I. ASYMPTOTIC BOUNDARY CONDITIONS FOR ORDINARY DIFFERENTIAL EQUATIONS

II. NUMERICAL HOPF BIFURCATION

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Abstract, Part I. "Asymptotic Boundary Conditions for Ordinary Differential Equations"

The numerical solution of two point boundary value problems on semi-infinite intervals is often obtained by truncating the interval at some finite point. In this thesis we determine a hierarchy of increasingly accurate boundary conditions for the truncated interval problem. Both linear and nonlinear problems are considered. Numerical techniques for error estimation and the determination of an appropriate truncation point are discussed.

A Fredholm theory for boundary value problems on semi-infinite intervals is developed, and used to prove the stability of our numerical methods.

Abstract, Part II. "Numerical Hopf Bifurcation"

Several numerical methods for locating a Hopf bifurcation point of a system of o.d.e.'s or p.d.e.'s are discussed. A new technique for computing the Hopf bifurcation parameters is also presented. Finally, well-known numerical techniques for simple bifurcation problems are adapted for Hopf bifurcation problems. This provides numerical techniques for computing the bifurcating branch of periodic solutions, possibly including turning points and simple bifurcation points. The stability of the periodic solutions is also discussed.

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PART I

Asymptotic Boundary Conditions for Ordinary Differential Equations

INTRODUCTION

We consider numerical techniques for solving two point boundary value problems (TPBVPs) on semi-infinite intervals. Several methods have been suggested for reducing these problems to TPBVPs on finite intervals. One method is to map the semi-infinite interval $[t_0,\infty)$ onto $[0,1)$, and then study the (irregular) singular point at 1 (see de Hoog and Weiss [2]). Another common technique is to truncate $[t_0,\infty)$ at some $t_\infty > t_0$ and then consider a suitable regular TPBVP on $[t_0,t_\infty]$. In [8] M. Lentini and H. B. Keller develop a theory for the proper asymptotic boundary conditions (ABCs) to be applied at t_∞.

In this thesis we only consider the truncation technique. We make several extensions of the Lentini-Keller theory. In particular, the Lentini-Keller ABCs are included in the first order boundary conditions of our hierarchy of increasingly accurate ABCs (Chapter I). For linear TPBVPs we estimate the difference between the solution of the truncated problem using ABCs and the solution of the full problem (Chapter II). Our results agree with the estimates for the class of first order boundary conditions obtained by Lentini and Keller. Furthermore, the hierarchy of ABCs leads to new numerical algorithms for estimating the error due to the approximate boundary conditions, and for finding a suitable t_∞.

We also consider first and higher order ABCs for nonlinear problems, along with some error estimates (Chapter IV). Most of our results for nonlinear problems are restricted to problems that are nearly linear for t sufficiently large. In general, even our first order ABCs for nonlinear problems are different from those obtained by Lentini and Keller [8]. The error estimation and t_∞-prediction procedures for linear problems can also be used on nonlinear problems that are nearly linear for t sufficiently large.

Example calculations using these new techniques are in progress, for both linear and nonlinear problems.

The error estimates obtained in Chapter II depend on the Fredholm theory for o.d.e.'s on semi-infinite intervals developed in Chapter III. The Fredholm theory is an easy application of the exact boundary conditions for $[t_0, t_\infty]$ (discussed in Chapter I). This is a simpler approach than that of F. de Hoog and R. Weiss [3]. Also, our Fredholm theory allows eigenvalues with zero real part in a much more general setting.

The stability and convergence of numerical schemes based on ABCs are briefly considered in Section II.3. Stronger results are an important area for future research. Also, the methods of this thesis can be applied to TPBVPs with (irregular) singular end-points. A careful comparison of mapping procedures with truncation procedures is another important research topic. Finally, automatic procedures for calculating the ABCs would be very worthwhile.

CHAPTER I: Asymptotic Boundary Condition Construction

I.1 Exact Boundary Conditions (EBCs) for Linear Homogeneous TPBPs on $[t_0,\infty)$

Consider the linear homogeneous TPBVP

$$\text{a)}\quad \frac{d\vec{y}}{dt} = t^p A(t)\vec{y}(t) \quad \text{for} \quad t \in (t_0,\infty),$$

1.1 $$\text{b)}\quad C_0\vec{y}(t_0) + \lim_{t\to\infty} C_\infty \vec{y}(t) = \vec{\gamma},$$

$$\text{c)}\quad \vec{y}(t) \quad \text{bounded on} \quad [t_0,\infty).$$

Here $y:[t_0,\infty) \to \mathbb{C}^n$, $p \geq -1$ is an integer, $t_0 > 0$, $A(t)$ is a smooth $n \times n$ matrix analytic at ∞, C_0, C_∞ are $m \times n$ matrices, $\vec{\gamma} \in \mathbb{C}^m$ and $m \leq n$. We proceed to define boundary conditions for 1.1a that are equivalent to 1.1b,c but that only depend on $\vec{y}(t_0)$ and $\vec{y}(t_\infty)$ for some <u>finite</u> t_∞. Using these boundary conditions we hope to reduce problem 1.1 to a regular TPBVP on $[t_0,t_\infty]$ and an initial value problem for $t > t_\infty$.

For $t_1 \in [t_0,\infty)$ define $Y(t,t_1)$ to be the fundamental solution matrix for 1.1a that satisfies $Y(t_1,t_1) = I$. For $t_\infty \in [t_0,\infty)$ define the <u>admissible space for 1.1 at t_∞</u>, $\mathcal{A}(t_\infty)$, to be

1.2 $$\mathcal{a}(t_\infty) \equiv \left\{ \vec{\xi} \in \mathbb{C}^n \,\middle|\, \vec{v}(t) \equiv Y(t,t_\infty)\vec{\xi} \Rightarrow \vec{v}(t) \text{ bounded on } [t_0,\infty) \text{ and } \lim_{t\to\infty} C_\infty \vec{v}(t) \text{ exists} \right\}.$$

Clearly $\mathcal{a}(t_\infty)$ is a subspace of $\mathbb{C}^n$ for each t_∞. Also we will need

<u>Definition 1.3:</u> Define $Q(t_\infty)$: $\mathbb{C}^n \to \mathcal{a}(t_\infty)$ to be a projection onto $\mathcal{a}(t_\infty)$ (here and in the sequel we do not distinguish between a projection operator and its matrix representation). Let

1.3

a) $$P(t_\infty) \equiv I - Q(t_\infty),$$

b) $$C(t_\infty) \equiv \lim_{t\to\infty} C_\infty Y(t,t_\infty)\, Q(t_\infty).$$

Notice that by the definition of $Q(t_\infty)$, the limit in 1.3b exists.

Using the above definitions we get the required boundary conditions

<u>Lemma 1.4</u>: With t_∞, $Q(t_\infty)$, $P(t_\infty)$, $C(t_\infty)$ as above, the TPBVP 1.1 is equivalent to the problem

1.4 a) $$\frac{d\vec{y}}{dt} = t^r A(t)\vec{y}(t) \quad \text{for } t \in [t_0,\infty),$$

with the <u>exact boundary conditions (EBCs)</u>,

$$\text{b)} \quad C_0 \vec{y}(t_0) + C(t_\infty)\vec{y}(t_\infty) = \vec{\gamma},$$

1.4

$$\text{c)} \quad P(t_\infty)\vec{y}(t_\infty) = \vec{0}.$$

In particular, the boundary conditions 1.4b,c depend only on $\vec{y}(t_0)$ and the value of $\vec{y}$ at the <u>internal boundary point</u> t_∞.

<u>Proof of Lemma 1.4</u>: A general solution of 1.4a, 1.1a is

1.5 $$\vec{y}(t;\vec{\xi}) \equiv Y(t,t_\infty)\vec{\xi}, \quad \vec{\xi} \in \mathbb{C}^n.$$

But from Definition 1.3, $\vec{y}(t;\vec{\xi})$ satisfies 1.1b,c iff

$$\text{a)} \quad P(t_\infty)\vec{\xi} = 0,$$

1.6

$$\text{b)} \quad [C_0 Y(t_0,t_\infty) + C(t_\infty)]\vec{\xi} = \vec{\gamma},$$

and the lemma follows. ■

We note that the construction of the EBCs for problem 1.1 would require a detailed knowledge of $Y(t,t_\infty)$. In the next section we consider one class of problems for which $Y(t,t_\infty)$ is easily calculated. However, for the general problem 1.1 we will only have an asymptotic expansion for $Y(t,t_\infty)$ at our disposal. In Section I.3 and in later sections we will use this asymptotic expansion for $Y(t,t_\infty)$ to approximate $C(t_\infty)$ and $P(t_\infty)$. By substituting these approximations into the EBCs 1.4b,c we will obtain <u>asymptotic boundary conditions (ABCs)</u> for problem 1.1.

Properties of the ABCs are discussed in Chapter II.

This principle of defining EBCs for a TPBVP on a semi-infinite interval, and then constructing asymptotic approximations for these EBCs, is the fundamental idea behind our ABCs.

I.2 Linear Homogeneous Problems with a Constant Tail

In this section we consider an illustrative special case of 1.1a for which the EBCs defined in Section I.1 are easily calculated (see also Keller [6, p.55]).

We assume that A(t) in 1.1a satisfies

2.1 $$A(t) = A_0 \text{ for } t > t_1 \geq t_0$$

(i.e. A(t) has a constant "tail"). Without loss of generality, assume A_0 is in Jordan form

2.2 $$A_0 = \begin{pmatrix} \mu_1 & \delta_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \delta_{n-1} \\ 0 & & & \mu_n \end{pmatrix},$$

with $\delta_i = 1$ or 0.

Example 2.3. Suppose $A_0 = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$ for $\alpha, \beta \in \mathbb{C}$ and p = 0. Then

2.3 $$Y(t,t_1) \equiv (\vec{y}_1, \vec{y}_2 \vec{y}_3) = \begin{pmatrix} e^{\alpha(t-t_1)} & (t-t_1)e^{\alpha(t-t_1)} & 0 \\ 0 & e^{\alpha(t-t_1)} & 0 \\ 0 & 0 & e^{\beta(t-t_1)} \end{pmatrix}$$

for $t \geq t_1$. Note that $\vec{y}_1(t)$ and $\vec{y}_3(t)$ are bounded iff $\mathrm{Re}(\alpha) \leq 0$ and $\mathrm{Re}(\beta) \leq 0$. However, $\vec{y}_2(t)$ is bounded iff $\mathrm{Re}(\alpha) < 0$.

Suppose $\alpha = i\omega$, $\omega > 0$, $\beta = 0$ and

$$C_\infty = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}.$$

Then from 2.3 we see that as $t \to \infty$, $\vec{y}_1(t)$ and $\vec{y}_3(t)$ are bounded, $\vec{y}_2(t)$ is not, and $\lim_{t\to\infty} C_\infty \vec{y}_1(t)$ does not exist. Thus we have for $t_\infty \geq t_1$

a) $$\mathcal{A}(t_\infty) = \mathcal{A}(t_1) = \mathrm{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\};$$

2.4 b) $$Q(t_\infty) = Q(t_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

c) $$C(t_\infty) = C(t_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

The EBCs can be given explicitly for $t_\infty \geq t_1$ using 2.4. ∎

For A_0 given in 2.2, and for $t, t_\infty \geq t_1$. the fundamental solution matrix of 1.1a is

2.5 $$Y(t, t_\infty) = \exp\left\{A_0 \int_{t_\infty}^{t} s^r\, ds\right\},$$

$$\equiv \left(\vec{y}_1(t; t_\infty), \ldots, \vec{y}_n(t; t_\infty)\right).$$

Here for i = 1, ..., n:

2.6a $\vec{y}_i\,(t,t_\infty) = \exp\left[\mu_i \int_{t_\infty}^{t} s^p ds\right]\left\{\vec{e}_i + \sum_{k=1}^{\ell_i} \frac{1}{k!}\left[\int_{t_\infty}^{t} s^p ds\right]^k \vec{e}_{i-k}\right\}$,

where e_i is the ith column of I and

2.6b $\ell_i \equiv i - \max\left\{ j \leq i \;\middle|\; \delta_{j-1} = 0 \;\; (\text{here } \delta_0 \equiv 0)\right\}$.

Notice that $\int_{t_\infty}^{t} s^p \, ds \to \infty$ as $t \to \infty$ since $p \geq -1$.

Using 2.6 we can prove (see Example 2.3):

<u>Lemma 2.7</u>: For $t_\infty \geq t_1$

2.7 $\mathcal{a}\,(t_\infty) = \mathcal{a}\,(t_1) = \operatorname{span}\left\{\vec{\xi}_i\right\}_{i=1}^{L}$.

Here $\left\{\vec{\xi}_i\right\}_{i=1}^{L}$ is a maximal linearly independent set in $\mathbb{C}^n$ satisfying

2.8a $\vec{\xi}_i = \vec{e}_{k_i}$

for some $k_i \in \{1, \ldots, n\}$ such that $\mathrm{Re}(\mu_{k_i}) < 0$; or

2.8b $A_0 \vec{\xi}_i = \mu_{k_i} \vec{\xi}_i$

for some $k_i \in \{1, \ldots, n\}$ such that $\mathrm{Re}(\mu_{k_i}) = 0$ and $\mu_{k_i} C_\infty \vec{\xi}_i = \vec{0}$. Furthermore, if we take $\left\{\vec{\xi}_i\right\}_{i=1}^{L}$ to be orthonormal and $\left\{\vec{\xi}_i\right\}_{i=1}^{L_0}$ to be a basis for the null space of A_0 ($L_0 \leq L$), then we have

$$\text{a)} \qquad Q(t_\infty) = Q(t_1) = UU^T, \qquad U \equiv (\vec{\xi}_1, \ldots, \vec{\xi}_L);$$

2.9

$$\text{b)} \qquad C(t_\infty) = C(t_1) = C_\infty U_0 U_0^T \qquad U_0 \equiv (\vec{\xi}_1, \ldots, \vec{\xi}_{L_0}).$$

Proof of Lemma 2.7: From 2.6, $\vec{\xi}_i \in \mathcal{a}(t_\infty)$ if $\vec{\xi}_i$ satisfies 2.8a. Recall that $\vec{y}_1(t;t_\infty)$ is bounded for $t \in [t_0, t_\infty]$ (see [1]). If $\vec{\xi}_i$ satisfies 2.8b then

$$Y(t,t_\infty)\vec{\xi}_i = \exp\left\{\mu_{k_i} \int_{t_\infty}^{t} s^p\, ds\right\} \vec{\xi}_i .$$

Therefore $Y(t,t_\infty)\vec{\xi}_i$ is bounded and $\lim_{t\to\infty} C_\infty Y(t,t_\infty)\vec{\xi}_i$ exists since either $\mu_{k_i} = 0$ or $C_\infty\vec{\xi}_i = \vec{0}$. Thus we have shown that

$$\text{span}\left\{\vec{\xi}_i\right\}_{i=1}^{L} \subset \mathcal{a}(t_\infty) .$$

The converse:

$$\mathcal{a}(t_\infty) \subset \text{span}\left\{\vec{\xi}_i\right\}_{i=1}^{L}$$

follows from the maximal property of $\left\{\vec{\xi}_i\right\}_{i=1}^{L}$. So we have proven 2.7. Also $Q(t_\infty) \equiv UU^T$ is a projection (matrix) onto $\mathcal{a}(t_\infty)$, as required.

Finally 2.9b gives $C(t_\infty)$ correctly since $\lim_{t\to\infty} C_\infty Y(t,t_\infty)\vec{\xi}_i = \vec{0}$ for $i > L_0$. To see this recall that for $i > L_0$, $\vec{\xi}_i$ satisfies 2.8a or 2.8b with $\mu_{k_i} \neq 0$. These imply, respectively, $Y(t,t_\infty)\vec{\xi}_i \to 0$ or $C_\infty Y(t,t_\infty)\vec{\xi}_i = e^{\mu_{k_i}\int_{t_\infty}^{t} s^p ds}\, C_\infty \vec{\xi}_i = 0$ for $t > t_\infty$. ■

Using 2.9 and $P(t_1) = I - Q(t_1)$, we have the EBCs 1.4b,c for problem 1.1 with $A(t)$ as in 2.1 explicitly in terms of $\{\vec{\xi}_i\}_{i=1}^{L}$ and C_∞. These EBCs allow us to reduce the singular TPBVP 1.1 to the <u>regular TPBVP</u>:

2.10

a) $\dfrac{d\vec{y}}{dt} = t^p A(t)\vec{y}(t)$ for $t \in (t_0, t_1)$,

b) $\begin{pmatrix} C_0 \\ 0 \end{pmatrix} \vec{y}(t_0) + \begin{pmatrix} C(t_1) \\ P(t_1) \end{pmatrix} \vec{y}(t_1) = \begin{pmatrix} \vec{\gamma} \\ \vec{0} \end{pmatrix}$;

and the <u>initial value problem</u>

2.11

a) $\dfrac{d\vec{y}}{dt} = t^p A(t)\vec{y}(t)$ for $t > t_1$,

b) $\vec{y}(t_1)$ given by the TPBVP 2.10.

Numerical procedures for calculating solutions of 2.10 and 2.11 are well known (see Keller [6]).

I.3 ABCs for Linear Homogeneous TPBVPs, Part I

We will assume that $A(t)$ in 1.1a satisfies

3.1

a) $A(t) = \sum_{k=0}^{\infty} A_k t^{-k}$ for $t \geq t_0$,

b) $A_0 = \mathrm{diag}(\mu_i \mid i = 1, \ldots, n)$, $\mu_i \neq \mu_j$ for $i \neq j$,

c) $p \geq 0$.

Wasow [12] shows that there exists a smooth $n \times n$ matrix $\mathcal{J}(t)$, nonsingular for $t \geq \tau \geq t_0$ s.t.

3.2 $$\vec{w}(t) \equiv \mathcal{J}^{-1}(t)\vec{y}(t) \quad \text{for } t \geq \tau$$

satisfies

3.3 $$\frac{d\vec{w}}{dt} = t^{p} D(t)\vec{w}(t) \quad \text{for } t > \tau.$$

Here

a) $D(t) = \operatorname{diag}(d_i(t) \mid i = 1, \ldots, n)$, bounded for $t \geq \tau$,

3.4 b) $D(t) \sim \sum_{i=0}^{\infty} D_i t^{-i}$ as $t \to \infty$,

c) $D_0 = A_0$.

However $\sum_{i=0}^{\infty} D_i t^{-i}$ need not converge for any finite t. Also, Wasow [12] (see Appendix A1) shows that there exists $\{T_k\}_{k=1}^{\infty}$, constant n $\times$ n matrices such that for each $q \geq 1$

3.5 $$\mathcal{J}_q(t) \equiv \left(I + \frac{1}{t} T_1\right) \cdots \left(I + \frac{1}{t^q} T_q\right) = \mathcal{J}(t) + O\left(t^{-1-q}\right)$$

as $t \to \infty$.

Using 3.2, we write the EBCs for problem 1.1 in the following useful form.

Lemma 3.6. Let A(t) satisfy 3.1. Using 3.2-3.4 define

a) $W(t, t_\infty) \equiv \exp\left[\int_{t_\infty}^{t} s^{\rho} D(s)\, ds\right]$ for $t, t_\infty \geq \tau$,

3.6 b) $I_W \equiv \{ i \,/\, \vec{w}_i(t) \equiv W(t, \tau)\vec{e}_i$ is bounded for $t \geq \tau$ and $\lim_{t\to\infty} C_\infty \vec{w}_i(t)$ exists $\}$,

c) $I_0 \equiv \{ i \,/\, d_i(t) = O(t^{-2-\rho})$ as $t \to \infty \}$.

We also define

3.7 a) $Q^W \equiv \mathrm{diag}\,(\chi_{I_W}(i) \,/\, i = 1, \ldots, n)$,

where χ_{I_W} (i) is the characteristic function of I_W, defined by

$$\chi_{I_W}(i) = \begin{cases} 1 \text{ if } i \in I_W\,; \\ 0 \text{ otherwise}. \end{cases}$$

Finally we define

b) $Q^0 \equiv \mathrm{diag}\,(\chi_{I_0}(i) \,/\, i = 1, \ldots, n)$

3.7

c) $W^0(t_\infty) \equiv \lim_{t\to\infty} W(t, t_\infty)\, Q^0$ for $t_\infty \geq \tau$.

Then we have for $t_\infty \geq \tau$

a) $Q(t_\infty) = J(t_\infty)\, Q^W J^{-1}(t_\infty)$,

3.8

b) $C(t_\infty) = C_\infty W^0(t_\infty) J^{-1}(t_\infty)$.

Before we prove Lemma 3.6 we will use 3.8 to construct asymptotic boundary conditions (ABCs) for problem 1.1. This is of great practical importance since in general we will not have $\mathcal{J}(t)$ in closed form. Therefore, the EBCs associated with $Q(t_\infty)$ and $C(t_\infty)$ in 3.8 cannot, in general, be used for a numerical calculation. However, as shown below, we can explicitly calculate asymptotic approximations to $Q(t_\infty)$ and $C(t_\infty)$.

The reader may find it helpful to compare the statement of the following theorem with Example 3.12 below.

Theorem 3.9. Let $A(t)$ satisfy 3.1. Using 3.4, 5, 6, 7 and 8 define

a) $Q_q(t_\infty) \equiv \mathcal{J}_q(t_\infty)\, Q^w \mathcal{J}_q^{-1}(t_\infty)$,

b) $P_q(t_\infty) \equiv I - Q_q(t_\infty)$,

3.9

c) $W_q^0(t_\infty) \equiv \lim_{t \to \infty} \left\{ \exp\left[\int_{t_\infty}^{t} s^p \sum_{k=0}^{\hat{q}} D_k s^{-k} ds \right] Q^0 \right\}$, $\hat{q} \equiv q + p + 1$,

d) $C_q(t_\infty) \equiv C_\infty W_q^0(t_\infty)\, \mathcal{J}_q^{-1}(t_\infty)$,

where t_∞ is sufficiently large and $q \geq 0$ $(\mathcal{J}_0(t_\infty) \equiv I)$.

Then for $Q(t_\infty)$, $C(t_\infty)$ as in 3.8 and $P(t_\infty) \equiv I - Q(t_\infty)$ we have

$$\text{a)}\quad Q_q(t_\infty) = Q(t_\infty) + O(t_\infty^{-1-q}),$$

$$\text{3.10 b)}\quad P_q(t_\infty) = P(t_\infty) + O(t_\infty^{-1-q}),$$

$$\text{c)}\quad C_q(t_\infty) = C(t_\infty) + O(t_\infty^{-1-q}),$$

as $t_\infty \to \infty$, for $q = 0, 1, \ldots .$

Notice that we only need $\{D_i\}_{i=0}^{p+1}$ to be able to calculate I_W and I_O and to determine Q^W and Q^O. Therefore $P_q(t_\infty)$, $C_q(t_\infty)$ are completely defined by $\{D_i\}_{i=0}^{\hat{q}}$ and $\{T_i\}_{i=1}^{q}$. In Appendix A1 we discuss a technique for calculating these D_i and T_i. Therefore, in principle we can calculate $P_q(t_\infty)$ and $C_q(t_\infty)$ for $q = 0, 1, 2,$

Proof of Theorem 3.9. The theorem is an easy consequence of Lemma 3.6, and of 3.4b, 5, and 9. ■

Definition 3.11. For C_q, P_q as in 3.9 with $q \geq 0$ define the $O(t^{-1-q})$ ABCs for problem 1.1 to be

$$\text{a)}\quad C_0\vec{y}(t_0) + C_q(t_\infty)\vec{y}(t_\infty) = \vec{\gamma},$$

3.11

$$\text{b)}\quad P_q(t_\infty)\vec{y}(t_\infty) = \vec{0},$$

where t_∞ is sufficiently large (i.e., $\mathcal{J}^{-1}(t')$ exists for $t' \geq t_\infty$).

Properties of the ABCs 3.11 are discussed in Chapter II. We present an example calculation below.

<u>Example 3.12.</u> Consider

a) $$\frac{d\vec{y}}{dt} = \left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{t}\begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 2i \\ 0 & 3i & 0 \end{pmatrix}\right\} \vec{y}(t) \quad \text{for } t > 1,$$

3.12 b) $$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \vec{y}(1) + \lim_{t \to \infty} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{y}(t) = \vec{\gamma} \in \mathbb{C}^2,$$

c) $\vec{y}(t)$ bounded on $[1, \infty)$.

Using Lemma A1.3 of Appendix A1,

$$T_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -2 \\ 0 & 3 & 0 \end{pmatrix}.$$

Then $\vec{w}_1(t) \equiv (I + \frac{1}{t} T_1)^{-1}\vec{y}(t)$ satisfies

$$\frac{d\vec{w}_1}{dt} = \left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{t}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{t^2}\begin{pmatrix} 0 & 3 & -2 \\ 0 & 6i & 2 \\ 0 & 9 & -6i \end{pmatrix} + O\left(\frac{1}{t^3}\right)\right\} \vec{w}_1(t),$$

determining D_0, D_1. Also, from Lemma A1.3,

$$D_2 = \text{diag. part of } \begin{pmatrix} 0 & 3 & -2 \\ 0 & 6i & 2 \\ 0 & 9 & -6i \end{pmatrix}.$$

Applying Lemma 3.6 and Theorem 3.9 we obtain

$$I_W = \{2, 3\}, \quad I_0 = \{3\},$$

$$Q_0 = Q_0(t_\infty) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C_0(t_\infty) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Also,

$$Q_1(t_\infty) = \left(I + \frac{1}{t_\infty} T_1\right) Q_0 \left(I - \frac{1}{t_\infty} T_1\right) + O\left(\frac{1}{t_\infty^2}\right).$$

Therefore

$$Q_1(t_\infty) = \begin{pmatrix} 0 & 0 & -1/t_\infty \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + O\left(1/t_\infty^2\right);$$

$$W_1^0(t_\infty) = \exp\left\{\int_{t_\infty}^{\infty} \frac{1}{s^2} D_2 \, ds\right\} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 - 6i/t_\infty \end{pmatrix};$$

$$C_1(t_\infty) = C_\infty W_1^0(t_\infty)\left(I - 1/t_\infty \, T_1\right) + O\left(t_\infty^{-2}\right),$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3/t_\infty & 1 - \frac{6i}{t_\infty} \end{pmatrix} + O\left(t_\infty^{-2}\right).$$

From 3.9, we need only compute $Q_q(t_\infty)$ and $C_q(t_\infty)$ up to and including $O(t^{-q})$ terms. Simplifying 3.11, we get the following $O(t_\infty^{-1})$ ABCs:

3.12

$$\text{b}') \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \vec{y}(t_0) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{y}(t_\infty) = \vec{\gamma},$$

$$\text{c}') \quad (1\ 0\ 0)\, \vec{y}(t_\infty) = 0;$$

and the $O(t_\infty^{-2})$ ABCs:

$$\text{b}'') \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \vec{y}\,(t_0) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3/t_\infty & 1-\frac{6i}{t_\infty} \end{pmatrix} \vec{y}\,(t_\infty) = \vec{\gamma},$$

3.12

$$\text{c}'') \quad \left(1 \;\; 0 \;\; \tfrac{1}{t_\infty}\right) \vec{y}\,(t_\infty) = 0.$$

In Chapter II we discuss the error produced by using ABCs 3.12b', c', or 3.12b'', c'' in numerical calculations. If 3.12 has a unique solution $\vec{y}(t)$ then we show that for t sufficiently large the TPBVPs 3.12a with 3.12b', c' or 3.12b'', c'' have unique solutions $\vec{u}_1(t)$ and $\vec{u}_2(t)$ respectively. Furthermore, we show that there exists constants K_i s.t.

$$\sup_{t\in[t_0,t_\infty]} \left| \vec{y}\,(t) - \vec{u}_i\,(t) \right| \leq \sup_{t\geq t_0} \left| \vec{y}\,(t) \right| K_i\, t_\infty^{-i}$$

for i = 1, 2. ■

We suggest that on the first reading the reader should either skim or skip the remainder of this chapter.

<u>Proof of Lemma 3.6</u>. By 3.2, 3, and 4 we can write the fundamental solution matrix $Y(t,t_\infty)$ of 1.1a as

$$3.13 \qquad Y(t, t_\infty) = J(t)\, W(t, t_\infty)\, J^{-1}(t_\infty).$$

Therefore, by the definition of $\mathcal{Q}(t_\infty)$ (Section I.1), $\vec{\xi} \in \mathcal{Q}(t_\infty)$ iff $\vec{\eta} \equiv J^{-1}(t_\infty)\vec{\xi}$ satisfies

$$\text{3.14} \quad \text{a)} \quad \sup_{t \geq t_\infty} | W(t, t_\infty) \vec{\eta} | < \infty ,$$

$$\text{b)} \quad \lim_{t \to \infty} C_\infty W(t, t_\infty) \vec{\eta} \text{ exists.}$$

Here we have used 3.5,13. But by 3.1, 4, 6 and 7, we see that $\vec{\eta}$ satisfies 3.14 iff

$$\text{3.15} \qquad (I - Q^W) \vec{\eta} = \vec{0}.$$

And finally from 3.15, $\vec{\xi} \in \mathcal{A}(t_\infty)$ iff

$$\text{3.16} \qquad \vec{\xi} = \mathcal{J}(t_\infty) Q^W \mathcal{J}^{-1}(t_\infty) \vec{\xi}.$$

Therefore $Q(t_\infty)$ defined in 3.8a is consistent with definition 1.1 (i.e., $\mathcal{J}(t_\infty) Q^W \mathcal{J}^{-1}(t_\infty)$ is a projection matrix onto $\mathcal{A}(t_\infty)$).

Finally, by 3.5, 6, 7, 8, 13,

$$\begin{aligned} \text{3.17} \quad \lim_{t \to \infty} C_\infty Y(t, t_\infty) Q(t_\infty) &= \lim_{t \to \infty} \left\{ C_\infty \mathcal{J}(t) W(t, t_\infty) Q^W \mathcal{J}^{-1}(t_\infty) \right\}, \\ &= \lim_{t \to \infty} C_\infty W(t, t_\infty) Q^W \mathcal{J}^{-1}(t_\infty), \\ &= C_\infty W^0(t_\infty) \mathcal{J}^{-1}(t_\infty) \equiv C(t_\infty). \end{aligned}$$

We have used $\lim_{t \to \infty} C_\infty W(t, t_\infty)(Q^W - Q^0) = 0$ in the last equality above (see the definitions of Q^W, Q^0 in 3.7a,b). ∎

I.4 ABCs for Linear Homogeneous TPBVPs, Part II

On the first reading we suggest that the reader either skip or skim this section.

We construct ABCs for the general case of 1.1 using a procedure very similar to the one in Section I.3. In particular, we begin by transforming 1.1a, and the EBCs 1.4b, c, to a simpler form. The ABCs are then obtained from the EBCs by using the known asymptotic behavior of the simplifying transformation.

The following lemma gives the key properties of the simplifying transformation.

Lemma 4.1. There exists a transformation of $\vec{y}(t)$ in 1.1a:

4.1 $$\vec{w}(s) = J^{-1}(s)\,\vec{y}(t), \quad s^r = t,$$

for $t \geq t_1 \geq t_0$, r an integer ≥ 1, such that

4.2 $$\frac{d\vec{w}}{ds} = \hat{A}(s)\,\vec{w}(s)$$

for $s \geq s_1 \equiv t_1^{1/r}$. Here $\hat{A}(s)$ has the form

4.3
a) $\hat{A}(s) = \mathrm{diag}\,(\hat{A}_i(s) \,/\, i = 1, \ldots, \ell)$,

b) $\hat{A}_i(s) = d_i(s)\, I_{n_i} + s^q B_i(s)$ for $s \geq s_1$,

with I_{n_i} the $n_i \times n_i$ identity matrix, $d_i(s)$ a polynomial in s, q an integer and

4.3 c) $B_i(s) \sim \sum_{k=0}^{\infty} B_{ik} s^{-k}$

as $s \to \infty$. Also for each $i = 1, \cdots, \ell$, and for s sufficiently large, either

4.4 a) $Re\,(d_i(s)) < 0, \ \tilde{s}^q = O\left(\tilde{s}^{-1}\, Re\,[d_i(\tilde{s})]\right.$ as $\tilde{s} \to \infty$;

or

4.4 b) $Re\,(d_i(s)) \equiv 0, \ q = -1,$ and $B_i(s) \equiv B_{i0}$.

Where B_{i0} is in Jordan form with no two eigenvalues μ_k, μ_j such that $\mu_k - \mu_k$ is a nonzero integer, or

4.4 c) $Re\,(d_i(s)) > 0, \ \tilde{s}^q = O\left(\tilde{s}^{-1}\, Re\,[d_i(\tilde{s}^q)]\right)$ as $\tilde{s} \to \infty$.

Furthermore, if $\hat{A}_i(s)$, $\hat{A}_j(s)$ both satisfy 4.4b and $i \neq j$ then

4.5 $d_i(s) \neq d_j(s)$.

Finally, for $s \to \infty$

4.6

a) $J(s) \sim \sum_{k=0}^{-\infty} T_k s^k$,

b) $J^{-1}(s) \sim \sum_{k=p}^{-\infty} \hat{T}_k s^k$, for some integer $p \geq 0$, and some constant $n \times n$ matrices $\{T_k\}_{k=0}^{-\infty}$, $\{\hat{T}_k\}_{k=p}^{-\infty}$.

A <u>constructive</u> proof of Lemma 4.1 is discussed in Appendix A1. We use Lemma 4.1 in proving

<u>Lemma 4.7</u>. Let $W(s,s_\infty)$ be the fundamental solution matrix of 4.2 with $W(s_\infty,s_\infty) = I$. Then for $t,t_\infty \geq s_1^r$, the fundamental solution matrix $Y(t,t_\infty)$ (see Section I.1) is given by

$$4.7 \qquad Y(t,t_\infty) = J(s(t))\, W(s(t), s_\infty)\, J^{-1}(s_\infty)$$

where $s(t) = t^{1/r}$, $s_\infty = s(t_\infty)$.

Let $P_i \equiv \operatorname{diag}(\delta_{ij} I_{n_j} \,|\, j=1,\ldots,\ell)$ for each $i = 1,\cdots,\ell$, where $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ otherwise. Then $\vec{\xi} \in \mathcal{A}(t_\infty)$ (see I.1) iff for each $i = 1,\cdots,\ell$:

$$4.8 \qquad \vec{y}_i(t,\vec{\xi}) \equiv J(s(t))\, W(s(t), s_\infty)\, P_i\, J^{-1}(s_\infty)\, \vec{\xi}$$

satisfies

4.9

a) $\sup_{t \geq t_\infty} |\vec{y}_i(t,\vec{\xi})| < \infty$,

b) $\lim_{t \to \infty} C_\infty \vec{y}_i(t,\vec{\xi})$ exists.

<u>Proof of Lemma 4.7</u>. Equation 4.7 follows from 4.1, 2. Also, from $Y(t,t_\infty)\vec{\xi} = \sum_{i=1}^{\ell} \vec{y}_i(t,\vec{\xi})$ we see that 4.8, 9 imply $\vec{\xi} \in \mathcal{A}(t_\infty)$.

So we are left with proving that 4.8, 9 imply $\vec{\xi} \in \mathcal{A}(t_\infty)$.

Notice that

4.10 $$W(s, s_\infty) = \operatorname{diag}\,(W_i(s, s_\infty) / i = 1, \ldots, \ell),$$

with $W_i(s,s_\infty)$ a $n_i \times n_i$ matrix. Coddington and Levinson [1] show that W_i satisfies (see also 4.2,3, and 4),

i) if $\hat{A}_i(s)$ satisfies 4.4a, then

4.11 a) $$|W_i(s, s_\infty)| \leq K_i\, e^{-\mu_i (s - s_\infty)} \quad \text{for } s \geq s_\infty,$$

for some $\mu_i > 0$; or

ii) if $\hat{A}_i(s)$ satisfies 4.4b then

4.11 b) $$W_i(s, s_\infty) = \exp\left\{\int_{s_\infty}^{s} d_i(x)\, I_{n_i} + \frac{1}{x} B_{io}\, dx\right\}$$

for $s \geq s_i$; or

iii) if $\hat{A}_i(s)$ satisfies 4.4c, then

4.11 c) $$|W_i^{-1}(s, s_\infty)| \leq K_i\, e^{-\mu_i (s - s_\infty)} \quad \text{for } s \geq s_\infty,$$

for some $\mu_i > 0$.

Using 4.8, let $\vec{\xi} \in \mathcal{A}(t_\infty)$ and
$$\vec{y}(t, \vec{\xi}) \equiv Y(t, t_\infty)\vec{\xi} = \sum_{i=1}^{\ell} \vec{y}_i(t; \vec{\xi}).$$

For i such that $\hat{A}_i(s)$ satisfies 4.4a,

$$4.12\ \text{a)} \quad \vec{y}_i\,(t, \vec{\xi}\,) = O\left(e^{-\mu_i (t^{1/r} - s_\infty)}\,|\,\vec{\xi}_i\,|\right) = o(1)$$

as $t \to \infty$ (see 4.11a). Therefore, 4.9 is satisfied for these i.

Next, suppose $\hat{A}_i(s)$ satisfies 4.4c. By the definition of $a(t_\infty)$, $\vec{y}(t, \vec{\xi}\,)$ is bounded. Therefore, by 4.6b,

$$J^{-1}(s(t))\,\vec{y}\,(t, \vec{\xi}\,) = O\,(t^{\rho/r})$$

as $t \to \infty$, Hence by 4,7, 8, and 10,

$$P_i\ J^{-1}(s(t))\,\vec{y}\,(t, \vec{\xi}\,) = J^{-1}(s(t))\,\vec{y}_i\,(t, \vec{\xi}\,) = O\,(t^{\rho/r})$$

as $t \to \infty$. So by 4,11c

$$\begin{aligned} P_i\,J^{-1}(s_\infty) &= P_i\ W^{-1}(s(t), s_\infty)\,J^{-1}(s(t))\,\vec{y}_i\,(t, \vec{\xi}\,), \\ &= O\left(t^{\rho/r}\,|\,W_i^{-1}(s(t), s_\infty)|\right) = o(1) \end{aligned}$$

as $t \to \infty$. Therefore

$$4.12\ \text{b)} \quad P_i\,J^{-1}(s_\infty) = \vec{0}\,, \quad \vec{y}_i\,(t, \vec{\xi}\,) = \vec{0},$$

so 4.9 is satisfied for these i.

Finally, if $\hat{A}_i(s)$ satisfies 4.4b then $W_i(s,s_\infty)$ is explicitly given by 4.11b. Now 4.5, 12a, and 12b imply 4.9 for these i also. ■

Lemma 4.7 implies that for $t_\infty \geq t_1$

4.13 $$Q(t_\infty) = J(s(t_\infty))\, Q^W J^{-1}(s(t_\infty)).$$

Here $Q^W = \text{diag}(Q_i^W \mid i=1,\cdots,\ell)$, and Q_i^W is a $n_i \times n_i$ projection matrix onto

4.14 $$\mathcal{A}_i^W \equiv \Big\{ \vec{\eta}_i \in \mathbb{C}^{n_i} \,\Big|\, \vec{y}(t,\vec{\eta}_i) \equiv J(s(t))W(s(t), s(t_\infty))\hat{P}_i\vec{\eta}_i$$

satisfies $\quad \vec{y}(t,\vec{\eta}_i) = O(1)$ as $t \to \infty \quad$ and

$$\lim_{t\to\infty} C_\infty \vec{y}(t,\vec{\eta}_i) \text{ exists} \Big\},$$

where $P_i \equiv (O_{n_i \times n_1} \cdots I_{n_i \times n_i} \cdots O_{n_i \times n_\ell})^T$. Furthermore Q^W and $\mathcal{A}_i^W$ are independent of t_∞, as we prove below.

<u>Lemma 4.15</u>. $\mathcal{A}_i^W$ is independent of t_∞, for $t_\infty \geq t_1$.

<u>Proof</u>. If $\hat{A}_i(s)$ satisfies 4.4a or 4.4c, then 4.11a,c imply that $\mathcal{A}_i^W = \mathbb{C}^{n_i}$ or $\{\vec{0}\}$ respectively.

If $\hat{A}_i(s)$ satisfies 4.4b, then 4.6a, 4.4b, (in particular, $\mu_i - \mu_j \neq k$ a nonzero integer) imply that we can consider the eigenspaces for each eigenvalue of B_{i0} separately. That is, we can assume without loss of generality that

4.15 $$B_{i0} = \mu_i I_{n_i \times n_i} + H_i$$

with $$H_i = \begin{pmatrix} 0 & \delta_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \delta_{n_i-1} \\ 0 & & & 0 \end{pmatrix}.$$

Then using 4.14, 15, we have $\vec{\eta} \in \mathcal{A}_i^w$ iff

4.16 $$\vec{y}\,(t(s), \vec{\eta}) \equiv J(s)\, \hat{P}_i \left(\frac{s}{s_\infty}\right)^{\mu} \left[\sum_{k=0}^{n_i-1} \frac{\log^k(s/s_\infty)}{k!} H_i^k\right] \vec{\eta}$$

satisfies

a) $\vec{y}\,(t(s), \vec{\eta}) = O(1)$ as $s \to \infty$,

4.17

b) $\lim\limits_{s\to\infty} C_\infty\, \vec{y}\,(t(s), \vec{\eta})$ exists.

Now 4.6a implies that 4.17 is equivalent to

a) $s^{\mu} J(s)\, \hat{P}_i\, \vec{\eta} = O(1)$,

4.18 b) $s^{\mu} J(s)\, \hat{P}_i\, H_i^k \vec{\eta} = o(1)$ for $k = 1, 2, \ldots, n_i - 1$,

c) if $\mathrm{Im}(\mu) \neq 0$ then
$$s^{\mu} C_\infty\, J(s)\, \hat{P}_i\, \vec{\eta} = o(1)$$

as $s \to \infty$. Notice that 4.18 is independent of s_∞. ■

From 4.6 and the proof of Lemma 4.15 we see that Q^W is completely determined by $\{T_k\}_{k=1}^{k_0}$ with $k_0 < \infty$ and by $d_i(s)$, B_{i0} for $i = 1, \ldots, \ell$. A constructive procedure for obtaining these is included in Appendix A1.

We now turn to $C(t_\infty)$. Notice that we can define

$$4.19 \quad C^W(t_\infty) \equiv \lim_{s\to\infty} C_\infty \mathcal{J}(s)\, W(s, s(t_\infty))\, Q^W .$$

And so by 4.7 for $Q(t_\infty)$ as in 4.13,

$$4.20 \quad C(t_\infty) \equiv \lim_{t\to\infty} C_\infty\, Y(t, t_\infty)\, Q(t_\infty) = C^W(t_\infty)\, \mathcal{J}^{-1}(s(t_\infty)).$$

Finally, to construct ABCs for problem 1.1, we first calculate asymptotic approximations for $\mathcal{J}(s)$ and $\mathcal{J}^{-1}(s)$ (see 4.6 and Appendix A1). By substituting these approximations into 4.13 and 4.20 we obtain approximations for $Q(t_\infty)$ and $C(t_\infty)$. Suitable ABCs for problem 1.1 are then obtained by using these approximations of $Q(t_\infty)$ and $C(t_\infty)$ in the EBCs 1.4b,c (recall $P(t_\infty) \equiv I-Q(t_\infty)$). Properties of these ABCs will be discussed in the next chapter.

We conclude this section with the following estimates obtained from 4.6, 13, 19, 20, and from the proof of Lemma 4.15:

$$\text{a)} \quad Q(t_\infty) = O(t_\infty^{\rho/r}) \text{ as } t_\infty \to \infty,$$

$$\text{b)} \quad C^W(t_\infty) = O(1) \text{ as } t_\infty \to \infty,$$

4.21

$$\text{c)} \quad C(t_\infty) = O(t_\infty^{\rho/r}) \text{ as } t_\infty \to \infty,$$

$$\text{d)} \quad \text{if } C(t_\infty) \not\equiv 0 \text{ then } |C(t_\infty)|^{-1} = O(1).$$

These estimates are sharp for the general case.

CHAPTER II: Error Estimates

II.1 Existence, Uniqueness and Error Estimates for Solutions of Linear Homogeneous TPBVPs with ABCs

In this section we will assume that II.1 has a unique solution $\vec{y}(t;\vec{\gamma})$ for each $\vec{\gamma} \in \mathbb{C}^m$. A useful characterization of this property is given below.

Lemma 1.1. The TPBVP II.1a,b,c has a unique solution $\vec{y}(t;\vec{\gamma})$ for each $\vec{\gamma} \in \mathbb{C}^m$ iff

a) $\begin{pmatrix} C_0 Y(t_0, t_\infty) + C(t_\infty) \\ P(t_\infty) \end{pmatrix}$ has rank n,

1.1

b) $\left(C_0\, Y(t, t_\infty) + C(t_\infty)\right) Q(t_\infty)$ has rank m.

Furthermore, if 1.1 is satisfied then there exists a constant K s.t.,

1.2 $$\|\vec{y}(t;\vec{\gamma})\|_\infty \equiv \sup_{t \geq t_0} |\vec{y}(t,\vec{\gamma})| \leq K |\vec{\gamma}|$$

for all $\vec{\gamma} \in \mathbb{C}^m$.

We prove Lemma 1.1 in Appendix A2.

Let $\mathcal{C}(t_\infty)$, $\tilde{P}(t_\infty)$ be asymptotic approximations for $C(t_\infty)$, $P(t_\infty)$ respectively (see Chapter I). Also, assume that $\tilde{P}(t_\infty)$ is a projection matrix. We attempt to approximate $y(t;\vec{\gamma})$ for

$t \in [t_0, t_\infty]$ by solving the following <u>regular</u> TPBVP:

$$\text{a)}\quad \frac{d\vec{u}}{dt} = t^p A(t)\,\vec{u}(t) \quad \text{for } t \in (t_0, t_\infty),$$

$$1.3\quad \text{b)}\quad C_0\,\vec{u}(t_0) + \tilde{C}(t_\infty)\,\vec{u}(t_\infty) = \vec{\gamma},$$

$$\text{c)}\quad \tilde{P}(t_\infty)\,\vec{u}(t_\infty) = \vec{0},$$

where $t_\infty < \infty$. (We call 1.3b,c, ABCs for 1.1.)

In order to investigate the TPBVP 1.3 we will need the following definitions.

<u>Definition 1.4</u>: For $t_1 \geq t_0$ define:

$$1.4\quad \text{a)}\quad \mathcal{A}_0(t_1) \equiv \left\{ \vec{\xi} \in \mathcal{A}(t_1) \;\middle|\; \lim_{t\to\infty} C_\infty Y(t,t_1)\vec{\xi} = \vec{0} \right\}.$$

Let $P_0(t_1): \mathbb{C}^n \to \mathcal{A}_0(t_1)$ be a projection (matrix) onto $\mathcal{A}_0(t_1)$, with $P_0(t_1) = Y(t_1,t_0)\,P_0(t_0)\,Y(t_0,t_1)$. Define

$$1.4\quad \text{b)}\quad \mathcal{A}_1(t_1) \equiv \left\{ \vec{\xi} \in \text{Range}\left\{ (I - P_0(t_1))\,Q(t_1) \right\} \right\}.$$

Let $P_1(t_1) : \mathbb{C}^n \to \mathcal{A}_1(t_1)$ be a projection matrix onto $\mathcal{A}_1(t_1)$ with $P_1(t_1) = Y(t_1,t_0)\,P_1(t_0)\,Y(t_0,t_1)$.

Finally, define:

$$1.4\quad \text{c)}\quad \phi(t_\infty; P) \equiv \sup\left\{ |C_0 Y(t_0,t_\infty)\vec{\xi}| \;\middle|\; \begin{array}{l} \vec{\xi} \in \mathbb{C}^n,\ P(t_\infty)\vec{\xi} = \vec{\xi}, \\ \text{and } |\vec{\xi}| = 1 \end{array} \right\}.$$

We state our main results on the properties of the TPBVP 1.3 below.

Theorem 1.5. Suppose that the TPBVP I1.1 has a unique solution for each $\vec{\gamma} \in \mathbb{C}^m$ (see Lemma 1.1). Suppose that C(t), P(t) are as in definition I1.3 and

$$\text{a)} \quad |C(t_\infty) - \tilde{C}(t_\infty)| = o(1),$$

1.5

$$\text{b)} \quad |P(t_\infty) - \tilde{P}(t_\infty)| = \left(\left[|P(t_\infty)|(1 + \phi(t_\infty; P))\right]^{-1}\right),$$

as $t_\infty \to \infty$, $\tilde{P}(t_\infty)$ a projection matrix. Let $\vec{\gamma} \in \mathbb{C}^m$.

Then for $t_\infty \geq T$, T independent of $\vec{\gamma}$, the TPBVP 1.3 has a unique solution $\vec{u}(t;t_\infty,\vec{\gamma})$. Furthermore the error $\vec{e}(t;t_\infty,\vec{\gamma}) \equiv \vec{u}(t,t_\infty,\vec{\gamma}) - \vec{y}(t;\vec{\gamma})$ for $t \in [t_0, t_\infty]$ can be written as

1.6 $$\vec{e}(t;t_\infty,\vec{\gamma}) = Y(t_1,t_0)\vec{\xi}_1 + Y(t,t_\infty)(\vec{\xi}_2 + \vec{\xi}_3),$$

with $\vec{\xi}_1 = P_0(t_0)\vec{\xi}_1$, $\vec{\xi}_2 = P_1(t_1)\vec{\xi}_2$ and $\vec{\xi}_3 = P(t_\infty)\vec{\xi}_3$.

Finally, $\vec{\xi}_1$, $\vec{\xi}_2$, $\vec{\xi}_3$ satisfy

1.7 $$\text{a)} \quad |\vec{\xi}_i| \leq K|\vec{y}(t_\infty;\vec{\gamma})| \left\{ |C(t_\infty) - \tilde{C}(t_\infty)| + \phi(t_\infty;P)|P(t_\infty) - \tilde{P}(t_\infty)| \right\},$$

for i = 1,2, and

1.7 b) $|\vec{\xi}_3| \leq K|\vec{y}(t_\infty;\vec{\gamma})| \{|P(t_\infty) - \tilde{P}(t_\infty)|\}$,

where K is a constant independent of t_∞, and $\vec{\gamma}$.

Applying Theorem 1.5 to Example I3.12 we obtain

$$\phi(t_\infty; P) = O\left(\frac{1}{t_\infty - t_0}\, e^{-(t_\infty - t_0)}\right)$$

as $(t_\infty - t_0) \to \infty$. And

$$a_0(t_\infty) = \text{span}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{1}{t_\infty}\begin{pmatrix}0\\0\\3\end{pmatrix} + O\left(\frac{1}{t_\infty^2}\right)\right\},$$

$$a_1(t_\infty) = \text{span}\left\{\begin{pmatrix}0\\0\\1\end{pmatrix} + \frac{1}{t_\infty}\begin{pmatrix}-1\\-2\\0\end{pmatrix} + O\left(\frac{1}{t_\infty^2}\right)\right\}.$$

Notice that for $\vec{\xi}_1$, $\vec{\xi}_2$ and $\vec{\xi}_3$ as in 1.6,

$$Y(t,t_0)\vec{\xi}_1 = O\left((t_0/t)^2 |\vec{\xi}_1|\right)$$

as $t \to \infty$, and

$$Y(t,t_\infty)\vec{\xi}_2 = O(|\vec{\xi}_2|),$$

$$Y(t,t_\infty)\vec{\xi}_3 = O\left(\frac{e^{-(t_\infty - t)}}{t_\infty - t}\,|\vec{\xi}_3|\right),$$

as $(t_\infty - t) \to \infty$.

Therefore Theorem 1.5 provides a good deal of information on $|\vec{e}(t;t_\infty,\vec{\gamma})|$ for $t \in [t_0,t_\infty]$. In particular, we see that the estimates given at the end of Example I3.12 are valid.

The proof of Theorem 1.5 is in Appendix A2, it can be left out on the first reading.

II.2 EBCs and ABCs for Forced Linear TPBVPs on $[t_0, \ )$

In this section we will extend the results of Section II.1 to problems of the form

$$\text{a)} \quad \frac{d\vec{y}}{dt} = t^p A(t)\vec{y}(t) + \vec{f}(t) \quad \text{for} \quad t > t_0 > 0,$$

$$2.1 \quad \text{b)} \quad C_0\vec{y}(t_0) + \lim_{t\to\infty} C_\infty \vec{y}(t) = \vec{\gamma} \in \mathbb{C}^m,$$

$$\text{c)} \quad \vec{y}(t) \text{ bounded on } [t_0,\infty).$$

Here $p, A(t), C_0, C_\infty, \vec{\gamma}$ are as in I1.1 and $\vec{f}(t)$ is continuous on $[t_0,\infty)$.

As in Section I.1, we wish to write 2.1 as a regular TPBVP on $[t_0,t_\infty]$, and an initial value problem for $t > t_\infty$. Let $C(t)$, $P(t)$ be as in Definition I1.3 for the homogeneous problem related to 2.1 (i.e., $\vec{f} \equiv 0$).

Assume that there exists a $\vec{y}_p(t)$ satisfying

$$\text{a)} \quad \frac{d\vec{y}_p}{dt} = t^p A(t)\,\vec{y}_p(t) + \vec{f}(t) \quad \text{for } t > t_\infty,$$

$$2.2 \quad \text{b)} \quad \vec{y}_p(t) \text{ is bounded for } t \geq t_\infty,$$

$$\text{c)} \quad \lim_{t\to\infty} C_\infty \vec{y}_p(t) = \vec{\gamma}_p$$

(i.e., the limit in 2.2c exists).

Definition 2.3: The following are EBCs for problem 2.1:

$$\text{a)} \quad C_0 \vec{y}(t_0) + C(t_\infty)\vec{y}(t_\infty) = \vec{\gamma} + [C(t_\infty)\vec{y}_p(t_\infty) - \vec{\gamma}_p],$$

2.3

$$\text{b)} \quad P(t_\infty)\vec{y}(t_\infty) = P(t_\infty)\vec{y}_p(t_\infty),$$

where $t_\infty \geq t_0$.

The following lemma justifies Definition 2.3.

Lemma 2.3. The TPBVP 2.1a,b,c is equivalent to the TPBVP 2.1a, 2.3.

If 1.1 is satisfied then

i) 2.1 has a unique solution;

ii) $\vec{y}_p(t)$ exists iff 2.1 has a solution.

Proof of Lemma 2.3. Notice that if $\vec{y}_p(t)$ satisfies 2.2 for some $t_\infty \geq t_0$ then we can assume it satisfies 2.2 for $t_\infty = t_0$.

Now $\vec{y}(t)$ is a solution of 2.1 iff $\vec{u}(t) = \vec{y}(t) - \vec{y}_p(t)$ satisfies

$$\text{a)}\quad \frac{d\vec{u}}{dt} = t^{\rho} A(t)\, \vec{u}(t) \quad \text{for } t > t_0 > 0,$$

$$\text{2.4}\quad \text{b)}\quad C_0\, \vec{u}(t_0) + \lim_{t\to\infty} C_\infty\, \vec{u}(t) = \vec{\gamma} - \left[C_0 \vec{y}_p(t_0) + \vec{\gamma}_p \right],$$

c) $\vec{u}(t)$ bounded for $t \geq t_0$.

Statements i) and ii) follow from 2.4 and Lemma 1.1. Furthermore, from Lemma I1.4, $\vec{u}(t)$ is a solution of 2.4 iff $\vec{u}(t)$ satisfies

$$\text{a)}\quad \frac{d\vec{u}}{dt} = t^{\rho} A(t)\, \vec{u}(t) \quad \text{for } t > t_0,$$

$$\text{2.5}\quad \text{b)}\quad C_0\, \vec{u}(t_0) + C(t_\infty)\, \vec{u}(t_\infty) = \vec{\gamma} - \left[C_0 \vec{y}_p(t_0) + \vec{\gamma}_p \right],$$

$$\text{c)}\quad P(t_\infty)\, \vec{u}(t_\infty) = \vec{0}.$$

Finally, $\vec{y}(t) = \vec{u}(t) + \vec{y}_p(t)$ implies that $\vec{u}(t)$ satisfies 2.5 iff $\vec{y}(t)$ satisfies 2.1a, 2.3. ■

In general, $\vec{y}_p(t)$, $C(t_\infty)$ and $P(t_\infty)$ will not be available explicitly. However, we will be able to compute asymptotic approximations for $\vec{y}_p(t)$, $C(t_\infty)$ and $P(t_\infty)$. (See Chapter I for the latter two). <u>Substituting these approximations into the EBCs 2.3a,b gives ABCs for problem 2.1</u>. The following theorem considers these ABCs.

<u>Theorem 2.6</u>. Consider the truncated problem

a) $\frac{d\vec{u}}{dt} = t^{p} A(t)\vec{u}(t) + \vec{f}(t)$ for $t \in (t_0, t_\infty)$,

2.6 b) $C_0 \vec{u}(t_0) + \tilde{C}(t_\infty)\vec{u}(t_\infty) = \vec{\gamma}_1(t_\infty) \equiv \{\tilde{C}(t_\infty)\tilde{\vec{y}}_p(t_\infty) - \vec{\gamma}_p\} + \vec{\gamma}$,

c) $\tilde{P}(t_\infty)\vec{u}(t_\infty) = \vec{\gamma}_2(t_\infty) = \tilde{P}(t_\infty)\tilde{\vec{y}}_p(t_\infty)$,

where 2.6b,c are ABCs for 2.1. That is $\tilde{C}(t)$, $\tilde{P}(t)$ satisfy 1.5a,b and $\tilde{\vec{y}}_p(t)$ satisfies

2.7 $|\vec{y}_p(t) - \tilde{\vec{y}}_p(t)| = o(1)$

as $t \to \infty$. Also we assume that $\tilde{P}(t)$ is a projection, and $\tilde{C}(t) = 0$ or $\tilde{P}(t) = 0$ if $C(t) = 0$ or $P(t) = 0$, respectively. Finally, we assume that 1.1 is satisfied.

Then for $t_\infty \geq T$, T a constant independent of $\vec{\gamma}$ and $\vec{f}$, the TPBVP 2.6 has a unique solution $\vec{u}(t;\, t_\infty)$. Furthermore, let the error function be given by

2.8 a) $\vec{e}(t; t_\infty) \equiv \vec{u}(t; t_\infty) - \vec{y}(t; \vec{\gamma}, \vec{f})$,

$$= Y(t,t_0)\,\vec{\xi}_1(t_\infty) + Y(t,t_\infty)\{\vec{\xi}_2(t_\infty) + \vec{\xi}_3(t_\infty)\},$$

with $\vec{y}(t;\vec{\gamma},\vec{f})$ the solution of 2.1, and

2.8 b) $(I - P_0(t_\infty))\,\vec{\xi}_1(t_\infty) = (I - P_1(t_\infty))\,\vec{\xi}_2(t_\infty)$

$$= (I - P(t_\infty))\,\vec{\xi}_3(t_\infty) = \vec{0}$$

(see Definition 1.4). Then for some constant K_0 independent of t_∞, $\vec{f}$ and $\vec{\gamma}$:

2.9 a) $|\vec{\xi}_i(t_\infty)| \le K_0 \,|y(t_\infty;\vec{\gamma},\vec{f}) - \vec{y}_p(t_\infty)|\,\{|\Delta C(t_\infty)| + \phi(t_\infty;P)\,|\Delta P(t_\infty)|\}$

$$+ K_0\,|\vec{y}_p(t_\infty) - \tilde{\vec{y}}_p(t_\infty)|\,\{|C(t_\infty)| + (|\Delta C(t_\infty)| + \phi(t_\infty;P))|P(t_\infty)|\}$$

for i = 1, 2, and

2.9 b) $|\vec{\xi}_3(t_\infty)| \le K_0\,|\vec{y}(t_\infty;\vec{\gamma},\vec{f}) - \vec{y}_p(t_\infty)|\,|\Delta P(t_\infty)|$

$$+ K_0\,|\vec{y}_p(t_\infty) - \tilde{\vec{y}}_p(t_\infty)|\,\{|P(t_\infty)| + |\Delta P(t_\infty)|\,|C(t_\infty)|\}.$$

Here $\phi(t_\infty;P)$ is as in Definition 1.4, and $\Delta C \equiv C - \tilde{C}$, $\Delta P \equiv P - \tilde{P}$.

See Appendix A2 for the proof of Theorem 2.6

II.3 Stability and Convergence of Finite Difference Methods Using ABCs

Using standard finite difference techniques we can compute the solution of the truncated problem 2.6. In this section we study the stability and convergence of these techniques as $t_\infty \to \infty$. We assume that 2.1 also satisfies

i) condition 1.1 (see Lemma 1.1);

3.1 ii) $A(t)$, $\vec{f}(t)$ smooth for $t \geq t_0$;

iii) $\| S(t)\, \vec{f}(t) \|_\infty < \infty$.

Here $S(t)$ is a smooth, possibly unbounded $n \times n$ matrix, as in Theorem III 4.1. An application of Theorem III 4.1 and Lemma 1.1 proves

Theorem 3.2. Let 2.1 satisfy 3.1. Then there exists a unique solution $\vec{y}(t;\vec{\gamma},\vec{f})$ of 2.1. Furthermore, there exists a constant K independent of $\vec{\gamma},\vec{f}$ such that

3.2 $$\| \vec{y}\,(t;\vec{\gamma},\vec{f}) \|_\infty \leq K \left\{ |\vec{\gamma}| + \| S(t)\, \vec{f}\,(t) \|_\infty \right\}.$$

Here $\| \vec{y} \|_\infty \equiv \sup_{t \geq t_0} | \vec{y} |$.

The Difference Scheme. (We follow Keller [6] very closely). We discretize 2.6 on meshes $\{t_j\}_{j=0}^{J}$ with $t_J = t_\infty$ and $h \equiv \max_{J \geq j \geq 1} (t_j - t_{j-1}) \leq \Theta \min_{J \geq j \geq 1} (t_j - t_{j-1})$. We hope to approximate $\{\vec{u}(t_j)\}_{j=0}^{J}$ by $w^h = \{\vec{w}_j\}_{j=0}^{J}$ where w^h is a solution of

3.3 a) $$\mathcal{L}_{h_j} w^h \equiv \sum_{k=0}^{J} C_{jk}(h)\, \vec{w}_k = \vec{F}_j(h,\vec{f})$$

for $j = 1,\cdots,J$, and

3.3 b) $$B_h \omega^h \equiv \begin{pmatrix} C_0 \vec{w}_0 + \tilde{C}(t_\infty)\vec{w}_J \\ \tilde{P}(t_\infty)\vec{w}_J \end{pmatrix} = \begin{pmatrix} \vec{\gamma}_1 (t_\infty) \\ \vec{\gamma}_2 (t_\infty) \end{pmatrix}.$$

Here $C_{jk}(h)$ are $n \times n$ matrices, $F_j(h,f) \in \mathbb{C}^n$.

We put the following restrictions on the difference scheme 3.3:

I) Accuracy Conditions. For each smooth solution $\vec{u}(t;\vec{\gamma},\vec{F})$ of 2.6, there exist constants $K_1(t_\infty;\vec{\gamma},\vec{f})$, $q>0$, $h_1(t_\infty)>0$ such that for all $h \leq h_1(t_\infty)$, $h > 0$;

3.4 a) $$\left| \mathcal{L}_{hj} \left\{ \vec{u}(t_i;\vec{\gamma},\vec{f}) \right\}_{i=0}^{J} - \vec{F}_j(h,\vec{f}) \right| \leq K_1(t_\infty;\vec{\gamma},\vec{f})\, h^q,$$

for $j = 1, \ldots, J$, and

3.4 b) $$\left| \mathcal{B}_h \left\{ u(t_j;\vec{\gamma},\vec{f}) \right\}_{j=0}^{J} - \begin{pmatrix} \vec{\gamma}_1(t_\infty) \\ \vec{\gamma}_2(t_\infty) \end{pmatrix} \right| \leq K_1(t_\infty,\vec{\gamma},\vec{f})\, h^q.$$

That is, the difference scheme 3.3 is accurate of order h^q.

II) Stability Condition for the Initial Value Problem

Consider the initial value problem (IVP):

3.5
a) $$\frac{d\vec{u}}{dt} = t^P A(t)\vec{u}(t) + \vec{f}(t) \quad \text{for } t > t_0,$$

b) $$\vec{u}(t_0) = \vec{\gamma}_0 \in \mathbb{C}^n.$$

Also consider the associated difference scheme for 3.5, in particular 3.3a with

3.6 $\vec{w}_0 = \vec{\gamma}_0$.

<u>We assume that 3.3a, 3.6 is stable</u>. That is, there exist constants $K_2(t_\infty)$, $h_2(t_\infty) > 0$ such that for all meshes $\{t_j\}_{j=0}^{J}$ with $h < h_2(t_\infty)$ and for all mesh functions $v^h \equiv \{\vec{v}_j\}_{j=0}^{J}$.

3.7 $|\vec{v}_j| \le K_2(t_\infty) \max\{|\vec{\gamma}_0|, \max_{1 \le k \le J} |\mathcal{L}_{hk} v^h|\}$

for $j = 0,1,\dots,J$.

III) <u>Uniformity Condition</u>. For each $t_1 \in (t_0, \infty)$ there exists a constant $K_3(t_1; \vec{\gamma}, \vec{f})$ such that

3.8 $\sup_{t \in [t_0, t_1]} \left\{ K_1(t; \vec{\gamma}, \vec{f}), K_2(t), \frac{1}{h_1(t)}, \frac{1}{h_2(t)} \right\} \le K_3(t_1; \vec{\gamma}, \vec{f})$.

Conditions I, II, and III are very reasonable conditions for any practical difference scheme. Under these conditions we have

<u>Theorem 3.9</u>. Suppose 2.1, 2.6 satisfy the hypotheses of Theorems 2.6 and 3.2, and suppose 3.3 satisfies conditions I, II, and III above. In addition, suppose

3.9 a) $|\vec{y}_p(t) - \tilde{\vec{y}}_p(t)| = o(\varepsilon_0(t_\infty))$,

with

3.9 b) $\varepsilon_0(t_\infty) \equiv \{|C(t_\infty)| + (1+\phi(t_\infty;P))|P(t_\infty)|\}^{-1}$.

Then there exists a constant T_0, and smooth functions $h(t)$, $K(t;\vec{\gamma},\vec{f})$ for $t \geq T_0$ such that for $t_\infty \geq T_0$

i) 3.3 is stable and consistent with 2.6 for $h \leq h(t_\infty)$;

ii) 3.3 has a unique solution w^h for $h \leq h(t_\infty)$;

iii) for $\vec{y}(t;\vec{\gamma},\vec{f})$ the solution of 2.1:

3.10 a) $|\vec{w}_j - \vec{y}(t_j;\vec{\gamma},\vec{f})| \leq K(t_\infty;\vec{\gamma},\vec{f})h^q + \sigma(t_\infty;\vec{\gamma},\vec{f})$

for $j = 0,\ldots,J$ and $h \leq h(t_\infty)$. Here

3.10 b) $\sigma(t_\infty;\vec{\gamma},\vec{f}) = o(1)$ as $t_\infty \to \infty$.

Furthermore, $h(t) > 0$ for $t \geq T$, and $\frac{1}{h(t)}$, $K(t;\vec{\gamma},\vec{f})$ are monotone increasing for $t \geq T$.

<u>Proof of Theorem 3.9</u>. We begin by applying

<u>Keller's Corollary 2.11 [6, p. 23]</u>: If 2.6 has a unique solution then the difference scheme 3.3a,b is stable and consistent iff the scheme 3.3a, 3.6 is stable and consistent for 3.5.

Statements i) and ii) follow from the corollary above, and its proof. Also, using Theorem 2.6 with $t_\infty \geq T_1 \geq T$, the above corollary provides the estimate

$$3.11 \qquad |\vec{w}_j - \vec{u}(t_j\,;\,t_\infty, \vec{\gamma}, \vec{f})| \leq K(t_\infty; \vec{\gamma}, \vec{f})\, h^q.$$

Here $\vec{u}(t; t_\infty, \vec{\gamma}, \vec{f})$ is the unique solution of 2.6. Finally, 3.9 and Theorems 2.6 and 3.1 imply

$$3.12 \qquad \sup_{t\in[t_0,t_\infty]} |\vec{u}(t; t_\infty, \vec{\gamma}, \vec{f}) - \vec{y}\,(t; \vec{\gamma}, \vec{f})| \leq \hat{K}\, \sigma_0\,(t_\infty)\, \{|\vec{\gamma}| + \|S(t)\vec{f}\|_\infty\}.$$

Here $\sigma_0(t_\infty) = o(1)$ as $t_\infty \to \infty$, and $\hat{K}$ is a constant. Finally, statement iii) follows from 3.11 and 3.12. ■

The problem with the estimate 3.10a is that $K(t_\infty; \vec{\gamma}, \vec{f})$ grows like $|Y(t_\infty, t_0)|$ as $t_\infty \to \infty$. (This is a consequence of calculating the stability bound for 3.3 using Keller's Cor. 2.11.) It is very likely that much better bounds can be obtained. Preliminary results of F. R. deHoog and H. B. Keller indicate that bounded $K(t_\infty; \vec{\gamma}, \vec{f})$ exist for at least some difference schemes. This is an important area for future research.

II.4 Numerical Techniques for Error Estimation and Choosing t (Linear Problems)

Theorem 2.6 can be used to motivate several numerical techniques for estimating the error due to the ABCs. Let $\vec{y}(t;\vec{\gamma},\vec{f})$ be the solution of 2.1. Under the hypotheses of Theorem 3.9 let $\vec{u}(t;t_\infty,\vec{\gamma},\vec{f})$ be the solution of 2.6 for some t_∞ sufficiently large. Then, using the EBCs 2.3 we see that $e(t;t_\infty) \equiv \vec{u}(t;t_\infty,\vec{\gamma},\vec{f}) - \vec{y}(t;\vec{\gamma},\vec{f})$ satisfies

a) $\dfrac{d\vec{e}}{dt} = t^p A(t)\,\vec{e}(t;t_\infty)$ for $t \in (t_0, t_\infty)$,

4.1 b) $C_0\,\vec{e}(t_0;t_\infty) + \tilde{C}(t_\infty)\,\vec{e}(t_\infty;t_\infty) = \vec{\gamma}_1(t_\infty)$,

c) $\tilde{P}(t_\infty)\,\vec{e}(t_\infty;t_\infty) = \vec{\gamma}_2(t_\infty)$.

Here

a) $$\vec{\gamma}_1(t) \equiv \left(C(t) - \tilde{C}(t)\right)\left[\vec{y}(t;\vec{\gamma},\vec{f}) - \vec{y}_p(t)\right] + \tilde{C}(t)\left[\tilde{\vec{y}}_p(t) - \vec{y}_p(t)\right],$$

4.2 b) $$\vec{\gamma}_2(t) \equiv \left(P(t) - \tilde{P}(t)\right)\left[\vec{y}(t;\vec{\gamma},\vec{f}) - \vec{y}_p(t)\right] + \tilde{P}(t)\left[\tilde{\vec{y}}_p(t) - \vec{y}_p(t)\right].$$

Notice that $\tilde{P}(t)\,\vec{\gamma}_2(t) = \vec{\gamma}_2(t)$ by 4.2b, 2.3b. Also, by Theorem 1.5, we see that 4.1 has a unique solution for t_∞

sufficiently large and for any $\vec{\gamma}_1(t_\infty), \vec{\gamma}_2(t_\infty)$ with $\tilde{P}(t_\infty)\,\vec{\gamma}_2(t_\infty) = \vec{\gamma}_2(t_\infty)$.

If we can approximate $\vec{\gamma}_1(t_\infty), \vec{\gamma}_2(t_\infty)$ then we can use 4.1 to compute an approximation for $\vec{e}(t;t_\infty)$. It is important in the applications to notice that the TPBVP 4.1 is of the same form as 2.6, only with different forcing terms. This means that if we solve the difference equations 3.3 using a LU decomposition of the linear system, then <u>the same LU decomposition</u> can be used to solve the difference equations for 4.1. Therefore, once we have an approximation for $\vec{\gamma}_1(t_\infty), \vec{\gamma}_2(t_\infty)$, we can compute an approximation for $\vec{e}(t;t_\infty)$ relatively cheaply.

We will only discuss one of the simplest methods to approximate $\vec{\gamma}_1(t), \vec{\gamma}_2(t)$ for a range of t values. This will allow us to make an estimate of an appropriate t_∞ for a given error tolerance.

We can estimate $\vec{y}_H(t) \equiv \vec{y}(t;\vec{\gamma},\vec{f}) - \vec{y}_p(t)$ in 4.2a,b by

$$4.3 \quad \hat{\vec{y}}_H(t) \equiv \hat{Y}(t,t_\infty)\,\hat{Q}(t_\infty)\Big(\vec{u}(t_\infty;t_\infty,\vec{\gamma},\vec{f}) - \hat{\vec{y}}_p(t_\infty)\Big)$$

where $\hat{P}(t), \hat{Q}(t), \hat{Y}(t,t_\infty), \hat{\vec{y}}_p(t), \hat{C}(t)$ are higher order asymptotic approximations of P,Q, etc. than $\tilde{P},\tilde{Q}$, etc. Then set

$$\text{4.4} \quad \begin{aligned} &\text{a)} \quad \hat{\vec{\gamma}}_1(t) \equiv \left(\hat{C}(t) - \tilde{C}(t)\right)\hat{\vec{y}}_H(t) + \tilde{C}(t)\left(\tilde{\vec{y}}_p(t) - \hat{\vec{y}}_p(t)\right), \\ &\text{b)} \quad \hat{\vec{\gamma}}_2(t) \equiv \tilde{P}(t)\left\{\left(\hat{P}(t) - \tilde{P}(t)\right)\hat{\vec{y}}_H(t) + \tilde{P}(t)\left[\tilde{\vec{y}}_p(t) - \hat{\vec{y}}_p(t)\right]\right\}. \end{aligned}$$

Then $\hat{\gamma}_1, \hat{\gamma}_2$ are estimates for $\vec{\gamma}_1, \vec{\gamma}_2$, for t sufficiently large. We can now estimate $\vec{e}(t;t_1)$ by $\hat{\vec{e}}(t;t_1)$, defined by

$$\begin{aligned} &\text{a)} \quad \frac{d\hat{\vec{e}}}{dt} = t^p A(t)\,\hat{\vec{e}}(t;t_1) \quad \text{for } t \in (t_0, t_1), \\ \text{4.5} \quad &\text{b)} \quad C_0\,\hat{\vec{e}}(t_0;t_1) + \tilde{C}(t_1)\,\hat{\vec{e}}(t_1;t_1) = \hat{\vec{\gamma}}(t_1), \\ &\text{c)} \quad \tilde{P}(t_1)\,\hat{\vec{e}}(t_1;t_1) = \hat{\vec{\gamma}}_2(t_1). \end{aligned}$$

Without solving 4.5 explicitly, we can bound $\hat{\vec{e}}(t;t_1)$ using lemma A2.14

$$\text{4.6} \qquad \sup_{t \in [t_0, t_1]} \left| \hat{\vec{e}}(t;t_1) \right| \leq K\hat{e}(t_1).$$

Finally, we can use 4.6 to provide a reasonable guess of t_∞ for a given error tolerance.

Example calculations using error estimation and error prediction techniques are in progress.

CHAPTER III: A Fredholm Theory for Linear TPBVPs on Semi-Infinite Intervals

III.1 Introduction. We consider TPBVPs of the form

a) $\frac{d\vec{y}}{dt} = t^p A(t)\vec{y}(t) + \vec{f}(t)$ for $t > t_0 > 0$,

1.1 b) $C_0\vec{y}(t_0) + \lim_{t\to\infty} C_\infty \vec{y}(t) = \vec{\gamma} \in \mathbb{C}^m$,

c) $\vec{y}(t)$ is bounded on $[t_0, \infty)$.

Here $A(t)$ is analytic at ∞; $p \geq -1$ is an integer; $\vec{y},\vec{f}:[t_0,\infty) \to \mathbb{C}^n$; $A(t),\vec{f}(t)$ are smooth. In particular, we are interested in the existence, uniqueness and bounds for solutions of 1.1. The Fredholm theory discussed below provides this information for a slightly restricted class of $\vec{f}(t)$'s.

Recently, F. de Hoog and R. Weiss [3] developed a Fredholm theory for 1.1 with the additional conditions that $\lim_{t\to\infty} \vec{y}(t)$ exists, and $t^p A(t)$ has several restrictions on any eigen-values $\alpha(t)$, such that $\mathrm{Re}(t\,\alpha(t)) \to 0$ as $t \to \infty$. Also in [10] Natterer presents a Fredholm theory for the regular singular case (i.e., $p = -1$). Our approach seems to be new.

III.2. Reduction of the Problem

Consider the problem for a particular solution $\vec{y}_p(t)$ of 1.1:

$$\text{a) } \frac{d\vec{y}_p}{dt} = t^p A(t)\vec{y}_p(t) + \vec{f}(t) \quad \text{for } t > t_0,$$

$$2.1 \qquad \text{b) } \lim_{t\to\infty} C_\infty \vec{y}_p(t) = \vec{\gamma}_p(\vec{y}_p) \quad (\text{i.e. } \vec{\gamma}_p(\vec{y}_p) \text{ exists}),$$

$$\text{c) } \vec{y}_p(t) \text{ is bounded on } [t_0, \infty).$$

Let $R \equiv \left\{ \vec{f}(t) \,|\, \vec{f}(t) \text{ smooth, and 2.1 has some solution } \vec{y}_p(t) \right\}$. Notice that if 1.1 has a solution then $\vec{f}(t) \in R$. Conversely, if $\vec{f}(t) \in R$ then we can use the EBCs defined in Section I.1 to reduce the problem of existence and uniqueness of solutions of 1.1 to the well-known theory for <u>regular</u> TPBVPs. In this way the Fredholm theory for 1.1 is reduced to classifying R, or subspaces of R, in terms of p, A(t) and C_∞ (see Section III.3).

The following lemmas give the details of this reduction procedure ($C(t_\infty)$, $P(t_\infty)$, $Q(t_\infty)$, $Y(t,t_\infty)$ are as defined in Section I.1):

<u>Lemma 2.2</u>. The TPBVP 1.1 has a solution $\vec{y}(t;\vec{\gamma},\vec{f})$ iff

i) there exists $\vec{y}_p(t;f)$ satisfying 2.1 (i.e. $\vec{f} \in R$); and for some $t_\infty \geq t_0$

ii) there exists $\vec{u}(t)$ satisfying the <u>regular</u> TPBVP

$$\text{a)}\quad \frac{d\vec{u}}{dt} = t^{p} A(t)\,\vec{u}(t) + \vec{f}(t) \quad \text{for } t \in (t_0, t_\infty),$$

$$2.2 \quad \text{b)}\quad C_0\,\vec{u}(t_0) + C(t_\infty)\,\vec{u}(t_\infty) = \vec{\gamma} + [C(t_\infty)\,\vec{y}_p(t_\infty;\vec{f}) - \vec{\gamma}_p(\vec{y}_p)],$$

$$\text{c)}\quad P(t_\infty)\,\vec{u}(t_\infty) = P(t_\infty)\,\vec{y}_p(t_\infty;\vec{f}).$$

Furthermore, for each $\vec{u}(t)$ a solution $\vec{y}(t;\vec{\gamma},\vec{f})$ of 1.1 is given by

$$2.3 \quad \vec{y}(t;\vec{\gamma},\vec{f}) = \begin{cases} \vec{u}(t) & \text{for } t \in [t_0, t_\infty], \\ \vec{w}(t) & \text{for } t > t_\infty. \end{cases}$$

Where $\vec{w}(t)$ is given by

$$\text{a)}\quad \frac{d\vec{w}}{dt} = t^{p} A(t)\,\vec{w}(t) + \vec{f}(t) \quad \text{for } t > t_\infty,$$

2.4

$$\text{b)}\quad \vec{w}(t_\infty) = \vec{u}(t_\infty).$$

Proof of Lemma 2.2. The proof is an immediate application of the properties of the EBCs discussed in Section I.1. ■

Next we present the well-known Fredholm theory for 2.2 (see [1], [5], or [11]).

Lemma 2.5. Suppose $\vec{f} \in R$ and $\vec{y}_p(t)$ is a solution of 2.1. For $t_\infty \geq t_0$ define

$$\text{a)} \quad \vec{K}(\vec{f}, \vec{\gamma}; \vec{y}_p) \equiv \begin{pmatrix} \vec{\gamma} - \vec{\gamma}_p(\vec{y}_p) - C_0 \vec{y}(t_0) \\ \vec{0} \end{pmatrix} \in \mathbb{C}^{m+n};$$

2.5

$$\text{b)} \quad B_1(t_\infty) \equiv \begin{pmatrix} C_0 Y(t_0, t_\infty) + C(t_\infty) \\ P(t_\infty) \end{pmatrix},$$

$$\text{c)} \quad k_1 = n - \mathrm{Rank}\{B_1(t_\infty)\} \geq 0;$$

$$\text{d)} \quad \{\vec{\beta}_i\}_{i=1}^{k_1+m} \quad \text{to be a basis of } \mathrm{Null}\,(B_1^*(t_\infty)).$$

Then the TPBVP 2.2 has a k_1-dimensional null space (i.e., the set of solutions of 2.2 for $(\vec{\gamma}, \vec{f}) = (\vec{0}, \vec{0})$). Also the TPBVP 2.2 has a solution $\vec{u}(t)$ iff

$$2.6 \quad \vec{\beta}_i^* \vec{K}(\vec{f}, \vec{\gamma}; \vec{y}_p) = 0$$

for $i = 1, \ldots, k_1+m$. Furthermore 2.6 does not depend on the particular $\vec{y}_p(t)$ chosen.

<u>Proof of Lemma 2.5</u>. We seek a solution of 2.2 in the form

$$2.7 \quad \vec{u}(t; \vec{\xi}) = Y(t, t_\infty)\vec{\xi} + \vec{y}_p(t).$$

Substituting 2.7 into 2.2 shows that $\vec{u}(t;\vec{\xi})$ satisfies 2.2 iff

$$2.8 \quad B_1(t_\infty)\vec{\xi} = \vec{K}(\vec{f},\vec{\gamma};\vec{y}_p) .$$

The first two statements of Lemma 2.5 now follow from 2.8 and the Fredholm theory for matrices.

To see that 2.6 is independent of $\vec{y}_p(t)$, let $\vec{y}_p^{\,1},\vec{y}_p^{\,2}$ both satisfy 2.1. Then

$$\vec{y}_p^{\,1} - \vec{y}_p^{\,2} = Y(t,t_\infty)\,Q(t_\infty)\vec{\eta},\quad \vec{\eta} \in \mathbb{C}^n.$$

Therefore, using 2.5a,b

$$2.9 \quad \vec{K}(\vec{\gamma},\vec{f};\vec{y}_p^{\,1}) - \vec{K}(\vec{\gamma},\vec{f};\vec{y}_p^{\,2}) = B_1(t_\infty)\,Q(t_\infty)\vec{\eta}.$$

But by the definition of $\vec{\beta}_i$

$$2.10 \quad \vec{\beta}_i^{\,*} B_1(t_\infty) = \vec{0}^T$$

for $i = 1,\ldots,k_1+m$. The result follows from 2.9,10. ■

Therefore Lemmas 2.2 and 2.5 reduce the problem of existence and uniqueness of solutions of 1.1 to determining whether $\vec{f} \in R$, $k_1 > 0$, and whether 2.6 is satisfied. We consider R in the next section.

III.3. Estimates on Particular Solutions

The complete description of R (see Section III.2) in terms of p, A(t), and C_∞ is difficult for the general case of 1.1. Here we will consider two subspaces of R that can be easily characterized.

We will need the simplifying transformation (see Lemma I4.1)

3.1 $$\vec{w}(s) = J^{-1}(s)\,\vec{y}(t), \quad s^r = t,$$

for $t \geq t_1$, $r \geq 1$ an integer. Here 3.1 is s.t. 1.1a implies

3.2 $$\frac{d\vec{w}}{ds} = \hat{A}(s)\,\vec{w}(s) + \vec{g}(s) \quad \text{for } s > s_1 \equiv t_1^{1/r},$$

with $\hat{A}(s) = \mathrm{diag}(\hat{A}_k(s)\,|\,k=1,\cdots,\ell)$, $\hat{A}_k(s)$ a $n_k \times n_k$ matrix. Also for $k = 1,\ldots,\ell$, and $s \geq s_1$

a) $$\hat{A}_k(s) = J_k(s) + s^{q_k} B_k(s),$$

3.3 b) $$\vec{g}(s) = J^{-1}(s)\,\vec{f}(s^r)\,(rs^{r-1})$$

c) $$J_k(s) = \left\{ i\omega_k p_k(s) + \mu_k s^{\sigma_k - 1} \right\} I_{n_k} + \frac{\delta_k}{s} H_k, \quad H_k = \begin{pmatrix} 0 & \delta_{k,1} & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \delta_{k,n_k-1} \\ 0 & & & 0 \end{pmatrix},$$

where $\omega_k, \mu_k \in \mathbb{R}$, $\sigma_k \geq 0$ is an integer, $sp_k(s)$ is a real polynomial, and $\delta_k, \delta_{kj} = 0$ or 1. Also

d) $\mu_k = 0 \Rightarrow \sigma_k = 0$;

e) $\sigma_k > 0 \Rightarrow \delta_k = 0$;

3.3

f) $B_k(s) \sim \sum_{j=0}^{\infty} B_{kj}\, s^{-j}$ is bounded on $[s_1, \infty)$;

g) $q_k = \sigma_k - 2$.

Furthermore for $1 \le j,\ k \le \ell$, let

3.4 $\Delta p_{jk}(s) \equiv s\left\{i\omega_k\, p_k(s) + \mu_k\, s^{\sigma_k - 1}\right\} - s\left\{i\omega_j\, p_j(s) + \mu_j\, s^{\sigma_j - 1}\right\}.$

Then $\Delta p_{jk}(s) = m'$, an integer, implies that $m' = 0$ and $j = k$. Finally for some integer $m_o \ge 0$

a) $\mathcal{J}(s) \sim \sum_{k=-m_o}^{\infty} T_k\, s^{-k}$,

3.5

b) $\mathcal{J}^{-1}(s) \sim \sum_{k=0}^{\infty} T_k\, s^{-k}$

as $s \to \infty$, where $\{T_k\}_{k=-p}^{\infty}$, $\{T_k\}_{k=0}^{\infty}$ are constant n×n matrices (see Appendix A1, note that we have multiplied the $\mathcal{J}(s)$ in Lemma I4.1 by s^{m_o}).

Using the transformation 3.1, 2.1 becomes

a) $\frac{d\vec{w}_p}{ds} = \hat{A}(s)\,\vec{w}_p + \vec{g}(s)$ for $s > s_1$,

3.6 b) $\lim_{s\to\infty} C_\infty \mathcal{J}(s)\,\vec{w}_p(s) = \zeta_p(\vec{w}_p)$ exists,

c) $\mathcal{J}(s)\,\vec{w}_p(s)$ is bounded on $[s_1, \infty)$.

In the following lemma we use 3.6 to obtain useful estimates on $\vec{y}_p(t;\vec{f})$.

Lemma 3.7. Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{R}^\ell$ satisfy

3.7 $$\alpha_j = \rho + 1 - \sigma_j + \varepsilon_j$$

for $j = 1, \ldots, \ell$, where $\varepsilon_j > 0$. Define

3.8 $$\mathcal{S}_{\vec{\alpha}}(t) = \begin{cases} \operatorname{diag}\left(t^{\alpha_j/r} I_{n_j} \mid j = 1, \ldots, \ell\right) \mathcal{J}^{-1}(t^{1/r}) \dfrac{1}{r t^{1-1/r}} & \text{for } t \ge t_1, \\ I_n & \text{for } t \in [t_0, t_1). \end{cases}$$

Define $R_{\vec{\alpha}}$ by $\vec{f} \in R_{\vec{\alpha}}$ iff $\vec{f}(t) \in C[t_0, \infty)$ and

3.9 $$\| \mathcal{S}_\alpha(t)\,\vec{f}(t) \|_\infty \equiv \sup_{t \ge t_0} | \mathcal{S}_\alpha(t)\,\vec{f}(t) | < \infty.$$

Then for $\varepsilon > 0$ sufficiently small there exists K_ε, depending only on ε, such that for $\vec{f} \in R_0$, 2.1 has a solution $\vec{y}_p(t;\vec{f})$ satisfying

3.10 $$|\vec{y}_p(t;\vec{f})| \leq K_\varepsilon \, \| \mathscr{J}_{\vec{\alpha}}(t) \vec{f}(t) \|_\infty \, t^{-\varepsilon},$$

for $t \geq t_0$. Notice $\vec{y}_p(t;\vec{f}) = o(1)$ as $t \to \infty$.

Proof of Lemma 3.7. The *straightforward* calculation leading to 3.10 is presented in Appendix A3. ∎

From Lemma 3.7 we see that $R_{\vec{\alpha}}$ is a subspace of R, and is easily characterized by using 3.5,8 and 9. Another subspace of R sometimes useful in practice is given in the next lemma.

Lemma 3.11. Let $\vec{\alpha}$, $\mathscr{J}_{\vec{\alpha}}(t)$ be as in 3.7,8. Let $\vec{f}(t) \in R^1_{\vec{\alpha}}$ iff $\vec{f}(t) = \vec{f}_1(t) + \vec{f}_2(t)$ with

i) $\vec{f}_1(t)$ such that there exists a $\vec{y}_1(t)$ satisfying

a) $t^\rho A(t) \vec{y}_1(t) = -\vec{f}_1(t)$ for $t \geq t_0$,

3.11 b) $\vec{y}_1(t) \in C_1[t_0, \infty)$, $\|\vec{y}_1(t)\|_\infty < \infty$, and

$\lim_{t \to \infty} C_\infty \vec{y}_1(t)$ exists,

c) $\| \mathscr{J}_{\vec{\alpha}}(t) \frac{d\vec{y}_1}{dt}(t) \|_\infty < \infty$;

ii) $f_2(t)$ such that there exists a $\vec{y}_2(t)$ satisfying

a) $\vec{y}_2(t) = -\int_t^\infty \vec{f}_2(x)\,dx \quad$ for $t \geq t_o$,

3.12 b) $\| \vec{y}_2(t) \|_\infty < \infty$,

c) $\| \mathscr{J}_{\vec{\alpha}}(t) t^p A(t) \vec{y}_2(t) \|_\infty < \infty$.

Then for $\varepsilon > 0$ sufficiently small there exists a K_ε such that for $\vec{f}(t) = \vec{f}_1(t) + \vec{f}_2(t) \in R'_{\vec{\alpha}}$, 2.1 has a solution $\vec{y}_p(t;\vec{f})$ satisfying

$$3.13 \quad |\vec{y}_p(t;\vec{f}) - \vec{y}_1(t) - \vec{y}_2(t)| \leq K_\varepsilon t^{-\varepsilon} \left\| \mathscr{J}_{\vec{\alpha}}(t) \left[\frac{d\vec{y}_1}{dt} - t^p A(t) \vec{y}_2(t) \right] \right\|_\infty$$

for $t \geq t_0$.

<u>Proof of Lemma 3.11</u>. Let $\vec{v}(t) = \vec{y}_p(t;\vec{f}) - \vec{y}_1(t) - \vec{y}_2(t)$ then by 2.1, 3.11,12

$$3.14 \quad \frac{d\vec{v}}{dt} = t^p A(t) \vec{v}(t) + \left\{ t^p A(t) \vec{y}_2(t) - \frac{d\vec{y}_1}{dt}(t) \right\}$$

for $t > t_0$. The lemma follows by applying Lemma 3.7 to 3.14.

III.4. <u>Fredholm Properties of 1.1</u>

Using the results of Section III.2 and III.3 we can easily prove

Theorem 4.1. Let $\vec{\alpha}, R_{\vec{\alpha}}, \mathcal{A}_{\vec{\alpha}}(t)$ be as in Section III.3. Let $\vec{f} \in R_{\vec{\alpha}}$, and $\vec{y}_p(t;\vec{f})$ satisfy 2.1, 3.10.

Then the TPBVP 1.1 has a solution iff

4.1 $$\vec{\beta}_i^{\,*} \begin{pmatrix} \vec{\gamma} - C_0 \vec{y}_p(t_0;\vec{f}) \\ \vec{0} \end{pmatrix} = 0$$

for $i = 1,\ldots,k_1+m$. Also 1.1, with $\vec{\gamma} = \vec{0}$, $\vec{f}(t) = \vec{0}$, has k_1 linearly independent solutions. Furthermore if 4.1 is satisfied then there exists $\vec{y}(t;\vec{\gamma},\vec{f})$ satisfying 1.1 (unique for $k_1 = 0$) such that

4.2 $$\|\vec{y}(t;\vec{\gamma},\vec{f})\|_\infty \le K_{\vec{\alpha}} \left(|\vec{\gamma}| + \|\mathcal{A}_{\vec{\alpha}}(t)\vec{f}(t)\|_\infty \right)$$

for $K_{\vec{\alpha}}$ independent of γ, f.

Proof. Write

4.3 $$\vec{y}(t;\vec{\gamma},\vec{f}) = Y(t,t_\infty)\vec{\xi} + \vec{y}_p(t;\vec{f}).$$

Then using the EBCs I1.4 shows that $\vec{y}(t;\vec{\gamma},\vec{f})$ satisfies 1.1 iff

4.4 $$B_1(t_\infty)\vec{\xi} = \vec{K}(\vec{f},\vec{\gamma};\vec{y}_p) = \begin{pmatrix} \vec{\gamma} - C_0 \vec{y}_p(t_0;\vec{f}) \\ \vec{0} \end{pmatrix}.$$

Here we have used 2.5a,b and 3.10. By the Fredholm theory for matrices, 4.4 has a solution iff 4.1 is satisfied. Furthermore, there exists a constant K_{t_∞} such that for each $\vec{\gamma}, \vec{y}_p$ satisfying 4.1, 4.4 has a solution $\vec{\xi}(\vec{\gamma},\vec{y}_p)$ such that

4.5 $$\left|\vec{\xi}\,(\vec{\gamma},\vec{y}_p)\right| \le K_{t_\infty}\left\{|\vec{\gamma}| + |C_0|\,|\vec{y}_p(t_0;\vec{f})|\right\}.$$

Finally, the theorem follows from 3.10, 4.3 and 4.5. ∎

Similar results hold for $R'_{\vec{\alpha}}$ (notice that there will be a $\vec{\gamma}_p(\vec{y}_p)$ term in 4.1). In applications we often consider the special case (see Chapter II and Lemma II1.1)

4.6
a) $\mathrm{Rank}\left\{B_1(t_\infty)\right\} = n,$

b) $\mathrm{Rank}\left\{\left[C_0\,Y(t_0,t_\infty) + C_\infty\right]Q(t_\infty)\right\} = m.$

We have

Lemma 4.7. If 4.6 is satisfied then

4.7
a) $k_1 = 0,$

b) 4.1 is identically satisfied.

Proof. 4.7a follows from 2.5c and 4.6a. Let $\vec{\zeta}\in\mathbb{C}^m$, then by 2.5b, 4.6b and Definition I1.3

4.8 $$\begin{pmatrix}\vec{\zeta}\\ \vec{0}\end{pmatrix} \in \mathrm{Range}\left\{B_1(t_\infty)\right\},\ \text{here}\ \begin{pmatrix}\vec{\zeta}\\ \vec{0}\end{pmatrix}\in\mathbb{C}^{m+n}.$$

Now 4.7b follows from 4.8 and 2.5d. ∎

So we have proven

<u>Theorem 4.9</u>. Let $\vec{\alpha}, R_{\vec{\alpha}}, \mathcal{J}_{\vec{\alpha}}(t)$ be as in Section III.3. Suppose 4.6 is satisfied and $\vec{f} \in R_{\vec{\alpha}}$. Then 1.1 has a unique solution $\vec{y}(t;\vec{\gamma},\vec{f})$. Furthermore $\vec{y}(t;\vec{\gamma},\vec{f})$ satisfies 4.2.

We can also use Theorem 4.1 to write the TPBVP1.1 as a Fredholm operator (see [5], [11]). To do this we could define a linear operator $L:\mathcal{D} \to \mathcal{B}$ by

4.10 $$L\vec{y}(t) = (\vec{\gamma},\vec{f}) \text{ iff } \vec{y}(t) = \vec{y}(t;\vec{\gamma},\vec{f}).$$

Here $\vec{y}(t;\vec{\gamma},\vec{f})$ is a solution of 1.1. The Banach spaces $\mathcal{D},\mathcal{B}$ must be chosen appropriately. For example, let $\vec{\alpha}$ satisfy 3.7 and $\mathcal{B} = \mathbb{C}^m \times R_{\vec{\alpha}}$, $\|(\gamma,\vec{f})\|_{\mathcal{B}} \equiv |\vec{\gamma}| + \|\mathcal{J}_{\vec{\alpha}}(t)\vec{f}(t)\|_{\infty}$, $\mathcal{D} = L^{-1}\mathcal{B}$, $\|\vec{y}(t)\|_{\mathcal{D}} = \|\vec{y}(t)\|_{\infty}$. We will not pursue the details here since all the useful results are contained in Theorem 4.1.

CHAPTER IV: Nonlinear Problems

IV.1. An Example Problem

Consider the nonlinear TPBVP

a) $\frac{d\vec{y}}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \vec{y}(t) + \begin{pmatrix} y_2^2(t) \\ y_1(t)\, y_2(t) \end{pmatrix}$ for $t > t_0$,

1.1 b) $C_0 \vec{y}(t_0) = \gamma \in \mathbb{R}$,

c) $\lim_{t \to \infty} \vec{y}(t) = \vec{0}$.

Here $\vec{y}: [t_0, \infty) \to \mathbb{R}^2$, C_0 is a 1×2 real matrix and $\vec{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$. The phase plane for 1.1 is sketched in Fig. 1.1.

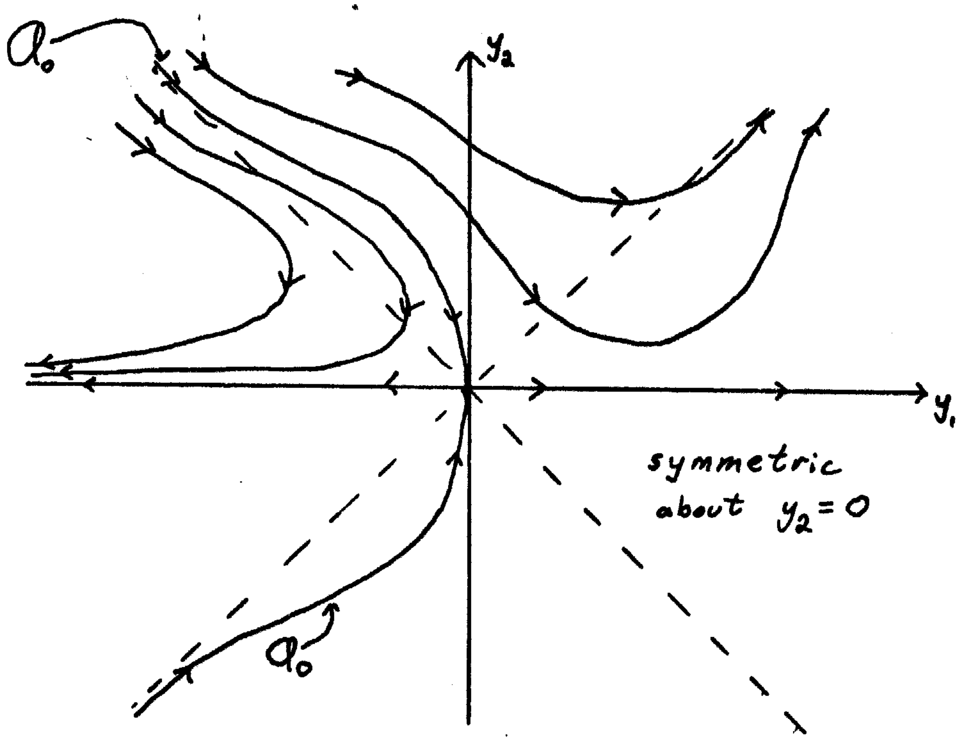

Fig. 1.1

The trajectories labeled $\mathcal{A}_0$ in Fig. 1.1 are analogous to the admissible space $\mathcal{A}(t_\infty)$ defined in Section 1.1 for linear problems. Here, for some $t_\infty \geq t_0$

1.2 $$\mathcal{A}_0 \equiv \{ \vec{y}_0 \in \mathbb{R}^2 \mid \text{if } \vec{y}(t;\vec{y}_0) \text{ satisfies 1.1a for } t > t_\infty \text{ and } \vec{y}(t_\infty;\vec{y}_0) = \vec{y}_0 \text{ then } \lim_{t\to\infty} \vec{y}(t;\vec{y}_0) = \vec{0} \}.$$

Note that $\mathcal{A}_0$ is independent of t_∞ since the right hand side of 1.1a does not depend explicitly on t.

For $t_\infty \geq t_0$ we have the following EBCs for problem 1.1

1.3

a) $C_0 \vec{y}(t_0) = \gamma$,

b) $\vec{y}(t_\infty) \in \mathcal{A}_0$.

It is easily shown that 1.1a,b,c is equivalent to 1.1a with the boundary conditions 1.3a,b.

In order to get ABCs for 1.1 we construct approximations of $\mathcal{A}_0$ for $|\vec{y}(t_\infty)| \ll 1$. Consider the expansion

1.4 $$\vec{y}(t) \sim \sum_{n=1}^{\infty} e^{-nt} \vec{y}_n \quad , \quad |\vec{y}(t)| \ll 1 .$$

Substituting 1.4 into 1.1a and then comparing coefficients of e^{-nt} for n=1,2,..., gives

1.5
a) $\vec{y}_1 = a\begin{pmatrix}0\\1\end{pmatrix}$, $|a| << 1$, $a \in \mathbb{R}$,

b) $\vec{y}_2 = \frac{-a^2}{3}\begin{pmatrix}1\\0\end{pmatrix}$,

and similarly for $\vec{y}_3, \vec{y}_4, \ldots$ Now 1.5 suggests approximating 1.3a for $|\vec{y}(t_\infty)| << 1$ by

1.6 a) $(1 \;\; 0)\,\vec{y}(t_\infty) = 0$;

or

1.6 b) $\vec{y}_1(t_\infty) + \frac{[y_2(t_\infty)]^2}{3} = 0$;

or higher order approximations.

It is easy to show that 1.4, 1.5 give a valid asymptotic expansion for $|a|$ sufficiently small. Therefore, 1.6a,b approximate 1.3b to $O(|\vec{y}(t_\infty)|^2)$ and $O(|\vec{y}(t_\infty)|^4)$ respectively. (We call 1.3a, 1.6a and 1.3a, 1.6b ABCs for 1.1 of $O(|\vec{y}(t_\infty)|^2)$ and $O(|\vec{y}(t_\infty)|^4)$, respectively.)

Using the $O(|\vec{y}(t_\infty)|^4)$ ABCs (for example) we can attempt to approximate a solution of 1.1 by solving the following truncated problem (a <u>regular</u> TPBVP)

1.7 $\frac{d\vec{u}}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \vec{u}(t) + \begin{pmatrix} u_2^2(t) \\ u_1(t)u_2(t) \end{pmatrix}$

for $t \in (t_0, t_\infty)$, and $\vec{u}(t) = (u_1(t), u_2(t))^T$;

b) $C_0 \vec{u}(t_0) = \gamma$,

1.7

c) $u_1(t_\infty) + [u_2(t_\infty)]^2/3 = 0$.

We also require $|\vec{u}(t_\infty)|$ to be "small."

The boundary conditions given by the procedure of M. Lentini and H. B. Keller (see [8]) are

a) $C_0 \vec{u}(t_0) = \gamma$,

1.8

b) $u_1(t_\infty) + u_2^2(t_\infty) = 0$.

This is an $O(|\vec{y}(t_\infty)|^2)$ boundary condition, and is different from the one we obtain using 1.6a.

IV.2. EBCs and ABCs for Nonlinear Problems

Consider

$$\text{a)}\quad \frac{d\vec{y}}{dt} = \vec{f}(t,\vec{y}) \quad \text{for } t > t_0 ,$$

$$2.1 \qquad \text{b)}\quad \vec{b}\left(\vec{y}(t_0), \vec{y}_\infty\right) = \vec{0} \in \mathbb{C}^m ,$$

$$\text{c)}\quad \lim_{t\to\infty} \vec{y}(t) = \vec{y}_\infty , \quad \lim_{t\to\infty} \frac{d\vec{y}}{dt}(t) = \vec{0} .$$

where $\vec{f}$ is smooth. Here $\vec{y}_\infty$ may or may not be given a priori. Let $t_\infty \geq t_0$. Define the admissible space $\mathcal{A}(t_\infty ; \vec{y}_\infty)$ for some $\vec{y}_\infty \in \mathbb{R}^n$ by

$$2.2 \qquad \mathcal{A}(t_\infty ; \vec{y}_\infty) \equiv \left\{ \vec{\xi} \in \mathbb{C}^n \;\middle|\; \begin{array}{l} \text{there exists } \vec{y}(t) \text{ for } t \geq t_\infty \\ \text{such that } \vec{y}(t) \text{ satisfies 2.1a} \\ \text{for } t > t_\infty \text{ and } \vec{y}(t) \\ \text{satisfies 2.1c} \end{array} \right\} .$$

(Compare $\mathcal{A}_0$, equation 1.2). Notice that $\mathcal{A}(t_\infty ; \vec{y}_\infty)$ may be the empty set for some $\vec{y}_\infty$.

Directly from 2.2 we see that we can define the following EBCs for 2.1

$$\text{a)}\quad \vec{b}\left(\vec{y}(t_\infty), \vec{y}_\infty\right) = \vec{0} ,$$

2.3

$$\text{b)}\quad \vec{y}(t_\infty) \in \mathcal{A}(t_\infty ; \vec{y}_\infty) .$$

In particular, the TPBVP 2.1 a,b,c is equivalent to the TPBVP 2.1a, 2.3a,b.

In general $\mathcal{a}(t_\infty;\vec{y}_\infty)$ will not be available in closed form. However, we might attempt to get approximate boundary conditions by approximating $\mathcal{a}(t_\infty;\vec{y}_\infty)$ for t_∞ large and for $|\vec{y}(t_\infty) - \vec{y}_\infty| << 1$ (see Section IV.1). In order to approximate $\mathcal{a}(t_\infty;\vec{y}_\infty)$ we attempt to find the behaviour of solutions of the IVP

2.4

a) $\frac{d\vec{y}}{dt} = f(t, \vec{y}(t))$ for $t \geq t_\infty > t_0$,

b) $\vec{y}(t_\infty) = \vec{y}_\infty + \vec{\xi}$,

where $|\vec{\xi}| << 1$ and $\vec{f}(t,\vec{y}_\infty) \to 0$ as $t \to \infty$. Without loss of generality we will assume that $\vec{y}_\infty = \vec{0}$.

In the sequel we will assume

2.5

a) $\vec{y}_\infty = \vec{0}$,

b) $\vec{f}(t,\vec{y}) = t^p A(t)\vec{y}(t) + \vec{h}(t) + \vec{g}(t, \vec{y}(t))$, for $t > t_0$.

Where

c) $A(t)$ is analytic at ∞, $p \geq -1$ is an integer;

2.5 d) $\vec{h}(t)$ smooth for $t > t_0$;

e) $\vec{g}(t, \vec{0}) = \vec{0}$ for $t \geq 0$, and
$|\vec{g}_{\vec{y}}(t, \vec{y})| \leq \Gamma(t)|\vec{y}(t)|$,

2.5 e) (continued)

$$\text{for } |\vec{y}(t)| < T_y \quad \text{where } T_y > 0 \text{ and } \Gamma(t) \in C[t_0, \infty).$$

With the assumptions in 2.5 we can use the linear theory given in Chapter III to investigate some solutions of 2.4 (i.e., we can study "nearly linear" solutions of 2.4). The reader may find it helpful to apply the rather technical conditions of the following theorem to the example treated in Section IV.1.

<u>Theorem 2.6</u>. Assume that $\vec{f}(t,\vec{y})$ satisfies 2.5. Let $\mathcal{J}(s), r, \rho$ be as given in Lemma I4.1 for the linearized homogeneous equation

2.6 a) $$\frac{d\vec{u}}{dt} = t^{\rho} A(t)\, \vec{u}(t) \quad , \quad \text{for } t > t_0 .$$

Let $\vec{u}(t) = \vec{u}(t; t_\infty, \vec{\xi})$ satisfy 2.6 and

2.6 b) $$\vec{u}(t_\infty) = \vec{\xi} \in \mathbb{C}^n .$$

Suppose $\vec{\xi}_0 \in \mathbb{C}^n$ is such that

2.7 $$|\vec{u}(t; t_\infty, \vec{\xi}_0)| \le K_0 |\vec{\xi}_0| \,\psi(t)$$

for $t \ge t_\infty$, where $K_0 < 0$, $\psi(t) > 0$ for $t \ge t_0$, and $\psi(t) = o(1)$ as $t \to \infty$.

In addition, we assume that for some $\beta > \frac{\rho+2}{r} - 1$ and $\phi(t) = t^{-\tilde{q}} e^{-\tilde{p}(t)}$, with $\tilde{q} \in \mathbb{R}$ and $\tilde{p}(t)$ a real polynomial in $t^{1/r}$, we have

2.8

a) $|\vec{h}(t)| = K_h \, \phi(t)$, for $t \geq t_0$;

b) $t^{\beta} \phi(t) (1 + t^{\beta} \Gamma(t)) = o(1)$,

c) $t^{\beta} \phi(t) = o(\psi(t))$,

d) $\Gamma(t) \psi^2(t) = O(\phi(t))$,

as $t \to \infty$. (For example, in Section 4.1 we could take $\vec{\xi}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\vec{h}(t) \equiv \vec{0}$, $\psi(t) = e^{-t}$, $\Gamma(t) \equiv 1$, $\beta = 1$ and $\phi(t) = e^{-2t}$).

Then there exist constants T, K_1 and $\varepsilon > 0$ such that for $t_\infty > T$ and for $|\vec{\xi}_0| < \varepsilon$ the nonlinear o.d.e. 2.4a has a solution $\vec{y}_p(t\,;\,t_\infty, \vec{\xi}_0)$ satisfying

2.9 $$\left| \vec{y}_p(t; t_\infty, \vec{\xi}_0) - \vec{u}(t; t_\infty, \vec{\xi}_0) \right| \leq K_1 \left\{ \left\| \frac{\vec{h}(t)}{\phi(t)} \right\|_{[t_\infty, \infty)} + |\xi_0|^2 \right\} t^{\beta} \phi(t)$$

for $t \geq t_\infty$.

The hypotheses of Theorem 2.6 are a setup for Lemma III 3.7 and a contraction technique. The proof of Theorem 2.6 is contained in Appendix A4.

Theorem 2.6 gives us useful information about $\mathcal{A}(t_\infty;\vec{0})$. In particular, from estimate 2.9 we see that $\vec{y}_p(t;t_\infty,\vec{\xi}_0) \to 0$ as $t \to \infty$. Therefore $\vec{y}_p(t_\infty;t_\infty,\vec{\xi}_0)\,t\,\mathcal{A}(t_\infty,\vec{0})$. Also by 2.9

2.10 $$\left| \vec{y}_p\,(t_\infty\,;\,t_\infty\,,\,\vec{\xi}_0) - \vec{\xi}_0 \right| \le K_1 \left\{ \left\| \frac{\vec{h}(t)}{\phi(t)} \right\|_{[t_\infty,\infty)} + \left|\vec{\xi}_0\right|^2 \right\} t_\infty^\beta \phi(t_\infty).$$

In this way we can estimate <u>some</u> of the elements of $\mathcal{A}(t_\infty;\vec{0})$. The proof of Theorem 2.6 can also be used to obtain and justify higher order approximations (including the approximations in Section 4.1).

Theorem 2.6 has a partial converse of the following form. Let $\delta > 0$ be sufficiently small, and $\vec{\xi} \in \mathbb{C}^n$ s.t. $\mathcal{U}(t;t_\infty,\vec{\xi}) \to \infty$ sufficiently quickly. Then for $\vec{h}(t)$ sufficiently small the initial value problem

2.11

a) $\dfrac{d\vec{y}}{dt} = \vec{f}(t,\vec{y})$ for $t > t_\infty$, $\vec{f}$ as in 2.5,

b) $\vec{y}\,(t_\infty) = \vec{\xi}$,

has a solution $\vec{y}(t;t_\infty,\vec{\xi})$ for $t \in [t_\infty, T_\delta]$ where $T_\delta \ge t_\infty$ and $\left|\vec{y}\,(T_\delta\,;\,t_\infty,\vec{\xi})\right| > \delta$ (T_δ depends on $\delta, t_\infty, \vec{\xi}$).

We will not pursue these results here. The interested reader can refer to Coddington and Levinson [1, p. 340].

Note that our results above come from a local analysis around $\vec{y}(t) \equiv \vec{0}$. As a result we cannot hope to determine $\mathcal{A}(t_\infty;0)$

in a neighborhood of $\vec{y} = \vec{0}$ for general systems with $\vec{f}(t,\vec{y})$ as in 2.5. To see the problem, consider

$$2.12 \qquad \frac{d\vec{y}}{dt} = \vec{f}(\vec{y}) \in \mathbb{R}^2, \quad \vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

where 2.12 has a phase plane as sketched in Fig. 2.1 below.

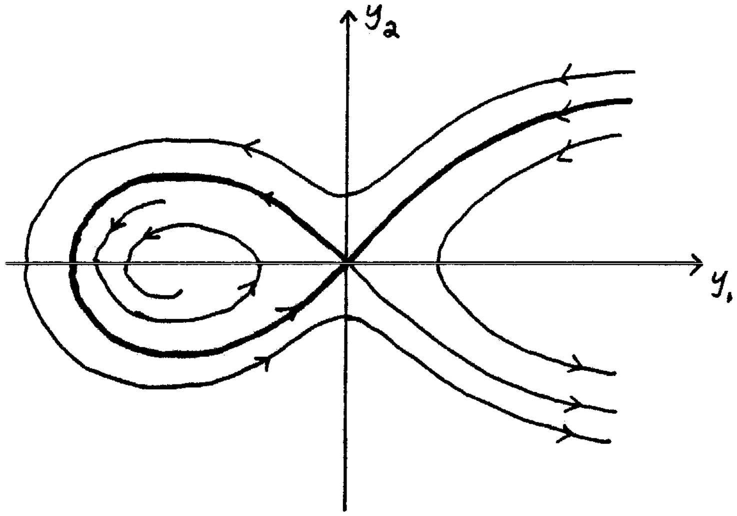

Fig. 2.1

$\mathcal{A}(\vec{0})$ is the heavy line in Fig. 2.1. Also, our local analysis depends on 2.1a being "nearly linear" for $|\vec{y}| \ll 1$. For example, we can obtain a <u>complete</u> local analysis around $\vec{y} = \vec{0}$ of problems of the form

$$2.13 \qquad \frac{d\vec{y}}{dt} = A_0\vec{y} + \vec{g}(\vec{y}), \quad \text{for } t > t_0.$$

Where $A_0 = \text{diag}(\mu_i \mid i=1,\ldots,n)$, $\text{Re}(\mu_i) \neq 0$, $|\vec{g}(\vec{y})| \leq K|\vec{y}|^2$ for some constant $K < \infty$ with $\vec{g}(\vec{y})$ smooth. (Apply Theorem 2.6 above and the converse results in [1, p. 340].) The phase portrait of 2.13 in a neighborhood of $\vec{y} = 0$ is a slight perturbation of the phase portrait of the linearized equation

$$2.14 \quad \frac{d\vec{u}}{dt} = A_o \vec{u}$$

in the same neighborhood. (Hence the term "nearly linear.") An example of a problem that is not nearly linear is presented in the next section

IV.3. A Nonlinear Example Problem

Consider the following example for which Theorem 2.6 does not apply:

$$3.1 \quad \text{a)} \quad \frac{d\vec{y}}{dt} = \begin{pmatrix} y_2(t) + a y_1^2(t) \\ y_2^2(t) \end{pmatrix} \quad \text{for } t > 0,$$

where $\vec{y}(t) = (y_1(t), y_2(t))^T$,

$$3.1 \quad \text{b)} \quad \vec{y}(t) \text{ bounded on } [0, \infty).$$

In particular, we are interested in the admissible space

$$3.2 \quad a(t_\infty ; \vec{0}) \equiv a_o .$$

From 3.1a we see that

3.3 $$y_2(t) = \frac{-1}{c+t} \quad \text{where} \quad c = -[y_2(0)]^{-1}.$$

Therefore $y_2(t)$ is bounded on $[0,\infty)$ iff $y_2(0) < 0$.

Substituting 3.3 into the equation for $y_1(t)$ gives

3.4 $$\frac{dy_1}{dt} = a\,y_1^2(t) - \frac{1}{c+t}, \qquad \text{for } t > 0$$

Here we take $c > 0$ (i.e., $y_2(0) < 0$).

Now if $a \leq 0$ then 3.4 has no bounded solution on $[0,\infty)$. For $a > 0$ we have sketched several solutions of 3.4 in Fig. 3.5.

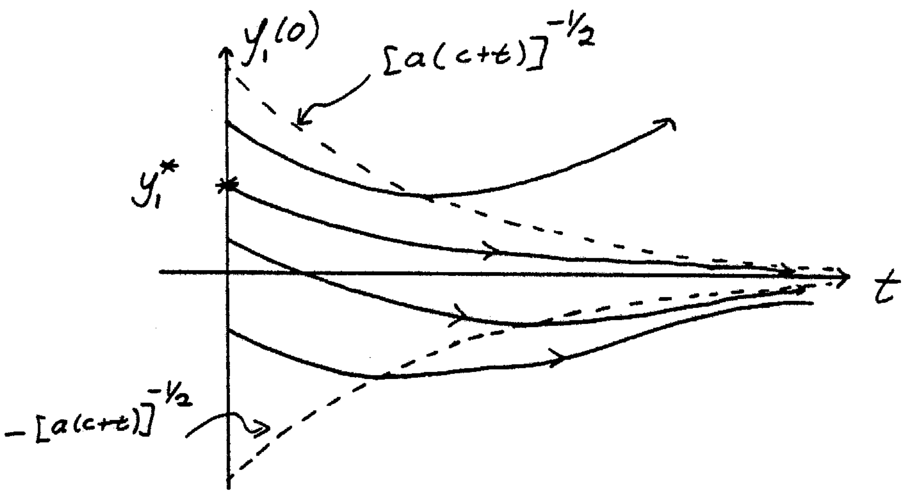

Fig. 3.5

Notice that there exists $y_1^*(a,y_2(0)) > 0$ for $a > 0$ and $y_2(0) < 0$ such that $y_1(t)$ is bounded on $[0,\infty)$ iff $y_1(0) \leq y_1^*(a,y_2(0))$. (Furthermore, $y_1(t) \to 0$ as $t \to \infty$ for these $y_1(0)$). Also, from Fig. 3.1 we see that

3.5 $$0 < y_1^*(a, y_2(0)) \leq \frac{1}{\sqrt{ac}} = \sqrt{\frac{-y_2(0)}{a}} \quad \text{for} \quad a, c > 0.$$

From 3.3, 3.5 and Fig. 3.5 we obtain $\mathcal{Q}_0$ for a > 0 as sketched in Fig. 3.6.

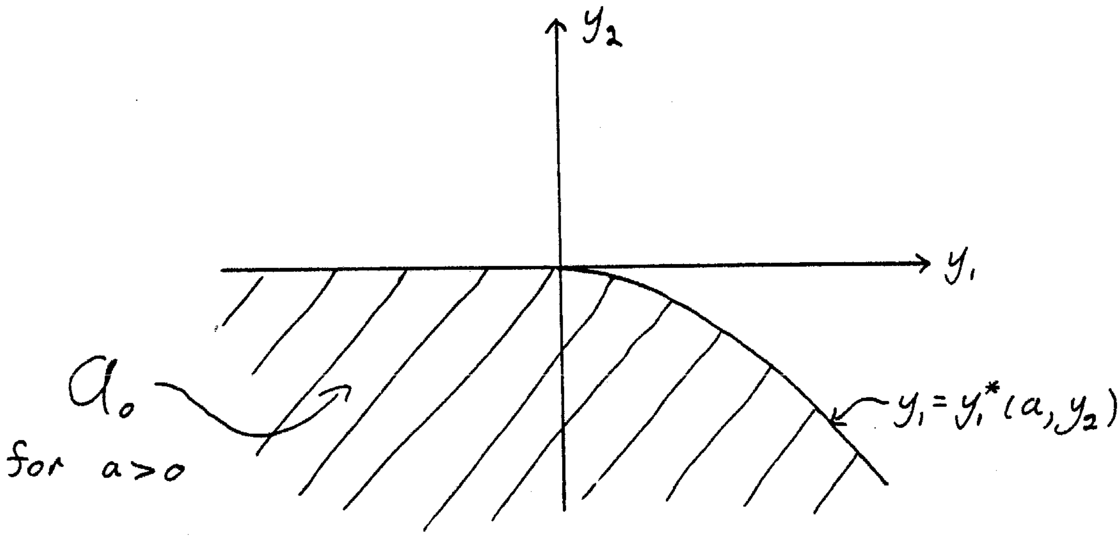

Fig. 3.6

Therefore, we have the following EBCs for problem 3.1 for $t_\infty > 0$:

a) $y_2(t_\infty) \leq 0$,

3.6

b) $y_1(t_\infty) \leq y_1^*(a, y_2)$.

By replacing $y_1^*(a,y_2)$ in 3.6b by an asymptotic approximation for $y_2 \leq 0$, $|y_2| << 1$, we get ABCs for problem 3.1. Notice that both the EBCs and the ABCs for problem 3.1 are <u>inequality constraints on $\vec{y}(t_\infty)$</u>. This is fundamentally different than the boundary conditions we have obtained previously.

A more general analysis of problems of the form 2.1, with $\vec{f}(t,\vec{y})$ as in 2.4 and 2.5, is an area for future research. In particular, we might consider $t^p A(t)$ in 2.4 to have some eigenvalue(s), $\mu_i(t)$, with the $\mathrm{Re}(\mu_i(t)) = O(1/t^2)$ as $t \to \infty$. The procedure outlined in Section IV.2 does not, in general, provide a reasonable $\mathcal{Q}(t_\infty; \vec{y}_\infty)$ in this case (see Section IV.3). The von Karman swirling flow is an example of an interesting practical problem with some $\mu_i(t) \equiv 0$ for certain parameter values. (See [7] and [9]).

IV.4 Existence and Error Estimates for Solutions of the Truncated Problem

Consider the TPBVP

a) $\dfrac{d\vec{y}}{dt} = A_0 \vec{y} + \vec{g}(\vec{y})$, for $t > t_0$,

4.1 b) $\vec{b}(\vec{y}(t_0)) = \vec{0} \in \mathbb{C}^{\ell}$,

c) $\lim\limits_{t \to \infty} \vec{y}(t) = \vec{0}$.

Here

a) $A_0 = \mathrm{diag}(\mu_i \mid i = 1, \ldots, n)$ with
$\mathrm{Re}(\mu_i) < 0$ for $i = 1, \ldots, \ell$ and
$\mathrm{Re}(\mu_i) > 0$ for $i = \ell+1, \ldots, n$;

4.2 b) $\vec{g}(\vec{y})$ smooth and $|\vec{g}(\vec{y})| \le K|\vec{y}|^2$ for

$\vec{y} \in \mathbb{C}^n$ s.t. $|\vec{y}| \le T_y$ where K, T_y are

constants, $T_y > 0$.

From Coddington and Levinson [1, p. 330. Thm. 4.1] we see that there exists a smooth $(n-\ell)$-vector valued function $\vec{\phi}(\vec{y})$ such that:

4.3 a) $\vec{\phi}(\vec{y}_0) = \vec{0} \in \mathbb{C}^{n-\ell} \Rightarrow$ there exists a unique solution

$\vec{y}(t;\vec{y}_0)$ of 4.1a satisfying $\vec{y}(t_0;\vec{y}_0) = \vec{y}_0$.

Furthermore, $\lim\limits_{t\to\infty} \vec{y}(t;\vec{y}_0) = \vec{0}$.

4.3 b) Suppose $\vec{y}_0 \in \mathbb{C}^n$ is such that $\vec{y}(t;\vec{y}_0) \to \vec{0}$ as

$t \to \infty$. Then there exists $t_\infty(\vec{y}_0) < \infty$

such that for all $t_\infty \ge t_\infty(\vec{y}_0)$, $\phi(\vec{y}(t_\infty;\vec{y}_0)) = \vec{0}$.

4.3 c) $\vec{\phi}(\vec{0}) = \vec{0}$, $\vec{\phi}_{\vec{y}}(\vec{0}) = \left(0_{(n-\ell)\times\ell} \quad I_{(n-\ell)\times(n-\ell)}\right)$.

Using this $\vec{\phi}(\vec{y})$ we see from 4.3 that the boundary condition 4.1c is equivalent to

4.4 $\phi(\vec{y}(t_\infty)) = \vec{0}$

for some $t_\infty < \infty$, t_∞ sufficiently large.

Suppose that for $|\vec{y}|$ sufficiently small we have $\tilde{\vec{\phi}}(\vec{y})$ an approximation of $\vec{\phi}(\vec{y})$. (See IV.1 and IV.2 for a discussion of techniques for constructing $\tilde{\vec{\phi}}$.) In particular, assume

4.5
a) $\left| \vec{\phi}(\vec{y}) - \tilde{\vec{\phi}}(\vec{y}) \right| = O(|\vec{y}|^2)$

b) $\tilde{\vec{\phi}}$ smooth

We wish to consider the following truncated problem

a) $\dfrac{d\vec{u}}{dt} = A_0 \vec{u} + \vec{g}(\vec{u})$ for $t \in (t_0, t_\infty)$,

4.6 b) $\vec{b}(\vec{u}(t_0)) = \vec{0}$,

c) $\tilde{\vec{\phi}}(\vec{u}(t_\infty)) = \vec{0}$,

where t_∞ is sufficiently large. We call 4.6b,c ABCs for problem 4.1 (see 4.5a).

We have the following theorem on the existence of solutions of the regular TPBVP 4.6:

<u>Theorem 4.7</u>. Suppose $\vec{y}_1(t)$ is a solution of 4.1 such that the linearized problem

$$\text{a)}\quad \frac{d\vec{v}}{dt} = \left[A_0 + \vec{g}_{\vec{y}}\,(\vec{y}_1\,(t))\right]\vec{v}\quad, \text{for } t \in (t_0, t_\infty),$$

$$4.7\qquad \text{b)}\quad \left\{\vec{b}_{\vec{y}}\,(\vec{y}_1\,(t_0))\right\}\vec{v}(t_0) = \vec{0} \in \mathbb{C}^{\ell},$$

$$\text{c)}\quad \left\{\vec{\phi}_{\vec{y}}\,(\vec{y}_1\,(t_\infty))\right\}\vec{v}(t_\infty) = \vec{\xi} \in \mathbb{C}^{n-\ell},$$

has a unique solution for each t_∞ sufficiently large, and $\vec{\xi} \in \mathbb{C}^{n-\ell}$. Furthermore, assume that for t_∞ sufficiently large there exists a constant K_0, independent of t_∞, s.t.:

$$4.8\qquad \sup_{t\in[t_0,t_\infty]} \left|\vec{v}\,(t\,;t_\infty, \vec{\xi})\right| \le K_0\,|\vec{\xi}|,$$

where $\vec{v}(t; t_\infty, \vec{\xi})$ is the solution of 4.7.

Then for t_∞ sufficiently large the TPBVP 4.6 has a solution $\vec{u}_1(t;t_\infty, \vec{y}_1)$ satisfying

$$4.9\qquad \sup_{t\in[t_0,t_\infty]} \left|\vec{u}_1\,(t;t_\infty, \vec{y}_1) - \vec{y}_1\,(t)\right| \le K_1\,\left|\vec{y}_1\,(t_\infty)\right|^2.$$

Where K_1 is a constant independent of t_∞.

<u>Proof of Theorem 4.7</u>. Consider the TPBVP

$$\text{a)}\quad \frac{d\vec{w}}{dt} = A_0\,\vec{w} + \vec{g}\,(\vec{w}),\quad \text{for } t \in (t_0, t_\infty),$$

$$4.10\qquad \text{b)}\quad \vec{b}\,(\vec{w}\,(t_0)) = \vec{0} \in \mathbb{C}^{\ell},$$

$$\text{c)}\quad \vec{\phi}\,(\vec{w}\,(t_\infty)) = \vec{\xi} \in \mathbb{C}^{n-\ell}.$$

By the Implicit Function theorem it follows from estimate 4.8 that for t_∞ sufficiently large and for $|\vec{\xi}|$ sufficiently small 4.10 has a solution $\vec{w}(t;t_\infty,\vec{\xi})$. Also, $\vec{w}(t;t_\infty,\vec{0}) \equiv \vec{y}_1(t)$ and

4.11 $$\sup_{t\in[t_0,t_\infty]} \left| \vec{w}(t;t_\infty,\vec{\xi}) - \vec{w}(t;t_\infty,\vec{\eta}) \right| \le K_2 \left|\vec{\xi}-\vec{\eta}\right|$$

where $\vec{\xi},\vec{\eta}\in\mathbb{C}^{n-1}, |\vec{\xi}|, |\vec{\eta}| < \delta$. Furthermore, $\delta > 0$ and K_2 are constants independent of t .

Now consider the iteration scheme

a) $$\vec{u}^{\,0}(t;t_\infty) = \vec{y}_1(t),$$

4.12 b) $$\vec{u}^{\,\nu}(t;t_\infty) \equiv \vec{w}(t;t_\infty,\vec{\xi}^{\,\nu-1}), \quad \text{for } \nu \ge 1,$$

c) $$\vec{\xi}^{\,\nu} = \phi(\vec{u}^{\,\nu}(t_\infty;t_\infty)) - \tilde{\Phi}(\vec{u}^{\,\nu}(t_\infty;t_\infty)).$$

Using estimate 4.11 (assuming for the moment that $\vec{\xi}^{\,\nu}, \vec{\xi}^{\,\nu-1}$ are sufficiently small)

$$\sup_{[t_0,t_\infty]} \left|\vec{u}^{\,\nu+1} - \vec{u}^{\,\nu}\right| = \Delta^{\nu+1} \le K_2 \left|\vec{\xi}^{\,\nu} - \vec{\xi}^{\,\nu-1}\right|.$$

However, by 4.3, 4.5, and 4.12c

$$\left|\vec{\xi}^{\,\nu} - \vec{\xi}^{\,\nu-1}\right| \le K_3 \left(\left|u^{\nu}(t_\infty,t_\infty)\right| + \left|u^{\nu-1}(t_\infty,t_\infty)\right| \Delta^{\nu}.\right.$$

Therefore, if $\vec{\xi}^{\,\nu}, \xi^{\nu}$ are sufficiently small, then

4.13 a) $\Delta^{\nu+1} \leq K_2 K_3 \left(|u^{\nu}(t_\infty ; t_\infty)| + |u^{\nu-1}(t_\infty ; t_\infty)|\right) \Delta^{\nu}$,

also

4.13 b) $\Delta^{1} \leq K_3 \, |y_1(t_\infty)|^2$.

Therefore, the theorem follows for $|\vec{y}_1(t_\infty)|$ sufficiently small, that is, t_∞ sufficiently large. ■

Similar results can be obtained (by similar techniques) for more general TPBVPs than 4.1. We will not pursue these results here.

APPENDIX A1

An Outline of a Constructive Proof of Lemma I4.1: Let A(t) in 1.1a satisfy

$$\text{a)}\quad A(t) = \sum_{k=0}^{\infty} A_k t^{-k} \quad \text{for } t > T;$$

A1.1 $$\text{b)}\quad A_0 = \operatorname{diag}\left(\mu_i I_{n_i} + H_i \,\middle|\, i = 1,\ldots,\ell\right),$$

$$H_i = \begin{pmatrix} 0 & \hat{\delta}_{i1} & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \hat{\delta}_{i(n_i-1)} \\ 0 & & & 0 \end{pmatrix},$$

with $\hat{\delta}_{ij} = 0$ or 1 and A_0 in Jordan form. Also we assume

$$\text{c)}\quad \mu_i - \mu_j \neq 0 \quad \text{for } i \neq j,$$

A1.1

$$\text{d)}\quad \text{if } p = -1 \text{ then } \mu_i - \mu_j \notin \{0, 1, 2, \ldots\} \text{ for } i \neq j.$$

Definition A1.2. For A_0 as above, a n×n matrix B is $\pi(A_0)$-diagonal iff it is block diagonal with

A1.2 $$B = \operatorname{diag}\left(B_i \,\middle|\, i = 1,\ldots,\ell;\ B_i \text{ a } n_i \times n_i \text{ matrix}\right).$$

Lemma A1.3. For A(t) as above, there exists $\{L_k\}_{k=1}^{\infty}$ a sequence of constant n×n matrices, and $\{t_k\}_{k=1}^{\infty}$ such that for $q \geq 0$

A1.3

a) $\mathcal{L}_q(t) \equiv (I + \frac{1}{t} L_1) \ldots (I + \frac{1}{t^q} L_q), \mathcal{L}_0(t) \equiv I,$

b) $\det(\mathcal{L}_q(t)) \neq 0$ for $t > t_q \geq T.$

Furthermore, $\vec{w}_q(t) = \mathcal{L}_q^{-1}(t)\vec{y}(t)$ satisfies (from 1.1a)

A1.4 $$\frac{d\vec{w}_q}{dt} = t^p A_q(t)\, \vec{w}_q(t)$$

for $t > t_q$. Here

A1.5

a) $A_q(t) = \sum_{k=0}^{\infty} A_{qk} t^{-k}$, for $t > t_q$,

b) $A_{q0} = A_0$.

And $\{L_k\}_{k=1}^{\infty}$ is chosen s.t.

i) if $p > -1$ then for $k = 0, 1, \ldots, q$,

A1.5 c) A_{qk} is $\pi(A_0)$ diagonal;

ii) if $p = -1$ then for $q = 1, 2, \ldots,$

A1.5 d) $A_{qk} = 0$.

Proof of Lemma A1.3. We see from A1.3a that

$$\text{A1.6} \quad \frac{d\vec{w}_q}{dt} = \left(I + \frac{1}{t^q} L_q\right)^{-1} \left\{ t^p A_{q-1}(t)\left(I + \frac{1}{t^q} L_q\right) + \frac{qL_q}{t^{q+1}} \right\} \vec{w}_q(t),$$

$$\equiv t^p A_q(t)\, \vec{w}_q(t)$$

for $t > t_q$, $q \geq 1$. Left multiplying A1.6 by $\left(I + \frac{1}{t^q} L_q\right)$ and comparing like powers of $1/t$ gives

$$\text{A1.7} \quad \text{a)} \quad A_{qk} = A_{(q-1)k}$$

for $k = 0,1,\ldots,q-1$, and

$$\text{A1.7} \quad \text{b)} \quad A_{q(q+k)} = A_{(q-1)(q+k)} + A_{(q-1)k} L_q - L_q A_k^q + qL_q\, \delta_{k,p+1}$$

for $k = 0,1,2,\ldots$. Here $\delta_{ij} = 1$ if $i = j$, $= 0$ otherwise.

We can use the recursion relations A1.7 to prove the lemma by induction on q. For each $q \geq 1$, we choose L_q using (from A1.7)

$$\text{A1.8} \quad A_{qq} = A_{(q-1)q} + A_0 L_q - L_q A_0 + qL_q\, \delta_{p,-1}$$

We will leave out the straightforward calculation showing that there exists L_q for $q \geq 1$ such that

i) if $p > -1$ then A_{qq} <u>equals</u> the $\pi(A_0)$-diagonal part of $A_{(q-1)q}$.

ii) if $p = -1$ and $\mu_i - \mu_j \notin \{0,1,2,\ldots\}$ for any i,j then $A_{qq} = 0$. ■

<u>Lemma A1.9</u>. For $A(t)$ as in A1.1, there exists $\mathcal{L}(t)$ for $t > T_1$ such that

A1.9 $$\mathcal{L}(t) = \mathcal{L}_q(t) + O(t^{-1-q})$$

as $t \to \infty$, for $q = 0,1,2,\ldots$; $\vec{w}(t) \equiv \mathcal{L}^{-1}(t)\vec{y}(t)$ for $t > T_1$ satisfies

A1.10 a) $$\frac{d\vec{w}}{dt} = t^p \hat{A}(t)\,\vec{w}(t), \quad \text{for } t > T_1 .$$

Here

b) $\hat{A}(t) = \text{diag}\,(\hat{A}_i(t) \mid i = 1,\ldots,l,$ with A_i a $n_i \times n_i$ matrix$)$,

A1.10

c) $$\hat{A}(t) = \sum_{k=0}^{q} A_{qk}\, t^{-k} + O(t^{-1-q})$$

as $t \to \infty$, for $q \geq 0$. Furthermore if $p = -1$ then

a) $$\mathcal{L}(t) = \lim_{q\to\infty} \mathcal{L}_q(t) \quad \text{for } t > T_1,$$

A1.11

b) $$\hat{A}(t) \equiv A_0 .$$

For the proof of Lemma A1.9 we refer the reader to Wasow [12, p. 55].

We can consider each diagonal block of A1.10 a separately, so without loss of generality we can take $\ell = 1$. Rewriting A1.10:

A1.12

a) $$\frac{d\vec{w}^{1}}{dt} = \left\{ t^{p}(\mu_{1} I + H_{1}) + t^{p-1} A^{1}(t) \right\} \vec{w}^{1}(t) \quad \text{for } t > T_{1}$$

b) $$A^{1}(t) \sim \sum_{k=0}^{\infty} A_{k}^{1} t^{-k} \quad \text{as } t \to \infty .$$

If $H_1 \equiv 0$ then we can transform A_0^1 to Jordan form and reapply Lemmas A1.3 and A1.9. If A_0^1 has a complete set of eigenvectors and $p > 0$, then we can apply Lemmas A1.3 and A1.9 once more to get

A1.13 a) $$\frac{d\vec{w}^{2}}{dt} = \operatorname{diag}\left(d_{i}(t) I_{m_{i}} + t^{p-2} A_{i}^{(2)}(t) \middle/ i = 1, \ldots, \ell_{2} \right) \vec{w}^{2}$$

with

A1.14 b) $$A_{i}^{2}(t) \sim \sum_{k=0}^{\infty} A_{ik}^{2} t^{-k} \quad \text{as } t \to \infty .$$

If we never encounter a nontrivial Jordan block, or a case for which A1.1d is not satisfied, then we see that Lemma I4.1 follows from several applications of Lemmas A1.3 and A1.9.

Finally we consider the two cases for which we need to do something more. First, suppose, that at some stage of the reduction procedure we have

$$\text{a)}\quad \frac{d\vec{v}}{dt} = \left\{ d(t)\, I_m + t^{-1} J(t) \right\} \vec{v}(t),$$

A1.15 $$\text{b)}\quad J(t) \sim \sum_{k=0}^{\infty} J_k t^{-k} \quad \text{as } t \to \infty,$$

$$\text{c)}\quad J_0 = \operatorname{diag}\left(\mu_i I_{m_i} + H_i \,\middle|\, i = 1, 2, \ldots, \ell \right),$$

with H_i as in A1.1b. Also, suppose that

$$\text{d)}\quad \mu_2 - \mu_1 = k \in \{1, 2, \ldots\},$$

A1.15

$$\text{e)}\quad \mu_i \neq \mu_j \quad \text{for } i \neq j.$$

Several applications of the transformation given in the following lemma will transform A1.15 to a form suitable for Lemma A1.3.

Lemma A1.16. Consider the transformation

A1.16 $$\text{a)}\quad \vec{v}_1(t) = \mathcal{J}_1^{-1}(t)\, \vec{v}(t) \qquad \text{for } t > t_0$$

with

$$\text{b)}\quad \mathcal{J}_1(t) = (I - P_1) + \frac{1}{t} P_1, \quad \text{a } m \times m \text{ matrix},$$

A1.16

$$\text{c)}\quad P_1 = \operatorname{diag}\left(\delta_{i,1} I_{m_i} \,\middle|\, i = 1, \ldots, \ell \right)$$

(see A1.15). Then there exists a constant m×m matrix T such that A1.15a implies that $\vec{w}(t) = T^{-1} \mathcal{J}_1^{-1}(t)\, \vec{v}(t)$ satisfies

A1.17 a) $$\frac{d\vec{w}}{dt} = \left\{ d(t)\, I_m + t^{-1} J_1(t) \right\} \vec{w}(t) \quad \text{for } t > t_0.$$

Here

b) $$J_1(t) \sim \sum_{k=0}^{\infty} J_{1k}\, t^{-k}, \quad .$$

A1.17 c) $$J_{10} = \operatorname{diag}\left(\mu_{1i}\, I_{m_i} + H_{1i} \mid i = 1, \ldots, l\right),$$

d) $$\mu_{1i} = \mu_i + \delta_{i1} \quad \text{for } i = 1, \ldots, l$$

(i.e., $\mu_{12} - \mu_{11} = k-1$).

The proof of Lemma A1.16 is an easy calculation, and it is omitted. Finally, we must consider the trouble noted above for a nontrivial Jordan block. For a complete treatment of this problem see Wasow [8]. Below we present an example to illustrate the procedure.

Example A1.18. Consider

A1.18 $$\frac{d\vec{y}}{dt} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{t} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\} \vec{y}(t) \quad \text{for } t > 1.$$

Notice that we cannot reduce A1.18 any further using Lemma A1.3. However, suppose we rescale the variables y_1, y_2, y_3 as follows:

A1.19 a) $$\vec{w}(s) = \mathscr{V}^{-1}(s)\, \vec{y}(t), \quad s = t^{1/3}$$

for $t \geq 1$, and

A1.19 b) $$\mathcal{J}(s) \equiv \begin{pmatrix} 1/s & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/s^2 \end{pmatrix}.$$

Then A1.18 becomes

A1.20 $$\frac{d\vec{w}}{ds} = \left\{ \frac{dt}{ds} \left[\mathcal{J}^{-1}(s) \begin{pmatrix} 0 & 1/t & 0 \\ 0 & 0 & 1 \\ 1/t & 0 & 0 \end{pmatrix} \mathcal{J}(s) \right] - \mathcal{J}^{-1} \frac{d\mathcal{J}}{ds} \right\} \vec{w}(s).$$

Simplifying A1.20, we get

A1.21 $$\frac{d\vec{w}}{ds} = \left\{ 3 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \frac{1}{s} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \right\} \vec{w}.$$

Now the leading order matrix in A1.21 can be diagonalized, with distinct eigenvalues. After transforming A1.21 to the form in A1.1 (a constant transformation) we can put A1.18 in the desired form with one application of Lemma A1.3.

The above techniques are not feasible for automatic computation of the $\{T_k\}_{k=0}^{\infty}$, etc. The development of an automatic procedure for calculating the ABCs is an important area for future research.

APPENDIX A2

Proof of Lemma 1.1. We begin by showing that 1.1 does not depend on t_∞, or the particular $P(t_\infty)$, $C(t_\infty)$ chosen (recall that $Q(t_\infty)$ is not uniquely defined by definition I1.3).

Notice that $C(t_\infty)P(t_\infty) = 0$ and $P(t_\infty) = I-Q(t_\infty)$. Therefore

$$\operatorname{Rank}\begin{pmatrix} C_0 Y(t_0,t_\infty)+C(t_\infty) \\ P(t_\infty) \end{pmatrix} = \operatorname{Rank}\left(\begin{array}{c:c} C_0Y(t_0,t_\infty)P(t_\infty) & [C_0Y(t_0,t_\infty)+C(t_\infty)]Q(t_\infty) \\ P(t_\infty) & 0 \end{array}\right)$$

$$= \operatorname{Rank}\left(P(t_\infty)\right) + \operatorname{Rank}\left([C_0 Y(t_0,t_\infty)+C(t_\infty)]\, Q(t_\infty)\right),$$

$$= n - \dim \mathcal{A}(t_\infty) + \dim\left\{[C_0 Y(t_0,t_\infty)+C(t_\infty)]\vec{\xi} \,/\, \vec{\xi}\in \mathcal{A}(t_\infty)\right\},$$

and $C(t_\infty)\vec{\xi}$ is uniquely defined for $\vec{\xi}\in \mathcal{A}(t_\infty)$. Therefore 1.1 is independent of the particular $P(t_\infty)$, $C(t_\infty)$.

Therefore without loss of generality we can assume that

a) $P(t) = Y(t,t_0)\,P(t_0)\,Y(t_0,t)$ for $t > t_0$,

A2.1

b) $C(t) = C(t_0)\,Y(t_0,t)$ for $t > t_0$.

Using A2.1 we see that 1.1 does not depend on t_∞.

Recall from Lemma I1.4 that the EBCs I1.4b,c are equivalent to I1.1b,c. Substituting the general solution of I1.1a

$$\vec{y}(t) = Y(t, t_\infty)\vec{\xi}\ , \quad \vec{\xi} \in \mathbb{C}^n,$$

into I1.4b,c implies

$$\begin{pmatrix} C_0 Y(t_0, t_\infty) + C(t_\infty) \\ P(t_\infty) \end{pmatrix} \vec{\xi} = \begin{pmatrix} \vec{\gamma} \\ \vec{0} \end{pmatrix}.$$

And so 1.1a and b follow.

Finally, if 1.1 is satisfied then the solution operator $S:\mathbb{C}^m \to C[t_0, \infty): \vec{\gamma} \to \vec{y}(t;\vec{\gamma})$ is a well-defined linear operator. By I1.1c S is pointwise bounded (i.e., $\sup_{t \geq t_0} |\vec{y}(t,\vec{\gamma})| < \infty$ for each $\vec{\gamma} \in \mathbb{C}^m$). So $m < \infty$ implies 1.1. ■

<u>Proof of Theorem 1.5</u>. We will first prove several related lemmas.

<u>Lemma A2.2</u>. The TPBVPs I1.1 and 1.3 are equivalent to the matrix problems

a) $B(t_\infty)\vec{\alpha}(t_\infty) = \begin{pmatrix} \vec{\gamma} \\ \vec{0} \end{pmatrix} \in \mathbb{C}^{4n+m}$,

A2.2

b) $\left(B(t_\infty) + E(t_\infty)\right)\vec{\beta}(t_\infty) = \begin{pmatrix} \vec{\gamma} \\ \vec{0} \end{pmatrix} \in \mathbb{C}^{4n+m}$,

respectively. Where $\vec{\alpha}(t_\infty), \vec{\beta}(t_\infty) \in \mathbb{C}^{3n}$ and

A2.3

a) $$B(t_\infty) = \left(\begin{array}{c:c:c} C_0 P_0(t_0) & \{C_0 Y(t_0, t_\infty) + C(t_\infty)\} P_1(t_\infty) & C_0 Y(t_0, t_\infty) P(t_\infty) \\ 0 & 0 & P(t_\infty) \\ I - P_0(t_0) & 0 & 0 \\ 0 & I - P_1(t_\infty) & 0 \\ 0 & 0 & I - P(t_\infty) \end{array}\right),$$

b) $$E(t_\infty) = \begin{pmatrix} \Delta C(t_\infty) Y(t_\infty, t_0) P_0(t_0) & \Delta C(t_\infty) & \Delta C(t_\infty) \\ \Delta P(t_\infty) Y(t_\infty, t_0) P_0(t_0) & \Delta P(t_\infty) & \Delta P(t_\infty) \\ 0 & 0 & 0 \end{pmatrix},$$

where $\Delta C(t_\infty) = \tilde{C}(t_\infty) - C(t_\infty)$, $\Delta P(t_\infty) = \tilde{P}(t_\infty) - P(t_\infty)$,

<u>Proof of Lemma A2.2</u>. A general solution of I1.1a is

A2.4 $$\vec{y}(t) = Y(t, t_0)\vec{\alpha}_1 + Y(t, t_\infty)(\vec{\alpha}_2 + \vec{\alpha}_3),$$

where $\vec{\alpha}_1 = P_0(t_0)\vec{\alpha}_1$, $\vec{\alpha}_2 = P_1(t_\infty)\vec{\alpha}_2$, $\vec{\alpha}_3 = P(t_\infty)\vec{\alpha}_3$.

To see this recall from Definition 1.4 that

$$a_0(t_\infty) = Y(t_\infty, t_0)\, a_0(t_0),$$

therefore $Y(t_\infty, t_0)\vec{\alpha}_1 \in a_0(t_\infty)$. So we can rewrite A2.4 as

$$\vec{y}(t) = Y(t, t_\infty)\left\{ Y(t_\infty, t_0)\vec{\alpha}_1 + \vec{\alpha}_2 + \vec{\alpha}_3 \right\}.$$

Finally, notice that Range $\left\{ P_0(t_\infty) + P_1(t_\infty) + P(t_\infty) \right\} = \mathbb{C}^n$

Substituting A2.4 into the EBCs I1.4b,c gives

A2.5

a) $$C_0 \left\{ \vec{\alpha}_1 + Y(t_0, t_\infty)(\vec{\alpha}_2 + \vec{\alpha}_3) \right\} + C(t_\infty)\vec{\alpha}_2 = \vec{\gamma},$$

b) $$P(t_\infty)\,\vec{\alpha}_3 = 0,$$

where we have used $C(t_\infty)\,P_0(t_\infty) = C(t_\infty)\,P(t_\infty) = 0$ in A2.5a. So A2.2a follows with $\vec{\alpha}(t_\infty) = (\vec{\alpha}_1^T, \vec{\alpha}_2^T, \vec{\alpha}_3^T)^T$. Similarly, set

A2.6 $$\vec{u}(t) = Y(t, t_0)\,\vec{\beta}_1 + Y(t, t_\infty)(\vec{\beta}_2 + \vec{\beta}_3),$$

where $\vec{\beta}_1 = P_0(t_0)\vec{\beta}_1$, $\vec{\beta}_2 = P_1(t_\infty)\vec{\beta}_2$, $\vec{\beta}_3 = P(t_\infty)\vec{\beta}_3$.

Substituting A2.6 into the boundary conditions I1.4b,c gives

A2.7

a) $$C_0 \left\{ \vec{\beta}_1 + Y(t_0, t_\infty)(\vec{\beta}_2 + \vec{\beta}_3) \right\} + \tilde{C}(t_\infty)\left\{ Y(t_\infty, t_0)\vec{\beta}_1 + \beta_2 + \beta_3 \right\} = \vec{\gamma},$$

b) $$\tilde{P}(t_\infty)\left\{ Y(t_\infty, t_0)\vec{\beta}_1 + (\vec{\beta}_2 + \vec{\beta}_3) \right\} = \vec{0}.$$

And then A2.2b is obtained by adding

A2.8

a) $$C(t_\infty)\vec{\beta}_2 - C(t_\infty)\left\{ Y(t_\infty, t_0)\vec{\beta}_1 + \vec{\beta}_2 + \vec{\beta}_3 \right\} = \vec{0},$$

b) $$P(t_\infty)\vec{\beta}_3 - P(t_\infty)\left\{ Y(t_\infty, t_0)\vec{\beta}_1 + \vec{\beta}_2 + \vec{\beta}_3 \right\} = \vec{0},$$

to the left hand sides of A2.7a,b, respectively. ■

Note that if 1.1 is satisfied then by Lemma 1.1, A2.2a has a unique solution $\vec{\alpha}(t_\infty)$ for each $\vec{\gamma} \in \mathbb{C}^n$. We would like to prove that A2.2b has a unique solution for t_∞ sufficiently large. We begin by rewriting equation A2.2b in a more useful form.

<u>Lemma A2.9</u>. Let $\hat{E}(t_\infty) = \begin{pmatrix} I_{m\times m} & 0 & 0 \\ 0 & P(t_\infty) & 0 \\ 0 & 0 & I_{3n\times 3n} \end{pmatrix} E(t_\infty)$

Assume that $\tilde{P}(t_\infty)$, $P(t_\infty)$ satisfy estimate 1.5b, with $\tilde{P}(t_\infty)$ a projection matrix.

Then there exists T_0 independent of $\vec{\gamma}$ s.t. for $t_\infty \geq T_0$, $\vec{\beta}$ satisfies A2.2b iff

A2.9 $$\left(B(t_\infty) + \hat{E}(t_\infty)\right)\vec{\beta}(t_\infty) = \begin{pmatrix} \vec{\gamma} \\ \vec{0} \end{pmatrix}.$$

<u>Proof of Lemma A2.9</u>. Clearly A2.2b $\Rightarrow$ A2.9. Conversely, if $\vec{\beta}(t_\infty)$ satisfies A2.9 then

A2.10 $$P(t_\infty)\,\tilde{P}(t_\infty)\,\vec{u}(t_\infty) = \vec{0},$$

where $\vec{u}(t_\infty) \equiv (Y(t_\infty, t_0)\, P_0(t_0) \mid I_{n\times n} \mid I_{n\times n})\vec{\beta}$ (see A2.3, A2.6, A2.7b, A2.8b, and the definition of $\hat{E}(t_\infty)$). Since $P = I-Q$ and since $\tilde{P}$ is a projection matrix, we see that we can write A2.10 as

A2.11 $$\left[Q(t_\infty)\left(\tilde{P}(t_\infty) - P(t_\infty)\right)\right]\tilde{P}(t_\infty)\,\vec{u}(t_\infty) = \tilde{P}(t_\infty)\,\vec{u}(t_\infty).$$

However, by 1.5b

$$\text{A2.12} \quad \left| Q(t_\infty)\left(\tilde{P}(t_\infty) - P(t_\infty)\right)\right| = o\left(1/[1+\phi(t_\infty; P)]\right) = o(1)$$

as $t_\infty \to \infty$. So A2.11, A2.12 $\Rightarrow \tilde{p}(t_\infty)\,\vec{u}(t_\infty) = \vec{0}$ for t_∞ sufficiently large $\Rightarrow \vec{\beta}(t_\infty)$ satisfies A2.2b. ■

We will show below that the columns of $\hat{E}(t_\infty)$ are in Range $(B(t_\infty))$ (not necessarily true for $E(t_\infty)$). This leads us to consider the following iteration scheme for solving A2.9

A2.13

a) $B(t_\infty)\,\vec{\beta}_\gamma = \begin{pmatrix} \vec{\gamma} \\ \vec{0} \end{pmatrix} - \hat{E}(t_\infty)\,\vec{\beta}_{\gamma-1}$ for $\gamma = 1, 2, \ldots;$

b) $\vec{\beta}_0 = \vec{0}$.

The following lemma gives us the key estimates for A2.13a.

<u>Lemma A2.14</u>. Suppose 1.1 is satisfied. Let $\vec{b} \in \mathbb{C}^{4n+m}$, $\vec{b} = (\vec{b}_1^T, \vec{b}_2^T, \vec{0}^T)^T$ where $\vec{b}_1 \in \mathbb{C}^m$, $b_2 \in \mathbb{C}^n$ and $P(t_\infty)\vec{b}_2 = \vec{b}_2$. Let $t_\infty \geq t_0$.

Then there exists a unique $\vec{\xi} \in \mathbb{C}^{3n}$ s.t.

$$\text{A2.14} \quad B(t_\infty)\,\vec{\xi} = \vec{b}.$$

Furthermore, setting $\vec{\xi} = (\vec{\xi}_1^T, \vec{\xi}_2^T, \vec{\xi}_3^T)^T$ with $\vec{\xi}_i \in \mathbb{C}^n$, there exists K_0 a constant independent of $\vec{b}$, $t_\infty \geq t_0$ such that

A2.15

a) $|\vec{\xi}_i| \le K_0 \left(|\vec{b}_1| + \phi(t_\infty; P)|\vec{b}_2|\right)$ for $i = 1, 2,$

b) $\vec{\xi}_3 = \vec{b}_3$,

where $\phi(t_\infty; P)$ is as in Definition 1.4.

<u>Proof of Lemma A2.14</u>. We can rewrite A2.14, using A2.3a, as

A2.16
$$C_0 P_0(t_0)\vec{\xi}_1 + \left\{\left(C_0 Y(t_0, t_\infty) + C(t_\infty)\right) P_1(t_\infty)\right\}\vec{\xi}_2 , = \vec{b}_1 - C_0 Y(t_0, t_\infty)\vec{b}_2 ,$$

with $\vec{\xi}_3 = \vec{b}_2$ and $(I-P_0(t_0))\vec{\xi}_1 = (I-P_1(t_\infty))\vec{\xi}_2 = \vec{0}$. Now using Definition 1.4 we can write A2.16 as follows

A2.17

a) $C_0 P_0(t_0)\vec{\xi}_1 + \left\{\left(C_0 + C(t_0)\right) P_1(t_0)\right\}\vec{\eta}_2 = \vec{b}_1 - C_0 Y(t_0, t_\infty)\vec{b}_2,$

b) $\vec{\xi}_2 = Y(t_\infty, t_0)\vec{\eta}_2$,

with $\vec{\xi}_3 = \vec{b}_2, (I-P_0(t_0))\vec{\xi}_1 = (I-P_1(t_0))\vec{\eta}_2 = \vec{0}$. Recall from Definition 1.4 that

a) Range $\left(P_0(t_0) + P_1(t_0)\right) = \mathcal{A}(t_0)$,

A2.18 b) $C(t_0) P_0(t_0) = 0$,

c) $P_1(t_0) P_0(t_0) = 0$.

Therefore, we can write A2.17 as follows

A2.19

a) $(C_0 + C(t_0))Q(t_0)\vec{\eta} = \vec{b}_1 - C_0 Y(t_0, t_\infty)\vec{b}_2,$

b) $\vec{\xi}_1 = P_0(t_0)\vec{\eta},$

c) $\vec{\xi}_2 = Y(t_\infty, t_0)\; P_1(t_0)\vec{\eta},$

d) $\vec{\xi}_3 = \vec{b}_2.$

Now by 1.1 we see that there exists a solution $\vec{\eta}$ of A2.19a. Notice that A2.19a does not have a unique solution in general, however we can assume that $\vec{\eta}$ satisfies

A2.20 $$|\vec{\eta}| \le K_1\left(|\vec{b}_1| + \phi(t_\infty; P)\,|\vec{b}_2|\right)$$

for some constant K_1 independent of $t_\infty, \vec{b}$. Notice that $|Y(t,t_0)P_1(t_0)| \le K_2|Y(t,t_0)Q(t_0)| \le K_3$ for $t \ge t_0$ for some constants K_2, K_3. Therefore, we see that A2.14 has a solution that satisfies A2.15.

Finally, $\vec{\xi}$ is unique by Lemmas 1.1 and A2.2. ■

Using Lemma A2.14 we can prove

<u>Lemma A2.20</u>. Suppose 1.1 is satisfied. Then there exists T_1 s.t. for each $\vec{\gamma} \in \mathbb{C}^m$, $t_\infty \ge T_1$

A2.9 $$\left(B(t_\infty) + \hat{E}(t_\infty)\right)\vec{\beta}(t_\infty) = \begin{pmatrix}\vec{\gamma}\\ \vec{0}\end{pmatrix}$$

has a unique solution $\vec{\beta}(t_\infty)$. Furthermore, for $\vec{\alpha}(t_\infty)$ representing the solution of A2.2a

A2.21 $$\left|\vec{\alpha}(t_\infty) - \vec{\beta}(t_\infty)\right| = o\left(\left|\vec{\alpha}(t_\infty)\right|\right) \quad \text{as } t_\infty \to \infty.$$

Proof of Lemma A2.20. We wish to use the contraction mapping theorem on the iteration scheme A2.13a. Note that for any initial guess $\vec{\beta}_0$, equation A2.13a uniquely defines $\{\vec{\beta}_\nu\}_{\nu=1}^{\infty}$ (see Lemma A2.14).

To use the contraction mapping theorem a standard calculation shows that we only need to show that

A2.22 $$B(t_\infty)\,\vec{x}_i(t_\infty) = \vec{f}_i(t_\infty) \equiv i^{th} \text{ column of } \hat{E}(t_\infty),$$

has (unique) solutions with $\left|\vec{x}_i(t_\infty)\right| = o(1)$ as $t_\infty \to \infty$. This estimate follows from 1.5, A2.3b, and from Lemma A2.14. Lemma A2.20 now follows from the contraction mapping theorem. ■

Now Lemmas A2.2 and A2.20 show that the TPBVP 1.3 has a unique solution $\vec{u}(t,\vec{\gamma})$ for t_∞ sufficiently large. Therefore, we are only left with proving 1.7 in Theorem 1.5.

We see from I1.4 and 1.3 that $\vec{e}(t;t_\infty,\vec{\gamma})$ satisfies

$$\text{a)}\quad \frac{d\vec{e}}{dt} = t^{p} A(t)\,\vec{e}(t) \quad \text{for } t \in (t_0, t_\infty),$$

$$\text{A2.23)}\quad \text{b)}\quad C_0\,\vec{e}(t_0) + C(t_\infty)\,\vec{e}(t_\infty) = \Delta C(t_\infty)\,\vec{u}(t_\infty;\vec{\gamma}),$$

$$\text{c)}\quad P(t_\infty)\,\vec{e}(t_\infty) = \Delta P(t_\infty)\,\vec{u}(t_\infty,\vec{\gamma}) = P(t_\infty)\,\vec{u}(t_\infty;\gamma),$$

where $\Delta C(t_\infty) = C(t_\infty) - \tilde{C}(t_\infty)$, $\Delta P(t_\infty) = P(t_\infty) - \tilde{P}(t_\infty)$.
Using Lemmas A2.2, A2.14 we see that

$$1.6\quad \vec{e}(t;t_\infty,\vec{\gamma}) = Y(t,t_0)\,\vec{\xi}_1 + Y(t,t_\infty)(\vec{\xi}_2 + \vec{\xi}_3),$$

with $(I - P_0(t_0))\vec{\xi}_1 = (I - P_1(t_\infty))\vec{\xi}_2 = (I - P(t_\infty))\vec{\xi}_3 = \vec{0}.$
and

$$\text{a)}\quad |\vec{\xi}_i| \le K_0\Big(|\Delta C(t_\infty)\,\vec{u}(t_\infty;\vec{\gamma})| + \varphi(t_\infty;P)\,|\Delta P(t_\infty)\,\vec{u}(t_\infty,\vec{\gamma})|\Big) \quad \text{for } i=1,2;$$

A2.24

$$\text{b)}\quad \vec{\xi}_3 = \Delta P(t_\infty)\,\vec{u}(t_\infty;\vec{\gamma}).$$

Let $\varepsilon_0(t_\infty) \equiv \big(|\Delta C(t_\infty)| + \phi(t_\infty;P)\,|\Delta P(t_\infty)|\big)K_0$.
Notice that from 1.6 $\vec{e}(t_\infty;t_\infty,\vec{\gamma}) \le K_2 \sum_{i=1}^{3} |\vec{\xi}_i|$ for some constant K_2 independent of $t_\infty, \vec{\gamma}$. Therefore, we can bound $\vec{\xi}_i$ in terms of $\vec{y}(t_\infty;\vec{\gamma})$ as follows: (from A2.24)

A2.25

a) $|\vec{\xi}_i| \le \varepsilon_0(t_\infty)\left\{|\vec{y}(t_\infty;\vec{\gamma})| + K_2 \sum_{k=1}^{3} |\vec{\xi}_k|\right\}$ for $i=1,2$;

b) $|\vec{\xi}_3| \le |\Delta P(t_\infty)|\left\{|\vec{y}(t_\infty;\vec{\gamma})| + K_2 \sum_{k=1}^{3} |\vec{\xi}_k|\right\}$.

It will be useful to write A2.25 in matrix notation. Define

A2.26

a) $\vec{x} \equiv \left(|\vec{\xi}_1|, |\vec{\xi}_2|, |\vec{\xi}_3|\right)^T \in \mathbb{R}^3$,

b) $\mathcal{E}(t_\infty) \equiv \begin{pmatrix} \varepsilon_0(t_\infty) & 0 & 0 \\ 0 & \varepsilon_0(t_\infty) & 0 \\ 0 & 0 & |\Delta P(t_\infty)| \end{pmatrix}$,

c) $\Gamma(t_\infty) \equiv K_2\, \mathcal{E}(t_\infty) \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$,

d) $\vec{b}(t_\infty;\vec{\gamma}) \equiv |\vec{y}(t_\infty;\vec{\gamma})|\, \mathcal{E}(t_\infty) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Then A2.25 can be written as

A2.27 $\vec{x} \le \vec{b}(t_\infty;\vec{\gamma}) + \Gamma(t_\infty)\vec{x}$,

where $\hat{\vec{x}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3) \le \hat{\vec{y}} = (\hat{y}_1, \hat{y}_2, \hat{y}_3)$ iff $\hat{x}_i \le \hat{y}_i$ for $i = 1,2,3$.
From A2.27:

A2.28 $\left(I - \Gamma(t_\infty)\right)\vec{x} \le \vec{b}(t_\infty;\vec{\gamma})$.

But $\Gamma(t_\infty) \to 0$ as $t_\infty \to \infty$, so for t_∞ sufficiently large (independent of $\vec{\gamma}$):

A2.29 $$(I - \Gamma(t_\infty))^{-1} = I + \Gamma(t_\infty) + \Gamma^{o}(t_\infty) + \ldots,$$

and $(I-\Gamma(t_\infty))^{-1}$ is positive definite. Therefore, multiplying A2.28 by $(I-\Gamma(t_\infty))^{-1}$ gives

A2.30 $$\begin{aligned} \vec{x} &\leq |\vec{y}(t_\infty, \vec{\gamma})| \, \mathcal{E}(t_\infty) \sum_{k=0}^{\infty} (\Gamma^T(t_\infty))^k \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{(see A2.26)}, \\ &\leq |\vec{y}(t_\infty, \vec{\gamma})| \, \mathcal{E}(t_\infty) \, K_3 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \end{aligned}$$

for $t_\infty \geq T$, where T, K_3 are constants independent of $\vec{\gamma}$ and t_∞. Estimate A2.30 implies 1.7. ■

<u>Proof of Theorem 2.6</u>. For t_∞ sufficiently large, the existence and uniqueness of the solution $\vec{u}(t;t_\infty)$ of 2.6 is an easy consequence of Theorem 1.5.

Notice that $\vec{e}(t;t_\infty)$ satisfies the homogeneous TPBVP:

a) $$\frac{d\vec{e}}{dt} = t^{\rho} A(t)\vec{e}(t) \quad \text{for } t \in (t_o, t_\infty),$$

A2.27 b) $$C_o \vec{e}(t_o) + C(t_\infty)\vec{e}(t_\infty) = \vec{\gamma}_1(t_\infty),$$

c) $$P(t_\infty)\vec{e}(t_\infty) = \vec{\gamma}_2(t_\infty).$$

Here

$$\text{a)}\quad \vec{\gamma}_1(t_\infty) = \Delta C(t_\infty)\vec{u}(t_\infty;t_\infty) + \tilde{C}(t_\infty)\tilde{\vec{y}}_p(t_\infty) - C(t_\infty)\vec{y}_p(t_\infty),$$

$$= \Delta C(t_\infty)\left(\vec{u}(t_\infty;t_\infty) - \tilde{\vec{y}}_p(t_\infty)\right) + C(t_\infty)\left(\tilde{\vec{y}}_p(t_\infty) - \vec{y}_p(t_\infty)\right),$$

A2.28

$$\text{b)}\quad \vec{\gamma}_2(t_\infty) = \Delta P(t_\infty)\left(\vec{u}(t_\infty;t_\infty) - \tilde{\vec{y}}_p(t_\infty)\right) + P(t_\infty)\left(\tilde{\vec{y}}_p(t_\infty) - \vec{y}_p(t_\infty)\right),$$

with $\Delta C(t_\infty) = C(t_\infty) - \tilde{C}(t_\infty)$ and $\Delta P(t_\infty) = P(t_\infty) - P(t_\infty)$.
Notice that since $\vec{u}(t_\infty;t_\infty)$ satisfies 2.6c

A2.29 $$P(t_\infty)\,\vec{\gamma}_2(t_\infty) = \vec{\gamma}_2(t_\infty).$$

Therefore we can apply Lemmas A2.2, A2.14 to get

$$\text{a)}\quad |\vec{\xi}_i| \le K_0\left(|\vec{\gamma}_1(t_\infty)| + \phi(t_\infty;P)\,|\vec{\gamma}_2(t_\infty)|\right) \quad \text{for } i = 1,2,$$

A2.30

$$\text{b)}\quad \vec{\xi}_3 = \vec{\gamma}_2(t_\infty).$$

As in the proof of Theorem 1.5, notice that $|\vec{e}(t_\infty;t_\infty,\vec{\gamma})| \le K_2 \sum_{k=1}^{3} |\vec{\xi}_k|$ (see 2.8), for some constant K_2 independent of t_∞ and $\vec{\gamma}$. Let $\epsilon_0(t_\infty) = K_0\left(|\Delta C(t_\infty)| + \phi(t_\infty;P)\,|\Delta P(t_\infty)|\right)$. Using the above, we can rewrite A2.30 as: (using A2.28)

$$\text{a)}\quad |\vec{\xi}_i| \le \varepsilon_o(t_\infty)\Big\{|\vec{y}(t_\infty;\vec{\gamma},\vec{f}) - \vec{y}_P(t_\infty)| + |\Delta y_P(t_\infty)| + K_2\sum_{k=1}^{3}|\vec{\xi}_k|\Big\}$$
$$+ \big(|C(t_\infty)| + \phi(t_\infty;P)\,|P(t_\infty)|\big)\,|\Delta\vec{y}_P(t_\infty)|$$

A2.31 for i = 1,2;

$$\text{b)}\quad |\vec{\xi}_3| \le |\Delta P(t_\infty)|\,\Big\{|\vec{y}(t_\infty;\vec{\gamma},\vec{f}) - \vec{y}_P(t_\infty)| + |\Delta\vec{y}_P(t_\infty)| + K_2\sum_{k=1}^{3}|\vec{\xi}_k|\Big\}$$
$$+ |\Delta\vec{y}_P(t_\infty)|\,\big(|P(t_\infty)|\big),$$

where $\Delta y_P(t_\infty) \equiv \vec{y}_P(t_\infty) - \tilde{\vec{y}}_P(t_\infty)$.

Notice that if $P(t_\infty) \ne 0$ then $1 = O(|P(t_\infty)|)$. ($P(t_\infty)$ is a projection matrix) therefore $|\Delta P(t_\infty)| = o(|P(t_\infty)|)$. Similarly, if $C(t_\infty) \ne 0$ then by I4.21d we know that $1 = O(|C(t_\infty)|)$, and therefore $|\Delta C(t_\infty)| = o(|C(t_\infty)|)$. Therefore K_3 s.t.: $(K_3-1)\big(|C(t_\infty)| + \phi(t_\infty;P)\,|P(t_\infty)|\big) \ge \varepsilon_o(t_\infty) + |\Delta P(t_\infty)|$ for t_∞ sufficiently large. This will simplify the $|\Delta\vec{y}_P(t_\infty)|$ terms in the bounds given in A2.31. Using $\varepsilon(t_\infty)$, $\Gamma(t_\infty)$, $\vec{x}$ as defined in A2.26, we see that A2.31 becomes

$$(I - \Gamma(t_\infty))\vec{x} \le |\vec{y}(t_\infty;\vec{\gamma},\vec{f}) - \vec{y}_P(t_\infty)|\,\varepsilon(t_\infty)\begin{pmatrix}1\\1\end{pmatrix}$$

A2.32
$$+ |\Delta\vec{y}_P(t_\infty)|\,K_3\begin{pmatrix}|C(t_\infty)| + \phi(t_\infty;P)\,|P(t_\infty)|\\ |C(t_\infty)| + \phi(t_\infty;P)\,|P(t_\infty)|\\ |P(t_\infty)|\end{pmatrix}$$

As in the proof of Theorem 1.5, for t_∞ sufficiently large

$(I-\Gamma(t_\infty))^{-1} = \sum_{k=0}^{\infty} \Gamma^k(t_\infty)$ is positive definite. So A2.32 implies

A2.33
$$\vec{\chi} \le \left|\vec{y}(t_\infty;\vec{\gamma},\vec{f}) - \vec{y}_p(t_\infty)\right| \varepsilon(t_\infty)\left(\sum_{k=0}^{\infty}(\Gamma^T(t_\infty))^k\right)\begin{pmatrix}1\\1\end{pmatrix}$$
$$+ \left|\Delta\vec{y}_p(t_\infty)\right| K_3\left(\sum_{k=0}^{\infty}(\Gamma(t_\infty))^k\right)\begin{pmatrix}|C(t_\infty)| + \phi(t_\infty;P)|P(t_\infty)|\\ |C(t_\infty)| + \phi(t_\infty;P)|P(t_\infty)|\\ |P(t_\infty)|\end{pmatrix}$$

But $\Gamma(t_\infty) = \varepsilon(t_\infty) K_2 \begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} = o(1)$ as $t \to \infty$. Therefore A2.33 implies

A2.34
$$\vec{\chi} \le \left|\vec{y}(t_\infty;\vec{\gamma},\vec{f}) - \vec{y}_p(t_\infty)\right| K_4\,\varepsilon(t_\infty)\begin{pmatrix}1\\1\end{pmatrix}$$
$$+ \left|\Delta\vec{y}_p(t_\infty)\right| K_3\left\{I + \varepsilon(t_\infty) K_4\begin{pmatrix}1&1&1\\1&1&1\end{pmatrix}\right\}\begin{pmatrix}|C(t_\infty)| + \phi(t_\infty;P)|P(t_\infty)|\\ |C(t_\infty)| + \phi(t_\infty;P)|P(t_\infty)|\\ |P(t_\infty)|\end{pmatrix}$$

for $t_\infty \ge T$, where T, K_4 are constants independent of $\vec{\gamma}, \vec{f}(t)$.

Finally, 2.9 follows from A2.34, completing the proof of Theorem 2.6. ■

APPENDIX A3

Proof of Lemma III3.7. By 3.3, 3.4 and 3.6 we can treat each diagonal block of $\hat{A}(s)$ separately. That is, we can assume without loss of generality that $\ell = 1$ in 3.2. We can write 3.7a as

A3.1 a) $$\frac{d\vec{w}}{ds} - J(s)\vec{w}(s) = s^{\sigma-2} B(s)\vec{w}(s) + \vec{g}(s) \quad \text{for } s > s_1 \, ,$$

Here

b) $$J(s) = \left\{ i\omega p(s) + \mu s^{\sigma-1} \right\} I_{n\times n} + \frac{\delta}{s} H \, ,$$

with $\omega, p, \mu, \sigma, \delta, H$ as in 3.3;

A3.2 c) $$\sup_{s \geq s_1} |B(s)| \leq K_B < \infty \, ,$$

d) $$\sup_{s \geq s_1} |s^{\alpha}\vec{g}(s)| \equiv \| s^{\alpha}\vec{g}(s) \|_{\infty} < \infty \, ,$$

where $\alpha = \rho + 1 - \sigma + \varepsilon_1$ and $0 < \varepsilon_1 < 1$. (Notice that we can take $\varepsilon_1 < 1$ w.l.o.g..) Define

a) $$Q_w \equiv \begin{cases} I_{n\times n} & \text{if } \mu < 0 \text{ and } \sigma > 0; \\ I_{n\times n} & \text{if } \mu < -\rho \text{ and } \sigma = 0; \\ 0_{n\times n} & \text{otherwise;} \end{cases}$$

A3.3

b) $$W(s_1, s_2) \equiv \exp\left\{ \int_{s_2}^{s} J(x)\,dx \right\} \quad \text{for } s, s_2 \geq s_1 \, .$$

Notice that if $\vec{w}_p(s;s_2)$ is a solution of

A3.4 $$\vec{w}_p(s,s_2) = W(s,s_2)\left\{\int_{s_2}^{s} Q_w W(s_2,x)\left[x^{\sigma-2} B(x)\vec{w}_p(x;s_2)+\vec{g}(x)\right]dx - \int_s^\infty dx(I-Q_w)\,W(s_2,x)\left[x^{\sigma-2}B(x)\vec{w}_p(x;s_2)+\vec{g}(x)\right]\right\}$$

for $s \geq s_1$ and for some fixed $s_2 \geq s_1$ then $\vec{w}_p(s;s_2)$ satisfies A3.1. We will attempt to solve A3.4 using the iteration scheme

A3.5 a) $$\vec{w}_\nu(s;s_2) = W(s,s_2)\left\{\int_{s_2}^{s} Q_w W(s_2,x)\left[x^{\sigma-2}B(x)\vec{w}_{\nu-1}(x;s_2)+\vec{g}(x)\right]dx - \int_s^\infty (I-Q_w)W(s_2,x)\left[x^{\sigma-2}B(x)\vec{w}_{\nu-1}(x;s_2)+\vec{g}(x)\right]dx\right\}$$

for $s \geq s_1$, $\nu = 1,2,\ldots;$

A3.5 b) $\vec{w}_0(s;s_2) \equiv \vec{0}$

for $s_1 s_2 \geq s_1$.

To deal with A3-5, we will need estimates for

a) $$I_1(s;s_2,\vec{h}) \equiv \left|\int_{s_2}^{s} W(s,s_2)Q_w W(s_2,x)\vec{h}(x)\,dx\right|,$$

A3.6

b) $$I_2(s;s_2,\vec{h}) \equiv \left|\int_s^\infty W(s,s_2)(I-Q_w)W(s_2,x)\vec{h}(x)\,dx\right|,$$

where $s \geq s_2 \geq s_1$ and $\| s^{\alpha} \vec{h}(s) \|_{[s_1,\infty)} < \infty$.

Lemma A3-7. Let $\varepsilon_1 > 0$ and $\alpha = \rho + 1 - \sigma + \varepsilon_1$. Let $\vec{h}(s) \in C[s_1,\infty)$, and

A3.7 $$\| s^{\alpha} \vec{h}(s) \|_{[s_1,\infty)} \equiv \sup_{s \geq s_1} | s^{\alpha} \vec{h}(s) | < \infty .$$

Then I_1, I_2 defined in A3.6 exist for $s \geq s_2 \geq s_1$. Furthermore, there exists an $\varepsilon_0 \in (0, \varepsilon_1)$, ε_0 independent of s, s_2 and $\vec{h}$, such that for any $\varepsilon > 0$, $\varepsilon < \varepsilon_0$ we have

A3.8 a) *if* $Q_w = I$ *then for* $s \geq s_2 \geq s_1$:

$$I_1(s; s_2, \vec{h}) \leq C_\varepsilon \| s^{\alpha} \vec{h}(s) \|_{[s_2,\infty)} \frac{1}{s^{\rho+\varepsilon_1-\varepsilon}} \left\{ 1 + \frac{s}{s_2}^{\varepsilon_1-\varepsilon_0} \right\} \frac{1}{s_2^{\varepsilon}}$$

$$\leq C_\varepsilon \| s^{\alpha} \vec{h}(s) \|_{[s_2,\infty)} \frac{2}{s_2^{\rho+\varepsilon_1}} \quad ;$$

A3.8 b) *if* $Q_w = 0$ *then for* $s \geq s_2 \geq s_1$.

$$I_2(s; s_2, \vec{h}) \leq C_\varepsilon \| s^{\alpha} \vec{h}(s) \|_{[s_2,\infty)} \, 1/s^{\rho+\varepsilon_1} \; .$$

Here $C_\varepsilon < \infty$ is a constant depending only on n, δ, s_1 and ε .

Before proving Lemma A3-7 we will use the estimates A3.8a and b to prove that for $0 < \varepsilon < \varepsilon_0$ and for s_2 sufficiently large, the iteration scheme A3.5 converges.

Let $0 < \varepsilon < \varepsilon_0$ and $\varepsilon_1 > 0$. Then by A3.8 and A3.5

A3.9 $$\left|\vec{w}_1(s;s_2)\right| \le C_\varepsilon \left\| s^\alpha \vec{g}(s)\right\|_\infty \frac{1}{s^{\rho+\varepsilon_1-\varepsilon}} \left\{1+\left(\frac{s}{s_2}\right)^{\varepsilon_1-\varepsilon_0}\right\} \frac{1}{s_2^{\varepsilon}}$$

$$\le C_\varepsilon \left\| s^\alpha \vec{g}(s)\right\|_\infty 2/s_2^{\rho+\varepsilon_1}$$

for $s \ge s_2 \ge s_1$.

Define $\vec{e}_\nu(s;s_2) = \vec{w}_\nu(s;s_2) - \vec{w}_{\nu-1}(s;s_2)$ for for $\nu = 1,2,\ldots.$ Then by A3.5

A3.10 a) $$\vec{e}_\nu(s;s_2) = W(s,s_2)\left\{\int_{s_2}^{s} Q_w W(s_2,x)x^{\sigma-2}B(x)\vec{e}_{\nu-1}(x;s_2)dx - \int_s^\infty (I-Q_w)W(s_2,x)x^{\sigma-2}B(x)\vec{e}_{\nu-1}(x;s_2)dx\right\}$$

for $\nu = 2,3,\ldots,$ and $s \ge s_1$;

A3.10 b) $$\vec{e}_1(s;s_2) = \vec{w}_1(s,s_2)$$

for $s \ge s_1$. Notice that $\lim_{\nu\to\infty} \vec{w}_\nu(s;s_2) = \sum_{\nu=1}^\infty e_\nu(s;s_2)$ if either limit exists. We wish to show that $\sum_{\nu=1}^\infty \vec{e}_\nu(s;s_2)$ converges uniformly for $s \ge s_2$, for some s_2 sufficiently large.

<u>Lemma A3.11</u>. For $0 < \varepsilon_1 < 1$ and $0 < \varepsilon < \varepsilon_0$, there exists $\vec{e}_\nu(s;s_2)$ for $\nu = 1,2,\ldots$ satisfying A3.10. Furthermore $\vec{e}_\nu(s;s_2)$ satisfies

A3.11 a) $$\sup_{s\ge s_2}\left| s^{\alpha+\sigma-2}B(s)\vec{e}_\nu(s;s_2)\right| \le \left\| s^\alpha \vec{g}(s)\right\|_\infty \gamma^\nu(s_2)$$

where $\mathcal{H}(s_2) = 2C_\varepsilon K_B / s_2$ for $s_2 \geq s_1$;

$$\text{b)}\ \left|\vec{e}_\nu(s;s_2)\right| \leq \frac{C_\varepsilon \|s^\alpha \vec{g}(s)\|_\infty \mathcal{H}^{\nu-1}(s_2)}{s_2^\varepsilon\, s^{\rho+\varepsilon_1-\varepsilon}} \left(1 + \left(\frac{s}{s_2}\right)^{\varepsilon_1-\varepsilon_0}\right),$$

$$\leq 2C_\varepsilon \|s^\alpha \vec{g}(s)\|_\infty \mathcal{H}^{\nu-1}(s_2) \big/ s_2^{\rho+\varepsilon_1}$$

for $s \geq s_2 \geq s_1$.

Proof of Lemma A3.11. By A3.9 we see that A3.11b is satisfied for $\nu = 1$. But if A3.11b is true for some $\nu \geq 1$ then

A3.12
$$\sup_{s \geq s_2} \left| s^{\alpha+\sigma-2} B(s) \vec{e}_\nu(s;s_2)\right| \leq K_B C_\varepsilon \|s^\alpha \vec{g}(s)\|_\infty \mathcal{H}^{\nu-1}(s_2)$$
$$\times \sup_{s \geq s_2} \left\{ \frac{s^{\rho+\varepsilon_1-1}}{s_2^\varepsilon s^{\rho+\varepsilon_1-\varepsilon}} \left(1 + \left(\frac{s}{s_2}\right)^{\varepsilon_1-\varepsilon_0}\right)\right\},$$

$$\leq \frac{\|s^\alpha \vec{g}(s)\|_\infty \mathcal{H}^{\nu-1}(s_2) C_\varepsilon K_B}{s_2^\varepsilon} \sup_{s \geq s_2} \left\{ \frac{1}{s^{1-\varepsilon}} + \frac{1}{s_2^{\varepsilon_1-\varepsilon_0}} \frac{1}{s^{1-\varepsilon_1+\varepsilon_0-\varepsilon}} \right\}$$

and A3.11a follows for this ν since $1 - \varepsilon_1 + \varepsilon_0 - \varepsilon > 0$.

We have shown that A3.11a,b holds for $\nu = 1$. Suppose A3.11a,b holds for some $\nu \geq 1$. (Our induction hypothesis.) Then by Lemma A3-1 and A3.10a we see that for $s \geq s_2 \geq s_1$, A3.13 $$\left|\vec{e}_{\nu+1}(s,s_2)\right| \le \frac{C_\varepsilon \left\| s^{\alpha+\sigma-2} B(s)\vec{e}_\nu(s;s_2)\right\|_{[s_2,\infty)}}{s_2^{\varepsilon}\, s^{\rho+\varepsilon_1-\varepsilon}} \left(1+\left(\frac{s}{s_2}\right)^{\varepsilon_1-\varepsilon_0}\right)$$

$$\le \frac{C_\varepsilon \left\| s^{\alpha}\vec{g}(s)\right\|_\infty \mathcal{H}^{\nu}(s_2)}{s_2^{\varepsilon}\, s^{\rho+\varepsilon_1-\varepsilon}} \left(1+\left(\frac{s}{s_2}\right)^{\varepsilon_1-\varepsilon_0}\right)$$

Where we have used the induction hypothesis. Estimate A3.13 is of the desired form. The lemma now follows by induction. ■

By A3.11 we see that for s_2 sufficiently large s.t. $\mathcal{H}(s_2)<1$, the series $\sum_{\nu=1}^{\infty}\vec{e}_\nu(s;s_2)$ converges uniformly for $s \ge s_2$. And $\vec{w}_p(s;s_2) \equiv \lim_{\nu\to\infty}\vec{w}_\nu(s,s_2) = \sum_{\nu=1}^{\infty}\vec{e}_\nu(s,s_2)$ satisfies A3.1 for $s \ge s_2$ and by A3.11b

A3.14 $$\left|\vec{w}_p(s;s_2)\right| \le \left[\frac{2C\ \left\|s^{\alpha}\vec{g}(s)\right\|_\infty}{(1-\mathcal{H}(s_2))\, s_2^{\varepsilon_1-\varepsilon_0+\varepsilon}}\right] \frac{1}{s^{\rho+\varepsilon_0-\varepsilon}} \quad \text{for } s \ge s_2,$$

where s_2 is sufficiently large.

Finally, since $\mathcal{J}(s) = O(s^{\rho})$ we see that for s_2 sufficiently large, $\vec{y}_p(t;s_2)$ defined by

a) $\vec{y}_p(t(s);s_2) \equiv \mathcal{J}(s)\vec{w}(s;s_2) \quad \text{for } s \ge s_2,$

A3.15

b) $t = s^{r},$

satisfies 2.1a for $t \ge s_2^{r}$, and

A3.16 $$\left|\vec{y}_p(t;s_2)\right| \le \left[\frac{2C_\varepsilon C_{\mathcal{J}}}{(1-\mathcal{H}(s_2))\, s_2^{\varepsilon_1-\varepsilon_0+\varepsilon}}\right] \left\|\mathcal{A}_\alpha(t)\,\vec{f}(t)\right\|_\infty \frac{1}{t^{(\varepsilon_0-\varepsilon)/r}}$$

for $t \geq s_2^r$ (where we have used $|J(s)| \leq C_J s^\rho$ for $s \geq s_1$). We can easily extend $\vec{y}_p(t;s_2)$ to a solution of 2.1a for $t \geq t_0$, which finishes the proof of Lemma 3.7. ■

We still have to prove Lemma A3-7.

Proof of Lemma A3-7. From A3.3b we see that there exists a constant, C, such that

A3.17 $$|W(s,x)| \leq C(1+\delta|\log(t/x)|^{n-1})\, e^{\mu\Phi(s,x)},$$

where $\Phi(s,x) = \int_x^s \tilde{x}^{\sigma-1}\, d\tilde{x}$. Suppose $Q_W = I$, then for $s \geq s_2 \geq s_1$

A3.18 $$\begin{aligned} I_1(s;s_2,\vec{h}) &= \left|\int_{s_2}^s W(s,x)\vec{h}(x)\,dx\right|, \\ &\leq C\,\|s^\alpha\vec{h}(s)\|_{[s_2,\infty)} \int_{s_2}^s (1+\delta|\log s/x|^{n-1})\, x^{-\alpha} e^{\mu\Phi(s,x)}\,dx, \end{aligned}$$

where we have used A3.17. Also for any $\varepsilon > 0$

A3.19 $$C \sup_{s \geq x \geq s_2 > 0} \left\{(1+\delta|\log s/x|^{n-1})\left(\frac{s}{x}\right)^{-\varepsilon}\right\} \equiv C_\varepsilon < \infty .$$

Therefore A3.18 becomes

A3.20 $$I_1(s;s_2,\vec{h}) \leq C_\varepsilon\,\|s^\alpha\vec{h}(s)\|_{[s_2,\infty)} \int_{s_2}^s \left(\frac{x}{s}\right)^{-\varepsilon} x^{-\alpha} e^{\mu\Phi(s,x)}\,dx.$$

It will be convenient to study the case $Q_W \equiv 0$ now. Let $s \geq s_1$ and $Q_W = 0$, then

$$\text{A3.21} \quad \left| \lim_{S_3\to\infty} \int_s^{S_3} W(s,x)\,\vec{h}(x)\,dx \right| = I_2(s; s_2, \vec{h}).$$

We need to show that the limit in A3.21 exists. Let $S_3 \geq s \geq S_2$, and $\varepsilon > 0$ then by A3.17 and A3.19

$$\text{A3.22} \quad \left|\int_s^{S_3} W(s,x)\,\vec{h}(x)\,dx\right| \leq C_\varepsilon \left\| s^\alpha \vec{h}(s) \right\|_{[s_2,\infty)} \int_s^{S_3} \left(\frac{x}{s}\right)^\varepsilon x^{-\alpha} e^{\mu\Phi(s,x)}dx.$$

Notice that in A3.20 and A3.22 we are only left with estimating for b ≥ a > 0, and $c > 0$

$$\text{A3.23} \quad \int_a^b x^{-\beta} e^{\mu\Phi(c,x)}\,dx = \int_a^b \frac{d}{dx}\left(e^{\mu\Phi(c,x)}\right)\frac{dx}{-\mu x^{\beta+\sigma-1}},$$

$$= \left[\frac{-e^{\mu\Phi(c,x)}}{+\mu x^{\beta+\sigma-1}}\right]_{x=a}^{b} - \frac{\beta+\sigma-1}{\mu}\int_a^b x^{-\beta-\sigma} e^{\mu\Phi(c,x)}\,dx.$$

From A3.23 we see

$$\text{A3.24} \quad \int_a^b x^{-\beta} e^{\mu\Phi(c,x)}\left(1+\frac{\beta+\sigma-1}{\mu x^\sigma}\right)dx = \left[\frac{e^{\mu\Phi(c,x)}}{\mu x^{\beta+\sigma-1}}\right]_{x=b}^{a}.$$

There exists a constant $X = X(\beta,\sigma,\mu) > 0$ s.t. $1+\frac{\beta+\sigma-1}{\mu x^\sigma} \geq \frac{1}{2}$ for $x \geq X$. Then for $b \geqslant a \geqslant X$ we have

$$\text{A3.25} \quad \int_a^b x^{-\beta} e^{\mu\Phi(c,x)}\,dx \leq 2\left[\frac{e^{\mu\Phi(c,x)}}{\mu x^{\beta+\sigma-1}}\right]_{x=b}^{a}.$$

Finally, it is easy to show that there exists a constant $K_0 > 0$ s.t. for $a \geq b \geq s_1$

A3.26 $$\int_a^b x^{-\beta} e^{\mu \Phi(c,x)} dx \le K_0 \left\{ \frac{e^{\mu \Phi(c,a)}}{a^{\beta+\sigma-1}} + \frac{e^{\mu \Phi(c,b)}}{b^{\beta+\sigma-1}} \right\}.$$

Returning to A3.20 and using A3.326, we see

A3.27 $$I_1(s;s_2,\vec{h}) \le K_0 C_\varepsilon \| s^\alpha \vec{h}(s) \|_{[s_2,\infty)} \, s^\varepsilon \left\{ \frac{e^{\mu \Phi(s,s_2)}}{s_2^{\alpha+\sigma-1+\varepsilon}} + \frac{1}{s^{\alpha+\sigma-1+\varepsilon}} \right\},$$

and from A3.3a (recall $Q_W = I$)

A3.28 $$e^{\mu \Phi(s,s_2)} \le K_1 \left(\frac{s}{s_2}\right)^{-\rho+\varepsilon_0)} \qquad \text{for } s \ge s_2 \ge s_1.$$

Where K_1, ε_0 are constants, $\varepsilon_0 > 0$ and $\mu + \rho \ge \varepsilon_0$ if $\sigma = 0$.
Therefore, using A3.28 in A3.27 (assuming $K_1 \ge 1$)

A3.29 $$I_1(s;s_2,\vec{h}) \le K_1 K_0 C_\varepsilon \| s^\alpha \vec{h}(s) \|_{[s_2,\infty)} \left\{ \frac{s^{-\rho-\varepsilon_0+\varepsilon}}{s_2^{\alpha+\sigma-1+\varepsilon-\rho-\varepsilon_0}} + \frac{1}{s^{\alpha+\sigma-1}} \right\},$$

and using $\alpha = \rho + 1 - \sigma + \varepsilon_1$

A3.30 $$\le K_1 K_0 C_\varepsilon \| s^\alpha \vec{h}(s) \|_{[s_2,\infty)} \frac{1}{s^{\rho+\varepsilon_1-\varepsilon}} \left\{ \frac{s^{\varepsilon_1-\varepsilon_0}}{s_2^{\varepsilon_1+\varepsilon-\varepsilon_0}} + \frac{1}{s^\varepsilon} \right\},$$

from which estimate A3.8a follows.

Next we consider A3.22. Using A3.26

A3.31 $$\left| \int_s^{s_3} W(s,x) \vec{h}(x) dx \right| \le K_0 C_\varepsilon \| s^\alpha \vec{h}(s) \|_{[s_2,\infty)} \left\{ \frac{e^{\mu \Phi(s,s_3)} s^{-\varepsilon}}{s_3^{\alpha+\sigma-1-\varepsilon}} + \frac{1}{s^{\alpha+\sigma-1}} \right\}$$

By A3.3 there exists a constant K_1 *s.t.*

A3.32 $$e^{\mu\Phi(s,s_3)} \leq K_1 \left(\frac{s_3}{s}\right)^{\rho}$$

for $s_1 \leq s \leq s_3$ (since we are treating the case $Q_W = 0$ (see A3.3)). Therefore A3.31 becomes (we assume $K_1 \geq 1$):

A3.33 $$\left|\int_s^{s_3} W(s,x)\vec{h}(x)\,dx\right| \leq K_0 K_1 C_\varepsilon \, \| s^\alpha \vec{h}(s)\|_{[s_2,\infty)} \left\{\frac{1}{s_3^{\varepsilon_1-\varepsilon}}\frac{1}{s^{\rho+\varepsilon}} + \frac{1}{s^{\rho+\varepsilon}}\right\}$$

for $s_2 \geq s \geq s_1$, where we have used $\alpha+\sigma-1 = \rho+\varepsilon_1$. By A3.33 we see that $\int_s^\infty W(s,x)\vec{h}(x)\,dx$ exists since $\varepsilon_1 - \varepsilon > \varepsilon_1 - \varepsilon_0 > 0$. Letting $s_3 \to \infty$ in A3.33 we obtain

A3.34 $$I_2(s;s_2,\vec{h}) \leq K_0 K_1 C_\varepsilon \, \| s^\alpha \vec{h}(s)\|_{[s_2,\infty)} \frac{1}{s^{\rho+\varepsilon_1}}$$

for $s \geq s_2 \geq s_1$, proving A3.8b. ■

APPENDIX A4

Proof of Theorem 2.6. We will construct a sequence $\{\vec{y}_\nu(t)\}_{\nu=0}^{\infty}$ such that

A4.1 a) $\dfrac{d\vec{y}_\nu}{dt} - t^p A(t)\vec{y}_\nu(t) = \vec{h}(t) + \vec{g}(t, \vec{y}_{\nu-1})$

for $t > t_\infty$ and for $\nu = 1,2,\ldots$; *with*

b) $\vec{y}_0(t) = \vec{u}(t; t_\infty, \vec{\xi}_0)$,

where $t_\infty \geq t_0$ is fixed. The sequence $\{\vec{y}_\nu\}_{\nu=0}^{\infty}$ will also satisfy

a) $|\vec{y}_\nu(t) - \vec{y}_0(t)| \leq C_\nu \left\{ \left\| \frac{1}{(t)} \vec{h}(t) \right\|_{[t_\infty, \infty)} + |\vec{\xi}_0| \right\}^2 t^\beta \phi(t)$,

A4.2

b) $|\vec{e}_\nu(t)| \leq \mathcal{H}^\nu \left\{ \left\| \frac{1}{\phi(t)} \vec{h}(t) \right\|_{[t_\infty, \infty)} + |\vec{\xi}_0|^2 \right\} t^\beta \phi(t)$,

for $t \geq t_\infty$, $\nu = 1,2,\ldots$. Where $\vec{e}_\nu \equiv y_\nu - y_{\nu-1}$ for $\nu \geq 1$ and $C_\nu, \mathcal{H}$ are constants (independent of t_∞) s.t.:

a) $C_\nu \leq 1 - \frac{1}{2\nu}$ for $\nu = 1, 2, \ldots$,

A4.3

b) $\mathcal{H} \leq \frac{1}{2}$.

We begin with $\nu = 1$. Let $\vec{y}_1(t) \equiv t^\beta \phi(t) \vec{v}_1(t)$. Then by A4.1a, $\vec{v}_1(t)$ satisfies

A4.4 $$\frac{d\vec{v}_1}{dt} - \left\{t^{\rho}A(t) - \frac{d}{dt}\log\left(t^{\beta}\phi(t)\right) I_{n\times n}\right\}\vec{v}_1(t),$$
$$= \frac{t^{-\beta}}{\phi(t)}\left[\vec{h}(t) + \vec{g}\left(t, \vec{y}_0(t)\right)\right]$$

for $t > t_\infty$. Now from the form of $\phi(t)$ we can easily see that the transformation

a) $\vec{w}_1(s) = \left\{s^{\rho}J(s)\right\}^{-1}\vec{v}_1(t)$

A4.5

b) $s^r = t$

for $t \geq t_\infty$ is of the form required by Lemma III3.7. Furthermore, let $\vec{\alpha} \equiv (\alpha_1, \ldots, \alpha_\ell)$ with

A4.6 $$\alpha_j = \rho + 1 + \varepsilon_1 r \quad \text{for } j = 1, \ldots, \ell,$$

where $\varepsilon_1 = \beta - \left\{\frac{(\rho+2)}{r} - 1\right\} > 0$. Then $\alpha_j > \rho+1 > \rho+1-\sigma_j$ and therefore $\vec{\alpha}$ satisfies the hypotheses of Lemma III3.7 for A4.4. Also, from the definition of $\mathcal{A}_{\vec{\alpha}}(t)$ in Lemma III3.7

$$\left|\mathcal{A}_{\vec{\alpha}}(t)\right| \equiv \left|\operatorname{diag}\left(t^{\alpha_j/r} I_{n_j\times n_j} \,\middle|\, j = 1, \ldots, \ell\right) s^{-\rho}J^{-1}(s)\frac{1}{r}t^{-1+\frac{1}{r}}\right|,$$
$$= O\left(t^{(\rho+1)/r + \varepsilon_1} s^{-\rho}s^{\rho}t^{-1+1/r}\right) = O\left(t^{\beta}\right) \text{ as } t \to \infty.$$

Therefore we have (assuming $\|\vec{y}_0(t)\|_{[t_\infty,\infty)} < T_y$)

A4.7 $$\left\| \mathscr{S}_{\vec{\alpha}}(t)\frac{t^{-\beta}}{\phi(t)}\left[\vec{h}(t)+\vec{g}(t,\vec{y}_0(t))\right]\right\|_{[t_\infty,\infty)},$$

$$\leq \left\| \mathscr{S}_{\vec{\alpha}}(t)t^{-\beta}\right\|_{[t_0,\infty)}\left\{\left\|\frac{1}{\phi(t)}\vec{h}(t)\right\|_{[t_\infty,\infty)}+\left\|\frac{\Gamma(t)\psi^2(t)}{\phi(t)}\right\|_{[t_0,\infty)}K_0^2|\vec{\xi}_0|^2\right\},$$

$$\leq K_{\vec{\alpha}}\left\{\left\|\frac{1}{\phi(t)}\vec{h}(t)\right\|_{[t_\infty,\infty)}+|\vec{\xi}_0|^2\right\},$$

where (see 2.8)

$$\left\| \mathscr{S}_{\vec{\alpha}}(t)t^{-\beta}\right\|_{[t_0,\infty)}\left\{1+K_0^2\left\|\frac{\Gamma(t)\psi^2(t)}{\phi(t)}\right\|_{[t_0,\infty)}\right\}\leq K_{\vec{\alpha}}<\infty.$$

By the preceding paragraph we see that we can apply Lemma III3.7 to A4.4. This implies that there exists $\vec{v}_1(t)$ satisfying A4.4 and

$$|\vec{v}_1(t)| \leq K_{\varepsilon_0}K_{\vec{\alpha}}\left\{\left\|\frac{1}{\phi(t)}\vec{h}(t)\right\|_{[t_\infty,\infty)}+|\vec{\xi}_0|^2\right\}t^{-\varepsilon_0}$$

for $t \geq t$. Here $\varepsilon_0 > 0$ is sufficiently small and K_{ε_0} is a constant independent from t_∞. Therefore, A4.1 for $\nu = 1$ has a solution $\vec{y}_1(t)$ satisfying

a) $\vec{y}_1(t) = \vec{y}_0(t) + t^{\beta}\phi(t)\vec{v}_1(t)$,

A4.8

b) $|\vec{y}_1(t) - \vec{y}_0(t)| \leq K_{\varepsilon_0}K_{\vec{\alpha}}t_\infty^{-\varepsilon_0}\left\{\left\|\frac{1}{\phi(t)}\vec{h}(t)\right\|_{[t_\infty,\infty)}+|\vec{\xi}_0|^2\right\}t^{\beta}\phi(t)$,

for $t \geq t_\infty$. This completes our estimates for $\nu = 1$.

From A4.8b we see that t_∞ must satisfy

A4.9 $$K_{\varepsilon_0} K_{\vec{\alpha}}\, t_\infty^{-\varepsilon_0} \le \tfrac{1}{2}$$

(since $C_1, \mathcal{H} \le \frac{1}{2}$). We proceed to construct $\{\vec{y}_\nu(t)\}_{\nu=0}^{\infty}$ by induction on ν. Suppose that for some $\nu \ge 1$, $\{\vec{y}_{\nu'}(t)\}_{\nu'=1}^{\nu}$ satisfies A4.1, A4.2, A4.3 for $\nu' - 0,1,\ldots,\nu$. We wish to construct a suitable $\vec{y}_{\nu+1}(t)$. (We will ignore the condition $|\vec{y}(t)| < T_y$ in 2.5e since this will be satisfied for t_∞ sufficiently large and $|\vec{\xi}_0|^2 + \left\| \frac{1}{\phi(t)} \vec{h}(t) \right\|_{[t_\infty,\infty)} < K_h + \varepsilon^2$ (see estimates A4.2a and 2.7)).

Notice that $\vec{e}_{\nu+1}(t)$ satisfies (by A4.1a)

A4.10 $$\frac{d\vec{e}_{\nu+1}}{dt} - t^{\rho} A(t)\, \vec{e}_{\nu+1} = \vec{k}_\nu(t)$$

for $t > t_\infty$, where $\vec{k}_\nu(t) \equiv \vec{g}(t, \vec{y}_\nu(t)) - \vec{g}(t, \vec{y}_{\nu-1}(t))$ for $t \ge t_\infty$ and $\nu = 1,2,\ldots.$

By 2.5e, 2.8b, A4.2, A4.3 (and the induction hypothesis)

A4.11 $$\begin{aligned} |k_\nu(t)| &\le \Gamma(t)\left(|\vec{y}_\nu(t)| + |\vec{y}_{\nu-1}(t)|\right) |\vec{e}_\nu(t)|, \\ &\le \left\{ 2\mathcal{H}^{\nu} (K_h + \varepsilon^2)\, t^{2\beta}\, \Gamma(t)\, \phi(t) \right\} \left(\left\|\tfrac{1}{\phi(t)} \vec{h}(t)\right\|_{[t_\infty,\infty)} + |\vec{\xi}_0|^2 \right) \phi(t), \\ &= o\left(\phi(t)\right) \text{ as } t \to \infty. \end{aligned}$$

By the same analysis using Lemma III3.7 that led to A4.8, we see that A4.10 has a solution $\vec{e}_{\nu+1}(t)$ s.t.

A4.12 $$|e_{\nu+1}(t)| \le \{2\mathcal{H}^{\nu}(K_h+\varepsilon^2)\,\|t^{2\beta}\Gamma(t)\phi(t)\|_{[t_0,\infty)}\,K_{\varepsilon_0}K_{\alpha}t_{\infty}^{-\varepsilon_0}\} \cdot \left(\left\|\frac{1}{\phi(t)}\vec{h}(t)\right\|_{[t_\infty,\infty)} + |\vec{\xi}_0|^2\right)t^{\beta}\phi(t).$$

Therefore, from A4.2b we require t_∞ sufficiently large so that

A4.13 $$\frac{1}{2} \ge \mathcal{H} \ge 2(K_h+\varepsilon^2)\,\|t^{2\beta}\Gamma(t)\|_{[t_0,\infty)}\,K_{\varepsilon_0}K_{\alpha}t_{\infty}^{-\varepsilon_0} = o(1)$$

as $t_\infty \to \infty$.

Finally, using $\vec{y}_{\nu+1} = \vec{e}_{\nu+1} + \vec{y}_\nu$, we see that (using A4.2)

A4.14 $$|\vec{y}_{\nu+1}(t)| \le (C_\nu + \mathcal{H}^{\nu+1})\left\{\left\|\frac{1}{\phi(t)}\vec{h}(t)\right\|_{[t_\infty,\infty)} + |\vec{\xi}_0|^2\right\}t^{\beta}\phi(t)$$

for $t \ge t_\infty$. By the induction hypothesis $C_\nu \le 1-\frac{1}{2}\nu$ and $\mathcal{H} \le \frac{1}{2}$. Therefore we can set

$$C_{\nu+1} = C_\nu + \mathcal{H}^{\nu+1} \le 1-\frac{1}{2}\nu + \frac{1}{2}\nu+1 = 1-\frac{1}{2}\nu+1,$$

as required. This completes the induction step, for t_∞ sufficiently large such that A4.9, 13 are satisfied and

A4.15 $$(K_h + |\vec{\xi}_0|^2)\,\|t^{\beta}\phi(t)\|_{[t_\infty,\infty)} + K_0\,\|\psi(t)\|_{[t_\infty,\infty)} < \frac{1}{2}T_y$$

(i.e., $\vec{y}_\nu(t) < \frac{1}{2} T_y \quad \forall \nu = 0, 1, \ldots$).

The convergence of $\vec{y}_\nu(t) \rightarrow \vec{y}(t; t_\infty, \vec{\xi}_0)$ *as* $\nu \rightarrow \infty$ follows easily from A4.1,2,3, where $\vec{y}(t; t_\infty, \vec{\xi}_0)$ is a solution of 2.4a satisfying estimate 2.9. ■

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PART II

Numerical Hopf Bifurcation

INTRODUCTION

onsider the autonomous system of o.d.e's

0.1 $$\vec{u}_t = \vec{f}(\vec{u},\lambda),$$

with $\vec{u}\in\mathbb{R}^n$, $\lambda\in\mathbb{R}$, and $\vec{f}$ a smooth function. Let $\vec{u}_0(\lambda)$ be a smooth steady state branch of 0.1, that is

0.2 $$\vec{f}(\vec{u}_0(\lambda),\lambda)=\vec{0}.$$

In this thesis we will be concerned with the Hopf bifurcation of a periodic solution of 0.1 from the branch $\vec{u}_0(\lambda)$.

Hopf bifurcation is an important mechanism for the birth of a periodic solution near a steady state branch. The following example illustrates this process.

Example 0.3. Let $\vec{u}(t) = (u_1(t),u_2(t))^T$ satisfy

0.3 $$\frac{d\vec{u}}{dt} = \begin{pmatrix} \lambda & 1 \\ -1 & \lambda \end{pmatrix}\vec{u} + \beta\{u_1^2+u_2^2\}\vec{u}$$

for $\beta<0$ and $\lambda\in\mathbb{R}$. Phase portraits of 0.3 for $\lambda\leq 0$ and $\lambda>0$ are given in Fig. 0.4.

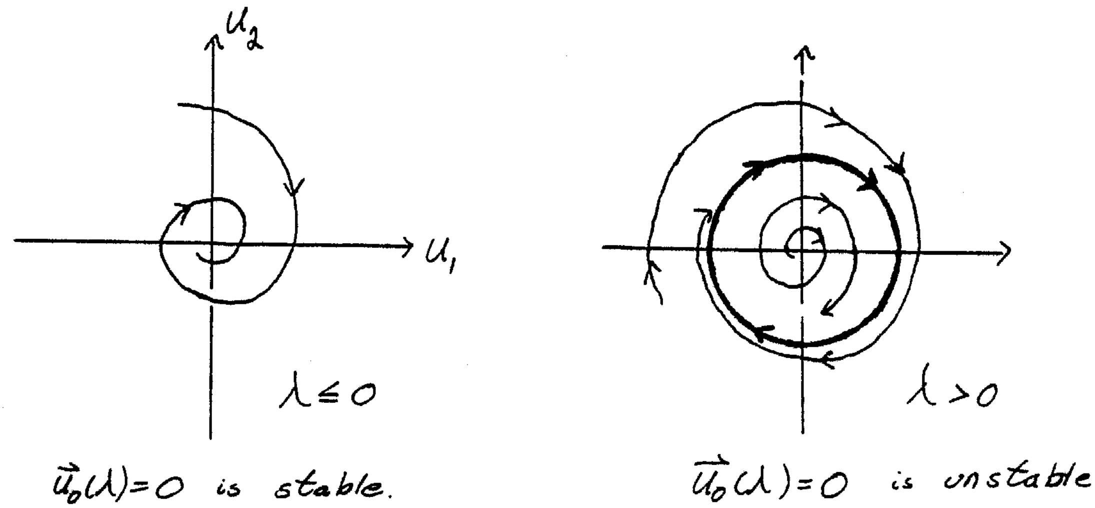


Fig. 0.4

We see that for $\lambda>0$ equation 0.3 has a periodic solution, given by

0.4
$$\vec{u}_L(t;\lambda) = \left|\frac{\lambda}{\beta}\right|^{1/2} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} .$$

Notice that the amplitude of the periodic solution, $a(\lambda) = \left|\frac{\lambda}{\beta}\right|^{1/2}$ increases from zero as λ increases from zero. The point $(\vec{u},\lambda) = (\vec{0},0)$ is a Hopf bifurcation point for 0.3. (The reader may find it instructive to consider the cases $\beta>0$, and $\beta=0$.)

Returning to the general system 0.1, we make the following definition:

<u>Definition 0.5</u>. (<u>H</u>opf <u>B</u>ifurcation <u>P</u>oint, <u>HBP</u>). The point $(\vec{u}_c,\lambda_c)$ is a HBP of 0.1 if there exist $\rho>0, \omega_c>0$ such that

a) there exists $\vec{u}_0(\lambda)$ satisfying 0.2, $\vec{u}_0(\lambda)$ a smooth function of λ for
$$\lambda \in B_\rho(\lambda_c) \equiv \{\lambda \mid |\lambda - \lambda_c| < \rho\},$$
and $\vec{u}_0(\lambda_c) = \vec{u}_c$;

b) $\vec{f}_{\vec{u}}(\vec{u}_0(\lambda), \lambda)$ has smooth, simple eigenvalues $\mu(\lambda) \pm i\omega(\lambda)$ with $\mu(\lambda_c) = 0$, and $\omega(\lambda_c) = \omega_c$;

0.5

c) $\left.\frac{d\mu}{d\lambda}\right|_{\lambda=\lambda_c} \neq 0$,

d) $\pm in\omega_c$ are <u>not</u> eigenvalues of $\vec{f}_{\vec{u}}(\vec{u}_c, \lambda_c)$ for $n = 0, 2, 3, 4, \ldots$.

With this definition we have the well known result (see Hopf [12], Marsden and McCracken [21]):

<u>Theorem 0.6 (The Hopf Bifurcation Theorem.)</u> Let $\vec{u}_0(\lambda)$ satisfy 0.2 and suppose $(\vec{u}_0(\lambda_c), \lambda_c)$ is a HBP for 0.1. Then there exists a family, $\vec{u}_L(t;\varepsilon)$, of real periodic solutions of 0.1 for $0 < \varepsilon < \varepsilon_0$. Here $\vec{u}_L(t;\varepsilon)$ has the form

a) $\vec{u}_L(t;\varepsilon) = \vec{u}_0(\lambda(\varepsilon)) + \varepsilon\, \vec{y}(t; \lambda(\varepsilon), \vec{\zeta}(\varepsilon), \varepsilon)$,

0.6 b) $\vec{y}(t + T(\varepsilon)) = \vec{y}(t)$,

c) $\vec{y}(0) = \vec{\zeta}(\varepsilon)$.

Where $\vec{u}_L, \lambda, \vec{\zeta}$, T and $\vec{y}$ are smooth functions of ε such that as $\varepsilon \downarrow 0$:

a) $\lambda(\varepsilon) = \lambda_c + O(\varepsilon^2)$,

0.7 b) $T(\varepsilon) = 2\pi/\omega_c + O(\varepsilon^2)$,

c) for some $\vec{\zeta}_1 \in \mathbb{R}^n$, $\vec{\zeta}(0+) + i\,\vec{\zeta}_1$ is an eigenvector of $\vec{f}_{\vec{u}}(\vec{u}_0(\lambda_c), \lambda_c)$ with eigenvalue $+\,i\omega_c$. Also $|\vec{\zeta}(0+)| = 1$.

Finally, any periodic solution of 0.1 near $(\vec{u}_0(\lambda_c), \lambda_c)$ and having a period near T(0) is given by 0.6 for some $\varepsilon > 0$ (up to a phase).

For a proof of Theorem 0.6 see Lemma V 4.15.

Relatively little work has been done on the automatic computation of HBPs and the bifurcating solution branches (see [4], [8], [9], [19], [20]). In this thesis we will present algorithms for locating a HBP, for calculating the Hopf bifurcation parameters (see Chapter III), for calculating the birth of a periodic solution, and for following a branch of periodic solutions. Many of the methods are new. We also consider similar algorithms for Hopf bifurcation in p.d.e's of the form

0.8 $$\vec{u}_t = \vec{f}(\vec{u}, \vec{u}_x, \vec{u}_{xx}, \ldots, \lambda).$$

Reaction diffusion equations are important problems of this form (see [17], [27], and [28]).

Example calculations, including applications to p.d.e's, are in progress.

CHAPTER I. Detecting a HBP While Continuing Along a Steady State Branch

I.1 Introduction

Techniques for the numerical computation of steady state branches of 0.1 are well known (see [14], [25]). For example, given a steady state $\vec{u}_0(\lambda_0)$ of 0.1, Euler-Newton continuation calculates $\vec{u}_0(\lambda_1)$ as follows. First calculate

1.1 $$\vec{w}_1 = \vec{u}_0(\lambda_0) - (\lambda_1 - \lambda_0)\left\{\vec{f}_{\vec{u}}(\vec{u}_0(\lambda_0),\lambda_0)\right\}^{-1}\vec{f}_\lambda(\vec{u}_0(\lambda_0),\lambda_0).$$

This equation is obtained by applying Euler's method (see [3]) to

1.2 $$\vec{f}_{\vec{u}}(\vec{u}_0(\lambda),\lambda)\vec{u}_{0\lambda}(\lambda) = -\vec{f}_\lambda(\vec{u}_0(\lambda),\lambda).$$

Therefore $|\vec{w}_1 - \vec{u}_0(\lambda_1)| = O((\lambda_1-\lambda_0)^2)$ as $\lambda_1 \to \lambda_0$. Next we apply Newton's method to

1.3 $$\vec{f}(\vec{w},\lambda) = 0,$$

using $\vec{w}_1$ as the initial guess. For $\lambda_1-\lambda_0$ sufficiently small, and for $\vec{f}_{\vec{u}}(\vec{u}_0(\lambda_0),\lambda_0) \equiv \vec{f}^{\,0}_{\vec{u}}$ nonsingular, the Newton iterates will converge to $\vec{w} = \vec{u}_0(\lambda_1)$ (see [24]).

One important property of Euler-Newton continuation is that singular points in $\vec{u}_0(\lambda)$ are easily detected. A singular point on the steady state branch $\vec{u}_0(\lambda)$ is a point $(\vec{u}_0(\lambda_c),\lambda_c)$ such that $\vec{f}_{\vec{u}}^{c} \equiv \vec{f}_{\vec{u}}(\vec{u}_0(\lambda_c),\lambda_c)$ is singular. In particular

$$\text{1.4} \qquad \det\left(\vec{f}_{\vec{u}}^{c}\right) = 0 .$$

Such points could be regular bifurcation points, or turning points (see [14]), and are therefore important points to detect. Notice that in the Euler Newton step above, the $\det(\vec{f}_{\vec{u}}(\vec{u}_0(\lambda_i),\lambda_i))$ for i = 0,1 could easily be calculated from the available LU-factorizations of $\vec{f}_{\vec{u}}$. Therefore we can check for a singular $\lambda_c \in (\lambda_0,\lambda_1)$ by checking if $\det(\vec{f}_{\vec{u}}^{0})\ \det(\vec{f}_{\vec{u}}^{1}) < 0$ (we will not consider multiple zeros here, see [14]).

In this chapter we discuss techniques for <u>detecting a HBP</u> $(\vec{u}_0(\lambda_c),\lambda_c)$ with $\lambda_0 < \lambda_c < \lambda_1$. A necessary condition for $(\vec{u}_0(\lambda_c),\lambda_c)$ to be a HBP is

$$\text{1.5} \qquad \det\left(\vec{f}_{\vec{u}}^{c} - i\,\omega_c\, I_{n\times n}\right) = 0 .$$

Unfortunately, in general we do not know ω_c a priori, and the determinant in 1.5 is not easy to obtain. This makes the detection of a HBP between λ_0 and λ_1 a much more difficult problem than the detection of a regular bifurcation point.

I.2 <u>Method 1. Calculating the Eigenvalues of $\vec{f}_{\vec{u}}$.</u>

Suppose that we compute $\vec{u}_0(\lambda_i)$ for $i = 0,1,2,\cdots$, with $\lambda_0 < \lambda_1 < \ldots$. At each stage of the continuation we are interested in detecting whether we have passed a HBP. Consider the following algorithm to do this.

<u>Method 1</u>. Suppose $(\vec{u}_0(\lambda_i),\lambda_i)$ is given and we have calculated the next steady state $\vec{u}_0(\lambda_{i+1})$. To check for a HBP with $\lambda_c \in (\lambda_i,\lambda_{i+1})$:

a) Calculate the eigenvalues $\{\alpha_k(\lambda_{i+1})\}_{k=1}^{n}$ of

$$\vec{f}_{\vec{u}}^{\,i+1} \equiv \vec{f}_{\vec{u}}(\vec{u}_0(\lambda_{i+1}), \lambda_{i+1}) \quad ;$$

2.1 b) Calculate $n_+^{i+1} \equiv$ the number of eigenvalues $\alpha_k(\lambda_{i+1})$ with $Re(\alpha_k(\lambda_{i+1})) > 0$;

c) Using n_+^i calculated in the previous continuation step, calculate $\Delta n_+^{i+1} \equiv n_+^{i+1} - n_+^i$.

From Method 1 we know that at least $|\Delta n_+^{i+1}|$ eigenvalues of $\vec{f}_{\vec{u}}(\vec{u}_0(\lambda),\lambda)$ cross the imaginary axis in the interval $(\lambda_i,\lambda_{i+1})$. If $|\Delta n_+^{i+1}| \geq 2$ we should further examine $[\lambda_i,\lambda_{i+1}]$ for a HBP (see Chapter II). If $|\Delta n_+^{i+1}| = 1$, there exists a singular point (see Section 1) of the path $\vec{u}_0(\lambda)$ in $[\lambda_i,\lambda_{i+1}]$. (There is a possibility that Method 1 might fail to detect a HBP if a pair of eigenvalues crosses the imaginary axis an even number of times in $[\lambda_i,\lambda_{i+1}]$, or if two (2 m) pairs cross in opposite directions. To detect these we would have to do some more work, for example, we could check $\frac{d}{d\lambda} Re(\alpha_k(\lambda))$ for eigenvalues $\alpha_k(\lambda)$ with $Re(\alpha_k(\lambda))$ small. We will not consider this further.)

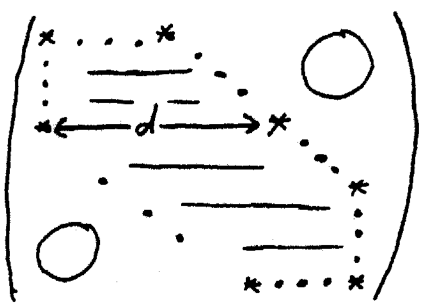
Estimates of the number of operations (i.e., multiplications and divisions) that Method 1 requires will be useful for comparisons with other methods. The estimates in Table 2.4 are based on the following algorithm for calculating the eigenvalues of a matrix (see Wilkinson [36] for details, or see [3], [34]):

2.2

a) Reduce $\vec{f}^{\,i+1}_{\vec{u}}$ to Hessenberg form using Householder transformations;

b) Apply the QR-algorithm to the reduced matrix.

The estimates are given for $\vec{f}^{\,i+1}_{\vec{u}}$ a full, unsymmetric $n \times n$ matrix and for $\vec{f}^{\,i+1}_{\vec{u}}$ having band width $d = 2m+1$ (i.e.,

2.3 $\vec{f}^{\,i+1}_{\vec{u}} =$ )

where "*" represents a possibly nonzero entry).

TABLE 2.4
Operation Counts for Method 1

Operation \ Type of Matrix	Full n×n, unsymmetric	banded n×n, band width d = 2m+1, nonsymmetric
Gaussian Elim. with partial pivoting	$\frac{1}{3} n^3$	$\frac{1}{2} nd^2$
Reduction to Hessenberg form	$\frac{5}{3} n^3$	-
QR - Algorithm	$\frac{8}{3} n^3$ †	$O(n^3)$
Total for Method 1	$\frac{13}{3} n^3$	$O(n^3)$

†Based on $\sum_{k=1}^{n} 8k^2$, where one QR-iteration on a k×k matrix takes $4k^2$ operations, and roughly two iterations are needed per eigenvalue.

As a rule of thumb, an efficient continuation scheme should require no more than three LU-factorizations of $\vec{f}_{\vec{u}}$ (see Perozzi [25]). From Table 2.4 we see that Method 1 is costly compared to continuation, taking roughly four times the work for $\vec{f}_{\vec{u}}$ a full n×n matrix. For banded $\vec{f}_{\vec{u}}$ the QR-algorithm fills in the upper triangular portion of the Hessenberg form, leading to an operation count of $O(n^3)$.

I.3 Method 2. Using the Routh-Hurwitz Theorem.

In this section we will use the Routh-Hurwitz theorem (see Gantmacher [5], [6]) to calculate n^i_+ (see Section I.2). We outline a special case of the theorem below.

Let

3.1 $$\rho^i(\alpha) \equiv (a_0\alpha^n + a_1\alpha^{n-2} + \dots) + (b_0\alpha^{n-1} + b_1\alpha^{n-3} + \dots)$$

be the characteristic polynomial for $f^{i}_{\bar{u}}$, with $a_0 \neq 0$ (note separation of odd and even degree terms in 3.1). Define the Hurwitz matrix H_1 to be

3.2 $$H_1 = \left.\begin{pmatrix} b_0 & b_1 & \cdots & b_{n-1} \\ a_0 & a_1 & \cdots & a_{n-1} \\ 0 & b_0 & b_1 \cdots & b_{n-2} \\ 0 & a_1 & a_1 \cdots & a_{n-2} \\ 0 & 0 & b_0 \cdots & b_{n-3} \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}\right\} n \text{ rows}$$

Here

3.3
a) $a_k = 0 \quad \text{if} \quad k > n/2$,

b) $b_k = 0 \quad \text{if} \quad k > (n-1)/2$.

Assume $b_0 \neq 0$, and subtract a_0/b_0 times the $(2k-1)$th row of H_1 from the $(2k)$th row of H_1, for $k = 1,2,\cdots, \lfloor n/2 \rfloor$. This leaves the matrix

3.4 $$\tilde{H}_2 = \left(\begin{array}{c|cccc} b_0 & b_1 & b_2 & \cdots & b_{n-1} \\ \hline \vec{0} & & H_2 & & \\ & & & & \end{array}\right), \quad H_2 \equiv \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} \\ b_0 & b_1 & \cdots & b_{n-2} \\ 0 & c_0 & \cdots & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \end{pmatrix},$$

where H_2 is $(n-1) \times (n-1)$. Assuming $c_0 \neq 0$ we can reduce H_1 as above. In <u>the regular case</u> we can continue this process, computing n nonzero divisors b_0, c_0, d_0, $\cdots$. Then we have

<u>Theorem 3.5. (Routh's Theorem (see [5])</u>. For $b_0, c_0, d_0, \cdots$ all nonzero, n_+^i equals the number of sign changes in the list of n+1 numbers,

3.5 $$a_0,\ b_0,\ c_0,\ \cdots.$$

From the structure of H_0, H_1, etc., we see that in the regular case it takes only $n + (n-1) + \cdots + 2 = \frac{n(n+1)}{2} - 1$ operations to calculate the divisors b_0, c_0, $\cdots$.

For irregular cases in which the kth divisor (say) is zero, we must apply the complete Routh-Hurwitz theorem. This requires computing each principal minor of the $(n-k+1) \times (n-k+1)$ matrix H_k. By a similar elimination procedure to that given above we can calculate all principle minors of H_k in roughly $\frac{(n-k+1)^2}{2}$ operations.

Next we examine the calculation of the coefficients of $p^i(\alpha)$. Krylov's method (see Wilkinson [36]) is based on $p^i(\vec{f}^{\,i}_{\vec{u}}) \equiv 0_{n\times n}$ (the Cayley-Hamilton theorem). Briefly, it involves choosing some $\vec{\xi}_0 \in \mathbb{R}^n$, calculating $(\vec{f}^{\,i}_{\vec{u}})^r \vec{\xi}_0$ for r = 1, ..., n, and then solving the n x n system

3.6 $$p^i(\vec{f}^{\,i}_{\vec{u}})\,\vec{\xi}_0 = \vec{0}$$

for the unknowns $a_0, a_1, \cdots, b_0, b_1, \cdots$. This method takes roughly 5/3 n^3 operations for full n×n matrices. Unfortunately, it is ill-conditioned for many matrices $\vec{f}^{\,i}_{\vec{u}}$ with n large (even $n \approx 10$) (see [36]).

Wilkinson [36] suggests that a reasonably stable method for calculating $p^i(\alpha)$ is by first reducing $\vec{f}^{\,i}_{\vec{u}}$ to tridiagonal form (taking roughly 5/6 n^3 single-precision operations and 1/6 n^3 double-precision operations when $\vec{f}^{\,i}_{\vec{u}}$ is a full matrix). The characteristic polynomial of a tridiagonal matrix can be obtained using a recursion relation in roughly 3/2 n^2 operations (see [36]).

TABLE 3.7

Operation Counts for Method 2

Operation \ Type of Matrix	Full $n \times n$, unsymmetric	$n \times n$, band width $d=2m+1$, nonsymmetric
Gaussian Elimination (partial pivoting)	$\frac{1}{3} n^3$	$\frac{1}{2} nd^2$
Char. Poly (Krylov)	$\frac{5}{3} n^3$	$\frac{1}{3} n^3$
Char. Poly (via Tridiagonal form)	$\frac{5}{6} n^3$ single prec. $+ \frac{1}{6} n^3$ double prec.*	–**
Total for Method 2	$\sim \frac{5}{6} n^3$ single prec. $+ \frac{1}{6} n^3$ double prec.*	at least $n^2/2$ (for Routh's Thm. alone)

* 1 double-precision operation $\approx$ 3 (or 4) single prec. ops.

** using Householder transformations to reduce $f^i_{\vec{u}}$ to upper Hessenberg form fills up much of the upper triangular portion of $f^i_{\vec{u}}$. This leads to $O(n^3)$ operations.

We see from Table 3.7 that for full matrices $f^i_{\vec{u}}$, Method 2 is a big improvement over Method 1. However, for large banded systems with d/n small (arising in p.d.e. applications) both methods are very expensive.

I.4 Method 3. Solving the Time-Dependent Problem

In this section we will assume that $(\vec{u}_0(\lambda_c), \lambda_c)$ is a HBP such that for $\lambda < \lambda_c$ the steady state $\vec{u}_0(\lambda)$ is linearly stable. That is, the eigenvalues $\{\alpha_k(\lambda)\}_{k=1}^{n}$ of $\vec{f}_{\vec{u}}(\vec{u}_0(\lambda), \lambda)$ satisfy

4.1 $$Re(\alpha_k(\lambda)) < 0 \quad \text{for } k = 1, \cdots, n; \ \lambda < \lambda_c.$$

From 0.5c, we can assume

4.2 $$Re(\alpha_k(\lambda)) > 0 \quad \text{for } k = 1, 2; \ \lambda_c < \lambda < \lambda_L.$$

for some $\lambda_L > \lambda_c$ that is, $\vec{u}_0(\lambda)$ is unstable for $\lambda \in (\lambda_c, \lambda_L)$.

H. B. Keller [17] has proposed that this change of stability can be detected by solving the initial value problem

4.3

a) $$\vec{u}_t = \vec{f}(\vec{u}, \lambda_i) \quad \text{for} \quad t > 0,$$

b) $$\vec{u}(0, \lambda_i) = \vec{u}_0(\lambda_i) + \vec{\omega}_0,$$

at each continuation point $(\vec{u}_0(\lambda_i), \lambda_i)$. Here $\vec{\omega}_0 \not\equiv \vec{0}$ and $|\vec{\omega}_0|$ is small.

If the periodic solution, bifurcating from $(\vec{u}_0(\lambda_c), \lambda_c)$, exists for $\lambda = \lambda(\varepsilon) = \lambda_c + \lambda_2 \varepsilon^2 + O(\varepsilon^3)$ with $\lambda_2 > 0$ (and with ε as in Theorem 0.6) then it is stable (see Example 0.3, or [21]). In this case an important by-product of computing 4.3 is that for $|\vec{w}_0|$ and $\lambda_i - \lambda_c$ small, the solution of 4.3 converges to the periodic solution that bifurcates from $(\vec{u}_0(\lambda_c), \lambda_c)$. (The periodic bifurcating branch for a reaction diffusion equation has been calculated in this way by Keller, Ferguson, and Reyna [17].) In view of the more general methods described in the next section, <u>the above case is the only case in which the solutions of 4.3 should be computed</u>. The problem of choosing an efficient difference scheme to solve 4.3 is beyond the scope of this work (see [7]).

I.5 <u>Method 4. Multiply Shifted Power Method</u>

Consider the explicit, one-step difference equation

5.1 $$\vec{\omega}_{k+1} = P(h A(\lambda))\, \omega_k$$

for $k \geq 0$. Here $A(\lambda) = \vec{f}_{\vec{u}}(\vec{u}_0(\lambda), \lambda)$, $h > 0$ and $P(z)$ is a polynomial,

5.2 $$P(z) = \prod_{\ell=1}^{m} (z - \zeta_\ell)$$

with $\zeta_\ell \in \mathbb{C}$ for $\ell = 1, \ldots, m$.

Notice that 5.1 includes all linear, explicit, one-step difference schemes for integrating the linearized form of 4.3, that is,

5.3
$$\text{a)}\quad \vec{v}_t = A(\lambda)\vec{v}(t) \quad \text{for} \quad t>0,$$
$$\text{b)}\quad \vec{v}(0) = \vec{w}_0.$$

The linear stability of $\vec{u}_0(\lambda)$ can be checked by solving 5.3 using a difference scheme of the form 5.1. Define the stability domain of 5.1 to be

5.4
$$\mathcal{D}_s = \{ z \in \mathbb{C} \mid |P(z)| < 1 \}.$$

In order for the difference scheme to provide a good check on the stability of $\vec{u}_0(\lambda)$ we require that for some small $\varepsilon > 0$,

5.5
$$\text{a)}\quad \{ z \in \mathbb{C} \mid |z| \le 1,\ Re(z) < -\varepsilon \} \subset \mathcal{D}_s,$$
$$\text{b)}\quad \mathcal{D}_s \cap \{ z \in \mathbb{C} \mid Re(z) > \varepsilon \} = \emptyset$$

(see Fig. 5.6).

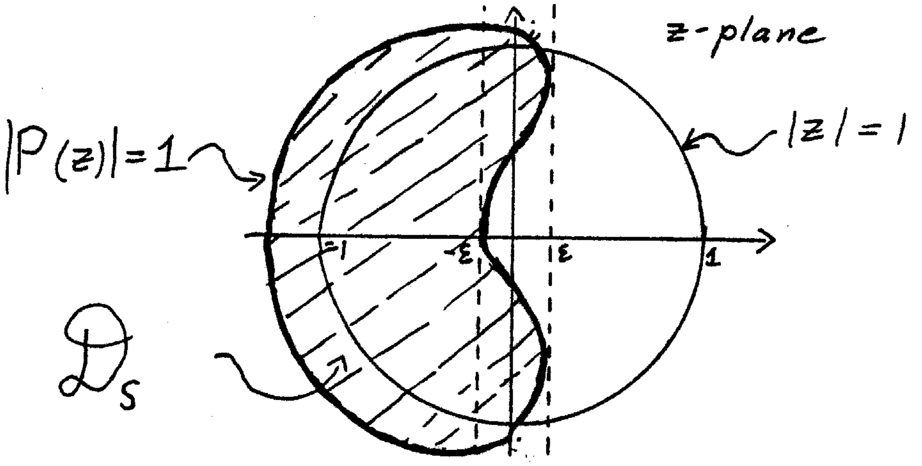


Fig. 5.6

For a P(z) satisfying 5.5 for some small $\varepsilon > 0$ we can use 5.1 to check the linear stability of $\vec{u}_0(\lambda)$. In particular, let h be sufficiently small so that the eigenvalues $\{h\alpha_i(\lambda)\}_{i=1}^{\ell}$ of $hA(\lambda)$ satisfy $|h\alpha_i| \le 1$ for $i = 1, \cdots, \ell$. Then $P(hA(\lambda))$ has eigenvalues $\{P(h\alpha_i)\}_{i=1}^{\ell}$. The iteration 5.1 is just the power method applied to $P(hA(\lambda))$ (see [36]). For most initial guesses w_0,

$$\frac{|\vec{\omega}_{k+1}|}{|\vec{\omega}_k|} \longrightarrow \max_{1 \le i \le \ell} |P(h\alpha_i)| \equiv |P(h\alpha_j)|$$

as $k \to \infty$. If $|P(h\alpha_j)| > 1$ then $Re(h\alpha_j) > -\varepsilon$, and if $|P(h\alpha_j)| < 1$ then $Re(h\alpha_i) < \varepsilon$ for $i = 1, \cdots, \ell$. For ε small, $|P(h\alpha_j)|$ is a sensitive test for the linear stability of $\vec{u}_0(\lambda)$.

The advantage of 5.1 over Method 3 applied to 5.3 is that we can ignore the accuracy of 5.1 for 5.3. The P(z) can be chosen simply to minimize the work involved in 5.1.

Given an $\varepsilon > 0$ and an integer $m > 0$, let $\mathcal{P}(\varepsilon, m) \equiv \{P(z)$ as in 5.2 such that 5.5 is satisfied$\}$. We want a $P(z) \in \mathcal{P}(\varepsilon, m)$ such that the number of iterations of 5.1 needed to detect the stability of $\vec{u}_0(\lambda)$ is minimized. For example, we could require that P(z) minimizes the ratio

5.7 $$\beta(\delta) = \frac{\max\{|P(z)| \mid |z| \le 1,\ Re(z) \le -\delta\}}{\min\{|P(z)| \mid |z| \le 1,\ Re(z) \ge \delta\}}$$

for some $\delta > \varepsilon$. $\beta(\delta)$ can be used as follows: Let $\vec{w}_0 = c_1\vec{a}_1 + c_2\vec{a}_2$ where $\vec{a}_1, \vec{a}_2$ are eigenvectors of $A(\lambda)$ with eigenvalues α_1 and α_2 respectively. Suppose $\mathrm{Re}(h\alpha_1) \le -\delta$, $\mathrm{Re}(h\alpha_2) \ge \delta$, and all the eigenvalues of $hA(\lambda)$ have moduli less than 1. Then $\vec{w}_k = c_{1k}\vec{a}_1 + c_{2k}\vec{a}_2$ where

5.8 $$\left|\frac{c_{1k}}{c_{2k}}\right| \le \beta^k(\delta)\left|\frac{c_1}{c_2}\right| .$$

The calculation of an optimal $P(z)$ is an important area for future work.

The method can be applied to multistep and/or implicit schemes. The trapezoidal method

5.9 $$\left(I - hA(\lambda)\right)\vec{w}_{k+1} = \left(I + hA(\lambda)\right)\vec{w}_k$$

has $\mathcal{D}_S \equiv \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\}$ (with $\mathcal{D}_S$ defined in the obvious way). The value of $h > 0$ should be chosen to minimize the number of iterations needed to detect an eigenvalue with positive real part. (See Franklin [37]).

If $A(\lambda)$ has eigenvalues in both the left and right half planes then it is still possible to use 5.1. The $P(z)$ used should have a ridge along the imaginary axis (see Fig. 5.10). However, the number of iterations of 5.1 needed is likely to be restrictively high.

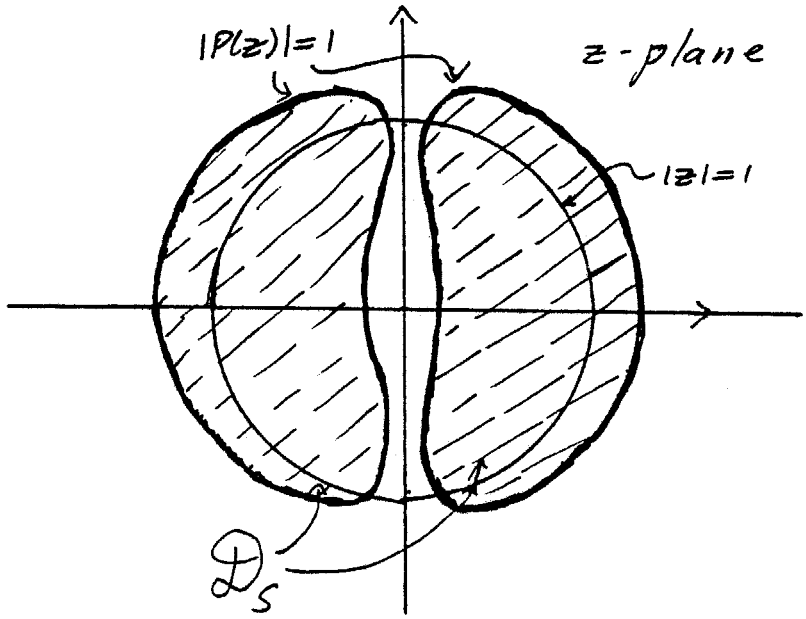


Fig. 5.10

I.6 Conclusions

Methods for detecting a HBP while continuing along a steady state branch are expensive compared to the cost of the continuation alone. For problems with full n×n Jacobians $\vec{f}_{\vec{u}}$ and unstable steady states (unstable for <u>both</u> $t \to \infty$ and $t \to -\infty$, otherwise use $\hat{t} = -t$) Method 2 is the best. For these problems Method 2 takes roughly twice the work of one continuation step. For problems with large Jacobians having a narrow bandwidth Methods 1 and 2 are unfeasible, requiring $O(n^3)$ operations (compared to $O(nd^2)$ for continuation). Method 3 is useful for bifurcations from a stable steady state to a <u>stable</u> periodic solution. Method 4 is an important area for future research; it should be useful for problems in which the eigenvalues of $\vec{f}_{\vec{u}}$ lie in a few known clusters. Certainly, Method 4 will prove to be more efficient than Method 3 for detecting a HBP.

Chapter II. Locating a HBP Accurately

II.1 Introduction

Define a candidate Hopf bifurcation point (CHBP) to be a steady state $(\vec{u}_0(\lambda_c),\lambda_c)$ such that for λ in a neighborhood of λ_c we have

a) $\det\left(\vec{f}_{\vec{u}}(\vec{u}_0(\lambda),\lambda)-[\mu(\lambda)+i\omega(\lambda)]I\right)=0,$

1.1 b) $\omega(\lambda_c)>0, \quad \mu(\lambda_c)=0,$

d) $\dfrac{d\mu}{d\lambda}(\lambda_c)\neq 0.$

In Chapter I we considered methods for detecting CHBPs. In this chapter we consider methods for accurately calculating $\vec{u}_0(\lambda_c),\lambda_c$ assuming that we are given a good initial guess.

The methods in Chapter I detect two steady states $(\vec{u}_0(\lambda_i),\lambda_i)$ and $(\vec{u}_0(\lambda_{i+1}),\lambda_{i+1})$ such that $\vec{f}_{\vec{u}}$ has a pair of eigenvalues $\alpha_\pm(\lambda)=\mu(\lambda)\pm i\omega(\lambda)$ with

1.2 $$\mu(\lambda_i)\mu(\lambda_{i+1})<0.$$

For $|\lambda_i-\lambda_{i+1}|$ small, a good initial guess for a CHBP is given by $\vec{u}_0(\lambda_i),\lambda_i$ and $\alpha_\pm(\lambda_i)$.

II.2 Inverse Iterations

We attempt to solve

$$\mu(\lambda) = 0 \tag{2.1}$$

using the secant method. The eigenvalues $\alpha_{\pm}(\lambda) = \mu(\lambda) \pm i\omega(\lambda)$ are calculated using inverse iterations. (A similar technique has been suggested by B. Hassard [9].)

In particular, suppose $\lambda^{\nu}, \lambda^{\nu-1}, \alpha_{\pm}(\lambda^{\nu})$, and $\alpha_{\pm}(\lambda^{\nu-1})$ have been calculated. Then applying the secant method to 2.1 we get

$$\lambda^{\nu+1} = \lambda^{\nu} - \left\{\frac{\lambda^{\nu} - \lambda^{\nu-1}}{\mu(\lambda^{\nu}) - \mu(\lambda^{\nu-1})}\right\} \mu(\lambda^{\nu}) \, . \tag{2.2}$$

We approximate $\alpha_{+}(\lambda^{\nu+1})$ by $\tilde{\alpha}_{+}(\lambda^{\nu+1}) = i\tilde{\omega}^{\nu+1}$ where

$$\tilde{\omega}^{\nu+1} = \omega^{\nu} + \left\{\frac{\omega^{\nu} - \omega^{\nu-1}}{\lambda^{\nu} - \lambda^{\nu-1}}\right\} (\lambda^{\nu+1} - \lambda^{\nu}) \tag{2.3}$$

(i.e., a linear interpolate). We calculate $\vec{u}_0(\lambda^{\nu+1})$ using a continuation method (see Chapter I, §1). Next we calculate an eigenvector $\vec{a}(\lambda^{\nu+1}) + i\vec{b}(\lambda^{\nu+1})$ of $\vec{f}_{\vec{u}}^{\,\nu+1} \equiv \vec{f}_{\vec{u}}(\vec{u}_0(\lambda^{\nu+1}), \lambda^{\nu+1})$, with eigenvalue $\alpha_{+}(\lambda^{\nu+1})$, by inverse iterations. That is, we solve

$$\text{a)}\quad \mathcal{M}^{\nu+1}\begin{pmatrix}\vec{\xi}_{k+1}\\ \vec{\eta}_{k+1}\end{pmatrix}=\begin{pmatrix}\vec{a}_k\\ \vec{b}_k\end{pmatrix}, \quad \mathcal{M}^{\nu+1}\equiv\begin{pmatrix}\vec{f}_{\vec{u}}^{\,\nu+1} & \tilde{\omega}^{\nu+1} I_{n\times n}\\ -\tilde{\omega}^{\nu+1} I_{n\times n} & \vec{f}_{\vec{u}}^{\,\nu+1}\end{pmatrix}$$

2.4

$$\text{b)}\quad \begin{pmatrix}\vec{a}_{k+1}\\ \vec{b}_{k+1}\end{pmatrix}=\frac{1}{\left\|\begin{pmatrix}\vec{\xi}_{k+1}\\ \vec{\eta}_{k+1}\end{pmatrix}\right\|}\begin{pmatrix}\vec{\xi}_{k+1}\\ \vec{\eta}_{k+1}\end{pmatrix}$$

for k = 0,1,2,$\cdots$,K (if $i\tilde{\omega}^{\nu+1}\approx \alpha_+(\lambda^{\nu+1})$ then K = 2 or 3 suffices). We solve 2.4a by calculating a LU-factorization of $\mathcal{M}^{\nu+1}$. The same LU-factorization can be used to calculate a left eigenvector $\vec{c}(\lambda^{\nu+1})+i\vec{d}(\lambda^{\nu+1})$ of $\vec{f}_{\vec{u}}^{\,\nu+1}$ with eigenvalue $\alpha_+(\lambda^{\nu+1})$.

Finally, we use the generalized Rayleigh quotient to calculate $\mu(\lambda^{\nu+1})+i\,\omega(\lambda^{\nu+1})$, that is,

$$2.5\quad \begin{pmatrix}\mu(\lambda^{\nu+1})\\ \omega(\lambda^{\nu+1})\end{pmatrix}\approx\left[\begin{pmatrix}\vec{c}_K^{\,T} & \vec{d}_K^{\,T}\\ -\vec{d}_K^{\,T} & \vec{c}_K^{\,T}\end{pmatrix}\begin{pmatrix}\vec{a}_K & -\vec{b}_K\\ \vec{b}_K & \vec{a}_K\end{pmatrix}\right]^{-1}\begin{pmatrix}\vec{c}_K^{\,T} & \vec{d}_K^{\,T}\\ -\vec{d}_K^{\,T} & \vec{c}_K^{\,T}\end{pmatrix}\begin{pmatrix}\vec{f}_{\vec{u}}^{\,\nu+1}\vec{a}_K\\ \vec{f}_{\vec{u}}^{\,\nu+1}\vec{b}_K\end{pmatrix}$$

(see Wilkinson [36]).

Most of the work is spent factoring $\mathcal{M}^{\nu+1}$. For full matrices $\vec{f}_{\vec{u}}$ Gaussian elimination with partial pivoting takes roughly $\frac{11}{6}\,n^3$

operations (treating $\mathcal{M}^{\nu+1}$ as a banded matrix with band width 2n+1). For banded $\vec{f}_{\vec{u}}$ with band width d we can reorder the rows and columns of $\mathcal{M}^{\nu+1}$ to form a matrix $\tilde{\mathcal{M}}^{\nu+1}$ with band width 2d-1 (the diagonal elements $1,2,\cdots,2n$ are rearranged as $1, n+1, 2, n+2, \cdots, n, 2n$). Now $\tilde{\mathcal{M}}^{\nu+1}$ can be factored in $\frac{1}{2}\, 2n(2d-1)^2 \doteq n(4d^2-4d+1)$ operations.

Another method for solving 2.4a is to calculate an LU-factorization of

$$A^{\nu+1} \equiv \left(\vec{f}_{\vec{u}}^{\,\nu+1} \vec{f}_{\vec{u}}^{\,\nu+1} + (\tilde{\omega}^{\nu+1})^2 I_{n\times n} \right), \tag{2.6}$$

and then solve

$$\text{2.7} \quad \begin{aligned} &\text{a)} \quad A^{\nu+1} \vec{\xi}_{k+1} = \vec{f}_{\vec{u}}^{\,\nu+1} \vec{a}_k + \tilde{\omega}^{\nu+1} \vec{b}_k , \\ &\text{b)} \quad \tilde{\omega}^{\nu+1} \vec{\eta}_{k+1} = \vec{b}_k - \vec{f}_{\vec{u}}^{\,\nu+1} \vec{\xi}_{k+1} . \end{aligned}$$

This is equivalent to 2.4a (see Wilkinson [36]). Roughly $\frac{4}{3} n^3$ operations are needed for this method when $\vec{f}_{\vec{u}}$ is full, compared to $\frac{11}{6} n^3$ operations for the first method. However, this method is worse for banded $\vec{f}_{\vec{u}}$, with $5nd^2$ operations compared to $n(4d^2 - 4d + 1)$.

II.3 Kubíček's Method

In this section we will briefly consider an algorithm for solving $\mu(\lambda) = 0$. The method is due to Kubíček [19].

<u>Kubíček's Method</u>. Suppose that at the nth iteration (n = 0,1,$\cdots$) we have $\mu_n = \mu(\lambda^n)$, $\omega_n = \omega(\lambda^n)$, $\vec{u}_n = \vec{u}_0(\lambda^n)$ and a guess for λ^{n+1}. Then we proceed as follows:

1) Calculate $\vec{u}_{n+1} = \vec{u}_0(\lambda_{n+1})$ using a continuation technique (Euler-Newton, say).

2) Calculate the characteristic polynomial $p(\alpha;\lambda_{n+1})$ of $\vec{f}_{\vec{u}}(\vec{u}_0(\lambda_{n+1}),\lambda_{n+1})$.

3) Calculate μ_{n+1}, ω_{n+1} using Bairstow's method (see Wilkinson [36, p. 449]).

4) Use secant method (using $\mu_{n+1}, \mu_n, \lambda_{n+1}, \lambda_n$) to determine a new guess λ_{n+2}.

Steps 1 and 2 of Kubíček's method take $O(n^3)$ operations for for problems with full $\vec{f}_{\vec{u}}$. If Krylov's method is used to obtain the characteristic polynomial (see Section I.3) then steps 2 and 3 take roughly $\frac{4}{3}n^3$. This is comparable to the inverse iteration technique of Section II.2. However, for $n \gtrsim 10$, Krylov's method is ill-conditioned for many matrices. As we discussed in Section I.3, a more stable method for calculating $p(\alpha;\lambda_{n+1})$ takes roughly $\frac{5}{6}n^3$ single precision operations and $\frac{1}{6}n^3$ double precision operations. Since a double precision operation is roughly equivalent to three or four single precision operations, the operation count is again comparable to the inverse iteration method.

II.4 Inflation Method

The HBP $(\vec{u}_0(\lambda_c), \lambda_c)$ satisfies

4.1 $$\vec{G}_0(\vec{u}, \vec{a}, \vec{b}, \omega, \lambda) \equiv \begin{pmatrix} \vec{f}(\vec{u}, \lambda) \\ \vec{f}_{\vec{u}}\vec{a} + \omega\vec{b} \\ \vec{f}_{\vec{u}}\vec{b} - \omega\vec{a} \\ \vec{N}(\vec{a}, \vec{b}) \end{pmatrix} = \vec{0}.$$

Where $\vec{u} = \vec{u}_0(\lambda_c)$, $\lambda = \lambda_c$, $i\omega = \alpha_+(\lambda_c) = i\omega(\lambda_c)$, and $\vec{a} + i\vec{b}$ is an eigenvector of $\vec{f}_{\vec{u}} \equiv \vec{f}_{\vec{u}}(\vec{u}, \lambda)$ with eigenvalue $i\omega(\lambda_c)$. Here $\vec{N}(\vec{a}, \vec{b}) = \vec{0}$ represents two normalization conditions on the eigenvector $\vec{a} + i\vec{b} = \rho e^{i\theta}(\vec{a}_0 + i\vec{b}_0)$; $\vec{N}(\vec{a}, \vec{b}) = \vec{0}$ fixes $\rho > 0$ and $\theta \in [0, 2\pi)$. The following lemma provides conditions on $\vec{N}$ such that a HBP is an isolated root of 4.1 (i.e., $\left.\frac{\partial G}{\partial(\vec{u}, \vec{a}, \vec{b}, \omega, \lambda)}\right|_{HBP}$ is nonsingular).

<u>Lemma 4.2</u>. Let $\vec{u}_0(\lambda)$ be a steady state branch of 0.1. Let $\alpha_\pm(\lambda) = \mu(\lambda) \pm i\omega(\lambda)$ be defined as in Section II.2. Suppose $\vec{z}_c \equiv (\vec{u}_c, \vec{a}_c, \vec{b}_c, \omega_c, \lambda_c)$ is a root of 4.1 such that

a) $\vec{u}_c = \vec{u}_0(\lambda_c)$, $\vec{f}_{\vec{u}}^{\,c} \equiv \vec{f}_{\vec{u}}(\vec{u}_c, \lambda_c)$ is nonsingular,

4.2 b) $i\omega_c$ is a simple eigenvalue of $\vec{f}_{\vec{u}}^{\,c}$,

c) $\frac{d\mu}{d\lambda}(\lambda_c) \neq 0$, $\mu(\lambda_c) = 0$, $\omega(\lambda_c) = \omega_c > 0$.

(Note that if $(\vec{u}_c, \lambda_c)$ is a HBP then 4.2 follows.)

Then $\frac{\partial \vec{G}_0}{\partial \vec{z}}\Big|_{\vec{z}=\vec{z}_c}$, $\vec{z} \equiv (\vec{u}, \vec{a}, \vec{b}, \omega, \lambda)$, is nonsingular iff the 2 2 matrix

4.3 $$\left(\frac{\partial \vec{N}}{\partial \vec{a}} \vec{a} + \frac{\partial \vec{N}}{\partial \vec{b}} \vec{b} \;\vdots\; \frac{\partial \vec{N}}{\partial \vec{a}} \vec{b} - \frac{\partial \vec{N}}{\partial \vec{b}} \vec{a} \right)\Bigg|_{(\vec{a},\vec{b})=(\vec{a}_c,\vec{b}_c)}$$

is nonsingular and $\vec{N}(\vec{0},\vec{0}) \neq \vec{0}$.

An example of a normalization $\vec{N}(\vec{a},\vec{b})$ that satisfies the conditions of Lemma 4.2 is

4.4 $$\vec{N}_1(\vec{a},\vec{b}) \equiv \begin{pmatrix} \vec{a}^T\vec{a} + \vec{b}^T\vec{b} - 1 \\ \vec{\ell}^T\vec{b} \end{pmatrix} .$$

Where $\vec{\ell} \in \mathbb{R}^n$, $\vec{\ell}$ is not perpendicular to the plane spanned by $\vec{a}_0$, $\vec{b}_0$; $\vec{f}^c_{\vec{u}}(\vec{a}_0+i\vec{b}_0) = i\omega_c(\vec{a}_0+i\vec{b}_0)$, and $\vec{a}_0$, $\vec{b}_0 \neq 0$.

<u>Proof of Lemma 4.2</u>. Let

4.5 $$A(\lambda) = \left(\begin{array}{c|c} \vec{f}_{\vec{u}}(\vec{u}_0(\lambda),\lambda) - \mu(\lambda) I_{n\times n} & \omega(\lambda) I_{n\times n} \\ -\omega(\lambda) I_{n\times n} & \vec{f}_{\vec{u}}(\vec{u}_0(\lambda),\lambda) - \mu(\lambda) I_{n\times n} \end{array} \right)$$

Since $i\omega_c$ is a simple eigenvalue of $\vec{f}^c_{\vec{u}}$, $\alpha_+(\lambda)$ is a simple eigenvalue of $\vec{f}_{\vec{u}}(\vec{u}_0(\lambda),\lambda)$ for λ in a neighborhood of λ_c. This implies that $A^k(\lambda)$ has a two-dimensional null space for k = 1,2, ... Therefore there exist $\vec{r}_+(\lambda)$, $\vec{r}_-(\lambda)$, $\vec{\ell}_+(\lambda)$, and $\vec{\ell}_-(\lambda)$ satisfying

a) $\vec{\ell}_{\pm}^{T}(\lambda) A(\lambda) = \vec{0}$; $\vec{\ell}_{+}(\lambda) = \begin{pmatrix} \vec{c}(\lambda) \\ \vec{d}(\lambda) \end{pmatrix}$; $\vec{c}(\lambda), \vec{d}(\lambda) \in \mathbb{R}^{n}$;

b) $A(\lambda) \vec{r}_{\pm}(\lambda) = \vec{0}$; $\vec{r}_{+}(\lambda) = \begin{pmatrix} \vec{a}(\lambda) \\ \vec{b}(\lambda) \end{pmatrix}$, $\vec{r}_{-}(\lambda) = \begin{pmatrix} \vec{b}(\lambda) \\ -\vec{a}(\lambda) \end{pmatrix} \in \mathbb{R}^{2n}$;

4.6 c) $\vec{a}(\lambda_c) = \vec{a}_c$, $\vec{b}(\lambda_c) = \vec{b}_c$;

d) $\begin{pmatrix} \vec{\ell}_{+}^{T}(\lambda) \\ \vec{\ell}_{-}^{T}(\lambda) \end{pmatrix} \left(\vec{r}_{+}(\lambda) \;\vdots\; \vec{r}_{-}(\lambda) \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

for λ in a nbhd. of λ_c.

Now by 4.1

4.7
$$\frac{\partial G_0^c}{\partial \vec{z}} \equiv \left. \frac{\partial G_0}{\partial \vec{z}} \right|_{\vec{z}=\vec{z}_c} = \left(\begin{array}{c:cc:c} \vec{f}_{\vec{u}}^{\,c} & & 0 & \vec{f}_{\lambda}^{\,c} \\ \hdashline \vec{f}_{\vec{u}\vec{u}}^{\,c} \vec{a}_c & & & \vec{f}_{\vec{u}\lambda}^{\,c} \vec{a}_c \\ \vec{f}_{\vec{u}\vec{u}}^{\,c} \vec{b}_c & & A^c & \vec{f}_{\vec{u}\lambda}^{\,c} \vec{b}_c \\ \hdashline 0 & \vec{N}_{\vec{a}}^{\,c} & \vec{N}_{\vec{b}}^{\,c} & \vec{0} \end{array} \right).$$

Here the superscript "c" means that the labeled function is evaluated at $\vec{z} = \vec{z}_c$.

Suppose $\delta\vec{z} \equiv (\delta\vec{u}, \delta\vec{a}, \delta\vec{b}, \delta\omega, \delta\lambda)$ is a right null vector of $\frac{\partial \vec{G}_0^c}{\partial \vec{z}}$, Then by 4.7

$$\text{a)}\ \delta\vec{u} = \delta\lambda\,\delta\vec{u}_1 \quad , \quad \delta\vec{u}_1 \equiv \left(\vec{f}^{\,c}_{\vec{u}}\right)^{-1}\left(-\vec{f}^{\,c}_{\lambda}\right) ,$$

4.8 $$\text{b)}\ A(\lambda_c)\begin{pmatrix}\delta\vec{a}\\ \delta\vec{b}\end{pmatrix} = +\delta\omega\begin{pmatrix}\vec{b}_c\\ -\vec{a}_c\end{pmatrix} - \delta\lambda\vec{\xi} \ , \quad \vec{\xi} \equiv \begin{pmatrix}\vec{f}^{\,c}_{\vec{u}\vec{u}}\vec{a}_c\delta\vec{u}_1 + \vec{f}^{\,c}_{\vec{u}\lambda}\vec{a}_c\\ \vec{f}^{\,c}_{\vec{u}\vec{u}}\vec{b}_c\delta\vec{u}_1 + \vec{f}^{\,c}_{\vec{u}\lambda}\vec{b}_c\end{pmatrix}$$

$$\text{c)}\ \vec{N}^{\,c}_{\vec{a}}\,\delta\vec{a} + \vec{N}^{\,c}_{\vec{b}}\,\delta\vec{b} = \vec{0}.$$

Therefore, from 4.8b and the Fredholm alternative, $\delta\omega, \delta\lambda$ must satisfy

4.9 $$\delta\omega\,\vec{\ell}^{\,T}_{\pm}(\lambda_c)\begin{pmatrix}\vec{b}_c\\ -\vec{a}_c\end{pmatrix} - \delta\lambda\,\vec{\ell}^{\,T}_{\pm}(\lambda_c)\vec{\xi} = 0 .$$

We simplify 4.9 as follows. By 4.6 we see that

$$\left.\frac{d\mu}{d\lambda}\right|_{\lambda=\lambda_c} = \left.\frac{1}{2}\frac{d}{d\lambda}\left\{\vec{\ell}^{\,T}_{+}(\lambda)\begin{pmatrix}\vec{f}_{\vec{u}}(u_0(\lambda),\lambda) & \omega(\lambda)I_{n\times n}\\ -\omega(\lambda)I_{n\times n} & \vec{f}_{\vec{u}}(\vec{u}_0(\lambda),\lambda)\end{pmatrix}\vec{r}_{+}(\lambda)\right\}\right|_{\lambda=\lambda_c},$$

$$= \frac{1}{2}\vec{\ell}^{\,T}_{+}(\lambda_c)\begin{pmatrix}\vec{f}^{\,c}_{\vec{u}\lambda} & 0\\ 0 & \vec{f}^{\,c}_{\vec{u}\lambda}\end{pmatrix}\vec{r}_{+}(\lambda_c) + \frac{1}{2}\frac{d\omega}{d\lambda}(\lambda_c)\,\vec{\ell}^{\,T}_{+}(\lambda_c)\,\vec{r}^{\,T}_{-}(\lambda_c),$$

4.10 $$= \frac{1}{2}\vec{\ell}^{\,T}_{+}(\lambda_c)\vec{\xi}.$$

Therefore, 4.9 becomes

4.11 $$\begin{pmatrix}0 & -2\frac{d\mu}{d\lambda}(\lambda_c)\\ 1 & x\end{pmatrix}\begin{pmatrix}\delta\omega\\ \delta\lambda\end{pmatrix} = \vec{0}$$

for some $x \in \mathbb{R}$. However, $\frac{d\mu}{d\lambda}(\lambda_c) \neq 0$ by 4.2c, therefore 4.11 implies $\delta\omega = \delta\lambda = 0$.

Setting $\delta\omega = \delta\lambda = 0$ in 4.8b we see

4.12 $$\begin{pmatrix} \delta\vec{a} \\ \delta\vec{b} \end{pmatrix} = x_+ \vec{r}_+(\lambda_c) + x_- \vec{r}_-(\lambda_c) ,$$

for some $x_+, x_- \in \mathbb{R}$. Substituting 4.12 into 4.8c we obtain

4.13 $$\left(\vec{N}^c_{\vec{a}} \vec{a}_c + \vec{N}^c_{\vec{b}} \vec{b}_c \;\vdots\; \vec{N}^c_{\vec{a}} \vec{b}_c - \vec{N}^c_{\vec{b}} \vec{a}_c \right) \begin{pmatrix} x_+ \\ x_- \end{pmatrix} = \vec{0}.$$

The lemma follows from 4.13. ■

Next we examine initial guesses for a root of 4.1. Suppose λ_i, λ_{i+1} are as described in Section II.1 (see eqn. 1.2). Let $\vec{a}_0 + i\vec{b}_0$ satisfy

4.14
a) $$\vec{f}_{\vec{u}}(\vec{u}_0(\lambda_i), \lambda_i)(\vec{a}_0 + i\vec{b}_0) = \alpha_+(\lambda_i)(\vec{a}_0 + i\vec{b}_0),$$

b) $$\vec{N}(\vec{a}_0, \vec{b}_0) = \vec{0}.$$

Here 4.14b sets the magnitude and phase of $\vec{a}_0 + i\vec{b}_0$. Then the initial guess $\vec{z}_0 = (\vec{u}_0(\lambda_i), \vec{a}_0, \vec{b}_0, \omega(\lambda_i), \lambda_i)$ satisfies

4.15 $$\left| \vec{G}_0(z_0) \right| = \left| \mu(\lambda_i) \begin{pmatrix} \vec{0} \\ \vec{a}_0 \\ \vec{b}_0 \\ \vec{0} \end{pmatrix} \right| = O(\lambda_i - \lambda_c)$$

as $\lambda_i \to \lambda_c$. The following lemma now follows easily from the Kantorovich theorem (Ortega and Rheinboldt [24]).

Lemma 4.16. Suppose $\vec{u}_0(\lambda_c), \lambda_c$ is a HBP and $\vec{N}(\vec{a},\vec{b})$ satisfies the conditions of Lemma 4.2. Consider Newton's method applied to 4.1 with the initial guess $\vec{z} = \vec{z}_0$ given above. Then for $|\lambda_i - \lambda_c|$ sufficiently small, the Newton iterates converge to $\vec{z} = \vec{z}_c = (\vec{u}_0(\lambda_c), \vec{a}_c, \vec{b}_c, \omega_c, \lambda_c)$.

Finally, we should examine the amount of work it takes to do one Newton step on 4.1. From 4.7 we see that the Jacobian depends on $\vec{f}_{\vec{u}\vec{u}}\vec{a}$, $\vec{f}_{\vec{u}\vec{u}}\vec{b}$, $\vec{f}_{\vec{u}\lambda}\vec{a}$, $\vec{f}_{\vec{u}\lambda}\vec{b}$. These quantities will be costly to obtain exactly for many practical problems. We might get around this by using difference approximations for $\vec{f}_{\vec{u}\vec{u}}\,\vec{a}$, etc. Another costly operation is calculating a factorization of $\frac{\partial \vec{G}}{\partial \vec{z}}$. Actually $\frac{\partial \vec{G}}{\partial \vec{z}}$ can be factored in roughly the same number of operations as one step of the inverse iteration method given in Section II.2 (see Chapter III).

In the next chapter we consider a very useful generalization of the method given above.

CHAPTER III. Continuation Methods for Locating a HBP

III.1 Introduction

In this chapter we consider an alternative method for locating HBPs. We suppose that 0.1 is imbedded in the system

1.1 $$\vec{w}_t = \vec{g}(\vec{w}, \lambda; \gamma).$$

Here $\vec{g}(\vec{w},\lambda;0) = \vec{f}(\vec{w},\lambda)$, and for $\gamma = 1$ a HBP, $(\vec{w}^1,\lambda^1)$, of 1.1 is known. We consider continuation techniques for following a path of candidate HBPs $(\vec{w}_c(\gamma),\lambda_c(\gamma))$ from $\gamma = 1$ to (hopefully) $\gamma = 0$.

Any of the local HBP finding methods given in Chapter II can be used in a suitable continuation method. Kubíček [19] has used the method given in Section II.3. Here we will only consider the method given in Section II.4.

In particular, we consider

1.2 $$\vec{G}(\vec{z}, \gamma) = \vec{0},$$

where

1.3 a) $$\vec{G}(\vec{z},\gamma) \equiv \begin{pmatrix} \vec{g}(\vec{w},\lambda;\gamma) \\ \vec{g}_{\vec{w}}(\vec{w},\lambda;\gamma)\vec{a} + \omega\vec{b} \\ \vec{g}_{\vec{w}}(\vec{w},\lambda;\gamma)\vec{b} - \omega\vec{a} \\ \vec{N}(\vec{a},\vec{b}) \end{pmatrix},$$

b) $$\vec{z} \equiv (\vec{w}, \vec{a}, \vec{b}, \omega, \lambda).$$

Also, we know $\vec{z}^1 = (\vec{w}^1, \vec{a}^1, \vec{b}^1, \omega^1, \lambda^1)$ a root of 1.2 for $\gamma = 1$. Many continuation methods can be applied to 1.2 (see Perozzi [25]) in an attempt to find a root for $\gamma = 0$. We will only consider Euler-Newton or Euler-Chord methods here.

III.2 Examples of Continuation Imbeddings

In this section we will consider several example imbeddings of the form 1.1.

Example 2.1. Consider 0.1 with a steady state branch $\vec{u}_0(\lambda)$. Suppose $\alpha_\pm(\lambda) = \mu(\lambda) \pm i\omega(\lambda)$ are eigenvalues of $\vec{f}_{\vec{u}}(\vec{u}_0(\lambda), \lambda)$ such that for some λ_c, $\mu(\lambda_c) = 0$. Suppose that $\vec{u}_0(\lambda_1), \alpha_\pm(\lambda_1)$, and the associated eigenvectors $\vec{a}^1 \pm i\, \vec{b}^1$ of $\vec{f}^1_{\vec{u}}$ are known, with $\mu(\lambda_1) \neq 0$. Then consider

2.1 $$\vec{g}(\vec{w}, \lambda; \gamma) = \vec{f}(\vec{w}, \lambda) - \gamma\mu(\lambda_1)(\vec{w} - \vec{u}_0(\lambda)).$$

With $\vec{g}$ as in 2.1, $\vec{G}(\vec{z}, \gamma)$ becomes

2.2 $$G(\vec{z}, \gamma) = \begin{pmatrix} \vec{f}(\vec{w}, \gamma) - \gamma\mu(\lambda_1)(\vec{w} - \vec{u}_0(\lambda)) \\ (\vec{f}_{\vec{w}}(\vec{w}, \lambda) - \gamma\mu(\lambda_1) I_{n\times n})\vec{a} + \omega\vec{b} \\ (\vec{f}_{\vec{w}}(\vec{w}, \lambda) - \gamma\mu(\lambda_1) I_{n\times n})\vec{b} - \omega\vec{a} \\ \vec{N}(\vec{a}, \vec{b}) \end{pmatrix}$$

So $\vec{z} = (\vec{u}_0(\lambda), \vec{a}, \vec{b}, \omega, \lambda)$ is a solution of 1.2 if $\gamma\mu(\lambda_1) \pm i\omega$ is an eigenvalue of $\vec{f}_{\vec{u}}(\vec{u}_0(\lambda), \lambda)$ with eigenvectors $\vec{a} \pm i\vec{b}$ (here we are assuming $\vec{N}(\vec{a}, \vec{b})$ is suitable, see Section II.4). Continuing the solution of 1.2 from $\gamma = 1$ to $\gamma = 0$ is therefore equivalent to following the eigenvalues $\alpha_{\pm}(\lambda)$, attempting to drive $\mu(\lambda)$ to zero.

This illustrates the main difficulty with the imbedding 2.1, the pair of eigenvalues $\alpha_{\pm}(\lambda)$ that eventually cross the imaginary axis must be known at $\lambda = \lambda_1$. The $\alpha_{\pm}(\lambda)$ could be calculated using the detection methods of Chapter I.

Another difficulty with 2.1 is that there may be extraneous steady state branches bifurcating from $\vec{u}_0(\lambda)$. (Note that $\vec{g}_{\vec{w}}(\vec{w}, \lambda; \gamma)$ is singular if $\vec{f}_{\vec{w}}(\vec{w}, \lambda)$ has the real eigenvalue $\gamma\mu(\lambda_1)$.) This difficulty is easily avoided by taking care at the bifurcation points of steady state branches of 1.1 (see Keller [14]).

Example 2.3. Many physical problems contain auxiliary parameters that can serve as a suitable continuation parameter γ. For example, the equations of motion of Watts' centrifugal governor are (see Pontryagin [26], or Hassard [9])

$$2.3 \qquad \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3^2 \sin(x_1)\cos(x_1) - \sin(x_1) - \nu x_2 \\ \gamma(\cos(x_1) - \rho) \end{pmatrix} = \vec{f}(\vec{x}, \vec{\gamma}),$$

where $\vec{\gamma} \equiv (\nu, \rho, \mathcal{H}) \in \mathbb{R}^3$. In example computations we used continuation on 1.2 with $\lambda = \rho$ and $\gamma \hat{=} \nu$ or $\mathcal{H}$ (with $\mathcal{H}$ or ν fixed, respectively). (B. Hassard [9] has also done similar computations using the inverse iteration method outlined in Section II.2.)

To start the continuation we need to know a HBP for 2.3. These can be obtained analytically for 2.3 (see [9]), however, we used an imbedding of the form described in Example 2.1. (Notice that $\vec{f}_{\vec{x}}$ has at most one complex conjugate pair of eigenvalues, so there was no problem with choosing $\alpha_{\pm}(\lambda_1)$.)

Example 2.4. It is often the case that asymptotic solutions can be obtained for 1.1 as some auxiliary parameter goes to zero. For example, consider the following reaction-diffusion equations modeling a simple, first order, exothermic reaction in a nonadiabatic tubular reactor (see A.B. Poore [27], [28], also see H. Hlavacek and H. Hofmann [11]):

2.4

$$\text{a)}\quad T_t = \frac{1}{\varepsilon} T_{xx} - T_x - \beta(T - T_c) + DB(1-C)\exp\left(\frac{T}{1+\frac{1}{\gamma_H}T}\right),$$

$$\text{b)}\quad C_t = \frac{1}{\varepsilon} C_{xx} - C_x + D(1-C)\exp\left(\frac{T}{1+\frac{1}{\gamma_H}T}\right),$$

$$\text{c)}\quad T_x(0,t) = \varepsilon T(0,t), \qquad C_x(0,t) = \varepsilon C(0,t),$$
$$T_x(1,t) = 0, \qquad Cx(1,t) = 0,$$

$$\text{d)}\quad T(x,0) = \phi(x), \qquad C(x,0) = \psi(x).$$

Here $t \geq 0$ and $x \in [0,1]$. Poore [27], [28] has calculated the location of a HBP for 2.4 with $0 < \varepsilon \ll 1$. Numerical calculations using 1.2 and 1.3 with $\gamma = \varepsilon$ are now in progress. The continuation procedure will provide HBPs of 2.4 for ε large, presumably up to $\varepsilon = O(1)$.

Example 2.5. We seek a root of

2.5 $$G_0(\vec{u}, \vec{a}, \vec{b}, \omega, \lambda) = 0$$

where G_0 is defined by II 4.1. The method of global homotopies (see H. B. Keller [16]) can be applied to imbed 2.5 in a system of the form 1.3. Smale's Boundary Conditions (see [16], Smale [32]) provide criteria on $\vec{f}(\vec{u},\lambda)$ such that the continuation method is almost guaranteed to find a root of 2.5. (Unfortunately, Smale's Boundary Conditions are not easily verified for many practical $\vec{f}$'s. But this is an important and useful area for further work.)

The above examples show the versatility of continuation methods using 1.3. In the following sections we consider some of the details of this technique.

III.3 Singularities in the Continuation Path

Singular points $(\vec{z}_0, \gamma_0)$ on a solution path of 1.2 are defined by

$$\text{a)} \quad \vec{G}(\vec{z}_0, \gamma_0) = 0,$$

$$\text{b)} \quad \vec{G}^0_{\vec{z}} \text{ is singular, where } \vec{G}^0_{\vec{z}} \equiv \vec{G}_{\vec{z}}(\vec{z}_0, \vec{\gamma}_0)$$

3.1
$$= \begin{pmatrix} \vec{g}^0_{\vec{\omega}} & 0 & 0 & & \vec{g}^0_{\lambda} \\ \vec{g}_{\vec{\omega}\vec{\omega}}\vec{a}_0 & \vec{g}^0_{\vec{\omega}} & \omega_0 I_{n\times n} & b_0 & \vec{g}^0_{\vec{\omega}\lambda}\vec{a}_0 \\ \vec{g}_{\vec{\omega}\vec{\omega}}\vec{b}_0 & -\omega_0 I_{n\times n} & \vec{g}^0_{\vec{\omega}} & -a_0 & \vec{g}^0_{\vec{\omega}\lambda}\vec{b}_0 \\ 0 & \vec{N}^0_{\vec{a}} & \vec{N}^0_{\vec{b}} & \vec{0} & \vec{0} \end{pmatrix}$$

We call $(\vec{z}_0, \gamma_0)$ a <u>normal limit point</u> if in addition to 3.1 we have

$$\text{a)} \quad \dim \mathcal{N}(\vec{G}^0_{\vec{z}}) = \operatorname{codim} \mathcal{R}(\vec{G}^0_{\vec{z}}) = 1,$$

3.2

$$\text{b)} \quad \vec{G}^0_{\gamma} \notin \mathcal{R}(\vec{G}^0_{\vec{z}}).$$

Also, necessary conditions for $(\vec{z}_0, \gamma_0)$ to be a <u>simple bifurcation point</u> of a solution branch of 1.2 are

a) $(\vec{z}_0, \gamma_0)$ satisfies 3.1 and 3.2a,

3.3

$$\text{b)} \quad \vec{G}^0_{\gamma} \in \mathcal{R}(\vec{G}^0_{\vec{z}})$$

(see Keller [14]).

Keller [14] has developed powerful numerical techniques for dealing with singular points. The key idea is to parameterize the solution branch by an approximation to arclength along the branch. This is done by solving

3.4
a) $\vec{G}(\vec{z}, \gamma) = \vec{0}$,

b) $L(\vec{z}, \gamma, s) = 0$,

where $s \in \mathbb{R}$ is the independent parameter on the solution branch. Keller [14] suggests the following choice of L for $s_0 \leq s < s_1$:

3.5
$$L(\vec{z}, \gamma, s) \equiv \Theta \dot{\vec{z}}^*(s_0)\{\vec{z}(s) - \vec{z}(s_0)\} + (1-\Theta)\dot{\gamma}(s_0)\{\gamma(s) - \gamma(s_0)\} - (s - s_0).$$

Here $(\dot{\vec{z}}(s_0), \dot{\gamma}(s_0)) = \frac{d}{ds}(\vec{z}(s), \gamma(s))\Big|_{s=s_0}$, $\Theta \in (0,1)$, and $\dot{\vec{z}}^*(s_0)$ is the dual element to $\dot{\vec{z}}(s_0)$ (i.e., $\dot{\vec{z}}^*(s_0) = \overline{[\dot{\vec{z}}(s_0)]}^T$ for $\dot{\vec{z}}(s_0) \in \mathbb{C}^n$ with the Euclidian norm). Notice that

3.6
a) $$\mathcal{Q}(s)\begin{pmatrix} \dot{\vec{z}}(s) \\ \dot{\gamma}(s) \end{pmatrix} = -\begin{pmatrix} \vec{0} \\ \frac{\partial L}{\partial s} \end{pmatrix} = \begin{pmatrix} \vec{0} \\ 1 \end{pmatrix},$$

b) $$\mathcal{Q}(s) \equiv \begin{pmatrix} \frac{\partial \vec{G}}{\partial \vec{z}} & \frac{\partial \vec{G}}{\partial \gamma} \\ \frac{\partial L}{\partial \vec{z}} & \frac{\partial L}{\partial \gamma} \end{pmatrix}\Bigg|_{(\vec{z}(s), \gamma(s), s)}.$$

Therefore, for L as in 3.5, $(\dot{\vec{z}}(s_0), \dot{\gamma}(s_0))$ is uniquely defined at regular points ($\vec{G}_{\vec{z}}$ nonsingular) and at normal limit points. The normalization 3.5 is called pseudo-arclength (see Keller [14] for details).

The advantage of using 3.4 is that at both regular points and normal limit points of 1.2, $\mathcal{A}(s)$ is nonsingular. That is, $(\vec{z}(s), \gamma(s), s)$ is a regular point of

3.7 $$\vec{P}(\vec{z}, \gamma, s) \equiv \begin{pmatrix} G(\vec{z}, \gamma) \\ L(\vec{z}, \gamma, s) \end{pmatrix} = \vec{0},$$

using s as the parameter. Using 3.4 also makes it relatively easy to switch branches at simple bifurcation points (see Keller [14]).

III.4 LU-Factorization of G_z and $\mathcal{A}(s)$

The matrices $\vec{G}_{\vec{z}}$ and $\mathcal{A}(s)$ (see Section 3) have the general form

4.1 $$\mathcal{M} = \begin{array}{c} \begin{array}{cccc} \overbrace{}^{n} & \overbrace{}^{n} & \overbrace{}^{n} & \overbrace{}^{k} \end{array} \\ \begin{pmatrix} A_1 & 0 & 0 & A_4 \\ B_1 & B_2 & \omega I & B_4 \\ C_1 & -\omega I & B_2 & C_4 \\ D_1 & D_2 & D_3 & D_4 \end{pmatrix} \end{array} \begin{array}{l} \\ \} \; n \\ \} \; n \\ \} \; n \\ \} \; k \end{array}$$

with k = 2 for $\vec{G}_{\vec{z}}$, k = 3 for $\mathcal{A}(s)$. In order to do a continuation step we will need to solve

$$\mathcal{M}\vec{x} = \vec{d} \tag{4.2}$$

for $\vec{x}$. In this section we consider two techniques for block LU-factoring the matrix $\mathcal{M}$.

Notice that 1.3a and 3.1b imply

$$\dim \mathcal{N}\left\{\begin{pmatrix} B_2 & \omega I \\ -\omega I & B_2 \end{pmatrix}\right\} \geq 2 \tag{4.3}$$

along the continuation path. Also, it is possible for A_1 and/or $\mathcal{M}$ to be singular. The factorization schemes discussed below are designed to handle these singularities.

We begin by considering an algorithm for the case in which A_1, B_1, B_2, C_1 are full $n \times n$ matrices. From the estimates in Section II.2 we are led to consider

$$\begin{aligned} \tilde{\mathcal{M}} &= \begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & B_2 & -\omega I_n & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_k \end{pmatrix} \mathcal{M}, \\ &\equiv \begin{array}{cl} \begin{pmatrix} A_1 & 0 & 0 & A_4 \\ \tilde{B}_1 & \tilde{B}_2 & 0 & \tilde{B}_4 \\ B_1 & B_2 & \omega I & B_4 \\ D_1 & D_2 & D_3 & D_4 \end{pmatrix} & \begin{matrix} \}n \\ \}n \\ \}n \\ \}n \end{matrix} \end{array} . \end{aligned} \tag{4.4}$$

For some $\varepsilon > 0$ let A_1, B_2 be partially LU-factored in the following form

4.5

$$\text{a)}\quad A_1 = \begin{pmatrix} L_1 & 0 \\ L_{12} & I_{\ell_1} \end{pmatrix} \begin{pmatrix} U_1 & U_{12} \\ 0 & E_1 \end{pmatrix}, \quad \|E_1\|_\infty < \varepsilon,$$

$$\text{b)}\quad \widetilde{B}_2 = \begin{pmatrix} L_2 & 0 \\ L_{22} & I_{\ell_2} \end{pmatrix} \begin{pmatrix} U_2 & U_{22} \\ 0 & E_2 \end{pmatrix}, \quad \|E_2\|_\infty < \varepsilon,$$

where E_i is $\ell_i \times \ell_i$ for $i = 1,2$. Then we factor $\widetilde{M}$ in the following form

$$\text{a)}\quad \widetilde{M} = \widetilde{P}^{-1} \begin{pmatrix} L_1U_1 & 0 & 0 & 0 \\ B_1 & L_2U_2 & 0 & 0 \\ B_2 & B_3 & \omega I & 0 \\ B_4 & B_5 & B_6 & L_4U_4 \end{pmatrix} \begin{pmatrix} I_{n-\ell_1} & 0 & 0 & C_1 \\ 0 & I_{n-\ell_2} & 0 & C_2 \\ 0 & 0 & I_n & C_3 \\ 0 & 0 & 0 & I_{\ell_1+\ell_2+k} \end{pmatrix} \widetilde{P},$$

4.6

$$\text{b)}\quad \widetilde{P} = \begin{pmatrix} I_{n-\ell_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n-\ell_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & I_{\ell_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\ell_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_k \end{pmatrix},$$

$$\text{c)}\quad L_4U_4 = E_4 \equiv \begin{pmatrix} E_1 & 0 & A_{42} \\ \widetilde{B}_{122} & E_2 & \widetilde{B}_{42} \\ D_{12} & D_{22} & D_4 \end{pmatrix} - \sum_{i=1}^{3} B_{3+i} C_i .$$

Where

4.6 d)

$$A_4 \equiv \begin{pmatrix} A_{41} \\ A_{42} \end{pmatrix} \begin{matrix} \}\, n-\ell_1 \\ \}\, \ell_1 \end{matrix}, \quad \widetilde{B}_4 \equiv \begin{pmatrix} \widetilde{B}_{41} \\ \widetilde{B}_{42} \end{pmatrix} \begin{matrix} \}\, n-\ell_2 \\ \}\, \ell_2 \end{matrix}, \quad \widetilde{B}_1 \equiv \begin{array}{c} \begin{matrix} \overbrace{\quad}^{n-\ell_1} & \overbrace{\quad}^{\ell_1} \end{matrix} \\ \begin{pmatrix} \widetilde{B}_{111} & \widetilde{B}_{112} \\ \widetilde{B}_{121} & \widetilde{B}_{122} \end{pmatrix} \end{array} \begin{matrix} \}\, n-\ell_1 \\ \}\, \ell_1 \end{matrix},$$

$$D_i \equiv (\overbrace{D_{i1}}^{n-\ell_i} \;\; \overbrace{D_{i2}}^{\ell_i}) \quad \text{for } i = 1, 2 .$$

Here the $\mathcal{B}_j$ and $\mathcal{C}_j$ satisfy relations easily obtained from 4.4 and 4.6a,b.

In the above factorization we avoid small pivots by interchanging the diagonal elements of A_1, $\tilde{B}_2$ so that small elements end up in E_1, E_2 respectively. Then E_1, E_2, D_2 are combined in E_4 (see 4.6c) and E_4 is factored using full pivoting $(\ell_1 + \ell_2 + k \leq 5$ or 6 usually$)$. Note that most of the computations are involved in calculating $\tilde{B}_2$ and in factoring A_1 and $\tilde{B}_2$, i.e., $O(\frac{5}{3} n^3)$ operations (see Section II.2).

Next we will consider factoring $\tilde{\mathcal{M}}$ for cases in which A_1, B_1, B_2, C_1 are $n \times n$ matrices with band width $d = 2m+1$. As in Section II.2, we permute the rows and columns of $\mathcal{M}$ so that the diagonal elements $1, 2, 3, \cdots, 3n+k$ are permuted to $n+1, 1, 2n+1, n+2, 2, 2n+2, \cdots, 2n, n, 3n, 3n+1, \cdots, 3n+k$. This permutation puts $\mathcal{M}$ into the form

$$4.7 \qquad \tilde{\mathcal{M}} = \begin{array}{cc} \overbrace{\phantom{\tilde{\mathcal{B}}}}^{3n} & \overbrace{\phantom{\tilde{\mathcal{C}}}}^{k} \end{array} \begin{pmatrix} \tilde{\mathcal{B}} & \tilde{\mathcal{C}} \\ \tilde{\mathcal{D}} & \tilde{\mathcal{E}} \end{pmatrix} \begin{matrix} \}\, 3n \\ \}\, k \end{matrix}$$

where $\tilde{B}$ has bandwidth 3d. We can factor $\tilde{B}$ using partial pivoting with an $\mathcal{E}$-cutoff (i.e., we never use a pivot less than ε in magnitude. Then we combine the rows and columns of B having small pivots with $\tilde{E}$, and factor the resulting small matrix (see 4.6c). The operation count is $O(\frac{9}{2} nd^2)$.

III.5 Nonlinear Modification

For many problems the computation of $\vec{g}_{\vec{w}\vec{w}}\vec{a}$, $\vec{g}_{\vec{w}\lambda}\vec{a}$, etc., in $\vec{G}_{\vec{z}}$ (see 3.1b) is expensive. We could avoid the direct computation by using difference approximations in $\vec{G}_{\vec{z}}$ (i.e., in place of $\vec{g}_{\vec{w}\vec{w}}(\vec{u},\vec{a},\vec{b},\omega,\lambda,\gamma)a$ we could use $\frac{1}{\varepsilon}[\vec{g}_{\vec{w}}(\vec{u}+\varepsilon\vec{a},\vec{a},\vec{b},\cdots,\gamma)$ $-\vec{g}_{\vec{w}}(\vec{u},\vec{a},\cdots,\gamma)$ Alternatively, we can use the difference approximations in the definition of $\vec{G}(\vec{z},\gamma)$. That is, define

$$
\text{5.1} \qquad \vec{F}(\vec{z},\gamma;\varepsilon) \equiv \begin{cases} \vec{G}(\vec{z},\gamma) & \text{for } \varepsilon = 0, \\ \begin{pmatrix} \vec{g}(\vec{\omega},\lambda;\gamma) \\ \frac{1}{\varepsilon}[\vec{g}(\vec{\omega}+\varepsilon\vec{a},\lambda;\gamma)-\vec{g}(\vec{\omega},\lambda;\gamma)]+\omega\vec{b} \\ \frac{1}{\varepsilon}[\vec{g}(\vec{\omega}+\varepsilon\vec{b},\lambda;\gamma)-\vec{g}(\vec{\omega},\lambda;\gamma)]-\omega\vec{a} \\ \vec{N}(\vec{a},\vec{b}) \end{pmatrix} & \text{for } \varepsilon > 0. \end{cases}
$$

Then we attempt to use a continuation method to solve

$$
\text{5.2} \qquad \vec{F}(\vec{z},\gamma;\varepsilon) = \vec{0}
$$

for some $\varepsilon > 0$. The Jacobian $\vec{F}_{\vec{z}}$ does not require $\vec{g}_{\vec{w}\vec{w}}$ or $\vec{g}_{\vec{w}\lambda}$. Therefore, we can use Euler-Newton continuation and recover quadratic convergence to the continuation path without computing $\vec{g}_{\vec{w}\vec{w}}\vec{a}$, $\vec{g}_{\vec{w}\lambda}\vec{a}$, etc. Also, in using 5.1 we do not have to be concerned about choosing an ε such that the difference approximations $\frac{1}{\varepsilon}[\vec{g}_{\vec{\omega}}(\vec{\omega}+\varepsilon\vec{a},\ldots)-\vec{g}_{\vec{\omega}}(\vec{\omega},\ldots\gamma)]$, etc., are good approximations.

The following lemma proves the existence of a solution of 5.2 for a regular point of 3.1a and for some $\varepsilon > 0$.

Lemma 5.3. Suppose $\vec{z}_0, \gamma_0$ satisfy

a) $\vec{G}(\vec{z}_0, \gamma_0) = \vec{0}$,

5.3

b) $\vec{G}^0_{\vec{z}} \equiv \dfrac{\partial G}{\partial \vec{z}}(\vec{z}_0, \gamma_0)$ is nonsingular.

Then there exists $\varepsilon_0 > 0$ such that 5.1 has a solution $(\vec{z}(\varepsilon), \gamma_0)$ for $0 < \varepsilon \leq \varepsilon_0$, $\vec{z}(\varepsilon)$ a smooth function of ε. Also, the root $(\vec{z}(\varepsilon), \gamma_0)$ of 5.1 is unique in a nbhd. of $(\vec{z}_0, \gamma_0)$.

Proof. Apply the Implicit Function Theorem (see Spivak [33]). ∎

Example computations using 5.1 are in progress.

CHAPTER IV. The Hopf Bifurcation Parameters

IV.1 Introduction

The Hopf Bifurcation Theorem (Thm. 0.6) shows that a periodic solution of the form

$$\text{a)}\quad \vec{u}_L(t;\varepsilon)=\vec{u}_0(\lambda(\varepsilon))+\varepsilon\,\vec{y}(t;\lambda(\varepsilon),\vec{\zeta}(\varepsilon),\varepsilon),$$

1.1

$$\text{b)}\quad \vec{y}(t;\lambda,\vec{\zeta},\varepsilon)=\vec{y}(t+T(\varepsilon);\lambda,\zeta,\varepsilon),\quad |\vec{y}(t;\ldots)|=O(1),$$

bifurcates from a HBP $(\vec{u}_0(\lambda_c),\lambda_c)$. Also $\lambda, T, \vec{\zeta}, \vec{y}$ are smooth, with

$$\text{c)}\quad \lambda(\varepsilon)=\lambda_c+\varepsilon^2\lambda_2+O(\varepsilon^4),$$

1.1

$$\text{d)}\quad T(\varepsilon)=T_0+\varepsilon^2 T_2+O(\varepsilon^4),\quad T_0={2\pi}/{\omega_0},$$

as $\varepsilon \to 0$. The constants λ_2, T_2 are called Hopf bifurcation parameters. There is a third parameter, β_2, giving the $O(\varepsilon^2)$ behavior of a Floquet exponent (characteristic exponent) of the periodic solution 1.1a,b. The Floquet multiplier related to β_2 gives the stability of the periodic solution with respect to perturbations within the center manifold (see [21], or Section IV.2 for a further discussion of β_2).

Recently B. Hassard and Y. H. Wan [8] have suggested automatic techniques for calculating λ_2, T_2, β_2 (and also the related λ_4, T_4, β_4). This technique requires the evaluation of derivatives $\vec{f}_{\vec{u}\vec{u}\vec{u}}$, $\vec{f}_{\vec{u}\vec{u}}$, etc. A much simpler technique for determining λ_2, T_2 and β_2 is presented in Section IV.3.

IV.2 Two-Timing

Our method for determining the Hopf bifurcation parameters is based on the well-known two-timing perturbation method (see Nayfeh [22], or O'Malley [23]). We review this method below for a HBP $(\vec{u}_0(\lambda_0), \lambda_0)$ (see Defn. 0.5).

It is convenient to write 0.1 as

2.1

a) $$\frac{d\vec{u}}{d\tau} = T\vec{f}(\vec{u}, \lambda) \qquad \text{for} \quad \tau \in (0,1),$$

b) $$\vec{u}(0) = \vec{u}(1).$$

Here $\tau = t/T$, T is the period of the bifurcating solution of 0.1. Let $\tau^* = \varepsilon^2\tau$ be the "slow time." Consider the ansatz:

2.2

a) $$\vec{u}(\tau, \tau^*) = \vec{u}_0(\lambda(\varepsilon)) + \varepsilon\vec{U}_1(\tau,\tau^*) + \varepsilon^2\vec{U}_2(\tau,\tau^*) + \ldots;$$

b) $$\vec{U}_k(0,\tau^*) = \vec{U}_k(1,\tau^*) \quad \text{for} \quad \tau^* \geq 0, \; k = 1, 2, \ldots;$$

c) $$T = T_0 + \varepsilon^2 T_2 + \cdots, \quad T_0 = 2\pi/\omega_0;$$

d) $$\lambda = \lambda_0 + \varepsilon^2\lambda_2 + \ldots.$$

The two-timing method is now specified by rewriting 2.1a as

2.3 $$\frac{\partial \vec{u}}{\partial \tau} = T\vec{f}(\vec{u},\lambda) - \varepsilon^2 \frac{\partial \vec{u}}{\partial \tau^*} .$$

Here τ, τ^* are treated as <u>independent</u> variables.

Substituting 2.2 into 2.3, 2.1b and collecting terms of $O(\varepsilon^k)$ together we get

$$\vec{f}(\vec{u}_0(\lambda(0)),\lambda(0)) = \vec{0}$$

for k = 0. Also

2.4

a) $$L\vec{U}_k \equiv \frac{\partial \vec{U}_k}{\partial \tau} - T_0 \vec{f}^{\,0}_{\vec{u}} \vec{U}_k = \vec{R}_k(\tau, \vec{U}_1, \ldots, \vec{U}_{k-1}) \text{ for } 0<\tau<1,$$

b) $$\vec{U}_k(0,\tau^*) = \vec{U}_k(1,\tau^*),$$

for $k \geqslant 1$. Here

2.5

a) $$\vec{R}_1 = \vec{0},$$

b) $$\vec{R}_2 = \frac{1}{2} T_0 \vec{f}^{\,0}_{\vec{u}\vec{u}} \vec{U}_1 \vec{U}_1 ,$$

c) $$\vec{R}_3 = -\vec{U}_{1\tau^*} + \frac{1}{6} T_0 \vec{f}^{\,0}_{\vec{u}\vec{u}\vec{u}} \vec{U}_1 \vec{U}_1 \vec{U}_1 + T_0 \vec{f}^{\,0}_{\vec{u}\vec{u}} \vec{U}_1 \vec{U}_2 + \lambda_2 T_0 \vec{f}^{\,0}_{\vec{u}\lambda} \vec{U}_1 + T_2 \vec{f}^{\,0}_{\vec{u}} \vec{U}_1 .$$

In order for $\vec{U}_1 \not\equiv$ const we set $\lambda(0) = \lambda_0$. Then

$$2.6 \qquad \vec{U}_1(\tau,\tau^*) = A(\tau^*)\{\vec{a}\cos(2\pi\tau) - \vec{b}\sin(2\pi\tau)\}$$
$$+ B(\tau^*)\{\vec{b}\cos(2\pi\tau) + \vec{a}\sin(2\pi\tau)\}.$$

Here $\vec{f}^{\,o}_{\vec{u}}(\vec{a}+i\vec{b}) = i\omega_0(\vec{a}+i\vec{b})$ with $\omega_0|\vec{a}+i\vec{b}| > 0$ (see Defn. 0.5). Substituting 2.6 into 2.5b we get

$$2.7 \qquad \vec{R}_2 = \vec{\phi}_0(A,B) + \vec{\phi}_1(A,B)\cos(4\pi\tau) + \vec{\phi}_2(A,B)\sin(4\pi\tau).$$

Then from 2.4 we get

$$2.8 \quad \text{a)} \quad \vec{U}_2 = \vec{\eta}_0(A,B) + \vec{\eta}_1(A,B)\cos(4\pi\tau) + \vec{\eta}_2(A,B)\sin(4\pi\tau),$$

where

$$\text{b)} \quad T_0\, \vec{f}^{\,o}_{\vec{u}}\, \vec{\eta}_0 = -\vec{\phi}_0\,,$$

2.8

$$\text{c)} \quad \begin{pmatrix} T_0 \vec{f}^{\,o}_{\vec{u}} & -4\pi I_n \\ 4\pi I_n & T_0 \vec{f}^{\,o}_{\vec{u}} \end{pmatrix} \begin{pmatrix} \vec{\eta}_1 \\ \vec{\eta}_2 \end{pmatrix} = -\begin{pmatrix} \vec{\phi}_1 \\ \vec{\phi}_2 \end{pmatrix} .$$

Notice that from the properties of $\vec{f}^{\,o}_{\vec{u}}$ (see Defn. 0.5), equations 2.8b,c uniquely determine $\vec{\eta}_i$. By scaling ε appropriately we do not need to include a homogeneous solution of 2.4 in $\vec{U}_2$.

Using 2.6, 8a in 2.5c we find

$$2.9 \qquad \vec{R}_3 = -\vec{U}_{1\tau^*} + \vec{\mathcal{H}}_1(A,B,\lambda_2,T_2)\cos(4\pi\tau) + \vec{\mathcal{H}}_2(A,B,\lambda_2,T_2)\sin(4\pi\tau)$$

$$+\left\{\cos(6\pi\tau),\ \sin(6\pi\tau)\ \text{terms}\right\}.$$

For $T_0 = 2\pi/\omega_0$, $T_0\ \vec{f}^{\,0}_{\vec{u}}$ has eigenvalues $\pm\, i2\pi$, therefore the general solution of 2.4 for k = 3 is given by

$$\vec{U}_3 = \vec{\zeta}_1(\tau^*)\,\tau\cos(2\pi\tau) + \vec{\zeta}_2(\tau^*)\,\tau\sin(2\pi\tau)$$

$$+\left\{\text{bounded 1-periodic terms}\right\}.$$

For $\vec{U}_3$ to be 1-periodic we require that

$$2.10 \qquad \vec{\zeta}_i(\tau^*) = 0 \qquad \text{for} \quad i = 1, 2.$$

Using 2.4, 6 and 9, a short calculation shows that 2.10 is satisfied iff

$$2.11 \qquad \begin{pmatrix} A_{\tau^*} \\ B_{\tau^*} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \vec{c}^{\,T}\vec{\mathcal{H}}_1 - \vec{d}^{\,T}\vec{\mathcal{H}}_2 \\ \vec{d}^{\,T}\mathcal{H}_1 + \vec{c}^{\,T}\vec{\mathcal{H}}_2 \end{pmatrix} .$$

Here

$$\text{a)}\quad \left(\vec{f}^{\,0}_{\vec{u}}\right)^T (\vec{c} - i\vec{d}) = i\omega_0 (\vec{c} - i\vec{d}) \ , \quad \vec{c}, \vec{d} \in \mathbb{R}^n,$$

2.12

$$\text{b)}\quad (\vec{c}\ \ \vec{d})^T (\vec{a}\ \ \vec{b}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(see Defn. 0.5).

In radial coordinates, that is (r,θ) with

2.13 $$\begin{pmatrix} A \\ B \end{pmatrix} = r \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}$$

the "secular equations" 2.11 become (after a lengthy calculation)

2.14 $$\begin{pmatrix} r_{\tau^*} \\ \theta_{\tau^*} \end{pmatrix} = \begin{pmatrix} \{\lambda_2 T_0 \mu^0_\lambda\} r + \alpha_1 r^3 \\ \{\lambda_2 T_0 \omega^0_\lambda + T_2 \omega_0\} + \alpha_2 r^2 \end{pmatrix} .$$

Here α_1, α_2 are real constants depending on $\vec{f}^{\,0}_{\vec{u}\vec{u}\vec{u}}$ and $\vec{f}^{\,0}_{\vec{u}\vec{u}}$. The parameters λ_2, T_2 are determined by 2.14 with

$$\text{a)}\quad r_{\tau^*} = \theta_{\tau^*} = 0,$$

2.15

$$\text{b)}\quad r = 1 .$$

Here 2.15b sets the scale for ε.

Writing $r = 1+\tilde{r}$ and linearizing 2.14 gives

2.16 $$\tilde{r}_{\tau^*} = \left[\lambda_2 T_0 \mu_\lambda^0 + 3\alpha_1\right]\tilde{r} \equiv \beta_2 \tilde{r}.$$

That is, we define

2.17 $$\beta_2 = 2\alpha_1 = -2\lambda_2 T_0 \mu_\lambda^0 = -4\pi\lambda_2 \mu_\lambda^0/\omega_0 \,,$$

where we have used the defn. of λ_2. Notice that the solution of 2.16 is

$$\tilde{r}(\tau^*) = e^{\beta_2 \tau^*}\tilde{r}(0).$$

Therefore, the periodic solution for $\varepsilon > 0$ sufficiently small is stable if $\beta_2 < 0$ and unstable if $\beta_2 > 0$.

IV.3 Averaging Method

The two-timing method suggests the following numerical algorithm for determining λ_2, T_2 and β_2. We assume that $\vec{u}_0(\lambda_0)$, λ_0, ω_0, $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are known where

a) $(\vec{u}_0(\lambda_0), \lambda_0)$ is a HBP of 0.1;

3.1 b) $f^0_{\vec{u}}(\vec{a} + i\vec{b}) = i\omega_0(\vec{a} + i\vec{b})$, $\omega_0 > 0$, $|\vec{a} + i\vec{b}| = 1$;

c) $\vec{c}, \vec{d}$ satisfy 2.12.

Define

$$\text{3.2}\quad \begin{aligned} &\text{a)}\quad \vec{U}_1(\tau) = \left(\vec{a}\cos(2\pi\tau) - \vec{b}\sin(2\pi\tau) \right) \\ &\text{b)}\quad T_0 = 2\pi/\omega_0 \end{aligned}$$

Using 2.5b, 2.6, 2.7 and 3.2, we approximate $\vec{\phi}_k(1,0)$ in 2.7 by

$$\text{3.3}\quad \begin{aligned} &\text{a)}\quad \tilde{\vec{\phi}}_0(\varepsilon) = \frac{T_0}{\varepsilon^2}\int_0^1 \vec{f}(\vec{u}_0(\lambda_0) + \varepsilon\vec{U}_1(\tau), \lambda_0)\, d\tau, \\ &\text{b)}\quad \tilde{\vec{\phi}}_1(\varepsilon) = \frac{2T_0}{\varepsilon^2}\int_0^1 \cos(4\pi\tau)\, \vec{f}(\vec{u}_0(\lambda_0) + \varepsilon\vec{U}_1(\tau), \lambda_0)\, d\tau, \\ &\text{c)}\quad \tilde{\vec{\phi}}_2(\varepsilon) = \frac{2T_0}{\varepsilon^2}\int_0^1 \sin(4\pi\tau)\, \vec{f}(\vec{u}_0(\lambda_0) + \varepsilon\vec{U}_1(\tau), \lambda_0)\, d\tau, \end{aligned}$$

for some $\varepsilon > 0$. Then approximations for $\vec{\eta}_k$ (see 2.8a) can be obtained by solving (see 2.8b,c)

$$\text{3.4}\quad \begin{aligned} &\text{a)}\quad T_0 \vec{f}^{\,0}_{\vec{u}}\, \tilde{\vec{\eta}}_0(\varepsilon) = -\tilde{\vec{\phi}}_0(\varepsilon), \\ &\text{b)}\quad \begin{pmatrix} T_0 \vec{f}^{\,0}_{\vec{u}} & -4\pi I_n \\ 4\pi I_n & T_0 \vec{f}^{\,0}_{\vec{u}} \end{pmatrix} \begin{pmatrix} \tilde{\vec{\eta}}_1(\varepsilon) \\ \tilde{\vec{\eta}}_2(\varepsilon) \end{pmatrix} = -\begin{pmatrix} \tilde{\vec{\phi}}_1(\varepsilon) \\ \tilde{\vec{\phi}}_2(\varepsilon) \end{pmatrix}. \end{aligned}$$

Define

$$\text{3.5}\quad \tilde{\vec{U}}_2(\tau;\varepsilon) = \tilde{\vec{\eta}}_0(\varepsilon) + \tilde{\vec{\eta}}_1(\varepsilon)\cos(4\pi\tau) + \tilde{\vec{\eta}}_2(\varepsilon)\sin(4\pi\tau).$$

Finally, we define

3.6 $$\tilde{\vec{R}}(\tau;\lambda_2,T_2,\varepsilon)=[T_0+\varepsilon^2 T_2]\,\vec{f}\left(\vec{u}(\lambda_0)+\varepsilon\vec{U}_1(\tau)+\varepsilon^2\vec{U}_2(\tau;\varepsilon),\ \lambda_0+\varepsilon^2\lambda_2\right).$$

We approximate the secular equations 2.11 with $A_{\tau^*} = B_{\tau^*} = 0$ by

3.7 $$\vec{S}(\lambda_2,T_2;\varepsilon)\equiv \begin{pmatrix} \frac{1}{\varepsilon^3}\int_0^1\{\vec{c}^{\,T}\cos(2\pi\tau)-\vec{d}^{\,T}\sin(2\pi\tau)\}\tilde{\vec{R}}(\tau;\lambda_2,T_2,\varepsilon)\,d\tau \\ \frac{1}{\varepsilon^3}\int_0^1\{\vec{d}^{\,T}\cos(2\pi\tau)+\vec{c}^{\,T}\sin(2\pi\tau)\}\tilde{\vec{R}}(\tau;\lambda_2,T_2,\varepsilon)\,d\tau+\frac{2\pi}{\varepsilon^2} \end{pmatrix} = \vec{0}.$$

The following two lemmas provide some important estimates for $\vec{S}(\lambda_2,T_2;\varepsilon)$.

<u>Lemma 3.8</u>. For $\vec{f}$ smooth

3.8 $$\vec{S}(\lambda_2,T_2;\varepsilon)=\frac{1}{2}\begin{pmatrix}\vec{c}^{\,T}\mathcal{H}_1(1,0,\lambda_2,T_2)-\vec{d}^{\,T}\mathcal{H}_2(1,0,\lambda_2,T_2)\\ \vec{d}^{\,T}\mathcal{H}_1(1,0,\lambda_2,T_2)+\vec{c}^{\,T}\mathcal{H}_2(1,0,\lambda_2,T_2)\end{pmatrix}+O(\varepsilon^2)$$

as $\varepsilon \to 0$.

<u>Proof</u>. An easy calculation using 3.3 shows that

3.9a $$\tilde{\vec{\phi}}_k(\varepsilon) = \vec{\phi}_k(1,0) + O(\varepsilon^2)$$

for k = 0,1,2. Using 2.8b,c, 3.4, 3.9a we find

3.9b $$\tilde{\vec{\eta}}_k(\varepsilon) = \vec{\eta}_k(1,0) + O(\varepsilon^2)$$

for k = 0,1,2. Also we see by 3.6 that

3.10
$$\begin{aligned}
&\int_0^1 e^{i2\pi\tau}\tilde{\vec{R}}(\tau;\lambda_2,T_2,\varepsilon)\,d\tau \\
&= \int_0^1 e^{i2\pi\tau}\Big(\varepsilon T_0\vec{f}^{\,0}_{\vec{u}}\vec{U}_1(\tau) + \varepsilon^3\Big\{\tfrac{1}{6}T_0\vec{f}^{\,0}_{\vec{u}\vec{u}\vec{u}}\vec{U}_1\vec{U}_1\vec{U}_1 + T_0\vec{f}^{\,0}_{\vec{u}\vec{u}}\vec{U}_1\tilde{\vec{U}}_2 \\
&\qquad + T_2\vec{f}^{\,0}_{\vec{u}}\vec{U}_1(\tau) + \lambda_2\vec{f}^{\,0}_{\vec{u}\lambda}\vec{U}_1(\tau)\Big\}\Big)d\tau + O(\varepsilon^5) \\
&= -\varepsilon\pi\{\vec{b}+i\vec{a}\} + \varepsilon^3\int_0^1 e^{i2\pi\tau}\Big\{\tfrac{1}{6}T_0\vec{f}^{\,0}_{\vec{u}\vec{u}\vec{u}}\vec{U}_1\vec{U}_1\vec{U}_1 + T_0\vec{f}^{\,0}_{\vec{u}\vec{u}}\vec{U}_1\vec{U}_2 \\
&\qquad + T_2\vec{f}^{\,0}_{\vec{u}}\vec{U}_1(\tau) + \lambda_2\vec{f}^{\,0}_{\vec{u}\lambda}\vec{U}_1(\tau)\Big\}d\tau + O(\varepsilon^5).
\end{aligned}$$

The lemma follows from 3.7, 3.10, 2.5c, 2.9 and 2.11. ■

<u>Lemma 3.11</u>. Suppose $\tilde{\lambda}_2(\varepsilon), \tilde{T}_2(\varepsilon)$ satisfy

3.11 $$\vec{S}(\tilde{\lambda}_2, \tilde{T}_2; \varepsilon) = \vec{0}$$

for $\varepsilon > 0$ sufficiently small. Then

$$\text{a)}\quad \tilde{\lambda}_2(\varepsilon) = \lambda_2 + O(\varepsilon^2),$$

3.12 $$\text{b)}\quad \tilde{T}_2(\varepsilon) = T_2 + O(\varepsilon^2),$$

$$\text{c)}\quad \tilde{\beta}_2(\varepsilon) \equiv -4\pi\, \tilde{\lambda}_2(\varepsilon)\, \mu_\lambda^0 / \omega_0 = \beta_2 + O(\varepsilon^2),$$

as $\varepsilon \to 0$.

Proof. Notice that in the radial coordinates 2.13, we have

$$r_{\tau^*} = \frac{1}{r}\left(A A_{\tau^*} + B B_{\tau^*}\right),$$

$$\theta_{\tau^*} = \frac{1}{r^2}\left(B A_{\tau^*} - A B_{\tau^*}\right).$$

Thus for (A,B) = (1,0) we have

$$r_{\tau^*} = A_{\tau^*}, \quad \theta_{\tau^*} = -B_{\tau^*}$$

Therefore by 2.11, 2.14 we see that 3.8 becomes

3.13 $$\frac{\partial \vec{S}(\lambda_2, T_2; \varepsilon)}{\partial(\lambda_2, T_2)} = \begin{pmatrix} T_0\, \mu_\lambda^0 & 0 \\ -T_0\, \omega_\lambda^0 & -\omega_0 \end{pmatrix} + O(\varepsilon^2).$$

The lemma follows from 2.17, 3.8, 3.13 and from

$$T_0\, \mu_\lambda^0\, \omega_0 \neq 0$$

by applying the implicit function theorem. ∎

In the numerical computation of the averages in 3.3 and 3.7 cancellation errors must be kept small. This means either using a fairly large ε ($\varepsilon \gtrsim 1/10$, say), or calculating the averages in double precision. Example calculations using the averaging method are proposed for the near future.

CHAPTER V. Computing Branches of Periodic Solutions

V.1 Basic Continuation

In this chapter we consider continuation schemes for calculating branches of periodic solutions of 0.1. The methods are based on solving

1.1 $$G(\vec{u},T,\lambda) \equiv \begin{pmatrix} \vec{u}_\tau - T\vec{f}(\vec{u}(\tau),\lambda) \\ \vec{u}(1)-\vec{u}(0) \\ \vec{\xi}^T(\lambda)\,[\vec{u}(0)-\vec{w}(0)] \end{pmatrix} = 0$$

where $0<\tau<1$. Here τ,T are as in Chapter IV, and $\vec{w}(t)$ is a T_1-periodic solution of 0.1 for $\lambda=\lambda_1$. Also $\vec{\xi}(\lambda) \in \mathbb{R}^n$ is such that $\vec{\xi}^T(\lambda)\,[\vec{u}(0) - \vec{w}(0)] = 0$ sets the phase of the periodic solution $\vec{u}(\tau)$ (see below).

In practice the o.d.e. in 1.1 can be approximated using a finite difference scheme. Well-known continuation techniques (see Keller [14], or Perozzi [25]) can be applied to the resulting system of algebraic equations. In this way both stable and unstable periodic solutions of 0.1 can be calculated. Furthermore, the powerful techniques due to H. B. Keller [14] for handling turning points and simple bifurcation points can be directly applied to the algebraic system. E. Doedel [4] has independently proposed this approach.

For theoretical purposes only, it is convenient to define

1.2a $$\vec{F}(\vec{z},\lambda) \equiv \begin{pmatrix} \vec{u}(1;\vec{z},\lambda) - \vec{u}(0;\vec{z},\lambda) \\ \vec{\xi}^T(\lambda)\left[\vec{u}(0;\vec{z},\lambda) - \vec{\omega}(0)\right] \end{pmatrix}.$$

Here

b) $$\vec{z} = \begin{pmatrix} \vec{\xi} \\ T \end{pmatrix},$$

1.2

c) $\vec{u}(\tau;\vec{z},\lambda)$ satisfies $\vec{u}_\tau = T\vec{f}(\vec{u},\lambda)$ for $0 \le \tau \le 1$ and $\vec{u}(0;\vec{z},\lambda) = \vec{\xi}$.

The equation

1.3 $$\vec{F}(\vec{z},\lambda) = \vec{0}$$

is equivalent to 1.1. Using $\vec{w}(t)$ as above, we see that $\vec{z}_1 \equiv (\vec{w}(0), T_1)$ is a solution of 1.3 for $\lambda = \lambda_1$.

Suppose $(\vec{u}^0(\tau), T^0, \lambda^0)$ is a solution of 1.1 Let $Y^0(\tau,0)$ be the fundamental solution matrix of

1.4 $$\vec{y}_\tau = T^0 \vec{f}_{\vec{u}}(\vec{u}^0(\tau),\lambda^0)\vec{y}(\tau) \quad \text{for } \tau \ge 0,$$

with $Y^0(0,0) \equiv I$. Then using 1.2 we can write

1.5 $$\frac{\partial \vec{F}^{\,0}}{\partial \vec{z}} \equiv \frac{\partial \vec{F}}{\partial \vec{z}}(\vec{z}^{\,0}, \lambda^0) = \begin{pmatrix} \left[\frac{\partial \vec{u}}{\partial \vec{\xi}}(1; \vec{z}^{\,0}, \lambda^0) - I_n\right] & \frac{\partial \vec{u}}{\partial T}(1; \vec{z}^{\,0}, \lambda^0) \\ \vec{\xi}^{\,T}(\lambda^0) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \left[Y^0(1,0) - I\right] & \vec{\eta}_0 \\ \vec{\xi}^{\,T}(\lambda^0) & 0 \end{pmatrix} .$$

Here $\vec{\eta}_0 = \frac{\partial \vec{u}}{\partial T}(1; \vec{z}^{\,0}, \lambda^0)$. Using $\vec{\omega}^{\,0}(T\tau) \equiv \vec{u}^{\,0}(\tau)$ and $t = T\tau$ we see that from 1.2

$$\frac{\partial \vec{u}}{\partial T}(\tau; \vec{z}^{\,0}, \lambda^0) = \frac{dt}{dT}\frac{d\vec{\omega}^{\,0}}{dt} = \tau \vec{f}(\vec{\omega}^{\,0}(T\tau), \lambda^0)$$

$$= \tau \vec{f}(\vec{u}(\tau; \vec{z}^{\,0}, \lambda^0), \lambda^0).$$

Therefore

1.6 $$\vec{\eta}_0 = \vec{f}(\vec{u}^{\,0}(1), \lambda^0) .$$

The following theorem gives conditions for $(\vec{u}^{\,0}(\tau), T^0, \lambda^0)$ to be a regular point of 1.1. For regular points the implicit function theorem gives the existence of a smooth branch $(\vec{u}(\tau; \lambda), T(\lambda), \lambda)$ of solutions of 1.1, with

$$(\vec{u}(\tau; \lambda^0), T(\lambda^0), \lambda^0) = (\vec{u}^{\,0}(\tau), T^0, \lambda^0).$$

This result is due to Poincaré (see [1]).

Theorem 1.7. Suppose $\frac{d\vec{u}^o}{d\tau} \not\equiv \vec{0}$ (i.e., $\vec{u}^o$ is a nontrivial periodic solution). Then the following three statements are equivalent:

i) $(\vec{u}^o(\tau), T^o, \lambda^o)$ is a regular point of 1.1;

ii) $(\vec{z}^o, \lambda^o)$ is a regular point of 1.3, with

$$\vec{z}^o \equiv \begin{pmatrix} \vec{u}^o(0) \\ T^o \end{pmatrix} ;$$

iii) $\vec{\xi}(\lambda^o)$, $\vec{\eta}_o$, $Y^o(1,0)$ satisfy:

a) $\vec{\xi}^T(\lambda^o)\vec{\eta}_o \neq 0$,

1.7

b) $Y^o(1,0)$ has 1 as a simple eigenvalue.

Proof. From the definitions of $\vec{F}$,G and their Frechét derivatives it is easily shown that i) and ii) are equivalent.

Notice that $\vec{y}(\tau) \equiv \frac{d\vec{u}^o(\tau)}{d\tau}$ satisfies 1.4. Therefore $\vec{\eta}_o = \frac{1}{T_o}\vec{u}^o_\tau(1) = \frac{1}{T_o}\vec{u}^o_\tau(0)$ and

1.8 $$Y^o(1,0)\vec{\eta}_o = \frac{1}{T_o}Y^o(1,0)\vec{u}^o_\tau(0) = \frac{1}{T_o}\vec{u}^o_\tau(1) = \vec{\eta}_o .$$

But $\vec{\eta}_o \neq \vec{0}$, so $Y^o(1,0)$ has the eigenvalue 1. The proof that ii) and iii) are equivalent now follows from 1.5, 1.8 and the following lemma.■

Lemma 1.9 (see H. B. Keller [14]). Let $\mathbb{B}$ be a Banach space and consider a linear operator $\mathcal{A} : \mathbb{B}\times\mathbb{R}^{\nu} \to \mathbb{B}\times\mathbb{R}^{\nu}$ of the form

$$1.9 \quad \mathcal{A} = \begin{pmatrix} A & B \\ C^* & D \end{pmatrix} \quad \text{where} \quad \begin{cases} A : \mathbb{B} \to \mathbb{B}, & B : \mathbb{R}^{\nu} \to \mathbb{B}, \\ C^* : \mathbb{B} \to \mathbb{R}^{\nu}, & D : \mathbb{R}^{\nu} \to \mathbb{R}^{\nu}. \end{cases}$$

i) If A is nonsingular then $\mathcal{A}$ is nonsingular iff

1.10 $\quad D - C^* A^{-1} B \quad$ is nonsingular.

ii) If A is singular and $\dim \mathcal{N}(A) = \operatorname{codim} \mathcal{R}(A) = \nu$ then $\mathcal{A}$ is nonsingular iff

1.11

a) $\dim \mathcal{R}(B) = \nu$, $\quad$ b) $\mathcal{R}(B) \cap \mathcal{R}(A) = \vec{0}$,

c) $\dim \mathcal{R}(C^*) = \nu$, $\quad$ d) $\mathcal{N}(A) \cap \mathcal{N}(C^*) = \vec{0}$.

(Here $\mathcal{R}(A)$ = range of A, and $\mathcal{N}(A)$ = null space of A.)

In practice we will use the solution of 1.1 calculated by the previous continuation step to define $(\vec{w}(0), T_1, \lambda_1)$. Then we set $\vec{\xi}(\lambda) = \vec{\xi}(\lambda_1) = \vec{f}(\vec{w}(0), \lambda_1)$. Setting $(\vec{u}^0(\tau), T^0, \lambda^0) = (\vec{w}(0), T_1, \lambda_1)$, then Theorem 1.7 gives necessary and sufficient conditions for $(\vec{w}(0), T_1, \lambda_1)$ to be a regular point of 1.1.

Singular points on solution paths of 1.1 are considered in the following sections. We also consider the linear stability of the periodic solution near the singular point. Recall that

$\vec{u}^0(\tau)$ is linearly stable iff the eigenvalues $\{\alpha_i\}_{i=1}^{\ell}$ of $Y^0(1,0)$ satisfy

1.12
a) $|\alpha_i| \leq 1$ for $i = 1, \ldots, \ell$;

b) if $|\alpha_i| = 1$ for some i then the eigenspace associated with α_i is nondegenerate

(see [1]).

V.2 Turning Points. The solution $(\vec{u}^0, T^0, \lambda^0)$ of 1.1 is called a normal limit point (for a branch of solutions parameterized by λ) if

2.1
a) $\dim \eta\left(\frac{\partial G^0}{\partial(\vec{u},T)}\right) = \operatorname{codim} \mathcal{R}\left(\frac{\partial G^0}{\partial(\vec{u},T)}\right) = 1$,

b) $\frac{\partial G^0}{\partial \lambda} \notin \mathcal{R}\left(\frac{\partial G^0}{\partial(\vec{u},T)}\right)$.

Here the superscript 0 denotes evaluation at $(\vec{u}^0, T^0, \lambda^0)$. A straightforward calculation shows that 2.1 is equivalent to

2.2
a) $\dim \eta\left(\frac{\partial \vec{F}^0}{\partial \vec{z}}\right) = 1$,

b) $\frac{\partial \vec{F}^0}{\partial \lambda} \notin \mathcal{R}\left(\frac{\partial \vec{F}^0}{\partial \vec{z}}\right)$.

Here the superscript denotes evaluation at $(\vec{z}^0, \lambda^0)$.

<u>Lemma 2.3</u>. Suppose $\vec{\xi}(\lambda^0)$ satisfies 1.7a. Then 2.1, 2.2 are satisfied iff either

a) $\eta(Y^0(1,0)-I) = \text{span}\{\vec{\eta}_0, \vec{\eta}_1\}$, $\dim \eta(Y^0(1,0)-I) = 2$,

2.3 b) $\vec{\eta}_k \notin \mathcal{R}(Y^0(1,0)-I)$ for $k = 0, 1$,

c) $\int_0^1 Y^0(1,x)\vec{f}^{\,0}_\lambda(x)\,dx \notin \mathcal{R}(\hat{A})$ (see 2.5 below);

or

a) $\eta(Y^0(1,0)-I) = \text{span}\{\vec{\eta}_0\}$,

2.4 b) $(Y^0(1,0)-I)\vec{\eta}_1 = \vec{\eta}_0$,

c) $\int_0^1 Y^0(1,x)\,\vec{f}^{\,0}_\lambda(x)\,dx \notin \mathcal{R}(\hat{A})$ (see 2.5).

Here

2.5 $$\vec{f}^{\,0}_\lambda(x) \equiv \vec{f}_\lambda(\vec{u}^{\,0}(x), \lambda^0), \quad \hat{A} \equiv \left([Y^0(1,0)-I] \quad \vec{\eta}_0\right),$$

$\hat{A}$ a $n\times(n+1)$ matrix.

<u>Proof</u>. Using 1.6, 1.7a, and 1.8 we can show that 2.2a is satisfied iff either 2.3a,b or 2.4a,b are satisfied.

From 1.2

2.6 $$\frac{\partial \vec{F}^0}{\partial \lambda} = \begin{pmatrix} \frac{\partial \vec{u}}{\partial \lambda}(1;\vec{z}^0,\lambda^0) \\ \frac{\partial}{\partial \lambda}\{\vec{\xi}^T(\lambda)[\vec{u}(0;\vec{z}^0,\lambda)-\vec{\omega}(0)]\}\big|_{\lambda=\lambda^0} \end{pmatrix}.$$

Using 1.2c, 1.4 we have

2.7
a) $$\vec{u}^0_{\lambda\tau}(\tau) = T^0 \vec{f}^0_{\vec{u}}\,\vec{u}^0_\lambda(\tau) + T^0\vec{f}^0_\lambda(\tau) \quad \text{for} \quad \tau\in(0,1),$$
b) $$\vec{u}^0_\lambda(0) = \vec{0}, \quad \vec{u}^0_\lambda(\tau) \equiv \frac{\partial \vec{u}}{\partial \lambda}(\tau;\vec{z}^0,\lambda^0).$$

Therefore

2.8 $$\vec{u}^0_\lambda(\tau) = \int_0^\tau Y^0(\tau,x)\,\vec{f}^0_\lambda(x)\,dx.$$

Also, notice that

2.9 $$\frac{\partial \vec{F}^0}{\partial \vec{z}}\begin{pmatrix}\vec{\eta}_0 \\ 0\end{pmatrix} = \begin{pmatrix}\vec{0} \\ \vec{\xi}^T(\lambda_0)\vec{\eta}_0\end{pmatrix} \neq \vec{0}.$$

Finally, from 2.6, 2.8 and 2.9 we see that 2.2b is satisfied iff 2.3c is satisfied.

In order to compute the solution arc through a normal limit point we can apply pseudo-arclength continuation (see Section III.3, or [14]). Using the pseudo-arclength s as a parameter, the normal limit point $(\vec{u}^0, T^0, \lambda^0) = (\vec{u}(\tau;s^0), T(s^0), \lambda(s^0))$ is a <u>regular</u> point of the branch $(\vec{u}(\tau;s), T(s), \lambda(s))$ (see

Section III.3). The following theorem shows that the linear stability of the periodic solution changes at turning points (i.e., $\lambda(s^0+\varepsilon)\,\lambda(s^0-\varepsilon) > 0$ for $\varepsilon > 0$ sufficiently small), but does not change at inflection points (i.e., $\lambda(s^0+\varepsilon)\,\lambda(s^0-\varepsilon) < 0$ for $\varepsilon > 0$ sufficiently small).

Theorem 2.10. Let $(\vec{z}(s), \lambda(s))$ be a solution arc of 1.3, where s is arclength and $(\vec{z}^0, \lambda^0) = (\vec{z}(s^0), \lambda(s^0))$ satisfies 2.2. Suppose

a) $\frac{\partial \vec{F}}{\partial \vec{z}}(s) \equiv \frac{\partial \vec{F}}{\partial \vec{z}}(\vec{z}(s), \lambda(s))$ is non-singular for $0 < |s - s^0| < \varepsilon$, for some $\varepsilon > 0$;

2.10 b) there exist smooth $\alpha(s)$, $\vec{\phi}_1(s)$, $\vec{\psi}_1(s)$ such that $\frac{\partial \vec{F}}{\partial \vec{z}}(s)\,\vec{\phi}_1(s) = \alpha(s)\,\vec{\phi}_1(s)$,
$\left[\frac{\partial \vec{F}}{\partial \vec{z}}(s)\right]^* \vec{\psi}_1(s) = \alpha^*(s)\,\vec{\psi}_1(s)$ and $\vec{\psi}_1^{\,*}(s)\,\vec{\phi}_1(s) = 1$;

c) $\alpha(s^0) = 0$.

Let $k < \infty$ be such that

a) $\frac{d^\ell \lambda}{d s^\ell}(s^0) = 0$ for $\ell = 1, 2, \ldots, k-1$,

2.11

b) $\frac{d^k \lambda}{d s^k}(s^0) \neq 0$.

Then $\hat{\alpha}(s) = 1 + \alpha(s)$ is an eigenvalue of $Y(1,0;\vec{z}(s),\lambda(s))$ and

$$\text{2.12} \quad \begin{aligned} &\text{a)}\ \frac{d^\ell \alpha}{ds^\ell}(s^0) = 0 \qquad \text{for} \quad \ell = 1, 2, \ldots, k-2, \\ &\text{b)}\ \frac{d^{k-1}}{ds^{k-1}} \alpha(s^0) \neq 0. \end{aligned}$$

Notice that for k even, $(\vec{z}^0, \lambda^0)$ is a turning point and $\alpha(s)$ changes sign as s passes through s^0. Also for k odd, $(\vec{z}^0, \lambda^0)$ is an inflection point and $\alpha(s)$ does not change sign near $s = s^0$.

<u>Proof of Theorem 2.10</u>. R. Szeto [35] has proved that 2.11 implies 2.12 under the conditions of Theorem 2.10. We present a shorter proof here.

By differentiating 1.3 w.r.t. s and using 2.10b we get

$$\text{2.13} \qquad \alpha(s) p_0(s) \vec{\phi}_1(s) = -\frac{\partial \vec{F}}{\partial \lambda}(s) \frac{d\lambda}{ds}(s) + \frac{\partial \vec{F}}{\partial \vec{z}}(s) \left\{ p_0(s) \vec{\phi}_1(s) - \frac{d\vec{z}}{ds}(s) \right\}.$$

Here $p_0(s)$ is any smooth real function. Evaluating 2.13 at $s = s^0$, premultiplying by $\vec{\psi}_1^*(s^0)$, and using 2.2b we obtain

$$\text{2.14} \qquad \frac{d\lambda}{ds}(s^0) = 0.$$

Substituting into 2.13 for $s = s^0$, and using $p_0(s) \equiv 1$ we get

$$\frac{\partial \vec{F}^0}{\partial \vec{z}} \left\{ \vec{\phi}_1(s^0) - \frac{d\vec{z}}{ds}(s^0) \right\} = \vec{0}.$$

Therefore

$$\vec{\phi}_1(s^0) - \frac{d\vec{z}}{ds}(s^0) = x_1\, \phi_1(s^0) .$$

Define $p_1(s) = 1 - x_1$, so

2.15 $$p_1(s^0)\, \vec{\phi}_1(s^0) = \frac{d\vec{z}}{ds}(s^0) .$$

Also $p_1(s^0) \neq 0$ by 2.14, 2.15 and the definition of arclength s.

Induction Hypothesis. For some ℓ with $1 \le \ell \le k-1$ we have

2.16 $$\frac{d^j\alpha}{ds^j}(s^0) = 0 \qquad \text{for} \quad j = 0, 1, \ldots, \ell-1.$$

Also there exists a polynomial $p_\ell(s)$, $p_\ell(s)$ nonzero for s near s^0, and

2.17 $$\frac{d^j}{ds^j}\left[p_\ell(s)\, \vec{\phi}(s) - \frac{d\vec{z}}{ds}(s) \right]\Bigg|_{s=s^0} = \vec{0}$$

for $j = 0, 1, \cdots, \ell-1$.

We have proved that we can satisfy the induction hypothesis for $\ell = 1$. Assume that 2.16, 17 are satisfied for some $\ell \ge 1$, $\ell \le k-1$. Then differentiating 2.13 w.r.t. s, with $p(s) = p_\ell(s)$ we find

2.18 $$\frac{d^\ell\alpha}{ds^\ell}(s^0)\, p(s^0)\, \vec{\phi}_1(s^0) = -\frac{\partial\vec{F}^0}{\partial\lambda}\frac{d^{\ell+1}\lambda}{ds^{\ell+1}}(s^0) + \frac{\partial\vec{F}^0}{\partial\vec{z}}\frac{d^\ell}{ds^\ell}\left[p_\ell\vec{\phi}_1 - \frac{d\vec{z}}{ds}\right]\Bigg|_{s=s^0} .$$

Premultiplying 2.18 by $\vec{\psi}_1^{\,*}(s^0)$ we find that

2.19 $$\frac{d^{\ell}\alpha}{ds^{\ell}}(s^0)\, p_{\ell}(s^0) = -\left(\vec{\psi}_1^{\,*}(s^0)\frac{\partial \vec{F}^{\,0}}{\partial \lambda}\right)\frac{d^{\ell+1}\lambda}{ds^{\ell+1}}(s^0).$$

If $\ell < k-1$ then 2.19 implies $\frac{d^{\ell}\alpha}{ds^{\ell}}(s^0) = 0$ and from 2.18 we obtain

$$\frac{d^{\ell}}{ds^{\ell}}\left[p_{\ell}(s)\vec{\phi}_1(s) - \frac{d\vec{z}}{ds}(s)\right]\Bigg|_{s=s^0} = x_{\ell}\,\vec{\phi}_1(s^0)$$

for some constant x_{ℓ}. Let

$$p_{\ell+1}(s) \equiv p_{\ell}(s) + \frac{x_1}{\ell!}(s-s^0)^{\ell}.$$

Then the induction hypothesis is satisfied for $\ell + 1$.

If $\ell = k-1$, 2.12b follows from 2.11b and 2.2b. This completes the proof of 2.12.

We must show that $\hat{\alpha}(s) = 1 + \alpha(s)$ is an eigenvalue of $Y(1,0;s) \equiv Y(1,0;\vec{z}(s),\lambda(s))$. Let

2.20 $$\vec{\phi}_1(s) = \begin{pmatrix}\vec{\phi}_{11}(s)\\ x(s)\end{pmatrix}\;;\quad \vec{\phi}_{11}(s) \in \mathbb{R}^n,\quad x(s) \in \mathbb{R}.$$

Then by 1.6, 2.10b

2.21

a) $$Y(1,0;s)\,\vec{\phi}_{11}(s) = \hat{\alpha}(s)\,\vec{\phi}_{11}(s) - x(s)\,\vec{\eta}_0(s),$$

b) $$\vec{\xi}^{\,T}(\lambda(s))\,\vec{\phi}_{11}(s) = \alpha(s)\,x(s).$$

Here $\vec{\eta}_0(s) \equiv T(s)\vec{f}(\vec{u}(0;\vec{z}(s),\lambda(s)),\lambda(s))$. For $s \neq s^0$ we see by the proof of Theorem 1.7 that

$$Y(1,0;s)\vec{\eta}_0(s) = \vec{\eta}_0(s) \, . \tag{2.22}$$

Using 2.20 define $\vec{\hat{\phi}}(s) = \vec{\phi}_{11}(s) - \frac{x(s)}{\hat{\alpha}(s)}\vec{\eta}_0(s)$ for s near s^0. Then by 2.21 we have

2.23

a) $Y(1,0;s)\vec{\hat{\phi}}(s) = \hat{\alpha}(s)\vec{\hat{\phi}}(s)$,

b) $\vec{\xi}^T(\lambda(s))\vec{\hat{\phi}}(s) = \alpha(s)x(s) - [\vec{\xi}^T(\lambda(s))\vec{\eta}_0(s)]\frac{x(s)}{\hat{\alpha}(s)}$

Finally $\vec{\hat{\phi}}(s^0) = 0$ implies (by 2.23b) that $x(s^0) = 0$, and $\vec{\hat{\phi}}(s^0) = \vec{\phi}_{11}(s^0) = \vec{0}$. From 1.6 and 2.10b, $\vec{\phi}_{11}(s^0) = \vec{0}$ implies $\vec{\eta}_0(s^0) = \vec{0}$ which contradicts 1.7a. Therefore $\vec{\hat{\phi}}(s^0) \neq \vec{0}$, and the theorem follows by 2.23a and the continuity of $\vec{\hat{\phi}}(s)$ near $s = s^0$. ■

Degree theory can be used to prove the results of Theorem 2.10 in greater generality. We will not pursue this here (see Sattinger [30]).

V.3 Simple Bifurcation Points

The point $(\vec{u}^0(\tau), T^0, \lambda^0) = (\vec{u}(\tau;s^0), T(s^0), \lambda(s^0))$ is a simple bifurcation point iff (see Keller [14]):

a) $\frac{\partial \vec{F}^0}{\partial \vec{z}}$ has zero as a <u>simple</u> eigenvalue,

b) $\frac{\partial \vec{F}^0}{\partial \lambda} \in \mathcal{R}\left(\frac{\partial \vec{F}^0}{\partial \vec{z}}\right)$,

3.1 c) $\vec{\psi}_1^* \left\{ \frac{\partial^2 \vec{F}^0}{\partial \vec{z}^2} \frac{d\vec{z}}{ds}(s^0) + \frac{\partial^2 \vec{F}^0}{\partial \vec{z}\,\partial \lambda} \frac{d\lambda}{ds}(s^0) \right\} \vec{\phi}_1 \neq 0$ if $\frac{d\lambda}{ds}(s^0) \neq 0$,

d) $\vec{\psi}_1^* \left\{ \frac{\partial^2 \vec{F}^0}{\partial \vec{z}^2} \vec{\phi}_0 + \frac{\partial^2 \vec{F}^0}{\partial \vec{z}\,\partial \lambda} \right\} \vec{\phi}_1 \neq 0$ if $\frac{d\lambda}{ds}(s^0) = 0$.

Here $\vec{\phi}_1 \in \mathcal{N}\left(\frac{\partial \vec{F}^0}{\partial \vec{z}}\right)$, $\vec{\psi}_1 \in \mathcal{N}\left(\frac{\partial \vec{F}^{0*}}{\partial \vec{z}}\right)$, $\vec{\psi}_1^* \vec{\phi}_1 = 1$, and

3.2 a) $\frac{\partial \vec{F}^0}{\partial \vec{z}} \vec{\phi}_0 = -\frac{\partial \vec{F}^0}{\partial \lambda}$,

b) $\vec{\psi}_1^* \vec{\phi}_0 = 0$.

The conditions 3.1 can also be written in terms of G simply by replacing $\vec{F}$ by G, and $\vec{z}$ by $(\vec{u},T)$.

The bifurcation conditions guarantee the existence of a nontangential second branch bifurcating at $(\vec{u}^0,T^0,\lambda^0)$ (see Keller [14], [15]). The algorithms given by Keller [14] for numerically computing both branches can be applied directly to a difference approximation for G.

Let $\alpha(s)$, $\vec{\phi}_1(s)$, $\vec{\psi}_1(s)$ be as in Theorem 2.10. If $\frac{d\lambda}{ds}(s^0) \neq 0$ then by 2.10b,c and 3.1c we see that

$$3.3 \qquad \frac{d\alpha}{ds}(s^0) = \vec{\psi}_1^{\,*}(s^0)\frac{d}{ds}\left[\frac{\partial \vec{F}}{\partial \vec{z}}(s)\,\vec{\phi}_1(s)\right]\Big|_{s=s^0},$$

$$= \vec{\psi}_1^{\,*}(s^0)\left[\frac{\partial^2 \vec{F}^0}{\partial \vec{z}^2}\frac{d\vec{z}}{ds}(s^0) + \frac{\partial^2 \vec{F}^0}{\partial \vec{z}\,\partial\lambda}\frac{d\lambda}{ds}(s^0)\right]\vec{\phi}_1(s^0),$$

$$\neq 0.$$

Exactly as is the proof of Theorem 2.10 we can show that $Y(1,0;s)$ has an eigenvalue $\hat{\alpha}(s)$ with $\hat{\alpha}(s^0) = 1$, $\frac{d\hat{\alpha}}{ds}(s^0) \neq 0$. That is, the periodic solution $\vec{u}(\tau;s)$ has a change in linear stability (or in the dimension of the space of linearly stable perturbations of $\vec{u}(0;s)$).

Let $\tilde{\alpha}(s)$ be the eigenvalue corresponding to $\alpha(s)$ for the other branch passing through $(\vec{z}^{\,0},\lambda^0)$. From topological degree theory (see [30]) the signs of $\alpha(s)$ and $\tilde{\alpha}(s)$ behave as in Fig. 3.4.

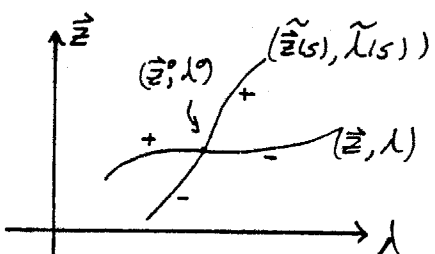


Fig 3.4a

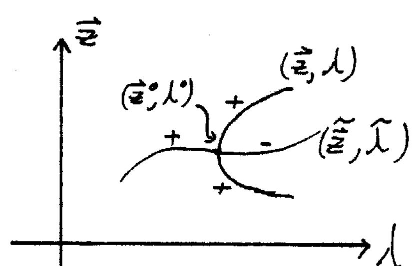


Fig 3.4b

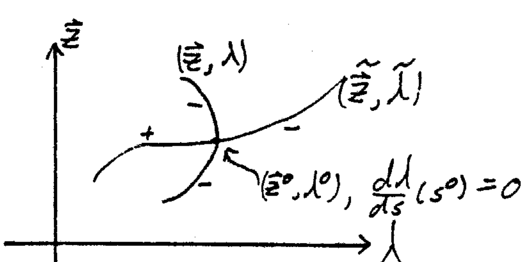


Fig 3.4c

The plus and minus signs in Fig. 3.4 give the signs of $\alpha(s)$ and $\tilde{\alpha}(s)$, or the signs of $-\alpha(s)$ and $-\tilde{\alpha}(s)$. As in Theorem 2.10, $Y(1,0;\ \vec{z}(s),\lambda(s))$ and $Y(1,0;\ \tilde{\vec{z}}(s),\tilde{\lambda}(s))$ have eigenvalues $1+\alpha(s)$ and $1+\tilde{\alpha}(s)$ respectively. Therefore Fig. 3.4 also represents changes in the stability of the periodic solutions.

V.4 Start Up at Hopf Bifurcation Points

We consider the problem of starting along the branch of periodic solutions bifurcating from a known HBP, $(\vec{u}_0(\lambda_0),\lambda_0)$, of 0.1. The nature of the singularity at $(\vec{u}_0(\lambda_0),\lambda_0)$ is discussed in

Lemma 4.1. Define

4.1

a) $$\frac{\partial G^0}{\partial(\vec{u},T)} \equiv \frac{\partial G}{\partial \vec{u}}(\vec{u}_0(\lambda_0),T_0,\lambda_0)\ ,\quad T_0 = \frac{2\pi}{\omega_0} \quad (\text{see } 0.5),$$

b) $$\frac{\partial \vec{F}^0}{\partial \vec{z}} \equiv \frac{\partial \vec{F}}{\partial \vec{z}}(\vec{z}_0,\lambda_0),\quad \vec{z}_0 \equiv (\vec{u}_0(\lambda_0),T_0).$$

Let $\vec{f}^{\,0}_{\vec{u}}\,(\vec{a}+i\vec{b}) = i\omega_0(\vec{a}+i\vec{b})$, where $\vec{f}^{\,0}_{\vec{u}} = \vec{f}_{\vec{u}}(\vec{u}_0(\lambda_0),\lambda_0)$ and $|\vec{a}+i\vec{b}| \neq 0$ (see 0.5). Then

4.2

a) $$\dim \eta\left(\frac{\partial G^0}{\partial(\vec{u},T)}\right) = \dim \eta\left(\frac{\partial \vec{F}^0}{\partial \vec{z}}\right) \geq 2\ ,$$

b) $$\dim \eta\left(\frac{\partial \vec{F}^0}{\partial \vec{z}}\right) = 2 \quad \text{iff} \quad \vec{\xi}^{\,T}(\lambda_0)(\vec{a}\ \ \vec{b}) \neq (0\ \ 0).$$

Proof. From 0.2, 1.4, 1.5 and 1.6 we have

$$4.3 \quad \text{a)} \quad \frac{\partial \vec{F}^{\,0}}{\partial \vec{z}} = \begin{pmatrix} [Y^0(1,0) - I] & \vec{0} \\ \vec{\xi}^{\,T}(\lambda_0) & 0 \end{pmatrix}$$

where,

$$4.3 \quad \text{b)} \quad Y^0(\tau, 0) \equiv Y(\tau, 0; \vec{u}_0(\lambda_0), \lambda_0) = e^{T_0 \vec{f}^{\,0}_{\vec{u}} \tau}.$$

Therefore from 0.5, 4.3b we see that

$$Y^0(1,0)\{\vec{a} + i\vec{b}\} = \vec{a} + i\vec{b}$$

and 1 is an eigenvalue of $Y^0(1,0)$ with geometric and algebraic multiplicity two. Therefore

$$4.4 \quad \text{a)} \quad \begin{pmatrix} \vec{0} \\ 1 \end{pmatrix} \in \eta\left(\vec{F}^{\,0}_{\vec{z}}\right),$$

$$\text{b)} \quad \begin{pmatrix} x_1\vec{a} + x_2\vec{b} \\ 0 \end{pmatrix} \in \eta\left(\vec{F}^{\,0}_{\vec{z}}\right) \quad \text{for} \quad x_1\left(\vec{\xi}^{\,T}(\lambda_0)\vec{a}\right) + x_2\left(\vec{\xi}^{\,T}(\lambda_0)\vec{b}\right) = 0.$$

The lemma follows from 4.4 and the definitions of $\frac{\partial G^0}{\partial(\vec{u},T)}$, $\frac{\partial \vec{F}^{\,0}}{\partial \vec{z}}$. ■

The methods discussed in Section V.1 can be expected to have difficulty near the singular point $(\vec{u}_0(\lambda_0), \lambda_0)$. W. Langford [20] has suggested an algorithm for calculating the bifurcating periodic solution near the bifurcation point. The method is based on singular perturbation techniques, and its accuracy improves as $(\vec{u}_0(\lambda_0), \lambda_0)$ is approached. The method requires $Y^0(1,0)$ explicitly, which can be costly to compute.

Also the method requires an entirely different program than the arclength continuation methods described previously.

E. Doedel [4] has suggested that arclength continuation can be used directly. That is, we attempt to solve

4.5 a) $$G(\vec{u}, T, \lambda) = 0$$

with $\vec{\xi}(\lambda) = \vec{\xi}(\lambda_0)$, $\vec{\xi}^T(\lambda_0)(\vec{a}\ \vec{b}) \neq (0, 0)$ and

4.5 b) $$L(\vec{u}, T, \lambda; s) \equiv \theta_1 \|\vec{u}(\tau) - \vec{u}_0(\lambda_0)\|^2 + \theta_2 |T - T_0|^2 + \theta_3 |\lambda - \lambda_0|^2 - s^2 = 0, \text{ with } \theta_k \geq 0.$$

Note that 4.5 has a trivial solution

$$(\vec{u}, T, \lambda) = (\vec{u}_0(\lambda_0), T_0 + \frac{1}{\sqrt{\theta_2}} s, \lambda_0)$$

for $\theta_2 \neq 0$. Also for $\frac{d\vec{u}_0}{d\lambda}(\lambda_0) \perp \vec{\xi}(\lambda_0)$ the steady state branch $(\vec{u}_0(\lambda), \lambda)$ is nearly a solution of 4.5 for $(\lambda - \lambda_0)$ small. This suggests that we should require

4.6
a) $\theta_2 = 0$,

b) $\vec{\xi}^T(\lambda_0) \frac{d\vec{u}_0}{d\lambda}(\lambda_0) \neq 0.$

Finally, 4.5b should be replaced by pseudo-arclength normalization, that is

4.7 $$\hat{L}(\vec{u},T,\lambda;s) \equiv \int_0^1 \frac{d\vec{u}^0}{ds}(x)\{\vec{u}(x) - \vec{u}_0(\lambda_0)\}dx - s = 0.$$

Here

a) $$\frac{d\vec{u}^0}{ds}(\tau) \equiv [x_1\vec{a} + x_2\vec{b}]\cos(2\pi\tau) + [x_2\vec{a} - x_1\vec{b}]\sin(2\pi\tau),$$

4.8 b) $$\vec{\xi}^T(\lambda_0)\frac{d\vec{u}^0}{ds}(0) = x_1[\vec{\xi}^T(\lambda_0)\vec{a}] + x_2[\vec{\xi}^T(\lambda_0)\vec{b}] = 0,$$

c) $$\frac{1}{2}\{|x_1\vec{a} + x_2\vec{b}| + |x_2\vec{a} - x_1\vec{b}|\} = 1.$$

Equations 4.8a,b,c determine $\frac{d\vec{u}^0}{ds}$, x_1, and x_2 uniquely. Notice that the parameter s in 4.7 is closely related to the amplitude parameter ε of the periodic solution (see 0.6).

For an initial guess of a solution of 4.5a, 4.7 we can take $(\vec{u},T,\lambda) = (\vec{w},T_0,\lambda_0)$ where

4.9 $$\vec{w}(\tau;s) \equiv \vec{u}_0(\lambda_0) + s\left\{\frac{d\vec{u}^0}{ds}(\tau)\right\}$$ (see 4.8).

It is easy to show that

4.10

a) $$G(\vec{w},T_0,\lambda_0) = O(s^2),$$

b) $$\hat{L}(\vec{w},T_0,\lambda_0;s) = 0$$

as $s \to 0$. Higher order initial guesses can be obtained using the Hopf bifurcation parameters λ_2, T_2 (see Chapter III).

Next we present an algorithm based on a method for simple bifurcation due to H. B. Keller (see Method IV in [14]). Let $(\vec{u}_0, \lambda_0)$ be a HBP of 0.1 with

4.11

a) $\vec{f}^{\,0}_{\vec{u}}(\vec{a}+i\vec{b}) = i\omega_0(\vec{a}+i\vec{b}), \quad (\vec{c}-i\vec{d})^T \vec{f}^{\,0}_{\vec{u}} = i\omega_0(\vec{c}-i\vec{d})^T,$

b) $\omega_0 > 0, \quad (\vec{c}\ \ \vec{d})^T(\vec{a}\ \ \vec{b}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$

Consider

$$4.12 \qquad H(\vec{v}, T, \lambda; \varepsilon) \equiv \begin{pmatrix} \frac{d\vec{v}}{d\tau}(\tau) - T\vec{g}(\vec{v}(\tau), \lambda; \varepsilon) \\ \vec{v}(1) - \vec{v}(0) \\ \vec{\xi}_0^{\,T}\, \vec{v}(0) \\ \int_0^1 \vec{w}^T(x)\, \vec{v}(x)\, dx - S_0 \end{pmatrix} = 0, \quad 0 < \tau < 1.$$

Here

4.13

a) $\vec{g}(\vec{v}, \lambda; \varepsilon) \equiv \begin{cases} \frac{1}{\varepsilon}\vec{f}(\vec{u}_0 + \varepsilon\vec{v}(\tau), \lambda) & \text{for } \varepsilon \neq 0, \\ \vec{f}_{\vec{u}}(\vec{u}_0, \lambda)\vec{v}(\tau) & \text{for } \varepsilon = 0; \end{cases}$

b) $\vec{w}(\tau) \equiv \vec{c}\cos(2\pi\tau) - \vec{d}\sin(2\pi\tau),$

c) $\vec{\xi}_0^{\,T}\vec{a} = 0, \quad \vec{\xi}_0^{\,T}\vec{b} = 1,$

d) $S_0 > 0.$

Notice that if $(\vec{v}(\tau;\varepsilon), T(\varepsilon), \lambda(\varepsilon))$ satisfies 4.12 for some

$\varepsilon \neq 0$ then

4.14 a) $\vec{u}_L(\tau;\varepsilon) = \vec{u}_0 + \varepsilon\vec{v}(\tau;\varepsilon)$

satisfies

b) $\vec{u}_\tau(\tau) = T(\varepsilon)\,\vec{f}(\vec{u}(\tau),\lambda(\varepsilon)) \quad$ for $\tau \in (0,1)$,

4.14

c) $\vec{u}(1) - \vec{u}(0) = \vec{0}$.

<u>Lemma 4.15</u>. For $\varepsilon = 0$ equation 4.12 has a solution

4.15 a) $(\vec{v}, T, \lambda) = (\vec{v}_0(\tau), T_0, \lambda_0)$,

for $T_0 = 2\pi/\omega_0$ and

4.15 b) $\vec{v}_0(\tau) = s_0\left\{\vec{a}\cos(2\pi\tau) - \vec{b}\sin(2\pi\tau)\right\}$.

The solution given above is a regular point of 4.12.

<u>Proof</u>. A straightforward calculation shows that 4.15 gives a solution of 4.12. Let $\vec{z} = (\vec{v},T,\lambda)$, $\vec{z}_0 = (\vec{v}_0,T_0,\lambda_0)$, and $\frac{\partial H^0}{\partial \vec{z}} = \frac{\partial H}{\partial \vec{z}}(\vec{z}_0;0)$. Without loss of generality we take $s_0=1$. Assume $(\Delta\vec{v},\Delta T,\Delta\lambda)$ is a null vector of $\frac{\partial H^0}{\partial \vec{z}}$, that is

4.16 $\frac{\partial H^0}{\partial \vec{z}}(\Delta\vec{v}, \Delta T, \Delta\lambda) =$

(4.16 cont.)

$$= \begin{pmatrix} \left[\frac{d}{d\tau} - T_0 \vec{f}^{\,o}_{\vec{u}}\right] \Delta\vec{v}(\tau) + \Delta T \vec{f}^{\,o}_{\vec{u}} \vec{v}_0(\tau) + T_0 \vec{f}^{\,o}_{\vec{u}\lambda} \vec{v}_0(\tau)\Delta\lambda \\ \Delta\vec{v}(1) - \Delta\vec{v}(0) \\ \vec{\xi}^{\,T}_0 \Delta\vec{v}(0) \\ \int_0^1 \vec{\omega}^T(x)\, \Delta\vec{v}(x)\, dx \end{pmatrix}$$

$$= 0.$$

Then by 4.16

4.17 a) $$\Delta\vec{v}(\tau) = Y(\tau,0)\vec{\eta}_0 + \Delta T \int_0^\tau Y(\tau,x)\, f_{\vec{u}}\, \vec{v}_0(x)\, dx + \Delta\lambda \int_0^\tau Y(\tau,x)\, T_0\, \vec{f}^{\,o}_{\vec{u}\lambda}\, \vec{v}_0(x)\, dx,$$

with

4.17

b) $$Y(\tau,x) = \exp\left\{ T_0\, \vec{f}^{\,o}_{\vec{u}} (\tau - x) \right\},$$

c) $$\vec{\eta}_0 \in \mathbb{R}^n.$$

We can write

4.18 $$\vec{v}_0(x) = Re\left\{ s_0(\vec{a} + i\,\vec{b})\, e^{i 2\pi x} \right\}.$$

Substituting 4.17b and 4.18 into 4.17a we get

4.19 $$\int_0^1 Y(1,x)\vec{f}^{\,o}_{\vec{u}}\vec{v}_o(x)\,dx = \int_0^1 Re\left\{e^{i\omega_o T_o(1-x)} s_o i\omega_o(\vec{a}+i\vec{b})e^{i2\pi x}\right\}dx$$

$$= -\omega_o s_o \vec{b}.$$

Also

4.20 $$(\vec{c}-i\vec{d})^T\int_0^1 Y(1,x)T_o\vec{f}^{\,o}_{\vec{u}\lambda}\vec{v}_o(x)\,dx$$

$$= T_o\int_0^1 e^{i2\pi(1-x)}(\vec{c}-i\vec{d})^T\vec{f}^{\,o}_{\vec{u}\lambda}s_o(\vec{a}\cos(2\pi x)-\vec{b}\sin(2\pi x))\,dx,$$

$$= s_o T_o\int_0^1 e^{-i2\pi x}\left\{\left[\mu^o_\lambda\cos(2\pi x)-\omega^o_\lambda\sin(2\pi x)\right]+i\left[\omega^o_\lambda (2\pi x)+\mu^o_\lambda\sin(2\pi x)\right]\right\}dx,$$

$$= s_o T_o(\mu^o_\lambda + i\omega^o_\lambda).$$

Here we have used $\vec{c}^{\,T}\vec{f}^{\,o}_{\vec{u}\lambda}\vec{a} = \vec{d}^{\,T}\vec{f}^{\,o}_{\vec{u}\lambda}\vec{b} = \mu^o_\lambda$ and $-\vec{d}^{\,T}\vec{f}^{\,o}_{u\lambda}\vec{a} = \vec{c}^{\,T}\vec{f}^{\,o}_{\vec{u}\lambda}\vec{b} = \omega^o_\lambda$ (see 0.5 for the definition of $\mu(\lambda)$, $\omega(\lambda)$ and see the proof of Theorem V 2.10 for the above relation).

Then from 4.17. 4.19 the boundary conditions $\Delta\vec{v}(1) - \Delta\vec{v}(0) = \vec{0}$ become

4.21 $$[Y(1,0)-I]\vec{\eta}_o = \Delta T\, s_o\omega_o\vec{b} - \Delta\lambda\int_0^1 Y(1,x)T_o\vec{f}^{\,o}_{\vec{u}\lambda}\vec{v}_o(x)\,dx.$$

The Fredholm theory for matrices and 4.11 (and 0.5 imply that 4.21 has a solution iff c^T and d^T postmultiplied by the right hand side of 4.21 equal zero. Using 4.20 this becomes

4.22 $$\begin{pmatrix} 0 & s_0 T_0 \mu_\lambda^0 \\ s_0 \omega_0 & s_0 T_0 \omega_\lambda^0 \end{pmatrix} \begin{pmatrix} \Delta T \\ \Delta \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

Finally, 4.22 and 0.5 imply that

4.23 $$\Delta T = \Delta \lambda = 0 .$$

From 4.17a,b, 4.21, 4.22 we have

4.24 $$\Delta \vec{v}(\tau) = Y(\tau, 0) \{ x_1 \vec{a} + x_2 \vec{b} \} .$$

From 4.12 we require $\vec{\xi}_0^T \Delta \vec{v}(0) = 0$, which by 4.13c implies that

$$0 = \vec{\xi}_0^T (x_1 \vec{a} + x_2 \vec{b}) = x_2 (\vec{\xi}_0^T \vec{b})$$

and therefore $x_2 = 0$. Finally from 4.12, 4.17b, 4.24, we have

$$0 = x_1 \int_0^1 \vec{\omega}^T(x) Y(x,0) \vec{a} \, dx = -x_1 \omega_0 \int_0^1 \vec{\omega}^T(x) \sin(2\pi x) \vec{b} \, dx$$

$$= \frac{1}{2} x_1 \omega_0 .$$

So we have shown $x_1 = x_2 = 0$, therefore $\Delta \vec{v} = 0$. ∎

Lemma 4.15 justifies using a standard continuation scheme to compute solutions of 4.12 for $\varepsilon > 0$ sufficiently small. For Euler-Newton continuation we will need

4.25 a) $$\frac{\partial H}{\partial \varepsilon}(\vec{z}^{\,0};0) = \begin{pmatrix} -\frac{1}{2} T_0 \vec{f}^{\,0}_{\vec{u}\vec{u}} \; \vec{v}_0(\tau)\,\vec{v}_0(\tau) \\ \vec{0} \end{pmatrix}.$$

This can be approximated by

4.25 b) $$\frac{\partial \tilde{H}}{\partial \varepsilon}(\vec{z}^{\,0};0) \equiv \begin{pmatrix} \frac{T_0}{\varepsilon^2}\left[\vec{f}(\vec{u}_0+\varepsilon\vec{v}_0(\tau),\lambda_0) - \varepsilon \vec{f}^{\,0}_{\vec{u}}\,\vec{v}_0(\tau)\right] \\ \vec{0} \end{pmatrix}.$$

Once we have computed a solution of 4.12 for $\varepsilon \neq 0$, the transformation 4.14a provides a nontrivial periodic solution 0.1. For ε sufficiently small this solution must lie on the bifurcating periodic solution branch.

Note that Lemma 4.15 and the implicit function theorem prove the Hopf bifurcation theorem (see [2]).

Other choices of $\vec{g}(v,\lambda;\varepsilon)$ might prove to be useful for some problems. For example, if $|\vec{f}^{\,0}_{\vec{u}\vec{u}}\,\vec{w}\vec{w}| >> 1$ then the first continuation step using the above method will require $|\Delta\varepsilon| << 1$. If $|\Delta\varepsilon\,\vec{w}(\tau)| << |\vec{u}_0|$ then large calculation errors will occur in calculating $\vec{u}_0 + \Delta\varepsilon\,\vec{w}(\tau)$ for $\vec{g}(\vec{w},\lambda;\Delta\varepsilon)$. This problem can be avoided by using

4.26 $$\vec{g}_1(\vec{v},\lambda;\varepsilon) = (1-\varepsilon)\vec{f}_{\vec{u}}(\vec{u}_0,\lambda)\vec{v} + \varepsilon\vec{f}(\vec{v},\lambda).$$

(We are only computing solutions of 0.1 when $\varepsilon = 1$!)

Example calculations are proposed for the near future.

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