THE DIFFERENTIAL GEOMETRY OF A SPACE WITH A TWO-POINT DIFFERENTIAL METRIC

Thesis by

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Summary

In this thesis we have generalized the Riemannian line element $ds^2 = g_{\alpha\beta}(x)dx^\alpha dx^\beta$ to the case where $g_{\alpha\beta}$ is a function of two points x_1 , x_2 , and we consider the differential geometry of the line element $ds^2 = g_{\alpha_1\beta}(x_1, x_2)dx_1^\alpha dx_2^\beta$.

The extremalizing of $L = \int_{t_0}^{t_1} \left| g_{\alpha_1\beta} \frac{dx_1^\alpha}{ds} \frac{dx_2^\beta}{ds} \right|^{1/2} ds$ leads to a pair of curves $x_1^i(s), x_2^i(s)$, called dyodesics, these curves being obvious generalizations of the geodesics of Riemannian geometry. A projective geometry of these paths is then investigated.

We then introduce a concept of parallel displacement of vectors relative to two paths $x_1^i(t), x_2^i(t)$ which is directly analagous to parallel displacement in a Riemannian space. Parallel displacement is found to depend in a very natural way on six fundamental two-point tensors, the vanishing of these tensors implying that the space is flat, and for this case the dyodesics take the simple forms $x_{1,2}^i = a_{1,2}^i s + b_{1,2}^i$ for special coordinate systems.

From the definition of parallel displacement arises a method for generating new two-point tensor fields by a process equivalent to covariant differentiation in Riemannian geometry. Parallel vector fields and ennuples of vectors are then introduced. It is shown that the ennuples $\frac{\partial^i \theta(x_1)}{\partial x_1^\alpha}$, $\frac{\partial^i \varphi(x_2)}{\partial x_2^\alpha}$, form parallel vector fields for the metric space $ds^2 = C_{ij} \frac{\partial^i \theta}{\partial x_1^\alpha} \frac{\partial^j \varphi}{\partial x_2^\beta} dx_1^\alpha dx_2^\beta$. We then define parallel displacement in sub-spaces and introduce a generalized covariant differentiation process, this last enabling us to develop second fundamental forms for hyper-surfaces.

It is found that special and important types of coordinate systems may be set up independently at the points M_1 and M_2. These coordinates enable us to generate new tensors by a method of extension. An equivalence problem is then studied.

Finally, a line element $ds^2 = g_{\alpha_1\beta} dx_1^\alpha dx_2^\beta$ is introduced for two masses at M_1, M_2, the $g_{\alpha_1\beta}$ satisfying $T_{\alpha_1\beta} = \overset{*}{T}_{\alpha_1\beta} = 0$, the T's corresponding to the Ricci tensor of Riemannian geometry. The dyodesics obtained for this space approximate the Einstein solution for the one body problem when the mass of the particle at M_1 is small compared with that at M_2. The motion for two equal masses differs from that obtained by Robertson in his solution of the equations of motion obtained by Einstein, Infeld, and Hoffman. The difference lies in the yet undetermined periastron effect for double stars.

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I. An Arithmetic space of n- dimensions.

An ordered set of n real numbers, $(x^1, x^2, \cdots, x^n)$, is called an arithmetic point. When no confusion results we will use the letter x to denote this point. The numbers $x^1, x^2, \cdots, x^n$ denote the components of the point x. The superscripts do not stand for powers, but are used to distinguish the various components of x, x^j standing for the j th component.

The totality of all arithmetic points, for a given value of n, is called an arithmetic space of n- dimensions.

The set of points x satisfying

$$(1.1) \qquad (x^1 - x_0^1)^2 + (x^2 - x_0^2)^2 + \cdots + (x^n - x_0^n)^2 < A^2, \quad A > 0$$

are said to be the interior points of the hypersphere, Σ, having its center at $(x_0^1, x_0^2, \cdots, x_0^n)$ with radius A.

II. The space S of n- dimensions.

By a space S of n- dimensions we shall mean a set of elements (undefined) with the following imposed topology. Certain subsets of S called neighborhoods will be the fundamental structure and will satisfy the following postulates:

A. Every element of S belongs to at least one neighborhood.

B. The elements of each neighborhood, N, of S, can be put into one-to-one reciprocal correspondence with the interior points of some hypersphere Σ of the arithmetic space of n- dimensions.

C. Let M be an element of the neighborhood N, m the corresponding point of Σ and let N' be any neighborhood of M.

There exists a hypersphere, σ, with center at m such that all points of N which correspond to points of σ in Σ lie in N'. We illustrate this graphically, (Fig. 1).

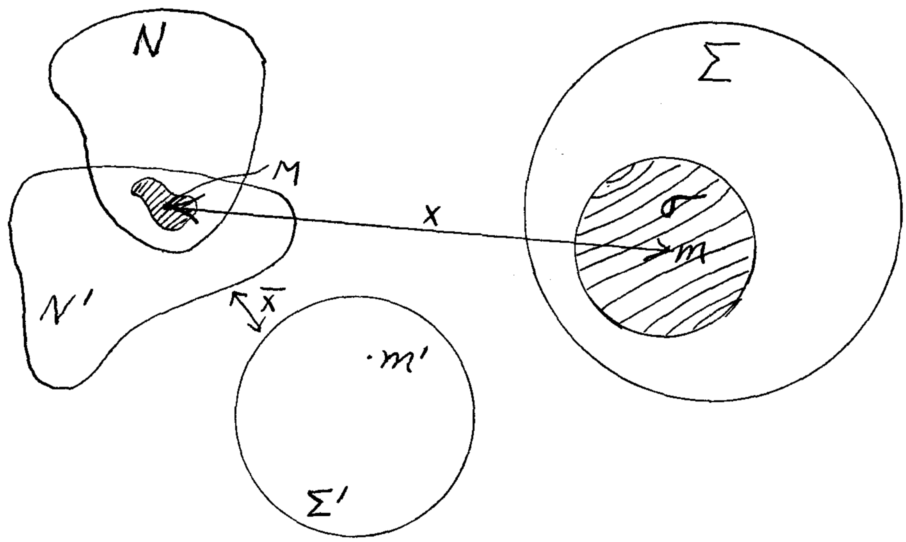

Fig. 1

The shaded areas correspond under $N \longleftrightarrow \Sigma$.

D. Let N be an arbitrary neighborhood corresponding to the hypersphere Σ, M a point of N, m its corresponding point of Σ. If σ is any hypersphere with center at m and entirely contained in Σ, then there exists a neighborhood N' of M, every element of N' belonging to N and such that every element of N' under the correspondence $N \longleftrightarrow \Sigma$ will correspond to some point of σ. See fig. 2.

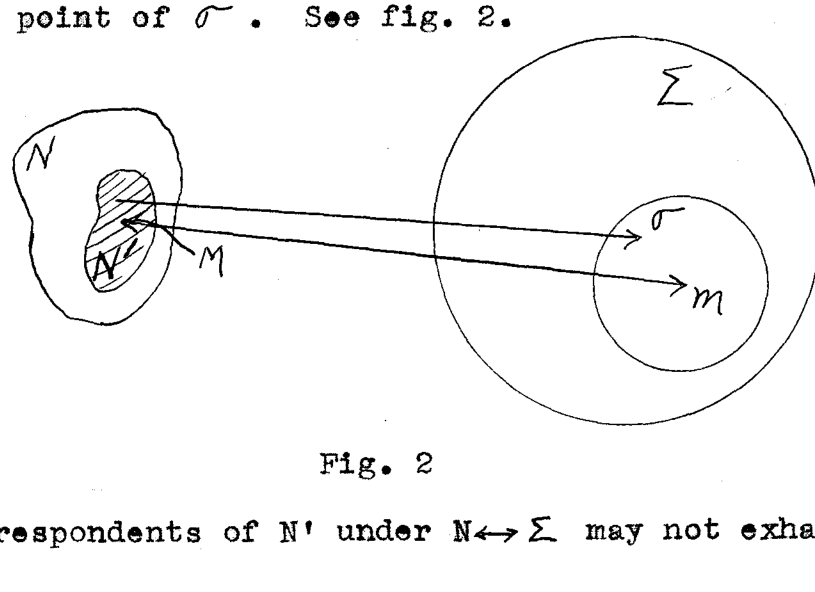

Fig. 2

The correspondents of N' under $N \longleftrightarrow \Sigma$ may not exhaust σ.

E. If M, M' are two distinct elements of S, there exists two neighborhoods N, N' containing M, M' respectively, and such that N and N' contain no elements in common.

E'. We can strengthen condition E by postulating that if M and M' are two distinct elements of S, there exists a neighborhood N of M not containing M'.

The one-to-one reciprocal correspondence between points of a neighborhood N and the points of a hypersphere Σ of the arithmetic n-space enables us to attach a set of n real numbers $(x^1, x^2, \ldots, x^n)$ to each element of N. Such a correspondence is called a coordinate system.

Thus every neighborhood has a coordinate system attached to it. The intersection of two neighborhoods (elements common to both neighborhoods) has two coordinate systems attached to it, see fig. 1. The relationship $\bar{x} \longleftrightarrow x$ is called a coordinate transformation, written

(2.1) $$\bar{x}^i = \bar{x}^i(x^1, x^2, \ldots, x^n), \quad i = 1, 2, \ldots, n,$$

and is a process by which we pass from one coordinate system to another.

In general, it will not be possible to find a coordinate system which covers S completely, so that coordinate systems are essentially localized. Finally, we impose the condition that the coordinate transformation (2.1) be analytic.*

* In most of what follows only differentiation of finite order r is required for the transformations (2.1), i.e., they are of class C^r.

Now in what follows we shall be interested in considering pairs of elements of S, say M_1 and M_2. If a single coordinate system x describes M_1 and M_2, then by $\frac{\partial \bar{x}_1^i}{\partial x_1^j}$ we shall mean $\frac{\partial \bar{x}^i}{\partial x^j}$ evaluated at M_1, and similarly $\frac{\partial \bar{x}_2^i}{\partial x_2^j}$ shall mean $\frac{\partial \bar{x}^i}{\partial x^j}$ evaluated at M_2. However, from postulate E, we can always find two independent coordinate systems, x_1 for a neighborhood of M_1 and x_2 for a neighborhood of M_2. When occasion demands we will distinguish between the two cases.

III. Two-point tensors.

We assume that the reader is acquainted with the summation notation convention, tensor analysis, and Riemannian geometry. The following is a generalization of one point tensors[1].

Definition. A set of quantities

$$(3.1) \qquad T^{a\cdots b,\, c\cdots d}_{\alpha\cdots\beta,\, \gamma\cdots\delta}(M_1, M_2)$$

which are functions of the two points $M_1(x_1^1, x_1^2, \cdots, x_1^n)$, $M_2(x_2^1, x_2^2, \cdots, x_2^n)$, whose law of transformation under the group of analytic transformations

$$(3.2) \qquad \begin{aligned} \bar{x}_1^i &= \bar{x}_1^i(x_1^1, x_1^2, \cdots, x_1^n) \\ \bar{x}_2^i &= \bar{x}_2^i(x_2^1, x_2^2, \cdots, x_2^n) \end{aligned}$$

is given by

$$(3.3) \qquad \bar{T}^{r\cdots s,\, m\cdots n}_{\rho\cdots\sigma,\, \mu\cdots\nu}(\bar{M}_1, \bar{M}_2) = \left|\frac{\partial x_1}{\partial \bar{x}_1}\right|^p \left|\frac{\partial x_2}{\partial \bar{x}_2}\right|^q T^{a\cdots b,\, c\cdots d}_{\alpha\cdots\beta,\, \gamma\cdots\delta}(M_1, M_2)\, \frac{\partial \bar{x}_1^r}{\partial x_1^a} \cdots \frac{\partial x_2^\delta}{\partial \bar{x}_2^\nu}$$

will be termed a two-point tensor, of weight p and contra-

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variant in the indices $a \cdots b$, covariant in the indices $\alpha \cdots \beta$ relative to the x_1- coordinate system, and of weight q and contravariant in the indices $c \cdots d$, covariant in the indices $\gamma \cdots \delta$ relative to the x_2- coordinate system. The determinants $\left|\frac{\partial x_1}{\partial \bar{x}_1}\right|$, $\left|\frac{\partial x_2}{\partial \bar{x}_2}\right|$ are the ordinary Jacobians studied in analysis.

The object obtained by abstraction from the above components with respect to the totality of all coordinate systems whose coordinates are related by (2.1) is called the tensor T.

Superscripts and subscripts preceding the comma in (3.1) refer to the tensorial character of T in the x_1- coordinate system. The indices following the comma refer to the x_2- coordinate system.

Example. The set of quantities $A^{\alpha,}$ which transfrom according to the law

$$\bar{A}^{\mu,}(\bar{M}_1, \bar{M}_2) = A^{\alpha,}(M_1, M_2)\,\frac{\partial \bar{x}_1^{\mu}}{\partial x_1^{\alpha}} \tag{3.4}$$

are the components of a contravariant vector field in the x_1- coordinate system and behave like a scalar invariant in the x_2- coordinate system. To see this more clearly, we first consider the transformations

$$\begin{aligned} \bar{x}_1^{i} &= \bar{x}_1^{i}(x_1^1, \cdots, x_1^n) \\ \bar{x}_2^{i} &= x_2^{i} \end{aligned} \tag{3.5}$$

which leave the coordinates of M_2 invariant. (3.4) becomes

$$\bar{A}^{\mu,}(\bar{M}_1, \bar{M}_2) = A^{\alpha,}(M_1, M_2)\,\frac{\partial \bar{x}_1^{\mu}}{\partial x_1^{\alpha}}\ , \tag{3.6}$$

so that the $A^{\alpha,}$ are the components of a contravariant vector relative to the transformations (3.5). By considering the transformations

$$\begin{aligned} \bar{x}_1^{i} &= x_1^{i} \\ \bar{x}_2^{i} &= \bar{x}_2^{i}(x_2^1, x_2^2, \cdots, x_2^n)\ , \end{aligned} \tag{3.7}$$

we see that (3.4) reduces to

$$\bar{A}^{\mu,}(M_1, \bar{M}_2) = A^{\alpha,}\,\delta^{\mu,}_{\alpha,} = A^{\mu,}(M_1, M_2)\ , \tag{3.8}$$

showing the scalar character of $A^{\alpha,}(M_1, M_2)$ when referred to the x_2 - coordinate system. In the above discussion we assumed the independence of the two coordinate systems x_1 and x_2, see § 2.

It is clear from the definition of a two-point tensor that most of the properties of the ordinary tensor algebra can be extended to this case. For example:

1. If all the components of a two-point tensor vanish identically for a particular coordinate system, they vanish identically in every coordinate system.

2. Two tensors of the same type (self-explanatory) may be added in the usual manner.

3. Any mixed tensor may be "contracted", providing the contraction is performed on two indices prededing the comma or following the comma.

4. The quotient law is easily verified; for example, if for arbitrary covariant vectors $g_{\alpha,}(M_1, M_2)$ the set of quantities

$$g_{\alpha,}(M_1, M_2)\ T^{\alpha\cdots\beta,\, a\cdots b}_{\sigma\cdots\tau,\, c\cdots d}\ (M_1, M_2) \tag{3.9}$$

are the components of a mixed two-point tensor, then the $T^{\alpha\cdots\beta,\, a\cdots b}_{\sigma\cdots\tau,\, c\cdots d}(M_1, M_2)$ are also the components of a mixed two-point tensor. A similar remark holds for a summation in M_2.

IV. The "metric".

We impose a "metric" on the space defined in §§ 2, 3, i.e., with the point $M_1(x_1^1, \cdots, x_1^n)$ and the point $M_2(x_2^1, \cdots, x_2^n)$, along with the tangent space of differentials dx_1^α at M_1, dx_2^α at M_2 we associate a scalar invariant

$$\phi = g_{\alpha,\beta}(M_1, M_2)\, dx_1^\alpha\, dx_2^\beta \ . \tag{4.1}$$

We call $g_{\alpha,\beta}(M_1, M_2)$ the fundamental "metric" tensor. Since ϕ is a scalar invariant we have

$$(4.2) \qquad \bar{g}_{\mu,\nu}(\bar{M}_1, \bar{M}_2)\, d\bar{x}_1^{\mu} d\bar{x}_2^{\nu} = g_{\alpha,\beta}(M_1, M_2)\, dx_1^{\alpha} dx_2^{\beta}$$

$$= g_{\alpha,\beta}(M_1, M_2) \frac{\partial x_1^{\alpha}}{\partial \bar{x}_1^{\mu}} \frac{\partial x_2^{\beta}}{\partial \bar{x}_2^{\nu}}\, d\bar{x}_1^{\mu} d\bar{x}_2^{\nu} .$$

Because of the independence of $d\bar{x}_1^{\mu}$, $d\bar{x}_2^{\nu}$, (4.2) yields

$$(4.3) \qquad \bar{g}_{\mu,\nu}(\bar{M}_1, \bar{M}_2) = g_{\alpha,\beta}(M_1, M_2) \frac{\partial x_1^{\alpha}}{\partial \bar{x}_1^{\mu}} \frac{\partial x_2^{\beta}}{\partial \bar{x}_2^{\nu}}$$

which shows that the $g_{\alpha,\beta}(M_1, M_2)$ are the components of a two-point tensor, being a covariant vector relative to M_1 and also covariant relative to M_2.

We do not assume $g_{\alpha,\beta}(M_1, M_2) = g_{\beta,\alpha}(M_2, M_1)$. We do assume that $|g_{\alpha,\beta}| \neq 0$.

It is at once apparent that ϕ can be both positive and negative for we have only to replace dx_2^{β} by $-dx_2^{\beta}$ in (4.1) to change the sign of ϕ. We now introduce a non-negative form given by

$$(4.4) \qquad ds^2 = e\, g_{\alpha,\beta}(M_1, M_2)\, dx_1^{\alpha} dx_2^{\beta} ,$$

where e is ± 1, so that $ds^2 \geq 0$.

It is clear that (4.4) reduces to

$$(4.5) \qquad \begin{aligned} ds^2 &= e\, g_{\alpha,\beta}(M_1, M_1)\, dx_1^{\alpha} dx_1^{\beta} \\ &= e\, h_{\alpha\beta}(M)\, dx^{\alpha} dx^{\beta} \end{aligned}$$

when M_1 and M_2 become the same point M, so that the line element (4.5) enables us to speak of the Riemannian geometry of our space.

Let x be a coordinate system which covers M_1 and M_2, and let ${}^{\sigma}\xi_{\alpha}(M)$ be an ennuple of n independent covariant vectors defined over this coordinate system. The superscript σ

distinguishes the n vectors. We can form a special type of $g_{\alpha,\beta}(M_1, M_2)$ by defining

(4.6) $$g_{\alpha,\beta}(M_1, M_2) = \sum_{\sigma=1}^{n} {}^{\sigma}\xi_{\alpha}(M_1)\, {}^{\sigma}\xi_{\beta}(M_2)$$

with $|{}^{\sigma}\xi_{\alpha}| \neq 0$.

We now show that the $g_{\alpha,\beta}$ defined by (4.6) are actually the components of a tensor. We have

(4.7) $$\bar{g}_{\alpha,\beta}(\bar{M}_1, \bar{M}_2) = \sum_{\sigma=1}^{n} {}^{\sigma}\bar{\xi}_{\alpha}(\bar{M}_1)\, {}^{\sigma}\bar{\xi}_{\beta}(\bar{M}_2)$$
$$= \sum_{\sigma=1}^{n} {}^{\sigma}\xi_{\mu}(M_1)\, {}^{\sigma}\xi_{\nu}(M_2) \frac{\partial X_1^{\mu}}{\partial \bar{X}_1^{\alpha}} \frac{\partial X_2^{\nu}}{\partial \bar{X}_2^{\beta}}$$
$$= g_{\mu,\nu}(M_1, M_2) \frac{\partial X_1^{\mu}}{\partial \bar{X}_1^{\alpha}} \frac{\partial X_2^{\nu}}{\partial \bar{X}_2^{\beta}}. \qquad \text{Q.E.D.}$$

From the form of (4.6) we see that $g_{\alpha,\beta}(M_1, M_2) = g_{\beta,\alpha}(M_2, M_1)$. In general, it will require two ennuples, at M_1 and M_2, and, moreover, $g_{\alpha,\beta}(M_1, M_2) \neq g_{\beta,\alpha}(M_2, M_1)$. We will have more to say about this type of metric in paragraph 13.

We can construct a two-point tensor $g_{\alpha,\beta}$ as follows. Let $\xi^{i}(t, x^1, x^2, x^3)$ be the contravariant velocity field of a fluid in motion, and let $h_{\alpha\beta}(x^1, x^2, x^3)$ be the Euclidean metric tensor in general coordinates x^i. Then $\xi_i(t, x^1, x^2, x^3) = h_{ij}(x)\xi^{j}(t,x)$ are the components of the covariant velocity vector field. Let $M[f(t)]$ denote the mean value of the function $f(t)$ over the time interval (t_0, t_1), i.e.,

(4.8) $$M[f(t)] = \int_{t_0}^{t_1} f(t)\,dt \Big/ t_1 - t_0 .$$

It is clear that

(4.9) $$M[\xi_i(t,x)] = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \xi_i(t, x)\,dt$$

is a covariant vector field, since the integration is performed over t. We define a set of quantities $g_{\alpha,\beta}(x_1, x_2)$ by

(4.10) $$g_{\alpha,\beta}(x_1, x_2) = \frac{M[\xi_{\alpha}(t, x_1)\, \xi_{\beta}(t, x_2)]}{\left\{h^{\lambda\mu}(x_1) M[\xi_{\lambda}(t, x_1)\xi_{\mu}(t, x_1)]\right\}^{1/2} \left\{h^{\lambda\mu}(x_2) M[\xi_{\lambda}(t, x_2)\xi_{\mu}(t, x_2)]\right\}}$$

It is evident that $g_{\alpha,\beta}(x_1, x_2)$ is a two-point tensor field of rank two, a covariant vector field with respect to each of the two points. The two point tensor field, $C^{i,j}$, defined by

$$C^{i,j}(x_1, x_2) = h^{i\alpha}(x_1)\, h^{j\beta}(x_2)\, g_{\alpha,\beta}(x_1, x_2) \tag{4.11}$$

is called the two-point correlation tensor of fluid mechanics[2]. Evidently $g_{\alpha,\beta}(x_1,x_2) = g_{\beta,\alpha}(x_2,x_1)$.

<u>Definition</u>. The quantities $g_*^{\alpha,\beta}(M_2, M_1)$ are defined by the matrix equations

$$\| g_*^{\alpha,\beta}(M_2, M_1) \| \cdot \| g_{\alpha,\beta}(M_1, M_2) \| = \| E \| , \tag{4.12}$$

where $\| E \|$ is the unit matrix, so that

$$g_*^{\alpha,\sigma}(M_2, M_1)\, g_{\sigma,\beta}(M_1, M_2) = \delta^{,\alpha}_{,\beta} , \tag{4.13}$$

where the $*$ is intended to point out the fact that the point M_2 is now associated with the superscript preceding the comma, while the point M_1 is associated with the superscript following the comma. We notice in (4.13) that the summation is performed over the index σ associated with M_1. We have no right to conclude that the Kronecker delta, $\delta^{,\alpha}_{,\beta}$, is as yet a tensor. This becomes apparent when we prove the following.

<u>Theorem 1</u>. The $g_*^{\alpha,\beta}(M_2, M_1)$ are the components of a two-point tensor, a contravariant vector at each point.

<u>Proof</u>. Let $A^{,\alpha}(M_1, M_2)$ be an arbitrary multiple point tensor which is a contravariant vector relative to M_2, and a scalar invariant relative to M_1, and whose law of transformation is

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given by

$$\bar{A}^{,\beta}(\bar{M}_1, \bar{M}_2) = A^{,\tau}(M_1, M_2)\,\frac{\partial \bar{x}_2^{\beta}}{\partial x_2^{\tau}} \quad . \tag{4.14}$$

The reader should realize that it is always possible to construct such tensors, for one may take any set $A^{,\tau}(x_1, x_2)$ and then the components in any other coordinate system are defined by (4.14).

Multiplying (4.13) by $A^{,\beta}(M_1, M_2)$ we see that

$$g_*^{\alpha,\sigma}(M_2, M_1)\, g_{\sigma,\beta}(M_1, M_2)\, A^{,\beta}(M_1, M_2) = A^{,\alpha}(M_1, M_2), \tag{4.15}$$

so that the left-hand side of (4.11) denotes an arbitrary contravariant vector field. It is easily seen that

$$B_{\sigma,}(M_1, M_2) \overset{\text{def.}}{\equiv} g_{\sigma,\beta}(M_1, M_2)\, A^{,\beta}(M_1, M_2) \tag{4.16}$$

is a covariant vector since

$$\begin{aligned} \bar{B}_{\sigma,}(\bar{M}_1, \bar{M}_2) &\equiv \bar{g}_{\sigma,\beta}(\bar{M}_1, \bar{M}_2)\, \bar{A}^{,\beta}(\bar{M}_1, \bar{M}_2) \\ &= g_{\mu,\nu}(M_1, M_2)\, A^{,\tau}(M_1, M_2)\, \frac{\partial x_1^{\mu}}{\partial \bar{x}_1^{\sigma}}\, \frac{\partial x_2^{\nu}}{\partial \bar{x}_2^{\beta}}\, \frac{\partial \bar{x}_2^{\beta}}{\partial x_2^{\tau}} \\ &= g_{\mu,\nu}(M_1, M_2)\, A^{,\nu}(M_1, M_2)\, \frac{\partial x_1^{\mu}}{\partial \bar{x}_1^{\sigma}} \\ &= B_{\mu,}(M_1, M_2)\, \frac{\partial x_1^{\mu}}{\partial \bar{x}_1^{\sigma}} \quad . \end{aligned} \tag{4.17}$$

Thus from (4.15), (4.16), (4.17) we see that $g_*^{\alpha,\sigma}(M_2, M_1)\, B_{\sigma,}(M_1, M_2)$ are the compoments of an arbitrary contravariant vector, so that

$$\bar{g}_*^{\alpha,\sigma}(\bar{M}_2, \bar{M}_1)\, \bar{B}_{\sigma,}(\bar{M}_1, \bar{M}_2) = g_*^{\mu,\nu}(M_2, M_1)\, B_{\nu,}(M_1, M_2)\, \frac{\partial \bar{x}_2^{\alpha}}{\partial x_2^{\mu}} \quad . \tag{4.18}$$

Using (4.17) we obtain

$$\bar{g}_*^{\alpha,\sigma}(\bar{M}_2, \bar{M}_1)\, B_{\nu,}(M_1, M_2)\, \frac{\partial x_1^{\nu}}{\partial \bar{x}_1^{\sigma}} = g_*^{\mu,\nu}(M_2, M_1)\, B_{\nu,}(M_1, M_2)\, \frac{\partial \bar{x}_2^{\alpha}}{\partial x_2^{\mu}} , \tag{4.19}$$

true for arbitrary $B_{\nu,}(M_1, M_2)$. Hence we immediately conclude that the law of transformation for the $g_*^{\alpha,\beta}(M_2, M_1)$ is given by

$$\bar{g}_*^{\alpha,\beta}(\bar{M}_2, \bar{M}_1) = g_*^{\mu,\nu}(M_2, M_1)\, \frac{\partial \bar{x}_2^{\alpha}}{\partial x_2^{\mu}}\, \frac{\partial \bar{x}_1^{\beta}}{\partial x_1^{\nu}} \quad . \qquad \text{Q.E.D.} \tag{4.20}$$

We now check that $\delta^{\alpha}_{\beta} \equiv \delta^{,\alpha}_{,\beta}$.

Proof.

(4.21)
$$\bar{\delta}^{\alpha}_{\beta} = \bar{g}_{*}^{\alpha,\sigma}(\bar{M}_2, \bar{M}_1)\, \bar{g}_{\sigma,\beta}(\bar{M}_1, \bar{M}_2)$$
$$= g_{*}^{\epsilon,\mu}(M_2, M_1)\, \frac{\partial \bar{X}_2^{\alpha}}{\partial X_2^{\epsilon}} \frac{\partial \bar{X}_1^{\sigma}}{\partial X_1^{\mu}}\, g_{\phi,\tau}\, \frac{\partial X_1^{\phi}}{\partial \bar{X}_1^{\sigma}} \frac{\partial X_2^{\tau}}{\partial \bar{X}_2^{\beta}}$$
$$= g_{*}^{\epsilon,\phi}(M_2, M_1)\, g_{\phi,\tau}(M_1, M_2)\, \frac{\partial \bar{X}_2^{\alpha}}{\partial X_2^{\epsilon}} \frac{\partial X_2^{\tau}}{\partial \bar{X}_2^{\beta}}$$
$$= \delta^{\epsilon}_{\tau}\, \frac{\partial \bar{X}_2^{\alpha}}{\partial X_2^{\epsilon}} \frac{\partial X_2^{\tau}}{\partial \bar{X}_2^{\beta}} \equiv \delta^{,\alpha}_{,\beta},$$

verifying that δ^{ϵ}_{τ} is a mixed tensor in M_2, and a scalar invariant relative to M_1.

Definition. We define the $g^{\alpha,\beta}(M_1, M_2)$ by the matrix equation

(4.22)
$$\| g^{\alpha,\beta}(M_1, M_2) \| \cdot \| {}_{*}g_{\alpha,\beta}(M_2, M_1) \| = \| E \|, \quad {}_{*}g_{\alpha,\beta}(M_2, M_1) = g_{\beta,\alpha}(M_1, M_2)$$

or by

(4.23)
$$g^{\alpha,\sigma}(M_1, M_2)\, g_{\beta,\sigma}(M_1, M_2) = \delta^{\alpha,}_{\beta,}\,.$$

By the same method as used above it is easy to show that

(4.24)
$$\bar{g}^{\alpha,\beta}(\bar{M}_1, \bar{M}_2) = g^{\mu,\nu}(M_1, M_2)\, \frac{\partial \bar{X}_1^{\alpha}}{\partial X_1^{\mu}} \frac{\partial \bar{X}_2^{\beta}}{\partial X_2^{\nu}},$$

the $g^{\alpha,\beta}(M_1, M_2)$ being components of a two-point tensor, a contravariant vector relative to both M_1 and M_2. It is also easy to show that $g^{\alpha,\beta}(M_1, M_2) = g_{*}^{\beta,\alpha}$.

V. The Dyodesics.

Let us return to (4.4) We assume the $g_{\alpha,\beta}(X_1, X_2)$ continuous in X_1 and X_2 and consider the curves $X_1^i(t)$, $X_2^i(t)$, having continuous first derivatives. Then certainly

(5.0)
$$\left(e\, g_{\alpha,\beta} \frac{dX_1^{\alpha}}{dt} \frac{dX_2^{\beta}}{dt} \right)^{1/2} \equiv \left| g_{\alpha,\beta}(X_1(t), X_2(t)) \frac{dX_1^{\alpha}}{dt} \frac{dX_2^{\beta}}{dt} \right|^{1/2}$$

is continuous and hence integrable. With the curves $X_1^i(t)$, $X_2^i(t)$, we associate a real non-negative number, L, given by

(5.1) $$L = \int_{t_0}^{t_1} \left(e\, g_{\alpha,\beta} \frac{dx_1^{\alpha}}{dt} \frac{dx_2^{\beta}}{dt} \right)^{1/2} dt \ .$$

If we perform a change of variable in the parameter t, say $u = f(t)$ such that $\frac{du}{dt} > 0$ for $t_0 \leq t \leq t_1$, then

$$e\, g_{\alpha,\beta} \frac{dx_1^{\alpha}}{dt} \frac{dx_2^{\beta}}{dt} = e\, g_{\alpha,\beta} \frac{dx_1^{\alpha}}{du} \frac{dx_2^{\beta}}{du} \left(\frac{du}{dt} \right)^2 > 0$$

so that $e\, g_{\alpha,\beta} \frac{dx_1^{\alpha}}{du} \frac{dx_2^{\beta}}{du} > 0$, and it is apparent that (5.1) remains invariant under such a change of parameter provided $\frac{du}{dt}$ is continuous. Hence the change of parameter given by

(5.2) $$s = \int_{t_0}^{t} \left(e\, g_{\alpha,\beta} \frac{dx_1^{\alpha}}{dt} \frac{dx_2^{\beta}}{dt} \right)^{1/2} dt$$

is a permissible change of parameter provided

(5.2') $$g_{\alpha,\beta} \frac{dx_1^{\alpha}}{dt} \frac{dx_2^{\beta}}{dt}$$

has the <u>same</u> sign throughout the interval $t_0 \leq t \leq t_1$. When this is done and the x_1^i, x_2^i describing the curves are expressed as functions of s, we have from (5.2),

(5.3) $$e\, g_{\alpha,\beta}(x_1(s), x_2(s)) \frac{dx_1^{\alpha}}{ds} \frac{dx_2^{\beta}}{ds} = 1 \ .$$

We now wish to extremalize (5.1) under the assumptions stated above. Geometrically, we wish to find the <u>paths</u>, $x_1^i(t)$, $x_2^i(t)$, for $t_0 \leq t \leq t_1$, which make (5.1) an extremal. We only consider the range $t_0 \leq t \leq t_1$ such that $g_{\alpha,\beta} \frac{dx_1^{\alpha}}{dt} \frac{dx_2^{\beta}}{dt}$ keeps the same sign.

A more general problem would be the following. Let $\phi(x_1^1, x_1^2, \cdots, x_1^n, x_2^1, x_2^2, \cdots, x_2^n, \dot{x}_1^1, \dot{x}_1^2, \cdots, \dot{x}_1^n, \dot{x}_2^1, \cdots, \dot{x}_2^n)$ be a non-negative integrable function. We look for the functions $x_1^i(t)$, $x_2^i(t)$ which make

(5.4) $$\int_{t_0}^{t_1} \phi \, dt$$

an extremal. The dots stand for the first derivatives with respect to the parameter t.

We assume $x_1^i = f_1^i(t)$, $x_2^i = f_2^i(t)$ are the required functions, and we consider the "neighborhing" functions

$$
(5.5) \qquad \begin{aligned} x_1^i(t) &= f_1^i(t) + \mu\, \psi_1^i(t) \\ x_2^i(t) &= f_2^i(t) + \nu\, \psi_2^i(t) \, , \end{aligned}
$$

where μ, ν are arbitrary and small, and $\psi_1^i(t) = \psi_2^i(t) = 0$ for $t = t_0$ and $t = t_1$. As such, the integral

$$
(5.6) \qquad I(\mu, \nu) = \int_{t_0}^{t_1} \phi(f_1 + \mu\psi_1, f_2 + \nu\psi_2, \dot{f}_1 + \mu\dot{\psi}_1, \dot{f}_2 + \nu\dot{\psi}_2)\, dt
$$

is an extremal for $\mu = \nu = 0$. From the calculus we must have $\frac{\partial I}{\partial \mu} = \frac{\partial I}{\partial \nu} = 0$, for $\mu = \nu = 0$. Now

$$
(5.7) \qquad \begin{aligned} \left.\frac{\partial I}{\partial \mu}\right|_{\mu,\nu=0} &= \int_{t_0}^{t_1} \left(\frac{\partial \phi}{\partial f_1^\alpha} \psi_1^\alpha + \frac{\partial \phi}{\partial \dot{f}_1^\alpha} \dot{\psi}_1^\alpha \right) dt \\ \left.\frac{\partial I}{\partial \nu}\right|_{\mu,\nu=0} &= \int_{t_0}^{t_1} \left(\frac{\partial \phi}{\partial f_2^\alpha} \psi_2^\alpha + \frac{\partial \phi}{\partial \dot{f}_2^\alpha} \dot{\psi}_2^\alpha \right) dt \, . \end{aligned}
$$

Integrating the second terms of (5.7) by parts and making use of the above hypothesis, we obtain

$$
(5.8) \qquad \begin{aligned} &\int_{t_0}^{t_1} \left[\frac{\partial \phi}{\partial f_1^\alpha} - \frac{d}{dt}\left(\frac{\partial \phi}{\partial \dot{f}_1^\alpha} \right) \right] \psi_1^\alpha \, dt = 0 \\ &\int_{t_0}^{t_1} \left[\frac{\partial \phi}{\partial f_2^\alpha} - \frac{d}{dt}\left(\frac{\partial \phi}{\partial \dot{f}_2^\alpha} \right) \right] \psi_2^\alpha \, dt = 0 \, . \end{aligned}
$$

These equations must be satisfied for arbitrary values of $\psi_1^\alpha(t)$, $\psi_2^\alpha(t)$, so that assuming continuity of the integrand, the $x_1^\alpha(t)$, $x_2^\alpha(t)$ must satisfy Euler's equations,

$$
(5.9) \qquad \begin{aligned} &\frac{d}{dt}\left(\frac{\partial \phi}{\partial \dot{x}_1^\alpha} \right) - \frac{\partial \phi}{\partial x_1^\alpha} = 0 \\ &\frac{d}{dt}\left(\frac{\partial \phi}{\partial \dot{x}_2^\alpha} \right) - \frac{\partial \phi}{\partial x_2^\alpha} = 0 \, . \end{aligned}
$$

Applying (5.9) to (5.1) and choosing s as parameter, see (5.2), we obtain

$$
(5.10) \qquad \begin{aligned} &\frac{d}{ds}\left(g_{\sigma,\beta}\, \dot{x}_2^\beta \right) - \frac{\partial g_{\alpha,\beta}}{\partial x_1^\sigma} \dot{x}_1^\alpha \dot{x}_2^\beta = 0 \\ &\frac{d}{ds}\left(g_{\alpha,\sigma}\, \dot{x}_1^\alpha \right) - \frac{\partial g_{\alpha,\beta}}{\partial x_2^\sigma} \dot{x}_1^\alpha \dot{x}_2^\beta = 0 \, , \end{aligned}
$$

or,

$$
(5.11)\qquad
\begin{aligned}
&g_{\alpha,\sigma}\ddot{X}_1^{\alpha} + \frac{\partial g_{\alpha,\sigma}}{\partial X_1^{\beta}}\dot{X}_1^{\alpha}\dot{X}_1^{\beta} + \left(\frac{\partial g_{\alpha,\sigma}}{\partial X_2^{\beta}} - \frac{\partial g_{\alpha,\beta}}{\partial X_2^{\sigma}}\right)\dot{X}_1^{\alpha}\dot{X}_2^{\beta} = 0\\
&g_{\sigma,\alpha}\ddot{X}_2^{\alpha} + \frac{\partial g_{\sigma,\beta}}{\partial X_2^{\alpha}}\dot{X}_2^{\alpha}\dot{X}_2^{\beta} + \left(\frac{\partial g_{\sigma,\beta}}{\partial X_1^{\alpha}} - \frac{\partial g_{\alpha,\beta}}{\partial X_1^{\sigma}}\right)\dot{X}_1^{\alpha}\dot{X}_2^{\beta} = 0 .
\end{aligned}
$$

Making use of (4.13) and (4.23) we obtain the system

$$
(5.12)\qquad
\begin{aligned}
&\ddot{X}_1^{i} + g^{i,\sigma}\frac{\partial g_{\alpha,\sigma}}{\partial X_1^{\beta}}\dot{X}_1^{\alpha}\dot{X}_1^{\beta} + g^{i,\sigma}\left(\frac{\partial g_{\alpha,\sigma}}{\partial X_2^{\beta}} - \frac{\partial g_{\alpha,\beta}}{\partial X_2^{\sigma}}\right)\dot{X}_1^{\alpha}\dot{X}_2^{\beta} = 0\\
&\ddot{X}_2^{i} + g_{*}^{i,\sigma}\frac{\partial g_{\sigma,\beta}}{\partial X_2^{\alpha}}\dot{X}_2^{\alpha}\dot{X}_2^{\beta} + g_{*}^{i,\sigma}\left(\frac{\partial g_{\sigma,\beta}}{\partial X_1^{\alpha}} - \frac{\partial g_{\alpha,\beta}}{\partial X_1^{\sigma}}\right)\dot{X}_1^{\alpha}\dot{X}_2^{\beta} = 0
\end{aligned}
$$

This system is of the type

$$
(5.13)\qquad
\begin{aligned}
\frac{d^2X_1^{i}}{ds^2} &= F^{i}(X_1, X_2, \dot{X}_1, \dot{X}_2)\\
\frac{d^2X_2^{i}}{ds^2} &= G^{i}(X_1, X_2, \dot{X}_1, \dot{X}_2) .
\end{aligned}
$$

The solutions of the 2n second-order differential equations in the 2n unknowns $X_1^i(s)$, $X_2^i(s)$ exist uniquely in the neighborhoods of $X_1^i = a_1^i$, $X_2^i = a_2^i$, when the initial conditions

$$
(5.14)\qquad
\begin{aligned}
X_1^{i}(s_0) &= a_1^{i}\\
X_2^{i}(s_0) &= a_2^{i}\\
\left.\frac{dX_1^{i}}{ds}\right|_{s=s_0} &= b_1^{i}\\
\left.\frac{dX_2^{i}}{ds}\right|_{s=s_0} &= b_2^{i}
\end{aligned}
$$

are given, if we assume the $g_{\alpha,\beta}$ are analytic. Less stringent conditions are actually needed for existence and uniqueness.

The curves $X_1^i(s)$, $X_2^i(s)$, which satisfy (5.12) will be called <u>dyodesics</u> or <u>dyo-paths</u>. They always occur in pairs and should not be confused with the geodesics obtained from the Riemannian metric tensor $g_{\alpha,\beta}(M,M)$.

Definition. A system (pair) of curves satisfying (5.12) and such that $g_{\alpha,\beta}\frac{dx_1^\alpha}{ds}\frac{dx_2^\beta}{ds}=0$ everywhere along the paths will be called minimal dyodesics.

In general, a pair of curves satisfying $g_{\alpha,\beta}\frac{dx_1^\alpha}{ds}\frac{dx_2^\beta}{ds}=0$ will not form minimal dyodesics, for we need only consider the case $x_1^\alpha \equiv$ constants, $x_2^\alpha(s)$ arbitrary. (5.12) will only be satisfied if

(5.15) $$\frac{d^2x_2^{\iota}}{ds^2}+g_*^{\iota,\sigma}\frac{\partial g_{\sigma,\beta}}{\partial x_2^\alpha}\frac{dx_2^\alpha}{ds}\frac{dx_2^\beta}{ds}=0,$$

which, in general, cannot be satisfied by arbitrary $x_2^\alpha(s)$.

VI. The Linear Connections.

Definition. The linear connection $\Gamma_{\alpha\beta,}^{\iota,}(M_1,M_2)$ is defined by

(6.1) $$\Gamma_{\alpha\beta,}^{\iota,}(M_1,M_2)=g^{\iota,\sigma}(M_1,M_2)\frac{\partial g_{\alpha,\sigma}(M_1,M_2)}{\partial x_1^\beta},$$

so that $$\frac{\partial g_{\alpha,\tau}}{\partial x_1^\beta}=g_{\iota,\tau}\Gamma_{\alpha\beta,}^{\iota,}.$$

Theorem 2. The law of transformation for the $\Gamma_{\alpha\beta,}^{\iota,}$ is given by

(6.2) $$\bar\Gamma_{\mu\sigma,}^{\alpha,}(\bar M_1,\bar M_2)=\Gamma_{\rho\tau,}^{\epsilon,}(M_1,M_2)\frac{\partial\bar x_1^\alpha}{\partial x_1^\epsilon}\frac{\partial x_1^\rho}{\partial\bar x_1^\mu}\frac{\partial x_1^\tau}{\partial\bar x_1^\sigma}+\frac{\partial^2x_1^\rho}{\partial\bar x_1^\sigma\partial\bar x_1^\mu}\frac{\partial\bar x_1^\alpha}{\partial x_1^\rho}.$$

Proof. Differentiating (4.3) we obtain

(6.3) $$\frac{\partial\bar g_{\mu,\nu}}{\partial\bar x_1^\sigma}=\frac{\partial g_{\alpha,\beta}}{\partial x_1^\tau}\frac{\partial x_1^\tau}{\partial\bar x_1^\sigma}\frac{\partial x_1^\alpha}{\partial\bar x_1^\mu}\frac{\partial x_2^\beta}{\partial\bar x_2^\nu}+g_{\alpha,\beta}\frac{\partial x_2^\beta}{\partial\bar x_2^\nu}\frac{\partial^2x_1^\alpha}{\partial\bar x_1^\sigma\partial\bar x_1^\mu}.$$

Now multiplying by $\bar g^{\epsilon,\nu}(\bar M_1,\bar M_2)$ we obtain

(6.4) $$\bar g^{\alpha,\nu}\frac{\partial\bar g_{\mu,\nu}}{\partial\bar x_1^\sigma}=g^{\epsilon,\phi}\frac{\partial\bar x_1^\alpha}{\partial x_1^\epsilon}\frac{\partial\bar x_2^\nu}{\partial x_2^\phi}\frac{\partial g_{\rho,\beta}}{\partial x_1^\tau}\frac{\partial x_1^\tau}{\partial\bar x_1^\sigma}\frac{\partial x_1^\rho}{\partial\bar x_1^\mu}\frac{\partial x_2^\beta}{\partial\bar x_2^\nu}$$
$$+g^{\epsilon,\phi}\frac{\partial\bar x_1^\alpha}{\partial x_1^\epsilon}\frac{\partial\bar x_2^\nu}{\partial x_2^\phi}g_{\rho,\beta}\frac{\partial x_2^\beta}{\partial\bar x_2^\nu}\frac{\partial^2x_1^\rho}{\partial\bar x_1^\sigma\partial\bar x_1^\mu}$$
$$=g^{\epsilon,\beta}\frac{\partial g_{\rho,\beta}}{\partial x_1^\tau}\frac{\partial\bar x_1^\alpha}{\partial x_1^\epsilon}\frac{\partial x_1^\tau}{\partial\bar x_1^\sigma}\frac{\partial x_1^\rho}{\partial\bar x_1^\mu}+\frac{\partial^2x_1^\rho}{\partial\bar x_1^\sigma\partial\bar x_1^\mu}\frac{\partial\bar x_1^\alpha}{\partial x_1^\rho},$$

by making use of (4.23). Our theorem follows at once. Moreover, if the x_1 coordinate system remains fixed, (6.4) reduces to $\bar{\Gamma}^{\alpha_1}_{\mu\sigma_1}(x_1, \bar{x}_2) = \Gamma^{\alpha_1}_{\mu\sigma_1}(x_1, x_2)$, showing the scalar invariance of the linear connection $\Gamma^{\alpha_1}_{\mu\sigma_1}$ relative to the point M_2.

Theorem 3. A necessary and sufficient condition that $\Gamma^{\iota_1}_{\alpha\beta_1}(M_1, M_2)$ be symmetric in its lower indices is that $g_{\alpha,\beta}(M_1, M_2) = \dfrac{\partial \phi_{,\beta}}{\partial x_1^{\alpha}}$, where $\phi_{,\beta}(M_1, M_2)$ is a scalar relative to M_1 and a covariant vector relative to M_2.

Proof. Since

(6.5) $$\bar{\phi}_{,\beta} = \phi_{,\mu} \frac{\partial x_2^{\mu}}{\partial \bar{x}_2^{\beta}} ,$$

we have

(6.6) $$\bar{g}_{\alpha,\beta} = \frac{\partial \bar{\phi}_{,\beta}}{\partial \bar{x}_1^{\alpha}} = \frac{\partial \phi_{,\mu}}{\partial x_1^{\sigma}} \frac{\partial x_1^{\sigma}}{\partial \bar{x}_1^{\alpha}} \frac{\partial x_2^{\mu}}{\partial \bar{x}_2^{\beta}}$$
$$= g_{\sigma,\mu} \frac{\partial x_1^{\sigma}}{\partial \bar{x}_1^{\alpha}} \frac{\partial x_2^{\mu}}{\partial \bar{x}_2^{\beta}} ,$$

so that $g_{\alpha,\beta}$ is a covariant two-point tensor. Moreover,

(6.7) $$\frac{\partial g_{\alpha,\beta}}{\partial x_1^{\sigma}} = \frac{\partial^2 \phi_{,\beta}}{\partial x_1^{\sigma} \partial x_1^{\alpha}} = \frac{\partial g_{\sigma,\beta}}{\partial x_1^{\alpha}} ,$$

which proves the sufficiency argument.

Conversely, assume $\Gamma^{\iota_1}_{\alpha\beta_1} = \Gamma^{\iota_1}_{\beta\alpha_1}$ so that

(6.8) $$g^{\iota,\sigma} \frac{\partial g_{\alpha,\sigma}}{\partial x_1^{\beta}} = g^{\iota,\sigma} \frac{\partial g_{\beta,\sigma}}{\partial x_1^{\alpha}} ,$$

and since $|g^{\iota,\sigma}| \neq 0$ we obtain $\dfrac{\partial g_{\alpha,\sigma}}{\partial x_1^{\beta}} = \dfrac{\partial g_{\beta,\sigma}}{\partial x_1^{\alpha}}$. Now relative to M_1 we have

(6.9) $$\bar{g}_{\alpha,\beta}(\bar{M}_1, M_2) = g_{\sigma,\mu}(M_1, M_2) \frac{\partial x_1^{\sigma}}{\partial \bar{x}_1^{\alpha}} \frac{\partial x_2^{\mu}}{\partial x_2^{\beta}}$$
$$= g_{\sigma,\beta}(M_1, M_2) \frac{\partial x_1^{\sigma}}{\partial \bar{x}_1^{\alpha}} ,$$

so that $g_{\alpha,\sigma}$ is a covariant vector relative to M_1. Since

(6.10) $$\frac{\partial g_{\alpha,\sigma}}{\partial X_1^{\beta}} - \frac{\partial g_{\beta,\sigma}}{\partial X_1^{\alpha}} = 0$$

we have that $g_{\alpha,\sigma}$ is a gradient relative to M_1, because of the vanishing of the curl of $g_{\alpha,\sigma}$. Hence

(6.11) $$g_{\alpha,\sigma}(M_1, M_2) = \frac{\partial \phi_\sigma(M_1,M_2)}{\partial X_1^{\alpha}}$$

where ϕ_σ is a scalar relative to M_1. We must now show that $\phi_\sigma \equiv \phi_{,\sigma}$, i.e., ϕ_σ is a covariant vector relative to M_2. Now

(6.12) $$\bar{g}_{\alpha,\sigma} = \frac{\partial \bar{\phi}_\sigma}{\partial \bar{X}_1^{\alpha}} = g_{\mu,\nu} \frac{\partial X_1^{\mu}}{\partial \bar{X}_1^{\alpha}} \frac{\partial X_2^{\nu}}{\partial \bar{X}_2^{\sigma}}$$

so that

(6.13) $$\frac{\partial \bar{\phi}_\sigma}{\partial \bar{X}_1^{\alpha}} = \frac{\partial \phi_\nu}{\partial X_1^{\mu}} \frac{\partial X_1^{\mu}}{\partial \bar{X}_1^{\alpha}} \frac{\partial X_2^{\nu}}{\partial \bar{X}_2^{\sigma}}$$
$$= \frac{\partial \phi_\nu}{\partial \bar{X}_1^{\alpha}} \frac{\partial X_2^{\nu}}{\partial \bar{X}_2^{\sigma}} .$$

Integrating with respect to $\bar{X}_1^{\alpha}$ we obtain

(6.14) $$\bar{\phi}_\sigma = \phi_\nu \frac{\partial X_2^{\nu}}{\partial \bar{X}_2^{\sigma}} + \psi_\sigma(\bar{X}_2)$$

Now relative to M_1 we have

(6.15) $$\bar{\phi}_\sigma(\bar{X}_1, X_2) = \phi_\sigma(X_1, X_2) + \psi_\sigma(\bar{X}_2)$$

and since ϕ_σ is a scalar invariant relative to M_1, we must have $\psi_\sigma(\bar{X}_2) \equiv 0$. Hence $\phi_\nu = \phi_{,\nu}$. Q.E.D.

<u>Definition.</u>

(6.16) $$\Gamma_{,\alpha\beta}^{,i}(M_1, M_2) = g_*^{i,\sigma}(M_2, M_1) \frac{\partial g_{\sigma,\beta}(M_1,M_2)}{\partial X_2^{\alpha}}$$

or
$$\frac{\partial g_{\tau,\beta}}{\partial X_2^{\alpha}} = g_{\tau,i} \Gamma_{,\alpha\beta}^{,i} .$$

<u>Theorem 4.</u> The linear connection $\Gamma_{,\alpha\beta}^{,i}$ transforms according to the law

(6.16') $$\bar{\Gamma}_{,\alpha\beta}^{,\lambda}(\bar{M}_1,\bar{M}_2) = \Gamma_{,\nu\tau}^{,\epsilon}(M_1,M_2)\frac{\partial \bar{X}_2^{\iota}}{\partial X_2^{\epsilon}}\frac{\partial X_2^{\nu}}{\partial \bar{X}_2^{\alpha}}\frac{\partial X_2^{\tau}}{\partial \bar{X}_2^{\beta}} + \frac{\partial^2 X_2^{\nu}}{\partial \bar{X}_2^{\beta}\partial \bar{X}_2^{\alpha}}\frac{\partial \bar{X}_2^{\iota}}{\partial X_2^{\nu}}.$$

Proof. Differentiating $\bar{g}_{\sigma,\beta} = g_{\mu,\nu}\frac{\partial X_1^{\mu}}{\partial \bar{X}_1^{\sigma}}\frac{\partial X_2^{\nu}}{\partial \bar{X}_2^{\beta}}$ we obtain

(6.17) $$\frac{\partial \bar{g}_{\sigma,\beta}}{\partial \bar{X}_2^{\alpha}} = \frac{\partial g_{\mu,\nu}}{\partial X_2^{\tau}}\frac{\partial X_2^{\tau}}{\partial \bar{X}_2^{\alpha}}\frac{\partial X_1^{\mu}}{\partial \bar{X}_1^{\sigma}}\frac{\partial X_2^{\nu}}{\partial \bar{X}_2^{\beta}} + g_{\mu,\nu}\frac{\partial X_1^{\mu}}{\partial \bar{X}_1^{\sigma}}\frac{\partial^2 X_2^{\nu}}{\partial \bar{X}_2^{\alpha}\partial \bar{X}_2^{\beta}}.$$

Multiplying (6.17) by $\bar{g}_*^{\iota,\sigma} = g_*^{\epsilon,\phi}\frac{\partial \bar{X}_1^{\sigma}}{\partial X_1^{\phi}}\frac{\partial \bar{X}_2^{\iota}}{\partial X_2^{\epsilon}}$ one obtains

(6.18)
$$\begin{aligned}\bar{g}_*^{\iota,\sigma}\frac{\partial \bar{g}_{\sigma,\beta}}{\partial \bar{X}_2^{\alpha}} &= g_*^{\epsilon,\phi}\frac{\partial \bar{X}_1^{\sigma}}{\partial X_1^{\phi}}\frac{\partial \bar{X}_2^{\iota}}{\partial X_2^{\epsilon}}\frac{\partial g_{\mu,\nu}}{\partial X_2^{\tau}}\frac{\partial X_2^{\tau}}{\partial \bar{X}_2^{\alpha}}\frac{\partial X_1^{\mu}}{\partial \bar{X}_1^{\sigma}}\frac{\partial X_2^{\nu}}{\partial \bar{X}_2^{\beta}} \\ &\quad + g_*^{\epsilon,\phi}\frac{\partial \bar{X}_1^{\sigma}}{\partial X_1^{\phi}}\frac{\partial \bar{X}_2^{\iota}}{\partial X_2^{\epsilon}}g_{\mu,\nu}\frac{\partial X_1^{\mu}}{\partial \bar{X}_1^{\sigma}}\frac{\partial^2 X^{\nu}}{\partial \bar{X}_2^{\alpha}\partial \bar{X}_2^{\beta}} \\ &= g_*^{\epsilon,\mu}\frac{\partial g_{\mu,\nu}}{\partial X_2^{\tau}}\frac{\partial \bar{X}_2^{\iota}}{\partial X_2^{\epsilon}}\frac{\partial X_2^{\nu}}{\partial \bar{X}_2^{\beta}}\frac{\partial X_2^{\tau}}{\partial \bar{X}_2^{\alpha}} + \frac{\partial^2 X_2^{\nu}}{\partial \bar{X}_2^{\alpha}\partial \bar{X}_2^{\beta}}\frac{\partial \bar{X}_2^{\lambda}}{\partial X_2^{\nu}}.\end{aligned}$$ Q.E.D.

Theorem 5. $\Gamma_{,\alpha\beta}^{,\iota}$ is symmetric in its lower indices if and only if $g_{\alpha,\beta} = \frac{\partial \theta_{\alpha,}}{\partial X_2^{\beta}}$ where $\theta_{\alpha,}(M_1,M_2)$ is a covariant vector relative to M_1 and a scalar relative to M_2.

The proof proceeds in much the same manner as that of Theorem 3 encountered above.

In order that $\Gamma_{\alpha\beta,}^{\iota,}$ and $\Gamma_{,\alpha\beta}^{,\iota}$ be simultaneously symmetric in their lower indices it follows from above that of necessity

(6.19) $$g_{\alpha,\beta} = \frac{\partial \phi_{,\beta}}{\partial X_1^{\alpha}} = \frac{\partial \theta_{\alpha,}}{\partial X_2^{\beta}}.$$

Now let $\psi(M_1,M_2)$ be a scalar relative to M_1 and to M_2. If

(6.20) $$g_{\alpha,\beta} = \frac{\partial^2\psi}{\partial X_1^{\alpha}\partial X_2^{\beta}},$$

then

(6.21) $$\begin{aligned}g_{\alpha,\beta} &= \frac{\partial}{\partial X_1^{\alpha}}\left(\frac{\partial\psi}{\partial X_2^{\beta}}\right) = \frac{\partial}{\partial X_2^{\beta}}\left(\frac{\partial\psi}{\partial X_1^{\alpha}}\right) \\ &= \frac{\partial \psi_{,\beta}}{\partial X_1^{\alpha}} = \frac{\partial \psi_{\alpha,}}{\partial X_2^{\beta}},\end{aligned}$$

so that both Γ's are symmetric in their lower indices.

Conversely, assume the Γ^{λ}_{2} symmetric in their lower indices. From (6.19) we have

(6.22) $$\varphi_{,\beta} = \int g_{\alpha,\beta}(x_1, x_2)\, dx_1^{\alpha} + h_{,\beta}(x_2)$$
$$= F_{,\beta}(x_1, x_2) + h_{,\beta}(x_2)$$

where from (6.19)

(6.22') $$F_{,\beta}(x_1, x_2) = \int g_{\alpha,\beta}\, dx_1^{\alpha} = \int \frac{\partial \theta_{\alpha,}}{\partial x_2^{\beta}}\, dx_1^{\alpha} .$$

Let us now see if it is possible to solve

(6.23) $$\frac{\partial \psi(x_1, x_2)}{\partial x_2^{\beta}} = F_{,\beta}(x_1, x_2) .$$

We need $$\frac{\partial F_{,\beta}}{\partial x_2^{\sigma}} = \frac{\partial F_{,\sigma}}{\partial x_2^{\beta}} ,$$

or $$\int \frac{\partial^2 \theta_{\alpha,}}{\partial x_2^{\beta} \partial x_2^{\sigma}}\, dx_1^{\alpha} = \int \frac{\partial^2 \theta_{\alpha,}}{\partial x_2^{\sigma} \partial x_2^{\beta}}\, dx_1^{\alpha} ,$$

which obviously holds. Hence

(6.24) $$\frac{\partial^2 \psi}{\partial x_2^{\beta} \partial x_1^{\alpha}} = \frac{\partial F_{,\beta}}{\partial x_1^{\alpha}} = g_{\alpha,\beta} .$$

We have demonstrated

Theorem 6. A necessary and sufficient condition that both linear connections $\Gamma^{i}_{\alpha\beta,}$, $\Gamma^{,i}_{,\alpha\beta}$, be symmetric in their lower indices is that there exist a scalar invariant $\psi(M_1, M_2)$ relative to M_1, M_2 such that $g_{\alpha,\beta} = \dfrac{\partial^2 \psi}{\partial x_1^{\alpha} \partial x_2^{\beta}}$.

From (6.2) we note that if a coordinate system exists for which $\Gamma^{i}_{\alpha\beta,} = 0$, then $\bar{\Gamma}^{r}_{st,} = \bar{\Gamma}^{r}_{ts,}$. Hence if the linear connection $\Gamma^{i}_{\alpha\beta,}$ is asymmetric no coordinate system can exist which will cause the symbols $\Gamma^{i}_{\alpha\beta,}$ to vanish. A similar remark holds for $\Gamma^{,i}_{,\alpha\beta}$.

If $\xi^{i_1}(X_1)$, $\xi^{j_1}(X_2)$ are contravariant vectors relative to X_1 and X_2 respectively, we may compare the two vectors as to magnitude, angle between the two, etc., provided we can join the points X_1, X_2 by a curve $X^i(t)$ in the space. To do so we introduce the Riemannian metric $g_{\alpha,\beta}(M,M)$, assuming symmetry in the indices. Let $X^i(t)$ be the geodesic joining X_1°, X_2° obtained by extremalizing

(6.25) $$\int_{t_0}^{t_1} \left(g_{\alpha,\beta}(M,M) \frac{dX^\alpha}{dt} \frac{dX^\beta}{dt} \right)^{1/2} dt \, ,$$

the parameter t satisfying $g_{\alpha,\beta} \frac{dX^\alpha}{dt} \frac{dX^\beta}{dt} = 1$.

We can now transport $\xi^{i_1}(X_1)$ along $X^i(t)$ through a parallel displacement from X_1° to X_2°. We need only integrate the system

(6.26) $$\frac{d\xi^{i_1}(x)}{dt} + \Gamma^i_{\alpha\beta}(x)\, \xi^{\alpha_1}(x) \frac{dX^\beta}{dt} = 0 ,$$

where $\Gamma^i_{\alpha\beta}(x)$ is the Christoffel symbol obtained from the Riemannian metric $g_{\alpha,\beta}(M,M)$. We now define the cosine of the angle between $\xi^{i_1}(X_2)$, $\xi^{j_1}(X_2)$ at X_2° as follows.

(6.27) $$\cos^2\theta = \frac{g_{\alpha,\beta}(X_2^\circ;X_2^\circ)\, \xi^{\alpha_1}(X_2^\circ)\, \xi^{j\beta}(X_2^\circ)}{\left[g_{\alpha,\beta}(X_2^\circ;X_2^\circ)\, \xi^{\alpha_1}(X_2^\circ)\, \xi^{\beta_1}(X_2^\circ) \right]^{1/2} \left[g_{\mu,\nu}(X_2^\circ,X_2^\circ)\, \xi^{j\mu}(X_2^\circ)\, \xi^{j\nu}(X_2^\circ) \right]}$$

where $\xi^{i_1}(X_2^\circ)$ is the value of $\xi^{i_1}(x)$ after parallel displacement along $X^i(t)$ to the point X_2°. The above results are relative to the path $X^i(t)$.

VII. Tensors connected with the dyodesics.

Interchanging σ and μ in (6.3) and subtracting we obtain

(7.1) $$\frac{\partial \bar{g}_{\mu,\nu}}{\partial \bar{X}_1^\sigma} - \frac{\partial \bar{g}_{\sigma,\nu}}{\partial \bar{X}_1^\mu} = \left(\frac{\partial g_{\alpha,\beta}}{\partial X_1^\tau} - \frac{\partial g_{\tau,\beta}}{\partial X_1^\alpha} \right) \frac{\partial X_1^\tau}{\partial \bar{X}_1^\sigma} \frac{\partial X_1^\alpha}{\partial \bar{X}_1^\mu} \frac{\partial X_2^\beta}{\partial \bar{X}_2^\nu} .$$

Definition.

$$C^{,\iota}_{\sigma,\nu}(M_1, M_2) \equiv g_*^{\iota,\mu}\left(\frac{\partial g_{\mu,\nu}}{\partial x_1^{\sigma}} - \frac{\partial g_{\sigma,\nu}}{\partial x_1^{\mu}}\right). \tag{7.2}$$

From (7.1) and (7.2) we obtain

$$\begin{aligned} \bar{C}^{,\iota}_{\sigma,\nu} &= g_*^{\epsilon,\phi}\frac{\partial \bar{x}_2^{\iota}}{\partial x_2^{\epsilon}}\frac{\partial \bar{x}_1^{\mu}}{\partial x_1^{\phi}}\left(\frac{\partial g_{\alpha,\beta}}{\partial x_1^{\tau}} - \frac{\partial g_{\tau,\beta}}{\partial x_1^{\alpha}}\right)\frac{\partial x_1^{\tau}}{\partial \bar{x}_1^{\sigma}}\frac{\partial x_1^{\alpha}}{\partial \bar{x}_1^{\mu}}\frac{\partial x_2^{\beta}}{\partial \bar{x}_2^{\nu}} \\ &= C^{,\epsilon}_{\tau,\beta}\frac{\partial x_1^{\tau}}{\partial \bar{x}_1^{\sigma}}\frac{\partial x_2^{\beta}}{\partial \bar{x}_2^{\nu}}\frac{\partial \bar{x}_2^{\iota}}{\partial x_2^{\epsilon}}. \end{aligned} \tag{7.3}$$

Similarly, we obtain

$$\bar{C}^{\iota,}_{\sigma,\alpha} = C^{\epsilon,}_{\mu,\tau}\frac{\partial \bar{x}_1^{\iota}}{\partial x_1^{\epsilon}}\frac{\partial x_1^{\mu}}{\partial \bar{x}_1^{\sigma}}\frac{\partial x_2^{\tau}}{\partial \bar{x}_2^{\alpha}}, \tag{7.4}$$

where

$$C^{\iota,}_{\sigma,\alpha}(M_1, M_2) = g^{\iota,\beta}\left(\frac{\partial g_{\sigma,\beta}}{\partial x_2^{\alpha}} - \frac{\partial g_{\sigma,\alpha}}{\partial x_2^{\beta}}\right). \tag{7.5}$$

Theorem 7. A necessary and sufficient condition that $\Gamma^{\iota,}_{\alpha\beta,}$ be symmetric in its lower indices is that $C^{,\iota}_{\sigma,\nu}$ vanish.

The proof follows from definition (7.2) and Theorem 3. A similar statement holds for $C^{\iota,}_{\sigma,\nu}$ in connection with the symmetry of $\Gamma^{,\iota}_{,\alpha\beta}$.

The C's as defined by (7.2) and (7.5) are two-point tensors. The transformation laws for the Γ's and C's could have been obtained from the invariant character of the differential equations of the dyodesics.

VIII. The dyodesics, continued.

Making use of (6.1), (6.15), (7.2), (7.5), the equations of the dyo-paths (5.12) take the forms

$$\begin{aligned} \ddot{x}_1^{\iota} + \Gamma^{\iota,}_{\alpha\beta,}\dot{x}_1^{\alpha}\dot{x}_1^{\beta} + C^{\iota,}_{\alpha,\beta}\dot{x}_1^{\alpha}\dot{x}_2^{\beta} &= 0 \\ \ddot{x}_2^{\iota} + \Gamma^{,\iota}_{,\alpha\beta}\dot{x}_2^{\alpha}\dot{x}_2^{\beta} + C^{,\iota}_{\alpha,\beta}\dot{x}_1^{\alpha}\dot{x}_2^{\beta} &= 0 \end{aligned}. \tag{8.1}$$

We notice in (8.1) that only the symmetric parts of $\Gamma^{i_1}_{\alpha\beta_1}$, $\Gamma^{\nu i}_{\nu\alpha\beta}$ contribute to the double sums, so that there is no loss in generality in replacing $\Gamma^{i_1}_{\alpha\beta_1}$ by $\frac{1}{2}(\Gamma^{i_1}_{\alpha\beta_1}+\Gamma^{i_1}_{\beta\alpha_1})$, etc.

<u>Theorem 8.</u> A first integral of the system (8.1) is

$$(8.2)\qquad g_{\alpha,\beta}\frac{dx_1^\alpha}{ds}\frac{dx_2^\beta}{ds} = \text{constant}.$$

<u>Proof.</u> We have

$$(8.3)\qquad \frac{d}{ds}\left(g_{\alpha,\beta}\frac{dx_1^\alpha}{ds}\frac{dx_2^\beta}{ds}\right) = g_{\alpha,\beta}\frac{d^2x_1^\alpha}{ds^2}\frac{dx_2^\beta}{ds} + g_{\alpha,\beta}\frac{dx_1^\alpha}{ds}\frac{d^2x_2^\beta}{ds^2} + \frac{\partial g_{\alpha,\beta}}{\partial x_1^\sigma}\frac{dx_1^\alpha}{ds}\frac{dx_1^\sigma}{ds}\frac{dx_2^\beta}{ds} + \frac{\partial g_{\alpha,\beta}}{\partial x_2^\sigma}\frac{dx_1^\alpha}{ds}\frac{dx_2^\beta}{ds}\frac{dx_2^\sigma}{ds}.$$

Replacing $\frac{d^2x_1^i}{ds^2}$, $\frac{d^2x_2^i}{ds^2}$ by their values from (8.1) we obtain

$$(8.4)\qquad \frac{d}{ds}\left(g_{\alpha,\beta}\frac{dx_1^\alpha}{ds}\frac{dx_2^\beta}{ds}\right) = g_{\alpha,\beta}\dot{x}_2^\beta\left[-\Gamma^{\alpha_1}_{\sigma\tau_1}\dot{x}_1^\sigma\dot{x}_1^\tau - C^{\alpha_1}_{\sigma,\tau}\dot{x}_1^\sigma\dot{x}_2^\tau\right] + g_{\alpha,\beta}\dot{x}_1^\alpha\left[-\Gamma^{\nu\beta}_{\nu\sigma\tau}\dot{x}_2^\sigma\dot{x}_2^\tau - C^{\nu\beta}_{\sigma,\tau}\dot{x}_1^\sigma\dot{x}_2^\tau\right] + \frac{\partial g_{\alpha,\beta}}{\partial x_1^\sigma}\dot{x}_1^\alpha\dot{x}_1^\sigma\dot{x}_2^\beta + \frac{\partial g_{\alpha,\beta}}{\partial x_2^\sigma}\dot{x}_1^\alpha\dot{x}_2^\beta\dot{x}_2^\sigma.$$

The right-hand side of (8.4) reduces to

$$(8.5)\qquad \left[\frac{\partial g_{\sigma,\beta}}{\partial x_1^\tau} - g_{\alpha,\beta}\Gamma^{\alpha_1}_{\sigma\tau_1} - g_{\gamma,\delta}C^{\nu\delta}_{\sigma,\beta}\right]\dot{x}_1^\sigma\dot{x}_1^\tau\dot{x}_2^\beta + \left[\frac{\partial g_{\alpha,\sigma}}{\partial x_2^\tau} - g_{\alpha,\beta}\Gamma^{\nu\beta}_{\nu\sigma\tau} - g_{\beta,\sigma}C^{\beta_1}_{\alpha,\tau}\right]\dot{x}_1^\alpha\dot{x}_2^\sigma\dot{x}_2^\tau$$

$$\equiv -\left(\frac{\partial g_{\alpha,\sigma}}{\partial x_2^\tau} - \frac{\partial g_{\alpha,\tau}}{\partial x_2^\sigma}\right)\dot{x}_1^\alpha\dot{x}_2^\sigma\dot{x}_2^\tau - \left(\frac{\partial g_{\sigma,\beta}}{\partial x_1^\tau} - \frac{\partial g_{\tau,\beta}}{\partial x_1^\sigma}\right)\dot{x}_1^\sigma\dot{x}_1^\tau\dot{x}_2^\beta,$$

by making use of (4.13), (4.23), (6.1), (6.15), (7.2), (7.5).

The right-hand side of (8.5) vanishes because of the skew-character of the coefficients of $\dot{X}_2^\sigma \dot{X}_2^\tau$ and $\dot{X}_1^\sigma \dot{X}_1^\tau$. Q.E.D.

The equations of the dyodesics take a more simple form when the Christoffel symbols (linear connections) are symmetric, for from Theorem 7 we have $C^{i_1}_{\alpha,\beta} = C^{2i}_{\alpha,\beta} = 0$ so that (8.1) becomes

(8.6)
$$\begin{aligned} \ddot{X}_1^i + \Gamma^{i_1}_{\alpha\beta_1} \dot{X}_1^\alpha \dot{X}_1^\beta &= 0 \\ \ddot{X}_2^i + \Gamma^{2i}_{2\alpha\beta} \dot{X}_2^\alpha \dot{X}_2^\beta &= 0 \end{aligned}$$

These differential equations are not independent since the Christoffel symbols are functions of X_1, X_2.

The particular form (8.1) for the differential equations of the dyodesics depends on the parameter s. If we make an arbitrary analytic substitution, $s \leftrightarrow t$, we have

(8.6')
$$\frac{dX_{1,2}^\alpha}{ds} = \frac{dX_{1,2}^\alpha}{dt}\frac{dt}{ds} \;;\; \frac{d^2X_{1,2}^\alpha}{ds^2} = \frac{d^2X_{1,2}^\alpha}{dt^2}\left(\frac{dt}{ds}\right)^2 + \frac{dX_{1,2}}{ds}\frac{d^2t}{ds^2}.$$

Substituting (8.6') into (8.1) we obtain

(8.7)
$$\begin{aligned} &\frac{d^2X_1^i}{dt^2}\left(\frac{dt}{ds}\right)^2 + \frac{dX_1^i}{dt}\frac{d^2t}{ds^2} + \Gamma^{i_1}_{\alpha\beta_1}\frac{dX_1^\alpha}{dt}\frac{dX_1^\beta}{dt}\left(\frac{dt}{ds}\right)^2 + C^{i_1}_{\alpha,\beta}\frac{dX_1^\alpha}{dt}\frac{dX_2^\beta}{dt}\left(\frac{dt}{ds}\right)^2 = 0 \\ &\frac{d^2X_2^i}{dt^2}\left(\frac{dt}{ds}\right)^2 + \frac{dX_2^i}{dt}\frac{d^2t}{ds^2} + \Gamma^{2i}_{2\alpha\beta}\frac{dX_2^\alpha}{dt}\frac{dX_2^\beta}{dt}\left(\frac{dt}{ds}\right)^2 + C^{2i}_{\alpha,\beta}\frac{dX_1^\alpha}{dt}\frac{dX_2^\beta}{dt}\left(\frac{dt}{ds}\right)^2 = 0, \end{aligned}$$

or

(8.8)
$$\frac{\dfrac{d^2X_1^i}{dt^2} + \Gamma^{i_1}_{\alpha\beta_1}\dfrac{dX_1^\alpha}{dt}\dfrac{dX_1^\beta}{dt} + C^{i_1}_{\alpha,\beta}\dfrac{dX_1^\alpha}{dt}\dfrac{dX_2^\beta}{dt}}{\dfrac{dX_1^i}{dt}} = \frac{-\dfrac{d^2t}{ds^2}}{\left(\dfrac{dt}{ds}\right)^2}$$

$$\frac{\dfrac{d^2X_2^i}{dt^2} + \Gamma^{2i}_{2\alpha\beta}\dfrac{dX_2^\alpha}{dt}\dfrac{dX_2^\beta}{dt} + C^{2i}_{\alpha,\beta}\dfrac{dX_1^\alpha}{dt}\dfrac{dX_2^\beta}{dt}}{\dfrac{dX_2^i}{dt}} = \frac{-\dfrac{d^2t}{ds^2}}{\left(\dfrac{dt}{ds}\right)^2}.$$

The right-hand sides of (8.8) are independent of **i**. These are the equations independent of the parameter. If the left-hand sides of (8.8) are set equal to $\Phi(t)$, then the analytic substitution $t \leftrightarrow s$ yields

(8.9)
$$\frac{\frac{d^2 X_{1,2}^{i}}{ds^2} + \cdots}{\frac{dX_{1,2}^{i}}{ds}} = \frac{\Phi(t)}{\frac{ds}{dt}} - \frac{\frac{d^2 s}{dt^2}}{\left(\frac{ds}{dt}\right)^2},$$

so that setting

(8.10)
$$\frac{\Phi(t)}{\frac{ds}{dt}} - \frac{\frac{d^2 s}{dt^2}}{\left(\frac{ds}{dt}\right)^2} = 0$$

and integrating, we obtain

(8.11)
$$\ln \frac{ds}{dt} = \int^t \Phi(u)\,du + \ln B$$

or

(8.12)
$$s = A + B \int^t e^{\int^v \Phi(u)\,du}\,dv,$$

which reduces (8.9) to (8.1).

Whenever the parameter s is used in connection with the minimal dyodesics we shall assume that s is the parameter identified with equations (8.1).

We next ask if there is a coordinate system for which the equations of the dyo-paths are linear in the parameter s, i.e., for which $\frac{d^2 X_{1,2}^{i}}{ds^2} = 0$ or $X_{1,2}^{i} = a_{1,2}^{i} s + b_{1,2}^{i}$? If such a coordinate system exists, we must have

(8.13)
$$\begin{aligned} \Gamma^{i_1}_{\alpha\beta_1} \dot{X}_1^{\alpha} \dot{X}_1^{\beta} + C^{i_1}_{\alpha,\beta} \dot{X}_1^{\alpha} \dot{X}_2^{\beta} &= 0 \\ \Gamma^{2i}_{2\alpha\beta} \dot{X}_2^{\alpha} \dot{X}_2^{\beta} + C^{2i}_{\alpha,\beta} \dot{X}_1^{\alpha} \dot{X}_2^{\beta} &= 0 \end{aligned}$$

for arbitrary $\dot{X}_1^{\alpha}$, $\dot{X}_2^{\alpha}$. Because of the independence of $\dot{X}_1^{\alpha}$, $\dot{X}_2^{\alpha}$ we immediately conclude that of necessity

(8.14)
$$\Gamma^{i_1}_{\alpha\beta_1} + \Gamma^{i_1}_{\beta\alpha_1} = \Gamma^{2i}_{2\alpha\beta} + \Gamma^{2i}_{2\beta\alpha} = C^{i_1}_{\alpha,\beta} = C^{2i}_{\alpha,\beta} = 0.$$

From Theorem 7 the vanishing of the C^{ν}_{λ} implies the symmetry of the Γ^{ν}_{λ} in their lower indices, and conversely, so that from (8.14) we have

(8.15) $$\Gamma^{\iota_1}_{\alpha\beta_1} = \Gamma^{\jmath\iota}_{\jmath\alpha\beta} = 0$$

as necessary conditions for the dyo-paths to be linear in the parameter s. From (6.1), (6.15) we see that (8.15) implies $g_{\alpha_1\beta} \equiv$ constants.

Conversely, if a coordinate system exists for which the linear connections vanish, then the Γ^{ν}_{λ} are symmetric in their lower indices in all other coordinate systems (see note after Theorem 6), so that from Theorem 7 we have $C^{\iota_1}_{\alpha_1\beta} = C^{\jmath\iota}_{\alpha_1\beta} = 0$. We have now proved

<u>Theorem 9</u>. A necessary and sufficient condition that the dyodesics be linear in the parameter s in some coordinate system X_1^{ι}, X_2^{ι} is that the linear connections $\Gamma^{\iota_1}_{\jmath\kappa_1}$, $\Gamma^{\jmath\iota}_{\jmath\jmath\kappa}$ vanish for this coordinate system.

Let us now return to the law of transformation for the linear connections. We have

(8.16) $$\overline{\Gamma}^{\lambda_1}_{\mu\nu_1}\frac{\partial X_1^{\alpha}}{\partial \overline{X}_1^{\lambda}} = \frac{\partial^2 X_1^{\alpha}}{\partial \overline{X}_1^{\mu}\partial \overline{X}_1^{\nu}} + \Gamma^{\alpha_1}_{\beta\gamma_1}\frac{\partial X_1^{\beta}}{\partial \overline{X}_1^{\mu}}\frac{\partial X_1^{\gamma}}{\partial \overline{X}_1^{\nu}}.$$

We now differentiate these equations with respect to $\overline{X}_1^{\sigma}$, and obtain

(8.17) $$\frac{\partial \overline{\Gamma}^{\lambda_1}_{\mu\nu_1}}{\partial \overline{X}_1^{\sigma}}\frac{\partial X_1^{\alpha}}{\partial \overline{X}_1^{\lambda}} + \overline{\Gamma}^{\lambda_1}_{\mu\nu_1}\frac{\partial^2 X_1^{\alpha}}{\partial \overline{X}_1^{\sigma}\partial \overline{X}_1^{\lambda}} = \frac{\partial^3 X_1^{\alpha}}{\partial \overline{X}_1^{\sigma}\partial \overline{X}_1^{\mu}\partial \overline{X}_1^{\nu}} + \frac{\partial \Gamma^{\alpha_1}_{\beta\gamma_1}}{\partial X_1^{\tau}}\frac{\partial X_1^{\tau}}{\partial \overline{X}_1^{\sigma}}\frac{\partial X_1^{\beta}}{\partial \overline{X}_1^{\mu}}\frac{\partial X_1^{\gamma}}{\partial \overline{X}_1^{\nu}} + \Gamma^{\alpha_1}_{\beta\gamma_1}\frac{\partial X_1^{\beta}}{\partial \overline{X}_1^{\mu}}\frac{\partial^2 X_1^{\gamma}}{\partial \overline{X}_1^{\sigma}\partial \overline{X}_1^{\nu}} + \Gamma^{\alpha_1}_{\beta\gamma_1}\frac{\partial X_1^{\gamma}}{\partial \overline{X}_1^{\nu}}\frac{\partial^2 X_1^{\beta}}{\partial \overline{X}_1^{\sigma}\partial \overline{X}_1^{\mu}}.$$

Now we interchange ν and σ and subtract, so that

$$(8.18)\quad \left(\frac{\partial \bar{\Gamma}^{\lambda_1}_{\mu\nu_1}}{\partial \bar{x}_1^{\sigma}} - \frac{\partial \bar{\Gamma}^{\lambda_1}_{\mu\sigma_1}}{\partial \bar{x}_1^{\nu}}\right)\frac{\partial x_1^{\alpha}}{\partial \bar{x}_1^{\lambda}} + \bar{\Gamma}^{\lambda_1}_{\mu\nu_1}\frac{\partial^2 x_1^{\alpha}}{\partial \bar{x}_1^{\sigma}\partial \bar{x}_1^{\lambda}} - \bar{\Gamma}^{\lambda_1}_{\mu\sigma_1}\frac{\partial^2 x_1^{\alpha}}{\partial \bar{x}_1^{\nu}\partial \bar{x}_1^{\lambda}}$$

$$= \left(\frac{\partial \Gamma^{\alpha_1}_{\beta\gamma_1}}{\partial x_1^{\tau}} - \frac{\partial \Gamma^{\alpha_1}_{\beta\tau_1}}{\partial x_1^{\gamma}}\right)\frac{\partial x_1^{\tau}}{\partial \bar{x}_1^{\sigma}}\frac{\partial x_1^{\beta}}{\partial \bar{x}_1^{\mu}}\frac{\partial x_1^{\gamma}}{\partial \bar{x}_1^{\nu}}$$

$$+ \Gamma^{\alpha_1}_{\beta\gamma_1}\frac{\partial x_1^{\gamma}}{\partial \bar{x}_1^{\nu}}\frac{\partial^2 x_1^{\beta}}{\partial \bar{x}_1^{\sigma}\partial \bar{x}_1^{\mu}} - \Gamma^{\alpha_1}_{\beta\gamma_1}\frac{\partial x_1^{\gamma}}{\partial \bar{x}_1^{\sigma}}\frac{\partial^2 x_1^{\beta}}{\partial \bar{x}_1^{\nu}\partial \bar{x}_1^{\mu}} .$$

We now eliminate the second derivatives by means of (8.16) and thus obtain

$$(8.19)\quad \bar{R}^{\lambda_1}_{\mu\nu\sigma_1}\frac{\partial x_1^{\alpha}}{\partial \bar{x}_1^{\lambda}} = R^{\alpha_1}_{\beta\gamma\tau_1}\frac{\partial x_1^{\beta}}{\partial \bar{x}_1^{\mu}}\frac{\partial x_1^{\gamma}}{\partial \bar{x}_1^{\nu}}\frac{\partial x_1^{\tau}}{\partial \bar{x}_1^{\sigma}}$$

where

$$(8.20)\quad R^{\alpha_1}_{\beta\gamma\tau_1} = \frac{\partial \Gamma^{\alpha_1}_{\beta\gamma_1}}{\partial x_1^{\tau}} - \frac{\partial \Gamma^{\alpha_1}_{\beta\tau_1}}{\partial x_1^{\gamma}} + \Gamma^{\alpha_1}_{\sigma\tau_1}\Gamma^{\sigma_1}_{\beta\gamma_1} - \Gamma^{\alpha_1}_{\sigma\gamma_1}\Gamma^{\sigma_1}_{\beta\tau_1} .$$

A similar result holds for $R^{,\alpha}_{,\beta\gamma\tau}$ in connection with $\Gamma^{,\alpha}_{,\beta\gamma}$.

The vanishing of $\Gamma^{\alpha_1}_{\beta\gamma_1}$ in some coordinate system implies that the tensor $R^{\alpha_1}_{\beta\gamma\tau_1}$ be a zero tensor. Thus

<u>Theorem 10</u>. A necessary condition that their exist some coordinate system for which the dyodesics are linear in the parameter s is that

$$(8.21)\quad R^{\alpha_1}_{\beta\gamma\tau_1} = R^{,\alpha}_{,\beta\gamma\tau} = 0 .$$

IX. <u>The Projective geometry of paths</u>.

We now inquire under what circumstances a set of differential equations

$$(9.1)\quad \frac{d^2 x_1^{i}}{ds^2} + \Delta^{i_1}_{\alpha\beta_1}\frac{dx_1^{\alpha}}{ds}\frac{dx_1^{\beta}}{ds} + D^{i_1}_{\alpha,\beta}\frac{dx_1^{\alpha}}{ds}\frac{dx_2^{\beta}}{ds} = 0$$

$$\frac{d^2 x_2^{i}}{ds^2} + \Delta^{,i}_{,\alpha\beta}\frac{dx_2^{\alpha}}{ds}\frac{dx_2^{\beta}}{ds} + D^{,i}_{\alpha,\beta}\frac{dx_1^{\alpha}}{ds}\frac{dx_2^{\beta}}{ds} = 0$$

can represent the same dyo-paths as (8.1)? If the pair of curves $x_1^i = \psi_1^i(t)$, $x_2^i = \psi_2^i(t)$ are dyo-paths for (8.1) and (9.1), then they satisfy equations of the type (8.8) and similar equations obtained from (9.1). Eliminating second derivatives we obtain

$$\frac{\left(\Gamma^{i}_{\alpha\beta,} - \Delta^{i}_{\alpha\beta,}\right)\frac{dx_1^\alpha}{dt}\frac{dx_1^\beta}{dt} + \left(C^{i}_{\alpha,\beta} - D^{i}_{\alpha,\beta}\right)\frac{dx_1^\alpha}{dt}\frac{dx_2^\beta}{dt}}{\frac{dx_1^i}{dt}} = \text{same equation with } i \text{ replaced by } j.$$

(9.2)

$$\frac{\left(\Gamma^{,i}_{,\alpha\beta} - \Delta^{,i}_{,\alpha\beta}\right)\frac{dx_2^\alpha}{dt}\frac{dx_2^\beta}{dt} + \left(C^{,i}_{\alpha,\beta} - D^{,i}_{\alpha,\beta}\right)\frac{dx_1^\alpha}{dt}\frac{dx_2^\beta}{dt}}{\frac{dx_2^i}{dt}} = \quad ''$$

Now define

(9.3)
$$\Gamma^{i}_{\alpha\beta,} - \Delta^{i}_{\alpha\beta,} = \varphi^{i}_{\alpha\beta,} \qquad \phi^{\epsilon}_{\beta\epsilon,} = (n+1)\,\phi_{\beta,}$$
$$C^{i}_{\alpha,\beta} - D^{i}_{\alpha,\beta} = \varphi^{i}_{\alpha,\beta} \qquad \phi^{\epsilon}_{\epsilon,\beta} = n\,\psi_{,\beta}$$
$$\Gamma^{,i}_{,\alpha\beta} - \Delta^{,i}_{,\alpha\beta} = \phi^{,i}_{,\alpha\beta} \qquad \phi^{,\epsilon}_{,\beta\epsilon} = (n+1)\,\phi_{,\beta}$$
$$C^{,i}_{\alpha,\beta} - D^{,i}_{\alpha,\beta} = \phi^{,i}_{\alpha,\beta} \qquad \phi^{,\epsilon}_{\alpha,\epsilon} = n\,\psi_{\alpha,} \quad .$$

Equations (9.2) now become

(9.4)
$$\varphi^{i}_{\alpha\beta,}\frac{dx_1^\alpha}{dt}\frac{dx_1^\beta}{dt}\frac{dx_1^j}{dt} + \varphi^{i}_{\alpha,\beta}\frac{dx_1^\alpha}{dt}\frac{dx_2^\beta}{dt}\frac{dx_1^j}{dt} = i,j \text{ interchanged.}$$
$$\varphi^{,i}_{,\alpha\beta}\frac{dx_2^\alpha}{dt}\frac{dx_2^\beta}{dt}\frac{dx_2^j}{dt} + \varphi^{,i}_{\alpha,\beta}\frac{dx_1^\alpha}{dt}\frac{dx_2^\beta}{dt}\frac{dx_2^j}{dt} = \quad '' \qquad ''$$

or,

(9.5)
$$\left(\varphi^{i}_{\alpha\beta,}\delta^{j}_{\epsilon,} - \varphi^{j}_{\alpha\beta,}\delta^{i}_{\epsilon,}\right)\frac{dx_1^\alpha}{dt}\frac{dx_1^\beta}{dt}\frac{dx_1^\epsilon}{dt} + \left(\varphi^{i}_{\alpha,\beta}\delta^{j}_{\epsilon,} - \varphi^{j}_{\alpha,\beta}\delta^{i}_{\epsilon,}\right)\frac{dx_1^\alpha}{dt}\frac{dx_2^\beta}{dt}\frac{dx_1^\epsilon}{dt} = 0$$
$$\left(\varphi^{,i}_{,\alpha\beta}\delta^{,j}_{,\epsilon} - \varphi^{,j}_{,\alpha\beta}\delta^{,i}_{,\epsilon}\right)\frac{dx_2^\alpha}{dt}\frac{dx_2^\beta}{dt}\frac{dx_2^\epsilon}{dt} + \left(\varphi^{,i}_{\alpha,\beta}\delta^{,j}_{,\epsilon} - \varphi^{,j}_{\alpha,\beta}\delta^{,i}_{,\epsilon}\right)\frac{dx_1^\alpha}{dt}\frac{dx_2^\beta}{dt}\frac{dx_2^\epsilon}{dt} = 0.$$

Equations (9.5) are to hold for arbitrary and independent $\frac{dx_1^{\alpha}}{dt}$, $\frac{dx_2^{\alpha}}{dt}$. This implies

$$
(9.6)\qquad
\begin{aligned}
&\left(\varphi^{i_1}_{\alpha,\beta}\,\delta^{j_1}_{\epsilon_1} - \varphi^{j_1}_{\alpha,\beta}\,\delta^{i_1}_{\epsilon_1}\right) + \left(\varphi^{i_1}_{\alpha,\beta}\,\delta^{j_1}_{\epsilon_1} - \varphi^{j_1}_{\epsilon,\beta}\,\delta^{i_1}_{\alpha_1}\right) = 0\\
&\left(\varphi^{,i}_{\alpha,\beta}\,\delta^{,j}_{,\epsilon} - \varphi^{,j}_{\alpha,\beta}\,\delta^{,i}_{,\epsilon}\right) + \left(\varphi^{,i}_{\alpha,\epsilon}\,\delta^{,j}_{,\beta} - \varphi^{,j}_{\alpha,\epsilon}\,\delta^{,i}_{,\beta}\right) = 0\\
&\left(\varphi^{i_1}_{\alpha\beta_1}\,\delta^{j_1}_{\epsilon_1} - \varphi^{j_1}_{\alpha\beta_1}\,\delta^{i_1}_{\epsilon_1}\right) + \left(\varphi^{i_1}_{\alpha\epsilon_1}\,\delta^{j_1}_{\beta_1} - \varphi^{j_1}_{\alpha\epsilon_1}\,\delta^{i_1}_{\beta_1}\right) + \left(\varphi^{i_1}_{\beta\epsilon_1}\,\delta^{j_1}_{\alpha_1} - \varphi^{j_1}_{\beta\epsilon_1}\,\delta^{i_1}_{\alpha_1}\right) = 0\\
&\left(\varphi^{,i}_{,\alpha\beta}\,\delta^{,j}_{,\epsilon} - \varphi^{,j}_{,\alpha\beta}\,\delta^{,i}_{,\epsilon}\right) + \left(\varphi^{,i}_{,\alpha\epsilon}\,\delta^{,j}_{,\beta} - \varphi^{,j}_{,\alpha\epsilon}\,\delta^{,i}_{,\beta}\right) + \left(\varphi^{,i}_{,\beta\epsilon}\,\delta^{,j}_{,\alpha} - \varphi^{,j}_{,\beta\epsilon}\,\delta^{,i}_{,\alpha}\right) = 0 .
\end{aligned}
$$

We have assumed $\Gamma^{i_1}_{\alpha\beta_1}$, $\Lambda^{i_1}_{\alpha\beta_1}$, $\Gamma^{,i}_{,\alpha\beta}$, $\Lambda^{,i}_{,\alpha\beta}$, symmetric since only their symmetric components contribute. Setting $j=\epsilon$ and summing we obtain

$$
(9.7)\qquad
\begin{aligned}
\Gamma^{i_1}_{\alpha\beta_1} - \Lambda^{i_1}_{\alpha\beta_1} &= \varphi_{\beta_1}\,\delta^{i_1}_{\alpha_1} + \varphi_{\alpha_1}\,\delta^{i_1}_{\beta_1}\\
\Gamma^{,i}_{,\alpha\beta} - \Lambda^{,i}_{,\alpha\beta} &= \varphi_{,\beta}\,\delta^{,i}_{,\alpha} + \varphi_{,\alpha}\,\delta^{,i}_{,\beta}\\
C^{i_1}_{\alpha,\beta} - D^{i_1}_{\alpha,\beta} &= \psi_{,\beta}\,\delta^{i_1}_{\alpha_1}\\
C^{,i}_{\alpha,\beta} - D^{,i}_{\alpha,\beta} &= \psi_{\alpha_1}\,\delta^{,i}_{,\beta} .
\end{aligned}
$$

Hence if equations (8.1) and (9.1) are to represent the same set of dyo-paths, equations (9.7) must hold. Conversely, let $[\varphi_{\beta_1}, \varphi_{,\beta}, \psi_{\beta_1}, \psi_{,\beta}]$ represent any arbitrary two-point functions satisfying (9.7). From (9.7) we have

$$
(9.8)\qquad
\begin{aligned}
&\frac{\left(\Gamma^{i_1}_{\alpha\beta_1} - \Lambda^{i_1}_{\alpha\beta_1}\right)\frac{dx_1^{\alpha}}{dt}\frac{dx_1^{\beta}}{dt} + \left(C^{i_1}_{\alpha,\beta} - D^{i_1}_{\alpha,\beta}\right)\frac{dx_1^{\alpha}}{dt}\frac{dx_2^{\beta}}{dt}}{\frac{dx_1^{i}}{dt}} \equiv 2\varphi_{\beta_1}\frac{dx_1^{\beta}}{dt} + \psi_{,\beta}\frac{dx_2^{\beta}}{dt}\\[2ex]
&\frac{\left(\Gamma^{,i}_{,\alpha\beta} - \Lambda^{,i}_{,\alpha\beta}\right)\frac{dx_2^{\alpha}}{dt}\frac{dx_2^{\beta}}{dt} + \left(C^{,i}_{\alpha,\beta} - D^{,i}_{\alpha,\beta}\right)\frac{dx_1^{\alpha}}{dt}\frac{dx_2^{\beta}}{dt}}{\frac{dx_2^{i}}{dt}} \equiv 2\varphi_{,\beta}\frac{dx_2^{\beta}}{dt} + \psi_{\beta_1}\frac{dx_1^{\beta}}{dt} ,
\end{aligned}
$$

for any pair of dyo-paths $x_1^{i}(t)$, $x_2^{i}(t)$ satisfying (8.8), so that equations (9.2) are satisfied. Subtracting equations (9.2) from (8.8) we obtain similar equations in the Λ's and D's. We thus obtain that every set of dyo-paths with respect to the Γ's, C's, is a set of dyo-paths with respect to the Λ's, D's, and conversely. Hence

<u>Theorem 11</u>. A necessary and sufficient condition that (8.1) and (9.1) shall represent the same system of dyo-paths is that a set of two-point functions $[\varphi_{\alpha,}, \varphi_{,\alpha}, \psi_{\alpha,}, \psi_{,\alpha}]$ shall exist satisfying (9.7).

If the Λ's, D's are related to the Γ's, C's by equations of the type (9.7) we say that they are obtainable one from the other by a projective change of linear and tensor connection.

Setting $i = \alpha$ in (9.7) and summing, we obtain

(9.9)
$$\begin{aligned}
\Gamma^{i,}_{i\beta,} - \Lambda^{i,}_{i\beta,} &= (n+1)\varphi_{\beta,} \\
\Gamma^{,i}_{,i\beta} - \Lambda^{,i}_{,i\beta} &= (n+1)\varphi_{,\beta} \\
C^{i,}_{i,\beta} - D^{i,}_{i,\beta} &= n\,\psi_{,\beta} \\
C^{,i}_{\alpha,i} - D^{,i}_{\alpha,i} &= n\,\psi_{\alpha,}
\end{aligned}$$

and eliminating $\varphi_{\beta,}, \ldots, \ldots$, etc., in (9.7), we obtain

(9.10)
$$\begin{aligned}
\Gamma^{i,}_{\alpha\beta,} - \Lambda^{i,}_{\alpha\beta,} &= \frac{\delta^{i,}_{\alpha,}}{n+1}\left(\Gamma^{\mu,}_{\mu\beta,} - \Lambda^{\mu,}_{\mu\beta,}\right) + \frac{\delta^{i,}_{\beta,}}{n+1}\left(\Gamma^{\mu,}_{\mu\alpha,} - \Lambda^{\mu,}_{\mu\alpha,}\right) \\
\Gamma^{,i}_{,\alpha\beta} - \Lambda^{,i}_{,\alpha\beta} &= \frac{\delta^{,i}_{,\alpha}}{n+1}\left(\Gamma^{,\mu}_{,\mu\beta} - \Lambda^{,\mu}_{,\mu\beta}\right) + \frac{\delta^{,i}_{,\beta}}{n+1}\left(\Gamma^{,\mu}_{,\mu\alpha} - \Lambda^{,\mu}_{,\mu\alpha}\right) \\
C^{i,}_{\alpha,\beta} - D^{i,}_{\alpha,\beta} &= \frac{\delta^{i,}_{\alpha,}}{n}\left(C^{\mu,}_{\mu,\beta} - D^{\mu,}_{\mu,\beta}\right) \\
C^{,i}_{\alpha,\beta} - D^{,i}_{\alpha,\beta} &= \frac{\delta^{,i}_{,\beta}}{n}\left(C^{,\mu}_{\alpha,\mu} - D^{,\mu}_{\alpha,\mu}\right),
\end{aligned}$$

so that the quantities

(911)
$$\begin{aligned}
\Pi^{i,}_{\alpha\beta,} &\equiv \Gamma^{i,}_{\alpha\beta,} - \frac{\delta^{i,}_{\alpha,}}{n+1}\Gamma^{\mu,}_{\mu\beta,} - \frac{\delta^{i,}_{\beta,}}{n+1}\Gamma^{\mu,}_{\mu\alpha,} \\
\Pi^{,i}_{,\alpha\beta} &\equiv \Gamma^{,i}_{,\alpha\beta} - \frac{\delta^{,i}_{,\alpha}}{n+1}\Gamma^{,\mu}_{,\mu\beta} - \frac{\delta^{,i}_{,\beta}}{n+1}\Gamma^{,\mu}_{,\mu\alpha} \\
\text{L}^{i,}_{\alpha,\beta} &\equiv C^{i,}_{\alpha,\beta} - \frac{\delta^{i,}_{\alpha,}}{n} C^{\mu,}_{\mu,\beta} \\
\text{L}^{,i}_{\alpha,\beta} &\equiv C^{,i}_{\alpha,\beta} - \frac{\delta^{,i}_{,\beta}}{n} C^{,\mu}_{\alpha,\mu}
\end{aligned}$$

are independent of projective changes of the affine and tensor components in the $\Gamma^{)}_{s}$ and $C^{)}_{s}$. From (9.11) we see that the $\Pi^{)}_{s}$, $\text{L}^{)}_{s}$ are projectively obtainable from the $\Gamma^{)}_{s}$, $C^{)}_{s}$.

We may now write the equations of the dyo-paths independent of projective changes as

(9.12)
$$\begin{aligned}
&\frac{d^2 x_1^i}{dp^2} + \Pi^{i,}_{\alpha\beta,}\frac{dx_1^\alpha}{dp}\frac{dx_1^\beta}{dp} + \text{L}^{i,}_{\alpha,\beta}\frac{dx_1^\alpha}{dp}\frac{dx_2^\beta}{dp} = 0 \\
&\frac{d^2 x_2^i}{dp^2} + \Pi^{,i}_{,\alpha\beta}\frac{dx_2^\alpha}{dp}\frac{dx_2^\beta}{dp} + \text{L}^{,i}_{\alpha,\beta}\frac{dx_1^\alpha}{dp}\frac{dx_2^\beta}{dp} = 0 .
\end{aligned}$$

X. A possible application : Motion of two bodies.

In the general theory of relativity we find that the motion of particles are the geodesics and that in free space the Einstein law, $R_{ij} = 0$, enables us to compute the g_{ij}, $R_{ij} = 0$ being a system of second order differential equations in the g_{ij}. This law was highly successful in predicting the motion of an infinitesimal particle in the field of a stationary mass M, provided spherical symmetry was assumed, plus the assumption that at great distances from M the orbits should resemble straight line paths.

In the case of two masses we cannot add the fields due to both objects since the Ricci tensor R_{ij} is not linear in the g_{ij}. Moreover, if this were possible, the geodesics would yield the motion of a third infinitesimal particle under the field of essentially two fixed masses, whereas we are concerned with the motion of the two gravitating objects. At first glance, the dyodesics lead one to believe that here is a possible method for attacking this problem. We shall attempt to obtain a geometric invariant solution of the two-body problem in paragraph 18.

Let us assume a line element

(10.1) $$ds^2 = g_{\alpha,\beta}\, dx_1^\alpha\, dx_2^\beta ,$$

where we assume
$$g_{\alpha,\beta} \approx 1 \quad \text{if} \quad \alpha = \beta$$
$$g_{\alpha,\beta} = 0 \quad \text{if} \quad \alpha \neq \beta ,$$

and furthermore we assume that
$$ds^2 \approx g_{4,4}\, dx_1^4\, dx_2^4 , \quad \text{i.e.,} \quad \frac{dx_{1,2}^i}{ds} \approx 0 \quad \text{for } i \neq 4.$$

Let us choose

(10.2) $$g_{4,4} = 1 - \frac{GM^2 m}{(m+M)^2}\frac{1}{r_1} - \frac{Gm^2 M}{(m+M)^2}\frac{1}{r_2}$$

where $r_1 = [(x_1^1)^2 + (x_1^2)^2 + (x_1^3)^2]^{1/2}$, $r_2 = [(x_2^1)^2 + (x_2^2)^2 + (x_2^3)^2]^{1/2}$, r_1 being associated with m, r_2 with M. If we assume $Mr_2 \approx mr_1$, the dyodesics satisfy

(10.3) $$\frac{d^2 x_1^i}{ds^2} - \frac{\partial g_{4,4}}{\partial x_2^i}\, \dot{x}_1^4\, \dot{x}_2^4 \approx 0$$
$$\frac{d^2 x_2^i}{ds^2} - \frac{\partial g_{4,4}}{\partial x_1^i}\, \dot{x}_1^4\, \dot{x}_2^4 \approx 0 ,$$

and calling $x_1^4 = x_2^4 = t$, we have approximately

(10.4) $$\frac{d^2 \mathbf{r}_1}{dt^2} = \nabla_2\, g_{4,4} = \frac{-GM}{\left(1 + \frac{m}{M}\right)^2}\frac{\mathbf{r}_1}{r_1^3}$$

$$\frac{d^2 \mathbf{r}_2}{dt^2} = \nabla_1\, g_{4,4} = \frac{-Gm}{\left(1 + \frac{M}{m}\right)^2}\frac{\mathbf{r}_2}{r_2^3} ,$$

the Newtonian equations of motion for earth and sun about their center of mass.

<u>Example</u>. As an example of the theory that has been set up, let us attempt to find the dyodesics for the line element

$$ds^2 = -e^{\lambda} dr_1 dr_2 - r_1 r_2 \, d\theta_1 d\theta_2 - r_1 r_2 \sin\theta_1 \sin\theta_2 \, d\varphi_1 d\varphi_2 + e^{\mu} dt_1 dt_2 \tag{10.5}$$

where $e^{\mu} = \left(1 - \frac{M}{4r_1} - \frac{M}{4r_2}\right)$, $e^{\lambda} = e^{-\mu}$.

We have

$$(g_{\alpha,\beta}) = \begin{pmatrix} -e^{\lambda} & 0 & 0 & 0 \\ 0 & -r_1 r_2 & 0 & 0 \\ 0 & 0 & -r_1 r_2 \sin\theta_1 \sin\theta_2 & 0 \\ 0 & 0 & 0 & e^{\mu} \end{pmatrix} \tag{10.6}$$

$$\left(g_*^{\alpha,\beta}\right) = \left(g^{\alpha,\beta}\right) = \begin{pmatrix} -e^{-\lambda} & 0 & 0 & 0 \\ 0 & -(r_1 r_2)^{-1} & 0 & 0 \\ 0 & 0 & -(r_1 r_2 \sin\theta_1 \sin\theta_2)^{-1} & 0 \\ 0 & 0 & 0 & e^{-\mu} \end{pmatrix} \tag{10.7}$$

We now compute the $\Gamma^{i,}_{\alpha\beta,}$; $\Gamma^{,i}_{,\alpha\beta}$; $C^{i,}_{\alpha,\beta}$; $C^{,i}_{\alpha,\beta}$.

We have

$$\Gamma^{i,}_{\alpha\beta,} = g^{i,\sigma} \frac{\partial g_{\alpha,\sigma}}{\partial x_1^{\beta}} = g^{i,i} \frac{\partial g_{\alpha,i}}{\partial x_1^{\beta}} \tag{10.8}$$

, i not summed.

Hence if α, i, β, all different, $\Gamma^{i,}_{\alpha\beta,} = 0$, and if $\alpha = i$,

$$\Gamma^{i,}_{i\beta,} = g^{i,i} \frac{\partial g_{i,i}}{\partial x_1^{\beta}} .$$

Similarly,

$$\Gamma^{,i}_{,\alpha i} = g_*^{i,i} \frac{\partial g_{i,i}}{\partial x_2^{\alpha}} .$$

We obtain

$$\begin{aligned} \Gamma^{1,}_{11,} &= \frac{\partial\lambda}{\partial r_1} \\ \Gamma^{2,}_{21,} &= \frac{1}{r_1} \\ \Gamma^{3,}_{31,} &= \frac{1}{r_1} \\ \Gamma^{4,}_{41,} &= \frac{\partial\mu}{\partial r_1} \\ \Gamma^{3,}_{32,} &= \cot\theta_1 \end{aligned} \tag{10.9}$$

$$\begin{aligned} \Gamma^{,1}_{,11} &= \frac{\partial\lambda}{\partial r_2} \\ \Gamma^{,2}_{,12} &= \frac{1}{r_2} \\ \Gamma^{,3}_{,13} &= \frac{1}{r_2} \\ \Gamma^{,4}_{,14} &= \frac{\partial\mu}{\partial r_2} \\ \Gamma^{,3}_{,23} &= \cot\theta_2 . \end{aligned} \tag{10.10}$$

Also,

$$\begin{aligned} C^{i,}_{\sigma,\alpha} &= g^{i,\beta}\left(\frac{\partial g_{\sigma,\beta}}{\partial x_2^{\alpha}} - \frac{\partial g_{\sigma,\alpha}}{\partial x_2^{\beta}}\right) \\ &= g^{i,i}\left(\frac{\partial g_{\sigma,i}}{\partial x_2^{\alpha}} - \frac{\partial g_{\sigma,\alpha}}{\partial x_2^{i}}\right), \text{ i not summed.} \end{aligned} \tag{10.11}$$

This implies

$$\begin{aligned} C^{i,}_{j,j} &= -g^{i,i}\frac{\partial g_{j,j}}{\partial x_2^{i}}, \quad i \neq j \\ C^{i,}_{i,\alpha} &= g^{i,i}\frac{\partial g_{i,i}}{\partial x_2^{\alpha}}, \quad i \neq \alpha, \qquad C^{i,}_{i,i} = 0 . \end{aligned} \tag{10.12}$$

We find that

$$\begin{aligned} C^{1,}_{2,2} &= -e^{-\lambda} r_1 \\ C^{1,}_{3,3} &= -e^{-\lambda} r_1 \sin\theta_1 \sin\theta_2 \\ C^{1,}_{4,4} &= e^{\mu-\lambda}\frac{\partial\mu}{\partial r_2} \\ C^{2,}_{3,3} &= -\sin\theta_1 \cos\theta_2 \\ C^{2,}_{2,1} &= \frac{1}{r_2} \\ C^{3,}_{3,1} &= \frac{1}{r_2} \\ C^{3,}_{3,2} &= \cot\theta_2 \\ C^{4,}_{4,1} &= \frac{\partial\mu}{\partial r_2} \end{aligned} \tag{10.13}$$

$$\begin{aligned} C^{,1}_{2,2} &= -e^{-\lambda} r_2 \\ C^{,1}_{3,3} &= -e^{-\lambda} r_2 \sin\theta_1 \sin\theta_2 \\ C^{,1}_{4,4} &= e^{\mu-\lambda}\frac{\partial\mu}{\partial r_1} \\ C^{,2}_{3,3} &= -\cos\theta_1 \sin\theta_2 \\ C^{,2}_{1,2} &= \frac{1}{r_1} \\ C^{,3}_{1,3} &= \frac{1}{r_1} \\ C^{,3}_{2,3} &= \cot\theta_1 \\ C^{,4}_{1,4} &= \frac{\partial\mu}{\partial r_1} . \end{aligned} \tag{10.14}$$

Two of the equations (8.1) for the dyodesics become,

$$\begin{aligned} &\frac{d^2\theta_1}{ds^2} + \frac{1}{r_1}\dot\theta_1 \dot r_1 - \sin\theta_1\cos\theta_2\,\dot\varphi_1\dot\varphi_2 + \frac{1}{r_2}\dot\theta_1 \dot r_2 = 0 \\ &\frac{d^2\theta_2}{ds^2} + \frac{1}{r_2}\dot\theta_2 \dot r_2 - \cos\theta_1\sin\theta_2\,\dot\varphi_1\dot\varphi_2 + \frac{1}{r_1}\dot\theta_2 \dot r_1 = 0 . \end{aligned} \tag{10.15}$$

If initially, $\theta_1 = \theta_2 = \pi/2$, $\dot{\theta}_1 = \dot{\theta}_2 = 0$, then

(10.16) $$\theta_1 = \theta_2 \equiv \pi/2$$

satisfies (10.15) and the initial conditions. From our general uniqueness theorems, we concur that θ_1 and θ_2 remain identically $\frac{\pi}{2}$.

We also obtain, using $\theta_1 = \theta_2 \equiv \frac{\pi}{2}$,

(10.17) $$\frac{d^2\varphi_1}{ds^2} + \frac{1}{r_1}\dot{r}_1\dot{\varphi}_1 + \frac{1}{r_2}\dot{r}_2\dot{\varphi}_1 = 0$$
$$\frac{d^2\varphi_2}{ds^2} + \frac{1}{r_2}\dot{r}_2\dot{\varphi}_2 + \frac{1}{r_1}\dot{r}_1\dot{\varphi}_2 = 0,$$

which imply

(10.17') $$\frac{d^2\varphi_1}{ds^2} + \frac{d\varphi_1}{ds}\cdot\frac{d\ln(r_1 r_2)}{ds} = 0$$
$$\frac{d^2\varphi_2}{ds^2} + \frac{d\varphi_2}{ds}\cdot\frac{d\ln(r_1 r_2)}{ds} = 0 .$$

Integrating equations (10.17') we obtain

(10.18) $$r_1 r_2 \frac{d\varphi_1}{ds} = h_1 = \text{const.}$$
$$r_1 r_2 \frac{d\varphi_2}{ds} = h_2 = \text{const.}$$

Also,

(10.19) $$\frac{d^2t_1}{ds^2} + \frac{\partial\mu}{\partial r_1}\frac{dt_1}{ds}\frac{dr_1}{ds} + \frac{\partial\mu}{\partial r_2}\frac{dt_1}{ds}\frac{dr_2}{ds} = 0$$
$$\frac{d^2t_2}{ds^2} + \frac{\partial\mu}{\partial r_2}\frac{dt_2}{ds}\frac{dr_2}{ds} + \frac{\partial\mu}{\partial r_1}\frac{dt_2}{ds}\frac{dr_1}{ds} = 0$$
$$\Longrightarrow \quad \frac{d^2t_1}{ds^2} + \frac{dt_1}{ds}\frac{d\mu}{ds} = 0, \qquad \frac{d^2t_2}{ds^2} + \frac{dt_2}{ds}\frac{d\mu}{ds} = 0 .$$

Integrating (10.19) we obtain

(10.20) $$\frac{dt_1}{ds} = c_1 e^{-\mu}$$
$$\frac{dt_2}{ds} = c_2 e^{-\mu} .$$

Finally we have

(10.21) $$\frac{d^2r_1}{ds^2} + \frac{\partial\lambda}{\partial r_1}\left(\frac{dr_1}{ds}\right)^2 - r_1 e^{-\lambda}\frac{d\varphi_1}{ds}\frac{d\varphi_2}{ds} + e^{\mu-\lambda}\frac{\partial\mu}{\partial r_2}\frac{dt_1}{ds}\frac{dt_2}{ds} = 0$$
$$\frac{d^2r_2}{ds^2} + \frac{\partial\lambda}{\partial r_2}\left(\frac{dr_2}{ds}\right)^2 - r_2 e^{-\lambda}\frac{d\varphi_1}{ds}\frac{d\varphi_2}{ds} + e^{\mu-\lambda}\frac{\partial\mu}{\partial r_1}\frac{dt_1}{ds}\frac{dt_2}{ds} = 0,$$

along with a first integral,

(10.22) $$g_{\alpha,\beta} \frac{dx_1^\alpha}{ds}\frac{dx_2^\beta}{ds} = 1 ,$$

or

(10.23) $$-e^\lambda \frac{dr_1}{ds}\frac{dr_2}{ds} - r_1 r_2 \frac{d\varphi_1}{ds}\frac{d\varphi_2}{ds} + e^\mu \frac{dt_1}{ds}\frac{dt_2}{ds} = 1 .$$

Making use of (10.18) and (10.20), (10.23) becomes

(10.24) $$-e^\lambda \frac{dr_1}{ds}\frac{dr_2}{ds} - \frac{h_1 h_2}{r_1 r_2} + c_1 c_2 e^{-\mu} = 1 ,$$

and since $\lambda = -\mu$,

(10.25) $$\frac{dr_1}{ds}\frac{dr_2}{ds} + \left(1 - \frac{M}{4r_1} - \frac{M}{4r_2}\right)\frac{h_1 h_2}{r_1 r_2} - c_1 c_2 = -\left(1 - \frac{M}{4r_1} - \frac{M}{4r_2}\right).$$

(10.21) now becomes

(10.26) $$\begin{aligned} &\frac{d^2 r_1}{ds^2} - \frac{M/4r_1^2}{1 - \frac{M}{4r_1} - \frac{M}{4r_2}}\left(\frac{dr_1}{ds}\right)^2 - \left(1 - \frac{M}{4r_1} - \frac{M}{4r_2}\right)\frac{h_1 h_2}{r_1 r_2^2} + \left(1 - \frac{M}{4r_1} - \frac{M}{4r_2}\right)^{-1}\frac{Mc_1c_2}{4r_1^2} = 0 \\ &\frac{d^2 r_2}{ds^2} - \frac{M/4r_2^2}{1 - \frac{M}{4r_1} - \frac{M}{4r_2}}\left(\frac{dr_2}{ds}\right)^2 - \left(1 - \frac{M}{4r_1} - \frac{M}{4r_2}\right)\frac{h_1 h_2}{r_2 r_1^2} + \left(1 - \frac{M}{4r_1} - \frac{M}{4r_2}\right)^{-1}\frac{Mc_1c_2}{4r_1^2} = 0. \end{aligned}$$

From (10.26) we see that if initially $r_1 = r_2$, $\frac{dr_1}{ds} = \frac{dr_2}{ds}$, then successive derivatives of r_1 , r_2 are equal at $s = 0$, so that

(10.27) $$r_1 \equiv r_2 = r .$$

More generally, if

(10.28) $$\begin{aligned} \ddot{x}_1 &= f(x_1, x_2, \dot{x}_1, \dot{x}_2) \\ \ddot{x}_2 &= f(x_2, x_1, \dot{x}_2, \dot{x}_1) \end{aligned}$$

such that $x_1 = x_2$, $\dot{x}_1 = \dot{x}_2$, at $s = 0$, then by considering

$$\ddot{x}_1 = f(x_1, x_1, \dot{x}_1, \dot{x}_1)$$

which has a unique solution $x_1 = \varphi_1(s)$ under the given initial conditions, we see immediately that $x_1 \equiv x_2 = \varphi_1(s)$ satifies (10.28) and the initial conditions.

Now as a further initial condition we take $\varphi_1 = \varphi_2 + \pi$, $\frac{d\varphi_1}{ds} = \frac{d\varphi_2}{ds}$ at $s=0$, so that $h_1 = h_2 = h$. Equation (10.25) now becomes

$$\left(\frac{dr}{ds}\right)^2 + \left(1 - \frac{M}{2r}\right)\frac{h^2}{r^2} - c_1 c_2 = 1 - \frac{M}{2r} \quad (10.29)$$

Using $ds = \frac{r^2 d\varphi}{h}$, (10.29) becomes

$$\left(\frac{h}{r^2}\frac{dr}{d\varphi}\right)^2 + \left(1 - \frac{M}{2r}\right)\frac{h^2}{r^2} - c_1 c_2 = 1 - \frac{M}{2r}, \quad (10.30)$$

and letting $u = \frac{1}{r}$, we obtain

$$\left(\frac{du}{d\varphi}\right)^2 + \left(1 - \frac{M}{2}u\right)u^2 - \frac{c_1 c_2}{h^2} = -\frac{1}{h^2}\left(1 - \frac{M}{2}u\right), \quad (10.31)$$

so that upon differentiation we obtain

$$\begin{aligned} \frac{d^2u}{d\varphi^2} + u &= \frac{M}{4h^2} + \frac{3}{4}Mu^2 \\ &= \frac{(M/4)}{h^2} + 3(M/4)u^2, \end{aligned} \quad (10.32)$$

the well-known Einstein equation for the motion of an infinitesimal planet moving in the field of a fixed mass $M/4$.

XI. Parallel displacement.

Consider the pair of curves $x_1^i(t)$, $x_2^i(t)$, and let $\xi^{1i}(M_1)$, $\xi^{2i}(M_2)$, be the components of two contravariant vectors with components ξ_1^{i} along $x_1^i(t)$, ξ_2^{i} along $x_2^i(t)$. There is no loss of generality in assuming that these two vectors are generated from a single contravariant vector field, ξ^i, with components ξ^{1i} along $x_1^i(t)$, and components ξ^{2i} along $x_2^i(t)$, so long as the two curves do not intersect one another. We shall only treat of this case by considering curves in the neighborhoods of M_1, M_2.

<u>Definition.</u> We say that ξ^i is parallely displaced relative to the curves $x_1^i(t)$, $x_2^i(t)$, if ξ^i satisfies the equations

$$\frac{d\xi^{i,}}{dt} + \Gamma^{i,}_{\alpha\beta,}\,\xi^{\alpha,}\frac{dx_1^{\beta}}{dt} + C^{i,}_{\alpha,\beta}\,\xi^{\alpha,}\frac{dx_2^{\beta}}{dt} = 0 \tag{11.1}$$
$$\frac{d\xi^{,i}}{dt} + \Gamma^{,i}_{,\alpha\beta}\,\xi^{,\alpha}\frac{dx_2^{\beta}}{dt} + C^{,i}_{\alpha,\beta}\,\xi^{,\beta}\frac{dx_1^{\alpha}}{dt} = 0 \,.$$

<u>Theorem 12.</u> The tangent vector $\frac{dx^\alpha}{ds}$ of the dyodesics is parallely displaced relative to these paths.

The proof is immediately evident if we replace t by s and $\xi^{i,}$ by $\frac{dx_1^i}{ds}$, $\xi^{,i}$ by $\frac{dx_2^i}{ds}$.

We now investigate the invariance of (11.1) under coordinate transformations. We have,

$$\frac{d\xi^{i,}}{dt} + \Gamma^{i,}_{\alpha\beta,}\,\xi^{\alpha,}\frac{dx_1^{\beta}}{dt} + C^{i,}_{\alpha,\beta}\,\xi^{\alpha,}\frac{dx_2^{\beta}}{dt} \equiv \frac{d}{dt}\left(\bar{\xi}^{\sigma,}\frac{\partial x_1^i}{\partial \bar{x}_1^{\sigma}}\right) \tag{11.2}$$
$$+\left(\bar{\Gamma}^{\epsilon,}_{\mu\nu,}\frac{\partial \bar{x}_1^{\mu}}{\partial x_1^{\alpha}}\frac{\partial \bar{x}_1^{\nu}}{\partial x_1^{\beta}}\frac{\partial x_1^{i}}{\partial \bar{x}_1^{\epsilon}} + \frac{\partial^2 \bar{x}_1^{\sigma}}{\partial x_1^{\alpha}\partial x_1^{\beta}}\frac{\partial x_1^{i}}{\partial \bar{x}_1^{\sigma}}\right)\left(\bar{\xi}^{\tau,}\frac{\partial x_1^{\alpha}}{\partial \bar{x}_1^{\tau}}\frac{d\bar{x}_1^{\psi}}{dt}\frac{\partial x_1^{\beta}}{\partial \bar{x}_1^{\psi}}\right)$$
$$+\ \bar{C}^{\epsilon,}_{\mu,\nu}\frac{\partial \bar{x}_1^{\mu}}{\partial x_1^{\alpha}}\frac{\partial \bar{x}_2^{\nu}}{\partial x_2^{\beta}}\frac{\partial x_1^{i}}{\partial \bar{x}_1^{\epsilon}}\,\bar{\xi}^{\sigma,}\frac{\partial x_1^{\alpha}}{\partial \bar{x}_1^{\sigma}}\frac{d\bar{x}_2^{\psi}}{dt}\frac{\partial x_2^{\beta}}{\partial \bar{x}_2^{\psi}}$$
$$\equiv \frac{\partial x_1^{i}}{\partial \bar{x}_1^{\epsilon}}\left(\frac{d\bar{\xi}^{\epsilon,}}{dt} + \bar{\Gamma}^{\epsilon,}_{\tau\psi,}\,\bar{\xi}^{\tau,}\frac{d\bar{x}_1^{\psi}}{dt} + \bar{C}^{\epsilon,}_{\sigma,\psi}\,\bar{\xi}^{\sigma,}\frac{d\bar{x}_2^{\psi}}{dt}\right)$$
$$+\ \bar{\xi}^{\sigma,}\frac{d\bar{x}_1^{\tau}}{dt}\left(\frac{\partial^2 x_1^{i}}{\partial \bar{x}_1^{\tau}\partial \bar{x}_1^{\sigma}} + \frac{\partial^2 \bar{x}_1^{\gamma}}{\partial x_1^{\alpha}\partial x_1^{\beta}}\frac{\partial x_1^{i}}{\partial \bar{x}_1^{\gamma}}\frac{\partial x_1^{\alpha}}{\partial \bar{x}_1^{\sigma}}\frac{\partial x_1^{\beta}}{\partial \bar{x}_1^{\tau}}\right)$$
$$\equiv \frac{\partial x_1^{i}}{\partial \bar{x}_1^{\epsilon}}\left(\frac{d\bar{\xi}^{\epsilon,}}{dt} + \bar{\Gamma}^{\epsilon,}_{\tau\psi,}\,\bar{\xi}^{\tau,}\frac{d\bar{x}_1^{\psi}}{dt} + \bar{C}^{\epsilon,}_{\sigma,\psi}\,\bar{\xi}^{\sigma,}\frac{d\bar{x}_2^{\psi}}{dt}\right).$$

We obtain a similar result for the second equation of (11.1). Hence the left-hand sides of (11.1) transform like contravariant vectors, so that (11.1) is an invariant under coordinate transformations.

Now let $u_\alpha(x)$ be the components of a covariant vector with components $u_{\alpha,}$ along $x_1^\alpha(t)$, $u_{,\alpha}$ along $x_2^\alpha(t)$, and let $u_\alpha \xi^\alpha$ be a scalar invariant under a parallel displacement along $x_1^i(t)$, $x_2^i(t)$. Then

$$\frac{d}{dt}\left(u_{\alpha,}\,\xi^{\alpha,}\right) = u_{\alpha,}\frac{d\xi^{\alpha,}}{dt} + \xi^{\alpha,}\frac{du_{\alpha,}}{dt} = 0, \tag{11.3}$$

along $x_1^i(t)$ so that using (11.1) we have

$$\xi^{\alpha,}\left(\frac{du_{\alpha,}}{dt} - \Gamma^{i,}_{\alpha\beta,}\,u_{i,}\frac{dx_1^\beta}{dt}\right) - \xi^{\alpha,}C^{i,}_{\alpha,\beta}\,u_{i,}\frac{dx_2^\beta}{dt} = 0, \tag{11.4}$$

alomg with

$$\xi^{,\alpha}\left(\frac{du_{,\alpha}}{dt} - \Gamma^{,i}_{,\alpha\beta}\,u_{,i}\frac{dx_2^\beta}{dt} - C^{,i}_{\beta,\alpha}\,u_{,i}\frac{dx_1^\beta}{dt}\right) = 0 \tag{11.5}$$

for arbitrary $\xi^{\alpha,}$, $\xi^{,\alpha}$. Hence u_α is parallely displaced along $x_1^i(t)$, $x_2^i(t)$ if

$$\begin{aligned} &\frac{du_{\alpha,}}{dt} - \Gamma^{\sigma,}_{\alpha\beta,}\,u_{\sigma,}\frac{dx_1^\beta}{dt} - C^{\sigma,}_{\alpha,\beta}\,u_{\sigma,}\frac{dx_2^\beta}{dt} = 0 \\ &\frac{du_{,\alpha}}{dt} - \Gamma^{,\sigma}_{,\alpha\beta}\,u_{,\sigma}\frac{dx_2^\beta}{dt} - C^{,\sigma}_{\beta,\alpha}\,u_{,\sigma}\frac{dx_1^\beta}{dt} = 0. \end{aligned} \tag{11.6}$$

If we are given two curves $x_1^i(t)$, $x_2^i(t)$ along with the initial values $(\xi_0^{i,}, \xi_0^{,i})$ for $t = t_0$, it follows from differential equation theory that we can solve (11.1) for the $\xi^{i,}$, $\xi^{,i}$. We must show that the $\xi^{i,}$, $\xi^{,i}$ thus uniquely determined are actually the components of contravariant vectors. Let the solutions of (11.1) be

$$\begin{aligned} \xi^{i,} &= \phi_1^{i,}(t) \\ \xi^{,i} &= \phi_2^{,i}(t) \end{aligned} \quad \text{with} \quad \begin{aligned} \phi_1^{i,}(t_0) &= \xi_0^{i,} \\ \phi_2^{,i}(t_0) &= \xi_0^{,i}. \end{aligned} \tag{11.7}$$

There are many ways by which we can choose

$$(11.8)\qquad \begin{aligned} \xi^{i_1} &= \xi^{i_1}(x_1^1, x_1^2, \ldots, x_1^n) \\ \xi^{\prime i} &= \xi^{\prime i}(x_2^1, x_2^2, \ldots, x_2^n) \end{aligned}$$

such that

$$(11.8')\qquad \begin{aligned} \xi^{i_1}\left(x_1^1(t), x_1^2(t), \ldots, x_1^n(t)\right) &= \phi_1^{i_1}(t) \\ \xi^{\prime i}\left(x_2^1(t), x_2^2(t), \ldots, x_2^n(t)\right) &= \phi_2^{\prime i}(t) . \end{aligned}$$

Let $\xi^{i_1}(M_1)$, $\xi^{\prime i}(M_2)$ be such a representation. Multiplying (11.1) by $\frac{\partial \bar{x}_1^{j}}{\partial x_1^{i}}$ we obtain

$$(11.9)\qquad \frac{d}{dt}\left(\xi^{i_1}\frac{\partial \bar{x}_1^{j}}{\partial x_1^{i}}\right) + \Gamma^{j_1}_{\alpha\beta_1}\left(\xi^{i_1}\frac{\partial \bar{x}_1^{\alpha}}{\partial x_1^{i}}\right)\frac{d\bar{x}_1^{\beta}}{dt} + C^{j_1}_{\alpha,\beta}\left(\xi^{i_1}\frac{\partial \bar{x}_1^{\alpha}}{\partial x_1^{i}}\right)\frac{d\bar{x}_2^{\beta}}{dt} = 0$$

by making use of (6.2). Hence $\bar{\xi}^{j_1} = \xi^{i_1}\frac{\partial \bar{x}_1^{j}}{\partial x_1^{i}}$, which establishes that ξ^{i_1} is a vector. A similar result holds for $\xi^{\prime i}$.

Now at a point we say that two vectors are parallel, or have the same direction, if their corresponding components are proportional. Thus the vector $\bar{\xi}^{i_1}(x_1)$ defined by

$$(11.10)\qquad \bar{\xi}^{i_1}(x_1) = \phi_1(x_1)\,\xi^{i_1}(x_1)$$

is said to be parallel to ξ^{i_1}. A similar remark holds for

$$(11.11)\qquad \bar{\xi}^{\prime i}(x_2) = \phi_2(x_2)\,\xi^{\prime i}(x_2) .$$

Now if $(\xi^{i_1}, \xi^{\prime i})$ are parallely displaced along $x_1^i(t)$, $x_2^i(t)$ we obtain from (11.1)

$$(11.12)\qquad \begin{aligned} \frac{d\bar{\xi}^{i_1}}{dt} + \Gamma^{i_1}_{\alpha\beta_1}\bar{\xi}^{\alpha_1}\frac{dx_1^{\beta}}{dt} + C^{i_1}_{\alpha,\beta}\bar{\xi}^{\alpha_1}\frac{dx_2^{\beta}}{dt} &= \bar{\xi}^{i_1}\frac{d\ln\phi_1}{dt} \\ \frac{d\bar{\xi}^{\prime i}}{dt} + \Gamma^{\prime i}_{\prime\alpha\beta}\bar{\xi}^{\prime\alpha}\frac{dx_2^{\beta}}{dt} + C^{\prime i}_{\alpha,\beta}\bar{\xi}^{\prime\beta}\frac{dx_1^{\alpha}}{dt} &= \bar{\xi}^{\prime i}\frac{d\ln\phi_2}{dt} . \end{aligned}$$

Dividing (11.12) by $\bar{\xi}^{i_1}$, $\bar{\xi}^{\prime i}$, respectively, we obtain expressions independent of <u>i</u>. Vectors satisfying (11.12) are said to be parallely displaced relative to $x_1^i(t)$, $x_2^i(t)$. Moreover, if $\bar{\xi}^{i_1}$, $\bar{\xi}^{\prime i}$ satisfy (11.12), then

$$\xi^{i_1} = \frac{1}{\phi_1}\bar{\xi}^{i_1}$$

$$\xi^{\prime i} = \frac{1}{\phi_2}\bar{\xi}^{\prime i}$$

will satisfy (11.1)

We now extend the definition of parallel displacement. Let $\xi^{i_1}(x_1,x_2)$ be a two-point contravariant vector, contravariant relative to x_1 , and a scalar relative to x_2 . Let $\xi^{,i}(x_1,x_2)$ be contravariant relative to x_2 and a scalar relative to x_1 .

Definition. We say that $\xi^{i_1}(x_1,x_2)$, $\xi^{,i}(x_1,x_2)$ are parallely displaced relative to the curves $x_1^i(t)$, $x_2^i(t)$ if

(11.13)
$$\frac{d\xi^{i_1}}{dt} + \Gamma^{i_1}_{\alpha\beta_1}\xi^{\alpha_1}\frac{dx_1^\beta}{dt} + C^{i_1}_{\alpha,\beta}\xi^{\alpha_1}\frac{dx_2^\beta}{dt} = 0$$
$$\frac{d\xi^{,i}}{dt} + \Gamma^{,i}_{,\alpha\beta}\xi^{,\alpha}\frac{dx_2^\beta}{dt} + C^{,i}_{\alpha,\beta}\xi^{,\beta}\frac{dx_1^\alpha}{dt} = 0$$

everywhere along $x_1^i(t)$, $x_2^i(t)$.

Equation (11.2) shows the invariance of (11.13) under coordinate transformations.

We say that $\xi^{i_1}(x_1,x_2)$, $\bar{\xi}^{i_1}(x_1,x_2)$ are parallel if

(11.14)
$$\bar{\xi}^{i_1}(x_1,x_2) = \phi_1(x_1,x_2)\,\xi^{i_1}(x_1,x_2)$$

where ϕ_1 is a scalar relative to x_1 and x_2 . If $\xi^{i_1}(x_1,x_2)$ satisfies (11.13) then $\bar{\xi}^{i_1}(x_1,x_2)$ will satisfy (11.12).

XII. Parallel displacement around infinitesimal closed paths.

Let us find the change in the components of a vector $(\xi^{i_1}, \xi^{,i})$ as we move around the infinitesimal closed paths $x_1^i(t)$, $x_2^i(t)$, $0 \le t \le 1$. From (11.1) we have

(12.1)
$$\delta\xi^{i_1} = \xi^{i_1}(1) - \xi^{i_1}(0) = -\oint \Gamma^{i_1}_{\alpha\beta_1}\xi^{\alpha_1}\frac{dx_1^\beta}{dt}dt - \oint C^{i_1}_{\alpha,\beta}\xi^{\alpha_1}\frac{dx_2^\beta}{dt}dt .$$

Expanding Γ , C , ξ in Taylor series' about $t=0$, we obtain

(12.2)
$$\Gamma^{i_1}_{\alpha\beta_1} = \left(\Gamma^{i_1}_{\alpha\beta_1}\right)_0 + \left(\frac{\partial\Gamma^{i_1}_{\alpha\beta_1}}{\partial x_1^\sigma}\right)_0 (x_1^\sigma - x_1^\sigma(0)) + \left(\frac{\partial\Gamma^{i_1}_{\alpha\beta_1}}{\partial x_2^\sigma}\right)_0 (x_2^\sigma - x_2^\sigma(0)) + \cdots$$
$$C^{i_1}_{\alpha,\beta} = \left(C^{i_1}_{\alpha,\beta}\right)_0 + \left(\frac{\partial C^{i_1}_{\alpha,\beta}}{\partial x_1^\sigma}\right)_0 (x_1^\sigma - x_1^\sigma(0)) + \left(\frac{\partial C^{i_1}_{\alpha,\beta}}{\partial x_2^\sigma}\right)_0 (x_2^\sigma - x_2^\sigma(0)) + \cdots$$
$$\xi^{\alpha_1} = \left(\xi^{\alpha_1}\right)_0 - \left(\Gamma^{\alpha_1}_{\mu\nu_1}\right)_0 \left(\xi^{\mu_1}\right)_0 (x_1^\nu - x_1^\nu(0)) - \left(C^{\alpha_1}_{\mu,\nu}\right)_0 \left(\xi^{\mu_1}\right)_0 (x_2^\nu - x_2^\nu(0)) + \cdots$$

where in the expansion of ξ^{α_1} we have used (11.13).

Hence (12.1) becomes (except for infinitesimals of higher order)

$$(12.3)\qquad \delta\xi^{\iota_1} = -\left(\Gamma^{\iota_1}_{\alpha\beta_1}\right)_0\left(\xi^{\alpha_1}\right)_0 \oint dx_1^{\beta} - \left(\xi^{\alpha_1}\right)_0\left(\frac{\partial\Gamma^{\iota_1}_{\alpha\beta_1}}{\partial x_1^{\sigma}}\right)_0 \oint\left(x_1^{\sigma}-x_1^{\sigma}(0)\right)dx_1^{\beta}$$

$$-\left(\xi^{\alpha_1}\right)_0\left(\frac{\partial\Gamma^{\iota_1}_{\alpha\beta_1}}{\partial x_2^{\sigma}}\right)_0 \oint\left(x_2^{\sigma}-x_2^{\sigma}(0)\right)dx_1^{\beta} + \left(\Gamma^{\iota_1}_{\tau\beta_1}\right)_0\left(\Gamma^{\tau_1}_{\alpha\sigma_1}\right)_0\left(\xi^{\alpha_1}\right)_0 \oint\left(x_1^{\sigma}-x_1^{\sigma}(0)\right)dx_1^{\beta}$$

$$+\left(\Gamma^{\iota_1}_{\tau\beta_1}\right)_0\left(C^{\tau_1}_{\alpha,\sigma}\right)_0\left(\xi^{\alpha_1}\right)_0 \oint\left(x_2^{\sigma}-x_2^{\sigma}(0)\right)dx_1^{\beta} - \left(C^{\iota_1}_{\alpha,\beta}\right)_0\left(\xi^{\alpha_1}\right)_0 \oint dx_2^{\beta}$$

$$+\left(C^{\iota_1}_{\tau,\beta}\right)_0\left(\Gamma^{\tau_1}_{\alpha\sigma_1}\right)_0\left(\xi^{\alpha_1}\right)_0 \oint\left(x_1^{\sigma}-x_1^{\sigma}(0)\right)dx_2^{\beta}$$

$$+\left(C^{\iota_1}_{\tau,\beta}\right)_0\left(C^{\tau_1}_{\alpha,\sigma}\right)_0\left(\xi^{\alpha_1}\right)_0 \oint\left(x_2^{\sigma}-x_2^{\sigma}(0)\right)dx_2^{\beta}$$

$$-\left(\frac{\partial C^{\iota_1}_{\alpha,\beta}}{\partial x_1^{\sigma}}\right)_0\left(\xi^{\alpha_1}\right)_0 \oint\left(x_1^{\sigma}-x_1^{\sigma}(0)\right)dx_2^{\beta}$$

$$-\left(\frac{\partial C^{\iota_1}_{\alpha,\beta}}{\partial x_2^{\sigma}}\right)_0\left(\xi^{\alpha_1}\right)_0 \oint\left(x_2^{\sigma}-x_2^{\sigma}(0)\right)dx_2^{\beta}.$$

Now, $\oint dx_1^{\beta} = \oint dx_2^{\beta} = \oint x_1^{\sigma}(0)\,dx_1^{\beta} = \cdots \equiv 0,$

so that (12.30 reduces to

$$(12.4)\qquad \delta\xi^{\iota_1} = -\left(\xi^{\alpha_1}\right)_0\left[\frac{\partial\Gamma^{\iota_1}_{\alpha\beta_1}}{\partial x_1^{\sigma}} - \Gamma^{\iota_1}_{\tau\beta_1}\Gamma^{\tau_1}_{\alpha\sigma_1}\right]_0 \oint x_1^{\sigma}dx_1^{\beta}$$

$$-\left(\xi^{\alpha_1}\right)_0\left[\frac{\partial\Gamma^{\iota_1}_{\alpha\beta_1}}{\partial x_2^{\sigma}} - \Gamma^{\iota_1}_{\tau\beta_1}C^{\tau_1}_{\alpha,\sigma}\right]_0 \oint x_2^{\sigma}dx_1^{\beta}$$

$$-\left(\xi^{\alpha_1}\right)_0\left[\frac{\partial C^{\iota_1}_{\alpha,\beta}}{\partial x_1^{\sigma}} - C^{\iota_1}_{\tau,\beta}\Gamma^{\tau_1}_{\alpha\sigma_1}\right]_0 \oint x_1^{\sigma}dx_2^{\beta}$$

$$-\left(\xi^{\alpha_1}\right)_0\left[\frac{\partial C^{\iota_1}_{\alpha,\beta}}{\partial x_2^{\sigma}} - C^{\iota_1}_{\tau,\beta}C^{\tau_1}_{\alpha,\sigma}\right]_0 \oint x_2^{\sigma}dx_2^{\beta}.$$

Also,

$$\oint d(x_1^\sigma x_1^\beta) = \oint x_1^\sigma dx_1^\beta + \oint x_1^\beta dx_1^\sigma = 0$$

(12.5)
$$\oint d(x_2^\sigma x_1^\beta) = \oint x_2^\sigma dx_1^\beta + \oint x_1^\beta dx_2^\sigma = 0$$

$$\oint d(x_2^\sigma x_2^\beta) = \oint x_2^\sigma dx_2^\beta + \oint x_2^\beta dx_2^\sigma = 0,$$

so that interchanging σ, β in (12.4) and using (12.5) we obtain

(12.6)
$$\begin{aligned}\delta\xi^{\iota,} = &-\tfrac{1}{4}(\xi^{\alpha,})_0 (R^{\iota,}_{\alpha\beta\sigma,})_0 \oint(x_1^\sigma dx_1^\beta - x_1^\beta dx_1^\sigma) \\ &-\tfrac{1}{4}(\xi^{\alpha,})_0 (S^{\iota,}_{\alpha,\beta\sigma})_0 \oint(x_2^\sigma dx_2^\beta - x_2^\beta dx_2^\sigma) \\ &-\tfrac{1}{4}(\xi^{\alpha,})_0 (T^{\iota,}_{\alpha\beta,\sigma})_0 \oint(x_2^\sigma dx_1^\beta - x_1^\beta dx_2^\sigma) \\ &-\tfrac{1}{4}(\xi^{\alpha,})_0 (T^{\iota,}_{\alpha\sigma,\beta}) \oint(x_1^\sigma dx_2^\beta - x_2^\beta dx_1^\sigma),\end{aligned}$$

where

(12.7)
$$\begin{aligned}R^{\iota,}_{\alpha\beta\sigma,} &= \frac{\partial\Gamma^{\iota,}_{\alpha\beta,}}{\partial x_1^\sigma} - \frac{\partial\Gamma^{\iota,}_{\alpha\sigma,}}{\partial x_1^\beta} + \Gamma^{\iota,}_{\tau\sigma,}\Gamma^{\tau,}_{\alpha\beta,} - \Gamma^{\iota,}_{\tau\beta,}\Gamma^{\tau,}_{\alpha\sigma,} \\ S^{\iota,}_{\alpha,\beta\sigma} &= \frac{\partial C^{\iota,}_{\alpha,\beta}}{\partial x_2^\sigma} - \frac{\partial C^{\iota,}_{\alpha,\sigma}}{\partial x_2^\beta} + C^{\iota,}_{\tau,\sigma}C^{\tau,}_{\alpha,\beta} - C^{\iota,}_{\tau,\beta}C^{\tau,}_{\alpha,\sigma} \\ T^{\iota,}_{\alpha\beta,\sigma} &= \frac{\partial\Gamma^{\iota,}_{\alpha\beta,}}{\partial x_2^\sigma} - \frac{\partial C^{\iota,}_{\alpha,\sigma}}{\partial x_1^\beta} + C^{\iota,}_{\tau,\sigma}\Gamma^{\tau,}_{\alpha\beta,} - C^{\tau,}_{\alpha,\sigma}\Gamma^{\iota,}_{\tau\beta,}.\end{aligned}$$

We now exhibit the tensor character of R, S, T. Referring back to § 8 we see that $R^{\iota,}_{jkl,}$ is a tensor. From (7.4) upon differentiation, we obtain

(12.8)
$$\frac{\partial\bar{C}^{\iota,}_{\sigma,\alpha}}{\partial\bar{x}_2^\mu} = \frac{\partial C^{\epsilon,}_{\mu,\tau}}{\partial x_2^\psi}\frac{\partial\bar{x}_1^\iota}{\partial x_1^\epsilon}\frac{\partial x_1^\mu}{\partial\bar{x}_1^\sigma}\frac{\partial x_2^\tau}{\partial\bar{x}_2^\alpha}\frac{\partial x_2^\psi}{\partial\bar{x}_2^\mu} + C^{\epsilon,}_{\omega,\tau}\frac{\partial\bar{x}_1^\iota}{\partial x_1^\epsilon}\frac{\partial x_1^\omega}{\partial\bar{x}_1^\sigma}\frac{\partial^2 x_2^\tau}{\partial\bar{x}_2^\mu\partial\bar{x}_2^\alpha}.$$

Interchanging α, μ and subtracting, we see that

(12.9)
$$\frac{\partial C^{\iota,}_{\alpha,\beta}}{\partial x_2^\sigma} - \frac{\partial C^{\iota,}_{\alpha,\sigma}}{\partial x_2^\beta}$$

are the components of a tensor. This proves the tensor character of $S^{\iota,}_{\alpha,\beta\sigma}$.

Finally, we can prove the tensor character of $T^{\iota,}_{\alpha\beta,\sigma}$ by appealing to the quotient law or by investigating $T^{\iota,}_{\alpha\beta,\sigma}$ directly as regards its law of transformation.

Similar results hold for $\delta\xi^{,\iota}$ which lead to the tensors $R^{,\iota}_{,\alpha\beta\sigma}$, $S^{,\iota}_{\alpha\sigma,\beta}$, $T^{,\iota}_{\sigma,\alpha\beta}$, where

$$
\begin{aligned}
R^{,\iota}_{,\alpha\beta\sigma} &= \frac{\partial \Gamma^{,\iota}_{,\alpha\beta}}{\partial x_2^{\sigma}} - \frac{\partial \Gamma^{,\iota}_{,\alpha\sigma}}{\partial x_2^{\beta}} + \Gamma^{,\iota}_{,\tau\sigma}\Gamma^{,\tau}_{,\alpha\beta} - \Gamma^{,\iota}_{,\tau\beta}\Gamma^{,\tau}_{,\alpha\sigma} \\
S^{,\iota}_{\alpha\sigma,\beta} &= \frac{\partial C^{,\iota}_{\alpha,\beta}}{\partial x_1^{\sigma}} - \frac{\partial C^{,\iota}_{\sigma,\beta}}{\partial x_1^{\alpha}} + C^{,\iota}_{\sigma,\tau}C^{,\tau}_{\alpha,\beta} - C^{,\iota}_{\alpha,\tau}C^{,\tau}_{\sigma,\beta} \qquad (12.9') \\
T^{,\iota}_{\sigma,\alpha\beta} &= \frac{\partial \Gamma^{,\iota}_{,\alpha\beta}}{\partial x_1^{\sigma}} - \frac{\partial C^{,\iota}_{\sigma,\alpha}}{\partial x_2^{\beta}} + C^{,\iota}_{\sigma,\tau}\Gamma^{,\tau}_{,\alpha\beta} - C^{,\tau}_{\alpha,\tau}\Gamma^{,\iota}_{,\sigma\beta} .
\end{aligned}
$$

If $\delta\xi^{\iota,} = \delta\xi^{,\iota} = 0$ for all infinitesimal closed paths we must have (and conversely),

$$
\begin{aligned}
R^{\iota,}_{\alpha\beta,\sigma} &= S^{\iota,}_{\alpha,\beta\sigma} = T^{\iota,}_{\alpha\beta,\sigma} = 0 \\
R^{,\iota}_{,\alpha\beta\sigma} &= S^{,\iota}_{\alpha\beta,\sigma} = T^{,\iota}_{\alpha,\beta\sigma} = 0 \qquad (12.10) .
\end{aligned}
$$

<u>Definition</u>. If (12.10) holds we say that the space is flat.

If the equations of the dyodesics are linear in the parameter s, we must have (Theorem 9) $\Gamma^{\iota,}_{\alpha\beta,} = \Gamma^{,\iota}_{,\alpha\beta} = 0$ which imply $C^{\iota,}_{\alpha,\beta} = C^{,\iota}_{\alpha,\beta} = 0$. Hence

<u>Theorem 13</u>. If the equations of the dyodesics are linear in the parameter s, the space is necessarily flat.

Let us now consider a space with symmetric $\Gamma^{\iota,}_{\alpha\beta,}$, $\Gamma^{,\iota}_{,\alpha\beta}$ which is flat. Since $C^{\iota,}_{\alpha,\beta} = C^{,\iota}_{\alpha,\beta} = 0$, the vanishing of $T^{\iota,}_{\alpha\beta,\sigma}$, $T^{,\iota}_{\alpha,\beta\sigma}$ imply

$$
\frac{\partial \Gamma^{\iota,}_{\alpha\beta,}}{\partial x_2^{\sigma}} = \frac{\partial \Gamma^{,\iota}_{,\beta\sigma}}{\partial x_1^{\alpha}} = 0 . \qquad (12.11)
$$

Hence

$$
\begin{aligned}
\Gamma^{\iota,}_{\alpha\beta,} &= \Gamma^{\iota,}_{\alpha\beta,}(x_1) \\
\Gamma^{,\iota}_{,\alpha\beta} &= \Gamma^{,\iota}_{,\alpha\beta}(x_2) \qquad (12.12) .
\end{aligned}
$$

From ordinary Riemannian geometry we know that a necessary and sufficient condition that coordinate systems X_1, X_2 exist such that $\Gamma^{i}_{\alpha\beta,}(X_1)$, $\Gamma^{,i}_{,\alpha\beta}(X_2)$ vanish is that $R^{i}_{jk\ell,}(X_1)$, $R^{,i}_{,jk\ell}(X_2)$ vanish. Hence we have shown

<u>Theorem 14.</u> A necessary and sufficient condition that a space with symmetric $\Gamma^{i}_{\alpha\beta,}$, $\Gamma^{,i}_{,\alpha\beta}$ consist of dyodesics linear in the parameter s for some coordinate system is that the space be flat.

From the definition of the Γ's and R's we can readily prove that

$$R^{i}_{jk\ell,} = R^{,i}_{,jk\ell} \equiv 0 \tag{12.13}$$

for all metrics $g_{\alpha,\beta}$. This does not affect any of the above theorems or statements. The proof of (12.13) now follows.

$$R^{i}_{\alpha\beta\sigma,} = \frac{\partial \Gamma^{i}_{\alpha\beta,}}{\partial X_1^{\sigma}} - \frac{\partial \Gamma^{i}_{\alpha\sigma,}}{\partial X_1^{\beta}} + \Gamma^{i}_{\tau\sigma,}\Gamma^{\tau}_{\alpha\beta,} - \Gamma^{i}_{\tau\beta,}\Gamma^{\tau}_{\alpha\sigma,}\,, \tag{12.14}$$

where $\Gamma^{i}_{\alpha\beta,} = g^{i,\epsilon}\dfrac{\partial g_{\alpha,\epsilon}}{\partial X_1^{\beta}}$.

Hence

$$R^{i}_{\alpha\beta\sigma,} = g^{i,\epsilon}\left(\frac{\partial^2 g_{\alpha,\epsilon}}{\partial X_1^{\sigma}\partial X_1^{\beta}} - \frac{\partial^2 g_{\alpha,\epsilon}}{\partial X_1^{\beta}\partial X_1^{\sigma}}\right) \tag{12.15}$$

$$+ \frac{\partial g_{\alpha,\epsilon}}{\partial X_1^{\beta}}\frac{\partial g^{i,\epsilon}}{\partial X_1^{\sigma}} - \frac{\partial g_{\alpha,\epsilon}}{\partial X_1^{\sigma}}\frac{\partial g^{i,\epsilon}}{\partial X_1^{\beta}} + \Gamma^{i}_{\tau\sigma,}\Gamma^{\tau}_{\alpha\beta,} - \Gamma^{i}_{\tau\beta,}\Gamma^{\tau}_{\alpha\sigma,}\,.$$

Now $g^{i,\epsilon}g_{i,\mu} = \delta^{,\epsilon}_{,\mu}$

so that
$$\frac{\partial g^{i,\epsilon}}{\partial X_1^{\sigma}}g_{i,\mu} + g^{i,\epsilon}\frac{\partial g_{i,\mu}}{\partial X_1^{\sigma}} = 0\,,$$

and
$$\frac{\partial g^{j,\epsilon}}{\partial X_1^{\sigma}} = -g^{j,\mu}g^{i,\epsilon}\frac{\partial g_{i,\mu}}{\partial X_1^{\sigma}}\,,$$

and hence (12.15) reduces to

$$R^{i}_{\alpha\beta\sigma,} = -g^{i,\mu}g^{\tau,\epsilon}\frac{\partial g_{\tau,\mu}}{\partial X_1^{\sigma}}\frac{\partial g_{\alpha,\epsilon}}{\partial X_1^{\beta}} + g^{i,\mu}g^{\tau,\epsilon}\frac{\partial g_{\tau,\mu}}{\partial X_1^{\beta}}\frac{\partial g_{\alpha,\epsilon}}{\partial X_1^{\sigma}} \tag{12.16}$$

$$+ \Gamma^{i}_{\tau\sigma,}\Gamma^{\tau}_{\alpha\beta,} - \Gamma^{i}_{\tau\beta,}\Gamma^{\tau}_{\alpha\sigma,}$$

$$= -\Gamma^{i}_{\tau\sigma,}\Gamma^{\tau}_{\alpha\beta,} + \Gamma^{i}_{\tau\beta,}\Gamma^{\tau}_{\alpha\sigma,} + \Gamma^{i}_{\tau\sigma,}\Gamma^{\tau}_{\alpha\beta,} - \Gamma^{i}_{\tau\beta,}\Gamma^{\tau}_{\alpha\sigma,}$$

$$\equiv 0\,.$$

XIII. Intrinsic derivatives.

Definition. Let ξ^{α} be a contravariant vector with components $\xi^{\alpha,}$, $\xi^{,\alpha}$ along $x_1^{\alpha}(t)$, $x_2^{\alpha}(t)$, respectively. We define the intrinsic derivatives of $\xi^{\alpha,}$, $\xi^{,\alpha}$ along these curves by the equations

$$
(13.1)\qquad
\begin{aligned}
\frac{\delta \xi^{i,}}{\delta t} &\equiv \frac{d\xi^{i,}}{dt} + \Gamma^{i,}_{\alpha\beta,}\,\xi^{\alpha,}\frac{dx_1^{\beta}}{dt} + C^{i,}_{\alpha,\beta}\,\xi^{\alpha,}\frac{dx_2^{\beta}}{dt} \\
\frac{\delta \xi^{,i}}{\delta t} &\equiv \frac{d\xi^{,i}}{dt} + \Gamma^{,i}_{,\alpha\beta}\,\xi^{,\alpha}\frac{dx_2^{\beta}}{dt} + C^{,i}_{\alpha,\beta}\,\xi^{,\beta}\frac{dx_1^{\alpha}}{dt} .
\end{aligned}
$$

From (13.1) we have

$$
(13.2)\qquad
\begin{aligned}
\frac{\delta \xi^{i,}}{\delta t} &= \left(\frac{\partial \xi^{i,}}{\partial x_1^{\beta}} + \Gamma^{i,}_{\alpha\beta,}\,\xi^{\alpha,}\right)\frac{dx_1^{\beta}}{dt} + C^{i,}_{\alpha,\beta}\,\xi^{\alpha,}\frac{dx_2^{\beta}}{dt} \\
\frac{\delta \xi^{,i}}{\delta t} &= \left(\frac{\partial \xi^{,i}}{\partial x_2^{\beta}} + \Gamma^{,i}_{,\alpha\beta}\,\xi^{,\alpha}\right)\frac{dx_2^{\beta}}{dt} + C^{,i}_{\alpha,\beta}\,\xi^{,\beta}\frac{dx_1^{\alpha}}{dt} .
\end{aligned}
$$

From (11.2) we know that $\frac{\delta \xi^{i,}}{\delta t}$ and $\frac{\delta \xi^{,i}}{\delta t}$ are contravariant vectors, and since $C^{i,}_{\alpha,\beta}\,\xi^{\alpha,}\frac{dx_2^{\beta}}{dt}$, $C^{,i}_{\alpha,\beta}\,\xi^{,\beta}\frac{dx_1^{\alpha}}{dt}$, are tensors, the quotient law tells us that

$$
(13.3)\qquad
\begin{aligned}
\xi^{i,}_{;\beta,} &\equiv \frac{\partial \xi^{i,}}{\partial x_1^{\beta}} + \Gamma^{i,}_{\alpha\beta,}\,\xi^{\alpha,} \\
\xi^{,i}_{;,\beta} &\equiv \frac{\partial \xi^{,i}}{\partial x_2^{\beta}} + \Gamma^{,i}_{,\alpha\beta}\,\xi^{,\alpha}
\end{aligned}
$$

are tensors.

Definition. The $\xi^{i,}_{;\beta,}$, $\xi^{,i}_{;,\beta}$, defined by (13.3) are called the covariant derivatives of $\xi^{i,}$ and $\xi^{,i}$ with respect to x_1^{β} and x_2^{β}, respectively. The semi-colon will denote covariant differentiation.

The above results could have been obtained from our knowledge of the transformation laws of the Γ's.

Let us find the intrinsic derivative of a general tensor $\xi^{r,}_{st,}$, along $x_1^{i}(t)$. Let $A^{s,}$, $B^{t,}$, $D_{r,}$, be general vectors along $x_1^{i}(t)$ whose intrinsic derivatives vanish. We now consider the scalar invariant $A^{s,}B^{t,}D_{r,}\,\xi^{r,}_{st,}$, and obtain

$$(13.4)\qquad \frac{d}{dt}\left(A^{s_1}B^{t_1}D_{r_1}\xi^{r_1}_{st_1}\right) = A^{s_1}B^{t_1}D_{r_1}\frac{d\xi^{r_1}_{st_1}}{dt}$$
$$- B^{t_1}D_{r_1}\xi^{r_1}_{st_1}\left(\Gamma^{s_1}_{\alpha\beta_1}A^{\alpha_1}\frac{dx_1^\beta}{dt} + C^{s_1}_{\alpha,\beta}A^{\alpha_1}\frac{dx_2^\beta}{dt}\right)$$
$$- A^{s_1}D_{r_1}\xi^{r_1}_{st_1}\left(\Gamma^{t_1}_{\alpha\beta_1}B^{\alpha_1}\frac{dx_1^\beta}{dt} + C^{t_1}_{\alpha,\beta}B^{\alpha_1}\frac{dx_2^\beta}{dt}\right)$$
$$+ A^{s_1}B^{t_1}\xi^{r_1}_{st_1}\left(\Gamma^{\sigma_1}_{r\beta_1}D_{\sigma_1}\frac{dx_1^\beta}{dt} + C^{\sigma_1}_{r,\beta}D_{\sigma_1}\frac{dx_2^\beta}{dt}\right),$$

by making use of (11.1), (11.6). Hence

$$(13.5)\qquad \frac{d}{dt}\left(A^{s_1}B^{t_1}D_{r_1}\xi^{r_1}_{st_1}\right) = A^{s_1}B^{t_1}D_{r_1}\Big[\frac{d\xi^{r_1}_{st_1}}{dt} + \xi^{\sigma_1}_{st_1}\left(\Gamma^{r_1}_{\sigma\beta_1}\frac{dx_1^\beta}{dt} + C^{r_1}_{\sigma,\beta}\frac{dx_2^\beta}{dt}\right)$$
$$- \xi^{r_1}_{\sigma t_1}\left(\Gamma^{\sigma_1}_{s\beta_1}\frac{dx_1^\beta}{dt} + C^{\sigma_1}_{s,\beta}\frac{dx_2^\beta}{dt}\right) - \xi^{r_1}_{s\sigma_1}\left(\Gamma^{\sigma_1}_{t\beta_1}\frac{dx_1^\beta}{dt} + C^{\sigma_1}_{t,\beta}\frac{dx_2^\beta}{dt}\right)\Big].$$

From the quotient theorem we obtain the intrinsic derivative

$$(13.6)\qquad \frac{\delta\xi^{r_1}_{st_1}}{\delta t} \equiv \frac{d\xi^{r_1}_{st_1}}{dt} + \xi^{\sigma_1}_{st_1}\left(\Gamma^{r_1}_{\sigma\beta_1}\frac{dx_1^\beta}{dt} + C^{r_1}_{\sigma,\beta}\frac{dx_2^\beta}{dt}\right)$$
$$- \xi^{r_1}_{\sigma t_1}\left(\Gamma^{\sigma_1}_{s\beta_1}\frac{dx_1^\beta}{dt} + C^{\sigma_1}_{s,\beta}\frac{dx_2^\beta}{dt}\right)$$
$$- \xi^{r_1}_{s\sigma_1}\left(\Gamma^{\sigma_1}_{\beta t_1}\frac{dx_1^\beta}{dt} + C^{\sigma_1}_{t,\beta}\frac{dx_2^\beta}{dt}\right).$$

It is immediately evident that

$$(13.7)\qquad \xi^{r_1}_{st_1;\beta_1} \equiv \frac{\partial\xi^{r_1}_{st_1}}{\partial x_1^\beta} + \xi^{\sigma_1}_{st_1}\Gamma^{r_1}_{\sigma\beta_1} - \xi^{r_1}_{\sigma t_1}\Gamma^{\sigma_1}_{s\beta_1} - \xi^{r_1}_{s\sigma_1}\Gamma^{\sigma_1}_{\beta t_1}$$

is a tensor (covariant derivative).

The generalization to any tensor is at once apparent from the forms of (13.6), (13.7). Thus $\xi^{\alpha_1}_{;\beta_1}$ denotes the covariant derivative of ξ^{α_1} with respect to x_1^β.

We also extend the definition of the intrinsic derivative as follows. Let $\xi^{\alpha}(x_1,x_2)$ be a contravariant vector in x_1, a scalar in x_2, and let $\xi'^{\alpha}(x_1,x_2)$ be a contravariant vector in x_2, and a scalar in x_1. The intrinsic derivatives of ξ^{α_1}, ξ'^{α}, relative to $x_1^i(t)$, $x_2^i(t)$ are defined by equations (13.1). From (13.1) we have

$$\frac{\delta \xi^{i,}}{\delta t} = \left(\frac{\partial \xi^{i,}}{\partial x_1^{\beta}} + \Gamma^{i,}_{\alpha\beta,}\, \xi^{\alpha,} \right) \frac{dx_1^{\beta}}{dt} + \left(\frac{\partial \xi^{i,}}{\partial x_2^{\beta}} + C^{i,}_{\alpha,\beta}\, \xi^{\alpha,} \right) \frac{dx_2^{\beta}}{dt}$$

(13.8)

$$\frac{\delta \xi^{,i}}{\delta t} = \left(\frac{\partial \xi^{,i}}{\partial x_2^{\beta}} + \Gamma^{,i}_{,\alpha\beta}\, \xi^{,\alpha} \right) \frac{dx_2^{\beta}}{dt} + \left(\frac{\partial \xi^{,i}}{\partial x_1^{\beta}} + C^{,i}_{\alpha,\beta}\, \xi^{,\beta} \right) \frac{dx_1^{\alpha}}{dt} .$$

Since $\frac{\partial \xi^{i,}}{\partial x_2^{\beta}}$ and $\frac{dx_2^{\beta}}{dt}$ are vectors, we can again conclude that (13.3) defines a tensor (covariant derivative). We may write (13.8) as

(13.8')

$$\frac{\delta \xi^{i,}}{\delta t} = \xi^{i,}_{;\beta,} \frac{dx_1^{\beta}}{dt} + \xi^{i,}_{*,\beta} \frac{dx_2^{\beta}}{dt}$$

$$\frac{\delta \xi^{,i}}{\delta t} = \xi^{,i}_{;,\beta} \frac{dx_2^{\beta}}{dt} + \xi^{,i}_{*\beta,} \frac{dx_1^{\beta}}{dt} ,$$

where

(13.8*)

$$\xi^{i,}_{*,\beta} = \frac{\partial \xi^{i,}}{\partial x_2^{\beta}} + C^{i,}_{\alpha,\beta}\, \xi^{\alpha,} , \qquad \xi^{,i}_{*\beta,} = \frac{\partial \xi^{,i}}{\partial x_1^{\beta}} + C^{,i}_{\beta,\tau}\, \xi^{,\tau} .$$

<u>Example</u>.

$$g_{\alpha,\beta;\sigma,} = \frac{\partial g_{\alpha,\beta}}{\partial x_1^{\sigma}} - g_{\tau,\beta}\, \Gamma^{\tau,}_{\alpha\sigma,}$$

$$= \frac{\partial g_{\alpha,\beta}}{\partial x_1^{\sigma}} - g_{\tau,\beta}\, g^{\tau,\mu} \frac{\partial g_{\alpha,\mu}}{\partial x_1^{\sigma}} .$$

Now $g_{\alpha,\sigma}\, g^{\beta,\sigma} = \delta^{\beta,}_{\alpha,}$. Let $h_{\beta,\tau}$ be the inverse of $g^{\beta,\sigma}$ so that $g^{\beta,\sigma} h_{\beta,\tau} = \delta^{,\sigma}_{,\tau}$. Hence

$$g_{\alpha,\sigma}\, g^{\beta,\sigma} h_{\beta,\tau} = \delta^{\beta,}_{\alpha,}\, h_{\beta,\tau}$$

$$g_{\alpha,\tau} = h_{\alpha,\tau}$$

so that $g_{\tau,\beta}\, g^{\tau,\mu} = \delta^{,\mu}_{,\beta}$. Thus

(13.9) $\quad g_{\alpha,\beta;\sigma,} = \frac{\partial g_{\alpha,\beta}}{\partial x_1^{\sigma}} - \frac{\partial g_{\alpha,\beta}}{\partial x_1^{\sigma}} \equiv 0$, and similarly $g_{\alpha,\beta;,\sigma} \equiv 0$.

Returning to parallelism, we see that (11.12) may be expressed as follows.

(13.10)

$$\xi^{j,} \frac{\delta \xi^{i,}}{\delta t} = \xi^{i,} \frac{\delta \xi^{j,}}{\delta t}$$

$$\xi^{,j} \frac{\delta \xi^{,i}}{\delta t} = \xi^{,i} \frac{\delta \xi^{,j}}{\delta t}$$

where $\xi^{i,} = \xi^{i,}(x_1, x_2)$, $\xi^{,i} = \xi^{,i}(x_1, x_2)$.

We may re-express (13.10) as

$$\xi^{j_1}\left[\xi^{i_1}_{;\beta_1}\frac{dx_1^\beta}{dt} + \left(\frac{\partial \xi^{i_1}}{\partial x_2^\beta} + C^{i_1}_{\alpha,\beta}\,\xi^{\alpha_1}\right)\frac{dx_2^\beta}{dt}\right] = i,j \text{ interchanged}.$$

(13.11)

$$\xi^{,j}\left[\xi^{,i}_{;,\beta}\frac{dx_2^\beta}{dt} + \left(\frac{\partial \xi^{,i}}{\partial x_1^\alpha} + C^{,i}_{\alpha,\beta}\,\xi^{,\beta}\right)\frac{dx_1^\alpha}{dt}\right] = \text{''} \qquad \text{''} \quad .$$

If ξ^{i_1}, $\xi^{,i}$ are to be parallely displaced along <u>all</u> curves $x_1^i(t)$, $x_2^i(t)$, (13.11) shows that we must have

(13.12)

$$\begin{aligned}
\xi^{j_1}\xi^{i_1}_{;\beta_1} &= \xi^{i_1}\xi^{j_1}_{;\beta_1}\\
\xi^{,j}\xi^{,i}_{;,\beta} &= \xi^{,i}\xi^{,j}_{;,\beta}\\
\xi^{j_1}\xi^{i_1}_{*,\beta} &= \xi^{i_1}\xi^{j_1}_{*,\beta}\\
\xi^{,j}\xi^{,i}_{*\alpha_1} &= \xi^{,i}\xi^{,j}_{*\alpha_1} \quad .
\end{aligned}$$

From (13.12) it follows that $\xi^{i_1}_{;\beta_1}/\xi^{i_1}$ would have to be independent of <u>i</u> so that

(13.13)
$$\xi^{i_1}_{;\beta_1} = {}_1\psi_{\beta_1}\,\xi^{i_1} \quad .$$

It can easily be shown that ${}_1\psi_\beta$ is a covariant vector, i.e., ${}_1\psi_\beta = {}_1\psi_{\beta_1}(x_1,x_2)$. It follows that

<u>Theorem 15</u>. A necessary condition that ξ^{i_1}, $\xi^{,i}$ be parallely displaced along all curves is that covariant vectors ${}_1\psi_{\beta_1}$, ${}_2\psi_{,\beta}$ ${}_2\chi_{,\beta}$, ${}_1\chi_{\beta_1}$, shall exist such that

(13.14)

$$\begin{aligned}
\xi^{i_1}_{;\beta_1} &= {}_1\psi_{\beta_1}\,\xi^{i_1}\\
\xi^{,i}_{;,\beta} &= {}_2\psi_{,\beta}\,\xi^{,i}\\
\xi^{i_1}_{*,\beta} &= {}_2\chi_{,\beta}\,\xi^{i_1}\\
\xi^{,i}_{*\alpha_1} &= {}_1\chi_{\alpha_1}\,\xi^{,i} \quad .
\end{aligned}$$

If ${}_{1}\psi_{\beta_1}$ and ${}_{2}\psi_{,\beta}$ are gradients, then

$$\xi^{i_1}_{;\beta_1} = \frac{\partial \ln \phi_1}{\partial x_1^{\beta}}\, \xi^{i_1} \tag{13.15}$$

$$\xi^{,i}_{;,\beta} = \frac{\partial \ln \phi_2}{\partial x_2^{\beta}}\, \xi^{,i} ,$$

where

$${}_{1}\psi_{\beta_1} = \frac{\partial \ln \phi_1(x_1, x_2)}{\partial x_1^{\beta}} , \qquad {}_{2}\psi_{,\beta} = \frac{\partial \ln \phi_2(x_1, x_2)}{\partial x_2^{\beta}} .$$

Hence

$$\left(\xi^{i_1}/\phi_1\right)_{;\beta_1} = \frac{1}{\phi_1}\left(\xi^{i_1}_{;\beta_1} - \xi^{i_1}\frac{\partial \ln \phi_1}{\partial x_1^{\beta}}\right) = 0 \tag{13.16}$$

$$\left(\xi^{,i}/\phi_2\right)_{;,\beta} = 0 .$$

Moreover, if ${}_{2}\chi_{,\beta}$, ${}_{1}\chi_{\alpha_1}$, are also gradients, we obtain

$$\frac{\delta}{\delta t}\left(\frac{\xi^{i_1}}{\phi_1}\right) = \left(\frac{\xi}{\phi}\right)\frac{\partial \ln \frac{\sigma_2}{\phi_1}}{\partial x_2^{\beta}}\frac{dx_2^{\beta}}{dt} \tag{13.17}$$

$$\frac{\delta}{\delta t}\left(\frac{\xi^{,i}}{\phi_2}\right) = \left(\frac{\xi}{\phi_2}\right)\frac{\partial \ln \frac{\sigma_1}{\phi_2}}{\partial x_1^{\beta}}\frac{dx_1^{\beta}}{dt} ,$$

where

$${}_{2}\chi_{,\beta} = \frac{\partial \ln \sigma_2(x_1, x_2)}{\partial x_2^{\beta}} , \qquad {}_{1}\chi_{\beta_1} = \frac{\partial \ln \sigma_1(x_1, x_2)}{\partial x_1^{\beta}} .$$

It immediately follows from (13.17) that (13.10) will be satisfied. In general the quantity $\frac{\partial \ln \sigma_1/\phi_2}{\partial x_1^{\beta}}\frac{dx_1^{\beta}}{dt}$ will depend on the path so that the vectors ξ^{i_1}/ϕ_1 , $\xi^{,i}/\phi_2$, will not return to their initial values after being parallely displaced around an infinitesimal closed path. If, however, it happens that $\frac{\sigma_1}{\phi_2} = f(x_1)$, and $\frac{\sigma_2}{\phi_1} = g(x_2)$, then (13.17) may be written

$$\frac{\delta}{\delta t}\left(\frac{\xi^{i_1}}{\phi_1}\right) = \frac{\xi^{i_1}}{\phi_1}\frac{d \ln f}{dt} \tag{13.18}$$

$$\frac{\delta}{\delta t}\left(\frac{\xi^{,i}}{\phi_2}\right) = \frac{\xi^{,i}}{\phi_2}\frac{d \ln g}{dt}$$

so that

$$\frac{\delta}{\delta t}\left(\frac{\xi^{i_1}}{\phi_1 f}\right) = 0 \tag{13.19}$$

$$\frac{\delta}{\delta t}\left(\frac{\xi^{,i}}{\phi_2 g}\right) = 0 ,$$

for arbitrary curves $x_1^i(t)$, $x_2^i(t)$.

Thus the vectors $\bar{\xi}^{i,}(x_1,x_2) \equiv \frac{\xi^{i,}}{\phi_1 f}$, $\bar{\xi}^{,i}(x_1,x_2) \equiv \frac{\xi^{,i}}{\phi_2 g}$ satisfy

$$(13.20)\qquad \begin{aligned} &\frac{\partial \bar{\xi}^{i,}}{\partial x_1^{\beta}} + \Gamma^{i,}_{\alpha\beta,}\,\bar{\xi}^{\alpha,} = 0 \qquad && \frac{\partial \bar{\xi}^{i,}}{\partial x_2^{\beta}} + C^{i,}_{\alpha,\beta}\,\bar{\xi}^{\alpha,} = 0 \\ &\frac{\partial \bar{\xi}^{,i}}{\partial x_2^{\beta}} + \Gamma^{,i}_{,\alpha\beta}\,\bar{\xi}^{,\alpha} = 0 \qquad && \frac{\partial \bar{\xi}^{,i}}{\partial x_1^{\beta}} + C^{,i}_{\beta,\alpha}\,\bar{\xi}^{,\alpha} = 0 \,. \end{aligned}$$

Since equations (13.20) are satisfied we must have

$$(13.21)\qquad \begin{aligned} \frac{\partial^2 \bar{\xi}^{i,}}{\partial x_1^{\beta}\partial x_1^{\sigma}} &= \frac{\partial^2 \bar{\xi}^{i,}}{\partial x_1^{\sigma}\partial x_1^{\beta}} \\ \frac{\partial^2 \bar{\xi}^{i,}}{\partial x_2^{\beta}\partial x_2^{\sigma}} &= \frac{\partial^2 \bar{\xi}^{i,}}{\partial x_2^{\sigma}\partial x_2^{\beta}} \\ &\vdots \\ \frac{\partial^2 \bar{\xi}^{,i}}{\partial x_1^{\beta}\partial x_2^{\sigma}} &= \frac{\partial^2 \bar{\xi}^{,i}}{\partial x_2^{\sigma}\partial x_1^{\beta}} \,, \end{aligned}$$

and these equations yield

$$(13.22)\qquad \bar{\xi}^{\alpha,} R^{i,}_{\alpha\beta\sigma,} = \bar{\xi}^{\alpha,} S^{i,}_{\alpha,\beta\sigma} = \cdots = \bar{\xi}^{,\alpha} T^{,i}_{\sigma,\alpha\beta} = 0 \,.$$

If (13.22) holds for arbitrary values of $\bar{\xi}^{\alpha,}$, $\bar{\xi}^{,\alpha}$, at M_1, M_2, we see that the space is flat and that the vectors $\bar{\xi}^{i,}$, $\bar{\xi}^{,i}$, when subjected to parallel displacements around infinitesimal closed paths, will return to their initial values. So $\xi^{i,} \equiv \phi_1 f\, \bar{\xi}^{i,}$ and $\xi^{,i} \equiv \phi_2 g\, \bar{\xi}^{,i}$ will do the same. Conversely, if the space is flat we can solve (13.20) for the $\xi^{i,}$, $\xi^{,i}$ uniquely with arbitrary initial conditions at M_1, M_2.

<u>Definition.</u> A set of vectors $\xi^{i,}(x_1,x_2)$, $\xi^{,i}(x_1,x_2)$, satisfying

$$(13.23)\qquad \begin{aligned} \xi^{i,}_{;\beta,} &\equiv \frac{\partial \xi^{i,}}{\partial x_1^{\beta}} + \Gamma^{i,}_{\alpha\beta,}\,\xi^{\alpha,} = 0 \\ \xi^{,i}_{,;\beta} &\equiv \frac{\partial \xi^{,i}}{\partial x_2^{\beta}} + \Gamma^{,i}_{,\alpha\beta}\,\xi^{,\alpha} = 0 \\ \xi^{i,}_{*,\beta} &\equiv \frac{\partial \xi^{i,}}{\partial x_2^{\beta}} + C^{i,}_{\alpha,\beta}\,\xi^{\alpha,} = \frac{\partial \ln \phi_2(x_2)}{\partial x_2^{\beta}}\,\xi^{i,} \\ \xi^{,i} &\quad \partial \xi^{,i} \quad \Gamma^{,i}\,\xi^{,\alpha} = \frac{\partial \ln \phi_1(x_1)}{\quad}\,\xi^{,i} \end{aligned}$$

are said to form parallel vector fields relative to the points of M_1, M_2. The substitution

$$(13.24)\qquad \begin{aligned} \bar{\xi}^{1i}(x_1,x_2) &= \xi^{1i}(x_1,x_2)/\phi_2(x_2) \\ \bar{\xi}^{2i}(x_1,x_2) &= \xi^{2i}(x_1,x_2)/\phi_1(x_1) \end{aligned}$$

reduces (13.23) to (13.20), so that if the parallel vector fields exist (with arbitrary initial values), the space is necessarily flat. Conversely, if the space is flat we can solve (13.23) for ξ^{1i}, ξ^{2i} by choosing $\phi_1 = \phi_2 = 1$. We have now proved

<u>Theorem 16.</u> A necessary and sufficient condition that there exist a set of contravariant vectors forming parallel vector fields with arbitrary components at the fixed points M_1, M_2, is that the space be flat.

Let ξ^{1i}, ξ^{2i} satisfy (13.23) and let us determine the curves $x_1^i(t)$, $x_2^i(t)$ by solving

$$(13.23^*)\qquad \frac{dx_1^1}{\xi^{11}} = \frac{dx_1^2}{\xi^{12}} = \cdots = \frac{dx_1^n}{\xi^{1n}} = \frac{dx_2^1}{\xi^{21}} = \cdots = \frac{dx_2^n}{\xi^{2n}} = dt .$$

Then multiplying (13.23) by $\frac{dx_1^\beta}{dt}$, $\frac{dx_2^\beta}{dt}$, and adding, replacing ξ^{1i} by $\frac{dx_1^i}{dt}$, ξ^{2i} by $\frac{dx_2^i}{dt}$ from (13.23*), we obtain

$$(13.24^*)\qquad \begin{aligned} \frac{d^2x_1^i}{dt^2} + \Gamma^{1i}_{\alpha\beta,}\frac{dx_1^\alpha}{dt}\frac{dx_1^\beta}{dt} + C^{1i}_{\alpha,\beta}\frac{dx_1^\alpha}{dt}\frac{dx_2^\beta}{dt} &= \frac{d\ln\phi_2(x_2(t))}{dt}\frac{dx_1^i}{dt} \\ \frac{d^2x_2^i}{dt^2} + \Gamma^{2i}_{2\alpha\beta}\frac{dx_2^\alpha}{dt}\frac{dx_2^\beta}{dt} + C^{2i}_{\alpha,\beta}\frac{dx_1^\alpha}{dt}\frac{dx_2^\beta}{dt} &= \frac{d\ln\phi_1(x_1(t))}{dt}\frac{dx_2^i}{dt}, \end{aligned}$$

so that from § 8 we have the following.

<u>Theorem 17.</u> The curves of a congruence of curves determined by a field of parallel vectors are dyodesics.

Returning to (13.22) we see that if 2n independent vectors ${}_\sigma\xi^{1i}$, ${}_\sigma\xi^{2i}$ exist, the index σ ranging from 1 to n and denoting the different vectors, such that (13.22) is satisfied, then the space is flat, for we may consider $\xi^{\alpha'} R^{i'}_{\alpha\beta\gamma'} = 0$ as n equations in the n unknowns $R_\alpha = R^{i'}_{\alpha\beta\gamma'}$, with determinant $|{}_\sigma\xi^{\alpha'}| \neq 0$ from the linear independence of the ${}_\sigma\xi^{\alpha'}$. Thus the only solution is $R_\alpha = R^{i'}_{\alpha\beta\gamma'} = 0$. Also $S^{i'}_{\alpha,\beta\sigma} = \cdots = T^{,i'}_{\alpha,\beta\sigma} = 0$. Hence

Theorem 18. A necessary and sufficient condition that the space admit n linearly independent parallel vectors $\xi^{i'}$ along with n linearly independent parallel vectors $\xi^{,i'}$ is that the space be flat.

The n linearly independent vectors ${}_\alpha\lambda^{i'}(M_1, M_2)$ are said to form an ennuple. Let us now define n linearly independent covariant vectors ${}^\alpha\lambda_{i'}$ by the equations

(13.25) $${}_\alpha\lambda^{i'}\, {}^\beta\lambda_{i'} = \delta^{\beta'}_{\alpha'} .$$

Moreover, $${}_\alpha\lambda^{i'}\, {}^\alpha\lambda_{j'} = \delta^{i'}_{j'}$$

(13.26)

for define $C^{i'}_{j'}$ by

(13.27) $$C^{i'}_{j'} = {}_\alpha\lambda^{i'}\, {}^\alpha\lambda_{j'}$$

and multiply by ${}^\beta\lambda_{i'}$ to obtain

(13.28) $${}^\beta\lambda_{i'}\, C^{i'}_{j'} = {}^\beta\lambda_{j'} ,$$

so that

(13.29) $$C^{i'}_{j'} = \delta^{i'}_{j'} \qquad \text{since} \qquad |{}^\beta\lambda_{i'}| \neq 0 .$$

Similarly, let ${}_\alpha\mu^{,i'}(M_1,M_2)$, ${}^\alpha\mu_{,i'}(M_1,M_2)$ be linearly independent vectors. Now let

(13.30) $$a^{i_1\cdots i_r,\, k_1\cdots k_t}_{j_1\cdots j_s,\, l_1\cdots l_u}(M_1, M_2)$$

denote the components of any mixed two-point tensor field.

The quantities

(13.31) $$C^{\alpha_1\cdots\alpha_r,\, \sigma_1\cdots\sigma_t}_{\beta_1\cdots\beta_s,\, \tau_1\cdots\tau_u}(M_1,M_2) = a^{i_1\cdots i_r,\, k_1\cdots k_t}_{j_1\cdots j_s,\, l_1\cdots l_u}\, {}^{\alpha_1}\lambda_{i_1} \cdots {}_{\beta_s}\lambda^{j_s}\, {}^{\sigma_1}\mu_{,k_1} \cdots {}_{\tau_u}\mu^{,l_u}$$

are invariants, i.e., $\bar{C}^{\cdots}_{\cdots} = C^{\cdots}_{\cdots}$.

Multiplying by ${}_{\alpha_1}\lambda^{i_1}$, ... , etc., we obtain

(13.32) $$a^{i_1\cdots i_n,\,k_1\cdots k_t}_{j_1\cdots j_s,\,l_1\cdots l_u} = C^{\alpha_1\cdots\alpha_n,\,\sigma_1\cdots\sigma_t}_{\beta_1\cdots\beta_s,\,\tau_1\cdots\tau_u}\ {}_{\alpha}\lambda^{i_1}\cdots{}^{\tau_u}\mu_{,l_u},$$

so that the components of any two-point tensor are expressible in terms of invariants and the components of an ennuple (contravariant, covariant) at M_1, and the components of an ennuple (contravariant, covariant) at M_2.

We now define the invariants (not to be confused with tensors) $\gamma^{\nu_1}_{\mu\sigma_1}(M_1,M_2)$, $\gamma^{,\nu}_{,\mu\sigma}(M_1,M_2)$, by the equations

(13.33) $$\begin{aligned}\gamma^{\nu_1}_{\tau\sigma_1} &= {}_{\tau}\lambda^{i_1}_{;j_1}\ {}^{\nu}\lambda_{i_1}\ {}_{\sigma}\lambda^{j_1}\\ \gamma^{,\nu}_{,\tau\sigma} &= {}_{\tau}\mu^{,i}_{;,j}\ {}^{\nu}\mu_{,i}\ {}_{\sigma}\mu^{,j}.\end{aligned}$$

These are the generalizations of Ricci's coefficients of rotation. From (13.26) we have

(13.34) $$\begin{aligned}{}_{\tau}\lambda^{i_1}_{;j_1} &= \gamma^{\nu_1}_{\tau\sigma_1}\ {}_{\nu}\lambda^{i_1}\ {}^{\sigma}\lambda_{j_1}\\ {}_{\tau}\mu^{,i}_{;,j} &= \gamma^{,\nu}_{,\tau\sigma}\ {}_{\nu}\mu^{,i}\ {}^{\sigma}\mu_{,j}.\end{aligned}$$

Now

(13.35) $${}_{\tau}\lambda^{i_1}_{;j_1} = \frac{\partial\,{}_{\tau}\lambda^{i_1}}{\partial x_1^{j}} + {}_{\tau}\lambda^{\alpha_1}\,\Gamma^{i_1}_{\alpha j_1},$$

so that on multiplying by ${}^{\tau}\lambda_{\sigma_1}$ we obtain

(13.36) $$\begin{aligned}\Gamma^{i_1}_{\sigma j_1} &= {}^{\tau}\lambda_{\sigma_1}\ {}_{\tau}\lambda^{i_1}_{;j_1} - {}^{\tau}\lambda_{\sigma_1}\,\frac{\partial_{\tau}\lambda^{i_1}}{\partial x_1^{j}}\\ &= -\,{}^{\tau}\lambda_{\sigma_1}\,\frac{\partial_{\tau}\lambda^{i_1}}{\partial x_1^{j}} + \gamma^{\nu_1}_{\tau\sigma_1}\ {}_{\nu}\lambda^{i_1}\ {}^{\sigma}\lambda_{j_1}\ {}^{\tau}\lambda_{\sigma_1}.\end{aligned}$$

A similar result holds for $\Gamma^{,i}_{,\sigma j}$.

Conversely, given an invariant $\gamma^{\nu_1}_{\tau\sigma_1}$, and defining $\Gamma^{i_1}_{\sigma j_1}$ by (13.36) for all coordinate systems, it can be shown that the $\Gamma^{i_1}_{\sigma j_1}$ transform in the usual manner. Hence any ennuple of independent vectors and any set of invariants $\gamma^{\nu,}_{\tau\sigma,}$ determine a linear connection $\Gamma^{\iota,}_{\sigma\delta,}$, and any linear connection is so determined from (13.36).

We now consider the $g_{\alpha,\beta}$ defined by

$$g_{\alpha,\beta}(x_1, x_2) = \sum_{\sigma=1}^{n} {}^{\sigma}\lambda_{\alpha,}(x_1)\; {}^{\sigma}\mu_{,\beta}(x_2) \tag{13.37}$$

where ${}^{\sigma}\lambda_{\alpha,}$, ${}^{\sigma}\mu_{,\beta}$ are ennuples at M_1, M_2, respectively, see (4.6). Using (13.25) we see that $g^{\alpha,\beta}$ defined by

$$g^{\alpha,\beta} = \sum_{\tau=1}^{n} {}_{\tau}\lambda^{\alpha,}\; {}_{\tau}\mu^{,\beta} \tag{13.38}$$

yields $g^{\alpha,\sigma} g_{\beta,\sigma} = \delta^{\alpha,}_{\beta,}$. Hence

$$\begin{aligned} \Gamma^{\iota,}_{\sigma\delta,} &\equiv g^{\iota,\rho} \frac{\partial g_{\sigma,\rho}}{\partial x_1^{\delta}} \\ &= \sum_{\tau,\phi=1}^{n} {}_{\tau}\lambda^{\iota,}\; {}_{\tau}\mu^{,\rho} \frac{\partial\, {}^{\phi}\lambda_{\sigma,}}{\partial x_1^{\delta}}\; {}^{\phi}\mu_{,\rho} \\ &= {}_{\phi}\lambda^{\iota,} \frac{\partial\, {}^{\phi}\lambda_{\sigma,}}{\partial x_1^{\delta}} = -\,{}^{\phi}\lambda_{\sigma,} \frac{\partial\, {}_{\phi}\lambda^{\iota}}{\partial x_1^{\delta}} , \end{aligned} \tag{13.39}$$

by differentiating ${}_{\phi}\lambda^{\iota,}\; {}^{\phi}\lambda_{\sigma,} = \delta^{\iota,}_{\sigma,}$. Comparing with (13.36), (13.34) we see that

$$ {}_{\tau}\lambda^{\iota,}_{;\delta,} = 0 . \tag{13.40}$$

From (13.21), (13.22) we conclude that the Riemann tensors $R^{\iota,}_{\alpha\beta\delta,}$, $R^{,\iota}_{,\alpha\beta\delta}$ must vanish for the metric defined by (13.37).

More generally than (13.37) we may consider the metric $g_{\alpha,\beta}$ defined by

$$g_{\alpha,\beta} = C_{\epsilon\phi}\; {}^{\epsilon}\lambda_{\alpha,}(x_1)\; {}^{\phi}\mu_{,\beta}(x_2) \tag{13.41}$$

where the $C_{\epsilon\phi}$ are constants such that the form $C_{\epsilon\phi}A^{\epsilon}A^{\phi}$ is positive definite. It is easy to show that the $g_{\alpha,\beta}$ transform like a two-point tensor. We define

(13.42) $$g^{\alpha,\beta} = C^{\sigma\tau}{}_{\sigma}\lambda^{\alpha_1}{}_{\tau}\mu^{,\beta},$$

where $C^{\sigma\tau}C_{\sigma\mu} = \delta^{\tau}_{\mu}$, so that $g^{\alpha,\sigma}g_{\beta,\sigma} = \delta^{\alpha_1}_{\beta_1}$. It is easy to show that

(13.43) $$\Gamma^{i_1}_{\sigma j_1} = -{}^{\psi}\lambda_{\sigma_1}\frac{\partial_{\psi}\lambda^{i_1}}{\partial x_1{}^{j}}$$

so that $R^{i_1}_{jk\ell_1} = 0$. Similarly $R^{,i}_{,jk\ell} = 0$.

Let us consider the ennuples $\frac{\partial\,{}^{i}\theta(x_1)}{\partial x_1{}^{\alpha}}$, $\frac{\partial\,{}^{i}\phi(x_2)}{\partial x_2{}^{\alpha}}$ and consider the metric

(13.44) $$g_{\alpha,\beta} = C_{ij}\frac{\partial\,{}^{i}\theta(x_1)}{\partial x_1{}^{\alpha}}\frac{\partial\,{}^{j}\phi(x_2)}{\partial x_2{}^{\beta}}$$
$$= \frac{\partial^2\chi}{\partial x_1{}^{\alpha}\partial x_2{}^{\beta}}$$

where $\chi = C_{ij}\,{}^{i}\theta(x_1)\,{}^{j}\phi(x_2)$.

From Theorems 6,7 we conclude that $C^{i_1}_{\alpha,\beta} = C^{,i}_{\alpha,\beta} = 0$.

Hence, see (12.7)

(13.45)
$$S^{i_1}_{\alpha,\beta\sigma} = S^{,i}_{\alpha\beta,\sigma} = 0$$
$$T^{i_1}_{\alpha\beta,\sigma} = T^{,i}_{\alpha,\beta\sigma} = 0,$$

since $\Gamma^{i_1}_{\alpha\beta_1} = \Gamma^{i_1}_{\alpha\beta_1}(x_1)$, see (13.43), which implies

(13.46) $$\frac{\partial\Gamma^{i_1}_{\alpha\beta_1}}{\partial x_2{}^{\sigma}} = 0, \quad \frac{\partial\Gamma^{,i}_{,\alpha\beta}}{\partial x_1{}^{\sigma}} = 0.$$

Hence the space is flat, so that from Theorem 16 we conclude that $\frac{\partial\,{}^{i}\theta(x_1)}{\partial x_1{}^{\alpha}}$, $\frac{\partial\,{}^{i}\phi(x_2)}{\partial x_2{}^{\alpha}}$ form parallel vector fields for a space defined by (13.45).

XIV. Parallel displacement in sub-spaces.

Let us consider the two distinct points M_1, M_2 of an n-dimensional space admitting the metric $g_{\alpha,\beta}(M_1,M_2)$. Let x_1 be a coordinate system for a neighborhood of M_1 and let x_2 be a coordinate system for a neighborhood of M_2. We now consider the transformations

$$\begin{aligned} x_1^{\alpha} &= x_1^{\alpha}(y_1^1, y_1^2, \cdots, y_1^m) \\ x_2^{\alpha} &= x_2^{\alpha}(y_2^1, y_2^2, \cdots, y_2^m), \qquad \alpha = 1, 2, \cdots, n \;;\; m < n. \end{aligned} \tag{14.1}$$

Each set $(y_1^1, y_1^2, \cdots, y_1^m)$ determines a point $(x_1^1, x_1^2, \cdots, x_1^n)$, but not conversely, since $m < n$, so that we may consider the allowable totality of $(y_1^1, y_1^2, \cdots, y_1^m)$ as determining a sub-space of the enveloping space described by x_1 in the neighborhood of M_1. Similarly the set $(y_2^1, y_2^2, \cdots, y_2^m)$ determines a sub-space of the same enveloping space in the neighborhood of M_2. There is obviously no loss in generality in assuming that the two sub-spaces are part of a single sub-space, elements in the neighborhood of M_1 having components $(y_1^1, y_1^2, \cdots, y_1^m)$, and elements in the neighborhood of M_2 having components $(y_2^1, y_2^2, \cdots, y_2^m)$.

We now define $h_{i,j}(y_1, y_2)$, the metric for the sub-space, by the equation

$$ds^2 = g_{\alpha,\beta}(x_1, x_2)\, dx_1^{\alpha}\, dx_2^{\beta} = h_{i,j}\, dy_1^{i}\, dy_2^{j}, \tag{14.2}$$

under the transformations (14.1).

We shall use Greek letters for the n-dimensional space and Latin letters for the m-dimensional sub-space given by (14.1).

From (14.2) we have

$$(14.3)\qquad g_{\alpha,\beta}\frac{\partial x_1^{\alpha}}{\partial y_1^{i}}\frac{\partial x_2^{\beta}}{\partial y_2^{j}}dy_1^{i}dy_2^{j} = h_{i,j}\,dy_1^{i}dy_2^{j},$$

so that

$$(14.4)\qquad h_{i,j}(y_1,y_2) = g_{\alpha,\beta}(x_1,x_2)\frac{\partial x_1^{\alpha}}{\partial y_1^{i}}\frac{\partial x_2^{\beta}}{\partial y_2^{j}}.$$

The dyodesics in the sub-space are obtained by extremalizing

$$(14.5)\qquad \int_{t_0}^{t_1}\left(e\,h_{i,j}\frac{dy_1^{i}}{dt}\frac{dy_2^{j}}{dt}\right)^{1/2}dt.$$

From the method of obtaining the dyodesics we immediately see that the differential equations of the dyodesics will have the same form as (8.1), with the linear connections and tensors of (8.1) replaced by the same symbols obtained from the metric tensor $h_{i,j}(y_1,y_2)$. Let us now find the relationship between these quantities. Differentiating (14.3) we obtain

$$(14.6)\qquad \frac{\partial h_{i,j}}{\partial y_1^{k}} = \frac{\partial g_{\alpha,\mu}}{\partial x_1^{\beta}}\frac{\partial x_1^{\beta}}{\partial y_1^{k}}\frac{\partial x_1^{\alpha}}{\partial y_1^{i}}\frac{\partial x_2^{\mu}}{\partial y_2^{j}} + g_{\alpha,\beta}\frac{\partial^2 x_1^{\alpha}}{\partial y_1^{k}\partial y_1^{i}}\frac{\partial x_2^{\beta}}{\partial y_2^{j}}.$$

Hence

$$(14.7)\qquad \begin{aligned} h_{r,s}\Gamma^{r}_{ik,} &= h_{r,s}h^{r,j}\frac{\partial h_{i,j}}{\partial y_1^{k}} = \frac{\partial h_{i,s}}{\partial y_1^{k}} \\ &= g_{\tau,\mu}\Gamma^{\tau}_{\alpha\beta,}\frac{\partial x_1^{\beta}}{\partial y_1^{k}}\frac{\partial x_1^{\alpha}}{\partial y_1^{i}}\frac{\partial x_2^{\mu}}{\partial y_2^{s}} + g_{\alpha,\beta}\frac{\partial^2 x_1^{\alpha}}{\partial y_1^{k}\partial y_1^{i}}\frac{\partial x_2^{\beta}}{\partial y_2^{s}}. \end{aligned}$$

Moreover,

$$(14.8)\qquad \frac{\partial h_{i,j}}{\partial y_2^{k}} - \frac{\partial h_{i,k}}{\partial y_2^{j}} = \left(\frac{\partial g_{\alpha,\mu}}{\partial x_2^{\nu}} - \frac{\partial g_{\alpha,\nu}}{\partial x_2^{\mu}}\right)\frac{\partial x_2^{\nu}}{\partial y_2^{k}}\frac{\partial x_1^{\alpha}}{\partial y_1^{i}}\frac{\partial x_2^{\mu}}{\partial y_2^{j}},$$

so that

$$(14.9)\qquad h_{r,s}C^{r,}_{i,j}(y_1,y_2) = g_{\tau,\beta}C^{\tau,}_{\alpha,\mu}(x_1,x_2)\frac{\partial x_2^{\beta}}{\partial y_2^{s}}\frac{\partial x_1^{\alpha}}{\partial y_1^{i}}\frac{\partial x_2^{\mu}}{\partial y_2^{j}}.$$

We are now in a position to find the relationship between the intrinsic derivatives of a vector as measured by observers in the full space and observers in the sub-space. Let a^{i_1} be the components of a contravariant vector in the sub-space and let A^{α_1} be its components in the full space. We have

(14.10) $$A^{\alpha_1}(x_1, x_2) = a^{i_1}(y_1, y_2)\, \frac{\partial x_1^{\alpha}}{\partial y_1^{i}} ,$$

so that

(14.11) $$\frac{dA^{\alpha_1}}{dt} = \frac{da^{i_1}}{dt}\frac{\partial x_1^{\alpha}}{\partial y_1^{i}} + a^{i_1}\frac{\partial^2 x_1^{\alpha}}{\partial y_1^{j}\partial y_1^{i}}\frac{dy_1^{j}}{dt} .$$

From (13.1) we have

(14.12) $$\frac{\delta A^{\alpha_1}}{\delta t} = \frac{da^{i_1}}{dt}\frac{\partial x_1^{\alpha}}{\partial y_1^{i}} + a^{i_1}\frac{\partial^2 x_1^{\alpha}}{\partial y_1^{j}\partial y_1^{i}}\frac{dy_1^{j}}{dt} + \Gamma^{\alpha_1}_{\mu\nu_1} A^{\mu_1}\frac{dx_1^{\nu}}{dt} + C^{\alpha_1}_{\mu,\nu} A^{\mu_1}\frac{dx_2^{\nu}}{dt}$$

$$= \frac{da^{i_1}}{dt}\frac{\partial x_1^{\alpha}}{\partial y_1^{i}} + a^{i_1}\frac{\partial^2 x_1^{\alpha}}{\partial y_1^{j}\partial y_1^{i}}\frac{dy_1^{j}}{dt} + \Gamma^{\alpha_1}_{\mu\nu_1}\frac{\partial x_1^{\mu}}{\partial y_1^{i}} a^{i_1}\frac{\partial x_1^{\nu}}{\partial y_1^{j}}\frac{dy_1^{j}}{dt} + C^{\alpha_1}_{\mu,\nu} a^{i_1}\frac{\partial x_1^{\mu}}{\partial y_1^{i}}\frac{\partial x_2^{\nu}}{\partial y_2^{j}}\frac{dy_2}{dt}$$

Hence

(14.13) $$g_{\alpha,\beta}\frac{\partial x_2^{\beta}}{\partial y_2^{a}}\frac{\delta A^{\alpha_1}}{\delta t} = \frac{da^{i_1}}{dt} g_{\alpha,\beta}\frac{\partial x_1^{\alpha}}{\partial y_1^{i}}\frac{\partial x_2^{\beta}}{\partial y_2^{a}}$$

$$+ a^{i_1}\frac{dy_1^{j}}{dt}\left[\frac{\partial^2 x_1^{\alpha}}{\partial y_1^{j}\partial y_1^{i}} g_{\alpha,\beta}\frac{\partial x_2^{\beta}}{\partial y_2^{a}} + \Gamma^{\gamma_1}_{\alpha\beta_1} g_{\gamma,\mu}\frac{\partial x_2^{\mu}}{\partial y_2^{a}}\frac{\partial x_1^{\alpha}}{\partial y_1^{i}}\frac{\partial x_1^{\beta}}{\partial y_1^{j}}\right]$$

$$+ g_{\alpha,\beta} C^{\alpha_1}_{\mu,\nu}\frac{\partial x_1^{\mu}}{\partial y_1^{i}}\frac{\partial x_2^{\nu}}{\partial y_2^{j}}\frac{dy_2^{j}}{dt}\frac{\partial x_2^{\beta}}{\partial y_2^{a}} a^{i_1} ,$$

and

(14.14) $$g_{\alpha,\beta}\frac{\partial x_2^{\beta}}{\partial y_2^{a}}\frac{\delta A^{\alpha_1}}{\delta t} = h_{i,a}\frac{da^{i_1}}{dt} + a^{r_1}\frac{dy_1^{j}}{dt} h_{i,a}\Gamma^{i_1}_{rj_1} + a^{r_1} C^{i_1}_{r,j} h_{i,a}\frac{dy_2^{j}}{dt}$$

$$= h_{i,a}\left[\frac{da^{i_1}}{dt} + \Gamma^{i_1}_{rj_1} a^{r_1}\frac{dy_1^{j}}{dt} + C^{i_1}_{r,j} a^{r_1}\frac{dy_2^{j}}{dt}\right]$$

$$= h_{i,a}\frac{\delta a^{i_1}}{\delta t} ,$$

by making use of (14.7), (14.9).

Similarly,

$$g_{\alpha,\beta} \frac{\partial x_1^{\alpha}}{\partial y_1^{s}} \frac{\delta A^{,\beta}}{\delta t} = h_{s,i} \frac{\delta a^{,i}}{\delta t} . \tag{14.15}$$

From (14.14), (14.15) we immediately have the following.

Theorem 19. If a vector field in a sub-space is parallely displaced along a pair of paths in the full space, and if the pair of paths lie in the sub-space, then the vector field is parallely displaced along the paths relative to the sub-space.

Theorem 20. If a pair of paths are dyodesics in the full space, they are also dyodesics in any sub-space in which they lie.

These theorems follow from the fact that $\frac{\delta A^{\beta,}}{\delta t} = 0$ implies $\frac{\delta a^{i,}}{\delta t} = 0$, see (14.14).

XV. Generalized covariant differentiation.

Let V_m be a sub-space of V_n , see §14, and let C_1 , C_2 be any two curves in V_m .

$$C_1 : \quad y_1^i = y_1^i(s) \quad ; \quad C_2 : \quad y_2^i = y_2^i(s) . \tag{15.1}$$

In V_n the curves are $x_1^{\alpha}(s)$, $x_2^{\alpha}(s)$. Now let $A_{\alpha,}(x_1, x_2)$, $A_{,\alpha}(x_1, x_2)$, $B^{\alpha,}(x_1, x_2)$, $B^{,\alpha}(x_1, x_2)$, be vector fields which are parallely displaced along C_1, C_2 relative to V_n. Similarly, let $C^{i,}(y_1, y_2)$, $C^{,i}(y_1, y_2)$ be vector fields in V_m parallely displaced along C_1 , C_2, respectively, relative to V_m. From (11.1), (11.6) we have

$$\begin{aligned} &\frac{dA_{\alpha,}}{ds} - \Gamma^{\sigma,}_{\alpha\beta,} A_{\sigma,} \frac{dx_1^{\beta}}{dt} - C^{\sigma,}_{\alpha,\beta} A_{\sigma,} \frac{dx_2^{\beta}}{dt} = 0 \\ &\frac{dB^{\sigma,}}{ds} + \Gamma^{\sigma,}_{\alpha\beta,} B^{\alpha,} \frac{dx_1^{\beta}}{ds} + C^{\sigma,}_{\alpha,\beta} B^{\alpha,} \frac{dx_2^{\beta}}{ds} = 0 \\ &\frac{dC^{i,}}{ds} + \Gamma^{i,}_{jk,} C^{j,} \frac{dy_1^{k}}{ds} + C^{i,}_{j,k} C^{j,} \frac{dy_2^{k}}{ds} = 0 , \end{aligned} \tag{15.2}$$

with similar equations for the $A_{,\alpha}$, $B^{,\sigma}$, $C^{,i}$.

Let $D^{\alpha_1}_{\sigma i_1}$ be a tensor field defined along C_1, C_2, such that $D^{\alpha_1}_{\sigma i_1}$ is a mixed tensor field in V_n relative to x_1 and a scalar relative to x_2, a covariant vector in V_m relative to y_1, a scalar relative to y_2. The product $A_{\alpha_1} B^{\sigma_1} C^{i_1} D^{\alpha_1}_{\sigma i_1}$ is a scalar invariant function of s along C_1 and C_2. Its derivative with respect to s is also a scalar invariant. Differentiating this product we obtain

$$(15.3) \qquad \frac{d}{ds}\left(A_{\alpha_1} B^{\sigma_1} C^{i_1} D^{\alpha_1}_{\sigma i_1}\right) = \frac{dA_{\alpha_1}}{ds} B^{\sigma_1} C^{i_1} D^{\alpha_1}_{\sigma i_1} + \frac{dB^{\sigma_1}}{ds} A_{\alpha_1} C^{i_1} D^{\alpha_1}_{\sigma i_1} + \frac{dC^{i_1}}{ds} A_{\alpha_1} B^{\sigma_1} D^{\alpha_1}_{\sigma i_1} + \frac{dD^{\alpha_1}_{\sigma i_1}}{ds} A_{\alpha_1} B^{\sigma_1} C^{i_1},$$

and using (15.2) we obtain

$$(15.4) \qquad \frac{d}{ds}\left(A_{\alpha_1} B^{\sigma_1} C^{i_1} D^{\alpha_1}_{\sigma i_1}\right) = A_{\alpha_1} B^{\sigma_1} C^{i_1} \Big[\frac{dD^{\alpha_1}_{\sigma i_1}}{ds} + D^{\mu_1}_{\sigma i_1}\left(\Gamma^{\alpha_1}_{\mu\beta_1} \frac{dx_1^\beta}{ds} + C^{\alpha_1}_{\mu_1\beta} \frac{dx_2^\beta}{ds}\right) - D^{\alpha_1}_{\mu i_1}\left(\Gamma^{\mu_1}_{\sigma\beta_1} \frac{dx_1^\beta}{ds} + C^{\mu_1}_{\sigma_1\beta} \frac{dx_2^\beta}{ds}\right) - D^{\alpha_1}_{\sigma j_1}\left(\Gamma^{j_1}_{ik_1} \frac{dy_1^k}{ds} + C^{j_1}_{i_1k} \frac{dy_2^k}{ds}\right)\Big].$$

From the quotient law we have that

$$(15.5) \qquad \frac{dD^{\alpha_1}_{\sigma i_1}}{ds} + \Gamma^{\alpha_1}_{\mu\beta_1} D^{\mu_1}_{\sigma i_1} \frac{dx_1^\beta}{ds} - \Gamma^{\mu_1}_{\sigma\beta_1} D^{\alpha_1}_{\mu i_1} \frac{dx_1^\beta}{ds} - \Gamma^{j_1}_{ik_1} D^{\alpha_1}_{\sigma j_1} \frac{dy_1^k}{ds}$$

is a tensor of the same type as $D^{\alpha_1}_{\sigma i_1}$. (15.5) may be written

$$(15.6) \qquad \left(\frac{\partial D^{\alpha_1}_{\sigma i_1}}{\partial y_1^k} + \Gamma^{\alpha_1}_{\mu\beta_1} D^{\mu_1}_{\sigma i_1} \frac{\partial x_1^\beta}{\partial y_1^k} - \Gamma^{\mu_1}_{\sigma\beta_1} D^{\alpha_1}_{\mu i_1} \frac{\partial x_1^\beta}{\partial y_1^k} - \Gamma^{j_1}_{ik_1} D^{\alpha_1}_{\sigma j_1}\right)\frac{dy_1^k}{ds} + \frac{\partial D^{\alpha_1}_{\sigma i_1}}{\partial y_2^k}\frac{dy_2^k}{ds},$$

and since $\frac{dy_1^k}{ds}$, $\frac{\partial D^{\alpha_1}_{\sigma i_1}}{\partial y_2^k}$ $\frac{dy_2^k}{ds}$ are tensors, the expression in parenthesis, denoted by $\underline{D^{\alpha_1}_{\sigma i_1 : k_1}}$ is a covariant tensor of second order in the y's. We call it the tensor derivative of $D^{\alpha_1}_{\sigma i_1}$ with respect to y_1^k, the colon representing this type of generalized covariant differentiation.

Definition.

(15.7) $$D^{\alpha_1}_{\sigma i_1 : k_1} \equiv \frac{\partial D^{\alpha_1}_{\sigma i_1}}{\partial y_1^k} + \Gamma^{\alpha_1}_{\mu\beta_1} D^{\mu_1}_{\sigma i_1} \frac{\partial x_1^\beta}{\partial y_1^k} - \Gamma^{\mu_1}_{\sigma\beta_1} D^{\alpha_1}_{\mu i_1} \frac{\partial x_1^\beta}{\partial y_1^k} - \Gamma^{j_1}_{ik_1} D^{\alpha_1}_{\sigma j_1} .$$

Example. x_1^α is an invariant for transformations of $y_1^1, \cdots, y_1^n$.

Hence

(15.8) $$x^\alpha_{1 : i_1} = \frac{\partial x_1^\alpha}{\partial y_1^i} \qquad \text{for each } \alpha .$$

For each α, $x^\alpha_{1:i_1}$ is a covariant vector relative to y_1.

Hence

(15.9) $$x^\alpha_{1 : ij_1} = \frac{\partial^2 x_1^\alpha}{\partial y_1^j \partial y_1^i} + \frac{\partial x_1^\mu}{\partial y_1^i} \Gamma^{\alpha_1}_{\mu\beta_1} \frac{\partial x_1^\beta}{\partial y_1^j} - \Gamma^{h_1}_{ij_1} \frac{\partial x_1^\alpha}{\partial y_1^h} ,$$

since $\frac{\partial x_1^\alpha}{\partial y_1^i}$ is a contravariant vector relative to x_1, since

(15.10) $$\frac{\partial \bar{x}_1^\alpha}{\partial y_1^i} = \frac{\partial \bar{x}_1^\alpha}{\partial x_1^\mu} \frac{\partial x_1^\mu}{\partial y_1^i} .$$

We re-write (15.9),

(15.9*) $$x^\alpha_{1 : ij_1} = x^\alpha_{1 : i_1 ; j_1} + \Gamma^{\alpha_1}_{\mu\beta_1} x^\mu_{1 : i_1} x^\beta_{1 : j_1} .$$

Returning to sub-spaces we have

(15.11) $$h_{ij : k_1} = \frac{\partial h_{ij}}{\partial y_1^k} - h_{hj} \Gamma^{h_1}_{ik_1} = 0$$
$$g_{\alpha\beta : k_1} = \frac{\partial g_{\alpha\beta}}{\partial x_1^\sigma} \frac{\partial x_1^\sigma}{\partial y_1^k} - g_{\mu\beta} \Gamma^{\mu_1}_{\alpha\sigma_1} \frac{\partial x_1^\sigma}{\partial y_1^k} = 0 .$$

We now take the generalized covariant derivative of

$$h_{ij} = g_{\alpha\beta} x^\alpha_{1 : i_1} x^\beta_{2 ; j}$$

with respect to y_1^{κ} and obtain

$$(15.12) \qquad g_{\alpha_1\beta}\, \frac{\partial x_2^{\beta}}{\partial y_2^{\partial}} \left(x_{1:\iota_1;\partial_1}^{\alpha} + \Gamma_{\mu\sigma_1}^{\alpha_1}\, x_{1:\iota_1}^{\mu}\, x_{1:\kappa_1}^{\sigma} \right) = 0 ,$$

by making use of (15.9*), (15.11) and the product rule for differentiating products of tensors.

We now define generalized normals to the hypersurfaces ($m = n-1$) at M_1, M_2.

<u>Definition.</u> The set of quantities $N^{\alpha_1}(M_1, M_2)$, $N^{\prime\alpha}(M_1, M_2)$ satisfying

$$(15.13) \qquad \begin{aligned} & g_{\alpha_1\beta}\, N^{\alpha_1} N^{\prime\beta} = 1 \\ & g_{\alpha_1\beta}\, N^{\alpha_1} \frac{\partial x_2^{\beta}}{\partial y_2^{\partial}} = 0 \\ & g_{\alpha_1\beta}\, N^{\prime\beta} \frac{\partial x_1^{\alpha}}{\partial y_1^{\partial}} = 0 \\ & g_{\beta_1\alpha}(M_2, M_1)\, N^{\prime\beta} N^{\alpha_1} = 1 \end{aligned}$$

are called the generalized normals to the hypersurface at M_1 , M_2.

We notice that (15.13) consists of 2n equations in the 2n unknowns $N^{\alpha_1}, N^{\prime\alpha}$.

From (15.13) we see that

$$(15.14) \qquad x_{1:\iota;\kappa_1}^{\alpha} + \Gamma_{\mu\sigma_1}^{\alpha_1}\, x_{1:\iota}^{\mu}\, x_{1:\kappa_1}^{\sigma} \equiv \Omega_{\iota\kappa_1}\, N^{\alpha_1}$$

will satisfy (15.12). Moreover, $\Omega_{\iota\kappa_1}$ are uniquely defined by

$$(15.15) \qquad \Omega_{\iota\kappa_1}(M_1, M_2) = g_{\alpha_1\beta}\, N^{\prime\beta} \left[x_{1:\iota;\kappa_1}^{\alpha} + \Gamma_{\mu\sigma_1}^{\alpha_1}\, x_{1:\iota_1}^{\mu}\, x_{1:\kappa_1}^{\sigma} \right],$$

from (15.13)

We notice that the $\Omega_{\iota\kappa_1}(M_1, M_2)$ are a generalization of the second fundamental form occuring in Riemannian geometry.

XVI. Normal coordinates. Dyodesic coordinates.

We consider the points $M_1(x_{1_0})$, $M_2(x_{2_0})$, with their local coordinate systems x_1^i, x_2^i, and assume that the linear connections $\Gamma^i_{jk_1}$, $\Gamma^{\prime i}_{jk}$ are symmetric in their lower indices. Let us now keep x_2 fixed, $x_2 \equiv x_{2_0}$, so that (8.1) reduces to

$$(16.1) \qquad \frac{d^2 x_1^i}{ds^2} + \Gamma^i_{jk_1}(x_1, x_{2_0}) \frac{dx_1^j}{ds} \frac{dx_1^k}{ds} = 0 .$$

Equation (16.1) has a unique solution $x_1^i(s)$ subject to the conditions $x_1^i(0) = x_{1_0}^i$, $\frac{dx_1^i(0)}{ds} = \xi^{i_1}$. If we perform a change of variable at x_2, we have $\Gamma^i_{jk_1}(x_1, x_{2_0}) = \bar{\Gamma}^i_{jk_1}(x_1, \bar{x}_{2_0})$, so that the solutions of (16.1) are independent of the coordinate system used at M_2.

The solutions of (16.1) are the minimal dyodesics through M_1 which correspond to the point M_2. In a small neighborhood of M_1 no two of these dyodesics will intersect from our general uniqueness theorem. Given any point M in a neighborhood of M_1, there will exist a unique minimal dyodesic joining M_1 to M. We can locate the point M by the two parameters ξ^{i_1} and s. Hence the equations

$$(16.2) \qquad y_1^i = \xi^{i_1} s , \quad i = 1, 2, \cdots, n$$

determine a one-to-one non-singular transformation of coordinates,

$$(16.3) \qquad y_1^i = y_1^i(x_1^1, x_1^2, \cdots, x_1^n) , \quad i = 1, 2, \cdots, n$$

in the neighborhood of M_1. We exhibit this transformation in (16.6). Now similarly by considering the minimal dyodesics through M_2 which correspond to the fixed point M_1, we can obtain

$$(16.4) \qquad y_2^i = \xi^{\prime i} s, \quad i = 1, 2, \cdots, n$$

where

(16.5) $$y_2^{\,i} = y_2^{\,i}(x_2^1, x_2^2, \dots, x_2^n), \quad i = 1, 2, \dots, n$$

is a non-singular transformation of coordinates in the neighborhood of M_2.

It is important to note that the dyodesics (16.2) do <u>not</u> correspond to the dyodesics (16.4).

<u>Definition</u>. The coordinate systems y_1, y_2 as defined above are called normal coordinates at M_1, M_2, respectively, with poles at M_1, M_2.

The solution of (16.1) may be written

(16.6) $$\begin{aligned} x_1^{\,i}(s) &= x_0^{\,i} + \xi^{i_1} s - \frac{1}{2!} \Gamma^{i_1}_{jk_1}(M_1, M_2)\, \xi^{j_1} \xi^{k_1} s^2 + \cdots \\ &= x_0^{\,i} + y_1^{\,i} - \frac{1}{2!} \Gamma^{i_1}_{jk_1}(M_1, M_2)\, y_1^{\,j} y_1^{\,k} + \cdots \end{aligned}$$

We immediately obtain

(16.7) $$\begin{aligned} \left.\frac{\partial x_1^{\,i}}{\partial y_1^{\,j}}\right|_{s=0} &= \delta^{i_1}_{j_1} \quad ; \quad \left.\frac{\partial^2 x_1^{\,i}}{\partial y_1^{\,k} \partial y_1^{\,j}}\right|_{s=0} = -\Gamma^{i_1}_{jk_1}(M_1, M_2) \\ \left.\frac{\partial y_1^{\,i}}{\partial x_1^{\,j}}\right|_{s=0} &= \delta^{i_1}_{j_1} \quad ; \quad \left.\frac{\partial^2 y_1^{\,i}}{\partial x_1^{\,\sigma} \partial x_1^{\,\tau}}\right|_{s=0} = \Gamma^{i_1}_{\sigma\tau_1}(M_1, M_2)\,. \end{aligned}$$

Let us now transform the x_1^i by a transformation (3.2) to coordinates $\bar{x}_1^{\,i}$ and let $\bar{y}_1^{\,i}$ be the normal coordinates at M_1 which correspond to the $\bar{x}_1^{\,i}$. The analytic relationship between the $y_1^{\,i}$ and $\bar{y}_1^{\,i}$ is found as follows.

Let $x_1^\alpha = \varphi_1^\alpha(s)$ be the solution of (16.1) subject to the initial conditions

(16.8) $$x_1^{\alpha} = x_0^{\alpha}, \quad \frac{dx_1^\alpha}{ds} = \xi^{\alpha_1} \quad \text{for } s = 0.$$

Under the transformation $x_1^i = f_1^i(\bar{x}_1)$, we obtain $\bar{x}_1^{\,i} = \psi_1^i(s)$ as the solutions of

(16.9) $$\frac{d^2 \bar{x}_1^{\,i}}{ds^2} + \bar{\Gamma}^{i_1}_{jk_1}(\bar{x}_1, M_2) \frac{d\bar{x}_1^{\,j}}{ds} \frac{d\bar{x}_1^{\,k}}{ds} = 0,$$

subject to the conditions

(16.10) $$\bar{x}_1^{\,i} = \bar{x}_{1_0}, \quad \frac{d\bar{x}_1^{\,i}}{ds} = \bar{\xi}^{i_1} \quad \text{for } s = 0,$$

where

(16.11) $$x_0^i = f_1^i(\bar{x}_0) \quad , \qquad \bar{\xi}^{i}_{,} = \xi^{\mu}_{,} \left(\frac{\partial \bar{x}_1^{\alpha}}{\partial x_1^{\mu}}\right)_{M_1} .$$

Multiplying (16.11) by s, we find

(16.12) $$\bar{\xi}^{\alpha}_{,} s = \xi^{\mu}_{,} s \; a^{\alpha}_{\mu}$$

or

(16.13) $$\bar{y}_1^{\alpha} = a^{\alpha}_{\mu} y^{\mu} \quad , \qquad a^{\alpha}_{\mu} = \left(\frac{\partial \bar{x}_1^{\alpha}}{\partial x_1^{\mu}}\right)_{M_1} ,$$

along a curve C, and since there exists a unique minimal path, C, joining M_1 to a neighborhing point M, (16.13) holds throughout a neighborhood of M_1.

All the above results which have been proved at M_1, will obviously hold at M_2 for the corresponding quantities involved. Moreover, even if the linear connections are not symmetric in their lower indices, (16.1) still holds, and we can obtain the results of (16.7) by considering only the symmetric parts of the connections.

We are now in a position to obtain new tensors from a given tensor by a process which we shall simply call '<u>extension</u>' of a tensor. Let us consider the two-point mixed tensor

(16.14) $$T^{\alpha\cdots\beta,\,a\cdots b}_{\sigma\cdots\tau,\,s\cdots t}(x_1, x_2) ,$$

and let its components be denoted by

(16.15) $$t^{\alpha\cdots\beta,\,a\cdots b}_{\sigma\cdots\tau,\,s\cdots t}(y_1, y_2)$$

when referred to normal coordinates at M_1, M_2.

If $\bar{t}^{\alpha\cdots\beta,\,a\cdots b}_{\sigma\cdots\tau,\,s\cdots t}(\bar{y}_1, \bar{y}_2)$ are the components of T with respect to the $\bar{y}_1$, $\bar{y}_2$ normal coordinate systems, then

(16.16) $$\bar{t}^{\mu\cdots\nu,\,a\cdots b}_{\eta\cdots\zeta,\,s\cdots t}(\bar{y}_1, \bar{y}_2) = \left|\frac{\partial y_1}{\partial \bar{y}_1}\right|^{p} \left|\frac{\partial y_2}{\partial \bar{y}_2}\right|^{q} t^{\alpha\cdots\beta,\,l\cdots m}_{\gamma\cdots\delta,\,\kappa\cdots\nu}(y_1, y_2) \frac{\partial y_1^{\gamma}}{\partial \bar{y}_1^{\eta}} \cdots \frac{\partial \bar{y}_2^{b}}{\partial y_2^{m}} ,$$

and in view of the fact that $\frac{\partial y_1^i}{\partial \bar{y}_1^j}, \cdots, \frac{\partial \bar{y}_2^i}{\partial y_2^j}$ are constants, see (16.13), we find

$$(16.17)\qquad \frac{\partial \bar{t}^{\mu\cdots\nu,a\cdots b}_{\eta\cdots\zeta,s\cdots t}}{\partial \bar{y}_1^{\epsilon}} = \left|\frac{\partial y_1}{\partial \bar{y}_1}\right|^p \left|\frac{\partial y_2}{\partial \bar{y}_2}\right|^q \frac{\partial t^{\alpha\cdots\beta,l\cdots m}_{\gamma\cdots\delta,u\cdots v}}{\partial y_1^{\phi}} \frac{\partial y_1^{\gamma}}{\partial \bar{y}_1^{\eta}} \cdots \frac{\partial \bar{y}_2^{b}}{\partial y_2^{m}} \frac{\partial y_1^{\phi}}{\partial \bar{y}_1^{\epsilon}}.$$

Evaluating at the origin of normal coordinates, using (16.13) we obtain

$$(16.18)\qquad \bar{T}^{\mu\cdots\nu,a\cdots b}_{\eta\cdots\zeta,s\cdots t|\epsilon_1}(\bar{x}_1,\bar{x}_2) = \left|\frac{\partial x_1}{\partial \bar{x}_1}\right|^p \left|\frac{\partial x_2}{\partial \bar{x}_2}\right|^q T^{\alpha\cdots\beta,l\cdots m}_{\gamma\cdots\delta,u\cdots v|\phi_1}(x_1,x_2) \frac{\partial x_1^{\gamma}}{\partial \bar{x}_1^{\eta}} \cdots \frac{\partial \bar{x}_2^{b}}{\partial x_2^{m}} \frac{\partial x_1^{1}}{\partial \bar{x}_1^{\epsilon}}$$

where

$$(16.19)\qquad \bar{T}^{\mu\cdots\nu,a\cdots b}_{\eta\cdots\zeta,s\cdots t|\epsilon_1}(\bar{x}_1,\bar{x}_2) \equiv \left.\frac{\partial \bar{t}^{\mu\cdots\nu,a\cdots b}_{\eta\cdots\zeta,s\cdots t}(\bar{y}_1,\bar{y}_2)}{\partial \bar{y}_1^{\epsilon}}\right|_{s=0},$$

the stroke $|_{\phi_1}$ denoting the first extension of T relative to M_1.

It is at once apparent that

$$(16.20)\qquad T^{\mu\cdots\nu,a\cdots b}_{\eta\cdots\zeta,s\cdots t|\epsilon_1\epsilon_2\cdots\epsilon_k,\psi_1\psi_2\cdots\psi_l} \equiv \left.\frac{\partial^{k+l} t^{\mu\cdots\nu,a\cdots b}_{\eta\cdots\zeta,s\cdots t}}{\partial y_1^{\epsilon_1}\cdots\partial y_1^{\epsilon_k}\partial y_2^{\psi_1}\cdots\partial y_2^{\psi_l}}\right|_{s=0}$$

is a tensor, the k^{th} extension relative to M_1 and the l^{th} extension relative to M_2.

The sum of the extensions of two tensors is an extension of their sum, and the extension of products is obtained by the rule for repeated differentiation of a product in the elementary calculus.

Making use of (16.7) we can find the explicit form of $T_{|\phi_1}$. Now

$$(16.21)\qquad t^{\mu\cdots\nu,a\cdots b}_{\sigma\cdots\tau,c\cdots d}(y_1,y_2) = \left|\frac{\partial x_1}{\partial y_1}\right|^p \left|\frac{\partial x_2}{\partial y_2}\right|^q T^{\alpha\cdots\beta,r\cdots s}_{\mu\cdots\nu,u\cdots v}(x_1,x_2) \frac{\partial x_1^{\mu}}{\partial y_1^{\sigma}} \cdots \frac{\partial y_2^{b}}{\partial x_2^{s}}$$

so that differentiating (16.21) and evaluating at the origin of normal coordinates, we obtain

(16.22) $$T^{\mu\cdots\nu,\,a\cdots b}_{\sigma\cdots\tau,\,c\cdots d\,|\,\epsilon,} \equiv \frac{\partial T^{\mu\cdots\nu,\,a\cdots b}_{\sigma\cdots\tau,\,c\cdots d}}{\partial x_1^{\epsilon}} - \Gamma^{\kappa,}_{\sigma\epsilon,} T^{\mu\cdots\nu,\,a\cdots b}_{\kappa\cdots\tau,\,c\cdots d} - \cdots + T^{\kappa\cdots\nu,\,a\cdots b}_{\sigma\cdots\tau,\,c\cdots d} \Gamma^{\mu,}_{\kappa\epsilon,} + \cdots + T^{\mu\cdots\kappa,\,a\cdots b}_{\sigma\cdots\tau,\,c\cdots d} \Gamma^{\nu,}_{\kappa\epsilon,} - p\, T^{\mu\cdots\nu,\,a\cdots b}_{\sigma\cdots\tau,\,c\cdots d} \Gamma^{\kappa,}_{\kappa\epsilon,} .$$

We note that the first extension of T relative to M_1 is the first covariant derivative of T relative to M_1.

Since in normal coordinates the components of the linear connections transform like tensors, we can obtain the extensions of these connections. We have

(16.23) $$\bar{\Gamma}^{\iota,}_{\jmath\kappa,}(\bar{y}_1, \bar{y}_2) = \Gamma^{\alpha,}_{\beta\gamma,}(y_1, y_2) \frac{\partial y_1^{\beta}}{\partial \bar{y}_1^{\jmath}} \frac{\partial y_1^{\gamma}}{\partial \bar{y}_1^{\kappa}} \frac{\partial \bar{y}_1^{\iota}}{\partial y_1^{\alpha}} ,$$

so that

(16.24) $$\frac{\partial \bar{\Gamma}^{\iota,}_{\jmath\kappa,}}{\partial \bar{y}_1^{\ell}} = \frac{\partial \Gamma^{\alpha,}_{\beta\gamma,}}{\partial y_1^{\epsilon}} \frac{\partial y_1^{\beta}}{\partial \bar{y}_1^{\jmath}} \frac{\partial y_1^{\gamma}}{\partial \bar{y}_1^{\kappa}} \frac{\partial \bar{y}_1^{\iota}}{\partial y_1^{\alpha}} \frac{\partial y_1^{\epsilon}}{\partial \bar{y}_1^{\ell}} ,$$

from which we can infer

(16.25) $$\bar{\Gamma}^{\iota,}_{\jmath k,|\ell,} = \Gamma^{\alpha,}_{\beta\gamma,/\epsilon,} \frac{\partial x_1^{\beta}}{\partial \bar{x}_1^{\jmath}} \frac{\partial x_1^{\gamma}}{\partial \bar{x}_1^{\kappa}} \frac{\partial \bar{x}_1^{\iota}}{\partial x_1^{\alpha}} \frac{\partial x_1^{\epsilon}}{\partial \bar{x}_1^{\ell}} .$$

We can thus generate a whole class of tensors

(16.26) $$\Gamma^{\iota,}_{\jmath k,|\ell,} , \quad \Gamma^{\iota,}_{\jmath k,|,\ell} , \quad \Gamma^{\iota,}_{\jmath k,|\ell,m} , \ldots, etc., \ldots$$

where

(16.27) $$\Gamma^{\iota,}_{\jmath\kappa,|\ell_1\cdots\ell_r,m_1\cdots m_s}(x_1 x_2) = \left(\frac{\partial^{r+s} \Gamma^{\iota,}_{\jmath\kappa,}(y_1, y_2)}{\partial y_1^{\ell_1}\cdots\partial y_1^{\ell_r}\partial y_2^{m_1}\cdots\partial y_2^{m_s}} \right)_{S=0} .$$

Now the $y_1^{\iota} = \xi^{\iota} s$ satisfy

(16.28) $$\frac{d^2 y_1^{\iota}}{ds^2} + \Gamma^{\iota,}_{\jmath\kappa,}(y_1, M_2) \frac{dy_1^{\jmath}}{ds} \frac{dy_1^{\kappa}}{ds} = 0 \quad , \; M_2 \text{ fixed},$$

so that

(16.29) $$\Gamma^{\iota,}_{\jmath\kappa,}(y_1, M_2)\, y_1^{\jmath} y_1^{\kappa} = 0 ,$$

on multiplying (16.28) by s . If (16.29) is expanded in a power series, we find

(16.30) $$0 = \Gamma^{i}_{jk}(y, M_2)\, y^j_1 y^k_1 = \Gamma^{i}_{jk}(y_1, y_2)\Big|_0 \xi^j \xi^k s^2 + \frac{\partial \Gamma^{i}_{jk}}{\partial y^l_1}\Big|_0 \xi^j \xi^k \xi^l s^3 + \cdots$$

the Γ's and their partial derivatives being evaluated at the origin of normal coordinates. Since (16.30) must hold for arbitrary ξ^i we infer at once that

$$\Gamma^{i}_{jk}\Big|_0 = 0$$

(16.31) $$\left[\frac{\partial \Gamma^{i}_{jk}}{\partial y^l_1} + \frac{\partial \Gamma^{i}_{lj}}{\partial y^k_1} + \frac{\partial \Gamma^{i}_{kl}}{\partial y^j_1} \right]_0 = 0$$

$$\vdots$$

and in general,

(16.32) $$S\left(\frac{\partial^m \Gamma^{i}_{jk}}{\partial y^{l_1}_1 \cdots \partial y^{l_m}_1} \right)_0 = 0,$$

in which S( ) stands for the sum of all the $\frac{(m+2)(m+1)}{2}$ terms obtainable from the one written in the parenthesis by replacing the pair of subscripts jk by any pair from the set $jkl_1 \cdots l_m$.

From (16.25), (16.26) we see that (16.32) may be replaced by

(16.33) $$S\left(\Gamma^{i}_{jk}|l_1 \cdots l_m \right) = 0.$$

The normal tensors (16.26) are represented by

(16.34) $$\begin{aligned} A^{i}_{jkl} &= \Gamma^{i}_{jk}|l, & \qquad A^{i}_{jk,l} &= \Gamma^{i}_{jk}|,l \\ A^{i}_{jklm} &= \Gamma^{i}_{jk}|lm, & \qquad A^{i}_{jk,lm} &= \Gamma^{i}_{jk}|,lm \\ &\vdots & &\vdots \\ A^{i}_{jkl,m} &= \Gamma^{i}_{jk}|l,m \\ &\vdots \end{aligned}$$

From

(16.34*) $$\Gamma^{i}_{jk,}(y_1,y_2) = \Gamma^{i}_{jk,}\Big|_0 + \left(\frac{\partial \Gamma^{i}_{jk,}}{\partial y_1^{\sigma}}\right)_0 y_1^{\sigma} + \left(\frac{\partial \Gamma^{i}_{jk,}}{\partial y_2^{\sigma}}\right)_0 y_2^{\sigma} + \cdots$$

we see that the linear connection $\Gamma^{i}_{jk,}$ is completely determined by the system of tensors in (16.34). It is rather obvious that similar results hold for $\Gamma^{i}_{,jk}$.

The A's obviously satisfy the identities

(16.35) $$A^{i}_{jk\,l\cdots s,} = A^{i}_{kj\,l\cdots s,}$$
$$A^{i}_{jk\,l\cdots s,} = A^{i}_{jk\,p\cdots q,}$$

where $p \cdots q$ is any permutation of $l \cdots s$.

We can now return to the explicit form of (16.20). Differentiating (16.21) the required number of times in y_1 and y_2 and evaluating at the origin will involve higher derivatives of x_1^i and x_2^i with respect to the y_1^j, y_2^j, respectively. We can use (16.6) and the corresponding equation in x_2^i, y_2^i to obtain these derivatives, these derivatives involving the A's. To obtain $A^{i}_{jkl,}$ we note that

(16.36) $$\bar{\Gamma}^{i}_{jk,}(y_1,y_2) = \Gamma^{\alpha}_{\beta\gamma,}(x_1,x_2)\frac{\partial x_1^{\beta}}{\partial y_1^{j}}\frac{\partial x_1^{\gamma}}{\partial y_1^{k}}\frac{\partial y_1^{i}}{\partial x_1^{\alpha}} + \frac{\partial^2 x_1^{\sigma}}{\partial y_1^{j}\partial y_1^{k}}\frac{\partial y_1^{i}}{\partial x_1^{\sigma}}$$

so that

(16.37) $$\frac{\partial \bar{\Gamma}^{i}_{jk,}}{\partial \bar{y}_1^{l}}\Big|_0 = \frac{\partial \Gamma^{i}_{jk,}}{\partial x_1^{l}} - \Gamma^{i}_{\beta k,}\Gamma^{\beta}_{jl,} - \Gamma^{i}_{j\beta,}\Gamma^{\beta}_{kl,} - \Gamma^{i}_{jkl,}$$

where

(16.38) $$\Gamma^{i}_{jkl,} = -\frac{\partial^3 x_1^{i}}{\partial y_1^{j}\partial y_1^{k}\partial y_1^{l}}\Big|_0$$

which is obtained from (16.6) and is not a tensor. It is easy to show that

(16.39) $$\Gamma^{i}_{jkl,} = \frac{1}{3}P\left(\frac{\partial \Gamma^{i}_{jk,}}{\partial x_1^{l}} - \Gamma^{i}_{\beta k,}\Gamma^{\beta}_{jl,} - \Gamma^{i}_{j\beta,}\Gamma^{\beta}_{kl,}\right)$$

where P( ) denotes the sum obtained by permuting cyclically the free indices jkl and adding the resulting terms.

Example. Let us compute $g_{\alpha,\tau/\beta,\sigma}$. We have

(16.40) $$\bar{g}_{\alpha,\tau}(y_1, y_2) = g_{\mu,\nu}(x_1, x_2) \frac{\partial x_1^\mu}{\partial y_1^\alpha} \frac{\partial x_2^\nu}{\partial y_2^\tau},$$

so that

(16.41) $$\frac{\partial \bar{g}_{\alpha,\tau}}{\partial y_1^\beta} = \frac{\partial g_{\mu,\nu}}{\partial x_1^\epsilon} \frac{\partial x_1^\epsilon}{\partial y_1^\beta} \frac{\partial x_1^\mu}{\partial y_1^\alpha} \frac{\partial x_2^\nu}{\partial y_2^\tau} + g_{\mu,\nu} \frac{\partial^2 x_1^\mu}{\partial y_1^\beta \partial y_1^\alpha} \frac{\partial x_2^\nu}{\partial y_2^\tau},$$

and

(16.42) $$\begin{aligned} \frac{\partial^2 \bar{g}_{\alpha,\tau}}{\partial y_2^\sigma \partial y_1^\beta} = & \frac{\partial^2 g_{\mu,\nu}}{\partial x_2^\phi \partial x_1^\epsilon} \frac{\partial x_1^\epsilon}{\partial y_1^\beta} \frac{\partial x_1^\mu}{\partial y_1^\alpha} \frac{\partial x_2^\nu}{\partial y_2^\tau} \frac{\partial x_2^\phi}{\partial y_2^\sigma} \\ & + \frac{\partial g_{\mu,\nu}}{\partial x_1^\epsilon} \frac{\partial x_1^\epsilon}{\partial y_1^\beta} \frac{\partial x_1^\mu}{\partial y_1^\alpha} \frac{\partial^2 x_2^\nu}{\partial y_2^\sigma \partial y_2^\tau} \\ & + g_{\mu,\nu} \frac{\partial^2 x_1^\mu}{\partial y_1^\beta \partial y_1^\alpha} \frac{\partial^2 x_2^\nu}{\partial y_2^\sigma \partial y_2^\tau} \\ & + \frac{\partial g_{\mu,\nu}}{\partial x_2^\epsilon} \frac{\partial^2 x_1^\mu}{\partial y_1^\beta \partial y_1^\alpha} \frac{\partial x_2^\nu}{\partial y_2^\tau} \frac{\partial x_2^\epsilon}{\partial y_2^\sigma}. \end{aligned}$$

Evaluating at the origin of normal coordinates, we infer that

(16.43) $$\begin{aligned} g_{\alpha,\tau/\beta,\sigma} = & \frac{\partial^2 g_{\alpha,\tau}}{\partial x_2^\sigma \partial x_1^\beta} - \frac{\partial g_{\alpha,\nu}}{\partial x_1^\beta} \Gamma^{,\nu}_{,\sigma\tau} \\ & + g_{\mu,\nu} \Gamma^{\mu,}_{\beta\alpha,} \Gamma^{,\nu}_{,\sigma\tau} - \frac{\partial g_{\mu,\tau}}{\partial x_2^\sigma} \Gamma^{\mu,}_{\beta\alpha,} \\ = & \frac{\partial^2 g_{\alpha,\tau}}{\partial x_2^\sigma \partial x_1^\beta} - \frac{\partial g_{\mu,\tau}}{\partial x_2^\sigma} \Gamma^{\mu,}_{\beta\alpha,}. \end{aligned}$$

Now

(16.44) $$T_{\alpha\beta,\sigma\tau} \equiv g_{\iota,\tau} T^{\iota,}_{\alpha\beta,\sigma} = g_{\iota,\tau} \frac{\partial \Gamma^{\iota,}_{\alpha\beta,}}{\partial x_2^\sigma},$$

since $C^{\iota,}_{\alpha\beta} = 0$. It is easy to show that (16.44) reduces to (16.43) so that

(16.45) $$T_{\alpha\beta,\sigma\tau} \equiv g_{\alpha,\tau/\beta,\sigma}.$$

The equations (16.29) may be used to completely characterize the normal coordinates y_1 at M_1 associated with the fixed point M_2. From (6.2) we infer

$$(16.46) \quad \frac{\partial^2 x_1^{\epsilon}}{\partial y_1^{\sigma} \partial y_1^{\mu}} = \overline{\Gamma}_{\mu\sigma_1}^{\alpha_1}(y_1, M_2) \frac{\partial x_1^{\epsilon}}{\partial y_1^{\alpha}} - \Gamma_{\rho\tau_1}^{\epsilon_1}(x_1, M_2) \frac{\partial x_1^{\rho}}{\partial y_1^{\mu}} \frac{\partial x_1^{\tau}}{\partial y_1^{\sigma}} ,$$

and from (16.29) we conclude that

$$(16.47) \quad \frac{\partial^2 x_1^{\epsilon}}{\partial y_1^{\sigma} \partial y_1^{\mu}} y_1^{\mu} y_1^{\sigma} = - \Gamma_{\rho\tau_1}^{\epsilon_1}(x_1, M_2) \frac{\partial x_1^{\rho}}{\partial y_1^{\mu}} \frac{\partial x_1^{\tau}}{\partial y_1^{\sigma}} y_1^{\mu} y_1^{\sigma} .$$

These differential equations uniquely determine a functional relation between the x_1^{α} and the y_1^{α} subject to the initial conditions $y_1^{\alpha} = 0$, $\frac{\partial x_1^{\alpha}}{\partial y_1^{\beta}} = \delta_{\beta_1}^{\alpha_1}$, for $x_1^{\alpha} = x_{1_0}^{\alpha}$. In fact successive differentiation of (16.47) leads to (16.6). We have, of course, assumed $\Gamma_{jk_1}^{i_1}$ symmetric.

We may also characterize the normal coordinates y_1^{α} by use of the metric $g_{\alpha_1\beta}(x_1, M_2)$. We first notice that

$$(16.48) \quad g_{\mu_1\beta}(x_1, M_2) \frac{dx_1^{\mu}}{ds} = \text{constants}$$

are first integrals of (16.1). If we differentiate the left-hand side of (16.48) and use the fact that

$$(16.49) \quad \left. \frac{\partial g_{\alpha_1\tau}}{\partial x_1^{\beta}} \right|_{x_1, M_2} = \left. \frac{\partial g_{\alpha_1\tau}(x_1, M_2)}{\partial x_1^{\beta}} \right|_{x_1} ,$$

we can infer that

$$(16.50) \quad \begin{aligned} \frac{d}{ds}\left[g_{\mu_1\beta}(x_1, M_2) \frac{dx_1^{\mu}}{ds} \right] &= g_{\mu_1\beta}(x_1, M_2) \frac{d^2 x_1^{\mu}}{ds^2} + \frac{\partial g_{\sigma_1\beta}(x_1, M_2)}{\partial x_1^{\tau}} \frac{dx_1^{\sigma}}{ds} \frac{dx_1^{\tau}}{ds} \\ &= g_{\mu_1\beta}\left[\frac{d^2 x_1^{\mu}}{ds^2} + \Gamma_{\sigma\tau_1}^{\mu_1} \frac{dx_1^{\sigma}}{ds} \frac{dx_1^{\tau}}{ds} \right] \\ &= 0 , \end{aligned}$$

by making use of (6.1), (16.1). Q.E.D. . We again assume the linear connections symmetric in their lower indices.

Now let $\psi_{\alpha,\beta}(y_1, y_2)$ be the components of the metric tensor in normal coordinates y_1, y_2. We shall keep the point M_2 fixed allowing only y_1 to vary. From (16.48) we conclude that along a minimal dyodesic satisfying (16.1) we must have

$$\psi_{\mu,\beta}(y_1, M_2)\frac{dy_1^\mu}{ds} = \left(\psi_{\mu,\beta}(y_1, M_2)\frac{dy_1^\mu}{ds}\right)_0 , \tag{16.51}$$

or

$$\psi_{\mu,\beta}(y_1, M_2)\,\xi_1^{\mu} = \left(\psi_{\mu,\beta}\,\xi^{\mu}\right)_0 \tag{16.52}$$

and multiplying by s,

$$\psi_{\mu,\beta}(y_1, M_2)\,y_1^\mu = \left(\psi_{\mu,\beta}\right)_0 y_1^\mu . \tag{16.53}$$

Similarly,

$$\psi_{\mu,\beta}(M_1, y_2)\,y_2^\beta = \left(\psi_{\mu,\beta}\right)_0 y_2^\beta . \tag{16.54}$$

The equations (16.53), (16.54) characterize the y_1, y_2 as being normal coordinates at M_1, M_2 respectively. To see this we first differentiate (16.53) with respect to y_1^σ and then multiply by y_1^τ. This leads to

$$\frac{\partial\psi_{\mu,\beta}(y_1, M_2)}{\partial y_1^\sigma}\,y_1^\mu y_1^\sigma + \psi_{\sigma,\beta}(y_1, M_2)\,y_1^\sigma = \left(\psi_{\sigma,\beta}\right)_0 y_1^\sigma , \tag{16.55}$$

and so from (16.53) we have

$$\frac{\partial\psi_{\mu,\beta}(y_1, M_2)}{\partial y_1^\sigma}\,y_1^\mu y_1^\sigma = 0 , \tag{16.56}$$

and similarly,

$$\frac{\partial\psi_{\mu,\beta}(M_1, y_2)}{\partial y_2^\sigma}\,y_2^\beta y_2^\sigma = 0 . \tag{16.57}$$

But for symmetric connections, (16.56), (16.57) lead to

$$\begin{aligned} &\Gamma^{i}_{jk,}(y_1, M_2)\,y_1^j y_1^k = 0 \\ &\Gamma^{,i}_{,jk}(M_1, y_2)\,y_2^j y_2^k = 0 , \end{aligned} \tag{16.58}$$

by (6.1) and the fact that $\frac{\partial g_{\alpha,\tau}(y_1,M_2)}{\partial y_1^\sigma} = \left.\frac{\partial g_{\alpha,\tau}(M_1,y_2)}{\partial y_1^\sigma}\right|_{y_1,M_2}$. Conversely, if (16.58) holds we can easily deduce (16.53), (16.54). Q.E.D.

We can generate an infinite class of metric tensor differential invariants, called the metric normal tensors, whose components $g_{\alpha,\beta|l_1 \cdots l_r, m_1 \cdots m_s}$ are given as functions of the coordinates x_1^α, x_2^α by the equations

$$(16.59) \qquad g_{\alpha,\beta|l_1 \cdots l_r, m_1 \cdots m_s} = \left(\frac{\partial^{r+s} \psi_{\alpha,\beta}}{\partial y_1^{l_1} \cdots \partial y_1^{l_r} \partial y_2^{m_1} \cdots \partial y_2^{m_s}} \right)_0 .$$

These quantities occur in a very natural manner in the expansion of $\psi_{\alpha,\beta}(y_1, y_2)$ in Taylor series about $y_1^i = y_2^i = 0$. Thus

$$(16.60) \qquad \psi_{\alpha,\beta}(y_1, y_2) = (\psi_{\alpha,\beta})_0 + \frac{1}{2!}\Big[(g_{\alpha,\beta|\gamma\delta,})_0 \, y_1^\gamma y_1^\delta + 2 (g_{\alpha,\beta|\gamma,\delta})_0 \, y_1^\gamma y_2^\delta + (g_{\alpha,\beta|,\gamma\delta})\, y_2^\gamma y_2^\delta \Big] + \cdots .$$

We see from (16.59) that the $g_{\alpha,\beta|l_1 \cdots l_r, m_1 \cdots m_s}$ satisfy the algebraic identities

$$(16.61) \qquad g_{\alpha,\beta|l_1 \cdots l_r, m_1 \cdots m_s} = g_{\alpha,\beta|p_1 \cdots p_r, q_1 \cdots q_s} ,$$

where $p_1 \cdots p_r$ are any permutations of $l_1 \cdots l_r$, and $q_1 \cdots q_s$ are any permutation of $m_1 \cdots m_s$. Furthermore, on further differentiation of (16.56), (16.57) followed by evaluation at the origin of normal coordinates, we find the identities

$$(16.62) \qquad \begin{aligned} &S_1 (g_{\alpha,\beta|l_1 \cdots l_r,}) = 0 \\ &S_2 (g_{\alpha,\beta|, m_1 \cdots m_s}) = 0 , \end{aligned}$$

where S_1 stands for the sum of terms obtained from the one in parenthesis by a cyclic permutation of the indices $\alpha\, l_1 \cdots l_r$. Similarly for S_2 by permuting $\beta\, m_1 \cdots m_s$.

Let us now consider any sequence of sets of numbers

$$(16.63) \qquad g_{\alpha,\beta} , \quad g_{\alpha,\beta|\sigma\tau,} , \quad g_{\alpha,\beta|\sigma,\tau} , \quad \cdots ,$$

satisfying the algebraic identities (16.61), (16.62), and such that (16.60) converges. We only consider those sequences which yield symmetric linear connections when we solve for $\psi_{\alpha,\beta}(y_1, y_2)$ from (16.60). From (16.60) we have

(16.64) $$\frac{\partial \psi_{\alpha,\beta}(y_1,0)}{\partial y_1^\sigma} = (g_{\alpha,\beta|\sigma\gamma,})_0\, y_1^\gamma + \frac{1}{2!}(g_{\alpha,\beta|\sigma\gamma\delta,})_0\, y_1^\gamma y_1^\delta + \cdots,$$

by making use of (16.62). Hence

(16.65) $$\frac{\partial \psi_{\alpha,\beta}(y_1,0)}{\partial y_1^\sigma} y_1^\alpha y_1^\sigma = y_1^\alpha\left[(g_{\alpha,\beta|\sigma\gamma,})_0\, y_1^\gamma y_1^\sigma + \frac{1}{2!}(g_{\alpha,\beta|\sigma\gamma\delta,})_0\, y_1^\sigma y_1^\gamma y_1^\delta\right]$$
$$= y_1^\alpha\left[\psi_{\alpha,\beta}(y_1,0) - \psi_{\alpha,\beta}(0,0)\right]$$

from (16.60). Again from (16.60) we obtain

(16.66) $$\psi_{\alpha,\beta}(y_1,0)\, y_1^\alpha = \psi_{\alpha,\beta}(0,0)\, y_1^\alpha + \frac{1}{2!}(g_{\alpha,\beta|\gamma\delta,})_0\, y_1^\gamma y_1^\delta y_1^\alpha + \cdots,$$

so that from (16.62) we infer that

(16.66*) $$\psi_{\alpha,\beta}(y_1,0)\, y_1^\alpha = \psi_{\alpha,\beta}(0,0)\, y_1^\alpha,$$

and similarly

(16.66**) $$\psi_{\alpha,\beta}(0,y_2)\, y_2^\beta = \psi_{\alpha,\beta}(0,0)\, y_2^\beta,$$

(16.66*) and (16.66**) exhibiting the fact that the y_1, y_2 are normal coordinates.

If I is a set of algebraic equations satisfied by the set (16.63), I not being a consequent of (16.61) and (16.62), then since the conditions of convergence of (16.60) are inequalities, we could choose the set (16.63) in such a way that I was not satisfied, while at the same time satisfying (16.61), (16.62). We have just shown that a metric space would exist such that the set (16.63) could be generated not satisfying I. Hence all algebraic identities satisfied by all the metric normal tensors are consequences of (16.61), (16.62). We say that the identities of (16.61) and (16.62) constitute a complete set of identities of the components $g_{\alpha,\beta|l_1\cdots l_r,m_1\cdots m_s}$ of the metric normal tensor.

We have shown that, see (16.45)

(16.67) $$T_{\alpha\beta,\sigma\tau} = g_{\alpha,\tau|\beta,\sigma}.$$

Hence the complete set of algebraic identities satisfied by $T_{\alpha\beta,\sigma\tau}$ must be the same as the complete set satisfied by $g_{\alpha,\tau/\beta,\sigma}$. From (16.62) it follows that

$$T_{\alpha\beta,\sigma\tau} + T_{\beta\alpha,\sigma\tau} = 0$$
$$T_{\alpha\beta,\sigma\tau} + T_{\alpha\beta,\tau\sigma} = 0. \tag{16.68}$$

Hence by covariant differentiation

$$T_{\alpha\beta,\sigma\tau;\gamma_1} + T_{\beta\alpha,\sigma\tau;\gamma_1} = 0$$
$$T_{\alpha\beta,\sigma\tau;;\gamma} + T_{\alpha\beta,\tau\sigma;;\gamma} = 0 \tag{16.69}$$
$$\vdots$$

Now

$$T_{\alpha\beta,\sigma\tau;\gamma_1} = \frac{\partial T_{\alpha\beta,\sigma\tau}}{\partial x^{\gamma}} + * \tag{16.70}$$

where $*$ is a polynomial in T and the Γ^{λ}. Evaluating at the origin of normal coordinates, we have

$$T_{\alpha\beta,\sigma\tau;\gamma_1} = \psi_{\iota,\tau}(0,0)\, A^{\iota,}_{\alpha\beta\gamma,\sigma} \tag{16.71}$$

or

$$T^{k,}_{\alpha\beta,\sigma;\gamma_1} = A^{k,}_{\alpha\beta\gamma,\sigma}\ . \tag{16.72}$$

Hence the complete set of algebraic identities satisfied by $T^{k,}_{\alpha\beta\sigma;\gamma_1}$ is the same as the complete set of identities satisfied by $A^{k,}_{\alpha\beta\gamma,\sigma}$. We can repeat this process for all T.

<u>Definition</u>. If a coordinate system y_1 and a coordinate system y_2 exist at M_1 and M_2 such that $\Gamma^{\iota}_{\alpha\beta,}(M_1,M_2) = \Gamma^{\iota}_{,\alpha\beta}(M_1,M_2) = 0$ in these coordinates, we call such a system of coordinates, dyodesic, with poles at M_1, M_2.

From (6.2), (6.16) we see that the existence of a dyodesic coordinate system implies that the Christoffel symbols be symmetric at the poles. Hence, in general, for asymmetric connection no dyodesic coordinate system will exist. It is very simple to show the existence of dyodesic coordinates for symmetric connections. The normal coordinates discussed above are special cases of dyodesic coordinates. We can generate dyodesic coordinate systems by considering the following transformations,

$$(16.73)\qquad \begin{aligned} x_1^i &= x_1^i(M_1) + y_1^i - \tfrac{1}{2!}\left(\Gamma^{i_1}_{jk_1}\right)_{M_1,M_2} y_1^j y_1^k + \Phi_1(y_1) \\ x_2^i &= x_2^i(M_2) + y_2^i - \tfrac{1}{2!}\left(\Gamma^{2i}_{2jk}\right)_{M_1,M_2} y_2^j y_2^k + \Phi_2(y_2), \end{aligned}$$

where $\Phi_1(y_1)$, $\Phi_2(y_2)$ are arbitrary subject to the conditions that the first and second partial derivatives vanish for $y_1^i = y_2^i = 0$. From (16.73) we immediately obtain

$$(16.74)\qquad \left.\frac{\partial x_1^i}{\partial y_1^j}\right|_{y_1=0} = \delta^{i_1}_{j_1} \qquad \left.\frac{\partial^2 x_{1,2}^i}{\partial y_{1,2}^j\, \partial y_{1,2}^k}\right|_{y_1=y_2=0} = -\left(\Gamma^{i_1}_{jk_1}\right)_{M_1,M_2} \text{ or }$$

$$\left.\frac{\partial x_2^i}{\partial y_2^j}\right|_{y_2=0} = \delta^{2i}_{2j} \qquad\qquad = -\left(\Gamma^{2i}_{2jk}\right)_{M_1,M_2}.$$

From (6.2), (6.16) we see that

$$(16.75)\qquad \begin{aligned} \Gamma^{i_1}_{jk_1}(y_1,y_2)\big|_{y_1=y_2=0} &= 0 \\ \Gamma^{2i}_{2jk}(y_1,y_2)\big|_{y_1=y_2=0} &= 0. \end{aligned}$$

From the definition of covariant differentiation it is clear that whenever the connections are symmetric, the components of the first covariant derivatives are the ordinary derivatives when evaluated at the poles of a dyodesic coordinate system.

When the connections are symmetric we have $C^{i_1}_{\alpha,\beta} = C^{2i}_{\alpha,\beta} = 0$, see Theorem 7, so that from (8.1) we obtain

$$(16.76)\qquad \frac{d^2 y_1^i}{ds^2} = \frac{d^2 y_2^i}{ds^2} = 0 \quad \text{at } s=0,$$

and the equations of the dyodesics are

$$(16.77)\qquad \begin{aligned} y_1^i &= \xi_1^i s + a_1^i s^3 + \cdots \\ y_2^i &= \xi_2^i s + a_2^i s^3 + \cdots ; \end{aligned}$$

XVII. An equivalence problem.

Let us suppose that a space admits of two metric forms

(17.1) $$g_{\alpha,\beta}\, dx_1^{\alpha}\, dx_2^{\beta} \qquad ; \qquad \bar{g}_{\alpha,\beta}\, d\bar{x}_1^{\alpha}\, d\bar{x}_2^{\beta} .$$

We ask if their exists a point transformation

(17.2) $$\begin{aligned} \bar{x}_1^{i} &= \bar{x}_1^{i}(x_1) \\ \bar{x}_2^{i} &= \bar{x}_2^{i}(x_2) \end{aligned}$$

which carries one form into the other? If this is possible we say that the two forms are equivalent.

Now if the two forms are equivalent, (17.2) must exist such that

(17.3) $$\bar{g}_{\alpha,\beta}(\bar{x}_1, \bar{x}_2)\, \frac{\partial \bar{x}_1^{\alpha}}{\partial x_1^{\sigma}}\, \frac{\partial \bar{x}_2^{\beta}}{\partial x_2^{\tau}} = g_{\sigma,\tau}(x_1, x_2) .$$

We define

(17.4) $$\begin{aligned} \frac{\partial \bar{x}_1^{\alpha}}{\partial x_1^{\sigma}} &= u_{\sigma}^{\alpha} \\ \frac{\partial \bar{x}_2^{\beta}}{\partial x_2^{\tau}} &= v_{\tau}^{\beta} , \end{aligned}$$

so that (17.3) becomes

(17.5) $$g_{\sigma,\tau} = \bar{g}_{\alpha,\beta}\, u_{\sigma}^{\alpha}\, v_{\tau}^{\beta} .$$

Now (17.2), (17.4), (17.5) imply

(17.6) $$\begin{aligned} \frac{\partial \bar{x}_1^{\alpha}}{\partial x_1^{\sigma}} &= u_{\sigma}^{\alpha} \\ \frac{\partial \bar{x}_1^{\alpha}}{\partial x_2^{\sigma}} &= 0 \\ \frac{\partial \bar{x}_2^{\alpha}}{\partial x_2^{\sigma}} &= v_{\sigma}^{\alpha} \\ \frac{\partial \bar{x}_2^{\alpha}}{\partial x_1^{\sigma}} &= 0 \\ \frac{\partial u_{\sigma}^{\alpha}}{\partial x_1^{\tau}} &= \Gamma_{\sigma\tau,}^{i,}\, u_i^{\alpha} - \bar{\Gamma}_{\beta\gamma,}^{\alpha,}\, u_{\sigma}^{\beta}\, u_{\tau}^{\gamma} \\ \frac{\partial u_{\sigma}^{\alpha}}{\partial x_2^{\tau}} &= 0 \\ \frac{\partial v_{\sigma}^{\alpha}}{\partial x_2^{\tau}} &= \Gamma_{,\sigma\tau}^{,i}\, v_i^{\alpha} - \bar{\Gamma}_{,\beta\gamma}^{,\alpha}\, v_{\sigma}^{\beta}\, v_{\tau}^{\gamma} \\ \frac{\partial v_{\sigma}^{\alpha}}{\partial x_1^{\tau}} &= 0 \end{aligned}$$

subject to

(17.7) $$g_{\sigma,\tau} = \bar{g}_{\alpha,\beta}\, u^{\alpha}_{\sigma}\, v^{\beta}_{\tau} ,$$

where $\Gamma^{i}_{jk_1} = g^{i,\sigma} \frac{\partial g_{j,\sigma}}{\partial x_1^{k}}$, $\Gamma^{ji}_{jk} = g_{*}^{i,\sigma} \frac{\partial g_{\sigma,k}}{\partial x_2^{j}}$.

Conversely, if solutions to (17.6) exist satisfying (17.7), then obviously (17.2), (17.4), (17.5) are satisfied. By a solution to (17.6) we mean that functions $\bar{x}_1^{\alpha}$, $\bar{x}_2^{\alpha}$, u^{α}_{σ} , v^{α}_{σ} , exist satisfying (17.6), these quantities being functions of the independent variables x_1^{i}, x_2^{i}. Hence our problem of equivalence becomes equivalent to the following.

Consider the system of differential equations

(17.8) $$\frac{\partial z^{\alpha}}{\partial x^{\beta}} = \psi^{\alpha}_{\beta}(z, x) , \qquad \alpha = 1, 2, \cdots, R \quad \beta = 1, 2, \cdots, n .$$

We seek solutions

(17.9) $$z^{\alpha} = z^{\alpha}(x^1, x^2, \cdots, x^n)$$

of (17.8) which satisfy a system of equations

(17.10) $$F^{(0)}_{\lambda}(z, x) = 0 .$$

It can be shown that the condition for the existence of such a solution is given by the following[3].

A necessary and sufficient condition that the system of differential equations (17.8) admit a solution (17.9) satisfying equations (17.10) is that there exist an integer $N \geqslant 1$ such that the first N sets of equations of the sequence

(17.11) $$F^{(1)}_{\lambda}(z, x) = 0 \quad ; \quad F^{(2)}_{\mu}(z, x) = 0 \quad ; \quad \cdots ,$$

are algebraically consistant considered as equations for the determination of the z^{α} as functions of the independent variables x^{β}, and that all their solutions satisfy the $(N+1)^{st}$ set of

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equations in (17.11), where $F_\lambda^{(1)}$ is the set of equations consisting of (17.10), the equations of integrability of (17.8), i.e., $\frac{\partial^2 Z^\alpha}{\partial x^\beta \partial x^\sigma} = \frac{\partial^2 Z^\alpha}{\partial x^\sigma \partial x^\beta}$, and the equations obtained by differentiating (17.10) with respect to x^β and eliminating the derivatives of Z^α by means of (17.8). $F_\sigma^{(k+1)}=0$ for $k \geqq 1$ is the set of equations obtained by differentiating the set of equations $F_\tau^{(k)}=0$ with respect to x^β and eliminating the derivatives of Z^α by means of (17.8).

Applying this theorem to (17.6), (17.7), we find that $F_\lambda^{(1)}$ consists of

(17.12)

$$(1) \quad g_{\sigma,\tau} = \bar{g}_{\alpha,\beta}\, u_\sigma^\alpha v_\tau^\beta$$

$$(2) \quad \left(\Gamma^{\iota,}_{\sigma\tau,} - \Gamma^{\iota,}_{\tau\sigma,}\right) u_i^\alpha = \left(\bar{\Gamma}^{\alpha,}_{\beta\gamma,} - \bar{\Gamma}^{\alpha,}_{\gamma\beta,}\right) u_\tau^\beta u_\sigma^\gamma$$

$$(3) \quad \left(\Gamma^{,i}_{,\sigma\tau} - \Gamma^{,i}_{,\tau\sigma}\right) v_i^\alpha = \left(\bar{\Gamma}^{,\alpha}_{,\beta\gamma} - \bar{\Gamma}^{,\alpha}_{,\gamma\beta}\right) v_\tau^\beta v_\sigma^\gamma$$

$$(4) \quad \frac{\partial \Gamma^{\iota,}_{\sigma\tau,}}{\partial x_2^\epsilon}\, u_i^\alpha = \frac{\partial \bar{\Gamma}^{\alpha,}_{\beta\gamma,}}{\partial \bar{x}_2^\psi}\, v_\epsilon^\psi u_\sigma^\beta u_\tau^\gamma$$

$$(5) \quad \frac{\partial \Gamma^{,i}_{,\sigma\tau}}{\partial x_1^\epsilon}\, v_i^\alpha = \frac{\partial \bar{\Gamma}^{,\alpha}_{,\beta\gamma}}{\partial \bar{x}_1^\psi}\, u_\epsilon^\psi v_\sigma^\beta v_\tau^\gamma$$

$$(6) \quad R^{\iota,}_{\sigma\tau\epsilon,}\, u_i^\alpha = \bar{R}^{\alpha,}_{\mu\nu\phi,}\, u_\sigma^\mu u_\tau^\nu u_\epsilon^\phi$$

$$(7) \quad R^{,i}_{,\sigma\tau\epsilon}\, v_i^\alpha = \bar{R}^{,\alpha}_{,\mu\nu\phi}\, v_\sigma^\mu v_\tau^\nu v_\epsilon^\phi$$

$$(8) \quad g_{\sigma,\tau;\epsilon,} = \bar{g}_{\mu,\nu;\psi,}\, u_\sigma^\mu v_\tau^\nu u_\epsilon^\psi$$

$$(9) \quad g_{\sigma,\tau;,\epsilon} = \bar{g}_{\mu,\nu;,\phi}\, u_\sigma^\mu v_\tau^\nu v_\epsilon^\phi$$

If the connections are symmetric then (2), (3) of (17.12) are automatically satisfied. Also (8), (9) are satisfied since $g_{\sigma,\tau;\epsilon_1}=0$, see (13.9). Moreover, (6), (7) are always satisfied since $R^{i_1}_{\sigma\tau\epsilon_1}\equiv 0$, see (12.13). Since $C^{i_1}_{\alpha,\beta}=C^{j_1}_{\alpha,\beta}=0$ for symmetric connections, $F^{(1)}_\lambda$ consists of

$$
(17.13)\qquad
\begin{aligned}
g_{\sigma,\tau} &= \bar{g}_{\alpha,\beta}\, u^\alpha_\sigma\, v^\beta_\tau\\
T^{i_1}_{\alpha\beta,\sigma}\, u^\lambda_i &= \bar{T}^{\lambda_1}_{\mu\nu,\phi}\, u^\mu_\alpha\, u^\nu_\beta\, v^\phi_\sigma\\
T^{\,j_1}_{\sigma,\alpha\beta}\, v^\lambda_i &= \bar{T}^{\,j_1}_{\phi,\mu\nu}\, u^\phi_\sigma\, v^\mu_\alpha\, v^\nu_\beta,
\end{aligned}
$$

see (12.7) for the definition of $T^{i_1}_{\alpha\beta,\sigma}$.

$F^{(2)}_\mu$ is obtained by differentiating (17.13) and eliminating $\frac{\partial u^\alpha_\sigma}{\partial x_1^\beta}$, etc., by means of (17.6).

We notice that if $\bar{g}_{\alpha,\beta}\equiv$ constants, then $g_{\alpha,\beta}\, dx_1^\alpha dx_2^\beta$ will be equivalent to $\bar{g}_{\alpha,\beta}\, d\bar{x}_1^\alpha d\bar{x}_2^\beta$ provided the $\Gamma^{i_1}_{jk_1}$ are symmetric in their lower indices and provided $T^{i_1}_{\alpha\beta,\sigma}=T^{\,j_1}_{\sigma,\alpha\beta}=0$, see (17.13), and compare with Theorem 14.

The set (17.13) may be replaced by

$$
(17.14)\qquad
\begin{aligned}
g_{\sigma,\tau} &= \bar{g}_{\alpha,\beta}\, u^\alpha_\sigma\, v^\beta_\tau\\
T_{\alpha\beta,\sigma\tau} &= \bar{T}_{ab,rs}\, u^a_\alpha\, u^b_\beta\, v^r_\sigma\, v^s_t\\
T^*_{\tau\sigma,\alpha\beta} &= \bar{T}^*_{ts,ab}\, u^t_\tau\, u^s_\sigma\, v^a_\alpha\, v^b_\beta
\end{aligned}
$$

where

$$
(17.15)\qquad T_{\alpha\beta,\sigma\tau} = g_{i,\tau}\, T^{i_1}_{\alpha\beta,\sigma}\quad ;\quad T^*_{\tau\sigma,\alpha\beta} = g_{\sigma,i}\, T^{\,j_1}_{\tau,\alpha\beta}\,.
$$

Repeated differentiation of (17.14) and elimination of $\frac{\partial u^a_\alpha}{\partial x_1^\tau}$, etc., by means of (17.6) leads to the sets

$$
(17.16)\qquad
\begin{aligned}
g_{\sigma,\tau} &= \bar{g}_{\alpha,\beta}\, u^\alpha_\sigma\, v^\beta_\tau &\qquad T^*_{\tau\sigma,\alpha\beta} &= \bar{T}^*_{ts,ab}\, u^t_\tau\, u^s_\sigma\, v^a_\alpha\, v^b_\beta\\
T_{\alpha\beta,\sigma\tau} &= \bar{T}_{ab,rs}\, u^a_\alpha\, u^b_\beta\, v^r_\sigma\, v^s_t &\qquad T_{\alpha\beta,\sigma\tau;\epsilon_1} &= \bar{T}_{ab,rs;\phi_1}\, u^a_\alpha\, u^b_\beta\, u^\phi_t \ldots\\
& & &\vdots \text{ etc.}
\end{aligned}
$$

Now by the same type of reasoning performed in the differential invariant theory of Riemannian geometry it can be shown that the set (17.16) may be replaced by the equivalent sequence

$$
(17.17)\qquad
\begin{aligned}
g_{\sigma,\tau} &= \bar{g}_{\alpha,\beta}\, u^\alpha_\sigma\, v^\beta_\tau\\
g_{\alpha,\beta|\sigma,\tau} &= \bar{g}_{\mu,\nu|\xi,\eta}\, u^\mu_\alpha\, u^\xi_\sigma\, v^\nu_\beta\, v^\eta_\tau
\end{aligned}
$$

XVIII. The two-body problem, continued.

We first investigate the contracted tensor $T^{\sigma_1}_{\alpha\sigma_1,\beta}$, see (12.7). We have

$$(18.1)\qquad T^{\sigma_1}_{\alpha\sigma_1\beta} = T_{\alpha,\beta} = \frac{\partial C^{\sigma_1}_{\alpha,\beta}}{\partial x_1^{\sigma}} - \frac{\partial \Gamma^{\sigma_1}_{\alpha\sigma_1}}{\partial x_2^{\beta}} + \Gamma^{\sigma_1}_{\tau\sigma_1} C^{\tau_1}_{\alpha,\beta} - C^{\sigma_1}_{\tau,\beta}\Gamma^{\tau_1}_{\alpha\sigma_1}.$$

Let us make the same assumptions for the $g_{\alpha,\beta}$ as in § 10, and assuming that the products ΓC are small, we obtain

$$(18.2)\qquad T_{4,4} \approx \frac{\partial C^{\sigma_1}_{4,4}}{\partial x_1^{\sigma}} - \frac{\partial \Gamma^{\sigma_1}_{4\sigma_1}}{\partial x_2^{4}}.$$

Now

$$(18.3)\qquad \begin{aligned} C^{\sigma_1}_{4,4} &= g^{\sigma,\beta}\left(\frac{\partial g_{4,\beta}}{\partial x_1^{4}} - \frac{\partial g_{4,4}}{\partial x_2^{\beta}}\right) \\ \Gamma^{\sigma_1}_{4\sigma_1} &= g^{\sigma,\beta}\frac{\partial g_{4,\beta}}{\partial x_1^{\sigma}}. \end{aligned}$$

This yields

$$(18.4)\qquad \begin{aligned} T_{4,4} &\approx -\frac{\partial}{\partial x_1^{\sigma}}\left(g^{\sigma,\beta}\frac{\partial g_{4,4}}{\partial x_2^{\beta}}\right) \\ &\approx -\sum_{\sigma=1}^{4}\frac{\partial^2 g_{4,4}}{\partial x_1^{\sigma}\partial x_2^{\sigma}}. \end{aligned}$$

If we now impose the condition, see (10.3), (10.4),

$$(18.5)\qquad \frac{\partial g_{4,4}}{\partial x_1^{\sigma}} = \text{Const.}\,\frac{\partial g_{4,4}}{\partial x_2^{\sigma}},$$

we obtain upon setting $T_{4,4}=0$, the equation

$$(18.6)\qquad \sum_{\sigma=1}^{4}\frac{\partial^2 g_{4,4}}{\partial x_1^{\sigma 2}} = 0. \qquad \text{(Laplace's equation).}$$

We notice that $g_{4,4}$ as given by (10.2) satisfies $\frac{\partial g_{4,4}}{\partial x_1^{\sigma}} = k\,\frac{\partial g_{4,4}}{\partial x_2^{\sigma}}$ provided $m_2 \equiv m_1$, and moreover, $Lap_1\, g_{4,4} = Lap_2\, g_{4,4} = 0$. Hence it seems reasonable to choose as our invariant law for determining the $g_{\alpha,\beta}$, the tensor laws

$$(18.7)\qquad \begin{aligned} T_{\alpha,\beta} &= 0 \\ T^{*}_{\alpha,\beta} &= 0, \end{aligned}$$

where $T^{*}_{\alpha,\beta} = T^{\sigma_1}_{\alpha,\beta\sigma}$.

For our line element we choose

$$(18.8)\quad ds^2 = -e^{\lambda(r_1,r_2)} dr_1 dr_2 - e^{\sigma(r_1,r_2)} r_1 r_2\, d\theta_1 d\theta_2 - e^{\mu(r_1,r_2)} r_1 r_2 \sin\theta_1 \sin\theta_2\, d\varphi_1 d\varphi_2 + e^{\nu(r_1,r_2)} dt_1 dt_2$$

Let us first investigate the differential equations of the dyodesics involving $\ddot\theta_1$, $\ddot\theta_2$. From (8.1) we have

$$(18.9)\quad \begin{aligned} &\ddot\theta_1 + \Gamma^{2}_{\alpha\beta,} \dot x_1^\alpha \dot x_1^\beta + C^{2,}_{\alpha,\beta} \dot x_1^\alpha \dot x_2^\beta = 0 \\ &\ddot\theta_2 + \Gamma^{,2}_{,\alpha\beta} \dot x_2^\alpha \dot x_2^\beta + C^{,2}_{\alpha,\beta} \dot x_1^\alpha \dot x_2^\beta = 0 . \end{aligned}$$

The only Γ's involved are

$$(18.10)\quad \begin{aligned} &\Gamma^{2}_{21,} = \frac{1}{r_1 r_2} e^{-\sigma} \frac{\partial}{\partial r_1}\left(e^\sigma r_1 r_2\right) = \frac{1}{r_1} + \frac{\partial\sigma}{\partial r_1} \\ &\Gamma^{,2}_{,21} = \frac{1}{r_2} + \frac{\partial\sigma}{\partial r_2} . \end{aligned}$$

Among the C's we have

$$(18.11)\quad \begin{aligned} &C^{2,}_{3,3} = -e^{-\sigma} e^{\mu} \sin\theta_1 \cos\theta_2 \\ &C^{2,}_{2,1} = \frac{1}{r_1 r_2} e^{-\sigma} \frac{\partial}{\partial r_2}\left(e^\sigma r_1 r_2\right) = \frac{1}{r_2} + \frac{\partial\sigma}{\partial r_2} \\ &C^{,2}_{3,3} = -e^{-\sigma} e^{\mu} \sin\theta_2 \cos\theta_1 \\ &C^{,2}_{1,2} = \frac{1}{r_1} + \frac{\partial\sigma}{\partial r_1} . \end{aligned}$$

Hence

$$(18.12)\quad \begin{aligned} &\ddot\theta_1 + \left(\frac{1}{r_1} + \frac{\partial\sigma}{\partial r_1}\right)\dot\theta_1 \dot r_1 + \left(\frac{1}{r_2} + \frac{\partial\sigma}{\partial r_2}\right)\dot\theta_1 \dot r_2 - e^{\mu-\sigma} \sin\theta_1 \cos\theta_2\, \dot\varphi_1 \dot\varphi_2 = 0 \\ &\ddot\theta_2 + \left(\frac{1}{r_2} + \frac{\partial\sigma}{\partial r_2}\right)\dot\theta_2 \dot r_2 + \left(\frac{1}{r_1} + \frac{\partial\sigma}{\partial r_1}\right)\dot\theta_2 \dot r_1 - e^{\mu-\sigma} \sin\theta_2 \cos\theta_1\, \dot\varphi_1 \dot\varphi_2 = 0 . \end{aligned}$$

If initially $\theta_1 = \theta_2 = \frac{\pi}{2}$, $\dot\theta_1 = \dot\theta_2 = 0$, then $\theta_1 \equiv \theta_2 \equiv \pi/2$ satisfies (18.12) and the initial conditions, so from our uniqueness theorem is the solution of (18.12). Our line element becomes

$$(18.13)\quad ds^2 = -e^\lambda dr_1 dr_2 - r_1 r_2 e^\mu d\varphi_1 d\varphi_2 + e^\nu dt_1 dt_2 .$$

We obtain

$$(18.14)\quad \begin{aligned} &\Gamma^{1}_{11,} = \frac{\partial\lambda}{\partial r_1} & &\Gamma^{,1}_{,11} = \frac{\partial\lambda}{\partial r_2} \\ &\Gamma^{3}_{21,} = \frac{1}{r_1} + \frac{\partial\mu}{\partial r_1} & &\Gamma^{,3}_{,12} = \frac{1}{r_2} + \frac{\partial\mu}{\partial r_2} \\ &\Gamma^{3}_{31,} = \frac{\partial\nu}{\partial r_1} & &\Gamma^{,3}_{,13} = \frac{\partial\nu}{\partial r_2} \\ &C^{1,}_{2,2} = -e^{\mu-\lambda}\left(1 + r_2 \frac{\partial\mu}{\partial r_2}\right) r_1 & &C^{,1}_{2,2} = -e^{\mu-\lambda}\left(1 + r_1 \frac{\partial\mu}{\partial r_1}\right) r_2 \\ &C^{1,}_{3,3} = e^{\nu-\lambda} \frac{\partial\nu}{\partial r_2} & &C^{,1}_{3,3} = e^{\nu-\lambda} \frac{\partial\nu}{\partial r_1} \\ &C^{2,}_{2,1} = \frac{\partial\mu}{\partial r_2} + \frac{1}{r_2} & &C^{,2}_{1,2} = \frac{\partial\mu}{\partial r_1} + \frac{1}{r_1} \\ &C^{3,}_{3,1} = \frac{\partial\nu}{\partial r_2} & &C^{,3}_{1,3} = \frac{\partial\nu}{\partial r_1} . \end{aligned}$$

The equations for the dyodesics take the form

$$(18.15)\qquad \begin{aligned} &\frac{d^2 r_1}{ds^2} + \frac{\partial\lambda}{\partial r_1}\left(\frac{dr_1}{ds}\right)^2 - r_1 e^{\mu-\lambda}\left(1+r_2\frac{\partial\mu}{\partial r_2}\right)\frac{d\varphi_1}{ds}\frac{d\varphi_2}{ds} + e^{\nu-\lambda}\frac{\partial\nu}{\partial r_2}\frac{dt_1}{ds}\frac{dt_2}{ds} = 0 \\ &\frac{d^2 r_2}{ds^2} + \frac{\partial\lambda}{\partial r_2}\left(\frac{dr_2}{ds}\right)^2 - r_2 e^{\mu-\lambda}\left(1+r_1\frac{\partial\mu}{\partial r_1}\right)\frac{d\varphi_1}{ds}\frac{d\varphi_2}{ds} + e^{\nu-\lambda}\frac{\partial\nu}{\partial r_1}\frac{dt_1}{ds}\frac{dt_2}{ds} = 0, \end{aligned}$$

$$(18.16)\qquad \begin{aligned} &\frac{d^2\varphi_1}{ds^2} + \left(\frac{1}{r_1} + \frac{\partial\mu}{\partial r_1}\right)\frac{d\varphi_1}{ds}\frac{dr_1}{ds} + \left(\frac{\partial\mu}{\partial r_2} + \frac{1}{r_2}\right)\frac{d\varphi_1}{ds}\frac{dr_2}{ds} = 0 \\ &\frac{d^2\varphi_2}{ds^2} + \left(\frac{1}{r_2} + \frac{\partial\mu}{\partial r_2}\right)\frac{d\varphi_2}{ds}\frac{dr_2}{ds} + \left(\frac{\partial\mu}{\partial r_1} + \frac{1}{r_1}\right)\frac{d\varphi_2}{ds}\frac{dr_1}{ds} = 0, \end{aligned}$$

$$(18.17)\qquad \begin{aligned} &\frac{d^2 t_1}{ds^2} + \frac{\partial\nu}{\partial r_1}\frac{dr_1}{ds}\frac{dt_1}{ds} + \frac{\partial\nu}{\partial r_2}\frac{dt_1}{ds}\frac{dr_2}{ds} = 0 \\ &\frac{d^2 t_2}{ds^2} + \frac{\partial\nu}{\partial r_2}\frac{dr_2}{ds}\frac{dt_2}{ds} + \frac{\partial\nu}{\partial r_1}\frac{dr_1}{ds}\frac{dt_2}{ds} = 0 \end{aligned}.$$

We see that (18.16) becomes

$$(18.18)\qquad \begin{aligned} &\frac{d^2\varphi_1}{ds^2} + \frac{d\varphi_1}{ds}\left(\frac{1}{r_1}\frac{dr_1}{ds} + \frac{1}{r_2}\frac{dr_2}{ds} + \frac{d\mu}{ds}\right) = 0 \\ &\frac{d^2\varphi_2}{ds^2} + \frac{d\varphi_2}{ds}\left(\frac{1}{r_1}\frac{dr_1}{ds} + \frac{1}{r_2}\frac{dr_2}{ds} + \frac{d\mu}{ds}\right) = 0, \end{aligned}$$

so that a single integration yields

$$(18.19)\qquad \begin{aligned} &r_1 r_2\frac{d\varphi_1}{ds} = h e^{-\mu} \\ &r_1 r_2\frac{d\varphi_2}{ds} = h e^{-\mu}, \end{aligned}$$

where we have chosen the constant of integration, h, to be the same in both terms of (18.19) by imposing the initial conditions $\frac{d\varphi_1}{ds} = \frac{d\varphi_2}{ds}$, $\varphi_2 = \varphi_1 + \pi$, at $s = 0$. The integration of (18.17) immediately yields

$$(18.20)\qquad \begin{aligned} &\frac{dt_1}{ds} = c_1 e^{-\nu} \\ &\frac{dt_2}{ds} = c_2 e^{-\nu}. \end{aligned}$$

Before determining the λ, μ, ν, we impose the following conditions

(18.21)

(1) $\lambda, \mu, \nu \to 0$ as $r_1, r_2 \to \infty$

(2) $g_{4,4} = e^{\nu}$ satisfies $\nabla_1^2 g_{4,4} = \nabla_2^2 g_{4,4} = 0$, along with $m \frac{\partial \nu}{\partial r_2} = M \frac{\partial \nu}{\partial r_1}$ when evaluated at $Mr_2 = mr_1$.

(3) The solutions to (18.15) shall yield $mr_1 \equiv Mr_2$.

Examining (18.15) we see that $mr_1 \equiv Mr_2$ will satisfy these equations provided

(18.22)

(1) $M \frac{\partial \lambda}{\partial r_1} = m \frac{\partial \lambda}{\partial r_2}$

(2) $M \frac{\partial \mu}{\partial r_1} = m \frac{\partial \mu}{\partial r_2}$, for $Mr_2 = mr_1$.

Now setting $T_{\alpha,\beta} = T^*_{\alpha,\beta} = 0$ we obtain

$$\frac{\partial^2 \lambda}{\partial r_1 \partial r_2} = 0$$

$$\frac{\partial^2 \mu}{\partial r_1 \partial r_2} = 0 \tag{18.23}$$

$$\frac{\partial^2 \nu}{\partial r_1 \partial r_2} = 0 .$$

The most general solutions of (18.23) are

$$\lambda, \mu, \nu = f_{\lambda,\mu,\nu}(r_1) + g_{\lambda,\mu,\nu}(r_2), \tag{18.24}$$

so that

$$g_{4,4} = e^{\nu} = e^{f+g} = \phi_1(r_1)\,\phi_2(r_2) . \tag{18.25}$$

We find that

$$g_{4,4} = e^{\nu} = \left(1 - \frac{mM^2}{(m+M)^2 r_1}\right)\left(1 - \frac{m^2 M}{(m+M)^2 r_2}\right) \tag{18.26}$$

satisfies (18.21).

Now we consider

$$e^{\mu} = \left(1 + \frac{M}{r_1}\right)\left(1 + \frac{m}{r_2}\right) \tag{18.27}$$

which satisfies

(18.28)

(1) $\dfrac{\partial^2 \mu}{\partial r_1 \partial r_2} = 0$

(2) $\mu \to 0$ as $r_1, r_2 \to \infty$

(3) $\dfrac{\frac{\partial \mu}{\partial r_1}}{\frac{\partial \mu}{\partial r_2}} = \dfrac{-\frac{M}{r_1^2}\left(1 + \frac{m}{r_2}\right)}{-\frac{m}{r_2^2}\left(1 + \frac{M}{r_1}\right)} = \dfrac{m}{M}$ if $M r_2 = m r_1$.

Hence (2) of (18.22) and (18.23) are satisfied by this (non-unique) μ .

Finally, we choose

$$\lambda = 2\mu - \nu \tag{18.29}$$

in order that for $m \ll M$, the Einstein one-body solution will be approximated. The λ of (18.29) satisfies

(18.30)

(1) $\dfrac{\partial^2 \lambda}{\partial r_1 \partial r_2} = 0$

(2) $\lambda \to 0$ as $r_1, r_2 \to \infty$

(3) $M \dfrac{\partial \lambda}{\partial r_1} = m \dfrac{\partial \lambda}{\partial r_2}$ for $M r_2 = m r_1$.

We now return to a first integral for the dyodesics, namely

$$g_{\alpha_1 \beta} \frac{dx_1^{\alpha}}{ds} \frac{dx_2^{\beta}}{ds} = 1, \quad \text{see } (8.2), \tag{18.31}$$

or

$$-e^{\lambda} \frac{dr_1}{ds} \frac{dr_2}{ds} - r_1 r_2 e^{\mu} \frac{d\varphi_1}{ds} \frac{d\varphi_2}{ds} + e^{\nu} \frac{dt_1}{ds} \frac{dt_2}{ds} = 1 . \tag{18.32}$$

Making use of (18.19), (18.20), $M r_2 \equiv m r_1$, we obtain

$$-e^{\lambda} \frac{m}{M} \left(\frac{dr_1}{ds}\right)^2 - \frac{M h^2}{m} \frac{e^{-\mu}}{r_1^2} + c_1 c_2 e^{-\nu} = 1, \tag{18.33}$$

and replacing ds by $\frac{m}{M} e^{\mu} \frac{r_1^2}{h} d\varphi_1$ from (18.19), we obtain

$$(18.34) \qquad -e^{\lambda-2\mu} \frac{M}{m} h^2 \left(\frac{1}{r_1^2}\frac{dr_1}{d\varphi_1}\right)^2 - \frac{M}{m} h^2 \frac{e^{-\mu}}{r_1^2} + C_1 C_2 e^{-\nu} = 1 .$$

Making use of (18.29) we infer

$$(18.35) \qquad \left(\frac{1}{r_1^2}\frac{dr_1}{d\varphi_1}\right)^2 + \frac{e^{\nu-\lambda}}{r_1^2} - C_1 C_2 \frac{m}{Mh^2} = -\frac{m}{Mh^2} e^{\nu} .$$

We now define

$$(18.36) \qquad \begin{aligned} V &\equiv \frac{1}{r_1} \\ h_1 &\equiv \frac{Mh}{m} , \end{aligned}$$

so that

$$(18.37) \qquad \left(\frac{dV}{d\varphi_1}\right)^2 + V^2 \left[\frac{\left(1 - \frac{M^2 m}{(m+M)^2} V\right)}{1+MV}\right]^2 - \frac{C_1 C_2 m}{Mh^2} = \frac{-M}{m h_1^2}\left[1 - \frac{M^2 m}{(m+M)^2} V\right]^2 .$$

Neglecting terms of the order of V^4, V^5, ---, etc., we obtain

$$(18.38) \qquad \left(\frac{dV}{d\varphi_1}\right)^2 + V^2\left(1 - \frac{2M^2 m}{(m+M)^2} V\right)(1 - 2MV) - \frac{C_1 C_2 m}{Mh^2} = -\frac{M}{m h_1^2}\left[1 - \frac{M^2 m}{(m+M)^2} V\right]^2 .$$

Differentiating with respect to φ_1 and factoring out $2\frac{dV}{d\varphi_1}$ we obtain

$$(18.39) \qquad \frac{d^2 V}{d\varphi_1^2} + \left(1 + \frac{Mm}{(1+\frac{m}{M})^4 h_1^2}\right) V = \frac{M}{(1+\frac{m}{M})^2 h_1^2} + 3MV^2\left[1 + \frac{Mm}{(m+M)^2}\right].$$

For $m \ll M$, (18.39) becomes

$$(18.40) \qquad \frac{d^2 V}{d\varphi_1^2} + V = \frac{M}{h_1^2} + 3MV^2, \quad \text{neglecting } \frac{Mm}{h_1^2} \ll 1 .$$

This is the Einstein solution for the motion of an infinitesimal particle moving in the field of a point gravitational mass M.

If $M = m$, we obtain, neglecting $\frac{M^2}{16 h_1^2} \ll 1$,

$$(18.41) \qquad \frac{d^2 V}{d\varphi_1^2} + V = \frac{\left(\frac{M}{4}\right)}{h_1^2} + 3\left(1 + \frac{1}{4}\right) M V^2 .$$

The orbit differs from the classical orbit only in the advance of periastron equal to that which an infinitesimal planet describing the same relative orbit would undergo in the field of a star having a mass equal to 5/4 the mass of one of the double stars.

XIX. Possible future work.

1. The investigation of a Finslerian two-point metric,

$$ds^2 = g_{\alpha,\beta}(x_1, \dot{x}_1, x_2, \dot{x}_2)\, dx_1^{\alpha} dx_2^{\beta},$$

where $g_{\alpha,\beta}$ is homogeneous of degree zero in $\dot{x}_1^{i}$, $\dot{x}_2^{i}$.

2. An investigation of conformal metrics,

$$\bar{g}_{\alpha,\beta}(x_1, x_2) = \phi(x_1, x_2)\, g_{\alpha,\beta}(x_1, x_2).$$

3. A generalization to a metric depending on m-points,

where $$ds^m = g_{\alpha_1, \alpha_2, \cdots, \alpha_m}(x_1, x_2, \cdots, x_m)\, dx_1^{\alpha_1} dx_2^{\alpha_2} \cdots dx_m^{\alpha_m}$$

4. Applications to physics.

The effects of a physical experiment depend on two points, the observer and the observed.

Applications to a unified field theory, see Annals of Math, 1944, A. Einstein.

The investigation of non-static line elements in § 18.

The investigation of the similarity of

$$\frac{d^2 x_1^{i}}{ds^2} + \Gamma^{i}_{jk_1} \frac{dx_1^{j}}{ds} \frac{dx_1^{k}}{ds} + \left(C^{i}_{\alpha,\beta} \frac{dx_2^{\beta}}{ds}\right) \frac{dx_1^{\alpha}}{ds} = 0$$

with the equations of motion of a charged particle given by

$$\frac{d^2 x^{i}}{ds^2} + \Gamma^{i}_{jk} \frac{dx^{j}}{ds} \frac{dx^{k}}{ds} + \frac{e}{m_0} F_{\alpha}^{i} \frac{dx^{\alpha}}{ds} = 0$$

where F_{α}^{i} is the electromagnetic field tensor.

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