G/H CONFORMAL FIELD THEORY

Thesis by

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To my mother, Nancy Douglas

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Abstract

We show that for every affine Lie algebra G and subalgebra H there exists an exactly solvable two-dimensional conformal field theory, and give a procedure for explicitly determining its correlation functions and partition function given those for the Wess-Zumino-Witten models with symmetry algebras G and H.

1. Introduction

A powerful general approach to understanding physical systems is to identify the symmetry group of the system, classify its representations and express all physical quantities as invariants under the group. Quantum field theories (QFTs) are systems with infinitely many degrees of freedom, and so one might hope to identify symmetry groups with infinitely many parameters, and perhaps even express the partition function and correlation functions in terms of a finite number of invariants, allowing an exact solution.

In 1983 Belavin, Polyakov and Zamolodchikov (BPZ) [1] worked out the consequences of conformal symmetry in two-dimensional quantum field theory. Two dimensions are special as the conformal group is infinite-dimensional; if we parametrize the plane with a single complex variable z, any map $z \to f(z)$ where $f(z)$ is analytic is a local conformal transformation, *i.e.*, preserves angles. They went on to define a series of examples, the "minimal models," in which the state space of the system decomposes into a finite sum of representations of the conformal group. Surprisingly, the structure of these representations leads directly to a set of linear partial differential equations which can be solved to determine all correlation functions in these models.

These results turned out to be of importance far beyond that of a new class of exactly solvable models, for two reasons. One is that conformal invariance is generic in two dimensions; the long-distance limit of a quantum field theory will always be scale invariant (if there is a mass gap, it will be trivial), and it is a non-trivial fact that scale invariance implies conformal invariance in two dimensions. The previously known scale invariant two-dimensional models have all been identified as exactly solvable conformal field theories (CFTs), or perturbations of exactly solvable CFTs.

The second reason is the connection with string theory [2] – the world-sheet of a string is described by a two-dimensional quantum field theory, and it is believed that the condition that for strings to be propagating in a background which is a solution of the string equations of motion is exactly that this QFT be conformally invariant.

Exactly solvable CFTs give easily studied examples of string compactifications which come surprisingly close to realistic phenomenology. Furthermore, the techniques developed to study CFT have been used to make general arguments which apply to any string compactification, at least to the extent that perturbation theory in the string coupling constant is qualitatively correct. The most striking example of this is an argument due to Dixon, Kaplunovsky and Vafa [35] that compactifications of the type II superstring can never have fermions with the appropriate quantum numbers for the standard model.

There have been two main areas of progress in the study of CFT following BPZ. One is the study of correlation and partition functions on arbitrary Riemann surfaces. This is primarily motivated by string theory, where these amplitudes are used to calculate quantum loop corrections to string S-matrix elements. There is also hope that consistency conditions for these amplitudes will be strong enough to give a complete classification of CFTs.

The second approach has been to find larger symmetry groups than the conformal group, work out their representation theory and find a series of "minimal models" analogous to those of BPZ. There are two main types of symmetry which were inspired by previous work on quantum field theory. One is supersymmetry and extended supersymmetry, and these minimal models have all been classified. The other is Lie group symmetry, and the simplest models with this symmetry, sigma models on the appropriate group manifolds, have been exactly solved.

Primarily through the work of Fateev and Zamolodchikov [11,13], it has been realized that there are many more exactly solvable CFTs with symmetry algebras of an essentially new type – they include operators of dimension higher than two, or operators of fractional spin. Fateev and Zamolodchikov explicitly constructed a few such models, but it seemed clear that this was only the tip of the iceberg.

The main result of this thesis is the construction of a general class of exactly solvable CFTs, the "G/H models," which includes the minimal models, FZ's models and many new models. The construction is an extension of Goddard, Kent and

Olive's coset construction of minimal model Virasoro algebras and partition functions [14]. The simplest description of the models is that they are sigma models on the Lie group manifold G, in which the subgroup H has been gauged, eliminating all degrees of freedom not invariant under H. Our analysis will not use this definition; rather it will be algebraic in character. We will give a non-constructive definition of the G/H model partition function, and give simple ways to determine the spectrum of operators of low dimension and the asymptotics of the partition function, the quantities of most physical interest. We will then define G/H correlation functions, and show that they satisfy a factorization relation which gives a practical method of computing the correlation functions and operator product coefficients given those for the sigma models on G and H. Finally, we will classify the G/H models with $N = 1$ or $N = 2$ supersymmetry.

A number of interesting questions about G/H models are not resolved by the present work. It would be very interesting to classify supersymmetric G/H models. So far we have partial results along this line, to be described in section 8. It would be useful to find an action which describes the models. There is a loose end in the construction for models which are not of the form $H \times H/H$ – we should understand the diagram automorphism and how it acts on the state space. It should not be hard to prove the "duality conjecture" made in Section 8. It seems likely that there is a factorization relation for higher genus correlation and partition functions. There is an alternate construction of many of these models known as the "Feigin-Fuchs" or "Coulomb gas" construction – the relation between this construction and ours is not understood. Finally, there are solvable lattice models which have critical points which we can identify as G/H models, whose definitions also involve Lie algebras and cosets – can we relate these constructions to the CFT constructions in a more precise way?

Perhaps the most important question to resolve in the general theory of CFTs is whether the G/H models, free models and their orbifolds, WZW models and combinations of these models exhaust the set of rational CFTs.

The outline of the thesis is as follows. Section 2 is an introduction to CFT following BPZ, which discusses the general consequences of conformal invariance, and introduces the representation theory of the Virasoro algebra and the minimal models of BPZ. Section 3 discusses the sigma models on group manifolds, known as WZW models, and derives their partition functions using the representation theory of Kac-Moody algebras. Section 4 discusses correlation functions in CFT and derives the Knizhnik-Zamolodchikov equation, the differential equation satisfied by correlation functions in the WZW models. Section 5 introduces the G/H models and discusses the Goddard, Kent and Olive construction of their stress-energy tensors and partition functions. Section 6 defines G/H correlation functions, and defines and proves the factorization relation which determines them. Section 7 describes the minimal models as G/H models and gives examples of the factorization of correlation functions. Finally, section 8 describes what is known about more general G/H models.

2. Introduction to Conformal Field Theory

The simplest QFTs are the theories of a free massless bosonic or fermionic field. "Free" means of course that modes with different spatial dependence do not interact and the system decouples into an infinite set of systems with one degree of freedom each. We can express this as the invariance of the system under variations of the free field ϕ satisfying the equation of motion

$$\Delta\phi(x) = 0. \tag{1}$$

In two dimensions with a Euclidean metric, it is convenient to parametrize the plane with the complex variable $z = x^1 + ix^2$ and its complex conjugate $\bar{z}$. The metric then has components $g_{zz} = g_{\bar{z}\bar{z}} = 0$, $g_{z\bar{z}} = g_{\bar{z}z} = 1$, and the Laplacian is $\Delta = 2\partial_z\partial_{\bar{z}}$. The symmetry is then

$$\delta\phi(z,\bar{z}) = f(z) + \bar{g}(\bar{z}) \tag{2}$$

for analytic functions f and g. This symmetry can be used to solve the theory not just on the plane but on any Riemann surface.

This is a very restrictive assumption and it is interesting to find subgroups of this symmetry group which might apply to more systems. The most interesting turns out to be the group of reparametrizations $z \to z'$ where z' is an analytic function of z, with the complex conjugate of this action on $\bar{z}$. These are the conformal transformations. The infinitesimal reparametrizations $z' = z + \delta z(z)$ are a subclass of (2) with $f(z) = \delta z \partial\phi(z)$.

To discuss symmetries in quantum field theory we want to define a state space associated with each space-like hypersurface, and an algebra of operators which represent the symmetry. The first non-trivial consequence of conformal invariance is that we can choose any closed one-dimensional contour as a space-like hypersurface. Riemann's mapping theorem assures us that they are all equivalent. The usual convention is to use "radial quantization": we let $z = re^{i\theta}$, then r becomes "time" and space-like hypersurfaces have constant r. A reasonable question at this point is, does this choice restrict us to considering systems of finite volume (since θ is a periodic variable), or is there a mapping from this choice to a choice of an open contour as space-like hypersurface (*e.g.*, constant x^2). There is certainly a conformal mapping which takes us from the one choice to the other; however it seems that the more usual quantization with x^2 as time gives different physics – for example the Hamiltonian has continuous spectrum. What is happening is that we allow modes e^{ikx^1} which would have essential singularities at ∞ considered as functions of z. This singles out the point $z = \infty$ and breaks the invariance under conformal transformations such as $z \longrightarrow 1/z$. Conformal field theory with this quantization has not been discussed in the literature. Since the relevant quantization for string theory is the finite volume one we restrict ourselves to this case.

Reparametrizations will be generated by the stress-energy tensor $T_{\mu\nu}$, with $z \longrightarrow z + \delta z(z)$ generated by

$$\int \delta z T_{zz} dz + \int \delta z T_{z\bar{z}} d\bar{z} \tag{3}$$

This will be a conserved charge if (using the conservation law for T)

$$\bar{\partial}(\delta z T_{zz}) + \partial(\delta z T_{z\bar{z}}) = (\partial \delta z) T_{z\bar{z}} = 0, \tag{4}$$

which requires $T_{z\bar{z}} = 0$. (There is some choice in the definition of the conformal generators which allows certain $T_{z\bar{z}} \neq 0$; one can always redefine T to set $T_{z\bar{z}} = 0$ in this case [36].)

At first sight this is a rather strong condition; much stronger than the condition for scale invariance, which is that

$$T_{z\bar{z}} = \partial_\mu K^\mu \tag{5}$$

for some operator K. Since it is easy to come up with scale invariant quantum field theories (just apply the renormalization group until one is as close to a fixed point as desired) it is interesting to ask which scale invariant theories are conformally invariant. Results of Zamolodchikov [3,36] imply that scale invariance in fact implies conformal invariance in two dimensions. The proof is computational. One might hope that such a simple fact would have an intuitive explanation, but no satisfying one is known.

We can now write a basis for the infinitesimal conformal reparametrizations

$$\begin{aligned} \delta z &= \epsilon_n z^{n+1} \\ \delta \bar{z} &= \bar{\epsilon}_n \bar{z}^{n+1} \end{aligned} \tag{6}$$

and define their generators in CFT

$$L_n = \int dz\, z^{n+1} T_{zz}(z) \tag{7}$$

$$\bar{L}_n = \int d\bar{z}\, \bar{z}^{n+1} T_{\bar{z}\bar{z}}(\bar{z}).$$

$L_0 + \bar{L}_0$ now generates dilatations, which correspond to time evolution in radial quantization; $iL_0 - i\bar{L}_0$ generates rotations.

In two dimensions, the components of a tensor each transform irreducibly under conformal transformations if we work in the $z, \bar{z}$ basis. A useful notation generalizes the usual differential forms, to write a tensor with Δ z indices and $\bar{\Delta}$ $\bar{z}$ indices as

$$\phi = \phi_{z_1 z_2 \ldots z_\Delta \bar{z}_1 \ldots \bar{z}_{\bar{\Delta}}}(z, \bar{z})(dz)^{\Delta}(d\bar{z})^{\bar{\Delta}}. \tag{8}$$

This is a "geometric object" in the sense that it does not depend on the choice of the coordinate system z, allowing us to write

$$\phi(z, \bar{z})(dz)^{\Delta}(d\bar{z})^{\bar{\Delta}} = \phi(z', \bar{z}')(dz')^{\Delta}(d\bar{z}')^{\bar{\Delta}}. \tag{9}$$

We can get the transformation law for the function ϕ by formally dividing both sides by the dz factors. Examining the special cases of dilatations and rotations we see that the dimension of the operator ϕ (the eigenvalue of $L_0+\bar{L}_0$) is $\Delta+\bar{\Delta}$ and the spin is $\Delta-\bar{\Delta}$. Scale and translation invariance now determine the two-point function,

$$\langle\phi(z_1)\phi(z_2)\rangle = c(z_1-z_2)^{-2\Delta}(\bar{z}_1-\bar{z}_2)^{-2\bar{\Delta}}. \tag{10}$$

for some constant c.

If we substitute in $z' = z+\delta z$, we will get the infinitesimal transformation of ϕ, which is generated by the stress-energy tensor:

$$[\int dw\, \delta\, z(w)\, T_{ww}, \phi(z)] = \Delta(\partial\delta z)\phi(z) + \delta z\partial\phi(z). \tag{11}$$

We now introduce a very useful technical device: the equivalence between commutators and the singular parts of operator product expansions. Since $\int dw\,\delta z(w) T_{ww}$ is a conserved charge, we can express this commutator in a correlation function as

$$\left\langle \int_A - \int_B dw\, \delta z(w) T_{ww}\phi(z)\ldots \right\rangle \tag{12}$$

where the contour A is slightly later in time than the point z and the contour B slightly earlier in time. The exact definition of time-ordering we use here is irrelevant (up to a sign), as it should be in Euclidean space. Now the correlation function will satisfy the conservation law $\bar{\partial}T_{ww}=0$ everywhere except where w coincides with the position of an operator, such as $\phi(z)$, so we are integrating a meromorphic function, and the result will be the residue of the pole at $w=z$. We can get this by using the operator product expansion between the operators T and ϕ. Only the singular terms will be relevant, and these are finite in number.

It is easy to see that the commutator determines this operator product expansion to be

$$T(w)\phi(z) \to \frac{\Delta}{(w-z)^2}\phi(z) + \frac{1}{w-z}\partial\phi(z) + \mathrm{non-singular\ terms}\ldots \tag{13}$$

Not all operators in CFT transform as tensors, however. The simplest counterexample is T_{zz} itself. Classically it is a tensor component with $\Delta=2, \bar{\Delta}=0$, but if we assume that it satisfies the o.p.e. derived above, we can calculate its two-point function using the o.p.e. and the consequence of scale invariance that the one-point functions of all the operators on the right hand side must vanish, since they have non-zero dimension. We are left with the answer zero, which in a unitary QFT implies that $T_{zz} = 0$ in all correlation functions. Evidently the o.p.e. must contain a term with the operator 1:

$$T(w)T(z) \to \frac{c/2}{(w-z)^4} + \frac{\Delta}{(w-z)^2}T(z) + \frac{1}{w-z}\partial T(z) + \mathrm{non-singular\ terms}\ldots \quad (14)$$

where c is a constant, positive (as we shall see) by unitarity, which can depend on the particular CFT in question. (The 1/2 is a convention which makes $c = 1$ for a free boson.)

The c term will modify the conformal transformation law of T. One can see that transformations $\delta z = \epsilon_0 + \epsilon_1 z + \epsilon_2 z^2$ will remain unchanged, so translation, scale invariance and special conformal invariance behave as they do classically, but the rest of the conformal group has an "anomaly" compared to the classical transformation law. (As with other anomalies, this one comes from the need to renormalize in the quantum theory. For example, the stress tensor in the free bosonic theory is quadratic in ϕ, and if one defines this product of operators carefully one finds that one loses either the strict tensorial character of $T_{\mu\nu}$ or else the condition $T^{\mu}_{\mu} = 0$.)

The next major consequence of conformal invariance is that there is a one-to-one correspondance between states and local operators. The argument is simply that we can evolve our space-like hypersurface $r = \mathit{constant}$ into one with $r = \epsilon$ for arbitrarily small ϵ, allowing us to create any state with an operator of arbitrarily small support. This shows that some local operator creates each state; the correspondence is one-to-one because a local operator which annihilated the vacuum would be zero in all correlation functions. This has the further consequence that the operator product expansion is an exact statement – this is simply because if we draw a contour around a pair of operators, they must create some state on that contour, which could have been created by some local operator.

We are now in a position to derive the consequences of conformal invariance on the state space of a CFT. Given an operator $\phi(z)$, define

$$L_n\phi(z) = \int_C dw\,(w-z)^{n+1}T(w)\phi(z) \tag{15}$$

where the contour C surrounds z (and no other operators, if we are using this operator in a correlation function). The L_n's satisfy a Lie algebra which is determined by substituting the o.p.e. we found for T above and doing the contour integrals. This is the Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}. \tag{16}$$

Note that

$$[L_m, \bar{L}_n] = 0 \tag{17}$$

since the conservation laws $\bar{\partial}T = \partial\bar{T} = 0$ and translation invariance force the singular terms in the o.p.e. $T\bar{T} \to \ldots$ to be zero. We thus have two commuting Virasoro algebras, and we can formally consider conformal transformations which treat z and $\bar{z}$ separately. This separation of the dependence of physical quantities on z and $\bar{z}$ will be an important feature of our analysis. In our discussion of the Virasoro algebra, we will generally consider only L_n; the discussion for $\bar{L}_n$ is exactly parallel.

Now an operator ϕ which transforms as a tensor will satisfy

$$L_0\phi = \Delta\phi \tag{18}$$

$$L_n\phi = 0, \quad n > 0 \tag{19}$$

We will refer to such operators as "primary fields with respect to the Virasoro algebra," or "conformal fields." In general QFT given an operator ϕ we can talk about an infinite series of operators $\partial_\mu \phi$, $\partial_{\mu\nu}\phi$, etc. Substituting in the definitions we find that

$$\partial_z \phi = L_{-1}\phi,$$

$$\partial_z \partial_z \phi = L_{-1}L_{-1}\phi,$$

and so forth, but now we can also talk about $L_{-2}\phi$, $L_{-2}L_{-1}\phi$, and so on. These are all eigenstates of L_0, as we can see from the commutation relations.

These operators are called descendants of ϕ under the Virasoro algebra. None of them transform as a tensor. All of them have extra, more singular terms in the o.p.e. with T. The complete set of such operators for a given ϕ is a linear representation of the Virasoro algebra.

As usual in the representation theory of Lie algebras, we will find a maximal commuting set of generators, and diagonalize them. We can then label a state by its vector of eigenvalues of these generators, or "weights". In the case of the Virasoro algebra, the operators L_0 and c form a maximal commuting set. c is a rather trivial operator, being constant on an entire representation, but it will be useful to consider it as part of the vector of weights.

In the representations of physical interest, there is a lower bound on the eigenvalue of L_0 (and of c), so we refer to them as "lowest weight representations." (The term "highest weight representation" is also used, and the difference is only one of convention – if we called the eigenvalue of $-L_0$ a weight, we would be interested in highest weight Virasoro representations). Not all representations are lowest weight representations – an example which is not is the adjoint representation of the Virasoro algebra.

We will then divide the remaining generators into raising and lowering operators – any state in a representation can be reached by applying some string of raising operators. We can easily write down an overcomplete basis for the representation –

it consists of the states

$$L_{-N}^{I_N} L_{-(N-1)}^{I_{N-1}} \cdots L_{-1}^{I_1} \phi(0)|0\rangle. \tag{20}$$

Ordering the raising operators has taken into account all the relations between states which follow from the Virasoro algebra, so why is this an overcomplete basis? We will define an inner product on this space for QFT, and it will turn out that in general some of these states have zero norm and are thus not in the state space. This innocent fact is responsible for the tremendous success of BPZ's analysis – it will turn out that the models with zero norm Virasoro descendants have a finite number of primary fields, and that the zero norm states will provide us with differential equations which can be solved to obtain all correlation functions.

In more detail, we can define the inner product of two states created by operators ϕ_1 and ϕ_2 as the two-point function

$$\lim_{z \to \infty} z^{2\Delta} \bar{z}^{2\bar{\Delta}} \left\langle \phi_1(z, \bar{z}) \phi_2(0) \right\rangle . \tag{21}$$

Conformal invariance guarantees that this will be non-zero only for two operators of the same dimension. We can then define the adjoint of a state $\phi(0)|0\rangle$ as the result of a coordinate transformation which maps $z = 0$ to $z = \infty$. A simple (and standard) choice is $z \to 1/z$. We now have enough information to compute the norms of descendant states, determining their two-point functions by using the Ward identity for T, or equivalently the Virasoro algebra. Consider the first non-trivial example, the state

$$L_{-n}\phi(0)|0\rangle \equiv L_{-n}|\phi\rangle. \tag{22}$$

Substituting in our definitions above and remembering the factor $(dz'/dz)^2$ in the transformation law for T we find that

$$(L_{-n}|\phi\rangle)^{+} = \langle\phi|L_n, \tag{23}$$

so the norm of this state is

$$\begin{aligned}\langle\phi|L_nL_{-n}|\phi\rangle \\ &= \left\langle\phi|(2nL_0 + c(n^3 - n)/12|\phi\right\rangle. \\ &= (2n\Delta + c(n^3 - n)/12)\,\langle\phi|\phi\rangle\end{aligned}$$

By considering the large n we see that a unitary theory must have $c > 0$.

The first example of a null vector is

$$(L_{-2} + \alpha L_{-1}^2)\phi(0)|0\rangle. \tag{24}$$

This will be null if it is annhilated by $L_n, n > 0$, which will be true if

$$\alpha = -\frac{3}{2(2\Delta + 1)}. \tag{25}$$

and

$$\Delta = \frac{1}{16}[5 - c \pm \sqrt{(c-1)(c-25)}]. \tag{26}$$

This Δ will be real and positive only for $c < 1$, so only in this case will this null vector be relevant.

This inner product is the only non-trivial structure of a Virasoro representation. It depends only on c and Δ, the weight of the lowest weight state, so we can label a Virasoro lowest weight representation by its lowest weight. This is a general feature of unitary Lie algebra representations.

The structure of unitary highest weight Virasoro algebra representations has been analyzed by Kac [1,4], who found that there are three cases. For $c > 1$ there are never any null vectors, and all positive Δ are allowed. We will see later that this implies that such theories have an infinite number of conformal fields. For $c = 1$ all Δ are allowed, and some have null vectors. Finally, for $c < 1$ there are a discrete series of allowed c's, and a finite list of allowed Δ's for each c. All of these representations have null vectors.

In section 7 we will derive the structure of all of the $c < 1$ representations from the G/H construction. The original analysis used properties of the Virasoro algebra; it has the advantage that one can show (following Friedan, Qiu and Shenker [4]) that the list of representations we will construct is a complete list.

3. The Wess-Zumino-Witten Model

The free bosonic field in two dimensions has a $U(1)_{\rm left} \times U(1)_{\rm right}$ symmetry

$$\delta\phi = f(z) + g(\bar{z}) \tag{27}$$

generated by the conserved currents

$$J_{\rm left} = \partial_z \phi \, dz \tag{28}$$

$$J_{\rm right} = \partial_{\bar{z}} \phi \, d\bar{z} \tag{29}$$

Just as we did for the stress-energy tensor, we can define an algebra of the normal modes of these operators and find primary fields, lowest weight representations, and so forth; this is all quite well-known but we repeat the definitions to establish notation. The operator product expansion and stress-energy tensor are

$$\partial\phi(z)\partial\phi(0) \to \frac{1}{z^2} + 2T(0) + \ldots, \tag{30}$$

$$T(z) = \frac{1}{2} \lim_{w \to z} \partial\phi(w)\partial\phi(z) - \frac{1}{(w-z)^2}, \tag{31}$$

$$\alpha_n = \int dz \, z^n \, \partial\phi(z). \tag{32}$$

Lowest weight states satisfy

$$\alpha_0 |k\rangle = k|k\rangle,$$

$$\alpha_n |k\rangle = 0, \quad n > 0,$$

so they are created by operators $V_k(z)$ satisfying

$$\partial\phi(z) V_k(0) \to \frac{ik}{z} V_k(0) + \ldots \tag{33}$$

Since $\partial\phi(z)$ generates $\delta\phi(z) = 1$, this implies

$$\frac{\delta}{\delta\phi} V_k = ikV_k, \tag{34}$$

which has the solution

$$V_k(z) =: e^{ik\phi(z)} : \tag{35}$$

where the colons indicate the usual normal ordering.

We could in fact understand the free boson completely as the generic system with conformal and $U(1) \times U(1)$ symmetry. All properties of the model are determined once we postulate the currents J and assert that no operators commute with them. One non-trivial consequence of this is boson-fermion equivalence. The theory of a Dirac fermion in two dimensions also has $U(1)_{\text{left}} \times U(1)_{\text{right}}$ symmetry; no operators commute with the currents; therefore it is the same theory.

It is natural to try to extend this to more general Lie algebras, and we will find that the picture extends simply and beautifully. For any Lie algebra G, there is a series of models labelled by an integer k with conformal and $G_{\text{left}} \times G_{\text{right}}$ symmetry, which are completely determined by the assumption that no operators commute with all the currents. These models are usually called the Wess-Zumino-Witten models, since the simplest path integral realization of these models is as sigma models on group manifolds with Wess-Zumino terms in the action, as was first shown by Witten [8]. Our analysis will be purely algebraic, following Knizhnik and Zamolodchikov [9], and will not use the path integral definition.

We can immediately generalize the o.p.e. above to

$$J^a(z)J^b(z') \rightarrow \frac{kg^{ab}}{(z-z')^2} + \frac{f^{abc}}{(z-z')} J^c(z') + \ldots \tag{36}$$

There is a current J^a for each generator of the G_{left} symmetry; they are necessarily dimension $(1,0)$ to be conserved currents whose charges, $\int dz J(z)$, commute with L_0 and $\bar{L}_0$. In a general 2D QFT, J would have two components satisfying the conservation law $\bar{\partial} J_z + \partial \bar{J}_{\bar{z}} = 0$. In CFT the condition $\bar{\Delta} = 0$ for J_z guarantees that $\bar{\partial} J_z = 0$ and that we can take $J_{\bar{z}} = 0$. The G_{right} symmetry will behave similarly with currents $\bar{J}(\bar{z})$; if we were willing to consider a chiral theory we could drop these operators altogether.

f^{abc} and g^{ab} are the structure constants and metric respectively of the Lie algebra G. We will mostly confine our explicit examples to the algebra $SU(2)$; we will choose the generators

$$\begin{aligned} T^{\pm} &= \frac{1}{2}(\sigma^1 \pm i\sigma^2) \\ T^3 &= \sigma^3 \\ g^{+-} &= \mathrm{Tr}\, T^+ T^- = 1 \\ g^{33} &= 2 \\ f^{+-3} &= \mathrm{Tr}[T^+, T^-]T^3 = 2. \end{aligned} \tag{37}$$

In these conventions the weights (T^3 eigenvalues) are always integers.

Using this o.p.e. we can compute any correlation function of currents. We only need the singular part of the o.p.e. We argued earlier that this was equivalent to knowing the commutator of currents, and together with the conservation law this will give us a Ward identity relating correlation functions with a current to correlation functions with the current extracted. A simple way to derive the Ward identity is to consider the correlation function as a function of the position of the current $J(z)$. It is an analytic function and therefore determined by its singularities, which are exactly given by the singular part of the o.p.e. We will state it in section 4, where we use it.

The stress-energy tensor must be expressible in terms of J and satisfy the correct o.p.e. with J. There is a unique candidate discovered by Sugawara:

$$T(z) = -\frac{1}{(2k + C_a)} : J^a(z) J^a(z) : \tag{38}$$

The normal ordering can be done by point-splitting as in equation (31) above. The constant is determined by checking $TJ \to J + \ldots$; $C_a = f^{abc} f^{abc}/\mathrm{dim}G$ is the Casimir of the adjoint representation. It is now straightforward to determine c from the two-point function of T:

$$c = \frac{k \dim G}{2k + C_a}. \tag{39}$$

As before we define normal modes of the currents

$$J^a_n \phi(z) = \int dw\, w^n J^a(w)\phi(z). \tag{40}$$

These satisfy the algebra

$$[J^a_m, J^b_n] = f^{ab}_c J^c_{m+n} + km\delta_{m,-n}. \tag{41}$$

This is known in the mathematical literature as an untwisted affine Kac-Moody algebra.

All operators in the theory will fall into representations of the algebra of charges J^a_0, which we will call G_0. An operator $\phi^i(z)$, on which J^a_0 acts as the matrix $(t^a)^i_j$, must have the term

$$J^a(w)\phi^i(z) \to \frac{(t^a)^i_{i'}}{w-z}\phi^{i'}(z) + \ldots \tag{42}$$

in its o.p.e. As with the Virasoro algebra, we will single out primary fields with respect to the current algebra, for which this is the only singular term in the o.p.e., and descendant fields (such as J itself), which also contain more singular terms. Any operator in the model will then fall into some representation of the current algebra whose lowest weight state is created by a primary field.

Primary fields with respect to the current algebra will also be primary with respect to the Virasoro algebra. We can easily compute the dimension of a primary field by using the expansion

$$L_0 = -\frac{1}{(2k + C_a)} \sum_{m=-\infty}^{\infty} : J^a_{-m} J^a_m : . \tag{43}$$

Since a primary is annihilated by $J_m, m > 0$, we get the dimension in terms of the second Casimir of the representation,

$$\Delta_R = \frac{C_R}{2k + C_a}. \tag{44}$$

To understand the state space of the WZW theory, we need to understand the lowest weight representations of affine Kac-Moody algebras. The structure of the representation is entirely determined once we specify G, k and the representation of G_0 in which the lowest weight states transform. Since the stress-energy tensor is a product of currents, the dimensions of the operators and their z dependence is also determined. Given a lowest weight multiplet $|i\rangle = \phi^i(0)|0\rangle$ satisfying

$$J_0^a|i\rangle = (t^a)^i_{i'}|i'\rangle$$
$$J_n^a|i\rangle = 0, \quad n > 0$$

we define a Verma module just as for the Virasoro algebra, consisting of states

$$J^{a_1}_{-n_1} J^{a_2}_{-n_2} \ldots \phi^i(z) \equiv J^A_{-I}\phi^i(z). \tag{45}$$

We can eliminate the relations which follow from the current algebra by ordering the a_i's and n_i's.

Just as for the Virasoro algebra, there is a natural inner product on the state space coming from the adjoint operation $z \to 1/z$, implying

$$(J_n^a)^+ = (g^{ab})^{-1} J^b_{-n}. \tag{46}$$

We can compute the norm of each state, and to completely determine the structure of the representation we must deal with the null vectors.

We will now classify the unitary representations of the Kac-Moody algebra $SU(2)_k$ (k is the central charge). The general algebra is no more difficult, the only real change being to use vectors instead of single integers as roots and weights. Each finite dimensional $SU(2)$ representation R will give us an $SU(2)_k$ representation whose ground state transforms as R under G_0; the question is which representations are unitary.

A simple warm-up is the theory of unitary representations of $SU(2)$. We will construct the appropriate Verma module, find its null vectors and set them to zero. Since we will analyze Kac-Moody algebras by considering their finite $SU(2)$ subalgebras, this warm-up will be very useful. A highest weight state would then be defined as

$$\begin{aligned} T^3|l,l\rangle &= l|l,l\rangle, \\ T^+|l,l\rangle &= 0, \\ \langle l,l|l,l\rangle &= 1, \end{aligned}$$

and the Verma module built on it will be the states

$$|l, l-2n\rangle \equiv (T^-)^n|l,l\rangle \tag{47}$$

of weight $l-2n$. We have defined the norm of the highest weight state; the others are determined by the relation $(T^-)^\dagger = T^+$ and the algebra:

$$\begin{aligned} \langle l,m|l,m\rangle &= \langle l,m+2|T^+(T^-)^n|l,l\rangle, \quad n=(l-m)/2 \\ &= \langle l,m+2|(\sum_{k=0}^{n-1}(l-2k))|l,m+2\rangle \\ &= \frac{1}{4}(l-m)(l+2+m)\langle l,m+2|l,m+2\rangle. \end{aligned} \tag{48}$$

If l is a non-negative integer, $|l,-l-2\rangle$ will be a null vector – setting it to zero gives us a unitary representation of dimension $l+1$. If l is not a non-negative integer, as T^3 decreases we will eventually get a negative norm state, and the representation is not unitary.

We will also use the formula for the Casimir operator below,

$$\begin{aligned} C_l|l,m\rangle &\equiv (g^{-1})_{ab}T^aT^b|l,m\rangle \\ &= \frac{1}{2}l(l+2)|l,m\rangle. \end{aligned} \tag{49}$$

For example, $C_a = 4$.

Two important concepts which we will generalize to the Kac-Moody case below are the "character" of a representation, and the Weyl group. The character of the $SU(2)$ representation with highest weight l is simply

$$\begin{aligned}\chi_l &\equiv \mathrm{Tr}_l\, z^{T^3} \\ &= z^l + z^{l-2} + \ldots + z^{-l} \\ &= \frac{z^l - z^{-l}}{1-z}.\end{aligned}$$

This equation has a simple interpretation in terms of our Verma module and null vector construction. The Verma module, consisting of all the descendants of $|l,l\rangle$, had character $z^l/(1-z)$. We then subtracted a term for the null vector and its descendants.

The Weyl group is a symmetry group in weight space, reflecting our arbitrary choice of raising and lowering operators. We defined weights to be eigenvalues of T^3 and worked down from the highest weight state; we could just have easily defined weights to be eigenvalues of $-T^3$, interchanged the roles of T^+ and T^-, and worked up. So, for every state of weight m, there is one of weight $-m$. We can express this symmetry as an operation on characters

$$\begin{aligned}w(z^l) &\equiv z^{-l}, \\ w(f(z)+g(z)) &= w(f(z)) + w(g(z)).\end{aligned}$$

So, $w(\chi_l) = \chi_l$. A larger Lie algebra will have a Weyl group generated by reflections in each of its $SU(2)$ subalgebras.

We now return to the Kac-Moody case. Since every Kac-Moody algebra contains finite $SU(2)$ subalgebras which do not commute with L_0, every unitary Kac-Moody representation has non-trivial null vectors. This is in contrast to the case for the Virasoro algebra. We consider the example of $SU(2)_k$, the relevant $SU(2)$ subalgebra is then J_1^-, J_{-1}^+ and their commutator $k - J_0^3$. We will refer to this subalgebra as "pseudo-spin" (following Gepner and Witten [10]). Unitary representations of this $SU(2)$, will have a highest weight $k - J_0^3$ which is a non-negative integer.

Consider the $SU(2)_k$ representation whose ground state is a spin $l/2$ representation of the charges J_0^+, J_0^- and J_0^3. Each component of the ground state will be a highest weight state of the pseudo-spin algebra, with weight $k-l$. The inner product is proportional to that of the pseudo-spin representation, so we see that representations of the algebra $SU(2)_k$ with a ground state of spin greater than $k/2$ are not unitary. This is a general feature of Kac-Moody representation theory – a given algebra has only finitely many distinct unitary representations. Just as for the minimal models, this feature will allow us to define CFT with a finite number of primary fields and make an exact solution possible. We also see that k must be a positive integer (so that the representation with $l = 0$ is allowed).

There is a general analysis due to Kac [23] which identifies all the null states and makes it possible to write an explicit formula for the character of any representation. It will turn out that all of the null states are images of the lowest weight state under the Weyl group.

The exact object we will determine is the character

$$\chi_\Lambda(q,z) = \mathrm{Tr}_\Lambda\, q^{L_0} z^{J_0^3}, \tag{50}$$

where we take the trace over the physical states in the representation whose lowest weight state $|\Lambda, 0\rangle$ satisfies

$$J_0^3|\Lambda,0\rangle = \Lambda|\Lambda,0\rangle \tag{51}$$

$$L_0|\Lambda,0\rangle = \Delta_\Lambda|\Lambda,0\rangle. \tag{52}$$

Δ_Λ is the dimension of the primary in this representation. The character will be a power series in q and z, with a term $q^{\Delta+n} z^\mu$ coming from a term in the trace

$$\langle\mu,n|q^{L_0} z^{J_0^3}|\mu,n\rangle.$$

It will be convenient to work with a Verma module obtained by applying $J_n^a, n < 0$

and J_0^- to a highest weight state; its character is

$$\chi_V(q,z) = (1-z^{-2})^{-1} \prod_{m=1}^{\infty} (1-q^m z^{-2})^{-1}(1-q^m)^{-1}(1-q^m z^2)^{-1}. \qquad (53)$$

We can write

$$\chi_\Lambda(q,z) = \sum c_{\mu,n} q^n z^\mu \chi_V(q,z) \qquad (54)$$

where $c_{\Lambda,\Delta} = 1$ and we allow other non-zero $c_{\mu,n}$ to subtract null states and their descendants. We may also have positive c's – it will turn out that a null state may have descendants which are linearly related only because the null state was a descendant of some other state, and so its character will not be given correctly by χ_V. All we know so far is that the c's are integers and non-zero only for μ, n pairs which can correspond to null states.

The first observation is that μ, n can correspond to a null state in the representation with highest weight Λ only if

$$\left(\frac{1}{(2k+C_a)} J_0^a J_0^a - n\right) |\mu, n\rangle \equiv \Omega|\mu,n\rangle = \Delta_\Lambda |\mu,n\rangle. \qquad (55)$$

This is because $L_0 = \Delta_0 + n$ on this state, and the terms in L_0 containing $J_m^a, m > 0$ will annihilate a null state. Substituting in C_a, C_μ and C_Λ we find

$$\frac{\mu(\mu+2)}{4(k+2)} - n = \frac{\Lambda(\Lambda+2)}{4(k+2)}. \qquad (56)$$

We now introduce the Weyl group. We will define it by its action on weight space, and later introduce a corresponding action on functions $f(q,z)$ such as the character. The Weyl group is generated by reflections in each of the $SU(2)$ subalgebras, or just those corresponding to simple roots. It suffices to consider the Weyl group of G_0 supplemented by the reflection in the pseudo-spin $SU(2)$. Here the Weyl group for

$SU(2)$ just consists of 1 and the reflection $J_0^3 \to -J_0^3$, which takes

$$w_1 : (\mu, n, k) \to (-\mu, n, k). \tag{57}$$

We have included the value of k for the representation in the weight vector, because we are going to want to use different values of k in our intermediate steps.

The new generator is the reflection $k - J_0^3 \to -(k - J_0^3)$, $J_1^+ \leftrightarrow J_{-1}^-$. This will also change the L_0 eigenvalue – a state with $J_0^3 = \mu$, $L_0 = n$ will be exchanged with the state with $L_0 = n + k - \mu$. So, we have

$$w_0 : (\mu, n, k) \to (2k - \mu, n + k - \mu, k). \tag{58}$$

The Weyl group acts on characters in a natural way, mapping a term with weight v into its image $w(v)$. Explicitly,

$$w(q^n z^\mu u^k) = q^{n'} z^{\mu'} u^{k'}, \tag{59}$$

where

$$(\mu', n', k') = w((\mu, n, k)). \tag{60}$$

Again, we have added a k-dependent term to our definition of character, for future use.

The Weyl group has several important properties. The most important is that the character for a representation is invariant under the action of the Weyl group – this is because the $SU(2)$ character is invariant under reflection, and a Weyl reflection acts on each irreducible representation of its $SU(2)$ independently.

Another important fact is that the quantity Ω defined above is invariant under a "shifted Weyl reflection." Let v be a weight vector (μ, n, k) and let

$$\rho = (1,0,2). \tag{61}$$

Then

$$\Omega(v) = \Omega(w(v+\rho)-\rho). \tag{62}$$

This will allow us to show that all the null states are images of the lowest weight state under the "shifted Weyl group." One can check this by explicit calculation; the easy way to see this is to define an inner product on weight space,

$$(\mu, n, k)\cdot(\mu', n', k') \equiv \frac{1}{2}\mu\mu' + nk' + n'k, \tag{63}$$

which is preserved by the Weyl group, and then write

$$\Omega(v) = (2k + C_a)(|v+\rho|^2 - |\rho|^2). \tag{64}$$

(Note that this inner product is not related to the inner product on our Hilbert space, it is a technical device only.)

For example, consider the $k=1$ representation whose ground state is a singlet. The lowest weight is $\Lambda = (0,0,1)$; acting with the shifted Weyl group we get

$$\begin{aligned} w_1(\Lambda+\rho)-\rho &= (-2,0,1), \\ w_0(\Lambda+\rho)-\rho &= (4,2,1), \\ w_0 w_1(\Lambda+\rho)-\rho &= (6,4,1), \end{aligned}$$

etc...

$$w_1 w_0((\lambda,n,k)+\rho)-\rho = (\lambda - 2(k+2), n+k+1-\lambda, k), \tag{65}$$

$$(w_1 w_0)^N((\lambda,n,k)+\rho)-\rho = (\lambda - 2N(k+2), n - N(\lambda+1) + N^2(k+2), k). \tag{66}$$

Let us compare our present construction of the $k = 1$ representation with the alternate representation using a free boson. The free boson construction defines the $SU(2)_1$ currents as

$$J^3(z) = \partial\phi(z), \qquad J^{\pm}(z) = e^{\pm i\sqrt{2}\phi}. \tag{67}$$

The Verma module built on $(0,0,1)$ has descendants which form a 3 of G_0 at $\Delta = 1$, and a $5+3+1$ at $\Delta = 2$. The null vector at $\Delta = 2$ is the highest weight of a 5 of G_0, so this representation actually has a $3 + 1$ at $\Delta = 2$. We can check this by using the representation of $SU(2)_1$ constructed from a free boson – the operators with $\Delta = 2$ are $\partial\phi e^{\pm i\sqrt{2}\phi}$, $\partial^2\phi$ and $(\partial\phi)^2$, forming a $3 + 1$. The first operator with $J_0^3 = 4$ is $e^{2i\sqrt{2}\phi}$ with $\Delta = 4$.

It remains to show that these are all possible weights of null states, and to find their multiplicities. Suppose we have a null state with weight (μ, n, k); by using the Weyl group we can relate it to a null state with $0 \le \mu \le k$. If it is a descendant of $(\lambda, 0, k)$ it must have $\mu \le \lambda + n$. It is a short exercise to show that these conditions and the condition $\Omega((\mu, n, k)) = \Omega((\lambda, 0, k))$ imply that $(\mu, n, k) = (\lambda, n, k)$.

To find the multiplicities we will use a formal argument due to Weyl. Adding our variable u^k which keeps track of the value of k in a representation, our character formula reads

$$\chi_\Lambda(q, z, u) = u^k \sum c_{\mu,n} q^n z^\mu \chi_V(q, z). \tag{68}$$

Since the Weyl group preserves the character of a unitary representation,

$$w(\chi_\Lambda(q, z, u)) = \chi_\Lambda(q, z, u). \tag{69}$$

Multiply both sides of (68) by $R \equiv u^2 z/\chi_V(q, z)$. R has the property that $w_0(R) = w_1(R) = -R$ (an easy calculation), Therefore both sides of the resulting equation have this property. On the right, we can relabel the sum to get

$$w\big(u^k \sum c_{\mu,n} q^n z^\mu\big) = u^k \sum c_{w((\mu,n,k)+\rho)-\rho} q^n z^\mu. \tag{70}$$

Since all of the terms in the sum are images of $(\Lambda, 0, k)$ under the Weyl group, this relation determines all of the c's.

We have arrived at the Weyl-Kac character formula for $SU(2)$ (setting $u = 1$ again),

$$\chi_\Lambda(q,z) = q^\Delta \sum_w \epsilon(w) w(z^{\Lambda+1} u^{k+2}) z^{-1} u^{-k-2} \chi_V(q,z), \tag{71}$$

where $\epsilon(w)$ is the parity of the Weyl group element w, ± 1 depending on whether it is the product of an even or odd number of reflections.

We have an explicit formula for all of the images of $(\Lambda, 0, k)$ under the Weyl group, equation (66). This allows us to write the sum in the numerator explicitly:

$$\chi_\Lambda(q,z) = \big(\Theta_{\Lambda+1,k+2}(q,z) - \Theta_{-\Lambda-1,k+2}(q,z)\big)\chi_V(q,z) \tag{72}$$

where

$$\Theta_{n,k}(q,z) \equiv \sum_{j \in Z+n/2k} q^{kj^2} z^{-kj}. \tag{73}$$

The functions Θ are the familiar Jacobi theta-functions from the theory of elliptic functions. These appear because the characters must have simple transformation properties under the modular group, for reasons explained below.

One final simplification can be made – for $k = 0$ all descendants of the ground state are null, so

$$\chi_{0,k=0}(q,z) = 1. \tag{74}$$

The Weyl-Kac formula is still valid in this case, so we find that

$$\frac{1}{\chi_V(q,z)} = \big(\Theta_{1,2}(q,z) - \Theta_{-1,2}(q,z)\big). \tag{75}$$

We have actually made a slight change in our definition of χ in passing to the expression in terms of Θ, multiplying by a factor $q^{-c/24}$ which will simplify the modular transformation – this factor will also be explained below.

Now that we have the characters for Kac-Moody representations, we will derive the partition function for the WZW model, following Gepner and Witten [10]. The partition function in a general system is

$$Z(\beta) = \sum_{\text{all states}} e^{-\beta E_s}. \tag{76}$$

In quantum field theory we can write this as

$$Z(\tau) = \text{Tr } e^{-\tau H} \tag{77}$$

since the Hamiltonian is an operator whose eigenvalues are the energies of all states. We could then express $< f|e^{-\tau H}|i >$ as a Euclidean path integral over a region of space-time which extends over a time τ with boundary conditions i and f on the initial and final space-like hypersurfaces, and finally express the trace as identifying the boundary conditions i and f and integrating over all possibilities, or equivalently considering our surface to be periodic in the time direction.

Since we are doing CFT with periodic boundary conditions, this prescription is equivalent to computing the value of the path integral on a toroidal surface. The path integral calculation has an additional symmetry which our original definition did not, however; we can exchange the space and time directions. Since the overall scale of the system is irrelevant by conformal invariance, we can describe our torus with a single number l, the ratio of the lengths of the two non-trivial closed cycles. We now have the relation

$$Z(l) = Z(1/l). \tag{78}$$

It will be convenient to allow not only time evolution in our trace but also an arbitrary twist in the space direction, since this will give us the most general torus.

We therefore write

$$Z(q,\bar{q}) = \mathrm{Tr}\, q^{L_0} \bar{q}^{\bar{L}_0} \tag{79}$$

as an analytic function of the complex variable q and its conjugate. This describes evolution from a space-like hypersurface defined at $|z| = 1$ to one at $|z| = |q|$ with a twist taking $z = 1$ to $z = q$, followed by taking the trace. This coordinate system can be related to one which treats time and space more symmetrically; write

$$z = e^{2\pi i \nu},$$

$$q = e^{2\pi i \tau}.$$

The torus is now a parallelogram in the ν plane with opposite sides identified, $\nu \simeq \nu + 1 \simeq \nu + \tau$.

Our partition function now has the symmetry of exchanging time and space

$$Z(\tau) = Z(-1/\tau) \tag{80}$$

(the sign comes because a time evolution with no twist corresponds to $\tau = il$). If all states in the theory have integer spin (as states created by measurable operators must; this condition is also a subset of the $L_0 = \bar{L}_0$ required for physical states in string theory), we also have the symmetry

$$Z(\tau) = Z(\tau + 1). \tag{81}$$

These two transformations (to be referred to as S and T respectively) generate a group isomorphic to $SL(2,Z)$, often referred to as the "modular group."

Since the left and right current algebras commute, a representation of both algebras is a tensor product of representations of the two algebras, and its character is just a product. The partition function of the WZW model therefore has the form

$$Z(q,\bar{q}) = \sum_{\Lambda,\bar{\Lambda}} h_{\Lambda,\bar{\Lambda}} \chi_\Lambda(q) \chi_{\bar{\Lambda}}(\bar{q}) \tag{82}$$

where the h's are the multiplicity of the representations.

We can express the constraint of modular invariance in terms of h if we know how the vector of characters transforms under S and T. We now show that S and T act as unitary matrices on the vector of $SU(2)_k$ characters, by explicit computation. The modular transformation law of the theta function is standard:

$$\Theta_{n,k}(q = e^{-2\pi i/\tau}, z = 1) = (-i\tau)^{1/2} \sum_{n' \in Z/2kZ} e^{-\pi i n n'/k} \Theta_{n',k}(e^{2\pi i \tau}, 1). \tag{83}$$

$$\Theta_{n,k}(q = e^{2\pi i(\tau+1)}, z = 1) = e^{2\pi i n^2/4k} \Theta_{n,k}(e^{2\pi i \tau}, 1). \tag{84}$$

Evidently T is unitary on χ_Λ; each character picks up the phase $e^{2\pi i(\Delta - c/24)}$, where Δ is the dimension of the primary field. The $c/24$ term may be a bit surprising; evidently our character is actually Tr $q^{L_0 - c/24}$. This can be thought of as an additional term in the vacuum energy. (Evidence for this is the standard calculation of the vacuum energy of a free boson as $\frac{1}{2}\sum n = -1/24$.) We can also determine S; the denominator of χ_Λ is invariant while the transformation of the numerator gives us

$$\chi_\Lambda(-1/\tau) = S_{\Lambda,\Lambda'} \chi'_\Lambda(\tau), \tag{85}$$

with

$$S_{\Lambda,\Lambda'} = \left(\frac{2}{k+2}\right)^{1/2} \sin\left(\frac{(\Lambda+1)(\Lambda'+1)}{k+2}\right), \tag{86}$$

which is unitary. (If we had not included the $q^{-c/24}$ factor, this transformation law would not be simple.)

The modular invariance condition is therefore

$$S_{\Lambda,\Lambda'} h_{\Lambda',\bar{\Lambda}'} S_{\bar{\Lambda}',\Lambda} = h_{\Lambda,\bar{\Lambda}} \tag{87}$$

and

$$T_{\Lambda,\Lambda'} h_{\Lambda',\bar{\Lambda}'} T_{\bar{\Lambda}',\Lambda} = h_{\Lambda,\bar{\Lambda}} \tag{88}$$

Since S and T are unitary, we always have at least one solution:

$$h_{\Lambda,\bar{\Lambda}} = \delta_{\Lambda,\bar{\Lambda}}. \tag{89}$$

We refer to this as the "left-right symmetric" solution.

There are often other solutions of these constraints. These have been completely classified for $SU(2)_k$ [38]. Besides the left-right symmetric solution, there is another solution whenever k is even, which is the spectrum of the sigma model with target space $SO(3)$. There are also three "exceptional" solutions, for $k = 10, 16$ or 28. Two of these can be understood as left-right symmetric partition functions of the WZW models $SO(5)_1$ and $G_2, k = 1$, which turn out to be quantum mechanically equivalent to $SU(2)$ sigma models – we will be able to understand this later as instances of the G/H construction in which the G/H model has $c = 0$ and is therefore trivial. The other can be obtained by decomposing an $E_7, k = 1$ partition function. A unified explanation of the $SU(2)$ modular invariant partition functions has been found by Nahm [37].

4. Correlation Functions in the WZW Model

In principle, all correlation functions on any genus Riemann surface for a conformal field theory are determined if we know the two and three-point functions for all operators in the theory. For correlation functions on the sphere we can see this by using the operator product expansion to calculate correlation functions. An N-point function can be reduced to a sum of $(N-1)$-point functions by using the o.p.e., and so it can be evaluated by a recursive procedure given the o.p.e. Since the o.p.e. is determined by the complete set of three-point functions, we are done. We note here that a two or three-point function in CFT is specified by a single complex number. This is because the conformal symmetry contains as a subgroup $SL(2,C)$, a group with enough parameters to fix three points on the Riemann sphere. It follows that the dependence of a correlation function on up to three positions of operators must be determined by the conformal symmetry.

On higher genus surfaces, we can imagine constructing any correlation function if we have the "three-string vertex" or "pants," the transition amplitude for a sphere with three holes cut in it and specific boundary conditions on each boundary, since we can form any Riemann surface by sewing together such surfaces at the boundaries. We now must deal with the complication that the amplitude will depend on the specific metric on the resulting surface through the conformal anomaly. If we are interested in the ratio of an N-point amplitude to the vacuum amplitude on a given surface, this dependence will cancel out. We could then get the vacuum amplitude by integrating the expectation value of the stress-energy tensor [33].

To define a CFT, it thus suffices to define the three-point functions between any primary fields of the theory. If the theory has a symmetry algebra larger than Virasoro for which "sufficiently powerful" Ward identities exist, we need only define three-point functions between primary fields with respect to this algebra. We will work through the example of the WZW models and show that the Ward identities for the current algebra do determine three-point functions of all descendants in terms of primaries. Later we will discuss G/H algebras briefly, and talk about problems in using them to determine correlation functions directly.

So far, the three-point functions of primary fields are undetermined. These will be determined by a consistency condition. To define an amplitude on a given Riemann surface in terms of three-point functions, we must make a choice of how to divide the surface into "pants," and the consistency condition is that the answer must not depend on this choice. The simplest example is the four-point function; there are three ways to divide four operators into two pairs of operators, and thus three "pants" decompositions. If we evaluated the amplitude by using the o.p.e., this would be the choice of which two operators to apply the o.p.e. on. This condition thus generalizes BPZ's "associativity of the o.p.e." In string theory it is what we mean by "duality."

It is not hard to see that duality of the four-point function implies duality for N-point functions on the sphere. If we evaluate the N-point function by successive o.p.e., duality of the four-point function allows the transposition of any pair of external legs. Whether duality and modular invariance on genus 1 imply modular invariance at arbitrary genus is not known.

We will be able to discuss the four-point functions in the WZW models explicitly, because we will derive a differential equation (first found by Knizhnik and Zamolodchikov [9]) which determines them. This will allow us to compute the o.p.e. coefficients of primary fields.

There are two steps to defining the WZW three-point functions. The first step is to relate all correlators of descendants of primary fields to a correlator of primary fields, by recursively using the Ward identity to express a correlation function involving N currents in terms of correlators involving fewer currents. This step determines all the three-point functions in terms of a finite set of coefficients, the three-point functions of primary fields. Explicitly, in

$$\left\langle J^{A}_{-I}\bar{J}^{\bar{A}}_{-\bar{I}}g^{i\bar{\imath}}(x,\bar{x})J^{B}_{-J}\bar{J}^{\bar{B}}_{-\bar{J}}g^{j\bar{\jmath}}(y,\bar{y})J^{C}_{-K}\bar{J}^{\bar{C}}_{-\bar{K}}g^{k\bar{k}}(z,\bar{z})\right\rangle \tag{90}$$

we can substitute the definition of $J^{A}_{-I}\bar{J}^{\bar{A}}_{-\bar{I}}g^{i\bar{\imath}}(z,\bar{z})$ in terms of contour integrals of the currents, and use the Ward identity [9]

$$\begin{aligned}\langle J^a(w)J^{a_1}(z_1)\ldots g^{a_M}(z_M)\ldots g^{a_N}(z_N)\rangle = \\ \sum_{n=1}^{M-1}\frac{k\delta^{a,a_n}}{(w-z_n)^2}\langle J^{a_1}(z_1)\ldots \widehat{J}^{a_n}(z_n)\ldots g^{a_M}(z_M)\ldots g^{a_N}(z_N)\rangle \\ +\sum_{n=1}^{N}\frac{(t^a)^{a_n}_{a_n'}}{w-z_n}\langle J^{a_1}(z_1)\ldots g^{a_n'}(z_n)\ldots g^{a_N}(z_N)\rangle\end{aligned} \tag{91}$$

(and its complex conjugate) repeatedly until we have expressed the correlator in terms of the correlator of the three primary fields.

Since the WZW primaries are also Virasoro primaries, the z and $\bar{z}$ dependence of the three-point function is fixed by conformal invariance, and we have expressed the general three-point function in terms of a finite number of coefficients, the three-point functions of primary fields. These are of course further constrained by invariance under the subalgebras of charges G_0 and $\bar{G}_0$. It will be useful to distinguish two cases. Given a choice of three primary fields, we can express the three-point function in terms of a basis of invariants formed from their group indices. For the algebras $SU(2)$ and $U(1)$, the product of three irreducible representations never contains more than one singlet, so each three-point function is determined by a single coefficient. For larger algebras, more than one invariant can be constructed in general, and the three-point function of primaries depends on more than one coefficient which is so far undetermined.

There is a selection rule which restricts the possible invariants which can appear in a three-point function [10]. In general, if we make some assumption for the three-point function of the primaries and work out the three-point functions of descendants, it can happen that a descendant with zero norm appears in a non-zero three-point function. This is an inconsistency and we must choose the coefficients of three-point functions of primaries so that it does not occur. For the model $SU(2)^k$, for example, the sum of the highest spins of the three primary fields in a non-zero three-point function must be less than or equal to k.

Having determined all three-point functions in terms of the three-point functions of primary fields, the next step is to determine the three-point functions of primary fields by the requirement that four-point functions of primary fields be dual. This problem turns out to be very like the problem of finding modular invariant partition functions. Since $SL(2;C)$ invariance allows us to fix the positions of three operators in a correlation function, a four-point function is effectively a function of one complex variable, and its complex conjugate. In the models we are concerned with, a four-point function can be written as a finite sum of terms

$$\sum C_{\alpha\bar{\alpha}} f^{\alpha}(z) f^{\bar{\alpha}}(\bar{z}) \tag{92}$$

where each term in the sum can be defined as follows:

$$f^{ixy}(z) = \sum_{I,J} \langle g_1(\infty) g_2(1)(J_{-I} g_i)(0)\rangle_x \, \langle (J_{-I} g_i)(J_{-J} g_i)\rangle^{-1} \, \langle (J_{-J} g_i)(\infty) g_3(z) g_4(0)\rangle_y \tag{93}$$

Here J_{-I} is a string of creation operators from the current algebra, indexed by the multiindex I. In other words, we decompose the four-point function into a sum over all intermediate states which are descendants of a particular primary field under the analytic current algebra. To write f^{α} we made three choices – the intermediate primary field i, and the group invariants x and y which are the values of the two three-point functions when their three arguments are all primary fields.

Multiplying $f^{\alpha}(z) f^{\bar{\alpha}}(\bar{z})$ gives as intermediate states all descendants under both symmetry algebras. Finally the sum allows us to include all primaries (and all invariants), with coefficients C. To write this expression we had to choose a channel (s, t or u) in which to insert the sum over intermediate states.

The individual functions $f^{\alpha}(z)$ are referred to as "WZW blocks," as they are determined entirely by WZW current algebra. They are meromorphic, but not single-valued functions of z. For example, under the transformation $z \rightarrow e^{2\pi i} z$, $f^{i}(z)$ picks up the phase $e^{2\pi i(\Delta_i - \Delta_3 - \Delta_4)}$, since all the intermediate states have dimension which is some integer plus Δ_i, the dimension of the primary g_i. This is exactly analogous to the transformation of genus 1 characters under $\tau \to \tau + 1$. We could also consider monodromy transformations which involve taking z around the other singularities at 1 and ∞. Our analogy with genus 1 characters would suggest that these also act linearly on the vector of WZW blocks, although not as diagonal matrices in this basis. This must be true for the whole picture to make sense, but the easiest way to see it in the specific case of the WZW models is to explicitly derive the WZW blocks, which we will now do.

We now show that the WZW blocks are determined by a differential equation derived from a null vector of the combined Kac-Moody and Virasoro algebras, the Knizhnik-Zamolodchikov equation. The relevant null vector is simply

$$(L_{-1} + \frac{1}{k + C_a} J^a_{-1} J^a_0) \phi^i(z) = 0 \tag{94}$$

which follows from the definition of L_n and the fact that J_m annihilates ϕ for $m > 0$. A four-point function of primary fields thus satisfies

$$\left\langle (L_{-1} + \frac{1}{k + C_a} J^a_{-1} J^a_0) \phi^i_0(z) \phi^{j_1}_1(z_1) \phi^{j_2}_2(z_2) \phi^{j_3}_3(z_3) \right\rangle = 0. \tag{95}$$

Now $L_{-1}\phi = \partial\phi$, and $J^a_0 \phi^i = (t^a)^i_{i'} \phi^{i'}$. Finally, we rewrite

$$J^a_{-1}\phi(z) = \int \frac{dw}{(w - z)} J^a(w) \phi(z), \tag{96}$$

and re-express this contour integral as a sum of contour integrals around the other three z_i's. We can substitute in the o.p.e. between $J^a(w)$ and the other $\phi(z_n)$'s; only the singular term

$$\frac{1}{w - z_n} (t^a)^{j_n}_{j'} \phi^{j'}_n(z_n) \tag{97}$$

will contribute. The result is

$$[(k + C_a)\partial_z + \sum_{n=1}^{3} \frac{t^a_0 t^a_n}{z - z_n}] \langle \phi_0(z) \phi_1(z_1) \phi_2(z_2) \phi_3(z_3) \rangle = 0, \tag{98}$$

where we have written t^a_n for the generator t^a acting on ϕ_n. We can simplify the equation slightly by using the $SL(2,C)$ invariance to set $z_3 = \infty$, which will eliminate one term from the equation.

This is a first order linear matrix differential equation of Fuchsian type. The theory of these equations is well understood. We can write the equation as

$$\partial_z M(z) = A(z)M(z), \tag{99}$$

where A is the matrix of coefficients and M a matrix of solutions – the different columns corresponding to different solutions of the equation. $M(z)$ is analytic except at the singularities of the coefficients of the equation, *i.e.*, at the z_i's. It is not single-valued, however. If we analytically continue z around a singularity z_i, we will get a new solution which must be linearly related to the old solution,

$$M(\gamma z) = M(z)C_\gamma. \tag{100}$$

Each path γ will thus be associated with a linear transformation C_γ, giving us a representation of the monodromy group.

We need to impose one more constraint on the solutions of this equation – they must satisfy charge conservation,

$$\sum_{n=0}^{3} t_n^a \langle \phi_0(z)\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = 0. \tag{101}$$

We can do this by expressing the solutions as sums

$$\sum_{x,y} I_x^{j_0 j_1 k} I_y^{k j_2 j_3} f^{xy}(z) \tag{102}$$

where the I's are group invariants of G_0 with three indices, where the index k is arbitrary. We can write the action of the matrices $t_0^a t_n^a$ as matrices on this basis of group invariants, and we are left with a matrix differential equation on $N \times N$ matrices, where N is the number of G_0 singlets in the product of the four representations of the fields ϕ_n.

The matrix $M(z)$ is ambiguous up to a linear transformation $M(z)L$, but if we make the right choice of M, its columns are exactly the WZW blocks which we defined earlier as sums over all intermediate states which are descendants of a given primary. To get the solution which corresponds to descendants of a particular primary in a particular channel, we choose one of the z_i (say z_1) and specify the lowest order term in a power series around $z = z_1$ to be

$$f^{pxy}(z) = I_x^{j_0 j_1 k} I_y^{k j_2 j_3} (z - z_1)^\alpha + \dots \tag{103}$$

where k is an index of the group representation p for our intermediate primary, and α is the eigenvalue of $A(z_1)$ acting on this group invariant – A will be diagonal in this basis since we can write

$$t_0^a t_1^a = [(t_0^a + t_1^a)^2 - (t_0^a)^2 - (t_1^a)^2]/2 \tag{104}$$

and our invariants combine a single representation of each of t_0^a, t_1^a and $t_0^a + t_1^a$ – so the Casimirs formed from these operators are constant on each invariant. We then use the differential equation to get a recursion relation between the matrix coefficients of the power series for f. There are two cases – either the recursion relation can be solved at each stage and we get a solution which must be equal to our original definition of the WZW block (since we satisfy all the symmetries which determined it), or else the recursion relation will fail to have a solution at some order. This later case can only happen if the WZW block does not exist, which will be the case if we start with a zeroth order term which was forbidden by the selection rule we described above. In this case we will eventually get a term in the sum over states which has a non-zero coupling to an intermediate null state, and we will not be able to normalize the null state. In the theory of LODE's it is shown that a first order matrix LODE with dimension n matrices always has n solutions, and that when the recursion relation fails one can still find a solution for those initial conditions which contains $\log z$ terms. These solutions are completely irrelevant in the WZW model, so the matrix of WZW blocks is not in general a square matrix – it can have more rows than columns. It is clear that terms without logarithms will form a closed set under monodromy, so the truncated matrix of solutions will still form a representation of the monodromy group.

It will be important for the G/H construction that we can truncate some rows from this matrix of correlators to get an invertible matrix. Our power series solution around the point $z_1 = 0$, written in a basis of invariants which diagonalizes $t_0^a t_1^a$, has the form

$$f(z) = \begin{pmatrix} z^\alpha & 0 & \dots \\ 0 & z^\beta & \dots \\ 0 & 0 & \ddots \end{pmatrix} (1 + O(z)). \tag{105}$$

Each column is associated with a row, which contains its most singular piece. If we truncate both the columns that contain *log* terms and their associated rows, the resulting $\tilde{f}(z)$ satisfies

$$\det \tilde{f}(z) = z^{\alpha+\beta+\cdots}(1 + O(z)). \tag{106}$$

$\det \tilde{f}$ is an analytic (though possibly not single-valued) function, so if it has zeroes, they will be at isolated points, and therefore $\tilde{f}$ is invertible, producing a matrix of meromorphic functions.

We now give an example of the differential equation and its solutions for the $SU(N)_k$ WZW model, taken from Knizhnik and Zamolodchikov [9]. We will consider the four-point function of primaries which transform in the fundamental of $SU(N)$. Let

$$f(z) = \left\langle \phi^{i_1}(z)\phi_{i_2}(0)\phi_{i_3}(1)\phi^{i_4}(\infty)\right\rangle, \tag{107}$$

where upper indices are N's and lower $\bar{N}$'s. One choice of basis of invariants is

$$I_1 = \delta_{i_2}^{i_1}\delta_{i_3}^{i_4} \tag{108}$$

$$I_2' = (t^a)_{i_2}^{i_1}(t^a)_{i_3}^{i_4} \tag{109}$$

corresponding to the intermediate states being descendants of 1 or K^a, the primary which transforms in the adjoint. We will use Knizhnik and Zamolodchikov's basis however, which is I_1 and

$$I_2 = \delta^{i_1}_{i_3}\delta^{i_4}_{i_2} = 2I_2' + \frac{1}{N}I_1. \tag{110}$$

In this basis

$$t_1^a t_2^a = \frac{-1}{N}\begin{pmatrix} N^2-1 & N \\ 0 & -1 \end{pmatrix} \tag{111}$$

$$t_1^a t_3^a = \frac{-1}{N}\begin{pmatrix} -1 & 0 \\ N & N^2-1 \end{pmatrix} \tag{112}$$

so

$$\frac{\partial f(z)}{\partial z} = \frac{1}{k+N}\frac{t_1^a t_2^a}{z} + \frac{1}{k+N}\frac{t_1^a t_3^a}{z-1}. \tag{113}$$

This has the solution

$$f_n^i = z^{-2\Delta}(1-z)^{\Delta_1-2\Delta}\times \\ \begin{pmatrix} F(-\frac{1}{2\kappa},\frac{1}{2\kappa};1+\frac{N}{2\kappa};z) & z^{\Delta_1}F(-\frac{(N-1)}{2\kappa},-\frac{(N+1)}{2\kappa},1-\frac{N}{2\kappa};z) \\ \frac{1}{k}zF(1-\frac{1}{2\kappa},1+\frac{1}{2\kappa};2+\frac{N}{2\kappa};z) & -Nz^{\Delta_1}F(-\frac{(N-1)}{2\kappa},-\frac{(N+1)}{2\kappa};-\frac{N}{2\kappa};z) \end{pmatrix} \tag{114}$$

where F is the hypergeometric function, $\kappa = -(N+k)/2$, $\Delta = (N^2-1)/2N(k+N)$ is the dimension of ϕ_i and $\Delta_1 = N/(k+N)$ is the dimension of K_a.

The leading behavior in z tells us which column corresponds to descendants of which primary field. The first column has leading singularity $z^{-2\Delta}I_1$ which is the contribution in the leading term of the o.p.e.

$$\phi^i(z)\phi_j(0) \to \delta^i_j z^{-2\Delta} + \frac{1}{k}(t_a)^i_j z^{1-2\Delta}J^a(0) + \ldots \tag{115}$$

The other invariant comes in with leading singularity $z^{1-2\Delta}$, since it first comes from the J^a term in the o.p.e. The second column is the WZW block with descendants of K^a.

The physical four-point function

$$G^{i,\bar{i}} = \left\langle \phi^{i_1\bar{i}_1}(z,\bar{z})\phi_{i_2\bar{i}_2}(0)\phi_{i_3\bar{i}_3}(1)\phi^{i_4\bar{i}_4}(\infty)\right\rangle \tag{116}$$

satisfies the differential equation above and also a complex conjugate equation in terms of the $\bar{z}$ and $\bar{i}$ dependence, derived from the null vector $(\bar{L}_{-1} - \bar{J}_{-1}\bar{J}_0)\phi$. We can thus write it as a product

$$G^{i,\bar{i}}(z,\bar{z}) = \sum_{n,\bar{n}} f_n^i(z)(C^{n,\bar{n}})^2 f_{\bar{n}}^{\bar{i}}(\bar{z}) \tag{117}$$

where the coefficients C are the o.p.e. coefficients

$$\phi\phi \to C^{11} + C^{21}K^a + C^{12}\bar{K}^{\bar{a}} + C^{22}K^{a\bar{a}}. \tag{118}$$

$C^{11} = 1$ defines the normalization of the field ϕ. If we take the usual left-right symmetric spectrum for the WZW model, the operators K^a and $\bar{K}^{\bar{a}}$ are not in the spectrum, so $C^{12} = C^{21} = 0$. Note that if we chose some other spectrum which gave a modular invariant partition function, the four-point function and o.p.e. coefficients might be different. Finally, we determine C^{22} by requiring invariance under the permutation $\phi_{i_2\bar{i}_2} \leftrightarrow \phi_{i_3\bar{i}_3}$, "crossing symmetry." This is non-trivial because in defining the WZW blocks we distinguished the $1-2$ channel to insert our complete set of intermediate states. This transformation acts by switching the group indices and taking $z \to 1-z$. The action of this transformation is (using Bateman for the transformations of the hypergeometric function)

$$f(1-z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} f(z) \begin{pmatrix} \alpha & \beta \\ (1+\alpha^2)/\beta & -\alpha \end{pmatrix}, \tag{119}$$

with

$$\alpha = \frac{sin\pi/2\kappa}{sinN\pi/2\kappa}, \tag{120}$$

$$\beta = -N \frac{\Gamma^2(-N/2\kappa)}{\Gamma(-(N-1)/2\kappa)\Gamma(-(N+1)/2\kappa)}. \tag{121}$$

The four-point function will now satisfy crossing symmetry if

$$(C^{22})^2 = \frac{1}{N^2} \frac{\Gamma((N-1)/(k+N))\Gamma((N+1)/(k+N))\Gamma(k/(k+N))^2}{\Gamma((k+1)/(k+N))\Gamma((k-1)/(k+N))\Gamma(N/(k+N))^2}. \tag{122}$$

5. G/H Models

We start our construction of G/H models by constructing the state space and stress-energy tensor. This is a minor generalization of work by Goddard, Kent and Olive [14]. The basic idea is simply that we start with the state space of the WZW model with algebra G, and keep only the states which are annihilated by the H currents, *i.e.*, satisfy

$$J_n^H|\rangle = 0, n > 0. \tag{123}$$

Each of these states is a lowest weight state for the H algebra and thus we can build an entire irreducible H representation from it. A priori it might be part of any representation of the algebra H_0 (we have not specified the action of J_0^H.) We know however that the H representation that we build on it is unitary, because it is a subspace of a unitary Hilbert space. We therefore infer that the H_0 representation must have been one of the finite set which give unitary H representations.

Let us call the G representation we start with $L(\Lambda)$, labelled by the G_0 representation of the ground state, Λ. We can decompose it into irreducible representations of H, which we will call $\dot{L}(\lambda)$. This can be written as

$$L(\Lambda) = \sum_\lambda \dot{L}(\lambda) \otimes U(\Lambda, \lambda), \tag{124}$$

where $U(\Lambda, \lambda)$ is the Hilbert space of states in $L(\Lambda)$ which are lowest weight states of an H representation $\dot{L}(\lambda)$. We can write an analogous equation for the characters:

$$\chi_\Lambda(q, \vec{0}) = \sum_\lambda \chi_\lambda(q, \vec{0}) b_\lambda^\Lambda(q), \tag{125}$$

where

$$b_\lambda^\Lambda(q) = Tr_{U(\Lambda,\lambda)} q^{L_0} \tag{126}$$

are called "branching functions." In writing this equation we have assumed that

$$L_0^G = L_0^H + L_0^{G/H},$$

which we will show is consistent below.

Unlike the situation for the Kac-Moody characters, no explicit formula is known for a general branching function; however, their modular transformation laws are determined from those for the G and H characters. The characters considered as a complex vector form a unitary representation of $SL(2,Z)$, the modular group – for the generators $S : \tau \rightarrow -1/\tau$ and $T : \tau \rightarrow \tau + 1$ we write

$$ch_\Lambda(-1/\tau) = S_{\Lambda,\Lambda'} ch_{\Lambda'}(\tau), \tag{127}$$

$$ch_\Lambda(\tau+1) = T_{\Lambda,\Lambda'} ch_{\Lambda'}(\tau). \tag{128}$$

We now show (following [19]) that

$$S^{(G)}_{\Lambda,\Lambda'} S^{(H)}_{\lambda',\lambda} = S^{(G/H)}_{\Lambda\lambda,\Lambda'\lambda'}. \tag{129}$$

(The analogous equation for T is obvious.) Apply $S^2 = 1$ to both sides of equation (125); on the left we get (summing over repeated indices),

$$\begin{aligned}
&(S\chi_\Lambda)(-1/\tau,\vec{0}) \\
&\qquad = S^{(G)}_{\Lambda,\Lambda'} \chi_{\Lambda'}(-1/\tau,\vec{0}) \\
&\qquad = S^{(G)}_{\Lambda,\Lambda'} \chi_{\lambda'}(-1/\tau,\vec{0}) b^{\Lambda'}_{\lambda'}(-1/\tau) \\
&\qquad = S^{(G)}_{\Lambda,\Lambda'} S^{(H)}_{\lambda',\lambda''} \chi_{\lambda''}(\tau,\vec{0}) b^{\Lambda'}_{\lambda'}(-1/\tau)
\end{aligned}$$

We then match coefficients of χ_λ on both sides and use $SS^\dagger = S^2 = 1$.

In principle this information (and the singularities of b as $\tau \rightarrow \infty$, which are determined by the dimensions of the lowest weight states) determines the branching functions uniquely. The simplest way to see this is to use the fact that the relevant representations are actually representations of finite quotients of the modular group Γ/Γ_N and are trivial on Γ_N, so the branching functions are functions on U/Γ_N (the upper half plane mod a congruence subgroup of the modular group), which is known to be a compact Riemann surface, on which functions are determined by their singularities.

These observations can be used to compute some branching functions (as is done in [19]). Physical applications generally do not require knowing the exact characters, however. One only needs the spectrum of low dimension operators – these give the massless modes in string theory and the relevant and marginal operators in statistical mechanics. It is usually easier to determine this by explicitly working out the G/H decomposition for the first few levels.

The branching functions for different Λ and λ are not all distinct – there are relations which can be understood in terms of automorphisms of the algebras G and H. The relevant automorphisms act on states in a representation $L(\Lambda)$ and in general produce states in a different representation $L(\Lambda')$. If we can find a combined automorphism of the G and H algebras which preserves the G/H symmetry algebra, we could identify states which are related by the automorphism in the G/H model. In fact this identification is forced on us if the vacuum, the dimension zero state in $U(0,0)$, is related to other dimension zero states by an automorphism, since the vacuum is assumed to be unique in conformal field theory. We will describe these automorphisms for models of the form $H \times H/H$ (where the different factors can have different central charges) in detail below.

The state space of our G/H model will now be a sum

$$\sum_{\Lambda,\lambda,\bar{\Lambda},\bar{\lambda}} h_{\Lambda\lambda,\bar{\Lambda}\bar{\lambda}} U(\Lambda,\lambda) \otimes U(\bar{\Lambda},\bar{\lambda}) \tag{130}$$

where the coefficients h are the multiplicities with which each representation appears in the physical Hilbert space. These can be any non-negative integers satisfying the requirement that the corresponding partition function be modular invariant. Equation (129) tells us that the characters $b_\lambda^\Lambda(q)$ transform under the modular group as a product of G and H modular representations, so the problem of classifying these modular invariant partition functions can be reduced to the problem for G and H, as was found for the minimal models in [18].

The simplest modular invariant partition function is the left-right symmetric one in which

$$h_{\Lambda\lambda,\bar{\Lambda}\bar{\lambda}} = \delta_{\Lambda,\bar{\Lambda}}\delta_{\lambda,\bar{\lambda}}. \tag{131}$$

This is always a solution since the matrix $S^{(G/H)}_{\Lambda\lambda,\Lambda'\lambda'}$ is a tensor product of unitary matrices and is therefore unitary. If there is an automorphism on Λ, λ which preserves the vacuum, as described above, there will be non-trivial relations

$$b_{\Lambda,\lambda} = b_{\Lambda',\lambda'}. \tag{132}$$

The correct procedure is to include such a b once only. This works because the unitarity of the matrices $S^{(G)}$ and $S^{(H)}$ implies that the matrix $S^{(G/H)}$ is unitary on the space of $b_{\Lambda,\lambda}$'s where we formally consider all the b's to be non-zero and distinct; the true space of functions b may be a subspace of this, but it is a subspace which is preserved by $S^{(G/H)}$, so S is unitary on the subspace as well.

This concludes our discussion of G/H partition functions, and we now move on to showing that we have actual G/H CFT's; the first step is to define a stress-energy tensor.

We can define the G/H stress-energy tensor by the construction of Goddard, Kent and Olive [14], which is simply

$$T_{G/H} = T_G - T_H. \tag{133}$$

This commutes with the H currents and with T_H, so

$$\begin{aligned}[T_{G/H}, T_{G/H}] &= [T_G, T_G] + [T_H, T_H] - [T_{G/H} + T_H, T_H] - [T_H, T_{G/H} + T_H] \\ &= [T_G, T_G] - [T_H, T_H],\end{aligned}$$

and the central extension is

$$c_{G/H} = c_G - c_H. \tag{134}$$

GKO noted that for the pair $G = SU(2)^1 \times SU(2)^k, H = SU(2)^{k+1}$, $c_{G/H}$ is one of the allowed values for a unitary conformal field theory with $c < 1$, and the b^Λ_λ are the (analytic) components of partition functions for that theory.

Unlike the case of the minimal models, for general G and H the spaces $U(\Lambda, \lambda)$ will not be irreducible representations of the Virasoro algebra. For example, the pair $G = SU(2)^2 \times SU(2)^k, H = SU(2)^{k+2}$ gives the $N = 1$ supersymmetric minimal models [5,6,7]; here the branching functions are finite sums of characters of representations of the superconformal algebra.

Although for purposes of defining the model and calculating correlation functions we will not need to know whether a given G/H model has symmetry algebras larger than the Virasoro algebra, it is interesting to find larger symmetry algebras in order to make contact with previous work on these models.

There is a simple way to define a larger algebra for which the G/H $U(\Lambda, \lambda)$ are representations. It consists of all operators formed as normal-ordered products of the G currents, which commute with the H currents. These clearly form an algebra, for which the space $U(\Lambda, \lambda)$ is clearly a representation, since the operators preserve the condition of being annihilated by the H annihilation operators. We will refer to this as "the G/H symmetry algebra." In the models which have been studied previously, the spaces $U(\Lambda, \lambda)$ are irreducible representations of this algebra, but there is no guarantee of this in general. An example where it is not true is the model $SU(3)^k/SU(2)^k$. The first excited level in the G singlet sector decomposes into the representations $8 = 3 + 2 + 2 + 1$ of $SU(2)$; both 2's are ground states in the sector $U(1, 2)$, and they are not transformed into each other by any operator in the symmetry algebra – such an operator would have to be bilinear in the currents (if it were order n it would transform a state generated by a single current into a state generated by $n - 1$ currents acting on the vacuum), and by looking at the branching function b^1_1

one sees that the only such operator in the symmetry algebra is the stress-energy tensor. So there are at least two irreducible representations of the symmetry algebra in $U(1,2)$.

We have no general method for determining the irreducible representations of the G/H symmetry algebras, and thus determining the complete set of fields primary with respect to the symmetry algebra. This will not be an important problem, as we will later describe a technique for calculating correlation functions which treats both primary and descendant fields on the same footing, so we do not have to know whether a field is primary with respect to the symmetry algebra.

The simplest description of a symmetry algebra would be the smallest set of operators whose closure under o.p.e. generates the whole algebra. In principle this could be found by an iterative procedure – take the lowest dimension operator and split $U(1,1)$ into irreducible representations under the algebra of its normal modes; then consider the irreducible representation whose ground state has the lowest non-zero dimension and add the operator which generates its ground state to the algebra; repeat until $U(1,1)$ is an irreducible representation.

The algebra one obtains by this procedure will not always be the most "natural" symmetry algebra for the model. For example, the pair $G = SU(2)^2 \times SU(2)^k, H = SU(2)^{k+2}$ is the series of $N = 1$ superconformal minimal models, with a symmetry algebra generated by the stress-energy tensor T and the supercurrent S. The representation $U(1,1)$ on the other hand consists of states generated from the (Neveu-Schwarz) vacuum with $(-1)^F = 1$ – this is clear because all the states have integral dimension. So the G/H symmetry algebra consists of T, $S\partial S$, $S\partial^3 S$, etc... It is not known whether this algebra is finitely generated. The operator S is the primary associated with the representation $U(3 \times 1, 1)$. In terms of G operators it is

$$(J^{(1)} + aJ^{(2)})^a K_a^{(2)}, \tag{135}$$

where a is a number chosen to make an operator which commutes with $J^H = J^{(1)} + J^{(2)}$.

In this example one might consider the more useful definition of the symmetry algebra to be the standard superconformal algebra including the operator S. We will refer to algebras which contain G/H primary fields as "extended" G/H symmetry algebras.

There is a sense in which these algebras will be different (and less useful) than the symmetry algebras of the previously studied conformal field theories. These symmetry algebras all have the property that in the o.p.e.s of the operators involved,

$$O_1(z)O_2(0) \to z^a(O_3(0) + c_1 z\partial O_3(0) + \ldots),$$

all the operators on the right-hand side appear with the same phase. This means that one can define an algebra of commutation relations (or "generalized commutation relations" [13] if the phase is not integral or half-integral) between the normal modes of the operators. In general one expects the o.p.e. of two fields in a conformal field theory to have several phases on the right-hand side; in this case there is no way of defining an algebra of commutation relations, and it is not clear how to generalize the standard construction of the highest weight modules. This will be true for the primary corresponding to S in the models $G = SU(2)^k \times SU(2)^l, H = SU(2)^{k+l}, k > 2, l > 2$ and $k \neq 4, l \neq 4$,* for example, and it is the generic situation.

Another property of the previously studied symmetry algebras for G/H theories is that the spaces $U(\Lambda, \lambda)$ (or finite sums of these spaces) are irreducible representations. We conjecture that a set of operators which generates an extended symmetry algebra with this property for the models with $G = H \times H$ is as follows: we generalize the Virasoro algebra to the algebra generated by all the operators

$$d^{i_1 i_2 \ldots} : J^{i_1} J^{i_2} \ldots :$$

where d is one of the higher Casimirs of the group H, and the J's are combinations of $J^{(1)}$ and $J^{(2)}$ chosen to make an operator which commutes with the diagonal

* WZW primaries will generally have "single-phase" o.p.e. in models with $k = 1$ [45]. We will relate the G/H o.p.e. to the WZW o.p.e. in section 6. The exceptional cases here are explained by the relations $SU(2)^2 = SO(3)^1$ and $SU(2)^4 = SU(3)^1$ [39].

subgroup, by a procedure described in Appendix A. We then add the primary field for $U(adjoint, 1)$. This is the algebra which arises in the Feigin-Fuchs construction of these models [13,25,32]. We have no similar conjecture for other possible G/H models, for example those with G simple, although the general arguments of this paper apply to these models as well.

Related work on G/H symmetry algebras can be found in [34].

6. Correlation Functions in G/H Models

Having defined the G/H state space, we would like to show that the model has well-defined correlation functions, and if possible compute them. We will restrict our considerations to N-point functions on the sphere, which we will be able to show exist and satisfy duality. We will derive a "factorization relation" between the G/H blocks, the contributions to the four-point function of G/H primaries coming from intermediate states which appear in a given component $U(\Lambda, \lambda)$ of the G/H state space, and the WZW blocks for the groups G and H. We will then verify this relation explicitly for some examples of minimal model correlation functions. The relation provides a simple way to derive G/H o.p.e. coefficients as well, and we give an example.

According to our earlier discussion, if we can define the G/H two and three-point functions between all operators in the model, we have defined the model completely. The two-point function is trivial given the dimensions of the operators. Now we have a linear relation between G, H and G/H operators,

$$g = h\phi. \tag{136}$$

We make a simple ansatz for the three-point function:

$$\langle g_1 g_2 g_3 \rangle = \langle h_1 h_2 h_3 \rangle \langle \phi_1 \phi_2 \phi_3 \rangle . \tag{137}$$

We must define this relation more carefully and show that it is independent of the choice of h_i when there is a choice.

It will be useful to distinguish two cases. Given a choice of three WZW primary fields, we can express the three-point function in terms of a basis of invariants formed from their group indices. For the algebras $SU(2)$ and $U(1)$, the product of three irreducible representations never contains more than one singlet, so each three-point function is determined by a single coefficient. For larger algebras, more than one invariant can be constructed in general – for example $Tr\ [T^a, T^b]T^c$ and $Tr\ \{T^a, T^b\}T^c$ in $SU(N)$. Therefore the three-point function of primaries depends on more than one coefficient. This will slightly complicate the construction of G/H correlation functions involving these algebras.

Suppose we are interested in a G/H three-point function in which this problem does not arise. We can thus write the relation

$$\begin{aligned}&\left\langle g_1^{i\bar{i}}(z_1,\bar{z}_1) g_2^{j\bar{j}}(z_2,\bar{z}_2) g_3^{k\bar{k}}(z_3,\bar{z}_3)\right\rangle = \\ &\left\langle h_1^{i\bar{i}}(z_1,\bar{z}_1) h_2^{j\bar{j}}(z_2,\bar{z}_2) h_3^{k\bar{k}}(z_3,\bar{z}_3)\right\rangle \langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\rangle\end{aligned} \tag{138}$$

which is meant as a definition of the G/H three-point function. The group dependence is clearly the same on both sides since there is only one invariant by assumption. The consistency conditions to check are as follows: Does it matter whether we use fields h which are primary under the algebra H? Does this definition of the G/H correlator respect G/H Ward identities, in particular the stress-tensor Ward identities? If both the G and H three-point functions are zero, is it consistent to take the G/H three-point function to be zero? Can it ever happen that the H three-point function is zero, but the G three-point function is not? If there are non-trivial automorphisms which preserve the G/H state space (as described in Section 5), are the correlation functions invariant under them? Finally, if we normalize the G/H three-point functions of primaries in this way, will the N-point functions satisfy duality?

Suppose we used fields $J_i h_i$, which were not primary, to get relation A. Consider the analogous relation which used the primaries h_i instead, relation B. We can act on the right-hand side with the currents J_i and get the right-hand side of relation A. The linear relation $g = h\phi$ will then be satisfied if we act on the left-hand side with exactly the same currents J_i, so this must give us the left-hand side of relation A. We can then use the Ward identities for H to extract the currents on the right-hand side, and the Ward identities for G to extract the currents on the left. The crucial point is that the Ward identities for H are exactly a subset of those for G – so we will get exactly the same result on both sides. We thus see that relation B implies relation A, and that it does not matter whether we use primaries of H.

Next we check the G/H Ward identities. We have defined a G/H symmetry algebra, including the Virasoro algebra, of operators which are products of G currents. We can therefore evaluate the three-point functions of G/H descendants by inserting these operators on both sides of the defining relation. Since G current algebra holds on the left, the G/H correlators will clearly satisfy the relations derived from it. For example, $T^{(G/H)}$ has the correct o.p.e., since we calculated it earlier assuming only that G current algebra was correct. This is all we needed to derive the Ward identities.

We now consider the case of vanishing G or H three-point functions. There are two reasons a WZW three-point function can vanish. One is that it might violate the general selection rule. Now the selection rule was that if the Ward identities implied that a descendant which was a null state had a non-zero three-point function, the original three-point function of the primary fields must have been zero. Our construction is clearly consistent with this – if the H three-point function violates the selection rule, the G three-point function will as well, since its primaries will have all of the descendants under the H algebra (as well as extra G descendants) with the same norms. A three-point function might also vanish although it respected the selection rule, because an o.p.e. coefficient between primary fields determined by duality just happened to vanish. This case is somewhat harder to deal with – we will show below that one can define dual G/H amplitudes by taking the relevant G/H o.p.e. coefficient to vanish when either the G or H coefficients vanish.

Suppose there were non-trivial automorphisms on the G/H state space, so that in particular there are several G/H operators 1_i which have dimension zero. We can then get the images of the arbitrary G/H operator ϕ by taking the o.p.e. of ϕ with 1_i. Now 1_i will satisfy some factorization relation $g_i = h_i 1_i$; g_i and h_i will in general not be singlets of G_0 and H_0. If we act on the operators in a correlation function with the automorphism, there will then be two cases – either the result will vanish by selection rules, or it will not, and we claim that if it does not, it is equal to the original correlator. This is because we can evaluate the correlator by first taking the o.p.e. of all operators 1_i which appear, leaving at most one. In general, the remaining operator 1_i can only be the original operator 1 satisfying $1_G = 1_H 1_{G/H}$, as any other choice would not satisfy G and H selection rules – this is not hard to check given the explicit form of the automorphism, which we have, at least for models $H \times H/H$. The exception to this is if any the operators in the correlator are transformed into operators in the same representations $U(\Lambda, \lambda)$ by the automorphism. In this case we can explicitly work out the o.p.e. of 1_i with the operator ϕ fixed by the automorphism, using the decomposition of the WZW model into a parafermionic theory times a free bosonic theory [12], since the highest charge operators g_i and h_i in $g_i = h_i 1_i$ turn out to be represented entirely in terms of free bosons. The result is that the automorphism preserves correlators in this case as well.

We now discuss the four-point function. We define G/H blocks as we did earlier for the WZW models, as four-point amplitudes where we sum over intermediate states which are descendants of a particular G/H primary. Rather than worry about the symmetry algebra and Ward identities, we will simply sum over all states in a specified $U(\Lambda, \lambda)$ and determine the three-point functions by the factorization relation above. So,

$$f_{\Lambda,\lambda}(z) = \sum_{\phi_a,\phi_b \in U(\Lambda,\lambda)} z^{\Delta_\phi} \langle \phi_1 \phi_2 \phi_a \rangle \langle \phi_a \phi_b \rangle^{-1} \langle \phi_b \phi_3 \phi_4 \rangle \tag{139}$$

where Δ_ϕ is the dimension of ϕ_a or ϕ_b. This equation uses the analytic parts of the G/H three-point functions only – the factorization relation for these follows from that for the complete three-point function. We are in fact using a weaker relation in that the four-point blocks have no a priori normalization, so the three-point functions of the primaries themselves are irrelevant – only the relations between these and the three-point functions of the descendants are relevant.

Let h_λ be the corresponding WZW block with intermediate states which are descendants of the primary λ. Now we can write

$$\sum_{\lambda} f_{\Lambda,\lambda}(z)h_{\lambda}(z) = \sum_{\lambda} \sum_{\phi_a,\phi_b \in U(\Lambda,\lambda)} z^{\Delta_\phi} \langle \phi_1\phi_2\phi_a \rangle \langle \phi_a\phi_b \rangle^{-1} \langle \phi_b\phi_3\phi_4 \rangle \sum_{h_a,h_b \in L(\lambda)} z^{\Delta_h} \langle h_1h_2h_a \rangle \langle h_ah_b \rangle^{-1} \langle h_bh_3h_4 \rangle . \tag{140}$$

But this sum over states is exactly equivalent to a sum over $L^G(\Lambda)$! Furthermore the two and three-point functions here appear in the combination which gives the G two and three-point functions. We thus have the exact equivalence

$$g^I_{\Lambda}(z) = \sum_{\lambda} f_{\Lambda,\lambda}(z)h^I_{\lambda}(z) \tag{141}$$

where $g_\Lambda(z)$ is the corresponding G WZW block, and I indexes the invariants we can form from the four operators in the G or H correlators. This matrix equation is our factorization relation. It was crucial in the derivation that the factorization definition for the three-point function was valid for any choice of descendant fields h_i.

Note that $g^I_\Lambda(z)$ and $h^I_\lambda(z)$ are not necessarily square matrices. They do not even necessarily have the same number of rows, because the rows of g are labelled by G_0 invariants, while the rows of H_0 are labelled by H invariants, which might be greater in number. To get a sensible equation we must express both matrices in the same basis, expressing G invariants in terms of H invariants. We then have an overdetermined system of equations in general. To solve for $f_{\Lambda,\lambda}(z)$, we can drop some arbitrary set of rows until $h^I_\lambda(z)$ is a square matrix – since the equations are overdetermined it does not matter which. We argued in section 4 that there was a choice of truncation which gave us an invertible matrix $\tilde{h}$. Therefore the factorization relation determines the G/H blocks explicitly (the matrix inversion can be done explicitly using Cramer's rule, for example).

It is a simple consequence of this that the monodromy transformation matrices C_γ for $f_{\Lambda,\lambda}(z)$ are the product $C^G_\gamma \otimes (C^H_\gamma)^{-1}$. ($C^H$ clearly has an inverse, since the monodromy matrices form a group). We now need to find the appropriate o.p.e. coefficients for a dual G/H four-point function. If all of the G and H o.p.e. coefficients are non-zero, these are just g_Λ/h_λ. If an H coefficient vanishes, this clearly does not work. Nevertheless, we expect that we can find consistent G/H coefficients. If D is a monodromy or duality transformation, the matrix of coefficients for H satisfies

$$DC_H D^\dagger = C_H. \tag{142}$$

The inverted H correlators will transform as

$$\tilde{h}^{-1} \rightarrow D^{-1}\tilde{h}^{-1} \tag{143}$$

and so the "inverted o.p.e. coefficients" will satisfy

$$(D^{-1})^\dagger C'_H D^{-1} = C'_H. \tag{144}$$

Even if C_H is not invertible, this can have solutions – consider the example

$$D = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}. \tag{145}$$

Then $C_H = diag(1,0)$, and $C'_H = diag(0,1)$.

We have no general proof that there are always solutions C'_H. This is a point for which it which it is useful to have an alternate definition of the G/H models, for example as a path integral, since we can argue that if the model is well defined, the coefficients must exist. A point worth mentioning is that if we know that the G/H model exists for some choice of G and H, it will exist for any choice of G with this H, since the construction of C'_H is independent of G. Various candidate actions exist for G/H models, using different ways to gauge away the H symmetry in a G WZW model [31,41].

We now consider the case of groups larger than $SU(2)$, for which the product of three representations can contain the singlet more than once. We can describe the three-point function of descendants of primaries $(Jg_i)^{x_i}$ as a linear transformation which acts on the choice of the three-point function for the primaries to give a vector in the space of group invariants formed from the G_0 indices of the descendants x_i. If we write a lower index α for the first choice and an upper index I for the second, we can write a factorization relation for the G/H three-point function as

$$\langle g_1 g_2 g_3 \rangle^I_\alpha = \sum_\beta \langle h_1 h_2 h_3 \rangle^I_\beta \langle \phi_1 \phi_2 \phi_3 \rangle^\beta_\alpha . \tag{146}$$

which simply postpones the choice of G and H three-point functions of primaries. As before, this equation will hold for any operators satisfying the G/H relation $g_i = h_i \phi_i$. Just as the physical WZW three-point functions are

$$\langle g_1 g_2 g_3 \rangle^I = \sum_\alpha \langle g_1 g_2 g_3 \rangle^I_\alpha C^\alpha \tag{147}$$

with C_α determined by duality of the four-point function, so we will write the physical G/H three-point functions as

$$\langle \phi_1 \phi_2 \phi_3 \rangle^I = \sum_{\alpha,\beta} \langle \phi_1 \phi_2 \phi_3 \rangle^I_{\alpha,\beta} C^{\alpha,\beta} \tag{148}$$

with coefficients to be determined.

This equation will always determine

$$\langle \phi_1 \phi_2 \phi_3 \rangle^\beta_\alpha ,$$

because if we choose the h_i's to be primaries, we have

$$\langle h_1 h_2 h_3 \rangle^I_\beta = \delta^I_\beta . \tag{149}$$

We now repeat our earlier argument to show that the four-point blocks satisfy a factorization relation. The choice of α and β is combined with the choice of intermediate primary field as the index of the vector of four-point blocks, and our previous arguments then go through unchanged.

7. The Minimal Models as G/H Models

We now describe the minimal models of BPZ as the G/H models

$$\frac{SU(2)_k \times SU(2)_1}{SU(2)_{k+1}}. \tag{150}$$

These have central extension

$$c = \frac{3k}{k+2} + 1 - \frac{3(k+1)}{k+3} = 1 - \frac{6}{(k+2)(k+3)}. \tag{151}$$

The primary fields are now labeled by a pair of $SU(2)$ representations; if we label a representation by its dimension, the primary is labeled by two integers, the $SU(2)^k$ representation p with $1 \leq p \leq k+1$ and the $SU(2)^{k+1}$ representation q with $1 \leq q \leq k+2$ (the $SU(2)^1$ representation is now 0 or 1 depending on whether $p-q$ is even or odd). This is the same labeling scheme as in BPZ.

Goddard, Kent and Olive [14] have shown that the branching functions are exactly the Virasoro algebra characters. This is an extra piece of information – our Virasoro representations are irreducible – but it is not necessary to know this to solve the model. We now have a one-to-one correspondence between pairs of representations and primary fields in this case, except for the well-known fact that

$$\phi_{(p,q)} = \phi_{(k+2-p,k+3-q)}. \tag{152}$$

There is a similar relation for the general G/H model which can be understood by the existence of a diagram automorphism of the algebras G and H. A diagram automorphism is an automorphism of a Lie algebra which is given by a permutation of the components of the roots which corresponds to a symmetry of the Dynkin diagram. The best-known example for finite Lie algebras is complex conjugation, which corresponds to the reflection symmetry of the diagrams $SU(N)$ and $SO(2N)$. Triality for $SO(8)$ is another example. Almost all affine Lie algebras have diagram automorphisms which exchange the "imaginary root" (roughly speaking, the one which corresponds to the "mode number" direction in root space) with the other roots; these will relate representations based on different representations of the finite subalgebra.

For $SU(2)^k$ there is a Z_2 diagram automorphism which acts on the algebra as

$$\begin{aligned} J_m^- &\leftrightarrow J_{m+1}^+ \\ J_m^3 &\rightarrow \frac{k}{2}\delta_{m,0} - J_m^3, \end{aligned} \tag{153}$$

and exchanges the representation λ with $k+2-\lambda$. The stress-energy tensor then transforms as

$$L_m \rightarrow L_m - J_m^3 + \frac{k}{4}\delta_{m,0}, \tag{154}$$

so we see that a combined automorphism on G and H will preserve $L_{G/H}$, and take the representation $U(\Lambda, \lambda)$ to $U(k+2-\Lambda, k+3-\lambda)$, explaining the relation between the primary fields.

The situation for the general model $H \times H/H$ is completely analogous. There is a diagram automorphism which generates the group *center*(H), which is Z_N for $SU(N)$, and in general some group Z_M. Its action on the representations of H is easily described in terms of the Dynkin label of the representation, and it preserves the G/H representations. There are no other equivalences between G/H representations.

We now discuss the minimal model correlation functions from the G/H point of view. The first consequence of our factorization relation is that the WZW selection rules are also valid in the G/H theory – if either the three G representations or the three H representations in a G/H three-point function are forbidden, the G/H correlator will be zero. Explicitly,

$$\langle \phi_{p,q}\phi_{r,s}\phi_{t,u} \rangle = 0$$

unless

$$p - r + 1 \leq t \leq p + r - 1$$

$$p+r+t \leq 2k+3$$
$$q-s+1 \leq u \leq q+s-1$$
$$q+s+u \leq 2k+5.$$

These are the selection rules found by Arnold and Mattis [40].

The simplest case of our relation between four-point functions is the subalgebra $\phi_{(p,1)}$ which transforms as H singlets – so our relation reduces to

$$\langle gggg \rangle = \langle \phi\phi\phi\phi \rangle \, ,$$

and we can represent these correlators entirely in terms of G operators. The first non-trivial operator is

$$\phi_{(2,1)} = g_i^{(1)} g_i^{(2)}, \tag{155}$$

(where i is an index in the fundamental of $SU(2)$), with dimension $1/4 + 3/4(k+2)$. We can check the relation between the correlation functions by using the expressions derived in [9] for the WZW correlators and checking that the product is the same as the correlator which satisfies the differential equation derived by the method of null vectors in [1] – this is done in Appendix B.

The next operator is

$$\phi_{(3,1)} =: (J^{(1)} + AJ^{(2)})^a K^{(2)a} : \tag{156}$$

where the coefficient A is chosen to make an operator which commutes with $J^{(H)}$. Similarly $\phi_{(2s+1,1)}$ will be made from the primary in the representation $2s+1$ of $SU(2)^k$ acted on by an appropriate combination of currents, which could be determined by checking the commutator with $J^{(H)}$ or by extracting the operator from the o.p.e. of $2s$ copies of $\phi_{(2,1)}$. Here we see a problem of this method for the practical calculation of correlation functions – it is not so easy to determine the precise operator in G that corresponds to a particular primary ϕ. Even determining the dimension of all the primary fields is not completely trivial. If we could explicitly compute the branching functions in all cases, the leading power of q that appears is of course the dimension of the primary, but in practice the branching functions are hard to determine. For the minimal models we can determine the dimension of all the primaries by the following considerations: the primary $\phi_{(p,p)}$ will correspond to the ground state of the G representation p, and its dimension will simply be the difference of the dimensions of the G and H primaries. A primary $\phi_{(p,q)}$ will appear at a higher dimension; at least $(p-q)/2$ higher since one needs to apply this many currents to get an H state in the representation q starting from the representation p. The dimension is in fact much higher than this because we will need to apply a sum of operators constructed from all possible choices of the currents from the two simple factors of G, and in the factor with $k=1$ simple products of the currents such as $J^+_{-1}J^+_{-1}$ create null vectors and hence vanish [10]. The lowest dimension operator in the representation $2s+1$ is not a product of s currents; it is easiest to see what it is by using the representation of $SU(2)^1$ in terms of a free boson. In this representation the lowest dimension operator with charge Λ is $:e^{i\Lambda\phi}:$, with dimension $\Lambda^2/4$. Thus the field $\phi_{(p,q)}$ has dimension

$$\frac{p^2-1}{4(k+2)}-\frac{q^2-1}{4(k+3)}+\frac{(p-q)^2}{4}, \tag{157}$$

which is the Kac formula, somewhat disguised. We have only really argued the case $p-q$ even, but one can see that this formula is also correct for $p-q$ odd.

A more interesting example of the factorization of correlation functions would involve both non-trivial G and H correlation functions. We will illustrate this by using the method described above to determine the o.p.e. coefficients for the field $\phi_{(2,2)}$. We can write its four-point correlation function given those for the primaries g_i and h_i which transform in the fundamental of $SU(2)$, and given the monodromy transformation properties of those correlators – the example we gave in section 4.

Let $G(z)$ be the matrix of WZW blocks for $SU(2)_k$, and C_G the matrix of squares of o.p.e. coefficients which gives us a single valued four-point function. H and C_H

are the same with k replaced by $k+1$.

To determine the o.p.e. coefficients we only need the lowest terms in the power series for the correlator in each channel, the contributions from each channel being distinguished by having terms with z dependence $c_i|z|^{2\Delta_c-4\Delta+i}$. Substituting in our equation

$$\langle\phi_1(\infty)\phi_2(1)\phi_3(z,\bar{z})\phi_4(0)\rangle = TrG(z)C_G G(z)^\dagger(H(z)C_H H(z)^\dagger)^{-1}, \tag{158}$$

and expanding in z, we get

$$\begin{aligned} < \phi\phi\phi\phi > = |z|^{-4\Delta}(1 + g(\frac{3}{2(k+4)(k+5)})^2|z|^{2+2\Delta_1} \\ + \frac{1}{h}(\frac{1}{2k(k+1)})^2|z|^{2-2\tilde{\Delta}_1} + \frac{g}{h}|z|^{2\Delta_1-2\tilde{\Delta}_1} + \ldots), \end{aligned} \tag{159}$$

where Δ_i are dimensions of G operators, $\tilde{\Delta}_i$ are dimensions of H operators, and h is g with k replaced by $k+1$. The powers of z which appear correspond to the contributions of the conformal families of $\phi_{(1,1)}$, $\phi_{(3,1)}$, $\phi_{(1,3)}$ and $\phi_{(3,3)}$ respectively – the $|z|^2$ factor in the contributions of $\phi_{(3,1)}$ and $\phi_{(1,3)}$ comes because of the cancellation of the leading term in z, and corresponds to the fact that these operators correspond to descendants of the G ground state in $L(1)$ and $L(3)$.

The coefficient in front of each term here is the square of the o.p.e. coefficient $\phi_{(2,2)}\phi_{(2,2)} \to \phi_{(p,q)}$; they are the same as those calculated by Dotsenko and Fateev by different methods [21].

Another way to determine these o.p.e. coefficients would be to consider three-point functions; to do this one must be careful to properly normalize all three operators involved, which can be slightly tricky.

8. Other G/H Models

All of the known "families" of CFTs solved using their symmetry algebras are G/H models. The $N = 1$ superconformal minimal models [5,6,7] are

$$\frac{SU(2)_2 \times SU(2)_k}{SU(2)_{k+2}}.$$

The Z_3 models [13] are

$$\frac{SU(3)_1 \times SU(3)_k}{SU(3)_{k+1}}$$

(as was noted by Fateev and Zamolodchikov in [13]). More recent work [25,32] constructing the models

$$\frac{SU(2)_k \times SU(2)_l}{SU(2)_{k+l}}$$

explicitly mentions the relation of their work to the coset construction.

A parafermionic model [11,12] can be considered to be a Lie algebra G mod its maximal torus; in this case one does not have to worry about vectors of correlation functions and summing over intermediate channels since all o.p.e.s in the $U(1)^r$ conformal field theory corresponding to the maximal torus have a single channel (labelled by the charge). Other known models (such as $N = 2$ superconformal minimal models [43]) can be built as products of these models.

The only known models which do not have any obvious relation to these models are orbifolds [42] and sigma models on Calabi-Yau manifolds [2]. Many orbifolds have certain values of their moduli for which they have enhanced symmetry such as a Kac-Moody algebra; it is possible that some orbifolds are equivalent to G/H theories for certain values of their moduli.

We briefly discuss discrete symmetries of G/H models. There are two general methods of identifying discrete symmetries in a conformal field theory. One way is to study the possible factors of the partition function, identifying those which correspond to twisted boundary conditions under the symmetry [24]. Another is to look for representations of the symmetry on the set of operators consistent with the o.p.e., which is what we shall do here. We will also identify only the "generic" symmetries of a particular class of models, analogous to the Z_2 symmetry possessed by all the minimal models, ignoring the possibility of other symmetries (for example, the minimal model $SU(2)^1 \times SU(2)^3/SU(2)^4$ with the non-standard combination of left and right-moving sectors is the three-state Potts model with a Z_3 symmetry – this exception can be understood as a consequence of the alternate representation of the model as $SU(3)^1 \times SU(3)^1/SU(3)^2$).

Assuming no "non-generic" vanishing of o.p.e. coefficients, one expects the o.p.e. for any pair of operators

$$\phi_\lambda^\Lambda \phi_{\lambda'}^{\Lambda'} \to \phi_{\lambda''}^{\Lambda''}$$

to be non-vanishing if the products of the representations $(\Lambda, \lambda) \times (\Lambda', \lambda')$ contains the representation (Λ'', λ''). Now representations fall into conjugacy classes which form a group under direct product which is isomorphic to the center of the Lie group associated with the algebra, so we might expect the symmetry of the G/H model to be the center of G times the center of H, or just the center of G in cases where the conjugacy class in H is determined by that in G, as for the minimal models. This is true up to the possibility of diagram automorphisms mentioned above, which will break some of this symmetry. For the minimal models, the center of G is $Z_2 \times Z_2$, but the diagram automorphism breaks one Z_2, since it exchanges conjugacy classes in the $SU(2)^1$ factor. For $SU(2)^k \times SU(2)^l/SU(2)^{k+l}$, k and l even, the diagram automorphism will not break either Z_2; taking the example of the $N = 1$ superconformal minimal models the Z_2 which is always present corresponds to changing the sign of all the Ramond operators; the additional Z_2 first shows up in the supersymmetric Gaussian model. The series $SU(2)^4 \times SU(2)^k/SU(2)^{k+4}$ was found in [36] to have an S_3 symmetry; this can be understood by the equivalence between WZW models $SU(2)^4$ and $SU(3)^1$ [39] and the S_3 symmetry of $SU(3)^1$ associated with the Z_3 center combined with charge conjugation.

For $SU(3)^2/SU(2)^2$, the G conjugacy class does not determine the H conjugacy class, so we would get $Z_3 \times Z_2$, but the diagram automorphism breaks the Z_3, leaving Z_2. As another example, the models $SU(N)^1 \times SU(N)^k/SU(N)^{k+1}$ have a Z_N symmetry from the center of G mod the Z_N diagram automorphism. For $N > 2$ this is extended to a D_N (dihedral) symmetry by including charge conjugation.

We briefly discuss the classification of supersymmetric G/H models. It was found in [14] that models of the form $H(k = C_{adjoint}) \times H_l/H_{k+l}$ were $N = 1$ supersymmetric – the primary in $U(adjoint, 0; 0)$ has dimension 3/2 and it is easy to verify that it generates the superconformal algebra. It was found in [43] that taking $l = C_{adjoint}$ gave $N = 2$ supersymmetry, with the new generator being the primary in $U(0, adjoint; 0)$, naturally enough. We have done a non-systematic search for further supersymmetric models and have not found any. For the $N = 2$ case we have tried to duplicate the construction in [46], which combined $SU(2)$ parafermions with a $U(1)$ current to construct the $N = 2$ minimal models. To satisfy the $N = 2$ algebra with operators $\psi_\pm e^{\pm ik\phi}$, where ψ is some operator from the G/H model and $\partial\phi$ is the $U(1)$ current, the dimension of ψ must take a specific value which is determined by c. We have ruled out a large class of G/H models by showing that no operator of this dimension exists.

There are non-trivial identities between G/H models. For example, the Z_N parafermions can be described either as $SU(2)_N/U(1)$ or as

$$SU(N)_1 \times SU(N)_1/SU(N)_2.$$

The former description is easier to use in practice, but the latter description at least makes the origin of the Z_N symmetry clearer. We conjecture that there is a general "duality relation"

$$\frac{SU(N)_k \times SU(N)_l}{SU(N)_{k+l}} = \frac{U(k+l)_N}{SU(k)_N \times SU(l)_N}. \tag{160}$$

The basic ingredient in this conjecture is the exact equivalence

$$U(NM)_1 = U(N)_M \times SU(M)_N \tag{161}$$

which follows simply from the equality of c on both sides. (A G/H-style relation between the characters has been shown by Frenkel [44]). We can use this relation to exchange numerator and denominator, up to additional $U(n)_1$ factors. These can be represented as free bosons on some lattice, and if we get the same lattice in numerator and denominator, we can cancel and the duality relation will hold.

Appendix A

We give here a generalization of the GKO construction of the stress-energy tensor for the $H \times H/H$ models [14] to a construction of a set of operators commuting with the diagonal H which are n-linear in the currents, one for each nth order Casimir of H. Let

$$W_x^{(n)}(z) = d^{i_1 i_2 \dots i_n} : J_x^{i_1} J_x^{i_2} \dots J_x^{i_n} : (z), \tag{A.1}$$

where d is a totally symmetric, traceless nth order invariant and

$$J_x^a(z) = J_{(1)}^a + x J_{(2)}^a. \tag{A.2}$$

Now, the commutator of $J_H = J_{(1)} + J_{(2)}$ with W has two sorts of terms – those coming from $[J_H, J_x] \sim J_x$, which will rotate all the J_x's (giving 0 since d is invariant) and require re-normal ordering (giving 0 since d is symmetric traceless), and terms $[J_H^a, J_x^b] = (k_1 + x k_2)\delta^{ab}$, which will give the non-zero result

$$J_H^a(z) W_x^{(n)}(0) \to \frac{1}{z^2}(k_1 + x k_2) d^{a i_1 i_2 \dots} : J_x^{i_1} J_x^{i_2} \dots J_x^{i_{n-1}} : (0). \tag{A.3}$$

Now there are n different combinations of $J_{(1)}$ and $J_{(2)}$ on the right-hand side, and $n+1$ combinations in $W_x^{(n)}$ each with a different power of x, so we can choose $f(x)$ in

$$W^{(n)}(z) = \int dx f(x) W_x^{(n)}(z), \tag{A.4}$$

to cancel the right-hand side completely without making $W^{(n)}(z)$ zero.

Appendix B

Here we check the relation

$$\begin{aligned}\langle \phi_{(2,1)}(\infty)\phi_{(2,1)}(1)\phi_{(2,1)}(z,\bar{z})\phi_{(2,1)}(0)\rangle = \\ \left\langle (g^{i_1}_{(1)}g^{i_1}_{(2)})(\infty)(g^{i_2}_{(1)}g^{i_2}_{(2)})(1)(g^{i_3}_{(1)}g^{i_3}_{(2)})(z,\bar{z})(g^{i_4}_{(1)}g^{i_4}_{(2)})(0)\right\rangle ,\end{aligned} \tag{B.1}$$

for the minimal models $SU(2)^1 \times SU(2)^k/SU(2)^{k+1}$. The minimal model correlator satisfies a second-order linear differential equation, equation 5.17 of [1]. It is derived from the Coulomb gas representation in [15] – the two solutions are ($m = k+2$)

$$M(z) = \begin{pmatrix} z^{-(m+3)/2m}(1-z)^{(m+1)/2m}F(\frac{m+1}{m}, -\frac{1}{m}; -\frac{2}{m}; z) \\ z^{1+2/m-(m+3)/2m}(1-z)^{(m+1)/2m}F(\frac{m+1}{m}, 2+\frac{3}{m}; 2+\frac{2}{m}; z) \end{pmatrix}. \tag{B.2}$$

The $SU(2)^k$ correlators $G_{(2)}(z)$ were given in section 4; the $SU(2)^1$ correlators can be simplified to

$$G_{(1)}(z) = \begin{pmatrix} z^{-1/2}(1-z)^{1/2} \\ z^{1/2}(1-z)^{-1/2} \end{pmatrix}. \tag{B.3}$$

Since the primary field which transforms in the adjoint K_a does not exist in this model, there is only one channel.

The relation between the correlators is now

$$G^t_{(1)}(z) \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} G_{(2)}(z) = \begin{pmatrix} 4 & 0 \\ 0 & 6\frac{m+3}{m+2} \end{pmatrix} M(z), \tag{B.4}$$

where the first matrix of constants comes from contracting the group invariants $\delta^{i_2}_{i_1}\delta^{i_4}_{i_3}$ and $\delta^{i_4}_{i_1}\delta^{i_2}_{i_3}$, and the second gets the normalizations of the two sides to agree. In more detail, there is an overall factor of 4 since we use the operator $\phi = g^i_1 g^i_2$, which satisfies $\langle\phi\phi\rangle = 2$; the second coefficient is a ratio of G/H and G o.p.e. coefficients.

We verified this equation by expanding both sides around $z = 0$ and comparing all the coefficients of the power series.

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