A General Differential Geometry with Two Types of Linear Connection

Thesis by

Max Wyman

In partial fulfilment of the requirements for the degree of Doctor of Philosophy

California Institute of Technology

Pasadena, California

1940

Summary

The object of this thesis was the study of a differential geometry for a Hausdorff space endowed with an affine linear connection and a non-holonomic linear connection. The coordinate spaces were taken to be Banach spaces. In Chapter II we define the notion of a non-holonomic contravariant vector field, and by means of the non-holonomic linear connection introduce the operation of covariant differentiation. It was then found that many of the formal tensor theorems carried over to such spaces.

For certain types of Hausdorff space it is possible to develop a normal representation theory, and by means of it to obtain normal non-holonomic vector forms. This then enables us to generalize the Michal-Hyers replacement theorem for differential invariants.

Chapter IV is concerned with the determination of non-holonomic linear connections. This leads to the consideration of interspace adjoints for linear functions.

In the main the results obtained in this thesis are generalizations of results obtained for finite dimensional spaces by A.D. Michal and J.L. Botsford. However the projective theory developed in Chapter V is new for spaces of finite dimension.

CONTENTS

INTRODUCTION

The study of a differential geometry for function spaces was begun in a paper(1) by A. D. Michal. From this Professor Michal was led to more general considerations in which the geometric spaces were taken to be Hausdorff spaces and the coordinate spaces were taken to be Banach spaces. Since these studies were first initiated the literature on the subject has become quite extensive, and for this reason it would be impossible to give even a reasonably short summary of the field. However I have tried to give short summaries of some of the definitions and theorems that it has been necessary for me to use. These theorems are stated without proof, and in many cases the statement of the theorems is taken over verbatim.

In one of his papers(2) Professor Michal points out that further generalisations are possible. He suggested the study of a geometry with a linear connection whose law of transformation is more general than that of an affine linear connection. It is this suggested generalisation that I have tried to carry out.

I should like to thank Professor Michal for not only suggesting this problem to me, but also for the many hours he has spent in helping me carry out this investigation.

(1). See Michal [4].
(2). See Michal [1].

CHAPTER I

This chapter will be devoted to giving a short summary of some of the concepts involved in obtaining a differential geometry for a Hausdorff space with coordinates in a Banach space. Most of these results can be found in Michal [1], Michal [2], and Michal-Hyers [1].

Section 1.1. Coordinate systems and transformations of coordinates.

Let H be a Hausdorff space, and let U_0 be a fixed neighborhood of H. We assume that U_0 can be mapped homeomorphically on an open set S of a Banach space E, and further that all neighborhoods $U \subset H$ can be mapped homeomorphically on open sets $\Sigma \subset S$. We call the mapping function x(P), (which maps a neighborhood U homeomorphically on an open set Σ) a coordinate system. The neighborhood U is called the geometric domain of x(P), and the open set Σ is called the coordinate domain of x(P).

Suppose that x(P) and $\bar{x}(P)$ are two coordinate systems with intersecting geometric domains U_1 and U_2 respectively. If Σ_1 and Σ_2 are the coordinate domains of x(P) and $\bar{x}(P)$ respectively, then the intersection U_1U_2 of U_1 and U_2 will be mapped onto a point set $\Sigma_1' \subset \Sigma_1$ by x(P), and onto a point set $\Sigma_2' \subset \Sigma_2$ by $\bar{x}(P)$. Since U_1U_2 is an open set, we have by Bohnenblust [1] pp. 42-44 that both Σ_1' and Σ_2' are also open sets. Thus we see that two coordinate systems with intersecting geometric domains induces a homeomorphism $\bar{x} = \bar{x}(x)$ taking the open set Σ_1' into the open set Σ_2'. We call this a transformation of coordinates, and say that Σ_1' and Σ_2' are the domains of definition of $\bar{x}(x)$ and $x(\bar{x})$ respectively.

Def. 1.11. A transformation $\bar{x} = \bar{x}(x)$ taking an open set of a Banach space E into an open set of E will be called a regular transformation if $\bar{x} = \bar{x}(x)$ and its inverse $x(\bar{x})$ are Fréchet differentiable throughout their respective domains.

The class of transformations of coordinates previously discussed does not allow us to provide a very extensive geometry for H. For this reason we assume that all transformations of coordinates are regular transformations, and further that $\bar{x}(x; \delta_1 x; \delta_2 x)$ exists continuous in x at each point of the domain of definition of $\bar{x}(x)$.

Def. 1.12. Let P_0 be any chosen point of the Hausdorff space H. A geometric object whose component ξ (the map of the object in E) in the coordinate system $x(P)$ is related to its component $\bar{\xi}$ in the coordinate system $\bar{x}(P)$ by

$$\bar{\xi} = \bar{x}(x(P_0); \xi) \tag{1.11}$$

will be called a contravariant vector associated with P_0.[3]

Def. 1.13. A geometric object whose component in the coordinate system $x(P)$ is $\xi(x)$ a function on Σ to E, and such that $\xi(x)$ is related to the component $\bar{\xi}(\bar{x})$ of the geometric object in $\bar{x}(P)$ by

(3) We assume of course that P_0 is in the geometric domains of both $x(P)$ and $\bar{x}(P)$.

$$\bar{\xi}(\bar{x}) = \bar{x}(x;\xi(x)) \tag{1.12}$$

is called a contravariant vector field.[4]

Def. 1.14. Let ξ_1 and ξ_2 be two arbitrary contravariant vectors. A geometric object whose components $\Gamma(x,\xi_1,\xi_2)$, $\bar{\Gamma}(\bar{x},\bar{\xi}_1,\bar{\xi}_2)$, are bilinear functions of the vectors will be called a linear connection if, in the intersection of two Hausdorff neighborhoods the components have the law of transformation

$$\bar{\Gamma}(\bar{x},\bar{\xi}_1,\bar{\xi}_2) = \bar{x}(x;\Gamma(x,\xi_1,\xi_2)) - \bar{x}(x;\xi_1;\xi_2). \tag{1.13}$$

With these definitions it is possible to prove the following two theorems.

Theorem 1.11. Let the bilinear function $\Gamma(x,\xi_1,\xi_2)$ in the contravariant vectors be the component in the $x(p)$ coordinate system of a geometric object. Then, a necessary and sufficient condition that

$$\delta\xi(x) + \Gamma(x,\xi(x),\delta x) \tag{1.14}$$

be the component in the $x(p)$ coordinate system of a contravariant vector field for every Fréchet differentiable contravariant vector field $\xi(x)$ is that $\Gamma(x,\xi_1,\xi_2)$ be the component of a linear connection.

Theorem 1.12. Let $\Gamma(x,\xi_1,\xi_2)$ be the components of a linear connection and $F(x,\xi_1,\xi_2,\cdots,\xi_n)$ a function with the following properties:

(4) Here we assume that $x(p)$, $\bar{x}(p)$ have intersecting geometrical domains which induces a regular transformation of coordinates $\bar{x} = \bar{x}(x)$.

(i) F is a contravariant vector field valued multilinear form in the "n" arbitrary vectors $\xi_1, \xi_2, \cdots, \xi_n$;

(ii) the partial Fréchet differential $F(x, \xi_1, \xi_2, \cdots, \xi_n; \delta x)$ exists and is continuous in x.

Then the function

$$F(x, \xi_1, \xi_2, \cdots; \xi_n / \delta x) = F(x, \xi_1, \xi_2, \cdots, \xi_n; \delta x) - \sum_{i=1}^{n} F(x, \xi_1, \cdots; \xi_{i-1}, \Gamma(x, \xi_i, \delta x), \xi_{i+1}, \cdots; \xi_n)$$
$$+ \Gamma(x, F(x, \xi_1, \xi_2, \cdots; \xi_n), \delta x) \tag{1.15}$$

is a contravariant vector field valued multilinear form in $\xi_1, \xi_2, \cdots, \xi_n, \delta x$.

We abbreviate expression (1.14) to $\xi(x / \delta x)$ and call it the covariant differential of ξ(x). Similarly $F(x, \xi_1, \cdots; \xi_n / \delta x)$ of (1.15) is called the covariant differential of F.

When the coordinate transformations possess continuous third Fréchet differentials and the component of the linear connection $\Gamma(x, \xi_1, \xi_2)$ possesses a continuous first Fréchet differential it is possible to show that

$$B(x, \xi_1, \xi_2, \xi_3) = \Gamma(x, \xi_1, \xi_2; \xi_3) - \Gamma(x, \xi_1, \xi_3; \xi_2)$$
$$+ \Gamma(x, \Gamma(x, \xi_1, \xi_2), \xi_3) - \Gamma(x, \Gamma(x, \xi_1, \xi_3), \xi_2). \tag{1.16}$$

is the component of a contravariant vector field valued trilinear form. We call $B(x, \xi_1, \xi_2, \xi_3)$ the curvature form.

When no ambiguity occurs we shall speak of the component $\Gamma(x, \xi_1, \xi_2)$ as being the linear connection. Similar terminology will be used in the case of other geometric objects.

Section 1.2. Abstract normal coordinates.

In order to introduce abstract normal coordinates it is necessary to place some further differentiability restrictions on our coordinate transformations.

Def. 1.21. Let E and E_1 be Banach spaces, and let X be a bounded convex region of E. A function F(x) on X to E_1 is said to be of class $C^{(n)}$ uniformly on X if,

1) $F(x; \delta_1 x; \delta_2 x; \cdots ; \delta_k x)$ exists and is uniformly continuous in x with respect to its entire set of arguments for $\|\delta_i x\| < 1$ $i = 1, \cdots, k$, $k = 1, \cdots n$, and x in X,

2) there exists a constant M_k such that

$$\| F(x; \delta_1 x; \delta_2 x; \cdots; \delta_k x) \| \leq M_k \; \|\delta_1 x\| \, \|\delta_2 x\| \cdots \|\delta_k x\|$$

for all x in X, and $k = 1, 2, \cdots, n.$ (5)

Def. 1.22. Let E and E_i $(i = 1, 2, \cdots n)$ be Banach spaces. A function F(x) on a subset $\Sigma \subset E$ to E_1 is said to be of class $C^{(n)}$ locally uniformly at x_0 if there exists a neighborhood of x_0 on which F(x) is of class $C^{(n)}$ uniformly. A function $F(x, y_1, y_2, \cdots, y_n)$ multilinear in $y_1, y_2, \cdots y_n$ on $\Sigma E_1 \cdots E_n$ to E_1 is said to be of class $C^{(n)}$ locally uniformly at x_0 if there exists a neighborhood X of x_0 such that $F(x, y_1, \cdots y_n)$ is of class $C^{(n)}$ uniformly on $X \Xi_1 \Xi_2 \cdots \Xi_n$,

(5) Although this is not the original definition as given in Hildebrandt and Graves [1], it is equivalent to it. See Michal-Hyers [2].

where Ξ_i is the open set $\|y_i\| < 1$.

Def. 1.23. A regular transformation $\bar{x} = \bar{x}(x)$ will be said to be of class $K^{(m)}$ if the functions $\bar{x}(x)$ and its inverse $x(\bar{x})$ are of class $C^{(m)}$ locally uniformly at each point of their respective domains.

Def. 1.24. A set of coordinate systems shall be said to form an allowable $K^{(m)}$ set of coordinate systems if the following postulates are satisfied.

1. The transformation of coordinates from one coordinate system of the set to another of the set is a regular transformation of class $K^{(m)}$.
2. Any coordinate system obtained by a regular transformation of class $K^{(m)}$ from a coordinate system of the set is in the set.
3. If $x(P)$ is a coordinate system of the set with geometric domain U and coordinate domain Σ, then the correspondence $x(P)$ taking a Hausdorff neighborhood $U_1 \subset U$ into an open set $\Sigma_1 \subset \Sigma$ is also a coordinate system of the set.
4. The coordinate system which maps the fundamental Hausdorff neighborhood U_0 onto the fixed open set S is in the set.

C onsider a Hausdorff space H which possesses allowable $K^{(n+2)}$ $(n \geq 2)$ coordinate systems and a linear connection $\Gamma(x, \xi_1, \xi_2)$. We also make the following restrictions.

(a) The fundamental open set S of E contains the zero of E.

(b) The linear connection is of class $C^{(n)}$ locally uniformly on the coordinate domain Σ of an allowable K^{n+2} coordinate system $x(P)$.

Michal-Hyers [1] then show that for any q of Σ there exists neighborhoods X_0 of q and Y_0 of 0 such that the differential system

$$\frac{d^2x}{ds^2} + \Gamma\left(x, \frac{dx}{ds}, \frac{dx}{ds}\right) = 0, \quad x(0) = p, \left(\frac{dx}{ds}\right)_0 = \xi. \tag{1.21}$$

has a unique solution $x = f(p, s\xi)$ for any $p \in X_0$ and any $\xi \in Y_0$. Further for any $0 \leq s \leq 1$ the function $f(p, s\xi)$ is of class $C^{(n)}$ uniformly in (p, ξ) on $X_0 Y_0$. The image $\mathcal{X}(P, Q, s)$ of $f(p, s\xi)$ in the coordinate system $\mathfrak{x}(P)$ is a parameterised curve in H. This is called a path.

<u>Def. 1.25</u>. A coordinate system $\mathfrak{y}(P)$ in which the equations of paths through a given point P_0 (with coordinate y = 0) take the form $y = s\xi$ is called a normal coordinate system with center P_0.

Michal-Hyers [1] prove the following two important theorems on normal coordinate systems.

<u>Theorem 1.21</u>. Suppose that the hypotheses (a) and (b) are satisfied. Then corresponding to each point $q \in \Sigma$ there is a constant $c > 0$ and a function h(p,x) of class $C^{(n)}$ uniformly on $E^2((q)_{2c})$ (1) such that for any p in $(q)_c$ the transformation y = h(p,x) is of class $K^{(n)}$ for $x \in (p)_c$ and defines a normal coordinate system $y(P)$ with center P_0, where $p = \mathfrak{x}(P_0)$.

The solution $f(p, s\xi)$ of (1.21) for s = 1 and ξ = y becomes f(p,y). The function h(p,x) of theorem 1.21 turns out to be the inverse of f(p,y). From this theorem we see that if P_0 is any point in the

(1) The open sphere $\|x - x_0\| < a$ is denoted by $(x_0)_a$ or $E((x_0)_a)$.

geometric domain of an allowable $K^{(n+2)}$ coordinate system $x(P)$, then $x(P)$ determines a normal coordinate system $y(P)$ with center P_o, where

$$y(P) = h(p,x(P)). \tag{1.22}$$

It should be noticed that $y(P)$ is not in general an allowable $K^{(n+2)}$ coordinate system.

<u>Theorem</u> <u>1.22</u>. Let assumptions (a), (b), be satisfied, and let $x(P)$, $\bar{x}(P)$ be two allowable $K^{(n+2)}$ coordinate systems whose geometric domains have a point P_o in common. Suppose that $y(P)$, $\bar{y}(P)$ are the normal coordinate systems with center P_o determined by $x(P)$ and $\bar{x}(P)$ respectively. Then there exists two open subsets S_y and $S_{\bar{y}}$ of the coordinate domains of $y(P)$ and $\bar{y}(P)$ respectively such that

1) $0 \in S_y$, $0 \in S_{\bar{y}}$,

2) the linear transformation

$$\bar{y} = \bar{x}(p;y) \tag{1.23}$$

takes S_y ~~in~~ into $S_{\bar{y}}$.

The latter theorem thus tells us that under a general transformation of allowable $K^{(n+2)}$ coordinate systems the corresponding transformation of normal coordinates is linear. This property and some other properties which normal coordinate systems have provide a great simplification of many of the proofs of tensor theorems. We list some of these properties which we shall make use of later on.

1. Let the linear connection $\Gamma(x, \xi_1, \xi_2)$ be symmetric in ξ_1, ξ_2. Further let $^{+}\Gamma(y, \xi_1, \xi_2)$ be the components of the linear connection in a normal coordinate system y(p). Then

$$^{+}\Gamma(0, \lambda_1, \lambda_2) = 0 \qquad \text{for all } \lambda_1, \lambda_2 \text{ in E.}$$

2. Let μ(y) = f(p,y) and ν(x) = h(p,x), then

   (a) $\mu(0; \delta y) = \delta y,$

   (b) $\mu(0; \delta_1 y; \delta_2 y) = -\Gamma(p, \delta_1 y, \delta_2 y),$

   (c) $\nu(p; \delta x) = \delta x,$

   (d) $\nu(p; \delta_1 x; \delta_2 x) = \Gamma(p, \delta_1 x, \delta_2 x).$

In concluding this chapter I should like to give a summary of the notation that shall be carried over to the succeeding chapters. That is the following letters shall always be used with the same meaning.

1. U is a geometric domain of a coordinate system.
2. U_0 is a fixed Hausdorff neighborhood which can be mapped homeomorphically on a fixed open set of S $\subset$ E.
3. Σ is the coordinate domain of a coordinate system.
4. S is a fixed open set of the Banach space E which contains all the coordinate domains of all coordinate systems.
5. $\mathfrak{X}$(P) is a coordinate system.
6. x is a point in the coordinate domain of the coordinate system $\mathfrak{X}$(P).
7. ξ is the component of an arbitrary contravariant vector.
8. ξ(x) is the component of a contravariant vector field.

9. $\Gamma(x, \xi_1, \xi_2)$ is the component of a linear connection.

10. $\delta f(x)$, or $f(x; \delta x)$ means the Fréchet differential of $f(x)$.

11. $B(x, \xi_1, \xi_2, \xi_3)$ shall mean the component of the curvature form based on the linear connection $\Gamma(x, \xi_1, \xi_2)$.

12. $f(p, s\xi)$ is the unique solution of

$$\frac{d^2x}{ds^2} + \Gamma\left(x, \frac{dx}{ds}, \frac{dx}{ds}\right) = 0, \quad x(0) = p, \quad \left(\frac{dx}{ds}\right)_0 = \xi.$$

13. $h(p,x)$ is the inverse of $f(p,y)$.

14. $y(p)$ is a normal coordinate system.

15. y is a point in the coordinate domain of the normal coordinate system $y(p)$.

16. ${}^{+}\xi(y)$ means the component of the arbitrary contravariant vector ξ in the normal coordinate system $y(p)$.

17. ${}^{+}\Gamma(y, {}^{+}\xi_1, {}^{+}\xi_2)$ is the component of the linear connection in the normal coordinate system $y(p)$.

In what follows we shall assume $\Gamma(x, \xi_1, \xi_2)$ to be symmetric in ξ_1 and ξ_2.

CHAPTER II

Section 2.1. Representations and changes of representation.

Def. 2.11. Two Banach spaces B_1, B_2 are said to be equivalent[1] if there exists a mapping function on B_1 to B_2 with the following properties.

1. $f(Y)$ is linear in Y and ranges over the whole of B_2.
2. $f(Y)$ is solvable in Y with inverse $f^{-1}(Z)$ a function on B_2 to B_1.

Suppose that we have a Hausdorff space H with coordinates in a Banach space E which satisfies all the assumptions of the previous chapter. Suppose moreover that to each point P of H we can associate a Banach space B_P such that,

(a) any two of the Banach spaces B_{P_1}, and B_{P_2} are equivalent to each other;

(b) each Banach space B_P is equivalent to a fixed Banach space E_1.

Def. 2.12. Let P be a fixed point of H, and suppose that B_P is the associated Banach space. We denote an element of B_P by Y_P. A vector coordinate system is a mapping function $X(P,Y_P)$ with the following properties.

(i). For fixed P, $X(P,Y_P)$ has values in E_1 and ranges over the whole of E_1.

(1) This definition is not the same as that given in Banach 1 . Banach requires $\|f(Y)\| = \|Y\|$.

(ii). $X(P,Y_p)$ is solvable linear in Y_p with inverse $Y_p = Y_p(P,X)$ a function on E_1 to B_p.

The values of $X(P,Y_p)$ which lie in E_1 we call the vector coordinates of B_p.

Suppose that $X(P_1,Y_{p_1})$ and $X(P_2,Y_{p_2})$ are two vector coordinate systems for the spaces B_{p_1} and B_{p_2} respectively. Since B_{p_1} is equivalent to B_{p_2} there exists a function $Y_{p_2} = f(Y_{p_1})$ which is solvable linear in Y_{p_1}. Thus

$$\bar{X} = \bar{X}(P_2, f(Y_{p_1}(P_1,X))). \tag{2.11}$$

If we let $\varphi(P_1,P_2,X) = \bar{X}(P_2,f(Y_{p_1}(P_1,X)))$, we see that $\varphi(P_1,P_2,X)$ is a solvable linear function of X on E_1 to E_1. That is any two vector coordinate systems induce a linear automorphism $\bar{X} = \varphi(P_1,P_2,X)$ of E. This we shall call a transformation of vector coordinates.

Def. 2.13. A representation shall mean,

(a). a coordinate system $x(P)$ with geometric domain U,

(b). for each point $P \in U$ we have associated one vector coordinate system $X(P,Y_p)$ with the associated space B_p.

To obtain a different representation we can change the coordinate system, change the vector coordinate system or do both.

Def. 2.14. A change of representation shall mean the composite of,

1. a change of coordinate systems $x(P)$ to $\bar{x}(P)$,

2. for each point P in the intersection U_1U_2 of the geometrical

domains of $x(P)$ and $\bar{x}(P)$ we change the vector coordinate system $X(P,Y_p)$ to $\bar{X}(P,Y_p)$.

For each point P of $U_1 U_2$ the change of vector coordinate systems induces a linear automorphism $\bar{X} = \varphi(P,X)$ of E_1. Since $P = P(x)$ under the coordinate system $x(P)$ we can let $\varphi(P(x),X) = M(x,X)$. Thus for each x in the domain of definition of $\bar{x}(x)$, $M(x,X)$ is a solvable linear function of X on E_1 to E_1. We call the transformations $\bar{x} = \bar{x}(x)$ and $\bar{X} = M(x,X)$ induced by a change of representation a transformation of representation.

Def. 2.15. Let $\bar{x}(x)$ be a regular transformation mapping an open set $\Sigma_1 \subset E$ homeomorphically onto an open set $\Sigma_2 \subset E$, and suppose that $M(x,X)$ is a function on $\Sigma_1 E_1$ to E_1. We shall call the pair of functions $\bar{x}(x)$, $M(x,X)$ a regular transformation of representation if,

(a) for each $x \in \Sigma_1$, $M(x,X)$ is a solvable linear function of X with inverse $N(\bar{x},X)$;

(b) $M(x,X)$ and $N(\bar{x},X)$ possess continuous Fréchet differentials at each point of their domains of definition.

We shall assume throughout that all transformations of representation arising from a change of representation are regular transformations of representation.

Def. 2.16 (a). Suppose that two transformations of representation induce a transformation of representation $\bar{x} = \bar{x}(x)$, $\bar{X} = M(x,X)$, and let P_0 be a point in the intersection of the geometrical domains of the coordinate systems $x(P)$ and $\bar{x}(P)$ which are involved. A non-holonomic contravariant vector[2] is a geometric object associated with P_0 whose maps in the given representations are elements $V(x(P_0))$ and $\bar{V}(\bar{x}(P_0))$ of E_1 respectively, and which are related by

$$\bar{V}(\bar{x}(P_0)) = M(x(P_0),V(x(P_0))). \tag{2.12}$$

We call the maps of geometric objects under a given representation the components of the geometric object. It will be noticed that the components may be elements of E or of E_1. A geometric object with components in E_1 will be called a non-holonomic geometric object.

Def. 2.16. A non-holonomic contravariant vector field[3] is a geometric object whose component in any given representation is $V(x)$ a function on Σ to E_1, and such that under a change of representation the components are related by

$$\bar{V}(\bar{x}) = M(x,V(x)). \tag{2.13}$$

Def. 2.17. Let V be an arbitrary n.h.c.v., and ξ an arbitrary contravariant vector. Consider a geometric object whose component $K(x,V,\xi)$ in ~~any given~~ every representation is a bilinear function of the

(2). This is abbreviated to n.h.c.v.

(3). This is abbreviated to n.h.c.v.f.

vectors on EE_1E to E_1. Such a geometric object is called a non-holonomic linear connection[4] if its components $K(x,V,\xi)$, $\bar{K}(\bar{x},\bar{V},\bar{\xi})$ in two representations are related by

$$\bar{K}(\bar{x},\bar{V},\bar{\xi}) = M(x,K(x,V,\xi)) - M(x,V;\xi) \qquad (2.14)$$

under a change of representation.

Theorem 2.11. Let $K(x,V,\xi)$ be a bilinear function of V,ξ on EE,E to E_1. Then a necessary and sufficient condition that

$$\delta V(x) + K(x,V,\delta x) \qquad (2.15)$$

be the component of a n.h.c.v.f. for every Fréchet differentiable n.h.c.v.f. $V(x)$ is that $K(x,V,\xi)$ be the component of a n.h. linear connection.

Proof of necessity. If (2.15) is the component of a n.h.c.v.f. we have

$$\delta\bar{V}(\bar{x}) + \bar{K}(\bar{x},\bar{V},\delta\bar{x}) = M(x,\delta V + K(x,V,\delta x)). \qquad (2.16)$$

But
$$\bar{V}(\bar{x}) = M(x,V(x)). \qquad (2.17)$$

Hence
$$\delta\bar{V} = M(x,\delta V) + M(x,V(x);\delta x). \qquad (2.18)$$

(2.16) then becomes

$$\bar{K}(x,\bar{V},\delta\bar{x}) = M(x,K(x,V,\delta x) - M(x\ V;\delta x).$$

(4). Abbreviated to n.h. linear connection.

This proves the necessity, and the sufficiency is proven by reversing the steps.

For the expression (2.15) we use the notation $V(x \mid \delta x)$ and call it the covariant differential of $V(x)$.

Theorem 2.12. Let $K(x, V, \delta x)$ be a n.h. linear connection, and $F(x, \xi_1, \xi_2, \cdots \xi_n, V_1, \cdots V_s)$ be a n.h.c.v.f. valued multilinear form in the arbitrary contravariant vectors $\xi_1, \xi_2, \cdots \xi_n$, and the arbitrary n.h.c.v. $V_1, \cdots, V_s$. Further suppose that $F(x, \xi_1, \cdots, \xi_n, V_1, \cdots V_s; \delta x)$ exists continuous in x. Then

$$\begin{aligned} F(x, \xi_1, \cdots \xi_n, V_1, \cdots, V_s \mid \delta x) &= F(x, \xi_1, \cdots, \xi_n, V_1, \cdots, V_s; \delta x) \\ &- \sum_{i=1}^{n} F(x, \xi_1, \cdots, \xi_{i-1}, \Gamma(x, \xi_i, \delta x), \xi_{i+1}, \cdots \xi_n, V_1, \cdots, V_s) \\ &- \sum_{i=1}^{s} F(x, \xi_1, \cdots, \xi_n, V_1, \cdots, V_{i-1}, K(x, V_i, \delta x), V_{i+1}, \cdots, V_s) \\ &+ K(x\, F(x, \xi_1, \cdots, \xi_n, V_1, \cdots, V_s), \delta x) \end{aligned} \tag{2.19}$$

is a n.h.c.v.f. valued multilinear form in $\xi_1, \xi_2, \cdots, \xi_n, \delta x, V_1, \cdots, V_s$. We give the proof for the case $n = s = 1$. The proof for the general case does not differ from this very much.

Proof. Since $F(x, \xi, V)$ is a n.h.c.v.f. we have

$$\bar{F}(\bar{x}, \bar{\xi}, \bar{V}) = M(x, F(x, \xi, V)). \tag{2.20}$$

Using the fact that ξ, V are both arbitrary we obtain on taking differentials of (2.20) that

$$\bar{F}(\bar{x}, \bar{\xi}, \bar{V}; \delta\bar{x}) = M(x, F(x, V, \xi; \delta x)) + M(x, F(x, \xi, V); \delta x)$$
$$- \bar{F}(\bar{x}, \delta\bar{\xi}, \bar{V}) - \bar{F}(\bar{x}, \bar{\xi}, \delta\bar{V}). \tag{2.21}$$

From the law of transformation of the linear connection $\Gamma(x, \xi, \delta x)$ we see that

$$\bar{F}(\bar{x}, \bar{\Gamma}(\bar{x}, \bar{\xi}, \delta\bar{x}), \bar{V}) = \bar{F}(\bar{x}, \bar{x}(x; \Gamma(x, \xi, \delta x)), \bar{V}) - \bar{F}(\bar{x}, \bar{x}(x; \xi; \delta x), V)$$
$$= M(x, F(x, \Gamma(x, \xi, \delta x), V)) - \bar{F}(\bar{x}, \delta\bar{\xi}, \bar{V}). \tag{2.22}$$

Similarly we can verify

$$\bar{F}(\bar{x}, \bar{\xi}, \bar{K}(\bar{x}, \bar{V}, \delta\bar{x})) = M(x, F(x, \xi, K(x, V, \delta x))) - \bar{F}(\bar{x}, \bar{\xi}, \delta\bar{V}), \tag{2.23}$$

and

$$\bar{K}(\bar{x}, \bar{F}(\bar{x}, \bar{\xi}, \bar{V}), \delta\bar{x}) = M(x, K(x, F(x, \xi, V), \delta x)) - M(x, F(x, \xi, V); \delta x). \tag{2.24}$$

Substituting in (2.19) we obtain

$$\bar{F}(\bar{x}, \bar{\xi}, \bar{V} \mid \delta\bar{x}) = M(x, F(x, \xi, V \mid \delta x)). \tag{2.25}$$

This shows that $F(x, \xi, V \mid \delta x)$ is a n.h.c.v.f. That $F(x, \xi, V \mid \delta x)$ is a trilinear function of ξ, V, δx follows from well known theorems on Fréchet differentials of multilinear forms.(5) We call $F(x, \xi_1, \cdots, \xi_n, V_1, \cdots, V_s \mid \delta x)$ the covariant differential of F.

<u>Theorem 2.13</u>. If in the hypotheses of theorem 2.12, F is taken to be a contravariant vector field valued multilinear form,(6) then

(5). See Michal [6].

(6). F will of course have values in E.

$$F(x, \xi_1, \cdots, \xi_n, V_1, \cdots V_s \mid \delta x) = F(x, \xi_1, \cdots \xi_n, V_1, \cdots V_s ; \delta x)$$
$$- \sum_{i=1}^{n} F(x, \xi_1, \cdots \xi_{i-1}, \Gamma(x, \xi_i, \delta x), \xi_{i+1}, \cdots, \xi_n, V_1, \cdots V_s)$$
$$- \sum_{i=1}^{s} F(x, \xi_1, \xi_2, \cdots, \xi_n, V_1, \cdots V_{i-1}, K(x, V_i, \delta x), V_{i+1}, \cdots, V_s)$$
$$+ \Gamma(x, F(x, \xi_1, \xi_2, \cdots, \xi_n, V_1, \cdots V_s), \delta x)$$

is a contravariant vector field valued multilinear form in

$\xi_1, \cdots, \xi_n, \delta x, V_1, \cdots, V_s$.

The proof is similar to that of the preceeding theorem.

Theorem 2.14. If in the hypotheses of theorem 2.12, F is taken to be a scalar valued multilinear form,(1) then

$$F(x, \xi_1, \cdots, \xi_n, V_1 \cdots, V_s \mid \delta x) = F(x, \xi_1, \cdots, \xi_n, V_1, \cdots, V_s ; \delta x)$$
$$- \sum_{i=1}^{n} F(x, \xi_1, \cdots \xi_{i-1}, \Gamma(x, \xi_i, \delta x), \xi_{i+1}, \cdots, \xi_n, V_1, \cdots, V_s)$$
$$- \sum_{i=1}^{s} F(x, \xi_1, \cdots \xi_n, V_1, \cdots, V_{i-1}, K(x, V_i, \delta x), V_{i+1}, \cdots, V_s)$$

is also a scalar valued multilinear form in $\xi_1, \cdots, \xi_n, V_1, \cdots, V_s, \delta x$.

Section 2.2. The non-holonomic curvature form.

Let V(x) be a n.h.c.v.f. possessing a continuous second Fréchet differential, and let $K(x, V, \xi)$ possess a continuous first Fréchet differential. In order that this latter condition be invariant to changes of representation it is necessary to assume that the $M(x, \mathfrak{X})$ of the transformations of representation all have continuous second Fréchet differentials. By means of theorems 2.11, 2.12 we can verify

(1) By a scalar valued form $F(x, \xi_1, \cdots \xi_n, V_1, \cdots V_s)$ we mean a form which has the law of transformation $\bar{F}(\bar{x}, \bar{\xi}_1, \cdots \bar{\xi}_n, \bar{V}_1, \cdots \bar{V}_s) = F(x, \xi_1, \cdots \xi_n, V_1, \cdots, V_s)$. The form can have values in E, E_1 or the real ~~members.~~ numbers.

that

$$V(x / \delta_1 x / \delta_2 x) - V(x / \delta_2 x / \delta_1 x) = H(x,V,\delta_1 x, \delta_2 x) \tag{2.21}$$

where

$$\begin{aligned} H(x,V, \delta_1 x, \delta_2 x) &= K(x,V,\delta_1 x; \delta_2 x) - K(x,V,\delta_2 x; \delta_1 x) \\ &+ K(x,K(x,V,\delta_1 x), \delta_2 x) - K(x,K(x,V,\delta_2 x), \delta_1 x). \end{aligned} \tag{2.22}$$

Thus $H(x,V,\delta_1 x, \delta_2 x)$ is a skew-symmetric n.h.c.v.f. valued trilinear form in V, $\delta_1 x$, $\delta_2 x$. This we shall call the non-holonomic curvature form.

The following theorems shall prove useful later on.

Theorem 2.21. Let $\rho(x,V)$ be the component of a contravariant vector field valued linear form in the arbitrary n.h.c.v. V. Further suppose that $\rho(x,V;\xi_1;\xi_2)$ exists continuous in x,[8] then

$$\begin{aligned} \rho(x,V/\xi_1/\xi_2) - \rho(x,V/\xi_2/\xi_1) &= -\rho(x,H(x,V,\xi_1,\xi_2)) + \\ &B(x,\rho(x,V),\xi_1,\xi_2). \end{aligned}$$

By theorem 2.13 we have

$$\rho(x,V/\xi_1) = \rho(x,V;\xi_1) - \rho(x,K(x,V,\xi_1)) + \Gamma(x,\rho(x,V),\xi_1) \tag{2.23}$$

Let $\varphi(x,V,\xi_1) = \rho(x,V/\xi_1)$, then

$$\begin{aligned} \rho(x,V/\xi_1/\xi_2) = \varphi(x,V,\xi_1/\xi_2) &= \varphi(x,V,\xi_1;\xi_2) - \varphi(x,K(x,V,\xi_2),\xi_1) - \\ &\varphi(x,V,\Gamma(x,\xi_1,\xi_2)) + \Gamma(x,\varphi(x,V,\xi_1),\xi_2). \end{aligned} \tag{2.24}$$

(8) The range of x is of course the coordinate domain of the representation in which we take the component.

Calculating these various terms we obtain

$$\rho(x,V/\xi_1/\xi_2) = -\rho(x,K(x,V,\xi_1;\xi_2)) + \rho(x,K(x,K(x,V\,\xi_2),\xi_1))$$
$$+\Gamma(x,\rho(x,V),\xi_1;\xi_2) + \Gamma(x,\Gamma(x,\rho(x,V),\xi_1),\xi_2) + (\text{terms symmetric}$$
$$\text{in } \xi_1, \xi_2). \tag{2.25}$$

Hence

$$\rho(x,V/\xi_1/\xi_2) - \rho(x,V/\xi_2/\xi_1) = -\rho(x,H(x,V,\xi_1,\xi_2)) +$$
$$B(x,\rho(x,V),\xi_1,\xi_2) \tag{2.26}$$

<u>Theorem</u> <u>2.22</u>. Let $G(x,\xi)$ be the component of a contravariant vector field valued linear form in the contravariant vector ξ, and let $F(x,V)$ be a contravariant vector field valued linear form in the n.h.c.v. V. Suppose that G and F both possess continuous first Fréchet differentials. Then the covariant differential of $\rho(x,V) = G(x,F(x,V))$ is given by

$$\rho(x,V/\xi) = G(x,F(x,V)/\xi) + G(x,F(x,V/\xi)). \tag{2.27}$$

<u>Proof</u>.

By theorem 2.13 we have

$$\rho(x,V/\xi) = \rho(x,V;\xi) - \rho(x,K(x,V,\xi)) + \Gamma(x,\rho(x,V),\xi), \tag{2.28}$$

and

$$G(x,\xi_1/\xi) = G(x,\xi_1;\xi) - G(x,\Gamma(x,\xi_1,\xi)) + \Gamma(x,G(x,\xi_1),\xi), \tag{2.29}$$

and

$$F(x,V/\xi) = F(x,V;\xi) - F(x,K(x,V,\xi)) + \Gamma(x,F(x,V),\xi). \tag{2.291}$$

From (2.28) we obtain

$$\rho(x,V/\xi) = G(x,F(x,V);\xi) + G(x,F(x,V;\xi)) - G(x,F(x,K(x,V,\xi))) + \rho(x,G(x,F(x,V)),\xi). \qquad (2.292)$$

By means of 2.29, 2.291 and 2.292 we can verify that

$$\rho(x,V/\xi) = G(x,F(x,V)|\xi) + G(x,F(x,V/\xi)).$$

Section 2.3. Allowable $K^{(m+1)}$ representations.

In order to develop a normal representation theory we shall have to assume that a certain subset of our representations possess certain additional properties.

Def. 2.31. A regular transformation of representation $\bar{x}(x), M(x,X)$ is said to be of class $K^{(m+1)}$ if,

(a) $\bar{x}(x)$ is a regular transformation of class K^{m+1} in the sense of Chapter I;

(b) $M(x,X)$ and its inverse $N(\bar{x},X)$ are of class $C^{(m)}$ locally uniformly on the domains of definition of $\bar{x}(x)$ and $x(\bar{x})$ respectively.

Def. 2.32. A set of representations shall be called an allowable $K^{(m+1)}$ set of representations if

I. Each coordinate system $x(P)$ of a representation of the set is an allowable $K^{(m+1)}$ coordinate system;

II. the transformation of representation arising from the change from one representation of the set to another of the set is a regular transformation of representation of class $K^{(m+1)}$;

III. any representation obtained from a representation of the set by a regular transformation of class $K^{(m+1)}$ is in the set.

<u>Theorem 2.31</u>. Let $x(p)$ and $\bar{x}(Q)$ be two allowable $K^{(m+1)}$ coordinate systems with intersecting geometrical domains. Further let $x(p), X(P, Y_p)$ and $\bar{x}(Q), \bar{X}(Q, Y)$ be two allowable $K^{(n+1)}$ representations generating the transformations of representation $\bar{x} = \bar{x}(x)$, $\bar{X} = M(x, X)$. If $V(x)$ the component of a n.h.c.v.f. is of class C^{n} locally uniformly on Σ_1 (the domain of definition of $\bar{x}(x)$), then the component $\bar{V}(\bar{x})$ is of the same class on Σ_2 (the domain of definition of $x(\bar{x})$), providing $n < m + 1$.

The proof follows from a theorem of Hildebrandt and Graves [1] on functions of class $C^{(n)}$ uniformly. If $n < m$ a similar theorem holds for $K(x, V, \xi)$ the component of the non-holonomic linear connection.

<u>Section 2.4</u>. Normal representation theory.

Let $x(p), X(P, Y_p)$ be an allowable $K^{(n+3)}$ representation ($n \geqq 3$). We shall assume that the n.h. linear connection $K(x, V, \xi)$ is of class $C^{(n)}$ locally uniformly on Σ the coordinate domain of $x(P)$. Thus for any point q of Σ there exists a neighborhood X_1 of q for which $K(x, V, \xi)$ is of class $C^{(n)}$ uniformly in x on X_1. Similarly there exists a neighborhood X_2 of q on which the affine connection $\Gamma(x, \xi_1, \xi_2)$ is of class $C^{(n)}$ uniformly in x. If X is the intersection of X_1 and X_2, then $K(x, V, \xi)$ and $\Gamma(x, \xi_1, \xi_2)$ are both of class $C^{(n)}$ uniformly in x on X. We shall be interested in the differential system

$$\frac{dV}{ds} + K\left(x, V, \frac{dx}{ds}\right) = 0, \quad V(p) = V_0, \tag{2.41}$$

along a path $x = f(p, s\xi)$.[9] From Michal-Hyers [1] we know that there exists neighborhoods $X_0 \subset X$ of q and Y_0 of 0 such that for any $p \in X_0$, $\xi \in Y_0$, and $0 \leq s \leq 1$ the functions $f(p, s\xi)$ and $\frac{df(p, s\xi)}{ds} = f'(p, s\xi)$ are of class $C^{(n)}$ uniformly in the pair (p, ξ). From this we can prove the following theorem.

Theorem 2.41. Let q, p, ξ, Y_0, X_0 be the same as in the preceeding discussion. Then for any choice of $p \in X_0$, $\xi \in Y_0$ there exists a function $R(\xi, p, s, V_0)$ with the following properties.

(i) $R(\xi, p, s, V_0)$ is the unique solution of (2.41) along the path $x = f(p, s\xi)$.

(ii) For each such ξ, p and any s $(0 \leq s \leq 1)$, $R(\xi, p, s, V_0)$ is solvable linear in V_0.

(iii) For any $0 \leq s \leq 1$, $R(\xi, p, s, V_0)$ is of class $C^{(n)}$ uniformly in the pair (p, ξ) on $X_0 Y_0$.

We can replace (2.41) by the equivalent integral equation.

$$V(s) = V_0 - \int_0^s K(f(p, t\xi), V(t), f'(p, t\xi))\,dt. \tag{2.42}$$

Equation (2.42) is of the same type as equation (3.9) of Michal-Hyers [5]. By an argument that is exactly the same as that used in this paper it will follow that a solution

$$V = R(\xi, p, s, V_0) \tag{2.43}$$

of (2.41) will exist and will have the properties stated in the theorem.

(9). See section 1.2.

It is easily seen that under a change of parameter $t = \frac{1}{\lambda} s$, the equation of the path becomes $x = f(p, t(\lambda\xi))$, and (2.41) remains invariant. We thus obtain

$$R(p, \lambda\xi, t, V_0) = R(p, \lambda\xi, \tfrac{1}{\lambda} s, V_0) = R(p, \xi, s, V_0). \tag{2.44}$$

Hence we see that the solution of (2.41) can be written $R(p, s\xi, V_0) = R(p, s\xi, 1, V_0)$. For $s = 1$ and $\xi = y$, $R(p, y, V_0)$ is a solvable linear function of V_0, and is of class $C^{(n)}$ uniformly in (p, y) on $X_0 Y_0$.

Let $x(\mathbf{p}), X(P, Y_P)$ be the allowable $K^{(n+2)}$ representation from which we started, and let P_0 be any point in the geometric domain of $x(\mathbf{p})$. Choosing $p = x(P_0)$ we have by Michal-Hyers [1] that $x(\mathbf{p})$ determines a normal coordinate system $y(P)$ with center P_0 and whose coordinate domain is contained in Y_0. It is easily verified that the function $^{+}X(P, Y_P)$ defined implicitly by

$$X(P, Y_P) = R(p, y(P), {}^{+}X(P, Y_P))$$

is for each P in the geometric domain of $y(P)$ a vector coordinate system for the associated space B_P. Thus any such normal coordinate system $y(P)$ and the vector coordinate systems $^{+}X(P, Y_P)$ form a representation.

__Def. 2.41.__ A representation $y(P)$, $^{+}X(P, Y_P)$ as obtained above shall be called a normal representation with center P_0 determined by the allowable $K^{(n+2)}$ representation $x(\mathbf{p}), X(P, Y_P)$.

We see that the transformation of representation induced by the change of representations $x(P)$, $X(P,Y_p)$ to the corresponding normal representation $y(P)$, ${}^{+}X(P,Y_p)$ is

$$x = f(p,y), \qquad X = R(p,y,{}^{+}X) \tag{2.45}$$

Since the point $p = x(P_0)$ is a fixed point in the discussion we shall delete it from (2.45) and write

$$x = \mu(y), \qquad X = R(y,{}^{+}X) \tag{2.46}$$

We shall write the inverse transformations as

$$y = \nu(x), \qquad {}^{+}X = R^{-1}(x, X) \tag{2.47}$$

Def. 2.42. A n.h.c.v.f. $Z(x)$ shall be said to be a vector field parallel to a n.h.c.v. Z_0 along a curve $x = x(s)$ if [10]

$$\frac{dZ(x)}{ds} + K\left(x, Z(x), \frac{dx}{ds}\right) = 0, \quad Z(p) = Z_0, \tag{2.47}$$

where $p = x(0)$.

Theorem 2.42. Let $x(P)$, $X(P,Y_p)$ and $\bar{x}(P)$, $\bar{X}(P,Y_p)$ be two allowable $K^{(n+2)}$ representations which induce the transformation of representations $\bar{x} = \bar{x}(x)$, $\bar{X} = M(x,X)$. Suppose moreover that these representations determine the normal representations $y(P)$, ${}^{+}X(P,Y_p)$ and $\bar{y}(P)$, ${}^{+}\bar{X}(P,Y_p)$ respectively, and suppose the latter have the same center P_0. Then the transformation of representation induced by the change from

(10). First discussed in Michal [6].

one normal representation to the other is

(i) $\bar{y} = x(p;y)$,

(ii) ${}^{+}\bar{X} = M(p, {}^{+}X)$,

where $p = x(\rho_0)$.

The proof of (i) is given in Michal-Hyers [1]. To prove (ii) we let $Z(x)$ be a vector field parallel to an arbitrarily chosen initial value Z_0 along a path $x = f(p, s\rho)$. The component ${}^{+}Z(y)$ of $Z(x)$ in the normal representation is given by

$$Z(x) = R(y, {}^{+}Z(y)). \tag{2.48}$$

But since $Z(x)$ is a parallel vector field we have

$$\frac{dZ(x)}{ds} + K\left(x, Z, \frac{dx}{ds}\right) = 0,\ Z(p) = Z_0,\ x = f(p, y). \tag{2.49}$$

Since the solution of (2.49) was shown to be unique, we must have $Z(x) = R(y, Z_0)$. By the solvability of $R(y, Z_0)$ we obtain that ${}^{+}Z(y) = Z_0$ along a path.

Under the change of allowable $K^{(n+2)}$ representations (2.49) becomes

$$\frac{d\bar{Z}(\bar{x})}{ds} + \bar{K}\left(\bar{x}, \bar{Z}, \frac{d\bar{x}}{ds}\right) = 0,\ \bar{Z}_0 = M(p, Z_0),\ \bar{x} = \bar{f}(\bar{p}, \bar{y}). \tag{2.491}$$

Repeating the argument we obtain ${}^{+}\bar{Z}(\bar{y}) = \bar{Z}_0$, and hence

$${}^{+}\bar{Z}(\bar{y}) = M(p, {}^{+}Z(y)), \tag{2.492}$$

along a path.

We can tabulate the transformations of representation that arise from the various changes of representation as follows.

1. $x(P), X(P, Y_\rho)$ to $y(P), {}^+X(P, Y_\rho)$ induces $x = \mu(y), X = R(y, {}^+X)$ (2.493)

2. $\bar{x}(P), \bar{X}(P, Y\ )$ to $\bar{y}(P), {}^+\bar{X}(P, Y_\rho)$ induces $\bar{x} = \bar{\mu}(\bar{y}), \bar{X} = \bar{R}(\bar{y}, {}^+\bar{X})$ (2.494)

3. $x(P), X(P, Y_\rho)$ to $\bar{x}(P), \bar{X}(P, Y_\rho)$ induces $\bar{x} = \bar{x}(x), \bar{X} = M(x, X)$. (2.495)

From these we can obtain

$$ {}^+\bar{X} = \bar{R}^{-1}(\bar{x}, M(x, R(y, {}^+X))) \tag{2.496}$$

Since ${}^+Z(y)$ is a n.h.c.v.f. (2.496) is the law of transformation from ${}^+Z(y)$ to ${}^+\bar{Z}(\bar{y})$. Thus for ${}^+Z(y)$, (2.496) must reduce to (2.492). By the arbitrariness of ${}^+Z(y) = Z_0$ we must have (2.496) reduces to (ii) for all values of ${}^+X$.

<u>Section 2.5.</u> The differentials of $R(y, V_0)$ and $R^{-1}(x, V_0)$.

In order to obtain explicit expressions for the differentials of $R(y, X_0)$ we define the functions $K_t(x, V, \xi_1, \xi_2, \cdots, \xi_t)$ by the following relation.

$$K_2(x, V, \xi_1, \xi_2) = \frac{1}{2} P\left\{K(x, V, \xi_1; \xi_2) - K(x, K(x, V, \xi_2), \xi_1) - K(x, V, \Gamma(x, \xi_1, \xi_2))\right\} \tag{2.51}$$

$$K_t(x, V, \xi_1, \cdots, \xi_t) = \frac{1}{t} P\Big\{K_{t-1}(x, V, \xi_1, \cdots, \xi_{t-1}; \xi_t) - K_{t-1}(x, K(x, V, \xi_t), \xi_1, \cdots, \xi_{t-1}) - \sum_{i=1}^{t-1} K_{t-1}(x, V, \xi_1, \cdots, \xi_{i-1}, \Gamma(x, \xi_i, \xi_t), \xi_{i+1}, \cdots, \xi_{t-1})\Big\}, \quad t \leq n.$$

$P\{\cdots\}$ means the sum of terms obtained by a cyclic permutation of the ξ's. From previous discussion we know that $R(y, V_0)$ and $R^{-1}(x, V_0)$ are of class $C^{(n)}$ uniformly on their domains of definition. Hence they have continuous Fréchet differentials of order "n". By a well known theorem on Fréchet differentials we have

$$R(0, V_0; \delta y) = \frac{d}{ds} R(s\delta y, V_0)\Big|_{s=0} . \tag{2.52}$$

Since $R(s\,\delta y, V_0)$ is a solution of (2.41) we obtain

$$R(0, V_0; \delta y) = -K(p, V_0, \delta y). \tag{2.53}$$

If we denote the t'th Frechet differential of $R(y, V)$ evaluated at $y = 0$ by $R_t(0, V_0; \delta_1 y; \delta_2 y; \cdots; \delta_t y)$ we similarly obtain

$$R_t(0, V_0; \delta y; \delta y; \cdots; \delta y) = \frac{d^t}{ds^t} R(s\delta y, V_0)\Big|_{s=0} \tag{2.54}$$

By induction it is easily verified that

$$R_t(0, V_0; \delta y; \delta y; \cdots; \delta y) = -K_t(p, V_0, \delta y, \delta y, \cdots, \delta y). \tag{2.55}$$

Each member of (2.55) is a homogeneous polynomial of degree t in δy. By the properties of the polar of a homogeneous polynomial,[11] it follows that

$$R_t(0, V_0; \delta_1 y; \delta_2 y; \cdots; \delta_t y) = -K_t(p, V_0, \delta_1 y, \cdots, \delta_t y). \tag{2.56}$$

To obtain the differential of the inverse $R^{-1}(x, V_0)$ we have the identity $R^{-1}(x, R(y, X_0)) = X_0$. From this we easily verify that

$$R^{-1}(p, V_0) = V_0 \tag{2.57}$$

(11). See Martin [1].

and $$R^{-1}(p, V_0; \delta x) = K(p, V_0, \delta x) \tag{2.58}$$

The following properties of normal representation are important in the application of normal representations.

If V(x) is the component of a n.h.c.v.f. in an allowable $K^{(n+2)}$ representation, and if ${}^{+}V(y)$ is its component in the corresponding normal representation, then $V(x) = R(y, {}^{+}V(y))$. From this we obtain $V(p) = {}^{+}V(0)$. That is the value of the component in an allowable K^{n+2} representation is equal to its value in the corresponding normal representation at the center of the normal representation. Similarly the components of the linear connection in the two representations are related by

$$K(x, V, \delta x) = R(y, {}^{+}K(y, {}^{+}V, \delta y)) - R(y, {}^{+}V; \delta y). \tag{2.59}$$

From this we obtain

$${}^{+}K(0, V_0, \delta y) = 0, \tag{2.591}$$

for all $V_0 \in E$, and $\delta y \in E$.

In concluding this section I would like to briefly summarize some of the properties of normal representations which will prove useful later on.

1. $R(0, V_0) = V_0$.
2. $R(0, V_0; \delta x) = -K(p, V_0, \delta x)$.
3. $R_t(0, V_0; \delta_1 x; \delta_2 x; \cdots; \delta_t x) = -K_t(p, V_0, \delta_1 x, \cdots, \delta_t x)$.

4. $R^{-1}(p, V_0) = V_0$.

5. $R^{-1}(p, V_0; \delta x) = K(p, V_0, \delta x)$.

6. $^{+}K(0, V_0, \delta y) = 0$.

7. If V is the component of an arbitrary n.h.c.v. , and $^{+}V(y)$ is its component in a normal representation, then $^{+}V(0) = V$, and $^{+}V(0; \delta y) = K(p, V, \delta y)$.

8. The component $K(x, V, \delta x)$ of the linear connection in an allowable $K^{(n+2)}$ representation is related to the component $^{+}K(y, {}^{+}V, \delta y)$ in the corresponding normal representation by

$$K(x, V, \delta x) = R(y, {}^{+}K(y, {}^{+}V, \delta y)) - R(y, {}^{+}V; \delta y).$$

Section 2.6. Tensor extensions of multilinear forms.

Let $F(x, V_1, \cdots, V_s, \xi_1, \cdots, \xi_n)$ be the component in an allowable $K^{(n+2)}$ representation $x(P)$, $X(P, Y_\rho)$ of a n.h.c.v.f. valued multilinear form in the arbitrary n.h.c.v. $V_1, \cdots, V_s$, and the arbitrary contavariant vectors $\xi_1, \cdots, \xi_n$. Further let $y(P)$, $^{+}X(P, Y_\rho)$ be the normal representation with center $p = x(P_0)$ determined by the representation $x(P)$, $X(P, Y_\rho)$.

Def. 2.61. The k'th extension $F(x, V_1, \cdots, V_s, \xi_1, \cdots, \xi_n | \xi_{n+1}, \cdots, \xi_{n+k})$ of F is defined at each point "p" of the coordinate domain of $x(P)$ by

$$F(p, V_1, \cdots, V_s, \xi_1, \xi_2, \cdots \xi_n | \xi_{n+1}, \cdots, \xi_{n+k}) = {}^{+}F(y, {}^{+}V_1, \cdots, {}^{+}V_s, {}^{+}\xi_1, \cdots, {}^{+}\xi_n; {}^{+}\xi_{n+1}; \cdots; {}^{+}\xi_{n+k})_{y=0} \quad (2.61)$$

where $^{+}F(y, {}^{+}V_1, \cdots, {}^{+}V_s, {}^{+}\xi_1, \cdots, {}^{+}\xi_n)$ are the components of F in the normal representation with center $p = x(P_0)$.

---

(12) The results in sections 2.6 and 2.7 will hold only for allowable $K^{(n+2)}$ representations

Theorem 2.61. The k'th extension of a n.h.c.v.f. values form F as defined above is a n.h.c.v.f. valued form in the arbitrary n.h.c.v. $V_1, \cdots, V_s$, and the arbitrary contravariant vectors $\xi_1, \cdots, \xi_{n+k}$.

Proof.

Let $^+F(y, {}^+V_1, \cdots, {}^+V_s, {}^+\xi_1, \cdots, {}^+\xi_n)$ and $^+\bar{F}(\bar{y}, {}^+\bar{V}_1, \cdots, {}^+\bar{V}_s, {}^+\bar{\xi}_1, \cdots, {}^+\bar{\xi}_n)$ be the components of F in two normal representations with the same center $p = x(P_0)$. By theorem 2.42 we have

$$^+\bar{F}(\bar{y}, {}^+\bar{V}_1, \cdots, {}^+\bar{V}_s, {}^+\bar{\xi}_1, \cdots, {}^+\bar{\xi}_n) = M(p, {}^+F(y, {}^+V_1, \cdots, {}^+V_s, {}^+\xi_1, \cdots, {}^+\xi_n)). \quad (2.62)$$

Taking differentials of (2.62) and evaluating at y = 0 we obtain

$$\bar{F}(\bar{p}, \bar{V}_1, \cdots \bar{V}_s, \bar{\xi}_1, \cdots, \bar{\xi}_n | \bar{\xi}_{n+1}, \cdots \bar{\xi}_{n+k}) = M(p, F(p, V_1, \cdots, V_s, \xi_1, \cdots, \xi_n | \xi_{n+1}, \cdots, \xi_{n+k})).$$

Since $p = x(P_0)$ is any point of the coordinate domain of $x(P)$ the theorem is proven.

Theorem 2.62. The first extension of F is equal to the covariant differential of F.

By equations (2.46) the components $F(x, V_1, \cdots V_s, \xi_1, \cdots, \xi_n)$ and $^+F(y, {}^+V_1, \cdots {}^+V_s, {}^+\xi_1, \cdots {}^+\xi_n)$ are related by

$$F(x, V_1 \cdots V_s, \xi_1, \cdots, \xi_n) = R(y, {}^+F(y, {}^+V_1, \cdots, {}^+V_s, {}^+\xi_1, \cdots, {}^+\xi_n)). \quad (2.63)$$

From this we see

$$F(x, V_1, \cdots, V_s, \xi_1, \cdots, \xi_n) = R(y, {}^+F(y, {}^+V_1, \cdots, {}^+V_s, {}^+\xi_1, \cdots, {}^+\xi_n); \delta y)$$

$$+ R(y, {}^+F(y, {}^+V_1, \cdots, {}^+V_s, {}^+\xi_1, \cdots, {}^+\xi_n; \delta y)) + \sum_{i=1}^{s} R(y, {}^+F(y, {}^+V_1, \cdots {}^+V_{i-1}, {}^+V_{i+1}(y; \delta y), V_{i+1}, \cdots {}^+V_s, {}^+\xi_1 \cdots {}^+\xi_n))$$

$$+ \sum_{i=1}^{n} R(y, {}^+F(y, {}^+V_1, \cdots, {}^+V_s, {}^+\xi_1, \cdots {}^+\xi_{i-1}, {}^+\xi_i(y; \delta y), {}^+\xi_{i+1}, \cdots, {}^+\xi_n)). \qquad (2.64)$$

Evaluating at y = 0, and using the properties listed in section (2.5) we obtain

$$\begin{aligned} {}^+F(y, {}^+V_1, \cdots, {}^+V_s, {}^+\xi_1, \cdots, {}^+\xi_n; \delta y)_{y=0} &= F(p, V_1, \cdots, V_s, \xi_1, \cdots, \xi_n; \delta x) \\ &- \sum_{i=1}^{s} F(p, V_1, \cdots, V_{i-1}, K(p, V_i, \delta x), V_{i+1}, \cdots, V_s, \xi_1, \cdots, \xi_n) \\ &- \sum_{i=1}^{n} F(p, V_1, \cdots, V_n, \xi_1, \cdots, \xi_{i-1}, \Gamma(p, \xi_i, \delta x), \xi_{i+1}, \cdots, \xi_n) \\ &+ K'(p, F(p, V_1, \cdots, V_s, \xi_1, \cdots, \xi_n), \delta x). \end{aligned} \qquad (2.65)$$

Thus the first extension evaluated at y = 0 is equal to the covariant differential evaluated at x = p. As before the center of our normal representation can be any point of the coordinate domain, and hence the theorem is proven.

__Def. 2.62__. Let ${}^+K(y, {}^+V, {}^+\xi)$ be the components of the linear connection in a normal representation with center $p = x(P_0)$. The functions $C_t(x, V, \xi_1, \cdots \xi_{t+1})$ defined at each point of the coordinate domain of $\chi(P)$ by

$$C_t(p, V, \xi_1, \cdots, \xi_{t+1}) = {}^{+}K(y, {}^{+}V, \overset{+}{\xi}_1; \overset{+}{\xi}_2; \cdots; \overset{+}{\xi}_{t+1})_{y=0} \tag{2.66}$$

are called non-holonomic normal vector forms.

By a proof similar to that of theorem 2.42 it can be shown that $C_t(x, V, \xi_1 \cdots \xi_{t+1})$ is a n.h.c.v.f. valued multilinear form in $V, \xi_1, \cdots, \xi_{t+1}$.

<u>Theorem 2.63.</u> The first non-holonomic normal form is given by

$$C_1(x, V, \xi_1, \xi_2) = \tfrac{1}{2} H(x, V, \xi_1, \xi_2). \tag{2.67}$$

<u>Proof.</u> From the law of transformation of linear connections we have

$$K(x, V, \xi_1) = R(y, {}^{+}K(y, {}^{+}V, \overset{+}{\xi}_1)) - R(y, {}^{+}V; \overset{+}{\xi}_1). \tag{2.68}$$

This leads to

$$\begin{aligned} K(x, V, \xi_1; \xi_2) &= R(y, {}^{+}K(y, {}^{+}V, \overset{+}{\xi}_1; \overset{+}{\xi}_2)) + R(y, {}^{+}K(y, {}^{+}V, \overset{+}{\xi}_1); \overset{+}{\xi}_2) \\ &+ R(y, {}^{+}K(y, {}^{+}V(y; \overset{+}{\xi}_2), \overset{+}{\xi}_1)) + R(y, {}^{+}K(y, {}^{+}V, \overset{+}{\xi}_1(y; \overset{+}{\xi}_2))) \\ &- R(y, {}^{+}V; \overset{+}{\xi}_1; \overset{+}{\xi}_2) - R(y, {}^{+}V(y; \overset{+}{\xi}_2); \overset{+}{\xi}_1) - R(y, {}^{+}V; \overset{+}{\xi}_1(y; \overset{+}{\xi}_2)). \end{aligned}$$

Evaluating at y = 0 we obtain

$$\begin{aligned} K(p, V, \xi_1; \xi_2) = {}^{+}K(y, {}^{+}V, \overset{+}{\xi}_1; \overset{+}{\xi}_2)_{y=0} &+ K_2(p, V, \xi_1, \xi_2) + K(p, K(p, V, \xi_2), \xi_1) \\ &+ K(p, V, \Gamma(p, \xi_1, \xi_2)). \end{aligned}$$

Substituting for $K_2(p, V, \xi_1, \xi_2)$ we have

$$C_1(p,V,\xi_1,\xi_2) = \overset{+}{K}(y,\overset{+}{V},\overset{+}{\xi}_1;\overset{+}{\xi}_2)_{y=0} = \frac{1}{2}H(p,V,\xi_1,\xi_2). \tag{2.69}$$

As before "p" is any point in the coordinate domain of x(P), hence the theorem is proven.

By the skew symmetry of $H(x,V,\xi_1,\xi_2)$ we can write (2.67) in the form

$$C_1(x,V,\xi_1,\xi_2) - C_1(x,V,\xi_1,\xi_2) = H(x,V,\xi_1,\xi_2). \tag{2.691}$$

Theorem 2.64. Let $F(x,V_1,\cdots,V_m)$ be the component of a n.h.c.v.f. valued multilinear form in the arbitrary n.h.c.v. $V_1,\cdots V_m$, in an allowable $K^{(n+2)}$ representation. Suppose moreover that F has a continuous second Fréchet differential. Then

$$F(x,V_1,\cdots,V_m|\xi_1,\xi_2) = \frac{1}{2}\{F(x,V_1,\cdots,V_m|\xi_1|\xi_2) + F(x,V_1,\cdots,V_m|\xi_2|\xi_1)\}.$$

Proof for m=1. Calculating $F(x,V|\xi_1|\xi_2)$ we obtain

$$\begin{aligned} F(x,V|\xi_1|\xi_2) &= F(x,V;\xi_1;\xi_2) - F(x,K(x,V,\xi_1);\xi_2) - F(x,K(x,V,\xi_2;\xi_1)) \\ &+ K(x,F(x,V),\xi_1;\xi_2) + K(x,F(x,V;\xi_2),\xi_1) - F(x,K(x,V,\xi_2);\xi_1) \\ &+ F(x,K(x,K(x,V,\xi_2),\xi_1)) - K(x,F(x,K(x,V,\xi_2)),\xi_1) - F(x,V;\Gamma(x,\xi_1,\xi_2)) \\ &+ F(x,K(x,V,\Gamma(x,\xi_1,\xi_2))) - K(x,F(x,V),\Gamma(x,\xi_1,\xi_2)) + K(x,F(x,V;\xi_1),\xi_2) \\ &- K(x,F(x,K(x,V,\xi_1)),\xi_2) + K(x,K(x,F(x,V),\xi_1),\xi_2). \end{aligned}$$

In the normal representation determined by our allowable $K^{(n+2)}$ representation we have

$$\overset{+}{F}(y, \overset{+}{V} | \overset{+}{\xi}_1 | \overset{+}{\xi}_2)_{y=0} = \overset{+}{F}(y, \overset{+}{V}; \overset{+}{\xi}_1; \overset{+}{\xi}_2)_{y=0} - \frac{1}{2} F(p, H(p, V, \xi_1, \xi_2)) + \frac{1}{2} H(p, F(p, V), \xi_1, \xi_2). \quad (2.692)$$

By the symmetry of the second Fréchet differential and the skew symmetry of $H(x, V, \xi_1, \xi_2)$ we obtain

$$\overset{+}{F}(y, \overset{+}{V}; \overset{+}{\xi}_1; \overset{+}{\xi}_2)_{y=0} = \frac{1}{2} \{ \overset{+}{F}(y, \overset{+}{V} | \overset{+}{\xi}_1 | \overset{+}{\xi}_2) + \overset{+}{F}(y, \overset{+}{V} | \overset{+}{\xi}_2 | \overset{+}{\xi}_1) \}_{y=0}.$$

This of course implies

$$\frac{1}{2} \{ F(x, V | \xi_1 | \xi_2) + F(x, V | \xi_2 | \xi_1) \} = F(x, V | \xi_1, \xi_2).$$

Theorem 2.65. Let $H(x, V, \xi_1, \xi_2)$ be the components of the non-holonomic curvature form in an allowable K^{n+2} representation. Then

$$H(x, V, \xi_1, \xi_2 | \xi_3) + H(x, V, \xi_2, \xi_3 | \xi_1) + H(x, V, \xi_3, \xi_1 | \xi_2) = 0.$$

Proof. In the normal representation with center $p = x(P_0)$ determined by our allowable K^{n+2} representation we have

$$\overset{+}{H}(y, \overset{+}{V}, \overset{+}{\xi}_1, \overset{+}{\xi}_2) = \overset{+}{K}(y, \overset{+}{V}; \overset{+}{\xi}_1; \overset{+}{\xi}_2) - \overset{+}{K}(y, \overset{+}{V}; \overset{+}{\xi}_2; \overset{+}{\xi}_1) \quad (2.693)$$

$$+ \overset{+}{K}(y, \overset{+}{K}(y, \overset{+}{V}, \overset{+}{\xi}_1), \overset{+}{\xi}_2) - \overset{+}{K}(y, \overset{+}{K}(y, \overset{+}{V}, \overset{+}{\xi}_2), \overset{+}{\xi}_1).$$

Taking Fréchet differentials of (2.693) and avaluating at $y = 0$ we obtain

$$\overset{+}{H}(y, \overset{+}{V}, \overset{+}{\xi}_1, \overset{+}{\xi}_2 ; \overset{+}{\xi}_3)_{y=0} = \{\overset{+}{K}(y, \overset{+}{V}, \overset{+}{\xi}_1 ; \overset{+}{\xi}_2 ; \overset{+}{\xi}_3) - \overset{+}{K}(y, \overset{+}{V}, \overset{+}{\xi}_2 ; \overset{+}{\xi}_1 ; \overset{+}{\xi}_3)\}_{y=0} \tag{2.694}$$

This implies

$$\{\overset{+}{H}(y, \overset{+}{V}, \overset{+}{\xi}_1, \overset{+}{\xi}_2 ; \overset{+}{\xi}_3) + \overset{+}{H}(y, \overset{+}{V}, \overset{+}{\xi}_2, \overset{+}{\xi}_3 ; \overset{+}{\xi}_1) + \overset{+}{H}(y, \overset{+}{V}, \overset{+}{\xi}_3, \overset{+}{\xi}_1 ; \overset{+}{\xi}_2)\}_{y=0} = 0.$$

Using theorem 2.62 we obtain

$$H(x, V, \xi_1, \xi_2 / \xi_3) + H(x, V, \xi_2, \xi_3 / \xi_1) + H(x, V, \xi_3, \xi_1 / \xi_2) = 0.$$

Section 2.7. A replacement theorem.

The following replacement theorem is a generalization of a replacement theorem given in Michal-Hyers [1]. We use a similar notation and state the theorem in a similar way.

Let $R_{\alpha\beta\cdots\delta}(f_1(\alpha_1, \alpha_2), \cdots f_k(\rho_1, \cdots, \rho_{k+1}), g_1(\tau_1, \gamma_1), \cdots ; g_n(\tau_n, \delta_1, \cdots ; \delta_n) \| \xi_1, \cdots ; \xi_n, V_1, \cdots ; V_m)$ by a functional whose arguments are multilinear functions $f_1, \cdots, g_n$, and whose value is a multilinear function of $\xi_1, \cdots ; \xi_n, V_1, \cdots ; V_m$.

Def. 2.71. Let

$$F(x, \xi_1, \cdots ; \xi_n, V_1, \cdots, V_m) =$$

$$R_{\alpha\beta\cdots\delta}(\Gamma(x, \alpha_1, \alpha_2), \Gamma(x, \beta_1, \beta_2 ; \beta_3), \cdots \Gamma(x, \rho_1, \rho_2 ; \rho_3 ; \cdots ; \rho_{k+1}), K(x, \tau_1, \gamma_1), K(x, \tau_2, \mu_1 ; \mu_2), \cdots ; K(x, \tau_n, \delta_1 ; \delta_2 ; \cdots ; \delta_n) \| \xi_1, \cdots, \xi_n, V_1, \cdots, V_m),$$

be the components of a n.h.c.v.f. valued linear form ~~or~~ of a contravariant vector field valued linear form in ~~an allowable K~~ any representation. F will be called a differential invariant if under a change of representation "R" as a functional retains its form.

Theorem 2.71. The component in an allowable $K^{(n+2)}$ representation of every differential invariant can be expressed in terms of the normal vector forms $A_k(x, \gamma_1, \cdots, \gamma_{k+2})$[13] and the non-holonomic normal vector forms $C_t(x, T, \rho_1, \cdots, \rho_{t+1})$ by the following process.

(i) $\Gamma(x, \alpha_1, \alpha_2)$ and $K(x, T, \mu)$ are replaced by zero.

(ii) $\Gamma(x, \gamma_1, \gamma_2; \gamma_3; \cdots; \gamma_{j+2})$ is replaced by $A_j(x, \gamma_1, \gamma_2, \cdots, \gamma_{j+2})$.

(iii) $K(x, T, \delta_1; \delta_2; \cdots; \delta_{t+1})$ is replaced by $C_t(x, T, \delta_1, \cdots, \delta_{t+1})$.

The proof follows directly by evaluating at the center of a normal representation.

As an example of this process we take the non-holonomic curvature form. Since

$$H(x, V, \xi_1, \xi_2) = K(x, V, \xi_1; \xi_2) - K(x, V, \xi_2; \xi_1) + K(x, K(x, V, \xi_1), \xi_2)$$

$$-K(x, K(x, V, \xi_2), \xi_1),$$

we obtain by replacement

$$H(x, V, \xi_1, \xi_2) = C_1(x, V, \xi_1, \xi_2) - C_1(x, V, \xi_2, \xi_1).$$

(13) See Michal-Hyers [1].

CHAPTER III

In this chapter we shall assume that the geometrical spaces satisfy all the assumptions of the previous two chapters. In addition we shall assume that both Banach spaces E and E_1 possess inner products. This will enable us to introduce the notions of covariant and non-holomic covariant vectors.(1)

The first section of this chapter will be devoted to briefly summarizing the properties of adjoints of linear transformations. A more complete discussion can be found in Michal [2], Michal-Hyers [3] and Stone [1]. We take over many of the definitions from these references verbatim.

__Section 3.1.__ Inner products and adjoints of linear functions.

__Def. 3.11.__ Let B be a Banach space. A function $[x,y]$ on B^2 to the real numbers shall be called an inner product if

1. $[x,y]$ is bilinear in x and y,
2. $[x,y] = [y,x]$,
3. $[x,x] \geqslant 0$, and $[x,x] = 0$ if and only if $x = 0$.

__Def. 3.12.__ A function $T^*(x)$ on B to B is said to be the adjoint of a linear function $T(x)$ on B to B if

1. $T^*(x)$ is linear in x,
2. $[T(x),y] = [x,T^*(y)]$ for all $x,y \epsilon B$.

(1) A more general treatment of these notions can be given by means of an inter-space inner product which Professor Michal has introduced.

Adjoints have the following properties.

(a) If $T(x)$ has and adjoint $T^*(x)$, then the adjoint is unique.

(b) If $T_1(x)$ and $T_2(x)$ have adjoints $T_1^*(x)$ and $T_2^*(x)$ respectively, then

(b_1) $\{T_1(T_2(x))\}^*$ exists and is equal to $T_2^*(T_1^*(x))$;

(b_2) $\{a\ T_1(x) + b\ T_2(x)\}^*$ exists and is equal to $a\ T_1^*(x) + b\ T_2^*(x)$.

(c) If $T(x)$ has adjoint $T^*(x)$, then $(T^*(x))^*$ exists equal to $T(x)$.

(d) Let $T_i(x)$ be a sequence of linear functions convergent to a limit function $T(x)$, and suppose that $T_i^*(x)$ exists for each i. If $T_i^*(x)$ converges to $M(x)$, then $T^*(x)$ exists and is equal to $M(x)$.

(e) Let $T(x)$ be a solvable linear function with adjoint $T^*(x)$. If one of $\{T^{-1}(x)\}^*$ or $\{T^*(x)\}^{-1}$ exists the other does also and the two are equal.

The following notation is used for adjoints of multilinear forms. If $F(x, \xi_1, \xi_2, \cdots, \xi_n)$ is a multilinear function of on B^{n+1} to B and if for each x, F considered as a linear function of ξ_i has an adjoint, then we denote the adjoint by $F_{(i)}^*(x, \xi_1, \xi_2, \cdots, \xi_{i-1}, \eta, \xi_{i+1}, \cdots, \xi_n)$.(2)
With this notation it is possible to show that adjoints have the following property.

(f) Let S be a subset of a Banach space B, and let $T(x,y)$ be a linear function of y on SB to B. If for each x of S, $T(x,y)$ has an adjoint

(2) I do not believe that it is necessarily true that $F_{(i)}^*(x, \xi_1, \xi_2, \cdots \xi_{i-1}, \eta, \xi_{i+1}, \cdots, \xi_n)$ preserves the continuity in $\xi_1, \xi_2, \cdots, \xi_{i-1}, \xi_{i+1}, \cdots \xi_n$.
Throughout this thesis this will be assumed to be true.

$L(x,y)$, and if both $T(x,y)$ and $L(x,y)$ have Fréchet differentials at each point of S, then $T^*_{(2)}(x,y;\delta x)$ exists and is equal to $L(x,y;\delta x)$.

Section 3.2. Non-holonomic covariant vector fields, and non-holonomic covariant linear connections.

We have assumed previously that the transformations of representation $\bar{x} = \bar{x}(x)$, $\bar{X} = M(x,X)$ arising from any change of representation were regular transformations of representation. We now place the following restrictions on all transformations of representation. These are to hold at each point of the domain of definition of $\bar{x}(x)$.

1. $\bar{x}(x;\delta x)$ has an adjoint $\bar{x}^*_{(2)}(x;\delta x)$.
2. $\bar{x}(x;\delta_1 x;\delta_2 x)$, and $\bar{x}^*_{(2)}(x;\delta_1 x;\delta_2 x)$ both exist continuous in x.
3. $\bar{x}^*_{(2)}(x;\delta x)$ is a solvable linear function with inverse $\chi^*_{(2)}(\bar{x};\delta x)$.
4. $M(x,X)$ has an adjoint $M^*_{(2)}(x,X)$.
5. $M^*_{(2)}(x,X)$ is a solvable linear function with inverse $N^*_{(2)}(\bar{x},X)$.
6. $M(x,X;\delta x)$, $N^*_{(2)}(\bar{x},X;\delta\bar{x})$, $M^*_{(2)}(x,X;\delta x)$, $\chi^*_{(2)}(\bar{x};\delta_1\bar{x};\delta_2\bar{x})$ exist and are continuous at each point of their domains of definition.

In order to save on terminology we shall now call any transformation of representation which has all of these properties a regular transformation of representation.

It will be noted that all the assumptions of Michal [2] are made here. Hence the notion of a covariant vector can be introduced. The law of transformation of a covariant vector η is

$$\bar{\eta} = \chi^*_{(2)}(\bar{x};\eta). \tag{3.21}$$

We shall assume that in ~~any~~ every coordinate system x(P) the component $\Gamma(x, \xi_1, \xi_2)$ of the affine linear connection has an adjoint $\Gamma^*_{(2)}(x, \eta, \xi)$. The law of transformation of $\Gamma^*_{(2)}(x, \eta, \xi)$ can be shown to be

$$\Gamma^*_{(2)}(\bar{x}, \bar{\eta}, \bar{\xi}) = x^*_{(2)}(\bar{x}; \Gamma^*_{(2)}(x, \eta, \xi)) + x^*_{(2)}(\bar{x}; \eta; \bar{\xi}). \tag{3.22}$$

Def. 3.21. Let $x(P)$, $X(P,Y_\rho)$ and $\bar{x}(P)$, $\bar{X}(P,Y_\rho)$ be any two representations generating the regular transformations of representation $\bar{x} = \bar{x}(x)$, $\bar{X} = M(x,X)$. A geometric object whose components $W(x(P_0))$, $\bar{W}(\bar{x}(P_0))$ in the two representations are elements of the Banach space E_1, and are related by

$$\bar{W}(\bar{x}(P_0)) = N^*_{(2)}(\bar{x}(P_0), W(x(P_0))) \tag{3.23}$$

is called a non-holonomic covariant vector(3) associated with P_0.

To distinguish between the inner products of E and E_1 we shall denote the inner product of the former by $[x,y]$, and that of the latter by $\{V,W\}$.

Def. 3.24. A geometric object whose component $W(x)$ (a function on Σ to E_1) in any representation is related to its component $\bar{W}(\bar{x})$ in any other representation by

$$\bar{W}(\bar{x}) = N^*_{(2)}(\bar{x}, W(x)) \tag{3.24}$$

is called a non-holonomic covariant vector field.(4)

Theorem 3.21. Let V be the component of an arbitrary n.h.c.v. . A

(3) This is abbreviated to n.h.$\bar{c}$.v.
(4) Abbreviated to n.h.$\bar{c}$.v.f.

necessary and sufficient condition for $\{V,W\}$ to be invariant under all changes of representation is that W be the component of a n.h.c.v.f.

Proof. If the transformation of representation is $\bar{x} = \bar{x}(x)$, $\bar{X} = M(x,X)$ and $\{V,W\}$ is invariant, then

$$\{V,W\} = \{\bar{V},\bar{W}\} = \{M(x,V),\bar{W}\} = \{V,M^*_{(a)}(x,\bar{W})\} . \tag{3.25}$$

By the arbitrariness of V we must have $W = M^*_{(a)}(x,\bar{W})$. This implies $\bar{W} = N^*_{(a)}(\bar{x},W)$. Thus the necessity is proven, the sufficiency is obtained by reversing the steps.

Def. 3.25. Consider a geometric object whose component $L(x,W,\xi)$ is a bilinear function of W,ξ on $\sum E, E$ to E, in every representation. This shall be called a non-holonomic covariant linear connection if, under a change of representation the components $L(x,W,\xi)$ and $\bar{L}(\bar{x},\bar{W},\bar{\xi})$ are related by,

$$\bar{L}(\bar{x},\bar{W},\bar{\xi}) = N^*_{(a)}(\bar{x},L(x,W,\xi)) + N^*_{(a)}(\bar{x},W;\bar{\xi}) . \tag{3.26}$$

Theorem 3.22. Suppose that in every representation the component $K(x,V,\xi)$ has an adjoint $K^*_{(a)}(x,W,\xi)$. Then $K^*_{(a)}(x,W,\xi)$ is the component of a non-holonomic covariant linear connection.

Proof. Let $x(P)$, $X(P,Y_p)$ and $\bar{x}(P)$, $\bar{X}(P,Y_p)$ be two representations generating the transformation of representation $\bar{x} = \bar{x}(x)$, $\bar{X} = M(x,X)$. Then

(5) As before we are assuming the linearity in ξ is preserved.

$$\{[\bar{K}(\bar{x}, \bar{V}, \bar{\xi}), \bar{W}]\} = \{V, M^*_{(2)}(x, \bar{K}^*_{(2)}(\bar{x}, \bar{V}, \bar{\xi}))\}.$$

By the law of transformation of linear connections we have

$$\{[\bar{K}(\bar{x}, \bar{V}, \bar{\xi}), \bar{W}]\} = \{V, K^*_{(2)}(x, W, \xi) - M^*_{(2)}(x, \bar{W}; \xi)\}$$

Hence $\{V, M^*_{(2)}(x, \bar{K}^*_{(2)}(\bar{x}, \bar{V}, \bar{\xi}))\} = \{V, K^*_{(2)}(x, W, \xi) - M^*_{(2)}(x, \bar{W}; \xi)\}.$ (3.27)

Since V is an arbitrary n.h.c.v. (3.27) implies

$$M^*_{(2)}(x, \bar{K}^*_{(2)}(\bar{x}, \bar{W}, \bar{\xi})) = K^*_{(2)}(x, W, \xi) - M^*_{(2)}(x, \bar{W}; \xi).$$

Solving we find that

$$\bar{K}^*_{(2)}(\bar{x}, \bar{W}, \bar{\xi}) = N'^*_{(2)}(\bar{x}, K^*_{(2)}(x, W, \xi)) - N'^*_{(2)}(\bar{x}, M^*_{(2)}(x, \bar{W}; \xi)). \qquad (3.28)$$

Since $N^*_{(2)}(\bar{x}, W)$ and $M^*_{(2)}(x, W)$ are inverses we obtain

$$\bar{K}^*_{(2)}(\bar{x}, \bar{W}, \bar{\xi}) = N'^*_{(2)}(\bar{x}, K^*_{(2)}(x, W, \xi)) + N^*_{(2)}(\bar{x}, \bar{W}; \bar{\xi}).$$

This proves the theorem.

Theorem 3.23. Let $L(x, W, \xi)$ be a bilinear function of W, ξ on $\Sigma E, E$ to E_1. Then a necessary and sufficient condition for

$$\delta W(x) - L(x, W(x), \delta x \qquad (3.29)$$

to be a n.h.$\bar{c}$.v.f. for every Fréchet differentiable n.h.c.v.f. $W(x)$, is that $L(x, W, \xi)$ be the component of a n.h. covariant linear connection.

We write (3.29) as $W(x \,|\, \delta x)$ and call this the covariant differential of W.

<u>Theorem 3.24</u>. Let $F(x, \xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_m, V_1, \cdots, V_s, W_1, \cdots, W_t)$ be a multilinear form in the arbitrary contravariant vectors $\xi_1, \xi_2, \cdots, \xi_n$, the arbitrary covariant vectors, $\eta_1, \cdots, \eta_m$, the arbitrary n.h.c.v. $V_1, \cdots, V_s$, and the arbitrary n.h.c̄.v. $W_1, \cdots, W_s$. Suppose also that F possesses a continuous first Fréchet differential. Consider

$$
\begin{aligned}
&F(x, \xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_m, V_1, \cdots, V_s, W_1, \cdots, W_t \,|\, \delta x) = F(x, \xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_m, V_1, \cdots, V_s, W_1, \cdots, W_t ; \delta x) \\
&- \sum_{i=1}^{n} F(x, \xi_1, \cdots, \xi_{i-1}, \Gamma(x, \xi_i, \delta x), \xi_{i+1}, \cdots, \xi_n, \eta_1, \cdots, \eta_m, V_1, \cdots, V_s, W_1, \cdots, W_t) \\
&+ \sum_{i=1}^{m} F(x, \xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_{i-1}, \Gamma^*_{(2)}(x, \eta_i, \delta x), \eta_{i+1}, \cdots \eta_m, V_1, \cdots, V_s, W_1, \cdots, W_t) \qquad (3.291) \\
&- \sum_{i=1}^{s} F(x, \xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_m, V_1, \cdots, V_{i-1}, K(x, V_i, \delta x), V_{i+1}, \cdots, V_s, W_1, \cdots, W_t) \\
&+ \sum_{i=1}^{t} F(x, \xi_1, \cdots, \xi_n, \eta_1, \cdots \eta_m, V_1, \cdots, V_s, W_1, \cdots, W_{i-1}, L(x, W_i, \delta x), W_{i+1}, \cdots W_t) \\
&+ R.
\end{aligned}
$$

For simplicity we write (3.291) as $F(x, \cdots | \delta x)$.

(a). If F is a contravariant vector field and $R = \Gamma(x, F, \delta x)$ then $F(x, \cdots | \delta x)$ is also a contravariant vector field valued multilinear form in the ξ 's, η 's, V's, W's and δx.

(b) If F is a covariant vector field and $R = -\Gamma^*_{(2)}(x, F, \delta x)$, then $F(x \cdots | \delta x)$ is also a covariant vector field valued multilinear form in the ξ's, η's, V's, W's and δx.

(c). If F is a n.h.c.v.f. and $R = K(x, F, \delta x)$, then $F(x, \cdots / \delta x)$ is a n.h.v.c.f. valued multilinear form in the ξ's, η's, V's, W's, and δx.

(d) If F is a n.h.$\bar{c}$.v.f. and $R = -L(x, F, \delta x)$ (a n.h. covariant linear connection), then $F(x, \cdots / \delta x)$ is a n.h.$\bar{c}$.v.f. valued multilinear form in the ξ's, η's, V's, W's and δx.

(e) If F is a scalar form and $R = 0$, then $F(x, \cdots / \delta x)$ is also a scalar valued multilinear form in the ξ's, η's, V's, W's and δx.

We call $F(x, \cdots / \delta x)$ the covariant differential of F.

Theorem 3.25. Let $F(x, W, \delta x)$ (a bilinear function of $W, \delta x$ on $\Sigma E, E$ to E_1) be the component in a given representation of a geometric object. Then a necessary and sufficient condition that the relation

$$-\{K(x, V, \delta x), W\} = \{V, F(x, W, \delta x)\}$$

be an invariant of all representations for all n.h.c.v. V and all n.h.$\bar{c}$.v. W is that $F(x, W, \delta x)$ be the components of a n.h. covariant linear connection.

Proof of necessity.

If

$$\{\bar{K}(\bar{x}, \bar{V}, \delta\bar{x}), \bar{W}\} = \{\bar{V}, \bar{F}(\bar{x}, \bar{W}, \delta\bar{x})\}$$

then

$$\{M(x, K(x, V, \delta x)), \bar{W}\} - \{M_1(x, V; \delta x), \bar{W}\} = \{M(x, V), \bar{F}(\bar{x}, \bar{W}, \delta\bar{x})\}.$$

Thus

$$\{V, F(x, W, \delta x) - M^*_{(2)}(x, W; \delta x)\} = \{V, M^*_{(2)}(x, \bar{F}(\bar{x}, \bar{W}, \delta\bar{x}))\}.$$

Since V is arbitrary we must have

$$F(x, W, \delta x) - M^*_{(2)}(x, W; \delta x) = M^*_{(2)}(x, \bar{F}(\bar{x}, \bar{W}, \delta\bar{x})).$$

Solving and using a well known theorem on Fréchet differentials of inverse functions[6] we obtain

$$\bar{F}(\bar{x}, \bar{W}, \delta\bar{x}) = N^*_{(2)}(\bar{x}, F(x, W, \delta x)) + N^*_{(2)}(\bar{x}, W; \delta\bar{x}).$$

This proves the necessity. The sufficiency is then easily obtained.

__Theorem 3.26__. A necessary and sufficient condition for

$$\delta\{V, W\} = \{V(x|\delta x), W\} + \{V, W(x|\delta x)\},$$

for all Fréchet differentiable n.h.c.v.f. $V(x)$ and n.h.$\bar{c}$.v.f. $W(x)$ is that the non-holonomic covariant linear connection $L(x, W, \delta x)$ be the adjoint of $K(x, V, \delta x)$.

The proof is easily obtained by substituting for $V(x|\delta x)$ and $W(x|\delta x)$, the expressions $\delta V + K(x, V, \delta x)$ and $\delta W - L(x, W, \delta x)$ respectively.

Because of theorems 3.22, 3.25 and 3.26 we shall assume that the n.h. linear connection $K(x, V, \delta x)$ has an adjoint $K^*_{(2)}(x, W, \delta x)$ in every representation, and shall take $K^*_{(2)}(x, W, \delta x)$ to be our n.h.$\bar{c}$. linear connection. Also we shall assume that $K^*_{(2)}(x, W, \delta x)$ possesses a continuous second Fréchet differential at each point of the coordinate domain.

__Section 3.3.__ Non-holonomic covariant curvature form.

(6). Michal-Elconin [1].

Let W(x) be a non-holonomic covariant vector field possessing a continuous second Fréchet differential. If we calculate

$W(x|\delta_1 x|\delta_2 x) - W(x|\delta_2 x|\delta_1 x)$ we obtain

$$W(x|\delta_1 x|\delta_2 x) - W(x|\delta_2 x|\delta_1 x) = -f(x, W, \delta_1 x, \delta_2 x),$$

where

$$f(x, W, \delta_1 x, \delta_2 x) = K^*_{(2)}(x, W, \xi_1 ; \xi_2) - K^*_{(2)}(x, W, \xi_2 ; \xi_1)$$
$$+ \Lambda^*_{(2)}(x, K^*_{(2)}(x, W, \xi_2), \xi_1) - K^*_{(2)}(x, K^*_{(2)}(x, W, \xi_1), \xi_2).$$

We define $f(x, W, \delta_1 x, \delta_2 x)$ to be the non-holonomic curvature form. It is obviously a n.h.$\bar{c}$.v.f. trilinear form in $W, \delta_1 x, \delta_2 x$.

<u>Theorem 3.31</u>. The adjoint $H^*_{(2)}(x, W, \xi_1, \xi_2)$ of the n.h. contravariant curvature form $H(x, V, \xi_1, \xi_2)$ exists and is equal to $f(x, W, \xi_1, \xi_2)$.

<u>Proof</u>.

Since

$$\{K(x, V, \xi_1), W\} = \{V, K^*_{(2)}(x, W, \xi_1)\},$$

we obtain

$$\{K(x, V, \xi_1 ; \xi_2), W\} = \{V, K^*_{(2)}(x, W, \xi_1 ; \xi_2)\}. \qquad (3.31)$$

Also

$$\{K(x, K(x, V, \xi_1), \xi_2), W\} = \{V, K^*_{(2)}(x, K^*_{(2)}(x, W, \xi_2), \xi_1)\}. \qquad (3.32)$$

Hence every term of

$$H(x,V;\xi_1,\xi_2) = K(x,V;\xi_1;\xi_2) - K(x,V;\xi_2;\xi_1)$$

$$+ K(x, K(x,V;\xi_1),\xi_2) - K(x, K(x,V;\xi_2),\xi_1)$$

has an adjoint. It is then easy to verify that

$$\{H(x,V;\xi_1,\xi_2),W\} = \{V, f(x,W;\xi_1,\xi_2)\}. \tag{3.33}$$

This of course implies $f(x,W;\xi_1,\xi_2) = H^*_{(2)}(x,W;\xi_1,\xi_2)$.

Section 3.4. Allowable $k^{(h+2)}$ representations.

In order to make the normal representation theory that we developed in the previous chapter apply to non-holonomic covariant vector fields it is necessary to show that the transformations of representation arising from a change to normal representations are regular transformations of representation.(1) To do this we shall assume that a certain subset of our representation possess certain properties.

Def. 3.41. Let Σ_1 be an open subset of our Banach space E and let $\bar{x} = \bar{x}(x)$, $\bar{X} = M(x,X)$ be functions with the following properties.

1. $\bar{x}(x)$ is a regular transformation of class $k^{(h+2)}$ in the sense of Michal-Hyers [3], taking Σ_1 homeomorphically into an open subset $\Sigma_2 \subset E$.

2. $M(x,X)$ is a solvable linear function of X on $\Sigma_1 E_1$ to E_1, with inverse $N(\bar{x},X)$ a function on $\Sigma_2 E_1$ to E_1.

3. For each point $x \in \Sigma_1$, $M(x,X)$ has an adjoint $M^*_{(2)}(x,X)$.

(1) Regular in the sense of this chapter.

4. For each point $x \in \Sigma_1$, $M^*_{(a)}(x,X)$ is a solvable linear function with inverse $N^*_{(a)}(\bar{x},X)$.

5. $M(x,X)$, $M^*_{(a)}(x,X)$, $N(\bar{x},X)$ and $N^*_{(a)}(\bar{x},X)$ are all of class $C^{(n+1)}$ locally uniformly at each point of there domain of definition.

If $\bar{x}(x)$, $M(x,X)$ possess all of these properties we shall call them a regular transfomation of representation of class $k^{(n+2)}$.

Def. 3.42. A set of allowable $K^{(n+2)}$ representations shall be called allowable $k^{(n+2)}$ if the following are true.

1. The transformations of representation arising from the change from one allowable $k^{(n+2)}$ representation to another is a regular transformation of representation of class $k^{(n+2)}$.

2. Each coordinate system $x(P)$ of an allowable $k^{(n+2)}$ representation is an allowable $k^{(n+2)}$ coordinate system in the sense of Michal-Hyers [3].

3. The representation obtained from an allowable $k^{(n+2)}$ representation by a regular transformation of representation of class $k^{(n+2)}$ is an allowable $k^{(n+2)}$ representation.

We shall assume that our geometric spaces possess a set of allowable $k^{(n+2)}$ representations. It should be pointed out that we do not assume the set T of all representations forms an allowable $k^{(n+2)}$ set of representations, but merely that there exists a subset of T that does form such a set.

Section 3.5. Extension of the normal representation theory.

We shall assume that the affine linear connection $\Gamma(x,\xi_1,\xi_2)$

satisfies hypotheses I - IV of Michal-Hyers [3]. Then all of their theory on normal coordinate systems of class k^{n} will apply to our geometrical Hausdorff space H. In addition we require $K^{*}_{(2)}(x,W,\xi)$ to be of class $C^{(n)}$ $(n \geq 3)$ locally uniformly on Σ the coordinate domain of an allowable $k^{(n+2)}$ coordinate system of an allowable $k^{(n+2)}$ representation. By the same method used in section (2.4) we can prove the following theorem.

Theorem 3.51. Let p, ξ, q, X_0, Y_0 be the same as in theorem 2.41. Then there exists a unique function $G(p, s\xi, W_0)$ with the following properties.

(i) For any choice of $\xi \in Y_0$ and $p \in X_0$, $G(p, s\xi, W_0)$ is the unique solution of

$$\frac{dW}{ds} - K^{*}_{(2)}\left(x, W, \frac{dx}{ds}\right) = 0, \quad W(p) = W_0, \tag{3.51}$$

along the path $x = f(p, s\xi)$.

(ii) For each such p, ξ and $0 \leq s \leq 1$, $G(p, s\xi, W_0)$ is solvable linear in W_0.

(iii) $G(p, s\xi, W)$ is of class $C^{(n)}$ uniformly in the pair (ξ, p) for each such ξ and p.

As before we delete the fixed point p and obtain by placing $\xi = y$ and $s = 1$ the function $G(y, W_0) = G(p, y, W_0)$. Consider now the function $R(y, V_0)$ of section 2.4 (expression 2.46). We note the following facts.

1. $R(sy, V_0)$ is a solution of

$$\frac{dV}{ds} + K\left(x, V, \frac{dx}{ds}\right) = 0, \quad V(p) = V_0$$

along the path $x = f(p, sy)$ for any $y \in Y_0$.

2. $G(sy, W_0)$ is a solution of

$$\frac{dW}{ds} - K^*_{(2)}\left(x, W, \frac{dx}{ds}\right) = 0, \quad W(p) = W_0$$

along the path $x = f(p, sy)$ for any $y \in Y_0$.

From these we can verify that

$$\frac{d}{ds}\left\{R(sy, V_0),\ G(sy, W_0)\right\} = 0.$$

Integrating from 0 to 1 we obtain

$$\left\{R(y, V_0),\ G(y, W_0)\right\} = \left\{V_0, W_0\right\}. \tag{3.52}$$

But $G(y, W_0)$ is solvable linear in W_0, and hence if we denote its inverse by $G^{-1}(y, W_0)$, (3.52) implies

$$\left\{R(y, V_0),\ W_0\right\} = \left\{V_0,\ G^{-1}(y, W_0)\right\} \tag{3.53}$$

This means that $R^*_{(2)}(y, W_0)$ exists and is equal to $G^{-1}(y, W_0)$. Obviously $R^*_{(2)}(y, W_0)$ is a solvable linear function of W_0.

Suppose $x(P)$, $X(P, Y_p)$ is the allowable k^{n+2} representation from which we started. By Michal-Hyers [3] the coordinate system $x(P)$ determines a normal coordinate system $y(P)$ with center $P_0 = P_1(p)$, and moreover the induced transformation of coordinates $x = f(p, y)$ is a regular transformation of coordinates of class k^n. The function

${}^{+}X(P,Y_P)$ defined implicitly by

$$X(P,Y_P) = R(y(P), {}^{+}X(P,Y_P))$$

is a vector coordinate system for the associated Banach space B_P, for each P in the geometric domain of y(P). Thus $y(P)$, ${}^{+}X(P,Y_P)$ is a representation and also is a normal representation in the sense of section 2.4. Moreover the induced transformations of representation $x = f(p,y)$, $X = R(y,{}^{+}X)$ will now be regular transformations of representation as defined in section 3.2. In what follows all normal representations referred to shall be of the type just discussed.

Theorem 3.52. Let $x(P),X(P,Y_P)$ and $\bar{x}(P),\bar{X}(P,Y_P)$ be two allowable $k^{(n+2)}$ representations with intersecting geometrical domains. Further suppose that ~~there~~ they generate the transformations of representation $\bar{x} = \bar{x}(x)$, $\bar{X} = M(x,X)$. If $y(P),{}^{+}X(P,Y_P)$ and $\bar{y}(P),{}^{+}\bar{X}(P,Y_P)$ are two normal representations with same center $p = x(P_0)$ determined by the two allowable $k^{(n+2)}$ representations respectively, then the components ${}^{+}W(y)$ and ${}^{+}\bar{W}(\bar{y})$ of a n.h.c̄.v.f. in the normal representations are related by

$${}^{+}\bar{W}(\bar{y}) = N^{*}_{(2)}(\bar{p},{}^{+}W(y)).$$

Proof:

The transformation of representation from $x(P),X(P,Y\ )$ to the normal representation determined by it, is

$$x = f(p,y), \quad X = R(y,{}^{+}X). \tag{3.54}$$

Since W(x) is a n.h.c.v.f. we have

$$W(x) = R_{(2)}^{*^{-1}}(x, {}^{+}W(y)). \tag{3.55}$$

Similarly

$$\bar{W}(\bar{x}) = \bar{R}_{(2)}^{*^{-1}}(\bar{x}, {}^{+}\bar{W}(\bar{y})) \tag{3.56}$$

Since $\bar{W}(\bar{x}) = N_{(2)}^{*}(\bar{x}, W(x))$ we can obtain

$${}^{+}\bar{W}(\bar{y}) = \bar{R}_{(2)}^{*}(\bar{y}, N_{(2)}^{*}(\bar{x}, R_{(2)}^{*^{-1}}(x, {}^{+}W(y)))). \tag{3.57}$$

But in theorem 2.42 we showed that

$$M(p, \bar{X}) = \bar{R}^{-1}(\bar{x}, M(x, R(y, \bar{X}))). \tag{3.58}$$

Thus

$$M_{(2)}^{*}(p, \bar{X}) = R_{(2)}^{*}(y, M_{(2)}^{*}(x, \bar{R}_{(2)}^{*^{-1}}(\bar{x}, \bar{X}))),$$

and $$N_{(2)}^{*}(\bar{p}, \bar{X}) = M_{(2)}^{*^{-1}}(p, \bar{X}) = \bar{R}_{(2)}^{*}(\bar{y}, N_{(2)}^{*}(\bar{x}, R_{(2)}^{*^{-1}}(x, \bar{X}))). \tag{3.59}$$

Applying this to (3.57) we obtain

$${}^{+}\bar{W}(\bar{y}) = N_{(2)}^{*}(\bar{p}, {}^{+}W(y)).$$

which proves the theorem.

Section 3.6. The differentials of $G(y, W_0)$ and of its inverse.

The remaining sections of this chapter are merely extensions of some of the theorems obtained in the previous chapter. As the proofs are almost identical they shall be omitted.

Let us define the functions $J_t(x,W,\xi_1,\xi_2,\cdots,\xi_t)$, $(t \leq n)$,

by the following recurrence relation.

$$J_2(x,W,\xi_1,\xi_2) = \frac{1}{2} P\Big[K^*_{(2)}(x,W,\xi_1;\xi_2) + K^*_{(2)}(x,K^*_{(2)}(x,W,\xi_1),\xi_2) - K^*_{(2)}(x,W,\Gamma(x,\xi_1,\xi_2))\Big],$$

(3.61)

$$J_t(x,W,\xi_1,\cdots,\xi_t) = \frac{1}{t} P\Big[J_{t-1}(x,W,\xi_1,\cdots,\xi_{t-1};\xi_t) + J_{t-1}(x,K^*_{(2)}(x,W,\xi_t),\xi_1,\cdots,\xi_{t-1}) - \sum_{i=1}^{t-1} J_{t-1}(x,W,\xi_1,\cdots,\xi_{i-1},\Gamma(x,\xi_i,\xi_t),\xi_{i+1},\cdots,\xi_{t-1})\Big],$$

where $P[\cdots]$ means the sum of terms obtained by a cyclic permutation of $\xi_1,\cdots,\xi_t$.

We can then obtain the following results.

(1) $G(o,W_o) = W_o$.

(2) $G(o,W_o;\delta x) = K^*_{(2)}(p,W_o,\delta x)$.

(3) $G(o,W_o;\delta_1 y;\delta_2 y;\cdots;\delta_t y) = J_t(p,W_o,\delta_1 y,\delta_2 y,\cdots,\delta_t y)$.

(4) ${}^+K^*_{(2)}(o,W_o,\xi) = o$, for all $W_o \in E_1$ and $\xi \in E$.

(5) $G^{-1}(p,W_o) = W_o$.

(6) $G^{-1}(p,W_o;\delta x) = -K^*_{(2)}(p,W_o,\delta x)$.

(7) If W is the component of an arbitrary n.h.c̄.v. in a given allowable $k^{(n+2)}$ representation, then its component ${}^+W(y)$ in the corresponding normal representation is such that

(a) ${}^+W(o) = W$,

(b) ${}^+W(o;y) = -K^*(p,W,\delta y)$.

---

(8) We write the inverse of $G(y,W_o)$ as $G^{-1}(x,W_o)$, where $x = f(p,y)$. Thus $x = p$ corresponds to $y = o$.

(8) By the law of transformations of linear connections we have

$$K^*_{(\omega)}(x, W, \xi x) = G(y, {}^+k^*_{(\omega)}(y, {}^+W, \xi y)) + G(y, {}^+W; \xi y).$$

Section 3.7. Tensor extensions of multilinear forms.

Let $x(P)$, $X(P, Y_p)$ be an allowable $k^{(n+s)}$ representation, and suppose that $F(x, V_1, \cdots, V_n, W_1, \cdots W_n, \eta_1, \cdots; \eta_s, \xi_1, \cdots; \xi_h)$ is a multilinear form of the arbitrary n.h.c.v. $V_1, \cdots V_n$, n.h.$\bar{c}$.v. $W_1, \cdots W_m$, contravariant vectors $\xi_1, \cdots, \xi_h$ and covariant vectors $\eta_1, \cdots, \eta_s$, in this representation. We shall consider the following cases.

(1). F is a n.h.c.v.f.

(2). F is a n.h.$\bar{c}$.v.f.

(3). F is a contravariant vector field.

(4). F is a covariant vector field.

(5). F is a scalar field.

Suppose also that $p = x(P_0)$ is any point of the coordinate domain of $x(P)$, and $y(P)$, ${}^+X(P, Y_p)$ is a normal representation with center $p = x(P_0)$ determined by the allowable $k^{(n+s)}$ representation. Further let the components of F in this representation be ${}^+F(y, {}^+V_1, \cdots, {}^+V_n, {}^+W_1, \cdots, {}^+W_m, {}^+\eta_1, \cdots, {}^+\eta_s, {}^+\xi_1, \cdots, {}^+\xi_h)$.

Def. 3.71. In all cases the k'th extension $F(x, V_1, \cdots V_n, W_1, \cdots, W_m, \eta_1, \cdots, \eta_s, \xi_1, \cdots, \xi_h \mid \xi_{h+1}, \cdots, \xi_{h+k})$ of F is defined at each point p of the coordinate domain of $x(P)$ by

$$F(p, V_1, \cdots, V_n, W_1, \cdots, W_m, \eta_1, \cdots, \eta_s, \xi_1, \cdots, \xi_h \mid \xi_{h+1}, \cdots, \xi_{h+k}) = {}^+F(y, {}^+V_1, \cdots {}^+V_n, {}^+W_1, \cdots, {}^+W_m, {}^+\eta_1, \cdots, {}^+\eta_s, {}^+\xi_1, \cdots {}^+\xi_h; {}^+\xi_{h+1}; \cdots; {}^+\xi_{h+k})_{y=0}.$$

Theorem 3.71. The k'th extension of any form F of type (1) - (5) is again a multilinear form in the V's, W's, ξ's and η's of the same type.

Theorem 3.72. In all cases the first extension of F is equal to its covariant differential.

Def. 3.72. The j'th normal covariant form $P_j(x, W, \xi_1, \xi_2, \cdots, \xi_{j+1})$ is defined at each point $p = x(P_0)$ by

$$P_j(p, W, \xi_1, \xi_2, \cdots, \xi_{j+1}) = \overset{+}{K}{}_{(2)}^{*}(y, \overset{+}{W}, \overset{+}{\xi}_1; \overset{+}{\xi}_2; \cdots; \overset{+}{\xi}_{j+1})_{y=0}.$$

Theorem 3.73. The normal non-holonomic covariant vector forms are all n.h.$\bar{c}$.v.f.

Theorem 3.74. The ~~second~~ first normal vector form $P_1(x, W, \xi_1, \xi_2)$ is equal to $\frac{1}{2} H_{(2)}^{*}(x, W, \xi_1, \xi_2)$.

Theorem 3.75. Let $F(x, V_1, \cdots V_n, W_1, \cdots W_m)$ be the components of a n.h.c.v.f. valued multilinear form in the arbitrary n.h.c.v. $V_1, \cdots V_n$, and the arbitrary n.h.c.v. $W_1, \cdots W_m$ in an allowable $k^{(n+2)}$ representation. If $F(x, V_1, \cdots V_n, W_1, \cdots W_m; \delta_1 x; \delta_2 x)$ exists continuous in x, then

$$F(x, V_1, \cdots, V_n, W_1, \cdots, W_m / \xi_1, \xi_2) = \frac{1}{2}\{F(x, V_1, \cdots, V_n, W_1, \cdots, W_m / \xi_1 / \xi_2)$$
$$+ F(x, V_1, \cdots, V_n, W_1, \cdots, W_m / \xi_2 / \xi_1)\}.$$

The same theorem is true if F is a n.h.$\bar{c}$.v.f.

Section 3.8. Replacement theorem.

Let

$$R_{\alpha\beta\cdots\delta}(f_1(\alpha_1, \alpha_2), f_2(\beta_1, \beta_2, \beta_3), \cdots f_{k+1}(\pi_1, \cdots \pi_{k+2}), g_1(\rho_1, \rho_2), \cdots g_{m+1}(\omega_1, \omega_2, \cdots \omega_{m+2}), p_1(\Sigma_1, \gamma), p_2(\Sigma_2, \mu_1, \mu_2),$$
$$\cdots, p_r(\Sigma_r, \nu_1, \cdots \nu_{r+1}), q_1(T_1, \lambda_1), \cdots q_t(T_t, \zeta_1, \cdots, \zeta_t) \,\|\, \xi_1, \cdots \xi_s, \eta_1, \cdots, \eta_r, V_1, \cdots V_v, W_1, \cdots W_w)$$

be a functional whose arguments are multilinear functions $f_1, \cdots, q_t$, and whose value is a multinear function of $\xi_1, \cdots, \xi_s, \eta_1, \cdots, \eta_\lambda, V_1, \cdots, V_r$, $W_1, \cdots, W_\omega$.

Def. 3.81. A multilinear form

$$F(x, \xi_1, \cdots, \xi_s, \eta_1, \cdots, \eta_\lambda, V_1, \cdots, V_r, W_1, \cdots, W_\omega) =$$

$$R_{\alpha\beta\cdots\delta}(\Gamma(x, \alpha_1, \alpha_2), \Gamma(x, \beta_1, \beta_2; \beta_3), \cdots, \Gamma(x, \pi_1, \pi_2; \pi_3; \cdots; \pi_{\omega}), \Gamma^*_{(2)}(x, \rho_1, \rho_2),$$
$$\Gamma^*_{(2)}(x, \varphi_1, \varphi_2; \varphi_3), \cdots, \Gamma^*_{(2)}(x, \psi_1, \psi_2; \psi_3; \cdots; \psi_{n+2}), K(x, \Sigma_1, \gamma), \cdots, K(x, \Sigma_n, \nu_1; \nu_2; \cdots; \nu_{n+1}),$$
$$K^*_{(2)}(x, T_1, \lambda), \cdots, K^*_{(2)}(x, T_\lambda, \delta_1; \delta_2; \cdots; \delta_{t+1}) \,||\, \xi_1, \cdots, \xi_s, \eta_1, \cdots, \eta_\lambda, V_1, \cdots, V_r, W_1, \cdots, W_\omega),$$

where F is of type (1) - (5), will be called a differential invariant if R as a functional is invariant to changes of representation.

Theorem 3.81. The component in an allowable $k^{(n+2)}$ representation of every differential invariant can be expressed in terms of the normal contravariant forms $A_i(x, \xi_1, \cdots, \xi_{i+2})$, the normal covariant vector forms $A^*_{(2)i}(x, \eta, \xi_1, \cdots, \xi_{i+2})$, the normal n.h. contravariant vector forms $C_j(x, V, \xi_1, \cdots, \xi_{j+1})$ and the normal n.h. covariant vector forms $P_j(x, W, \xi_1, \cdots, \xi_{j+1})$ by the following replacement process.

1. $\Gamma(x, \alpha_1, \alpha_2)$, $\Gamma^*_{(2)}(x, \rho_1, \rho_2)$, $K(x, \Sigma, \gamma)$ and $K^*_{(2)}(x, T, \mu)$ are replaced by zero.
2. $\Gamma(x, \alpha_1, \alpha_2; \alpha_3; \cdots; \alpha_{j+2})$ is replaced by $A_j(x, \alpha_1, \cdots, \alpha_{j+2})$.
3. $\Gamma^*_{(2)}(x, \rho_1, \rho_2; \rho_3; \cdots; \rho_{j+2})$ is replaced by $A^*_{(2)j}(x, \rho_1, \rho_2, \cdots, \rho_{j+2})$.
4. $K(x, \Sigma_j, \nu_1; \nu_2; \cdots; \nu_{j+1})$ is replaced by $C_j(x, \Sigma_j, \nu_1, \cdots, \nu_{j+1})$.
5. $K^*_{(2)}(x, T_j, \gamma_1; \gamma_2; \cdots; \gamma_{j+1})$ is replaced by $P_j(x, T_j, \gamma_1, \cdots, \gamma_{j+1})$.

(9). See Michal-Hyers [3].

CHAPTER IV

Section 4.1. Interspace adjoints.[1]

Def. 4.11. Let E and E_1 be Banach spaces with inner products $[x,y]$ and $\{V,W\}$ respectively. A linear function $T(x)$ on E to E_1 is said to have an interspace adjoint $T^*(V)$ if

1. $T^*(V)$ is a linear function on E_1 to E;
2. $\{T(x),V\} = [x,T^*(V)]$ for every $x \in E$ and $V \in E_1$.

Def. 4.12. Let E and E_1 be Banach spaces with inner products $[x,y]$ and $\{V,W\}$ respectively. A linear function $T(V)$ on E_1 to E is said to have an interspace adjoint $T^*(x)$ if

1. $T^*(x)$ is a linear function on E_1 to E_1;
2. $[x,T(V)] = \{T^*(x),V\}$ for every $x \in E$ and $V \in E_1$.

With these definitions we can show that all the properties listed for ordinary adjoints are also true for interspace adjoints.

Theorem 4.11. Let the components of a n.h.$\bar{c}$.v.f. be in any representation a linear function $T(\xi)$ of the arbitrary contravariant vector on E to E_1. If $T^*(V)$ exists in every representation, then $T^*(V)$ is the component of a covariant vector field.

Proof.

[1] The interspace adjoints defined in this section are different to those that can be defined by means of the Michal inter-space inner product.

$$[\bar{\xi}, \bar{T}^*(\bar{V})] = \{\bar{T}(\bar{\xi}), \bar{V}\} = \{N^*_{(2)}(\bar{x}, T(\xi)), \bar{V}\} = \{T(\xi), V\} = [\xi, T^*(V)].$$

But

$$[\bar{\xi}, \bar{T}^*(\bar{V})] = [\bar{x}(x; \xi), \bar{T}^*(\bar{V})] = [\xi, \bar{x}^*_{(2)}(x; \bar{T}^*(\bar{V}))].$$

Hence

$$T^*(V) = \bar{x}^*_{(2)}(x; \bar{T}^*(\bar{V})).$$

This implies

$$\bar{T}^*(\bar{V}) = x^*_{(2)}(\bar{x}; T^*(V))$$

which proves the theorem.

Theorem 4.12. Let the components of a covariant vector be a linear function $T(V)$ on E_1 to E in any representation. If $T^*(\xi)$ exists in every representation, then $T^*(\xi)$ is a n.h.$\bar{c}$.v.f.

The proof is similar to that of theorem 4.11.

In this chapter we shall consider the same geometrical spaces, and shall carry over all of the assumptions made in the previous chapters. We shall restrict our representations to yield only regular transformations of representation $\bar{x}(x)$, $M(x,X)$ such that;

1. $\bar{x}(x)$ and $x(\bar{x})$ have continuous Fréchet differentials of order $n + 2$ at each point of their domain of definition;

2. the functions $M(x,X)$, $M^*_{(2)}(x,X)$, $N(\bar{x},X)$ and $N^*_{(2)}(\bar{x},X)$ each have continuous Fréchet of order $n + 1$ at each point of their domain of definition.

This will of course make the Hausdorff space H an $(n + 2)$ differentiable manifold in the sense of Michal [3]. In addition we shall assume the existence of a non-holonomic covariant vector field with component $R(x,V)$ and a covariant vector field with component $\gamma(x,W)$ such that each point x of the coordinate domain of every representation the following things are true.

A_1. $R(x,V)$ is a solvable linear function of the arbitrary n.h.c.v. V on ΣE_1 to E_1, with inverse $R^{-1}(x,W)$.

A_2. $\{R(x,V),V\} \geq 0$, and $\{R(x,V),V\} = 0$ implies $V = 0$.

A_3. $\{R(x,V_1),V_2\} = \{V_1, R(x,V_2)\}$ for all V_1, $V_2 \in E_1$.

A_4. $R(x,V;\delta x)$ exists and has an interspace adjoint $R^*_{(3)}(x,V_1;V_2)$.[2]

A_5. $R(x,V;\delta x)$ and $R^*_{(3)}(x,V_1;V_2)$ possess continuous Fréchet differentials of order $(p-1)$, where $p < n+1$.

B_1. $\gamma(x,W)$ is a linear function of the arbitrary n.h.c.v. W on ΣE_1 to E.

B_2. $\gamma(x,W) = 0$ has only a finite number of linearly independent solutions, and there exists a fundamental set $\tau_r(x)$, $r = 1,2,\cdots,s$, such that

(a) $\tau_r(x;\xi_1;\xi_2)$ exists continuous in x,

(b) $\{R^{-1}(x,\tau_r),\tau_k\} = \delta_{rk}$ (Kroenecker delta).

B_3. $\gamma(x,W)$ has an interspace adjoint $\gamma^*_{(2)}(x,\xi)$.

B_4. $\gamma(x,R(x,\gamma^*_{(2)}(x,\xi)))$ is a solvable linear function of ξ.

---

(2) We carry over the notation $F^*_{(i)}(x,\xi_1,\cdots,\xi_{i-1},\eta,\xi_{i+1},\cdots,\xi_n)$ to mean the adjoint (either kind) of the function considered as a linear function of the i'th place. We also retain the assumption that the adjoint remains linear in all the ξ's.

B_5. $\gamma(x,W)$, and $\gamma^{*}_{(2)}(x,\xi)$ both possess continuous Fréchet differentials of order p.

B_6. $\gamma(x,W;\xi)$ has an adjoint $\gamma^{*}_{(3)}(x,W;\eta)$ which has continuous Fréchet differentials of order (p - 1).

<u>Theorem 4.13</u>. If postulates A_1 - B_6 are satisfied,(3) then our geometric space H is a general Riemannian differential geometry in the sense of Michal [3].

<u>Proof</u>. To prove this theorem we merely have to show the existence of a metric form $g(x,\xi)$ with the following properties.

1. $g(x,\xi)$ has continuous Fréchet differentials of order p.
2. $[\xi, g(x,\xi)]$ is positive definite in ξ.
3. $g(x,\xi)$ is solvable linear in ξ.
4. $g(x,\xi)$ is self adjoint.
5. $g^{*}_{(3)}(x,\xi;\xi x)$ exists and is continuously differentiable up to order p - 1.

(3). We shall give infinite dimensional examples where this is true.

We choose

$$g(x,\xi) = \gamma(x, R(x, \gamma_{\omega}^{*}(x,\xi))) \tag{4.11}$$

(1), (3), (4), (5) are all easy to verify. To prove (2) we must show that

$$[\gamma(x, R(x, \gamma_{\omega}^{*}(x,\xi))), \xi] = 0 \tag{4.12}$$

implies $\xi = 0$. But

$$[\gamma(x, R(x, \gamma_{\omega}^{*}(x,\xi))), \xi] = \{R(x, \gamma_{\omega}^{*}(x,\xi)), \gamma_{\omega}^{*}(x,\xi)\} \tag{4.13}$$

Thus by A_2 we have that (4.12) implies $\gamma_{\omega}^{*}(x,\xi) = 0$. But if $\gamma_{\omega}^{*}(x,\xi) = 0$, then $\gamma(x, R(x, \gamma_{\omega}^{*}(x,\xi))) = 0$. Hence by B_4 we have $\xi = 0$. This completes the proof of the theorem.

We shall now take the metric $g(x, \xi)$ of the general Riemannian geometry to be (4.11), and we shall write the inverse of $g(x, \xi)$ as $G(x, \eta)$. Henceforth also the affine linear connection $\Gamma(x, \xi_1, \xi_2)$ of H shall be taken to be the abstract Christoffel symbols based on $g(x,\xi)$.(4)

<u>Theorem 4.14</u>. The function

$$\rho(x,V) = G(x, \gamma(x, R(x,V))) \tag{4.14}$$

has the following properties.

(4). See Michal [3].

1. $\rho(x,V)$ is the component of a contravariant vector field and is is a linear function of the arbitrary n.h.c.v. V.

2. $\rho(x,V)$ possesses an interspace adjoint

$$\rho^*_{(2)}(x,\eta) = R(x, \gamma^*_{(2)}(x, \sigma(x,\eta))).$$

3. $\rho(x, \gamma_2^*(x,\xi)) = \xi.$

4. $\gamma(x, \rho^*_{(2)}(x,\eta)) = \eta.$

5. $\rho(x,V; \xi x)$ possesses an adjoint $\rho^*_{(3)}(x, V; \xi x)$.

It is easy to verify (1), (2) and (5). To show (3) we have

$$g(x,\xi) = \gamma(x, R(x, \gamma^*_{(2)}(x,\xi))).$$

Thus

$$\xi = \sigma(x, g(x,\xi)) = \sigma(x, \gamma(x, R(x, \gamma^*_{(2)}(x,\xi)))) = \rho(x, \gamma^*_{(2)}(x,\xi)).$$

Similarly we can show (4).

Theorem 4.15. The equation $\rho(x,V) = 0$ has only a finite number of linearly independent solutions. Further $T^r = R^{-1}(x, t_r)$, $(r=1,\dots,s)$, are a set of linearly independent solutions of $\rho(x,V) = 0$, and

$$\{T^r, R(x, T^k)\} = \delta_{rk}$$

Proof.

$$\rho(x, T^r) = \rho(x, R^{-1}(x, t_r)) = \sigma(x, \gamma(x, t_r)) = 0, \quad (r=1,2,\dots,s).$$

Thus T^r, $(r = 1,\dots,s)$, is a solution. Since

$$\sum_{i=1}^{s} c_i T^i = R^{-1}(x, \sum_{i=1}^{s} c_i t_i),$$

we have the T^λ are linearly independent because the τ_λ are. All that remains to show is that $\rho(x,V) = 0$ can have no other solution which is linearly independent of the T^λ. This we do by contradiction. Suppose V_1 is a solution of $\rho(x,V) = 0$ which is linearly independent of the T^λ. Then it is easily verified that $W_1 = R(x,V_1)$ is a solution of $\gamma(x,W) = 0$. But

$$\sum_{i=1}^{s} c_i \tau_i + c_{s+1} W_1 = R\left(x, \sum_{i=1}^{s} c_i T^i + c_{s+1} V_1\right) \tag{4.15}$$

Hence (4.15) would say that we have (s + 1) linearly independent solutions of $\gamma(x,W) = 0$. This contradicts B_2.

Finally $\{\tau_\lambda, R^{-1}(x, \tau_k)\} = \delta_{\lambda k}$ implies

$$\{\tau_\lambda, T^k\} = \{R(x, T^\lambda), T^k\} = \delta_{\lambda k}.$$

Theorem 4.16. The following identities are true

(a) $\gamma^*_{(a)}(x, \rho(x,V)) = V - \sum_{\lambda=1}^{s} \{\tau_\lambda, V\} T^\lambda$

(b) $\rho^*_{(b)}(x, \gamma(x,W)) = W - \sum_{\lambda=1}^{s} \{T^\lambda, W\} \tau_\lambda.$

Proof of (a). Let $\gamma^*_{(a)}(x, \rho(x,V)) = \ell$. Thus $\rho(x, \gamma^*_{(a)}(x, \rho(x,V))) = \rho(x, \ell)$. By theorem (4.14) we obtain $\rho(x,V) = \rho(x,\ell)$. That is $\ell - V$ is a solution of $\rho(x,V) = 0$. By theorem (4.15) we must have

$$\ell - V = \sum_{\lambda=1}^{s} a_\lambda T^\lambda, \qquad \text{where } a_\lambda\text{'s are constants.}$$

That is

$$\gamma^*_{(a)}(x, \rho(x,V)) = V + \sum_{\lambda=1}^{s} a_\lambda T^\lambda. \tag{4.16}$$

By taking inner products we obtain

$$\{\gamma^{*}_{(\omega)}(x, \rho(x, V)), c_k\} = \{V, c_k\} + u_k. \tag{4.17}$$

But

$$\{\gamma^{*}_{(\omega)}(x, \rho(x,V)), c_k\} = [\rho(x, V), \gamma(x, c_k)] = 0. \tag{4.18}$$

Hence

$$\gamma^{*}_{(\omega)}(x, \rho(x,V)) = V - \sum_{i=1}^{s} \{V, c_k\} \Gamma^k.$$

The proof of (b) is similar to that of (a).

<u>Theorem 4.17</u>. The following identities are true,

1. $G(x, \eta) = \rho(x, R^{-1}(x, \beta^{*}_{(\omega)}(x, \eta)))$.
2. $R(x, V) = \beta^{*}_{(\omega)}(x, g(x, \rho(x, V))) + \sum_{i=1}^{s} \{c_i, V\} c_i$.
3. If V is a solution of $\rho(x,V) = \xi$, then

$$V = \gamma^{*}_{(\omega)}(x, \xi) + \sum_{i=1}^{s} [c_i, V] \Gamma^i$$

4. If W is a solution of $\eta = \gamma(x,W)$, then

$$W = \beta^{*}_{(\omega)}(x, \eta) + \sum_{i=1}^{s} [\Gamma^i, W] c_i.$$

<u>Proof of 1</u>.

If $G(x, \eta)$ is defined by (1) we have

$$g(x, G(x, \eta)) = \gamma(x, R(x, \gamma^{*}_{(\omega)}(x, \rho(x, R^{-1}(x, \beta^{*}_{(\omega)}(x, \eta)))))). \tag{4.19}$$

By theorem 4.16 we have

$$\gamma^*_{(2)}(x, \rho(x, K^{-1}(x, \zeta^*_{(2)}(x,\eta)))) = K^{-1}(x, \zeta^*_{(2)}(x,\eta)) - \sum_{r=1}^{s}\{\tau_r, K^{-1}(x, \zeta^*_{(2)}(x,\eta))\}\,\Gamma^r \quad (4.20)$$

But

$$\{\tau_r, K^{-1}(x, \zeta^*_{(2)}(x,\eta))\} = \{\Gamma^r, \zeta^*_{(2)}(x,\eta)\} = \{\rho(x,\Gamma^r), \eta\} = 0.$$

Hence

$$g(x, G(x,\eta)) = \gamma(x, R(x, K^{-1}(x, \zeta^*_{(2)}(x,\eta)))) = \eta.$$

Similarly $G(x,g(x,\xi)) = \xi$, and this proves 1.

The proof of the other identities is similar to the proof of 1.

__Section 4.2.__ Determination of a n.h. linear connection.

In this section we shall take the linear forms $\gamma(x,W)$, $\rho(x,V)$ and $R(x,V)$ to be the same as in the previous section. Let $F_r(x,\xi)$ $(r = 1,\cdots,s)$ be the components of "s" arbitrary covariant vector fields in a given representation. Our problem will be to determine a n.h. linear connection in terms of these linear forms.

__Theorem 4.21.__ There exists a unique n.h. linear connection with components $K(x,V,\xi)$ such that at any point "x" of the coordinate domain of the given representation the following things are true.

1. $K(x,V,\xi)$ has an adjoint $K^*_{(2)}(x,W,\xi)$.
2. $R(x,V|\xi) = R(x,V;\xi) - R(x, K(x,V,\xi)) - K^*_{(2)}(x, R(x,V),\xi) = 0.$
3. $\gamma(x,W|\xi) = \gamma(x,W;\xi) + \gamma(x, K^*_{(2)}(x,W,\xi)) - \Gamma^*_{(2)}(x,\gamma(x,W),\xi) = \sum_{r=1}^{s}\{\Gamma^r, W\}\, F_r(x,\xi).$
4. $$\{\tau_r(x|\xi), \Gamma^k\} = \{\tau_r(x;\xi) - K^*_{(2)}(x,\tau_r,\xi), \Gamma^k\} = 0, \quad (r,k = 1,2,\cdots,s).$$

Moreover $K(x,V,\xi)$ is given by

$$K(x,V,\xi) = -\sum_{r=1}^{s}\{\tau_r, V\}\Gamma^r(x;\xi) - \sum_{r=1}^{s}\{\tau_r, V\}R^{-1}(x, \beta^*_{(\omega)}(x, F_r(x,\xi))) \tag{4.21}$$
$$+ \sum_{r=1}^{s}\{\beta^*_{(\omega)}(x, F_r(x,\xi)), V\}\Gamma^r + \gamma^*_{(\omega)}(x, \Gamma(x, \rho(x,V), \xi)) - \gamma^*_{(\omega)}(x, \rho(x,V); \xi).$$

It should be pointed out that $\Gamma(x, \xi_1, \xi_2)$ is the abstract Christoffel symbol[5] based on the metric form $g(x,\xi)$.

<u>Proof</u>. We must first prove that $K(x,V,\xi)$ as defined by (4.21) is the component of a n.h. linear connection. In any other representation we have

$$\bar{K}(\bar{x},\bar{V},\bar{\xi}) = -\sum_{r=1}^{s}\{\bar{\tau}_r, \bar{V}\}\bar{\Gamma}^r(\bar{x};\bar{\xi}) - \sum_{r=1}^{s}\{\bar{\tau}_r, \bar{V}\}\bar{R}^{-1}(\bar{x}, \bar{\beta}^*_{(\omega)}(\bar{x}, \bar{F}_r(\bar{x},\bar{\xi}))) \tag{4.22}$$
$$+ \sum_{r=1}^{s}\{\bar{\beta}^*_{(\omega)}(\bar{x}, \bar{F}_r(\bar{x},\bar{\xi})), \bar{V}\}\bar{\Gamma}^r + \bar{\gamma}^*_{(\omega)}(\bar{x}, \bar{\Gamma}(\bar{x}, \bar{\rho}(\bar{x},\bar{V}), \bar{\xi})) - \bar{\gamma}^*_{(\omega)}(\bar{x}, \bar{\rho}(\bar{x},\bar{V}); \bar{\xi}).$$

From the laws of transformation of the terms involved we can show the following things.

$$-\sum_{r=1}^{s}\{\bar{\tau}_r, \bar{V}\}\bar{\Gamma}^r(\bar{x};\bar{\xi}) = -\sum_{r=1}^{s}\{\tau_r, V\}M(x, \Gamma^r(x;\xi)) - \sum_{r=1}^{s}\{\tau_r, V\}M(x, T_r^*; \xi). \tag{4.23}$$

$$-\sum_{r=1}^{s}\{\bar{\tau}_r, \bar{V}\}\bar{R}^{-1}(\bar{x}, \bar{\beta}^*_{(\omega)}(\bar{x}, \bar{F}_r(\bar{x},\bar{\xi}))) = M(x, -\sum_{r=1}^{s}\{\tau_r, V\}R^{-1}(x, \beta^*_{(\omega)}(x, F_r(x,\xi)))). \tag{4.24}$$

$$\sum_{r=1}^{s}\{\bar{\beta}^*_{(\omega)}(\bar{x}, \bar{F}_r(\bar{x},\bar{\xi})), \bar{V}\}\bar{T}^r = M(x, \sum_{r=1}^{s}\{\beta^*_{(\omega)}(x, F_r(x,\xi)), V\}T^r). \tag{4.25}$$

$$\bar{\gamma}^*_{(\omega)}(\bar{x}, \bar{\Gamma}(\bar{x}, \bar{\rho}(\bar{x},\bar{V}), \bar{\xi})) = M(x, \gamma^*_{(\omega)}(x, \Gamma(x, \rho(x,V), \xi))) - \bar{\gamma}_{(\omega)}(\bar{x}, \bar{x}(x; \rho(x,V); \xi)). \tag{4.26}$$

$$\bar{\gamma}^*_{(\omega)}(\bar{x}, \bar{\rho}(\bar{x},\bar{V}); \bar{\xi}) = M(x, \gamma^*_{(\omega)}(x, \rho(x,V); \xi)) + M(x, \gamma^*_{(\omega)}(x, \rho(x,V)); \xi) \tag{4.27}$$
$$+ M(x, \gamma^*_{(\omega)}(x, \rho(x, V; \xi))) - \bar{\gamma}^*_{(\omega)}(\bar{x}, \bar{\rho}(\bar{x}, \bar{V}; \bar{\xi})) - \bar{\gamma}^*_{(\omega)}(\bar{x}, \bar{\rho}(\bar{x}, \bar{V}(\bar{x};\bar{\xi}))).$$

(5). See Michal [3].

By use of these and also of theorem 4.16 we can show

$$\bar{K}(\bar{x},\bar{V},\bar{\xi}) = M(x,K(x,V,\xi)) - M(x,V;\xi) + \bar{\gamma}^*_{(\omega)}(\bar{x},\bar{\rho}(\bar{x},\bar{V};\bar{\xi}))$$

$$+\bar{\gamma}^*_{(\omega)}(\bar{x},\bar{\rho}(\bar{x},\bar{V}(\bar{x};\bar{\xi}))) - \bar{\gamma}^*_{(\omega)}(\bar{x},\bar{x}(x;\rho(x,V);\xi)) - M(x,\gamma^*_{(\omega)}(x,\rho(x,V;\xi))). \tag{4.28}$$

Since

$$\bar{\rho}(\bar{x},\bar{V}) = \bar{x}(x;\rho(x,V)), \tag{4.29}$$

we have

$$\bar{\rho}(\bar{x},\bar{V};\bar{\xi}) + \bar{\rho}(\bar{x},\bar{V}(\bar{x};\bar{\xi})) - \bar{x}(x;\rho(x,V);\xi) = \bar{x}(x;\rho(x,V;\xi)). \tag{4.291}$$

Hence

$$\bar{\gamma}^*_{(\omega)}(\bar{x},\bar{\rho}(\bar{x},\bar{V};\bar{\xi})) + \bar{\gamma}^*_{(\omega)}(\bar{x},\bar{\rho}(\bar{x},\bar{V}(\bar{x};\bar{\xi}))) - \bar{\gamma}^*_{(\omega)}(\bar{x},\bar{x}(x;\rho(x,V);\xi)) = M(x,\gamma^*_{(\omega)}(x,\rho(x,V;\xi))). \tag{4.292}$$

Hence

$$\bar{K}(\bar{x},\bar{V},\bar{\xi}) = M(x,K(x,V,\xi)) - M(x,V;\xi),$$

and $K(x,V,\xi)$ is the component of a n.h. linear connection. We must further verify that $K(x,V,\xi)$ as defined by (4.21) has the properties (1) - (4) of this theorem. To prove (1) we merely note that

$$K^*_{(\omega)}(x,W,\xi) = -\sum_{\lambda=1}^{4}\{T^{\lambda}(x;\xi),W\}\tau_{\lambda} - \sum_{\lambda=1}^{4}\{R^{-1}(x,\beta^*_{(\omega)}(x,F_{\lambda}(x,\xi))),W\}\tau_{\lambda}$$

$$+\sum_{\lambda=1}^{4}\{T^{\lambda}_{;},W\}\beta^*_{(\omega)}(x,F_{\lambda}(x,\xi)) + \beta^*_{(\omega)}(x,\beta^*_{(\omega)}(x,\gamma(x,W),\xi)) - \beta^*_{(\omega)}(x,\gamma(x,W;\xi)). \tag{4.293}$$

I shall omit the proof of (2) as it is long but straightforward. To prove three we have by means of (4.293) that

$$\gamma(x, K_{(a)}^*(x, W, \xi)) = \sum_{\lambda=1}^{s} \{\Gamma^\lambda, W\} F_\lambda(x, \xi) + \beta_{(a)}^*(x, \gamma(x, W), \xi) - \gamma(x, W; \xi).$$

Hence

$$\gamma(x, W / \xi) = \sum_{\lambda=1}^{s} \{\Gamma^\lambda, W\} F_\lambda(x, \xi).$$

Since

$$K_{(a)}^*(x, \bar{\zeta}_\lambda, \xi) = -\sum_{k=1}^{s} \{T^k(x;\xi), \bar{\zeta}_\lambda\} \bar{\zeta}_k + \beta_{(a)}^*(x, F_\lambda(x,\xi)) - \beta_{(a)}^*(x, \gamma(x, \zeta_\lambda; \xi)), \quad (4.294)$$

we have

$$\{\bar{\zeta}_\lambda(x;\xi) - K_{(a)}^*(x, \bar{\zeta}_\lambda, \xi), T^k\} = \{\bar{\zeta}_\lambda(x;\xi), T^k\} + \{T^k(x;\xi), \bar{\zeta}_\lambda\} = 0. \quad (4.295)$$

Hence (4) has been shown.

Thus there does exist one linear connection which possesses properties (1) - (4). In order to show the linear connection is unique let us suppose that another linear connection $K_1(x, V, \xi)$ exists and has these properties. We shall have to make use of the following lemmas.

Lemma 1. (a) $\gamma(x, \bar{\zeta}_\lambda(x/\xi)) = -F_\lambda(x, \xi)$, $\lambda = 1, 2, \cdots s$,

(b) $\bar{\zeta}_\lambda(x/\xi) = -\beta_{(a)}^*(x, F_\lambda(x, \xi))$.

Proof of (a)

Since $\gamma(x, \zeta_\lambda(x)) = 0$, we have

$$\gamma(x, \zeta_\nu; \xi) + \gamma(x, \bar{\zeta}_\nu(x;\xi)) = 0.$$

From this we can show

$$\gamma(x, \tau_2/s) = -\gamma(x, \tau_2(x/s)).$$

By property (3) we have

$$\gamma(x, \tau_2(x/s)) = -F_2(x, s).$$

The proof of (b) follows directly from (a), theorem 4.16 and our fourth property.

Lemma 2. $K_1(x, \Gamma^2_j s) = -\Gamma^2(x; s) - R^{-1}(x, \mathcal{Q}^*_{(2)}(x, F_2(x, s)))$.

Since

$$\tau_2(x) = R(x, \Gamma^2(x)),$$

hence

$$\tau_2(x/s) = R(x, \Gamma^{-2}(x/s)) + R(x, \Gamma^2/s). \qquad (4.296)$$

Using our second property we have

$$\Gamma^2(x/s) = \Gamma^{-2}(x; s) + K_1(x, \Gamma^2_j s) = R^{-1}(x, \tau_2(x/s)).$$

Thus by means of lemma 1 we obtain

$$K_1(x, \Gamma^2_j s) = -\Gamma^2(x; s) - R^{-1}(x, \mathcal{Q}^*_{(2)}(x, F_2(x, s))),$$

which proves lemma 2.

By property three we have

$$\gamma(x, W_j s) + \gamma(x, K^*_{1(2)}(x, W_j s)) - \Gamma^{-1*}_{(2)}(x, \gamma(x, W_j s)) = \sum_{i=1}^{4} \{\Gamma^2_j W\} F_2(x, s).$$

Hence

$$\mathcal{Q}^*_{(2)}(x, \gamma(x, K^*_{1(2)}(x, W_j s))) = \sum_{i=1}^{4} \{\Gamma^2_j W\} \mathcal{Q}^*_{(2)}(x, F_2(x, s)) \qquad (4.297)$$

$$+ \mathcal{Q}^*_{(2)}(x, \Gamma^{-1*}_{(2)}(x, \gamma(x, W), s)) - \mathcal{Q}^*_{(2)}(x, \gamma(x, W_j s))).$$

By theorem 4.16 we obtain

$$K^*_{1(a)}(x,W,\xi) = \sum_{r=1}^{s}\{K^*_{1(a)}(x,W,\xi),T^r\}\tau_r + \sum_{r=1}^{s}\{T^r,W\}\rho^*_{(a)}(x,E_r(x,\xi))$$
$$+ \rho^*_{(a)}(x,\Gamma^*_{(a)}(x,\gamma(x,W),\xi)) - \rho^*_{(a)}(x,\gamma(x,W;\xi)). \tag{4.298}$$

Calculating the adjoint we obtain

$$K_1(x,V,\xi) = \sum_{r=1}^{s}\{\tau_r,V\}K_1(x,T^r,\xi) + \sum_{r=1}^{s}\{\rho^*_{(a)}(x,E_r(x,\xi)),V\}T^r$$
$$+ \gamma^*_{(a)}(x,\Gamma(x,\rho(x,V),\xi)) - \gamma^*_{(a)}(x,\rho(x,V);\xi).$$

By means of lemma two we can finally show

$$K_1(x,V,\xi) = -\sum_{r=1}^{s}\{\tau_r,V\}T(x;\xi) - \sum_{r=1}^{s}\{\tau_r,V\}R^{-1}(x,\rho^*_{(a)}(x,E_r(x,\xi)))$$
$$+\sum_{r=1}^{s}\{\rho^*_{(a)}(x,E_r(x,\xi)),V\}T^r + \gamma^*_{(a)}(x,\Gamma(x,\rho(x,V),\xi) -$$
$$-\gamma^*_{(a)}(x,\rho(x,V);\xi). \tag{4.299}$$

Comparing (4.299) with (4.21) we see $K_1(x,V,\xi) = K(x,V,\xi)$ and hence the linear connection is unique.

Theorem 4.22. The following identity holds.

$$\rho(x,V|\xi) = G(x,\sum_{r=1}^{s}\{T^r,R(x,V)\}E_r(x,\xi)).$$

Proof. By definition

$$\rho(x,V) = G(x,\gamma(x,R(x,V))).$$

Thus by theorem 2.22 we have

$$\rho(x,V|\xi) = G(x,\gamma(x,R(x,V))|\xi) + G(x,\gamma(x,R(x,V)|\xi))$$
$$+ G(x,\gamma(x,R(x,V|\xi))).$$

By Michal ~~Hyers~~ [3] $G(x, \eta / \xi) = 0$,[(6)] and by theorem 4.21 $R(x, V/\xi) = 0$. Hence

$$\rho(x, V/\xi) = G(x, \gamma(x, R(x,V))/\xi)). \tag{4.2991}$$

Using property three of theorem 4.21 we obtain

$$\rho(x, V/\xi) = G(x, \sum_{i=1}^{s} \{ \Gamma_j^i R(x,V) \} F_i(x, \xi)).$$

Section 4.3.

It would be of interest to calculate the curvature form based on the linear connection of theorem 4.21. In order to do this it is necessary to assume that $F_i(x, \xi_1; \xi_2)$, $i = 1, 2, \cdots s$, exist continuous in x.

Theorem 4.31. The curvature form $H(x, V, \xi_1, \xi_2)$ based on the linear connection of theorem 4.21 is given by

$$H(x, V, \xi_1, \xi_2) = \sum_{i=1}^{s} \{ V, \gamma_{(2)}^{*}(x, F_i(x, \xi_1/\xi_2) - F_i(x, \xi_2/\xi_1)) \} \Gamma^i$$
$$+ \sum_{i=1}^{s} \sum_{m=1}^{s} \{ [F_i(x, \xi_2), G(x, F_m(x, \xi_1))] - [F_i(x, \xi_1), G(x, F_m(x, \xi_2))] \} \{ \tau_i, V \} \Gamma^m \tag{4.31}$$
$$+ \sum_{j=1}^{s} \{ \tau_j(x/\xi_1), V \} \gamma_{(2)}^{*}(x, G(x, F_j(x, \xi_2))) - \sum_{j=1}^{s} \{ \tau_j(x/\xi_2), V \} \gamma_{(2)}^{*}(x, G(x, F_j(x, \xi_1)))$$
$$+ \sum_{j=1}^{s} \{ \tau_j, V \} \gamma_{(2)}^{*}(x, G(x, F_j(x, \xi_2/\xi_1) - F_j(x, \xi_1/\xi_2)))$$
$$+ \gamma_{(2)}^{*}(x, B(x, \eta(x, V), \xi_1, \xi_2)),$$

(6). This is true of course since we are using the abstract Christoffel symbol as our affine connection.

where $\beta(x, \xi_1, \xi_2, \xi_3)$ is the curvature form based on the Christoffel symbol $\Gamma(x, \xi_1, \xi_2)$.

To prove this theorem we need the following lemmas. Throughout we shall be using the linear connection of theorem 4.21.

Lemma 1.

$$R^{-1}(x, W \mid \xi) = 0.$$

The proof of this is a direct consequence of theorem 4.21.

Lemma 2.

$$H(x, \Gamma^{2}_{j}\, \xi_1, \xi_2) = R^{-1}(x, \rho^{*}_{\omega}(x, F_2(x, \xi_2 \mid \xi_1) - F_2(x, \xi_1 \mid \xi_2))) \tag{4.32}$$
$$+ \sum_{k=1}^{s}\left\{ [F_k(x, \xi_1), G(x, F_2(x, \xi_2))] - [F_k(x, \xi_2), G(x, F_2(x, \xi_1))]\right\} T^{k}.$$

Proof.

From the definition of the n.h. curvature form we have

$$H(x, \Gamma^{2}_{j}\, \xi_1, \xi_2) = \Gamma^{-2}(x \mid \xi_1 \mid \xi_2) - \Gamma^{-2}(x \mid \xi_2 \mid \xi_1). \tag{4.33}$$

By lemma 2 of theorem 4.21 we have

$$T^{\nu}(x \mid \xi_1) = -R^{-1}(x, \rho^{*}_{\omega}(x, F_2(x, \xi_1))). \tag{4.34}$$

Lemma 1 of this theorem tells us that

$$T^{\nu}(x \mid \xi_1 \mid \xi_2) = -R^{-1}(x, \rho^{*}_{\omega}(x, F_2(x, \xi_1 \mid \xi_2))) - R^{-1}(x, \rho^{*}_{\omega}(x, F_2(x, \xi_1) \mid \xi_2)). \tag{4.35}$$

By means of theorem (4.22) we have

$$\rho_{(2)}^{*}(x,\eta\,|\,\xi) = \sum_{i=1}^{s}\left[F_i(x,\xi),\, G(x,\eta)\right] c_i .$$

Hence (4.35) becomes

$$T^{i}(x\,|\,\xi_1\,|\,\xi_2) = -R^{-1}\left(x,\, \rho_{(2)}^{*}(x, F_i(x,\xi_1\,|\,\xi_2))\right) - \sum_{k=1}^{s}\left[F_k(x,\xi_2),\, G(x, F_i(x,\xi_1))\right] T^{k},$$

and therefore

$$H(x, T^{i}, \xi_1, \xi_2) = R^{-1}\left(x,\, \rho_{(2)}^{*}(x,\, F_i(x,\xi_2\,|\,\xi_1) - F_i(x,\xi_1\,|\,\xi_2))\right)$$
$$+ \sum_{k=1}^{s}\left\{\left[F_k(x,\xi_1),\, G(x, F_i(x,\xi_2))\right] - \left[F_k(x,\xi_2),\, G(x, F_i(x,\xi_1))\right]\right\} T^{k}$$

Lemma 3.

$$H_{(2)}^{*}(x, c_i, \xi_1, \xi_2) = -\rho_{(2)}^{*}\left(x,\, F_i(x,\xi_2\,|\,\xi_1) - F_i(x,\xi_1\,|\,\xi_2)\right) \tag{4.36}$$
$$- \sum_{k=1}^{s}\left\{\left[F_k(x,\xi_1),\, G(x, F_i(x,\xi_2))\right] - \left[F_k(x,\xi_2),\, G(x, F_i(x,\xi_1))\right]\right\} c_k .$$

By definition

$$H_{(2)}^{*}(x, c_i, \xi_1, \xi_2) = -\left\{\bar{\tau}_i(x\,|\,\xi_1\,|\,\xi_2) - \bar{\tau}_i(x\,|\,\xi_2\,|\,\xi_1)\right\}. \tag{4.37}$$

Since

$$\tau_i(x) = R(x, T^{i}(x)),$$

we have $$\tau_i(x\,|\,\xi_1\,|\,\xi_2) = R(x, T^{i}(x\,|\,\xi_1\,|\,\xi_2)). \tag{4.38}$$

Hence

$$H_{(2)}^{*}(x, \iota_i; \zeta_1, \zeta_2) = -R(x, H(x, \Gamma^i; \zeta_1, \zeta_2)). \tag{4.39}$$

By use of lemma 2 of this theorem we obtain the result (4.36).

<u>Proof of theorem 4.31</u>.

By theorem 2.21 we have

$$\rho(x, V|\zeta_1|\zeta_2) - \nu(x, V|\zeta_2|\zeta_1) = -\nu(x, H(x, V, \zeta_1, \zeta_2)) + B(x, \nu(x,V), \zeta_1, \zeta_2). \tag{4.391}$$

Thus

$$\gamma_{(2)}^{*}(x, \nu(x, H(x, V, \zeta_1, \zeta_2))) = \gamma_{(2)}^{*}(x, B(x, \nu(x,V), \zeta_1, \zeta_2)) - \gamma_{(2)}^{*}(x, \nu(x, V|\zeta_1|\zeta_2) - \nu(x, V|\zeta_2|\zeta_1)). \tag{4.392}$$

By theorem 4.16 we obtain

$$H(x, V, \zeta_1, \zeta_2) = \sum_{i=1}^{s} [\iota_i, H(x, V, \zeta_1, \zeta_2)] \Gamma^i + \gamma_{(2)}^{*}(x, B(x, \nu(x,V), \zeta_1, \zeta_2)) - \gamma_{(2)}^{*}(x, \nu(x, V|\zeta_1|\zeta_2) - \nu(x, V|\zeta_2|\zeta_1)). \tag{4.393}$$

By theorem 4.22 we have

$$\rho(x, V|\zeta_1) = G(x, \sum_{i=1}^{s} [\iota_i, V] F_i(x, \zeta)).$$

Hence

$$\rho(x, V|\zeta_1|\zeta_2) = G(x, \sum_{j=1}^{s} [\iota_j(x|\zeta_2), V] F_j(x, \zeta_1)) + G(x, \sum_{j=1}^{s} [\iota_j, V] F_j(x, \zeta_1|\zeta_2)). \tag{4.394}$$

Also

$$\{\tau_i, H(x, V, \zeta_1, \zeta_2)\} = \{H_{(2)}^{*}(x, \tau_i, \zeta_1, \zeta_2), V\}. \qquad (4.395)$$

By lemma (3), (4.394) and (4.395) we obtain that $H(x,V,\zeta_1,\zeta_2)$ is equal to the expression given in (4.31).

It might be pointed out that if $\gamma(x,W) = 0$ has only one linearly independent solution then the second term on the right hand side of (4.31) is zero.

CHAPTER V

In this chapter I should like to consider projective non-holonomic tensor theory. Most of this theory is new for finite dimensional spaces.

Section 5.1. Curl differential equations.

Let B be a Banach space, and let S denote the open set $\|x\| < K$. Suppose that $F(x, \xi_1, \xi_2)$ is a function on SB^2 to B satisfying the following conditions.

1. For each $x \in S$, $F(x, \xi_1, \xi_2)$ is a bilinear function of ξ_1 and ξ_2.
2. $F(x, \xi_1, \xi_2; \xi_3; \xi_4)$ exists continuous in x for each $x \in S$.

We shall be interested in differential equations of the form

$$\varphi(x, \xi_1; \xi_2) - \varphi(x, \xi_2; \xi_1) = F(x, \xi_1, \xi_2). \tag{5.11}$$

We seek solutions $\varphi(x, \xi)$ of (5.11) which are linear in ξ, and which will possess certain differentiability conditions.

Theorem 5.11. A necessary and sufficient condition for a solution $\varphi(x, \xi)$ of (5.11) to exist which is linear in ξ, and for which $\varphi(x, \xi_1; \xi_2; \xi_3)$ exists continuous in x for $x \in S$ is,

(a) $F(x, \xi_1, \xi_2) = -F(x, \xi_2, \xi_1)$,

(b) $F(x, \xi_1, \xi_2; \xi_3) + F(x, \xi_2, \xi_3; \xi_1) + F(x, \xi_3, \xi_1; \xi_2) = 0$.

The necessity of the conditions is obvious. To prove the sufficienty we show that

$$\varphi(x, \xi) = \int_0^1 \sigma F(\sigma x, \xi, x)\, d\sigma \tag{5.12}$$

is a solution of (5.11) which meets the required conditions.

Let x be any element of S, and let ξ_2 be any element of B. Then for any t such that $|t| < \frac{K-\|x\|}{\|\xi_2\|}$ and any σ such that $0 \leq \sigma \leq 1$, we have $\sigma(x+t\xi_2)$ is in S. By the hypotheses on F we have $\frac{d}{dt}F(\sigma(x+t\xi_2), \xi_1, x+t\xi_2)$ exists and is continuous in (σ, t) for all such σ and t. Thus by theorem 1.8 of Michal-Elconin [2] we must have

$$\varphi(x, \xi_1; \xi_2) = \int_0^1 \sigma \frac{d}{dt} F(\sigma(x+t\xi_2), \xi_1, x+t\xi_2)\Big|_{t=0} d\sigma. \qquad (5.13)$$

By a well known theorem on Fréchet differentials we obtain

$$\varphi(x, \xi_1; \xi_2) = \int_0^1 \sigma^2 F(\sigma x, \xi_1, x; \xi_2) d\sigma + \int_0^1 \sigma F(\sigma x, \xi_1, \xi_2) d\sigma. \qquad (5.14)$$

By the skew-symmetry of F we have

$$\varphi(x, \xi_1; \xi_2) - \varphi(x, \xi_2; \xi_1) = \int_0^1 \sigma^2 (F(\sigma x, \xi_1, x; \xi_2) + F(\sigma x, x, \xi_2; \xi_1)) d\sigma + \int_0^1 2\sigma F(\sigma x, \xi_1, \xi_2) d\sigma \qquad (5.15)$$

By integrating by parts we see

$$\int_0^1 2\sigma F(\sigma x, \xi_1, \xi_2) d\sigma = F(x, \xi_1, \xi_2) + \int_0^1 \sigma^2 F(\sigma x, \xi_2, \xi_1; x) d\sigma. \qquad (5.16)$$

By (5.15), (5.16) and condition (b) of this theorem we see

$$\varphi(x, \xi_1; \xi_2) - \varphi(x, \xi_2; \xi_1) = F(x, \xi_1, \xi_2). \qquad (5.17)$$

Hence $\varphi(x, \xi)$ as defined by (5.12) is a solution of (5.11). It is obvious from the hypotheses on F that this solution meets the differentiability conditions of the theorem.

Theorem 5.12. The most general solution $\varphi(x, \xi)$ of (5.11) which meets the conditions of theorem 5.11 is of the form

$$\varphi(x, \xi) = f(x; \xi) + \int_0^1 \sigma F(\sigma x, \xi, x)\, d\sigma, \tag{5.18}$$

where f(x) is any function possessing a continuous third Fréchet differential for each $x \in S$.

To prove this theorem we need the following lemma.

Lemma. The most general solution $\psi(x, \xi)$ of

$$\psi(x, \xi_1; \xi_2) - \psi(x, \xi_2; \xi_1) = 0 \tag{5.19}$$

which is linear in ξ, and which possesses a continuous second Fréchet differential for each x in S, is of the form

$$\psi(x, \xi) = f(x; \xi) \tag{5.191}$$

where f(x) is any function possessing a continuous third Fréchet differential for each $x \in S$.

Proof.

By the symmetry of the second Fréchet differential it is readily seen that $\psi(x, \xi)$ as given by (5.191) is a solution of (5.19) which meets the required conditions. Suppose now that $\psi_1(x, \xi)$ is any solution of (5.19) whihh meets these conditions. Let us then consider the differential equation

$$d_{\xi}^{x} f(x) = \psi_1(x, \xi)^{(1)}. \tag{5.192}$$

By theorem 3.2 of Michal-Elconin [2] we have that (5.192) always has a solution f(x), and that this solution will have a continuous third Fréchet differential for each $x \in S$. Thus every solution $\psi_1(x, \xi)$ of (5.19) which meets the differentiability condition must be of the form

$$\psi_1(x, \xi) = f(x; \xi).$$

Proof of theorem 5.12.

By theorem 5.11 we have

$$\varphi_1(x, \xi) = \int_0^1 \sigma F(\sigma x, \xi, x)\, d\sigma$$

is a solution of equation (5.11). If $\varphi(x, \xi)$ is any other solution of (5.11), then $\varphi(x, \xi) - \varphi_1(x, \xi)$ is a solution of (5.19). Thus by our lemma we must have

$$\varphi(x, \xi) - \varphi_1(x, \xi) = f(x; \xi). \tag{5.193}$$

Thus the most general solution of (5.11) which satisfies the differentiability conditions must be of the form

$$\varphi(x, \xi) = f(x; \xi) + \int_0^1 \sigma F(\sigma x, \xi, x)\, d\sigma,$$

where f(x) is any function possessing a continuous third Fréchet differential for each $x \in S$.

(1). $d_{\xi}^{x} f(x)$ is another notation for the Fréchet differential of f(x) with increment .

It is interesting to know under what conditions will equation (5.11) have a unique solution. Suitable boundary conditions are suggested by the following theorem.

Theorem 5.13. Let $\varphi(x,\xi)$ be any linear function of ξ on SB to B which possesses a continuous first Fréchet differential for each $x \in S$. There always exists a function f(x) on S to B such that

1. $f(x;\xi)$ exists continuous in x for each $x \in S$,
2. $f(x;x) = \varphi(x,x)$ for each x of S.

Proof.

Define

$$f(x) = \int_0^1 \varphi(\sigma x, x)\, d\sigma. \tag{5.194}$$

Then

$$f(x;\xi) = \int_0^1 \sigma\varphi(\sigma x, x;\xi)\, d\sigma + \int_0^1 \varphi(\sigma x, \xi)\, d\sigma.$$

Hence

$$f(x;x) = \int_0^1 \sigma\varphi(\sigma x, x;x)\, d\sigma + \int_0^1 \varphi(\sigma x, x)\, d\sigma. \tag{5.195}$$

Thus

$$f(x;x) = \int_0^1 \frac{d}{d\sigma}(\sigma\varphi(\sigma x, x))\, d\sigma = \varphi(x,x). \tag{5.196}$$

Property (1) is obvious from the hypotheses of the theorem.

Theorem 5.14. Let g(x) be any given function on S to B and for which $g(x;\xi_1;\xi_2;\xi_3)$ exists continuous in x for each $x \in S$, then there exists a unique solution $\varphi(x,\xi)$ of (5.11) which meets the differentiability conditions of theorem 5.11, and such that $\varphi(x,x) = g(x;x)$ for each $x \in S$.

Proof. Let

$$\varphi(x,\xi) = g(x;\xi) + \int_0^1 \sigma F(\sigma x, \xi, x)\, d\sigma \tag{5.197}$$

By theorem 5.12, $\varphi(x,\xi)$ is a solution of (5.11). Moreover by the skew-symmetry of $F(x, \xi_1, \xi_2)$ we have $\varphi(x,x) = g(x;x)$. To show $\varphi(x,\xi)$ as given by (5.197) is unique we assume there exists another solution $\varphi_1(x,\xi)$ of (5.11) which meets the conditions of theorem 5.14. By theorem 5.12 we have that $\varphi_1(x,\xi)$ must have the form

$$\varphi_1(x,\xi) = f(x;\xi) + \int_0^1 \sigma F(\sigma x, \xi, x)\, d\sigma \tag{5.198}$$

By assumption $\varphi_1(x,x) = g(x;x)$. Hence we must have

$$f(x;x) = g(x;x).$$

For any $0 < \sigma \leq 1$ we have

$$f(\sigma x; x) = g(\sigma x; x) \tag{5.199}$$

Thus

$$\frac{d}{d\sigma} f(\sigma x) = \frac{d}{d\sigma} g(\sigma x) \tag{5.1991}$$

Hence by integrating from 0 to 1 we obtain

$$f(x) = g(x) + C,$$

where C is an additive Banach constant. This of course implies $f(x;\xi) = g(x;\xi)$ and hence $\varphi_1(x,\xi) = \varphi(x,\xi)$.

Section 5.2. Normed ring with contraction.(2)

Suppose that B is a Banach space, and let us consider the totality R of linear functions $\ell(x)$ on B to B. The following notation is used. The linear function $\ell(x)$ when thought of as an element of R shall be denoted by L. If N(x) is a linear function on B to the real numbers, then $N(x)\alpha$ is a linear function on B to B for any fixed α of B. All linear functions of the form $N(x)\alpha$ will be denoted by $N(*)\alpha$ when thought of as elements of R. Professor Michal points out that R can be made into a Banach ring with a unit in the following way. The norm function of R is taken to be the modulus of the linear transformation, and the ring product L_1L_2 is defined by the iteration of the two linear transformations in B. With these definitions the unit I of the ring is the identity transformation.

Def. 5.21. A Banach space B will be called a space with a contraction ring R if there exists a linear function $[L]$ on R to the real numbers with the following properties.

1. $[L_1L_2] = [L_2L_1]$
2. $[N(*)\alpha] = N(\alpha)$, where N(x) is a linear function on B to the real numbers.

Section 5.3. Projective non-holonomic theory.

In this section we shall assume that our geometrical spaces

(2). This section is a short summary of Michal [7].

satisfy the assumptions made in Chapters one and two. In addition we assume that the Banach space E is a space with a contraction ring R and also that E possesses an inner product $\{V, W\}$

Def. 5.31. Let x = x(t) be the coordinate equation of a curve in the Banach space E, and suppose that V(x) is the component of a n.h.c.v.f. in a given representation. If $K(x,V,\xi)$ is the component of a n.h. linear connection in this representation, then the defining equations of parallelism along the given curve are

$$\frac{d}{dt}V + K(x, V, \frac{dx}{dt}) = \alpha(x)\, V \tag{5.31}$$

where $\alpha(x)$ is an arbitrarily chosen numerically valued scalar field.

Theorem 5.31. The defining equation of parallelism (5.31) can be written in the equivalent form

$$\{V, W\}\left(\frac{dV}{dt} + K(x, V, \frac{dx}{dt})\right) = \{W, \frac{dV}{dt} + K(x, V, \frac{dx}{dt})\}\, V, \tag{5.32}$$

where W is an arbitrary element of E .

Proof. Taking the inner of each side of (5.31) with W we obtain

$$\{W, \frac{dV}{dt} + K(x, V, \frac{dx}{dt})\} = \alpha(x)\{V, W\} \tag{5.33}$$

Eliminating $\alpha(x)$ between (5.31) and (5.33) we obtain (5.32).

Conversely if equation (5.32) holds in every representation then we take

$$\alpha(x) = \frac{\{W, \frac{dV}{dt} + K(x, V, \frac{dx}{dt})\}}{\{V, W\}},$$

and it is quite easy to show from the laws of transformation that $\alpha(x)$ is a numerically valued scalar field.

Theorem 5.32. Let Σ be the coordinate domain of any given representation, and suppose that the component of a $T(x,V,\xi)$ of a n.h.c.v.f. in ~~this~~ every representation is a bilinear function of the n.h.c.v. V and the contravariant vector ξ on $\Sigma E, E$ to E_1. If we think of $T(x,V,\xi)$ as an element of the contraction ring R we denote it by $T(x,*,\xi)$. Then $[T(x,*,\xi)]$ is a scalar valued numerical linear form in ξ.

Proof.

Since

$$\bar{T}(\bar{x}, \bar{V}, \bar{\xi}) = M(x, T(x, N(\bar{x}, \bar{V}), \xi)) \tag{5.34}$$

we have

$$\bar{T}(\bar{x}, *, \bar{\xi}) = M(x, *)\, T(x, *, \xi)\, N(\bar{x}, *) \tag{5.35}$$

By the properties of the contraction function we have

$$[\bar{T}(\bar{x}, *, \bar{\xi})] = [M(x, *)\, N(\bar{x}, *)\, T(x, *, \xi)] \tag{5.36}$$

Since $M(x,V)$ and $N(\bar{x},V)$ are inverse functions we must have $M(x,*)N(\bar{x},*) = I$, where I is the unit of the normed ring R. Thus (5.36) implies

$$[\bar{T}(\bar{x}, *, \bar{\xi})] = [T(x, *, \xi)], \tag{5.37}$$

and hence $T(x, *, \xi)$ is a scalar valued numerical form.
Since the function $T(x, *, \xi)$ is obviously additive in ξ, we have only to show that $T(x, *, \xi)$ is continuous in ξ in order to complete the proof of the theorem. Since $T(x, V, \xi)$ is bilinear in V and we have for each x of Σ'

$$\| T(x, V, \xi) \| \leq g(x) \| V \| \| \xi \| \tag{5.38}$$

Since $T(x, V, \xi)$ is a linear function of V for each $x \in \Sigma'$ and each $\xi \in E$, we have

$$\| T(x, V, \xi) \| \leq Q(x, \xi) \| \xi \| \tag{5.39}$$

where $Q(x, \xi)$ is the least number which will make (5.39) true. Thus (5.38) and (5.39) imply

$$Q(x, \xi) \leq g(x) \| \xi \| . \tag{5.391}$$

By definition

$$\| T(x, *, \xi) \| = Q(x, \xi) . \tag{5.392}$$

Thus

$$\| T(x, *, \xi) \| \leq g(x) \| \xi \| . \tag{5.393}$$

This implies $T(x, *, \xi)$ is linear in ξ, and hence $[T(x, *, \xi)]$ is also linear in ξ.

From the defining equations of parallelism it is obvious that this notion depends on the choice of our linear connection. It is of interest to know under what conditions we can change our linear connection and still have the parallelism of our n.h.c.v.f. invariant. A change of linear connections which preserves parallelism is called a projective change of non-holonomic linear connections.

Theorem 5.33. The most general projective change of linear connection is of the form

$$K'(x,V,\xi) = K(x,V,\xi) + \varphi(x,\xi)V$$

where $\varphi(x,\xi)$ is a scalar valued numerical linear form in ξ.

Proof. Let $K(x,V,\xi)$ and $K'(x,V,\xi)$ be two linear connections which have exactly the same parallel n.h.c.v.f. If we define

$$a(x,V,\xi) = K'(x,V,\xi) - K(x,V,\xi), \tag{5.394}$$

then it is easily shown that $a(x,V,\xi)$ is the component of a n.h.c.v.f. and also that it is bilinear in V and ξ. If V(x) is any parallel n.h.c.v.f. along a curve $x = x(t)$ we have

$$[V,W]\left\{\frac{dV}{dt} + K(x,V,\frac{dx}{dt})\right\} = [W,\frac{d}{dt}V + K(x,V,\frac{dx}{dt})]V, \tag{5.395}$$

and

$$[V,W]\left\{\frac{dV}{dt} + K'(x,V,\frac{dx}{dt})\right\} = [W,\frac{dV}{dt} + K'(x,V,\frac{dx}{dt})]W. \tag{5.396}$$

Subtracting these equations we obtain

$$\{V, W\}\, a(x, V, \tfrac{dx}{dt}) = \{a(x, V, \tfrac{dx}{dt}), W\}\, V. \tag{5.397}$$

Since (5.397) must hold for all curves, we have $\frac{dx}{dt}$ is arbitrary. We replace $\frac{dx}{dt}$ by ξ in (5.397). That is

$$\{V, W\}\, a(x, V, \xi) = \{a(x, V, \xi), W\} \tag{5.398}$$

must hold for all $V, W \in E_1$, $x \in \Sigma$ and $\xi \in E$. Placing $V = V_1 + V_2$ we obtain

$$\{V_1, W\}\, a(x, V_2, \xi) + \{V_2, W\}\, a(x, V_1, \xi) = \{a(x, V_1, \xi), W\}\, V_2 + \{a(x, V_2, \xi), W\}\, V_1. \tag{5.399}$$

Thinking of this as a linear function of V_2 we have in the contraction ring R

$$\{V_1, W\}\, a(x, *, \xi) + \{*, W\}\, a(x, V_1, \xi) = \{a(x, V_1, \xi)\}\, I + \{a(x, *, \xi), W\}\, V_1. \tag{5.3991}$$

By pefforming contraction to each side of (5.3991) we obtain

$$\{V_1, W\}\, [a(x, *, \xi)] = \{a(x, V, \xi), W\}\, [I] \tag{5.3992}$$

If we let $\alpha = [I]$, we have by the positive definiteness of the inner product that if $\alpha \neq 0$, then

$$a(x, V_1, \xi) = \frac{[a(x, *, \xi)]}{\alpha}\, V_1. \tag{5.3993}$$

Thus

$$K'(x, V, \xi) = K(x, V, \xi) + \varphi(x, \xi)\, V, \tag{5.3994}$$

where $\varphi(x,\xi) = \frac{[a(x,*,\xi)]}{\alpha}$. By theorem 5.32, $\varphi(x,\xi)$ will have the properties stated in theorem 5.33.

Section 5.4. Projective non-holonomic invariants.

Definition 5.41. A geometric object whose component remains invariant under a projective change of non-holonomic linear connections is called a projective non-holonomic invariant.

Theorem 5.41. Let $H(x,V,\xi_1,\xi_2)$ be the non-holonomic curvature form based on the non-holonomic linear connection $K(x,V,\xi)$. If

$\beta(x,\xi_1,\xi_2) = [H(x,*,\xi_1,\xi_2)]$, then

$$Q(x,V,\xi_1,\xi_2) = H(x,V,\xi_1,\xi_2) - \frac{\beta(x,\xi_1,\xi_2)}{\alpha}V$$

is the component of a projective non-holonomic invariant.

Proof.

By theorem 5.33 the most general projective change of non-holonomic linear connections is of the form

$$K'(x,V,\xi) = K(x,V,\xi) + \varphi(x,\xi)V.$$

Thus the non-holonomic curvature form $H'(x,V,\xi_1,\xi_2)$ based on $K'(x,V,\xi)$ is given by

$$H'(x,V,\xi_1,\xi_2) = H(x,V,\xi_1,\xi_2) + (\varphi(x,\xi_1;\xi_2) - \varphi(x,\xi_2;\xi_1))V. \tag{5.41}$$

Thus

$$\beta'(x,\xi_1,\xi_2) = \beta(x,\xi_1,\xi_2) + (\varphi(x,\xi_1;\xi_2) - \varphi(x,\xi_2;\xi_1))\alpha. \tag{5.42}$$

From (5.41) and (5.42) we obtain

$$Q'(x,V,\xi_1,\xi_2) = H'(x,V,\xi_1,\xi_2) - \underset{\alpha}{\underline{\beta'(x,\xi_1,\xi_2)}}V = Q(x,V,\xi_1,\xi_2).$$

Hence $Q(x,V,\xi_1,\xi_2)$ is the component of a projective non-holonomic invariant.

Theorem 5.42. If the function $K(x,*,\xi)$ has a continuous third Fréchet differential, then the following identities are true.

1. $[Q(x,*,\xi,\xi)] = 0.$
2. $\beta(x,\xi_1,\xi_2) = -\beta(x,\xi_2,\xi_1).$
3. $Q(x,V,\xi_1,\xi_2) = -Q(x,V,\xi_2,\xi_1).$
4. $\beta(x,\xi_1,\xi_2|\xi_3) = [H(x,*,\xi_1,\xi_2|\xi_3)]$.
5. $\beta(x,\xi_1,\xi_2;\xi_3) + \beta(x,\xi_2,\xi_3;\xi_1) + \beta(x,\xi_3,\xi_1;\xi_2) = 0$.

The proof of the first three of these identities follows directly from the definitions of the functions involved. To prove (4) we have

$$H(x,V,\xi_1,\xi_2/\xi_3) = H(x,V,\xi_1,\xi_2;\xi_3) - H(x,K(x,V,\xi_3),\xi_1,\xi_2)$$
$$+ K(x,H(x,V,\xi_1,\xi_2),\xi_3) - H(x,V,\Gamma(x,\xi_1,\xi_3),\xi_2) - H(x,V,\xi_1,\Gamma(x,\xi_2,\xi_3)) .$$

By the second property of the contraction function we obtain

$$[H(x,*,\xi_1,\xi_2/\xi_3)] = [H(x,*,\xi_1,\xi_2;\xi_3)] - [H(x,*,\Gamma(x,\xi_1,\xi_3),\xi_2)]$$
$$-[H(x,*,\xi_1,\Gamma(x,\xi_2,\xi_3))] .$$

This implies the fourth identity. The fifth identity follows directly from the Bianchi identity of the curvature form.

Def. 5.42. Our geometrical spaces will be called locally projectively flat if there exists a projective change of non-holonomic linear connections which yields a non-holonomic linear connection whose non-holonomic curvature form $H'(x,V,\xi_1,\xi_2)$ vanishes locally.(3)

<u>Theorem 5.43</u>. If the function $K(x,*,\xi)$ possesses a continuous third Fréchet differential, then a necessary and sufficient condition that our geometrical spaces be locally projectively flat is that $Q(x,V,\xi_1,\xi_2)$ vanish locally.

The condition is obviously necessary because $H'(x,V,\xi_1,\xi_2)$ vanishing locally implies the same for $\beta'(x,\xi_1,\xi_2)$. But $Q(x,V,\xi_1,\xi_2)$ is a projective non-holonomic invariant, and hence $Q(x,V,\xi_1,\xi_2)$ also vanishes locally. To prove the sufficiency we consider the differential equation

$$\varphi(x,\xi_1;\xi_2)-\varphi(x,\xi_2;\xi_1)=-\frac{1}{\alpha}\beta(x,\xi_1,\xi_2). \tag{5.43}$$

By theorem (5.42) $-\frac{\beta(x,\xi_1,\xi_2)}{\alpha}$ satisfies the hypotheses of theorem 5.11. Hence (5.43) will always have locally a solution $\varphi(x,\xi)$. If we make the projective change of non-holonomic linear connections

$$K'(x,V,\xi)=K(x,V,\xi)+\varphi(x,\xi)V \tag{5.44}$$

we have

$$H'(x,V,\xi_1,\xi_2)=H(x,V,\xi_1,\xi_2)+(\varphi(x,\xi_1;\xi_2)-\varphi(x,\xi_2;\xi_1))V \tag{5.45}$$

By (5.43) we obtain

(3). By locally we mean that for any point x, there exists a neighborhood for which $H'(x,V,\xi_1,\xi_2)=0$.

$$H'(x, V, \xi_1, \xi_2) = H(x, V, \xi_1, \xi_2) - \underset{\alpha}{\underline{B(x, \xi_1,}}\, \xi_2) V = Q(x, V, \xi_1, \xi_2) = 0. \tag{5.46}$$

Hence the condition is also sufficient.

Section 5.4. Ennuple theory.

Def. 5.41. Let the component B(x,V) of a Banach invariant linear form[4] be a linear function of the arbitrary n.h.c.v. V on $\sum$ E, to E, in any representation. If B(x,V) is a solvable linear function of V with inverse $A(x,\gamma)$, we shall call the pair of functions $B(x,V), A(x,\gamma)$ the components of an ennuple.

Theorem 5.41. If $B(x,V), A(x,\gamma)$ are the components of an ennuple, then $A(x,\gamma)$ is the component of a n.h.c.v.f. valued linear form in the Banach invariant γ.

Proof.

In any other representation we have that the components $\bar{B}(\bar{x},\bar{V}), \bar{A}(\bar{x},\bar{\gamma})$ are inverse function. This means

$$\bar{A}(\bar{x}, \bar{B}(\bar{x}, \bar{V})) = \bar{V} = M(x, V). \tag{5.41}$$

If we let

$$\bar{\gamma} = \bar{B}(\bar{x}, \bar{V}) = B(x, V) = \gamma \tag{5.42}$$

we have

$$\bar{A}(\bar{x}, \bar{\gamma}) = M(x, A(x, \gamma)). \tag{5.43}$$

(4). By Banach invariant we mean $\bar{B}(\bar{x},\bar{V}) = B(x,V)$ under any change of representation.

which proves the theorem.

Theorem 5.42. The component $F(x,V_1,\cdots V_n,\xi_1,\cdots\xi_m)$ of any n.h.c.v.f. valued linear form in the arbitrary n.h.c.v. $V_1,\cdots V_n$ and the arbitrary contravariant vectors $\xi_1,\cdots,\xi_m$ can be expressed in terms of the components of a Banach invariant and the components $B(x,V)$, $A(x,\gamma)$ of an ennuple.

Proof.

Define the function $R(x,V_1,\cdots V_n,\xi_1,\cdots,\xi_m)$ by

$$R(x,V_1,\cdots,V_n,\xi_1,\cdots,\xi_m)=B(x,F(x,V_1,\cdots,V_n,\xi_1,\cdots\xi_m)) \tag{5.44}$$

Clearly R is the component of a Banach invariant linear form in $V_1,\cdots V_n$, $\xi_1,\cdots,\xi_m$. Solving (5.44) we obtain

$$F(x,V_1,\cdots,V_n,\xi_1,\cdots,\xi_m)=A(x,R(x,V_1,\cdots,V_n,\xi_1,\cdots,\xi_m)), \tag{5.45}$$

which proves the theorem.

Theorem 5.43. Let $B(x,V)$, $A(x,\gamma)$ be the components of an ennuple and suppose that both of these functions possess continuous first Fréchet differentials. Then the components $K(x,V,\xi)$ of any non-holonomic linear connection can be expressed in terms of the components of the ennuple and the components $R(x,\gamma,\xi)$ of a Banach invariant valued linear form in the Banach invariant γ and the contravariant vector ξ. Conversely every such ~~example~~ ennuple and every such Banach invariant form determine a non-holonomic linear connection.

<u>Proof</u>. Let $K(x,V,\xi)$ be the component of any linear n.h. linear connection. Obviously

$$R(x,\gamma,\xi) = B(x, A(x,\gamma|\xi)) = B(x, A(x,\gamma;\xi) + K(x, A(x,\gamma),\xi)) \qquad (5.46)$$

is the component of a Banach invariant linear form in the Banach invariant γ, and the contravariant vector ξ. But (5.46) implies

$$K(x, A(x,\gamma),\xi) = -A(x,\gamma;\xi) + A(x, R(x,\gamma,\xi)). \qquad (5.47)$$

Letting $V = A(x,\gamma)$, we have $\gamma = B(x,V)$ and hence

$$K(x,V,\xi) = -A(x,B(x,V);\xi) + A(x, R(x,B(x,V),\xi)). \qquad (5.48)$$

This proves the first part of the theorem. To prove the second we merely have to show that $K(x,V,\xi)$ as defined by (5.48) is the component of a n.h. linear connection for any ennuple and any Banach invariant $R(x,\gamma,\xi)$. This can be shown from the transformation laws of the terms involved.

By the preceding theorem we see that if we choose $R(x,\gamma,\xi) = 0$, then $K(x,V,\xi) = -A(x,B(x,V);\xi)$ is the component of a non-holonomic linear connection. Using this non-holonomic linear connection we have

$$A(x,\gamma|\xi) = A(x,\gamma;\xi) - A(x, B(x,A(x,\gamma));\xi) = 0.$$

Since

$$H(x, A(x,\gamma),\xi_1,\xi_2) = A(x,\gamma|\xi_1|\xi_2) - A(x,\gamma|\xi_2|\xi_1) = 0,$$

we have by the solvability of $A(x,\gamma)$ that the non-holonomic curvature form based on this linear connection vanishes.

CHAPTER VI

In this chapter we shall give a few instances to which the theory developed in the preceeding section applies. Since many of the finite dimensional instances are well known I shall give only examples in which one or more of the spaces is infinite dimensional.

Example 1.

Let T be any arbitrary real Hilbert space whose elements we shall denote by $f, g, \cdots$, and let T_0 be the Hilbert space of one rowed infinite matrices $X = (X^1, X^2, \cdots\ )$, where X^α is a sequence of real numbers such that $\sum_{\alpha=1}^{\infty}(X^\alpha)^2$ converges. Stone [1] has shown that T and T_0 are equivalent.

We choose the Hausdorff space H to be the affine space A_n of n dimensions, and also take $E = A_n$. An element P of A_n will then be $P = (x^1, x^2, \cdots, x^n)$ where the x^i are real numbers. To each point P of A_n we associate the space T, and choose $E_1 = T_0$. We use the following notation.

1. Small greek letters $\alpha, \beta, \cdots$ shall have the range $1, 2, 3, \cdots \infty$.
2. Small latin letters $a, b, c, \cdots$ shall have the range $1, 2, \cdots n$.
3. The summation convention will apply in each case.
4. The inner product of T_0 is $[X, Y] = \sum_{\alpha=1}^{\infty} X^\alpha Y^\alpha$.
5. The inner product of A_n is $[x, y] = \sum_{a=1}^{n} x^a y^a$.
6. Capital letters $X, Y, Z, \cdots$ are elements of T_0.

By definition a vector coordinate system X(P,f) for the space T is a linear homeomorphic mapping of T on T_0. Such will of course exist because T is equivalent to T_0. A representation is simply a coordinate system x(P) for A_n and a family of vector coordinate systems X(P,f). We have seen that any two representations induce transformations of representation of the form

$$\bar{x}^i = \bar{x}^i(x^1, \cdots, x^n), \quad \bar{X} = M(P, X), \tag{6.1}$$

where M(P,X) is for each P a solvable linear transformation on T_0 to T_0. If $\bar{X} = (\bar{X}^1, \bar{X}^2, \cdots)$, and $X = (X^1, X^2, \cdots)$, then by Riesz [1] we must have

$$\bar{X}^\alpha = M^\alpha_\beta(x^1, \cdots, x^n) X^\beta \tag{6.2}$$

The totality of all representations is too general a class to support much of a differential geometry. For this reason we restrict ourselves to representations for which the following things are true.

1. The partial derivatives $\dfrac{\partial^r M^\alpha_\beta(x^1 \cdots x^n)}{\partial x^{a_1} \cdots \partial x^{a_r}}$ all exist continuous in $(x^1, \cdots, x^n)$ at each point of the coordinate domain, where $r = 1, 2, \cdots, m+3$.
2. The series $\sum_{\alpha=1}^{\infty} \sum_{a_1 \cdots a_r = 1}^{n} \left(\dfrac{\partial^r M^\alpha_\beta}{\partial x^{a_1} \cdots \partial x^{a_r}} \right)^2$ converge uniformly in $x = (x^1, \cdots, x^n)$ for x in any closed region of the coordinate domain and $r = 1, 2, \cdots m+3$.

(1) Since the transformation is solvable we also must have

$$m^2 \sum_{\alpha=1}^{\infty} (X^\alpha)^2 \le \sum_{\alpha=1}^{\infty} (M^\alpha_\beta X^\beta)^2 \le M^2 \sum_{\beta=1}^{\infty} (X^\beta)^2$$

3. $\bar{x}^{i} = x^{i}(x^1, \dots, x^n)$ is a regular transformation of class $C^{(m+2)}$ in the sense of Veblen and Whitehead.

With these restrictions it can easily be verified that the transformations of representation (6.1) are regular transformations of class $k^{(m+2)}$ (2).

Clearly in this case all elements of T will be non-holonomic contravariant vectors. If $\varphi(X^1, X^2, \dots)$ is the component of a scalar field in a given representation, and if $\frac{\partial \varphi}{\partial X^{\alpha}}$ exists such that $\sum_{\alpha=1}^{\infty}\left(\frac{\partial \varphi}{\partial X^{\alpha}}\right)^2$ converges, then $W = \left(\frac{\partial \varphi}{\partial X^1}, \frac{\partial \varphi}{\partial X^2}, \dots\right)$ is the component of a non-holonomic covariant vector field.

Let $\overset{(\alpha)}{W} = \left(\overset{(\alpha)}{W_1}(x^1, \dots, x^n), \overset{(\alpha)}{W_2}(x^1, \dots, x^n), \dots\right)$

be an infinite number of covariant vector fields for which;

4. $m_1^2 \sum_{\alpha=1}^{\infty}(X^{\alpha})^2 \leq \sum_{\alpha=1}^{\infty}\left(\sum_{\tau=1}^{\infty} \overset{(\alpha)}{W_\tau} X^{\tau}\right)^2 \leq M_1^2 \sum_{\tau=1}^{\infty}(X^{\tau})^2$ for all X of T_0;

5. $\dfrac{\partial^s \overset{(\alpha)}{W_\tau}(x^1, \dots, x^n)}{\partial x^{a_1} \dots \partial x^{a_s}}$ exist continuous in $x = (x^1, \dots, x^n)$ for s=1,2, m+1;

6. $\sum_{\alpha,\tau=1}^{\infty} \sum_{a_1,\dots,a_s=1}^{n}\left(\dfrac{\partial^s W^{\alpha}}{\partial x^{a_1} \dots \partial x^{a_s}}\right)^2$ converges uniformly in $x = (x^1, \dots, x^n)$

for s = 1,2, --- m+1, and x in any closed region of the coordinate domain.

By Riesz [1] there will exist quantities $A^{\tau}_{(\beta)}(x^1, x^2, \dots x^n)$

such that

$$A^{\tau}_{(\alpha)} \overset{(\alpha)}{W_\beta} = \overset{(\tau)}{W_\alpha} A^{\alpha}_{(\beta)} = \delta^{\tau}_{\beta}.$$

Then we can show that

$$K(x, V, \xi) = -\left(\frac{\partial A^1_{(\alpha)}}{\partial x^{\sigma}} \overset{(\alpha)}{W_\tau} \xi^{\sigma} V^{\tau}, \frac{\partial A^2_{(\alpha)}}{\partial x^{\sigma}} \overset{(\alpha)}{W_\tau} \xi^{\sigma} V^{\tau}, \dots\right)$$

(2). Clearly M^{α}_{β} constant will satisfy all of these conditions. This case would correspond to an infinite flat affine space in the sense of Veblen and Whitehead [1].

is the component of a non-holonomic linear connection which is of class $C^{(m)}$ locally uniformly in x. The theory of chapters I, II, III and V will apply to this example. It can be seen that it will not be too difficult to construct another example of this type where the Hausdorff space H is also taken to be a Hilbert space.

<u>Example 2</u>.

Let us suppose that all spaces involved are the same space namely the Banach space E. Moreover let us assume that E is a general Riemannian space in the sense of Michal [3], p. 55. By definition a coordinate system x(P) for E is a homeomorphic mapping of an open set Σ_1 of E onto an open set Σ_2 of E. Suppose we restrict our class of admissable coordinate systems to those for which $x(P;\delta P)$ and $P(x;\delta x)$ exist and are continuous in their arguments over their respective domains of definition.(3) With this restriction it is clear that $x(P), x(P;\delta P)$ is a representation. Moreover given two such representations $x(P), X = x(P;\delta P)$, and $\bar{x}(P), \bar{X} = \bar{x}(P;\delta P)$, then these will induce transformations of representation of the form

$$\bar{x} = \bar{x}(x), \quad \bar{X} = \bar{x}(x;X) \tag{6.3}$$

If we consider only representations of this type then our non-holonomic

(3) Clearly the totality of all linear homeomorphisms of E will be included in this class. Moreover if we assume that the identity transformation of E is an allowable K^{m} coordinate system we can show that all allowable $K^{(m)}$ coordinate systems satisfy this condition.

contravariant vectors are contravariant vectors, and similarly non-holonomic linear connections become affine linear connections. In this case most of the theory of Chapters I, II, III and V will be equivalent to theory developed by Professor Michal for such spaces. Chapter IV however will be new theory for such spaces. Moreover for spaces of this type the hypotheses made in that chapter can always be satisfied. We can see this in the following way.

We now use ξ and η to denote non-holonomic contravariant vectors and non-holonomic covariant vectors respectively. $R(x, \xi)$ of Chapter IV is chosen to be the metric form $g(x, \xi)$ of the Riemannian differential geometry, and $\gamma(x, \eta)$ is chosen to be η. In this case all of the postulates of Chapter IV are satisfied and the linear connection determined in this chapter reduces to the abstract Christoffel symbol of the second kind.

Professor Michal has given several infinite dimensional examples of spaces of this type.(4) It might be pointed out that even though all the spaces are taken to be the same the theory developed here is not equivalent to that of Professor Michal unless we restrict our representation in a manner similar to what we did in this example.

Example 3.

We take E to be the Banach space of continuous functions on a closed interval (a,b) as given in Michal-Hyers [3] pp. 329-332. The

(4) See Michal [4], and Michal-Hyers [3].

associated spaces and the Banach space E_1 are chosen to be the Banach space of continuous functions $X(\mu)$ defined on a closed interval (c,d).[5] We use the following notation.

1. Latin letters $m, n, \cdots$ shall be variables ranging over (a,b).
2. Greek letters $\mu, \nu, \cdots$ shall be variables ranging over (c,d).
3. An element $x(m)$ of E is written by the Michal convention as x^m or x_m, and similarly $X(\mu)$ of E is written X^μ or X_μ.
4. A repetition of an index once as a superscript and once as a subscript shall mean Riemann integration over the corresponding interval.
5. The inner product $\{X,Y\}$ of E_1 is $X^\alpha Y_\alpha$.

We shall restrict our representations to induce transformations of representation of the form

$$\bar{x}^m = \bar{x}^m(x^n), \quad \bar{X}^\alpha = M^\alpha(x^n)X^\alpha + M^\alpha_\beta(x^n)X^\beta = M(x,X), \tag{6.4}$$

where the following things are true.

1. $\bar{x}^m(x^n)$ satisfies the conditions of Michal-Hyers [3], p. 330. Thus $\delta\bar{x}^m = \bar{K}^m(x^n)\delta x^m + \bar{K}^m_n(x^n)\delta x^n$.
2. The rth Fréchet differentials of the coefficient $M^\alpha(x^m), M^\alpha_\beta(x^n)$ exist uniformly in their parameters, are continuous functions of x^τ uniformly in their parameters, and are of Volterra type where $r = 1, 2, \cdots (m+2)$.

(5). $\|X(\mu)\|$ is taken to be max. $|X(\mu)|$. Other operations are defined in the usual way.

3. The Fredholm determinant $D\left(\frac{M^{\alpha}_{\beta}(x^m)}{M^{\alpha}(x^m)}\right) \neq 0$, and $M^{\alpha}(x^m) \neq 0$. Thus M(x,X) will be solvable linear in X.

Since $M^{*}_{(2)}(x,W) = M^{(\alpha)}W_{\alpha} + M^{\beta}_{\alpha}W_{\beta}$ [(1)], and $D\left(\frac{M^{\beta}_{\alpha}}{M^{\alpha}}\right) = D\left(\frac{M^{\alpha}_{\beta}}{M^{\alpha}}\right) \neq 0$, it follows that $M^{*}_{(2)}(x,W)$ is a solvable linear function of W. This will imply that the transformations of representation (6.4) are regular transformations of representation of class $k^{(m+2)}$.

We choose the affine linear connection $\Gamma(x, \xi, \xi)$ to be that given in Michal-Hyers [3] p. 331 and assume the non-holonomic linear connection to have the form

$$K(x,V,\xi) = R^{\alpha}_{\beta m}(x^n)V^{\beta}\xi^m + R^{\alpha}_m(x^n)\xi^m V, \tag{6.5}$$

where $R^{\alpha}_{\beta m}(x^n)$, $R^{\alpha}_m(x^n)$ have Fréchet differentials of order "r", which exist uniformly in their parameters, are of Volterra type and are continuous functionals of x^m uniformly in their parameters, $(r = 1,2, \cdots, m+1)$.

With these restrictions it can be shown that the form of (6.5) is invariant to changes of representation, and that the theory of chapters I, II, III and V will imply to these infinite dimensional function spaces.

Example 4.

Let us consider the spaces of example 3 and suppose that we take the intervals (a,b) and (c,d) to be the same. Moreover we shall choose $\bar{K}^m(x^n)$ and $M^{\alpha}(x^n)$ to be both equal to one. $\gamma(x,w)$ of Chapter 4 is taken to be of the form

(1). The parenthesis in $M^{(\alpha)}W_{\alpha}$ means the ~~numeration~~ summation convention does not apply.

$$\gamma(x,W) = W_\alpha + \varphi_\alpha^\beta(x^n)\bar{W}_\beta \tag{6.6}$$

where $\varphi_\alpha^\beta(x^n)$ has Fréchet differentials of order $(\bar{p} + 1)$ which exist uniformly in their parameters and are of Volterra type. In order for $\gamma(x,\mathbf{W})$ to be a covariant vector field $\varphi_\alpha^\beta(x^m)$ must have the law of transformation

$$\bar{\varphi}_\alpha^\beta = \varphi_\alpha^\beta + k_\alpha^\tau \varphi_\tau^\beta + M_\tau^\beta \varphi_\alpha^\tau + M_\rho^\beta k_\alpha^\tau \varphi_\tau^\rho + k_\alpha^\beta + M_\alpha^\beta + M_\tau^\beta k_\alpha^\tau \tag{6.7}$$

where k_α^τ is the resolvent kernel of $\bar{K}_\beta^\alpha\ (x^n)$. It might be noticed that if we choose $M_\beta^\alpha = \bar{K}_\beta^\alpha$, then $k_\alpha^\beta + M_\alpha^\beta + M_\tau^\beta k_\alpha^\tau = 0$, and (6.7) would say that $\varphi_\alpha^\beta W_\beta \xi^\alpha$ is an invariant to changes of representation.

We can see that

$$\gamma_{(2)}^*(x,\xi) = \xi^\alpha + \varphi_\beta^\alpha \xi^\beta. \tag{6.8}$$

We choose $R(x,\mathbf{V})$ to have the form

$$R(x,V) = V^\mu + R_{\mu\nu} V^\nu, \quad R_{\mu\nu} = R_{\nu\mu} \tag{6.9}$$

and shall assume $\int_a^b (V^\mu)^2 du + R_{\mu\nu} V^\mu V^\nu \geq 0$. This will of course imply that the Fredholm determinant $D(R_{\mu\nu}) \neq 0$ and hence $R(x,V)$ will be a solvable linear function of V. The coefficient functional $R_{\mu\nu}(x^n)$ will be assumed to have Fréchet differentials of order (p + 1) each of which exist uniformly in the parameters, are continuous functionals of x^m uniformly in the parameters, and are of Volterra type.

We can calculate $\gamma(x, R(x, \gamma_{(a)}^*(x, \xi)))$ to be of the form

$$\gamma(x, R(x, \gamma_{(a)}^*(x, \xi))) = \xi^\mu + g_{\mu\beta}\, \xi^\beta, \tag{6.91}$$

where

$$g_{\mu\beta} = \varphi_\beta^\mu + \varphi_\mu^\beta + R_{\mu\beta} + \varphi_\mu^\rho R_{\rho\beta} + R_{\mu\nu}\varphi_\beta^\nu + R_{\rho\nu}\varphi_\mu^\rho \varphi_\beta^\nu + \int_a^b \varphi_\mu^\rho \varphi_\beta^\rho \, d\rho,$$

and shall assume $\Delta(g_{\mu\beta}) \neq 0$.

It is of course well known that the integral equation $W_\alpha + \varphi_\alpha^\beta W_\beta = 0$ can have only a finite number of linearly independent solutions for W_α, and these can be made orthonormal by the usual orthogonalisation process.

In this case all the hypotheses of chapter IV will be satisfied and the theory of the whole thesis will apply to this example.

Other instances of this type can be given, and ~~also~~ instances in which E_1 is a finite dimensional space also exist.

Bibliography

S. Banach.

[1]. Théorie des Opérations Linéaires, 1932.

H. F. Bohnenblust.

[1]. Real Variables, Princeton Lectures.

E. Bortolotti and V. Hlavaty.

[1]. Contributi alla teoria delle connessioni, Annali Di Matematica, Ser. 4, vol. 15 (1939), pp. 127-154.

L. P. Eisenhart.

[1]. Riemannian Geometry, 1926.

[2] Non-Riemannian Geometry, 1928.

T. H. Hildebrandt and L. M. Graves.

[1]. Implicit functions and their differentials in general analysis, Transactions of American Mathematical Society, vol. 29 (1927), pp. 127-153.

F. Hausdorff .

[1]. Mengenlehre, 1927.

R. Konig.

[1]. Beitrage zu einer allgemeinen linearen Mannigfaltigkeitslehre, Jahresberichte der Deutschen Mathematiker-Vereinigung, vol. 28 (1918), pp. 213-228.

[2]. Funktionen und Zahlentheoretische Analogien, (same as preceeding paper) pp. 208-213.

A. D. Michal.

[1]. General tensor analysis, Bulletin of American Mathematical Society, vol. 43 (1937), pp. 394-401.

[2] Abstract covariant vector fields in a general absolute calculus, American Journal of Mathematics, vol. 59 (1937), pp. 306-314.

[3] General differential geometries, Bulletin of American Mathematical Society, (1939), pp. 529-563.

[4] Affinely connected function space manifolds, American Journal of Mathematics, vol. 50 (1928), pp. 473-517.

[5] Projective functional tensors and other allied functionals, Proceedings of the National Academy of Sciences, vol. 17 (1931), pp. 217-225.

[6] Postulates for a linear connection, Annali Di Matematica (1936), pp. 197-220.

[7] Projective differential geometry, Proceedings of the National Academy of Sciences, vol. 23 (1937), pp. 546-548.

[8] Differential calculus in linear topological spaces, Proceedings of the National Academy of Sciences, vol. 24 (1938), pp. 340-342.

[9] Riemannian differential geometries in abstract spaces, Proceedings of the National Academy of Sciences, vol. 15 (1935), pp. 526-529.

A. D. Michal and V. Elconin.

[1] Differential properties of abstract transformation groups with abstract parameters, American Journal of Mathematics, vol. 59 (1937), pp. 129-143.

[2] Completely integrable differential equations in abstract spaces, Acta Mathematica, vol. 68 (1937), pp. 71-107.

A. D. Michal and J. L. Botsford.

[1] Geometries involving affine connections and general linear connections, Annali di Matematica, Serie IV, Tomo XII (1933-34), pp. 13-32.

A. D. Michal and D. H. Hyers.

[1] Theory and applications of abstract normal coordinates in a general differential geometry, Annali della R. Scuola Normale Superiore di Pisa, vol. 7 (1938), pp. 157-176.

[2]. Second order differential equations with two point boundary conditions in general analysis, American Journal of Mathematics, vol. 58 (1936), pp. 646-660.

[3]. Differential invariants in a general differential geometry, Mathematische Annalen, Sept. 3 (1939), pp. 310-333.

A. D. Michal and A. B. Mewborn .

[1] Abstract flat projective geometry, Proceedings of the National Academy of Sciences, vol. 25 (1939), pp. 440-443.

A. D. Michal and T. S. Peterson .

[1]. The invariant theory of functional forms under the group of linear functional transformations of the third kind, Annals of Mathematics, vol. 32 (1931), pp. 431-450.

R. S. Martin.

[1]. California Institute of Technology Thesis, 1932.

F. Riesz .

[1]. Les Systemes D'Équations Linéaires, 1913.

J. A. Schouten and D. J. Struick .

[1]. Einfuhrung in die neuern methoden der Differentialgeometrie, volumes I, II, 1935.

M. H. Stone.

[1]. Linear Transformations in Hilbert Spaces and their applications to analysis.

O. Veblen .

[1] Invariants of Quadratic Differential Forms, 1933.

O. Veblen and J. H. C. Whitehead .

[1]. The Foundations of Differential Geometry, 1932.

J. H. C. Whitehead .

[1]. On linear connections, Transactions of American Mathematical Society, vol. 33 (1931), pp. 191-209.