclude that Case 1 is the best and Case 3 is the worst.

3.6.1 Discussion

The analysis of the sum-rate capacity of DPC shows that increasing the number of receive antennas beyond $N = 1$ does not substantially increase the total throughput [58]. Therefore, one may ask whether it is beneficial for any user to have more than one antenna. Thus, assume that some users have $N > 1$ antennas, and that we are employing the scheme of Case 1. It is quite clear that a user with N antennas will receive N times the rate of a user with one antenna simply because the probability that it will be the strongest user and be transmitted to increases N-fold. Thus users with more antennas will receive higher rates. However, since more receive antennas does not increase the throughput, this will come at the expense of all users in the system.

3.7 Fairness in the Scheduling

So far, we have assumed a homogeneous network in the sense that the SNR for all users was equal, namely $\rho = \rho_i,\ i = 1, \ldots, n$. In practice, however, due to the different distances of the users from the base station and the corresponding different path losses, the users will experience different SNRs so that ρ_i's will not be identical. Such networks are called heterogeneous.

In heterogeneous networks, there is usually tension between the gains obtained from employing multiuser diversity and the fairness of the system. More explicitly, if we transmit only to the best user to maximize the throughput, the system may be dominated by users who are closest to the base station. On the other hand, if we insist on transmitting to users in a fair way (for example, by insisting on proportional fairness [57]), then we will be sacrificing throughput since we will not always be transmitting to the strongest user.

A fortunate consequence of our random multi-beam method is that, if the number of transmit antennas is large enough, then the system becomes interference dominated. In other words, even though the closest users will receive strong signal, they will also receive strong interference. In this case, being the best user will depend not so much on how close one is to the base station, but rather on how one's channel vector H_i aligns with the closest beam direction ϕ_m, $m = 1, \ldots, M$. Therefore, one would expect that the probability that any user is the strongest will not depend on its SNR ρ_i.

In what follows we will make this observation more precise. We will show that if the number of transmit antennas M grows faster than or equal to $\log n$ then the system will be fair; thus we achieve maximum throughput and fairness simultaneously.

As usual, we consider M transmit antennas and $N = 1$ receive antennas at each user (recall that for $N > 1$, the best policy is to have no cooperation between N antennas, which basically changes the problem to nN single antenna users). Denoting the signal to noise ratio of the i'th user by ρ_i, then the PDF of $\text{SINR}_{i,m}$ can be written as

$$f_{s,i}(x) = \frac{1/\rho_i e^{-x/\rho_i}}{(1+x)^{M-1}} + \frac{(M-1)e^{-x/\rho_i}}{(1+x)^M}, \qquad x \geq 0. \tag{3.56}$$

We are interested in computing the probability of transmitting the m-th signal to the $i'th$ user with SNR of ρ_i (denoted by $P_{\rho_i,m}$), i.e.,

$$\begin{aligned} P_{\rho_i} = P_{\rho_i,m} &= \Pr\{\text{SINR}_{i,m} > \text{SINR}_{1,m}, \ldots, \text{SINR}_{i-1,m}, \text{SINR}_{i+1,m}, \ldots, \text{SINR}_{n,m}\} \\ &= \int_0^\infty \int_0^{x_i} \cdots \int_0^{x_i} f_{\rho_i}(x_i) \prod_{j=1, j\neq i}^{n} f_{\rho_j}(x_j) dx_1, \ldots, dx_n. \end{aligned} \tag{3.57}$$

Note that due to the fact that $\text{SINR}_{i,m}$ for $m = 1, \ldots, M$ have the same distribution, $P_{\rho_i,m}$ does not depend on the index m and $P_{\rho_i} = P_{\rho_i,m}$ for $m = 1, \ldots, M$. The following theorem obtains bounds on the probability of choosing the weakest user with $\rho_{\min} = \min\limits_{1 \le i \le n} \rho_i$ and the strongest user with $\rho_{\max} = \max\limits_{1 \le i \le n} \rho_i$.

Theorem 3.10 *Let M be the number of transmit antennas and ρ_i is the SINR of the i'th user. Define $\rho_{\min} = \min\limits_{1 \le i \le n} \rho_i$ and $\rho_{\max} = \max\limits_{1 \le i \le n} \rho_i$, the SNR corresponding to the weakest and strongest user, respectively. Then,*

$$P_{\rho_{\min}} \ge \frac{M - 1 + \frac{1}{\rho_{\min}}}{M - 1 + \frac{1}{\rho_{\max}}} \frac{e^{-(\frac{1}{\rho_{\min}} - \frac{1}{\rho_{\max}})(e^{\frac{2 \log n}{M-1}} - 1)}}{n} + O\left(\frac{1}{n^2}\right), \tag{3.58}$$

and

$$P_{\rho_{\max}} \le \frac{e^{(\frac{1}{\rho_{\min}} - \frac{1}{\rho_{\max}})(e^{\frac{2 \log n}{M-1}} - 1)}}{n} + \frac{1}{n^2}, \tag{3.59}$$

where $P_{\rho_{\min}}$ and $P_{\rho_{\max}}$ are the probability of choosing users with minimum and maximum SNR, respectively.

Proof: Let $\epsilon = \frac{1}{\rho_{\min}} - \frac{1}{\rho_{\max}} \ge 0$. We first find a lower bound for the probability of choosing the user with minimum SNR by assuming all the other users have the maximum SNR. Therefore, using (8.25), we get

$$\begin{aligned} P_{\rho_{\min}} &\ge \int_0^\infty \frac{e^{-(\frac{1}{\rho_{\max}} + \epsilon)x}}{(1+x)^M} \left\{ (\frac{1}{\rho_{\max}} + \epsilon)(1+x) + M - 1 \right\} \left(1 - \frac{e^{-\frac{x}{\rho_{\max}}}}{(1+x)^{M-1}}\right)^{n-1} dx \\ &\ge \int_0^\infty e^{-\epsilon x} \frac{M - 1 + (\frac{1}{\rho_{\max}} + \epsilon)(1+x)}{M - 1 + \frac{(1+x)}{\rho_{\max}}} \left(1 - \frac{e^{-\frac{x}{\rho_{\max}}}}{(1+x)^{M-1}}\right)^{n-1} \frac{d}{dx}\left(-\frac{e^{-\frac{x}{\rho_{\max}}}}{(1+x)^{M-1}}\right) dx. \end{aligned} \tag{3.60}$$

Also we can use the following inequality for $x > 0$,

$$\frac{M - 1 + \frac{1}{\rho_{\max}} + \epsilon}{M - 1 + \frac{1}{\rho_{\max}}} \le \frac{M - 1 + (\frac{1}{\rho_{\max}} + \epsilon)(1+x)}{M - 1 + \frac{1+x}{\rho_{\max}}} \le \frac{\frac{1}{\rho_{\max}} + \epsilon}{\frac{1}{\rho_{\max}}}, \tag{3.61}$$

where we used the fact that the function being bounded is monotonically increasing for $x \ge 0$. Now we define u_0 to be the solution to $\frac{e^{-\frac{u_0}{\rho_{\max}}}}{(1+u_0)^{M-1}} = \frac{1}{n^2}$. Clearly $\frac{e^{-\frac{x}{\rho_{\max}}}}{(1+x)^{M-1}}$ is monotonically decreasing and there is a unique solution for u_0. Then for $\epsilon > 0$,

$$P_{\rho_{\min}} \geq \frac{M-1+\frac{1}{\rho_{\max}}+\epsilon}{M-1+\frac{1}{\rho_{\max}}}\left\{e^{-\epsilon u_0}\int_0^{u_0}\left(1-\frac{e^{-\frac{x}{\rho_{\max}}}}{(1+x)^{M-1}}\right)^{n-1}\frac{d}{dx}\left(-\frac{e^{-\frac{x}{\rho_{\max}}}}{(1+x)^{M-1}}\right)dx \right.$$
$$\left. +e^{-\epsilon u_0}\int_{u_0}^{\infty}\left(1-\frac{e^{-\frac{x}{\rho_{\max}}}}{(1+x)^{M-1}}\right)^{n-1}\frac{d}{dx}\left(-\frac{e^{-\frac{x}{\rho_{\max}}}}{(1+x)^{M-1}}\right)dx\right\}.$$

In order to find a lower bound, we ignore the second integral and we use the fact that $e^{-\epsilon x} > e^{-\epsilon u_0}$ for $\epsilon > 0$ and $0 \leq c \leq u_0$. Therefore, we get

$$\begin{aligned} P_{\rho_{\min}} &\geq \frac{M-1+\frac{1}{\rho_{\max}}+\epsilon}{M-1+\frac{1}{\rho_{\max}}}\left\{\frac{e^{-\epsilon u_0}}{n}\left(1-\frac{e^{-\frac{u_0}{\rho_{\max}}}}{(1+u_0)^{M-1}}\right)^{n}\right\} \\ &= \frac{M-1+\frac{1}{\rho_{\max}}+\epsilon}{M-1+\frac{1}{\rho_{\max}}}\frac{e^{-\epsilon u_0^u}}{n}(1-1/n^2)^{n-1}, \end{aligned}$$

where u_0^u is an upper bound for u_0, i.e., $u_0^u > u_0$, and can be calculated as

$$\frac{1}{\rho_{\max}}u_0+(M-1)\log(1+u_0)=2\log n \Longrightarrow u_0 \leq n^{\frac{2}{M-1}}-1=e^{\frac{2\log n}{M-1}}-1=u_0^u. \quad (3.62)$$

Therefore for any M and n, the lower bound can be written as

$$P_{\rho_{min}} \geq \frac{M-1+\frac{1}{\rho_{\min}}}{M-1+\frac{1}{\rho_{\max}}}\frac{e^{-(\frac{1}{\rho_{\min}}-\frac{1}{\rho_{\max}})(e^{\frac{2\log n}{M-1}}-1)}}{n}(1-1/n^2)^{n-1}, \quad (3.63)$$

which leads to (3.58). We can also find an upper bound for $P_{\rho_{\max}}$ by considering that all the other receivers have the minimum SNR. Therefore similar to (3.60), we may write

$$P_{\rho_{\max}} \leq \int_0^{\infty}\frac{e^{-\frac{x}{\rho_{\max}}}}{(1+x)^M}\left\{\frac{(1+x)}{\rho_{\max}}+M-1\right\}\left(1-\frac{e^{-(\frac{1}{\rho_{\max}}+\epsilon)x}}{(1+x)^{M-1}}\right)^{n-1}dx, \quad (3.64)$$

where we used the definition of ϵ. We can further define $h(x)=\frac{e^{-\frac{x}{\rho_{\max}}}}{(1+x)^{M-1}}$ and we let

u_o be the solution to $h(u_0) = 1/n^2$. Therefore, separating the integral to two regions, we get

$$\begin{aligned} P_{\rho_{\max}} &\leq \int_0^{u_0} (-h'(x)) \left(1 - h(x)e^{-\epsilon x}\right)^{n-1} dx + \int_{u_0}^{\infty} (-h'(x)) \left(1 - h(x)e^{-\epsilon x}\right)^{n-1} dx \\ &\leq \int_0^{u_0} (-h'(x)) \left(1 - h(x)e^{-\epsilon u_0}\right)^{n-1} dx + \int_{u_0}^{\infty} (-h'(x)) dx, \end{aligned}$$

where we used the fact that $1 - h(x)e^{-\epsilon x} < 1 - h(x)e^{-\epsilon u_0}$ for $0 \leq x \leq u_0$ and $\epsilon > 0$. Similarly for the second integral we used $1 - h(x)e^{-\epsilon x} \leq 1$. Noting that $h(u_0) = \frac{1}{n^2}$, the upper bound can be written as

$$\begin{aligned} P_{\rho_{\max}} &\leq \frac{e^{\epsilon u_0}}{n} \left\{(1 - h(u_0)e^{-\epsilon u_0})^n - (1 - e^{-\epsilon u_0})^n\right\} + h(u_0) \\ &\leq \frac{e^{\epsilon u_0}}{n} + \frac{1}{n^2}, \end{aligned}$$

where $u_0 \leq u_0^u$. We can therefore use the fact that $u_0 \leq e^{\frac{2\log n}{M-1}} - 1$ as shown in (3.62) to get

$$P_{\rho_{\max}} \leq \frac{e^{\left(\frac{1}{\rho_{\min}} - \frac{1}{\rho_{\max}}\right)\left(e^{\frac{2\log n}{M-1}} - 1\right)}}{n} + \frac{1}{n^2}, \tag{3.65}$$

which leads to (3.59). □

Based on the result of Theorem 3.10, we can state the following corollary:

Corollary 3.11 *If $\frac{M}{\log n} = \alpha$ then by increasing the average transmit power, we have $P_{\rho_{\min}} \rightarrow \frac{1}{n}$, and so the system becomes more and more fair. Alternatively, if we fix the SNR and increase α, $P_{\rho_{\min}} \rightarrow \frac{1}{n}$ and the system becomes fair.*

Proof: It is clear from Theorem 3.10 that if $\frac{M}{\log n}$ is fixed and we increase the average power, $P_{\rho_{\min}}$ is going to $\frac{1}{n}$. Moreover, as $\frac{M}{\log n}$ goes to infinity, again $P_{\rho_{\min}}$ is approaching $\frac{1}{n}$. Therefore the system becomes fair. □

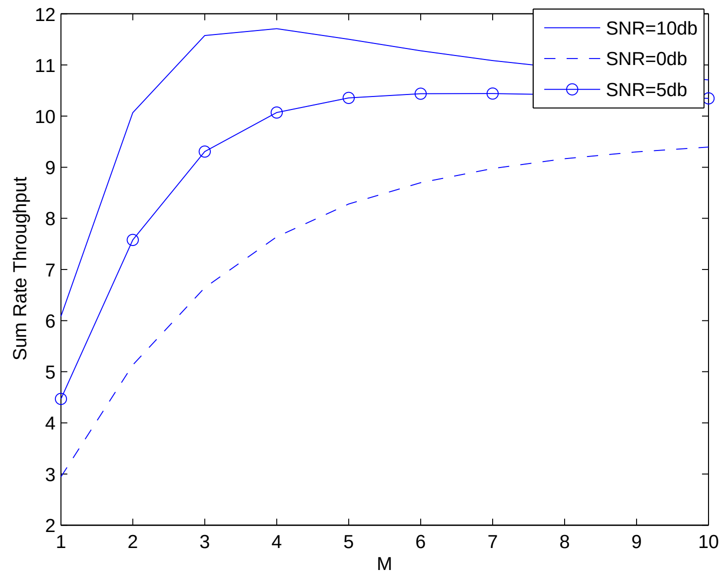

Figure 3.1: Throughput versus the number of transmit antennas for different SNRs and $n = 500$.

3.8 Simulation Results

In this section we verify our asymptotic results with simulations and numerical evaluation. As Lemma 3.2 states, bounds on the throughput can be evaluated for any n, M, and ρ. We also proved in Theorems 3.5 and 3.6 that the upper bound is tight when $M \leq \alpha \log n$, which is the region that we are interested in, therefore, we plot Eq. (3.7) as a good approximation for the throughput. Fig. 3.1 and 3.2 show the throughput versus the number of transmit antennas M, for different SNRs. Clearly for $M \leq 4$ the curve behaves linearly and as M becomes $\log n \approx 4$ the throughput curves become saturated.

We also investigate the fairness of the scheduling by simulations. We compare the fairness of our scheduling with multiple transmit antennas with that of the case with one antenna in the base station $M = 1$, in which the base scheduling strategy (in terms of maximizing the throughput) is to transmit to the user with the maximum SNR. Suppose users have SNRs uniformly distributed from 6 dB to 15 dB, therefore

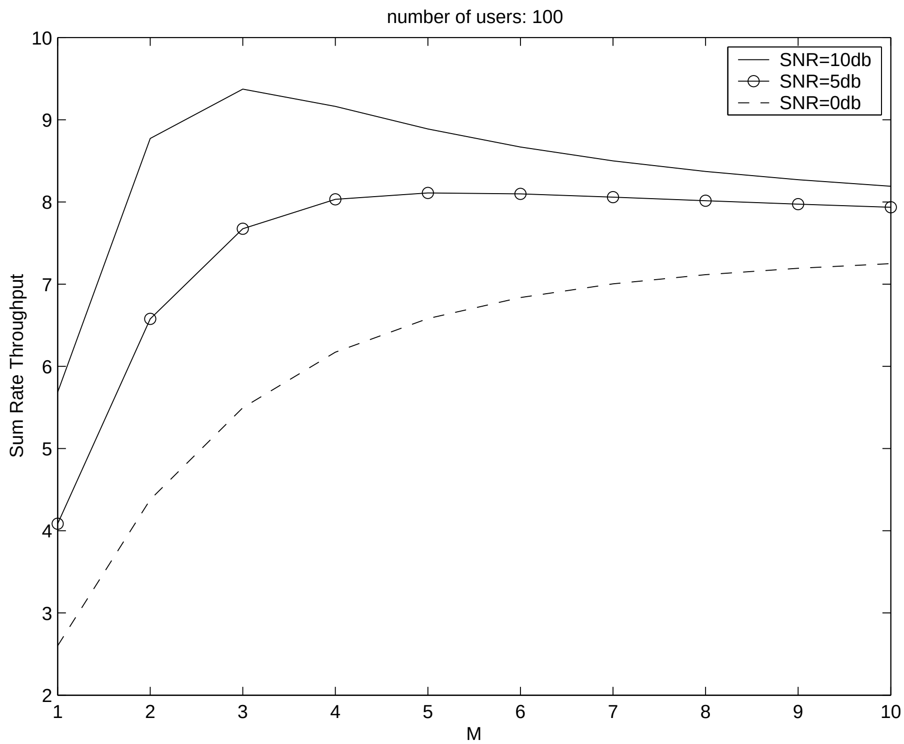

Figure 3.2: Throughput versus the number of transmit antennas for different SNRs and $n = 100$.

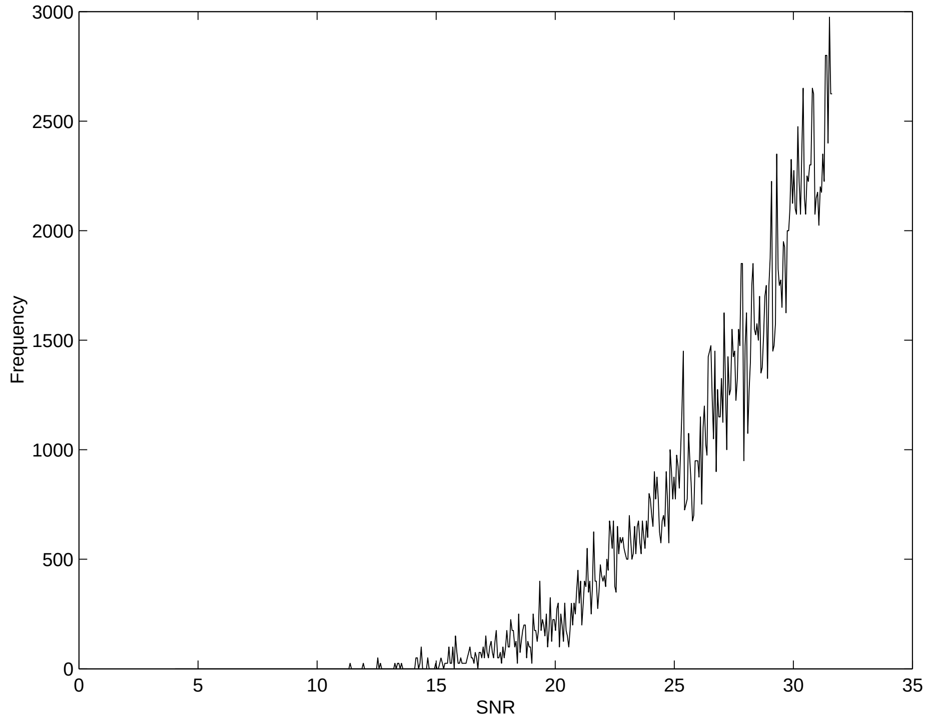

Figure 3.3: The number of times that each user with the corresponding SNR is chosen for 50000 iterations with $M = 1$ and $n = 500$.

the users corresponding to the SNR of 15 dB and 6 dB are the strongest and the weakest users, respectively. Fig. 3.3 shows the number of times that each user with the corresponding SNR is chosen out of 50,000 iterations. Clearly the user with the minimum SNR rarely gets to be transmitted to. On the other hand, Fig. 3.4 shows the fairness of our proposed algorithm by using $M = 5(\approx \log n)$ antennas in the base station. As Fig. 3.3 and 3.4 show, the fairness has been significantly improved by using multiple transmit antennas. For instance, the ratio of the number of times that the strongest user is chosen to the number of times that the weakest user is chosen is 700 for the case with $M = 1$ as opposed to 2.5 for the case with $M = 5$ using our scheduling.

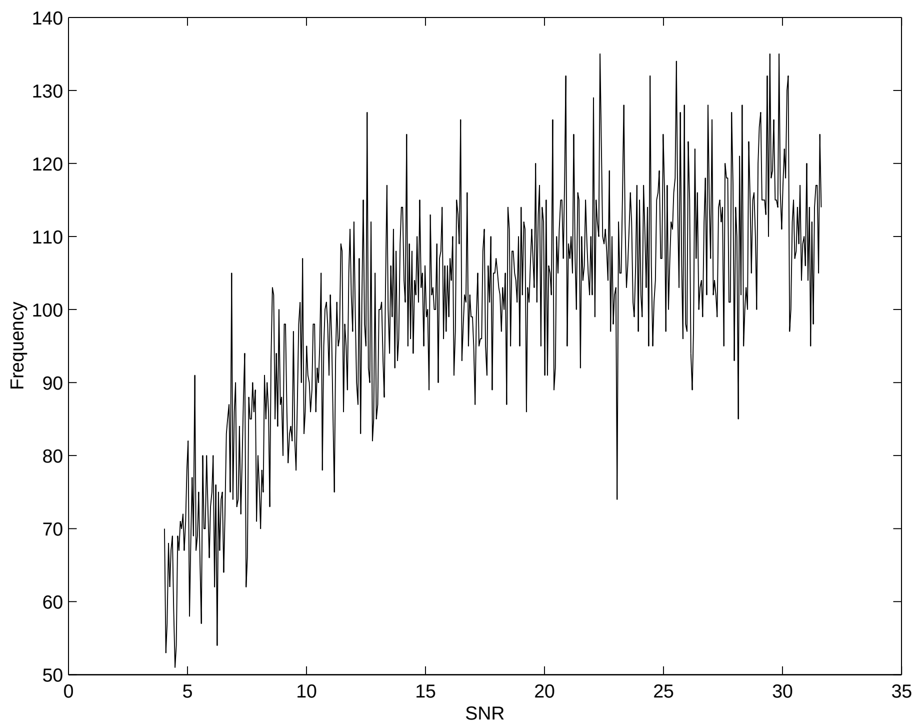

Figure 3.4: The number of times that each user with the corresponding SNR is chosen for 10000 iterations with $M = 5$ and $n = 100$.

3.9 Conclusion

This chapter dealt with multiple antenna broadcast channels where due to rapid time variations of the channel, limited resources, and imperfect feedback, full channel state information for all users cannot be provided at the transmitter. Since having no channel state information does not lead to gains, it is important to study MIMO broadcast channels with partial CSI. In this chapter, we proposed using random beams and choosing the users with the highest SINR. When the number of users grows and M is fixed, we proved that the throughput scales like $M \log\log n$, which coincides with the scaling law of the sum-rate capacity assuming perfect CSI and using dirty paper coding. We further showed that with our scheme, the throughput scales linearly with M, provided that M does not grow faster than $\log n$. Moreover, we considered different scenarios for the case with more than one receive antenna $N > 1$, and we showed that by using random beamforming, throughput of our scheme scales as $M \log\log nN$ when M is fixed and for any N that is precisely the same as the scaling of the sum rate capacity using dirty paper coding. This implies that increasing N has no significant impact on the throughput.

Another issue that we addressed was to analyze the fairness in our scheduling when the users are heterogeneous. We proved that as M becomes large the scheduling becomes more and more fair and when $M > \alpha \log n$ the scheduling will be fair irrespective of the signal to noise ratio of the users. We conclude that using $M = \alpha \log n$ emerges as a desirable operating point, both in terms of guaranteeing fairness as well as providing linear scaling of the throughput with M.

3.10 Appendix

3.10.1 Proof of Eq. (3.36)

In this appendix, we compute a lower bound for $\Pr\{B|A\}$ where A is the event that for all m, $c-\epsilon \le \max_{1\le i\le n} \text{SINR} \le c$, and B is the event that each user can be the maximum for at most one signal s_m. Let's assume $p_m = \arg\max_i \text{SINR}_{i,m}$. Therefore, we can upper bound $1-\Pr\{B|A\}$ by the probability of the event that there exists an index p_m such that the corresponding user is the maximum for at least two signals s_{m_1} and s_{m_2}. Since this event is conditioned on event A, both the maximums should be between c and $c-\epsilon$ and clearly one of them should be $\max_{1\le m\le M} \text{SINR}_{p_1,m}$. Therefore

$$\begin{aligned}
1-\Pr\{B|A\} &\le \Pr\left\{\exists p \in \{p_1,\ldots,p_M\} : c-\epsilon \le \text{SINR}_{p_1,m_1}, \max_{1\le m\le M} \text{SINR}_{p_1,m} \le c\right\} \\
&\le M\Pr\left\{c-\epsilon \le \text{SINR}_{1,m_1}, \max_{1\le m\le M} \text{SINR}_{1,m} \le c\right\}, \qquad (3.66)
\end{aligned}$$

where we used the union bound and the fact that all SINRs have the same distribution over i, and m_1 is not the index corresponding to the maximum SINR over m. In order to compute the probability in (3.66), we define the random variables $\beta_m = |h_1\phi_m|^2$ for $m = 1,\ldots,M$, and let $r = \arg\max \beta_i$. Therefore, we want to compute the probability that

$$c \ge \max_{1\le m\le M} \text{SINR}_{1,m} = \frac{\max \beta_i}{\frac{1}{\rho} + \sum_{i\ne r}^{M} \beta_i} = \frac{\max \beta_i}{\frac{1}{\rho} + \beta_{m_1} + D} \qquad (3.67)$$

and that there exists $m_1 \in \{1,\ldots,M\}$ and $m_1 \ne r$ such that

$$c-\epsilon \le \text{SINR}_{1,m_1} = \frac{\beta_{m_1}}{\frac{1}{\rho} + \max \beta_i + D}, \qquad (3.68)$$

where $D = \sum_{i=1}^{M} \beta_i - \max \beta_i - \beta_{m_1}$. Eq. (3.67) and (3.68) imply that $\beta_{m_1} \ge (1-\frac{\epsilon}{c})\max\beta_i$. This probability can be computed by integrating over a the probability that there are at least two β_i's in the region $[a(1-\frac{\epsilon}{c}),\ a]$ and all the β_i's are less than a. Therefore,

$$\begin{aligned} 1-\Pr\{B|A\} &\leq M\Pr\{\exists m_1 \in \{1,\ldots,M\}, m_1 \neq r : \beta_{m_1} \geq (1-\frac{\epsilon}{c})\max\beta_i\} \\ &= M\int_0^{\infty}\binom{M}{2}\Pr\left\{a(1-\frac{\epsilon}{c}) \leq \beta_1 \leq a\right\}^2 (\Pr\{\beta_1 < a\})^{M-2}\,da \\ &= \frac{M^2}{2}(M-1)\int_0^{\infty}\left(e^{-(1-\frac{\epsilon}{c})a}-e^{-a}\right)^2\left(1-e^{-a}\right)^{M-2}da \\ &= \frac{M^2}{2}(M-1)\int_0^1 (u^{-\epsilon}-1)^2 u(1-u)^{M-2}du, \end{aligned} \tag{3.69}$$

where in the second step we use the fact that two β_i's must be large (in fact in the region $[a(1-\frac{\epsilon}{c}),\ a]$) and β_i's are i.i.d random variables. In order to compute (3.69), we define the function $T(\epsilon)=\int_0^1 u^{1-\epsilon-x}(1-u)^{M-2}du$. Clearly, the integral in (3.69) can be written as $T(\epsilon)+T(-\epsilon)-2T(0)$. By mean value theorem, we can use the Taylor expansion of the integral to get

$$\int_0^1 (u^{-\epsilon}-1)^2 u(1-u)^{M-2}du = T(\epsilon)+T(-\epsilon)-2T(0) = T''(\zeta)\epsilon^2, \tag{3.70}$$

where $-\epsilon \leq \zeta \leq \epsilon$. Now we can write the second derivative of $T(\zeta)$ as

$$\begin{aligned} T''(\gamma) &= \int_0^1 (\log u)^2 u^{1-\epsilon-\zeta}(1-u)^{M-2}du \\ &\leq \int_0^1 (\log u)^2 u^{0.9}(1-u)^{M-2}du, \end{aligned} \tag{3.71}$$

where we used the fact that ϵ is very small and $-\epsilon \leq \zeta \leq \epsilon$. We know that $u^{0.4}(\log u)^2$ is a bounded function for $u \in [0, 1]$, therefore,

$$\begin{aligned} T''(\gamma) &\leq O\left(\int_0^1 u^{0.5}(1-u)^{M-2}du\right) \\ &= O\left(\frac{\Gamma(M-1)\Gamma(1.5)}{\Gamma(M+0.5)}\right) \\ &= O\left(\frac{1}{M^{1.5}}\right), \end{aligned} \tag{3.72}$$

where we used the asymptotic expansion of the gamma functions [63]. Replacing (3.72) in (3.70) and then into (3.69), we get

$$1 - \Pr\{B|A\} \leq O(\epsilon^2 M^{1.5}) = O\left(\frac{(\log\log n)^2}{\sqrt{\log n}}\right). \tag{3.73}$$

Chapter 4

Differentiated Rate Scheduling for Gaussian Broadcast Channels

4.1 Introduction

The downlink scheduling in cellular systems is known to be one major bottleneck for future broadband wireless communications. Information-theoretic results on broadcast channels provide the limits for the maximum achievable rates[1] for each receiver [22, 25, 26]. For example in a homogeneous network, if the transmitter wants to maximize the throughput (or the sum of the rates to all the receivers[2]), the best strategy is to transmit to the user with the best channel condition at each channel use. This is the so called "opportunistic" transmission strategy as described in Chapter 2 (see also [28]).

In homogenous networks, opportunistic scheduling treats all the users equally. In systems that are provisioned to provide differentiated services to different users, the transmitter has to give different services (or rates) to different subsets of receivers, and yet at the same time, maximize the throughput [24].

In this chapter, we are interested in analyzing differentiated rate scheduling schemes for broadcast channels. In particular we assume receivers are divided into M groups

[1]Here we assume the channel is ergodic and rate refers to the average rate over all channel realizations.

[2]We use users and receivers interchangeably.

where the ratios of the rates of users in different groups are given. Given these ratios, the overriding question in this chapter is to devise schemes that provide the rate constraints and yet at the same time maximize the throughput of the system. We are also interested to see how much throughput loss the transmitter would incur by imposing such constraints.

In the first part of the chapter, we consider channels with a small number of users (i.e., $n = 2, 3$). It turns out that the problem of determining a schedule that satisfies the rational rate constraints becomes analytically intractable as the number of users grows beyond 3. Therefore, in the second part of the chapter, we assume that the number of users is large. This is, of course, of practical interest since many systems operate in such a regime. Furthermore, it allows us to obtain explicit results in the asymptote of large n.

We should also mention that in this chapter we will only be dealing with homogenous networks, in the sense that the SNRs of the different users are assumed to have the same probability distribution. Of course many networks are, in fact, heterogenous, with different users having different distributions for their SNRs. The methodology of this chapter (and many of the results, we suspect) can be carried over to the heterogenous case, with the caveat that the development will be much more involved and cumbersome. For this reason, and for reasons of space, although quite important in practice, we deem the heterogenous case beyond the scope of the current chapter. We only remark that by appropriate power control any heterogenous network can be made to look homogenous (after which all our results will directly apply).

4.2 Problem Formulation

We consider a scalar Gaussian broadcast system with n receivers. We also assume the channels between the transmitter and the receivers are block fading, i.e., the channels remain fixed over any block of data transmission. After any such block the channels will change (perhaps to independent values). For the purposes of this chapter it will not matter *how* the channels change from block to block, other than the fact that they vary in some stationary and ergodic way.[3]

The relationship between X, the transmitted signal, and Y_i, the received signal at receiver i, can be written as

$$Y_i = h_i X + W_i, \tag{4.1}$$

where h_i is the channel coefficient between the transmitter and the i-th receiver and W_i is the additive noise. h_i and W_i are i.i.d. complex circularly symmetric Gaussian random variables with zero mean and variance one, $CN(0, 1)$. In terms of channel knowledge, we assume that h_i is known perfectly at the transmitter and the receiver. We further assume that the transmitter is subject to a short-term power constraint, so that the transmitted signal in every state must satisfy the power constraint P. We denote the (average) rate of the i'th user, $i = 1, \ldots, n$, over the different channel realizations by R_i.

In this chapter, we are interested in analyzing differentiated rate scheduling schemes for broadcast systems with n users. We consider a partitioning of the users into M groups $\mathcal{G}_1, \ldots, \mathcal{G}_M$, where different groups require different rates from the transmitter. We also assume that the sizes of the groups are all of the same order and hence, the cardinality of $\mathcal{G}_k$ is $\alpha_k n$ where M and α_k's are fixed numbers such that $\sum_{i=1}^{M} \alpha_i = 1$.

Assuming that the average rate of a user in the k'th group is denoted by R^k,[4]

[3] This is because our focus is on the rate. If we had focused on other performance measures, such as delay, then how the channels vary with time would have been important.

[4] Throughout the chapter, we use superscript k to refer to any user in $\mathcal{G}_k$.

without loss of generality, we may assume $R^1 < \ldots < R^M$. We further impose the constraint that the average rate of a user in the i-th group is β_i times the average rate of a user in the M-th group. In general we are interested in the following optimization problem:

$$\max \quad \sum_{i=1}^{n} R_i \tag{4.2}$$

$$\text{subject to} \quad \frac{R^k}{R^M} = \beta_k, \qquad k = 1, \ldots, M, \tag{4.3}$$

where $\beta_1 < \ldots < \beta_{M-1} < \beta_M = 1$ are fixed numbers independent of n. It is clear that the best operating point in the ergodic capacity region is the intersection of the boundary of the capacity region $\mathcal{C}_{BC}(R_1, \ldots, R_n)$ with the line defined in (4.3). While this is easy enough to state, it is not so easy to do since $\mathcal{C}_{BC}$ is implicitly defined as the convex hull of an infinite set of rates. Moreover, it is not so easy to see how any point on the boundary of the capacity region can be mapped back to a schedule.

In Section 4.3, we consider a channel with a small number of users, namely $n = 2$ and $n = 3$. We also focus on the rate region achieved by weighted-opportunisitc (WO) scheduling in which we transmit to the user that has the maximum weighted SNR. We obtain the relationship between the weights for WO scheduling and the ratio of the rates. It turns out that finding an explicit relationship between the weights as a function of the given ratios is analytically intractable for $n > 3$, even if we allow for simplifying assumptions such as considering the low SNR regime. We further look into the throughput loss due to the rate constraints in (4.3).

In order to obtain explicit solutions, in the second part of the chapter, we consider a system with many users and, rather than attempt to solve (4.2)-(4.3) directly, we look at the performance of three specific scheduling schemes to provide the rational rate constraints in (4.3). These are weighted-opportunistic, time-division opportunistic, and superposition coding. In the time-division scheme we allow the transmitter to divide each channel use to non-equal time slots. At the k-th slot, the transmitter sends to the receiver with the best channel condition from group $\mathcal{G}_k$. Finally, superposition coding is the one that achieves the ergodic capacity region [22, 25].

4.3 Channels with a Small Number of Users

In this section, we start with characterizing the achievable rate region using weighted-opportunistic (WO) scheduling. In WO, at each channel use we send to only the user that has the maximum weighted signal to noise ratio, i.e., the user for which

$$\max_{1\leq i\leq n} \mu_i |h_i|^2. \tag{4.4}$$

Let us look at the rate transmitted to one of the users, say the first user (denoted by R_1^w). Clearly,

$$\begin{aligned} R_1^w &= \Pr(\text{user 1 is transmitted to}) \times \\ &\quad \int_0^\infty \log(1+Px)\Pr(x|x=|h_i|^2, \text{user 1 is transmitted to})dx. \end{aligned}$$

Defining $x_i = |h_i|^2$, $i = 2, \ldots, n$, the probability density function inside the integral can be written as

$$\begin{aligned} \Pr(x|x > \frac{\mu_i}{\mu_1}, x_i, i = 2, \ldots, n) &= \frac{\Pr(x, x > \frac{\mu_i}{\mu_1}, x_i, i = 2, \ldots, n)}{\Pr(x > \frac{\mu_i}{\mu_1} x_i, i = 2, \ldots, n)} \\ &= \frac{\Pr(x > \frac{\mu_i}{\mu_1} x_i, i = 2, \ldots, n|x)p(x)}{\Pr(x > \frac{\mu_i}{\mu_1} x_i, i = 2, \ldots, n)} \\ &= \frac{\prod_{i=2}^n \left(1 - e^{-\frac{\mu_1}{\mu_i} x}\right) e^{-x}}{\Pr(\text{user 1 is transmitted to})}. \end{aligned}$$

Replacing this last expression into the formula for R_1^w yields

$$R_1^w = \int_0^\infty \log(1+Px) \prod_{i=2}^{n} \left(1 - e^{-\frac{\mu_1}{\mu_i} x}\right) e^{-x} dx. \tag{4.5}$$

The rates to the other users can be found in a similar fashion.

Let us now focus on (4.5) for the case of two and three users.

4.3.1 Case 1: Two-User Channels

Eq. (4.5) now simply reduces to

$$R_1^w = \int_0^\infty \log(1+Px) \left(1 - e^{-\frac{\mu_1}{\mu_2} x}\right) e^{-x} dx. \tag{4.6}$$

Similarly, the rate for the second user is as in (4.6) with the only difference that μ_1 should be exchanged by μ_2. We would like to find μ_1 and μ_2[5] such that the rate constraint, i.e., $\frac{R_1^w}{R_2^w} = \beta$, is satisfied.

We can simplify (4.6) as

$$\begin{aligned} R_1^w &= \int_0^\infty \left\{ \log(1+Px)e^{-x}dx - \frac{\mu_2}{\mu_1+\mu_2} \log\left(1 + \frac{P\mu_2}{\mu_1+\mu_2} x\right) \right\} e^{-x} dx \\ &= -e^{\frac{1}{P}} Ei(-\frac{1}{P}) + \frac{\mu_2}{\mu_1+\mu_2} e^{\frac{\mu_1+\mu_2}{P\mu_2}} Ei\left(-\frac{\mu_1+\mu_2}{P\mu_2}\right), \end{aligned} \tag{4.7}$$

where we used the definition of the exponential integral function defined as $-Ei(-x) = \int_1^\infty \frac{e^{-tx}}{t} dt$. We can similarly write the rate for the second user as

$$R_2^w = -e^{\frac{1}{P}} Ei(-\frac{1}{P}) + \frac{\mu_1}{\mu_1+\mu_2} e^{\frac{\mu_1+\mu_2}{P\mu_1}} Ei\left(-\frac{\mu_1+\mu_2}{P\mu_1}\right). \tag{4.8}$$

In order to find the μ_1 that satisfies the rate constraint of (4.3), we need to solve the

[5] It is worth mentioning that the sum of μ_i's is equal to 1.

following non-polynomial equation,

$$\frac{-e^{\frac{1}{P}}Ei(-\frac{1}{P}) + (1-\mu_1)e^{\frac{1}{P(1-\mu_1)}}Ei\left(-\frac{1}{P(1-\mu_1)}\right)}{-e^{\frac{1}{P}}Ei(-\frac{1}{P}) + \mu_1 e^{\frac{1}{P\mu_1}}Ei\left(-\frac{1}{P\mu_1}\right)} = \beta. \tag{4.9}$$

It does not seem that (4.9) has a closed form solution because it involves the exponential integral. We can numerically evaluate β versus μ_1 as shown in Fig. 4.1. Since $\mu_1 + \mu_2 = 1$, here we assume that μ_1 is varying between zero and one.

The generalization to a system with $n > 2$ users is straightforward: We simply need to expand the products in (4.5) into a summation of exponentials and then repeatedly use the exponential integral. Applying the rate constraints will lead to a non-polynomial system of equations with $n-1$ equalities and $n-1$ variables. Although it may be possible to solve numerically such a system of equations, it gives us little insight into the problem.

In order to find more explicit results, in the next section, we simplify the system (4.9) by assuming that P is small (low SNR regime).

4.3.2 Case 2: Low SNR Regime

Assuming that the system is working in the regime of small P, the instantaneous rate can be approximated to first order as $P|h_i|^2$ (instead of $\log(1 + P|h_i|^2)$). It turns out that this leads to a system of polynomial equations, which can be theoretically dealt with using Groebner bases.[6] Given a finite set of multivariate polynomials over a field, a new set of polynomials with good properties can be found by an algorithm of Buchberger, called the Groebner basis, which can be used to find the solutions of the polynomial system. This method has been extensively studied, developed and has been implemented on all major computer algebra systems.

[6]Groebner bases method was introduced by Bruno Buchberger in 1965 [64].

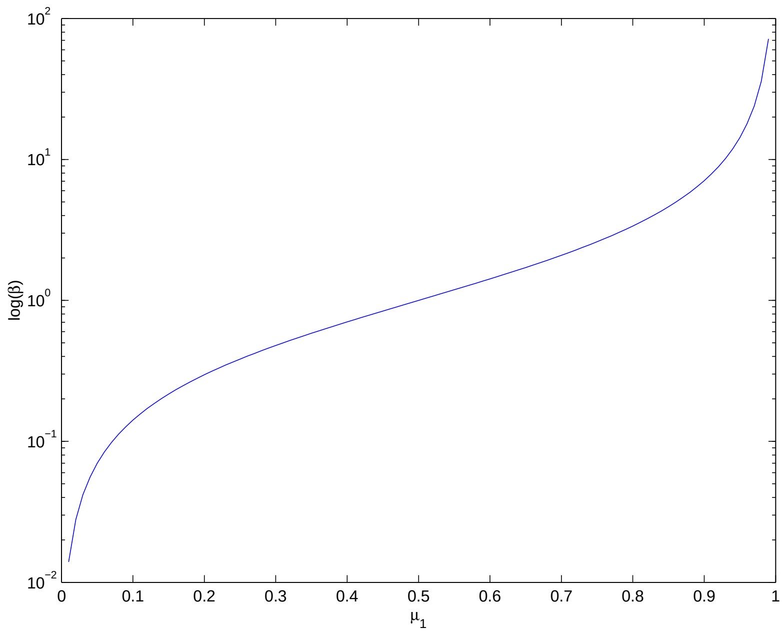

Figure 4.1: β versus μ_1 for a channel with $n = 2$.

In a channel with two users, the rates can be written as

$$R_1^w = \int_0^\infty Px(1 - e^{-\frac{\mu_1}{\mu_2}x})e^{-x}dx = P(2 - \mu_1)\mu_1. \tag{4.10}$$

Similarly,

$$R_2^w = P(1 - \mu_1^2). \tag{4.11}$$

Therefore the boundary of the rate region is characterized by (4.10) and (4.11). This parametric characterization can be made explicit by eliminating μ_1 from (4.10) and (4.11) as

$$\frac{1}{4}\left(1 - \frac{R_2^w}{P} + \frac{R_1^w}{P}\right)^2 = 1 - \frac{R_2^w}{P}. \tag{4.12}$$

Now given the ratio of the rates and (4.12), we can easily obtain μ_1 such that the ratio of the rates will be equal to β.

This framework can be easily generalized to the case of more than two users. We omit the details and simply state that for $n = 3$, we may write the rates as

$$\begin{aligned}
\frac{R_1^w}{P} &= 1 - \frac{\mu_2}{\mu_1+\mu_3} - \frac{\mu_3}{\mu_1+\mu_2} + \frac{\mu_2\mu_3}{\mu_2\mu_3+\mu_1\mu_3+\mu_1\mu_2}, \\
\frac{R_2^w}{P} &= 1 - \frac{\mu_1}{\mu_2+\mu_3} - \frac{\mu_3}{\mu_2+\mu_1} + \frac{\mu_1\mu_3}{\mu_2\mu_3+\mu_1\mu_3+\mu_1\mu_2}, \\
\frac{R_3^w}{P} &= 1 - \frac{\mu_1}{\mu_2+\mu_3} - \frac{\mu_2}{\mu_3+\mu_2} + \frac{\mu_1\mu_2}{\mu_2\mu_3+\mu_1\mu_3+\mu_1\mu_2}, \\
1 &= \mu_1+\mu_2+\mu_3,
\end{aligned}$$

To find the explicit characterization of the rate region, we have to eliminate the μ_i's from the above set of polynomial equations. This can be done with the aid of Groebner bases using Mathematica, say. However the complexity of the algorithm becomes formidable, even for the case of $n = 3$.

On the other hand, it is possible to attempt to solve the above system of equations numerically. This, in principle, will allow us to map a set of rate constraints to a set of weights for the schedule. However, as mentioned earlier, this gives little insight and, moreover, it too can be quite complex for large n.

4.3.3 Throughput Loss

It is clear that there is a price to pay in terms of throughput (sum-rate) to maintain the rate constraints. This is due to the fact that we are not working on the sum-rate capacity point and therefore, the throughput will be reduced compared to the case where we had no rate constraint. In this part, we numerically evaluate the throughput degradation due to imposing the rate constraint of β for a channel with two users. Assuming that the rate of the first user is β times the rate of the second user, Fig. 4.2 shows the ratio of the throughput of the WO scheduling over the sum-rate capacity versus β.

Clearly, when β equals one, the WO scheduling achieves the sum-rate capacity,

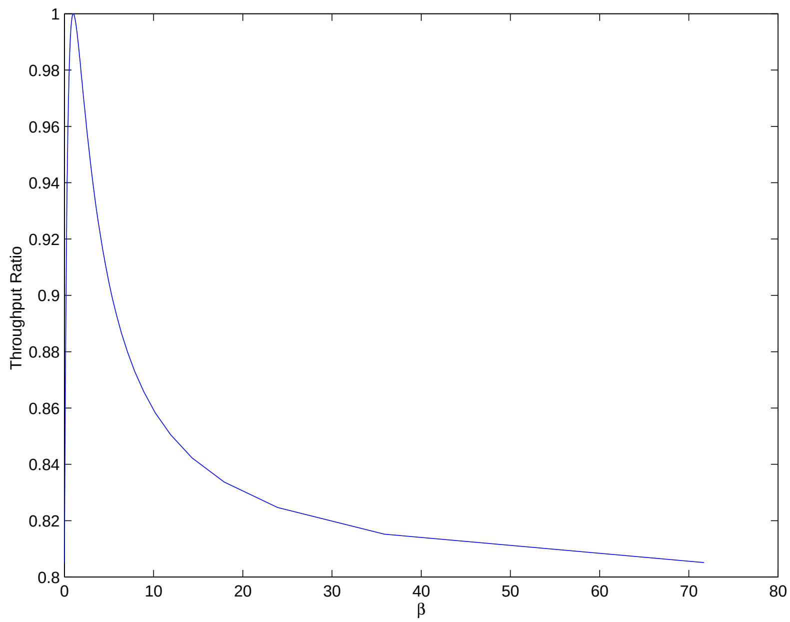

Figure 4.2: Ratio of the throughput with rate constraints over the sum-rate capacity versus β for a channel with $n = 2$.

and therefore, throughput will be equal to the sum-rate capacity. As β increases the throughput loss will be more. It is quite interesting to observe that even for very large β (e.g., close to 70), the throughput is above 80 percent of the sum-rate capacity. Therefore, the throughput does not seem to be too much affected by the differentiated rate scheduling. In the next section, we look into this throughput loss in the regime of large number of users.

4.4 Channels with Many Users

In Section 4.3, we observed that finding an explicit relationship between β_i's and the μ_i's in WO scheduling becomes very complicated even for the case of $n = 3$.

Therefore, for the remainder of the chapter, we look into the regime of a large number of users. We consider three different schedulings, namely, WO, TO and superposition coding. It will turn out that having a large number of users will simplify the derivations and lead to explicit results.

4.4.1 Weighted-Opportunistic Scheduling

In this part, we consider the scheduling described in (4.4). Since the rates of the users within the same group are the same, it is clear that we only need to set M different μ_i's corresponding to each of M different groups. The first question would be to figure out the mapping between the β_k's in (4.3) and μ_k's. In other words, what values of μ_i's lead to the desired ratios β_i's? Secondly, we are also interested to obtain the loss we incur on the throughput (i.e., sum of the rates) of the system by imposing (4.3) and using this scheduling.

In order to find the rate of a user in $\mathcal{G}_1$, i.e. $R^{1,w}$, using weighted-opportunistic scheduling, we may use (4.5) to write

$$R^{1,w} = \int_0^\infty \log(1+Px)e^{-x}(1-e^{-x})^{\alpha_1 n-1}\prod_{j=2}^{M}(1-e^{-\frac{\mu_1}{\mu_j}x})^{\alpha_j n}dx. \tag{4.13}$$

Equation (4.13) follows by noting that a user in $\mathcal{G}_1$ is chosen if its own channel is better than $\alpha_1 n - 1$ users in $\mathcal{G}_1$ and its weighted channel is better than $\alpha_i n$ users in $\mathcal{G}_i$ for $i = 2, \ldots, M$.

As observed in Section 4.3, analyzing $R^{1,w}$ for any n involves the exponential integral function and therefore explicitly finding the solution for μ_i's to guarantee β_i's becomes cumbersome and numerically intractable. Therefore, we consider the regime of large number of users and find μ_i's that satisfy (4.3) in this regime.

Lemma 4.1 *Suppose M and α_i's are fixed and n grows. Then if μ_i's are chosen such that*

$$\mu_i = 1 + \frac{\log \beta_i}{\log n}, \qquad i = 1, \ldots, M, \tag{4.14}$$

then $\lim_{n\to\infty} \frac{R^{i,w}}{R^{M,w}} = \beta_i$ *for* $i = 1, \ldots, M-1$. *Furthermore*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} R_i^w}{\log\log n} = 1. \tag{4.15}$$

Proof: Assuming (4.14) holds, we first prove that $R^{1,w} = \frac{\beta_1 \log\log n}{\sum_{k=1}^{M} \alpha_k \beta_k}$. We can write (4.13) as

$$\begin{aligned} R^{1,w} &= \int_0^\infty \log(1+Px) e^{-x} e^{\sum_{k=1}^{M} \alpha_k n \log(1-e^{-\frac{\mu_1}{\mu_k}x})} dx \\ &= \int_0^\infty \log(1+Px) e^{-x-\sum_{k=1}^{M} \alpha_k n e^{-\frac{\mu_1}{\mu_k}x} + O(ne^{-2\frac{\mu_1}{\mu_M}x})} dx \\ &= \int_0^\infty \log(1+Px) e^{-x-ne^{-x}\left(\sum_{k=1}^{M} \alpha_k e^{-\frac{x\log(\beta_1/\beta_k)}{\log n}} + O(n^{-\gamma})\right)} dx, \end{aligned} \tag{4.16}$$

where γ is some positive constant and where Eq. (4.16) is obtained by expanding the logarithm and using (4.14).

We now consider three regions for the integral in (4.16), namely, between $\log an \pm 4\log\log n$ ($\mathcal{H}_1$), greater than $\log an + 4\log\log n$ ($\mathcal{H}_2$), and less than $\log an - 4\log\log n$ ($\mathcal{H}_3$) where $a = \frac{\sum_{k=1}^{M} \alpha_k \beta_k}{\beta_1}$. We can bound the integral over $\mathcal{H}_2$ by noting that the average rate to the first group is at most of the order $\log\log n$ and therefore an upper would be $\log\log n$ multiplied by the probability that $|h_1|^2$ is greater than $\log an + 4\log\log n$ condition on the fact that $|h_1|^2 \geq \frac{\mu_i}{\mu_1}|h_i|^2$ for $i = 2, \ldots, n$. Hence the integral over $\mathcal{H}_2$ can be written as

$$\begin{aligned} &\int_{\log an + 4\log\log n}^\infty \log(1+Px) e^{-x-ne^{-x}\sum_{k=1}^{M} \alpha_k e^{-\frac{x\log(\beta_1/\beta_k)}{\log n}} + O(ne^{-2\frac{\mu_1}{\mu_M}x})} dx \\ &= O\left(\log\log n \int_{\log an + 4\log\log n}^\infty e^{-x-ne^{-x}\sum_{k=1}^{M} \alpha_k e^{-\log(\beta_1/\beta_k)\frac{x}{\log n}}} dx\right) \\ &= O\left(\frac{\log\log n}{n(\log n)^4}\right). \end{aligned}$$

Similarly the integral over $\mathcal{H}_3$ can be obtained and is equal to $O\left(\frac{\log\log n}{n}e^{-(\log n)^4}\right)$.

We now focus on evaluating the integral over $\mathcal{H}_1$. We may change the variables to $y = x - \log an$ to get

$$\begin{aligned}
&\int_{\log an-4\log\log n}^{\log an+4\log\log n} \log(1+Px)e^{-x-e^{-x}\sum_{k=1}^{M}\alpha_k e^{-\log(\beta_1/\beta_k)\frac{x}{\log n}}}dx \\
=& \left(\log\log n + O\left(\frac{1}{\log n}\right)\right)\int_{\log an-4\log\log n}^{\log an+4\log\log n} e^{-x-e^{-x}\sum_{k=1}^{M}\alpha_k e^{-\log(\beta_1/\beta_k)\frac{x}{\log n}}}dx \\
=& \frac{\log\log n + O(1/\log n)}{an}\int_{-4\log\log n}^{4\log\log n} e^{-y}e^{-e^{-y}}dy \\
=& \frac{\beta_1\log\log n}{n\sum_{k=1}^{M}\alpha_k\beta_k}\left(1 - O\left(\frac{1}{\log n}\right)\right).
\end{aligned}$$

It is quite straightforward to write the rate for users in the other groups in a similar way. Therefore for any group, we obtain

$$R^{k,w} = \frac{\beta_k}{\sum_{j=1}^{M}\alpha_j\beta_j}\frac{\log\log n}{n} + O\left(\frac{\log\log n}{n\log^4 n}\right). \tag{4.17}$$

Clearly the ratios of the rates satisfy (4.3) in the limit of large n and also the first order term in the throughput of this scheme is $\log\log n$ that leads to the second part of the lemma. □

Remark 4.1 Lemma 4.1 asserts that the average rates of users are quite sensitive to the change of μ_i's. In order to further understand the impact of a change in μ_i's on the rates, we consider a two-group system. Following the methodology in the proof of Lemma 4.1, we can prove the following results. If

$$\frac{\mu_1}{\mu_2} = 1 - o\left(\frac{1}{\log n}\right) \Longrightarrow \lim_{n\to\infty}\frac{R^{1,w}}{R^{2,w}} = 1, \tag{4.18}$$

where $R^{i,w}$ is as defined in Lemma 4.1. Moreover, if

$$\frac{\mu_1}{\mu_2} = c < 1 \Longrightarrow \lim_{n\to\infty}\frac{R^{1,w}}{R^{2,w}} = 0, \tag{4.19}$$

where c is a constant independent of n.

We further look into the throughput loss that we would incur by differentiated rate scheduling. From Lemma 4.1, it is clear that the first order of the sum-rate (i.e., $\log\log n$) remains unchanged. In the next lemma we show that the difference of the sum-rate capacity (maximum throughput) and the throughput of this schemes converges to zero. We also obtain the convergence rate.

Lemma 4.2 *Suppose M and α_i's are fixed and n grows. Then,*

$$\int_0^\infty n\log(1+Px)e^{-x}(1-e^{-x})^{n-1} - \sum_{i=1}^{n} R_i^w = \Theta\left(\frac{(\log\log n)^3}{(\log n)^2}\right), \tag{4.20}$$

where the first term denotes the sum-rate capacity achieved by sending to the user with the best channel condition at each channel use.

Proof: We prove this lemma for the case of $M = 2$ for the sake of brevity, however it is quite straightforward to generalize to $M > 2$. We first divide the integral that represents the difference of the throughputs into three regions as we did in the proof of Lemma 4.1. The first term would be the integral over the region $\mathcal{H}_1$. We can then write the integral as

$$\begin{aligned} &\int_{\mathcal{H}_1} \frac{n}{2}\log(1+Px)e^{-x}(1-e^{-x})^{n/2-1}\left\{(1-e^{-x})^{n/2-1} -\right. \\ &\quad \left.(1-e^{-\frac{\mu_1}{\mu_2}x})^{n/2} - (1-e^{-\frac{\mu_1}{\mu_2}x})^{n/2}\right\} dx \\ = \;& \Theta\left(\log\log n\int_{-4\log\log n}^{4\log\log n} e^{-y}\left(1-\frac{e^{-y}}{n/2}\right)^{n/2-1}\left\{\left(1-\frac{e^{-y}}{n/2}\right)^{n/2}(1-e^{-\frac{\mu_1}{\mu_2}(y+\log n/2)})^{n/2}\right.\right. \\ &\quad \left.\left. -(1-e^{-\frac{\mu_1}{\mu_2}(y+\log n/2)})^{n/2} \right\} dy\right), \end{aligned} \tag{4.21}$$

where $y = x - \log n/2$. Assuming that $\epsilon = \frac{\log\log n}{\log n}$, we can write (4.21) as

$$\begin{aligned} & \Theta\left(\log\log n\left(2e^{-\frac{1}{\log^4 n}} - \frac{e^{-\frac{1}{\log^{\gamma+1} n}(1+\beta_1(1+\epsilon))}}{1+\beta_1(1+\epsilon)} - \frac{e^{-\frac{1}{\log^{\gamma+1} n}(1+\frac{1}{\beta_1}(1-\epsilon))}}{1+\frac{1}{\beta_1}(1-\epsilon)}\right)\right) \\ = & \Theta(\epsilon^2 \log\log n) \\ = & \Theta\left(\frac{(\log\log n)^3}{(\log n)^2}\right). \end{aligned} \tag{4.22}$$

It is quite straightforward to show that the other terms would contribute $O\left(\frac{\log\log n}{(\log n)^{2+2\gamma}}\right)$. This completes the proof. □

4.4.2 Time-Division Opportunistic Scheduling

Another, in fact simpler, approach to guarantee the rate constraints is to do time-sharing between different groups by dividing each channel use of duration T into M slots of different lengths.[7] The i'th slot is dedicated to the i'th group and the transmitter chooses the receiver with the best channel conditions for transmission from $\mathcal{G}_i$. Intuitively, we should be able to achieve the rational rate constraints if we divide the slots into the same ratios. (Lemma 4.3 shows this is the case.)

Here we denote the rate of a user in $\mathcal{G}_k$ using this scheme by $R^{k,t}$, similarly the rate of the i'th user will be denoted by R_i^t. In the next lemma, we show that if the cardinality of all groups is of the order of n, we can construct the length of slots such that (4.3) is satisfied and the throughput of the scheme scales like $\log\log n$.

Lemma 4.3 *Suppose M and α_i's are fixed. Also, let l_i be the length of the i'th slot and is equal to*

$$\frac{l_i}{T} = \frac{\alpha_i\beta_i}{\sum_{i=1}^M \alpha_i\beta_i} \quad i = 1, \ldots, M. \tag{4.23}$$

[7] Instead of one channel use, one might divide every K channel uses into slots of different lengths

Then $\lim_{n\to\infty} \frac{R^{i,t}}{R^{M,t}} = \beta_i$ *for* $i = 1, \ldots, M-1$. *Moreover,*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} R_i^t}{\log\log n} = 1. \tag{4.24}$$

Proof: It is quite easy to show that at the i'th slot in which the transmitter sends information to the user with the best channel condition among users in $\mathcal{G}_i$, the total rate of information sent to $\mathcal{G}_i$ is equal to

$$\frac{l_i}{T} \log\log \alpha_i n = \frac{\alpha_i \beta_i}{\sum_{i=1}^{M} \alpha_i \beta_i} \log\log \alpha_i n. \tag{4.25}$$

Therefore since users in each group are equally likely to be chosen, the rate to a user in the i'th group is equal to $\frac{1}{\alpha_i n}$ times the rate to $\mathcal{G}_i$ as in (4.25). This proves the first part of the lemma. The second part of the lemma follows by noting that $\sum_{k=1}^{M} \alpha_k = 1$ and α_k's are fixed and are not vanishing to zero. □

It is worth mentioning that in the time-division scheme at the i'th slot, the transmitter will certainly not transmit to any user outside group i even if such a user has a much better channel condition. Therefore although simpler, one may guess that the time-division scheme has a lower throughput than the weighted opportunistic one. The next lemma proves again the throughput of this scheme converges to the sum-rate capacity, however, the convergence rate for the time-division scheme is polynomially slower than that of the weighted-opportunistic scheduling (i.e., $\Theta\left(\frac{\log\log n}{\log n}\right)$ versus $\Theta\left(\frac{(\log\log n)^3}{(\log n)^2}\right)$).

Lemma 4.4 *Suppose* M *and* α_i*'s are fixed. Also, let* l_i *be chosen as in (4.23). Then*

$$\int_0^\infty n\log(1+Px)e^{-x}(1-e^{-x})^{n-1} - \sum_{i=1}^{n} R_i^t = \Theta\left(\frac{\log\log n}{\log n}\right). \tag{4.26}$$

Proof: Here we present the proof for the special case of having two groups of the same size. The proof however can be generalized easily and we omit it for the sake of brevity.

The difference of the throughputs can be written as

$$\int_0^\infty n\log(1+Px)e^{-x}\left\{(1-e^{-x})^{n-1}-\frac{1}{2}(1-e^{-x})^{n/2-1}\right\}dx.$$

We can now follow the same approach as in the proof of Lemma 4.1 and expand the integral to three parts, namely, $\log n/2 \pm 4\log\log n$, larger than $\log n/2 + 4\log\log n$, and smaller than $\log n/2 - 4\log\log n$, which were named as $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, respectively. It is quite straightforward to show that the last two integrals over $\mathcal{H}_\in$ and $\mathcal{H}_3$ lead to $O(\frac{\log\log n}{\log^4 n})$.

The integral over $\mathcal{H}_1$ can be then evaluated as

$$\begin{aligned}
&\int_{\log n/2-4\log\log n}^{\log n/2+4\log\log n} n\log(1+Px)e^{-x}\left\{(1-e^{-x})^{n-1}-\frac{1}{2}(1-e^{-x})^{n/2-1}\right\}dx \\
&= \Theta\left(\log\log n\int_{-4\log\log n}^{4\log\log n} e^{-y}\left(1-\frac{e^{-y}}{n/2}\right)^{n/2-1}\left\{\left(1-\frac{e^{-y}}{n/2}\right)^{n/2}-\frac{1}{2}\right\}dy\right) \\
&= \Theta\left(\log\log n\int_{-4\log\log n}^{4\log\log n} e^{-y}e^{-e^{-y}}\left(e^{-e^{-y}}-\frac{1}{2}\right)dy\right) \\
&= \Theta\left(\frac{\log\log n}{\log n}\right).
\end{aligned} \tag{4.28}$$

This completes the proof for the two-group case. The generalization to the M group case follows by using the same technique and we omit for the sake of brevity. □

In the next subsection, we look into a scheme that employs superposition coding and clearly leads to the best throughput as we actually work on the boundary of the capacity region. As the analysis becomes complicated, we just consider two groups and obtain a scheduling that maximizes the throughput while the rational rate constraints of (4.3) are satisfied. It should be mentioned that the ergodic capacity region of a broadcast channel with two users has been studied in [25], here we look at a generalization of the result of [25] in which we have n users divided into two groups with different rate demands.

4.4.3 Superposition Coding

In this section, we analyze the performance of superposition coding for the case when there are only two groups of users $\mathcal{G}_1, \mathcal{G}_2$ with equal sizes that have different rate demands. We assume that the average rate provided to a user in the first group is required to be $\beta > 1$ times the rate provided to a user in the second group.

In order to maximize the rate (sum-rate) while keeping the ratio of different group rates fixed and equal to β, we need to find the point on the boundary of the capacity region of the Gaussian broadcast channel with short-term power constraint P that satisfies the differentiated rate constraint. We know that every boundary point is the solution to the maximization problem

$$\max_{(R_1,\ldots,R_n)\in\mathcal{C}_{BC}} \sum_{i=1}^{n} \mu_i R_i$$

for some positive values $\mu_1, \ldots, \mu_n$. In our case because of the symmetry among the users in each group, the values of μ_i's will be the same for the users in the same group. Therefore, we only need to characterize the boundary points that are the maximizing solution to the problem

$$\max_{(R_1,\ldots,R_n)\in\mathcal{C}_{BC}} \mu_1(\sum_{i\in\mathcal{G}_1} R_i) + \mu_2(\sum_{i\in\mathcal{G}_2} R_i)$$

for $\mu_1, \mu_2 > 0$. The following lemma characterizes such boundary points. The proof of this lemma uses the duality of the broadcast channel and the multi-access channel for scalar channels [65].

Lemma 4.5 *Consider a scalar Gaussian broadcast system with the model described in Section 4.2 . Consider the following optimization problem*

$$\max_{(R_1,\ldots,R_n)\in\mathcal{C}_{BC}} \mu_1(\sum_{i\in\mathcal{G}_1} R_i) + \mu_2(\sum_{i\in\mathcal{G}_2} R_i) \tag{4.29}$$

where $\mathcal{C}_{BC}$ is the ergodic capacity region of broadcast channel with short-term power constraint P and $\mu_1 \geq \mu_2$ are two positive numbers. Then the solution of the above optimization problem is

$$\frac{n}{2}R_i = E(\log(1+Px)|\mu_1 x \geq \mu_2 y) + E(\log(\frac{(\mu_1-\mu_2)y(1+Px)}{\mu_1(y-x)})|(x,y)\in\mathcal{R})$$

for $i \in \mathcal{G}_1$. Similarly, for $i \in \mathcal{G}_2$, we have

$$\frac{n}{2}R_i = E(\log(1+Py)|\mu_1 x \leq \mu_2 y) - E(\log(\frac{(\mu_1-\mu_2)x(1+Py)}{\mu_2(y-x)})|(x,y)\in\mathcal{R}), \tag{4.31}$$

where $x = \max_{i\in\mathcal{G}_1}|h_i|^2$, $y = \max_{i\in\mathcal{G}_2}|h_i|^2$ and region $\mathcal{R}$ is defined as

$$\mathcal{R} = \{(x,y)\in R^+\times R^+ | 0 \leq \frac{\mu_2}{(\mu_1-\mu_2)x} - \frac{\mu_1}{(\mu_1-\mu_2)y} \leq P\}.$$

Proof: The duality between the broadcast channel and the multi-access channel for the scalar case in [65] states that

$$\mathcal{C}_{BC} = \bigcup_{\sum_i P_i(\cdot)=P} \mathcal{C}_{MAC}(P_1(\underline{h}),\ldots,P_n(\underline{h}))$$

where $\underline{h} = (h_1,\ldots,h_n)$, $P_i(\underline{h})$ is the power allocation function of user i and the union is over all the permissible power allocation functions. Also

$$\mathcal{C}_{MAC}(P_1(\underline{h}),\ldots,P_n(\underline{h})) = \{\underline{R} : \sum_{i\in\mathcal{S}} R_i \leq \log(1+\sum_{i\in\mathcal{S}} P_i(\underline{h})|h_i|^2), \forall\mathcal{S}\subset\{1,\ldots,n\}\}.$$

Using this we can rewrite (4.29) as a maximization over all the power allocation functions and all the corresponding rate vectors in the dual multi-access capacity region. Based on this, it can be verified that the maximum of (4.29) occurs when we send only to the users with the best channel in each group. Therefore,

$$V_o = E \max_{\substack{P_x,P_y\\ P_x+P_y=P}} (\mu_1-\mu_2)\log(1+P_x x)+\mu_2\log(1+P_x x+P_y y),$$

where $x=\max_{i\in\mathcal{G}_1}|h_i|^2$, $y=\max_{i\in\mathcal{G}_2}|h_i|^2$. Performing the maximization over P_x, P_y we have one of the following possibilities:

1. If $\mu_1 x \geq \mu_2 y$, we assign all the power to the best user of the first group.

2. If $0 \leq \frac{\mu_2}{(\mu_1-\mu_2)x} - \frac{\mu_1}{(\mu_1-\mu_2)y} \leq P$ then we split the power between the two best users in the two groups as

$$P_x = \frac{(\mu_1-\mu_2)Pxy+\mu_1 x-\mu_2 y}{\mu_1(y-x)x},\;\; P_y = P-P_x. \tag{4.33}$$

3. If $\frac{\mu_2}{(\mu_1-\mu_2)x} - \frac{\mu_1}{(\mu_1-\mu_2)y} > P$, all the power is assigned to the best user in the second group.

We have plotted the decision region for power allocation in the $(\frac{1}{x},\frac{1}{y})$ region in Fig. 3. In the weighted-opportunistic scheduling the power allocation policy would be to send to the best user in the first group if (x,y) is in $\mathcal{R}_1$ and to send to the best user in the second group if (x,y) is in $\mathcal{R}\cup\mathcal{R}_2$.□

The question that remains to be answered is to figure out how to choose μ_1 and μ_2 such that the rate constraint in (4.3) satisfied. This is answered in the next lemma.

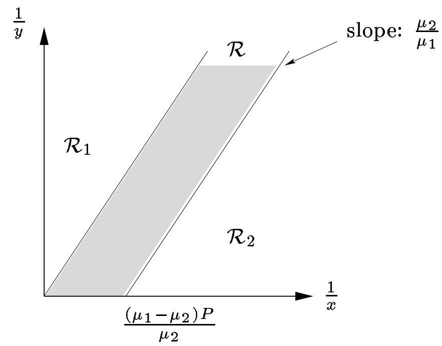

Figure 4.3: The decision region for power allocation in the superposition coding in two-group case: If $(x, y) \in \mathcal{R}_1$, all the power is allocated to best user of group one. If $(x, y) \in \mathcal{R}_2$, all the power is allocated to best user of group two. If $(x, y) \in \mathcal{R}$ then power is split between the best users of the two groups as in (4.33).

Lemma 4.6 *Suppose $\beta < 1$ is fixed, $\mu_1 = 1$, and $\mu_2 = 1 - \frac{1}{(\log n)^\alpha}$ where $\alpha = \frac{\beta+3}{\beta+1}$ (i.e., $1 \le \alpha \le 2$), then*

$$\lim_{n\to\infty} \frac{R^1}{R^2} = \beta. \tag{4.34}$$

Proof: The lemma follows by using the result of Lemma 4.5 and asymptotically analyzing the ratio of the rates. The techniques are similar to the ones we used in the proof of Lemma 4.1 and we omit the proof for the sake of brevity. □

Finally we look into the throughput loss due to the constraint of (4.3) using superposition coding. Using Lemma 4.1, it is clear that the loss should tend to zero for large n and also the convergence rate should be faster than $\frac{(\log\log n)^3}{(\log n)^2}$. In the next lemma we prove that the convergence rate cannot be faster than $\frac{(\log\log n)^{1+2\alpha}}{(\log n)^{2\alpha}}$ where $1 \le \alpha \le 2$ is a fixed number.

Lemma 4.7 *Suppose $\beta > 1$ is fixed and μ_1, μ_2 are chosen as in Lemma 4.6. Then*

$$\int_0^\infty n\log(1+Px)e^{-x}(1-e^{-x})^{n-1} - \sum_{i=1}^n R_i = \Omega\left(\frac{(\log\log n)^{1+2\alpha}}{(\log n)^{2\alpha}}\right). \tag{4.35}$$

Proof: Here is the outline of the proof. We can write the throughput under constraints of (4.3) using (4.30) as

$$\begin{aligned} & \alpha_1 n \int_0^{\infty} \log(1+Px) e^{-x} (1-e^{-x})^{\alpha_1 n - 1} (1 - e^{-\frac{\mu_1}{\mu_2} x})^{\alpha_2 n} dx \\ + \; & \alpha_2 n \int_0^{\infty} \log(1+Px) e^{-x} (1-e^{-x})^{\alpha_2 n - 1} (1 - e^{-\frac{\mu_2}{\mu_1} x})^{\alpha_1 n} dx \\ + \; & E \log \left(\frac{(1+Px)y}{(1+Py)x} | (x,y) \in \mathcal{R} \right). \end{aligned}$$

In fact the first two terms are the same as the throughput of the weighted-opportunistic scheduling with $\mu_1 = 1$ and μ_2 are chosen as in Lemma 4.6. Therefore similar to the proof of Lemma 4.1 we can show that the difference of the sum-rate capacity and the first two terms tends to zero like $\frac{(\log \log n)^{1+2\alpha}}{(\log n)^{2\alpha}}$. The third term however can be easily shown to be a positive when $(x, y) \in \mathcal{R}$. Therefore, the difference of the sum-rate capacity and the throughput of this scheme cannot tend to zero faster than $\frac{(\log \log n)^{1+2\alpha}}{(\log n)^{2\alpha}}$. This completes the proof of the lemma. □

4.5 Simulation Results

In this section we present some simulation results of the three scheduling schemes studies in this chapter. For ease of demonstration of the results, and since the superposition coding was only explicitly constructed for the case of $M = 2$ groups, we will only present simulations for $M = 2$.

The first set of simulations are for $\beta = 2$, i.e., one group requires twice the rate of the second group. Figure 4.4 shows the sum of the transmitted rate for WO, TO, and SC as a function of the number of users. As expected, all show a $\log \log n$ growth rate. In fact, the sum of the transmitted rates of WO and SC are quite close to the actual sum-rate capacity, signifying that that the rate constraints do not lead to much of a rate hit on the throughput.

Figure 4.5 shows the ratio of the rates transmitted to the two groups as a function

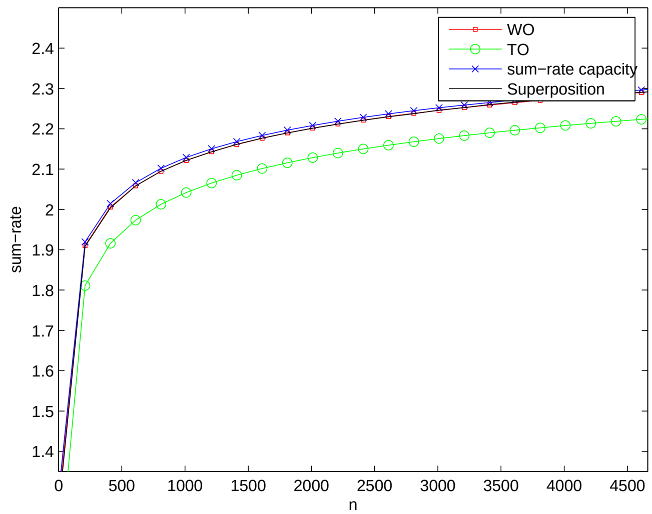

Figure 4.4: The sum of the transmitted rates for WO, TO, and SC, as well as the sum-rate capacity of the broadcast channel as a function of the number of users for a system with $M = 2$ and $\beta = 2$.

of of the number of users for the WO and SC schedules. These slowly converge to the desired values. TO is not shown as it clearly gives the correct ratio of $\beta = 2$.

The second set of simulations are for $\beta = 4$ and are shown in Figures 4.6 and 4.7.

4.6 Conclusion

In this chapter, we consider the downlink of a wireless cellular system (in information-theoretic terms, a broadcast channel with fading) where users have different rate demands. In particular, we assume n homogenous users are divided into M groups, each group of which requires the same rate, and where the ratio of the groups' rates are given. The transmitter would like to maximize the throughput (sum of the rates to all users) while maintaining the rational rate constraints. In general, this problem appears to be computationally intractable since the ergodic capacity region is described as the convex hull of (an infinite) set of rates. To illustrate this, we first consider systems where $n = 2$ and $n = 3$ and where each user requires a different

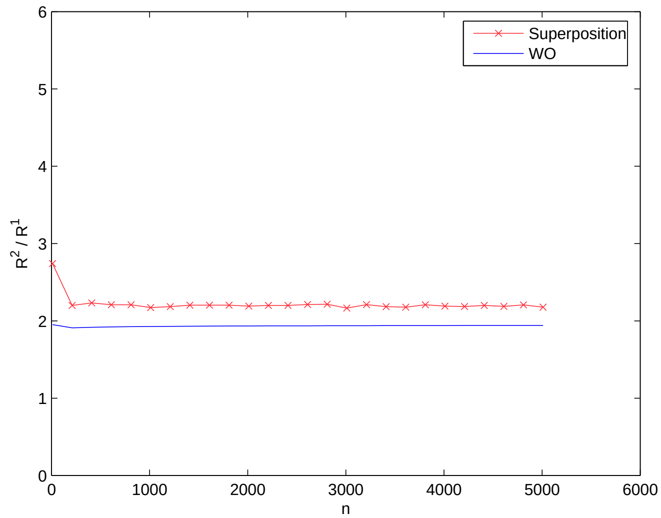

Figure 4.5: The ratio of the rates transmitted to the two groups of users as a function of the number of users for WO and SC for a system with $\beta = 2$.

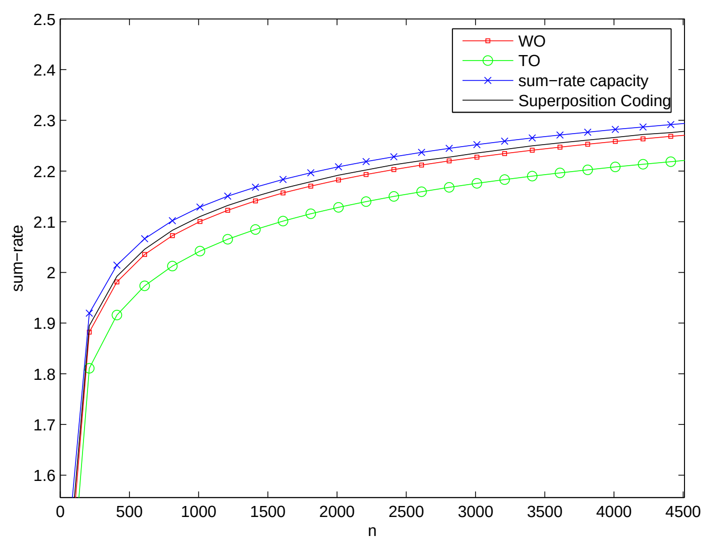

Figure 4.6: The sum of the transmitted rates for WO, TO, and SC, as well as the sum-rate capacity of the broadcast channel as a function of the number of users for a system with $M = 2$ and $\beta = 4$.

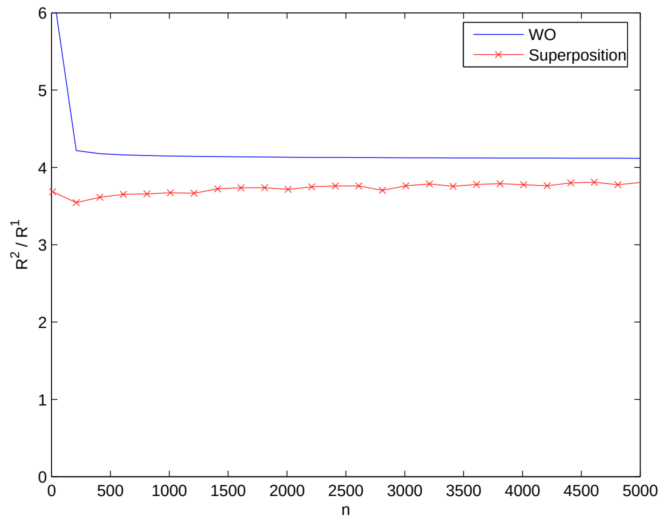

Figure 4.7: The ratio of the rates transmitted to the two groups of users as a function of the number of users for WO and SC for a system with $\beta = 4$.

rate. We focus on the achievable region by using weighted-opportunistic scheduling (WO)—a generalization of opportunistic scheduling—in which we transmit to only the user that has the largest "weighted" signal to noise ratio (SNR). It turns out that determining the explicit relationship between the appropriate weights of the schedule and the desired ratio of the rates is analytically intractable even for the case of $n = 3$. For this reason, and also because most practical systems have many users, much of the chapter focuses on the asymptotic regime of large n where explicit results can be found. In particular, we propose three scheduling schemes to provide the rational rate constraints namely, the aforementioned WO, time-division opportunistic (TO), and superposition coding (SC). In TO, each group has its own time slot in which the transmitter chooses the user with the best SNR from the corresponding group. Superposition coding is the scheme that achieves the information-theoretic capacity region. For each scheduling we give an explicit scheme to guarantee the rational rate constraints. We also analyze the throughput loss due to the rate constraints for all three different schemes. In particular, we show that the throughput loss compared to the maximum throughput (i.e., the sum rate capacity without any rate constraints) tends to zero for large n. Thus, there is not much of a penalty in providing different levels of service to different users. We also analyze the convergence rate of all the schemes and provide simulations supporting the theoretical analysis.

Chapter 5

Delay in Broadcast Channels

5.1 Introduction

Resource allocation in wireless systems aims for two conflicting goals, firstly providing quality of service such as delay and fairness to users, and secondly maximizing the throughput of the system. A fundamental property of wireless channels is their time variation due to multi-path effects and the mobility of the users. This implies that at each channel use some users have favorable channel conditions and other users incur deep fades. In fact, assuming a block fading model for the channel and having full CSI in the transmitter, it can be shown that sending to the user with the best channel conditions maximizes the sum rate (or throughput) of the single antenna broadcast channel.

In order to exploit this multiuser diversity, the base station (or the transmitter) has to know the channel state information (CSI) of all the users. In fact, this opportunistic way of transmission has been proposed in Qualcomm's High Data Rate (HDR) system (1xEV-DO). Other variations of this scheduling that do not require full CSI in the transmitter are studied in [76, 77].

However, there is a price to pay for maximizing the throughput, which is fairness among users and delay in sending packets. Assuming users have different signal-to-noise ratios, the throughput optimal scheduling will provide much less service to the user with the lowest signal-to-noise ratio (SNR) compared to that of the user with the highest SNR. Even in a homogeneous network where users have equal SNRs and so the system is long-term fair, there is no delay guarantee for transmitting a packet to a specific user as the transmission is probabilistic, i.e., at each channel use each user will be chosen with some probability. The other extreme would be to use a round-robin type of scheduling that fairly gives service to all users and can guarantee a fixed delay for transmitting a packet to each user. In applications with delay constraints, one may wonder how bad the worst case delay (or the delay for the most unfortunate user) for the throughput optimal strategy is.

In this chapter, we consider a broadcast channel with n backlogged users. The transmission is packet based and the channel is assumed to be block Rayleigh fading and changes independently from one block to the other. We also assume packets are dropped if outage capacity occurs, i.e., the instantaneous capacity goes below the amount of information in the packet. Given the probability of outage P_e, we assume packets carry a fix amount of information C_0, which only depends on the scheduling. For example, opportunistic scheduling is the one that maximizes the throughput given P_e. This will be further discussed in Section 5.2.

We define the delay as the minimum number of transmissions that guarantees all the users will receive m packets successfully. This notion of delay is clearly stronger than the average delay in the sense that it guarantees the reception of m packets by *all* users. Disregarding the throughput, the minimum delay of mn can be achieved by round-robin scheduling. However, the throughput optimal strategy has to contend with delay hits. The overriding question in this chapter is to characterize the delay for the throughput optimal strategy, e.g., to determine its mean and other moments. Finally, we propose an algorithm to reduce the delay at the expense of a little hit in the throughput of the system. The results in this chapter imply that opportunistic transmission increases the delay by a factor of $\log n$ compared to that of delay optimal strategies.

Previously, the question of the delay-throughput trade-off has been addressed by several authors in different contexts. In single link systems, the problem of how to optimally allocate the power among channel uses such that the capacity is maximized while guaranteeing the delay for sending bits remains bounded has been considered in [78, 79]. Also, the trade-off between average power and delay has been addressed by Berry and Gallager for single link systems [80]. In multiuser channels, traditionally delay and throughput were considered separately and therefore, access schemes such as ALOHA [81] were proposed to avoid collisions without exploiting multiuser diversity. As noted later in [82, 83], there has been a large body of work to combine the physical layer and multiple access layer (see [84, 85, 86, 87, 88] and references therein). For multiple access channels, a decentralized variation of the ALOHA algorithm is proposed that exploits multiuser diversity [76]. In [89], the authors consider the problem of characterizing the capacity region under a stability condition for queues. Stability here is in the sense that the probability of the queue overflow can be made arbitrarily small by making the buffer size sufficiently large [89].

Scheduling in broadcast channels has been also considered by several authors [90, 91, 92, 93]. In [91], stabilizing parallel queues in the transmitter is considered, where the connectivity of queues are random to capture deep fades in the wireless channel. In [93], the authors incorporate the channel state information in their scheduling while providing delay constraints for packets. Analyzing the average delay (over the users) can be also done using the results for the general independent input/output (GI/GI/1) queues and it can be shown that the average delay is of the order of the number users [94, 95]. However, in order to provide delay guarantee for all users, we have to study the delay for *the most unfortunate user* in the system. Clearly the worst case delay is a function of the number of users and their SNRs (or the probability of being chosen as the best user at each channel use). While these works give many insights and algorithms, they leave open the question of how large the worst case delay is as a function of the number of users and their SNRs for using throughput optimal strategies. This is the main goal of this chapter.

This chapter is organized as follows. Section 5.2 introduces our channel model and our notation. Section 5.3 deals with characterizing the delay for single antenna broadcast fading channels. Section 5.4 generalizes the results of Section 5.3 to multi-antenna broadcast channels. Finally Section 5.5 proposes an algorithm to reduce the delay at the expense of a little reduction in the throughput and Section 5.6 concludes the chapter.

5.2 System Model and Assumptions

We consider a single antenna broadcast channel with n receivers. We assume a block fading channel with a coherence interval of T, and where the channel changes independently after T seconds. The transmission is assumed to be packet-based and the length of each packet is T.[1]

For each block of length T, the received signal at the i'th user at time t can be written as

$$y_i(t) = \sqrt{\rho_i} h_i(t) S(t) + n_i(t), \qquad i = 1, \ldots, n, \tag{5.1}$$

where $h_i(t)$ is the effect of channel and $n_i(t)$ is additive white noise and that both are i.i.d. circularly symmetric complex Gaussian distributed with zero mean and variance of one. Here ρ_i is the SNR of the i'th user and $S(t)$ is the transmitted symbol at time t. We further assume an independent memoryless channel, which implies that the channel changes independently to another value after the coherence interval of T.

In the transmitter we assume there are n queues corresponding to each receiver and that there is always a packet available to be transmitted to any user (i.e., backlogged

[1] If the length of the packet is smaller than T, the results in this chapter can be easily generalized.

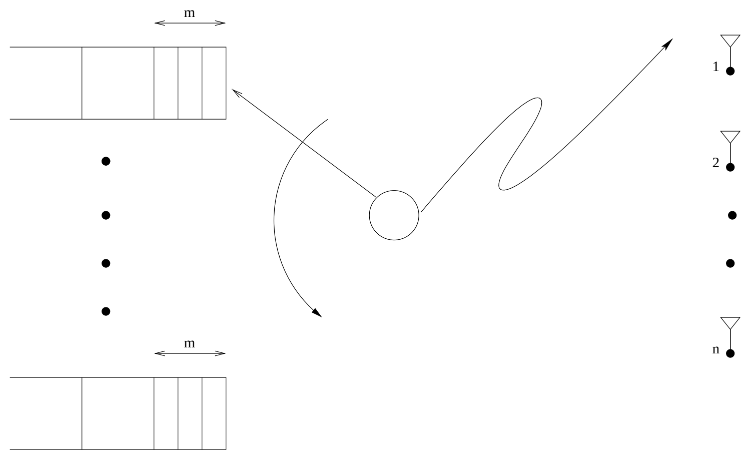

Figure 5.1: n parallel queues in the transmitter corresponding to n users; we are interested in the behavior of the longest queue.

users). Fig. 5.1 illustrates the arrangement of queues in the transmitter. In fact, the main challenges for the scheduler are first to balance the service among all the users and second to exploit the multiuser diversity in the channel in order to maximize the throughput of the system. Any scheduling strategy implies a probability for choosing each user at each channel use that may depend on the signal-to-noise ratio (SNR) of all users, the length of the queue of users, and the statistics of the channel. For the throughput optimal strategy, this probability only depends on the SNR of the user and the channel statistics. For i.i.d. channels, it is clear that these probabilities are only functions of users' SNRs.

Assuming that all packets have C_0 information bits for a homogeneous network (i.e., $\rho_i = \rho$), we consider a packet to be dropped if outage occurs, i.e., if the instantaneous capacity C goes below C_0 at the time of the transmission [96]. The instantaneous capacity however depends on the scheduling. For the round-robin scheduling, $C = \log(1 + \rho|h_i|^2)$, which does not depend on n. For the throughput optimal strategy,[2] C however is the maximum of $\log(1 + \rho|h_i|^2)$ over $1 \leq i \leq n$,

[2]In this chapter, we use the terms opportunistic scheduling and throughput optimal strategy interchangeably.

i.e., $C = \max_{1 \leq i \leq n} \log(1 + \rho |h_i|^2)$. We assume if a packet is dropped, the transmitter will be notified and the packet will be considered for retransmission whenever the corresponding user has the best channel conditions.

If we assume that the error probability is simply the outage probability (a reasonable assumption for long packets [79]), we have $P_e = \Pr(C < C_0)$. The throughput is therefore $R = C_0(1 - P_e) = C_0 \Pr(C \geq C_0)$. Given P_e, any scheduling would lead to a different C_0. Note that for any value of C_0, the throughput optimal strategy is to send to the best user as this would minimize P_e. Conversely, for any fixed value of P_e, sending to the strongest user maximizes the throughput as this would allow for the largest possible C_0. It is also worth mentioning that the maximum of n i.i.d. exponential random variables (the $|h_i|^2$) behaves almost surely as $\log n$. Therefore for large n, we do not need to use power control to compensate for the channel variation as the maximization automatically prevents having deep fades for large number of users with high probability. Thus, for the throughput optimal scheduling, it is reasonable to assume that all the packets have the same amount of information, i.e., C_0 roughly about $\log(1 + \rho \log n)$, independent of the time and channel condition.

In this thesis, we define the delay in the broadcast channel as the number of channel uses (denoted by $D_{m,n}$) required to guarantee that all the users will receive m packets successfully. It is clear from the definition of $D_{m,n}$ that this notion of delay refers to the worst case delay among users (or the delay for the most unfortunate user). Of course, $D_{m,n}$ is a random variable and depends on the number of users n, the number of packets m and also the scheduling algorithm. A delay-optimal strategy is round-robin scheduling, which clearly achieves the optimal delay of mn. However, round-robin is not throughput optimal, which requires transmitting to the user with the best channel conditions at each channel use. Throughput optimal strategies, on the other hand, will have to contend with delay hits. The following section deals with the delay for the throughput optimal scheduling.

5.3 Delay Analysis for Single Antenna Broadcast Channels

Opportunistic transmission is a probabilistic scheduling that implies that each user will be given service with some given probability. Assuming that the outage probability P_e is given, the opportunistic scheduling maximizes the throughput or equivalently C_0 (the amount information bits per packet). Analyzing the average delay (over all the users) can be done as the queue of each user can be considered as an i.i.d. input/output queue [94]. In particular, it can be shown that the average delay is of the order n [95]. However analyzing the worst case delay (or the delay for the most unfortunate user in the system) requires considering n parallel queues of n users alltogether [97]. In this section, assuming that at each channel use the transmitter sends to the i'th user with the probability p_i, which only depends on the SNR of all users, and drops the packet with probability P_e, we obtain the moment generating function of the random variable $D_{m,n}$.

We first consider the simple case in which the network is homogeneous and $P_e = 0$. Then we generalize the result to the case where we have a non-zero P_e and/or a heterogeneous network where users are chosen with different probabilities. We obtain the mean and variance of the delay $D_{m,n}$ for any m and n. We further look into the asymptotic behavior of $D_{m,n}$ for different regions of m and n at the end of this section.

5.3.1 Homogeneous Networks with No Dropping Probability

When users are homogeneous and assuming throughput optimal scheduling, the transmitter chooses the i'th user with probability $\frac{1}{n}$ from the pool of n users since it is equally likely for each user to have the best channel condition. The random variable $D_{m,n}$ is basically the minimum number of channel uses to guarantee all n users have been chosen at least m times.

This problem can be restated as the coupon-collector problem [98], which is studied by several authors in the mathematics literature (see also chapter 6 of [99]). To be more precise, users can be seen as people carrying coupons and the transmitter is the collector who chooses randomly and uniformly from the n people and collects his/her coupon. The question is how many times should the collector choose to guarantee that everybody has given at least m coupons. In fact we can state the mean value of $D_{m,n}$ based on a result found in [100].

Theorem 5.1 *(Newman and Shepp [100]) Consider a homogeneous broadcast system with n users. We assume that at each channel use, the transmitter sends to the user with the best channel condition. Then, we have,*

$$E(D_{m,n}) = n \int_0^\infty \left(1 - \left(1 - S_m(t)e^{-t}\right)^n\right) dt, \tag{5.2}$$

for any m and n where $S_m(t) = \sum_{k=0}^{m-1} \frac{t^k}{k!}$.

Proof: Since the network is homogeneous, the probability of choosing the i'th users is $\frac{1}{n}$. Therefore, the problem is the same as the problem considered by Newman and Shepp [100]. See [100] for the proof. □

Inspired by the proof of Theorem 5.1, we can derive the moment-generating function of $D_{m,n}$ defined as

$$F(z) = \sum_{i=0}^{\infty} z^i \Pr\{D_{m,n} > i\} = \sum_{i=0}^{\infty} z^i b_i. \tag{5.3}$$

Using the generating function $F(z)$ in (5.3), we can obtain all the moments of $D_{m,n}$ with a little effort and by taking higher derivatives of $F(z)$ at $z = 1$ [**?**]. For example, using the definition of $F(z)$ in (5.3), we can write

$$\begin{aligned} E(D_{m,n}) &= F(1), \\ \sigma^2(D_{m,n}) &= 2F'(1) + F(1) - (F(1))^2. \end{aligned} \quad (5.4)$$

The next theorem obtains $F(z)$ and generalizes the result of Theorem 5.1.

Theorem 5.2 *Considering the setting of Theorem 5.1, we can write the moment generating function of* $D_{m,n}$ *defined in (5.3) as*

$$F(z) = \frac{n}{z} \int_0^\infty e^{-\frac{n}{z}t} \left(e^{nt} - (e^t - S_m(t))^n \right) dt. \quad (5.5)$$

Proof: We evaluate $F(z)$ by the same trick as [100] in which the mean of $D_{m,n}$ is derived. In fact, $F(z)$ can be evaluated by noting that b_i is the probability of failure in obtaining m packets at all the n users up to and including the i'th trial. Therefore, b_i is simply the polynomial $(\frac{1}{n}x_1 + \ldots + \frac{1}{n}x_n)^i$ evaluated at $x_1 = \ldots = x_n = 1$ after excluding all terms which have all x_i's with exponent larger than $m-1$. Therefore, we may write

$$F(z) = \sum_{i=0}^{\infty} z^i \frac{\{(x_1 + \ldots + x_n)^i\}}{n^i}, \quad (5.6)$$

where $\{\cdot\}$ denotes the operator that removes all the terms that have all x_i's with exponent less than $m-1$. Considering the following identities [100],

$$\frac{z^i i!}{n^i} = \frac{n}{z} \int_0^\infty e^{-\frac{n}{z}t} t^i dt, \quad (5.7)$$

$$\{e^{x_1 + \ldots + x_n}\} = \sum_{i=0}^{\infty} \frac{\{(x_1 + \ldots + x_n)^i\}}{i!} = e^{x_1 + \ldots + x_n} - \prod_{i=1}^{n} \left(e^{x_i} - S_m(x_i) \right), \quad (5.8)$$

where the first equality in Eq. (5.8) is the definition of the exponential function and the second equality follows by noting that the second term in the right hand side just subtracts out the terms with all x_i's larger than m. We may then replace the integral form for $\frac{1}{n^i}$ using (5.7) in (5.6) to get

$$\begin{aligned} F(z) &= \sum_{i=0}^{\infty} \int_0^{\infty} \frac{n}{z} e^{-\frac{n}{z}t} t^i dt \times \frac{\{(x_1 + \ldots + x_n)^i\}}{i!} \\ &= \frac{n}{z} \int_0^{\infty} e^{-\frac{n}{z}t} \sum_{i=0}^{\infty} \frac{\{(x_1 + \ldots + x_n)^i\}}{i!} dt \\ &= \frac{n}{z} \int_0^{\infty} e^{-\frac{n}{z}t} \left(e^{tx_1 + \ldots + tx_n} - \prod_{i=0}^{n} \left(e^{tx_i} - S_m(tx_i) \right) \right) dt \\ &= \frac{n}{z} \int_0^{\infty} e^{-\frac{n}{z}t} \left(e^{nt} - (e^t - S_m(t))^n \right) dt, \qquad (5.9) \end{aligned}$$

where we replaced $x_i = 1$ for $i = 1, \ldots, n$ and we used (5.8) to get the second equality and we replaced $x_i = 1$ for $i = 1, \ldots, n$ to obtain the last equation. □

It is now quite straightforward to derive the variance of $D_{m,n}$ using $F(z)$ and (5.4) as

$$\sigma^2(D_{m,n}) = 2n^2 \int_0^{\infty} t \left(1 - \left(1 - S_m(t)e^{-t}\right)^n\right) dt - E(D_{m,n}) - \left(E(D_{m,n})\right)^2. \qquad (5.10)$$

5.3.2 Heterogeneous Networks with Dropping Probability

For the special case of a homogeneous network, we derived the moment generating function of $D_{m,n}$ in Theorem 5.2. In what follows, we generalize the results to a more general setting in which users may have different SNRs and also a packet may be dropped if outage occurs. We assume the transmitter will be notified in case a packet is dropped and it will be considered for retransmission whenever the corresponding user has the best SNR. Here, we assume a memoryless i.i.d. channel and that the transmitter chooses the i'th user with probability p_i that depends on the SNR of all users and their channel conditions for the throughput optimal strategy.

The following theorem states the mean and variance of $D_{m,n}$ for this general setting and for any m and n. The theorem is another generalization of the result of Newman and Shepp [100] stated in Theorem 5.1.

Theorem 5.3 *Suppose we have n users such that the probability of choosing the i'th user is $p_i = \frac{\alpha_i}{n}$ and the probability of dropping a packet is P_e. Then the moment generating function for $D_{m,n}$ defined in (5.3) is*

$$F(z) = \frac{n}{z} \int_0^\infty e^{-\frac{n}{z}t} \left(e^{nt} - e^{nP_e t} \prod_{i=1}^{n} (e^{t\beta_i} - S_m(t\beta_i)) \right) dt, \tag{5.11}$$

where $\beta_i = (1 - P_e)\alpha_i$. In particular, assuming $S_m(t)$ is as defined in Theorem 5.1, we have

$$E(D_{m,n}) = n \int_0^\infty \left(1 - \prod_{i=1}^{n} \left(1 - S_m(\beta_i t) e^{-\beta_i t}\right) \right) dt, \tag{5.12}$$

and

$$\sigma^2(D_{m,n}) = 2n^2 \int_0^\infty t \left(1 - \prod_{i=1}^{n} \left(1 - S_m(\beta_i t) e^{-\beta_i t}\right) \right) dt - E(D_{m,n}) - \left(E(D_{m,n})\right)^2. \tag{5.13}$$

Proof: Similar to the proof of Theorem 5.2, we derive the moment generating function of $D_{m,n}$ as defined in (5.3). Since we have a non-zero probability of dropping a packet, we may assume that there is a fictitious user ($n + 1$'th one) corresponding to the case where the packet is lost; therefore whenever a packet is dropped, we may assume that $n+1$'th user has been chosen to be transmitted to. Assuming that P_e is the probability of dropping a packet, the probability of choosing the fictitious user is P_e and the probability of choosing the i'th user and sending successfully is $\frac{\alpha_i}{n}(1-P_e)$ for $i = 1, \ldots, n$. Therefore, the delay $D_{m,n}$ is the number of channel uses that guarantees having m packets in all the n users (i.e., except for the fictitious user).

The moment generating function $F(z) = \sum_{i=0}^{\infty} z^i \Pr(D_{m,n} > i) = \sum_{i=0}^{\infty} z^i b_i$ where b_i is the probability of failure in sending m packets to n users up to and including the i channel uses and is equal to the polynomial $(\frac{\beta_1}{n}x_1 + \ldots, \frac{\beta_n}{n}x_n + P_e x_{n+1})^i$ evaluated at $x_1 = \ldots = x_{n+1} = 1$ after removing all the terms that have all $x_1, \ldots, x_n$ exponents larger than m. Therefore, we can write $F(z)$ as

$$
\begin{aligned}
F(z) &= \sum_{i=0}^{\infty} \frac{z^i}{n^i} \{(\beta_1 x_1 + \ldots, \beta_n x_n + nP_e x_{n+1})\} \\
&= \sum_{i=0}^{\infty} \frac{\{(\beta_1 x_1 + \ldots, \beta_n x_n + nP_e x_{n+1})\}}{i!} \frac{n}{z} \int_0^{\infty} e^{-\frac{n}{z}t} t^i dt \\
&= \frac{n}{z} \int_0^{\infty} e^{-\frac{n}{z}t} \sum_{i=0}^{\infty} \frac{\{(\beta_1 x_1 + \ldots, \beta_n x_n + nP_e x_{n+1})\}}{i!} dt \\
&= \frac{n}{z} \int_0^{\infty} e^{-\frac{n}{z}t} \left\{ e^{t\beta_1 x_1 + \ldots + t\beta_n x_n + nP_e x_{n+1}} - e^{nP_e x_{n+1}} \prod_{i=1}^{n} (e^{t\beta_1 x_1} - S_m(t\beta_1 x_1)) \right\} dt \\
&= \frac{n}{z} \int_0^{\infty} \left(e^{nt} - e^{nP_e t} \prod_{i=1}^{n} (e^{t\beta_i} - S_m(t\beta_i)) \right) dt, \qquad (5.14)
\end{aligned}
$$

where we used the identity in (5.7) to deduce the second equality. We also used the following identity (which is analogous to (5.8)) to obtain the third equality in (5.14):

$$
\left\{ e^{x_1 + \ldots + x_n + x_{n+1}} \right\} = e^{x_1 + \ldots + x_n + x_{n+1}} - e^{x_{n+1}} \prod_{i=1}^{n} \left(e^{x_i} - S_m(x_i) \right). \qquad (5.15)
$$

In (5.15), $S_m(t)$ is as defined in Theorem 1 and the operator $\{\cdot\}$ removes the terms that have the exponents of $x_1, \ldots, x_n$ larger than m. Eq. (5.15) can be easily proved by noting that the polynomials on the left remove all the terms from the exponential function that have all x_i's for $i = 1, \ldots, n$ with exponents larger than $m - 1$.

Using the relationship between $F(z)$ and its moments shown in Appendix 5.7.1 and (5.4) and having $F(z)$ derived in (5.14), we can obtain the mean and variance of $D_{m,n}$ as stated in the theorem. □

For example, as a simple consequence of (5.12), we can obtain the expected delay

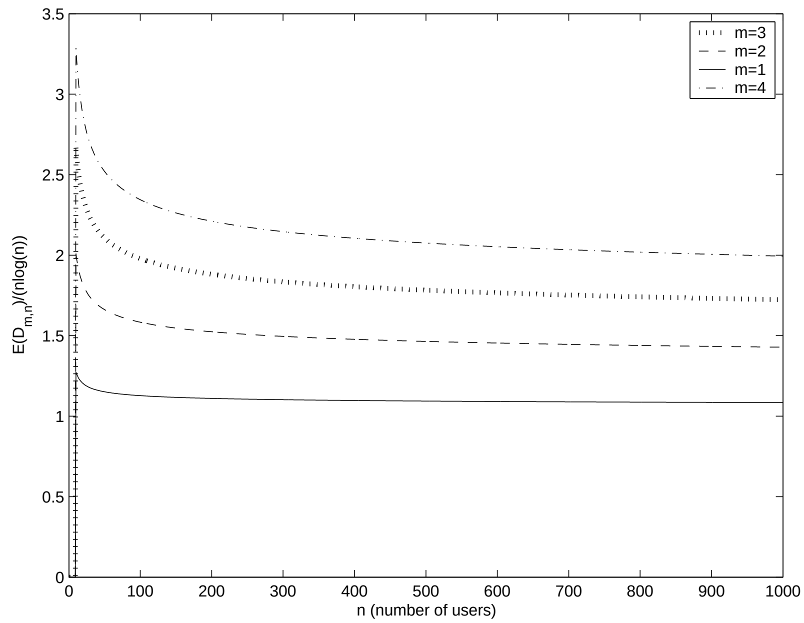

Figure 5.2: Expected delay $\frac{E(D_{m,n})}{n \log n}$ for different values of m and n

for the case where n users are equally likely and that the probability of dropping a packet is P_e, as

$$E(D_{m,n}) = \frac{1}{1-P_e}(n+1)\int_0^\infty \left(1 - \left(1 - S_m(x)e^{-x}\right)^n\right) dx, \tag{5.16}$$

by a simple change of variable in the integral stated in (5.12).

Fig. 5.2 shows the expected delay for $m = 1, 2, 3, 4$ and for different numbers of users for a homogeneous network. It is clear that when n is large and $m = 1$, the growth in the expected delay is like $n \log n$. Also Fig. 5.2 implies that the expected delay does not grow linearly with m (for small values of m). In fact it converges to $n \log n$ although the convergence seems to be quite slow. The next subsection deals with the asymptotic analysis of the delay for different regions of m and n.

Remark 5.1: It is worth mentioning that we can consider the delay in sending m_i packets to the i'th user for $i = 1, \ldots, n$. In particular, considering the setting of Theorem 5.3, and we are interested in sending m_j packets to the j'th user for

$j = 1, \ldots, i$ where $i \leq n$. Defining $\mathbf{m} = (m_1, \ldots, m_i)$ and $D_{\mathbf{m}}$ as the minimum number of channel uses guarantees the receipt of m_j packets at the j'the user for $j = 1, \ldots, i$, we can write the moment generating function for $D_{\mathbf{m}}$ as

$$\begin{aligned} F(z) &= \sum_{i=0}^{\infty} z^i \Pr(D_{\mathbf{m}} > i) \\ &= \frac{n}{z} \int_0^{\infty} e^{-\frac{n}{z}t} \left(e^{nt} - e^{nP_e t + \sum_{k=i+1}^{n} \beta_k t} \prod_{p=1}^{i} (e^{t\beta_p} - S_{m_p}(t\beta_p)) \right) dt. \quad (5.17) \end{aligned}$$

5.3.3 Asymptotic Analysis of the Moments of $D_{m,n}$

In the previous subsection, we obtained the moments of $D_{m,n}$ for a general setting and for any m and n in closed form. However, it is hard to speculate how the mean and variance of the delay behave as functions of m and n. In order to get a better insight into the behavior of the delay, we derive some asymptotic results for the moments of $D_{m,n}$ and for different regions of m and n.

Theorem 5.4 *Assuming a homogeneous network and that a packet will be dropped with probability* P_e*,*

1. *For* m *fixed and* $n \to \infty$*, we have*[3]

$$\begin{aligned} E(D_{m,n}) &= \frac{1}{1-P_e} n \log n + n(m-1) \log\log n + o(n \log\log n), \quad (5.18) \\ \sigma^2(D_{m,n}) &= O(n^2). \quad (5.19) \end{aligned}$$

2. *For* $m = \log n$ *and* $n \to \infty$*, we have*

$$E(D_{m,n}) = \alpha \frac{1}{1-P_e} n \log n + O(n \log\log n), \quad (5.20)$$

where $\alpha = 3.146$ *is the solution to the equation* $\alpha - \log \alpha = 2$.

[3] This case has been also proved in [100], however we present another proof that leads to results for other regions of m and n as well.

3. For $m = (\log n)^r$ where $r > 1$ is fixed and $n \to \infty$, then

$$E(D_{m,n}) = \frac{1}{1-P_e} n(\log n)^r + o(n(\log n)^r) = \frac{1}{1-P_e} mn + o(mn). \qquad (5.21)$$

4. For n fixed and $m \to \infty$,

$$E(D_{m,n}) = \frac{1}{1-P_e} nm + o(m). \qquad (5.22)$$

Proof: Refer to appendix 5.7.5 for the proof. □

Assuming $m = 1$ and using the result of Theorem 5.4, we can state that the delay converges to the mean almost surely using Chebychev's inequality as

$$\Pr\left\{ \left| D_{m,n} - \frac{1}{1-P_e} n \log n + O(n \log\log n) \right| \le n\sqrt{\log n} \right\} \ge 1 - \frac{1}{\log n}, \qquad (5.23)$$

for large n. This implies that the delay hit for sending the first packet successfully to all the users is increased from the minimum of n for the round-robin scheduling to $n \log n$ for the opportunistic transmission for large n. So the delay degradation due to exploiting the channel variation and maximizing the throughput of the system is a multiplicative factor of $\log n$. It would be also interesting to investigate the scaling law of the variance of $D_{m,n}$ when m also grows to infinity; this would then imply the type of convergence to the mean for different regions of m and n.

Remark 5.2: For a homogeneous network, as opportunistic transmission is long-term fair (i.e., the probability of choosing all the users is the same), we know that for sufficiently large m, the expected delay should behave like mn. This is confirmed by the fourth part of Theorem 5.4. Interestingly, Theorem 5.4 further implies that if m grows faster than $(\log n)^r$ where r is fixed and greater than one the expected delay behaves like mn. This has implications for the timescale after which the system behaves fairly. Moreover, if m grows logarithmically with n, the expected delay is only off by a constant factor of $\alpha = 3.14$, compared to the minimum delay mn. Therefore, our result can be seen as the short-term behavior of the delay for any m.

As mentioned, the largest delay hit is when we focus on sending a few packets, i.e., $m = 1$ or m is small. The delay hit gets less when we focus on sending more and more packets (i.e., when m gets larger). Therefore, in the rest of the chapter, we mainly focus on the delay for sending the first packet, i.e. $D_{1,n}$.

5.4 Delay in Multi-Antenna Broadcast Channels

Multiple transmit antennas have been shown to significantly improve the throughput of a broadcast channel. It is shown that dirty paper coding achieves the sum-rate capacity of a Gaussian broadcast channel [10, 9, 12]. However, beamforming has long been proposed as a heuristic method to mitigate the interference in the transmitter and to send multiple beams to different users. Although beamforming is not optimal in achieving the sum-rate capacity, its throughput does scale the same as that of dirty paper coding for a system with many users and has much less complexity than that of dirty paper coding [101, 102].

In this chapter, for a system with M transmit antennas, we assume a simple model in which the base station transmits to M different receivers at each channel use. This is certainly a valid model for beamforming or channel inversion, though it does not fit the dirty paper scheduling in which the transmitter sends information to all the users at each time. However, as far as the scaling law of the sum-rate is concerned, when M is either fixed or growing logarithmically with n, it can be shown that beamforming, channel inversion, and random beamforming all give the optimal scaling law for the sum-rate [27].

For a homogeneous network, our model for the multiple antenna transmitter implies that, at each channel use, the transmitter sends to M *different* users uniformly chosen from the pool of n users (see [27]). In this scheduling the transmitter sends M beams, each one assigned to the user with the best signal-to-noise and interference ratio (SINR) for the corresponding beam. As shown in Chapter 3, the best SINR behaves like $\log n$ with high probability for large n. Therefore, we may again assume that each packet carries a fix amount information (roughly about $\log(1+\rho\log n)$).

This scheduling is certainly more balanced compared to the case where we have a single antenna system that works M times faster. This can be justified by noticing the fact that we exclude the possibility of sending to one user twice (or more) in each block of M transmissions and hence the scheduling is more balanced. In particular, assuming that there is no packet dropped as in Theorem 5.1, then we have

$$D_{m,n}(M) \leq \frac{1}{M} D_{m,n}, \tag{5.24}$$

where $D_{m,n}(M)$ is the delay for sending m packets successfully to n users in an M-transmit antenna system and where $D_{m,n}$ is the delay for a single antenna broadcast system as in Theorem 5.1.

In fact we can compute exactly the expected delay in transmitting the first packet successfully, i.e., $E\left(D_{1,n}(M)\right)$, for any n and M. Further generalization of the result to $m > 1$ is non-trivial and we have not been able to do this; however, it is quite easy to show that $D_{m,n}(M) \leq m D_{1,n}(M)$. The next theorem presents the result for $m = 1$ and for any n and M.

Theorem 5.5 *Consider a broadcast channel with M transmit antennas and n users. Assuming that no packet is dropped, we can write the expected delay in sending one packet to all users for any m and n as*

$$E\left(D_{1,n}(M)\right) = \sum_{k=0}^{\infty}\sum_{r=1}^{n}\sum_{i=0}^{n-r}(-1)^{n-r-i}\frac{\binom{n}{r}}{\binom{n}{M}^k}\binom{n-r}{i}\binom{i}{M}^k. \tag{5.25}$$

Proof: Similar to the proof of Theorem 5.3, we first note that the mean of $D_{1,n}(M)$ can be written as

$$E\left(D_{1,n}(M)\right) = \sum_{k=0}^{\infty} \Pr\left(D_{1,n}(M) > k\right). \tag{5.26}$$

In order to compute the probability of $D_{1,n} > k$, we define the auxiliary random variable $\mu_n^M(k)$ as the number of users that have received no packets after k channel uses in which the transmitter sends to M different users. From the definition of μ_n^M, it is clear that $\mu_n^M \leq n$ and that $D_{1,n}(M) > k$ is equivalent to $\mu_n^M(k) > 0$. Therefore, Eq. (5.26) can be written as

$$E\left(D_{1,n}(M)\right) = \sum_{k=0}^{\infty} \Pr\left(\mu_n^M(k) > 0\right) = \sum_{k=0}^{\infty}\sum_{r=1}^{n} \Pr\left(\mu_n^M(k) = r\right). \tag{5.27}$$

The probability that $\mu_n^M(k) = r$ can be computed as follows. Assuming $\mu_n^M(k) = r$ implies that *only* $n-r$ users have received at least one packet in k channel uses. We then define the event S_i for $i = 0, 1, \ldots, n-r$ as the event that at least $n-r-i$ users have not received any packets among $n-r$ users who are supposed to receive a packet. This implies that there are at most i users that the transmitter sends packets to. It is clear that for $1 \leq i \leq M$ probability of S_i is zero, since the transmitter certainly can transmit to M different users at each channel use. For $i > M$, however we can write the probability of S_i as

$$\Pr\{S_i\} = \binom{n}{r, i} \frac{\binom{i}{M}^k}{\binom{n}{M}^k} = \binom{n}{r}\binom{n-r}{i} \frac{\binom{i}{M}^k}{\binom{n}{M}^k} \qquad i = 0, 1, \ldots, n-r, \tag{5.28}$$

where we first chose two sets of users with cardinality r and i from the set of n users and then we distributed packets among i of them k times by choosing M different users at each time.

Considering the definition of $\mu_n^M(k) = r$ and the S_i's, we can use the inclusionexclusion principle (see chapter 4 of [98]) to obtain

$$\begin{aligned}\Pr\left(\mu_n^M(k)=r\right) &= \Pr(S_{n-r})-\Pr(S_{n-r-1})+\ldots+\Pr(S_0)\\ &= \sum_{i=0}^{n-r}(-1)^{n-r-i}\Pr(S_i)\\ &= \frac{\binom{n}{r}}{\binom{n}{M}^k}\sum_{i=M}^{n-r}(-1)^{n-r-i}\binom{n-r}{i}\binom{i}{M}^k. \end{aligned} \tag{5.29}$$

Substituting (5.29) in (5.27), we can write the expected delay as

$$\begin{aligned}E\left(D_{1,n}(M)\right) &= \sum_{k=0}^{\infty}\Pr\left(\mu_n^M(k)>0\right)\\ &= \sum_{k=0}^{\infty}\sum_{r=1}^{n}\frac{\binom{n}{r}}{\binom{n}{M}^k}\sum_{i=M}^{n-r}(-1)^{n-r-i}\binom{n-r}{i}\binom{i}{M}^k. \end{aligned} \tag{5.30}$$

This completes the proof for the theorem. □

Remark 5.3: It is worth mentioning that we can also obtain the generating function $F(z)$ that would lead to the moments of $D_{1,n}(M)$ for any M and n. In fact, $F(z)$ is equal to

$$F(z)=\sum_{k=0}^{\infty}z^k\Pr(D_{1,n}(M)>k)=\sum_{k=0}^{\infty}\sum_{r=1}^{n}\sum_{i=0}^{n-r}(-1)^{n-r-i}\frac{z^k\binom{n}{r}}{\binom{n}{M}^k}\binom{n-r}{i}\binom{i}{M}^k. \tag{5.31}$$

Using (5.4) and (5.31), we can easily obtain the variance (and other moments) of $D_{1,n}(M)$.

Although Theorem 5.5 gives us the exact value of the expected delay for any number of users, it does not make clear how much improvement on the delay we can get in using a multi-antenna transmitter over that of the single antenna system. We can in fact asymptotically analyze the expected delay derived in Theorem 5.5 for large numbers of users to get a better intuition about this result.

Theorem 5.6 *Consider the setting of Theorem 5.5. Then the expected delay in send-*

ing at least one packet to all n users using an M-antenna transmitter derived in (5.25) behaves like

$$E\left(D_{1,n}(M)\right) = \frac{\sum_{k=1}^{n}\frac{1}{k}}{\sum_{r=0}^{M-1}\frac{1}{n-r}} + O(1), \tag{5.32}$$

for large n and when M grows no faster than $\log n$.

Proof: The interested reader is referred to Appendix 5.7.6 for the proof. □

For the special case of $M = 1$, the problem reduces to the coupon-collector problem when $m = 1$ (one packet). It can be easily shown that the expected delay is equal to $n\sum_{i=1}^{n}\frac{1}{i} \approx n\log n$. Clearly the result of Theorem 5.5 confirms this result for one transmit antenna, i.e., $M = 1$.

Remark 5.4: As mentioned in (5.24), using multiple transmit antennas in the transmitter should improve the delay. We may write the improvement on the expected delay by using M transmit antennas over that of single antenna case as

$$\frac{1}{\sum_{r=0}^{M-1}\frac{n}{n-r}} = M + O\left(\frac{M^2}{n}\right). \tag{5.33}$$

Eq. (5.33) implies that when M is not growing faster than $\log n$, the gain in delay is a factor of M, which comes from the fact that we are transmitting packets M times faster. Therefore, multiple transmit antenna systems incur pretty much the same delay as that of a single antenna transmitter that operates M times faster when there is no channel correlation.

Although the gain on delay in using multiple transmit antennas is not that much, multiple transmit antennas can significantly improve the long-term fairness in a heterogeneous network. More precisely, in [27], it is proven that if M grows logarithmically with the number of users, the probability of choosing each user becomes independent of its SNR and approaches $\frac{1}{n}$. Moreover, when there is channel correlation, multiple antenna systems can significantly reduce the delay by "decorrelating in time" the effective channel through means such as random beamforming [27, 57].

5.5 Trading Delay with the Throughput: d-Algorithm

Previously, we showed the delay hit in using the optimal throughput scheduling is a $\log n$ fold increase compared to the minimum achievable delay. In this section, we propose an algorithm that can reduce the expected delay for sending the first packet at the price of a little throughput degradation. The goal is to improve the $\log n$-fold degradation in the delay without too much reduction in the throughput of the system.

In order to improve the delay, we have to introduce more options to the scheduler at each channel use. For single antenna systems, this can be done by looking at the d best users in terms of capacity and transmitting to the user among those d users who has received the least number of packets. We call this scheduling the *d-algorithm.* For a large number of users and fixed d, it is quite easy to show that the capacity of the best user and that of the d'th best user is quite close almost surely. This in fact guarantees that the throughput degradation using our algorithm is not that much. The next theorem quantifies the performance of the d-algorithm precisely.

Theorem 5.7 *Consider the setting of Theorem 5.1 and suppose the transmitter uses the d-algorithm. We denote the expected delay in sending the first packet by $E(D^d_{1,n})$. Then, for any d,*

$$E(D^d_{1,n}) = n \int_0^{1-\frac{d}{n}} \frac{1}{1-x^d} dx + O(1). \tag{5.34}$$

Asymptotically, we can further prove that if d is fixed,

$$\lim_{n\to\infty} \frac{E(D_{1,n}(d))}{E(D_{1,n})} = \lim_{n\to\infty} \frac{E(D^d_{1,n})}{n \log n} = \frac{1}{d}. \tag{5.35}$$

Proof: In order to compute the expected delay, we again define the variable r_i as the number of channel uses after sending at least one packet to $i-1$ users and before completing the transmission of at least one packet to i users. Clearly r_i has a geometric distribution as

$$\Pr(r_i = k) = (1-p_i)^{k-1} p_i \qquad k = 1, 2, \ldots, \tag{5.36}$$

where p_i is the probability that all the d best users have been chosen before, therefore

$$\begin{aligned} p_i &= 0 \qquad 1 \leq i \leq d-1 \\ p_i &= 1 - \frac{\binom{i}{d}}{\binom{n}{d}}, \qquad d \leq i \leq n-1. \end{aligned} \tag{5.37}$$

Noting that $D_{1,n} = \sum_{i=0}^{n-1} r_i$, and also using the fact that the mean value of r_i is $\frac{1}{p_i}$, we can obtain the expected value of $D_{1,n}$ as

$$E(D_{1,n}^d) = \sum_{i=d}^{n-1} \frac{1}{p_i} = \sum_{i=d}^{n-1} \frac{1}{1 - \frac{i(i-1)\ldots(i-d+1)}{n(n-1)\ldots(n-d+1)}} \leq \sum_{i=d}^{n-1} \frac{1}{1 - \left(\frac{i-d+1}{n}\right)^d}, \tag{5.38}$$

where we used a simple upper bound for $\binom{i}{d}/\binom{n}{d}$. To evaluate the summation in the right hand side of (5.38), we may take integrals from $x = 1$ to $x = n-d+1$ from both sides of

$$\frac{1}{1-(x/n)^d} \geq \frac{1}{1-(\lfloor x \rfloor/n)^d} \geq \frac{1}{1-((x-1)/n)^d}, \tag{5.39}$$

to obtain

$$E(D_{1,n}^d) = n \int_0^{1-d/n} \frac{dx}{1-x^d} + O(1), \tag{5.40}$$

which completes the proof for the first part of the theorem. To prove the second part, we define the integral on the right hand side of (5.40) as $G(n)$. Then it is quite easy to show that when d is fixed, we have

$$\lim_{n\to\infty} \frac{G(n)}{\log n} = \lim_{n\to\infty} \frac{d}{n(1-(1-\frac{d}{n})^d)} = \frac{1}{d}, \tag{5.41}$$

where we used the L'Hopital's rule in (5.41). Considering that $E(D_{1,n})$ scales like

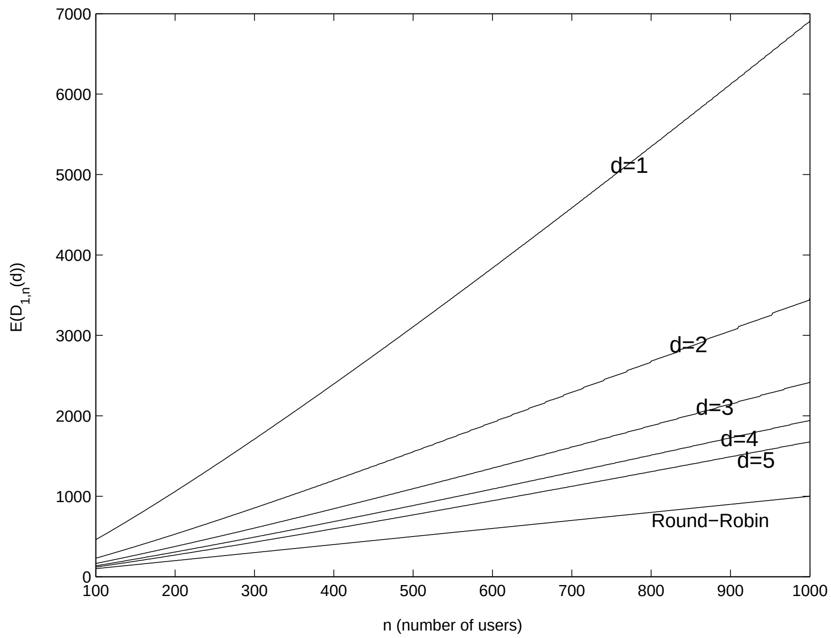

Figure 5.3: Expected delay $E(D^d_{1,n})$ for different values of d and n

$n \log n$ as proved in Theorem 5.4, the second part of the theorem immediately follows from (5.41). □

Fig. 5.3 shows the delay improvement for different values of d and for different number of users. As d increases, the delay improves though with less pace. Clearly, we can get most of the improvement by just checking the best two users ($d = 2$) and further increasing d will not improve the expected delay as much as before.

There is of course a price to pay on the rate for the delay improvement. In order to see the throughput hit, we look into the ergodic throughput of the channel (denoted by $R(d)$) using the d algorithm defined as

$$R(d) = E \log \left(1 + \rho \max_{1 \le i \le n}{}^{k} |h_i|^2 \right), \tag{5.42}$$

where $\max^k$ denotes the k'th maximum and k is a random variable uniformly distributed between 1 and d. Using results on the extreme value theory, it is quite straightforward to show that

$$\lim_{n\to\infty} R(d) - R(1) = 0, \tag{5.43}$$

when d is fixed. The proof is based on the fact that is d is fixed, the first and the d best user both have SNR of about $\log n$(see [60, 27]. Eq. (5.43) implies that in the limit of large n, the difference of the throughput of the d-algorithm and the maximum throughput converges to zero.

Remark 5.5: It is worth mentioning that the transmitter may use a round-robin type of scheduling and also exploits the channel. This can be done by sending to the best user among n users at the first channel use, and then sending to the best user among $n-1$ users who have not been chosen and so on. This method can ensure that the worst case delay is equal to n. The ergodic throughput of this scheme can be written as

$$R_{RR} = E\left\{\frac{1}{n}\sum_{k=1}^{n}\log\left(1+\rho\max_{1\le i\le k}|h_i|^2\right)\right\}. \tag{5.44}$$

Assuming that the channel is Rayleigh fading, we can show that in the limit the ratio of R_{RR} over $R(1)$ is one. Of course, the convergence in (5.43) for the d-algorithm holds in a stronger sense. Moreover, it is worth mentioning that this scheduling may require packets with different amounts of information.

Remark 5.6: Another approach to trade the delay with throughput is to consider a threshold for the capacity and to send to the user who has received the least number of packets among the users with instantaneous capacity above the threshold value (C_{Th}). In this case, we basically have a random d that has a binomial distribution where the binomial parameter q depends on the threshold value C_{Th}. We can in fact bound the delay for sending one packet to all users using the d-algorithm as

$$E(D_{1,n}) = E_d\{E\{D_{1,n}|d\}\} \le E_d\left\{\sum_{l=1}^{n-d+1}\frac{1}{1-\left(\frac{l-1}{n}\right)^d}\right\}, \tag{5.45}$$

where d has binomial distribution with parameter $q = \Pr\{\log(1 + \rho_i|h_i|^2) \geq C_{Th}\}$.

5.6 Conclusion

Providing quality of service (QoS) and also maximizing the throughput in a cellular system are the main challenges that require designing the physical layer and multiple access layer together. In this chapter, we considered the downlink of a cellular system (i.e., a broadcast channel) and we also considered a notion of worst case delay, which is defined as the delay $D_{m,n}$ incurred in receiving m packets by *all* the n users in the system. Clearly this definition of the delay is stronger than the average delay and represents the worst case delay among the users. In order to maximize the throughput, the transmitter has to send a packet to the user with the best channel condition, which increases the delay. The main goal of this chapter was to analyze this delay increase.

Assuming a block fading i.i.d. channel and a single antenna broadcast system with n backlogged users, we derived the moment generating function of the delay for any m and n and for a general hetereogeouns network where a packet can be dropped if outage capacity occurs. Asymptotically, for a homogeneous network where the throughput optimal scheduling is long-term fair (i.e., the probability of choosing users are equal), the result implies that the average delay in sending one packet to all users behaves like $n \log n$ as opposed to n for a round-robin scheduling. We also proved that when m grows like $(\log n)^r$, for some $r > 1$, then to the first order the delay scales as mn. This roughly determines the timescale required for the system to behave fairly. We also looked into the delay analysis for a system equipped with multiple transmit antennas. Finally we proposed an algorithm that without sacrificing too much on the throughput can significantly improve the delay. The algorithm always considers the first d user with the best channel conditions and transmits to the one who has received the least number of packets.

There are still questions that remain to be answered. For example, in the model we considered, all the users always have packets of equal size for transmission; it would be quite interesting to generalize the results to the case where each user has a random rate of arrival or a different transmission rate and analyze the behavior of the length of the longest queue among n users.

5.7 Appendices

5.7.1 Properties of $F(z)$

In this appendix we prove that using $F(z)$, we can generate all the moments of $D_{m,n}$ as in (5.4). Defining a_i as the probability of success in sending at least m packet to all users in i channel uses, we may write the mean of $D_{m,n}$ as

$$E\left(D_{m,n}\right) = \sum_{i=0}^{\infty} i a_i. \tag{5.46}$$

As b_i was the probability of failure in obtaining m packets in all the receivers up to and including the i'th channel uses, it is clear that $a_i = b_{i-1} - b_i$. Therefore, we may write the mean value of $D_{m,n}$ as

$$E\left(D_{m,n}\right) = \sum_{i=1}^{\infty} i(b_{i-1} - b_i) = \sum_{i=1}^{\infty} b_i = F(1), \tag{5.47}$$

which leads to the first identity in (5.4). We can also write the second moment of $D_{m,n}$ as

$$\begin{aligned} E\left(D_{m,n}^2\right) = \sum_{i=1}^{\infty} i^2 a_i &= \sum_{i=1}^{\infty} i^2 (b_{i-1} - b_i) \\ &= \sum_{i=0}^{\infty} (i+1)^2 b_i - \sum_{i=1}^{\infty} i^2 b_i \\ &= b_0 + \sum_{i=1}^{\infty} (2i+1) b_i \\ &= \sum_{i=0}^{\infty} (2i+1) b_i = 2F'(1) + F(1), \end{aligned} \tag{5.48}$$

which completes the proof of (5.4). We can similarly prove that the i'th moment of $D_{m,n}$ can be written as

$$E\left(D_{m,n}^k\right) = \sum_{i=0}^{\infty} (i+1)^k b_i - \sum_{i=0}^{\infty} i^k b_i = \sum_{j=0}^{k-1} \binom{k}{j} \sum_{i=0}^{\infty} i^k b_i. \tag{5.49}$$

It is worth noting that the inner summation can be found by taking the derivatives of $F(z)$. For example,

$$\sum_{i=0}^{\infty} i^2 b_i = \frac{d}{dz} \left(z \frac{dF(z)}{dz} \right) |_{z=1}. \tag{5.50}$$

Therefore, Eqs. (5.49) and (6.40) imply that all the moments of $D_{m,n}$ can be obtained in terms of derivatives of $F(z)$ at $z = 1$.

5.7.2 Proof of (5.78)

In this appendix, we prove Eq. (5.78) and we study the behavior of the maximum n i.i.d. $\chi^2(2m)$ random variables (x_i's) when $m = (\log n)^r$ where r is a constant larger than one. We first note that the CDF of x_i is the incomplete gamma function.

Therefore,

$$\Pr\left\{\max_{1\le i\le n} x_i \le x\right\} = \left(1 - S_m(x)e^{-x}\right)^n = \left(\frac{1}{m!}\gamma(m,x)\right)^n, \tag{5.51}$$

where $\Gamma(m+1,x) = 1 - \gamma(m+1,x) = (m-1)!e^{-x}S_m(x)$. We first compute the following probability:

$$\Pr\left\{\max_{1\le i\le n} x_i \le (\log n)^r + (\log n)^{r/2-1}\right\} = \left(m!\gamma\left(m+1, (\log n)^r + (\log n)^{r/2-1}\right)\right)^n. \tag{5.52}$$

In [4, 8], it is shown that when both of the arguments of the incomplete gamma function grow equally fast, we can expand the incomplete gamma function as

$$\gamma(m+1, m+\sqrt{2my}) = \frac{1}{2}m!\left(1 + \mathrm{erf}(y) + O(\frac{1}{m})\right), \tag{5.53}$$

when m and y are real and y is bounded and erf is the error function. Therefore, we may use (5.53) to compute (5.52) as

$$\Pr\left\{\max_{1\le i\le n} x_i \le (\log n)^r + (\log n)^{r/2-1}\right\} = (1 - O(1/(\log n)^r))^n = O(e^{-n/(\log n)^r}). \tag{5.54}$$

We can also evaluate the following probability as

$$\Pr\left\{\max_{1\le i\le n} x_i \le (\log n)^r + (\log n)^{r/2+1}\right\} = \left(1 - \frac{\Gamma(m+1, (\log n)^r + (\log n)^{r/2+1})}{m!}\right)^n. \tag{5.55}$$

We may now use the expansion for $\Gamma(a+1,x)$ mentioned in (5.76) when the argument satisfies the fact that $\frac{\sqrt{a}}{x-a}$ tends to zero. Therefore, using Sterling's formula for $m!$, we obtain

$$\Pr\left\{\max_{1\le i\le n} x_i \le (\log n)^r + (\log n)^{r/2+1}\right\} = (1-O(e^{-(\log n)^{r/2+1}}))^n = 1-O(1/n). \tag{5.56}$$

Using the union bound, Eq. (5.56) and (5.54) imply that

$$\Pr\left\{(\log n)^r - (\log n)^{r/2-1}) \le \max_{1\le i\le n} x_i \le (\log n)^r + (\log n)^{r/2+1})\right\} = 1 - O(1/n). \tag{5.57}$$

which completes the proof of (5.78). □

5.7.3 Roots of the Polynomial $r(x)$

In this appendix we investigate the zeros of the polynomial $r(x) = \frac{1}{f(x)} = (x - M + 1)_M - (n - M + 1)_M$ when n is large and M is not growing faster than $\log n$. It is clear that $r(x)$ has m roots and one if them is clearly $x_0 = n$. Intuitively, we would expect that if n is large the roots of $r(x)$ are very close to the circle $|x| = n$. We can in fact prove that if $|x| > n$ then $|x - i| > n - i$ for any n and i. Therefore if $|x| > n$ then

$$|\prod_{i=0}^{M-1}(x-i)| > \prod_{i=0}^{M-1}(n-i). \tag{5.58}$$

Therefore, $r(x) = 0$ cannot have any root outside $|x| = n$.

In other words, we proved $g(x) = \frac{1}{Mn^M} r(x/n)$ has all its roots inside the unit circle. Therefore, we can use the method of iteration to find the x_i's. To use the iteration method we write the equation $g(x) = 0$ as $h(x) = \frac{1}{M}g(x) - x = x$, then start with a guess such as x_0, and compute $x_{i+1} = h(x_i)$. Then the x_i 's converge to the root if the sequence x_i's belong to an interval I and $|h'(x)| < 1$ in the same interval for all $x \in I$.

We first show that for $|x| < 1$, then $|h'(x)| < 1$. This can be proved by expanding $h(x)$ as

$$h(x) = \frac{1}{M}\left(\prod_{i=0}^{M-1}(x - \frac{i}{n}) - \prod_{i=0}^{M-1}(1 - \frac{i}{n})\right) = \frac{1}{M}\left(x^M - \frac{M(M-1)}{2n}x^{M-1} + \dots\right). \tag{5.59}$$

It is pretty clear from (5.59) that $|h'(x)| = |x| + O\left(\frac{M^2}{n}\right) < 1$.

Now we can apply the iteration method by starting with the roots of $x^M = n^M$ as the first approximation. We can then obtain

$$\frac{x_i}{n} = e^{j\frac{2\pi i}{M}}\left(1 - \frac{M-1}{2n}\right) + \frac{M-1}{2n} + o\left(\frac{M}{n}\right) \quad ,i = 1, \ldots, M-1, \tag{5.60}$$

where x_i's are the poles of $f(x)$. Having derived the roots, we can obtain the partial fraction expansion of $f(x)$ as

$$\frac{1}{f(x)} = \frac{\alpha_0}{x-n} + \sum_{i=1}^{M-1} \frac{\alpha_i}{x - x_i}, \tag{5.61}$$

where x_i's are as defined in (5.60) and α_i can be obtained as

$$\alpha_i = \lim_{x \to x_i} (x - x_i) f(x) = \frac{1}{(x_i - M + 1)_M} \frac{1}{\sum_{k=0}^{M-1} \frac{1}{x_i - k}}. \tag{5.62}$$

For example, Eq. (5.62) implies that $\alpha_0 = \frac{1}{(n-M+1)_M \sum_{k=0}^{M-1} \frac{1}{n-k}}$.

5.7.4 Roots of (5.60)

In this appendix we show that assuming x_p's are as in (5.60), then $\frac{n!}{x_p(x_p-1)\ldots(x_p-n+1)} = o(1)$. Using the definition of x_p we get

$$\begin{aligned}
\mathrm{Re}(x_i) &= n\left(1 - \frac{M-1}{2n}\right)\cos\frac{2\pi p}{M} + \frac{M-1}{2} + O(M^2/n), \\
\mathrm{Im}(x_i) &= n\left(1 - \frac{M-1}{2n}\right)\sin\frac{2\pi p}{M} + \frac{M-1}{2} + O(M^2/n).
\end{aligned} \tag{5.63}$$

We can then easily show that for all $1 \leq p \leq M-1$, we have

$$\begin{aligned} |x_p - i| &= \left(1 - \frac{M}{2n}\right)\sqrt{\left(\cos\frac{2\pi p}{M} - \frac{i}{n-M/2}\right)^2 + \sin^2\frac{2\pi p}{M}} \\ &> n - M/2 - i. \end{aligned} \quad (5.64)$$

We may denote the upper bound for $\frac{|x_p - i|}{n-M/2-i} \leq \alpha < 1$ where α is a constant independent of n and less than 1. Therefore

$$\begin{aligned} \left|\frac{n!}{x_p(x_p-1)\ldots(x_p-n+1)}\right| &\leq \prod_{i=1}^{n-M/2-1}\left|\frac{n-M/2-i}{x_p-i}\right| n^M \left|\prod_{i=n-M/2}^{n}\frac{1}{x_p-i}\right| \\ &\leq \alpha^{n-M/2-1} n^M \prod_{i=n-M/2}^{n}\frac{1}{n(1-\frac{M}{2n})\sin\frac{2\pi}{M}} \\ &= O\left(\alpha^n (\log n)^{\log n}\right) = o(1), \end{aligned} \quad (5.65)$$

where we used the fact that M is not growing faster than $\log n$. This complete the proof for $\frac{n!}{x_p(x_p-1)\ldots(x_p-n+1)} = o(1)$.

5.7.5 Proof of Theorem 5.4

In this appendix we present the proof for Theorem 5.4.

Proof: *i*) The expected value of $D_{m,n}$ is shown to be equal to (5.16). We first prove that the integral in (5.16) is in fact proportional to the expected value of the maximum of n i.i.d. $\chi^2(2m)$ random variables. To prove that, we assume x_i's for $i = 1, \ldots, n$ are i.i.d. random variables with $\chi^2(2m)$ distribution. We can then write the expected value of the maximum of x_i's as

$$E\{\max_{1\leq i\leq n} x_i\} = \int_0^\infty x f_{max}(x)dx = \int_0^\infty (1 - F_{max}(x))dx = \int_0^\infty (1 - F^n(x))dx, \quad (5.66)$$

where $f_{max}(x)$ and $F_{max}(x)$ are probability distribution and cumulative distribution functions (CDF) of the maximum of x_i's and $F(x)$ is the CDF of x_i. We further know that x_i's are i.i.d. and have $\chi^2(2m)$ distribution and therefore their CDF is the incomplete gamma function and can be written as $F(x) = 1 - S_m(x)e^{-x}$. Therefore, we may write (5.66) as

$$E(D_{m,n}) = \frac{(n+1)}{1-P_e} E\{\max_{1\le i\le n} x_i\} = \frac{(n+1)}{1-P_e} \int_0^\infty \left(1 - (1 - S_m(x)e^{-x})^n\right) dx. \tag{5.67}$$

Therefore to analyze the mean of $D_{m,n}$, we investigate the behavior of the maximum of x_i's. In [27], it is shown that for m fixed, $\max_{1\le i\le n} x_i$ behaves like $\log n + (m-1)\log\log n$. More precisely,

$$\Pr\left\{\left|\frac{\max_{1\le i\le n} x_i - \log n - (m-1)\log\log n}{\log n}\right| \le O\left(\frac{\log\log n}{\log n}\right)\right\} = 1 - O\left(\frac{1}{\log n}\right). \tag{5.68}$$

We can therefore state a lower bound for $D_{m,n}$ as

$$\begin{aligned} E(D_{m,n}) &\ge \frac{n+1}{1-P_e} \int_{\log n + (m-1)\log\log n}^\infty \left(1 - (1 - S_m(x)e^{-x})^n\right) dx \\ &= \frac{n+1}{1-P_e} \Pr\left\{\max_{1\le i\le n} x_i \ge \log n + (m-1)\log\log n\right\} \\ &\ge \frac{1}{1-P_e}\left(n\log n + n(m-1)\log\log n + o(n\log\log n)\right)\left(1 - O\left(\frac{1}{\log n}\right)\right) \\ &= \frac{1}{1-P_e} n\log n + n(m-1)\log\log n + o(n\log\log n). \end{aligned} \tag{5.69}$$

We can also obtain an upper bound for the expected delay as

$$\begin{aligned}
E(D_{m,n}) &\leq \frac{n+1}{1-P_e}\int_0^{\log n+(m-1)\log\log n}\left(1-(1-S_m(x)e^{-x})^n\right)dx \\
&\quad +\frac{n+1}{1-P_e}\int_{\log n}^{\infty}\left(1-(1-S_m(x)e^{-x})^n\right)dx \\
&= \frac{n+1}{1-P_e}\Pr\left\{\max_{1\leq i\leq n} x_i \leq \log n+(m-1)\log\log n\right\}+\frac{n}{1-P_e}\int_{\log n}^{\infty}\frac{x^m}{m!}e^{-x}dx \\
&\leq \frac{1}{1-P_e}(n\log n+n(m-1)\log\log n+o(n\log\log n)+\frac{n}{1-P_e}\int_{\log n}^{\infty}\frac{x^m}{m!}e^{-x}dx \\
&= \frac{1}{1-P_e}(n\log n+n(m-1)\log\log n+o(n\log\log n)+O\left(n\Gamma(m+1,\log n)\right) \\
&= \frac{1}{1-P_e}\left(n\log n+n(m-1)\log\log n\right)+o(n\log\log n), \qquad (5.70)
\end{aligned}$$

where the third inequality follows from (5.66). Noting that when m is fixed, we have $\Gamma(m,x)=O(x^{m-1}e^{-x})$ [63] and thus the last term in the fourth line of (5.70) is of the order $(\log n)^m$. Clearly, the upper bound in (5.70) and the lower bound of (5.69) are tight to the first order and lead to the proof of (5.18) for the expected value of $D_{m,n}$.

To obtain the variance, we first note that $D_{m,n}\leq mD_{1,n}$, which is clear from the definition of $D_{m,n}$. Now we first derive the variance of $D_{1,n}$ and, since m is fixed, the variance of $D_{m,n}$ has the same order. Denote by r_i, for $i=1,\ldots,n$, the number of transmissions after transmitting at least one packet to $i-1$ users and before i users receive their first packet. Clearly the r_i's are independent and have geometric distribution

$$\Pr\{r_i=k\}=\left(\frac{i-1}{n}\right)^{k-1}\left(1-\frac{i-1}{n}\right). \qquad (5.71)$$

The distribution of r_i is obtained by noting that r_i equals k if in the last $k-1$ trials the packet is transmitted to the $i-1$ users who have already been chosen and then in the k'th channel use, one user will be transmitted to from the pool of $n-i+1$ users who have already been chosen.

Using the definition of $D_{1,n}$, it is clear that $D_{1,n}=\sum_{i=1}^n r_i$ and therefore the variance of $D_{1,n}$ can be written as

$$\sigma^2_{D_{1,n}} = n^2 \sum_{i=1}^{n} \frac{1}{i^2} - n \sum_{i=1}^{n} \frac{1}{i}. \tag{5.72}$$

It is quite straightforward to prove that the first term on the right hand side of (5.72) behaves like $O(n^2)$ and the second term behaves like $n \log n$. Therefore the variance of $D_{m,n}$ can be written as

$$\sigma^2_{D_{m,n}} \leq m^2 \sigma^2_{D_{1,n}} = O(n^2). \tag{5.73}$$

ii) To prove the second asymptotic result, we mention a result on the behavior of the maximum of n i.i.d. $\chi^2(2m)$ random variables proved in [101]. It is shown that if $m = \log n$, then

$$\Pr\left\{\alpha \log n - O(\log\log n) \leq \max_{1\leq i\leq n} x_i \leq \alpha \log n + O(\log\log n)\right\} = 1 - O\left(\frac{1}{\log n}\right), \tag{5.74}$$

where α satisfies $\alpha - \log\alpha = 2$ (i.e. $\alpha \approx 3.14$). Therefore, using the same methodology as in (5.70), we may write an upper bound for the delay as

$$\begin{aligned} E(D_{m,n}) &\leq \frac{1}{1-P_e} n \left(1 - O\left(\frac{1}{\log n}\right)\right) \alpha \log n + O\left(\int_{\alpha \log n}^{\infty} xnf(x)F^{n-1}(x)dx\right) \\ &= \frac{1}{1-P_e} \alpha n \log n + O\left(\int_{\alpha\log n}^{\infty} xn\frac{x^{m-1}}{m!}e^{-x}dx\right) \\ &= \frac{1}{1-P_e} \alpha n \log n + O\left(\frac{n}{m!}\Gamma(m+1, \alpha\log n)\right), \end{aligned} \tag{5.75}$$

where $m = \log n$. In [4, 8], the asymptotic behavior of the incomplete gamma function when both arguments are going to infinity has been considered and it is shown that

$$\Gamma(a+1, x) = e^{-x}\frac{x^{a+1}}{x-a}\left(1 - O(\frac{a}{(x-a)^2})\right), \tag{5.76}$$

when the fraction $\frac{\sqrt{a}}{x-a}$ is positive and tends to zero. Using (5.76), (5.75) and using Sterling's formula for the factorial, we obtain

$$E(D_{m,n}) \leq \frac{1}{1-P_e}\alpha n \log n + O(n \log\log n). \tag{5.77}$$

A lower bound for $E(D_{m,n})$ can be obtained from (5.74), and it is quite easy to show that $E(D_{m,n}) \geq \alpha n \log n$ using the same approach as in (5.69). Since the upper and lower bounds for the expected delay are tight to the first order, the second part of the theorem follows immediately.

iii) Proof of this part is along the same line as that of the second part. In Appendix 5.7.2, it is proved that if $m = (\log n)^r$ for r fixed and greater than one, then

$$\Pr\left\{(\log n)^r - (\log n)^{r/2-1} \leq \max_{1\leq i\leq n} x_i \leq (\log n)^r + (\log n)^{r/2+1}\right\} = 1 - O\left(\frac{1}{\log n}\right). \tag{5.78}$$

Writing the upper and lower bounds for the expected delay as in (5.70) and (5.69) and using (5.78) and following along the same line as what we did in the proof of the second part leads to the proof for this part as well.

iv) The proof of the fourth region is a bit different as n is fixed and m grows to infinity. Using Chebychev's inequality and noting that the x_i's have $\chi^2(2m)$ distribution and therefore have mean and variance of m, we obtain

$$\Pr\left\{m - m^{3/4} \leq x_i \leq m + m^{3/4}\right\} = 1 - O\left(\frac{1}{\sqrt{m}}\right) \tag{5.79}$$

for any $1 \leq i \leq n$. We can then use union bound to get

$$\Pr\left\{m - m^{3/4} \leq \max_{1\leq i\leq n} x_i \leq m + m^{3/4}\right\} = 1 - n \times O\left(\frac{1}{\sqrt{m}}\right) = 1 - O\left(\frac{1}{\sqrt{m}}\right), \tag{5.80}$$

where we used the fact that n is fixed. Similarly to the upper bound derived in (5.70), we may then obtain an upper bound for the expected delay as

$$
\begin{aligned}
E(D_{m,n}) &\leq \frac{1}{1-P_e}(m+m^{3/4})n\left(1-O\left(\frac{1}{\sqrt{m}}\right)\right)+O\left(\int_{m+m^{3/4}}^{\infty}\frac{n}{m!}x^m e^{-x}dx\right) \\
&= \frac{1}{1-P_e}mn+O\left(\frac{n}{m!}\Gamma(m,m+m^{3/4})\right)+o(mn) \\
&= \frac{1}{1-P_e}mn+o(mn). \qquad (5.81)
\end{aligned}
$$

The last equality follows by using the asymptotic expansion of the incomplete gamma function stated in (5.76) [4, 8]. Clearly mn is the lower bound for the expected delay and therefore the lower and upper bounds are tight to the order of mn. □

5.7.6 Proof of Theorem 5.6

Here we present the proof for Theorem 5.6.

Proof: To asymptotically analyze the expected delay, we may write Eq. (5.25) as

$$
\begin{aligned}
E\left(D_{1,n}(M)\right) &= \sum_{r=1}^{n}\binom{n}{r}\sum_{i=0}^{n-r}(-1)^{n-r-i}\binom{n-r}{i}\sum_{p=0}^{\infty}\left(\frac{\binom{i}{M}}{\binom{n}{M}}\right)^p \\
&= \sum_{k=0}^{n-1}\binom{n}{k}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{k-i}n(n-1)\dots(n-M+1)}{n(n-1)\dots(n-M+1)-i(i-1)\dots(i-M+1)} \\
&= (n-M+1)_M\sum_{k=0}^{n-1}\binom{n}{k}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{k-i}}{(n-M+1)_M-(i-M+1)_M}, \qquad (5.82)
\end{aligned}
$$

where the second equality follows by noting that the inner summation is a geometric sum and also we let $k = n - r$ in the second equality. In (5.82), $(n)_M$ is the Pochhammer symbol and is equal to $n(n+1)\dots(n+M-1)$ [63].

To compute the inner summation in (5.82), we define $f(x) = \frac{1}{(n-M+1)_M-(x-M+1)_M}$

and expand the inside fraction using the partial fraction expansion around $x = i$ as

$$f(x)|_{x=i} = \left\{ \frac{1}{(n-M+1)_M - (x-M+1)_M} \right\}_{x=i} = \left\{ \sum_{p=0}^{M-1} \frac{a_p}{x - x_p} \right\}_{x=i} = \sum_{p=0}^{M-1} \frac{a_p}{i - x_p} \tag{5.83}$$

where x_p's for $p = 0, \ldots, M-1$ are the poles of $f(x)$. The trivial pole of $f(x)$ is $x_0 = n$. In Appendix 5.7.3, it is shown that when n is large, the poles of $f(x)$ can be written as

$$\frac{x_p}{n} = e^{j\frac{2\pi p}{M}} \left(1 - \frac{M-1}{2n}\right) + \frac{M-1}{2n} + O\left(\frac{(\log n)^2}{n^2}\right) \qquad p = 1, \ldots, M-1, \tag{5.84}$$

as long as M does not grow faster than $\log n$. Moreover, the a_p's can be computed as in (5.62) in Appendix 5.7.3. Replacing (5.83) in (5.82), we get

$$E\left(D_{1,n}(M)\right) = (n-M+1)_M \sum_{p=0}^{M-1} a_p \sum_{k=0}^{n-1} \binom{n}{k} \sum_{i=0}^{k} (-1)^{k-i} \frac{1}{i - x_p} \binom{k}{i}. \tag{5.85}$$

Clearly the inner summation is just a function of x_p and k. Fortunately we can obtain a closed form function for the inner summation by noting that following identity on the partial fraction expansion,

$$g(x) \triangleq \frac{k!}{x(x-1)\ldots(x-k)} = \sum_{i=0}^{k} \frac{\beta_i}{x - i}, \tag{5.86}$$

where the β_i's can be obtained as

$$\beta_i = \lim_{x \to i} (x - i) g(x) = (-1)^{k-i} \binom{k}{i}. \tag{5.87}$$

Therefore considering (5.86) and the β_i's derived in (5.87), the inner summation in (5.85) is equal to $g(x) = \frac{k!}{(x-k)_{k+1}}$. Replacing $g(x)$ by the inner summation, we can write Eq. (5.85) as

$$E\left(D_{1,n}(M)\right) = (n-M+1)_M \sum_{p=0}^{M-1} a_p \sum_{k=0}^{n-1} \binom{n}{k} \frac{k!}{(x_p-k)_{k+1}}. \tag{5.88}$$

Again we can simplify (5.88) by first noting that the inner summation is just a function of x_p. Moreover

$$\begin{aligned} h(x) &\triangleq \sum_{k=0}^{n-1} \binom{n}{k} \frac{k!}{(x-k)_{k+1}} \\ &= \frac{1}{x} \sum_{k=0}^{n-1} \frac{(n-k+1)_k}{(x-k)_k} \\ &= \frac{1}{x} \left\{ \sum_{k=0}^{n} \frac{(n-k+1)_k}{(x-k)_k} - \frac{(1)_n}{(x-n)_n} \right\} \\ &= \frac{1}{x} \left\{ F(-n,1;1-z;1) - \frac{n!}{(x-n)_n} \right\}, \end{aligned} \tag{5.89}$$

where $F(a,b;c;z)$ is the Gauss hypergeometric function and is equal to $\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$. It is further quite easy to see that

$$F(-n,1;1-x;1) = \sum_{k=0}^{n} \frac{(n-k+1)_k}{(x-k-1)_k}. \tag{5.90}$$

So the third equality in (5.89) follows from (5.90). It is worth mentioning that $h(n) = \sum_{k=0}^{n-1} \frac{1}{n-k}$ by just replacing x by n in the definition of $h(x)$; this will be used later in the proof.

We can further simplify the hypergeometric function by noting that $F(x,y;z;1) = \frac{\Gamma(z)\Gamma(z-x-y)}{\Gamma(z-x)\Gamma(z-y)}$ [63], and therefore we can write $h(x)$ in (5.89) as

$$\begin{aligned} h(x) &= \frac{1}{x} \frac{\Gamma(1-x)\Gamma(n-x)}{\Gamma(n+1-x)\Gamma(-x)} - \frac{n!}{(x-n)_{n+1}} \\ &= -\frac{n!}{(x-n)_{n+1}} + \frac{1}{x-n}, \end{aligned} \tag{5.91}$$

where the equality follows using from the fact that $\Gamma(1+x) = x\Gamma(x)$. We can now replace $h(x)$ in (5.88) and write the expected delay as

$$
\begin{aligned}
E\left(D_{1,n}(M)\right) &= (n-M+1)_M \sum_{p=0}^{M-1} a_p h(x_p) \\
&= \frac{\sum_{k=1}^{n} \frac{1}{k}}{\sum_{r=0}^{M-1} \frac{1}{n-r}} - (n-M+1)_M \left(\sum_{p=1}^{M-1} \frac{a_p}{x_p - n} + \sum_{p=1}^{M-1} \frac{a_p n!}{(x_p - n)_{n+1}} \right),
\end{aligned} \tag{5.92}
$$

where we used the fact that $a_0 = \frac{1}{(n-M+1)_M \sum_{r=0}^{M-1} \frac{1}{n-r}}$ proved in Appendix 5.7.3 and $h(x=n) = \sum_{k=1}^{n} \frac{1}{k}$, which follows from the definition of $h(x)$ in (5.89). We have to evaluate the second and third terms in (5.92) for large n. The second term can be obtained by defining the polynomial $r(x) = \frac{1}{f(x)} = (x-M+1)_M - (n-M+1)_M$. Noting that x_p's are the roots of $r(x)$, we can write

$$
\sum_{p=1}^{M-1} \frac{a_p}{x - x_p} = \frac{1}{r(x)} - \frac{a_0}{x-n} = \frac{1}{r(x)} - \frac{1}{r'(n)(x-n)}, \tag{5.93}
$$

where the first equality follows from the definition of $r(x)$ and its partial fraction expansion and a_0 is basically $\lim_{x \to n} (x-n) f(x) = \frac{1}{r'(n)}$. Now we can replace $x = n$ in (5.93) to get

$$
\begin{aligned}
\sum_{p=1}^{M-1} \frac{a_p}{n - x_p} &= \lim_{x \to n} \left\{ \frac{1}{r(x)} - \frac{1}{r'(n)(x-n)} \right\} \\
&= -\lim_{x \to n} \frac{r(x) - r'(n)(x-n)}{r(x)(x-n)r'(n)} \\
&= -\lim_{x \to n} \frac{\frac{1}{2} r''(n)(x-n)^2 + O((x-n)^3)}{x(x-1)\ldots(x-n+1)(x-n)^2 r'(n)} \\
&= -\frac{r''(n)}{2(r'(n))^2},
\end{aligned} \tag{5.94}
$$

where we used the Taylor expansion of $r(x)$ around $x = n$. Using the definition of

$r(x)$ it is quite straightforward to show that $\frac{r''(n)}{(r'(n))^2} \leq \frac{1}{(n-M+1)_M}$. Therefore the second term is of the order one.

To analyze the third term in (5.92), we first note that as x_p's are close to the circle of radius n and $|x_p - i|$ is roughly always greater than $n - i$, it can be shown that $\frac{n!}{(x_p-n+1)_n} = O(1)$ for $p = 1, \ldots, M-1$ (see Appendix 5.7.4 for the proof). Using that, we can write the third term as

$$\left|(n-M+1)_M \sum_{p=1}^{M-1} \frac{a_p n!}{(x_p-n)_{n+1}}\right| = O\left(\sum_{p=1}^{M-1} \left|\frac{n!}{(x_p-n+1)_n}\right| \left|\frac{a_p(n-M+1)_M}{(x_p-n)}\right|\right) = O(1). \tag{5.95}$$

Replacing (5.95) and (5.94) in (5.92) leads to the proof. □

Chapter 6

OFDM Systems and the High Peak-to-Average-Power Problem

6.1 Introduction

Multicarrier modulation has been proposed in different broadband wireless and wire-line applications such as wireless local area networks (WLAN) and digital subscriber line (DSL). Even though multicarrier modulation is very well-suited in a multipath fading environment, it suffers from high amplitude variation, which is unfavorable from a practical point of view. Different schemes have been proposed to reduce the peak to mean envelope power ratio (PMEPR) [1] such as coding methods, clipping, reserved carriers, and probabilistic methods such as selective mapping and partial transmit sequence [34, 36, 39, 46, 68, 37, 38].

Unfortunately, the worst case PMEPR of multicarrier signals is rather high and is of the order of n where n is the number of subcarriers. On the other hand, the numerical evaluation of the distribution of PMEPR shows that encountering the worst case n is highly unlikely [33, 47, 69, 70, 71, 72]. This in fact motivates the problem of finding the PMEPR distribution to quantify how severe that is. In [33, 47], by assuming that the multicarrier signal is a Gaussian process, an expression for the

[1] In this thesis, the term PMEPR refers to the peak to average power of the complex baseband OFDM signal and PAPR denotes the the peak to average power ratio of the transmit real signal [37].

probability distribution of PMEPR is derived. This is a very strong assumption, and, when the codewords are chosen from fixed constellations, is mathematically not valid for the joint distribution of n or more samples [73]. Recently, in [71], an upper bound for the PMEPR distribution is shown for QAM/PSK with M^2 points and uniform distribution over the constellation points, and it is shown that the probability of encountering a PMEPR of greater than $(1+\epsilon)\log n$ is going to zero as n increases. On the other hand, in [72] using techniques different from ours, a lower bound for the distribution of PMEPR is obtained when codewords are uniformly distributed over a complex sphere. However, [72] does not perform an asymptotic analysis, which is what we do here. In this chapter, we generalize the results to a larger class of constellations with even distribution over the constellation points, and we show a stronger result, namely with high probability the PMEPR *behaves* like $\log n + O(\log\log n)$. In other words, encountering a PMEPR of less than $\log n + O(\log\log n)$ is also highly unlikely.

The results are based on a generalization of the well-known result of Halasz [74] for Littlewood trigonometric polynomials with equiprobable coefficients chosen independently from $\{+1, -1\}$ [69, 37, 71]. In summary, we show that, with probability approaching one, any codeword either with entries chosen independently from the symmetric QAM/PSK constellations or chosen uniformly from a complex sphere has PMEPR of $\log n + O(\log\log n)$ for a large number of subcarriers. We then use this result to determine the achievable rate of codes with given minimum distance and bounded PMEPR.

The rest of the chapter is outlined as follows. Section 6.2 introduces the notation, multicarrier signals, and the PMEPR of a codeword. The lower and upper probability bounds for the PMEPR distribution are derived in Section 6.3. In Section 6.4, we discuss the consequences of the bounds and we obtain a Varsharmov-Gilbert type bound for the achievable rate of codes with bounded PMEPR and with given minimum Hamming distance.

6.2 Definition

The complex envelope of a multicarrier signal with n subcarriers may be represented as

$$s_C(t) = \sum_{i=1}^{n} c_i e^{j2\pi i f_0 t}, \qquad 0 \le t \le 1/f_0, \tag{6.1}$$

where f_0 is the subchannel spacing and $C = (c_1, \ldots, c_n)$ is the complex modulating vector with entries from a given complex constellation. The admissible modulating vectors are called codewords and the ensemble of all possible codewords constitute the code $\mathcal{C}$. For mathematical convenience, we define the normalized complex envelope of a multicarrier signal as

$$s_C(\theta) = \sum_{i=1}^{n} c_i e^{j\theta i}, \qquad 0 \le \theta < 2\pi. \tag{6.2}$$

Then, the PMEPR of each codeword C in the code $\mathcal{C}$ may be defined as

$$\text{PMEPR}_{\mathcal{C}}(C) = \max_{0\le\theta<2\pi} \frac{|s_C(\theta)|^2}{E\{\|C\|^2\}}. \tag{6.3}$$

Similarly, the PMEPR of the code $\mathcal{C}$, denoted by $\text{PMEPR}_{\mathcal{C}}$, is defined as the maximum of (7.3) over all codewords in $\mathcal{C}$. It is clear from the definition of PMEPR that if all the carriers add up coherently, the PMEPR can be of the order of n.

Even though the worst case PMEPR is of the order n when c_i's are chosen from a constellation such as QAM, it is shown that with high probability the PMEPR of a random codeword is $\log n$ almost surely [75, 74, 35]. This implies that the PMEPR is not as bad as what is predicted by the worst case and its distribution should be taken into consideration. Fig. 6.1 compares the complementary cumulative distribution function (CCDF) of PMEPR for a multicarrier system with $n = 128$ and using a 64QAM constellation with that of a single carrier system. By ignoring peaks with

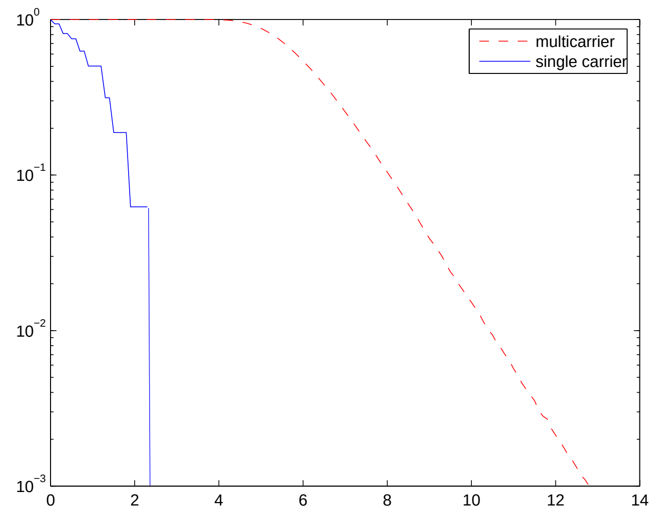

Figure 6.1: Comparison of Pr(PMEPR $> \lambda$) for a multicarrier system with $n = 128$ and a single carrier system using a 64QAM constellation for 5000 random codewords.

probability below 10^{-3},[2] the PMEPR of the multicarrier system is 12.5 as opposed to 2.3 for the single carrier system. This shows a 7.35 dB gap between the PMEPR of these two systems.

In this chapter, we will consider two classes of codes, namely, complex symmetric q-ary codes in which each coordinate is chosen from a complex QAM or PSK constellation with alphabets of cardinality q, and spherical codes in which codewords are points on a complex n-dimensional sphere defined as $\Omega_n = \{(c_1, \ldots, c_n) : \sum_{i=1}^{n} |c_i|^2 = n\}$.

It is worth noting that for a random q-ary code with i.i.d. entries chosen from a constellation, $E\{\|C\|^2\} = nE_{av}$ where E_{av} is the average energy of the constellation. Also, for spherical codes chosen from Ω_n, $E\{\|C\|^2\} = n$ since all the codewords have constant norms.

Throughout the thesis, we will use the following notations: $\mathcal{C}$ and C represent the code and codeword, c_i denotes the i'th coordinate of the modulating vector C, $\log\{\cdot\}$

[2] Throughout the paper, in order to compare the simulation results, we approximate the PMEPR of a scheme by the value η such that Pr(PMEPR $> \eta$) $= 10^{-3}$. We basically ignore peaks with probability below 10^{-3} in our simulations.

is the natural logarithm, and $H_q(x) = -x\log_q(x) - (1-x)\log_q(x)$. We use α and β as arbitrarily constants and $f(n) = O(g(n))$ denotes that $\lim_{n\to\infty}\left|\frac{f(n)}{g(n)}\right| \le |\alpha|$.

6.3 Bounds on Distribution of PMEPR

It is a commonly assumed in the literature that when the c_i's in (7.2) are independently chosen, $s_C(\theta)$ can be approximated as a Gaussian process (for example see [33, 47]). However, this is not mathematically rigorous for spherical codewords and codewords with entries from a QAM/PSK constellation. In other words, by assuming that the c_i's are i.i.d., even though it is conceivable that any finite samples of $s_C(\theta)$ are jointly Gaussian for large values of n, this statement is not valid for n samples of $s_C(\theta)$.

In this section without using any Gaussian assumption, we derive upper and lower bounds for the PMEPR distribution for different schemes. The derivation of the bounds are the generalization of a result of Halasz [74] for the asymptotic distribution of the maximum of $|\sum_{i=1}^{n} a_i \cos i\theta|$ when a_i's are chosen independently from $\{+1, -1\}$ with equal probability. This result is extended in [75] to the maximum of the modulus of polynomials over the unit circle[3] with real independent coefficients, c_k, and with characteristic function $E\{e^{jtc_k}\} = e^{-\alpha_2 t^2 + \alpha_3 t^3 + O(t^4)}$ for t in some nontrivial interval $[-d, d]$.

Based on our application for OFDM signals, we generalize the result of [75] and [74] to polynomials over the unit circle when its coefficients are chosen from the following three general cases: (i) c_i's are i.i.d. chosen from a complex QAM constellation in which the real and imaginary parts of c_i each has i.i.d. and even distribution, (ii) c_i's are i.i.d. chosen from a PSK constellation where the distribution function over the constellation points is invariant under rotation by $\pi/2$, and (iii) when the modulating

[3]By polynomials over the unit circle, here we mean polynomials over the complex field evaluated on the unit circle.

codeword, C, is chosen from a complex n-dimensional sphere in which c_k's are no longer independent.

6.3.1 Lower Bound for the PMEPR

In this subsection, we obtain a bound for the probability of having a PMEPR slightly less than $\log n$ and we show that asymptotically this probability goes to zero. Theorem 6.1 derives the bound for QAM constellations and it is later generalized to PSK and spherical codes in Theorem 6.2 and 6.4.

Since scaling the constellation does not affect the PMEPR, for mathematical convenience, we assume the maximum energy of the constellation is one and therefore the resulting E_{av} is less than one and denotes the normalized average energy of the constellation.

Theorem 6.1 *[Lower Bound: QAM Case] Let $s_C(\theta)$ be as in (6.2) where $c_i = a_i + jb_i$ and the a_i's and b_i's each has i.i.d and even distributions. Also, let $\mathcal{C}_q^{QAM}$ be the ensemble of all the admissible codewords C. Then,*

$$\Pr\left\{\text{PMEPR}_{\mathcal{C}_q^{QAM}}(C) \leq \log n - 6.5 \log\log n\right\} \leq O\left(\frac{1}{\log^4 n}\right). \tag{6.4}$$

Proof: Since we are looking for an upper bound for $\Pr\{\text{PMEPR}_{\mathcal{C}}(C) \leq \lambda\}$, hence a lower bound for $\Pr\{\text{PMEPR}_{\mathcal{C}}(C) > \lambda\}$, instead of considering the maximum of $s_C(\theta)$ over all θ, we may consider the maximum of $s_C^R(\theta) = \text{Re}\{s_C(\theta)\}$ over its n samples $\theta_m = \pi(2m+1)/n$ for $m = 1, \ldots, n$. Following the proof of [74], we also define $0 \leq u(x) \leq 1$ as

$$u(x) = \begin{cases} 0 & |x| \leq M, \\ 1 & |x| \geq M + \Delta \end{cases} \tag{6.5}$$

where $\Delta = \sqrt{\frac{n}{\log n}}$ and assume $M = \sqrt{nE_{av}\log n - 6.5nE_{av}\log\log n} - \sqrt{\frac{n}{\log n}}$. We also assume $u(x)$ be a function that is ten times differentiable such that $u^{(r)}(x) = O(\Delta^{-r})$ for $1 \leq r \leq 10$ [4]. Based on these assumptions on $u(x)$, in Appendix 6.5.1, we proved other properties of $u(x)$ that will be used later in the proof. We then define the random variable η as

$$\eta = \sum_{m=1}^{n} u(\mathrm{Re}\{s_C(\theta_m)\}) = \sum_{m=1}^{n} u(s_C^R(\theta_m)) = \sum_{m=1}^{n} \int_{-\infty}^{\infty} e^{jts_C^R(\theta)} v(t)dt, \tag{6.6}$$

where we replaced $u(x)$ by its Fourier transform $v(t)$. To find a lower bound, we use the following inequalities:

$$\begin{aligned} \Pr\{\max_{0\leq\theta\leq 2\pi} |s_C(\theta)| \geq M\} \geq \Pr\{\max_{0\leq m\leq n} |s_C^R(\theta_m)| \geq M\} &= 1 - \Pr\{\eta = 0\} \\ &\geq 1 - \Pr\{\eta = 0, \eta \geq 2E\{\eta\}\} \\ &= 1 - \Pr\{|\eta - E\{\eta\}| \geq E\{\eta\}\} \\ &\geq 1 - \frac{\sigma_\eta^2}{E^2\{\eta\}}. \end{aligned} \tag{6.7}$$

The first inequality follows from the definition of η, which is zero when $|s_C^R(\theta)|$ is less than M. The second inequality follows from the fact that η is a non-negative random variable, and the last inequality is Chebychev's inequality.

Therefore, the evaluation of the lower bound boils down to the asymptotic analysis of the first and second moments of η. In Appendix 6.5.2, it is shown that $E\{\eta\} \geq O(\log^6 n)$, and $\sigma_\eta^2 = E\{\eta^2\} - E^2\{\eta\} \leq O(E\{\eta\}\log^2 n + \log^5 n)$. Therefore, the above results imply that

$$\Pr\{\max_{0\leq\theta\leq 2\pi} |s_C(\theta)| \geq \sqrt{nE_{av}\log n - 6.5nE_{av}\log\log n} - \sqrt{\frac{n}{\log n}}\} \geq 1 - O\left(\frac{1}{\log^4 n}\right). \tag{6.8}$$

Using the definition of PMEPR and normalizing both sides of (6.8) to $P_{av} = nE_{av}$,

[4] Note that $u^{(r)}(x) = O(\Delta^{-r})$ means that for all x, $\lim_{n\to\infty} |\frac{u^{(r)}(x)}{\Delta^{-r}}| \leq \alpha$, which implies that the maximum of $|u^{(r)}(x)|$ is less than $\alpha\Delta^{-r}$ for large n.

the theorem follows. □

As mentioned in Appendix 6.5.2, the derivation of the lower bound relies on the characteristic function of $s_C^R(\theta)$. For the PSK case the real and imaginary parts of c_i are not independent, however, we can still use a similar argument to generalize the result to PSK constellations in which the distribution over the constellation points is invariant under $\pi/2$ rotation.

Theorem 6.2 *[Lower Bound: PSK Case] Let $c_i = e^{j\beta_i}$ where the c_i's i.i.d. chosen from a q-ary PSK constellation in which the* PDF *of c_i is invariant under rotation by $\pi/2$ and $\mathcal{C}_q^{PSK}$ is the ensemble of all codewords C. Then,*

$$\Pr\{\mathrm{PMEPR}_{\mathcal{C}_q^{PSK}}(C) \leq \log n - 6.5 \log\log n\} \leq O\left(\frac{1}{\log^4 n}\right). \tag{6.9}$$

Proof: In the PSK case, we can write the characteristic function of $s_C^R(\theta)$ as

$$\begin{aligned}\Phi_{\mathrm{PSK}}(t) = E\{e^{jts_C^R(\theta)}\} = \prod_{i=1}^{n} E\{e^{jt\mathrm{Re}\{c_i e^{j\theta i}\}}\} &= \prod_{i=1}^{n} E\{\cos(t\cos(i\theta + \beta_i))\} \\ &= \prod_{i=1}^{n} E\{\cos t\beta_i'\},\end{aligned} \tag{6.10}$$

where $\beta_i' = \cos(i\theta + \beta_i)$ has an even distribution since the β_i's are chosen from a PSK constellation such that β_i, $\beta_i + \pi/2$, and consequently $\beta_i + \pi$ are equiprobable. Furthermore, for $|t| < 1$ the characteristic function is positive. Therefore, using the result of Appendix 6.5.3, we can then write $E\{\cos(t\beta_i')\} = e^{-E\{(\beta_i')^2\}t^2/2 + \alpha t^4 + O(t^6)}$ for $|t| < 1$, where the second moment of β_i' can be evaluated as

$$E\{(\beta_i')^2\} = E\{\cos^2(i\theta + \beta_i)\} = \frac{1}{2} + \frac{1}{2}E\{\cos(2i\theta + 2\beta_i)\} = \frac{1}{2} \tag{6.11}$$

and the second term is zero since $2\beta_i$ has the same probability as $\pi + 2\beta_i$ due to the fact that the PDF of β_i is invariant under rotation by $\pi/2$. Therefore, replacing

(6.11) into $E\{\cos(t\beta_i')\}$ and then into (6.10), we get, $\Phi_{\rm PSK}(t) = e^{-nt^2/4+n\alpha t^4+O(nt^6)}$ for $|t| \leq 1$. Now, we can use the same argument as that of Theorem 6.1 to find the mean and variance of η as in (6.54) and (6.66), respectively. The theorem follows similarly by setting $E_{av} = 1$ for PSK constellations. □

To generalize the result to spherical codes, we initially need to find the characteristic function of $s_C^R(\theta)$ when the codeword C is uniformly distributed over Ω_n. Clearly all c_i's are dependent, however the following lemma provides the characteristic function of $s_C^R(\theta)$ when C is uniformly distributed over Ω_n.

Lemma 6.3 *Let $C = (c_1, \ldots, c_n)$ be a random complex vector uniformly distributed over Ω_n and $s_C^R(\theta)$. Then*

$$E\{e^{jts_C^R(\theta)}\} = \frac{2^n\Gamma(n)}{|t|^n n^{n-1}} J_n(n|t|).$$

Proof: Let $c_i = a_i + jb_i$ for $i = 1, \ldots, n$. As a first step to find the characteristic function of $s_C^R(\theta)$, since $\sum_{k=1}^n \sin^2 k\theta + \cos^2 k\theta = n$, we can state that

$$\begin{aligned} p(s_C^R(\theta)) &= p(\langle(a_1, \ldots, a_n, b_1, \ldots, b_n), (\cos\theta, \ldots, \cos n\theta, \sin\theta, \ldots, \sin n\theta)\rangle) \\ &= p(\langle U(a_1, \ldots, a_n, b_1, \ldots, b_n), (\sqrt{n}, 0, \ldots, 0)\rangle) = p(\sqrt{n}a_1'), \qquad (6.13) \end{aligned}$$

where $p(x)$ denotes the PDF of the random variable x, $(a_1', \ldots, a_n', b_1', \ldots, b_n') = U(a_1, \ldots, a_n, b_1, \ldots, b_n)$, $\langle\cdot,\cdot\rangle$ denotes the inner product of two vectors, and U is any orthogonal matrix such that $U(\cos\theta, \cos 2\theta, \ldots, \sin n\theta) = (\sqrt{n}, 0 \ldots, 0)$.

Moreover, since the vector $(a_1, \ldots, a_n, b_1, \ldots, b_n)$ has an isotropic distribution [20], the distribution of the vector remains the same under multiplication by orthogonal matrices, and therefore $p(s_C^R(\theta)) = p(\sqrt{n}a_1') = p(\sqrt{n}a_1) = p(\sqrt{n}r_1\phi_1)$. Now we can use Eq. (6.13) to write

$$E\{e^{jts_C^R(\theta)}\} = E\{e^{jt\sqrt{n}r_1\cos\phi_1}\} = E\left\{\int_0^{2\pi}\frac{1}{2\pi}e^{jt\sqrt{n}r_1\cos\phi_1}d\phi_1\right\} = E\{J_0(t\sqrt{n}r_1)\}, \tag{6.14}$$

where we used the definition of the Bessel function and the fact that ϕ_1 has the uniform distribution proved in Appendix 6.5.4. Since $J_0(x)$ is an even function, the characteristic function is an even function of t and we can therefore focus on $t > 0$. Using the distribution of r_1 computed in Appendix 6.5.4, we can write Eq. (6.14) as

$$\begin{aligned} E\{e^{jts_C^R(\theta)}\} = \frac{2}{n^{n-1}}\int_0^{\sqrt{n}} rJ_0(t\sqrt{n}r)(n-r^2)^{n-1}dr &= 2n\int_0^1 uJ_0(tnu)(1-u^2)^{n-1}du \\ &= \frac{2^n\Gamma(n)}{t^n n^{n-1}}J_n(nt) \end{aligned} \tag{6.15}$$

for $t > 0$, where we used the identity $\int_0^1 x(1-x^2)^{n-1}J_0(bx)dx = \frac{2^{n-1}\Gamma(n)}{b^n}J_n(bx)$ for $b > 0$ [63]. Since the characteristic function is even, the lemma follows from (6.15). □

For large values of n and $0 < t < 1$, we can use the asymptotic expansion of the Bessel function of order n as [63],

$$J_n(nt) = \frac{e^{-n(\cosh^{-1}\frac{1}{t}-\sqrt{1-t^2})}}{\sqrt{2\pi n(1-t^2)^{1/2}}}\left(1+O\left(1/n\right)\right) \tag{6.16}$$

for $0 < t < 1$. Therefore, we can use the asymptotic expansion

$$\Gamma(n) = e^{-n}n^{n-1/2}\sqrt{2\pi}\left(1+O\left(1/n\right)\right)$$

for large n [63], together with (6.16) and replace them into (6.15), to obtain

$$E\{e^{jts_C^R(\theta)}\} = E\{J_0(t\sqrt{n}r_1)\} = e^{-nf(t)}\left(1+O\left(1/n\right)\right) = e^{-n(\alpha_1t^2+\alpha_2t^4+O(t^6))}(1+O(1/n)), \tag{6.17}$$

for $|t| < 1$ and large n. Therefore, $\log E\{e^{jts_C^R(\theta)}\}$ is an even positive function of t for large n and $|t| < 1$. Now using the Taylor expansion of $\log E\{e^{jts_C^R(\theta)}\}$ as in Appendix 6.5.3, we can write α_1 in (6.17) as

$$n\alpha_1 = \frac{1}{2}E\left\{\left(s_C^R(\theta)\right)^2\right\} = \frac{1}{2}E\{na_1^2\} = E\{nr_1^2\cos^2\phi_1\} = \frac{n}{4}. \qquad (6.18)$$

Therefore, the characteristic function of $s_C^R(\theta)$ can be written as

$$E\{e^{jts_C^R(\theta)}\} = e^{-nt^2/4+n\alpha t^4+O(nt^6)}(1+O(1/n)) \qquad (6.19)$$

for large n and $|t| < 1$. This in fact allows us to generalize the lower bound for random spherical codes in the following theorem.

Theorem 6.4 *[Lower Bound: Spherical Codes] Let $s_C(\theta)$ be as in (6.2) where C is chosen uniformly from Ω_n. Also, let $\mathcal{C}_s$ be the ensemble of all the admissible codewords. Then,*

$$\Pr\{\text{PMEPR}_{\mathcal{C}_s}(C) \le \log n - 6.5\log\log n\} \le O\left(\frac{1}{\log^4 n}\right). \qquad (6.20)$$

Proof: Using Lemma 6.3 in which the characteristic function of $s_C^R(\theta)$ is computed for $|t| < 1$, and using the identity $e^{-a} = e^{-b} + O(|b-a|)$, we may write $E\{e^{jts_C^R(t)}\} = e^{-nt^2/4} + O(nt^4)$, where C is uniformly distributed over Ω_n. We can now follow the same line as the proof of Theorem 6.1 to write the mean of η as in Eq. (6.54). Similarly, in order to calculate the second moment of η, we have to compute

$$\Phi_s(t,\tau;\theta_m,\theta_l) = E\{e^{j\sum_{k=1}^n a_k(t\cos k\theta_m+\tau\cos k\theta_l)-b_k(t\sin k\theta_m+\tau\sin k\theta_l)}.\} \qquad (6.21)$$

Since $\sum_{k=1}^n (t\cos k\theta_m + \tau\cos k\theta_l)^2 + (t\sin k\theta_m + \tau\sin k\theta_l)^2 = n(t^2+\tau^2)$ for $m \ne l$

and $m+l \neq n$, by using a similar argument as in the proof of Lemma 6.3, we get

$$\Phi_s(t,\tau;\theta_m,\theta_l) = E\{e^{j\sqrt{n(t^2+\tau^2)}a_1}\}, \tag{6.22}$$

where a_1 is as defined in Lemma 6.3. Consequently, it can be then concluded that for $m \neq l$ and $m+l \neq n$, we have

$$\Phi_s(t,\tau;\theta_m,\theta_l) = e^{-n(t^2+\tau^2)/4+n\alpha(t^2+\tau^2)^2+nO((t^2+\tau^2)^3)}(1+O(1/n)). \tag{6.23}$$

We can similarly prove that $E\{\eta\}$ and $E\{\eta^2\}$ are as in Eqs. (6.54) and (6.66), respectively. So using Chebychev's inequality as in Theorem 6.1, we can complete the proof for random spherical codes. □

6.3.2 Upper Bound for the PMEPR

In this subsection, Theorem 6.5 obtains the probability of having PMEPR slightly greater than $\log n$ for the QAM case and shows that this probability goes to zero as n tends to infinity. This will be extended to PSK and spherical codes in Theorem 6.6 and 6.7, respectively.

Theorem 6.5 *[Upper Bound: QAM Case] Consider the setting of Theorem 6.1. Then,*

$$\Pr\left\{\mathrm{PMEPR}_{\mathcal{C}_q^{QAM}}(C) \geq \log n + 5.5\log\log n\right\} \leq O\left(\frac{1}{\log^4 n}\right). \tag{6.24}$$

Proof: We first define the real function $s_C(\gamma,\theta)$ as

$$s_C(\gamma,\theta) = \mathrm{Re}\{e^{j\gamma}s_C(\theta)\} = \sum_{i=1}^{n} a_i\cos(i\theta+\gamma) - b_i\sin(i\theta+\gamma), \tag{6.25}$$

consequently we define K as $K = \max_{\gamma,\theta} |s_C(\gamma,\theta)| = |s_C(\gamma_0,\theta_0)| = \max_{\theta} |s_C(\theta)|$.

As mentioned in [74] and used in [75], the point of introducing γ is that we can now deal with the maximum of a real function $s_C(\gamma,\theta)$ and generalize the result of Halasz to complex polynomials over the unit circle. Let $u(x)$ be as defined in (6.5) with the only difference that here $M = \sqrt{nE_{av}\log n + 5.5nE_{av}\log\log n}$.We also assume E_{av} is the normalized average energy of the constellation. Consequently, we define the random variable η as

$$\eta = \int_0^{2\pi}\int_0^{2\pi} u(s_C(\gamma,\theta))d\theta d\gamma = \int_0^{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} e^{jtu(s_C(\gamma,\theta))}v(t)dtd\theta d\gamma,$$

where we substitute $u(x)$ by its Fourier transform $v(t)$. As the second step, by using the Taylor expansion of $s_C(\gamma,\theta)$ around its maximum absolute value, it is shown in [75, 74] that if $\eta \le \frac{1}{n\log^2 n}$ then $K \le M + 2\Delta$. Therefore,

$$\Pr\{K \le M + 2\Delta\} \ge \Pr\{\eta \le \frac{1}{n\log^2 n}\} \ge 1 - n\log^2 nE\{\eta\}, \tag{6.27}$$

where we used Markov's inequality to deduce (6.27). Therefore the derivation of the upper bound boils down to computation of the mean of η. As in the derivation of the lower bound, we start with computing the characteristic function of $s_C(\gamma,\theta)$,

$$\begin{aligned} E\{e^{jt\sum_{i=1}^n a_i\cos(i\theta+\gamma)-b_i\sin(i\theta+\gamma)}\} &= e^{-\frac{t^2}{2}E_{av1}\sum_{i=1}^n\cos^2(i\theta+\gamma)-\frac{t^2}{2}E_{av2}\sum_{i=1}^n\sin^2(i\theta+\gamma)} \\ &+ t^4\left\{\alpha_1\sum_{i=1}^n\cos^4(i\theta+\gamma)+\alpha_2\sum_{i=1}^n\sin^4(i\theta+\gamma)\right\}\times \\ &\quad e^{-\frac{t^2}{2}E_{av1}\sum_{i=1}^n\cos^2(i\theta+\gamma)-\frac{t^2}{2}E_{av2}\sum_{i=1}^n\sin^2(i\theta+\gamma)} \\ &+ O(nt^6+n^2t^8) \end{aligned} \tag{6.28}$$

for $|t| \le 1$ where we used $e^{-a} = e^{-b} + (b-a)e^{-b} + O((b-a)^2)$ as in Eq. (6.59) and E_{av1} and E_{av2} are as defined in Theorem 6.5. We can now take the expectation of η

as

$$\begin{aligned} E\{\eta\} &= \int_0^{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} E\{e^{jt\sum_{i=1}^n a_i\cos^2(i\theta+\gamma)-b_i\sin^2(i\theta+\gamma)}\}v(t)dtd\theta d\gamma \\ &= \int_0^{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}E_{av1}\sum_{i=1}^n\cos^2(i\theta+\gamma)-\frac{t^2}{2}E_{av2}\sum_{i=1}^n\sin^2(i\theta+\gamma)}v(t)dtd\gamma d\theta \\ &+ O\left(n\int_0^{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} t^4 e^{-\frac{t^2}{2}E_{av1}\sum_{i=1}^n\cos^2(i\theta+\gamma)-\frac{t^2}{2}E_{av2}\sum_{i=1}^n\sin^2(i\theta+\gamma)}|v(t)|dtd\gamma d\theta\right) \\ &+ O\left(\int_{-\infty}^{\infty}(nt^6+n^2t^8)|v(t)|dt\right). \end{aligned} \tag{6.29}$$

Using the results in Appendix 6.5.1, we can simplify the expectation of η as follows: the first and second terms follow from (6.43) and $p = 0, 4$, and the last term can be computed by (6.40), to get

$$E\{\eta\} = O\left(\frac{\sqrt{n}e^{-\frac{M^2}{nE_{av}}}}{M}\right) + O\left(\frac{n\sqrt{n}e^{-\frac{M^2}{nE_{av}}}}{M\Delta^4}\right) + O\left(\frac{n}{\Delta^6}+\frac{n^2}{\Delta^8}\right). \tag{6.30}$$

Therefore, by setting the value of M and Δ, and using the Markov inequality, we conclude that $E\{\eta\} = O\left(\frac{1}{n\log^6 n}\right)$ to get

$$\Pr\{K \le M+2\Delta\} \ge \Pr\left\{\eta \le \frac{1}{n\log^2 n}\right\} \ge 1-\log^2 nO\left(\frac{1}{n\log^6 n}\right) = 1-O\left(\frac{1}{\log^4 n}\right). \tag{6.31}$$

The theorem follows by using the definition of PMEPR for large values of n.□

The next theorem presents the same asymptotic result for the PSK constellations.

Theorem 6.6 *[Upper Bound: PSK Case] Consider the setting of Theorem 6.2. Then,*

$$\Pr\left\{\mathrm{PMEPR}_{\mathcal{C}_q^{PSK}}(C) \ge \log n + 5.5\log\log n\right\} \le O\left(\frac{1}{\log^4 n}\right). \tag{6.32}$$

Proof: We first compute the characteristic function of $s_C(\gamma,\theta)=\sum_{i=1}^n \cos(i\theta+\beta_i+\gamma)$,

$$E\{e^{jts_C(\gamma,\theta)}\}=\prod_{i=1}^n E\{\cos(t\cos(i\theta+\beta_i+\gamma))\}=\prod_{i=1}^n E\{\cos(t\beta_i^{''})\}, \tag{6.33}$$

where $\beta_i^{''}=\cos(i\theta+\beta_i+\gamma)$ has an even distribution since β_i is chosen from a symmetric PSK constellation. Since the distribution of c_i is invariant under rotation by $\pi/2$, using the result in Appendix 6.5.2, and following the same line as in the proof of Theorem 6.2, we can write the characteristic function as $E\{e^{jts_C(\gamma,\theta)}\}=e^{-nt^2/4+n\alpha t^4+O(nt^6)}$ for $|t|<1$. The theorem follows by using the characteristic function of $s_C^R(\theta)$ and following the same line as in the proof of Theorem 6.5. □

Theorem 6.7 *[Upper Bound: Spherical Codes] Let C be a codeword uniformly chosen from Ω_n and $\mathcal{C}_s$ be the ensemble of all those codewords C. Then*

$$\Pr\{\text{PMEPR}_{\mathcal{C}_s}(C)\geq \log n+5.5\log\log n\}\leq O\left(\frac{1}{\log^4 n}\right). \tag{6.34}$$

Proof: First of all, we derive the characteristic function of $s_C(\gamma,\theta)$ when the codeword C is uniformly distributed over Ω_n. We can use the result of Lemma 6.3 to show that

$$\begin{aligned}
E\{e^{jts_C(\gamma,\theta)}\} &= e^{-nt^2/4+\alpha nt^4+O(nt^6)}(1+O(1/n)) \\
&= e^{-nt^2/4}+\alpha(nt^4+O(nt^6))e^{-nt^2/4}+O(n^2t^8) \\
&= e^{-nt^2/4}+\alpha nt^4e^{-nt^2/4}+O(nt^6+n^2t^8),
\end{aligned} \tag{6.35}$$

where we used $e^{-a}=e^{-b}+(b-a)e^{-b}+O((b-a)^2)$ for $a,b>0$. Fortunately the characteristic function of $s_C(\gamma,\theta)$ allows us to use a similar approach as in Theorem

6.5 to evaluate the mean of η as

$$\begin{aligned} E\{\eta\} &= \int_0^{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} E\{e^{jts_C(\gamma,\theta)}\}v(t)dtd\gamma d\theta \\ &= \int_0^{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} e^{-nt^2/4}v(t)dtd\gamma d\theta + n\int_0^{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} t^4 e^{-nt^2/4}v(t)dtd\gamma d\theta \\ &+ O\left(\int_{-\infty}^{\infty}(nt^6+n^2t^8)|v(t)|dt\right). \end{aligned}$$

Using the result of Appendix 6.5.1 and similar to Theorem 6.5, we can simplify (6.36) to get $E\{\eta\} = O\left(\frac{1}{n\log^6 n}\right)$. The theorem follows using a similar argument as in Theorem 6.5 and setting the value of M and Δ. □

6.4 Summary and Discussion

To get a better insight into the above results, let $\mathcal{C}_t$ correspond to $\mathcal{C}_q^{PSK}$, $\mathcal{C}_q^{QAM}$, or $\mathcal{C}_s$ as random codes over the corresponding constellations. Using the inequality $\Pr(A) + \Pr(B) - 1 \leq \Pr(A \cap B) \leq \Pr(A)$, and Theorems 6.1 to 6.7, we may write

$$1 > \Pr\{\log n + 5.5\log\log n > \mathrm{PMEPR}_{\mathcal{C}_t}(C) > \log n - 6.5\log\log n\} > 1 - O\left(\frac{1}{\log^4 n}\right). \tag{6.36}$$

Eq. (6.36) shows that with probability approaching unity the PMEPR of any codewords randomly chosen from symmetric QAM/PSK or Ω_n behaves like $\log n + O(\log\log n)$ asymptotically. This result implies that for large number of subcarriers, clipping the signal with a threshold value less than $\log n$ may cause severe distortion in the signal. On the other hand, by using probabilistic methods [36] in which we randomly map the data to different codewords and choose the best one in terms of PMEPR to transmit, we cannot further reduce the PMEPR below $\log n$. Meanwhile these methods performs very well for moderate values of n since $\log n$ is reasonably small.

Another class of methods to reduce the PMEPR is to use coding not only to introduce a large minimum distance but also to reduce PMEPR [37]. It has been shown in [37] that the Varsharmov-Gilbert (VG) bound for spherical codes with PMEPR less than $8\log n$ remains the same as that of spherical codes without PMEPR restriction. This shows that there exist high rate and large minimum distance spherical codes with PMEPR of $8\log n$.

In fact, we can use the result of Section 6.3 to derive a VG-type bound on the rate of code with given minimum distance and PMEPR of less than $\log n$. Here, we use the minimum Hamming distance, which is defined as the minimum number of coordinates in which any two codewords are different [42]. The rate of $\mathcal{C}$, is also defined as $R = \frac{1}{n}\log_q|\mathcal{C}|$ where $|\mathcal{C}|$ is the cardinality of the set $\mathcal{C}$.

Corollary 6.8 *Let $\mathcal{Q}_q$ be a complex q-ary symmetric PSK or QAM constellation, $R > 0$, and $0 \leq \delta \leq \frac{q-1}{q}$. If*

$$R \leq 1 - H_q(\delta) - O(\frac{1}{n\log^4 n}), \tag{6.37}$$

then, asymptotically, there exists a code $\mathcal{C}$ of length n, with entries chosen from $\mathcal{Q}_q$, rate R, minimum Hamming distance $d_{min} = \lfloor \delta n \rfloor$, and $\mathrm{PMEPR}_{\mathcal{C}} < \log n + 5.5\log\log n$.

Proof: The proof follows by first excluding codewords with PMEPR larger than $\log n + 5.5\log\log n$, and then using a VG-type argument to construct a code with the minimum Hamming distance $d_{min} = \lfloor \delta n \rfloor$ [42]. $\square$

According to Corollary 6.8, it follows that not only do there exist spherical high rate codes with PMEPR of $8\log n$, but there also exist codes chosen from usual constellations like QAM and PSK with the same asymptotic. On the other hand, this result does not contradict the existence of exponentially many codewords with constant PMEPR. However the ratio of the number of these codewords to q^n has to tend to zero asymptotically. So there still remains an open problem of what is the rate of codes with constant PMEPR?

6.5 Appendices

6.5.1 Properties of $u(x)$

We adopt the following lemma from [74] with modifications to the fifth and sixth inequalities that are required for the generalization to polynomials over the unit circle with complex coefficients.

Lemma 6.9 *Let $u(x)$ be a continuous differentiable function as defined in the proof of Theorem 6.1 and $v(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x)e^{jtx}dx$. Then we have the following properties,*

$$i)|t^r v(t)| = O\left(\frac{1}{\Delta^{r-1}}\right) \qquad 1 \le r \le 10 \tag{6.38}$$

$$ii)\int_{-\infty}^{\infty} |v(t)|dt = O\left(\frac{M}{\Delta}\right) \tag{6.39}$$

$$iii)\int_{-\infty}^{\infty} |t^p v(t)|dt = O\left(\frac{1}{\Delta^p}\right) \qquad 1 \le p \le 8 \tag{6.40}$$

$$iv)\int_{|t|>l_0} |v(t)|dt = O(1/\Delta^9) \qquad \text{for any constant } \; l_0 > 0 \tag{6.41}$$

$$v)\left|\int_{-\infty}^{\infty} e^{-nE_{av}t^2/4} t^p v(t)dt\right| = O\left(\frac{\sqrt{n}e^{-\frac{M^2}{nE_{av}}}}{M\Delta^p}\right) \qquad 1 \le p \le 8 \tag{6.42}$$

$$vi)\int_0^{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}\left(E_{av1}\sum_{i=1}^n \cos^2(i\theta+\gamma)+E_{av2}\sum_{i=1}^n \sin^2(i\theta+\gamma)\right)} t^p v(t)dtd\gamma d\theta$$

$$= O\left(\frac{\sqrt{n}e^{-\frac{M^2}{nE_{av}}}}{M\Delta^p}\right), \tag{6.43}$$

where $M = \sqrt{nE_{av}\log n + O(\log\log n)}$, $\Delta = \sqrt{\frac{n}{\log n}}$, and $E_{av} = E_{av1} + E_{av2}$.

Proof: For the proof of (6.38) to (6.41) refer to [74]. In order to prove (6.42), we use Parseval's theorem and the properties of Fourier transform to obtain

$$\left|\int_{-\infty}^{\infty} e^{-nt^2/4} t^p v(t) dt\right| = \left|\frac{1}{\sqrt{\pi n}} \int_{-\infty}^{\infty} e^{-x^2/n} u^{(p)}(x) dt\right| \tag{6.44}$$

Now we can use the fact that $u^{(p)}(x)$ is zero for $|x| < M$ and equals to $O(1/\Delta^r)$ for $|x| > M$ to rewrite the integral as,

$$O\left(\left|\int_{-\infty}^{\infty} e^{-nt^2/4} t^p v(t) dt\right|\right) = O\left(\frac{1}{\sqrt{n}\Delta^p} \int_{|x|>M} e^{-x^2/n} dx\right) = O\left(\frac{Q\left(\frac{M}{\sqrt{n/2}}\right)}{\Delta^p}\right) \tag{6.45}$$

where $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-x^2/2} dx$. Using the asymptotic expansion $Q(x) = \frac{e^{-x^2/2}}{x\sqrt{2\pi}}(1 - O(1/x^2))$ [63], we get,

$$O\left(\frac{Q\left(\frac{M}{\sqrt{n/2}}\right)}{\Delta^p}\right) = O\left(\frac{\sqrt{n} e^{-\frac{M^2}{nE_{av}}}}{M\Delta^p}\right) \tag{6.46}$$

(6.42) follows from (6.45) and (6.46). To prove (6.43), we first use (6.42) to write the inner integral in (6.43) as,

$$\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}(E_{av1}\sum_{i=1}^{n}\cos^2(i\theta+\gamma)+E_{av2}\sum_{i=1}^{n}\sin^2(i\theta+\gamma))} t^p v(t) dt$$
$$= O\left(\frac{\sqrt{n}}{M\Delta^p} e^{-\frac{M^2}{2(E_{av1}\sum_{i=1}^{n}\cos^2(i\theta+\gamma)+E_{av2}\sum_{i=1}^{n}\sin^2(i\theta+\gamma))}}\right) \tag{6.47}$$

We then use the following inequality for large values of n similar to [74],

$$\begin{aligned}\sum_{i=1}^{n} E_{av1}\cos^2(i\theta+\gamma) \quad &+ \quad E_{av2}\sin^2(i\theta+\gamma) \\ &= \frac{(E_{av1}+E_{av2})n}{2} + \frac{(E_{av1}-E_{av2})}{2}\sum_{i=1}^{n}\cos 2(i\theta+\gamma) \\ &\leq \begin{cases} \frac{E_{av}n}{2} + \frac{n|E_{av1}-E_{av2}|}{2\log n} & \frac{\log n}{n} \leq |\theta| \leq \pi/2, \\ nE_{av} & \text{everywhere} \end{cases}\end{aligned}$$

where we used the inequality $\sum_{i=1}^{n}\cos 2(i\theta+\alpha) \leq \frac{1}{2|\sin\theta|}$ and considering that $|\sin\theta| > \theta/2$ for $\theta < 0.1$, the inequality follows for large n. Therefore we may write,

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi} & e^{-\frac{M^2}{2(E_{av1}\sum_{i=1}^{n}\cos^2(i\theta+\gamma)+E_{av2}\sum_{i=1}^{n}\sin^2(i\theta+\gamma))}}d\theta d\gamma \\ &\leq \frac{4\pi\log n}{n}e^{-\frac{M^2}{2nE_{av}}} + 4\pi^2 e^{-\frac{M^2}{nE_{av}+\frac{n|E_{av1}-E_{av2}|}{\log n}}}.\end{aligned}$$

We can now use the fact that $\frac{\log n}{n} \leq e^{-\frac{M^2}{2nE_{av}}}$ for $M = \sqrt{nE_{av}\log n + O(\log\log n)}$ and also using,

$$e^{-\frac{M^2}{nE_{av}+\frac{n|E_{av1}-E_{av2}|}{\log n}}} = e^{-M^2/nE_{av}} \times e^{\frac{M^2}{E_{av}n\log n+|\alpha_1-\alpha_2|n}} = O(e^{-M^2/nE_{av}}) \qquad (6.48)$$

to bound (6.48) by $O(e^{-\frac{M^2}{nE_{av}}})$. Eq. (6.43) follows from (6.47) and (6.48). $\square$

6.5.2 Mean and Variance of η

In order to calculate the mean and variance of η, we substitute $s_C^R(\theta)$ in (6.6) to get

$$\eta = \sum_{m=1}^{n}\int_{-\infty}^{\infty} e^{jt\sum_{k=1}^{n} a_k\cos k\theta_m - b_k\sin k\theta_m} v(t)dt. \qquad (6.49)$$

Using the independence of a_k's and b_k's, we obtain,

$$E\{e^{jt\sum_{k=1}^{n} a_k \cos k\theta_m - b_k \sin k\theta_m}\} = \prod_{k=1}^{n} E\{e^{jta_k \cos k\theta_m}\}E\{e^{jtb_k \sin k\theta_m}\} \triangleq \Phi_{QAM}(t;\theta_m). \tag{6.50}$$

It is shown in Appendix 6.5.3 that for $|t| < 1$, $E\{e^{ja_k t}\} = e^{-E_{av1}t^2/2-\alpha_1 t^4+O(t^6)}$ and similarly $E\{e^{jb_k t}\} = e^{-E_{av2}t^2/2-\alpha_2 t^4+O(t^6)}$ where E_{av1} and E_{av2} are the average energy of a_k and b_k, and therefore, $E_{av} = E_{av1} + E_{av2}$ is the average energy of c_k. Now using $e^{-a} = e^{-b} + O(|b-a|)$ for $a, b \geq 0$, we can write Eq. (6.50) as,[5]

$$\Phi_{QAM}(t;\theta_m) = e^{-\frac{t^2}{2}(\sum_{k=1}^{n} E_{av1}\cos^2 k\theta_m + E_{av2}\sin^2 k\theta_m) + nO(t^4)} = e^{-nE_{av}t^2/4} + O(nt^4) \tag{6.51}$$

for $|t| < 1$, where we used the identities: $\sum_{k=1}^{n} \cos^2 k\theta_m = \sum_{k=1}^{n} \sin^2 k\theta_m = n/2$ for $\theta_m = \pi(2m+1)/n$. To evaluate the expectation of η in (6.49), we replace (6.51) in (6.49) for $|t| < 1$, and use one as the upper bound for the absolute value of the characteristic function for $|t| > 1$ to get

$$E\{\eta\} = n\int_{-1}^{1} e^{-nE_{av}t^2/4}v(t)dt + O\left(n^2\int_{-1}^{1} t^4|v(t)|dt\right) + O\left(n\int_{|t|>1} |v(t)|dt\right). \tag{6.52}$$

We may then extend the first integral to infinity and include the resulting error in the third term, also by extending the second integral to infinity the third term can be included in the second integral. Finally, Eq. (6.52) simplifies to

$$E\{\eta\} = n\int_{-\infty}^{\infty} e^{-nE_{av}t^2/4}v(t)dt + O\left(n^2\int_{-\infty}^{\infty} t^4|v(t)|dt\right). \tag{6.53}$$

Using the property (6.40) of $u(x)$ shown in appendix 6.5.1, we may substitute the

[5]Note that since the characteristic function is less than 1 as shown in Appendix 6.5.3, a has to be non-negative.

second term by $O\left(\frac{n^2}{\Delta^4}\right)$ and using $\Delta = \sqrt{\frac{n}{\log n}}$, we get

$$E\{\eta\} = n \int_{-\infty}^{\infty} e^{-nE_{av}t^2/4} v(t)dt + O\left(\log^2 n\right). \tag{6.54}$$

In order to find the second moment of η, we may write η^2 as

$$\eta^2 = \sum_{m=1}^{n} \sum_{l=1}^{n} u(s_C^R(\theta_m))u(s_C^R(\theta_l)). \tag{6.55}$$

Therefore, after substituting the Fourier transform of $u(x)$ in (6.55), to evaluate each term of the double summation of (6.55), we should compute

$$\begin{aligned} & u(s_C^R(\theta_m))u(s_C^R(\theta_l)) = \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} & E\left\{e^{j\sum_{k=1}^{n} a_k(t\cos k\theta_m + \tau\cos k\theta_l) - b_k(t\sin k\theta_m + \tau \sin k\theta_l)}\right\} v(t)v(\tau)dtd\tau. \end{aligned} \tag{6.56}$$

The inner expectation in (6.56) can be split using the independence of a_k's and b_k's as,

$$\begin{aligned} & E\{e^{j\sum_{k=1}^{n} a_k(t\cos k\theta_m + \tau\cos k\theta_l) - b_k(t\sin k\theta_m + \tau\sin k\theta_l)}\} \\ & = \prod_{k=1}^{n} E\{e^{ja_k(t\cos k\theta_m + \tau\cos k\theta_l)}\} \times \prod_{k=1}^{n} E\{e^{-jb_k(t\sin k\theta_m + \tau\sin k\theta_l)}\} \\ & \triangleq \Phi'_{QAM}(t, \tau; \theta_m, \theta_l) \end{aligned} \tag{6.57}$$

As we stated for the calculation of $E\{\eta\}$, for $|t| \leq 1$, we have

$$E\{e^{ja_k t}\} = e^{-E_{av1}t^2/2 - \alpha_1 t^4 + O(t^6)}$$

and $E\{e^{jb_k t}\} = e^{-E_{av2}t^2/2 - \alpha_2 t^4 + O(t^6)}$. Therefore, for $|t|, |\tau| < 1/2$, each expectation in

(6.57) can be written as

$$E\{e^{ja_k(t\cos k\theta_m+\tau\cos k\theta_l)}\} =$$
$$e^{-1/2\sum_{k=1}^{n}\left\{E_{av1}(t\cos k\theta_m+\tau\cos k\theta_l)^2+\alpha_1(t\cos k\theta_m+\tau\cos k\theta_l)^4\right\}+O(n(|t|+|\tau|)^6)}. \tag{6.58}$$

where we used $O((t\cos k\theta_m + \tau\cos k\theta_l)^6) = O((|t| + |\tau|)^6)$ for the last term in the exponent. We can also write a similar equation for b_k. After substituting (6.58) in (6.57), we can use the second-order approximation $e^{-a} = e^{-b} + (b-a)e^{-b} + O((b-a)^2)$ for $a, b > 0$, to write Eq. (6.57) as

$$\begin{aligned}
\Phi'_{QAM}(t,\tau;\theta_m,\theta_l) &= e^{-\frac{1}{2}\sum_{k=1}^{n}E_{av1}(t\cos k\theta_m+\tau\cos k\theta_l)^2+E_{av2}(t\sin k\theta_m+\tau\sin k\theta_l)^2} \\
&+ \sum_{k=1}^{n}\Big\{ \alpha_1(t\cos k\theta_m+\tau\cos k\theta_l)^4+\alpha_2(t\sin k\theta_m+\tau\sin k\theta_l)^4 \\
&\qquad + O(n(|t|+|\tau|)^6)\Big\} \\
&\qquad \times e^{-\frac{1}{2}\sum_{k=1}^{n}E_{av1}(t\cos k\theta_m+\tau\cos k\theta_l)^2+E_{av2}(t\sin k\theta_m+\tau\sin k\theta_l)^2} \\
&+ O(n^2(|t|+|\tau|)^8)
\end{aligned} \tag{6.59}$$

for $|t|, |\tau| \leq 1/2$. We can further simplify (6.59) by using the identities,

$$\sum_{k=1}^{n}(t\cos k\theta_m+\tau\cos k\theta_l)^2 = \sum_{k=1}^{n}(t\sin k\theta_m+\tau\sin k\theta_l)^2 = n(t^2+\tau^2)/2$$

for $m \neq l$ and $m + l \neq n$, and $E_{av} = E_{av1} + E_{av2}$, to get

$$\begin{aligned}
\Phi'_{QAM}(t,\tau;\theta_m,\theta_l) &= e^{-\frac{1}{2}nE_{av}(t^2+\tau^2)} + O(n(|t|+|\tau|)^4)e^{-\frac{1}{2}nE_{av}(t^2+\tau^2)} \\
&\quad + O(n(|t|+|\tau|)^6) + O(n^2(|t|+|\tau|)^8)
\end{aligned} \tag{6.60}$$

for $|t|, |\tau| < 1/2$, $m \neq l$, and $m + l \neq n$. For the other $2n$ terms (i.e. $m = l$ or

$m + l = n$) in (6.49), we can use the following inequality

$$2\sum_{m=1}^{n} u(s_C^R(\theta_m))u(s_C^R(\theta_l)) \leq 2\sum_{m=1}^{n} u(s_C^R(\theta_m)) = 2\eta \qquad (6.61)$$

since $0 \leq u(x) \leq 1$. Now replacing (6.60) into (6.56) and then into (6.55) for $|t|, |\tau| < 1/2$, using one as an upper bound for $|t|, |\tau| > 1/2$, and using (6.61) for $2n$ terms with $m = l$ or $m + l = n$, we obtain

$$\begin{aligned} E\{\eta^2\} \;&\leq (n^2 - 2n)\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} e^{-nE_{av}(t^2+\tau^2)/4} v(t)v(\tau)dtd\tau \\ &+ O\left(n^3 \sum_{k=1}^{n}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} (|t| + |\tau|)^4 e^{-n(t^2+\tau^2)/4} v(t)v(\tau)dtd\tau\right) \\ &+ O\left(n^3 \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} (|t| + |\tau|)^6 |v(t)||v(\tau)|dtd\tau\right) \\ &+ \;\; O\left(n^4 \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} (|t| + |\tau|)^8 |v(t)||v(\tau)|dtd\tau\right) \\ &+ O\left(n^2 \int_{|t|>1/2}\int_{|\tau|>1/2} |v(t)||v(\tau)|dtd\tau\right) + 2E\{\eta\}. \end{aligned} \qquad (6.62)$$

To evaluate (6.62), we may extend the integrals in the first four terms from $-\infty$ to ∞ to find an upper bound for $E\{\eta^2\}$. So we may write Eq. (6.62) as

$$\begin{aligned} E\{\eta^2\} \;&\leq (n^2 - 2n)\left(\int_{-\infty}^{\infty} e^{-nE_{av}(t^2+\tau^2)/4} v(t)v(\tau)dtd\tau\right)^2 \\ &+ O\left(n^3 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (|t| + |\tau|)^4 e^{-nE_{av}(t^2+\tau^2)/4} |v(t)||v(\tau)|dtd\tau\right) \\ &+ O\left(n^3 \int_{-\infty}^{\infty}\int_{-1/2}^{1/2} (|t| + |\tau|)^6 |v(t)||v(\tau)|dtd\tau\right) \\ &+ \;\; O\left(n^4 \int_{-\infty}^{\infty}\int_{-1/2}^{1/2} (|t| + |\tau|)^8 |v(t)||v(\tau)|dtd\tau\right) \\ &+ O\left(n \int_{|t|>1/2} |v(t)|dt\right)^2 + 2E\{\eta\}. \end{aligned} \qquad (6.63)$$

Now we can use (6.41) in the Appendix 6.5.1 to write the fourth term in (6.63) as

$O\left(\frac{n^2}{\Delta^{18}}\right)$. The second term in (6.63) will be also simplified to

$$\begin{aligned}
& O\left(n^3 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (|t|+|\tau|)^4 e^{-nE_{av}(t^2+\tau^2)/4} |v(t)||v(\tau)| dt d\tau\right) \\
= \; & O\left(n^3 \sum_{p=0}^{4} \int_{-\infty}^{\infty} t^p e^{-nE_{av}t^2/4} dt \int_{-\infty}^{\infty} \tau^{4-p} e^{-nE_{av}\tau^2/4} d\tau\right) \\
= \; & O\left(\frac{n^4 e^{-2M^2/nE_{av}}}{\Delta^4 M^2}\right),
\end{aligned} \tag{6.64}$$

where we used the identities (6.42) with $p = k$ and $p = 4-k$. The third term similarly can be evaluated as

$$\begin{aligned}
& O\left(n^3 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (|t|+|\tau|)^6 |v(t)||v(\tau)| dt d\tau\right) = \\
& = O\left(n^3 \left\{\sum_{k=1}^{5} \int_{-\infty}^{\infty} |t^k v(t)| dt \int_{-\infty}^{\infty} |t^{6-k} v(t)| dt + 2\int_{-\infty}^{\infty} |v(\tau)| d\tau \int_{-\infty}^{\infty} |t^6 v(t)| dt\right\}\right) \\
& = O\left(\frac{n^3}{\Delta^6}\right) + O\left(n^3 \times \frac{1}{\Delta^6} \times \frac{M}{\Delta}\right) = O\left(\frac{n^3}{\Delta^6}\right) + O\left(\frac{n^3 M}{\Delta^7}\right),
\end{aligned} \tag{6.65}$$

where we again used (6.39) and (6.40) to evaluate both terms in (6.65). Along the same line as the evaluation of the third term, the fourth term can be also shown to be $O\left(\frac{n^4 M}{\Delta^9}\right)$. Therefore, setting the value of M and Δ, we may write

$$\begin{aligned}
E\{\eta^2\} \;\leq\; & 2E\{\eta\} + n^2 \left(\int_{-\infty}^{\infty} e^{-nE_{av}t^2/4} v(t) dt\right)^2 + O\left(\frac{n^4 e^{-2M^2/nE_{av}}}{\Delta^4 M^2}\right) \\
& + O\left(\frac{n^3 M}{\Delta^7}\right) + O\left(\frac{n^4 M}{\Delta^9}\right) \\
= \; & 2E\{\eta\} + n^2 \left(\int_{-\infty}^{\infty} e^{-nE_{av}t^2/4} v(t) dt\right)^2 + O(\log^5 n) + O(\log n) + O(\log^2 n).
\end{aligned}$$

On the other hand, it is easy to find a lower bound for $E\{\eta\}$ by using the definition of $u(x)$ and Parseval's theorem and show that $E\{\eta\} \geq O(\log^6 n)$ [74]. Equivalently, this implies that

$$\sigma_\eta^2 = E\{\eta^2\} - E^2\{\eta\} = O\left(E\{\eta\}\log^2 n + \log^5 n\right). \tag{6.66}$$

6.5.3 Properties of $\Phi_c(x)$

In this appendix we calculate the characteristic function of a bounded random variable with an even probability distribution function.

Lemma 6.10 *Let c be a real random variable with even PDF (probability density function), variance E_{av}, and maximum energy 1, i.e., $|c|^2 \leq 1$. Then for $|t| < 1$, we have*

$$\log \Phi_c(t) = -E_{av}t^2/2 + a_4 t^4 + O(t^6) \tag{6.67}$$

where $\Phi_c(t) = E\{e^{jtc}\}$.

Proof: It is clear that when the PDF of c is even then the characteristic function is real, so $\Phi_c(t) = E\{\cos(tc)\}$. Since the PDF of c is non-negative and the maximum energy of c is one, $\Phi_c(t)$ is a real positive function. $\Phi_c(t)$ is also inifnitely differentiable for $|t| < 1$ since $E\{c^k \cos(tc)\}$ is bounded for any k. Now we can write the Taylor expansion $\log \Phi_c(t) = \sum_{i=0}^{\infty} a_i t^i$. Since $\Phi_c(0) = 1$, a_0 will be zero. Furthermore, since the PDF of c is even, $\Phi_c(t)$ and $\log \Phi_c(t)$ will be even and therefore $a_{2k+1} = 0$, for $k = 0, 1, \ldots$. The values of a_2 can be computed as $a_2 = \frac{-1}{2}\frac{\Phi''(0)}{\Phi(0)} = -\frac{1}{2}E\{c^2\}$. Therefore, for $|t| < 1$, we can write the Taylor expansion $\log(\Phi(t)) = -E\{c^2\}t^2/2 + a_4 t^4 + O(t^6)$.

6.5.4 Distribution of c_1

Lemma 6.11 *Let $C = (c_1, \ldots, c_n)$ be a random complex vector uniformly distributed over Ω_n. Let $c_i = r_i e^{j\phi_i}$. Then r_1 and ϕ_1 are independent with the following distribu-*

tion,

$$p(r_1) = \frac{2}{n^{n-1}} r_1 (n - r_1^2)^{n-1}, \quad (6.68)$$

$$p(\phi_1) = \frac{1}{2\pi}. \quad (6.69)$$

Proof : Since the vector C is uniformly distributed over Ω_n, $p(C) = \frac{1}{V}\delta(CC^* - n)$ where $V = \frac{\pi^n n^n}{\Gamma(n)}$. Now let's define $C' = (c_2, \ldots, c_n)$. Then, we may write $p(c_1)$ as

$$p(c_1) = \frac{1}{V}\int \delta(CC^* - n)dC' = \frac{1}{2\pi V}\int\int e^{j\omega(CC^* - n)}dC'd\omega, \quad (6.70)$$

where we used the definition of $\delta(x)$. In order to make the integral converging, we multiply the integral by $1 = e^{n\beta - CC^*\beta}$ in which $\beta > 0$. Therefore,

$$p(c_1) = \frac{1}{2\pi V}\int\int e^{n(\beta - j\omega)} e^{-r_1^2(\beta - j\omega)} e^{-C'C'^*(\beta - j\omega)} dC'd\omega \quad (6.71)$$

It is shown in [20] that if P and Q are Hermition $M \times M$ matrices and $P > 0$, $\int e^{-y^*(P+jQ)y}dy = \frac{\pi^M}{\det(P+jQ)}$. So setting $z = \beta - j\omega$, we get

$$p(c_1) = \frac{\pi^{n-1}}{2\pi V j}\int \frac{e^{(n-r_1^2)z}}{z^{n-1}}dz = \frac{\pi^{n-1}}{V\Gamma(n-1)}(n - r_1^2)^{n-1}. \quad (6.72)$$

Therefore, we can now compute the probability distribution of r_1 and ϕ_1 as follows,

$$p(r_1) = \int r_1 p(r_1, \phi_1)d\phi_1 = \frac{2\pi^n}{V\Gamma(n-1)} r_1 (n - r_1)^{n-1} = \frac{2}{n^{n-1}} r_1 (n - r_1^2)^{n-1},$$

$$p(\phi_1) = \int r_1 p(r_1, \phi_1) dr_1 = \frac{1}{2\pi}.$$

Also since $p(r_1, \phi_1) = p(r_1)p(\phi_1)$, r_1 and ϕ_1 are independent.□

Chapter 7

Existence of Codes with Constant PMEPR

7.1 Introduction

Recently, there has been considerable interest in using multicarrier modulation for high speed communications [31]. As wireline applications, we can mention Discrete Multitone (DMT) in the asymmetric digital subscriber line (ADSL) and the very high rate digital subscriber line (VDSL). Similarly, Orthogonal Frequency Division Multiplexing (OFDM) has been proposed for different wireless scenarios such as Wireless Local Area Network (WLAN) and digital video broadcasting (DVB) [31].

In this modulation, information is carried on several narrowband orthogonal subcarriers, each subcarrier being modulated by a complex constellation like QAM or PSK. A major drawback of using several subcarriers is spurious high amplitude peaks of the transmitted signal when all of the subcarriers add constructively. To be more specific, considering a multicarrier system with n subcarriers and each subcarrier being modulated by BPSK constellation, the worst case peak to mean envelope power ratio (PMEPR) of this system is n as all the subcarriers add up coherently.

However, several authors noticed that this worst case PMEPR rarely occurs [32, 33, 34, 35]. For large values of n, it is shown in the previous chapter that almost surely, the PMEPR of any randomly chosen modulating vector, carved from any symmetric

QAM/PSK constellation, is $\log n$ asymptotically [35]. Therefore, even though the worst case PMEPR can be as bad as n, in the probability sense and in the limit, PMEPR behaves as $\log n$.

Several schemes have been proposed to reduce the high PMEPR of multicarrier signals including probabilistic methods (e.g., selective mapping), coding, clipping, and reserved subcarriers [36, 37, 38, 34, 39]. While existing coding methods give a guarantee on the PMEPR of the system with a large rate hit for large n, probabilistic methods usually improve the *statistical* properties of the PMEPR with a little redundancy and using side information [37, 36]. The basic idea behind the probabilistic methods is to lower the probability of occurrence of a peak, and in fact, these methods use the limited redundancy not to eliminate the peaks, but only to make them less frequent. Therefore, there is no guarantee on the PMEPR similar to the coding methods.

Recently, Paterson and Tarokh have raised the question of what the trade-off is between rate and minimum distance of a code with bounded PMEPR [37]. It is also proved that the Varsharmov-Gilbert upper bound remains the same for spherical codes with PMEPR less than $8 \log n$ for large n. In [35], based on the asymptotic analysis of PMEPR, it is further shown that the PMEPR of spherical codes and symmetric QAM/PSK constellations is $\log n$. However, without contradicting the result of [35], there still might be exponentially many codewords with constant PMEPR, even though the probability of randomly choosing one of them goes to zero, and therefore, they are rare!

In this chapter, we start with addressing the achievable PMEPR reduction by choosing an optimum sign for each subcarrier. Based on an elegant result of Spencer on bounded linear forms [40], we prove that by choosing an optimum sign for each subcarrier, we can indeed achieve constant (independent of n) bounded PMEPR for sufficiently large n. Moreover, we find an upper bound for the best constant and we then use this result and prove the existence of codes carved from a symmetric q-ary constellation with constant bounded PMEPR and rate greater than $1-\log_q 2$. We also derive a Varsharmov-Gilbert upper bound on the rate of a code given its minimum Hamming distance and with constant bounded PMEPR. A scheme with more degrees of freedom is then considered to reduce the PMEPR at the price of further reducing the rate of the code. It is also worth mentioning that this scheme can be interpreted as reducing the PMEPR by expanding the constellation by a factor of two. Of course, in order to preserve the minimum distance of the constellation, the average power has to be increased by 3dB.

This chapter is outlined as follows: Section 7.2 introduces our notations and definitions. Section 7.3 discusses the peak reduction methods by choosing an optimum sign for each carrier and elaborates the statement of the problem. Furthermore, Section 7.3 reviews the mathematical results on bounded linear forms that will be used in this thesis. Then, in Section 7.4, we address the achievable PMEPR reduction by choosing signs; we prove the existence of codes with constant bounded PMEPR for sufficiently large n. Finally, Section 7.5 concludes this chapter.

7.2 Definition

Assuming no pulse shaping, we may represent the complex envelope of a multicarrier signal with n subcarriers as

$$s_C(t) = \sum_{i=1}^{n} c_i e^{j2\pi i f_0 t}, \qquad 0 \le t \le 1/f_0, \tag{7.1}$$

where f_0 is the subchannel spacing and $C = (c_1, \ldots, c_n)$ is the complex modulating vector with entries from a given complex constellation $\mathcal{Q}$. The admissible modulating vectors are called codewords and the ensemble of all possible codewords constitute the code $\mathcal{C}$. For mathematical convenience, we define the normalized complex envelope of a multicarrier signal as

$$s_C(\theta) = \sum_{i=1}^{n} c_i e^{j\theta i}, \qquad 0 \leq \theta < 2\pi. \tag{7.2}$$

Then, the PMEPR of each codeword C in the code family $\mathcal{C}$ may be defined as

$$\text{PMEPR}_{\mathcal{C}}(C) = \max_{0 \leq \theta < 2\pi} \frac{|s_C(\theta)|^2}{E\{\|C\|^2\}}. \tag{7.3}$$

Similarly, $\text{PMEPR}_{\mathcal{C}}$ is defined as the maximum of Eq. (7.3) over all codewords in $\mathcal{C}$. Clearly, when $\mathcal{C}$ is a random code such that the c_i's are chosen independently from a constellation with average power E_{av}, the average power of C is nE_{av}. Throughout this thesis, whenever we drop $\mathcal{C}$ from PMEPR, we mean $\mathcal{C}$ is the random code with average power nE_{av}.

We will use the following notation: $\mathcal{C}$ and C represent the code family and codeword, respectively, c_i denotes the i'th coordinate of the modulating vector C, and $\log\{\cdot\}$ is the natural logarithm. We also define the rate of a code $\mathcal{C}$ chosen from a q-ary constellation as

$$R = \frac{1}{n} \log_q |\mathcal{C}|, \tag{7.4}$$

where $|\mathcal{C}|$ is the cardinality of the set $\mathcal{C}$. Hamming distance is also defined as the number of coordinates in which two codewords are different and consequently the minimum Hamming distance of $\mathcal{C}$ is its minimum over all pairs of codewords [42].

7.3 A Peak Reduction Scheme and Bounded Linear Forms

In this section, for any codeword $C = (c_1, \ldots, c_n)$, we study designing optimum signs for each subcarrier in order to reduce the PMEPR of C. We initially motivate and elaborate the statement of the problem and then we introduce results on bounded linear forms that will be used in the subsequent sections to prove the existence of codes with high rate and constant bounded PMEPR.

7.3.1 Peak Reduction by Choosing Optimum Signs

Given the codeword $C = (c_1, \ldots, c_n)$, we consider the design of optimum sign, $\epsilon_i \in \{+1, -1\}$, for each c_i in order to minimize the PMEPR of the resulting codeword $C_\epsilon = (\epsilon_1 c_1, \ldots, \epsilon_n c_n)$. Clearly the worst case PMEPR of a codeword C is of the order of n. We also know that a randomly chosen codeword C will have a PMEPR of $\log n$ for large values of n [35], and therefore randomly choosing signs should work well for large n. In fact, randomly choosing signs has shown to be an effective method of PMEPR reduction for moderate values of n [36]. As an example of similar techniques, in the selective mapping method (SLM) there are M statistically independent codewords representing the same information, and the codeword resulting in the lowest PMEPR is selected for transmission and therefore it needs $\log_2 M$ bits of side information. This approach was first proposed in [43] for $M = 2$ and it is generalized in [32, 36]. For implementation purposes, the M independent codewords are generated by element-by-element product of the codeword by M pseudorandom sequences with entries from $\{+1, -1\}$ or $\{\pm 1, \pm j\}$, for instance Hadamard vectors or m sequences [44].

This raises the question of how much further reduction in PMEPR we can get by choosing the best sign for each subcarrier. Moreover is there any deterministic algorithm to design an optimum sign for each subcarrier? Since changing signs does not affect the average power, we can focus on minimizing the peak of $s_C(\theta)$ over the ϵ_i's. Here is the statement of the problem:

Problem Statement: For any given complex vector $C = (c_1, \ldots, c_n)$ where

$|c_i| \in \mathcal{Q}$ and $c_i \leq \sqrt{E_{\max}}$, consider the following minimization problem:

$$\min_{\epsilon} \max_{0\leq\theta\leq 2\pi} |\sum_{i=1}^{n} \epsilon_i c_i e^{j\theta i}|, \tag{7.5}$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ and $\epsilon_i \in \{+1, -1\}$.

- What is the value of the min-max problem of (7.5) for any codeword C where $c_i \in \mathcal{Q}$?
- How much further improvement can we get by considering more elaborate schemes?
- How can we design the optimum vector ϵ efficiently?

The goal of this chapter is to address the first two questions. The third question is the topic of Chapter 8 and will be addressed later. In order to answer the above problem, we need the following lemma.

Lemma 7.1 *Let $s_C^R(\theta)$ and $s_C^I(\theta)$ be the real and imaginary parts of $s_C(\theta)$, respectively. Also let $\theta_i = \frac{2\pi i}{kn}$ for $i = 1, \ldots, kn$ where $k > 1$ is such that kn is an integer. Then*

$$\max_{0\leq\theta\leq 2\pi} |s_C(\theta)| \leq \frac{1}{\cos \pi/2k} \sqrt{\max_{1\leq i\leq kn} |s_C^R(\theta_i)|^2 + \max_{1\leq i\leq kn} |s_C^I(\theta_i)|^2}.$$

Proof: It is shown by Ehlich and Zeller that a real trigonometric polynomial with n subcarriers satisfies the following inequality [45]:

$$\max_{0\leq\theta\leq 2\pi} |s_C^R(\theta)| \leq \frac{1}{\cos \pi/2k} \max_{1\leq i\leq kn} |s_C^R(\theta_i)|, \tag{7.7}$$

where the θ_i's are kn uniform samples in $[0, 2\pi]$. It is worth noting that various versions of the inequality in (7.7) have appeared in [46]. Similarly, the same inequality is valid for the real trigonometric polynomial $s_C^I(\theta)$. Now, considering that

$$\max_{0 \le \theta \le 2\pi} |s_C(\theta)| \le \sqrt{\max_{1 \le i \le kn} |s_C^R(\theta_i)|^2 + \max_{1 \le i \le kn} |s_C^I(\theta_i)|^2}, \tag{7.8}$$

the lemma follows immediately from (7.7) and (7.8). □

Lemma 7.1 reformulates the problem in (7.5), and allows us to consider the optimum ϵ to minimize $2kn$ linear forms constructed by $s_C^R(\theta_i)$ and $s_C^I(\theta_i)$ for $i = 1, \ldots, kn$ instead of minimizing the maximum of $|s_C(\theta)|$ over a continuous variable $\theta \in [0, 2\pi]$. Therefore, instead of Eq. (7.5), we may then consider the following minimization problem:

$$\min_{\epsilon} \max_{1 \le p \le 2kn} \Big| \sum_{i=1}^{n} \epsilon_i a_{pi} \Big|, \tag{7.9}$$

where a_{pi} is defined as

$$a_{pi} = \begin{cases} \mathrm{Re}\{c_i e^{j\theta_p i}\} & 1 \le p \le kn, \\ \mathrm{Im}\{c_i e^{j\theta_p i}\} & kn + 1 \le p \le 2kn, \end{cases} \tag{7.10}$$

where $\theta_p = \frac{2\pi p}{kn}$. Of course, the optimization problems in (7.9) and (7.5) are identical in the limit for large k. In the following subsection we briefly review some results on bounded linear forms that will be used to solve the problem in (7.9).

7.3.2 Bounded Linear Forms

In what follows, we define bounded linear forms and introduce the result of J. Spencer [40] on the discrepancy of sets with two colorings that can be interpreted as bounding linear forms by using optimum signs.

Definition 1: For any $1 \leq l \leq m$, linear forms L_l in n variables $(x_1, \ldots, x_n)$ are defined as

$$L_l(x_1, \ldots, x_n) = \sum_{j=1}^{n} a_{lj} x_j, \qquad 1 \leq l \leq m, \tag{7.11}$$

where all a_{lj} are real and $|a_{lj}| \leq 1$.

Throughout this chapter, we assume $m \geq n$ and we define $\alpha = n/m$. We quote the following result from [40] to bound the linear forms assuming $x_j \in \{+1, -1\}$ in Definition 1.

Theorem 7.2 (Spencer [40]) *Let L_l be $m = \alpha n$ linear forms as defined in Definition 1 where α is a constant independent of n. Then for sufficiently large values of n, there exist $\epsilon_1, \ldots, \epsilon_n \in \{+1, -1\}$ such that*

$$|L_l(\epsilon_1, \ldots, \epsilon_n)| \leq \mathcal{K}(\alpha)\sqrt{m} = \frac{\mathcal{K}(\alpha)}{\sqrt{\alpha}}\sqrt{n}, \tag{7.12}$$

where $\mathcal{K}(\alpha)$ is a constant independent of n and is bounded by

$$\mathcal{K}(\alpha) \leq 11\sqrt{\alpha \log(2\alpha^{-1})}. \tag{7.13}$$

Proof: Refer to Theorem 4 and 7 of [40]. □

Theorem 7.2 states that there exists a sign vector $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ that can reduce the maximum of m linear forms to as much as $\mathcal{K}(\alpha)\sqrt{m}$ and also provides an upper bound for this best constant $\mathcal{K}(\alpha)$. In the Appendix 7.6.1, we obtain tighter bounds for $\mathcal{K}(\alpha)$, for instance, the bound for $\mathcal{K}(\alpha)$ for $\alpha = 0.5$ in Eq. (7.13) is improved from 9.15 to 4.03. Fig. 7.1 compares the upper bound of $\frac{\mathcal{K}(\alpha)}{\sqrt{\alpha}}$ derived in Appendix 7.6.1 with Eq. (7.13) for different values of α.

It is also worth mentioning that considering the matrix $[a_{ij}]$ as a Hadamard matrix,

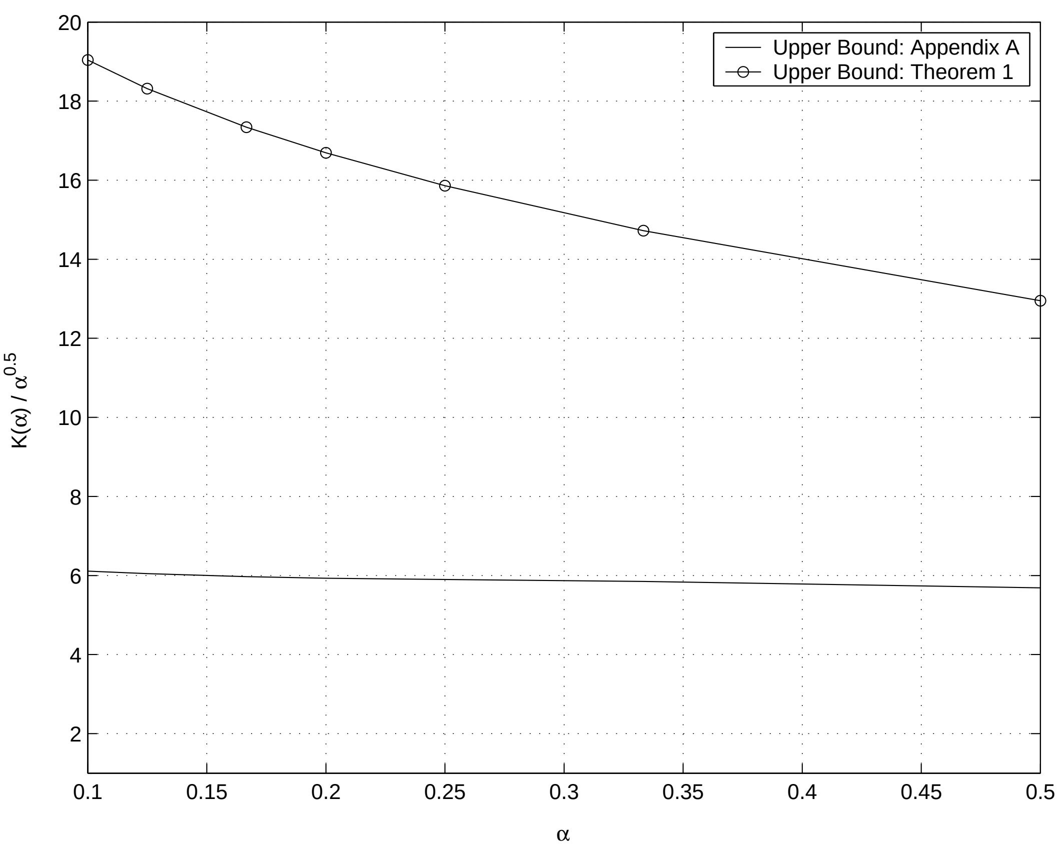

Figure 7.1: Upper bounds for $\mathcal{K}(\alpha)/\sqrt{\alpha}$ for different values of α

it is clear that there is no sign vector to further reduce the linear forms below $\sqrt{n}$. Therefore an easy lower bound for $\mathcal{K}(\alpha)/\sqrt{\alpha}$ in Theorem 7.2 would be one for any α. Another implication of this is that Hadamard vectors can be considered as good candidates for generating pseudorandom sign vectors for probabilistic methods such as SLM.

Remark 7.1: Note that each codeword C generates a matrix $[a_{ij}]$ according to (8.26). In [40], it is shown that for any $[a_{ij}]$, hence any codeword C, there exists exponentially many vectors ϵ, $(1+\delta_\alpha)^n$ of them say, such that (7.12) holds. If one chooses only one of these ϵ vectors, then the rate hit is $\log_q 2$. However, if we have the choice of choosing different vectors ϵ for each codeword, this choice will carry information and therefore the rate hit will be $\log_q 2 - \log_q (1+\delta_\alpha)$. Characterization of δ_α will further reveal the trade-off between PMEPR and rate. Although having the option of choosing different vector ϵ increases the rate, it greatly complicates the encoding and even more so the decoding, and therefore, we will not consider it further.

7.4 Codes with Constant PMEPR

In this section based on the scheme presented in subsection 3.1 and the results in subsection 3.2, we address the problem stated in (7.5). We study the existence of codes with constant PMEPR for sufficiently large values of n. We first derive a lower bound on the rate of codes with constant bounded PMEPR and then we obtain a Varsharmov-Gilbert bound for the rate and minimum distance of such codes. We further reduce the best achievable constant for PMEPR by choosing the optimum sign for each subcarrier at the price of reducing the rate by using a scheme with more degrees of freedom. The next theorem answers the first question raised in (7.5).

Theorem 7.3 *For any codeword* $C = (c_1, \ldots, c_n)$ *chosen from a constellation* $\mathcal{Q}$

with maximum and average power E_{max} and E_{av}, respectively, there exists a vector $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ with $\epsilon_i \in \{+1, -1\}$ such that

$$\mathrm{PMEPR}(C_\epsilon) \le \min_k \left\{ \frac{\sqrt{2k}\mathcal{K}(1/2k)}{\cos \pi/2k} \right\}^2 \frac{2E_{max}}{E_{av}} \tag{7.14}$$

for sufficiently large n, where $C_\epsilon = (\epsilon_1 c_1, \ldots, \epsilon_n c_n)$ and $k > 1$ is such that kn is an integer.

Proof: To prove the theorem, we use Lemma 7.1 and then Theorem 7.2 to bound the linear forms in (7.9). Therefore, setting $\alpha = \frac{1}{2k} < 1$ where k is as in Lemma 7.1,we can use Theorem 7.2 to get

$$\max_{0 \le \theta \le 2\pi} |s_{C_\epsilon}(\theta)| \le \frac{\sqrt{2k}\mathcal{K}(1/2k)}{\cos \pi/2k} \sqrt{2E_{max} n}. \tag{7.15}$$

Theorem 7.3 follows by using the definition of PMEPR. □

The following corollary is an immediate consequence of Theorem 7.3 and states that there exist exponentially many codewords with constant bounded PMEPR for any n sufficiently large.

Corollary 7.4 *For any q-ary symmetric constellation in which both A and $-A$ are in the constellation, there exists a code with rate $1 - \log_q 2$ and with constant* PMEPR, *for any n sufficiently large.*

Proof: We first consider half of the points in the constellation by choosing one of each two symmetric points, i.e., A or $-A$. We then use Theorem 7.3 to state the existence of a choice between any constellation point and its symmetric point such that the PMEPR of the resulting codeword is less than

$$\min_k \left\{ \frac{\sqrt{2k}\mathcal{K}(1/2k)}{\cos \pi/2k} \right\}^2 \frac{2E_{max}}{E_{av}}, \tag{7.16}$$

where k, E_{av}, and E_{max} are as defined in Theorem 7.3. Therefore, we have at least $(q/2)^n$ codewords with PMEPR less than (7.16), so $R \geq 1 - \log_q 2$. □

In the following corollary, we obtain a region for the rate and minimum Hamming distance of codes with constant PMEPR.

Corollary 7.5 *Let $\mathcal{Q}_q$ be a symmetric q-ary constellation as in Corollary 7.4, $R > 0$, $q > 2$, and $0 \leq \delta \leq \frac{q-2}{q}$, if we have*

$$R \leq \left(1 - H_{q/2}(\delta)\right)(1 - \log_q 2), \tag{7.17}$$

where H is the entropy function, then asymptotically, there exists a code $\mathcal{C}$ of length n, with entries from $\mathcal{Q}_q$, rate R, minimum Hamming distance $d_{min} = \lfloor \delta n \rfloor$, and constant bounded PMEPR.

Proof: We first consider one point from every two symmetric points of the constellation, and then we use the Varsharmov-Gilbert argument for the $q/2$-ary constellation to construct a code with rate $1 - H_{q/2}(\delta)$ and minimum Hamming distance of $\lfloor \delta n \rfloor$ [42]. Therefore, we can state that if Eq. (7.17) is valid, then there exists a code $\mathcal{C}'$ of length n, with entries from the $q/2$-ary constellation, rate R, and minimum Hamming distance $d_{min} = \lfloor \delta n \rfloor$.

Now we construct the code $\mathcal{C}$ by modifying the code $\mathcal{C}'$. For every codeword $C = (c_1, \ldots, c_n)$ in $\mathcal{C}'$, we choose between c_i and $-c_i$ in the constellation in order to minimize the PMEPR, and clearly this does not decrease the minimum Hamming distance of the code $\mathcal{C}$ from that of $\mathcal{C}'$. From Theorem 7.3 we know that for any codeword in $\mathcal{C}'$, there exists such a choice that has PMEPR of less than Eq. (7.16). This completes the proof. □

We can also consider more sophisticated modifications to a codeword, i.e., choosing between four symmetric points with respect to the imaginary and real axis instead of two as in Theorem 7.3. To clarify the idea, let's assume $c_i = a_i + jb_i$ and also assume

ϵ_i's and ϵ_i''s are chosen from $\{+1, -1\}$ and let

$$s_{C_{\epsilon,\epsilon'}}(\theta) = \sum_{i=1}^{n} \epsilon_i a_i \cos i\theta - \epsilon_i b_i \sin i\theta + j \sum_{i=1}^{n} \epsilon_i' a_i \sin i\theta + \epsilon_i' b_i \cos i\theta. \tag{7.18}$$

In this scheme, when we want to transmit the symbol $a_i + jb_i$, we are allowed to use $\pm a_i$ and $\pm b_i$ independently in order to reduce the PMEPR. In other words, since this scheme certainly has more degrees of freedom, we can further reduce the PMEPR at the price of reducing the number of codewords to $(q/4)^n$ for symmetric constellations with respect to the x and y axis. The following corollary addresses this trade-off between rate and PMEPR, which is analogous to Corollary 7.4 and **??**.

Corollary 7.6 *For any q-ary constellation such that the constellation points are symmetric with respect to both real and imaginary axes, there exists a code with rate at least* $1 - \log_q 4$ *and with constant bounded* PMEPR*, for sufficiently large values of n.*

Proof: We prove this along the same line as Corollary 7.4. Since in this case we have twice as many degrees of freedom, Eq. (7.15) can be rewritten as

$$\max_{0 \le \theta \le 2\pi} |s_{C_{\epsilon,\epsilon'}}(\theta)| \le \frac{\sqrt{k}\mathcal{K}(1/k)}{\cos \pi/2k} \sqrt{2nE_{max}}. \tag{7.19}$$

This can be then optimized over k defined as in Theorem 7.3, and therefore the PMEPR is less than

$$\min_k \left\{ \frac{\sqrt{k}\mathcal{K}(1/k)}{\cos \pi/2k} \right\}^2 \frac{2E_{max}}{E_{av}}. \tag{7.20}$$

Clearly (7.20) is less than (7.16) and this scheme can improve the best achievable constant PMEPR of (7.16). On the other hand, since at each time when we want to choose $a_i + jb_i$, we use any of the four combinations of $\pm a_i \pm jb_i$, the number of codewords will be at least $(q/4)^n$ and therefore, the rate should be greater than $1 - \log_q 4$. □

As shown in Corollary 7.4, using the optimum signs for each carrier, $\epsilon_i \in \{+1, -1\}$, we can reduce the PMEPR from n to a constant of Eq. (7.16) by the rate hit of $\log_q 2$. Another interpretation of this is that by expanding the constellation by a factor of two, we can reduce the PMEPR from n to a constant for asymptotically large values of n. By using more degrees of freedom as in Corollary 7.6, we can further reduce the constant bound for PMEPR from (7.16) to (7.20) at the price of the same rate hit, i.e., $\log_q 2$. As Fig. 7.1 suggests, since the upper bound for $\mathcal{K}(\alpha)$ is quite close to $\mathcal{K}(2\alpha)$, further reduction of the PMEPR by using more degrees of freedom does not seem to be very efficient.

Remark 7.2: Corollary 7.4 has an interesting implication for the PMEPR distribution. Assuming $\mathcal{Q}$ to be a symmetric constellation, it is shown in [47] that if the c_i's are independently chosen from $\mathcal{Q}$, then the OFDM signal tends to a Gaussian process for large n. Based on this, it is claimed in [47] that the distribution of the PMEPR can be approximated by

$$\Pr\{\text{PMEPR}(C) < \lambda\} \cong e^{-e^{-\lambda} n \sqrt{\frac{\pi}{3} \log n}} \tag{7.21}$$

for large n and where $C = (c_1, \ldots, c_n)$. However, Eq. (7.21) cannot be true since it implies that the number of codewords that have a constant PMEPR of λ (independent of n) is given by

$$q^{n\left(1 - \frac{e^{-\lambda}}{\log q} \sqrt{\pi/3 \log n}\right)}, \tag{7.22}$$

which clearly goes to zero as n tends to infinity. This contradicts Corollary 7.4. Therefore, even though the OFDM signal is a Gaussian process (any finite number of time samples are jointly Gaussian), this does not say anything about the distribution of the peak, since it involves an infinite number of samples.

7.5 Conclusion

We proved the existence of q-ary codes with rate greater than $1-\log_q 2$ with constant PMEPR when the number of subcarriers n, is large. In fact we can achieve this region by using the optimum sign for each subcarrier to reduce PMEPR. We also obtained a Varsharmov-Gilbert upper bound on the rate of a code given its minimum Hamming distance with constant PMEPR. We then considered a scheme to choose between four constellation points (rather than two) to further reduce PMEPR at the price of reducing the rate. In the second part of the chapter we considered the design of signs to reduce the PMEPR. We proposed a deterministic algorithm that computes the optimum sign vector efficiently and guarantees a PMEPR of $c\log n$ for any n. This scheme allows us to reduce the PMEPR at the price of expanding the constellation. Simulation results show a large improvement in the PMEPR by expanding the constellation by a factor of two.

7.6 Appendices

7.6.1 Bounds on $\mathcal{K}(\alpha)$

In this appendix we obtain a tighter values for $\mathcal{K}(\alpha)$ than the bound in Theorem 7.2. We first quote the following theorem from [40] that will be used to find better bounds.

Theorem 7.7 (Spencer [40]-Theorem 10) *Let L_l be m linear forms as defined in Definition 1. Let t, an infinite positive sequence $\gamma_1, \gamma_2, \ldots$, β, and p be given satisfying*

$$\begin{array}{ll} 1) & \beta = \frac{1}{\alpha}\sum_{s=1}^{\infty} H(2\gamma_s Q(t(2s-1))) + 2\gamma_s Q(t(2s-1)) \\ 2) & \sum_{s=1}^{\infty} \gamma_s^{-1} = 1 - c < 1 \\ 3) & H(\frac{1}{2}-p) < 1-\beta \end{array} \tag{7.23}$$

then there exist $\epsilon_1, \ldots, \epsilon_n \in \{-1, 0, +1\}$ *such that*

$$\begin{array}{ll} 1) & |\{i: \quad \epsilon_i = 0\}| \leq 2pn \\ 2) & |L_l(\epsilon_1, \ldots, \epsilon_n)| \leq t\sqrt{\alpha}\sqrt{m} \quad\quad 1 \leq l \leq m \end{array} \tag{7.24}$$

It is also worth noting that in the proof of the above theorem, sufficiently large n means that

$$n \geq \frac{-\log c}{1 - \beta - H(\frac{1}{2} - p)},$$

where β, c, and p are as defined in Theorem 7.7 [40]. Now we can state the following lemma:

Lemma 7.8 *Let* $\mathcal{K}(\alpha)$ *be defined as in Theorem 7.2. Then, for any* $t > 3$*,*

$$\mathcal{K}(\alpha) \leq t\sqrt{\alpha} + \mathcal{K}\left(\sqrt{-3.05\alpha Q(t) \log_2 0.39Q(t)}\right), \tag{7.25}$$

where $Q(t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-x^2/2} dx$.

Proof: As Theorem 7.2 states, there exists a vector ϵ with entries chosen from $\{+1, -1\}$ such that all m linear forms are bounded by $\mathcal{K}(\alpha)\sqrt{m}$ for sufficiently large values of n. Now to bound the linear forms, we first use Theorem 7.7 and then use Theorem 7.2 to assign a sign to the remaining coefficients, which were assigned zero by Theorem 7.7. Therefore, using the definition of $\mathcal{K}$ and triangle inequality, we get

$$\mathcal{K}(\alpha) \leq t\sqrt{\alpha} + \mathcal{K}(2p\alpha) \tag{7.26}$$

for sufficiently large n. From now on, we use Theorem 7.7 to compute p in Eq. (7.26) as a function of t and α. As a first step, we use the inequality $H(1/2-p) \leq 1-2.88p^2$

for $p < 0.5$ to choose p as a function of β as

$$H\left(\frac{1}{2} - p\right) \leq 1 - 2.88p^2 = 1 - \beta^{up} \leq 1 - \beta, \tag{7.27}$$

where β^{up} is an upper bound for β. Therefore Eq. (7.27) guarantees the third condition and we only need to find an upper bound for β and simplify the first condition in (7.23). In Appendix 7.6.2, it is further shown that the first and second conditions of (7.23) imply that

$$\alpha\beta \leq -2.2Q(t)\log_2 0.39Q(t) = \alpha\beta^{up} \qquad t \geq 3. \tag{7.28}$$

Inserting the upper bound of β in (7.27), we deduce that $p = \sqrt{\frac{0.763}{\alpha} Q(t)\log_2 0.39Q(t)}$. Lemma follows by substituting p in (7.26). □

We can now numerically compute each $\mathcal{K}(\alpha)$ by using the recursion of Lemma 7.8. For example for $\alpha = 0.5$, letting $t = 5$, we get

$$\mathcal{K}(0.5) \leq 5\sqrt{0.5} + \mathcal{K}(0.0032),$$

again using the lemma by $t = 5$,

$$\mathcal{K}(0.5) \leq 5\sqrt{0.5} + 6\sqrt{0.0032} + \mathcal{K}(1.75 \times 10^{-5}). \tag{7.29}$$

For the last term we can use the bound in Theorem 7.2 (i.e., $\mathcal{K}(\alpha) \leq 11\sqrt{\alpha \log 2\alpha^{-1}}$), and hence Eq. (7.29) implies that $\mathcal{K}(0.5) \leq 4.03$, which is much better than 9.15 as suggested by Theorem 7.2.

7.6.2 Proof of Eq. (7.23)

In this appendix, for $t > 3$, we simplify the first condition of (7.23), and reveal the relationship between β, α, and t. The idea is to show that for $t > 3$ the dominant term is the first term with $s = 1$ in (7.23). As a first step, we assume $\gamma_1 = 1.1$ and $\gamma_i = 20^{i-1}$ for $i \geq 2$ to satisfy the second condition in Eq. (7.23). Therefore, in the derivation of Lemma 7.8 we will not optimize over the values of γ_i's. Now inserting the value of γ_i's in the first inequality of (7.23), we may write

$$\begin{aligned}\alpha\beta \;&\leq\; H(2.2Q(t)) + 2.2Q(t) + \sum_{s=2}^{\infty}\left\{-2Q(t(2s-1))20^{s-1}\log_2\left\{2Q(t(2s-1))20^{s-1}\right\}\right.\\ &\left. +4Q(t(2s-1))20^{s-1}\right\}, \qquad (7.30)\end{aligned}$$

where we used the fact that $H(\epsilon) \leq -\epsilon\log_2\epsilon + \epsilon\log_2 e$, which follows from $\log_2(1-x) \leq x\log_2 e$. Now we can further simplify (7.30), by using $e^{-t^2} \leq Q(t) \leq 1/2e^{-t^2/2}$ for $t \geq 3$, to obtain

$$\begin{aligned}\alpha\beta \leq\; & H(2.2Q(t)) + 2.2Q(t) + \sum_{s=2}^{\infty}\left\{-20^{s-1}e^{-\frac{t^2}{2}(2s-1)^2}\log_2\left\{2e^{-t^2(2s-1)^2}20^{s-1}\right\}\right.\\ &\left. + 2.43e^{-\frac{t^2}{2}(2s-1)^2}20^{s-1}\right\}\\ \leq\; & H(2.2Q(t)) + 2.2Q(t) + \sum_{s=2}^{\infty}\left\{20^{s-1}e^{-t^2(2s-1)}\left(-(s-1/2)\log_2 20 + t^2(2s-1)^2\right)\right.\\ &\left. +1.43e^{-t^2(2s-1)}20^{s-1}\right\},\end{aligned}$$

where we used $s - 1/2 < s - 1$ and $2s - 1 > 2$ for $s \geq 2$. The above summation can be easily worked out by defining

$$f(t^2) = \sum_{s=2}^{\infty} 20^{s-1}e^{-t^2(2s-1)} = \frac{20e^{-3t^2}}{1-20e^{-2t^2}}. \qquad (7.31)$$

Therefore, Eq. (7.31) can be written as

$$\begin{aligned}\alpha\beta &\leq H(2.2Q(t)) + 2.2Q(t) + 1.43f(t^2) + \frac{1}{2}\log 20\left(-\frac{df(a)}{da}|_{a=t^2}\right) \\ &\quad +t^2\left(\frac{d^2f(a)}{da^2}|_{a=t^2}\right) \\ &\leq H(2.2Q(t)) + 2.2Q(t) + 28.6e^{-3t^2} + 60\log_2\sqrt{20}e^{-3t^2} + 180t^2e^{-3t^2} \\ &\leq H(2.2Q(t)) + 2.3Q(t), \end{aligned} \tag{7.32}$$

where the second inequality is valid for $t \geq 3$ by differentiating (7.31) and letting $t = 3$ in the denominator. For the third inequality, we can simply observe that the last three terms are less than e^{-2t^2} and since $0.1Q(t) > e^{-2t^2}$ for $t \geq 3$, the third inequality follows. We can further simplify (7.32) by using $H(\epsilon) \leq -\epsilon\log\epsilon + \epsilon\log_2 e$ to obtain

$$\begin{aligned}\alpha\beta &\leq H(2.2Q(t)) + 2.3Q(t) \\ &\leq -2.2Q(t)\log_2\{2.2Q(t)\} + 5.43Q(t) \\ &= -2.2Q(t)\log_2\{0.39Q(t)\}, \end{aligned} \tag{7.33}$$

which yields (7.28).

Chapter 8

PMEPR Reduction Techniques

8.1 Introduction

The high peak to mean envelope power ratio (PMEPR) of multicarrier signals is one of the major obstacles in implementing OFDM, xDSL, and other broadband multicarrier systems. The occurrence of large peaks in the signal seriously hampers the efficiency of the power amplifier.

Due to the importance of this problem, over the years, different schemes have been proposed for PMEPR reduction such as coding, deliberate clipping, selective mapping (SLM), reserved carriers, and tone injection [37, 2, 34, 36, 39, 3]. In all these schemes, there is always a trade-off between PMEPR and other parameters in the systems, including coding rate, average power, signal distortion, and bandwidth. Methods like coding usually give a worst case guarantee on the PMEPR; on the other hand, there are other methods such as SLM that improve the probability distribution of PMEPR, i.e., reduce the probability of encountering large PMEPR.

The results in Chapter 7 shed light into this problem and prove the power of coding on reducing the PMEPR of multicarrier signals. One important result was to show that balancing each codeword by choosing the optimum sign for each subcarrier can lead to significant PMEPR reductions. The main goal of this chapter is to seek practical schemes to reduce the PMEPR of OFDM signals. We first focus on constructing codes with low PMEPR by balancing each codeword using the sign of each subcarrier. Clearly, the complexity of the reduction algorithms are of paramount importance and the main challenge is to find the signs efficiently while significantly reducing the PMEPR. We propose three different algorithms for choosing the signs with polynomial time complexity. In the second part, we look into schemes to modify the constellation for PMEPR reduction. The resulting algorithms are mainly based on convex optimization techniques. Simulation results are presented at the end of this chapter.

8.2 Design of Signs to Reduce the PMEPR

In this section, we are seeking an answer to the third question raised in the previous chapter, namely the efficient design of signs to reduce the PMEPR. This is in fact well-motivated by the result of Theorem 7.3 in which it is proved that there exists a sign vector that yields constant bounded PMEPR for sufficiently large values of n. The following subsections present three main techniques namely, derandomization, greedy methods, and algorithms based on sphere decoding. These algorithms seek a sign vector with a worst-case guarantee on the PMEPR of the resulting codeword.

8.2.1 An Algorithm Based on Derandomization

Based on the results of Chapter 6, any random sign vector should have PMEPR of $\log n$ for large values of n with probability one, and therefore, random methods should work well in the probability sense. It should be also mentioned that searching over all the possible 2^{n-1} sign vectors has exponential complexity. Therefore, there is no hope to find the *best* sign vector as it is an NP hard problem

In what follows, we propose deterministic and efficient algorithms that basically search for a *good* set of sign vector ϵ and then we prove that our algorithm guarantees deterministically (*not probabilistically*) that the PMEPR is less than $c\log n$ for any n (*not asymptotically*) where c is a constant independent of n. We further quantify how good the resulting sign vector is in terms of PMEPR reduction.

We restate the problem here. For any given complex vector $C = (c_1, \ldots, c_n)$ where $|c_i| \in \mathcal{Q}$ and $c_i \leq \sqrt{E_{\max}}$, consider the following minimization problem:

$$\min_{\epsilon} \max_{0\leq\theta\leq 2\pi} |\sum_{i=1}^{n} \epsilon_i c_i e^{j\theta i}|, \tag{8.1}$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ and $\epsilon_i \in \{+1, -1\}$.

We again use Lemma 7.1 to reformulate the problem in (8.1) and instead of designing the vector ϵ to minimize the maximum of $|s_C(\theta)|$ over a continuous variable θ, we find the optimum ϵ_i's to minimize $2kn$ linear forms corresponding to $s_C^R(\theta_i)$ and $s_C^I(\theta_i)$ for $i = 1, \ldots, n$ and defined as in Eq. (7.9). Therefore, we may consider the following problem,

$$\min_{\epsilon} \max_{1\leq p\leq 2kn} |\sum_{i=1}^{n} \epsilon_i a_{pi}|, \tag{8.2}$$

where a_{pi} is defined as

$$a_{pi} = \begin{cases} \mathrm{Re}\{c_i e^{j\theta_p i}\} & 1 \leq p \leq kn, \\ \mathrm{Im}\{c_i e^{j\theta_p i}\} & kn+1 \leq p \leq 2kn, \end{cases} \tag{8.3}$$

where $\theta_p = \frac{2\pi p}{kn}$.

In order to solve (8.2), we consider a more general setting for our problem. Let's consider the set of equiprobable vectors $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, $\epsilon_i \in \{-1, +1\}$. Then, for any codeword C, we define A_p^λ as the event that the p'th linear form defined in (8.2) is greater than λ. Furthermore, assume λ is chosen such that $\sum_{i=1}^{2kn} \Pr\{A_i^\lambda\}$ is less than 1, and therefore, there exists a vector ϵ with the above property. We would like to efficiently find the vector ϵ, such that none of the bad events A_p^λ occur.

This problem has been considered in mathematics and is usually referred to as the derandomization of random algorithms [30, 48]. In this approach, we assume that we can compute the conditional probability $\Pr\{A_p^\lambda|\epsilon_1, \ldots \epsilon_j\}$, and we find the ϵ_i's sequentially. At the j'th step, given the optimally chosen signs $\epsilon_1^*, \ldots, \epsilon_{j-1}^*$,[1] we choose $\epsilon_j^* \in \{+1, -1\}$ such that

$$\sum_{i=1}^{2kn} \Pr\{A_i^\lambda|\epsilon_1^*, \ldots, \epsilon_{j-1}^*, \epsilon_j^*\} = \min\left\{\sum_{i=1}^{2kn} \Pr\{A_i^\lambda|\epsilon_1^*, \ldots, \epsilon_{j-1}^*, \epsilon_j = +1\}, \sum_{i=1}^{2kn} \Pr\{A_i^\lambda|\epsilon_1^*, \ldots, \epsilon_{j-1}^*, \epsilon_j = -1\}\right\}. \tag{8.4}$$

Due to the above recursive minimization and assuming that $\epsilon_1^*, \ldots, \epsilon_{j-1}^*$ are determined, we can write the following inequality,

$$\begin{aligned}
&\sum_{i=1}^{2kn} \Pr\{A_i^\lambda|\epsilon_1^*, \ldots, \epsilon_{j-1}^*\} \\
&= \frac{\sum_{i=1}^{2kn} \Pr\{A_i^\lambda|\epsilon_1^*, \ldots, \epsilon_{j-1}^*, \epsilon_j = 1\} + \sum_{i=1}^{2kn} \Pr\{A_i^\lambda|\epsilon_1^*, \ldots, \epsilon_{j-1}^*, \epsilon_j = -1\}}{2} \\
&\geq \min\left\{\sum_{i=1}^{2kn} \Pr\{A_i^\lambda|\epsilon_1^*, \ldots, \epsilon_{j-1}^*, \epsilon_j = 1\}, \sum_{i=1}^{2kn} \Pr\{A_i^\lambda|\epsilon_1^*, \ldots, \epsilon_{j-1}^*, \epsilon_j = -1\}\right\} \\
&= \sum_{i=1}^{2kn} \Pr\{A_i^\lambda|\epsilon_1^*, \ldots, \epsilon_{j-1}^*, \epsilon_j^*\},
\end{aligned} \tag{8.5}$$

and finally since $\sum_{i=1}^{2kn} \Pr\{A_i^\lambda\} < 1$, and after determining each ϵ_j sequentially according to (8.4), we will end up with

$$\sum_{i=1}^{2kn} \Pr\{A_i^\lambda|\epsilon_1^*, \ldots, \epsilon_n^*\} < 1. \tag{8.6}$$

Since there is no randomness in the conditional events of (8.6) when all the ϵ_i^*'s

[1] We use the superscript star to denote that the ϵ_j's are optimally chosen not arbitrary.

are determined, each $\Pr\{A_i^\lambda|\epsilon_1^*,\ldots,\epsilon_n^*\}$ is either one or zero. Therefore, Eq. (8.6) implies that all of the probabilities are zero, and consequently, the resulting vector ϵ^* guarantees that none of the events A_i^λ will occur.

The difficulty here is now in the efficient computation of the conditional probabilities. Instead of using the exact conditional probability functions, we can use upper bounds for conditional probabilities defined as

$$\Pr\{A_i^\lambda|\epsilon_1,\ldots,\epsilon_j\} \le F_i^\lambda(\epsilon_1,\ldots,\epsilon_j), \tag{8.7}$$

if the upper bounds satisfy the following conditions:

$$\begin{array}{ll} i) & \sum_{i=1}^{2kn} F_i^\lambda < 1 \\ ii) & F_i^\lambda(\epsilon_1,\ldots,\epsilon_j) \ge \min\limits_{\epsilon_j\in\{+1,-1\}} F_i^\lambda(\epsilon_1,\ldots,\epsilon_{j-1},\epsilon_j). \end{array} \tag{8.8}$$

Obviously by the same reasoning used for the original algorithm, we can use the upper bound to find the vector ϵ^* such that none of the events A_i^λ occur. Fortunately, as will be proved in Theorem 8.1, Chernoff's bound does the work for us,

$$\begin{aligned} \Pr\{|\sum_{i=1}^n a_{pi}\epsilon_i| > \lambda|\epsilon_1,\ldots,\epsilon_j\} &\le 2e^{-\gamma\lambda}\cosh\left\{\gamma\sum_{r=1}^j \epsilon_r a_{pr}\right\}\prod_{r=j+1}^n \cosh\gamma a_{pr} \\ &\stackrel{\text{def.}}{=} F_p^\lambda(\epsilon_1,\ldots,\epsilon_j), \end{aligned} \tag{8.9}$$

for any $\gamma > 0$ and $1 \le p \le 2kn$. We will show in the proof of Theorem 8.1 that $F_p^\lambda(\epsilon_1,\ldots,\epsilon_j)$, as defined in (8.9), satisfies both conditions in (8.8). Now we return to our problem and present the following algorithm.

Algorithm 8.1. *For any codeword $C = (c_1,\ldots,c_n)$, let a_{pi} be as in (7.9), k be as* *in Lemma 1, and* $|c_i| \le \sqrt{E_{max}}$*. Then* $\epsilon_1 = 1$*, and* ϵ_j*'s are recursively determined as*

$$\epsilon_j = -\text{sign}\left\{\sum_{p=1}^{2kn} \sinh\left\{\gamma^* \sum_{r=1}^{j-1} \epsilon_r a_{pr}\right\} \sinh(\gamma^* a_{pj}) \prod_{r=j+1}^{n} \cosh\left\{\gamma^* a_{pr}\right\}\right\} \tag{8.10}$$

for $j = 2, \ldots, n$*, where* $\gamma^* = \sqrt{\frac{2\log 4kn}{nE_{max}}}$. □

The following theorem gives the worst case guarantee on the PMEPR of the codeword $C_\epsilon = (\epsilon_1 c_1, \ldots, \epsilon_n c_n)$.

Theorem 8.1 *Let* $C = (c_1, \ldots, c_n)$ *be a given codeword where* $|c_i| \le \sqrt{E_{max}}$ *and* $E_{av} = E\{|c_i|^2\}$*. Also, let* $C_\epsilon = (\epsilon_1 c_1, \ldots, \epsilon_n c_n)$ *where* $\epsilon_i \in \{+1, -1\}$ *is determined according to Algorithm 8.1. Then the* PMEPR *of the resulting codeword,* C_ϵ*, will be less than* $\frac{4E_{max}}{\cos^2(\pi/2k)E_{av}} \log 4kn$ *where* k *is as in Lemma 7.1.*

Proof: The proof relies on the derandomization method illustrated before and uses the Chernoff bound to evaluate the conditional probability distributions. As a first step, we derive the upper bound for the conditional probability in (8.7) as

$$\begin{aligned}
\Pr\left\{\left|\sum_{i=1}^{n} a_{pi}\epsilon_i\right| > \lambda \middle| \epsilon_1, \ldots, \epsilon_j\right\} &= \Pr\left\{\sum_{i=1}^{n} a_{pi}\epsilon_i > \lambda | \epsilon_1, \ldots, \epsilon_j\right\} \\
&\quad + \Pr\left\{\sum_{i=1}^{n} a_{pi}\epsilon_i < -\lambda | \epsilon_1, \ldots, \epsilon_j\right\} \\
&= \Pr\left\{\sum_{i=j+1}^{n} a_{pi}\epsilon_i > \lambda - \sum_{i=1}^{j} a_{pi}\epsilon_i | \epsilon_1, \ldots, \epsilon_j\right\} \\
&\quad + \Pr\left\{-\sum_{i=j+1}^{n} a_{pi}\epsilon_i > \lambda + \sum_{i=1}^{j} a_{pi}\epsilon_i | \epsilon_1, \ldots, \epsilon_j\right\} \\
&\le e^{\gamma \sum_{i=1}^{j} a_{pi}\epsilon_i} \times e^{-\gamma\lambda} E\left\{e^{\gamma \sum_{i=j+1}^{n} a_{pi}\epsilon_i}\right\} \\
&\quad + e^{-\gamma \sum_{i=1}^{j} a_{pi}\epsilon_i} \times e^{-\gamma\lambda} E\left\{e^{-\gamma \sum_{i=j+1}^{n} a_{pi}\epsilon_i}\right\} \\
&= 2e^{-\gamma\lambda} \cosh\left\{\gamma \sum_{r=1}^{j} \epsilon_r a_{pr}\right\} \prod_{r=j+1}^{n} \cosh \gamma a_{pr} \\
&= F_p^{\lambda}(\epsilon_1, \ldots, \epsilon_j),
\end{aligned} \tag{8.11}$$

where we used Chernoff's bound and the fact that $\epsilon_i \in \{+1, -1\}$ is equiprobable and where γ is the Chernoff's bound coefficient to be optimized. We then show that the upper bound satisfies the conditions in (8.8). Using $\cosh(a+b) + \cosh(a-b) = 2\cosh a \cosh b$ and using the definition of F_p^λ, Eq. (8.11) can be written as

$$\begin{aligned} F_p^\lambda(\epsilon_1, \ldots, \epsilon_{j-1}) &= 2e^{-\gamma\lambda} \cosh\left\{\gamma \sum_{r=1}^{j-1} \epsilon_r a_{pr}\right\} \cosh \gamma a_{pj} \prod_{r=j+1}^{n} \cosh \gamma a_{pr} \\ &= e^{-\gamma\lambda} \prod_{r=j+1}^{n} \cosh \gamma a_{pr} \left(\cosh\left\{\gamma \sum_{r=1}^{j-1} \epsilon_r a_{pr} + \gamma a_{pj}\right\}\right. \\ &\qquad \left. + \cosh\left\{\gamma \sum_{r=1}^{j-1} \epsilon_r a_{pr} - \gamma a_{pj}\right\}\right) \\ &= \frac{F_p^\lambda(\epsilon_1, \ldots, \epsilon_{j-1}, \epsilon_j = +1) + F_p^\lambda(\epsilon_1, \ldots, \epsilon_{j-1}, \epsilon_j = -1)}{2} \\ &\geq \min_{\epsilon_j \in \{+1,-1\}} F_i^\lambda(\epsilon_1, \ldots, \epsilon_{j-1}, \epsilon_j). \end{aligned} \tag{8.12}$$

That ensures the second condition in (8.8) is satisfied. To verify the first condition, we use the following inequalities:

$$\begin{aligned} \sum_{i=1}^{2kn} F_i^\lambda = \sum_{i=1}^{2kn} 2e^{-\gamma\lambda} \prod_{r=1}^{n} \cosh \gamma a_{pr} &< \sum_{i=1}^{2kn} 2e^{-\gamma\lambda} \prod_{r=1}^{n} e^{\frac{\gamma^2 a_{pr}^2}{2}} \\ &\leq \sum_{i=1}^{2kn} 2e^{-\gamma\lambda} \times e^{\frac{\gamma^2}{2} nE_{max}} \\ &\leq 4kne^{-\gamma\lambda + \frac{\gamma^2}{2} nE_{max}}, \end{aligned}$$

where we used $\cosh x < e^{x^2/2}$ for $x \neq 0$. Now we can optimize over γ to get $\gamma^* = \lambda / nE_{max}$ and setting $\lambda = \sqrt{2nE_{max} \log 4kn}$, Eq. (8.13) can be written as

$$\sum_{i=1}^{2kn} \Pr\{|\sum_{i=1}^{n} a_{pi}\epsilon_i| > \lambda\} \leq \sum_{i=1}^{2kn} F_i^\lambda < 4kne^{-\frac{\lambda^2}{2nE_{max}}} = 1. \tag{8.13}$$

Therefore, the first condition will be also satisfied. Based on (8.4), we can compute

ϵ_j^* in each step given $\epsilon_1^*, \ldots, \epsilon_{j-1}^*$ as

$$\epsilon_j^* = -\text{sign}\left\{\sum_{i=1}^{kn} F_i^\lambda(\epsilon_1^*, \ldots, \epsilon_{j-1}^*, \epsilon_j = +1) - \sum_{i=1}^{kn} F_i^\lambda(\epsilon_1^*, \ldots, \epsilon_{j-1}^*, \epsilon_j = -1)\right\}. \quad (8.14)$$

Substituting the definition of $F_i^\lambda(\epsilon_1^*, \ldots, \epsilon_{j-1}^*, \epsilon_j)$ in (8.14), ϵ_j^* can be written as in (8.10). Lemma 7.1 then completes the proof by setting the value of λ and relating PMEPR to the maximum of $2kn$ linear forms, and it can be easily shown that the PMEPR of the resulting codeword is less than $\frac{4E_{max}}{\cos^2(\pi/2k)E_{av}} \log 4kn$ where k is as in Lemma 7.1. □

In order to get better insight into the above result, we define the rate of a q-ary code family $\mathcal{C}$ as in (7.4). In fact Theorem 8.1 implies that, by using an optimum sign for each subcarrier, we can construct a code with rate $1-\log_q 2$ and PMEPR of $c \log n$ for any n. The rate and PMEPR of this code is much higher than those of the previous codes proposed in [37] (and references therein) whose PMEPR is $O(\log^2 n)$ and whose rate is approaching zero as n increases. It is also worth mentioning that finding optimum signs in the transmitter side can be done very efficiently, and the decoding is very simple since the decoder simply ignores the sign of each subcarrier. On the other hand, this scheme can be interpreted as a scheme to reduce the PMEPR by expanding the constellation. For instance, by expanding the number of constellation points by a factor of two and making it symmetric, the resulting PMEPR can be reduced from n to the order of $\log n$ for any n. Of course, in order to preserve the minimum distance of the constellation, the average power should be increase by 3dB.

Now we present simulation results for different constellations including QPSK and 16QAM and for $n = 64$ and $n = 128$. Fig. 8.1 shows the actual complementary cumulative distribution function (CCDF) of PMEPR, $\Pr\{\text{PMEPR} > a\}$, and compares with the PMEPR distributions after using the SLM method by Hadamard vectors as psuedorandom sequences for $M = 2, 4, 8$, i.e., the transmitter sends the best code-

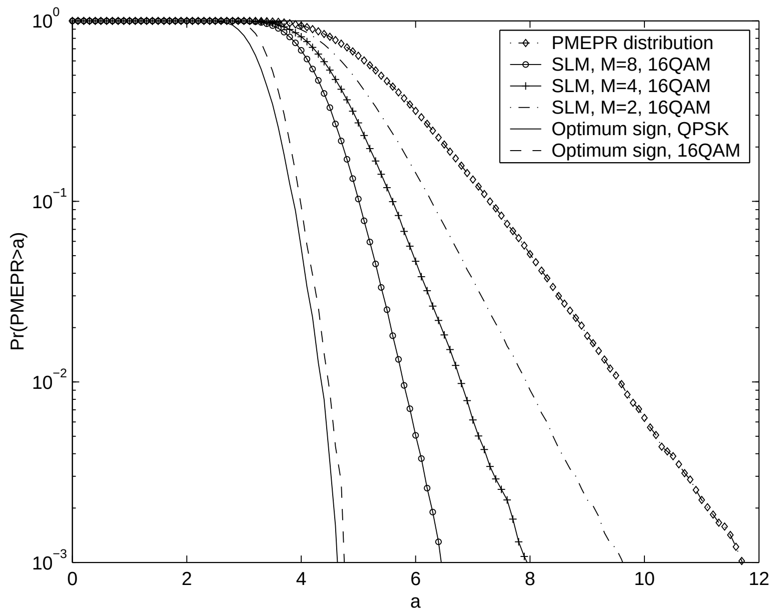

Figure 8.1: PMEPR distributions for $n = 64$ and for different schemes: SLM with $M = 2, 4, 8$ and using signs based on Algorithm 1 for each subcarrier

word out of M in terms of PMEPR. Fig. 8.1 also shows the CCDF of PMEPR after using the signs derived by Algorithm 8.1. Clearly the distribution function improves significantly. For instance the probability of having PMEPR of 4.6 is almost one, however by using signs based on Algorithm 8.1 this probability will go down to 10^{-3}.

Fig. 8.2 compares similar schemes when the number of subcarriers is 128. Interestingly the gain in PMEPR reduction here is much more. Furthermore the PMEPR distribution after using the designed signs for $n = 64$ and $n = 128$ are very close. It is also worth noting that the PMEPR drop is much more abrupt after optimizing the signs. For example, the probability of having PMEPR greater than four is almost one, however probability of having PMEPR greater than 4.8 goes down to 10^{-3}.

8.2.2 A Greedy Algorithm to Choose the Signs

Following Section 8.2.1, we just consider the kn uniform samples of the signal $s_C(\theta)$ at $\theta_p = \frac{2\pi p}{kn}$ for $p = 1, \ldots, kn$ where $k > 1$ is the oversampling factor. Therefore, the

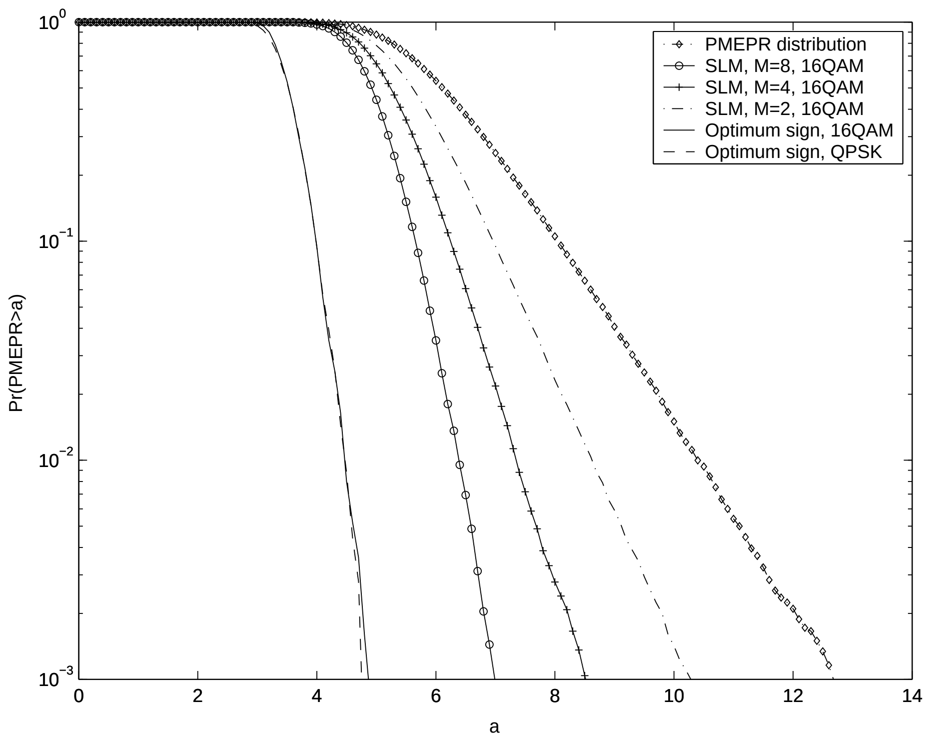

Figure 8.2: PMEPR distributions for $n = 128$ and for different schemes: SLM with $M = 2, 4, 8$ and using signs based on Algorithm 1 for each subcarrier

problem can be stated as in (8.2).

In other words, we would like to solve

$$\min_{\epsilon_i^2=1, i=1,\ldots,n} \|A\epsilon\|_\infty = \min_{\epsilon_i^2=1, i=1,\ldots,n} \|\sum_{i=1}^{n} a_i \epsilon_i\|_\infty, \tag{8.15}$$

where $A^t = [a_i]$ is a $2kn \times n$ real matrix and $a_i = [a_{p,i}]$ and $a_{p,i}$ is as defined in (8.26). Without loss of generality we assume that $|a_{i,p}| < 1$ (which can be done by scaling the constellation).

In Section 8.2.1, a deterministic algorithm was proposed to design the signs using derandomization. The algorithm chooses the signs recursively based on the knowledge of all a_i's. In fact, at the j'th step, we choose the sign that minimizes the conditional probability that $\|A\epsilon\|_\infty$ is greater than some threshold λ and given $\epsilon_1, \ldots, \epsilon_{j-1}$. Since finding the conditional probability is quite messy, we used the Chernoff bound instead. This leads to Algorithm 8.1 as obtained in Section 8.2.1.

The only drawback of Algorithm 8.1 is that the computation at each step involves taking the cosine hyperbolic kn times, which may increase the computation. In order to simplify the computation of Algorithm 8.1 at each step, one may try choosing the signs in a greedy manner in which at each step the sign that minimizes $\|\sum_{i=1}^{j} a_j \epsilon_j\|_\infty$ is chosen given $\epsilon_1, \ldots, \epsilon_{j-1}$. Interestingly, we can improve the performance by changing the infinity norm to norm p. Fig. 8.3 shows the performance of this method using different norms. It is clear that for $n = 128$, using $p = 6$ or 7 leads to quite a large improvement.

We can in fact justify this behavior analytically. The main result of this section is to obtain a bound on the PMEPR obtained from greedily minimizing the metric $\|\sum_{j=1}^{i} \epsilon_j a_j\|_p^p$. In particular, we show that the optimal p is $\log 2kn$, which yields a PMEPR of $c \log n$ for any n. Here is the algorithm:

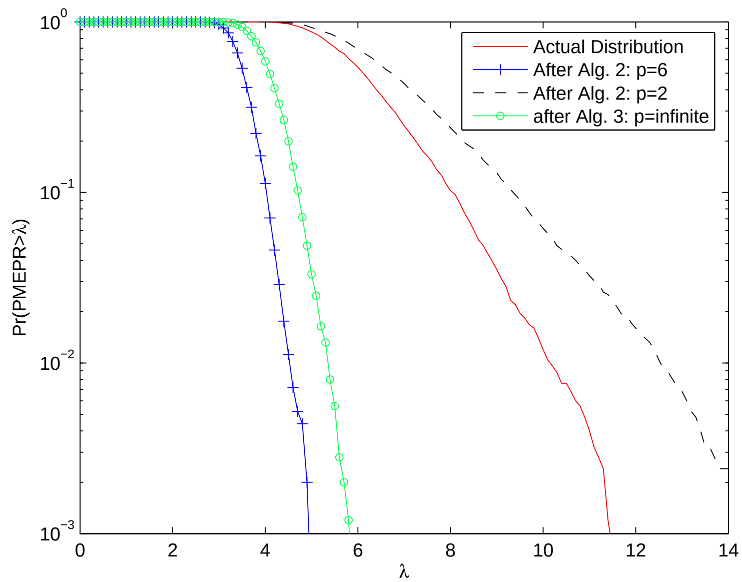

Figure 8.3: Comparison of $\Pr(\text{PMEPR} > \lambda)$ for $n = 128$ and using Algorithm 2 for different value of p and for 5000 codewords.

Algorithm 8.2: Let $\epsilon_1 = 1$, and having chosen $\epsilon_2, \ldots, \epsilon_{k-1}$, then

$$\epsilon_k = \arg \min_{\epsilon_k \in \{+1,-1\}} \| \sum_{j=1}^{k} a_i \epsilon_i \|_p^p. \tag{8.16}$$

The next theorem provides a worst case guarantee on the PMEPR when p is even. We conjecture that the result holds for p odd as well.

Theorem 8.2 *For any p greater than 2, and assuming all the entries of $A = [a_{i,j}]$ are $|a_{i,j}| \leq 1$, Algorithm 2 ensures that*

$$\|A\epsilon\|_\infty \leq (2kn)^{1/p} \sqrt{pn} \tag{8.17}$$

for any n. If $p = \log 2kn$, then the upper bound is $e\sqrt{n \log kn}$.

Proof: We present the proof when p is even for simplicity. If p is odd, we can follow a similar approach. Assume $\epsilon_1, \ldots, \epsilon_{k-1}$ have already been determined. We define the sequence $B_{r-1}^p = \frac{1}{2kn} \| \sum_{j=1}^{r-1} a_j \epsilon_j \|_p^p$. Using Algorithm 8.2, we now find a bound on B_k based on B_{k-1}. We first denote $\sum_{j=1}^{k-1} a_j \epsilon_j = (y_1, \ldots, y_{2kn})^t$ and $a_k = (x_1, \ldots, x_{2kn})^t$. Hence we may write

$$\begin{aligned} 2knB_r^p &= \min \left\{ \sum_{j=1}^{2kn} (y_j - x_j)^p, \sum_{j=1}^{kn} (y_j + x_j)^p \right\} \\ &\leq \frac{1}{2} \left(\sum_{j=1}^{2kn} (y_j + x_j)^p + (y_j - x_j)^p \right) \\ &\leq \frac{1}{2} \left(\sum_{j=1}^{2kn} (y_j + 1)^p + (y_j - 1)^p \right). \end{aligned} \tag{8.18}$$

The last inequality follows from the fact that $|x_i| \leq 1$ and also using the inequality

$$(y_j + x_j)^p + (y_j - x_j)^p \leq (y_j + 1)^p + (y_j - 1)^p \tag{8.19}$$

for $p \geq 1$ and $|x_j| \leq 1$. The bound can be proved using the convexity of the left hand side of (8.19) and therefore its maximum is attained on the boundary.

We can further bound (8.18) by using the inequality

$$\begin{aligned} \frac{1}{2kn} \sum_{j=1}^{2kn} (x_j + 1)^p + (x_j - 1)^p &= \frac{1}{2kn} \sum_{j=1}^{2kn} \sum_{r=0}^{p} \binom{s}{r} x_j^r (1 + (-1)^{p-r}) \\ &\leq \sum_{r=0}^{p} \binom{s}{r} \left(\frac{1}{2kn} \sum_{j=1}^{2kn} x_j^p \right)^{r/p} (1 + (-1)^{p-r}) \\ &\leq \left(\left(\frac{1}{2kn} \sum_{j=1}^{2kn} x_j^p \right)^{1/p} + 1 \right)^p \\ &\quad + \left(\left(\frac{1}{2kn} \sum_{j=1}^{2kn} x_j^p \right)^{1/p} - 1 \right)^p. \end{aligned}$$

Therefore,

$$knB_r^p \leq kn\left\{(B_{r-1}+1)^p + (B_{r-1}-1)^p\right\} \tag{8.20}$$

$$\leq 2kn\left(B_{r-1}^2 + p\right)^{p/2}, \tag{8.21}$$

where the last inequality follows by expanding the right hand side of (8.20) and using the fact that

$$\binom{2n}{2j} \leq \binom{n}{n-j} \times (2n)^{k-j} = \binom{n}{j} \times (2n)^{k-j}.$$

We can therefore obtain a recursive bound for $B_r^2 \leq B_{r-1}^2 + p$. Noting that $B_1 \leq 1$, we conclude that $B_n \leq \sqrt{np}$, and therefore,

$$\|\sum_{i=1}^{n} a_i\epsilon_i\|_\infty \leq \|\sum_{i=1}^{n} a_i\epsilon_i\|_p$$

$$\leq \left(knp^{p/2}n^{p/2}\right)^{1/p}$$

$$= (2kn)^{1/p}\sqrt{pn}. \tag{8.22}$$

Finally, letting $p = \log 2kn$, the theorem follows. □

Theorem 8.2 implies that if the norm p is properly chosen, the PMEPR of the resulting codeword is guaranteed to be less than $c \log n$ where c is a constant independent of n.

In fact, if we just allow the designer to find ϵ_i causally, i.e., based on $a_1, \ldots, a_i$ and not using $a_{i+1}, \ldots, a_n$, the problem of choosing the signs can be formulated as a mathematical game [1]. Following Spencer's terminology, at the k'th stage the "pusher" chooses a_k such that $\|a_k\|_\infty \leq 1$ and then the "chooser" decides on the sign ϵ_k. The value of the game at the k'th stage is $\|\sum_{j=1}^{k} a_j\epsilon_j\|_\infty$. Based on a result of [30], we can state the following corollary.

Corollary 8.3 *Considering any real $kn \times n$ matrix A with entries bounded by one,* *any algorithm that chooses ϵ_i's causally, cannot achieve a PMEPR of less than* $\log n$ *for large n.*

In fact any suboptimal algorithm for the pusher to find a_k's leads to a lower bound for the problem of causally choosing ϵ_i's. In [30], an algorithm is also proposed to design the signs causally. Here is the algorithm:

Algorithm 8.3: Let $\epsilon_1 = 1$, and having chosen $\epsilon_2, \ldots, \epsilon_{k-1}$, then

$$\epsilon_k = \underset{\epsilon_k}{\operatorname{argmin}} \cosh\left(\sum_{i=1}^{k} a_i \epsilon_i\right), \tag{8.23}$$

where $\cosh(X)$ for the vector $X^t = (x_1, \ldots, x_m)$ is defined as $\sum_{i=1}^{m} \cosh x_i$.

In [30], it is further proved that for a square $n \times n$ matrix, the algorithm can guarantee that $\|A\epsilon\|_\infty \leq \sqrt{2n \log n}$. The proof can be easily extended to the case of a $kn \times n$ matrix.

8.2.3 Pruning Algorithms

As shown in Fig. 8.4, there is still a pretty large gap between the PMEPR of the multicarrier system (i.e., 4.8) and that of the single carrier systems (i.e., 2.3). More precisely, we would like to see whether we can efficiently find a better choice of the signs that further reduces the PMEPR and approaches the CCDF of the single carrier system. Here we consider two variations of algorithm 8.2.

Pruning Algorithm 8.4: In the first approach, we search over all the possible signs for the first m subcarriers and then we use Algorithm 8.2 to find the choice of the signs for the remaining $n - m$ signs. Finally we choose the sign vector (out of 2^{m-1} possible choices as $\epsilon_1 = 1$) that has the least PMEPR. This of course has the complexity of order $O(2^m n \log n)$ as it requires searching for the best vector by performing 2^m IFFTs with size n. Fig. 8.5 shows the performance of this scheme for different m's. It can be seen that the PMEPR has been reduced from 4.8 to 3.4 at

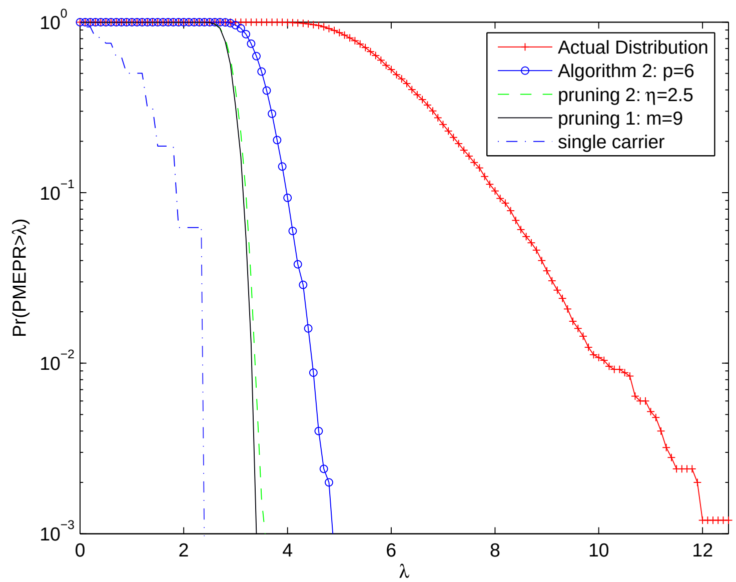

Figure 8.4: Comparison of $\Pr(\text{PMEPR} > \lambda)$ for $n = 128$ using the pruning algorithms compared to Algorithm 8.2 with $p = 6$ for 5000 random codewords.

the cost of additional computational complexity at the transmitter.

Pruning Algorithm 8.5: In the second approach, we consider the metric at the j'th stage to be $\| \sum_{i=1}^{j} a_i \epsilon_i \|_p^p$. Instead of just looking at the choice of sign that minimizes the metric at each stage, we keep the sign choices as long as the metric is less than some threshold value. One legitimate choice of the threshold would be the value of the metric by running Algorithm 8.2. In order to allow for more sign vectors, we may increase the threshold at each stage by some value (say η). At the end of the algorithm, we choose the best sign vector in terms of PMEPR. Fig. 8.4 shows the resulting PMEPR improvement for different values of η.

Fig. 8.5 is the rescaled version of Fig. 8.4 to see better the difference in the CCDF of PMEPR for the pruning algorithms and Algorithm 8.1. Clearly, the PMEPR is improved from 12.5 to 3.4 for the multicarrier system with 128 subcarriers and its PMEPR is just 1.6dB= $10\log(3.4/2.3)$ worse than the single carrier system. This motivates further investigation for more effective algorithms to choose the sign vector

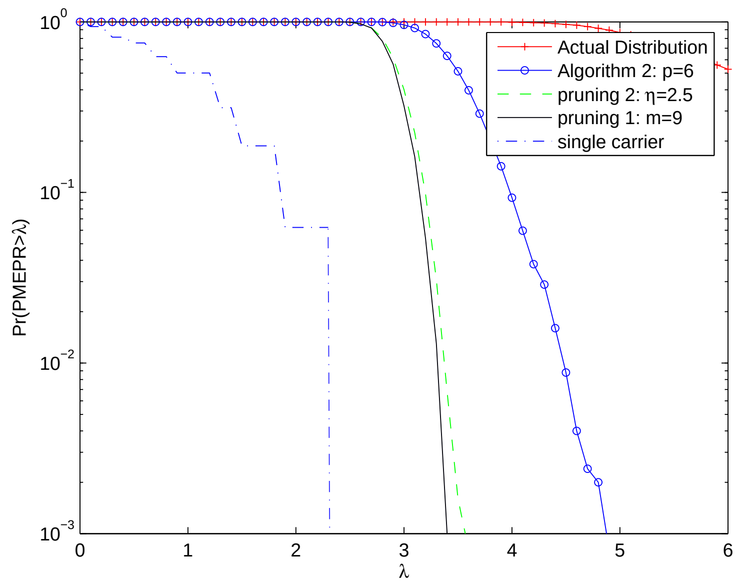

Figure 8.5: Comparison of $\Pr(\text{PMEPR} > \lambda)$ for $n = 128$ using the pruning algorithms compared to Algorithm 8.2 with $p = 6$ for 5000 random codewords.

with less complexity. Moreover, the question of how much further we can improve the PMEPR remains open.

8.3 Codes for BPSK Constellations

Unfortunately, the scheme proposed in Section 8.2 cannot be extended to BPSK constellation as all the information sent to the receiver is over the sign of each subcarrier. This problem is further motivated by the fact that for BPSK there is no construction for a high rate code with PMEPR of less than $O(\log n)$ (see [2] and [1]). In this section, we try to extend the result to the BPSK constellation by only using a fraction of all the signs to reduce the PMEPR.

In order to construct BPSK codewords with bounded PMEPR, for any integer r, we use the sign of $\frac{n}{r}$ equally spaced subcarriers indexed from 1 to n to minimize the

PMEPR.[2] Therefore, for any given codeword C, we would like to solve the following non-convex problem:

$$\min_{\epsilon_1,\epsilon_2,\ldots} \max_{0\le\theta\le 2\pi} \left| \sum_{i=0}^{\frac{n}{r}-1} \epsilon_{i+1} \left(\sum_{s=1}^{r} c_{ir+s} e^{j\theta(ir+s)} \right) \right|, \tag{8.24}$$

where $\epsilon_i \in \{+1,-1\}$. It is worth mentioning that setting $r = 1$ reduces the problem to the one addressed in [73].

Following [73], instead of maximizing over $0 \le \theta \le 2\pi$, we can minimize the maximum over kn samples of θ [**?**, 45]. Therefore the min-max problem of (8.24) can be written as

$$\min_{\epsilon_1,\epsilon_2,\ldots} \max_{1\le p\le 2kn} \left| \sum_{i=0}^{\frac{n}{r}-1} \epsilon_{i+1} a_{pi} \right|, \tag{8.25}$$

where

$$a_{pi} = \begin{cases} \mathrm{Re}\{\sum_{s=1}^{r} c_{ir+s} e^{j\theta(ir+s)}\} & 1 \le p \le kn, \\ \mathrm{Im}\{\sum_{s=1}^{r} c_{ir+s} e^{j\theta(ir+s)}\} & kn+1 \le p \le 2kn, \end{cases} \tag{8.26}$$

and $\theta_p = \frac{2\pi p}{kn}$.

Fortunately, the machinery used in section 8.2.1 can be generalized to this case and the following algorithm can be deduced by using a derandomization method as we did in Algorithm 8.1.

Algorithm 8.6. *For any $C = (c_1, \ldots, c_n)$, let k be an integer greater than 1 and $|c_i| \le \sqrt{E_{max}}$. Then $\epsilon_1 = 1$, and ϵ_s's are recursively determined as*

$$\epsilon_j = -\mathrm{sign} \left\{ \sum_{p=1}^{2kn} \sinh\left\{ \alpha^* \sum_{r=1}^{s-1} \epsilon_r a_{pr} \right\} \sinh(\alpha^* a_{ps}) \prod_{r=s+1}^{n/r} \cosh\{\alpha^* a_{pr}\} \right\},$$

for $s = 2, \ldots, \frac{n}{r}$, where $\alpha^* = \sqrt{\frac{2\log 4kn}{nrE_{max}}}$.

[2] For simplicity, we always assume r divides n. This condition is not necessary and it is just for simplifying the notations.

The next theorem gives a guarantee on the PMEPR of the resulting codeword by using Algorithm 8.6.

Theorem 8.4 *Let C be a given codeword where $c_i \leq \sqrt{E_{max}}$ and $E_{av} = E\{|c_i|^2\}$. Also let $C_\epsilon = (\epsilon_1 c_1, \ldots, \epsilon_1 c_{r-1}, \epsilon_2 c_r, \ldots)$ where ϵ_i's are determined according to Algorithm 8.6. Then the PMEPR of C_ϵ is less than $\frac{4rE_{max}}{\cos^2(\pi/2k)E_{av}} \log 4kn$ for any n and r where k is a positive integer such that kn is an integer.*

Proof: The proof is along the same line as the proof of Theorem 8.1. The only difference here is that we are minimizing the maximum of $2kn$ linear forms over n/r signs as opposed to n in Section 8.2. □

Remark 8.1: It is worth mentioning that our scheme is similar to the PTS method in that we search for the optimum sign for each group to minimize the PMEPR. The difference however is that we do not require side information in the receiver as the signs that we used for PMEPR reduction do not carry any information. Moreover, we propose a simple deterministic algorithm that provides a guarantee on the PMEPR without performing any Fourier transformation.

Remark 8.2: Since k is a constant, Theorem 1 implies that the resulting codeword has a PMEPR of less than $c \log n$ where c is a constant independent of n and r and that c can be determined by optimizing over k.

We can now construct a code set $\mathcal{C}$ such that the PMEPR of all its codewords is less than $cr \log n$ when the c_i's are chosen from a symmetric q-ary constellation. This can be done by reserving the sign of only n/r subcarriers (indexed $i = 1, r+1, 2r+1, \ldots, n-r+1$) to minimize the PMEPR over those signs. Given all the c_i's, Algorithm 1 can be used to determine the signs in polynomial time. Therefore, we end up having $(q/2)^{n/r} q^{n-n/r}$ codewords with the PMEPR of less than $cr \log n$ for any n and r. That leads to the following corollary:

Corollary 8.5 *If c_i's are chosen from a q-ary constellation, the code $\mathcal{C}$ constructed* *using Algorithm 1 has a rate of* $1 - \frac{1}{r}\log_q 2$ *and its PMEPR is less than* $cr\log n$ *for any* n *and* r*, where* c *is a constant independent of* n *and* r*.*

It is worth noting that the decoding of $\mathcal{C}$ is quite simple as the decoder can infer the signs (ϵ_i's) from the corresponding subcarriers since they do not convey any information over their signs.

Remark 8.3: The extension of our algorithm to the case where ϵ_i's can be chosen from $\{\pm 1 \pm j\}$ is straightforward. In this case ϵ_i's in (8.24) should be replaced by $\epsilon_i + j\epsilon_i'$ where ϵ_i and ϵ_i' are chosen from $\{+1, -1\}$. Therefore using the same argument as in (8.25), the problem can be again written in a similar form as in (8.25) and can be solved using Algorithm 8.6.

This result has an interesting implication on the famous result of Halasz [74]. In [74], Halasz states that almost all BPSK codewords have a PMEPR of less than $\log n + O(\log\log n)$ for large n. The design of such a code has been recently addressed in [2] where codewords with PMEPR of less than $\frac{3}{4}n$ have been characterized for any n.

Corollary 8.5 in fact constructs $2^{n(1-1/r)}$ codewords with PMEPR of less than $cr\log n$ for any n. For fixed r and large n, this implies a construction of exponentially many codewords (in fact $2^{n(1-1/r)}$) such that their PMEPR is $O(\log n)$.

8.3.1 Large Number of Subcarriers

In [35], the result of Halasz is extended to many other constellations including symmetric QAM, PSK constellations, and spherical codes. Therefore, the existence of codes with rate approaching one and the PMEPR of less than $\log n + O(\log\log n)$ has been established, although there is no construction close to this result [1].

Since Algorithm 1 and Theorem 1 work for any n and r, we may choose r to be $\log n$. We can therefore use Corollary 1 to prove that we can construct a code with rate $1 - O(\frac{1}{\log n})$ and a PMEPR of less than $c \log^2 n$. We can make a more precise argument in the following corollary.

Corollary 8.6 *For large n, using the construction as in Corollary 1, the code $\mathcal{C}$ has a rate $1 - O(\frac{1}{f(n)})$ and a PMEPR of less than $cf(n)\log n$ where $1 \le f(n) \le n$ such that $\lim_{n\to\infty} f(n) = \infty$.*

Therefore in an attempt to construct almost rate-one codes with PMEPR of $\log n$, we have been able to construct codes with almost rate-one (in fact, $R = 1 - O(\frac{1}{f(n)})$) and PMEPR of less than $cf(n) \times \log n$ where c is a constant independent of n. Recent results by Litsyn et. al [3] have provided codes with less redundancy and pretty much similar bounds on the PMEPR. Th only catch is additional complexity (which is in the order of $n^{\log n}$).

8.4 Constellation Shaping for PMEPR Reduction

In this section, we consider the PMEPR reduction by adjusting the sign and amplitude of each subcarrier. This would give more freedom to balance the IFFT of the codewords and further reduce the PMEPR. This method is a more general version of the scheme that has been recently proposed in [50, 73].

We first consider MPSK constellations and we then generalize the idea to other constellations as well. Here is the statement of the problem: For any given complex vector $C = (c_1, \dots, c_n)$ where c_i's are chosen from any MPSK constellation, find the solution to the following optimization problem:

$$\begin{array}{lll} \underset{\epsilon,\ u}{\text{minimize}} & \max\limits_{0\le\theta\le2\pi} \left| \sum_{i=1}^{n} \epsilon_i (1+u_i) c_i e^{j\theta i} \right| & \\ \text{subject to} & \epsilon_i \in \{+1,-1\} & \\ & 0 \le u_i \le u_{\max} & \\ & \sum_{i=1}^{n} |c_i|^2 (1+u_i)^2 \le (1+\eta) n P_{av}, & \end{array} \tag{8.27}$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ and $u \in^n$ are the optimization variables, u_i denotes the amplitude variations of the i'th subcarrier, and η denotes the average power increase. It is of great practical interest to reduce the peak of $s_C(\theta)$ without excessively increasing the average power and therefore η cannot be too large. In order to limit the variation of the new modulating vector $C_{\epsilon,u} = (\epsilon_1(1+u_1)c_1, \ldots, \epsilon_n(1+u_n)c_n)$, we further constrain u_i to be less than $u_{\max}$. The last constraint also implies that the average power increase is controlled by the parameter η. Clearly, the bound on u_i's limits the dynamic range of the quantizer in the transmitter. On the other hand, η limits the total variations of the constellation points while the minimum distance between the constellation points is fixed.

In summary, the price for reducing the PMEPR with our scheme is a slight increase in the average power, η or $10\log(1+\eta)$ dB, and sending no information over the sign of each subcarrier. For instance, Fig. 8.6 shows the modified QPSK constellation. For constellations other than MPSK, we let outer points in the constellation move such that the minimum distance of the constellation points does not change. This is shown for the 16QAM constellation in Fig. 8.7.

It is also worth noting that the receiver is not required to know the vectors ϵ and u. Therefore, for the decoding, the receiver may ignore the sign of each subcarrier as it does not convey any information. Furthermore, the receiver may use the same decision region for the decoding of the constellation points as for the case where $u = 0$. This is due to the fact that constellation points are only allowed to move outward as shown in Fig. 8.6 and 8.7.

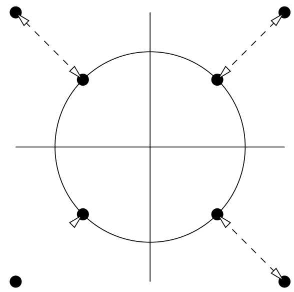

Figure 8.6: Modified QPSK constellation

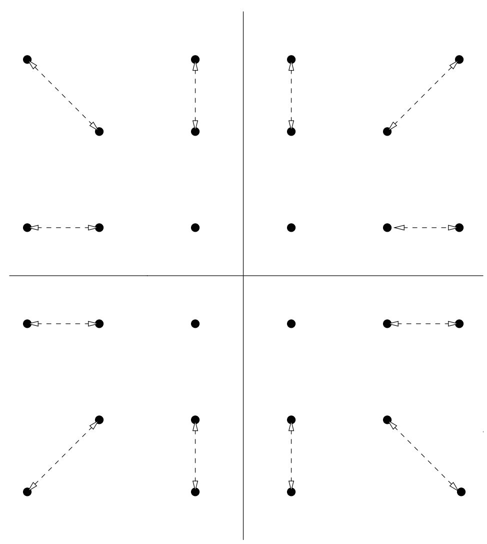

Figure 8.7: Modified 16QAM constellation

In order to compensate the rate loss due to not sending information over the signs, the transmitter can double the constellation size at the expense of a 3dB average power increase (to preserve the minimum distance of the constellation). Therefore, the total cost for amplitude and sign adjustment will be $3 + 10\log(1+\eta)$dB average power increase.

Clearly the optimization problem as stated in (8.27) is not convex due to having an integer constraint, i.e., $\epsilon_i \in \{+1, -1\}$. However assuming that $u_i = 0$ for all i, we can find a suboptimal solution for the signs using the result of [50, 73]. Afterwards, given c_i's and ϵ_i's, we show that the optimization over u_i's is convex and can be done very efficiently.

In fact, a suboptimal solution to the optimization over the sign can be efficiently evaluated using the results in Section 8.2 (e.g., Algorithm 8.1). Even though, Algorithm 8.1 does not give us the *best* signs, it is shown in Section 8.2 that it can significantly reduce the PMEPR. Now by further optimizing over the u_i, we can further reduce the PMEPR at the price of a slight increase in the average power. This gives us another degree of freedom to trade the PMEPR with a negligible average power increase and without deteriorating the minimum distance of the constellation.

One might ask whether changing the order of the optimization might improve the PMEPR reduction. Intuitively, balancing the maximum of a multicarrier signal that is already fairly balanced by optimizing over the sign of each subcarrier requires less average power increase η than the case where we first optimize the constellation over u_i's. Simulation results also confirm this.

8.4.1 Amplitude Adjustment Using Convex Optimization

In this section, we solve the problem of minimizing the peak of the multicarrier signal over u_i's given the signs and the information symbols c_i's and we show that it is a convex problem with a linear matrix inequality (LMI) constraint [7]. We further present a relaxation of the problem that leads to an approximate solution with less complexity by minimizing the maximum of the samples of the multicarrier signal.

First of all we notice the fact that

$$\sum_{i=1}^{n} c_i \epsilon_i (1+u_i) z^{-i} = H(zI-F)^{-1}G, \tag{8.28}$$

where $H^t = [c_1\epsilon_1(1+u_1), \ldots, c_n\epsilon_n(1+u_n)]$, $z = e^{j\theta}$, $G^t = [1, 0, \ldots, 0]$, and

$$F = \begin{bmatrix} 0 & \ldots & 0 & \\ 1 & 0 & \ldots & \\ \vdots & \vdots & & \\ 0 & \ldots & 1 & 0 \end{bmatrix}.$$

Given the ϵ_i's and c_i's, we can then restate (8.27) as the following optimization problem,

$$\begin{aligned} \text{minimize} \quad & \gamma \\ \text{subject to} \quad & \|H(zI-F)^{-1}G\|_\infty \leq \gamma \\ & 0 \leq u_i \leq u_{\max} \\ & \textstyle\sum_{i=1}^{n} |c_i|^2 (1+u_i)^2 \leq (1+\eta) n P_{av}. \end{aligned} \tag{8.29}$$

In order to show that the above problem is convex, we use the bounded real lemma [5]:

Lemma 8.7 *(Bounded Real Lemma) Suppose $\gamma > 0$ and F is stable. Then the following two statements are equivalent:*

(i)

$$\|H(zI-F)^{-1}G\|_\infty < \gamma.$$

(ii) There exists a Hermitian Y such that

$$\mathcal{H} = \begin{bmatrix} -Y + F^*YF & H^* & F^*YG \\ H & \gamma I & 0 \\ G^*YF & 0 & \gamma I + G^*YG \end{bmatrix} > 0.$$

Since the matrix $\mathcal{H}$ is linear in the entries of the matrix H, we can see that the constraint in the optimization problem of (8.29) is a linear matrix inequality [7].

Therefore, given the values of ϵ_i's and c_i's, we can state the minimization in (8.27) over u_i's as the following convex optimization problem:

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & \mathcal{H} \geq 0 \\ & 0 \leq u_i \leq u_{\max} \\ & \sum_{i=1}^{n} |c_i|^2 (1+u_i)^2 \leq (1+\eta) n P_{av}. \end{array} \tag{8.30}$$

where $H^t = [c_1\epsilon_1(1+u_1), \ldots, c_n\epsilon_n(1+u_n)]$, and the matrix Y in $\mathcal{H}$ is Hermitian.

This problem is a semi-definite program (SDP) and can be solved globally and efficiently using interior-point methods. Software packages exist that implement these methods; we use the recent package SeDuMi 1.02 [6].

Since the size of the LMI in the above SDP is relatively large, the computational load is still high for practical purposes. In fact the complexity is $O(n^6)$ even though exploiting the structure of the LMI can lead to faster implementations [7]. Another way to lower the computational load with very little loss in accuracy is to discretize $s_C(\theta)$ and then solve the discretized problem. That is, instead of minimizing the maximum of $s_{C_{\epsilon,u}}(\theta)$ over $1 \leq \theta \leq 2\pi$, we consider minimization of the maximum of kn uniform samples of $s_{C_{\epsilon,u}}(\theta)$ at $\theta_p = \frac{2\pi p}{kn}$ for $p = 1, \ldots, kn^3$. This certainly has

[3]It is worth mentioning that the resulting $s_{C_{\epsilon,u}}(\theta_p)$'s are the oversampled IFFT of the vector $C_{\epsilon,u}$.

much less complexity and can be written as a quadratically-constrained quadratic program [7], which is solved much more efficiently than the original SDP. We use SeDuMi for solving this problem as well. Furthermore, using the relationship between the maximum of $s_{C_{\epsilon.u}}(\theta)$ over θ and the maximum over θ_i, we can make our approximation practically accurate by choosing $k = 4$ [46].

More specifically, this optimization problem can be written as

$$\begin{aligned} \text{minimize} \quad & \gamma \\ \text{subject to} \quad & 0 \leq u_i \leq u_{\max} \\ & |\text{Re}\{s_C(\theta_p)\}|^2 + |\text{Im}\{s_C(\theta_p)\}|^2 \leq \gamma \\ & \text{for } p = 1, \ldots, kn, \\ & \textstyle\sum_{i=1}^{n} |c_i|^2 (1 + u_i)^2 \leq (1 + \eta) n P_{av}. \end{aligned} \tag{8.31}$$

Simulation results show that the solution to the problems in (8.30) and (8.31) are very close by choosing $k = 4$. Therefore, in the simulations part we solve the problem in (8.31) to optimize over u_i's instead of solving (8.30), which requires more computation.

8.4.2 Simulation Results

As we discussed in the previous sections, there is a trade-off between PMEPR reduction and average power increase, η, and also the range of variation for u_i's, i.e., $u_{\max}$. In this section we carry out simulations to explore this trade-off for $n = 64$ and $n = 128$ and for QPSK and 16QAM constellations. The algorithm for designing the signs is applicable to any symmetric constellation. For the amplitude variation of the constellation points, we use the schemes shown in Fig. 8.6 and Fig. 8.7, for QPSK and 16QAM, respectively.

Fig. 8.8 shows the CCDF (complementary cumulative distribution function) of PMEPR when c_i's are chosen from QPSK constellation and for different average

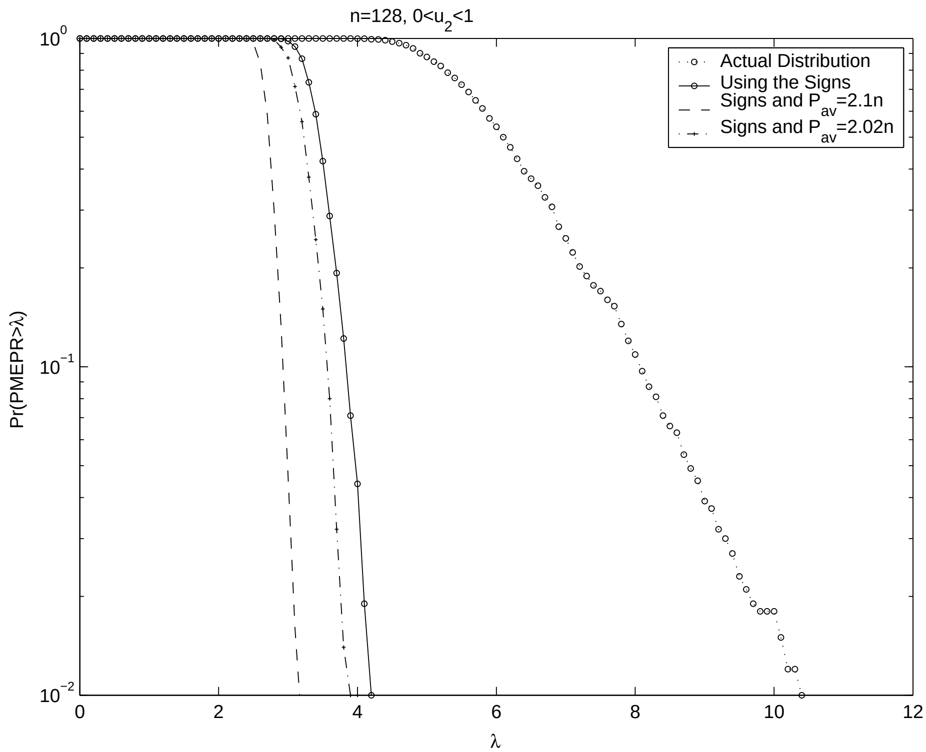

Figure 8.8: CCDF of the PMEPR for QPSK by optimizing over the ϵ_i's and u_i's for $n = 128$, $u_{\max} = 1$, and $\eta = 0.01, 0.05$.

power increases. Clearly, we need at least 0.21db ($\eta = 0.05$) average power increase, to get a noticeable PMEPR reduction after optimizing over the signs. As Fig. 8.8 suggests for $n = 128$, the PMEPR can be decreased from 10.5 to 4.5 with just using the signs and this can be further pushed down to 3.1 by also optimizing over the u_i's with a little average power increase.

We can further do the simulations for a wider range of u_i, i.e., $u_{\max} = 2$. As Fig. 8.8 shows for $n = 128$, we can further reduce the PMEPR by allowing more degrees of freedom to each point, however, this causes large peak to average power ratios for the c_i's, which is not practically favorable.

Fig. 8.9 also shows the PMEPR reduction when c_i's are chosen from 16QAM constellation and the variation of the constellation points is as in Fig. 8.7. In summary, simulation results suggest that by expanding the constellation and increasing

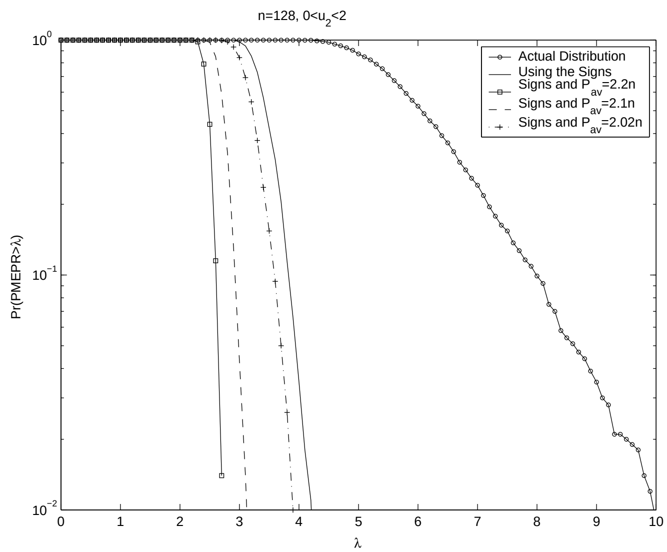

Figure 8.9: CCDF of the PMEPR for QPSK by optimizing over the ϵ_i's and u_i's for $n = 128$, $u_{\max} = 2$, and $\eta = 0.01, 0.05, 0.1$.

the average power by 0.21db, the PMEPR of multicarrier signals can be decreased dramatically, i.e., from 10.5 to 3.1 for $n = 128$.

8.5 Other Applications of the Sign Algorithms

The algorithms proposed to choose the signs to reduce the PMEPR for multicarrier signals in Section 8.2 can be applied to more general signals. In fact, we can address the PMEPR problem for non-harmonic multicarrier signals as a straightforward generalization. We can also consider the joint PMEPR and intercarrier interference (ICI) reduction in OFDM systems that has been recently addressed in [49]. In the following subsection we consider the peak to average power ratio (PAPR) in multiple antenna systems employing LD codes [41] .

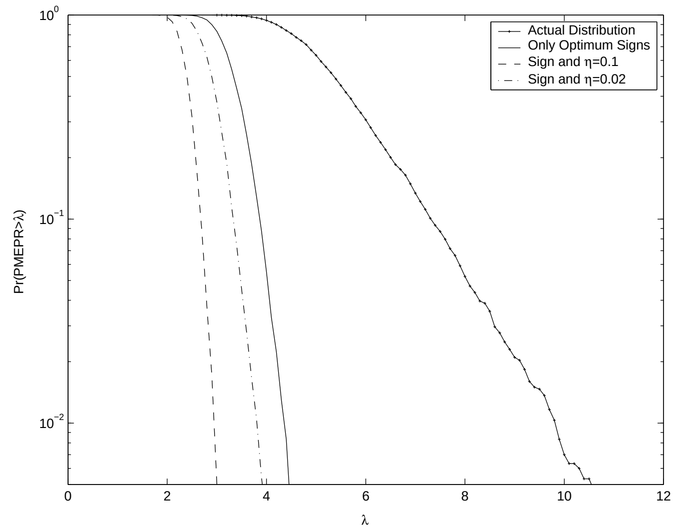

Figure 8.10: CCDF of the PMEPR for 16QAM constellation by optimizing over the ϵ_i's and u_i's for $n = 64$, $\eta = 0.02, 0.1$, and the constellation modification is according to Fig. 2 with $u_{\max} = 0.3$ for outer points.

8.5.1 PAPR in Multiple Antenna Systems Using LD Codes

Another issue of interest is the peak value analysis of the transmitted signal in multiple antenna systems. We consider LD codes for two reasons: first, LD codes are very general and include many proposed codes, and second, LD codes are linear as a function of the information symbols and this allows us to use our approach to address this problem.

We consider the following systems: Assume M and N are the number of transmit and receive antennas and α_i's and β_i's are chosen from a constellation. Then using LD codes, the transmit matrix $S = [s_{ij}]_{T\times M}$ over T time slots is defined as

$$S = \sum_{q=1}^{Q} \alpha_q A_q + j\beta_q B_q, \tag{8.32}$$

where A_q's are constant $T \times M$ matrices to be optimized to achieve capacity and where $Q \leq \min(M, N)T$. Since each entry of the matrix S is the linear combination of $2Q$ independent information symbols, $|s_{ij}|$, in the worst case, can be in the order of $2Q$ as all of the terms add up coherently. To mathematically quantify this effect, we define the peak to average power ratio (PAPR) of the multiple antenna system as

$$\mathrm{PAPR}(C) = \frac{\max_{i,j} |s_{ij}|^2}{E\left\{\sum_{i,j} |s_{ij}|^2\right\}} = \frac{\max_{i,j} |s_{ij}|^2}{MT}, \tag{8.33}$$

since the average transmit power is MT, i.e., $E\{tr(SS^*)\} = E\left\{\sum_{i,j} |s_{ij}|^2\right\} = MT$, and where $C = (\alpha_1, \ldots, \alpha_Q, \beta_1, \ldots, \beta_Q)$ is the information vector that is mapped to the transmit matrix S as in (8.32). Similarly the maximum of PAPR(C) over all the admissible vectors of C is the PAPR of the multiple antenna system.

To get a better insight on how severe this problem can be for multiple antenna systems, let the α_i's and the β_i's be chosen from a BPSK constellation, and therefore Eq. (8.32) can be written as

$$S = \sum_{i=1}^{Q} \alpha_q A_q^R - \beta_q B_q^I + j \sum_{i=1}^{Q} \alpha_q A_q^I + \beta_q B_q^R, \tag{8.34}$$

where A^R and A^I correspond to the real and imaginary parts of the matrix A. Furthermore, we assume

$$\sum_{j=1}^{Q} (a_{ij}^R)^2 + (b_{ij}^I)^2 \leq Q \quad \text{and} \quad \sum_{j=1}^{Q} (a_{ij}^I)^2 + (b_{ij}^R)^2 \leq Q. \tag{8.35}$$

Clearly Eq. (8.35) is not valid; we can simply scale a_{ij}^R's and b_{ij}^I's to satisfy (8.35) without affecting the peak to average power ratio as defined in (8.33). This assumption simplifies our derivations and makes the result more clear. Clearly the worst case analysis can give us the PAPR of $O(Q)$ as all the terms add up constructively. However, the following theorem shows that encountering a PAPR of greater than $O(\log Q)$ is highly unlikely as Q is getting large and N/M is a constant.

Lemma 8.8 *Under the assumption of (8.35) and when the α_i's and β_i's are chosen* *from a BPSK constellation,*

$$\Pr\{\text{PAPR}(C) > \frac{4Q}{MT}\log MT + \frac{4Q}{MT}\log\log MT\} \leq \frac{1}{\log 2MT}. \qquad (8.36)$$

Proof: Since α_i's and β_i's are independent and have uniform distribution over $\{+1, -1\}$, we can use (8.34) to write

$$\Pr\{|s_{ij}^R| > \lambda\} \leq 2\Pr\{s_{ij}^R > \lambda\} \leq 2\frac{E\{e^{ts_{ij}^R}\}}{e^{t\lambda}} = 2e^{-t\lambda}E\left\{e^{t\sum_{i=1}^{Q}\alpha_i a_{ij}^R - \beta_i b_{ij}^I}\right\}, \qquad (8.37)$$

where we used the union bound and Chernoff's bound for the first and second inequality. We can now further use the distribution of α_i's and β_i's to bound the characteristic function as

$$E\{e^{ts_{ij}^R}\} = \prod_{i=1}^{Q}\{\cosh ta_{ij}^R \times \cosh tb_{ij}^I\} \leq e^{\frac{t^2}{2}\sum_{i=1}^{Q}(a_{ij}^R)^2 + (b_{ij}^I)^2} \leq e^{t^2 Q/2}, \qquad (8.38)$$

where we used the inequality $\cosh\alpha \leq e^{\alpha^2/2}$ and (8.35). Therefore, Eq. (8.37) can be simplified to

$$\Pr\{|s_{ij}^R| > \lambda\} \leq 2e^{t^2 Q/2 - t\lambda} \qquad (8.39)$$

for any $t \geq 0$ and similarly the same inequality holds for s_{ij}^I. We can now optimize over t to set $t^* = \frac{\lambda}{Q}$, and use the following inequalities to get

$$\begin{aligned}\Pr\{\max_{i,j}|s_{ij}| > \sqrt{2}\lambda\} &\leq MT\Pr\{|s_{ij}| > \sqrt{2}\lambda\} \\ &\leq MT\Pr\{|s_{ij}^R| > \lambda\} + MT\Pr\{|s_{ij}^I| > \lambda\} \\ &\leq 2MTe^{-\lambda^2/2Q}.\end{aligned}$$

The lemma follows by letting $\lambda = \sqrt{2Q\log MT + 2Q\log\log MT}$. □

Lemma 8.8 states that even though the worst case PAPR can be of the order of Q, it is highly unlikely to encounter peaks greater than $O\left(\frac{N}{M}\log Q\right)$ since $Q = \min\{M, N\} \times T$.

Now since the problem here again can be reduced to bounding linear forms, we can follow along the same line as the previous section and show that there exist codes with a constant PAPR by just choosing optimum signs for each α and β. In this case, we have a sign vector with $2Q$ elements and $2MT$ linear forms. We can also use the algorithm in [50], to find the vector ϵ by simply using $2MT$ linear forms corresponding to the real and imaginary parts of the entries of S and Q signs to be chosen to reduce $\max_{i,j}|s_{ij}|$.

Chapter 9

Future Work

The results in the thesis have brought up a few interesting open problems. In what follows, we discuss the open problems and opportunities for future research in the areas of scheduling in broadcast channels and code design for PMEPR reduction of OFDM signals.

Sum-Rate of MIMO Broadcast Channels

While the results presented in Chapters 2 and 3 have shown that beamforming is a promising technique to increase the sum-rate of MIMO broadcast channels, these results rely on simplifying assumptions on the behavior of the channels. For instance, the outdoor wireless channel will have shadowing on top of the Rayleigh fading behavior; this certainly would impact the multiuser diversity as the channel variations of users will be magnified by the shadowing.

Another simplifying assumption is the fact that we did not assume any Doppler spread in the system. This would affect the authenticity of the feedback and would cause the transmitter to mistakenly choose the user that does not have the true maximum SINR (signal to noise and interference ratio). This would also raise another interesting question about what we should do when the base station (or the transmitter) receives a noisy version of the true SINR. One simple scheme would be to back off from the received maximum SINR; this certainly decreases the sum-rate. Understanding the amount of back off and the effect of the noise variance on the sum-rate are challenging open problems.

Finally, in practice there is a correlation between the transmit antennas because of the distance between antennas in the base station. It is certainly worthwhile understanding the sum-rate sensitivity to the correlation for dirty paper coding and our proposed beamforming scheme.

Scheduling in Broadcast Channel

In Chapter 4, we considered a single antenna homogeneous broadcast channel where different users demand for different sets of rates. It would be intriguing to generalize the set up in two different directions, namely, a heterogeneous network, or a broadcast channel with multiple antennas in the transmitter.

We could also consider the effect of noisy feedback on the scheduling. In practice, we would not have perfect feedback, and therefore, the designer has to take into account the noise variance in the scheduling as well.

In Chapter 5, we obtained the delay hit using opportunistic scheduling by assuming that whenever a user has the best channel conditions, he/she would certainly have a packet for transmission. In other words, the users are backlogged. Although it seems to be quite difficult, we would be interested in removing this assumption. The general problem here in fact is to come up with an algorithm that depends on both channel state conditions and the queue lengths while minimizing the delay. Here delay can be defined as average delay or the maximum delay among users. In fact, there are results stating algorithms to stabilize the queues, however, the delay behavior has not been analyzed [89].

It is also worth mentioning that we considered a channel with no temporal correlation in our analysis of delay. Wireless channel do have temporal correlations which can affect the delay dramatically. Therefore, analysis of the effect of temporal correlation would be of interest.

Coding for PMEPR reduction

The results in Chapters 6, 7 and 8 state that there exist high rate codes with very good PMEPR properties. Although we have been able to prove the existence of high rate codes with constant PMEPR and constructed codes with PMEPR of order $\log n$, we have not been able to find codes with PMEPR of less than order $\log n$. Recently there has been a line of work by S. Litsyn [3] to establish a relationship between strength of binary codes over $\{+1, -1\}$ and its ability to reduce the PMEPR. This further proves the power of coding for PMEPR reduction. Therefore, finding codes with small PMEPR remains as an interesting open problem which requires further research.

Chapter 10

Appendix

10.1 On Extreme Value Theory

In this appendix, we review some results on the asymptotic behavior of the maximum of n i.i.d. random variables when n is sufficiently large. This problem has been addressed in several papers and books (see, e.g., [60, 62] and references therein). It is known that for an arbitrary distribution, the density of the maximum does not necessarily have a limit as n goes to infinity. In [60], necessary and sufficient conditions for the existence of a limit for the distribution of the maximum is established.

In what follows, Theorem 10.1 presents all possible limiting distributions for the cumulative distribution of the maximum of n i.i.d. random variables. Theorem 10.2 focuses on the class of distributions that are of interest in this thesis and establishes the convergence rate to the limiting distribution. Finally, using Theorem 10.2, we deduce Corollary 10.3, which is the main result.

Theorem 10.1 *(Gnedenko '47) Let $x_1, \ldots, x_n$ be a sequence of i.i.d. random variables and $x_{\max} = \max(x_1, \ldots, x_n)$. Suppose that for some sequences $\{a_n > 0\}$, $\{b_n\}$ of real constants, $a_n(x_{\max} - b_n)$ converges in distribution to a random variable with* *distribution function $G(x)$. Then $G(x)$ must be one of the following three types:*

$$
\begin{aligned}
i) \quad G(x) &= e^{-e^{-x}} \\
ii) \quad G(x) &= e^{-x^{-\alpha}} u(x), \qquad \alpha > 0 \\
iii) \quad G(x) &= \begin{cases} e^{-(-x)^{\alpha}}, & \alpha > 0 \quad x \leq 0, \\ 1 & \qquad\quad\; x \geq 0, \end{cases}
\end{aligned}
$$

where $u(\cdot)$ is the step function.

Proof: Refer to [60, 67].

It turns out that the class of distribution functions we encounter in this chapter are of type i. Therefore, we further look into sufficient conditions on the distribution of x_i such that the distribution of the maximum is of type i.

We shall need the following definitions: let x_i's be positive random variables with continuous and strictly positive distribution function $f_X(x)$ for $x > 0$ and CDF of $F_X(x)$, and define the growth function as $g_X(x) = \frac{1-F_X(x)}{f_X(x)}$. Further define u_n to be the unique solution to

$$1 - F_X(u_n) = \frac{1}{n}, \tag{10.1}$$

(note that u_n is unique due to the fact that $F_X(\cdot)$ is continuous and strictly increasing for $x \geq 0$). We now state the following result from [61]:

Theorem 10.2 *(Uzgoren '56) Let $x_1, \ldots, x_n$ be a sequence of i.i.d. positive random variables with continuous and strictly positive PDF $f_X(x)$ for $x > 0$ and CDF of $F_X(x)$. Let also $g_X(x)$ be the growth function. Then if $\lim_{x\to\infty} g(x) = c > 0$, then*

$$\log\{-\log F^n(u_n + ug(u_n))\} = -u - \frac{u^2 g'(u_n)}{2!} - \ldots - \frac{u^m g^{(m)}(u_n)}{m!} + O\left(\frac{e^{-u+O(u^2 g'(u_n))}}{n}\right), \tag{10.2}$$

where u_n is as defined in (10.1).

Proof: Refer to the proof of Eq. (19) in [61].

Consider, for example, a $\chi^2(2)$ random variable with $f(x) = e^{-x}u(x)$. Then, it is quite easy to see that $g(x) = 1$, $u_n = \log n$, and all the derivatives of $g(x)$ are zero. Then, Theorem 10.2 simplifies to

$$\Pr\{\max x_i \leq \log n + u\} \to e^{-e^{-u+O(\frac{e^{-u}}{n})}}. \tag{10.3}$$

Letting $u = \log\log n$ and $u = -\log\log n$ and using (10.3) and (10.2), we can easily show that

$$\Pr\{\log n - \log\log n \leq \max x_i \leq \log n + \log\log n\} \geq 1 - O(1/\log n). \tag{10.4}$$

Imposing a constraint on the derivatives of the growth function, we can use Theorem 10.2 to state the following corollary, which is used throughout the thesis.

Corollary 10.3 *Let $x_1, \ldots, x_n$ be as defined in Theorem 10.2. If $u_n = O(\log n)$ and $g(x)$ is such that $\lim_{x\to\infty} g(x) = c > 0$ and $g^{(m)}(u_n) = O(1/u_n^m)$, then,*

$$\Pr\{u_n - c\log\log n \leq \max x_i \leq u_n + c\log\log n\} \geq 1 - O\left(\frac{1}{\log n}\right). \tag{10.5}$$

Proof: Since the distribution of x_i's satisfies the conditions of Theorem 10.2, and $g(u_n) = c + o(1)$, we can choose $u = \log\log n$ and write the expansion of the distribution of max x_i as

$$\begin{aligned} \Pr\{\max x_i \leq u_n + c\log\log n\} &= F^n(u_n + c\log\log n) \\ &= e^{-e^{-\log\log n + O\left(\frac{\log^2\log n}{\log n}\right)}} \\ &= 1 - O(\frac{1}{\log n}), \end{aligned} \tag{10.6}$$

where we used the identity $e^x = 1 + O(x)$ for small x and also we used the fact that $u_n = O(\log n)$. Similarly we may write

$$\Pr\{\max x_i \le u_n - c \log\log n\} = e^{-\log n(1+O(\log\log n/\log n))} = O\left(\frac{1}{n}\right). \qquad (10.7)$$

Combining (10.7) and (10.6) completes the proof for this corollary. □

Example 1. Suppose the x_i's have a $\chi^2(2m)$ distribution and we apply Corollary 10.3 to obtain the asymptotic behavior of max x_i. We can write $g(x)$ as

$$g(x) = \frac{1-F(x)}{f(x)} = \frac{(m-1)! e^{-x} \sum_{i=0}^{m-1} \frac{x^i}{i!}}{e^{-x} x^{m-1}} = (m-1)! \sum_{i=0}^{m-1} \frac{x^{i-(m-1)}}{i!}. \qquad (10.8)$$

In order to find u_n, we use the asymptotic expansion of the incomplete gamma function to get [63],

$$1 - F(u_n) = \Gamma(m, u_n) = u_n^{m-1} e^{-u_n}(1 + O(1/u_n)) = \frac{1}{n}. \qquad (10.9)$$

This would then imply that

$$u_n = \log n + (m-1)\log\log n + O(\log\log\log n). \qquad (10.10)$$

We can also observe that $g^{(m)}(u_n) = O(1/u_n^m)$. Therefore, the maximum value of n i.i.d. $\chi^2(2m)$ random variables satisfies

$$\Pr\left\{\log n + (m-2)\log\log n + O(\log\log\log n) \le \max_{1\le i\le n} x_i \le \log n + m\log\log n + O(\log\log\log n\right\} > 1 - O\left(\frac{1}{\log n}\right).$$

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