Forced Generation of Solitary Waves in a Rotating Fluid and Their Stability

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Wooyoung Choi

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Abstract

The primary objective of this graduate research is to study forced generation of solitary waves in a rotating fluid and their stability properties. For axisymmetric flow of a non-uniformly rotating fluid within a long cylindrical tube, an analysis is presented to predict the periodic generation of upstream-advancing vortex solitons by axisymmetric disturbance steadily moving with a transcritical velocity as a forcing agent. The phenomenon is simulated using the forced Korteweg-de Vries (fKdV) equation to model the amplitude function of the Stokes stream function for describing this family of rotating flows of an inviscid and incompressible fluid. The numerical results for the weakly nonlinear and weakly dispersive wave motion show that a sequence of well-defined axisymmetrical recirculating eddies is periodically produced and emitted to radiate upstream of the disturbance, soon becoming permanent in the form as a procession of vortex solitons, which we call vortons. Two primary flows, the Rankine vortex and the Burgers vortex, are adopted to exhibit in detail the process of producing the upstream vortons by the critical motion of a slender body moving along the central axis, with the Burgers vortex being found the more effective of the two in the generation of vortons. To investigate the evolution of free or forced waves within a tube of non-uniform radius, a new forced KdV equation is derived which models the variable geometry with variable coefficients. A set of section-mean conservation laws is derived specially for this class of rotational tube flows of an inviscid and incompressible fluid, in both differntial and integral forms. A new aspect of stability theory is analyzed for possible instabilities of the axisymmetric solitary waves subject to *non-axisymmetric* disturbances. The present linear analysis based on the model equation involving the bending mode shows that the axisymmetric solitary wave is neutrally stable with respect to small bending mode disturbances. To study nonlinear interactions between the axisymmetric mode and bending mode, a new model is derived which consists of two coupled equations for disturbances of the two modes. The numerical results of the coupled equations show that the primary axisymmetric soliton appears to maintain its own entity, with some oscillations of its amplitude and an undular tail, inferring an interchange of energy between the two modes, when subject to small non-axisymmetric perturbations.

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Chapter 1

Introduction

The striking phenomena of weakly nonlinear and weakly dispersive waves generated by resonant forcing have recently attracted much attention. One of their remarkable characteristics is that a steadily moving transcritical disturbance can produce, continuously and periodically, a succesion of solitons advancing upstream of the moving disturbance, and this phenomenon is deemed possible to occur in all soliton-bearing systems under resonant excitation. In particular, when a pressure distribution on the free surface or a submerged topography is moving with a transcritical speed in a water layer of constant depth, the upstream solitons have been identified first numerically (Wu & Wu, 1982) and then validated experimentally (Lee, 1985; Lee *et al.*, 1989). Some theoretical model equations for simulating this phenomenon have been proposed and, among these, the forced Korteweg-de Vries (fKdV) equation has been adopted as an effective and convenient model by Akylas (1984), Lee (1985), Cole (1985), Wu (1987) and Lee *et al.* (1989). In connection with the exploration of the basic mechanism underlying this phenomenon, the hydrodynamic instability of several forced steady solitons of the fKdV family has been investigated by Camassa (1990), Camassa & Wu (1991) and Yates & Wu (1992).

In this thesis, we consider the analogous phenomenon arising in axisymmetric flow of a non-uniformly rotating fluid confined within a long cylindrical tube resonantly excited by a body centered at the axis and/or by a constriction of the tube wall which is moving along the axis of rotation with a critical velocity. The primary objective is to determine the general criteria in terms of the domain of pertinent parameters in which the phenomenon of periodic production of upstream advancing solitary waves can manifest under external forcing sustained at resonance. In a more searching quest, we make an attempt to explore the basic mechanism underlying the phenomenon in question and to examine the effect of periodic production of vortex solitons on the variation of vorticity distribution and its transport.

Determination of flow induced by bodies moving steadily and axially in uniformly rotating fluids is one of the classical problems in fluid mechanics and such flows have many interesting features. When the inertial forces are negligible compared with the Coriolis force, the resulting flow is two-dimensional and it is well-known that an axial column of fluid is pushed ahead of a body moving parallel to the axis of rotation (Taylor, 1922). However, in the case with a body moving in a non-uniformly rotating fluid, the problem becomes more complicated. Near the critical state at which the axial velocity of body is very nearly the speed of the longest wave, the nonlinear and dispersive effects jointly play an important role. The interplay of these two important effects with the net linear dynamics can give rise to the remarkable phenomenon of succesive production of vortex solitons in a process which would be impossible on linear theory.

It has been known that free solitary waves can occur in non-uniformly rotating fluids as enunciated by Benjamin (1967) and Leibovich (1970). They showed that a stationary circulating eddy permanent in form can appear on the axis of rotation when the ratio of the axial velocity to the swirl velocity reaches a certain critical value such that the long wave speed vanishes (reaching the critical condition) and this was interpreted by them as a mild axisymmetric vortex breakdown. Experimentally, Pritchard (1970) made an observation of such waves of finite amplitude propagating, permanent in form, inside a long cylindrical tube. In his experiment, the waves of inward displacement in a vortex core was generated by moving an annular body. He further attempted to generate the waves of outward displacement by plunging a rigid body into the fluid but was not able to generate any waves of permanent form.

In chapter 2, we present the basic equations for describing three-dimensional motions of an incompressible and inviscid fluid within a long tube which may be gradually divergent or convergent. For this class of flows, a set of the conservation laws are derived for sectional mean flow in both differential and integral forms.

From this system of the basic equations, we derive in §3.1 the forced Korteweg-de Vries (fKdV) equation to simulate weakly nonlinear and weakly dispersive long waves propagating through a rotating fluid, with possibly the presence of an axisymmetric slender body and/or an axisymmetrical topographical slender deformation of an otherwise uniform tube wall which moves with a transcritical velocity along the axis of rotation as forcing agencies.

Also considered are tubes with non-uniform but stationary wall with the objective to investigate the deformation of free or forced nonlinear waves within a gradually divergent or convergent tube. For this case, we derive in §3.1 the fKdV equation with variable coefficients, which is analogous with the equation first derived by Kakutani (1971) and Johnson (1972) for free waves in an open channel of gradually varying depth.

In §3.2, we adopt two primary flows: namely, the Rankine vortex and the Burgers vortex, to evaluate the rotating flow perturbed by a critically moving slender body and/or an imposed tube wall deformation. Using the slender body as a forcing agency, we determined the existence of a parameter domain in which eddies with closed stream surfaces, which we call the *vortex solitons* or *vortons*, are produced sequentially to move upstream along the axis of rotation, as simulated by the numerical results of the fKdV equation. Each of these vortons is permanent in form and behaves like a free solitary wave solution of the KdV equation after having advanced a small distance ahead of the forcing region, thus attaining their own entity after being produced in turn, and forming a procession of ever-growing length. Of the two primary flows, the Burgers vortex is found through our numerical experiments to be the more effective in generating forward moving vortons. During the preparation of a paper on the subject in chapter 3, we became aware of the articles by Grimshaw (1990) and Hanazaki (1991) who had previously made the observation on the phenomenon in question. The present work may be regarded as a complementary and further development in the following new aspects. The Rankine vortex employed as a new primary flow as presented in §3.3.1 has enabled us to attain considerable amount of analytical results in closed form that may be of value to provide a standard reference for assessing results from numerical approaches. The results exhibiting the processes of generating upstream-advancing vortons illustrated with streamlines are new, giving a vivid flow visualization of this remarkable phenomenon. In addition, the local and global conservation laws presented in chapter 2, which are exact for incompressible and inviscid fluids, should be of value for practical applications of the result and for assessing errors of numerical results. The resulting wave resistance that must be overcome by forcing agency furnishes a very sensitive measure of periodicity and rate of growth for unstable modes that may be discernible only on very slow time scales.

In chapter 4, we examine the stability of axisymmetric free solitary wave governed by the Kortweg-de Vries (KdV) equation and subject to non-axisymmetric disturbances. Although in real flows, asymmetric disturbances are inevitable, they have not received as much attention compared with axisymmetric ones. For axisymmetric perturbations only, the stability of the axisymmetric solitary wave has been well understood from the works of Benjamin (1972) for free waves and from Camassa and Wu (1991) for the specific forced waves based on the KdV and the fKdV equations, respectively. However, for non-axisymmetric perturbations, the stability property of these primary waves has not been examined.

For non-axisymmetric perturbations, the difficulty of the stability analysis for solitary wave lies in that the KdV equation derived on the assumption of axisymmetric flow is no longer the appropriate governing equation. Moreover, since the primary flow is a superposition of the basic swirling flow and the axisymmetric soliton, it belongs to the family of non-parallel flows to which the classical stability theory cannot be directly applied. Under some assumptions appropriate for the problem without losing any essential physical elements, we try to get an idea to better understand the full three-dimensional flow features.

By assuming small perturbations with respect to the axisymmetric solitary waves, a model equation for non-axisymmetric disturbances is derived in §4.2. Particular attention has been paid to the disturbances with the non-zero radial velocity component on the axis, which may cause a considerable change in the flow field. On linear theory, the model equation becomes the Schrodinger equation with a complex potential and the resulting eigenvalue problem is solved by applying both a perturbation method and the numerical method described in §4.3. Also a nonlinear interaction mechanism between the axisymmetric modes and the bending modes is suggested and the coupled two model equations are solved numerically to exhibit their interactions between the two modes in §4.4.

Chapter 2

Basic Equations and Conservation Laws

In this chapter, the basic equations for motions of an inviscid and incompressible fluid will be reviewed and the conservation laws for general three-dimensional flows are considered in both differential and integral forms.

2.1 Basic equations

For an inviscid and incompressible flow, the velocity vector $\mathbf{u}$ and the pressure p satisfy the Euler equation and the continuity equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p, \qquad \nabla \cdot \mathbf{u} = 0, \tag{2.1}$$

where ρ is the fluid density. For the cylindrical coordinate system (x, r, θ), (2.1) can be written as

$$\frac{\partial u}{\partial t} + \mathbf{u} \cdot \nabla u = -\frac{1}{\rho} \frac{\partial p}{\partial x}, \tag{2.2a}$$

$$\frac{\partial v}{\partial t} + \mathbf{u} \cdot \nabla v - \frac{w^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r}, \tag{2.2b}$$

$$\frac{\partial w}{\partial t} + \mathbf{u} \cdot \nabla w + \frac{vw}{r} = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta}, \tag{2.2c}$$

$$\frac{\partial u}{\partial x}+\frac{1}{r}\frac{\partial (rv)}{\partial r}+\frac{1}{r}\frac{\partial w}{\partial \theta}=0, \qquad (2.2d)$$

where

$$\mathbf{u}\cdot\nabla = u\frac{\partial}{\partial x}+v\frac{\partial}{\partial r}+\frac{w}{r}\frac{\partial}{\partial \theta}. \qquad (2.2e)$$

The vorticity $\boldsymbol{\Omega}$ defined by

$$\boldsymbol{\Omega}=\nabla\wedge\mathbf{u}, \qquad (2.3)$$

is governed by the following vorticity equation obtained by taking the curl of (2.1)

$$\frac{\partial \boldsymbol{\Omega}}{\partial \mathbf{t}}+\mathbf{u}\cdot\nabla\boldsymbol{\Omega}=\boldsymbol{\Omega}\cdot\nabla\mathbf{u}, \qquad (2.4)$$

and has the components in the cylindrical coordinate systems given by

$$\Omega_x=\frac{1}{r}\frac{\partial (rw)}{\partial r}-\frac{1}{r}\frac{\partial v}{\partial \theta},\quad \Omega_r=\frac{1}{r}\frac{\partial u}{\partial \theta}-\frac{\partial w}{\partial x},\quad \Omega_\theta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial r}. \qquad (2.5)$$

For axisymmetric flows ($\partial(\cdot)/\partial\theta=0$), the governing equations can be expressed in terms of two dependent variables: the Stokes stream function $\psi(x,r,t)$ such that

$$u=\frac{1}{r}\frac{\partial\psi}{\partial r},\qquad v=-\frac{1}{r}\frac{\partial\psi}{\partial x}, \qquad (2.6)$$

and the circulation about the x-axis, $\Gamma(x,r,t)$, or the angular momentum density

$$\Gamma(x,r,t)=rw(x,r,t), \qquad (2.7)$$

where w is the azimuthal velocity. From (2.4) and (2.5), the azimuthal vorticity Ω_θ is related to ψ by

$$r\Omega_\theta=-D^2\psi=-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}\right)\psi, \qquad (2.8)$$

and satisfies the vorticity equation (2.4) as

$$\frac{\partial (D^2\psi)}{\partial t}-\frac{1}{r}\frac{\partial(\psi,D^2\psi)}{\partial(x,r)}+\frac{2}{r^2}(D^2\psi)\frac{\partial\psi}{\partial x}+\frac{2\Gamma}{r^2}\frac{\partial\Gamma}{\partial x}=0, \qquad (2.9a)$$

where

$$\frac{\partial(\psi,\phi)}{\partial(x,r)} = \frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial\phi}{\partial x}. \tag{2.9b}$$

The θ-momentum equation (2.2c) can be shown to state the conservation of material circulation

$$\frac{D\Gamma}{Dt} = 0, \tag{2.10}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial r}. \tag{2.11}$$

For axisymmetric flows, (2.9) and (2.10) constitute of the governing equtions for ψ and Γ. In chapter 3 where we consider axisymmetric flows, (2.9) and (2.10) will be used, and in chapter 4, where we study some non-axisymmetric flows, the original set of equations (2.2) will be used.

2.2 Conservation laws for sectional mean flow

We are primarily interested in analyzing rotating flows contained within a long duct of radius $r = r_1(x,\theta,t)$ past an elongated body whose surface is at $r = r_0(x,\theta,t)$, the flow being not necessarily axisymmetric. We shall use non-dimensional coordinates based on a length scale R and time scale R/W_m, where R and W_m characterize the tube radius and the basic swirl velocity, and so the velocity scale is W_m and the pressure scale is ρW_m^2. With the continuity equation and the Euler equation for the velocity components (u,v,w) and the pressure p, so non-dimensionalized, we shall first explore some conservation laws for flow quantities averaged over a cross-section of the duct for this class of rotating flows of an inviscd and incompressible fluid. For a flow quantity $f(x, r, \theta, t)$, we define $< f >$, the *section-mean* value of f, as

$$< f >\equiv \int_0^{2\pi} \int_{r_0(x,\theta,t)}^{r_1(x,\theta,t)} r f(x, r, \theta, t)\ dr\ d\theta. \tag{2.12}$$

By using the Leibniz rule for the derivative of a definite integral and invoking the kinematical boundary conditions at $r = r_0$ and $r = r_1$, one can readily show that the material derivative of f satisfies the equation

$$\left\langle \frac{df}{dt} \right\rangle = \frac{\partial}{\partial t} < f > + \frac{\partial}{\partial x} < uf >, \tag{2.13a}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial r} + \frac{w}{r}\frac{\partial}{\partial \theta}, \tag{2.13b}$$

upon further using the continuity equation. Taking $f = 1, u, \Gamma$ and, the kinetic energy density, $e \equiv \frac{1}{2}(u^2 + v^2 + w^2)$ in turn, using the continuity equation and the Euler equation with (2.13) readily yield the following four equations in a non-dimensionalized form

$$\frac{\partial}{\partial t}(S) + \frac{\partial}{\partial x} < u >= 0, \tag{2.14}$$

$$\frac{\partial}{\partial t} < u > + \frac{\partial}{\partial x} < (u^2 + p) >= \int_0^{2\pi} \left(r_1 r_{1x} p_{r_1} - r_0 r_{0x} p_{r_0} \right) d\theta, \tag{2.15}$$

$$\frac{\partial}{\partial t} < \Gamma > + \frac{\partial}{\partial x} < u\Gamma >= \int_0^{2\pi} \left(r_1 r_{1\theta} p_{r_1} - r_0 r_{0\theta} p_{r_0} \right) d\theta, \tag{2.16}$$

$$\frac{\partial}{\partial t} < e > + \frac{\partial}{\partial x} < u(e + p) >= - \int_0^{2\pi} \left(r_1 r_{1t} p_{r_1} - r_0 r_{0t} p_{r_0} \right) d\theta, \tag{2.17}$$

where $S(x,t) = \frac{1}{2}\int_0^{2\pi}\left(r_1^2(x,\theta,t) - r_0^2(x,\theta,t)\right)\,d\theta$ is the cross-sectional area and p_{r_0} and p_{r_1} denote the pressure p at $r = r_0$ and at $r = r_1$, respectively. These four equations represent the section-mean conservation of mass, x-momentum, angular momentum and kinetic energy, respectively. They are exact for inviscid, incompressible fluids; and their physical significances are as quite evident as expected on the first principles of mechanics.

From these differential expressions for local conservation laws, a set of integral conservation laws can be derived under the assumptions that all physical variables except those for the primary flow vanish fast enough as $|x| \to \infty$ (for the following longitudinal integrals to exist) and by using the following Reynolds transport theorem

$$\frac{d}{dt}\left\{\int_{-\infty}^{\infty} dx\left(\int_0^{2\pi}\int_{r_0}^{r_1} rf\ drd\theta\right)\right\} = \int_{-\infty}^{\infty} dx\left(\int_0^{2\pi}\int_{r_0}^{r_1} r\frac{df}{dt}\ drd\theta\right)$$

$$= \int_{-\infty}^{\infty} dx\left(\frac{\partial}{\partial t} < f > + \frac{\partial}{\partial x} < uf >\right). \quad (2.18)$$

By integrating both sides of the differential form of conservation equations (2.14)-(2.17) with respect to x and using (2.18), we can derive the conservation laws in an integral form. However, in the following, only the integal conservation laws for axisymmetric flows will be examined. Then (2.14) implies the conservation of mass and we have, from (2.15), the conservation of x-momentum M

$$\frac{dM}{dt} = \frac{d}{dt}\left\{\int_{-\infty}^{\infty} dx\left(\int_{r_0}^{r_1} ru\ dr\right)\right\} = \int_{-\infty}^{\infty}\left(r_1 r_{1x} p_{r_1} - r_0 r_{0x} p_{r_0}\right)dx. \quad (2.19)$$

From (2.16), the conservation of angular momentum J can be obtained as

$$\frac{dJ}{dt} = \frac{d}{dt}\left\{\int_{-\infty}^{\infty} dx\left(\int_{r_0}^{r_1} r\Gamma\ dr\right)\right\} = 0. \quad (2.20)$$

And from (2.17), the energy conservation is given by the relation

$$\frac{dE}{dt} = \frac{d}{dt}\left\{\int_{-\infty}^{\infty} dx \left(\int_{r_0}^{r_1} re\ dr\right)\right\} = -\int_{-\infty}^{\infty} \left(r_1 r_{1t} p_{r_1} - r_0 r_{0t} p_{r_0}\right) dx. \qquad (2.21)$$

Here, clearly, (2.19) asserts that the total longitudinal momentum M is increased at a rate equal to the longitudinal integral of the x-component of pressure force acting on the flow boundary. By (2.20), the total angular momentum of axisymmetric rotating flow is conserved. Equation (2.21) shows that dE/dt is equal to the rate of working by the surface pressure.

When an axisymetrical body is moving through a tube, it experiences a resistance due to the generation and radiation of waves. The wave resistance D_w non-dimensionalized by $\pi\rho W_m^2 R^2$ is given by

$$D_w = -\int_{-\infty}^{\infty} (r_1 r_{1x} p_{r_1} - r_0 r_{0x} p_{r_0}) dx. \qquad (2.22)$$

Up to this point, all the above expressions in this chapter are exact. In the following chapters, some approriate approximations will be introduced to facilitate analysis of various physical problems of interest.

Chapter 3

Forced Generation of Vortex Solitons

In this chapter, we derive the governing equations for weakly nonlinear and weakly dispersive waves in a rotating fluid propagating along a long cylindrical tube under resonant forcing. The family of the forced KdV equation is derived for different cases of basic flows and tube boundaries. Two primary flows, the Rankine vortex and the Burgers vortex, are specifically considered for the applications.

3.1 A model for weakly nonlinear long waves in a rotating fluid

By introducing the variable $y = r^2$ for convenience, the axial and radial velocity components (u, v) are given by

$$u = 2\psi_y, \qquad rv = -\psi_x, \tag{3.1}$$

and the circulation about the x-axis, $\Gamma(x, y, t)$, or the angular momentum density, is

$$\Gamma(x, y, t) = rw(x, y, t), \tag{3.2}$$

where w is the azimuthal velocity. With the fluid assumed inviscid, the stream function ψ and the circulation Γ satisfy equations (2.9) and (2.10), or

$$D^2\psi_t + 2\psi_y D^2\psi_x - 2y\psi_x \frac{\partial}{\partial y}(\frac{D^2\psi}{y}) + \frac{2\Gamma\Gamma_x}{y} = 0, \tag{3.3}$$

$$\Gamma_t - 2\psi_x\Gamma_y + 2\psi_y\Gamma_x = 0, \tag{3.4}$$

where

$$D^2 = \frac{\partial^2}{\partial x^2} + 4y\frac{\partial^2}{\partial y^2}, \tag{3.5}$$

subject to the kinematic boundary conditions at the axis and at the wall as will be specified.

For the primary flow field, we assume that the axial and azimuthal velocity components may both be arbitrarily sheared in the radial direction

$$\mathbf{U}(y) = (U_0(y), 0, W_0(y)), \tag{3.6}$$

where $U_0(y)$ and $W_0(y)$ are arbitrary functions of y which represent a primary cylindrical flow of the inviscid fluid undisturbed by waves, assumed to be sufficiently smooth and stable to axisymmetric disturbances, i.e., they satisfy the linear stablity criterion of Howard & Gupta (1962), $\Gamma\Gamma_y \geq y^2\psi_{yy}^2$ in the present notation. In this connection, we note that the occurrence of the remarkable phenomenon of resonantly forced rotating flows actually requires W_0 to be sheared because the nonlinear effects disappear, so does the resonant forcing, when W_0 is constant.

Concerning the boundary conditions, we have three cases of interest. The first is when the medium is uniform, i.e., when the tube radius is constant throughout and the fluid is free of any other boundaries, in which case we speak of all the wave motions being free. The second is the case of forced motion produced by a slender body, or a tube wall constriction as a local departure from the constant tube radius, or an annular body or a combination of them moving along the tube axis as forcing agencies, especially when they move with a transcritical speed. The third is when the free or forced waves are propagating through a tube of stationary non-uniform wall (see figure 3.1).

Before we proceed with consideration of forced wave motions, we first recapitulate the derivation of a long-wave model, of the KdV family for nonlinear and dispersive waves in rotating fluids.

3.1.1 Free solitary waves

Weakly nonlinear long waves have been extensively studied and the procedure to obtain appropriate evolution equations is well known (e.g., Whitham, 1974; Dodd *et al.*, 1982). If we follow the procedure adopted by Leibovich (1970) for the present problem, we may use the tube radius R to scale radial distances, some typical wave length λ for axial distances, the maximum swirl speed W_m for velocities and λ/W_m for time. The long waves will then be characterized by two important parameters:

$$\epsilon = R^2/\lambda^2, \qquad \alpha = a/R, \tag{3.7}$$

where a is a typical wave amplitude and ϵ is by definition small for long waves. We are interested in weakly nonlinear and weakly dispersive motions, in particular the Boussinesq family with $\alpha = O(\epsilon)$.

It is however somewhat simpler to adopt R for scaling all distances and introduce the following dimensionless variables (with primes)

$$x = Rx', \quad y = R^2 y', \quad t = \frac{R}{W_m} t', \quad \psi = W_m R^2 \psi', \quad \Gamma = W_m R \Gamma', \tag{3.8}$$

so that, after dropping the primes, (3.3) and (3.4) remain unaltered in form. To describe evolution of unidirectional (left-going) long waves of the Boussinesq family, we adopt the following stretched coordinates in the wave frame moving with velocity c_0 in the negative x-direction (after Gardner & Morikawa, 1969)

$$\xi = \epsilon^{\frac{1}{2}}(x + c_0 t), \qquad \tau = \epsilon^{\frac{3}{2}} t, \tag{3.9}$$

where the phase velocity c_0 of infinitesimal long waves depends on the primary flow velocity field and will have to be determined. Here, the factor $\epsilon^{\frac{1}{2}}$ signifies the smallness of the ratio R/λ and the factor $\epsilon^{\frac{3}{2}}$ arises in accordance with the condition of $\alpha = O(\epsilon)$.

We further assume for the stream function ψ and the circulation Γ the expansion:

$$\psi(\xi, \tau; y) = \psi_0(y) + \epsilon \psi_1(\xi, \tau; y) + \epsilon^2 \psi_2(\xi, \tau; y) + O(\epsilon^3), \tag{3.10}$$

$$\Gamma(\xi, \tau; y) = \Gamma_0(y) + \epsilon \Gamma_1(\xi, \tau; y) + \epsilon^2 \Gamma_2(\xi, \tau; y) + O(\epsilon^3), \tag{3.11}$$

where

$$\psi_0 = \frac{1}{2} \int_0^y U_0(y) dy, \qquad \Gamma_0 = r W_0(y). \tag{3.12}$$

Substituting (3.9)-(3.11) into (3.3) and (3.4) yields the first-order equations in the form

$$\psi_1(\xi, \tau; y) = \phi_1(y) A_1(\xi, \tau), \qquad \Gamma_1 = \gamma_1(y) A_1(\xi, \tau), \tag{3.13a}$$

$$\gamma_1 = \frac{2\Gamma_0{}'}{U_0 + c_0} \phi_1, \tag{3.13b}$$

$$L\phi_1 = 0, \tag{3.13c}$$

where

$$L = \frac{d^2}{dy^2} + q(y), \qquad q(y) = \frac{1}{(U_0 + c_0)}\left[\frac{\Gamma_0\Gamma_0{}'}{(U_0 + c_0)y^2} - U_0{}''\right], \tag{3.13d, e}$$

and the prime means differentiation with respect to y. The boundary conditions, based on the assumed regularity condition for ψ and Γ at infinity, can be prescribed as $\psi(\xi, \tau; 0) = 0$ and $\psi(\xi, \tau; 1) = \psi_0(1)$, whence, accordingly,

$$\phi_1(0) = 0, \qquad \phi_1(1) = 0, \tag{3.14}$$

the corresponding kinematic conditions being $v(\xi, \tau; 0) = 0$ and $v(\xi, \tau; 1) = 0$. The system of homogeneous equations (3.13) and (3.14) constitute an eigenvalue problem whose solution determines the eigenvalue c_0. As (3.13) invovles a variable coefficient $q(y)$, we might encounter singular $q(y)$ with $U_0 + c_0 = 0$ in the interval $(0, 1)$. However, Chandrasekhar (1961, §78b) has shown that there exist at least two eigenvalues, say c_{0m} and c_{0M}, such that

$$-c_{0m} < \min_{y \epsilon (0,1)} U_0(y), \qquad -c_{0M} > \max_{y \epsilon (0,1)} U_0(y), \tag{3.15}$$

provided the flow is stable. Thus the eigenvalue problem is a regular one (in the Sturm-Liouville sense) for which some practical procedures of solution can be found in Leibovich (1970) and Chandrasekhar (1961).

The second-order terms in (3.3) and (3.4) yield the equation for ψ_2

$$L\psi_{2\xi} = f_1(y)\phi_1 A_{1\tau} + f_2(y)\phi_1^2 A_1 A_{1\xi} + f_3(y)\phi_1 A_{1\xi\xi\xi}, \tag{3.16a}$$

where

$$f_1(y) = \frac{1}{U_0 + c_0}\left(2q + \frac{U_0''}{U_0 + c_0}\right), \tag{3.16b}$$

$$f_2(y) = -\frac{2}{U_0 + c_0}\left[q' + \frac{{\Gamma_0'}^2}{(U_0+c_0)^2 y^2} + \frac{\Gamma_0}{y^2(U_0+c_0)}\left(\frac{\Gamma_0'}{U_0+c_0}\right)'\right], \qquad (3.16c)$$

$$f_3(y) = -\frac{1}{4y}. \qquad (3.16d)$$

The solvability condition for (3.16) for ψ_2 can be obtained by taking the inner product of (3.16a) with ϕ_1, the solution of (3.13c), where the inner product of any two functions $f(y)$ and $g(y)$, both being real on y for $0 \leq y \leq 1$, is defined as

$$(f, g) = \int_0^1 f(y)g(y)dy. \qquad (3.17)$$

This operation readily gives for the wave amplitude function $A_1(\xi, \tau)$ the equation

$$A_{1\tau} + c_1 A_1 A_{1\xi} + c_2 A_{1\xi\xi\xi} = 0, \qquad (3.18a)$$

$$c_1 = \frac{(f_2, {\phi_1}^3)}{(f_1, {\phi_1}^2)}, \qquad c_2 = \frac{(f_3, {\phi_1}^2)}{(f_1, {\phi_1}^2)}, \qquad (3.18b)$$

since the inner product between ϕ_1 and the left-hand side of (3.16a) gives the result

$$(\phi_1, L\psi_{2\xi}) = \int_0^1 \phi_1(\partial_y^2 + q)\psi_{2\xi}dy = (\phi_1\psi_{2\xi y} - \phi_{1y}\psi_{2\xi})\Big|_0^1 + (L\phi_1, \psi_{2\xi}) = 0 \quad (3.19)$$

on the account of (3.13c), the operator L being self-adjoint, and both ϕ_1 and ψ_2 satisfying (3.14). Thus we have the KdV equation which was first derived by Leibovich (1970) for long waves in rotating fluids.

Finally, absorbing the order factor by setting $A = \epsilon A_1 + O(\epsilon^2)$ and restoring the original coordinates (x, t) used for the specific primary flow, the KdV equation has for the left-going waves the form

$$A_t - c_0 A_x + c_1 A A_x + c_2 A_{xxx} = 0. \qquad (3.20)$$

This equation has the classical solitary wave solution

$$A(x,t) = a \cdot \mathrm{sech}^2 \left[\frac{1}{2}\sqrt{\frac{ac_1}{3c_2}}(x + (c_0 - \frac{ac_1}{3})t) \right], \tag{3.21}$$

which is a one-parameter family in a as a constant wave amplitude, and also the cnoidal wave solution as waves of permanent form. These solutions were found by Benjamin (1967) and Leibovich (1970). In particular, when a wave amplitude reaches the critical value at which the stagnation point appears at the axis of rotation, the streamsurfaces of the free solitary wave solution have the recirculating pattern of axisymmetrical vortex eddies enclosed by a stream surface.

3.1.2 Forced nonlinear waves in rotating fluids

Here we introduce a slender body aligned at the tube axis, or a localized small perturbation of the tube wall radius, or a thin annular body or a combination of them as weak forcing disturbances, moving parallel to the tube axis. The governing equations, (3.3) and (3.4), remain unchanged. The only difference from the free wave case is in the kinematic boundary conditions at these new surfaces. Disregarding any annular body for the moment, the boundary conditions at the tube axis and tube wall due to the imposed forcing disturbances are

$$(U + 2\psi_y)b_{0x} + 2\psi_x = 0 \qquad \text{at} \quad y = r_0^2 = b_0(x + Ut), \tag{3.22a}$$

$$-(U + 2\psi_y)b_{1x} + 2\psi_x = 0 \qquad \text{at} \quad y - 1 = r_1^2 - 1 = -b_1(x + Ut). \tag{3.22b}$$

Further we assume that the left-going disturbances $y = r_0^2 = b_0(x + Ut)$ and $y - 1 = r_1^2 - 1 = -b_1(x + Ut)$ are of $O(\epsilon^2)$ and the transcritical velocity U of the forcing disturbances can be detuned from the criticality by a margin of $O(\epsilon)$,

$$b_0(x+Ut) = \epsilon^2 \tilde{b}_0(x+Ut), \qquad b_1(x+Ut) = \epsilon^2 \tilde{b}_1(x+Ut), \tag{3.23}$$

$$U = c_0(1+\epsilon\delta+O(\epsilon^2)), \tag{3.24}$$

where δ is $O(1)$. The scale of weak forcing is dictated by the solvability of the amplitude function as first found (Wu, 1987) for the channel flow problem and the same as will be seen below to hold for the present case. The range permitted for the detuning will also become clear later. For simplicity of analysis, we shall adopt the stretched coordinates (3.9) and the asymptotic expansion (3.10) and (3.11). Substituting (3.10), (3.23) and (3.24) in (3.22a,b) and expanding the functions involved into their Taylor series about $y=0$ and $y=1$ respectively, we find that the boundary conditions are unaltered in the first order and in the second order yield

$$\psi_2(\xi,\tau;0) = -\frac{1}{2}(U_0(0)+c_0)\tilde{b}_0(\xi,\tau) \qquad \text{at} \quad y=0, \tag{3.25a}$$

$$\psi_2(\xi,\tau;1) = \frac{1}{2}(U_0(1)+c_0)\tilde{b}_1(\xi,\tau) \qquad \text{at} \quad y=1, \tag{3.25b}$$

where (ξ,τ) are defined by (3.9), and b_0 and b_1 are taken positive (or negative) when they reduce (or enlarge) the cross-sectional area of the tube. We note that the forcing bodies have radial dispacement at the tube axis $r_0 = b_0^{\frac{1}{2}} = O(\epsilon)$ and at the tube wall $r_1 - 1 = O(b_1) = O(\epsilon^2)$, both being of the assigned orders so as to induce boundary perturbations in the second-order term ψ_2. This implies that the blockage ratio defined as the ratio of the cross-sectional area of disturbance to that of tube are of the same order for both forcing bodies.

The first-order problem for ψ_1 therefore remains intact while the second-order expansion again yields equation (3.16),

$$L\psi_{2\xi} = f_1(y)\phi_1 A_{1\tau} + f_2(y)\phi_1^2 A_1 A_{1\xi} + f_3(y)\phi_1 A_{1\xi\xi\xi}, \tag{3.16a}$$

where f_1,f_2 and f_3 are given in (3.16b,c,d). To satisfy the solvability condition for ψ_2, we take inner product between the above equation and ϕ_1 in which operation the left-hand side of (3.16a) under conditions (3.14) and (3.25) gives

$$
\begin{aligned}
(\phi_1, L\psi_{2\xi}) &= \int_0^1 \phi_1(\partial_y^2 + q)\psi_{2\xi}dy = (\phi_1\psi_{2\xi y} - \phi_{1y}\psi_{2\xi})\Big|_0^1 + (L\phi_1, \psi_{2\xi}) = -\phi_{1y}\psi_{2\xi}|_0^1 \\
&= -\frac{1}{2}(U_0(0) + c_0)\phi_{1y}(0)\tilde{b}_{0\xi} - \frac{1}{2}(U_0(1) + c_0)\phi_{1y}(1)\tilde{b}_{1\xi} \\
&\equiv (f_1, \phi_1^2)\tilde{F}_\xi, \quad \text{say}, \qquad (3.26)
\end{aligned}
$$

so that, when combined with the contribution from the right-hand side of (3.16a), we obtain the forced KdV equation as

$$A_{1\tau} + c_1 A_1 A_{1\xi} + c_2 A_{1\xi\xi\xi} = \tilde{F}_\xi \qquad (3.27)$$

where c_1 and c_2 are given by (3.18b) and $\tilde{F}$ by (3.26). After recovering the orginal physical variables x and t, we have for the amplitude function $A = \epsilon A_1 + O(\epsilon^2)$ and the forcing function $F = \epsilon^2 \tilde{F}$ the equation

$$A_t - c_0 A_x + c_1 A A_x + c_2 A_{xxx} = F_x(x + Ut). \qquad (3.28)$$

Finally, we may express the equation with respect to the body frame defined by $X = x + Ut$, $T = t$ in which the steadily moving disturbance appears stationary, and we have

$$A_T + (U - c_0)A_X + c_1 A A_X + c_2 A_{XXX} = F_X(X), \qquad (3.29)$$

where

$$F(X) = c_3 b_0(X) + c_4 b_1(X), \qquad (3.30a)$$

$$c_3 = -\frac{\frac{1}{2}(U_0(0)+c_0)\phi_1'(0)}{(f_1,\phi_1^2)}, \qquad c_4 = -\frac{\frac{1}{2}(U_1(1)+c_0)\phi_1'(1)}{(f_1,\phi_1^2)}. \tag{3.30b}$$

From this expression it is readily seen that $\mid U - c_0 \mid = O(\epsilon)$ is required to conform with the order estimate of the other terms of (3.29).

From the earlier momentum consideration (see (2.19), (2.22)), we have seen that, when a body moves in a rotating fluid, it experiences a resistance due to the generation and radiation of waves, which can be determined as follows. First, we assume for the pressure p the same asymptotic expansion as (3.10) and (3.11),

$$p(x,y,t) = P_0(y) + \epsilon p_1(x,y,t) + \epsilon^2 p_2(x,y,t) + O(\epsilon^3). \tag{3.31}$$

With (3.10), (3.11), (3.31) and $\partial(\cdot)/\partial x = O(\epsilon^{\frac{1}{2}})$, we have from the Euler equation the expressions for P_0 and p_1 in terms of their derivatives as

$$P_{0y} = \frac{\Gamma_0^2}{2y^2}, \tag{3.32}$$

$$p_{1x} = -2(\psi_{1yt} + U_0\psi_{1yx} - U_0'\psi_{1x}), \tag{3.33}$$

$$p_{1y} = \frac{\Gamma_0\Gamma_1}{y^2}. \tag{3.34}$$

The first-order pressure p_1 can be easily integrated, by use of (3.13), to give

$$p_1 = 2\{U_0'\phi_1 - (U_0+c_0)\phi_1'\}A(x,t). \tag{3.35}$$

On account of $b_0 = O(\epsilon^2)$ and $b_1 = O(\epsilon^2)$, the wave resistance D_w from (2.22) can be written as

$$D_w = \int_{-\infty}^{\infty} \{b_{1x}(p)_{y=1} + b_{0x}(p)_{y=0}\}\,dx + O(\epsilon^4) \tag{3.36a}$$

$$= -\int_{-\infty}^{\infty} \{b_1(p_x)_{y=1} + b_0(p_x)_{y=0}\}\,dx + O(\epsilon^4). \tag{3.36b}$$

Using (3.14), (3.30) and (3.35), the wave resistance D_w is given, to leading order, by

$$D_w = 4(f_1, \phi_1^2) \int_{-\infty}^{\infty} F_x(x+Ut)A(x,t)dx = 4(f_1, \phi_1^2) \int_{-\infty}^{\infty} F_X(X)A(X,t)dX. \quad (3.37)$$

As the previous numerical solutions of the fKdV equation obtained for the open channel flows have shown the process of periodic generation of upstream-advancing solitons has been determined under the general premise of resonant forcing and found to be in broad agreement with the experimental observations by Lee (1985) and Lee *et al* (1989). Since the fKdV equation here seems to be generic, we should expect, in principle, that an analogous process of periodic generation of forward-moving eddies would have to arise, permanent in form, in rotating fluids under resonant forcing.

3.1.3 Non-uniform tube wall

In this section, we shall consider the more general case of free or forced non-linear waves propagating within a long variable cylindrical tube, with stationary, gradually divergent or convergent sections. The forcing agents at the tube axis and the uniform tube wall considered in the preceeding section are stationary in the frame moving with the linear long wave speed. But the non-uniform tube wall now under consideration is no longer stationary in the wave frame. The boundary condition at the central axis due to the steadily moving disturbance is still given by (3.22a) as before and the boundary condition at the stationary tube wall becomes

$$-\psi_y b_{1x} + \psi_x = 0 \qquad \text{at} \quad y - 1 = r_1^2 - 1 = -b_1(x). \qquad (3.38)$$

In this section, we use the frame in which the non-unifrom tube boundary is stationary. Further we assume the wall boundary variation to have an amplitude of $O(\epsilon)$, which is lower in order than the amplitude of steadily moving disturbances at the wall, $O(\epsilon^2)$, as considered in the previous section, and can be written as

$$b_1(x) = \epsilon\tilde{b}_1(\eta), \qquad \eta = \epsilon^{1/2}x. \tag{3.39}$$

The disturbance at the tube axis is assumed to be moving with the critical speed and to have the same form (see 3.23) as before. Then we can seperate the stream functions for disturbances into steady and unsteady ones, in place of (3.10), as

$$\psi = \psi_0(y) + \psi^U(\xi,\tau;y) + \psi^S(\eta;y), \tag{3.40a}$$

$$\psi^U(\xi,\tau;y) = \epsilon\psi_1^U(\xi,\tau;y) + \epsilon^2\psi_2^U(\xi,\tau;y) + O(\epsilon^3), \tag{3.40b}$$

$$\psi^S(\eta;y) = \epsilon\psi_1^S(\eta;y) + \epsilon^2\psi_2^S(\eta;y) + O(\epsilon^3), \tag{3.40c}$$

where the unsteady disturbances, denoted by $\psi^U(\xi,\tau;y)$, moves with the speed of the linear long wave and the similar expressions for $\Gamma^U(\xi,\tau;y)$ and $\Gamma^S(\eta;y)$ are assumed. The steady solutions $\psi^S(\eta;y)$ are necessary in this analysis since the primary axial velocity (3.6) is now not quite (to order ϵ) an exact solution to the Euler equation for non-uniform wall boundary.

Substituting (3.40) into (3.22a) and (3.38) yields the boundary conditions in the first and second order

$$\psi_1^U(\xi,\tau;0) = 0, \qquad \psi_1^S(\eta;0) = 0 \qquad \text{at} \quad y = 0, \tag{3.41a}$$

$$\psi_1^U(\xi,\tau;1) = 0, \qquad \psi_1^S(\eta;1) = \frac{1}{2}U_0(1)\tilde{b}_{1\eta} \qquad \text{at} \quad y = 1, \tag{3.41b}$$

$$\psi_2^U(\xi,\tau;0) = -\frac{1}{2}(U_0(0) + c_0)\tilde{b}_0(\xi,\tau) \qquad \text{at} \quad y = 0, \tag{3.42a}$$

$$\psi_{2\xi}^U(\xi,\tau;1) = \psi_{1\xi y}^U(\xi,\tau;1)\tilde{b}_1 + \psi_{1y}^U(\xi,\tau;1)\tilde{b}_{1\eta} \qquad \text{at} \quad y=1. \tag{3.42b}$$

Substituting (3.40) into (3.3)-(3.4) gives the same equation for ψ_1^U as (3.13) and the following equation for ψ_1^S in the leading order

$$\psi_1^S(\eta;y) = \varphi_1(y)\tilde{b}_1(\eta), \qquad L_0\varphi_1 = 0, \tag{3.43a}$$

$$\varphi_1(0) = 0, \qquad \varphi_1(1) = \frac{1}{2}U_0(1), \tag{3.43b}$$

where

$$L_0 = \frac{d^2}{dy^2} + q_0(y), \qquad q_0(y) = \frac{1}{U_0}\left[\frac{\Gamma_0\Gamma_0'}{U_0 y^2} - U_0''\right]. \tag{3.43c, d}$$

Here we remark that φ_1 in (3.43) is different from the ϕ_1 in (3.13). Since (3.43) reduces to (3.13) for $\psi_1^U(\xi,\tau;y)$ when $c_0 = 0$ and the boundary conditions (3.43b) are non-homogeneous, we assume that $c_0 = 0$ is not an eigenvalue of (3.13) for solutions to (3.43) to exist. When $c_0 = 0$ is an eigenvalue, which means that waves are stationary in the presence of the non-zero axial velocity, the fKdV equation derived in the preceeding section could be the more appropriate mathematical model in this situation. Since our interest is here primarily in unsteady wave generation due to steady disturbances moving with the critical speed rather than in stationary waves which have been studied in connection with the vortex breakdown phenomenon, non-zero c_0 is assumed hereafter. To study the effects of non-uniform tube wall on the evolution of $A_1(\xi,\tau)$, we will seek the higher order equations for unsteady disturbances $\psi_2^U(\xi,\tau;y)$. For the steady part of solutions which are the leading order corrections to the primary flow due to the non-uniformity of stationary tube wall, the higher order terms $\psi_n^S(\eta;y)$ $(n>1)$ can be obtained by further expansions but it will not be pursued here. The second-order terms of

(3.3) and (3.4) yield the following equation for $\psi_2^U(\xi,\tau;y)$

$$L\psi_{2\xi} = f_1(y)\phi_1 A_{1\tau} + f_2(y)\phi_1^2 A_1 A_{1\xi} + f_3(y)\phi_1 A_{1\xi\xi\xi} + f_4(y)\tilde{b}_1 A_\xi + f_5(y)\tilde{b}_{1\eta} A_1 \tag{3.44}$$

where f_1,f_2 and f_3 are given in (3.16b,c,d) and

$$\begin{aligned} f_4(y) = &-\frac{2}{U_0+c_0}\left[q_0' + \frac{\Gamma_0'^2}{y^2U_0(U_0+c_0)} + \frac{\Gamma_0}{y^2(U_0+c_0)}\left(\frac{\Gamma_0'}{U_0}\right)'\right]\varphi_1\phi_1 \\ &-\frac{2}{U_0+c_0}\left[(q_0-q) + \frac{c_0\Gamma_0\Gamma_0'}{y^2U_0(U_0+c_0)^2}\right]\varphi_1'\phi_1, \end{aligned} \tag{3.45a}$$

$$\begin{aligned} f_5(y) = &-\frac{2}{U_0+c_0}\left[q' + \frac{\Gamma_0'^2}{y^2U_0(U_0+c_0)} + \frac{\Gamma_0}{y^2(U_0+c_0)}\left(\frac{\Gamma_0'}{U_0+c_0}\right)'\right]\varphi_1\phi_1 \\ &-\frac{2}{U_0+c_0}\left[(q-q_0) - \frac{c_0\Gamma_0\Gamma_0'}{y^2U_0(U_0+c_0)^2}\right]\varphi_1\phi_1'. \end{aligned} \tag{3.45b}$$

By using the same solvability condition as (3.26), we readily obtain the evolution equation for waves due to steadily moving disturbances along the central axis of the non-uniform tube as

$$A_{1\tau} + c_1 A_1 A_{1\xi} + c_2 A_{1\xi\xi\xi} + c_3\tilde{b}_1(\eta)A_{1\xi} + c_4\tilde{b}_{1\eta}(\eta)A = \tilde{F}_\xi, \tag{3.46}$$

where c_1 and c_2 are given by (3.18b), $\tilde{F}$ by (3.26) and the coefficients of two new terms are given by

$$c_3 = \left[(f_4,\phi_1) + \left(\phi_1'(1)\right)^2\right]/(f_1,\phi_1^2), \qquad c_4 = \left[(f_5,\phi_1) + \left(\phi_1'(1)\right)^2\right]/(f_1,\phi_1^2). \tag{3.47}$$

Upon recovering the original physical variables (x,t) for the amplitude function $A = \epsilon A_1 + O(\epsilon^2)$ and the forcing function $F = \epsilon^2\tilde{F}$, the equation becomes

$$A_t - (c_0 - c_3 b_1)A_x + c_1 A A_x + c_2 A_{xxx} + c_4 b_{1x} A = F_x(x+Ut). \tag{3.48}$$

By introducing the local wave velocity $c(x) = c_0 - c_3 b_1(x)$, (3.48) becomes

$$A_t - c(x)A_x + c_1 A A_x + c_2 A_{xxx} - \bar{c}_4 c(x)(\log c(x))_x A = F_x(x + Ut), \qquad (3.49)$$

where $\bar{c}_4 = c_4/c_3$. This is analogous to the various forms of equations derived by Kakutani (1971), Johnson (1972), Shuto (1974) and others to study solitary waves in an open channel of gradually varying depth. According to this model, the waves governed by (3.49) are one-directional and the reflected waves are neglected. However, it has been argued that the waves reflected by a gradually varying channel would be much weaker than the incident wave and hence the energy must be adiabatically invariant (see, e.g. Miles 1979). To determine the reflected waves and their effects on energy conservation, the Boussinesq equation model often used in study of an open channel flow is desirable in this case.

It is also of interest to consider the special case of basic flow with $U_0 = 0$, then (4.43a) has the $\varphi_1 = 0$ under the homogeneous boundary conditions (3.43b). Therefore, the effects of non-uniform wall disappear in equation (3.49) since $f_4 = 0$ in (3.44) and some new equations will be required. By assuming $b_w(x) = 1 - b_1(x) = O(1)$ and $b_{wx} = O(\epsilon^{3/2})$, which are the original assumption of Kakutani (1971) and others for variable open channel flows, the effects of non-uniform wall can be taken into consideration. Without showing the details of derivations, the resulting evolution equation for A, in case of $U_0 = 0$, can be shown to have the same form of (3.49) but non-constant coefficients as

$$A_t - c(x)A_x + c_1(x) A A_x + c_2(x) A_{xxx} - c(x)(\log g(x))_x A = \frac{1}{c(x)} F_t(x + Ut), \qquad (3.50)$$

where $g(x)$ is given by

$$g(x) = \frac{1}{c(x)} \int_0^{b_w(x)} \frac{\Gamma_0 \Gamma_0'}{y^2} \phi_1^2 \, dy, \qquad (3.51)$$

and $c_1(x)$ and $c_2(x)$ are functions of x which can be determined from (3.16) and (3.18) with $U_0 = 0$ by replacing c_0 by the local long wave speed $c(x)$. The long wave speed $c(x)$ can be found by solving the eigenvalue problem (3.13) with $c(x)$ in place of c_0 with the boundary conditions

$$\phi_1(0) = 0, \qquad \phi_1(b_w(x)) = 0, \tag{3.52}$$

where x can be regarded as a parameter in solving the eigenvalue problem for gradually varying tube radius.

For free waves ($F = 0$), by solving an equation similar to (3.49) and (3.50), Johnson (1972) studied the fisson of solitary wave climbing up a shelf, first discovered by Mei and Madsen (1968), in an open channel flow. In the following sections, we will concentrate on the phenomenon of forced generation of vortex solitons rather than the deformation of free waves in non-homogeneous media.

3.2 Applications to two primary flows

Two basic flows are chosen as examples to illustrate the phenomenon of generation of upstream-advancing vortex solitons. One is the Rankine vortex which is the simplest model flow in which the swirl velocity is non-uniform in the radial direction. The Rankine vortex is a rather idealized mathematical model but sometimes it is a good approximation to the more realistic vortex flow. The other basic flow is the Burgers vortex whose velocity profile has been observed in experiments of swirling flow (Harvey, 1962; Sarpkaya, 1971) as being nearly realistic. Although the Rankine vortex model cannot be directly applied to the previous analysis due to a discontinuity of vorticity inside the tube, we choose the Rankine vortex as the first example since we can determine the coefficients of the fKdV equation explicitly and to compare these results with the numerical results for the Burgers vortex.

3.2.1 The Rankine vortex

This primary flow has a vortex core of solid-body rotation with constant angular velocity Ω and an irrotaional potential vortex motion outside the vortex core and within the tube wall at $r = R$. Accordingly we assume that (in this section, the original coordinate r instead of y will be used for convenience)

$$U_0(r) = 0, \tag{3.53a}$$

$$W_0(r) = \begin{cases} r & \text{for } 0 \le r \le 1 \\ \frac{1}{r} & \text{for } 1 \le r \le R \end{cases} \tag{3.53b}$$

where we use the vortex core radius r_c to scale all distances and the maximum swirl velocity Ωr_c as the velocity scale.

Since this flow has an additional free boundary which is the vortex surface originally at $r = 1$, the analysis is more complicated than before. Inside the vortex core (the inner region, $0 \le r \le 1 + \zeta$ where ζ is the displacement of the vortex boundary interface), the stream function ψ and the circulation Γ still satisfy equations (3.3) and (3.4) with r as an independent variable. In the irrotational flow region (the outer region, $1 + \zeta \le r \le R$), the stream function Ψ satisfies the vorticity-free equation:

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} = 0, \tag{3.54}$$

and the circulation rW_0 is constant (equal to 1) in the outer region.

The boundary conditions at the interface ($r = 1 + \zeta$) require the continuity of axial, radial and swirl velocities which can be written as

$$\psi = \Psi, \qquad \psi_r = \Psi_r, \qquad \Gamma = 1 \qquad \text{at} \qquad r = 1 + \zeta(x, t). \tag{3.55a, b, c}$$

These three kinematic boundary conditions (3.55) will ensure that the surface of discontinuity in vorticity is a material surface and the pressure is continuous across this interface. For the external forcing excitation, we consider a slender body centered at the tube axis and a topographic disturbance at the wall, both moving to the left in this primary flow. Then the boundary conditions at the body surface ($r = r_0(x + Ut)$) and the disturbance at the tube wall ($r = r_1(x + Ut)$) are

$$(Ur_0 + \psi_r)r_{0x} + \psi_x = 0 \qquad \text{at} \qquad r = r_0(x + Ut), \tag{3.56a}$$

$$(Ur_1 + \Psi_r)r_{1x} + \Psi_x = 0 \qquad \text{at} \qquad r = r_1(x + Ut). \tag{3.56b}$$

Adopting the stretched coordinate system (3.9), we asssume the same expansions for Ψ and ζ as ψ in (3.10)

$$\Psi = \epsilon \Psi_1(\xi, \tau; r) + \epsilon^2 \Psi_2(\xi, \tau; r) + O(\epsilon^3), \tag{3.57}$$

$$\zeta = \epsilon \zeta_1(\xi, \tau) + \epsilon^2 \zeta_2(\xi, \tau) + O(\epsilon^3) \tag{3.58}$$

and r_0 and $|R - r_1|$ are assumed to be $O(\epsilon)$ and $O(\epsilon^2)$, respectively, for the consistency of analysis, as previously shown. Substituting (3.57) into (3.54) and using (3.13), we have for ψ_1, Ψ_1 and Γ_1 the first-order equations in the form

$$\Gamma_1 = \gamma_1(r)A_1(\xi, \tau), \qquad \psi_1 = \phi_1(r)A_1(\xi, \tau), \qquad \Psi_1 = \Phi_1(r)A_1(\xi, \tau), \tag{3.59a}$$

$$\gamma_1 = \frac{2}{c_0}\phi_1, \qquad L_1\phi_1 = 0, \qquad L_2\Phi_1 = 0, \tag{3.59b}$$

$$L_1 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{4}{{c_0}^2}, \qquad L_2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}. \tag{3.59c}$$

Substituting (3.10-3.11) and (3.57-3.58) into (3.55-3.56) yields the boundary conditions in the first-order

$$\phi_1(0) = 0, \qquad \Phi_1(R) = 0, \tag{3.60a}$$

$$\phi_1(1) - \Phi_1(1) = 0, \qquad \phi_1'(1) - \Phi_1'(1) = 0, \tag{3.60b}$$

and, from (3.55c), the displacement of interface ζ_1 is given by

$$\zeta_1 = -\frac{1}{2}\gamma_1(1)A_1(\xi,\tau). \tag{3.60c}$$

From (3.59), with boundary conditions (3.60a,c), we have the first-order solutions

$$\psi_1 = rJ_1(\beta r)A_1(\xi,\tau), \tag{3.61a}$$

$$\Gamma_1 = \frac{2}{c_0}rJ_1(\beta r)A_1(\xi,\tau), \tag{3.61b}$$

$$\Psi_1 = \sigma(r^2 - R^2)A_1(\xi,\tau), \tag{3.61c}$$

$$\zeta_1 = -\frac{1}{c_0}J_1(\beta)A_1(\xi,\tau), \tag{3.61d}$$

where J_n denotes the n-th order Bessel function and $\beta = |2/c_0|$. Two unknowns, σ and β (or c_0), can be determined by making use of (3.60b), giving

$$\frac{J_1(\beta)}{\beta J_0(\beta)} = \frac{1}{2}(1 - R^2), \tag{3.62a}$$

$$\sigma = \frac{1}{2}\beta J_0(\beta). \tag{3.62b}$$

An infinite number of roots for β can be found for a given tube radius R by solving equation (3.62) and each β lies between $j_{0,n}$ and $j_{1,n}$, the n-th zeros of J_0 and J_1, respectively. By choosing the first root β for which the eigenfunction has no zero in $0 < r < R$,

$$j_{0,1}(= 2.4048) < \beta < j_{1,1}(= 3.8317), \tag{3.63a}$$

the phase velocity of the long wave, c_0 non-dimensionalized by Ωr_c lies in

$$\frac{2}{j_{1,1}}(= 0.5220) < |c_0| < \frac{2}{j_{0,1}}(= 0.8317). \tag{3.63b}$$

To determine the amplitude function $A_1(\xi, \tau)$, the second-order problems must be considered. The stream function ψ_2 for the inner region has to satisfy equation (3.16a) in terms of r (instead of y)

$$L_1\psi_{2\xi} = \frac{8}{c_0^3}\phi_1 A_{1\tau} - \phi_1 A_{1\xi\xi\xi}, \tag{3.64a}$$

while Ψ_2 for the outer region, from (3.54), satisfies the following equation

$$L_2\Psi_{2\xi} = -\Phi_1 A_{1\xi\xi\xi}. \tag{3.64b}$$

The second-order expansions of (3.55) and (3.56) yield the relations

$$\psi_2 = -\frac{c_0}{2}\tilde{r}_0^2 - \psi_{1r}\tilde{r}_0 \qquad \text{at} \qquad r = 0, \tag{3.65a}$$

$$\psi_{2\xi} + \zeta_1\psi_{1\xi r} = \Psi_{2\xi} + \zeta_1\Psi_{1\xi r} \qquad \text{at} \qquad r = 1, \tag{3.65b}$$

$$\psi_{2r} + \zeta_1\psi_{1rr} = \Psi_{2r} + \zeta_1\Psi_{1rr} \qquad \text{at} \qquad r = 1, \tag{3.65c}$$

$$\Psi_2 = c_0 R(\tilde{r}_w) \qquad \text{at} \qquad r = R, \tag{3.65d}$$

where $r_0 = \epsilon\tilde{r}_0$ and $R - r_1 = \epsilon^2\tilde{r}_w$ have been used. Substituting the first-order solutions (3.61) into (3.65), we have the boundary conditions of the second order

$$\psi_2 = -\frac{c_0}{2}\tilde{r}_0^2 \qquad \text{at} \qquad r = 0, \tag{3.66a}$$

$$\psi_2 - \Psi_2 = 0 \qquad \text{at} \qquad r = 1, \tag{3.66b}$$

$$\psi_{2r} - \Psi_{2r} = -\frac{4}{c_0^3} J_1^2(\beta) A_1^2 \qquad \text{at} \qquad r = 1, \tag{3.66c}$$

$$\Psi_2 = c_0 R \tilde{r}_w \qquad \text{at} \qquad r = R. \tag{3.66d}$$

Define the following inner products

$$(f,g)^i = \int_0^1 \frac{1}{r} f(r) g(r) dr, \qquad (f,g)^o = \int_1^R \frac{1}{r} f(r) g(r) dr. \tag{3.67a, b}$$

To satisfy the solvability condition for ψ_2 and Ψ_2, we take the inner products between (3.64a) and ϕ_1 for $0 \le r \le 1$ and between (3.64b) and Φ_1 for $1 \le r \le R$,

$$(\phi_1, L_1 \psi_{2\xi})^i + (\Phi_1, L_2 \Psi_{2\xi})^o = \frac{8}{c_0^3}(1, \phi_1^2)^i A_{1\tau} - \{(1, \phi_1^2)^i + (1, \Phi_1^2)^o\} A_{1\xi\xi\xi}. \tag{3.68}$$

The factor $1/r$ in the definition of the inner products is to make the operators L_1 and L_2 formally self-adjoint. The left-hand side of (3.68) with use of (3.59) and (3.60) gives the result

$$\begin{aligned}
&(\phi_1, L_1 \psi_{2\xi})^i + (\Phi_1, L_2 \Psi_{2\xi})^o \\
&= \left(\frac{1}{r}\phi_1 \psi_{2\xi r} - \frac{1}{r}\phi_{1r}\psi_{2\xi}\right)\Big|_0^1 + (L_1\phi_1, \psi_{2\xi})^i + \left(\frac{1}{r}\Phi_1 \Psi_{2\xi r} - \frac{1}{r}\Phi_{1r}\Psi_{2\xi}\right)\Big|_1^R + (L_2\Phi_1, \Psi_{2\xi})^o \\
&= -\frac{c_0}{2}\beta \tilde{b}_{0\xi} - c_0 \beta J_0(\beta) R \tilde{r}_{w\xi} - \frac{8}{c_0^3} J_1^3(\beta) A_1 A_{1\xi}.
\end{aligned} \tag{3.69}$$

Combining the contributions from both sides of (3.68), we have the forced KdV equation (3.27)

$$A_{1\tau} + c_1 A_1 A_{1\xi} + c_2 A_{1\xi\xi\xi} = \tilde{F}_\xi, \tag{3.27}$$

of which the coefficients can be evaluated by substituting (3.61) into (3.68) and (3.69), with the results as

$$c_1 = \frac{2J_1^3(\beta)}{\Delta}, \quad \Delta = J_0^2(\beta) + J_1^2(\beta) - \frac{2}{\beta} J_0(\beta) J_1(\beta), \tag{3.70a}$$

$$c_2 = -\frac{c_0^3}{8}\left[\frac{2\sigma^2(R^4 \ln R + R^2 - \frac{3}{4}R^4 - \frac{1}{4})}{\Delta} + 1\right], \qquad (3.70b)$$

$$\tilde{F} = c_3\tilde{b}_0 + c_4\tilde{b}_1 = c_3\tilde{r}_0^2 + c_4(2R\tilde{r}_w), \quad c_3 = -\frac{2}{\beta^3}\frac{1}{\Delta}, \quad c_4 = -\frac{2}{\beta^3}\frac{J_0(\beta)}{\Delta}. \qquad (3.70c)$$

Upon recovering the original physical variables and using the body frame (X, T) defined by $X = x + Ut$ and $T = t$ as before, we finally obtain (3.29)

$$A_T + (U - c_0)A_X + c_1 A A_X + c_2 A_{XXX} = F_X(X), \qquad (3.29)$$

where c_1,c_2, and F are given by (3.70a-c).

Since the signs of c_1 and c_3 in (3.70) are independent of the sign of c_0 (the direction of propagation), $-A(-X, T)$ is a solution for the negative U and c_0 if $A(X, T)$ is a solution of the fKdV equation (3.29) for the positive U and c_0. Therefore we need only consider the positive U and c_0 (the left-going waves) in this section.

As an example, we choose a tube radius $R = 2, 3$ and 4 (normalized by the vortex core radius) and obtain β from (3.62a) by applying Newton-Rapson's method. Subsequently, we can determine σ from (3.62b) and all coefficients in fKdV equation from (3.62b) and (3.70). The results are shown in table 3.1.

R	β	σ	c_0	c_1	c_2	c_3	c_4
2	2.6411	-.1531	.7573	.7318	-.0816	-.4104	.0476
3	2.5025	-.0621	.7992	.9111	-.1315	-.4751	.0236
4	2.4584	-.0338	.8135	.9686	-.1694	-.4999	.0137

Table 3.1 The coefficients of the fKdV equation (3.29) for tube radius R=2, 3 and 4.

In the following numerical calculations, $R = 2$ is chosen as a representative case. The r-dependence of the first-order stream function and axial velocity are shown in figure 3.2. Without any forcing disturbance ($F = 0$ and $U = c_0$), we have the classical free solitary wave solution (3.21). For the streamlines to form a recirculating eddy at the axis of rotation, we first find the minimum amplitude of solitary wave a by requiring that the axial velocity at the axis vanish, which gives

$$|a| \geq \left| \frac{c_0}{\beta - \frac{c_1}{3}} \right| (= 0.3159). \tag{3.71}$$

The streamlines of a free solitary wave corresponding to the amplitude $a = -0.4$ in a moving frame in which the eddy appears stationary are shown in figure 3.3 and this eddy was interpreted as a mild axisymmetric vortex breakdown by Benjamin (1967).

For the case of forced waves, we consider a cosine-shaped body moving along the tube axis, with the body radius

$$r_0(x,t) = r_b \cos^2\left[\frac{\pi}{L}(x+Ut)\right] \qquad \text{for} \qquad -\frac{L}{2} \leq (x+Ut) \leq \frac{L}{2}, \tag{3.72}$$

and $r_0(x,t) = 0$ elsewhere. We could choose a typical blunt body (like a sphere) as a forcing agent which might be more effective in generating upstream-progressing solitons, but the additional jump discontinuities in body slope at the leading and trailing edges would require special consideration as such singularities of body geometry are not consistent with our original assumptions. For the class of smooth body geometry as given by (3.72), with say $r_{0x}/r_0 = O(\epsilon^{\frac{1}{2}})$ uniformly bounded, we choose a typical maximum radius of body $r_b = 0.35$ (the ratio of body radius to tube radius=0.175) and a body length $L = 2$. To solve the fKdV equation

(3.29) numerically, we shall adopt the scheme of Zabusky and Kruskal (1965), i.e., the leap-frog method in time and the central difference in space. As can be seen in figure 3.4a, the time sequence of evolution of $-A(X,T)$ from the rest, for the detuning parameter $\delta = 0$, show the solution with salient features known to be characteristic of the numerical solutions of the fKdV equation. After the disturbance has moved with the critical speed for a certain time, a solitary wave emerges in front of the disturbance, breaks away to run ahead of the disturbance as a free wave and is later followed by another new solitary wave similarly produced and radiated. This whole process of generation of upstream solitons seems to continue periodically and indefinitely while a uniform depression just behind the disturbance is being prolonged and is followed by a train of cnoidal-like trailing waves. The corresponding streamlines in a moving coordinate (X,T) at several time instants are shown in figure 3.4b. More numerical results of the fKdV equation will be disscussed over a range of pertinent parameters in the next section. So far we have obtained only the solution to the first-order problem (modelled by the nonlinear evolution equation (3.29) for A) of this specific phenomenon.

3.2.2 The Burgers vortex

The Burgers vortex is a frequently observed velocity profile that seems to occur in non-uniformly rotating fluids in a long cylindrical tube in studies for simulating various vortex generation mechanisms (see Leibovich, 1984). Here we consider the following primary flow which is non-uniform in the radial direction and linearly stable with respect to axisymmetrical perturbation by the criteria of

Howard & Gupta (1962)

$$U_0(y) = U_m e^{-\mu_1 y}, \tag{3.73a}$$

$$\Gamma_0(y) = \Gamma_m(1 - e^{-\mu_2 y}), \tag{3.73b}$$

where all variables are non-dimensionalized by the maximum azimuthal velocity at $y = 1.2565/\mu_2$ (or $r = 1.1209/\sqrt{\mu_2}$) and the tube radius. Thus this flow has three free parameters (U_m, μ_1, μ_2) noting that, for a given μ_2 which measures the concentration of the axial vorticity, Γ_m is scaled out. Also, U_m is the ratio of the maximum axial velocity to the maximum azimuthal velocity. Substituting (3.73) into (3.13) and solving the resulting eigenvalue problem numerically, we can determine the first eigenvalue c_0 and the corresponding eigenfunction ϕ_1 (which can be normalized). Since this eigenvalue problem has a regular singular point at $y = 0$, we may obtain a series solution about $y = 0$ with a suitable normalization and find the values of ϕ_1 and ϕ_1' at $y = y_0 << 1$. Taking these as initial values, we find the solution using the 4-th order Runge-Kutta method and the corresponding eigenvalue using the secant method. Subsequently, all the coefficients in the fKdV equation can be determined by using (3.18) and (3.30).

For numerical computations, we choose $\mu_1 = 5$, $\mu_2 = 12$ and $U_m = 0.6$, with the corresponding velocity profiles of the primary flow shown in figure 3.5. Due to the non-uniformity of the basic axial velocity, the right-going (RG) wave is not symmetric with the left-going (LG) wave. The absolute values of c_1 and c_3 are found to depend on the choice of normalization constant for the eigenfunction but the final values of the stream function are invariant. Here, the value of $\phi_1' = 1$ at $y = 0$ is used and the corresponding eigenfunctions of the stream function and axial velocity are shown in figure 3.6. The coefficients of the fKdV equations are shown in table 3.2.

	μ_1	μ_2	U_m	c_0	c_1	c_2	c_3	c_4
LG	5.	12.	0.6	.4557	.7316	-.0137	-11.5056	.1530
RG	5.	12.	0.6	-1.4906	.7790	.0195	-15.4165	1.0225

Table 3.2 The coefficients of the fKdV equation (3.29) associated with the Burgers vortex

The numerical solutions of the fKdV equation are obtained for the same axisymmetric forcing as (3.72), now with $r_b = 0.1$ and $L = 2$. The perspective view of $A(X,T)$ and the drag D_w given by (3.37) are calculated for the detuning parameter $\delta = 0$; the results are shown in figure 3.7a,b with the corresponding streamlines shown in figure 3.8a,b. The distance traversed by the body, denoted by ξ_b, with respect to the tube radius is $\xi_b = c_0 t$. At $t = 12$ ($\xi_b = 5.35$ for LG waves accompanying the left-going body and $\xi_b = 17.89$ for RG waves with the right-going body), the first forward-progressing eddy is completely seperated from the body forcing effect and, by $t = 30$ ($\xi_b = 13.37$ for LG waves and $\xi_b = 44.72$ for RG waves), the first three identical forward-progrssing vortons have been generated and the periodicity, as evidenced in the wave resistance data, is remarkable. As can be seen, the LG vortons, whose direction of propagation is opposite to the basic axial velocity, U_0, are generated in much shorter travelling distance of the body as compared with the case of RG ones. More details of numerical solutions for a broad numerical range of the parameters in the fKdV equation can be found in Lee *et al.* (1989) for the analogous shallow water problem. Generally, as the speed of the disturbance increases, the amplitude and the generation period of upstream-advancing solitons will increase. Compared with the results for the Rankine vortex, the Burgers vortex with $U_m = 0$ is found to be more effective in generating the forward moving vortons due to a concentration of axial vorticity near the axis of rotation since the absolute value of c_3 increases with increasing μ_2 which measures the concentration of axial vorticity. Also, as can be seen in table 3.2, a body situated at the tube axis as compared with a tube wall constriction with the same magnitude is the more effective of the two forcing agencies in generating vortons in the highly concentrated vortical flow since the absolute value of c_3 is much greater than that of c_4. In other words, whilst the upstream solitary waves are generated, the wave amplitude may be too small to form the recirculating eddies which we call vortons.

For these axisymmetric vortons to be physically significant, their stability characteristics must be investigated. The hydrodynamic stability analysis of several forced steady solitons of the fKdV equation developed by Camassa (1990) can be directly applicable to our problem. For convenience, we transform the fKdV equation (3.29) into one for the open channel flow using the following new variables

$$\hat{t} = \gamma_1 T, \qquad \hat{x} = \gamma_2 X, \qquad \eta = \gamma_3 A, \tag{3.74a}$$

$$\gamma_1 = c_0 \gamma_2, \qquad \gamma_2^2 = -\frac{1}{6}\frac{c_0}{c_2}, \qquad \gamma_3 = -\frac{2}{3}\frac{c_1}{c_0}, \tag{3.74b}$$

where c_0/c_2 is always negative as shown in Table 3.1 or Table 3.2. Then the fKdV equation can be written as (after dropping the hat)

$$\eta_t + (F-1)\eta_x - \frac{3}{2}\eta\eta_x - \frac{1}{6}\eta_{xxx} = P_x, \tag{3.75a}$$

where

$$F = \frac{U}{c_0}, \qquad P = -\frac{2}{3}\frac{c_1}{c_0^2}B. \tag{3.75b}$$

The stationary solutions of (3.75) that vanish at infinity satisfy the following equation

$$(F-1)\eta - \frac{3}{4}\eta^2 - \frac{1}{6}\eta_{xx} = P(x), \tag{3.76}$$

which is obtained from (3.75) by dropping the time derivative and integrating it once in x. One possible solution of (3.76) whose instability was investigated in details by Camassa & Wu (1991) is

$$\eta_s(x) = \frac{4}{3}k^2\mathrm{sech}^2(kx), \qquad P(x) = \frac{4}{3}k^2(F-1-\frac{2}{3}k^2)\mathrm{sech}^2(kx), \tag{3.77}$$

where the Froude number F can be regarded as a free parameter in addition to the parameter k for the length of disturbance and k is $O(\epsilon^{1/2})$. By considering small perturbations to this particular steady solution, Camassa & Wu (1991) showed how instability near critical speed leads to the periodic generation of upstream-advancing solitons when the eigenvalues are complex with a positive real part . Another question of whether or not the free solitary wave subject to, at least, axisymmetric perturbations is stable in addition to the stability of forced steady solitary waves should be addressed since each upstream-advancing solitary wave behaves as a free soliton after having gained a small distance ahead of the forcing region. But the stability of the free solitary wave solution of the KdV equation has been confirmed by Jefferey and Kakutani (1971) and Benjamin (1972).

Therefore we readily draw a conclusion that this vortex soliton is stable under the axisymmetric perturbations although the stability characteristic under non-axisymmetrical perturbations is still an open question and will be examined in chapter 4.

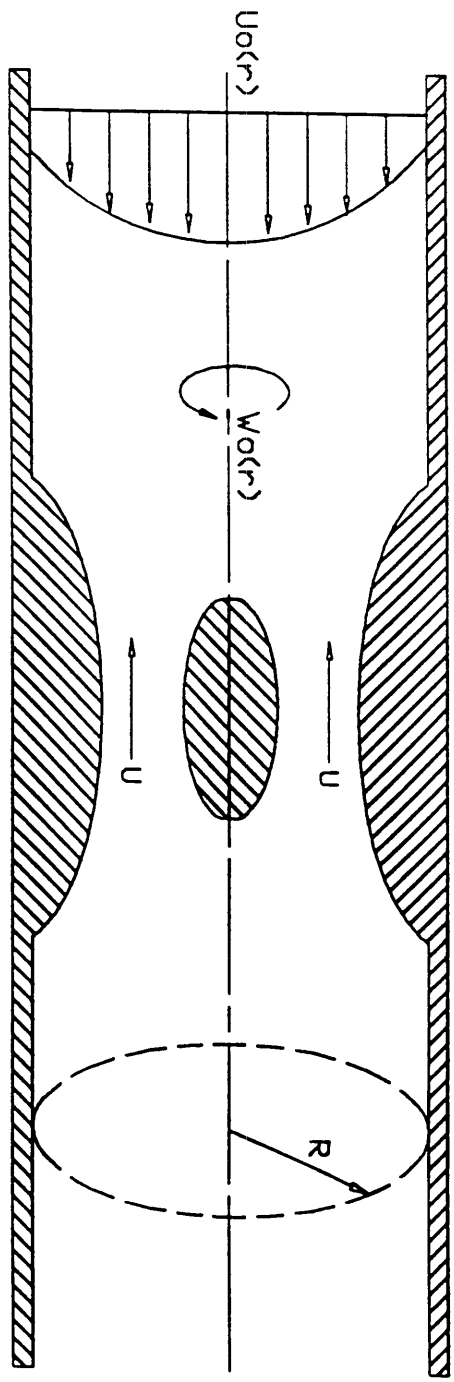


Figure 3.1 Schematic view of the problem.

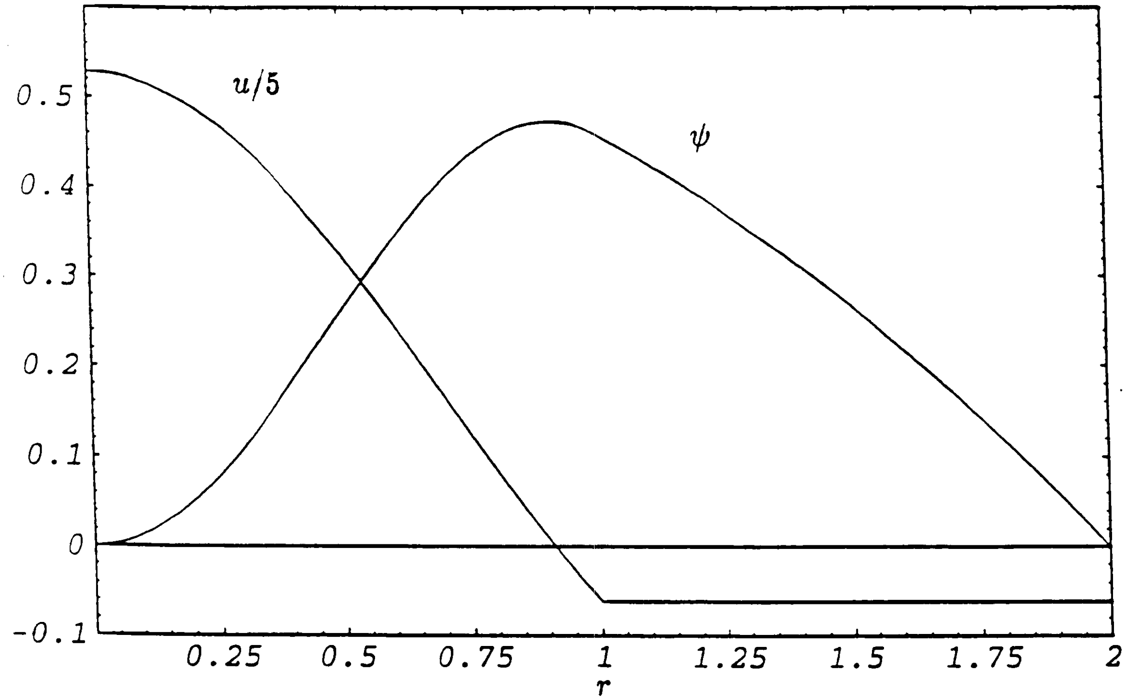


Figure 3.2 The eigenfunctions of stream function (ψ) and axial velocity (u) for the Rankine vortex with the core radius $r_c = 1$ and tube radius $R = 2$.

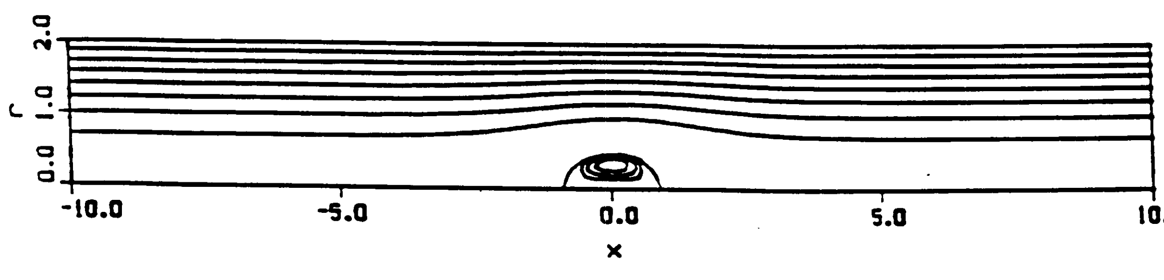


Figure 3.3 The streamlines of a free solitary vortex wave solution of amplitude $a = -0.4$ in the Rankine vortex with the core radius $r_c = 1$ and tube radius $R = 2$.

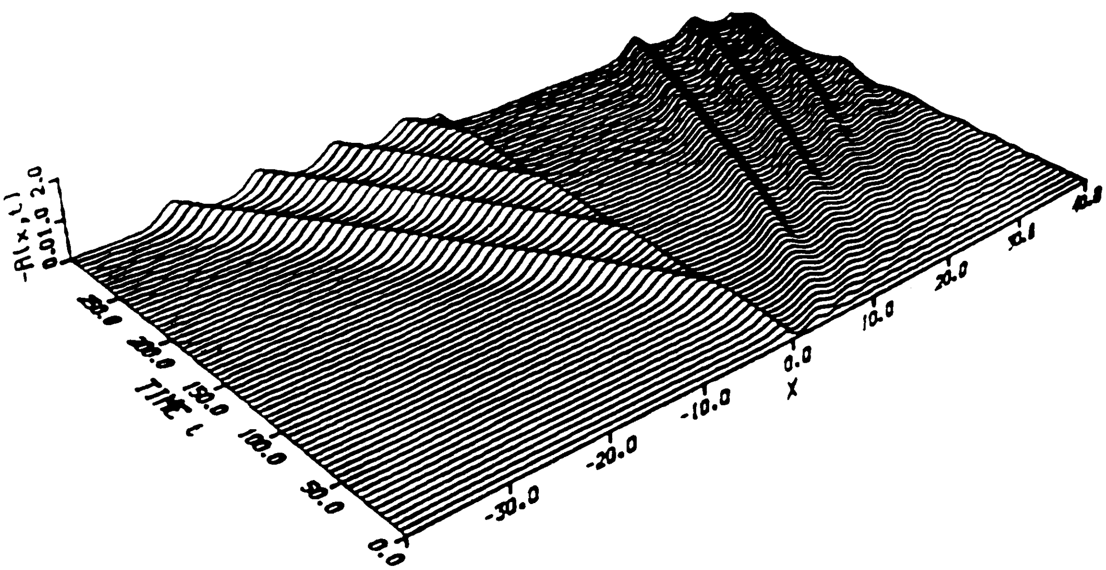


Figure 3.4a The numerical solution of the fKdV equation for the amplitude function $A(X,T)$ due to the body r_0 in (3.72) at the central axis with $r_b = 0.35$ and $L = 2$ moving at the critical speed ($\delta = 0$). The primary flow is the Rankine vortex with the core radius $r_c = 1$ and tube radius $R = 2$.

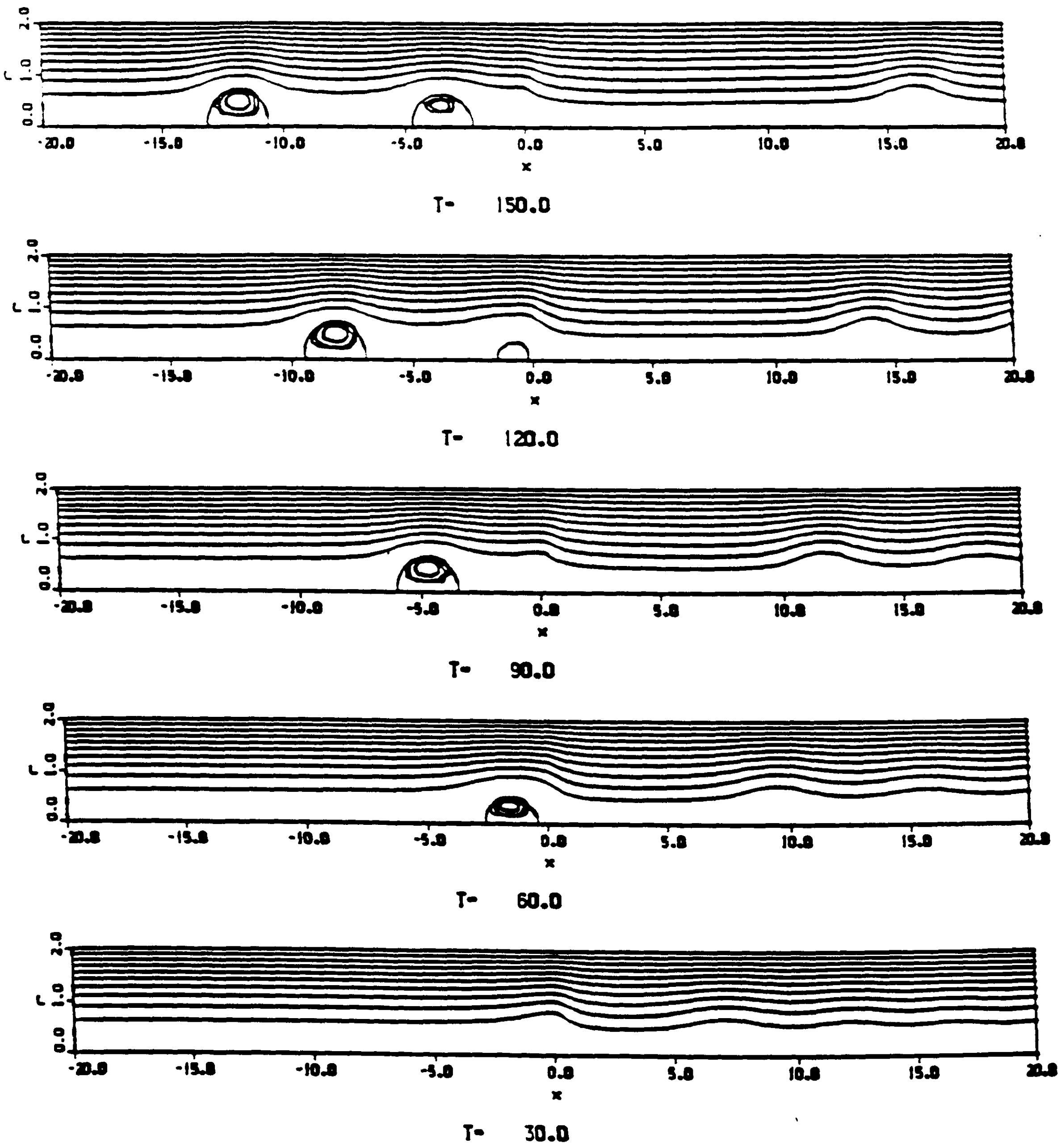


Figure 3.4b The streamlines corresponding to the solution in (a) at several time instants as specified.

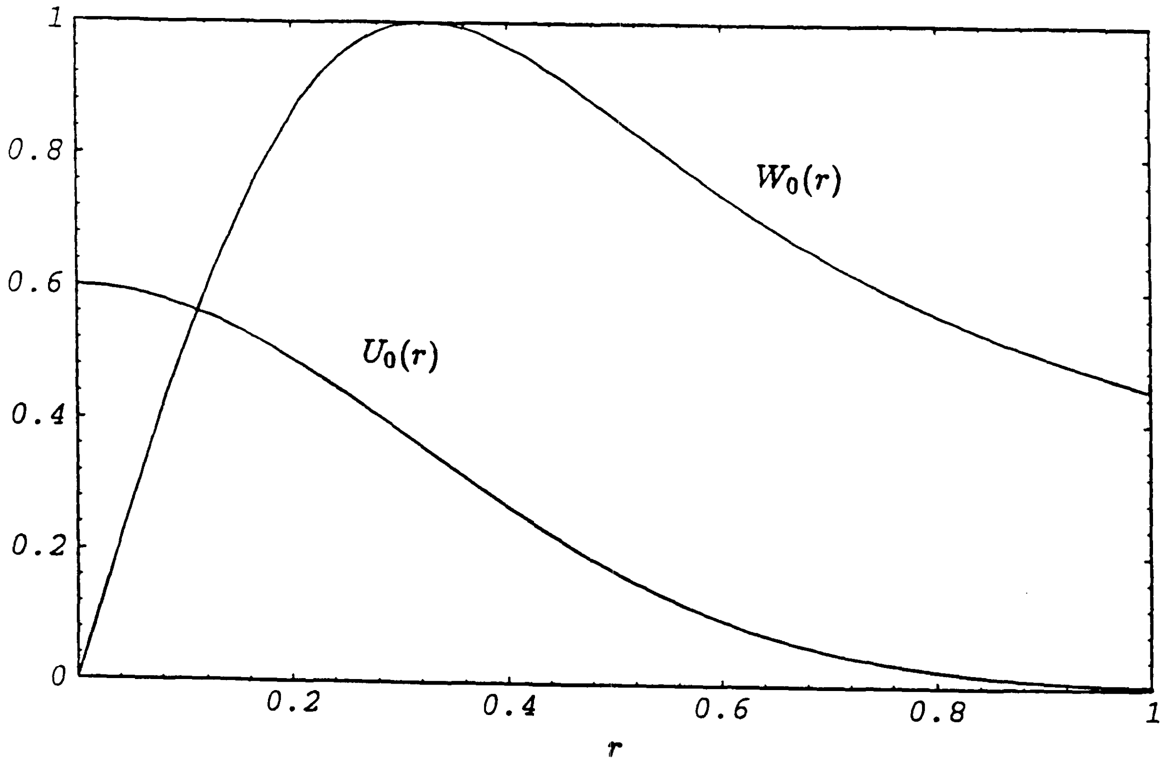
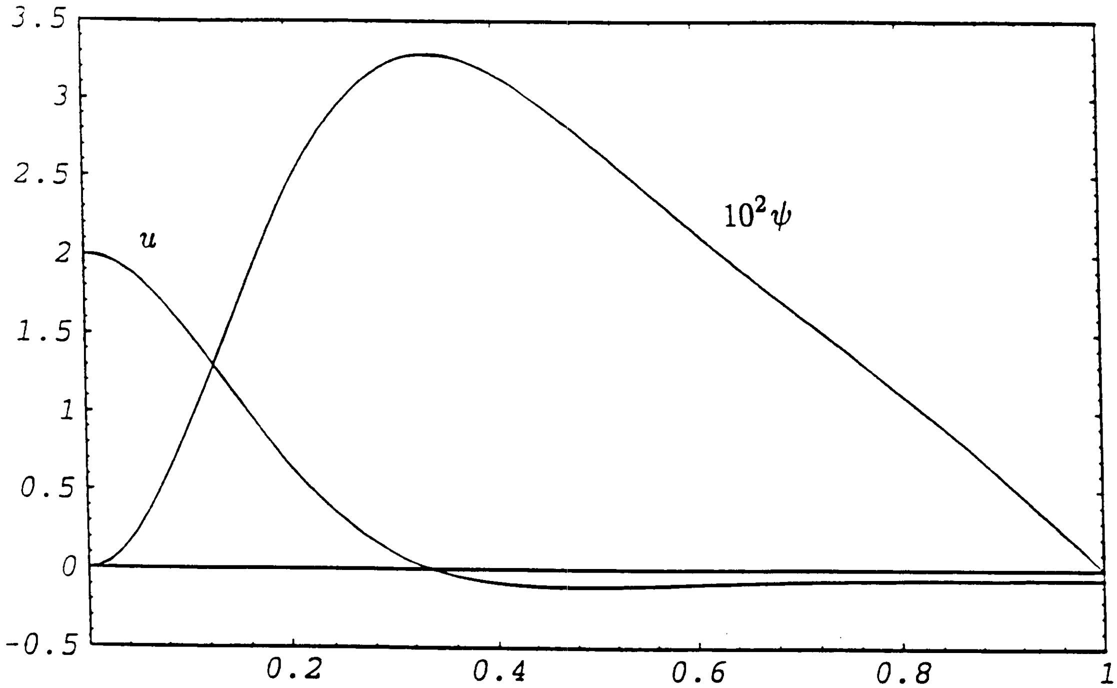
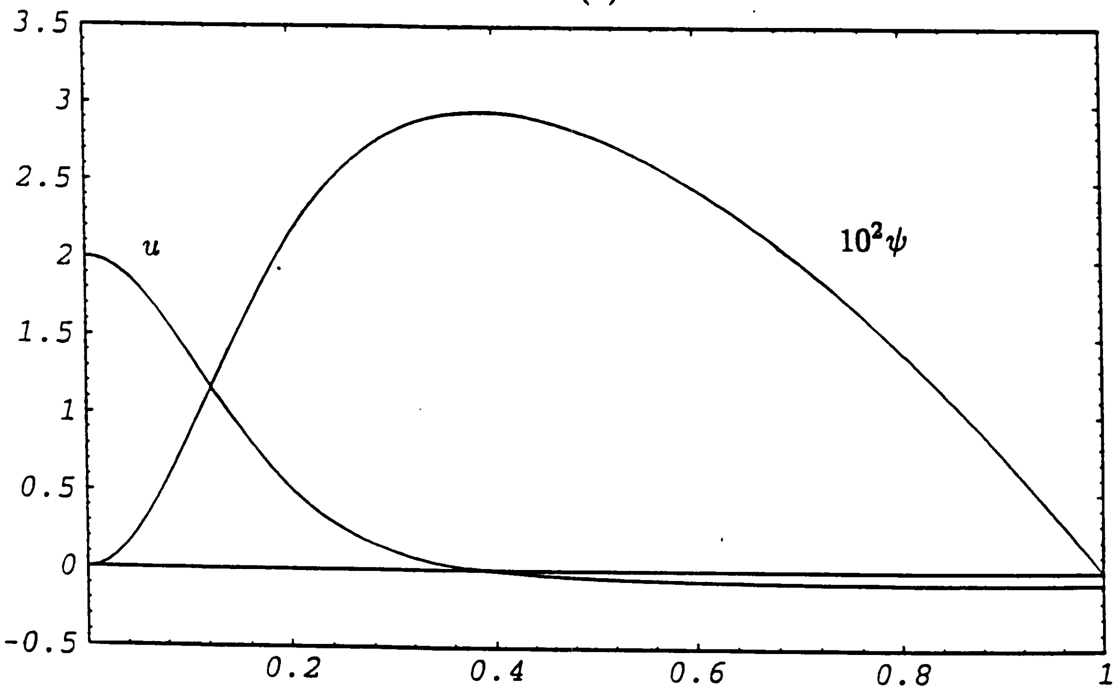


Figure 3.5 The velocity profiles of $U_0(r)$ and $W_0(r)$ of the Burgers vortex, (3.73a,b), with $\mu_1 = 5$, $\mu_2 = 12$ and U_m=0.6.



(a)



(b)

Figure 3.6 The eigenfunctions of the stream function (ψ) and axial velocity (u) for the Burgers vortex with $\mu_1 = 5$, $\mu_2 = 12$ and U_m=0.6, pertaining to (a) the left-going waves, (b) the right-going waves.

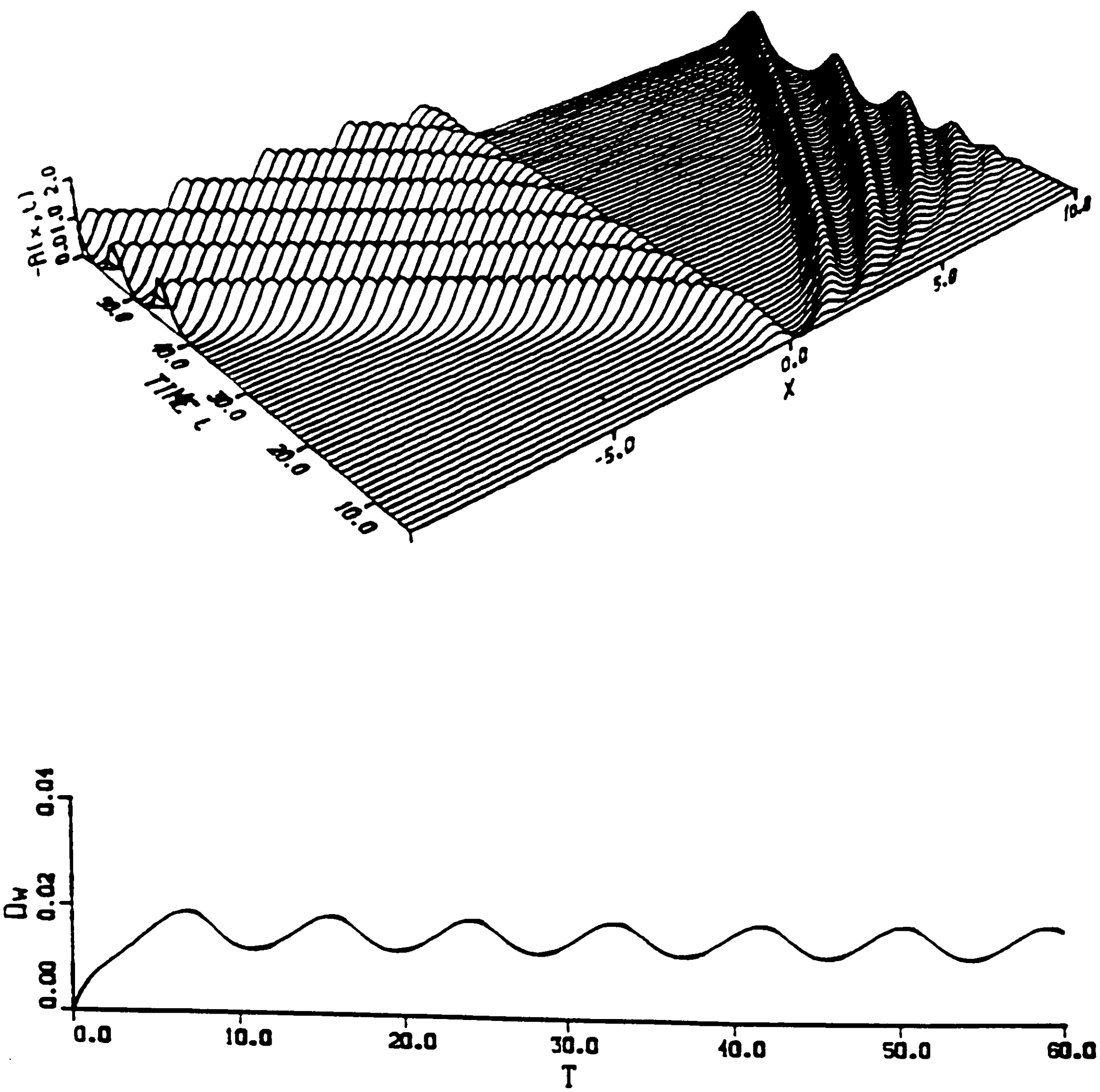


Figure 3.7a The numerical solution of the fKdV equation for the amplitude function $A(X,T)$ and the wave resistance D_w induced by the body r_0 in (3.72) moving along the central axis, with $r_b = 0.1$, $L = 2$ and with the critical speed ($\delta = 0$). The primary flow is the Burgers vortex with $\mu_1 = 5$, $\mu_2 = 12$ and $U_m = 0.6$. (a) the left-going waves.

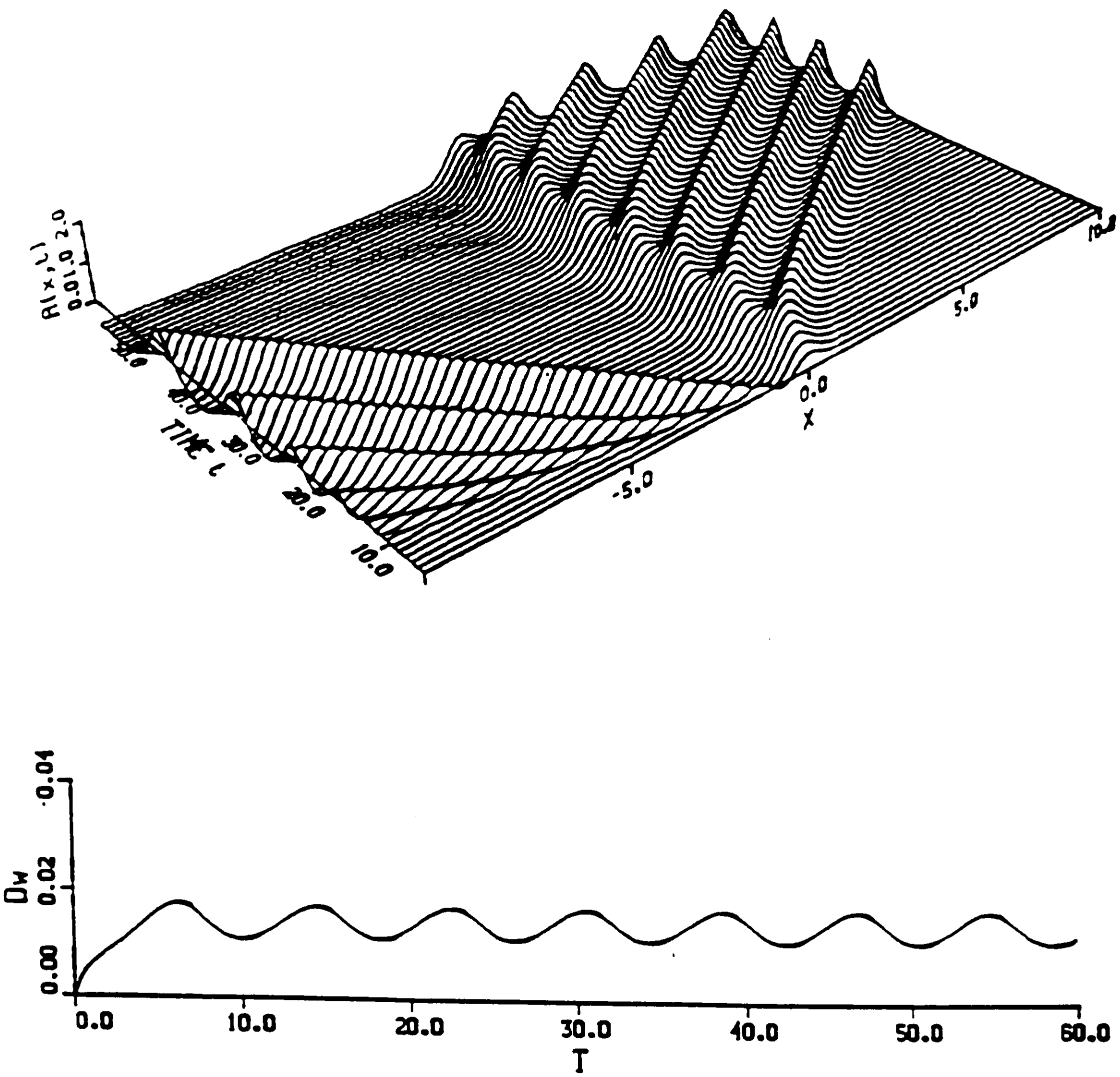


Figure 3.7b The numerical solution of the fKdV equation for the amplitude function $A(X,T)$ and the wave resistance D_w induced by the body r_0 in (3.72) moving along the central axis, with $r_b = 0.1$, $L = 2$ and with the critical speed ($\delta = 0$). The primary flow is the Burgers vortex with $\mu_1 = 5$, $\mu_2 = 12$ and $U_m = 0.6$. (b) the right-going waves.

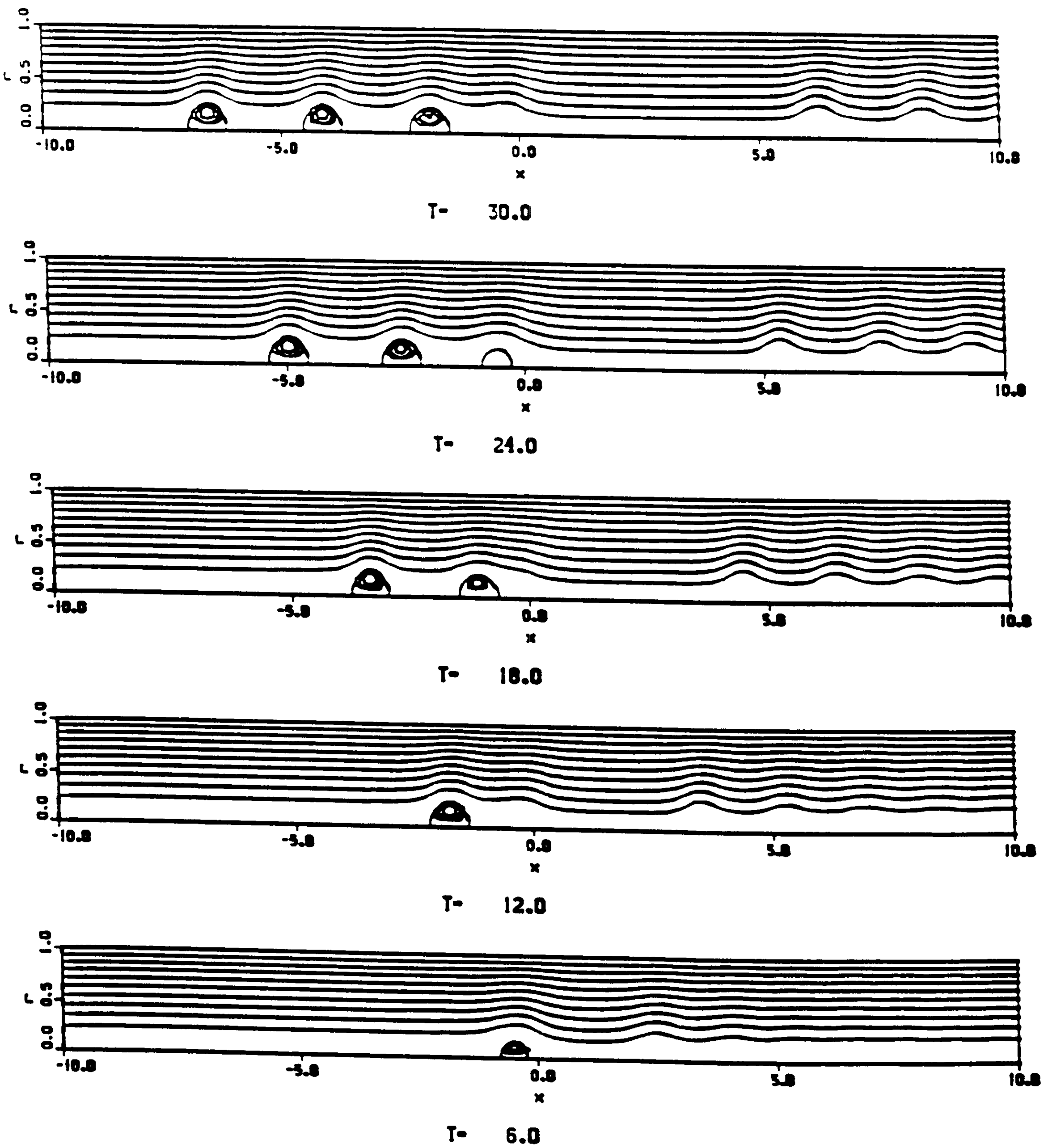


Figure 3.8a The streamlines corresponding to the solution shown in Figure 3.7, for (a) the left-going waves.

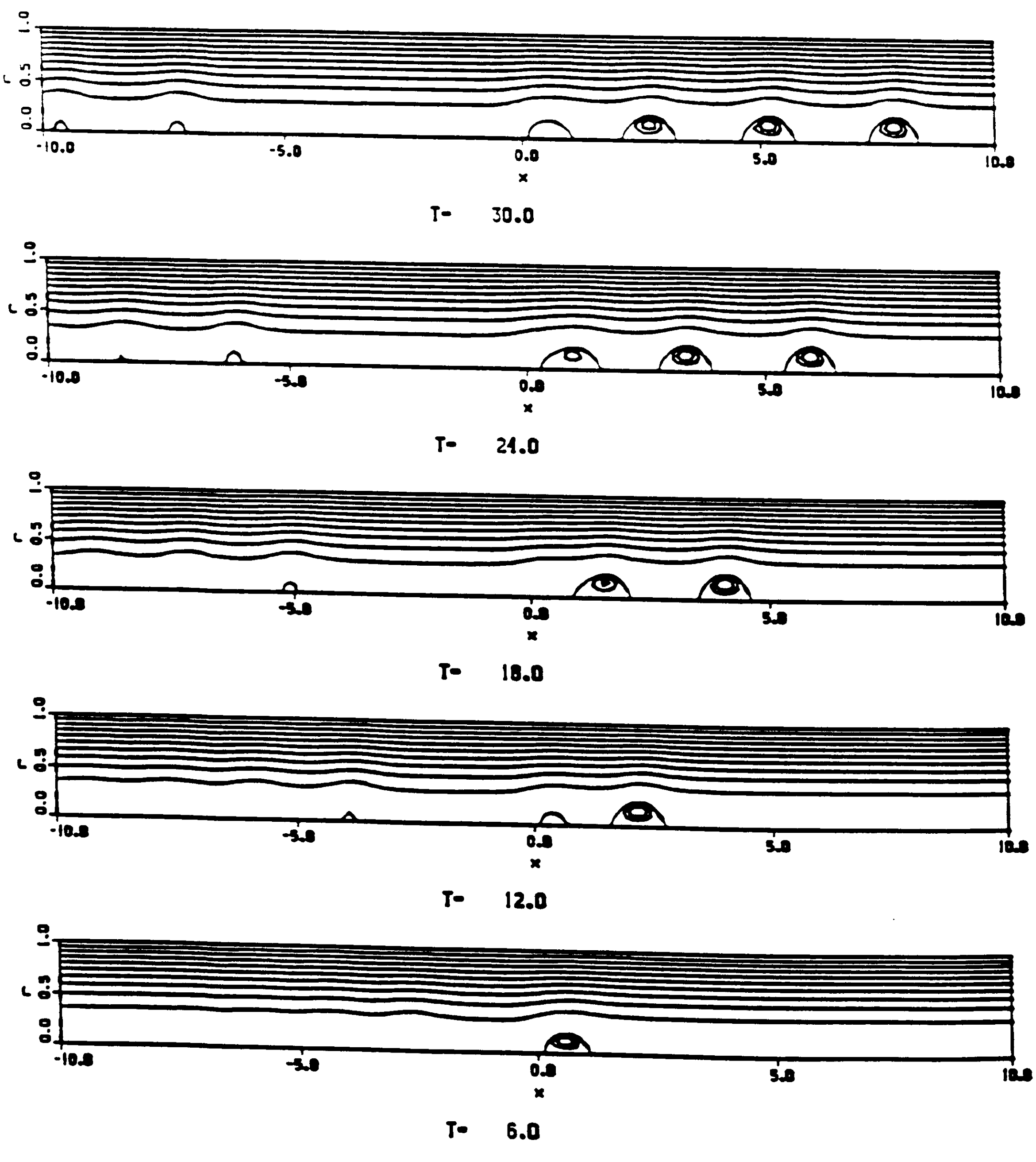


Figure 3.8b The streamlines corresponding to the solution shown in Figure 3.7, for (b) the right-going waves.

Chapter 4

Linear Stability of Free Soliton

In the preceeding chapter, we found the free axisymmetric solitary wave in a rotating fluid confined in a long cylindrical tube. When an axisymmetric flow, a superposition of the cylindrical basic flow and the axisymmetric solitary wave, is slightly perturbed by non-axisymmetric disturbances, the stability characteristics are examined by using the model equation to investigate the role of non-axisymmetric flow features to axisymmetric flows.

4.1 Mathematical formulation

For the primary flow field, we consider a cylindrical flow inside a tube of radius R, with the velocity distribution

$$\mathbf{U}_0(r) = (U_0, 0, W_0(r)) \tag{4.1}$$

where U_0 is the axial velocity component, here taken to be uniform, and $W_0(r)$ is the azimuthal velocity component, with a shear in r, as in the preceeding chapters. When this axisymmetric basic flow is subject to arbitrary small three-dimensional perturbations, the total velocity field can be written as

$$\mathbf{U}(x, r, \theta, t) = \mathbf{U}_0(r) + \mathbf{u}(x, r, \theta, t), \tag{4.2}$$

where $\mathbf{u} = (u, v, w)$ is the perturbation velocity and a similar expression for the pressure p is assumed. For an inviscid and incompressible fluid, $\mathbf{u}$ and p satisfy the Euler equation and the continuity equation (2.2):

$$Lu + \frac{1}{\rho}p_x = -Q, \tag{4.3a}$$

$$Lv - \frac{2W_0}{r}w + \frac{1}{\rho}p_r = -R, \tag{4.3b}$$

$$Lw + (D_* W_0)v + \frac{1}{\rho r}p_\theta = -S, \tag{4.3c}$$

$$u_x + \frac{1}{r}(rv)_r + \frac{1}{r}w_\theta = 0, \tag{4.3d}$$

where

$$L = \frac{\partial}{\partial t} + U_0\frac{\partial}{\partial x} + \frac{W_0}{r}\frac{\partial}{\partial \theta}, \tag{4.4a}$$

$$D = \frac{\partial}{\partial r}, \qquad D_* = \frac{\partial}{\partial r} + \frac{1}{r}. \tag{4.4b}$$

All the terms on the left-hand sides of equations (4.3a-d), linear in (u, v, w, p), represent the leading order of the perturbation equations of motion and the nonlinear terms Q, R and S on the right-hand sides, given by

$$Q = uu_x + vu_r + \frac{w}{r}u_\theta, \tag{4.5a}$$

$$R = uv_x + vv_r + \frac{w}{r}v_\theta - \frac{w^2}{r}, \tag{4.5b}$$

$$S = uw_x + vw_r + \frac{w}{r}w_\theta + \frac{vw}{r}, \tag{4.5c}$$

represent the nonlinear effects, or alternatively regarded as the corresponding source terms. In this chapter, we shall focus our study on the stabilities (or instabilities) of free solitary waves being perturbed by small perturbations which may not be axisymmetric; no forcing is applied to the flow field and all the wave motions are free.

We follow the approach that the perturbation velocity $\mathbf{u}$ can be expanded in the following Fourier series in θ,

$$\mathbf{u} = \mathbf{u}_0(x,r,t) + \sum_{n=1}^{\infty}(\mathbf{u}_n(x,r,t)e^{in\theta} + C.C.), \tag{4.6}$$

where $C.C.$ denotes the complex conjugate. In (4.6), the axisymmetric component $\mathbf{u}_0$ is real but the non-axisymmetric component $\mathbf{u}_n$ is generally complex. Also, the same expression for the pressure p is assumed.

With the dimensionless variables scaled by the tube radius R and the maximum swirl speed W_m as before, substituting (4.6) into (4.3) yields an infinite number of coupled nonlinear equations for u_n, v_n, w_n and p_n $(n = 0, 1, 2, \cdots, \infty)$ as

$$L_n u_n + p_{nx} = -Q_n, \tag{4.7a}$$

$$L_n v_n - \frac{2W_0}{r} w_n + p_{nr} = -R_n, \tag{4.7b}$$

$$L_n w_n + (D_* W_0) v_n + \frac{in}{r} p_n = -S_n, \tag{4.7c}$$

$$u_{nx} + \frac{1}{r}(r v_n)_r + \frac{in}{r} w_n = 0, \tag{4.7d}$$

where

$$L_n = \frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x} + \frac{inW_0}{r}, \tag{4.8}$$

and Q_n, R_n and S_n on the right-hand sides of (4.7) are the terms proportional to $e^{in\theta}$ of Q, R and S resulting form the nonlinear interaction of different modes. For an example, Q_n for $n = 0$ and 1 can be written as

$$Q_0 = Q_{00} + Q_{01} + \cdots, \tag{4.9a}$$

$$Q_1 = Q_{10} + Q_{12} + \cdots, \tag{4.9b}$$

where Q_{0m} is the term from the cross product of $\mathbf{u}_m$ and it's $C.C.$, $\mathbf{u}_m^*$; Q_{1m} is the term from the cross product of $\mathbf{u}_m^*$ and $\mathbf{u}_{m+1}$; etc. And R_n and S_n are assumed to have the similar expressions. For future references, some of Q_{nm}, R_{nm} and S_{nm} are given in the followings

$$Q_{00} = u_0 u_{0x} + v_0 u_{0r}, \tag{4.10a}$$

$$R_{00} = u_0 v_{0x} + v_0 v_{0r} - \frac{w_0^2}{r}, \tag{4.10b}$$

$$S_{00} = u_0 w_{0x} + v_0 w_{0r} + \frac{v_0 w_0}{r}, \tag{4.10c}$$

$$Q_{01} = u_1 u_{1x}^* + v_1 u_{1r}^* - \frac{i}{r} w_1 u_1^* + C.C, \tag{4.11a}$$

$$R_{01} = u_1 v_{1x}^* + v_1 v_{1r}^* - \frac{i}{r} w_1 v_1^* - \frac{1}{r} w_1 w_1^* + C.C, \tag{4.11b}$$

$$S_{01} = u_1 w_{1x}^* + v_1 w_{1r}^* + \frac{1}{r} v_1 w_1^* + C.C, \tag{4.11c}$$

$$Q_{10} = u_0 u_{1x} + u_{0x} u_1 + v_0 u_{1r} + u_{0r} v_1 + \frac{i}{r} w_0 u_1, \tag{4.12a}$$

$$R_{10} = u_0 v_{1x} + v_{0x} u_1 + v_0 v_{1r} + v_{0r} v_1 + \frac{i}{r} w_0 v_1 - \frac{2}{r} w_0 w_1, \tag{4.12b}$$

$$S_{10} = u_0 w_{1x} + w_{0x} u_1 + v_0 w_{1r} + w_{0r} v_1 + \frac{i}{r} w_0 w_1 + \frac{1}{r}(v_0 w_1 + w_0 v_1). \tag{4.12c}$$

The boundary conditions on the solutions to (4.7) at the tube axis ($r = 0$) and at the tube wall ($r = 1$) are given by (Leibovich et al., 1986)

$$u_n(x, 0, t) = 0 \qquad (n \neq 0), \tag{4.13a}$$

$$Du_n(x, 0, t) = 0 \qquad (n = 0), \tag{4.13b}$$

$$v_n(x,0,t) = w_n(x,0,t) = 0 \qquad (|n| \neq 1), \tag{4.13c}$$

$$Dv_n(x,0,t) = Dw_n(x,0,t) = 0 \qquad (|n| = 1), \tag{4.13d}$$

$$v_n(x,1,t) = 0 \qquad (\text{for} \quad \text{all} \quad n), \tag{4.13e}$$

$$u_n(x,1,t) \quad \text{and} \quad w_n(x,1,t) \quad \text{are} \quad \text{finite} \quad (\text{for} \quad \text{all} \quad n). \tag{4.13f}$$

Although our formulation up to this point is exact, the governing equations (4.7) is very difficult to solve since an infinite number of modes are coupled in a very complicated way. To carry out the analysis further theoretically, we will introduce some assumptions appropriate for our current interests. First of all, the most important assumption is that the major perturbations are axisymmetric, i.e., the axisymmeric mode dominates any non-axisymmetric mode. Therefore, to the leading order approximation, we assume $(u_n, v_n, w_n, p_n) = 0$ for all non-zero integers. Also, assuming all the physical variables to be slowly-varying so that the long wave approximations are applicable, we can obtain the axisymmetric soliton solution in the leading order approximation which has been already found in §3.1.1. After axisymmetric solitary wave is generated, the secondary perturbations, which are non-axisymmetric, will be studied in the next section.

4.2 Model equations

In this section, the model equation for studying the linear stability of axisymmetric solitary waves subject to non-axisymmetric perturbations will be derived. In this regard, the primary flow will consist of the cylindrical flow (4.1) and an axisymmetric solitary wave. We choose a reference frame in which the solitary wave is stationary, in other words, the uniform stream of velocity U_0 is introduced just to cancel the propagation speed of the solitary wave relative to the fluid. Since the primary velocity vector depends on (x, r) in the presence of the solitary wave, it is no longer the classical hydrodynamic stability problem of parallel flows. To deal with this typical example of non-parallel flows and the nonlinear interactions of various modes, the model equations will be first derived in this section.

In place of (4.1), the primary velocity vector in this section is assumed to be

$$\mathbf{U}(x,r) = \mathbf{U}_0(r) + \mathbf{U}_s(x,r), \tag{4.14}$$

where $\mathbf{U}_0(r)$ is given in (4.1) and $\mathbf{U}_s = (u_s, v_s, w_s)$ represents the velocity components correponding to steady axisymmetric solitary wave characterized by one parameter, $\epsilon_s = R/\lambda$ where λ is the characteristic wave length of axisymmetric wave, which is assumed to be small but finite. Note that the smallness parameter ϵ defined in (3.7) is equal to ϵ_s^2 in this chapter.

To incorporate asymmetric disturbances with the first-order solitary wave solution found in §3.1.1 using the stream function ψ and the circulation Γ, let us briefly review the procedure of finding the solution with $(\mathbf{u}, p)$ so as to be consistent with the current formulation. The steady axisymmetric solitary wave solutions (u_s, v_s, w_s) and p_s can be expanded in terms of ϵ_s as

$$f_s = (u_s, w_s, p_s), \tag{4.15a}$$

$$f_s(r,\xi) = \epsilon_s^2[f_{s1}(r,\xi) + \epsilon_s^2 f_{s2}(r,\xi) + O(\epsilon_s^4)], \tag{4.15b}$$

$$v_s(r,\xi) = \epsilon_s^3[v_{s1}(r,\xi) + \epsilon_s^2 v_{s2}(r,\xi) + O(\epsilon_s^4)], \tag{4.15c}$$

and the uniform axial velocity U_0 can be expanded as follows, for solitary wave to be stationary in this frame,

$$U_0 = \beta_0 + \epsilon_s^2\beta_1 + O(\epsilon_s^4), \tag{4.15d}$$

where $\xi = \epsilon_s x$. The details of the procedure to find the leading order solution can be found in Appendix A by neglecting all the terms associated with the effects of the bending waves (the terms propotional to $|B|^2$). In (4.15d), β_0 is the phase velocity of axisymmetric linear long wave and, from (A.13b), β_1 is given by

$$\beta_1 = -4c_2, \tag{4.16}$$

where c_2, a coefficient of the KdV equation, is given by (A.12). From (A.4) and (A.13a), the leading order terms in (4.15) are found to be

$$(u_{s1}, w_{s1}, p_{s1}) = \big(\hat{u}_s(r), \hat{w}_s(r), \hat{p}_s(r)\big)\mathcal{A}_{s1}(\xi), \quad v_{s1} = \hat{v}_s(r)\mathcal{A}_{s1\xi}(\xi), \tag{4.17a}$$

$$\mathcal{A}_{s1}(\xi) = 12\left(\frac{c_2}{c_1}\right) \cdot \mathrm{sech}^2(\xi), \tag{4.17b}$$

where c_1, c_2 are given in (A.12) and $\mathcal{A}_{s1}$, the amplitude function of the radial velocity v_{s1}, is equal to $-A_1$, the amplitude function of the stream function defined in (3.13). It will be noted that we need not continue to search for the higher order solutions, say f_{sm} for $m \geq 2$, since the first-order solution obtained in chapter 3 or Appendix A turns out to be sufficient for the present analysis. In fact, the exact numerical solution to the Euler equation for axisymmetric solitary wave by Leibovich and Kribus (1990) shows that the flow field of the first-order solution (4.17) gives a good qualitative agreement with the exact numerical solution and the differences are less than 10 percent even for large ϵ_s. Therefore, (4.17) can be used for reasonable approximations for small to finite values of ϵ_s.

In the following analysis for the secondary perturbations, the $|n| = 1$ (bending) modes will be selected for a focused study for the following two reasons. First, for the bending modes, the explicit solutions have been found at each order and the analysis is much simpler than any other modes. For the other reason, which is more important, the bending modes are the only modes that have a non-zero radial velocity component on the axis of tube that gives the spiral type of streaklines, originally released at the axis, commonly observed in real experiments. Therefore, we assume that the first two modes ($n = 0$ and $|n| = 1$) can be the minimal representation of all salient features of real flows. The analysis for other modes is parallel to that for $|n| = 1$ but the non-homogeneous ordinary differential equations arising in the analysis will have to be solved numerically.

Assuming small perturbations to the axisymmetric solitary wave, we can linearize the Euler equation (4.7) with respect to perturbation velocity as

$$L_n u_n + p_{nx} = -Q_{ns}, \tag{4.18a}$$

$$L_n v_n - \frac{2W_0}{r} w_n + p_{nr} = -R_{ns}, \tag{4.18b}$$

$$L_n w_n + (D_* W_0) v_n + \frac{in}{r} p_n = -S_{ns}, \tag{4.18c}$$

$$u_{nx} + \frac{1}{r}(r v_n)_r + \frac{in}{r} w_n = 0, \tag{4.18d}$$

where

$$L_n = \frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x} + \frac{inW_0}{r}, \tag{4.19}$$

and Q_{ns}, R_{ns} and S_{ns} represent the effects of the solitary wave field $\mathbf{U}_s(x, r)$,

$$Q_{ns} = u_s u_{nx} + u_{sx} u_n + v_s u_{nr} + u_{sr} v_n + \frac{in}{r} w_s u_n, \tag{4.20a}$$

$$R_{ns} = u_s v_{nx} + v_{sx} u_n + v_s v_{nr} + v_{sr} v_n + \frac{in}{r} w_s v_n - \frac{2}{r} w_s w_n, \tag{4.20b}$$

$$S_{ns} = u_s w_{nx} + w_{sx} u_n + v_s w_{nr} + w_{sr} v_n + \frac{in}{r} w_s w_n + \frac{1}{r}(v_s w_n + w_s v_n), \tag{4.20c}$$

where (u_s, v_s, w_s) are the velocity components for the solitary wave given by (4.15). When we consider the secondary perturbations for $n = 0$, we are expected to encounter the linearized KdV equation with respect to the steady solitary wave solution (4.17b) for which only a neutral mode exists (Jefferey and Kakutani, 1970). For the non-axisymmetric disturbances ($n \neq 0$), we also assume that the characteristic wave length is greater than the tube radius R. So, for non-axisymmetric long waves, two dimensional motions in (r,θ)-plane are dominant in the leading order approximation. But this is impossible for axisymmetric perturbations since any expansion or contraction of stream tube must be accompanied by an axial flow.

For linear waves of small wave number k (as $k \to 0$) for the n-th mode, we have the following dispersion relation for the wave frequencies $\bar{\omega}$ (see appendix B)

$$\bar{\omega} = \omega_n + \gamma_0 k + \gamma_1 k^2 + O(k^4), \tag{4.21}$$

where the first term ω_n which arising from the tube boundary effects gives the frequency of two-dimensional perturbations and γ_j are constants to be determined. By making the following substitutions (Whitham, 1974) of a *correspondence rule*,

$$(\bar{\omega} - \omega_n) \to -i\frac{\partial}{\partial t}, \qquad k \to i\frac{\partial}{\partial x}, \tag{4.22}$$

an evolution equation for the amplitude function $\mathcal{A}_n(x,t)\exp(-i\omega_n t)$ of the n-th mode can be expected to assume the form

$$\mathcal{A}_{nt} + \gamma_0 \mathcal{A}_{nx} + i\gamma_1 \mathcal{A}_{nxx} + \mathcal{F}(\mathcal{A}_n, \mathcal{A}_s) = 0, \tag{4.23}$$

where $\mathcal{F}(\mathcal{A}_n, \mathcal{A}_s)$ is a certain functional representing the effect of the primary solitary wave which is to be determined. Although the wave number k need not be of the same order in magnitude as ϵ_s, $k = O(\epsilon_s)$ is assumed for definiteness in the following analysis.

Since the axial velocity can be assumed to be small such that the dominant perturbation is two-dimensional, the perturbation velocity components for non-axisymmetric disturbances can be expanded as

$$f_n = (v_n, w_n, p_n), \tag{4.24a}$$

$$f_n(r,t,\xi,\tau_m) = \alpha_n \left[f_{n1} + \epsilon_s f_{n2} + \epsilon_s^2 f_{n3} + \epsilon_s^3 f_{n4} + O(\epsilon_s^4) \right], \tag{4.24b}$$

$$u_n(r,t,\xi,\tau_m) = \alpha_n \epsilon_s \left[u_{n1} + \epsilon_s u_{n2} + \epsilon_s^2 u_{n3} + \epsilon_s^3 u_{n4} + O(\epsilon_s^4) \right], \tag{4.24c}$$

$$\tau_m = \epsilon_s^m t \tag{4.24d}$$

where α_n is the amplitude of the n-th mode disturbance assumed to be $\alpha_n < O(\epsilon_s^2)$ is assumed. In (4.24b,c) the first subscript of all variables indicates the different mode and the second gives the order of magnitude. By substituting (4.24) into (4.20), the coupled terms, (Q_{ns}, R_{ns}, S_{ns}), in the right-hand sides of (4.18) can be also expanded as

$$(Q_{ns}, R_{ns}, S_{ns}) = \alpha_n [\epsilon_s^2 (Q_{ns}^1, R_{ns}^1, S_{ns}^1) + \epsilon_s^3 (Q_{ns}^2, R_{ns}^2, S_{ns}^2) + O(\epsilon_s^4)], \tag{4.25}$$

where

$$Q_{ns}^1 = u_{s1r} v_{n1}, \tag{4.26a}$$

$$R_{ns}^1 = \frac{in}{r} w_{s1} v_{n1} - \frac{2}{r} w_{s1} w_{n1}, \tag{4.26b}$$

$$S_{ns}^1 = (w_{s1r} + \frac{1}{r} w_{s1}) v_{n1} + \frac{in}{r} w_{s1} w_{n1}, \tag{4.26c}$$

and

$$Q_{ns}^2 = [\frac{in}{r} w_{s1} u_{n1}] + [u_{s1r} v_{n2}], \tag{4.27a}$$

$$R_{ns}^2 = [u_{s1} v_{n1x} + v_{s1} v_{n1r} + v_{s1r} v_{n1}] + [\frac{in}{r} w_{s1} v_{n2} - \frac{2}{r} w_{s1} w_{n2}] \tag{4.27b}$$

$$S_{ns}^2 = [u_{s1} w_{n1x} + v_{s1} w_{n1r} + \frac{1}{r} v_{s1} w_{n1}] + [w_{s1r} v_{n2} + \frac{in}{r} w_{s1} w_{n2} + \frac{1}{r} w_{s1} v_{n2}]. \tag{4.27c}$$

By writing

$$f_{nj} = \bar{f}_{nj}(r, \xi, \tau_m)\mathrm{e}^{-in\omega_n t}, \qquad f_{nj} = (u_{nj}, v_{nj}, w_{nj}, p_{nj}), \tag{4.28}$$

to eliminate the fast time proportional scaled by $1/\omega_n$ in (4.21) and substituting (4.24) into (4.18), we have the first-order equations at $O(\alpha_n)$ (after dropping the bar)

$$-in(\omega_n - \Omega)u_{n1} + p_{n1\xi} = 0, \tag{4.29a}$$

$$-in(\omega_n - \Omega)v_{n1} - \frac{2W_0}{r}w_{n1} + p_{n1r} = 0, \tag{4.29b}$$

$$-in(\omega_n - \Omega)w_{n1} + (D_* W_0)v_{n1} + \frac{in}{r}p_{n1} = 0, \tag{4.29c}$$

$$D_* v_{n1} + \frac{in}{r}w_{n1} = 0, \tag{4.29d}$$

where $\Omega(r) = W_0(r)/r$. Taking the first-order solutions as

$$(v_{n1}, w_{n1}, p_{n1}) = \big(\hat{v}_{n1}(r), (in)\hat{w}_{n1}(r), (in)\hat{p}_{n1}(r)\big)\mathcal{A}_n(\xi, \tau_m), \tag{4.30a}$$

$$u_{n1} = \hat{u}_{n1}(r)\mathcal{A}_{n\xi}(\xi, \tau_m), \tag{4.30b}$$

and substituting (4.30) into (4.29b-d), we have for $\hat{v}_{n1}(r) \equiv \phi_{n1}(r)$ the equation

$$\mathcal{L}_n \phi_{n1} = 0, \tag{4.31a}$$

where

$$\mathcal{L}_n \equiv r^2\frac{d}{dr^2} + 3r\frac{d}{dr} - (n^2 - 1) + q_n(r), \quad q_n(r) = \frac{rD(D_* W_0)}{\omega_n - \Omega}. \tag{4.31b}$$

From (4.13), the boundary conditions on the solution to (4.31) can be written as

$$\phi'_{n1}(0) = 0, \qquad \phi_{n1}(1) = 0 \quad \text{for} \quad n = 1, \tag{4.32a}$$

$$\phi_{n1}(0) = 0, \qquad \phi_{n1}(1) = 0 \quad \text{for} \quad n \neq 1. \tag{4.32b}$$

By solving the eigenvalue problem (4.31) with the boundary conditions (4.32), we can determine the frequency ω_n of the two dimensional disturbances. For $|n| = 1$, the explicit solution to (4.31) can be found as

$$\phi_{11} = \Omega(r) - \omega_1, \qquad \omega_1 = \Omega(1). \tag{4.33}$$

For simplicity of further analysis, we impose $\Omega(1) = 0$ to the primary flow and, then, $\omega_1 = 0$.

After having solved the eigenvalue problem (4.31) with the homogeneous boundary conditions (4.32), we can find from (4.29) the first-order solutions in (4.30) as

$$\hat{v}_{n1}(r) = \phi_{n1}, \qquad \hat{u}_{n1}(r) = \frac{r}{n^2}\left[r(D_*\phi_{n1}) + \frac{(D_*W_0)}{(\omega_n - \Omega)}\phi_{n1}\right], \tag{4.34a, b}$$

$$\hat{w}_{n1}(r) = \frac{r}{n^2}(D_*\phi_{n1}), \quad \hat{p}_{n1}(r) = \frac{r}{n^2}\left[r(\omega_n - \Omega)(D_*\phi_{n1}) + (D_*W_0)\phi_{n1}\right]. \tag{4.34c, d}$$

The evolution equation for $\mathcal{A}_n(\xi, \tau_m)$ can be obtained successively in the higher order equations by imposing the solvability conditions as will be shown below. All of the higher order equations obtained by substituting (4.24) into (4.18) have the following forms

$$\mathcal{L}_{nx}(u_{nm}, p_{nm}) \equiv -in(\omega_n - \Omega)u_{nm} + p_{nm\xi} = -F^m_{nx}, \tag{4.35a}$$

$$\mathcal{L}_{nr}(v_{nm}, w_{nm}, p_{nm}) \equiv -in(\omega_n - \Omega)v_{nm} - \frac{2W_0}{r}w_{nm} + p_{nmr} = -F^m_{nr}, \tag{4.35b}$$

$$\mathcal{L}_{n\theta}(v_{nm}, w_{nm}, p_{nm}) \equiv -in(\omega_n - \Omega)w_{nm} + (D_*W_0)v_{nm} + \frac{in}{r}p_{nm} = -F^m_{n\theta}, \tag{4.35c}$$

$$\mathcal{L}_{nc}(v_{nm}, w_{nm}) \equiv D_*v_{nm} + \frac{in}{r}w_{nm} = -F^m_{nc}. \tag{4.35d}$$

Then, by eliminating u_{nm}, w_{nm}, p_{nm} from (4.35b-d), we have the equations for v_{nm} as

$$\mathcal{L}_n v_{nm} = \frac{1}{(\omega_n - \Omega)}\Big[inF^m_{nr} - D(rF^m_{n\theta}) + 2r\Omega F^m_{nc} - D\big(r^2(\omega_n - \Omega)F^m_{nc}\big)\Big] \equiv F^m_n, \tag{4.36}$$

where $\mathcal{L}_n$ is defined in (4.31). Since the linear operator $\mathcal{L}_n$ of the left-hand side of (4.36) is the same as that of the homogeneous equation (4.31), for particular solutions of (4.36) to exist, the right-hand side of (4.36) must be othogonal to the homogeneous solution of (4.36). This orthogonality gives the solvability condition for the non-homogenous equation (4.36) as

$$\int_0^1 r\phi_{n1} F^m_n dr = 0. \tag{4.37}$$

Since the higher order expansions to obtain the model equation is lengthy, the details of the procedure are presented in Appendix C.

In what follows, only disturbances for the bending mode which has non-zero radial velocity will be considered for the reasons mentioned earlier. From (C.19), we find the evolution equation for the amplitude function of bending waves $\mathcal{A}_1 \equiv B$ as

$$B_\tau + \gamma_0 B_\xi + i\epsilon_s\gamma_1 B_{\xi\xi} + \epsilon_s^2\gamma_2\left(\mathcal{A}_{s1}(\xi)B\right)_\xi = 0, \tag{4.38}$$

where $\gamma_0 = U_s = \beta_0 + \epsilon_s^2\beta_1$ is the propagation speed of axisymmetric solitary wave and γ_1, γ_2 are given in (C.9), (C.17), respectively. In (4.38), the $B_{\xi\xi}$-term represents the dispersive effects (see the dispersion relation (4.21)) and the last term of the left-hand side gives the effect of interaction with the primary solitary wave. Also $B^*(\xi,\tau)$, the complex conjugate of $B(\xi,\tau)$, satisfies the conjugate equation of (4.38) as

$$B^*_\tau + \gamma_0 B^*_\xi - i\epsilon_s\gamma_1 B^*_{\xi\xi} + \epsilon_s^2\gamma_2\left(\mathcal{A}_{s1}(\xi)B^*\right)_\xi = 0. \tag{4.39}$$

When the original variables (x, t) is recovered, (4.38) can be written as

$$B_t + c(x)B_x + i\gamma_1 B_{xx} + \gamma_2 \mathcal{A}_s'(x)B = 0, \tag{4.40}$$

where $c(x)$ given by

$$c(x) = \gamma_0 + \gamma_2 \mathcal{A}_s(x), \tag{4.41}$$

is the local wave velocity in the presence of the solitary wave and the last term of the left-hand side in (4.40) represent the effect of interaction with the axisymmetric solitary wave which is given by

$$\mathcal{A}_s(x) = \epsilon_s^2 \mathcal{A}_{s1}(x) = 12\epsilon_s^2 \left(\frac{c_2}{c_1}\right) \cdot \operatorname{sech}^2(\xi), \tag{4.42}$$

(see (4.17b)). To study (4.40), the boundary conditions at both infinities are required. Assuming that there are no disturbances at far upstream and downstream sides, we invoke that $B \to 0$ and $B_x \to 0$ as $x \to \pm\infty$.

4.3 Linear Stability Analysis

In this section, the evolution equation (4.40) derived in the preceeding section for the amplitude function B of the bending modes will be considered for analyzing the stability of the primary flow. When the axisymmetric solitary wave is slightly perturbed by bending waves, the primary flow (4.14), a superposition of the basic swirl velocity and the axisymmetric solitary wave, is said to be unstable if the amplitude of the bending waves increases in time. Otherwise, the flow is regarded as linearly stable except when the result indicates a neutral stability on linear analysis, in which case no conclusion can be made on linear theory. The instability, if exists, is rendered possibly by the presence of the primary axisymmetric solitary wave since the cylindrical flow itself is stable with respect to the long wave perturbations in (4.28).

By use of the transformation

$$B(x,t) = G(x,t)H(x,t), \qquad G(x,t) = \exp\left[\frac{i}{2\gamma_1}\int(\gamma_0 + \gamma_2 \mathcal{A}_s)dx - \frac{i\gamma_0^2}{4\gamma_1}t\right], \tag{4.43}$$

equation (4.40) can be further simplified as

$$H_t + i\gamma_1 H_{xx} + iV(x)H = 0, \tag{4.44}$$

where $V(x)$ is given by

$$V(x) = \frac{\gamma_0\gamma_2}{2\gamma_1}\mathcal{A}_s - i\frac{\gamma_2}{2}\mathcal{A}_{sx} + \frac{\gamma_2^2}{4\gamma_1}\mathcal{A}_s^2, \tag{4.45}$$

and $\mathcal{A}_s$ is given by (4.42). Mutiplying equation (4.44) and its conjugate equation by H^* and H, respectively, and integrating the sum of the resulting two equations, we have the conservation law

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}|H|^2 dx = -\gamma_2\int_{-\infty}^{\infty}\mathcal{A}_{sx}|H|^2 dx, \tag{4.46}$$

where the following boundary conditions are used at both infinities

$$H, \quad H_x \to 0 \qquad \text{as} \quad |x| \to \infty. \tag{4.47}$$

Hereafter we will consider the transformed equation (4.44) for the stability analysis. Note that equation (4.44) becomes the Schrodinger equation when the *potential* $V(x)$ is purely real and no unstable solution exists. In the present case, however, the potential $V(x)$ in (4.44) is a complex function and it is of interest to ascertain the criteria for instability of B. For a complex function $V(x)$, equation (4.44) becomes the fourth-order differential equation in space with all real variables.

With $H(x,t) = a(x,t)\exp(iS(x,t))$, seperating the real and imaginary parts of (4.44) gives

$$a_t - \gamma_1 S_{xx} a - 2\gamma_1 S_x a_x - (\Im V)a = 0, \tag{4.48a}$$

$$S_t - \gamma_1 S_x^2 + \gamma_1 \frac{a_{xx}}{a} + (\Re V) = 0. \tag{4.48b}$$

Multiplying (4.48a) by $2a$, we have for the amplitude a the equation

$$\frac{\partial a^2}{\partial t} + \frac{\partial}{\partial x}\big[-2\gamma_1 S_x a^2\big] = 2(\Im V)a^2. \tag{4.49}$$

For the regular Schrodinger equation, the amplitude does not grow in time but here the amplitude of function $H(x,t)$ in (4.44) might grow in time, since $V(x)$ in (4.44) has an imaginary component, as can be seen from (4.46) or (4.49).

By using the slow variables to deal with the slowly-varying potential (4.45)

$$\xi = \epsilon_s x, \qquad T = \epsilon_s^2 \gamma_1 t, \tag{4.50}$$

(4.44) becomes

$$H_T + iH_{\xi\xi} + i\bar{V}(\xi)H = 0, \tag{4.51}$$

where, upon noting $\mathcal{A}_s(\xi) = \epsilon_s^2 \mathcal{A}_{s1}$, the potential can be written as

$$\bar{V}(\xi) = \frac{\bar{\gamma}_0\bar{\gamma}_2}{2}\mathcal{A}_{s1} - i\epsilon_s\frac{\bar{\gamma}_2}{2}\mathcal{A}_{s1\xi} + \epsilon_s^2\frac{\bar{\gamma}_2^2}{4}\mathcal{A}_{s1}^2, \tag{4.52a}$$

$$\bar{\gamma}_j = \frac{\gamma_j}{\gamma_1}, \quad j = 0, 2 \tag{4.52b}$$

$$\mathcal{A}_{s1}(\xi) = 12\left(\frac{c_2}{c_1}\right)\cdot \mathrm{sech}^2(\xi). \qquad (4.52c)$$

To solve the evolution equation (4.51), we assume a separable solution as

$$H(\xi, T) = \mathrm{e}^{\sigma T}\chi(\xi), \qquad (4.53)$$

where σ is a complex number whose real part provides the growth rate of H and $\chi(\xi)$ is also complex. Then, from (4.51), we have for $\chi(\xi)$ the equation

$$\sigma\chi + i\chi'' + i\bar{V}(\xi)\chi = 0, \qquad (4.54)$$

where $\bar{V}$ is given by (4.52). By (4.47) and (4.53), $\chi(\xi)$ satisfies the boundary conditions:

$$\chi, \quad \chi_\xi \to 0 \qquad \text{as} \quad |\xi| \to \infty. \qquad (4.55)$$

From (4.46), we have the expression for the growth rate, σ_r, as

$$\sigma_r = -\epsilon_s \frac{\gamma_2}{2} \frac{\int_{-\infty}^{\infty} \mathcal{A}'_{s1}|\chi|^2 d\xi}{\int_{-\infty}^{\infty} |\chi|^2 d\xi}. \qquad (4.56)$$

From (4.56), it is obvious that the eigenfunction χ must have both even and odd components for a non-zero real part of σ to exist since $\mathcal{A}'_{s1}$ is purely odd in ξ. From (4.54), it follows that, if $\chi(\xi)$ is an eigenfunction and σ is the corresponding eigenvalue, then $\chi^*(-\xi)$ and $-\sigma^*$ are also an eigenfunction and its eigenvalue of (4.54) since $\Re(\bar{V})$ is purely even and $\Im(\bar{V})$ is purely odd (see (4.52)). Therefore, as soon as the eigenvalue σ has a non-vanishing real part, it implies instability.

Perturbation solutions of (4.54) for small ϵ_s.

Let us first examine the solution behaviour for small ϵ_s. For this case, we may solve the eigenvalue problem (4.54) by the perturbation method. By expanding the eigenvalue σ and the eigenfunction χ in terms of ϵ_s as

$$\sigma = \sigma_0 + \epsilon_s \sigma_1 + \epsilon_s^2 \sigma_2 + O(\epsilon_s^3), \tag{4.57a}$$

$$\chi = \chi_0 + \epsilon_s \chi_1 + \epsilon_s^2 \chi_2 + O(\epsilon_s^3), \tag{4.57b}$$

$$\bar{V} = \bar{V}_0 + \epsilon_s \bar{V}_1 + \epsilon_s^2 \bar{V}_2 + O(\epsilon_s^3), \tag{4.57c}$$

we obatin, at the first order, the equation for χ_0 as

$$\frac{d^2\chi_0}{d\xi^2} + [\lambda_0 + \bar{V}_0(\xi)]\chi_0 = 0, \tag{4.58}$$

where, by (4.52) and (4.54), the eigenvalue λ_0 and the potential $\bar{V}_0(\xi)$ are given by

$$\lambda_0 = -i\sigma_0, \qquad \bar{V}_0(\xi) = \bar{V}_{0M}\text{sech}^2(\xi), \qquad \bar{V}_{0M} = 12\left(\frac{\bar{\beta}_0\bar{\gamma}_2}{2}\right)\left(\frac{c_2}{c_1}\right). \tag{4.59}$$

Then, (4.58) is the regular Schrodinger equation in quantum mechanics. It is well-known that σ_0 in (4.58) is purely imaginary (λ_0 purely real) subject to the given boundary conditions at both ends. The real eigenvalue λ_0 has a discrete spectrum for $-\bar{V}_{0M} < \lambda_0 < 0$, and a continuous spectrum for $\lambda_0 > 0$, and the solution becomes unbounded for $\lambda_0 < -\bar{V}_{0M}$. Since the first-order potential $\bar{V}_0(\xi)$ is an even function, the eigenfunction must be purely even or purely odd. Only the eigenfunctions corresponding to discrete eigenvalues are of interest since the specified boundary conditions can be satisfied only for that case. The number of eigenvalues depends on $\bar{V}_{0M}$, the maginute of $\bar{V}_0$. The first few eigenfunctions are given by

$$\chi_0^{(0)} = \text{sech}^{p_0}(\xi), \tag{4.60a}$$

$$\chi_0^{(1)} = \tanh(\xi)\text{sech}^{p_1}(\xi), \tag{4.60b}$$

$$\lambda_0^{(j)} = -p_j^2, \tag{4.60c}$$

where

$$p_j = s - j, \qquad s = \frac{1}{2}\left(-1 + \sqrt{1 + 4\bar{V}_{0M}}\right), \tag{4.61a}$$

$$j = 0, 1, .., \qquad j < s. \tag{4.61b}$$

At the second order, we have, for χ_1, the equation:

$$\chi_1'' + \left[\lambda_0 + \bar{V}_0(\xi)\right]\chi_1 = -\lambda_1\chi_0 - \bar{V}_1(\xi)\chi_0, \tag{4.62}$$

where

$$\lambda_1 = -i\sigma_1, \quad \bar{V}_1(\xi) = -\frac{i\bar{\gamma}_2}{2}\mathcal{A}_{s1\xi} = i\bar{V}_{1M}\tanh(\xi)\text{sech}^2(\xi), \quad \bar{V}_{1M} = 12\bar{\gamma}_2\left(\frac{c_2}{c_1}\right). \tag{4.63}$$

Invoking the solvability condition for (4.62), we can show that $\lambda_1 = 0$ and the particular solutions corresponding to $\chi_0^{(0)}$ and $\chi_0^{(1)}$ are

$$\chi_1^{(0)} = \frac{i\bar{V}_{1M}}{2(1+p_0)}\tanh(\xi)\text{sech}^{p_0}(\xi), \tag{4.64a}$$

$$\chi_1^{(1)} = -\frac{i\bar{V}_{1M}}{2}\text{sech}^{p_1}(\xi)\left[\frac{1}{1+p_1} - \frac{1}{2+p_1}(1 + \tanh^2(\xi))\right]. \tag{4.64b}$$

When the higher order problems are considered, it will be found that σ_m is pure imaginary for m even and zero for m odd. Therefore, for small amplitude ϵ_s, only imaginary eigenvalues exist which means that the flow is neutrally stable on linear theory for small disturbances.

From (4.54) and (4.56), we are left with the question if non-zero σ_r may exist in some domain of pertinent parameters of our problem, namely the amplitude of the primary axisymmetric solitary wave and the basic swirl velocity profile. Since, as mentioned in §4.2, the first-order soliton solution can be a good approximation of the exact solution to the Euler equation for a good range of ϵ_s, the magnitude of ϵ_s can be chosen to have a modest finite value. For such small but finite ϵ_s, the eigenvalue problem can be solved by numerical methods.

Numerical Solutions of (4.54) for finite ϵ_s

The equation to be solved, by (4.54), is

$$\chi'' + \left[\lambda + \bar{V}(\xi)\right]\chi = 0, \tag{4.65a}$$

with the boundary conditions

$$\chi(\pm\infty) = 0, \qquad \chi'(\pm\infty) = 0, \tag{4.65b}$$

where $\lambda = -i\sigma$. Using (4.52) and the transformation

$$\eta = \tanh(\xi), \tag{4.66}$$

(4.54) becomes

$$\chi_{\eta\eta} - \frac{2\eta}{(1-\eta^2)}\chi_\eta + \left[\frac{f_0 + (f_1 + f_2\eta)(1-\eta^2) + f_3(1-\eta^2)^2}{(1-\eta^2)^2}\right]\chi = 0, \tag{4.67a}$$

where

$$f_0 = \lambda, \quad f_1 = \frac{1}{2}\bar{\gamma}_0\bar{\gamma}_s, \quad f_2 = i\epsilon_s\bar{\gamma}_s, \quad f_3 = \frac{1}{4}\epsilon_s^2\bar{\gamma}_s^2, \quad \bar{\gamma}_s = \bar{\gamma}_2\left(12\frac{c_2}{c_1}\right). \tag{4.67b}$$

For simplicity of analysis, we write (4.67) as

$$\chi_{\eta\eta} + \frac{P(\eta)}{(1-\eta^2)}\chi_\eta + \frac{Q(\eta)}{(1-\eta^2)^2}\chi = 0. \tag{4.68}$$

The solutions near the regular singular points at $\eta = \pm 1$ can be determined in indicial form by expanding χ, P and Q about $\eta = -1$ as

$$\chi = (1+\eta)^s[h_0 + h_1(1+\eta) + h_2(1+\eta)^2 + \cdots], \tag{4.69a}$$

$$P = P_0 + P_1(1+\eta) + P_2(1+\eta)^2 + \cdots, \tag{4.69b}$$

$$Q = Q_0 + Q_1(1+\eta) + Q_2(1+\eta)^2 + \cdots. \tag{4.69c}$$

The equation for the index s is found as

$$s^2 + (P_0/2 - 1)s + Q_0/4 = 0. \tag{4.70}$$

In our problem, $P_0 = 2$ and $Q_0 = \lambda$. To satisfy the boundary conditions at $\eta = -1$, the solution with a positive real part, say s_1, is taken. Also, near $\eta = 1$, the same indicial equation holds. To remove the derivative singularity of χ at $|\eta| = 1$, we adopt the following transformation for χ

$$\chi = (1-\eta^2)^{s_1}\varphi, \tag{4.71}$$

where s_1 is the root of (4.70) having a positive real part. Substituting (4.71) into (4.68) gives for φ the equation

$$\varphi_{\eta\eta} + \frac{p(\eta)}{(1-\eta^2)}\varphi_\eta + \frac{q(\eta)}{(1-\eta^2)^2}\varphi = 0. \tag{4.72a}$$

where

$$p(\eta) = P(\eta) - 4s_1\eta, \tag{4.72b}$$

$$q(\eta) = Q(\eta) - 2s_1\eta P(\eta) + 4s_1(s_1-1)\eta^2 - 2s_1(1-\eta^2). \tag{4.72c}$$

Therefore, the solutions at $\eta = -1$ is given by

$$\varphi = h_0(k_0 + k_1(1+\eta) + k_2(1+\eta)^2 + \cdots), \tag{4.73a}$$

and, at $\eta = 1$,

$$\varphi = H_0(K_0 + K_1(1+\eta) + K_2(1+\eta)^2 + \cdots), \tag{4.73b}$$

where $k_0 = 1+i$, $K_0 = 1+i$ as selected and k_i, K_i for $i \geq 1$ are complex constants to be determined. After first finding a few terms in (4.73) by substituting (4.73) into (4.72a) and taking them as initial conditions, equation (4.72a) is numerically integrated from both ends toward the center by applying the fourth-order Runge-Kutta method and, then, match two solutions at the center by use of the matching condition as follows,

$$\varphi_L = \varphi_R, \qquad \varphi_l' = \varphi_R', \qquad \text{at} \quad \eta = 0 \tag{4.74}$$

where φ_L and φ_R are the solutions to (4.72) originated from the left end and the right end, respectively. These two matching conditions will ensure the continuity of the higher order derivatives as is implied by equation (4.65). Further, we note that the continuity of φ and its derivatives implies, by (4.71), that the same must hold for our solution $\chi(\xi)$. After setting $h_0 = 1$ without loss of generality, H_0 can be determined by one of the two conditions in (4.75). In numerical computations, the matching conditions are fulfilled by iterations using the Newton-Rapson method.

To understand the behaviour of solutions to the Schrodinger equation with a complex potential, we consider an example similar to equation (4.54) in Appendix D, in which we find that the imaginary component in the potential must be large enough to yield a non-zero growth rate. This can be understood from the fact that the potential must have, in leading order, a non-vanishing odd component for the eigenfunction to have both even and odd components, as we can see from (4.54) and (4.56).

In this chapter, the basic swirl velocity in (4.1) is chosen as

$$W_0(r) = \frac{\Gamma_0}{r}(1 - \mathrm{e}^{-\alpha r^2}) + \Gamma_1 r^m, \tag{4.75}$$

where the first term on the right-hand side is the Burgers vortex and the second one, with the term r^m, is introduced to satisfy the conditions $W_0(1) = 0$ and $(W_0)_{max} = 1$. The first condition is used for simplicity of the analysis in §4.2, which gives

$$\Gamma_1 = -\Gamma_0 \left(1 - \mathrm{e}^{-\alpha}\right), \tag{4.76}$$

and the second one for making the normalization which gives

$$\Gamma_0 = 1/\left[(1 - \mathrm{e}^{-\alpha r_0^2})/r_0 - (1 - \mathrm{e}^{-\alpha})r_0^m\right], \tag{4.77}$$

where $r = r_0$ is the position of the maximum swirl velocity. For given α (or r_0) and m, all other parameters in the primary swirl velocity in (4.75) are determined by use of (4.76) and (4.77).

The basic swirl velocity profiles of (4.75) for $m = 1$ and $m = 11$ are compared with the experimental data by Faler and Leibovich (1977) as shown in figure 4.1. For $m = 1$, the basic swirl velocity (4.75) becomes the superposition of the Burgers vortex and a rigid body rotation which has been the velocity profile observed inside a tube by Escudier et al. (1982); and for $m = 11$, the velocity profile fits well with the observed profiles of Faler and Leibovich. In the following, most of the results are attained for $m = 1$ or $m = 11$. The coefficients of equations, c_i for $\mathcal{A}_s$ and b_i for B, for varying α are shown in figure 4.2 and figure 4.3. For a specific model, $\alpha = 15$ is taken for highly concentrated vortical flows. The coefficients of the governing equations, (A.11) and (4.40), are given in table 4.1.

m	c_0	c_1	c_2	β_1	γ_1	γ_2	s
1	.9599	-.7826	-.0128	.0513	.1397	.0459	.187
11	.9706	-.7717	-.0151	.0602	.0395	.0586	1.6263

Table 4.1 The coefficients of equations for $\mathcal{A}_s$ and B given by (A.11) and (4.40) ($\alpha = 15$)

i) m=1

The eigenfunctions of velocity components given by (A.6) and (4.30) are shown in figure 4.4 for the axisymmetric modes and in figure 4.5 for the bending modes. First, to compare the numerical solutions with the solutions evaluated by the perturbation method, $\epsilon_s = 0.1$ is taken. With $s = .187$ (see (4.61)), only one mode is known to exist, whose eigenvalue, from (4.60c), is given by

$$\lambda = \lambda_0 + O(\epsilon_s^2) = -.03498 \qquad \text{(perturbation solution)}, \qquad (4.78a)$$

$$\lambda = -.03501 \qquad \text{(numerical solution)}. \qquad (4.78b)$$

Two eigenvalues and eigenfunctions given in figure 4.6 show good agreement between two different methods. With this eigenvalue as a starting point, we continue to calculate the eigenvalues as ϵ_s increases. The only real λ (imaginary σ) that can be found up to $\epsilon_s = 2$ are shown in figure 4.7. The real and imaginary parts of potential $\bar{V}(\xi)$ for $\epsilon_s = 1, 2$ are shown in figure 4.8.

ii) m=11

The eigenfunctions of velocity components are shown in figure 4.9 for the axisymmetric modes and in figure 4.10 for the bending modes. With $s = 1.6263$, two modes exist. The eigenvalue for the symmetric modes is given by

$$\lambda^{(0)} = \lambda_0^{(0)} + O(\epsilon_s^2) = -2.6449 \qquad \text{(perturbation solution)}, \tag{4.79a}$$

$$\lambda^{(0)} = -2.6473 \qquad \text{(numerical solution)}. \tag{4.79b}$$

and, for the anti-symmetric modes,

$$\lambda^{(1)} = \lambda_0^{(1)} + O(\epsilon_s^2) = -.3923 \qquad \text{(perturbation solution)}, \tag{4.80a}$$

$$\lambda^{(1)} = -.3932 \qquad \text{(numerical solution)}, \tag{4.80b}$$

where $\epsilon_s = 0.1$ is taken. The eigenfunctions in figure 4.11 exhibit good agreement between two different methods as before. As for $m = 1$, the only real λ is found as ϵ_s increases up to $\epsilon_s = 2$ as shown in figure 4.12.

Although no unstable bending modes have been found for our primary flow, (4.75), unstable modes may still exist for some special primary flows since the existence of unstable modes is possible for equation (4.65a) as can be seen in appendix D. In our calculations, no unstable mode is found since the numerical values of γ_1 are found to be so small that the real even component of $\bar{V}$ is much greater than the imaginary odd component. However, as the primary flow (4.75) adopted here appear to be quite realistic compared with what have been commonly observed in experiment, we will stop searching for any unstable modes that may arise with some other velocity profiles. But it should be noted that, whenever the non-axisymmetric flow features are present, the solitary wave will not be steady or in equilibrium due to the nonlinear interactions between various modes even though the basic flow is neutrally stable on linear theory. Therefore, we will consider the nonlinear interaction mechanism which affects the basic axisymmetric flows (4.14).

4.4 Discussion

The result of the foregoing linear stability analysis shows that the axisymmetric solitary wave generated in the highly sheared primary flow of (4.75) is neutrally stable with respect to small perturbations in the sense that no bending waves will grow (or decay) in time. Hence, once the soliton is generated, its own entity seems to be never lost provided that non-axisymmetric disturbances can be controlled to be sufficiently small.

However, it may be argued that the result of the linear analysis concerning small perturbations of the primary axisymmetric solitary wave cannot give a definite conclusion for the stability question without considering the nonlinear effects if all eigenvalues are purely imaginary. For the class of non-axisymmetric disturbances of finite amplitude which is comparable with the primary axisymmetirc soliton, the question concerning the stability characteristics or the mode selection mechanism between differnt modes of finite amplitude is very important for the future study. However, this problem is beyond the scope of the present project to pursue in full context. Nevetheless, we will try to better understand the role of non-axisymmetric flow features embedded in an axisymmetric flow by discussing some of the nonlinear interaction mechanism, without further performing detailed analysis but with some simple argument and reasoning, and further verified with numerical simulations.

Suppose that only one mode (e.g., $|n| = 1$) of disturbances is initially assigned, then, all the other modes will be excited in their due times, because of the nonlinear effects of the system. As pointed out in the preceeding section, when the nonlinear interactions are taken into consideration, the axisymmetric solitary wave may suffer some deformations and its further development is now of interest in this section. Although this result cannot be used to infer the long-time asymptotic behaviour of the flow, we at least may gain some qualitative picture about the flow developments in the early stage.

First, the existence of the solitary wave without any non-axisymmetric effects is assumed as before. Then, the disturbances of the $|n| = 1$ mode are superimposed on the flow field and other modes for $|n| \neq 1$ are initially so small that they can be ignored for a finite time. With this assumption, the main nonlinear interaction can be expected to occur between two modes, $n = 0$ and $|n| = 1$, for a finite time. The leading order effects of the bending waves on the axisymmetric waves are considered in Appendix A where we derive a form of the fKdV equation with the squre of the amplitude of bending waves as a forcing term which reads

$$\mathcal{A}_{st} + \beta_1 \mathcal{A}_{sx} + c_1 \mathcal{A}_s \mathcal{A}_{sx} + c_2 \mathcal{A}_{sxxx} = c_5 \left(|B|^2 \right)_x, \tag{4.81}$$

where β_1 and c_i are given by (A.12). When $|B| = 0$, we have a stationary solitary wave given by (A.13). In this derivation, $|B| = O(\mathcal{A}_s)$ is assumed but it may still be applied to the case of $|B| = o(\mathcal{A}_s)$ since the right-hand side of (4.81) is the first term appearing in the interaction process and the contributions from terms of the higher nonlinearity than $\mathcal{A}_s \mathcal{A}_{sx}$ in the axisymmetric motion can be assumed to be small if $\mathcal{A}_{s1}$ is a good approximation to the exact solitary wave solution of the Euler equation shown by Leibovich and Kribus (1990).

For the bending waves, it can be shown that equation (4.40) for B with a steady soliton $\mathcal{A}_s(x)$ can be also generalized for the unsteady form of $\mathcal{A}_s$. Since the disturbances correponding to the bending mode are assumed to be secondary, we shall take $B = o(\mathcal{A}_s)$, hereafter. When we assume $B = O(\mathcal{A}_s)$, a nonlinear term in the equation for B is expected to arise, probably cubic, and some other modes like $|n| = 2$ should be included in the analysis. With $B = o(\mathcal{A}_s)$, the equation governing B becomes (4.40) with $\mathcal{A}_s(x,t)$ instead of $\mathcal{A}_s(x)$, namely,

$$B_t + c(x,t)B_x + i\gamma_1 B_{xx} + \gamma_2 \mathcal{A}_{sx}(x,t)B = 0, \tag{4.82}$$

where

$$c(x,t) = \gamma_0 + \gamma_2 \mathcal{A}_s(x,t). \tag{4.83}$$

With the solitary wave solution (A.13) ascribed as initial data for $\mathcal{A}_s$ and with the eigenfunction (4.60a or b) for $|B|$, we have solved the coupled equations (4.81) and (4.82) numerically by use of the finite differencing scheme developed in chapter 3. During the calculations, we also monitored the rates of energy transfer defined by

$$\dot{e}_0 \equiv \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \left(\mathcal{A}_s^2/2\right) dx = c_5 \int_{-\infty}^{\infty} \mathcal{A}_s \left(|B|^2\right)_x dx, \tag{4.84}$$

$$\dot{e}_1 \equiv \frac{\partial}{\partial t}\int_{-\infty}^{\infty} |B|^2 dx = \gamma_2 \int_{-\infty}^{\infty} \mathcal{A}_s \left(|B|^2\right)_x dx. \tag{4.85}$$

Although e_0 and e_1 are not exactly the kinetic energy but the signs of $\dot{e}_0$ and $\dot{e}_1$ indicate the directions of the energy transfer.

For numerical computations, we choose $\alpha = 15$ and $m = 11$ for the basic swirl velocity (4.75). First, to examine the effects of the right-hand side terms in (4.81), we solved equation (4.81) (not coupled equations) with (4.60a) for $|B|$. Under the assumption of $B = o(\mathcal{A}_s)$, initial amplitudes of $\mathcal{A}_s$ and B are chosen as $\mathcal{A}_m = 0.4$ and $B_m = 0.06$. As simulated in figure 4.13a, the axisymmetric waves $\mathcal{A}_s$ shows the periodic oscillations in amplitude without noticeble change in the location. The amplitude increases and decreases, repeatedly, in time and the process seems to continue with radiation of small trailing waves. The periodicity of this process can be seen in figure 4.13b for $\dot{e}_0$. But this solution has been obtained by neglecting a process of feedback to the bending waves from the deformed axisymmetric waves. To consider the deformations of both waves, we now solve the coupled equations (4.81) and (4.82) in which both waves are found to interact continuously. With the same initial conditions ascribed above, the axisymmetric solitons suffer the same periodic oscillations in amplitude but with larger trailing waves as shown in figure 4.14a. On the other hand, the bending wave B is washed away from its initial position, leaving a set of periodically decaying disturbances at the original location as shown in figure 4.14b. The rates of energy transfer, $\dot{e}_0$ and $\dot{e}_1$, between the two modes decrease continuously with increasing time as shown in figure 4.14c. From the numerical solutions of the coupled equations (4.81) and (4.82), the effects of bending waves on the axisymmetric solitary wave can be summarized to assert that (i) the unsteady responses of the solitary wave proceed with some periodicity and (ii) a set of large trailing axisymmetric waves produced as the bending wave moves downstream. Although the primary solitary wave seems to be able to maintain its inital shape and location with small unsteadiness, the flow field on the downstream side is considerably distorted by interaction with the bending waves. Similar conclusions for larger $\mathcal{A}_m$ and B_m can be drawn as shown in figure 4.15a,b,c.

Since the analysis is based on the assumption that the primary disturbances are axisymmetirc and the secondary disturbances are smaller than the primary ones, the results of the present analysis cannot be applied to the case when both axisymmetric waves and non-axisymmetric waves are initially comparable in amplitude.

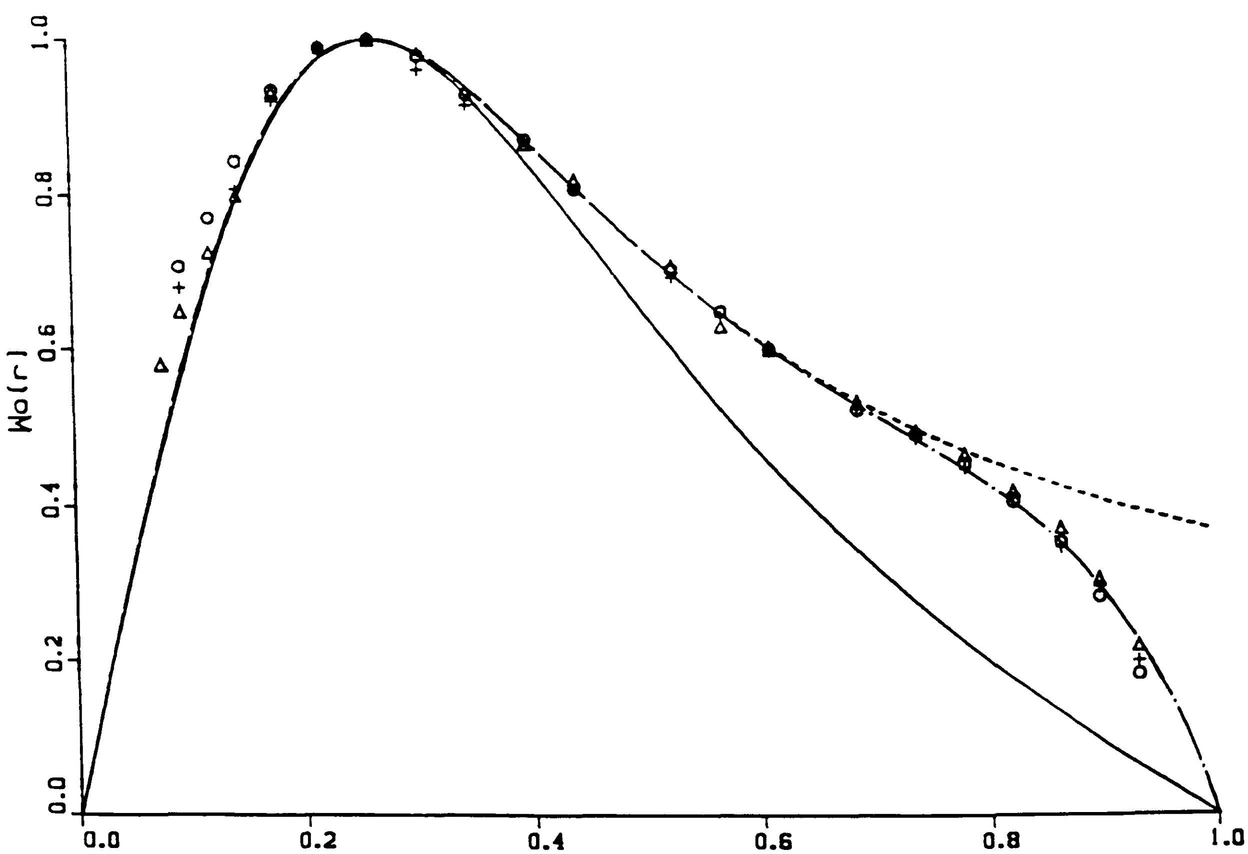


Figure 4.1 The velocity profile of $W_0(r)$ in (4.75) with the position of the maximum swirl velocity r_0 obtained by experiment data by Faler and Leibovich (1977) : —, m=1 ; – · – · , m=11 ; - - - -, the Burgers vortex; ∘, +, △, experiment by Faler and Leibovich.

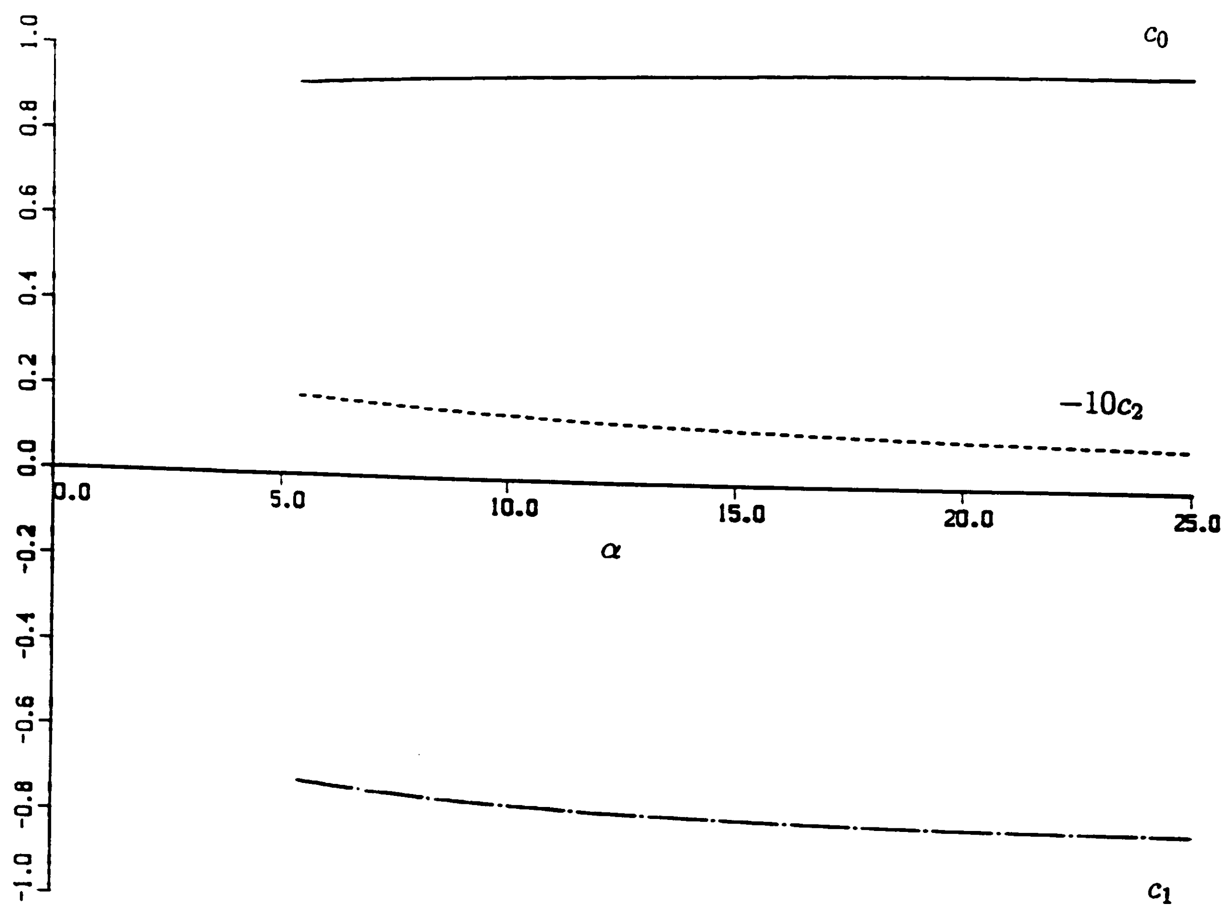


Figure 4.2a The coefficients of the KdV equation for $\mathcal{A}_s$ with varying α and $m = 1$ for the basic swirl velocity (4.75) .

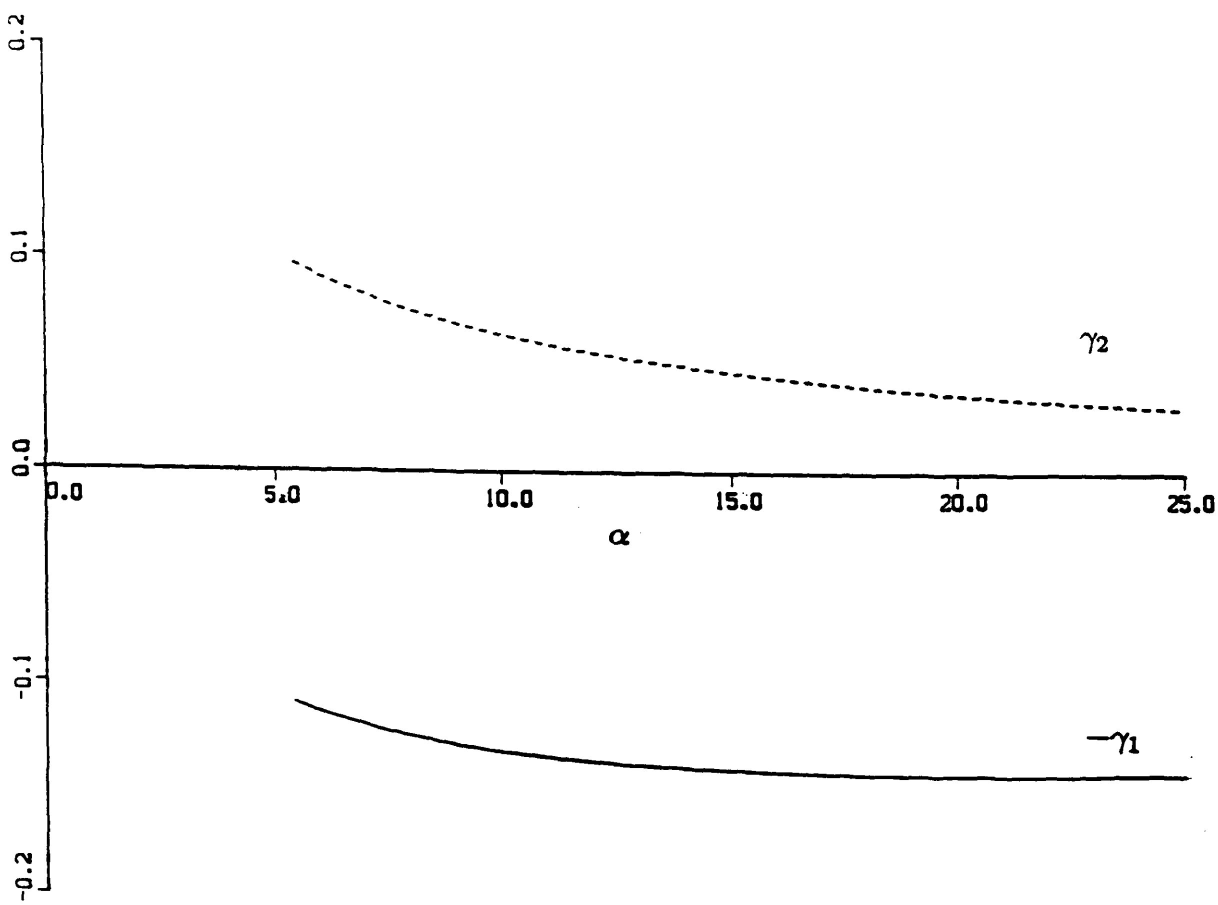


Figure 4.2b The coefficients of equation (4.40) for B with varying α and $m = 1$ for the basic swirl velocity (4.75) .

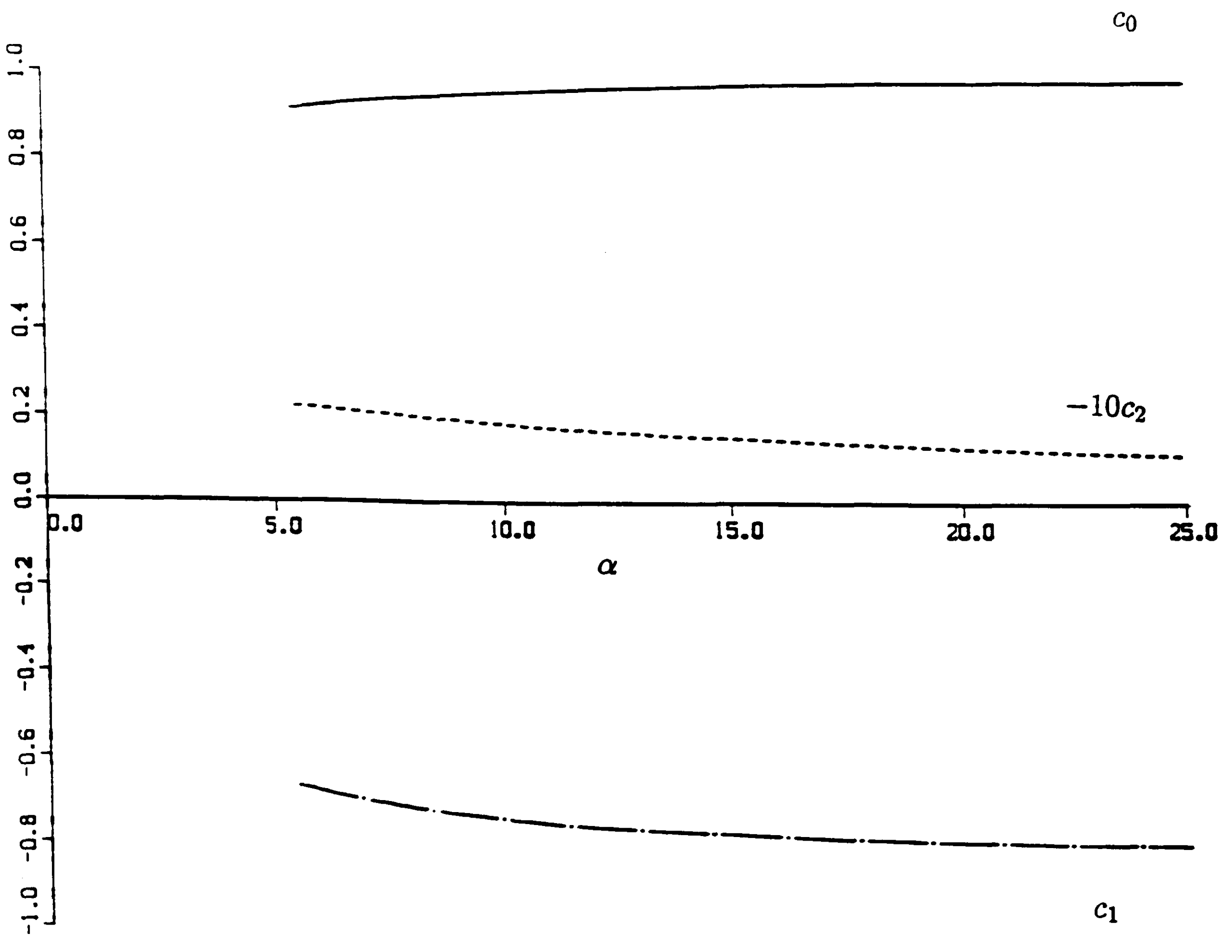


Figure 4.3a The coefficients of the KdV equation (A.10) for $\mathcal{A}_s$ with varying α and $m = 11$ for the basic swirl velocity (4.75).

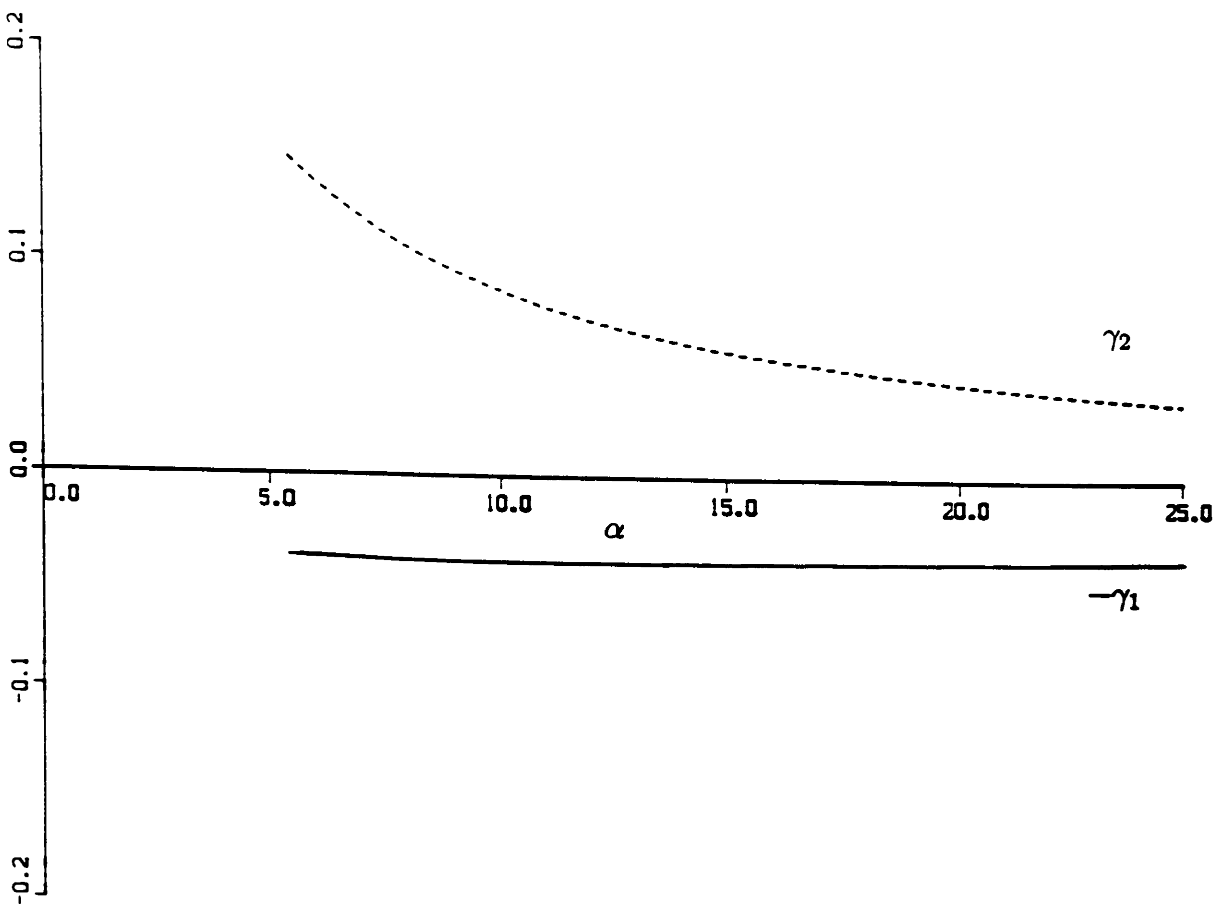


Figure 4.3b The coefficients of equation (4.40) for B with varying α with $m = 11$ for the basic swirl velocity (4.75).

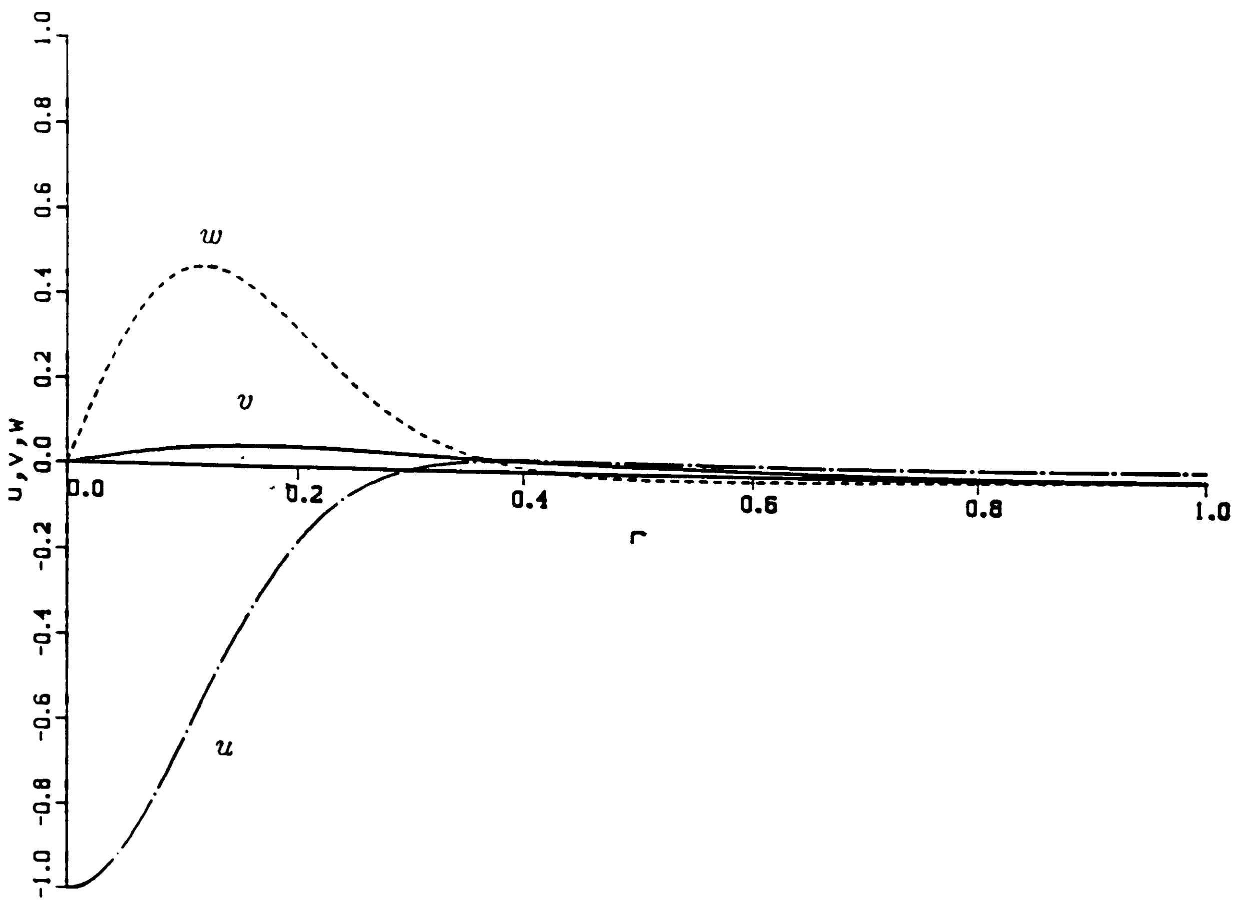


Figure 4.4 The eigenfunctions of the velocity components (u_s, v_s, w_s) for the axisymmetric modes with $\alpha = 15$ and $m = 1$ for the basic swirl velocity.

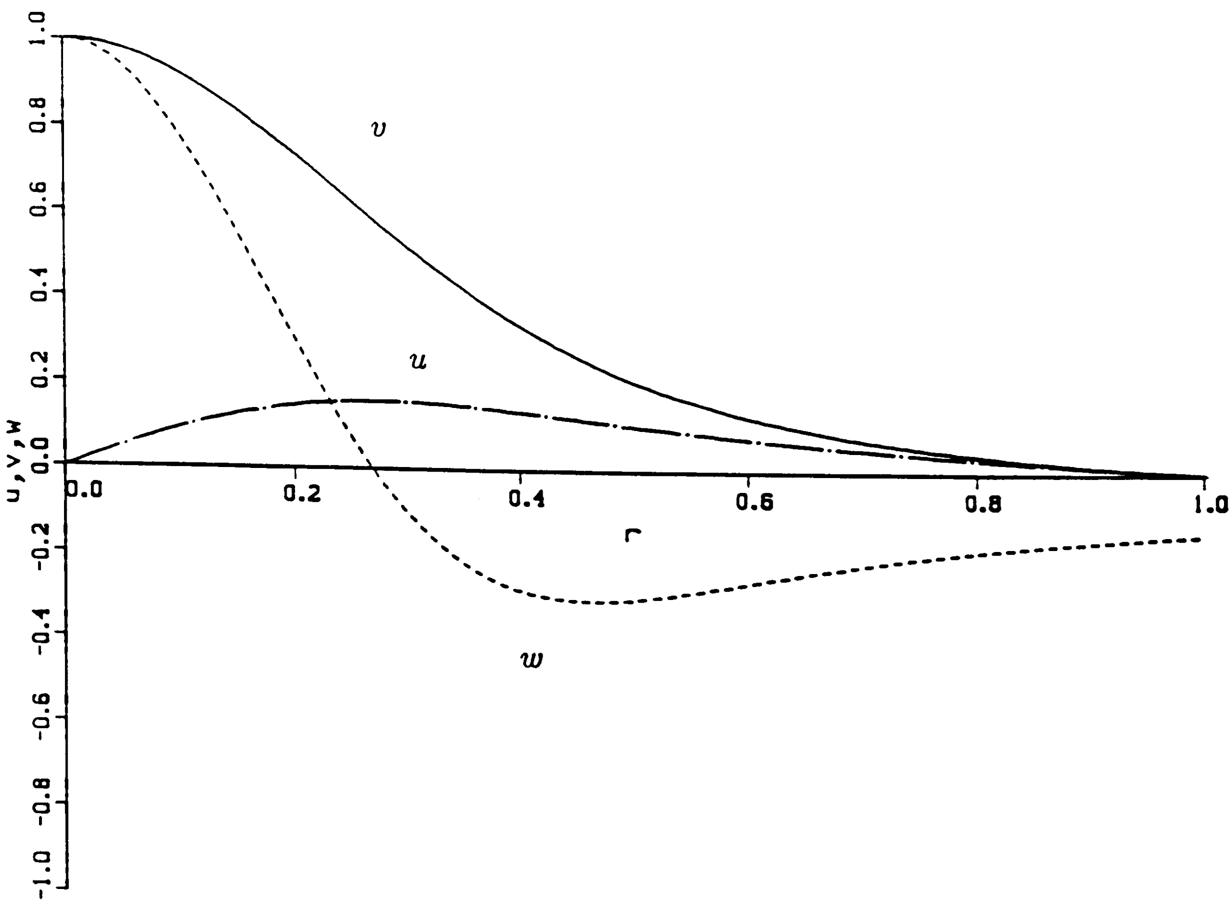


Figure 4.5 The eigenfunctions of the velocity components (u_1, v_1, w_1) for the bending modes with $\alpha = 15$ and $m = 1$ for the basic swirl velocity.

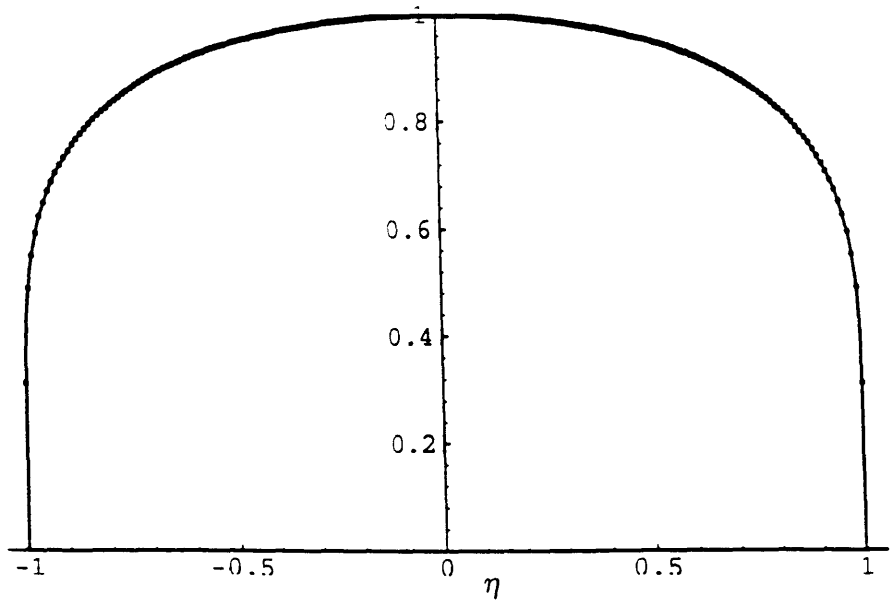


Figure 4.6 The solution of the eigenvalue problem (4.72), $|\varphi|$, for $\epsilon_s = 0.1$ and the basic swirl velocity (4.75) with $\alpha = 15$, $m = 1$: ——, the numerical solution ; $\cdots$, the perturbation solution given by (4.60).

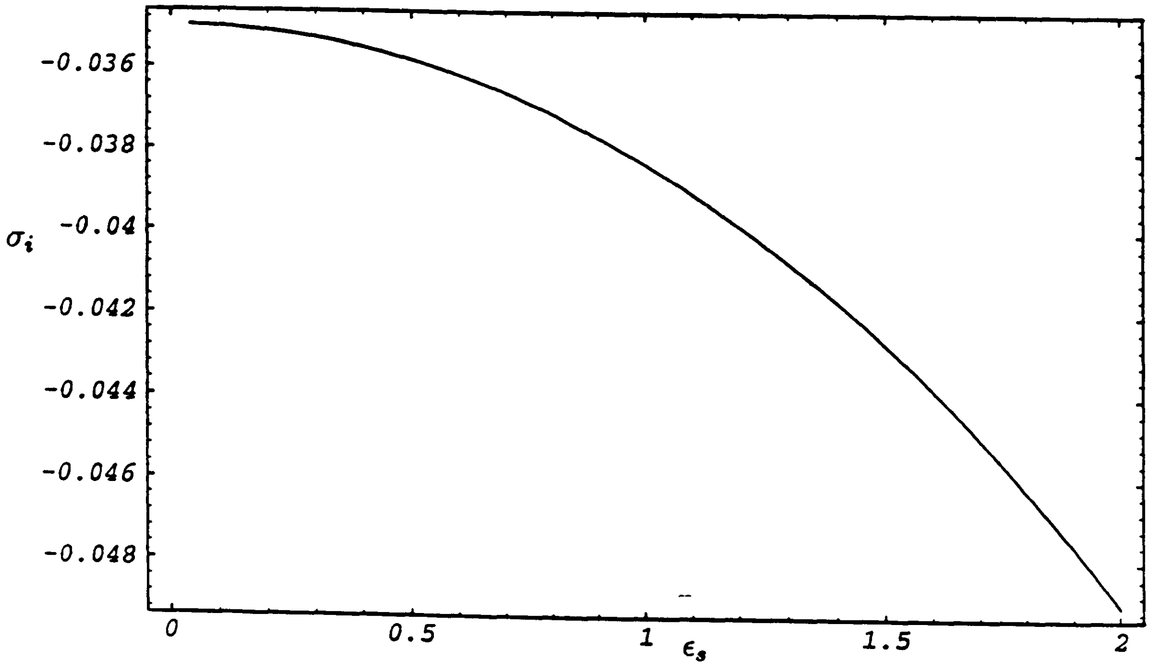
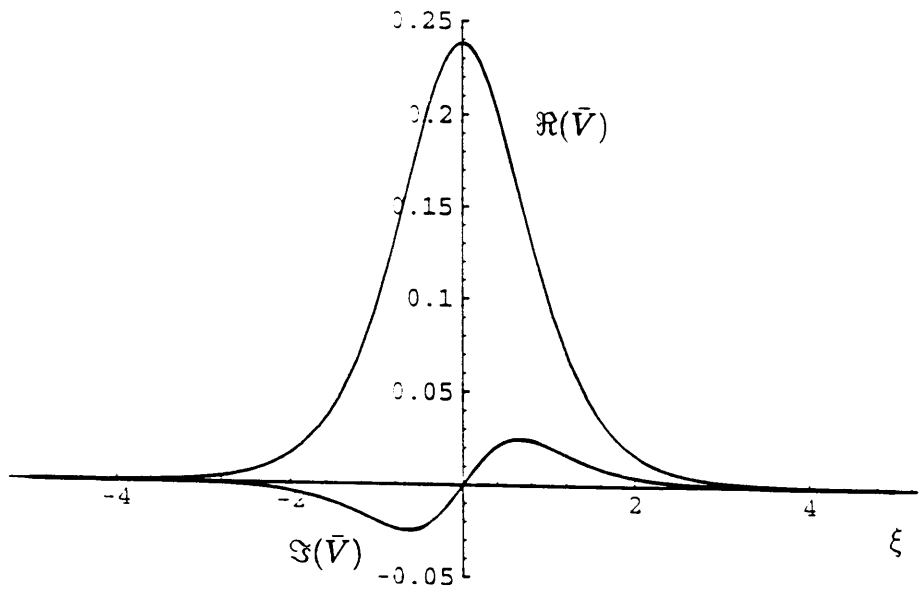
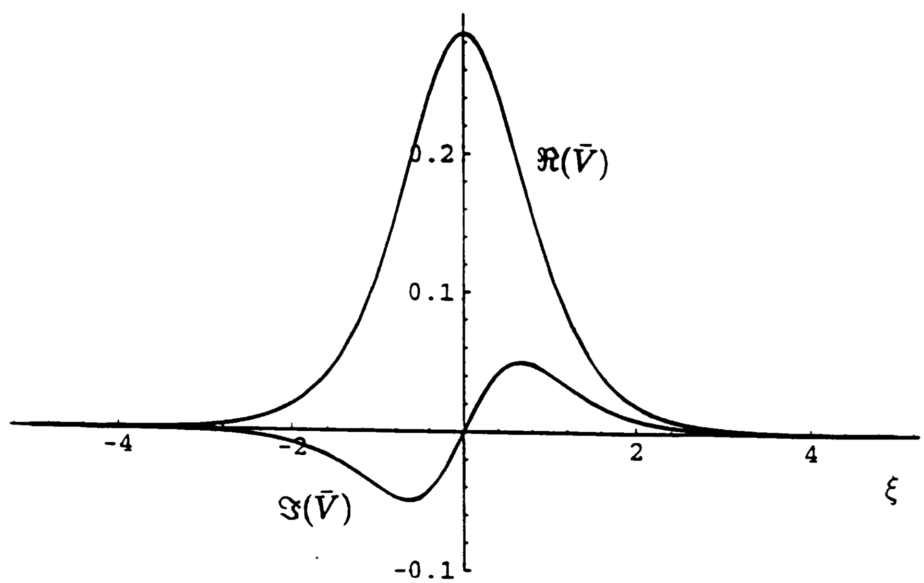


Figure 4.7 The imaginary part of the eigenvalue, $\sigma_i (= \lambda_r)$ for varying ϵ_s, and the basic velocity (4.75) with $\alpha = 15$ and $m = 1$.



(a) $\epsilon_s = 1$



(b) $\epsilon_s = 2$

Figure 4.8 Real and imaginary components in the potential $\bar{V}(\xi)$ given by (4.52a) for $\epsilon_s = 1, 2$ and the primary swirl velocity (4.75) with $\alpha = 15$ and $m = 1$.

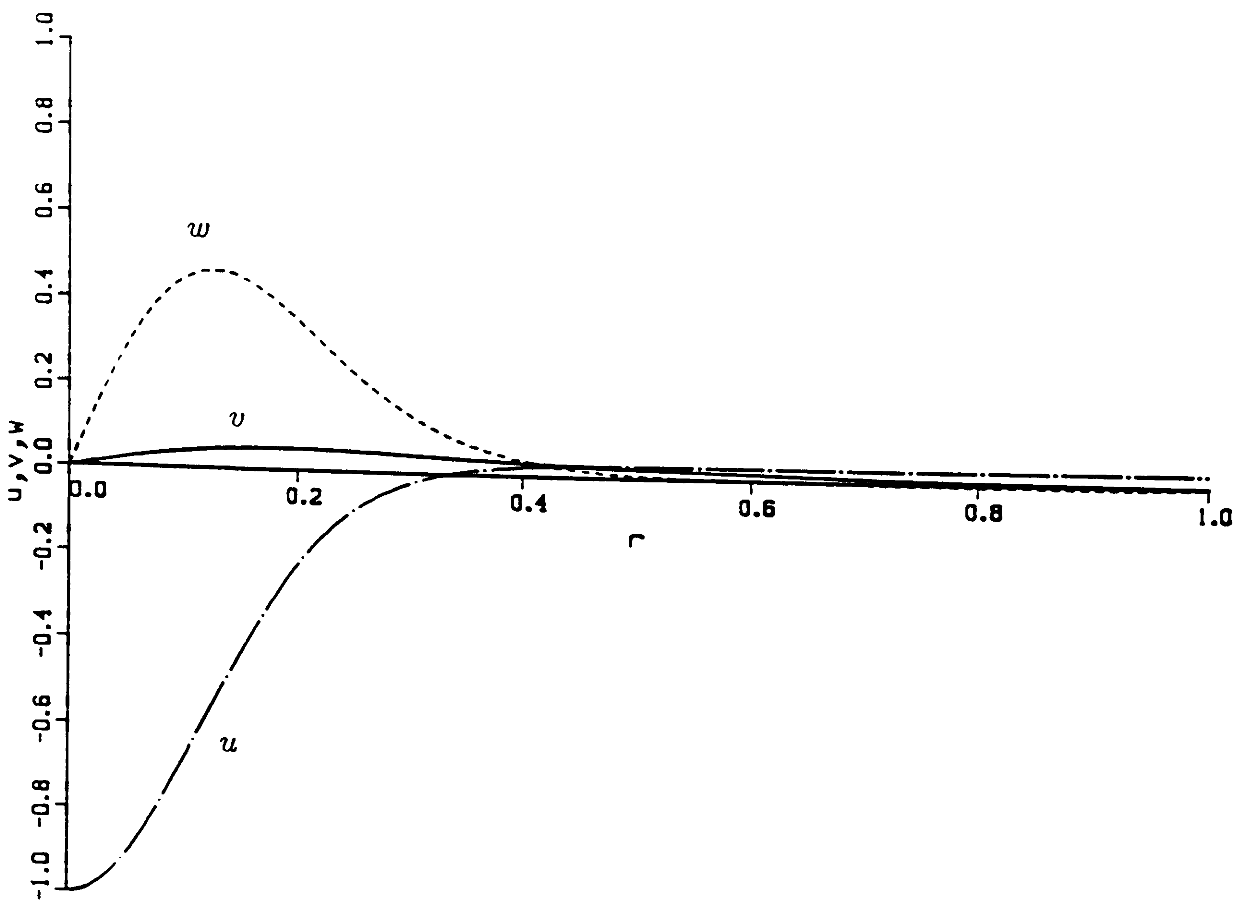


Figure 4.9 The eigenfunctions of velocity components (u_s, v_s, w_s) for axisymmetric modes with $\alpha = 15$ and $m = 11$.

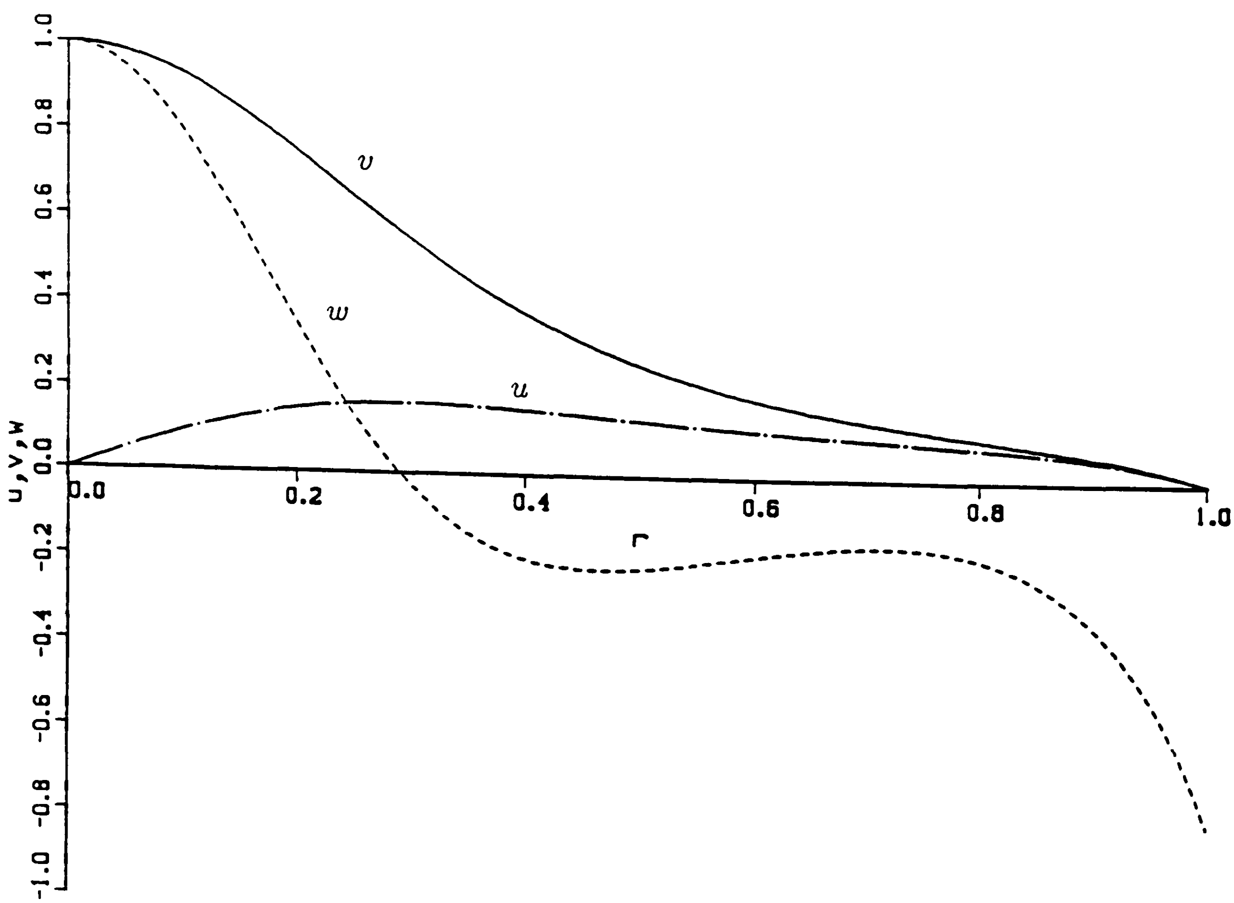
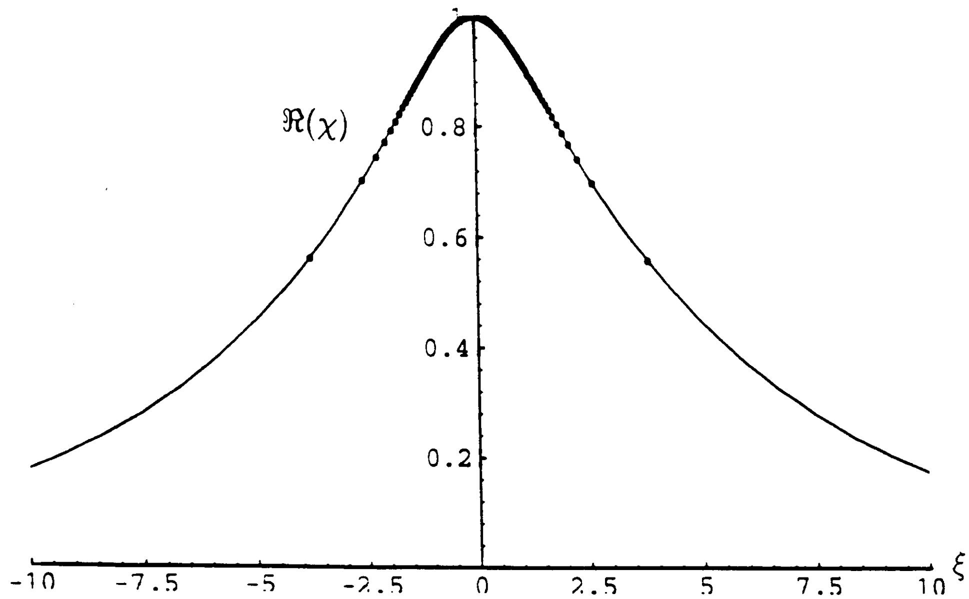
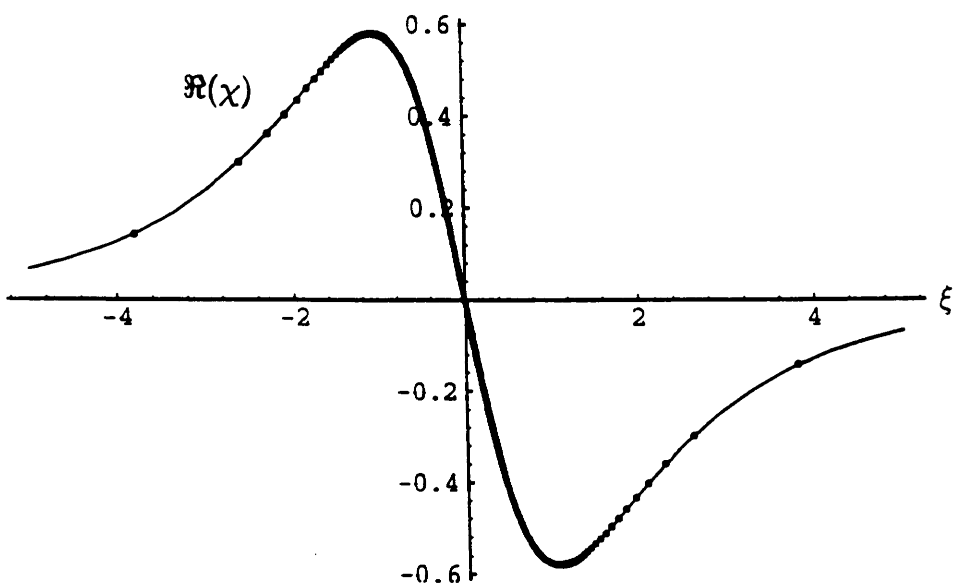


Figure 4.10 The eigenfunctions of velocity components (u_1, v_1, w_1) for bending modes with $\alpha = 15$ and $m = 11$.

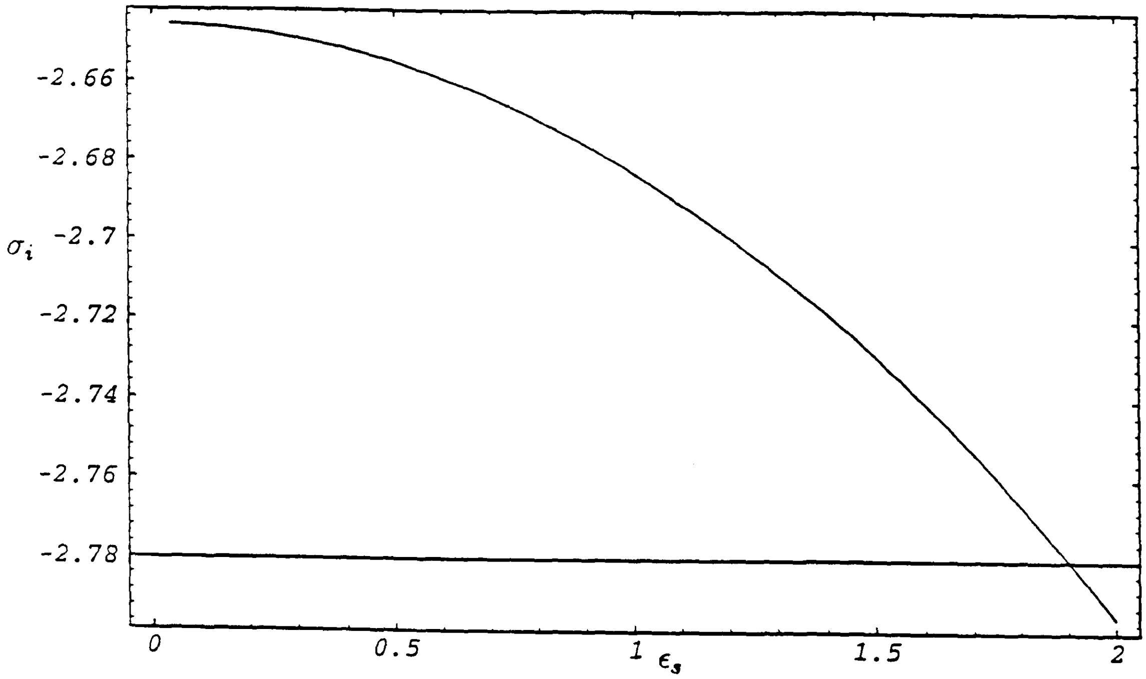


(a) symmetric mode

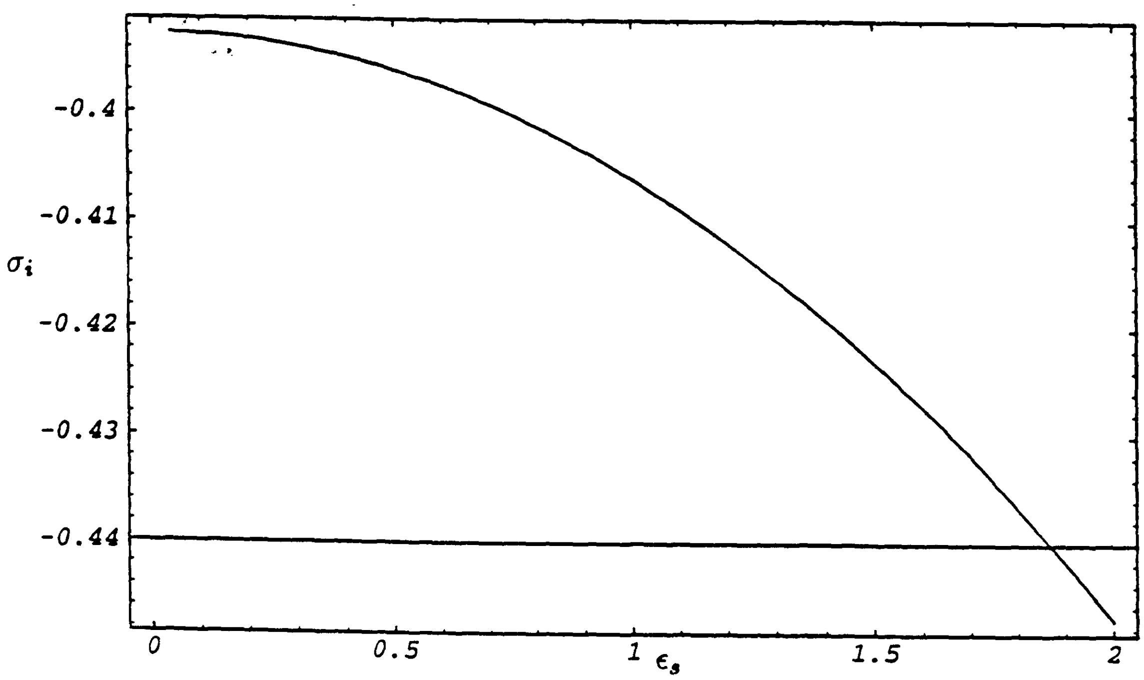


(b) antisymmetric mode

Figure 4.11 The solutions of the eigenvalue problem (4.54), χ, for $\epsilon_s = 0.1$ and the basic swirl velocity (4.75) with $\alpha = 15$, $m = 11$: ——, the numerical solution; ···, the perturbation solution given by (4.60).

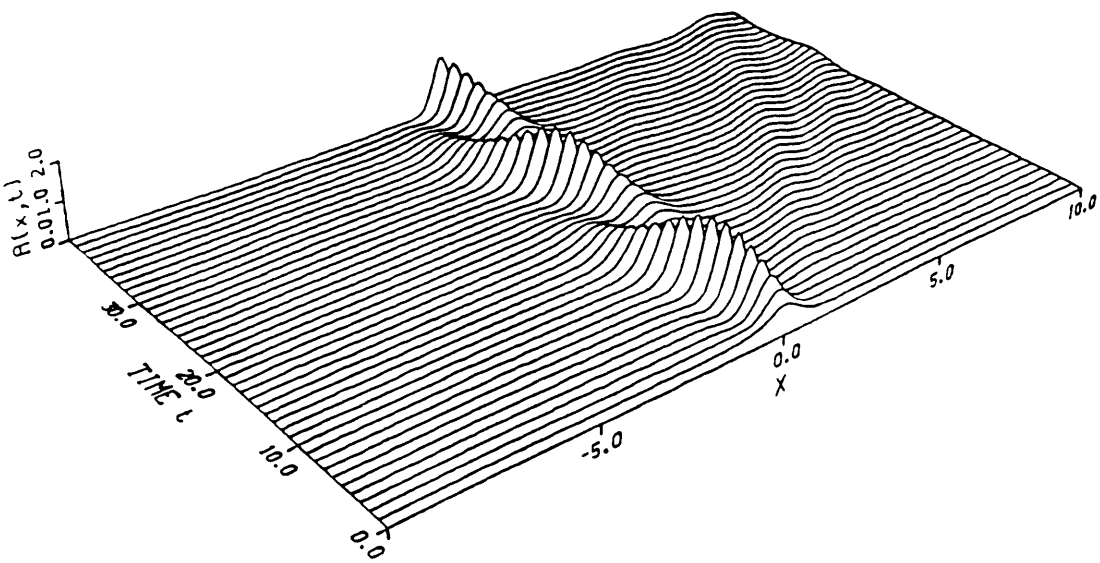


(a) symmetric mode



(b) antisymmetric mode

Figure 4.12 The imaginary part of the eigenvalue, $\sigma_i(=\lambda_r)$ for varying ϵ_s and the basic velocity (4.75) with $\alpha = 15$ and $m = 1$.



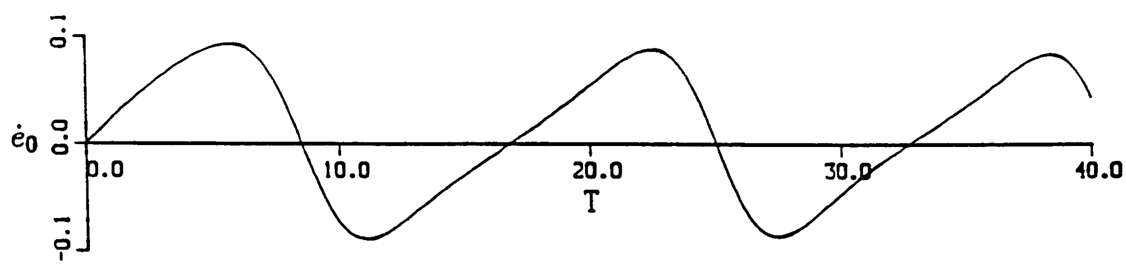


Figure 4.13 a) The numerical solution of (4.81) for $\mathcal{A}_s$ with $\mathcal{A}_s = \mathcal{A}_m \text{sech}^2(\xi)$ and $|B| = B_m \chi_0^{(0)}$ as initial conditions where $\mathcal{A}_m = 0.4$ and $B_m = 0.06$. The parameters in the primary flow (4.75) are $\alpha = 15$ and $m = 11$. b) The time history of $\dot{e}_0$ corresponding to the solution in a).

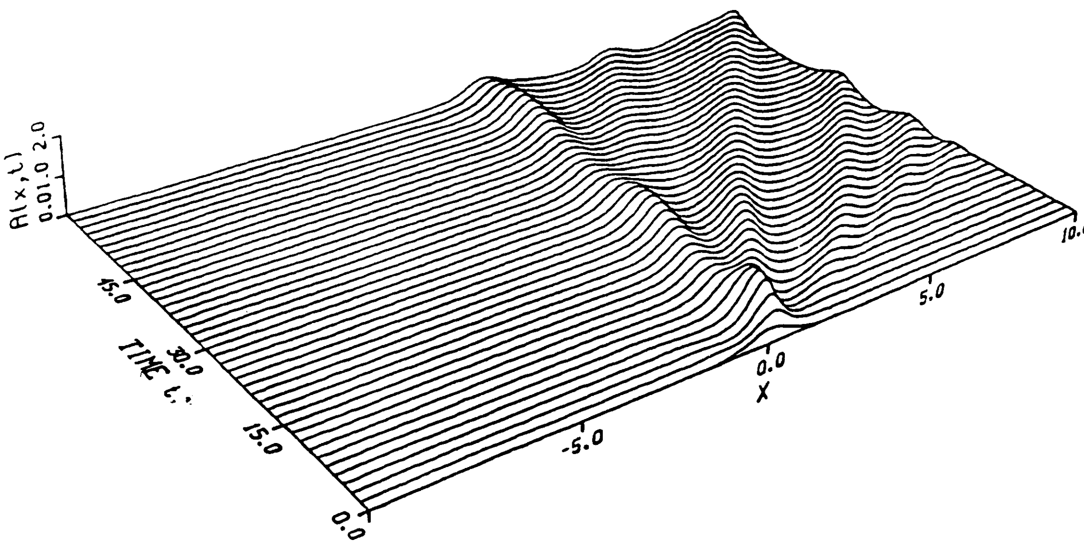


Figure 4.14a The numerical solution of the coupled equations for $\mathcal{A}_s$ with $\mathcal{A}_s = \mathcal{A}_m \text{sech}^2(\xi)$ and $|B| = B_m \chi_0^{(0)}(\xi)$ as initial conditions where $\mathcal{A}_m = 0.4$ and $B_m = 0.06$. The parameters in the primary flow (4.75) are $\alpha = 15$ and $m = 11$.

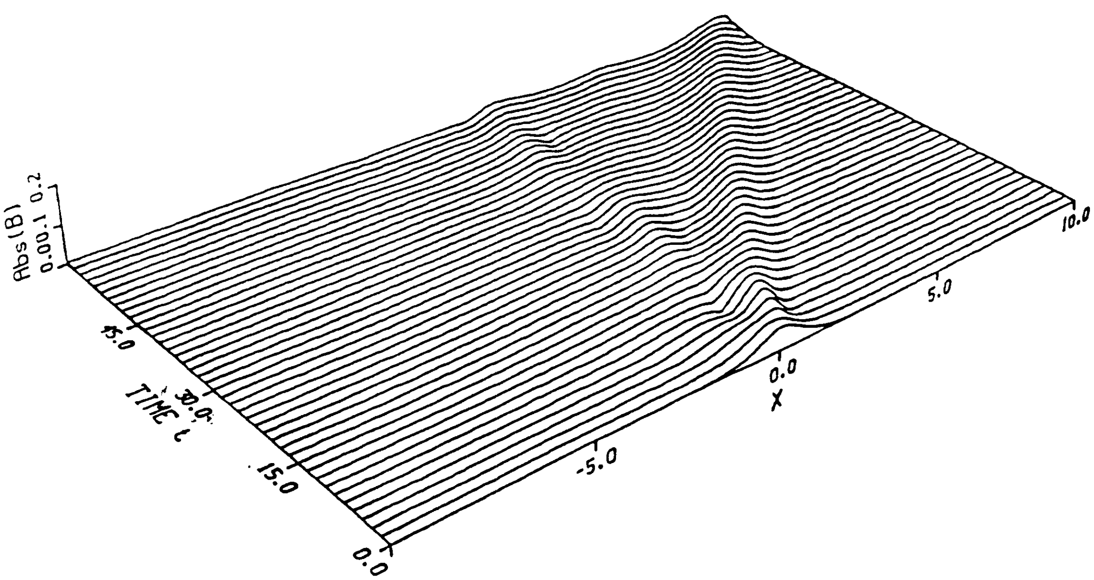


Figure 4.14b The numerical solution for B corresponding to the solution (a).

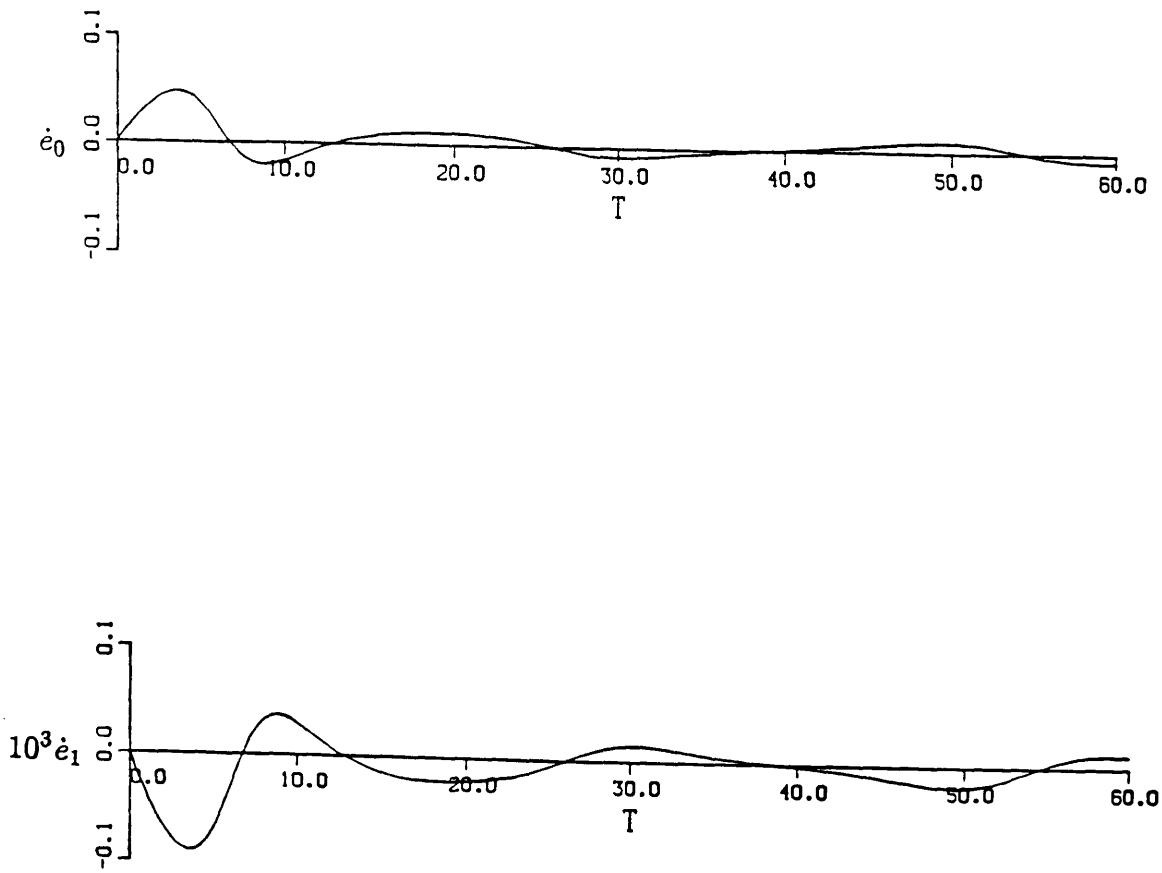


Figure 4.14c $\dot{e}_0$ and $\dot{e}_1$ defined by (4.84) and (4.85) corresponding to the solutions in (a) and (b).

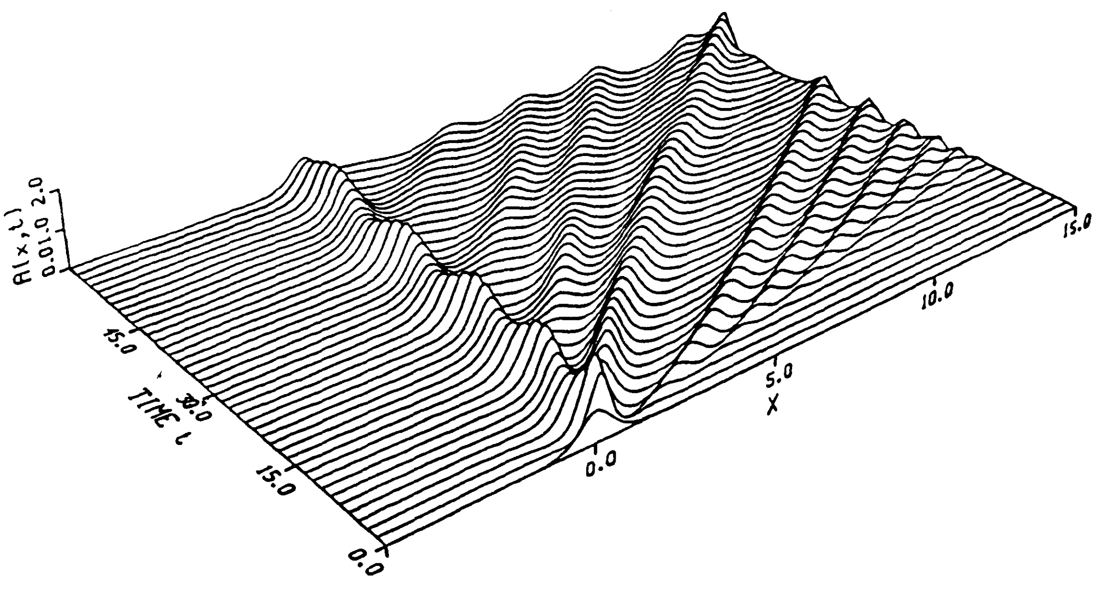


Figure 4.15a The numerical solution of the coupled equations for $\mathcal{A}_s$ with $\mathcal{A}_s = \mathcal{A}_m \text{sech}^2(\xi)$ and $|B| = B_m \chi_0^{(0)}(\xi)$ as initial conditions where $\mathcal{A}_m = 0.8$ and $B_m = 0.1$. The parameters in the primary flow (4.75) are $\alpha = 15$ and $m = 11$.

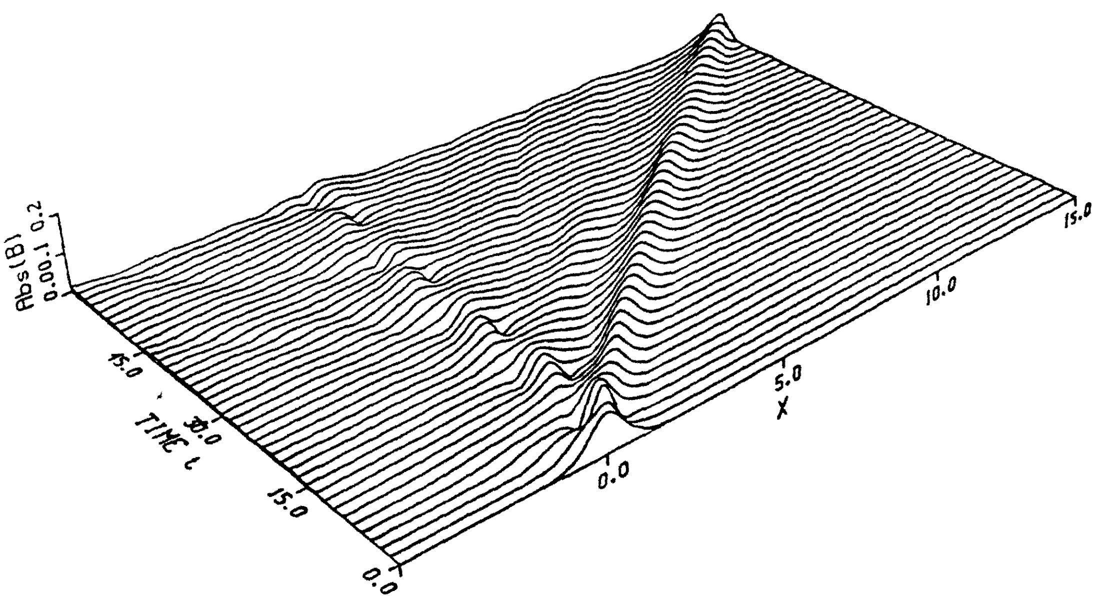


Figure 4.15b The numerical solution for B corresponding to the solution (a).

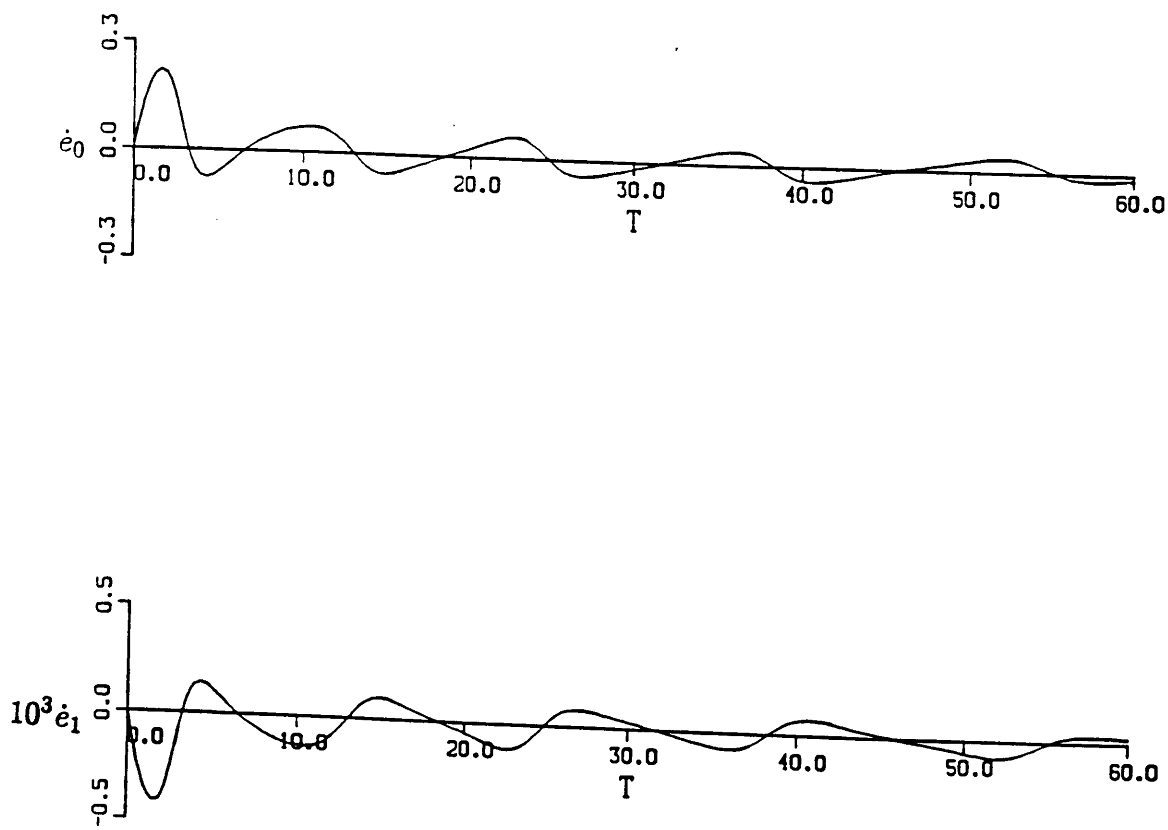


Figure 4.15c $\dot{e}_0$ and $\dot{e}_1$ defined by (4.84) and (4.85) corresponding to the solutions in (a) and (b).

Chapter 5

Conclusions

We consider axisymmetric flows of a non-uniformly rotating fluid in a long cylindrical tube. For an inviscid and incompressible fluid, we derive the forced Korteweg-de Vries equation for weakly nonlinear and weakly dispersive waves when a slender body at the central axis or an axisymmetrical topographical slender deformation of tube wall is moving with a transcritical speed close to the phase velocity of linear long waves. We also derive the fKdV equation with variable coefficients for a tube with non-uniform but stationary wall to investigate the deformation of free or forced nonlinear waves within a gradually convergent or divergent tube. Using two primary flows, the Rankine vortex and the Burgers vortex, we demonstrate the remarkable phenomenon of periodic production of upstream-advancing solitary waves under resonant forcing. The solitary waves in a form of the well defined axisymmetrical recirculating eddies are periodically produced and emitted to keep moving upstream of the disturbance, soon becoming permanent in form. The Burgers vortex is found to be more effective in generations of upstream propagating vortex soliton due to the concentration of axial vorticity in the basic swirl velocity. The section-mean conservation laws for three-dimensional flows in inviscid and incompressible fluids are also derived.

Next, we consider the stability of the axisymmetric flow which is a superposition of a basic swirl velocity (the Buregers vortex with a polynomial function of the radial coordinate) and the free vortex solitary wave. By assuming small perturbations of the bending mode with respect to the axisymmetric solitary waves,, we derive the model equation, which has the terms representing the dispersive effect and the effect of interaction between the axisymmetric mode and the bending mode.

On linear theory, the model equation becomes the Schrodinger equation with a complex potential. By assuming a separable solution to the model equation, the resulting eigenvalue problem is solved with the zero boundary conditions at both infinities. After having found the solution behavior, by use of a perturbation method, for small ϵ_s which is the wave number of the solitary wave, we solve the eigenvalue problem by using the series solutions near both infinities and integrating the complex second-order differential equation numerically toward the center to match two solutions originated from both ends. The result of linear theory shows that the axisymmetric vortex soliton in a rotating fluid is neutrally stable with respect to small disturbances of the bending mode since no unstable (or stable) solutions are found in the parameter domain which consists of ϵ_s and the basic swirl velocity profile.

To examine the further development of waves of the two differnt modes, we consider the nonlinear interaction between the axisymmetric mode and the bending mode. The coupled equations for the amplitude functions of the two modes are derived by assuming that the amplitude of the bending mode is smaller than the axisymmetric solitary wave. The equation for the axisymmetric waves has a form of the fKdV equation with the square of the amplitude of the bending waves as a forcing term. The equation for the bending waves is the same as that considered in linear stability analysis but with the unsteady complex potential which represents the nonlinear interaction with the axisymmetric soliton. The numerical solutions of the coupled equations show that the axisymmetric solitary wave seems to maintain its own entity except small unsteadiness like oscillations in amplitude and position. But the flow fields in the downstream side are distorted by the trailing waves in the process of interaction. The full descriptions on the three-dimensional flow fields in a rotating fluid are beyond the scope of our analysis but our observations are still applicable to the axisymmetric flow with small non-axisymetric flow features.

Appendix A

Axisymmetric Solitons Coupled with Bending Waves

For axisymmetric flows, it is more convenient to deal with the stream function and the circulation as in chapter 3 but we rederive the evolution equation for axisymmetric nonlinear waves using the velocity vector and the pressure for comparison with the formulations in chapter 4. Furthermore, we consider the nonlinear interaction, as described in §4.1, between the axisymmetric waves and the bending waves. By introducing the small parameter $\epsilon_s(=\epsilon^{1/2}$ in chapter 3) signifying the smallness of the ratio of R/λ where R is the tube radius and λ is a charactristic wave length, the axisymmetric flow quantities (u_s, v_s, w_s, p_s) for weakly nonlinear and weakly dispersive waves can be expanded as

$$f_s = (u_s, w_s, p_s), \tag{A.1a}$$

$$f_s(r,\xi,\tau) = \epsilon_s^2[f_{s1}(r,\xi,\tau) + \epsilon_s^2 f_{s2}(r,\xi,\tau) + O(\epsilon_s^4)], \tag{A.1b}$$

$$v_s(r,\xi,\tau) = \epsilon_s^3[v_{s1}(r,\xi,\tau) + \epsilon_s^2 v_{s2}(r,\xi,\tau) + O(\epsilon_s^4)], \tag{A.1c}$$

$$U_0 = \beta_0 + \epsilon_s^2\beta_1 + O(\epsilon_s^4), \tag{A.1d}$$

where U_0 is the basic uniform velocity which is assumed to be the same as the propagation speed of the free solitary wave such that the primary flow of an axisymmetric solitary wave is stationary in the absence of bending waves and (ξ,τ) are defined by

$$\xi = \epsilon_s x, \qquad \tau = \epsilon_s^3 t. \tag{A.2a, b}$$

In (A.1d), β_0, the linear long wave speed, and β_2, the nonlinear correction to the propagation speed of solitary wave, are to be determined. With (A.2a), the same length scale for both axisymmetric and bending waves are assumed and the amplitude of bending waves is assumed to be $O(\epsilon_s^2)$, the same order of amplitude as that of the axisymmetric wave, so that the unsteady behaviors of the axisymmetric waves due to the nonlinear interaction with bending waves appear in the second order. This scale for time t will be clear later .

Therefore, when we ignore the effects of bending waves, the time dependence disappears in the expansion (A.1) as a result of choosing the axial uniform velocity as the propágation speed of the solitary wave.

Substituting (A.1) into (4.7) yields the following first-order equations

$$\beta_0 u_{s1\xi} + p_{s1\xi} = 0, \tag{A.3a}$$

$$-\frac{2W_0}{r} w_{s1} + p_{s1r} = 0, \tag{A.3b}$$

$$\beta_0 w_{s1\xi} + (D_* W_0) v_{s1} = 0, \tag{A.3c}$$

$$u_{s1\xi} + D_* v_{s1} = 0. \tag{A.3d}$$

By taking the first-order solutions in the form of

$$(u_{s1}, w_{s1}, p_{s1}) = \big(\hat{u}_s(r), \hat{w}_s(r), \hat{p}_s(r)\big)\mathcal{A}_{s1}(\xi,\tau), \quad v_{s1} = \hat{v}_s(r)\mathcal{A}_{s1\xi}(\xi,\tau), \tag{A.4}$$

and eliminating (u_{s1}, w_{s1}, p_{s1}) from (A.3), we have for $\hat{v}_s \equiv \phi_s$ the equation

$$\mathcal{L}_s\phi_s \equiv (DD_* \ + q_0(r))\phi_s = 0, \qquad q_0(r) = \frac{2W_0(D_*W_0)}{r\beta_0^2}, \tag{A.5a}$$

and, by (4.13), the boundary conditions are given by

$$\phi_s(0) = 0, \qquad \phi_s(1) = 0. \tag{A.5b}$$

Note that the amplitude function for the radial velocity, $\mathcal{A}_{s1}$, is equal to the amplitude function for the stream function, $-A_1$ defined in chapter 3. This eigenvalue problem (A.5) determines the linear long wave speed β_0, with which we obtain, from (A.3), the solution for $(\hat{u}_s, \hat{w}_s, \hat{p}_s)$ as

$$\hat{v}_s(r) = \phi_s(r), \quad \hat{u}_s(r) = -(D_*\phi_s), \tag{A.6a}$$

$$\hat{w}_s(r) = -\frac{(D_*W_0)}{\beta_0}\phi_s, \quad \hat{p}_s(r) = \beta_0(D_*\phi_s). \tag{A.6b}$$

For the second order, we find from (A.1) and (4.7) the equations, in which the leading-order effects of bending waves first appear, as

$$\beta_0 u_{s2\xi} + p_{s2\xi} = -u_{s1\tau} - \beta_1 u_{s1\xi} - Q_{00}^1 - Q_{01}^1, \tag{A.7a}$$

$$-\frac{2W_0}{r} w_{s2} + p_{s2r} = -c_0 v_{s1\xi} - R_{00}^1 - R_{01}^1, \tag{A.7b}$$

$$\beta_0 w_{s2\xi} + (D_*W_0)v_{s2} = -w_{s1\tau} - \beta_1 w_{s1\xi} - S_{00}^1 - S_{01}^1, \tag{A.7c}$$

$$u_{s2\xi} + D_* v_{s2} = 0, \tag{A.7d}$$

where, by substituting (A.1) into (4.10), Q_{00}^1, R_{00}^1 and S_{00}^1, the leading order terms of Q_{00}, R_{00} and S_{00}, are obtained as

$$Q_{00}^1 = u_{s1}u_{s1\xi} + v_{s1}u_{s1r}, \tag{A.8a}$$

$$R_{00}^1 = -\frac{w_{s1}^2}{r}, \tag{A.8b}$$

$$S_{00}^1 = u_{s1}w_{s1\xi} + v_{s1}w_{s1r} + \frac{v_{s1}w_{s1}}{r}, \tag{A.8c}$$

and, with (4.11), the leading order terms of Q_{01}, R_{01} and S_{01} are found as

$$Q^1_{01} = v_1 u^*_{1r} - \frac{i}{r} w_1 u^*_1 + C.C = \left(\hat{v}_1 \hat{u}'_1 + \frac{1}{r}\hat{w}_1 \hat{u}_1\right)\left(|B|^2\right)_\xi, \tag{A.8d}$$

$$R^1_{01} = v_1 v^*_{1r} - \frac{i}{r} w_1 v^*_1 - \frac{1}{r} w_1 w^*_1 + C.C = 2\left(\hat{v}_1 \hat{v}'_1 + \frac{\hat{v}_1 \hat{w}_1}{r} - \frac{\hat{w}^2_1}{r}\right)|B|^2, \tag{A.8e}$$

$$S^1_{01} = o\Big((|B|^2)_\xi\Big). \tag{A.8f}$$

Here, (u_1, v_1, w_1) are the first-order velocity components of the bending mode, given by (4.30). By eliminating (u_{s2}, v_{s2}, w_{s2}) from (A.7), we have the equation for v_{s2} as

$$\mathcal{L}_s v_2 = f_1(r)\phi_s(\mathcal{A}_{s1\tau} + \beta_1 \mathcal{A}_{s1\xi}) + f_2(r)\phi^2_s \mathcal{A}_{s1}\mathcal{A}_{s1\xi} - \phi_s \mathcal{A}_{s1\xi\xi\xi} + \frac{1}{\beta_0}\Big(Q^1_{01r} - R^1_{01\xi}\Big), \tag{A.9}$$

where $f_1(r)$ and $f_2(r)$ are defined by

$$f_1(r) = \frac{4W_0(D_* W_0)}{r\beta^3_0}, \tag{A.10a}$$

$$f_2(r) = \frac{1}{\beta_0}\left[\left(q'_0 - \frac{2q_0}{r}\right) + \frac{2(D_* W_0)^2}{r\beta^2_0} + \frac{2W_0 D(D_* W_0)}{r\beta^2_0}\right]. \tag{A.10b}$$

The evolution equation governing $\mathcal{A}_{s1}(\xi, \tau)$ can be determined by invoking the solvability condition for the differential equation (A.9) which requires the orthogonality between the homogeneous solution of (A.9) and the right-hand side of (A.9). This orthogonality condition gives the evolution equation for $\mathcal{A}_{s1}$ as

$$\mathcal{A}_{s1\tau} + \beta_1 \mathcal{A}_{s1\xi} + c_1 \mathcal{A}_{s1}\mathcal{A}_{s1\xi} + c_2 \mathcal{A}_{s1\xi\xi\xi} = c_5\left(|B|^2\right)_\xi, \tag{A.11}$$

where

$$c_1 = \frac{(rf_2, \phi^3_s)}{(rf_1, \phi^2_s)}, \qquad c_2 = -\frac{(r, \phi^2_s)}{(rf_1, \phi^2_s)}, \qquad c_5 = \frac{(rf_5, \phi_s)}{(rf_1, \phi^2_s)}, \tag{A.12a, b, c}$$

$$f_5(r) = \frac{1}{\beta_0}\left[rq_0 \left(r\phi_1\phi_1' + \phi_1^2 - \frac{(D_* W_0)}{\Omega}\phi_1^2 \right) - 2r\left(\phi_1'\right)^2 \right]. \qquad (A.12d)$$

When $|B| = 0$, we have the classical KdV equation which has the stationary solitary wave solution, in the original variables (x,t), as

$$\mathcal{A}_s(x) = \mathcal{A}_m \operatorname{sech}^2(\epsilon_s x), \qquad (A.13a)$$

where

$$\mathcal{A}_m = 12\left(\frac{c_2}{c_1}\right)\epsilon_s^2, \qquad U_0 = \beta_0 + \epsilon_s^2\beta_1, \qquad \beta_1 = -4c_2. \qquad (A.13b)$$

When $|B| \neq 0$, the axisymmetric solitary wave (A.13a) will generally experience variations from (A.13a) according to (A.11) which involves the effects of bending waves. Since equation (A.11) has exactly the same form as the fKdV equation (3.29), the numerical code developed in chapter 3 can be used to study the nonlinear interactions between the two wave modes specified in §4.4.

Appendix B

Dispersion Relation for Linear Long Waves

In this appendix, the dispersion relations for linear long waves propagating freely in a long cylindrical tube for both axisymmetric ($n = 0$) and non-axisymmetric ($|n| \neq 0$) modes will be determined. For this linear problem, the wave solutions of equation (4.7) with $Q_n = R_n = S_n = 0$ assumes the form

$$f_n(x, r, \theta, t) = \hat{f}_n(r) e^{i(kx+n\theta-\bar{\omega}t)}, \qquad (B.1)$$

where $f_n = (u_n, v_n, w_n, p_n)$. By eliminating $\hat{u}_n, \hat{w}_m, \hat{p}_n$, we obtain a single equation for $\hat{v}_n$ as (after dropping the hat)

$$D\left[\eta(D_* v_n)\right] - \left[1 + \frac{\alpha_1}{\gamma} + \frac{\alpha_2}{\gamma^2}\right] v_n = 0, \qquad (B.2)$$

where D and D_* are defined by (4.4b) and

$$\eta = \frac{r^2}{n^2 + k^2 r^2}, \qquad \gamma = \bar{\omega} - kU_0 - n\frac{W_0}{r}, \qquad (B.3a, b)$$

$$\alpha_1 = -rD\left[\frac{n(D_* W_0)}{n^2 + k^2 r^2}\right], \qquad \alpha_2 = -k^2 \frac{2W_0(D_* W_0)}{r}\eta. \qquad (B3c, d)$$

By (4.13), the boundary conditions for the solution to (B.2) are given by

$$v_n(0) = 0 \quad (|n| \neq 1), \qquad Dv_n(0) = 0 \quad (|n| = 1), \qquad (B.4a)$$

$$v_n(1) = 0. \tag{B.4b}$$

This is the equation of Howard & Gupta (1962), by which the linear stability of a cylindrical flow (4.1) can be studied. For axisymmetric linear long waves (n=0, $k \to 0$), we can expand v_0 and $\bar{\omega}$ as

$$v_0(r) = v_{01}(r) + k^2 v_{02}(r) + \cdots, \tag{B.5a}$$

$$\bar{\omega} = kc_0 - k^3 c_2 + \cdots. \tag{B.5b}$$

Here, (B.5b) is the dispersion relation for axisymmetric long waves, where c_0 is the linear long wave speed and c_2 also appears as the coefficient of term with $A_{\xi\xi\xi}$ in the KdV equation (A.11), representing the dispersive effects.

For non-axisymmetric long waves, i.e., with $n \neq 0$, we have to include a term which is independent of k in the expansion of $\bar{\omega}$ due to the tube wall. According to (B.2), the correct expansions for v_n and $\bar{\omega}$ are found to be

$$v_n(r) = v_{n1}(r) + kv_{n2}(r) + k^2 v_{n3}(r) + \cdots, \tag{B.6a}$$

$$\bar{\omega} = \omega_n + k\gamma_0 + k^2\gamma_1 + \cdots. \tag{B.6b}$$

Substituting (B.6) into (B.2) gives the coefficients of linear terms in the evolution equation for non-axisymmetric waves as can be seen in appendix C.

Appendix C

Higher Order Expansions for Non-axisymmetric Waves

In this appendix, the details of the procedure to obtain the model equation for non-axisymmetric waves will be shown. After having solved the first-order equation (4.31), we can proceed to find the higher order solutions. All the notations used in this appendix are the same as the ones used in §4.2

At $O(\alpha_n \epsilon_s)$, we have the second-order equations

$$\mathcal{L}_{nx}(u_{n2}, p_{n2}) = -[u_{n1\tau_1} + \beta_0 u_{n1\xi}] - Q^1_{ns}, \qquad (C.1a)$$

$$\mathcal{L}_{nr}(v_{n2}, w_{n2}, p_{n2}) = -[v_{n1\tau_1} + \beta_0 v_{n1\xi}], \qquad (C.1b)$$

$$\mathcal{L}_{n\theta}(v_{n2}, w_{n2}, p_{n2}) = -[w_{n1\tau_1} + \beta_0 w_{n1\xi}], \qquad (C.1c)$$

$$\mathcal{L}_{nc}(v_{n2}, w_{n2}) = 0, \qquad (C.1d)$$

where, by substituting the first-order solution (4.34) into (4.26), Q^1_{ns} is found to be

$$Q^1_{ns} = -\big[D(D_* \phi_s)\phi_{n1}\big]\mathcal{A}_{s1}\mathcal{A}_n = [q_0(r)\phi_s\phi_{n1}]\mathcal{A}_s\mathcal{A}_n, \qquad (C.2)$$

where $q_0(r)$ is given in (A.5a). By substituting the first-order solution (4.34) into (C.1) and using (4.36), we have for v_{n2} the equation

$$\mathcal{L}_n v_{n2} = -\frac{i}{n(\omega_n - \Omega)}[r^2\phi''_{n1} + 3r\phi'_{n1} - (n^2-1)\phi_{n1}](\mathcal{A}_{n\tau_1} + \beta_0\mathcal{A}_{n\xi}), \qquad (C.3)$$

and, invoking the solvability condition (4.37), we can obtain the equation for $\mathcal{A}_n$ as

$$\mathcal{A}_{n\tau_1} + \beta_0 \mathcal{A}_{n\xi} = 0. \qquad (C.4)$$

Since we have only homogeneous solutions in this order which can be absorbed into the first-order solution, we therfore have the second-order solution as

$$v_{n2} = w_{n2} = p_{n2} = 0, \qquad u_{n2} = -\frac{iq_0(r)\phi_s\phi_{n1}}{n(\omega_n - \Omega)}\mathcal{A}_{s1}\mathcal{A}_n. \qquad (C.5)$$

At $O(\alpha_n \epsilon_s^2)$, we have the third-order equations

$$\mathcal{L}_{nx}(u_{n3}, p_{n3}) = -u_{n1\tau_2} - [u_{n2\tau_1} + \beta_0 u_{n2\xi}] - Q^2_{ns}, \qquad (C.6a)$$

$$\mathcal{L}_{nr}(v_{n3}, w_{n3}, p_{n3}) = -v_{n1\tau_2} - R^1_{ns}, \qquad (C.6b)$$

$$\mathcal{L}_{n\theta}(v_{n3}, w_{n3}, p_{n3}) = -w_{n1\tau_2} - S^1_{ns}, \qquad (C.6c)$$

$$\mathcal{L}_{nc}(v_{n3}, w_{n3}) = -u_{n1\xi}, \qquad (C.6d)$$

where, from (4.26) and (4.27), $Q^2_{ns}, R^1_{ns}, S^1_{ns}$ are given by

$$Q^2_{ns} = \frac{in}{r}\hat{w}_s\hat{u}_{n1}\mathcal{A}_{s1}\mathcal{A}_{n\xi}, \qquad (C.7a)$$

$$R^1_{ns} = \frac{in}{r}\hat{w}_s(\hat{v}_{n1} - 2\hat{w}_{n1})\mathcal{A}_{s1}\mathcal{A}_n, \qquad (C.7b)$$

$$S^1_{ns} = \left[\left(\hat{w}'_s + \frac{\hat{w}_s}{r}\right)\hat{v}_{n1} - \frac{n^2}{r}\hat{w}_s\hat{w}_{n1}\right]\mathcal{A}_{s1}\mathcal{A}_n. \qquad (C.7c)$$

By using the same method as before, we can find the solvability condition at this order as

$$\mathcal{A}_{n\tau_2} + i\gamma_{n1}\mathcal{A}_{n\xi\xi} = 0, \qquad (C.8)$$

where

$$\gamma_{n1} = \frac{(rg_2, \phi_{n1})}{(rg_1, \phi_{n1})}, \qquad g_1(r) = \frac{q_n(r)}{(\omega_n - \Omega)}\phi_{n1}, \qquad (C.9a)$$

$$g_2(r) = -\frac{r^2}{n}\left[(\omega_n - \Omega)(2r\phi'_{n1} + (n^2+2)\phi_{n1}) + 2(D_*W_0)\left(1 - \frac{\Omega}{(\omega_n - \Omega)}\right)\phi_{n1}\right]. \qquad (C.9b)$$

With $\mathcal{A}_n$ so determined, v_{n3} and w_{n3}, which are needed for the next order expansion, can be obtained from (C.6) as

$$v_{n3} = \phi_{n3}(r)\mathcal{A}_{s1}\mathcal{A}_n + \chi_{n3}(r)\mathcal{A}_{n\xi\xi}, \qquad w_{n3} = \frac{ir}{n}(D_*v_{n3}), \qquad (C.10a,b)$$

where $\phi_{n3}(r)$ satisfies the following equation

$$\mathcal{L}_n\phi_{n3} = -\frac{\phi_{n1}}{r(\omega_n - \Omega)}\left(r^2\hat{w}''_s + r\hat{w}'_s - \hat{w}_s + q_n(r)\hat{w}_s\right), \qquad (C.11)$$

and χ_{n3} satisfies a similar equation but is not needed for the following analysis. In particular, for $n = 1$, ϕ_{13} satisfying the boundary conditions (4.32a) becomes, by making use of (4.33),

$$\phi_{13} = \frac{\hat{w}_s}{r}. \qquad (C.12)$$

The term coupled with $\mathcal{A}_{s1}$ in the evolution equation for $\mathcal{A}_n$ appears in the fourth order equations which are

$$\mathcal{L}_{nr}(v_{n4}, w_{n4}, p_{n4}) = -[v_{n3\tau_1} + \beta_0 v_{n3\xi}] - v_{n1\tau_3} - \beta_1 v_{n1\xi} - R^2_{ns}, \quad (C.13a)$$

$$\mathcal{L}_{n\theta}(v_{n4}, w_{n4}, p_{n4}) = -[w_{n3\tau_1} + \beta_0 w_{n3\xi}] - w_{n1\tau_3} - \beta_1 w_{n1\xi} - S^2_{ns}, \quad (C.13b)$$

$$\mathcal{L}_{nc}(v_{n4}, w_{n4}) = -u_{n2\xi}, \qquad (C.13c)$$

where

$$R^2_{ns} = (\hat{u}_s\hat{v}_{n1})\mathcal{A}_{s1}\mathcal{A}_{n\xi} + (\hat{v}_s\hat{v}_{n1})'\mathcal{A}_{s1\xi}\mathcal{A}_n, \qquad (C.14a)$$

$$S^2_{ns} = in\left[(\hat{u}_s\hat{w}_{n1})\mathcal{A}_{s1}\mathcal{A}_{n\xi} + \hat{v}_s\left(\hat{w}'_{n1} + \frac{\hat{w}_{n1}}{r}\right)\mathcal{A}_{s1\xi}\mathcal{A}_n\right]. \qquad (C.14b)$$

Next, the bending mode ($n = 1$) will be considered. By substituting (4.30) and (C.10) into (C.13) with (C.4), we obtain, after some lengthy manipulation (with $\omega_1 = 0$ by (4.33)), for v_{14} the equation

$$\mathcal{L}_n v_{14} = -\frac{i}{\Omega}\phi_{11}\left[(r^2\hat{u}_s'' + 3r\hat{u}_s' + q_1(r)\hat{u}_s)\mathcal{A}_{s1}\mathcal{A}_{1\xi} - (q_1(r)\phi_s' - (r^2q_0(r)\phi_s)')\mathcal{A}_{s1\xi}\mathcal{A}_1\right] - \frac{i}{\Omega}\phi_{11}q_1(r)(\mathcal{A}_{1\tau_3} + \beta_1\mathcal{A}_{1\xi}). \qquad (C.15)$$

By either using (4.37) or noting that

$$v_{14}(r) = -i\left[\hat{u}_s(r)\mathcal{A}_{s1}\mathcal{A}_{1\xi} - \hat{v}_s'(r)\mathcal{A}_{s1\xi}\mathcal{A}_1 + (\mathcal{A}_{1\tau_3} + \beta_1\mathcal{A}_{1\xi})\right], \qquad (C.16)$$

we have, from (3.6),

$$\mathcal{A}_{1\tau_3} + \beta_1\mathcal{A}_{1\xi} + \gamma_2(\mathcal{A}_{s1}\mathcal{A}_1)_\xi = 0, \qquad \gamma_2 = -\phi_s'(1). \qquad (C.17)$$

It is to be noted that, for the next order expansion, we need to find the next higher order steady solitary wave, say f_{s2} in (4.15). But the first-order solution for the steady solitary wave, namely f_{s1}, is sufficient for performing linear stability analysis up to $O(\alpha_n\epsilon_s^3)$. By adding (C.4), (C.8) and (C.17) and using

$$\frac{\partial}{\partial\tau} = \frac{\partial}{\partial\tau_1} + \epsilon_s\frac{\partial}{\partial\tau_2} + \epsilon_s^2\frac{\partial}{\partial\tau_3}, \qquad (C.18)$$

we finally have for $\mathcal{A}_1(\xi,\tau) \equiv B(\xi,\tau)$ the evolution equation

$$B_\tau + \gamma_0 B_\xi + i\epsilon_s\gamma_1 B_{\xi\xi} + \epsilon_s^2\gamma_2\left(\mathcal{A}_{s1}(\xi)B\right)_\xi = 0, \qquad (C.19)$$

where

$$\gamma_0 = \beta_0 + \epsilon_s^2\beta_1, \qquad (C.20)$$

and γ_1 and γ_2 are given in (C.9) and (C.17), respectively.

Appendix D

The Scrodinger Equation with a Complex Potential

In this appendix, we will consider the numerical solutions to the Scrodinger equation with a complex potential. The equation to be studied has the same form as (4.65a) and can be written

$$\chi'' + \Big[\lambda + V(x)\Big]\chi = 0, \qquad (D.1)$$

where the complex potential $V(x)$ assumes the following form

$$V(x) = V_r\ \mathrm{sech}^2(x) + iV_i\ \mathrm{sech}^2(x)\ \tanh(x) \qquad (D.2)$$

and the zero boundary conditions are imposed on χ at both infinities. We will examine the solution behavior for fixed V_r as V_i increases from zero. We use the computer code described in §4.3 for numerical calculations of the eigenvalue problem (D.1).

Two numerical values of V_r are chosen, namely $V_r = 2, 6$. First, for $V_r = 6$, we have two solutions for $V_i = 0$, as shown in (4.60), which are

$$\chi^{(0)} = \mathrm{sech}^2(x), \qquad (D.3a)$$

$$\chi^{(1)} = \tanh(x)\mathrm{sech}(x), \qquad (D.3b)$$

$$\lambda^{(j)} = -(2-j)^2, \qquad j = 0, 1. \tag{D.3c}$$

As V_i increases, the two modes in (D.3) approach each other and become identical after the the imaginary component of λ becomes non-zero as shown in figure D.1. We recall that, as pointed out in §4.3, λ^*, the complex conjugate of λ, is also an eigenvalue, with the corresponding eigenfunction $\chi^*(-x)$, since $\Re(V)$ is even and $\Im(V)$ is odd.

For $V_r = 2$, there exists only one solution with $V_i = 0$, which is given by

$$\chi = \mathrm{sech}(x), \qquad \lambda = -1. \tag{D.4}$$

As the imaginary component V_i passes a certain critical value, the eigenvalue λ becomes complex as shown in figure D.2.

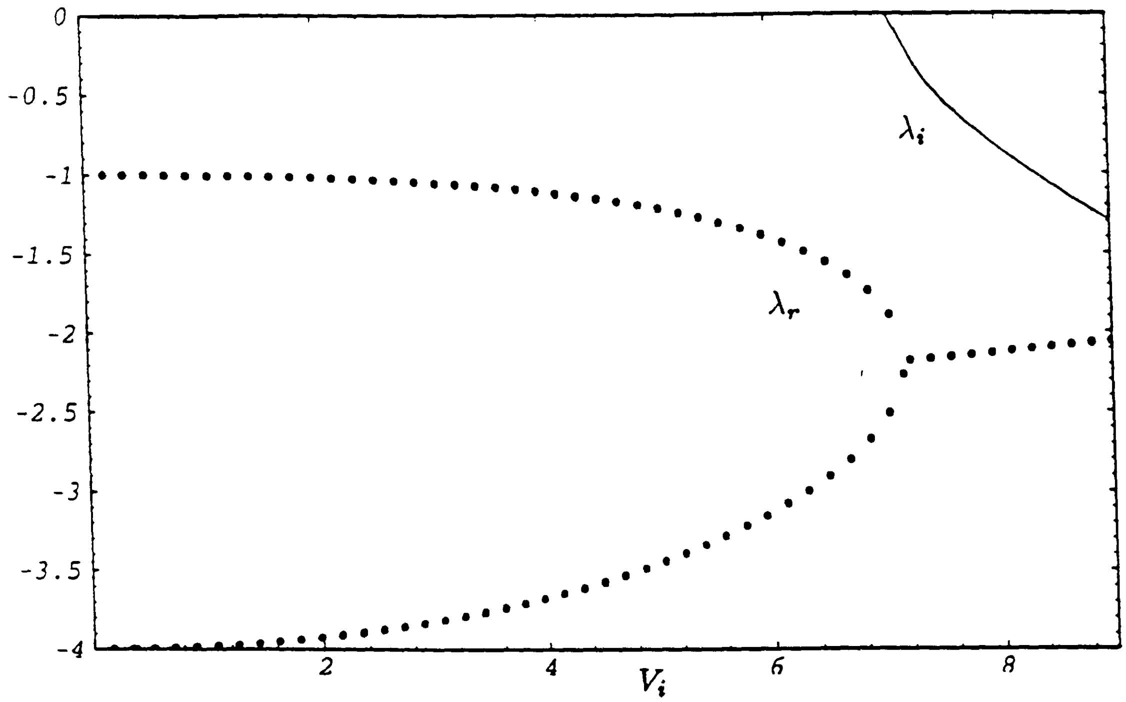


Figure D.1 The real and imaginary parts of eigenvalue λ of equation (D.1) for varying V_i with $V_r = 6$.

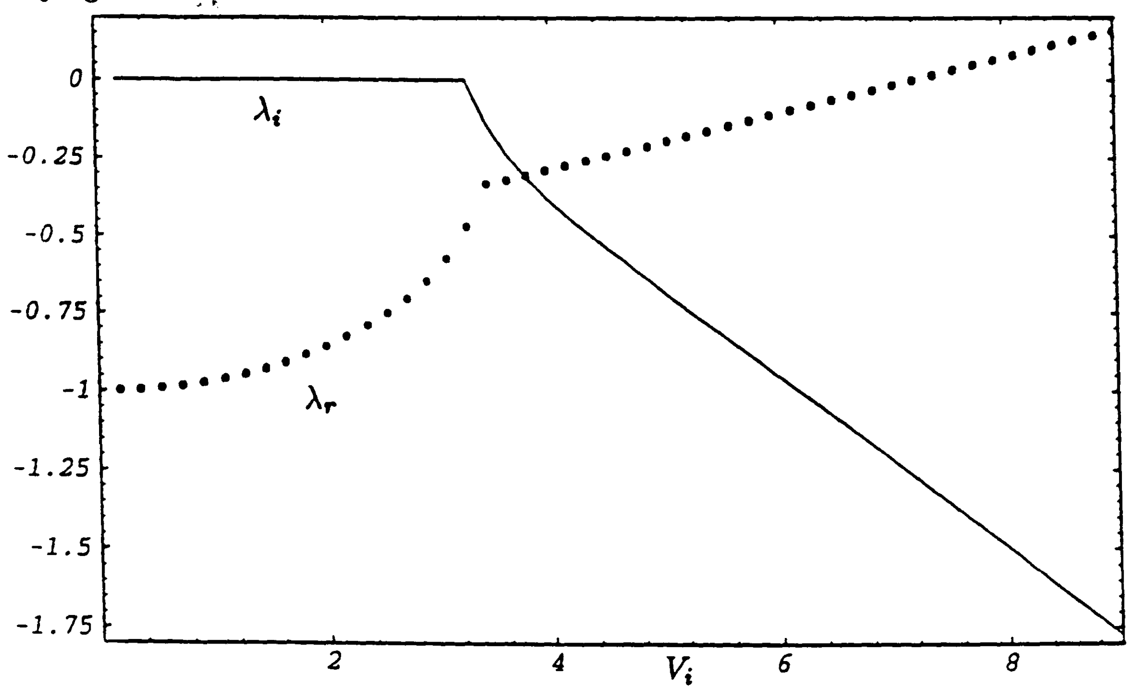


Figure D.2 The real and imaginary parts of eigenvalue λ of equation (D.1) for varying V_i with $V_r = 2$.

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