THE LUMINOSITY FUNCTION FOR GALAXIES
AND
THE CLUSTERING OF GALAXIES

Thesis by

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ABSTRACT

An analytic representation for the luminosity function for galaxies is proposed. Best fits of this function to counts of nearby bright galaxies and to counts of galaxies in rich clusters have been obtained. The results are marginally consistent with a single luminosity function valid for both samples. The proposed representation contains a characteristic magnitude M^* which exhibits an equivalent dispersion of only .24 magnitudes from cluster to cluster. The narrow dispersion in absolute magnitude observed for the brightest members of clusters is understood in large part as statistical fluctuation about a universal luminosity function, but the correlation of absolute magnitude with richness expected from the proposed representation is not observed.

It is shown that galaxies will condense into clusters of the sizes presently observed if the perturbations giving rise to galaxies were randomly distributed at recombination. A model for the origin of clusters is proposed which assumes (a) that galaxies collapse without dissipation, (b) that the perturbations giving rise to galaxies are centrally condensed, and (c) that most of the matter density in the universe is in galaxies.

The problem of the distribution of cluster sizes for Poisson distributed points is discussed and an analytic approach to a solution is developed. Numerical experiments show factor of two agreement with the solution obtained.

It is shown that if galaxies were randomly distributed at some early epoch massive galaxies are less likely to be isolated than less massive galaxies. An observational definition of an "isolated galaxy" is offered and an observational test of the model is proposed.

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General Introduction

It may be argued that the luminosity function for galaxies and the clustering of galaxies are two aspects of the same problem, the distribution of galaxies with respect to luminosity and with respect to position. These two aspects are treated here as separate questions. We present in Chapter I an analytic representation of the luminosity function. It was developed primarily as a tool in simplifying the calculations in Chapters II and IV, but the proposed representation turns out to be interesting in its own right, offering a convenient parameterization of the properties of different samples of galaxies. We find, for example, that the luminosity function contains a characteristic luminosity which is remarkably constant from one cluster of galaxies to the next. We are led to interpret the narrow dispersion in the absolute magnitude of the brightest members of clusters of galaxies as a reflection of the constancy of this characteristic luminosity.

In Chapter II we take galaxies for granted, arguing that there must have been density perturbations in an otherwise homogeneous universe in order for us to observe galaxies at the present. We then argue that if these pregalactic perturbations were Poisson distributed (a rather strong assumption) the presently observed clustering of galaxies may be understood entirely in terms of fluctuations in the density of pregalactic perturbations at recombination. To do this we borrow a result from Chapter III, which deals exclusively with the clustering of random points. We examine in Chapter III the frequency distribution of clusters of points and develop an analytic approximation to the expected distribution of clusters. The approximation developed in Chapter III is useful in a variety of problems, including the apparent clustering of certain asteroids and the apparent flaring of X-ray sources.

We discuss in Chapter IV a slightly different model for the clustering of galaxies, assuming that galaxies rather than pregalactic perturbations were randomly distributed at some early epoch. We offer an operational definition of an "isolated galaxy", and predict that isolated galaxies will be substantially fainter than average.

The central thread running through this thesis is that the masses (and luminosities) of galaxies were determined at recombination by the mass distribution of pregalactic perturbations. These perturbations are assumed to be randomly distributed. A region with extra pregalactic perturbations collapses to form a cluster, but in the limit of rich clusters the distribution of luminosities in a cluster is a "fair sample" of the universal distribution of pregalactic perturbations. Isolated galaxies, however, are very strongly biased in favor of less massive galaxies, since a massive galaxy is more likely to bind a companion galaxy than a less massive one.

Chapter I

THE LUMINOSITY FUNCTION FOR GALAXIES

1. Introduction

The theory of gravitational condensation of galaxies from statistically independent subcondensations (Press and Schechter 1974a) predicts that the number of galaxies of mass $\mathfrak{M}$ will show an exponential cutoff at some characteristic mass $\mathfrak{M}^*$. If we assume that the mass to luminosity ratio for galaxies is independent of mass (Morton and Chevalier 1973, Roberts 1969) then the luminosity function for galaxies will exhibit an exponential cutoff beyond some characteristic luminosity L^*. The luminosity function predicted by the statistical theory is then

$$\phi(L)\mathrm{d}L = \phi^* \exp[-L/L^*](L/L^*)^{-3/2}\, \mathrm{d}(L/L^*) \qquad (1)$$

where $\phi(L)$ is the number of galaxies per luminosity interval, $\mathrm{d}L$, per unit volume, dV.

While the predicted luminosity function exhibits a power law dependence of number versus luminosity at luminosities fainter than the characteristic luminosity L^*, the negative three halves power shown in equation (1) depends upon a simplified assumption about how one identifies potential galaxies. A more careful treatment shows that the power is almost certainly not three halves (Press and Schechter 1974b). Nonetheless, the proposition of an exponential cutoff in luminosity, if correct, is extremely important. It "explains" in a natural way the remarkably constant luminosity found for the very brightest galaxies in rich clusters of galaxies. We shall therefore examine the analytic representation

$$\phi(L)dL = \phi^* \exp[-L/L^*] \; (L/L^*)^{\alpha} \; d(L/L^*) \qquad (2)$$

where α is taken as a free parameter to be determined from fits to observational data.

In Section 3 we show that for a suitable choice of the parameters α and L^*, equation (2) shows agreement with Oemler's (1974) observations of the luminosity function for galaxies in rich clusters. We also show that a suitable choice of these parameters yields similarly good agreement with the luminosity function for field galaxies, which we determine in Section 2. Furthermore, the values of α and L^* found respectively for field and cluster galaxies are consistent with a single luminosity function for both samples of galaxies. For the purpose of discussion we adopt a standard value of α and in Section 4 we examine the variability of L^* for a sample of fourteen rich clusters, finding that L^* is a useful standard candle. The existence of such a standard candle has frequently been stressed by Abell (Abell 1962, Bautz and Abell 1973). The results presented here differ chiefly in the definition of the characteristic luminosity and in using the method of least squares to determine the parameter L^* rather than relying upon "eyeball" estimates. In Section 5 we show that the narrow dispersion in luminosity exhibited by the brightest members of rich clusters can be understood (in the manner suggested by Peebles 1968) as statistical fluctuations about a universal luminosity function. Section 6 presents several brief calculations which help demonstrate the utility of the proposed analytic representation.

2. The Local Luminosity Function

The term "luminosity function" is applied to galaxies by extension of the terminology of stellar astronomy. The space density $\psi(L,\Delta L,\underset{\sim}{r},\Delta V)$ is defined to be the density of galaxies (or stars) in the volume-luminosity interval $\Delta L \Delta V$. If $n(L,\Delta L,\underset{\sim}{r},\Delta V)$ is the number of galaxies in that interval then

$$\psi(L,\Delta L,\underset{\sim}{r},\Delta V) \equiv \frac{n(L,\Delta L,\underset{\sim}{r},\Delta V)}{\Delta L \Delta V} \tag{3}$$

In discussions of the space density of stars it is usually assumed that ψ is only weakly dependent on ΔV and ΔL for a large range of values. (ΔV must be large enough to smooth over the inhomogeneity due to individual stars but smaller than the scale on which the Galaxy exhibits spatial structure). A further assumption is frequently made that $\psi(L,\underset{\sim}{r})$ is separable into a dimensionless density function $D(\underset{\sim}{r})$ and a luminosity function $\phi(L)$ with the units of number density per luminosity interval:

$$\phi(L)\ D(\underset{\sim}{r})\ \Delta L \Delta V \equiv \psi(L,\underset{\sim}{r})\ \Delta L \Delta V \tag{4}$$

The space density for galaxies, however, is strongly dependent on ΔV ; even for the richest clusters of galaxies the scales ΔV for which one can obtain a well determined space density are of the same size as the scales on which the cluster exhibits spatial structure. If the universe is homogeneous on large scales, then the luminosity function can be redefined as the limit of the space density for large volumes:

$$\phi(L)\Delta L \equiv \lim_{\Delta V \to \infty} \psi(L, \Delta L, \Delta V) \quad (5)$$

Our observations are limited, however, to finite volumes in the vicinity of our own galaxy, and we must be careful to consider the scales on which inhomogeneities may occur in a local determination of the luminosity function.

Local determination of the luminosity function is made more difficult by the fact that the sample of galaxies used in such a determination is necessarily limited by apparent magnitude. If all galaxies down to some apparent magnitude are included, then smaller volumes are being sampled for low luminosity galaxies than for brighter ones. For every volume one expects fluctuations in the space density of some finite size. As the volume gets smaller, one expects larger fluctuations. The faint end of the luminosity function will therefore be relatively poorly determined. The situation is further complicated by the fact that we conduct our counts of galaxies from a privileged position, namely our own galaxy. Peebles (1974) has shown that the probability of finding a galaxy in a volume element chosen close to another galaxy is higher than in a volume element chosen at random. If dP is the probability of finding a galaxy in a volume dV at a distance r from another galaxy, then

$$dP = N[1 + \xi(r)]\, dV \quad (6)$$

where N is the mean density and $\xi(r)$ is the covariance function. Peebles finds that $\xi(r) \sim r^{-1.8}$ and is of order unity at distances of 5 Mpc. We shall assume that any local density enhancement exhibits the same power law dependence but we shall take the amplitude of such an enhancement to be indeterminate. Further assuming that such an enhancement is independent of luminosity, the expected space density $\psi(L,r)$ in the vicinity of our galaxy is given by

$$\psi(L,r) = \phi(L)[1 + \xi(r)] \quad (7)$$

$$\xi(r) \equiv \xi^*(\frac{r}{r^*})^{-1.8} \quad (8)$$

where ξ^* indicates the amplitude of any local density enhancement and r^* is a convenient reference radius. The space density may then be integrated over the volume sampled for a given limiting apparent magnitude to give the expected number of galaxies $n(L)dL$ in a magnitude limited sample of galaxies:

$$n(L)dL = V^*\phi(L)[(L/L^*)^{3/2} + 2.5\ \xi^*(L/L^*)^{.6}]\ dL \quad (9)$$

where L^* is a reference luminosity and V^* is the volume sampled at luminosity L^*. The reference radius r^* is related to L^* by the relation

$$r^* = \text{dex}[(m_c - M^* - 25)/5]\ \text{Mpc} \quad (10)$$

where m_c is the limiting apparent magnitude and M^* is the absolute magnitude corresponding to L^*. The volume V^* is given by

$$V^* = \frac{4}{3}\pi r^{*3} \int_{|b^{II}_{lim}|}^{90^0} \text{dex}[-.6A \csc\ z] \sin z\ dz \quad (11)$$

where $|b^{II}_{lim}|$ is the limiting galactic latitude for the sample and A is the absorption coefficient appropriate to a cosecant absorption law.

Equation (9) allows us to obtain the luminosity function $\phi(L)$ from observed counts of galaxies $n(L)$ in an apparent magnitude limited survey. We must be extremely careful that our survey sample is complete, not only with respect to identification and measurement of apparent magnitude but also with respect to measurement of the distance modulus. This is particularly important when radial velocities are used as distance indicators, since the low surface brightness of intrinsically faint galaxies makes measurement of these galaxies more difficult. Any galaxies in a magnitude limited sample which lack redshifts are likely to be the least luminous ones.

We present here a new determination of the luminosity function from counts of bright galaxies. The sample consisted of all galaxies listed in the Reference Catalog of Bright Galaxies (de Vaucouleurs and de Vaucouleurs 1964) with apparent B(0) magnitudes brighter than 11.75. Galaxies closer than 30^{o} to the galactic plane and within 6^{o} of the center of the Virgo cluster of galaxies ($\alpha = 12^{h}27^{m}$, $\delta = +13.5^{o}$) were excluded from the sample. Distances were determined using radial velocities (corrected for solar motion and galactic rotation according to de Vaucouleurs and de Vaucouleurs (1964)) and assuming a Hubble constant $h_0 = 50$ km $sec^{-1}Mpc^{-1}$. Absolute magnitudes were computed using an absorption coefficient $A_B = 0.12$ magnitude (Peterson 1970a). Radial velocities were taken from the Reference Catalog, with the following exceptions: (1) Neutral hydrogen velocities of Lewis and Robinson (1973) were used for members of the South Polar Group of galaxies. (2) Radial velocities were found for twelve galaxies without velocities in the Reference Catalog and are listed in Table 1.

TABLE 1

Reference Catalog Galaxies Brighter than $m_{B(0)}$ = 11.75, and with $|b^{II}| > 30^o$, without Catalogued Radial Velocities

ID	Velocity	Reference†
NGC1326	1233	(b)
NGC1532	1587	(c)
NGC1559	1284	(c)
NGC1672	1034	(c)
NGC1792	1035	(a)
NGC4096	540	(b)
NGC4145	1035	(e)
NGC4236	186	(f)
NGC4651	685	(a)
NGC4654	960	(a)
NGC4939	2862	(a)
NGC5247	1530	(d)

ID	Group Velocity	Group Identification
NGC1448	665	G21
NGC1617	999	G16
NGC4395	342	G3
NGC5054	2597	NGC5049
NGC7424	1561	G27
IC5332	142	G1
IC5201	...	None
A58	...	Local

† (a) de Vaucouleurs and de Vaucouleurs (1967)
(b) Bottinelli et al (19 70)
(c) Carranza (1967)
(d) Balkowski et al (1973)
(e) Chincarini and Rood (1972)
(f) Rogstad et al (1967)

(3) No radial velocity was found for eight galaxies. For five of these, the mean radial velocities of the de Vaucouleurs group (1968) with which each has been identified have been used. One galaxy, NGC5054, appears to be a member of a group with NGC5049. The radial velocity of that galaxy has been used. No radial velocity was assigned either to IC5201 or A58 (Sculptor).

The sample contains 192 galaxies, of which two have no measured redshifts and five show blueshifts. Of the remaining 185, all but one, NGC1313, are brighter than $M_{B(0)} = -16$. The counts $n(L)$ of galaxies brighter than $M_{B(0)} = -16$ are shown in Figure 1.

It is obvious from the observation of negative radial velocities that there is some component Δv in each velocity which cannot be due to the Hubble law. The luminosities computed using radial velocities are therefore subject to some random error. This situation is analogous to that encountered in stellar astronomy when parallaxes have a statistical uncertainty. The uncertainty ΔL in the luminosity is given by

$$\Delta L = \frac{2\Delta v}{v} L \qquad (12)$$

For each magnitude bin in Figure 1 we can compute an uncertainty averaged over all apparent magnitudes and over all galactic latitudes, assuming a root-mean-squared uncertainty in the radial velocity given by $<\Delta v^2>^{1/2}$. We find that

$$\sigma(L) = 3.28 \frac{<\Delta v^2>^{1/2}}{v^*} (LL^*)^{1/2} \qquad (13)$$

where $v^* = h_o r^*$. We can therefore use the Eddington method (Trumpler and Weaver 1953) to correct the observed number of counts to the "true"

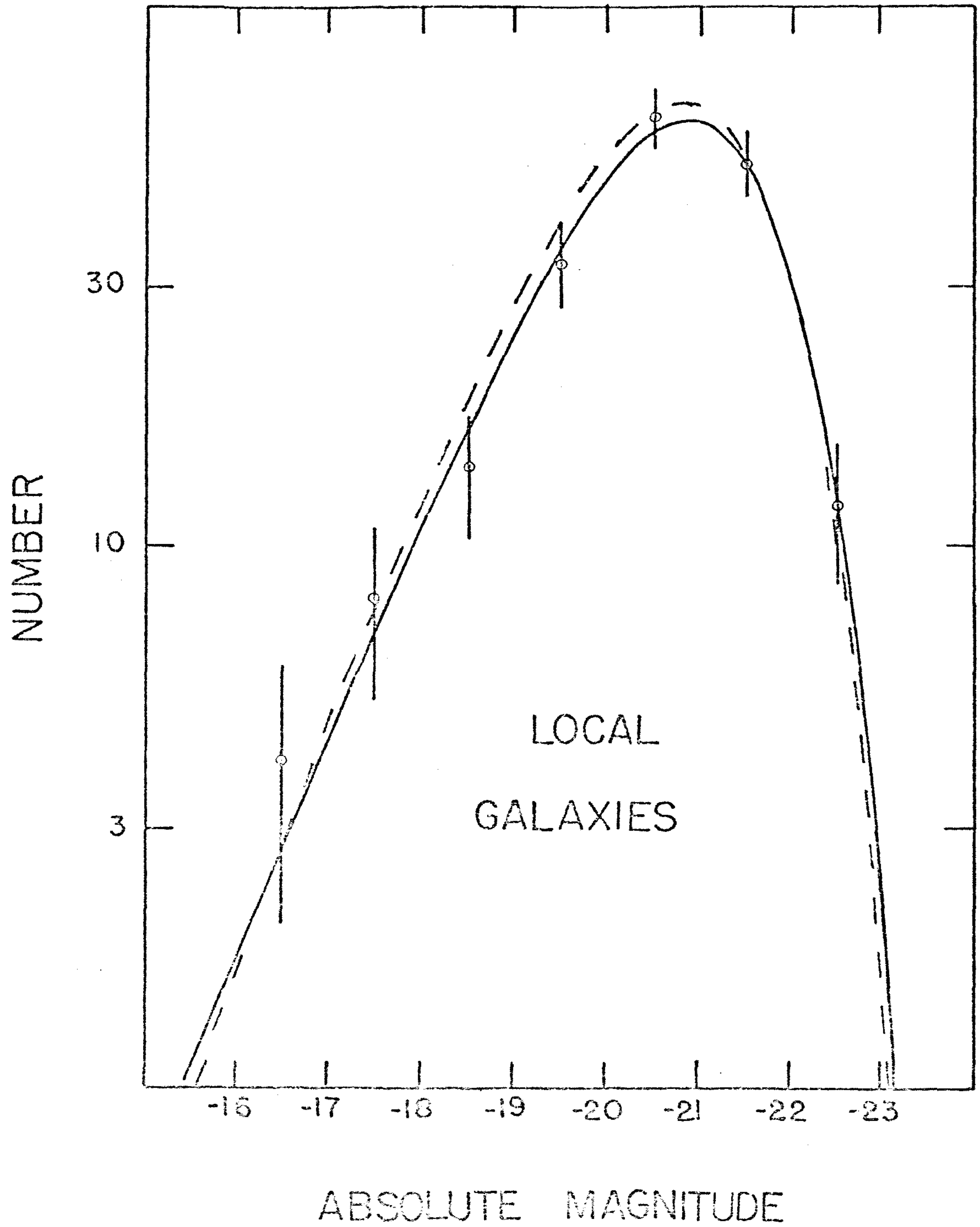


Fig. 1. Counts of local galaxies brighter than $m_{B(0)} = 11.75$. Solid line shows best fit of proposed representation to data. Broken line shows correction for non-Hubble component of radial velocities.

counts

$$n_{true}(L) = n_{obs}(1 - \sigma'^2 - \sigma''\sigma) - 2n'_{obs}\sigma'\sigma - n''_{obs}\,\sigma^2/2 + \cdots \qquad (14)$$

where primes indicate derivatives with respect to luminosity. It is clear from equation (14) that we need a smooth curve through the observed counts in order to take derivatives. The solid line shown in Figure 1 shows reasonably good agreement with the data. The broken line gives the "true" counts applying equations (13) and (14) and assuming an rms non-Hubble component $<\Delta v^2>^{1/2} = 200$ km sec^{-1}, a value typical of the velocity dispersions in de Vaucouleurs' groups. It is remarkable that the uncorrected and corrected curves differ by so little. This is an accident of the observed distribution: were the solid line either steeper or less steep, the corrections would be much larger.

The luminosity function $\phi(L)$ obtained from the counts shown in Figure 1 is in substantial agreement with the results of van den Bergh (1961), Christiansen (1968), and Shapiro (1971). The use of radial velocity as the sole distance indicator, however, allows for a more careful treatment of errors in derived absolute magnitudes. The attention paid to the completeness of radial velocity data limits the size of possible systematic effects.

3. Comparison of Proposed Analytic Representation with Counts of Field and Cluster Galaxies

We have an analytic expression for the expected number of galaxies $n(L)$ in an apparent magnitude limited sample, equation (9), and we have the observed counts shown in Figure 1. We shall use the method of least squares to obtain the best agreement of the proposed representation and the observations. The proposed representation has three free parameters ϕ^*, L^* , and α . In addition there are two additional parameters about which we have only limited information. ξ^* characterizes the amplitude of some local density enhancement, assumed to show an $r^{-1.8}$ dependence, and $<\Delta v^2>^{1/2}$ characterizes the size of the non-Hubble component of the observed radial velocities. The data are not sufficiently strong to determine five parameters. We shall therefore constrain ξ^* and $<\Delta v^2>^{1/2}$ to reasonable values and examine how changes in them affect the solution for ϕ^* , L^* , and α .

Before solving for these parameters, we must recast the correction for non-Hubble velocities into a correction to the theory rather than a correction to the data. If $n(L)$ is the expected number of galaxies of luminosity L , then $n_{c1}(L)$, the number expected after correcting for uncertainties in the luminosity $\sigma(L)$, is given by

$$n_{c1}(L) = n(1 + \sigma'^2 + \sigma''\sigma) + 2n'\sigma'\sigma + n''\sigma^2/2 + \cdots \qquad (15)$$

which differs only in signs from the correction as applied to observation (equation (14)). A second correction must be applied because a finite bin width, $\Delta M = 1$ magnitude, has been used in assembling the data. The correction is given by Trumpler and Weaver (1953)

$$n_{c2}(M) = n_{c1}(M) + n''_{c1}(M)(\Delta M)^2/24 + \cdots \qquad (16)$$

where we have for the first time cast the theory in terms of absolute magnitude rather than luminosity. Assuming values for ξ^* and $<\Delta v^2>^{1/2}$ we may fit the data using equations (16), (15), (13), (9) and (2).

The method of least squares not only yields values for the parameters, but also provides an "unbiased estimate" of the covariance matrix (Wolberg 1967). The diagonal elements give the squares of the uncertainties in the parameters while the off-diagonal elements determine how strongly correlated the parameters are. Using the least squares method and assuming $<\Delta v^2>^{1/2} = \xi^* = 0$, we find the following values for α , $M^*_{B(0)}$, $\phi^* V^*$ and the covariance matrix Σ

$$\begin{aligned} \phi^* V^* &= 188 \pm 11 \\ M^*_{B(0)} &= -20.89 \pm .10 \\ \alpha &= -1.47 \pm .08 \end{aligned} \qquad \Sigma = \begin{bmatrix} 1.14(2) & 5.68(-1) & 5.17(-1) \\ 5.68(-1) & 9.40(-3) & 6.25(-3) \\ 5.17(-1) & 6.25(-3) & 5.70(-3) \end{bmatrix} \tag{17}$$

We have solved for the product $\phi^* V^*$ rather than ϕ^* alone to facilitate comparison with the results obtained from cluster galaxies. V^* is found to be $6.40 \times 10^4 \mathrm{Mpc}^3$ from equation (11). The value of χ^2 obtained from the least squares fit is 1.4 for four degrees of freedom. The solid line in Figure 1 shows equation (16) evaluated using the least squares solution for the free parameters.

The effects of non-Hubble velocities and of a local density enhancement, characterized by $<\Delta v^2>^{1/2}$ and ξ^* respectively, may be found by postulating non-zero values and again solving for ϕ^*, L^* , and α . The effects of non-zero values for these quantities is shown by broken lines in Figure 3. A value of $<v^2>^{1/2} = 200$ km sec^{-1} changes M^* by .34 magnitudes. A value of $\xi^* = .04$, corresponding to a density enhancement at $M_{B(0)} = -16.5$ of a factor of four, changes the parameter α

by .29 . Such a local enhancement may be possible. The Local Group may have decelerated the two nearest neighboring groups (the South Polar Group and the M81 Group) leaving an unbound density enhancement of that order.

We would also like to compare the proposed analytic representation with counts of galaxies in rich clusters. We can define a luminosity function for clusters of galaxies if we can agree on a definition for a cluster of galaxies. If V_i is the volume associated with the i^{th} cluster of galaxies, then

$$\phi_c(L)\Delta L \equiv \lim_{k \to \infty} \frac{\sum_{i=1}^{k} n(L,\Delta L,V_i)}{\sum_{i=1}^{k} V_i} \tag{18}$$

$\phi_c(L)\Delta L$ can be determined with arbitrary precision if a large enough unbiased sample of clusters is used. There is unfortunately, no universal agreement as to the definition of a cluster, and in particular there is disagreement as to the physical extent of clusters (for example, see Yahil 1974). It is possible, however, that for a reasonable range of definitions of clusters, the derived luminosity function differs only by a multiplicative factor. The shape of the luminosity function would then be independent of the cluster definition. The parameter ϕ^* would depend upon the definition used but the parameters α and L^* would be independent of the definition used. For a sample of clusters of galaxies, the expected number of galaxies is then

$$n(L)dL = n^* \exp(-L/L^*)(L/L^*)^{\alpha} \, d(L/L^*) \tag{19}$$

where n^* depends upon the number of clusters included and the cluster definition used.

We have constructed a composite set of counts from counts in thirteen of the fifteen clusters studied by Oemler (1974). The clusters Abell 2670 and ZwCl 1545.1 + 2104 were not used, since counts in these clusters were obtained at very faint apparent magnitudes and are extremely sensitive to the number of background galaxies subtracted. Furthermore, the five galaxies classified as cD galaxies by Oemler have been deleted on the hypothesis that some extraordinary phenomenon has greatly increased their luminosity. Statistical uncertainties were computed under the assumption that no background correction has been subtracted from the data. Since background corrections have in fact been applied, the uncertainties have been underestimated. The composite set of counts is presented in Figure 2.

Before obtaining a best fit of the expected counts to the observed counts, we note that since the data presented in Figure 2 have been binned twice, once by Oemler and again here, the correction for finite binning becomes

$$n_c(M) = n(M) + n''(M)(\Delta M)^2/8 \qquad (20)$$

The least squares solution for the parameters n^*, $M^*_{J(24.1)}$, and α yields the following results

$$n^* = 910 \pm 120 \qquad M^*_{J(24.1)} = -21.41 \pm .10 \qquad \alpha = -1.24 \pm .05$$

$$\Sigma = \begin{bmatrix} 1.33(4) & 1.12(1) & 5.22(0) \\ 1.12(1) & 1.09(-2) & 4.13(-3) \\ 5.22(0) & 4.13(-3) & 2.32(-3) \end{bmatrix} \qquad (21)$$

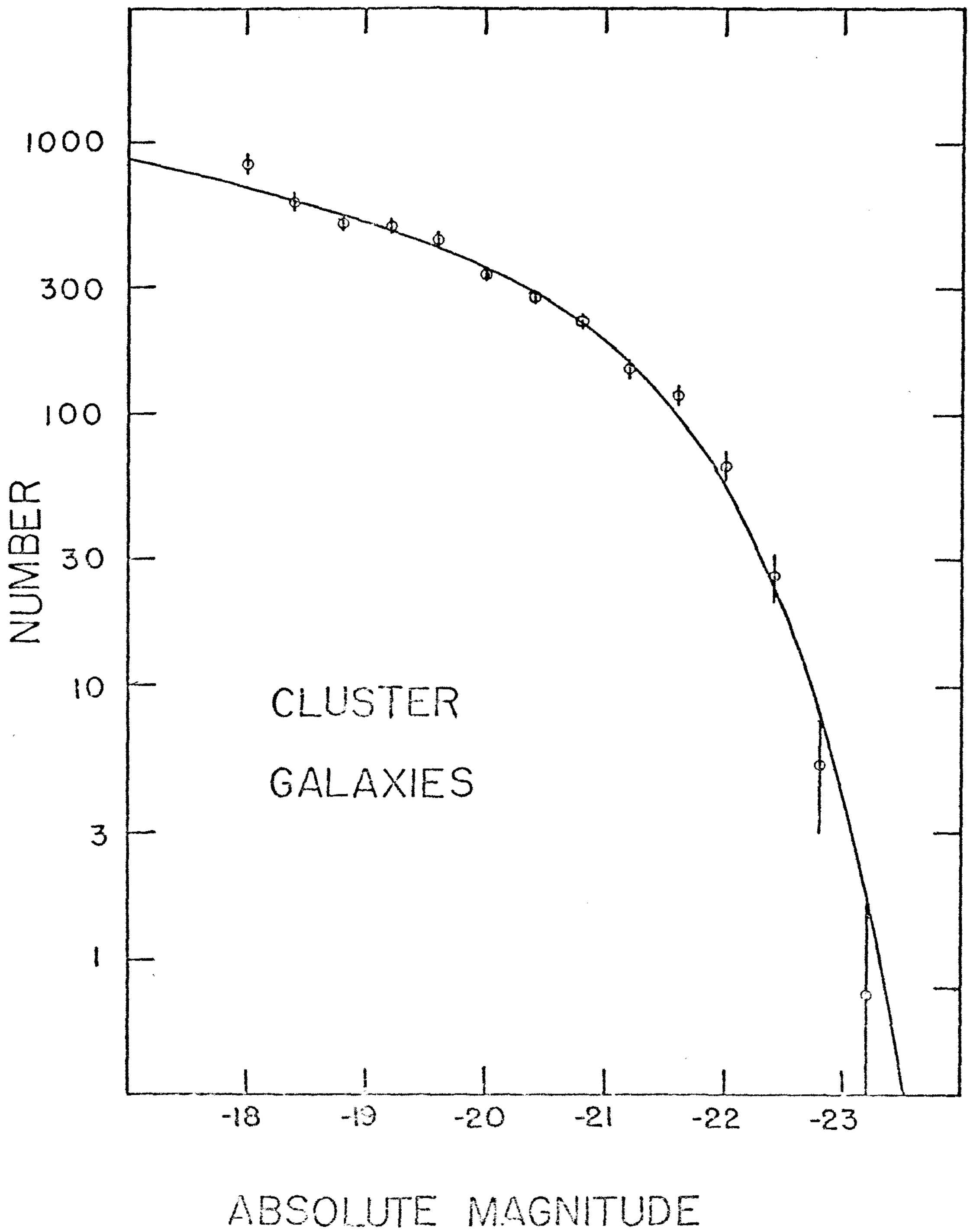


Fig. 2. Composite counts of cluster galaxies as a function of $M_{J(24.1)}$. Solid line shows best fit of proposed representation to data.

A value of $\chi^2 = 16.5$ for 11 degrees of freedom was obtained, which indicates reasonably good agreement considering the underestimation of the uncertainties. The solid line in Figure 2 shows equation (20) evaluated using the least squares solution for the three parameters.

We are in a position to compare the luminosity functions for field and cluster galaxies by comparing the solutions for the parameters M^* and α . There are strong reasons, however, to expect the luminosity functions to be substantially different for the two samples. The field galaxies are dominated by spirals, while the cluster galaxies are dominated by lenticulars and ellipticals. The characteristic magnitude M^* might be different either because the masses of spirals and ellipticals are very different (Page 1965), or because the mass to luminosity ratios are very different for the two classes. If we assume for the moment that the two classes of galaxies have the same distribution as a function of mass, the parameter α will still depend first upon the variation of the mass-to-luminosity ratio with mass, and second, with the variation of isophotal magnitude with luminosity. There is no reason to expect these to be the same for the two classes of galaxies.

Before the solutions for M^* for field and cluster galaxies may be compared, they must be reduced to a common system of magnitudes. The value obtained for field galaxies in the B band may be corrected to the J band using Oemler's conversion formula

$$J = B - .65\ (B-V) \tag{22}$$

along with the mean color for the galaxies in the local sample $\langle(B-V)\rangle = .75$ magnitude. A correction should also be made so that the absolute magnitudes refer to the same limiting isophote, but this is difficult to accomplish with any degree of reliability. We proceed, however, noting that Oemler (1974) gives a correction to total magnitude of the order of two-tenths of a magnitude, while de Vaucouleurs and de Vaucouleurs (1964) give corrections to total magnitude of .65 magnitude for "early" type galaxies and .33 magnitude for "late" type galaxies. Since the local samples contain equal numbers of early and late types, a correction of the order of five-tenths of a magnitude seems reasonable.

Figure 3 shows the solutions for α and $M_J^*(\infty)$ for field and cluster galaxies. The ellipses show formal 50% confidence intervals obtained from the covariance matrices. The uncertainties in absolute magnitude are somewhat larger due to the uncertainty in conversion to total magnitude. The broken lines show the results of least squares fits to the local data assuming non-zero values for the non-Hubble component of radial velocities and the local density enhancement.

The agreement between the field and cluster galaxies is rather remarkable given the difference in the two types of galaxies. (Such agreement has been noted before by Peebles 1971). The agreement improves if one takes account of the possible systematic effects in the local determination. The effects of local density enhancement and of non-Hubble radial velocities will be substantially diminished when radial velocities are available for a complete sample of galaxies to a fainter limiting magnitude. Until such data are available, we suggest that a "standard" value of $\alpha = -5/4$ be used in calculations using the proposed analytic representation. Constraining α to this value and again fitting the local data, we find $M^*_{B(0)} = -20.68$ and $\phi^* = 4.23 \times 10^{-3} \mathrm{Mpc}^{-3}$. Likewise a constrained fit to the cluster

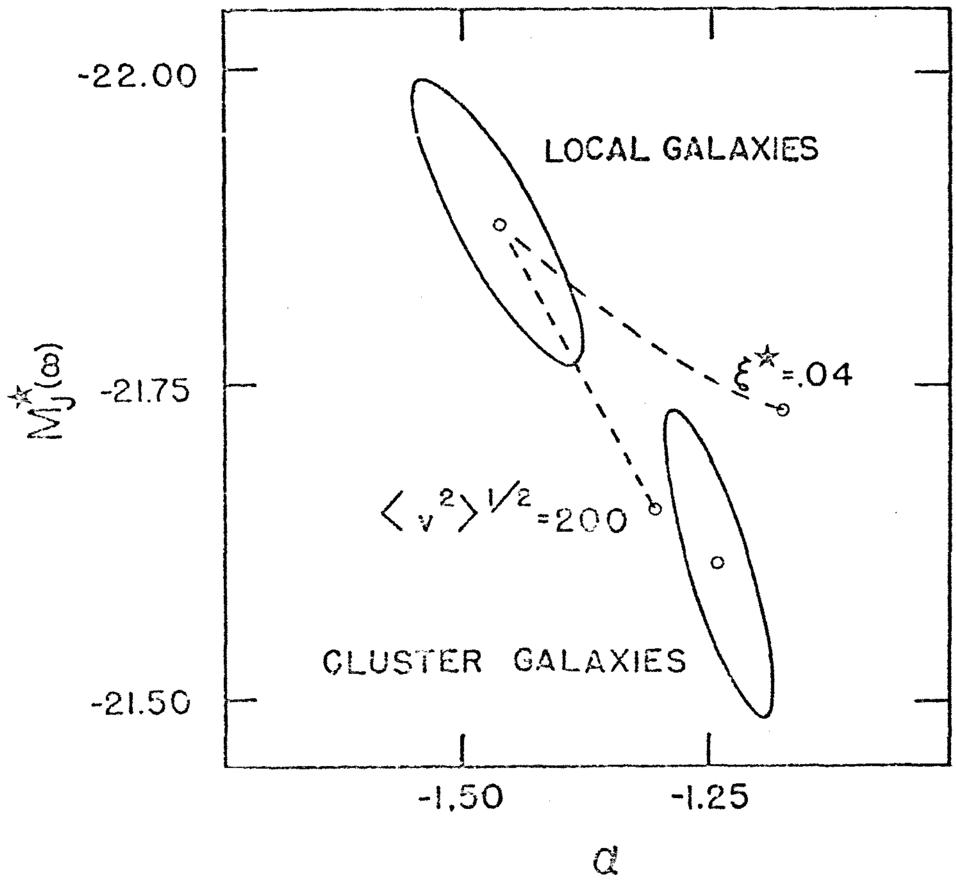


Fig. 3. Results of least squares solutions for parameters $M^*_{J(\infty)}$ and α. Ellipses show 50 percent confidence contours. Broken lines show effects of local density enhancement and non-Hubble component of radial velocities.

luminosity function yields $M^*_{J(24.1)} = -21.43$.

4. L^* as a Standard Candle

The existence of a characteristic luminosity L^* was first noted by Abell (1962). He represented the integrated luminosity function $N(\geq L))$ for galaxies in clusters by two power laws:

$$N(\geq L) = \begin{cases} N^*\left(\frac{L}{L^*_A}\right)^{\alpha+1} & L < L^*_A \\ N^*\left(\frac{L}{L^*_A}\right)^{\beta+1} & L > L^*_A \end{cases} \qquad (23)$$

where L^*_A is Abell's characteristic luminosity, and where $\alpha \simeq -1.5$ and $\beta \simeq -3$. He further noted that L^*_A seemed remarkably constant from cluster to cluster. Bautz and Abell (1972) report a dispersion $\sigma(M^*_A)$ in the absolute magnitude of the characteristic luminosity

$$\sigma(M^*_A) = 0.16 \text{ magnitude} \qquad (24)$$

for a sample of 8 rich clusters of galaxies. The integrated luminosity function $N(\geq L)$ is not suitable for least squares analysis, however, since the points are not statistically independent. When differentiated, equation (23) exhibits a "hump" which is present in the Coma Cluster (Abell 1962) but absent in most other clusters (Oemler 1974).

If we consider the purpose of L^*_A as defining the luminosity at which the luminosity function exhibits a rapid change in logarithmic slope, then L^*_A and L^* as defined in this work serve the same purpose. We are naturally led to ask whether L^* is the same for all clusters. The method of least squares is particularly useful in this respect, since it gives a value for L^* and an uncertainty in that value. The less well the proposed representation fits, the more uncertain the solution for L^*. In such fits the value of α is constrained to -5/4 since the data are not sufficient to determine three parameters.

It is possible to use the composite luminosity function to predict the expected dispersion in M^*. When α is constrained to the "standard" value, M^* is determined for the composite luminosity function with an uncertainty of .058 magnitude. Since thirteen clusters went into that composite, we expect the average uncertainty for a typical individual cluster to be of the order of $\sqrt{13}$ times as large, or .21 magnitude.

The results of the least squares analysis for each of fourteen clusters are given in Table 2. Column (2) of that table gives the solution for $M^*_{J(24.1)}$ for each cluster, and column (4) gives the uncertainty in that value. Column (3) gives the residual from the mean value. Column (5) gives the value of n^* found by constraining $M^*_{J(24.1)}$ to the mean value, -21.37, and column (6) gives the uncertainty in that value. The values in parenthesis for Abell 2670 were obtained deleting those galaxies fainter than $M_{J\,(24.1)} = -19$.

For the purpose of cosmological tests, one is less interested in the dispersion about the mean value of M^* than one is in the accuracy with which that mean value of M^* can be determined. (The deceleration parameter would be well determined if we had a large number of clusters at $z = .4$ and a large number at $z = 0$. The mean value of M^* at each redshift would then determine q_0.) We can use the uncertainties in Table 3 to compute a mean characteristic magnitude $\langle M^* \rangle = -21.37$

TABLE 2

Magnitude Residuals for Fourteen Rich Clusters of Galaxies.

Cluster	M^*	$M^* - \langle M^* \rangle$	$\sigma(M^*)$	n^*	$\sigma(n^*)$
194	-22.32	- 0.95	0.64	22	4
400	-21.41	- .04	.33	35	2
539	-21.26	+ .11	.22	44	7
665	-22.52	- 1.15	.66	120	32
1228	-21.32	+ .05	.25	42	2
1314	-21.41	- .04	.11	47	2
1367	-21.30	+ .07	.37	50	6
1413	-21.36	+ .01	.24	125	5
Coma	-21.26	+ .11	.20	117	4
1904	-21.50	- .15	.17	84	8
Herc	-21.52	- .15	.17	58	5
2197	-20.63	.74	.44	66	4
2199	-21.06	.31	.16	75	10
2670	-22.09	- .72	.53	57	11
(2670)	(-21.45)	(- .08)	(.47)	(93)	(13)

and an uncertainty in the mean of .065 magnitude. Multiplying this by $\sqrt{14}$ to obtain an "equivalent" dispersion in M^* we find

$$\sigma_{eq}(M^*) = 0.24 \text{ magnitude} \tag{25}$$

in good agreement with that expected from the composite luminosity function. While this result is larger than that found by Bautz and Abell, it is assuredly free from any bias since the solution is accomplished by an automatic method without human intervention. Furthermore, there is hope of improving the uncertainties by restricting attention to the central regions of clusters, minimizing the effect of uncertainties in the subtracted background.

5. Significance of the Brightest Member of a Rich Cluster of Galaxies

Peebles (1968) has suggested that the narrow dispersion in absolute magnitude of the brightest members of a rich cluster of galaxies can be understood if one assumes that a single universal luminosity function applies to all clusters of galaxies. He has also reversed the argument and used the narrow dispersion in M_1 to compute the logarithmic slope of such a luminosity function at the bright end (Peebles 1969). Peterson (1970b) has used the analytic representation proposed by Abell to show that one expects a correlation of cluster richness and the absolute magnitude of the brightest member. He extended Abell's representation with two additional power laws (Peterson 1970c) to obtain a more refined prediction of such a correlation.

The proposed analytic representation of the cluster luminosity function may be used to compute the most probable value of the absolute magnitude of the j^{th} brightest galaxy as a function of n^* under the assumption that L^* is a universal constant. The probability that the j^{th} brightest galaxy has absolute magnitude M is given by

$$P_j(M)dM = \frac{[N(\leq M)]^{j-1}}{(j-1)!} \exp[-N(\leq M)]\, n(M)\, dM \qquad (26)$$

where $N(\leq M)$ is the expected number of galaxies brighter than M. The most probable value of the absolute magnitude of the j^{th} brightest galaxy is found by setting the first derivative $P_j(M)$ equal to zero while the second derivative gives a reasonably good estimate of the root-mean-squared dispersion about that value. (The use of the most probable value of M_j rather than the mean makes only a few hundredths of a magnitude difference, but greatly simplifies the numerical computation.) A good estimate of the richness (as defined by Abell 1958) of a cluster with a given n^* is made by computing the most probable value of the absolute magnitude of the third brightest galaxy and the expected number of galaxies in the subsequent two magnitude interval. We have used the proposed representation with $\alpha = -5/4$ to compute the most probable values of M_j^* for $j=1,2$ and 3 and an estimate of the richness for a range of values of n^*. The results are shown in Figure 4. The broken lines show the estimated rms dispersion of M_1 about the most probable value. Also shown is an estimate of the "population", N_c^{48}, defined by Sandage and Hardy (1973) to be the number of galaxies in the 2-1/2 magnitude interval following the third brightest galaxy in a circle of radius 2 Mpc centered on the cluster.

The proposed representation predicts a range of M_1 with richness which over a reasonable range of values of n^* $(15 \leq n^* \leq 150)$

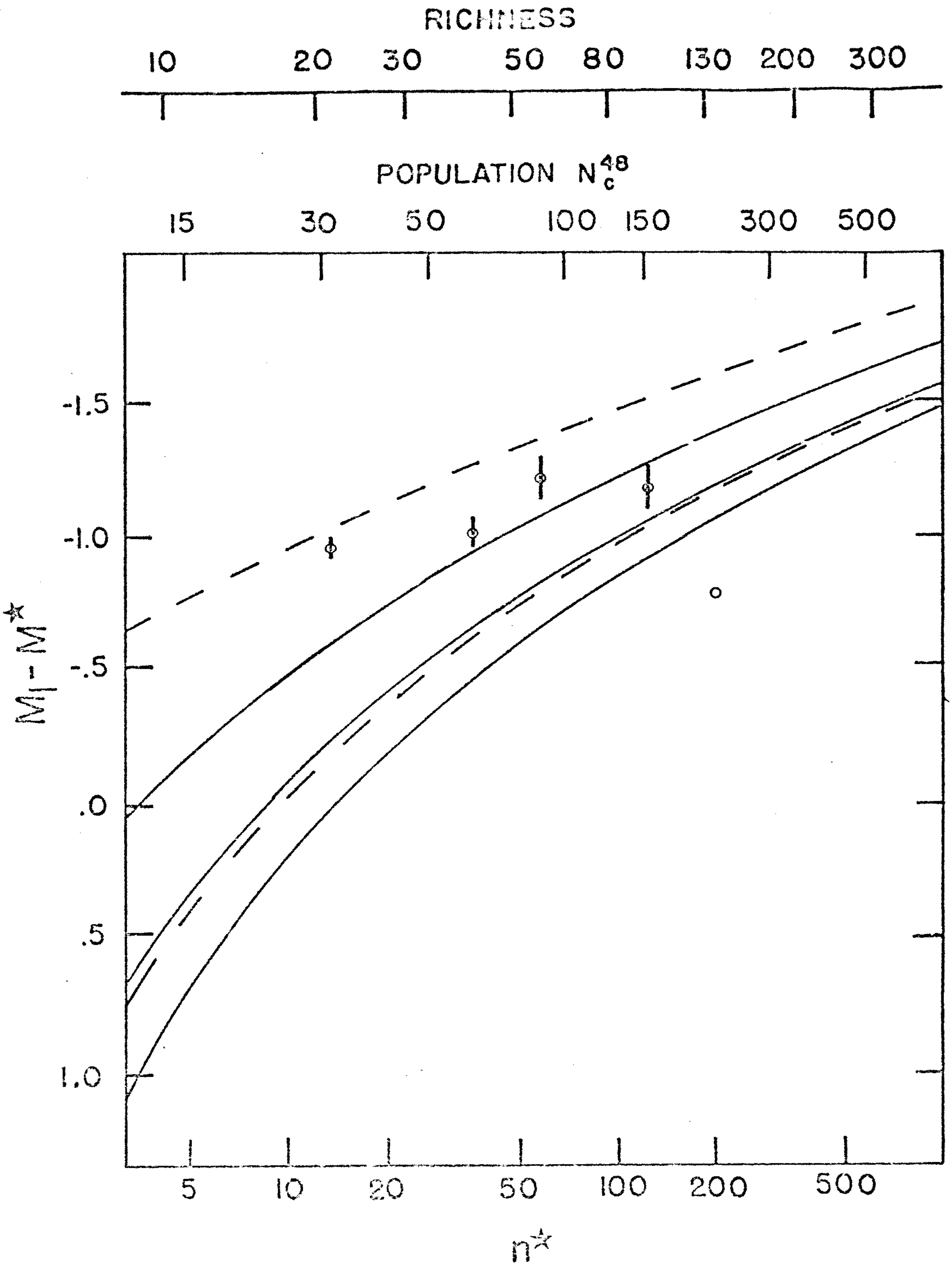


Fig. 4. Solid lines give most probable values of absolute magnitude for first, second and third ranked cluster galaxies as a function of n^* . Broken lines show expected rms deviation in magnitude of brightest member. Open circles give mean absolute magnitudes for brightest members of richness classes 0-4. Upper scales allow for conversion from n^* to richness or population.

is of the same order as the internal dispersion in M_1. Sandage and Hardy (1973) have estimated N_c^{48} for 79 rich clusters of galaxies. Breaking these up into Abell richness classes they compute a mean absolute magnitude for each richness class. A mean "population" has been computed for each class and the resulting points have been plotted in Figure 4, using an arbitrary vertical normalization. It may be seen that there is little or no observed correlation of M_1 with N_c^{48}. We hasten to point out, however, that the data plotted in Figure 4 are a composite of photometry by Sandage (1972) with an 87 kpc aperture and photometry by Peterson (1970a) with a 41 kpc aperture. Peterson's data show a marginal anti-correlation of luminosity and cluster richness, while Sandage's data show a marginal positive correlation (for comparison, see Sandage 1972). It is clear that one must be very careful to specify whether one means total, standard metric or isophotal magnitudes. Since the analytic representation was fitted to Oemler's isophotal magnitudes, it is not strictly applicable. For example, we might expect a weaker correlation of standard metric absolute magnitude with richness if brighter galaxies have lower average surface brightness, as found by Oemler (1974).

Figure 4 may be used to compute the expected dispersion in M_1 for a given sample of clusters. There are two sources of dispersion in M_1. The first is due to statistical fluctuations in M_1 at a given value of n^* (or N_c^{48}). The second is due to the dispersion in n^* (or N_c^{48}) for a given sample of clusters. The mean value for N_c^{48} for the sample of Sandage and Hardy is 68 ± 42. The dispersion in M_1 at $N_c^{48} = 68$ is .32 magnitude. The approximate range of the most probable value of M_1 over the range of N_c^{48} is also .32 magnitude. The expected dispersion in M_1 is then .45 magnitude. This is somewhat larger than the value obtained by Sandage and Hardy

$$\sigma(M_1) = 0.34 \text{ magnitude} \tag{27}$$

It appears that in large part the narrow dispersion in M_1 can be understood as a reflection of the constancy of M^*. There may well be, however, surface brightness effects which further narrow this dispersion in M_1.

6. Simple Calculations Employing the Proposed Analytic Representation

The proposed analytic representation has the advantage that its moments may be written as products of gamma functions and powers of L^*. For example, the mean luminosity density in galaxies is readily calculated:

$$\lambda_{tot} = \int_0^\infty L\phi(L)\, dL = \Gamma(\alpha+2)\ \phi^* L^* \tag{28}$$

where we have assumed that the representation is valid beyond the range $5L^* > L > (L^*/20)$ for which its validity has been established. It is important to note that while the number density of galaxies diverges for $\alpha = -5/4$, the luminosity density does not. Furthermore, the contribution to total luminosity from outside the established range is rather small: only eleven percent from galaxies fainter than $L^*/20$ and four-tenths of a percent from galaxies brighter than $5L^*$.

Another simple calculation is the mean luminosity of galaxies in an apparent magnitude limited sample:

$$\langle L \rangle = \frac{\int_0^\infty L^{5/2}\ \phi(L)\ dL}{\int_0^\infty L^{3/2}\ \phi(L)\ dL} = \frac{\Gamma(\alpha + \frac{7}{2})}{\Gamma(\alpha + \frac{5}{2})}\ L^* = (\alpha + \tfrac{5}{2})\ L^* \qquad (29)$$

If $\alpha = -5/4$, then the average luminosity in such a sample is $5/4\ L^*$, or .25 magnitude brighter than M^*. Taking the "standard" value $M^*_{B(0)} = -20.68$, we see that Messier 31 with $M_{B(0)}$ approximately equal to -20 (Gunn 1974) is not even an "average" galaxy.

7. Conclusions

We find that the proposed analytic representation adequately represents the observed luminosity function over a range of roughly six magnitudes. The data appear to be marginally consistent with the hypothesis that the luminosity functions for field and cluster galaxies are identical. The characteristic luminosity L^* can be determined for a rich cluster of galaxies with an uncertainty of roughly .24 magnitude. The constancy of L^* from cluster to cluster indicates that it is a valuable standard candle. It remains to be seen, however, with what accuracy L^* can be determined for the more distant clusters. The narrow dispersion in absolute magnitude for brightest cluster members can largely be understood as a consequence of the exponential luminosity cutoff exhibited at the bright end of the luminosity function of galaxies. The absence of a correlation of absolute magnitude with richness remains unexplained. The proposed analytic representation can simplify calculations requiring the luminosity function, and provides a useful reference magnitude for discussions of absolute magnitudes of galaxies.

Chapter II

THE CLUSTERING OF RANDOM PREGALACTIC PERTURBATIONS

1. Introduction

Not only is the universe inhomogeneous on the scale of galaxies, it is inhomogeneous on scales including large numbers of galaxies. The groups of galaxies catalogued by de Vaucouleurs (1968) and the clusters of galaxies catalogued by Abell (1958) are not mere fluctuations in a homogeneous distribution of galaxies. The spatial covariance function for galaxies (Peebles 1974) shows that on scales of order ten megaparsecs, galaxies exhibit significant clustering.

In marked contrast to the inhomogeneous distribution of galaxies stands the extreme isotropy of the cosmic microwave background radiation (Boynton and Partridge 1973). If this radiation was last scattered during the epoch at which, according to standard Friedmann cosmology, the primeval plasma recombined, then the radiation distribution at that epoch was remarkably homogeneous. The discovery of the cosmic background radiation (Penzias and Wilson 1965) sparked new efforts at explaining how substantial inhomogeneities could have condensed out of a homogeneous matter distribution.

Several important results have emerged from these efforts (see Field 1968 for a review). First, adiabatic perturbations to a Friedmann universe undergo severely damped acoustic oscillations in the period prior to recombination (Silk 1967,1968). Adiabatic perturbations smaller than 10^{12} solar masses in a critically bound ($\Omega = 1$) universe and smaller than 10^{14} solar masses in a cosmologically open ($\Omega = .03$) universe cannot have survived recombination (Peebles and Yu 1970).

Second, the photon fluid and matter fluid are strongly coupled via Compton scattering. The time scale for the damping of bulk motion of matter with respect to the photon fluid is short (Peebles 1971), therefore those isothermal perturbations smaller than the above limits which did survive recombination can have had little peculiar velocity with respect to similar neighboring perturbations. Finally, this same photon drag strongly limits the rate at which isothermal perturbations may grow prior to recombination. Any isothermal perturbations which existed at recombination must therefore have existed for long times prior to recombination.

Field (1968) gives rough estimates (depending upon Ω and the collapse time for galaxies) of the density perturbations required at recombination to give rise to galaxies. Peebles (1972) has argued that randomly distributed inhomogeneities on a given scale lead through gravitational instability to inhomogeneities on a larger scale. Combining these two lines of attack, we argue here that granted the density perturbations required to make galaxies, and granted the assumption that these pregalactic perturbations were independently, randomly distributed at recombination, one expects clustering of galaxies on the scales presently observed. We ignore the question of the origin of these pregalactic perturbations, and argue only that whatever their origin, they will form clusters.

The assumption of independently distributed pregalactic perturbations is rather strong. It is hard to imagine an evolution of such perturbations from a completely homogeneous universe which would not leave anti-correlations: if there is a positive density perturbation, there ought to be a negative one nearby. But it is justified observationally by Peebles' (1974) determination of the spatial covariance function for galaxies and warrants theoretical examination as a natural origin for clusters of galaxies.

2. Preliminary Considerations

Field (1968) and Gunn and Gott (1972) have shown how to use estimates of the dynamical parameters of collapsed systems to estimate the size of density fluctuations required to yield such systems. For the case of a radiation free Friedmann universe, the value of the Hubble parameter h at epoch $(1+z)$ is related to its present value, h_0, and the present value of the deceleration parameter, q_0, according to the relation

$$h^2 = h_0^2[2q_0(1+z)^3 + (1-2q_0)(1+z)^2] \tag{1}$$

(Misner, Thorne, and Wheeler 1973). The matter density at any epoch is related to the deceleration parameter and the Hubble parameter according to the relation

$$qh^2 = \frac{4}{3}\pi\rho \tag{2}$$

where all quantities are expressed in "geometrized" units with $G = c = 1$. Suppose that a point mass m is introduced into an otherwise uniform Friedmann universe with density ρ. Following Gunn and Gott (1972) the binding energy dE of a spherical shell of radius r surrounding the point mass is given in the Newtonian approximation by

$$dE = \left[\frac{(\frac{4}{3}\pi r^3 \rho + m)}{r} - \frac{1}{2} h^2 r^2 \right] 4\pi r^2 \rho dr \tag{3}$$

Letting $\mathcal{M} \equiv \frac{4}{3}\pi R^3 \rho$ and integrating this energy over all shells out to radius R we find

$$E = \frac{3}{10} [(2q_o - 1) + 5 \frac{m}{\mathcal{M}} q_o (1+z)] (\frac{\mathcal{M}}{q_o h_o^2})^{2/3} h_o^2 \mathcal{M} \tag{4}$$

It should be noted that this expression does not include any binding energy associated with the point perturbation. We can also calculate a binding energy when the perturbing mass is uniformly distributed out to a radius R , yielding

$$E = \frac{3}{10} [(2q_o - 1) + 2 \frac{m}{\mathcal{M}} q_o (1+z)] h_o^2 (\frac{\mathcal{M}}{q_o h_o^2})^{2/3} (1+\delta) \mathcal{M} \tag{5}$$

where δ is the fractional density perturbation, defined as the ratio of the perturbing mass m to the unperturbed mass $\mathcal{M}$. It is straightforward to calculate the time required for the shell at radius R to expand to maximum extent and then to collapse to zero volume. Defining t_c as the time required to collapse from maximum to zero volume, we find

$$t_c = \frac{\pi q_o}{h_o 2\sqrt{2} \, [(2q_o - 1) + 2q_o \delta (1+z)]^{3/2}} \tag{6}$$

These three relations are identical to those derived by Gunn and Gott, but differ from those of Field in that we have assumed that the perturbing mass has not perturbed the Hubble flow. Radiation pressure will have prevented an isothermal fluctuation from decelerating at a rate different from the universe.

3. The Amplitude of Pregalactic Density Perturbations

We wish to determine the amplitude required of pregalactic perturbations existing at recombination. We must therefore recast our relations in terms of presently observable properties of galaxies. Dividing both sides of equations (4) and (5) by $\mathcal{M}$ we get a binding energy per unit mass. We shall identify this with $\frac{1}{2}\langle v^2 \rangle$ where $\langle v^2 \rangle$ is the mean squared velocity of stars in a galaxy of mass $\mathcal{M}$ +m. This identification will be valid only if there is little dissipation in the collapse of galaxies. This in turn will be true if the time scale for star formation t_s is short compared to the collapse time scale t_c. Gott (1974) has elaborated arguments in favor of dissipationless collapse. Then neglecting terms of order δ we have

$$\langle v^2 \rangle \simeq \frac{3}{5}\left[(2q_0-1) + fq_0 \frac{m}{\mathcal{M}}(1+z)\right]h_0^2\left(\frac{\mathcal{M}}{q_0 h_0^2}\right)^{2/3} \qquad (7)$$

where f takes on a value somewhere between 2 for smooth perturbations and 5 for point perturbations.

We are faced with the problem that both the masses $\mathcal{M}$ of galaxies and the deceleration parameter q_0 are known only to within a factor of ten. When these two quantities appear as a ratio we may lump our ignorance into a single parameter β, the present ratio of the present matter density to the density in the form of galaxies. Assuming that the mass distribution of galaxies is given by

$$\phi(\mathcal{M})\, d\mathcal{M} = \phi^* \exp\left[-\frac{\mathcal{M}}{\mathcal{M}^*}\right]\left(\frac{\mathcal{M}}{\mathcal{M}^*}\right)^{\alpha} d\left(\frac{\mathcal{M}}{\mathcal{M}^*}\right) \qquad (8)$$

we have

$$\beta \frac{4}{3} \pi \rho_{gal} = q_o h_o^2 = \beta \frac{4}{3} \pi \Gamma(\alpha+2)\ \phi^*\ \mathfrak{M}^* \tag{9}$$

which may be solved for the ratio $\mathfrak{M}^*/q_o$.

Let $\langle v^{*2} \rangle$ be defined as the mean squared velocity in a galaxy of mass $\mathfrak{M}^*$ and m^* be the associated perturbing mass. Then ignoring the term $(2q_o-1)$ we have from equations (4) and (5)

$$f q_o \delta^*(1+z) \simeq \frac{5}{3} \langle v^{*2} \rangle \left[\frac{4\pi\Gamma(\alpha+2)\beta\phi^*}{h_o^3}\right]^{2/3} \tag{10}$$

and

$$m^* \simeq \frac{5}{3f} \frac{\langle v^{*2} \rangle}{(1+z)} \left[\frac{3}{4\pi\beta\ \Gamma(\alpha+2)\ \phi^*}\right]^{1/3} \tag{11}$$

where f takes on the value 2 for smooth perturbations and 5 for point perturbations. The fractional density enhancement δ^* required to form a characteristic galaxy is inversely proportional to q_o and proportional to $\beta^{2/3}$. Furthermore, since $\phi^* \sim h_o^3$, δ^* is independent of the Hubble constant. The mass m^* of such a perturbation does not contain q_o explicitly and depends only weakly upon β . It seems then that regardless of cosmological model, the mass required to form galaxies is remarkably well determined by observable quantities: $\langle v^{*2} \rangle$ and ϕ^* .

4. The Amplitude of Precluster Perturbations

There is no general agreement on the definition of a cluster or group of galaxies. What is certain is that they constitute a substantial enhancement in the number density of galaxies. We can nonetheless obtain a rough estimate of the size of perturbations required to produce clusters. We let equation (6) define a characteristic dynamical time. Setting δ equal to zero, we get a universal characteristic time. In the event of an open universe, this time is imaginary, but this is of little concern. What does concern us is the fact that in order for a perturbation to show up as a substantial density enhancement at the present, its characteristic dynamical time must be substantially different from the universal characteristic time. Assuming that the term $(2q_0-1)$ is of order unity, we get

$$2q_0\delta_c(1+z) \simeq 1 \tag{12}$$

where the subscript c refers to clusters of galaxies. The approximate equality indicates that the exact value of the quantity depends upon one's definition of a cluster. Note that by dividing equation (12) by equation (10) we obtain the ratio of δ^* to δ_c independent of q_0.

Suppose that we chose at random a volume V at recombination. The density of pregalactic perturbations must have been $(1+z)^3$ times greater than the present density of galaxies. Letting $n^* \equiv \phi^* V(1+z)^3$, such a volume would contain a mass $\beta\Gamma(\alpha+2)n^*\mathcal{M}^*$. This volume will form a cluster if there is a mass perturbation sufficiently large to satisfy the cluster formation criterion, equation (12). We shall express the perturbing mass in units of m^* letting it equal some unknown quantity $\Delta(n^*)$ times $\Gamma(\alpha+2)m^*$. Substituting into equation (12) yields

$$2q_o \frac{\Delta(n^*)}{\beta n^*} \frac{m^*}{\mathcal{M}^*} (1+z) \simeq 1 \tag{13}$$

But $m^*/\mathcal{M}^* = \delta^*$, and we have already solved for $q_o\delta^*(1+z)$. We therefore have

$$\frac{\Delta(n^*)}{n^*} \simeq \frac{3f\ \beta^{1/3}}{10\langle v^{*2}\rangle} \left[\frac{3h_o^3}{4\pi\ \Gamma(\alpha+2)\ \phi^*}\right]^{2/3} \tag{14}$$

The quantity on the left hand side is just the fractional increase in the density of pregalactic perturbations required to form a cluster.

5. Model for the Origin of Clusters of Galaxies

We have as yet said nothing about the nature of the perturbations giving rise to clusters. But expressing the criterion for the formation of clusters in terms of a fractional increase in the density of pregalactic perturbations anticipates our hypothesis concerning the origin of clusters: that the pregalactic perturbations were independently, randomly, uniformly distributed and that statistical fluctuations in the density of such perturbations gave rise to clusters of galaxies. Peebles (1974) has interpreted the slope of the spatial covariance function for galaxies as indicating a white noise perturbation spectrum at recombination, which is just what would obtain from random pregalactic perturbations. The amplitude of the perturbation spectrum would then be fixed by the amplitude and density of pregalactic perturbations. Since our calculations have assumed spherical symmetry, we shall only consider spherical clusters in our model. Any spherical region for which the fractional density enhancement in pregalactic perturbations at recombination $\Delta(n^*)/n^*$ exceeds the value given by equation (14) is assumed to form a cluster. Every pregalactic perturbation will end up in a cluster of some size, even if it is only a "cluster" of one. We ignore dynamical effects and assume that the final members of each cluster can be predicted at recombination by locating an appropriate overdense spherical region at recombination. In the event that several pregalactic perturbations are members of two overlapping overdense regions, we shall assume that they end up as members of the larger cluster.

A specific test of our model would be the comparison of the observed distribution of cluster sizes with that predicted by the model. But we have not yet calculated this expected distribution. We could, of course, compute this distribution by Monte Carlo methods: we could distribute points randomly and search for clusters of all sizes, starting with the largest size of interest. But the fractional density enhancement $\Delta(n^*)/n^*$ depends upon the assumed or observed values of f, β, $\langle v^{*2}\rangle$ and ϕ^*. We would therefore need to perform such searches for different values of $\Delta(n^*)/n^*$. Searching for clusters by computer is unfortunately extremely time consuming. It would therefore be desirable to have an analytic expression for the distribution of clusters.

An analytic treatment of the problem is greatly simplified by treating randomly distributed objects with equal mass. A cluster characterized by n^* will then have some effective number of randomly distributed equal mass pregalactic perturbations associated with it.

We assume that the distribution (8) is valid for the masses m of pregalactic perturbations as well as for galaxies, with m^* replacing $\mathcal{M}^*$ as the characteristic mass. This will be so if $\delta = \delta^*$ for all galaxies, independent of mass. We can then calculate the root-mean-squared fractional mass fluctuation for a volume characterized by n^*

$$\frac{\Delta(n^*)}{n^*} = \frac{\left\{\int_0^\infty m^2 n^* \exp[-\, m/m^*](m/m^*)^\alpha d(m/m^*)\right\}^{1/2}}{\int_0^\infty m\, n^* \exp[-m/m^*](m/m^*)^\alpha\, d(m/m^*)} = \left[\frac{(\alpha+2)}{\Gamma(\alpha+2)n^*}\right]^{1/2} \quad (15)$$

The root-mean-squared fluctuations for a volume characterized by n^* would be the same if one expected $\Gamma(\alpha+2)n^*/(\alpha+2)$ points of equal mass to be contained in such a volume. We therefore set

$$n_{eff} = \frac{\Gamma(\alpha+2)}{(\alpha+2)}\, n^* \quad (16)$$

We have developed an approximate solution to the problem of the clustering of random points into clusters of points with average density contrast $\gamma = (1 + \Delta n/n)$, which we present in detail in Chapter III. If $n_\gamma(k)$ is the density of clusters of k points with density contrast γ, we obtain an approximate solution for $n_\gamma(k)$ by solving the equation

$$\sum_{m=k}^{\infty} n_\gamma(m) \left(m^{1/3} \{1 + 2/[(\gamma-1)m/\gamma + 1]\} - k^{1/3}\right)^3$$

$$= \gamma\rho \sum_{m=k}^{\infty} \frac{(k/\gamma)^m}{m!} e^{-k/\gamma} \quad (17)$$

where ρ is the average point density. The solution to this equation shows factor of two agreement with Monte Carlo experiments for $\gamma = 3$ and $\gamma = 9$, and we are therefore tempted to use it to predict the expected distribution of clusters for a value of γ appropriate to the formation of clusters.

For the purpose of evaluating γ, we shall assume that the entire present mass density is in the form of galaxies, letting $\beta = 1$. This seems reasonable considering the limits on possible density of many different forms of intergalactic matter (Gott, Gunn, Schramm and Tinsley 1974). We shall further assume that the perturbations giving rise to galaxies are centrally condensed, using the value $f = 5$. We adopt the standard values $\alpha = -5/4$ and $\phi^* = 4.23 \times 10^{-3} \mathrm{Mpc}^{-3}$ developed in Chapter I. And finally we shall use a mean squared velocity dispersion $\langle v^{*2} \rangle = 10^{-6}$ ($\langle v^{*2} \rangle^{1/2} = 300$ km sec^{-1} in conventional units). This value is of the order of the velocity dispersions observed in bright elliptical galaxies by Morton and Chevalier (1973). It is smaller than the dispersions observed in giant ellipticals by Minkowski (1962) but larger than the typical maximum rotation velocity observed in spirals (Brosche 1971). Evaluating equation (14) we have

$$\frac{\Delta(n^*)}{n^*} \simeq .53 \qquad (18)$$

indicating that an increase in the density of pregalactic perturbations of slightly more than 50% is required to form a cluster of galaxies.

We have solved equation (17) for the cases $\gamma = 1.48$ and $\gamma = 1.58$ and present the results of such a solution in Figure 1. The ordinate shows $f_\gamma(\geq n_{eff})$, the fraction of points in clusters greater than n_{eff}, where

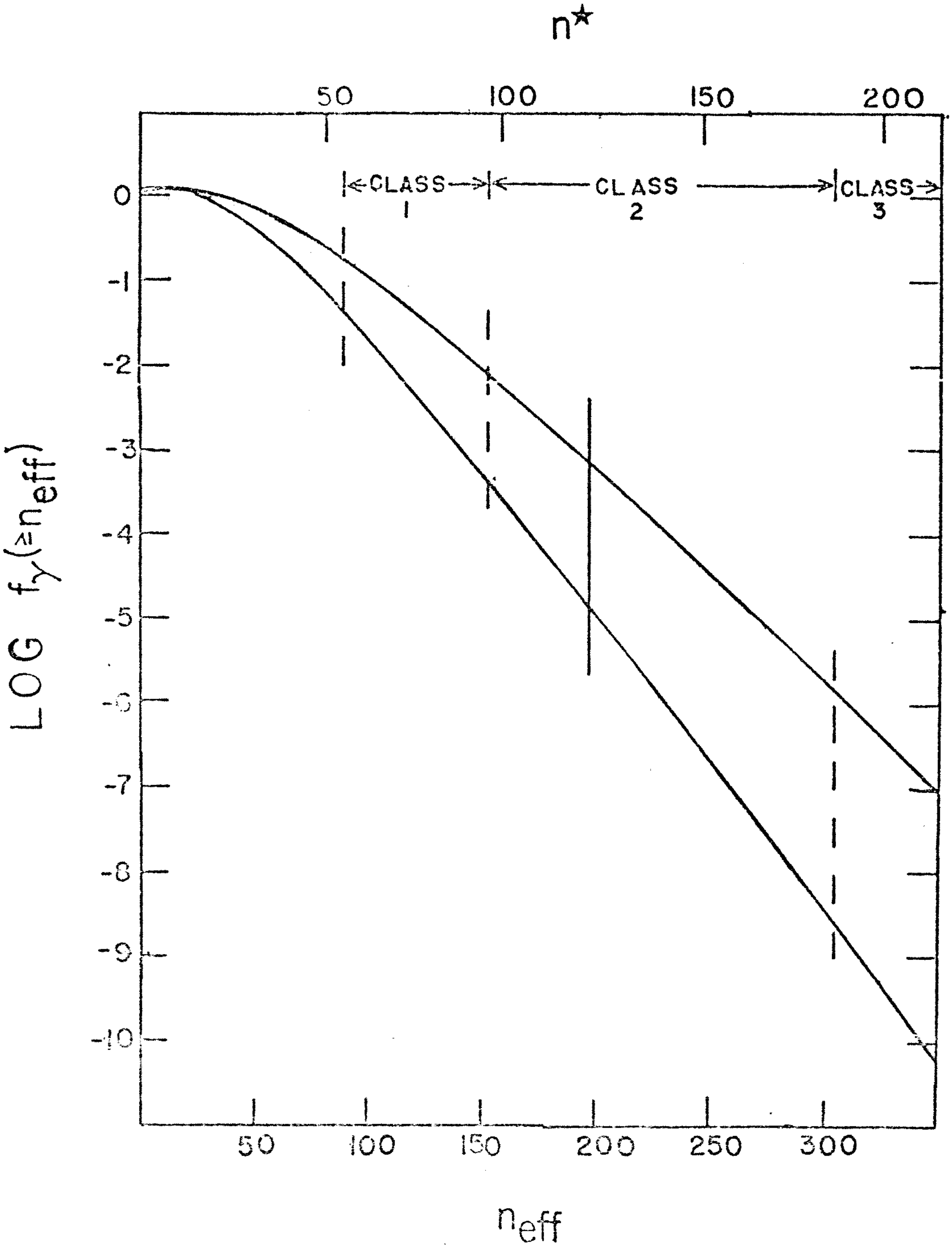


Fig. 1. Fraction of galaxies in clusters of size n_{eff} or larger for $\gamma = 1.48$ and $\gamma = 1.58$. Broken lines indicate boundaries of Abell richness classes. Solid vertical line indicates richest clusters observed by Oemler.

$$f_\gamma(\geq n_{eff}) = \frac{1}{\rho} \sum_{k=n_{eff}}^{\infty} k n_\gamma(k) \tag{19}$$

The abscissa shows n_{eff} and also the corresponding value of n^*. We have indicated the values of n^* appropriate to Abell's (1958) richness classes 1, 2, and 3, and have also drawn a line at $n^* = 120$, the value appropriate to the richest clusters measured by Oemler (1974) (see Table 2, Chapter I). It is unfortunate that the approximate solution for $n_\gamma(k)$ obtained from equation (17) breaks down when γ approaches unity; we discuss the nature of the approximations going into equation (17) in Chapter 3. We nonetheless believe that the solution gives a reasonable order of magnitude estimate for $k >> 1$.

There is little data available against which to test the predicted distribution. Abell's (1958) richness classifications are not sufficiently accurate for the purpose. We note, however, that the Coma cluster, the richest nearby cluster, has $n^* \simeq 117$ and lies at a distance of roughly 140 Mpc (Rood et al 1972). The volume V occupied by a sphere of that radius is $1.15 \times 10^7 Mpc^3$. Assuming that Coma is the largest cluster in a sphere of twice that volume, and taking the ratio of n^* to $2\phi^* V$, we obtain an estimate of the fraction of galaxies in clusters the size of Coma or larger: $f_\gamma(\geq \text{Coma}) = 1.2 \times 10^{-3}$. This result seems reasonably consistent with the predicted 10^{-4} of Figure 1. We also note that for the chosen range of γ, clusters of richness class 3 are extremely unlikely.

The model is conveniently summarized by enumerating the assumptions and uncertainties involved. First, we have assumed that pregalactic density perturbations were randomly distributed at recombination. Second, we assume that galaxies collapse without substantial dissipation. Third, we assume that all matter is in the form of galaxies. Fourth, we assume that pregalactic perturbations were centrally condensed. Fifth, we assume that the mass to luminosity ratio for galaxies is independent of mass. And sixth, we assume that the fractional density perturbations δ giving rise to galaxies were independent of mass. The uncertainties include a substantial uncertainty in the accuracy of the approximate solution for $n_\gamma(k)$, the observational uncertainties in $\langle v^{*2} \rangle$, ϕ^*, and α, and the uncertainty in translating our criterion for cluster formation into a definition of cluster size or radius. In spite of these uncertainties (or more probably because of them) the model makes reasonable predictions concerning the distribution of rich clusters and galaxies.

Chapter III

THE CLUSTERING OF RANDOM POINTS

1. Introduction

The hypothesis of random pregalactic perturbations led us to pose the following problem: Suppose that point masses are Poisson distributed in Euclidean 3 space with average density ρ . Choose a critical density $\gamma\rho$ where γ will be referred to as the density contrast. Identify as k-clusters sets of k mass points which may be contained in a sphere with volume $k/\gamma\rho$ but which are not elements of m-clusters where $m > k$. What then is the density $n_\gamma(k)$ of k-clusters?

Our problem is not yet well posed. While it is certain that for a finite volume V we can determine the average density $n_\gamma(k,V)$, it is not clear that $n_\gamma(k,V)$ converges to a limiting $n_\gamma(k)$ as V grows larger. We shall assume without proof that such a limit exists. Further, we must specify a means of choosing between two or more intersecting sets of k points which are not elements of larger clusters.

The problem turns out to be of fairly general interest. For example, a two-dimensional variant of the problem is considered by Danielsson (1969), who considers the possible clustering of asteroids in the π - θ plane. The question is whether two apparent associations of asteroids can be explained as a chance fluctuation. One needs to know the probability of observing chance associations of a given size with some specified density contrast.

A one-dimensional variant of the problem arises in the interpretation of counts arriving from an X-ray source (Rothschild et al 1974). Suppose one observes a "flare" with a higher than average count rate.

Is such a flare statistically significant, or is it a chance cluster in Poisson data? Again the distribution $n_\gamma(k)$ will give the possibility of a chance cluster. The distinguishing characteristic in these situations is that the cluster boundary is identified post-hoc, after the points have been distributed, and is carefully chosen to include the maximum number of points.

Similar problems have been treated by several authors. Melzak (1968) has developed a formalism for the probability of observing a cluster of k or more points in a constant volume a (rather than a cluster with constant density ρ) when N points are distributed in a volume V . The formalism grows unmanageable, however, when k and N grow large. Mack (1948) has treated the problem of clustering in one and two dimensions, but his approach fails to take account of the possible overlapping of clusters.

In Section 2 we develop an analytic approach to a solution for $n_\gamma(k)$ in a Euclidean space of arbitrary dimension. The method requires that we guess the analytic form of a function, $h_\gamma(m,k)$. We shall make several successive guesses for this functional form. We present the reasons for each ansatz and the reasons for questioning the resulting solution. We present in Section 3 the results of numerical experiments searching for clusters in one, two and three dimensions along with the solution for $n_\gamma(k)$ obtained from our best guess of $h_\gamma(m,k)$. The numerical experiments and the derived solution agree within roughly a factor of two.

2. Analytic Approach to a Solution for the Density $n_\gamma(k)$ of k-Clusters

As a concrete example, let us suppose that we are interested in two-dimensional square clusters. We insist that the squares have some specified orientation. Our results will apply to cube-shaped clusters in all dimensions, and we shall later discuss the applicability to clusters with other shapes.

We suppose then, that we have a large area A in which mass points have been Poisson distributed in the density ρ. Suppose further that we have already conducted a careful search for all clusters starting with the largest and working down to the smallest. Around each m-cluster we have drawn a square with area $m/\gamma\rho$. It is still true that if we choose a point in the plane at random and place the "lower left" corner of a square of area $k/\gamma\rho$ on that point, the probability that such a square contains k or more mass points is given by

$$P_\gamma(k) = \sum_{m=k}^{\infty} \frac{(k/\gamma)^m}{m!} e^{-k/\gamma} \tag{1}$$

We shall call such a point a "k-corner". The fraction of all points in the plane which are k-corners is just $P_\gamma(k)$. Therefore the fraction of the total area A covered by k-corners is also $P_\gamma(k)$.

But any k-corner must have an m-cluster with $m \geq k$ nearby, since there are k or more nearby mass points which may be contained in a square of area $k/\gamma\rho$. Every m-cluster will have some average area covered by k-corners associated with it, which we shall call $h_\gamma(m,k)$. The fraction of the plane covered by k-corners is then

$$P_\gamma(k) = \sum_{m=k}^{\infty} h_\gamma(m,k)\ n_\gamma(m) \tag{2}$$

If we knew the functional form of $h_\gamma(m,k)$, we would be able to solve equations (1) and (2) for $n_\gamma(k)$.

Our first guess of the functional form of $h_\gamma(m,k)$ is obtained as follows: We approximate an m-cluster by a square of area $m/\gamma\rho$ with a smooth distribution of finely subdivided mass points with point density $\gamma\rho$. The area outside is assumed to have no points. This approximation fails when $\gamma \simeq 1$. A square of area $k/\gamma\rho$ will contain exactly k points only if it is wholly contained in the m-cluster. The lower left corner of the smaller square sweeps out a square with a side of length $(m/\gamma\rho)^{1/2} - (k/\gamma\rho)^{1/2}$. The area of this square gives us our estimate of $h_\gamma(m,k)$. The situation is illustrated in Figure 1. This analysis is readily generalized to the case of r-cubic clusters in an r-dimensional space. The functional form of $h_\gamma(m,k)$ is then given by

$$h_\gamma(m,k) = \frac{(m^{1/r} - k^{1/r})^r}{\gamma\rho} \tag{3}$$

Combining equations (1)-(3), we obtain

$$\sum_{m=k}^{\infty} n_\gamma(m)(m^{1/r} - k^{1/r})^r = \gamma\rho \sum_{m=k}^{\infty} \frac{(k/\gamma)^m}{m!}\, e^{-k/\gamma} \tag{4}$$

We were able to solve this equation for $n_\gamma(k)$ only by approximating the left hand side by an integral and by using the integral approximation of Wilson and Hilferty (1931) for the right hand side. Then

$$\int_k^{\infty} n_\gamma(m)(m^{1/r} - k^{1/r})^r\ dm = \gamma\rho\ \frac{1}{\sqrt{\pi}} \int_g^{\infty} \exp(-x^2)\ dx \tag{5}$$

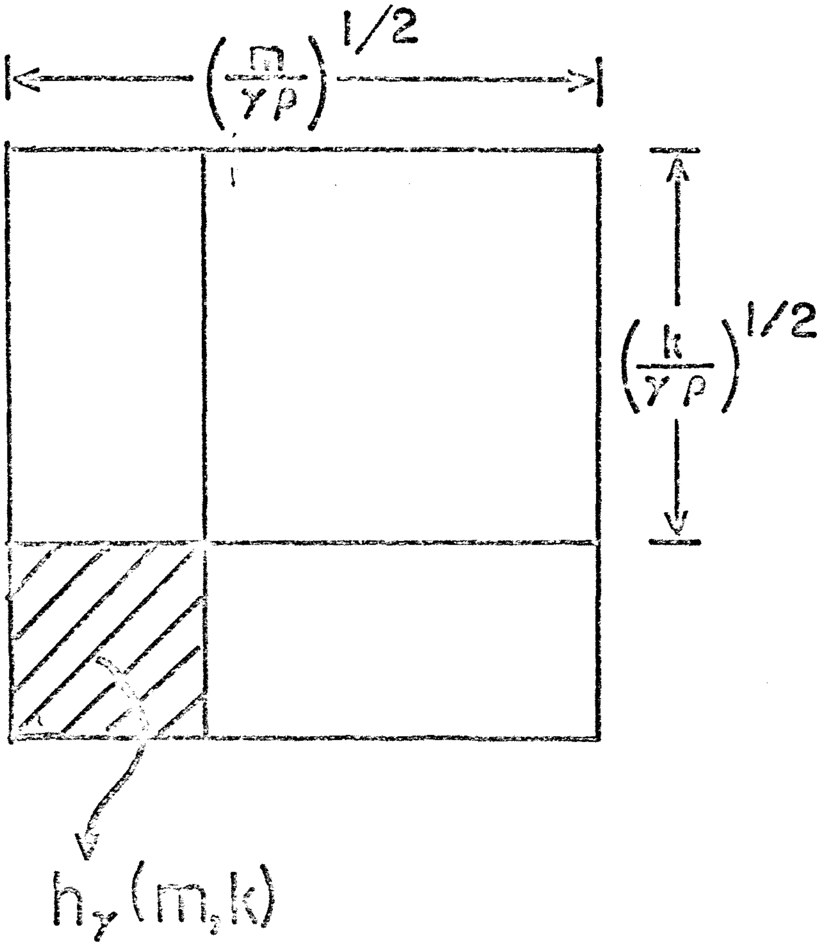


Fig. 1. Schematic drawing of area swept out by k-corners associated with an $\bar{m}$-cluster

where

$$g = [3(1 - 1/\gamma^{1/3})k^{1/2} - k^{-1/2}/3] / \sqrt{2} \qquad (6)$$

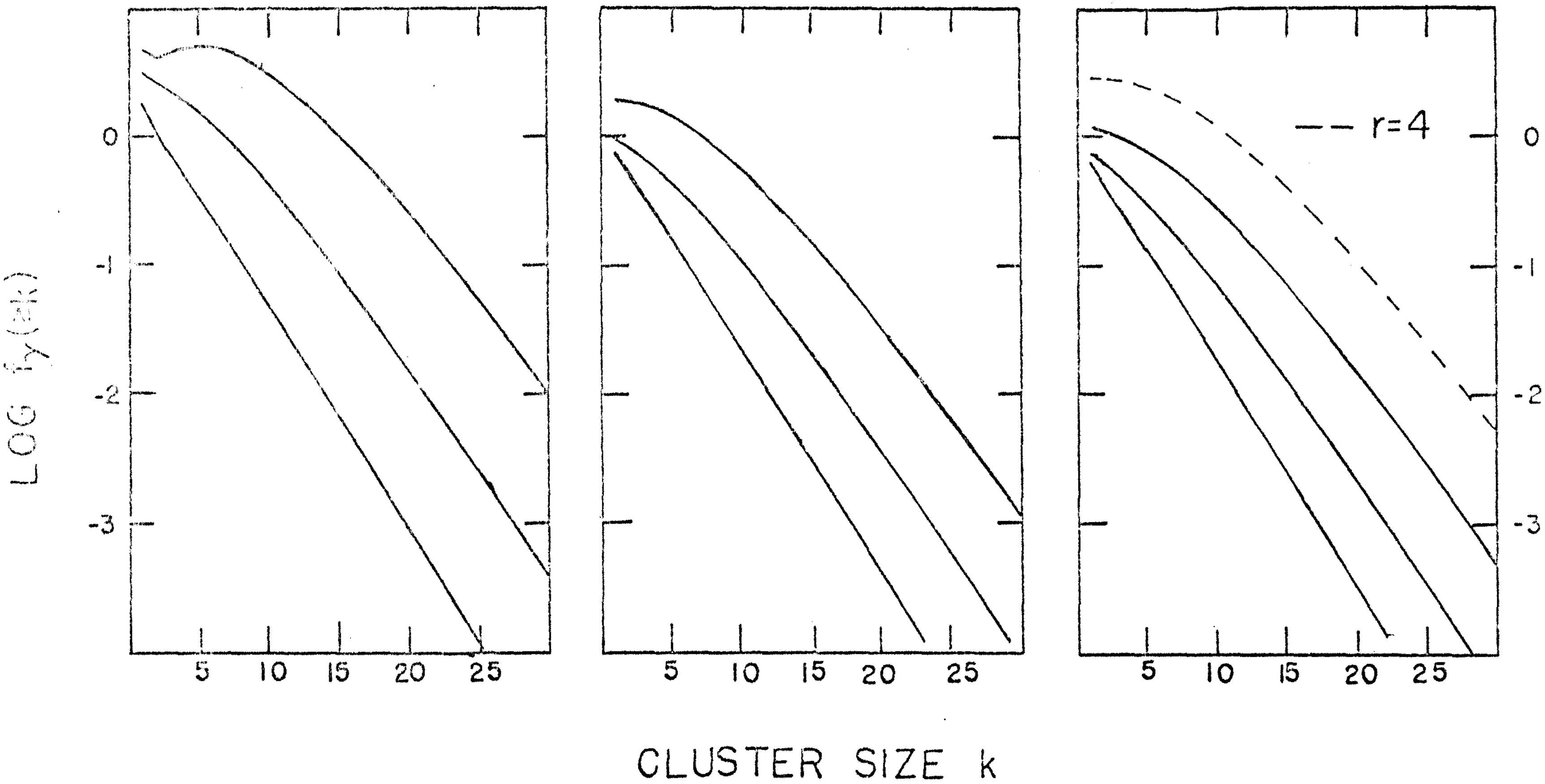
Differentiating both sides r times with respect to $k^{1/r}$ and once with respect to k, we obtain a solution for $n_\gamma(k)$. It is convenient to present such results in terms of the fraction of all points in clusters of size k or greater, $f_\gamma(\geq k)$, defined by

$$f_\gamma(\geq k) = \frac{1}{\rho} \sum_{m=k}^{\infty} m\, n_\gamma(m) \qquad (7)$$

Once we have solved for $n_\gamma(k)$, we shall therefore present our results in terms of $f_\gamma(\geq k)$.

Shown in Figure 2a are the results of solutions of equation (5) for the case $\gamma = 3$ in one, two and three dimensions. Figure 2a immediately illustrates the difficulties with our solution for $n_\gamma(k)$. First, the fraction of all points in clusters exceeds unity which is impossible. Second, the curve $f_\gamma(\geq k)$ exhibits a positive slope, indicating a negative density of clusters, which is also impossible. Both of these difficulties indicate that our function $h_\gamma(m,k)$ is an underestimate of the average area covered by k-corners associated with an m-cluster.

Given a k-cluster, there must be some finite area covered by k-corners: if the k-mass points may be contained in a square of area $k/\gamma\rho$, it must be possible to displace the square some small distance in any direction and still contain all k points. Just how much the square may be displaced depends upon how the mass points are distributed. If, for example, the k-mass points were randomly distributed



Figs. 2a,b,c. Fraction of points in clusters of size k or greater in one, two and three dimensions for three successive guesses of function $h_\gamma(m,k)$. Broken line in Fig. 2c shows $f_\gamma(\geq k)$ for four dimensions using best guess of $h_\gamma(m,k)$

inside a square of area $k/\gamma\rho$, the average distance to the nearest point from a given side would be $1/(k+1)$ times the length of a side. On average, the lower left corner of the square would sweep out an area

$$h_\gamma(k,k) = \frac{k}{\gamma\rho}\left(\frac{2}{k+1}\right)^2 \qquad (8)$$

without any of the mass points falling outside the square. There is reason to expect, however, that the k-mass points are not distributed randomly inside a square of area $k/\gamma\rho$. We know that the points are not contained in a larger cluster, which places conditions on the distribution of mass-points inside and outside the cluster. It seems likely that equation (8) gives an upper limit: that the points of a k-cluster are distributed in some fashion which is smoother than random.

Our second guess of $h_\gamma(m,k)$ is an attempt to modify our initial guess with a term which compensates for the discrete distribution of mass-points in an m-cluster. We try

$$h_\gamma(m,k) = \{m^{1/r}[1 + 2/(m+1)] - k^{1/r}\}^r / \gamma\rho \qquad (9)$$

which reduces to equation (8) when $m = k$ and $r = 2$. Equation (9) is clearly makeshift: it uses a result derived assuming that the points inside an m-cluster are extremely smoothly distributed and tacks on a result assuming that all points inside an m-cluster are completely random. If we are now assuming that all points are randomly distributed, we should modify our original guess (equation (3)) to take account of this randomness. The result of such a correction would be to lower our guess of $h_\gamma(m,k)$ to something of the order of half that given by equation (3) in the limit when $m >> k >> 1$. But since our first guess of $h_\gamma(m,k)$ was clearly an underestimate (giving $f_\gamma(\geq k) > 1$), we allow ourselves the liberty of taking the maximum estimate based on a completely smooth distribution of points and tacking on a maximum estimate based on a random distribution.

Combining equations (1), (2) and (9), we obtain

$$\sum_{m=k}^{\infty} n_\gamma(m)\ \{m^{1/r}[1+2/(m+1)] - k^{1/r}\}^r = \gamma\rho \sum_{m=k}^{\infty} \frac{(k/\gamma)^{-m}}{m!} e^{-k/\gamma} \qquad (10)$$

We have obtained solutions for $n_\gamma(k)$ by assuming that $n_\gamma(m) = 0$ for $m > \ell$ and solving stepwise from $m = \ell$ to $m = 1$. Using different values of ℓ, the solutions appear to converge to a single value of $n_\gamma(k)$ when $\ell - k >> 1$. Figure 2b shows the results of numerical solutions of equation (10) in all three dimensions for $\gamma = 3$. Figure 3 shows the results of three-dimensional solutions for $\gamma = 2$, 3, and 4. We are confronted with the same difficulties previously encountered: the fraction of mass points in clusters of size k or greater, $f_\gamma(\geq k)$, exceeds unity and for $\gamma \leq 2$ exhibits a positive slope.

It is important to note that the solution is better behaved for the case $\gamma = 4$ than for $\gamma = 2$. This tendency is what we expect from the fact that we have ignored those k-corners which include mass points outside m-clusters. There are in fact mass points outside m-clusters, and it may be possible to "trade" a point inside an m-cluster for a point outside. We shall compensate for this by approximating an m-cluster with a square containing $m(\gamma-1)/\gamma$ random mass points superimposed on a smoothed background point density of ρ. The analog of equation (9) is then

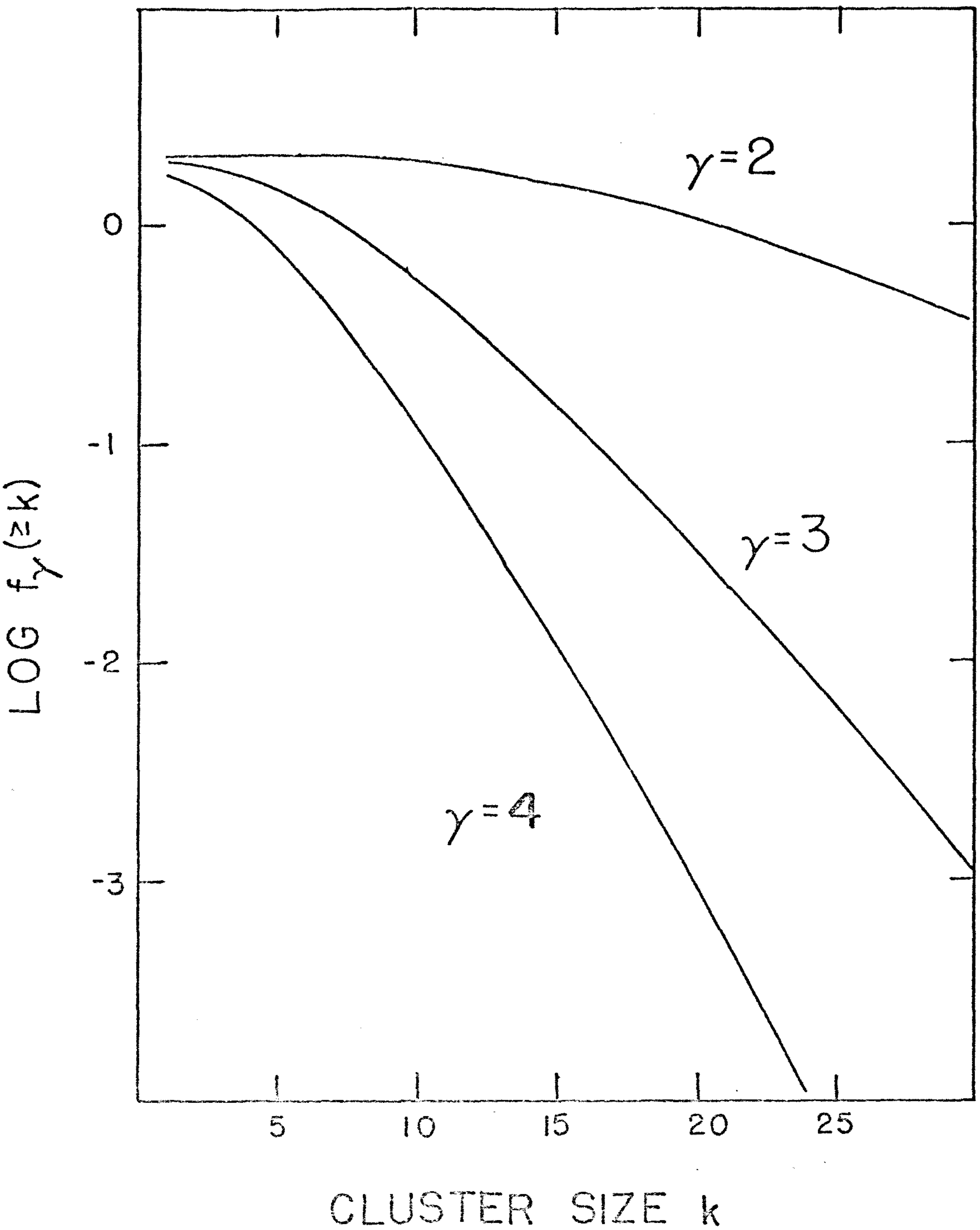


Fig. 3. Fraction of points in clusters of size k or greater from solution of equation (10) with $r = 3$

$$h_\gamma(m,k) = (m^{1/r}\{1 + 2/[(\gamma-1)m/\gamma + 1]\} - k^{1/r})^r \,/\, \gamma\rho \qquad (11)$$

which reduces to it in the limit as $\gamma \to \infty$.

We have still not dealt adequately with the problem of the randomness of points outside an m-cluster: all we have done is to increase the correction for the randomness of points inside an m-cluster. Our justification is that our previous estimate of $h_\gamma(m,k)$ was clearly an overestimate, as evidenced by the fact that $f_\gamma(\geq k)$ exceeded unity.

Combining equations (1), (2) and (11), we obtain

$$\sum_{m=k}^{\infty} n(m) \left(m^{1/r} \{1 + 2/[(\gamma-1)m/\gamma] + 1\} - k^{1/r}\right)^r = \gamma\rho \sum_{m=k}^{\infty} \frac{(k/\gamma)^m}{m!} e^{-k/\gamma} \qquad (12)$$

which may be solved by a procedure similar to that used for equation (10).

Figure 2c shows the results of solutions to equation (12) with $\gamma = 3$ for $r = 1$-4. While the solutions in one, two and three dimensions now seem fairly well behaved, it is clear from the four-dimensional solution that we have not yet dealt adequately with the problem of k-corners including points outside m-clusters. Furthermore, the fact that the one and two-dimensional solutions show $f_\gamma(\geq k)$ less than unity indicates that having worked so hard to obtain an expression valid in three dimensions, we are now underestimating the number of clusters in one and two dimensions. Before making any further guesses of $h_\gamma(m,k)$, we shall test the solutions of equation (12) against Monte Carlo experiments.

3. Numerical Experiments in One, Two, and Three Dimensions

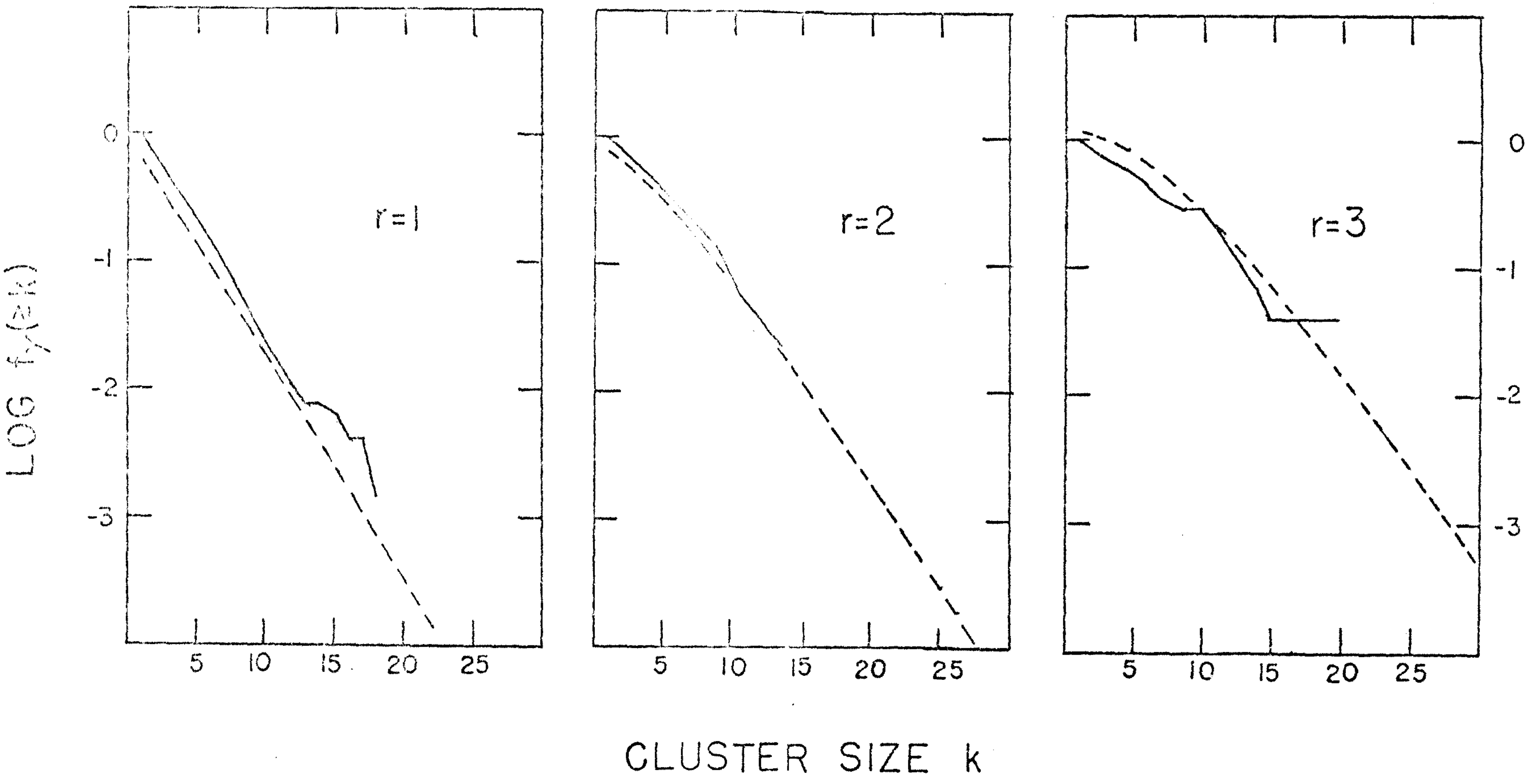
We have determined the density $n_\gamma(k)$ of k-clusters with overdensity γ by numerical experiments in one, two, and three dimensions. The experiments are most easily understood in one dimension. N mass points are distributed randomly in a line of length N/ρ. While such a distribution is not Poisson, for intervals much smaller than $N\rho$ the difference between the Poisson and binomial probabilities is small. The line was divided into γN bins, so that a cluster of size k would span k bins. Starting with a large integer ℓ, the line was scanned for clusters of size $k \leq \ell$ by keeping a running total over k bins, adding a new bin and subtracting an old bin, identifying a k-cluster whenever the running total equalled k (or on the first pass exceeded ℓ).

For the two and three-dimensional experiments, N points were distributed randomly in a square or cubic volume N/ρ. The full positions of the points were saved, but on each scan for k-clusters the volume was divided into $k\gamma N$ bins for the two-dimensional case and $k^2\gamma N$ bins for the three-dimensional case, insuring that each square (or cubic) k-cluster had an edge k bins long. This elaborate procedure was necessary to insure that the fraction of clusters not identified due to the binning of points was comparable in one, two, and three dimensions. We expect that the probability of missing a cluster is roughly twice as great in two dimensions as in one, since each cluster is scanned in two directions.

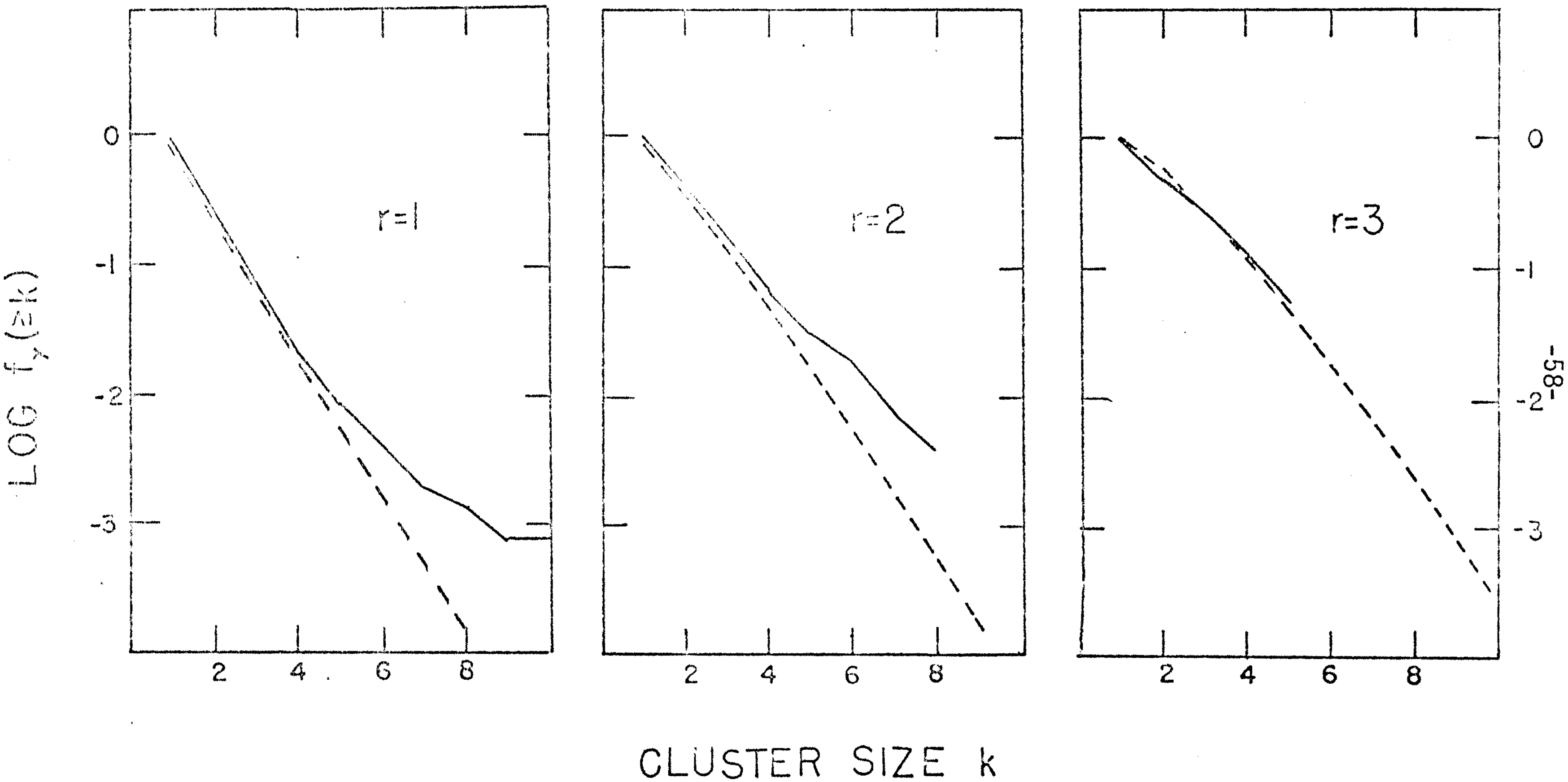
Searching for clusters by computer is extremely time consuming. For our algorithm, the number of operations is of order $N\gamma\ell^r$ where ℓ is the length of the edge of the largest cluster expressed in bins. By binning our points we gain speed in identifying clusters at the expense of losing some fraction of them. As a test of the efficiency of the cluster finding algorithm, the numerical experiment for the case $r = 2$ and $\gamma = 9$ was repeated with the edge of each square 2k bins long rather than k. The fraction of points in clusters $f_\gamma(\geq k)$ did not increase by more than 30%.

Edge effects are negligible only when the length of the edge of a cluster is much less than the edge of the large volume. To compensate for edge effects in one dimension, the ends of the long line were joined to make a ring. In two dimensions the edges of the large square area were joined to make a torus (as in Danielsson 1969) and faces were likewise treated in three dimensions.

The results of these numerical experiments are presented in Figures 4 and 5. The solid lines give experimental values of $f_\gamma(\geq k)$ for $\gamma = 3$ and $\gamma = 9$ while the broken lines give the values predicted using our best guess of $h_\gamma(m,k)$. The experiments were performed with 12500 points, 2500 points and 500 points respectively for the one, two and three-dimensional experiments. For the one and two-dimensional experiments our estimate of $n_\gamma(k)$ is clearly an underestimate, while for the three-dimensional case we are overestimating the number of clusters. Even so, the computed and the experimental curves appear to have roughly the same shape, and at no point does the $f_\gamma(\geq k)$ appear to be significantly wrong by greater than a factor of two.



Figs. 4a,b,c. Solid lines show results of numerical experiments. Broken lines give solutions of equation (12) with $\gamma = 3$



Figs. 5a,b,c. Solid lines show results of numerical experiments. Broken lines give solutions of equation (12) with $\gamma = 9$

4. Discussion

Our method for approximating the density of k-clusters rapidly deteriorates as the number of dimensions increases and as γ approaches unity. It nonetheless offers an order of magnitude estimate of the relative numbers of k-clusters expected. It is clear that we have not yet exhausted the possible functional forms for $h_\gamma(m,k)$. A better guess may yield better solutions for $n_\gamma(k)$.

While our analysis and experiments assumed square or cubic clusters, our first guess of $h_\gamma(m,k)$ (equation (3)) is valid for circular or spherical and triangular or tetrahedral clusters as well. Though the subsequent correction (equations (8) and (9)) explicitly used the assumption of square or cubic clusters, the correction will be of the same order in m for other geometries. Careful two-dimensional numerical experiments may settle the question of whether the distribution of clusters differs for different cluster shapes.

Chapter IV
ISOLATED GALAXIES

1. Introduction

Although there is considerable discussion of systematic properties of those galaxies which are members of pairs, groups and clusters, relatively little attention is paid to those galaxies which are not members of larger systems. This is not surprising since galaxies which are not members of larger systems are of little use in dynamical studies. But they are interesting in their own right precisely because they are not members of larger associations. Any theory which attempts to explain the clustering of galaxies ought to take account as well of the non-clustering of some galaxies.

The model for the clustering of galaxies presented in Chapter II assumed that pregalactic perturbations were randomly distributed at some early epoch. We assume here that galaxies were randomly distributed at some early epoch, but the model is otherwise the same. Granted this assumption, we show that isolated galaxies ought to be fainter than the average for all galaxies. We also show that if galaxies are distributed in a subrandom fashion, with a zone of avoidance around each galaxy, the size of the effect will be diminished. The angular momentum and gas content of isolated galaxies also deserve serious attention, but for the present we restrict our attention to predicting the luminosity of such isolated galaxies.

2. Pairwise Unbound Galaxies

We assume a matter-dominated Friedmann cosmology, that galaxies were randomly distributed at some early epoch, and that all matter is in galaxies. The precise epoch is not important as long as the deceleration parameter q at that epoch is approximately equal to one-half (or equivalently, as long as the fraction of critical density at that epoch is approximately unity). The galaxies are assumed to have no peculiar velocities with respect to local comoving coordinates at this initial epoch. The clustering of galaxies is assumed to result from the gravitational condensation of fluctuations in the density of galaxies.

An analytic calculation of the probability that any given galaxy will end up as a member of a bound association of any size is a difficult problem (see Chapter III). But it is straightforward to calculate the probability that a given galaxy is not bound to any other galaxy when the two are considered as a pair. We shall discuss later the problems associated with identifying such pairs with isolated galaxies.

For simplicity, suppose that at the initial epoch all galaxies have mass $\mathcal{M}^*$ and were distributed with density ϕ. Letting h be the value of the Hubble parameter at the initial epoch, we have

$$\frac{4}{3}\pi\phi\,\mathcal{M}^* = \frac{1}{2}h^2 \qquad (1)$$

where we have again set $G = c = 1$. Suppose that one test galaxy has mass $\mathcal{M}$. A second galaxy of mass $\mathcal{M}^*$ will be pairwise bound to this test galaxy if it is located within a sphere of radius r around it where

$$\frac{\mathcal{M} + \mathcal{M}^*}{r} = \frac{1}{2} hr^2 \qquad (2)$$

The probability that no galaxy of mass $\mathcal{M}^*$ is located within this sphere is given by $P(\mu)$ where

$$P(\mu) = \exp[-\frac{4}{3}\pi r^3 \phi] = \exp[-(1+\mu)] \qquad (3)$$

and where the dimensionless mass μ is defined by

$$\mu \equiv \mathcal{M}/\mathcal{M}^* \qquad (4)$$

It is immediately clear that the more massive the test galaxy, the more likely it is to be pairwise bound in this picture.

For comparison, suppose that the presence of a galaxy of mass μ precluded the possibility of a second galaxy being located within a sphere of radius $(3/4\pi\phi\ \mu)^{1/3}$. This would be the case if all the mass were drawn up from a homogeneous medium. The probability that a second galaxy lies within radius r is $S(\mu)$ where

$$S(\mu) = \exp[-1] \qquad (5)$$

independent of the mass of the test galaxy. This picture is somewhat inconsistent since we ought to insist that the second galaxy be drawn up from the surrounding medium as well. In spite of this inconsistency, this alternative proves useful for the purpose of comparison. We shall call this model the "subrandom" picture in contrast to the "completely random" model.

These results are readily generalized to deal with a complete distribution of masses $\phi(\mu)$. We shall assume that the distribution of galaxy masses has the same distribution as the distribution of galaxy luminosities,

$$\phi(\mu)\, d\mu = \phi^* \exp[-\mu]\mu^\alpha\, d\mu \qquad (6)$$

The analog of equation (1) is then

$$\frac{4}{3}\pi\, \Gamma(\alpha+2)\, \phi^*\, \mathcal{M}^* = \frac{1}{2} h^2 \qquad (7)$$

We may further generalize the problem by introducing a binding factor κ such that if μ and μ' are the dimensionless masses of two galaxies, they are pairwise bound by the factor κ if they are closer than r where

$$\frac{\mathcal{M}^*(\mu+\mu')}{r} = \frac{1}{2}\kappa\, h^2 r^2 \qquad (8)$$

in analogy to equation (2). We must make one further generalization, necessitated by the fact that for the purpose of observations, the mass μ' of the companion galaxy must be some fraction χ of the mass of the first galaxy or it will escape detection. The probability that a galaxy of mass μ will not be pairwise bound by the factor κ to a galaxy of mass $\mu' \geq \chi\mu$ is given by $P_\kappa(\mu,\chi)$ where

$$P_\kappa(\mu,\chi) = \exp\left\{ - \frac{\int_{\chi\mu}^{\infty} (\mu+\mu')\, \phi(\mu')\, d\mu'}{\kappa \int_0^{\infty} \mu'\, \phi(\mu')\, d\mu'} \right\} \qquad (9)$$

Using equation (6), this may be expressed in terms of incomplete gamma functions as

$$P_\kappa(\mu,\chi) = \exp\{-\frac{\Gamma[(\alpha+2),\chi\mu] + \mu\Gamma[(\alpha+1),\chi\mu]}{\kappa\Gamma(\alpha+2)}\} \qquad (10)$$

We can likewise generalize the subrandom picture giving the analog of equation (5)

$$S_\kappa(\mu,\chi) = \exp\{-\frac{\Gamma[(\alpha+2),\chi\mu]}{\kappa\Gamma(\alpha+2)} + \mu\frac{(\kappa-1)[(\alpha+1),\chi\mu]}{\kappa\Gamma(\alpha+2)}\} \qquad (11)$$

which reduces to equation (5) when $\chi = 0$ and $\kappa = 1$.

3. Definition of Isolated Galaxies

We would like to choose an observational definition of an isolated galaxy which corresponds to a pairwise unbound galaxy at the initial epoch. We must first decide upon a value of the binding factor κ which is as large as possible so that the dynamics of the pair is not likely to be influenced by nearby neighbors, but small enough so that the factor κ can be determined unambiguously for a given pair.

Two galaxies of masses μ and μ' bound by the factor κ at the initial epoch will reach a maximum radius r^{max} at time t^{max} where

$$r^{max} = [\frac{2\mathcal{M}^*(\mu+\mu')}{(\kappa-1)^3h^2}]^{1/3} \qquad (12)$$

and

$$t^{max} \simeq \frac{\pi}{2}[\frac{1}{(\kappa-1)^3h^2}]^{1/2} \qquad (13)$$

Suppose we choose κ_{lim} such that t_{lim}^{max} is one-third t_o , the present age of the universe. Then any pair of galaxies with masses μ and μ' separated at the present by r_{lim}^{max} has had either one close encounter or no close encounters. If they have had one close encounter, they are bound by the factor κ_{lim} . If they are just approaching for the first time, then they are bound by the factor κ_c where $(\kappa_c - 1) = .679(\kappa_{lim} - 1)$. Any pair of galaxies of masses μ and μ' separated by more than r_{lim}^{max} must be approaching for the first time and <u>must</u> be bound by a factor less than κ_c . Any such pair of galaxies separated by less than r_{lim}^{max} may be approaching for the second time and <u>may</u> be bound by a factor greater than κ_{lim}. Therefore r_{lim}^{max} is the smallest radius such that κ for a pair separated by greater than that radius can be determined unambiguously.

We shall drop the superscript "max" and let the "limiting radius" r_{lim} be defined by

$$r_{lim} \equiv [(\frac{2}{3\pi})^2 \mathcal{M}^*(\mu+\mu')t_o^2]^{1/3} \tag{14}$$

We shall call a galaxy of mass μ "isolated by x magnitudes" if there is no galaxy of mass $\mu' \geq \chi\mu$ within a limiting radius, and where

$$\chi = 10^{-.4x} \tag{15}$$

We have chosen the smallest possible definition of r_{lim} to minimize dynamical effects due to third galaxies. Close binary systems are less likely to be affected by encounters with third galaxies than less tightly bound systems. Nonetheless, it seems likely that particularly in rich clusters of galaxies, pairwise bound galaxies may have been split apart, and new close pairs may have been formed. Dynamical effects may therefore affect the relative numbers of pairwise unbound galaxies. In spite of this we shall identify isolated galaxies as defined in this section with the pairwise unbound galaxies of the previous section. We still expect that among isolated galaxies, the more massive galaxies will be underrepresented.

4. Selection Effects in the Identification of Isolated Galaxies

Having chosen a definition of isolated galaxies to agree as well as possible with the definition of pairwise unbound galaxies, we may use equations (10) and (11) to calculate the expected mass distribution of pairwise unbound galaxies and compare this with the observed distribution of isolated galaxies. We must be careful, however, to eliminate selection effects in the identification of isolated galaxies.

Owing to the difficulty in obtaining redshifts, it is likely that an isolated galaxy will be identified only if there is no galaxy within some angular separation. In particular, a galaxy with apparent magnitude m and at distance D will be identified as isolated by x magnitudes only if there is no second galaxy of apparent magnitude $m' < m+x$ separated by angle θ from the first galaxy, where

$$\theta < \frac{r^*_{lim}}{D} (\mu+\mu')^{1/3} \tag{16}$$

and

$$r^*_{lim} \equiv [(\frac{2}{3\pi})^2 \mathcal{M}^* t_o^2]^{1/3} \tag{17}$$

Tne dimensionless masses μ and μ' are calculated assuming a constant mass-to-luminosity ratio and that the second galaxy is at the same distance as the first.

There is always the chance that a background or foreground galaxy may be separated by only a small angle from a galaxy which is in fact isolated. Such a galaxy would not be identified by the above procedure. Suppose that there are N galaxies per steradian brighter than $m+x$ and assume that they are randomly distributed on the sky. Then the probability that a given galaxy has no other galaxy of magnitude $m' < m+x$ within angle θ is given by

$$p(D,\mu,\chi) = \exp\left\{- \frac{\pi r^{*2}_{lim}\, \mu^{2/3}}{D^2}\, \frac{3}{2} N \chi^{3/2} \int_{\chi}^{\infty} y^{-5/2}(1+y)^{2/3}\, dy \right\} \qquad (18)$$

where the integral must be performed numerically. It is important to note that at a given apparent magnitude (μ/D^2 constant) massive isolated galaxies are less likely to have spurious projected companions than less massive isolated galaxies. Therefore selection effects favor the <u>identification</u> of massive isolated galaxies.

5. Predicted Mean Luminosity of Isolated Galaxies

Suppose we have a sample of galaxies complete to some limiting magnitude m with measured redshifts. Suppose further that we have a second list of galaxies, complete to magnitude $m+x$ with N galaxies per steradian. The second list may be used to identify isolated galaxies on the first list. The expected number of galaxies isolated by magnitudes, $n(x)$, is then

$$n(x) = \int_0^{\infty} d\mu\, \phi(\mu)\, P_K(\mu,\chi) \int_0^{\mu^{1/2}D^*} 4\pi D^2 dD\, p(D,\mu,\chi) \qquad (19)$$

where D^* is the greatest distance at which a galaxy of mass $\mathfrak{m}^*$ will be included in the first list. Substituting equation (17) we find

$$n(x) = \int_0^\infty \phi(\mu)\, P_\kappa(\mu,\chi) \times \left\{ \int_0^{\mu^{1/2} D^*} 4\pi D^2 \exp\left[-\frac{\pi r_{lim}^{*2} \mu^{2/3}}{D^2} N\left(\frac{\chi D^2}{\mu D^{*2}}\right)^{3/2} \int_\chi^\infty y^{-5/2}(1+y)^{2/3} dy \right] dD \right\} d\mu \tag{20}$$

For a given value of x , the integral with respect to y is constant and the integral with respect to D may be expressed as an incomplete gamma function. Note that if we increase D^* (N/D^{*3} remaining constant) there is a finite limit to the number of galaxies isolated by x magnitudes that one can identify by this process. This is because beyond a certain distance, virtually all isolated galaxies will have spurious apparent companions.

The expected total mass in the $n(x)$ isolated galaxies in units of $\mathfrak{M}^*$ may be found by multiplying the integrand in equation (19) by an additional factor of μ . Dividing the expected mass by the expected number gives a predicted mean mass in units of $\mathfrak{M}^*$. We have already assumed a constant mass-to-light ratio, which gives us the average luminosity in units of L^* . We can just as well use $S_\kappa(\mu,\chi)$ in equations (19) and (20) instead of $P_\kappa(\mu,\chi)$ to obtain the mean luminosity of isolated galaxies in the subrandom model.

Before we can actually do the integrals in equation (20) we must specify a mass-to-light ratio. Given the luminosity function of Chapter I, we can then calculate the present mass density, which gives a value of q_0 and hence t_0 . We can either compute N down to a given limiting apparent magnitude or we can allow N to be determined by the comparison list. κ_c may also be computed once a mass $\mathcal{M}^*$ is specified. If the time at which galaxies are presumed to be random is very much smaller than the present age of the universe, then κ will be very close to unity.

While the predicted mean luminosity depends upon the depth D^* of the sample of galaxies, it is interesting to compute the mean luminosity for the special case when spurious apparent companions are unimportant. The exponential in equation (20) is then unity, and the results are independent of the assumed mass-to-luminosity ratio. The results of such a computation are shown in Figure 1, where the broken line gives the mean luminosity in the subrandom case. We see that if we consider galaxies isolated by 3 magnitudes, the mean luminosity in the random model is almost 3/2 of a magnitude fainter than the subrandom model. The horizontal line at $5/4L^*$ shows the mean luminosity of all galaxies in a magnitude limited survey. It is remarkable that for small values of x , the mean luminosity of isolated galaxies exceeds that of all galaxies. But this is due to the fact that the most luminous galaxies are rare, and a galaxy only x magnitudes fainter, where x is small, is not likely to be nearby at the initial epoch. When the selection effect due to background and foreground galaxies is included, both curves will move to brighter absolute magnitudes.

The completely random model presented here differs from the model presented in Chapter II in that we assume here that galaxies themselves were randomly distributed rather than pregalactic perturbations. Nonetheless we would expect a similar effect for pregalactic

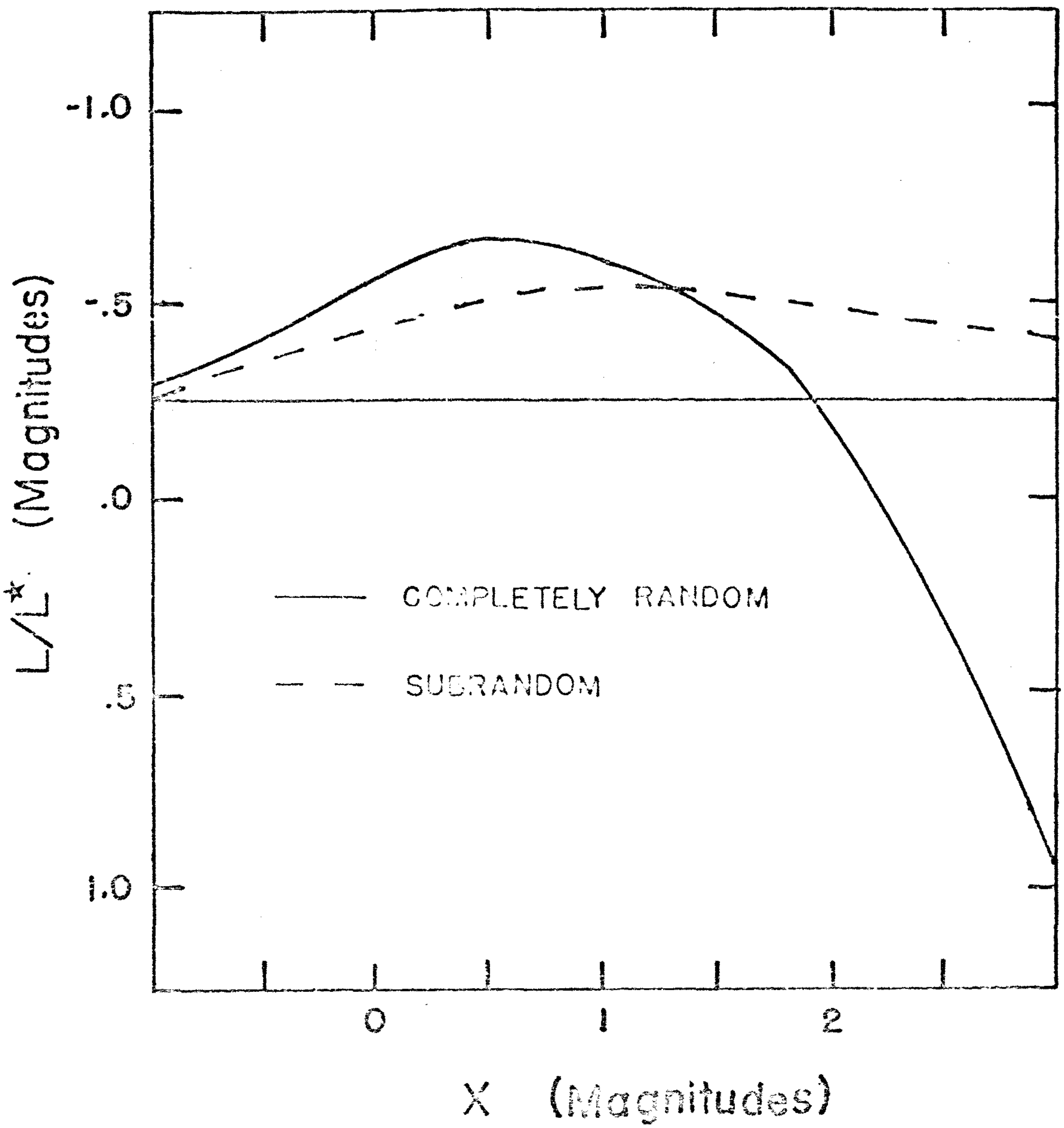


Fig. 1. Mean luminosity of galaxies isolated by x magnitudes as a function of x .

perturbations: the more massive ones would be more likely to be bound to companions than the less massive ones.

REFERENCES

Abell, G. O. 1958, Ap. J. (Supplement) 3, 211.

Abell, G. O. 1962, in Problems of Extra-Galactic Research, ed. G. C. McVittie (New York: Macmillan), p. 213.

Balkowski, C., Bottinelli, L., Goughenheim, L. and Heidemann, J. 1973, Astr. and Ap. 23, 139.

Bautz, L. P. and Abell, G. O. 1973, Ap. J. 184, 709.

Bergh, S. van den 1961, Z. Astrophys. 53, 19.

Bottinelli, L., Chamaraux, P., Gougenheim, L. and Lauqué, R. 1970, Astr. and Ap. 6, 453.

Boynton, P. E. and Partridge, R. B. 1973, Ap. J. 181, 243.

Brosche, P. 1971, Astr. and Ap. 13, 293.

Carranza, G. J. 1967, Observatory 87, 38.

Chincarini, G. and Rood, H. J. 1972, A. J. 77, 4.

Christiansen, C. 1968, unpublished paper, California Institute of Technology.

Danielsson, L. 1969, Astrophys. and Space Sci. 5, 53.

Field, G. B. 1968, in Stars and Stellar Systems, Vol. 9, ed. A. and M. Sandage (Chicago: University of Chicago Press) in press.

Gott, J. R. 1974, Preprint: "On the Formation of Elliptical Galaxies".

Gott, J. R., Gunn, J. E., Schramm, D. N. and Tinsley, B. M. 1974, Preprint: "An Unbound Universe".

Gunn, J. E. 1974, Preprint: "A Worms Eye View of the Mass Density in the Universe".

Gunn, J. E. and Gott, J. R. 1972, Ap. J. 176, 1.

Lewis, B. M. and Robinson, B. J. 1973, Astr. and Ap. 23, 295.

Mack, C. 1948, Phil. Mag. (Series 7) 39, 778.

Minkowski, R. 1962, in Problems of Extra-Galactic Research, ed. G. C. McVittie (New York: Macmillan), p. 112.

Melzak, Z. A. 1968, Bell System Tech. J. 47, 1105.

Misner, C. W., Thorne, K. S. and Wheeler, J. A. 1973, Gravitation (San Francisco: W. H. Freeman).

Morton, D. C. and Chevalier, R. A. 1973, Ap. J. 179, 55.

Oemler, A., Jr. 1974, Ap. J. 194, 000.

Page, T. 1965, Smithsonian Ap. Obs. Spec. Rept., No. 195, pp. 1-38.

Peebles, P.J.E. 1968, Ap. J. 153, 13.

Peebles, P.J.E. 1969, Nature 224, 1093.

Peebles, P.J.E. 1971, Physical Cosmology (Princeton: Princeton University Press).

Peebles, P.J.E. 1972, Comm. Ap. and Sp. Phys. 4, 53.

Peebles, P.J.E. 1974, Ap. J. (Letters) 189, L51.

Peebles, P.J.E. and Vu. J. T. 1970, Ap. J. 162, 815.

Penzias, A. A. and Wilson, R. W. 1965, Ap. J. 142, 419.

Peterson, B. A. 1970a, A. J. 75, 695.

Peterson, B. A. 1970b, Ap. J. 159, 333.

Peterson, B. A. 1970c, Nature 227, 54.

Press, W. H. and Schechter, P. 1974a, Ap. J. 187, 425.

Press, W. H. and Schechter, P. 1974b, Ap. J. 193, 000.

Roberts, M. S. 1969, A. J. 74, 859 .

Rogstad, D. H., Rougoor, G. W. and Whiteoak, J. B. 1967, Ap. J. 150, 6.

Rood, H. J., Page, T. L., Kintner, E. C. and King, I. R. 1972, Ap. J. 175, 627.

Rothschild, R. E., Boldt, E. A., Holt, S. S. and Serlemitsos, P. J. 1974, Ap. J. (Letters), 189, L13.

Sandage, A. 1972, Ap. J. 178, 1.

Sandage, A. and Hardy, E. 1973, Ap. J. 183, 743.

Shapiro, S. L. 1971, A. J. 76, 291.

Silk, J. 1967, Nature 215, 1155.

Silk, J. 1968, Ap. J. 151, 459.

Trumpler, R. J. and Weaver, H. F. 1953, Statistical Astronomy (Berkeley: University of California Press).

Vaucouleurs, A. de and Vaucouleurs, G. de 1967, A. J. 72, 733.

Vaucouleurs, G. de 1968, in Stars and Stellar Systems, Vol. 9, ed. A. and M. Sandage (Chicago: University of Chicago Press) in press.

Vaucouleurs, G. de and Vaucouleurs, A. de 1964, Reference Catalogue of Bright Galaxies (Austin: University of Texas Press).

Wilson, E. B. and Hilferty, M. M. 1931, Proc. N.A.S. 17, 684.

Wolberg, J. R. 1967, Prediction Analysis (Princeton: D. Van Nostrand).

Yahil, A. 1974, Ap. J. 191, 623.