SU(3) Chiral Symmetry in Non-Relativistic Field Theory

Thesis by

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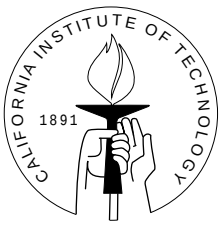

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Abstract

Applications imposing SU(3) chiral symmetry on non-relativistic field theories are considered. The first example is a calculation of the self-energy shifts of the spin-$\frac{3}{2}$ decuplet baryons in nuclear matter, from the chiral effective Lagrangian coupling octet and decuplet baryon fields. Special attention is paid to the self-energy of the Δ baryon near the saturation density of nuclear matter. We find contributions to the mass shifts from contact terms in the effective Lagrangian with coefficients of unknown value. As a second application, we formulate an effective field theory with manifest SU(2) chiral symmetry for the interactions of K and η mesons with pions at low energy. SU(3) chiral symmetry is imposed on the effective field theory by a matching calculation onto three-flavor chiral perturbation theory. The effective Lagrangian for the πK and $\pi\eta$ sectors is worked out to order Q^4; the effective Lagrangian for the KK sector is worked out to order Q^2 with contact interactions to order Q^4. As an application of the method, we calculate the KK s-wave scattering phase shift at leading order and compare with chiral perturbation theory. We conclude with a discussion of the limitations of the approach and propose new directions for work where the matching calculation may be useful.

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Chapter 1

Introduction

I once heard a wise man say "In the history of scientific endeavor, no problem has consumed as much of mankind's resources as the understanding of nuclear forces," or something like that. In any case, for all the considerable effort poured into solving the mysteries of the strong interaction, a number of significant problems remain. The fundamental theory of the strong interactions, Quantum Chromodynamics (QCD), is solidly established as a pillar of the Standard Model of particle physics and, to the extent that QCD is a renormalizable gauge theory, is well understood in the perturbative regime. As a theory with asymptotic freedom [1, 2], QCD is perturbative in the high-energy regime; for low energies the coupling constant of the theory becomes large and perturbative treatments break down. Some features of non-perturbative QCD which are still not fully understood are quark structure of hadrons, dynamical symmetry breaking, and quark confinement.

The special difficulties accompanying the non-perturbative regime require special methods for working within the theory. One direct approach is to formulate QCD on a lattice of space-time points and use numerical techniques to perform the functional integrals. The lattice QCD method [3, 4] has great potential for shedding light on many of the unanswered questions of QCD and has become an industry unto itself. A complementary approach is to focus on the long-distance physics, and base the field theory description of the physics on the directly observed degrees of freedom. Approaches of the second type are generically called effective field theories and rely on a two-part foundation. I refer to the first important concept as the Weinberg Hypothesis [5]: the only content of quantum field theory (apart from the choice of degrees of freedom) is analyticity, unitarity, cluster decomposition, and the assumed *symmetry principles*. As a consequence, we can describe

the strong interactions in terms of hadron degrees of freedom provided we write down the most general Lagrangian consistent with the symmetries of QCD. The second key concept is that we must identify some expansion parameter, typically a small momentum or energy scale, which permits us to calculate to any given order in the expansion with a finite amount of work. The predictive power of an effective field theory arises from the combination of the two underlying concepts; the symmetry restricts the parameters of the theory to a meaningful set and the expansion parameter allows a systematic framework in which we can include all contributions of a given order and estimate the size of the contributions we have neglected. For a more detailed discussion of effective field theory in general, see references [6, 7, 8].

In the case of QCD, the hope is that by experimental determination of the parameters of the effective Lagrangian, we can learn about the underlying theory in the non-perturbative regime, possibly through lattice QCD as an intermediary. An alternative possibility will also be considered in this work. If the 'fundamental' theory at higher-energy is known and calculable, then at a momentum scale where the theories meet the parameters of the effective theory can be determined by matching onto the fundamental theory. This is sometimes done because certain calculations are more easily performed in the effective theory, either because of additional approximate symmetry in the low-energy limit or because a non-relativistic framework may be used. For instance this sort of matching calculation has been performed and applied with success in non-relativistic QED (NRQED) [9] and non-relativistic QCD (NRQCD) [10].

In this thesis we consider two applications of effective field theory to exploit the $SU(3)_L \times SU(3)_R$ chiral symmetry of QCD. In Chapter 2 we cover the theoretical framework upon which the effective field theories will be built. We discuss the symmetries of QCD which constrain the effective Lagrangian, the principles for constructing an effective Lagrangian for the hadron degrees of freedom, and the power counting schemes that apply to the sectors of the theory with only light fields, one heavy field (static case), or more than one heavy field (non-relativistic case). In Chapter 3 we present an effective field

theory calculation of the self-energy shift of spin-$\frac{3}{2}$ decuplet baryons in nuclear matter. With the exception of an expanded discussion of the Δ-baryon self-energy, the material in this chapter has already been published [11]. In Chapter 4 we present new material on an effective field theory for low-energy interactions of pions with kaons or an eta meson as an alternative to the standard SU(3) chiral perturbation theory. The objective is to achieve better convergence for very low energies by treating the kaon in a non-relativistic framework. To determine the parameters of the low-energy theory we perform a matching calculation from the heavy kaon/eta theory onto SU(3) chiral perturbation theory in the spirit of NRQED or NRQCD. Finally, we consider KK scattering in the heavy kaon formalism and discuss the utility of the matching calculation.

Chapter 2

Theoretical Background

In this chapter chiral perturbation theory (χPT) is introduced as the foundation for describing interactions involving the light pseudoscalar meson octet. As an effective field theory, χPT gets predictive power from the symmetries of the underlying theory (QCD) and a consistent scheme for counting powers of 'small' momenta. Section 2.1 reviews the relevant symmetries of the QCD Lagrangian. In sections 2.2 and 2.3 we outline the formulation of SU(3) χPT and its extension to include heavy fields. The momentum power counting for diagrams with one or two heavy particles is briefly discussed in section 2.4.

2.1 Symmetries of QCD

An appropriate starting point is the Lagrangian of QCD,

$$\mathcal{L}_{\rm QCD} = -\tfrac{1}{4} G^A_{\mu\nu} G^{A\mu\nu} + \overline{q}_L i \not{D} q_L + \overline{q}_R i \not{D} q_R - \overline{q}_R \mathcal{M}_q q_L - \overline{q}_L \mathcal{M}_q q_R + \cdots \qquad (2.1)$$

in which the ellipsis denotes gauge-fixing and ghost terms, renormalization counterterms, and the θ term. As written $\mathcal{L}_{\rm QCD}$ is invariant under the Poincaré group, $\mathrm{SU(3)_C}$ gauge transformations, and charge conjugation (C). In addition the coefficient of the θ term is known to be small [12], so we neglect it throughout; in this approximation $\mathcal{L}_{\rm QCD}$ is also invariant under parity (P) and time reversal (T).

In the limit that N of the quark masses vanish, the chiral limit, $\mathcal{L}_{\rm QCD}$ acquires additional symmetries under independent U(N) rotations of the left- and right-handed quark fields;

$$q_L \to L q_L, \qquad q_R \to R q_R, \qquad (2.2)$$

where L and R are unitary matrices restricted to acting in the massless sector. The $\mathrm{U(1)_V}$ subgroup of this symmetry corresponds to the conservation of quark number in the massless flavors. A second subgroup, $\mathrm{U(1)_A}$, is broken in the quantum theory by the axial-vector anomaly. The symmetries of $\mathcal{L}_{\mathrm{QCD}}$ relevant for our purposes are the remaining $\mathrm{SU}(N)_{\mathrm{L}} \times \mathrm{SU}(N)_{\mathrm{R}}$ chiral symmetry and the discrete symmetries C, P, and T.

The quark masses appearing in $\mathcal{L}_{\mathrm{QCD}}$ are non-zero and explicitly break the chiral symmetry. Even in the limit of massless quarks the vacuum expectation value of the quark bilinear

$$\langle 0 | \, \overline{q}_{La} q_{Rb} \, | 0 \rangle = -\Delta \, \delta_{ab} \tag{2.3}$$

spontaneously breaks the chiral symmetry $\mathrm{SU}(N)_{\mathrm{L}} \times \mathrm{SU}(N)_{\mathrm{R}}$ down to the vector subgroup $\mathrm{SU}(N)_{\mathrm{V}}$. The scale $\Lambda_\chi \sim 1$ GeV, associated with spontaneous chiral symmetry breaking, determines the relative importance of the quark masses in breaking the chiral symmetry [13]. Quark masses m_q which are much less than Λ_χ, specifically $m_{u,d} \lesssim 10$ MeV and 75 MeV $\lesssim m_s \lesssim 170$ MeV [14], can be treated as perturbations about the chiral limit $m_q \to 0$ by expanding in powers of m_q/Λ_χ. For the rest of this chapter we identify the chiral symmetry group as $G_\chi = \mathrm{SU(3)_L} \times \mathrm{SU(3)_R}$ and drop any explicit reference to the heavy quark flavors.

Invariance under C, P, and G_χ is imposed on the Lagrangian of the effective field theory; invariance under T follows automatically from the CPT theorem.

2.2 Chiral Perturbation Theory

The spontaneous breaking of chiral symmetry to the $\mathrm{SU(3)_V}$ subgroup implies the existence of eight massless Goldstone scalar fields. However, because the chiral symmetry is explicitly broken by the quark masses, these fields acquire (small) finite masses and are commonly referred to as pseudo-Goldstone bosons. The pseudo-Goldstone bosons of spontaneously broken chiral symmetry are identified as the light pseudoscalar meson octet of pions (π^+, π^-, and π^0), kaons (K^+, K^-, K^0, and $\overline{K}^0$), and the eta (η). Chiral perturba-

tion theory is the effective field theory for describing the interactions of this meson octet at energies much lower than the chiral symmetry breaking scale, Λ_χ. For reviews of this subject see references [15, 16].

The pseudo-Goldstone bosons are represented by a 3×3 special unitary matrix, $U = e^{i\Phi/F_0}$, with

$$\Phi = \phi_a \lambda_a = \sqrt{2} \begin{pmatrix} \pi^0/\sqrt{2} + \eta/\sqrt{6} & \pi^+ & K^+ \\ \pi^- & -\pi^0/\sqrt{2} + \eta/\sqrt{6} & K^0 \\ K^- & \overline{K}^0 & -2\eta/\sqrt{6} \end{pmatrix} \tag{2.4}$$

and $F_0 \simeq F_\pi \simeq 93$ MeV. The pion decay constant F_π is determined from

$$\langle 0 | \, J_5^\mu \, | \pi^-(p) \rangle = i\sqrt{2} F_\pi p^\mu, \tag{2.5}$$

where $J_5^\mu = \overline{u}\gamma^\mu\gamma_5 d$ is an octet axial-vector current associated with chiral symmetry. Under the symmetries of section 2.1 the field U transforms as

$$U \xrightarrow{G_\chi} RUL^\dagger, \qquad U \xrightarrow{P} U^\dagger, \qquad U \xrightarrow{C} U^T. \tag{2.6}$$

We introduce the quark mass matrix $\mathcal{M}_q$ through the field $\chi = 2B_0\mathcal{M}_q$, where the constant B_0 is related to the vacuum expectation value in equation (2.3) and is approximately $B_0 \simeq 2\Delta/F_0^2 \sim 1300$ MeV [16]. For the purpose of constructing effective Lagrangians χ is assumed to transform in a way that preserves the symmetries of the QCD Lagrangian;

$$\chi \xrightarrow{G_\chi} R\chi L^\dagger, \qquad \chi \xrightarrow{P} \chi^\dagger, \qquad \chi \xrightarrow{C} \chi^T. \tag{2.7}$$

In treating $\mathcal{M}_q$ we neglect the quark mass difference $m_u - m_d$ and replace both m_u and m_d with the average $\hat{m} = \frac{1}{2}(m_u + m_d)$. Because the corresponding SU(2) subgroup of $\mathrm{SU(3)_V}$ is isospin symmetry, this approximation ignores isospin violation in the strong interaction.

Calculations in χPT are organized as expansions in powers of m_q and Q, the charac-

teristic momentum scale of the interaction. The terms in the Lagrangian of χPT ($\mathcal{L}_{\chi\mathrm{PT}}$) are grouped by the number of powers of m_q and Q each contributes to diagrams when it is present. For a suitable choice of regulator and subtraction scheme, such as dimensional regularization with modified minimal subtraction ($\overline{\mathrm{MS}}$), each derivative contributes one power of Q. In the expansions of $\mathcal{L}_{\chi\mathrm{PT}}$ (in powers of Q and m_q respectively), the first terms are given by

$$\mathcal{L}_{\chi\mathrm{PT}} = \frac{F_0^2}{4}\mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger] + \frac{F_0^2}{4}\mathrm{Tr}[\chi U^\dagger + U\chi^\dagger] + \cdots \tag{2.8}$$

The first term yields a canonically normalized kinetic term for the pseudo-Goldstone bosons ϕ_a. The role of the arbitrary coefficient for the second term is played by the empirically-determined constant B_0. The second term gives the leading contribution to the pseudoscalar masses,

$$\begin{aligned} m_\pi^2 &= 2B_0\hat{m} + \cdots, \\ m_K^2 &= B_0(\hat{m} + m_s) + \cdots, \\ m_\eta^2 &= \tfrac{2}{3}B_0(\hat{m} + 2m_s) + \cdots, \end{aligned} \tag{2.9}$$

represented collectively as m_Φ^2.

The result for the pseudoscalar masses indicates that the dual expansion in m_q and Q is unnecessary. Since equation (2.9) shows $m_\Phi^2 \sim B_0 m_q \sim \chi$ and we work in the relativistic regime where $Q^2 \sim m_\Phi^2$, one factor of m_q (i.e., χ) counts as a contribution of order Q^2. Thus, the dual expansion is replaced with a scheme counting powers of Q only. The expansion of the Lagrangian is written $\mathcal{L}_{\chi\mathrm{PT}} = \mathcal{L}_2 + \mathcal{L}_4 + \cdots$ where $\mathcal{L}_d$ contains all terms of order Q^d and $\mathcal{L}_2$ is given by equation (2.8). The conventional parameterization of $\mathcal{L}_4$

i	$L_i^r(m_\rho) \times 10^3$	source
1	0.4 ± 0.3	K_{e4}, $\pi\pi \to \pi\pi$
2	1.35 ± 0.3	K_{e4}, $\pi\pi \to \pi\pi$
3	-3.5 ± 1.1	K_{e4}, $\pi\pi \to \pi\pi$
4	-0.3 ± 0.5	Zweig rule (large N_c)
5	1.4 ± 0.5	ratio $F_K : F_\pi$
6	-0.2 ± 0.3	Zweig rule (large N_c)
7	-0.4 ± 0.2	Gell-Mann–Okubo, with L_5^r, L_8^r
8	0.9 ± 0.3	ratio $(m_s - \hat{m}) : (m_d - m_u)$, $(m_{K^0}^2 - m_{K^+}^2)$, with L_5^r

Table 2.1: Phenomenological values of the coefficients $L_i^r(\mu)$, renormalized in $\overline{\rm MS}$ at $\mu = m_\rho$, taken from J. Bijnens *et al.* [21] (also see reference [20] for discussion).

is

$$\begin{aligned}
\mathcal{L}_4 \;=\; & L_1 \mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger]^2 + L_2 \mathrm{Tr}[\partial_\mu U \partial_\nu U^\dagger]\mathrm{Tr}[\partial^\mu U \partial^\nu U^\dagger] \\
& + L_3 \mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger \partial_\nu U \partial^\nu U^\dagger] + L_4 \mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger]\mathrm{Tr}[\chi U^\dagger + U \chi^\dagger] \\
& + L_5 \mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger (\chi U^\dagger + U \chi^\dagger)] + L_6 \mathrm{Tr}[\chi U^\dagger + U \chi^\dagger]^2 \\
& + L_7 \mathrm{Tr}[\chi U^\dagger - U \chi^\dagger]^2 + L_8 \mathrm{Tr}[\chi U^\dagger \chi U^\dagger + U \chi^\dagger U \chi^\dagger]
\end{aligned} \tag{2.10}$$

as first worked out by Gasser and Leutwyler [17, 18, 19, 20]. Table 2.1 presents phenomenological values of the coefficients L_i renormalized in $\overline{\rm MS}$. In Appendix A we present *Mathematica* routines to symbolically expand $\mathcal{L}_{\chi\rm PT}$ in terms of the meson fields to the order necessary for this work.

The power counting rules for Feynman diagrams, determined by Weinberg [5], establish the relative importance of diagrams in an arbitrary process. A diagram with N_L loops, N_π meson propagators, and constructed from N_d vertices derived from $\mathcal{L}_d$ will contribute

at order Q^D where

$$\begin{aligned} D &= 2 + 2N_L + \sum_d N_d(d-2) \\ &= 4N_L - 2N_\pi + \sum_d dN_d. \end{aligned} \tag{2.11}$$

The leading-order contribution is given by the sum of all tree diagrams constructed exclusively from operators in $\mathcal{L}_2$. At next to leading order one must include all one-loop diagrams constructed exclusively from operators in $\mathcal{L}_2$ and all tree diagrams with one vertex from $\mathcal{L}_4$ and any number of vertices from $\mathcal{L}_2$. Diagrams with more loops and more powers of Q^2 at the vertices must be included at higher order.

On dimensional grounds, the powers of Q^2 which suppress higher-order contributions must be accompanied by a mass scale squared (Λ^2) in the denominator. For powers of Q^2 generated by a loop integral, the factor which appears is $\Lambda^2 \sim (4\pi F_0)^2$. When a factor of Q^d arises from an operator in $\mathcal{L}_d$, the compensating powers of Λ are implicit in the dimension of the coefficient of the operator. We can represent the typical size of coefficients appearing in $\mathcal{L}_d$ as $C_d = \alpha\Lambda^{4-d}$, where α is a dimensionless constant of 'natural' size (discussed below). Since each successive order in the expansion of $\mathcal{L}_{\chi\mathrm{PT}}$ serves to approximate physics at short distance better than the preceding order, the scale relating coefficients in different orders of $\mathcal{L}_{\chi\mathrm{PT}}$ is characteristic of the short-distance physics approximated. Thus, Λ represents scales like the mass of the ρ meson (m_ρ) and the chiral symmetry breaking scale (Λ_χ). We treat all of the scales which suppress powers of Q (e.g., $4\pi F_0$, Λ_χ, m_ρ, ...) equally in terms of the power counting and refer to the common scale as $\Lambda = \Lambda_\chi \simeq 4\pi F_0$.

The convergence of calculations in χPT depends on the assumption of naturalness, that the coefficients appearing in $\mathcal{L}_{\chi\mathrm{PT}}$ are not much larger than their natural size. An anomalously large coefficient, by a factor of order $(\Lambda_\chi/Q)^2$, would indicate that the corresponding operator violates the power counting in equation (2.11) and the systematic expansion breaks down. A priori, there is no reason to expect such large coefficients

and the occurrence of such an anomaly would indicate either the omission of a relevant degree of freedom or fine-tuning in the underlying theory. We get an extremely simple estimate of the natural size of the dimensionless parameter α by recognizing $C_2 \sim F_0^2$ or $\alpha \sim (F_0/\Lambda_\chi)^2 \sim (4\pi)^{-2}$. For instance, we expect the coefficients in $\mathcal{L}_4$ to be of about the natural scale, $L_i \sim C_4 = \alpha$. Manohar and Georgi [22] give a more rigorous derivation of the same estimate based on naive dimensional analysis. The phenomenological values presented in Table 2.1 are slightly smaller than this estimate.

2.3 Coupling to Matter Fields

The so-called matter fields are strongly interacting particles with masses which do not vanish in the chiral limit. Further, in the applications we consider, the masses of the matter fields are large enough that the fields can be treated in a non-relativistic formalism. The matter fields do not form representations of the full chiral symmetry group G_χ, but instead form irreducible representations under the approximate $\mathrm{SU(3)_V}$ symmetry of the vacuum. In this section we describe a prescription for including the baryon octet fields (the N, Σ, and Ξ isomultiplets and the Λ) and decuplet fields (the Δ, Σ^*, and Ξ^* isomultiplets and the Ω^-) in the effective Lagrangian. For a review of this subject see reference [13].

Under a G_χ transformation (L, R), fields in the fundamental representation of $\mathrm{SU(3)_V}$ transform as $\psi \to H\psi$ where $H \in \mathrm{SU(3)}$ is a function of L, R, and the pseudo-Goldstone bosons U. The pure $\mathrm{SU(3)_V}$ transformations correspond to taking $H = L = R$. The form of H is determined by specifying the transformation property of $u = \sqrt{U} = e^{i\Phi/2F_0}$ as

$$u \xrightarrow{G_\chi} RuH^\dagger = HuL^\dagger \tag{2.12}$$

which, when solved for H, gives

$$H = \sqrt{RUL^\dagger}\, L\sqrt{U^\dagger} = \sqrt{LU^\dagger R^\dagger}\, R\sqrt{U}. \tag{2.13}$$

(The field u is an alternative representation for the pseudo-Goldstone bosons, corresponding to a different choice for the spontaneously broken generators of G_χ in the CCWZ prescription [7, 23, 24].) Because the matrix H is a function of the pseudo-Goldstone boson field $U(x)$, H also implicitly depends on the space-time coordinate x^μ. To compensate for the x^μ-dependence of H, we introduce a vector field

$$V_\mu = \tfrac{1}{2}(u^\dagger \partial_\mu u + u \partial_\mu u^\dagger) \tag{2.14}$$

which transforms under G_χ as $V_\mu \to H V_\mu H^\dagger - (\partial_\mu H) H^\dagger$ and construct the covariant derivative $D_\mu \psi = \partial_\mu \psi + V_\mu \psi$ such that $D_\mu \psi \to H(D_\mu \psi)$.

Armed with the $\mathrm{SU(3)_V}$ formalism for the fundamental representation, incorporating the spin-$\frac{1}{2}$ octet and spin-$\frac{3}{2}$ decuplet fields is relatively straight forward. The octet baryons are encoded as a 3×3 matrix of Dirac fields

$$B = \begin{pmatrix} \Sigma^0/\sqrt{2} + \Lambda/\sqrt{6} & \Sigma^+ & p \\ \Sigma^- & -\Sigma^0/\sqrt{2} + \Lambda/\sqrt{6} & n \\ \Xi^- & \Xi^0 & -2\Lambda/\sqrt{6} \end{pmatrix} \tag{2.15}$$

and transform in the adjoint representation of $\mathrm{SU(3)_V}$, i.e.,

$$B \xrightarrow{G_\chi} H B H^\dagger, \tag{2.16}$$

$$D_\mu B = \partial_\mu B + [V_\mu, B]. \tag{2.17}$$

The spin-$\frac{3}{2}$ decuplet baryons form a fully-symmetric rank-3 tensor under $\mathrm{SU(3)_V}$ characterized by

$$T^{abc} \xrightarrow{G_\chi} H^{aa'} H^{bb'} H^{cc'} T^{a'b'c'}, \tag{2.18}$$

$$(D_\mu T)^{abc} = \partial_\mu T^{abc} + V_\mu^{aa'} T^{a'bc} + V_\mu^{bb'} T^{ab'c} + V_\mu^{cc'} T^{abc'}. \tag{2.19}$$

The components of T are Rarita-Schwinger fields subject to the auxiliary constraint

$\gamma_\mu T^\mu = 0$ and are identified as follows:

$$\begin{array}{lll} T^{111} = \Delta^{++}, & T^{113} = \Sigma^{*+}/\sqrt{3}, & T^{133} = \Xi^{*0}/\sqrt{3}, \\ T^{112} = \Delta^{+}/\sqrt{3}, & T^{123} = \Sigma^{*0}/\sqrt{6}, & T^{233} = \Xi^{*-}/\sqrt{3}, \\ T^{122} = \Delta^{0}/\sqrt{3}, & T^{223} = \Sigma^{*-}/\sqrt{3}, & \\ T^{222} = \Delta^{-}, & & T^{333} = \Omega^{-}. \end{array} \tag{2.20}$$

The minimal-coupling Lagrangian for the B and T fields is

$$\mathcal{L}_0 = \overline{B}(i\not{D} - m_B)B - \overline{T}_\mu(i\not{D} - m_T)T^\mu \tag{2.21}$$

with implied pair-wise summation of the dangling chiral indices.

The effective Lagrangian for matter fields will also explicitly include U and χ which transform under G_χ as $O \to ROL^\dagger$. For convenience, we define fields $\widetilde{O} = u^\dagger O u^\dagger$ such that $\widetilde{O} \to H\widetilde{O}H^\dagger$ under G_χ transformations and coupling B and T to the new fields $\widetilde{O}$ becomes transparent. The result for U is trivial, $\widetilde{U} = u^\dagger U u^\dagger = 1$; for $\partial_\mu U$ we get the more interesting result

$$\begin{aligned} A_\mu &= \tfrac{i}{2}u^\dagger\partial_\mu U u^\dagger \;=\; \tfrac{-i}{2}u\partial_\mu U^\dagger u \\ &= \tfrac{i}{2}(u^\dagger\partial_\mu u - u\partial_\mu u^\dagger), \end{aligned} \tag{2.22}$$

where the factor of $\frac{i}{2}$ is included so A_μ is hermitian. To replace χ (and $\chi^\dagger$) we choose the (anti-)hermitian combinations

$$\chi_\pm = u^\dagger\chi u^\dagger \pm u\chi^\dagger u. \tag{2.23}$$

As an example the order-Q^2 Lagrangian of χPT, equation (2.8), can be rewritten in terms of A_μ and $\chi_\pm$ as $\mathcal{L}_2 = F_0^2\mathrm{Tr}[A_\mu A^\mu] + \frac{1}{4}F_0^2\mathrm{Tr}[\chi_+]$.

We briefly mention two relations which help to identify a minimal set of terms which

are needed at higher order, see Fearing and Scherer [25] for a more complete account.[1] The first is the chain rule in the form

$$u^\dagger \partial_\mu O u^\dagger = D_\mu \widetilde{O} - i\{A_\mu, \widetilde{O}\} \tag{2.24}$$

which allows the replacement of multiple derivatives of U with factors of A_μ and covariant derivatives of A_μ, e.g., $u^\dagger \partial_\mu \partial_\nu U u^\dagger = -2iD_\mu A_\nu - 2\{A_\mu, A_\nu\}$. Two consequences of equation (2.24) are 1) $D_\mu A_\nu$ is symmetric in (μ, ν) and 2) despite the dependence of $\chi_\pm$ on $u(x)$, covariant derivatives of $\chi_\pm$ are unnecessary since $D_\mu \chi_\pm = i\{A_\mu, \chi_\mp\}$. The second simplifying relation gives the field strength associated with V_μ in terms of the field A_μ

$$G^{(V)}_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu] = [A_\mu, A_\nu]. \tag{2.25}$$

Because antisymmetric covariant derivatives result in factors of the field strength, like $(D_\mu D_\nu - D_\nu D_\mu)\psi = G^{(V)}_{\mu\nu}\psi$ in the fundamental representation, we can treat all multiple covariant derivatives of any field as implicitly symmetric without loss of generality. To summarize, the G_χ-invariant effective Lagrangian for the baryon fields B and T is written in terms of the building blocks B, T, A_μ, $\chi_\pm$, and fully-symmetric covariant derivatives of them. The field strength $G^{(V)}_{\mu\nu}$ and covariant derivatives of $\chi_\pm$ may be omitted in favor of terms involving more factors of A_μ.

2.4 Non-Relativistic Power Counting

The appearance of baryon masses in equation (2.21) wrecks the power counting of χPT. Derivatives of heavy fields would contribute factors of 'hard' momenta, where $p_0 \geq m_B$, and loop diagrams could result in explicit factors of m_B or m_T in the numerator. These diagrams are not suppressed relative to 'lower order' diagrams since $m_B \sim m_T \sim \Lambda_\chi$. We use a non-relativistic approach adapted from the heavy baryon chiral perturbation

[1] When using the external field method to impliment *local* chiral symmetry, these relations are modified by additional terms involving external (i.e., non-propagating) gauge fields.

theory (HBχPT) developed by Jenkins and Manohar [26, 27, 28, 29] to address this problem. We start with the velocity-dependent baryon fields of HBχPT then specialize to working in the rest frame of the baryons.

The momentum of a heavy baryon field is decomposed as $p^\mu = m_B v^\mu + k^\mu$ in HBχPT, where the residual momentum k^μ is assumed small and reflects how much the baryon is off mass-shell. The velocity-dependent baryon fields are defined using v^μ as

$$\begin{aligned} B_v(x) &= \frac{1+\not{v}}{2}\, e^{i m_B v\cdot x}\, B(x), \\ T_v(x) &= \frac{1+\not{v}}{2}\, e^{i m_B v\cdot x}\, T(x), \end{aligned} \tag{2.26}$$

where the octet mass m_B is used in the definition of the decuplet T_v to avoid x^μ-dependent phases in the Lagrangian coupling B_v to T_v. The factor of $\frac{1}{2}(1+\not{v})$ projects out the particle components of the Dirac spinors. The anti-particle components are implicitly integrated out of the theory and the effects of virtual baryon loops are absorbed into terms of the effective Lagrangian suppressed by powers of $1/m_B$.

This representation of the heavy baryon fields permits a sensible power counting scheme. Derivatives of the velocity-dependent fields give factors of the small residual momenta k^μ in place of the hard momenta p^μ. The minimal-coupling Lagrangian corresponding to equation (2.21) becomes

$$\mathcal{L}_v = \overline{B_v}(iv\cdot D)B_v - \overline{T_{v_\mu}}(iv\cdot D)T_v{}^\mu + \Delta m \overline{T_{v_\mu}} T_v{}^\mu + \cdots, \tag{2.27}$$

where $\Delta m = m_T - m_B \simeq 300$ MeV is considered of order Q and the ellipsis denotes higher-order terms, such as $-\overline{B_v} D^2 B_v / 2m_B$, induced by the integration over anti-particle degrees of freedom. By removing m_B and m_T from the baryon propagators, loop diagrams will not introduce positive powers of the masses in diagrams (except in a case considered toward the end of the section).

The Dirac structure of the fields can be eliminated in favor of two-component spinors

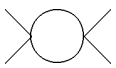
because we have integrated out the anti-particle components. We choose to explicitly work in the frame $v^\mu = (1,0,0,0)$ and drop the subscript v on heavy fields. This choice of frame simplifies the spinor-related notation; for instance, the auxiliary condition on Rarita-Schwinger fields, $\gamma_\mu T^\mu = 0$, implies constraints which reduce to $T^0 = 0$ and $\vec{\sigma}\cdot\vec{T} = 0$. In this frame the resulting non-relativistic framework is equivalent to a Lagrangian formulation of the time-ordered approach discussed by Weinberg [30, 31].

The power counting for diagrams with a single heavy field generalizes equation (2.11). Since $m_B \sim m_T \sim \Lambda_\chi$ we do not need to distinguish between corrections suppressed by Q/m_B and Q/Λ_χ. As before, each meson propagator counts as Q^{-2} and each loop integration gives a factor of Q^4. Heavy field propagators each contribute Q^{-1} as seen from equation (2.27). We let N_i represent the number of vertices in a diagram which contribute d_i factors of Q and contain n_i heavy fields. A diagram in the single heavy-particle sector with N_L loops, N_π meson propagators, and N_I baryon propagators contributes at order Q^D where [30]

$$\begin{aligned} D &= 4N_L - 2N_\pi - N_I + \sum_i d_i N_i \\ &= 1 + 2N_L + \sum_i N_i(d_i + \tfrac{1}{2}n_i - 2). \end{aligned} \tag{2.28}$$

The primary difference from equation (2.11) is that each order is suppressed by Q/Λ_χ rather than $(Q/\Lambda_\chi)^2$ relative to the preceeding order.

For diagrams with two or more heavy particles, the power counting is complicated by infrared divergences in some loop integrals [31]. Consider a heavy-particle bubble diagram; for octet fields the loop integral takes the form



$$\longrightarrow \int \frac{d^d q}{(2\pi)^d}\, \frac{i}{E + q_0 + i\epsilon}\, \frac{i}{E - q_0 + i\epsilon} = \frac{i}{2E}\int \frac{d^{d-1}q}{(2\pi)^{d-1}}, \tag{2.29}$$

where the incoming particles have energies E and momenta $\pm\vec{p}$. The divergence arises as $E \to 0$ because the q_0 contour is pinched between the poles of the static propagators. By

resumming the kinetic energy operator, e.g., $B^\dagger \nabla^2 B/2m_B$, into the heavy field propagator, the infrared divergences are removed. In dimensional regularization the result for the modified loop integral becomes

$$\begin{aligned} \int \frac{d^d q}{(2\pi)^d} \, & \frac{i}{E + q_0 - |\vec{q}\,|^2/2m_B + i\epsilon} \, \frac{i}{E - q_0 - |-\vec{q}\,|^2/2m_B + i\epsilon} \\ &= \int \frac{d^{d-1} q}{(2\pi)^{d-1}} \, \frac{i}{2E - q^2/m_B + 2i\epsilon} \\ &= \frac{-i m_B}{4\pi} \left(\frac{-2m_B E - i\epsilon'}{4\pi} \right)^{\frac{d-3}{2}} \Gamma(\tfrac{3-d}{2}) = \frac{m_B \sqrt{2m_B E}}{4\pi} \end{aligned} \tag{2.30}$$

which is well-defined in the infrared. Setting $E = p^2/2m_B + \cdots$, the leading behavior of the loop integral is $\sim m_B Q$, not $\sim Q^2$ as expected from equation (2.28).

By including the kinetic energy in the heavy field propagator we are treating the operator $\nabla^2/2m_B$ on an equal footing with ∂_0. For consistency, powers of the integration variable q_0 should be counted as $q_0 \sim Q^2/m_B$ in loops leading to nearly-infrared-divergent behavior. Heavy field propagators count as m_B/Q^2, loop integrals now give $dq_0\, d^3q \sim Q^5/m_B$, and meson propagators contribute $1/Q^2$ as before. Thus the loop integral above counts as $\sim (Q^5/m_B)(m_B/Q^2)^2 = m_B Q$ which reconciles the power counting scheme with the result of equation (2.30). A general Feynman diagram with heavy particles in the initial and final states may contribute at order Q^D where

$$D = 5N_L - 2N_\pi - 2N_I + \sum_i d_i N_i \tag{2.31}$$

with the same notation as used in equation (2.28). If the loop integrals in a diagram are not infrared divergent, as with crossed pion exchange, the actual contribution will be of higher order consistent with equation (2.28). The systematics of the power counting for diagrams with two or more heavy particles is developed in references [10, 31, 32, 33, 34, 35, 36]. The presence of a bound state or resonance near threshold can also complicate the power counting [37, 38, 39]. This complication arises in nucleon-nucleon scattering and the

power counting was studied in that context by several authors, in particular Kaplan, Savage, and Wise [40, 41]. In this work we do not consider systems requiring this special treatment and refer the reader to the literature for a discussion of the relevant power counting schemes [42, 43, 44].

Chapter 3

Decuplet Self-Energy in Nuclear Matter

Strong interaction effects shift the self-energy of hadrons in nuclear matter from the free-space values. This effect for the spin-$\frac{1}{2}$ octet baryons was studied in an effective field theory framework by Savage and Wise [45]. Using the formalism reviewed in Chapter 2, we calculate the self-energy shifts of the spin-$\frac{3}{2}$ decuplet baryons in nuclear matter at leading order. The self-energy shifts of the decuplet baryons, particularly of the Δ isomultiplet, are relevant in studies of meson-nucleus scattering [46, 47] and of stellar and neutron star matter [48, 49]. For the Δ isomultiplet the self-energy shifts have also been examined in various phenomenological models [49, 50, 51] and in QCD sum rules [52]. The work described here differs from the earlier approaches by extending the calculation to multiplets of chiral $SU(3)_V$ and by including contact diagrams necessary for a consistent and systematic momentum expansion. This chapter makes a minor correction to a prior publication [11] and includes a more detailed discussion of the self-energy of the Δ isomultiplet in nuclear matter near saturation density.

The first section discusses how effects of nuclear matter are described in the effective field theory. In section 3.2 we determine what Feynman diagrams contribute to the self-energy shifts at leading order and construct the relevant effective Lagrangian. Section 3.3 presents the main results of the calculation. Finally, a discussion and interpretation of the results is contained in section 3.4.

3.1 Effects of Nuclear Matter

In nuclear matter the propagation of decuplet baryons is effected by interactions with the background medium. In particular the self-energy $\mathcal{E}$, i.e., the location of the k_0 pole

in the decuplet two-point function, is shifted relative to free space. As an additional consideration, the background medium breaks Lorentz boost invariance by specifying a unique frame, the zero-momentum frame of the nuclear matter. The self-energy in free space of a decuplet baryon with momentum $k = |\vec{k}\,|$ is

$$\begin{aligned} \mathcal{E}_{\rm vac}(k^2) &= \sqrt{m_T^2 + k^2 - i m_T \Gamma_{\rm vac}} - m_B \\ &= \left(\Delta m + \frac{k^2}{2m_T} + \cdots\right) - \frac{i}{2}\Gamma_{\rm vac}\left(1 - \frac{k^2}{2m_T^2} + \cdots\right) + \mathcal{O}(\Gamma_{\rm vac}^2), \end{aligned} \tag{3.1}$$

where $\Gamma_{\rm vac}$ is the free-space decay rate and $\Delta m = m_T - m_B$. Throughout this chapter, the octet mass m_B has been implicitly subtracted whenever we refer to the decuplet self-energy, as discussed in section 2.4. Equation (3.1) contains only two decuplet parameters; the full k^2-dependence of the self-energy is determined by Lorentz invariance. The corresponding expression for a decuplet baryon in nuclear matter can be written

$$\mathcal{E}_{\rm nm}(k^2) = (\Delta m^* + \alpha k^2 + \cdots) - \tfrac{i}{2}\Gamma^*(1 + \beta k^2 + \cdots) \tag{3.2}$$

in which the k^2-dependence (e.g., α, β, ...) cannot be determined from symmetry arguments alone.

Although invariance under Lorentz boosts is lost, the remaining rotational symmetry constrains the spin-dependence of the self-energy $\mathcal{E}_{\rm nm}$. Because the nuclear medium is rotationally invariant, the only preferred spacial directions are along the decuplet baryon three-momentum $\vec{k}$ and spin $\vec{S}$. As a function of $\vec{k}$ and $\vec{S}$, the self-energy depends on only the combinations k^2 and $\vec{k}\cdot\vec{S}$. ($S^2 = \frac{15}{4}$ is trivial.) Further, parity invariance of the strong interaction dictates that the self-energy depends on even powers of $\vec{k}$, which means replacing $\vec{k}\cdot\vec{S}$ with $(\vec{k}\cdot\vec{S}\,)^2 = k^2h^2$ in terms of the baryon helicity h. Consequently, the self-energy in nuclear matter is diagonal in the baryon helicity states and takes the values $\mathcal{E}_{\rm nm}^{(1/2)}(k^2)$ for $h = \pm\frac{1}{2}$ and $\mathcal{E}_{\rm nm}^{(3/2)}(k^2)$ for $h = \pm\frac{3}{2}$. In the limit of vanishing momentum, the rotational symmetry is elevated to full SU(2) invariance, and the self-energy must be independent of the decuplet spin projection along any direction. In terms of the

parameters in equation (3.2), Δm^* and Γ^* are helicity-independent, while the coefficients of powers of k^2 (e.g., α, β, ...) depend on $|h|$.

The lowest order in a density expansion for nuclear matter is a Fermi gas of non-interacting protons and neutrons with Fermi momenta $p_F^{(p)}$ and $p_F^{(n)}$ respectively. In this framework, the characteristic momenta relevant in the chiral derivative expansion are $p_F^{(p)}$, $p_F^{(n)}$, and k the decuplet baryon momentum. Since the density of a degenerate Fermi gas is given by $d_F = p_F^3/3\pi^2$, the density expansion for nuclear matter is consistent with the chiral derivative expansion. The static nucleon propagator in nuclear matter with Fermi momentum p_F is [45]

$$\widetilde{S}_{\rm nm}(q_0,\vec{q}\,) = \frac{i\Theta(|\vec{q}\,|-p_F)}{q_0+i\epsilon} + \frac{i\Theta(p_F-|\vec{q}\,|)}{q_0-i\epsilon} \tag{3.3}$$

at lowest order in the nuclear density. The modified nucleon propagator reflects the presence of the background medium through two effects, nucleon states inaccessible due to Pauli-blocking and nucleon-hole intermediate states allowed for $|\vec{q}\,| \leq p_F$.

As the location of the k_0 pole in the exact decuplet two-point function, the self-energy in nuclear matter is given by the solution of

$$\mathcal{E}_{\rm nm}(k^2) - \Delta m - \Sigma_{\rm nm}(\mathcal{E}_{\rm nm}(k^2),\vec{k}\,) = 0, \tag{3.4}$$

where $\Sigma_{\rm nm}$ is the *proper* self-energy for nuclear matter, i.e., $-i\Sigma_{\rm nm}$ is the sum of connected one-particle-irreducible diagrams in the two-point function. What we calculate is the self-energy shift, $\delta\mathcal{E} = \mathcal{E}_{\rm nm} - \mathcal{E}_{\rm vac}$, obtained from equation (3.4) by expanding the proper self-energy for nuclear matter about the free-space pole $\mathcal{E}_{\rm vac}$,

$$\begin{aligned} \delta\mathcal{E}(k^2) &= \Sigma_{\rm nm}(\mathcal{E}_{\rm vac}(k^2),\vec{k}\,) - \Sigma_{\rm vac}(\mathcal{E}_{\rm vac}(k^2),\vec{k}\,) \\ &\quad + \Big\{\mathcal{E}_{\rm nm}(k^2) - \mathcal{E}_{\rm vac}(k^2)\Big\}\,\frac{\partial}{\partial k_0}\Sigma_{\rm nm}(k_0,\vec{k}\,)\,\Big|_{k_0=\mathcal{E}_{\rm vac}(k^2)} + \cdots \end{aligned} \tag{3.5}$$

and in turn expanding $\mathcal{E}_{\rm vac}(k^2)$ in powers of Q as shown in equation (3.1). Note that

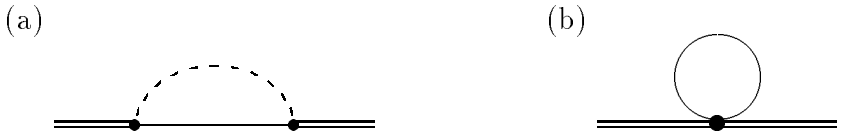


Figure 3.1: Feynman diagrams for the self-energy shifts at leading order of decuplet baryons in nuclear matter, (a) meson-nucleon loop diagrams and (b) contact diagrams. Double lines represent decuplet baryons, single lines represent nucleons, and dashed lines represent pseudo-Goldstone bosons.

only Feynman diagrams with an internal nucleon propagator contribute to the difference on the first line. Because the chiral expansion of the proper self-energy $\Sigma_{\rm nm}$ starts at order Q^2, successive terms in equation (3.5) only contribute at higher order. (A very important exception to the last point is discussed in detail in subsection 3.4.2.) The real part of the self-energy shift $\delta E^{(h)}(k^2)$ modifies the decuplet baryon energy-momentum dispersion relation from the free-space form. The change in the decuplet decay rate is given by $\delta\Gamma^{(h)}(k^2) = -2\,{\rm Im}[\delta\mathcal{E}^{(h)}(k^2)]$ at leading order.

3.2 The Effective Lagrangian

At leading order in the chiral expansion the self-energy shifts $\delta\mathcal{E}$ coincide with the difference $\Sigma_{\rm nm}(\mathcal{E}_{\rm vac},\vec{k}\,) - \Sigma_{\rm vac}(\mathcal{E}_{\rm vac},\vec{k}\,)$. For diagrams contributing to the proper self-energies $\Sigma_{\rm nm}$ and $\Sigma_{\rm vac}$, the power counting is given by equation (2.28) for the single heavy-particle sector. Because a nucleon propagator is required for a non-zero difference, the leading contribution arises from one-loop diagrams in which the vertices satisfy $d + \frac{1}{2}n - 2 = 0$ and contributes at order Q^3. Topologically, the one-loop diagrams are constructed from either two three-leg vertices or a single four-leg vertex; the two possibilities with a nucleon propagator are shown in Figure 3.1.

The vertices of meson-nucleon loop diagrams are derived from operators coupling the fields $TB^\dagger A_\mu$ in the order-Q Lagrangian. The most general Lagrangian of that form,

invariant under G_χ and parity, is [27, 28]

$$\mathcal{L}_{\rm a} = -\mathcal{C}\epsilon^{abc}\left(\vec{A}^{ad}\cdot[B^{\dagger be}\,\vec{T}^{cde}] + \vec{A}^{da}\cdot[\vec{T}^{\dagger cde}\,B^{eb}]\right), \tag{3.6}$$

where square brackets denote summation on spinor indices. The value of the coefficient, $|\mathcal{C}| \simeq 1.53$, is empirically determined from $T \to B\pi$ decays [29]. The meson-nucleon loop diagrams contribute to the self-energy shifts of the Δ and Σ^* isomultiplets only; the Lagrangian $\mathcal{L}_{\rm a}$ does not couple Ξ^* or Ω^- baryons to a nucleon and single pseudo-Goldstone boson, a consequence of strangeness conservation.

For contact diagrams the vertex contributes $d = 0$ powers of Q, so the relevant Lagrangian contains only simple products of field operators $TB(TB)^\dagger$. To construct the four-baryon operators we start with the spin and chiral structures of the product TB. The operator product decomposes under rotational SU(2) as $\frac{3}{2}\otimes\frac{1}{2} = \mathbf{2}\oplus\mathbf{1}$ and under chiral $\mathrm{SU(3)_V}$ as $\mathbf{10}\otimes\mathbf{8} = \mathbf{35}\oplus\mathbf{27}\oplus\mathbf{10}\oplus\mathbf{8}$. By coupling products TB and $(TB)^\dagger$ to form chiral and rotational singlets, we find eight linearly-independent four-baryon operators which contribute to the self-energy shifts at leading order. We choose to write the effective Lagrangian which contains these operators as[1]

$$\begin{aligned}
\mathcal{L}_{\rm b} \;=\; & -\frac{d_1}{2F_0^2}[T^{\dagger jabc}\,T^{jabc}][B^{\dagger ed}B^{de}] - \frac{d_5}{2F_0^2}[T^{\dagger jabc}\sigma^k\,T^{jabc}][B^{\dagger ed}\sigma^k B^{de}] \\
& -\frac{d_2}{2F_0^2}[T^{\dagger jabc}\,T^{jabd}][B^{\dagger ed}B^{ce}] - \frac{d_6}{2F_0^2}[T^{\dagger jabc}\sigma^k\,T^{jabd}][B^{\dagger ed}\sigma^k B^{ce}] \\
& -\frac{d_3}{2F_0^2}[T^{\dagger jabc}\,T^{jabd}][B^{\dagger ce}B^{ed}] - \frac{d_7}{2F_0^2}[T^{\dagger jabc}\sigma^k\,T^{jabd}][B^{\dagger ce}\sigma^k B^{ed}] \\
& -\frac{d_4}{2F_0^2}[T^{\dagger jabc}\,T^{jade}][B^{\dagger bd}B^{ce}] - \frac{d_8}{2F_0^2}[T^{\dagger jabc}\sigma^k\,T^{jade}][B^{\dagger bd}\sigma^k B^{ce}],
\end{aligned} \tag{3.7}$$

where (j,k) are vector indices, $(a$–$e)$ are chiral indices, and square brackets indicate sums over spinor indices. Factors of F_0^{-2} are included in $\mathcal{L}_{\rm b}$ to make the coefficients d_i dimensionless.

[1] In our prior publication we used a different convention for the pion decay constant, specifically $f = \sqrt{2}F_0 \simeq 132$ MeV.

The values of the eight coefficients d_i have not yet been experimentally determined. However, because the baryon helicity is conserved in the self-energy diagrams, the terms in $\mathcal{L}_{\rm b}$ of the form $[T^{\dagger j}\sigma^k T^j][B^\dagger\sigma^k B]$ do not generate self-energy shifts and the results are independent of d_5–d_8. Values for the remaining four coefficients d_1–d_4 are important for quantitative predictions. Unfortunately, knowledge of the values will likely have to wait until low-energy octet-decuplet scattering data become available. In section 3.4 we discuss constraints on the coefficients d_i from the further assumption of SU(6) spin-flavor symmetry.

Two characteristics of the contact diagrams allow some predictions which are independent of the coefficients d_i. In a contact diagram there is no 'intermediate state' which corresponds to an allowed decay of the decuplet baryon so the d_i do not appear in the imaginary part of the self-energy shift. Also the contact diagrams are independent of the momentum on the external line so the d_i-dependence is restricted to the k^2-independent parts of the self-energy shifts. In terms of the parameters in equation (3.2), at leading order only the helicity-independent quantity Δm^* depends on the coefficients d_i. Quantities independent of the coefficients d_i, for which we present quantitative results, are the helicity-splitting of the self-energy shifts $\Delta^{(h)}E = \delta E^{(1/2)} - \delta E^{(3/2)}$ and the decuplet decay rates in nuclear matter $\Gamma^{(h)}_{\rm nm} = \Gamma_{\rm vac} + \delta\Gamma^{(h)}$.

3.3 Self-Energy Shift Results

From the effective Lagrangians, equations (3.6, 3.7), we calculate the self-energy shifts $\delta E^{(h)}$ and $\delta\Gamma^{(h)}$ of the spin-$\frac{3}{2}$ decuplet baryons in nuclear matter to leading order ($\sim Q^3$) in the chiral momentum expansion. For convenience we introduce the two 'threshold' mass scales

$$\begin{aligned} \mu &= \sqrt{(m_\Delta - m_N)^2 - m_\pi^2} \simeq 255 \text{ MeV}, \\ \tilde{\mu} &= \sqrt{m_K^2 - (m_{\Sigma^*} - m_N)^2} \simeq 215 \text{ MeV}, \end{aligned} \tag{3.8}$$

and present the results in terms of the functions

$$
\begin{aligned}
g_{\frac{1}{2}}(k,p,m) &= \frac{1}{16m^3k^3}(m^2-p^2+k^2)\left\{(m^2-p^2+k^2)^2+4m^2k^2\right\}, \qquad (3.9)\\
\mathcal{G}_{\frac{1}{2}}(k,p,m) &= \frac{-1}{12m^2k^2}\left\{3(m^2-p^2+k^2)^2-2k^2(p^2+3k^2-9m^2)\right\}\\
&\quad+\frac{m}{p}g_{\frac{1}{2}}(k,p,m)\ln\left|\frac{m^2-(p-k)^2}{m^2-(p+k)^2}\right|,\\
g_{\frac{3}{2}}(k,p,m) &= \frac{-1}{16m^3k^3}(m^2-p^2+k^2)\left\{(m^2-p^2+k^2)^2-12m^2k^2\right\}, \qquad (3.10)\\
\mathcal{G}_{\frac{3}{2}}(k,p,m) &= \frac{1}{12m^2k^2}\left\{3(m^2-p^2+k^2)^2-2k^2(p^2+3k^2+15m^2)\right\}\\
&\quad+\frac{m}{p}g_{\frac{3}{2}}(k,p,m)\ln\left|\frac{m^2-(p-k)^2}{m^2-(p+k)^2}\right|,
\end{aligned}
$$

which are constructed such that $\mathcal{G}_h(k,p,m)\to 0$ as $k\to 0^+$.

For the negatively-charged member of each isomultiplet we find the real parts of the self-energy shifts are

$$
\begin{aligned}
\delta E^{(h)}_{\Delta^-} &= \frac{1}{18\pi^2F_0^2}\left\{(3d_1){p_F^{(p)}}^3+(3d_1+3d_2){p_F^{(n)}}^3\right\} \qquad (3.11)\\
&\quad+\frac{\mathcal{C}^2}{48\pi^2F_0^2}\left\{\mu^3\ln\left|\frac{(\mu-p_F^{(n)})^2-k^2}{(\mu+p_F^{(n)})^2-k^2}\right|\right.\\
&\qquad\left.+\frac{4}{3}{p_F^{(n)}}^3+4\mu^2p_F^{(n)}+\mu^2p_F^{(n)}\mathcal{G}_h(k,p_F^{(n)},\mu)\right\},\\
\delta E^{(h)}_{\Sigma^{*-}} &= \frac{1}{18\pi^2F_0^2}\left\{(3d_1+d_3){p_F^{(p)}}^3+(3d_1+2d_2+d_3+d_4){p_F^{(n)}}^3\right\} \qquad (3.12)\\
&\quad+\frac{\mathcal{C}^2}{144\pi^2F_0^2}\left\{2\tilde{\mu}^3\arccos\left(\frac{\tilde{\mu}^2-{p_F^{(n)}}^2+k^2}{\sqrt{(\tilde{\mu}^2+{p_F^{(n)}}^2+k^2)^2-4k^2{p_F^{(n)}}^2}}\right)\right.\\
&\qquad\left.+\frac{4}{3}{p_F^{(n)}}^3-4\tilde{\mu}^2p_F^{(n)}-\tilde{\mu}^2p_F^{(n)}\mathcal{G}_h(k,p_F^{(n)},i\tilde{\mu})\right\},\\
\delta E_{\Xi^{*-}} &= \frac{1}{18\pi^2F_0^2}\left\{(3d_1+2d_3){p_F^{(p)}}^3+(3d_1+d_2+2d_3+d_4){p_F^{(n)}}^3\right\}, \qquad (3.13)\\
\delta E_{\Omega^-} &= \frac{1}{18\pi^2F_0^2}(3d_1+3d_3)\left\{{p_F^{(p)}}^3+{p_F^{(n)}}^3\right\}. \qquad (3.14)
\end{aligned}
$$

The helicity splitting of the Δ^- and Σ^{*-} self-energies are graphed as functions of $p_F^{(n)}$ and k in Figure 3.2.

At leading order the imaginary part of the self-energies, or resonance widths, are shifted in nuclear matter for the Δ isomultiplet only. We find the Δ^- resonance widths are

$$\Gamma^{(h)}_{\mathrm{nm},\Delta^-} = \Big\{ \tfrac{1}{2}\left(1 + g_h(k, p_F^{(n)}, \mu)\right) \Theta\left(k + p_F^{(n)} - \mu\right) \Theta\left(\mu^2 - (p_F^{(n)} - k)^2\right) + \Theta\left(\mu - p_F^{(n)} - k\right) + \Theta\left(k - p_F^{(n)} - \mu\right) \Big\} \Gamma_{\mathrm{vac},\Delta^-} \tag{3.15}$$

where $\Gamma_{\mathrm{vac},\Delta^-} = \mu^3 \mathcal{C}^2 / 12\pi F_0^2$ is the Δ^- resonance width in free space at leading order. Figure 3.3 presents the resonance widths as functions of $p_F^{(n)}$ and k.

Results for the other baryons in the decuplet can be determined from the expressions for the negatively-charged members. For the member of each isomultiplet with the most positive charge (Δ^{++}, Σ^{*+}, and Ξ^{*0}) the self-energy shifts are obtained by exchanging $p_F^{(p)}$ and $p_F^{(n)}$ in equations (3.11–3.13, 3.15). The self-energy shifts of the remaining decuplet baryons are given by the following relations:

$$\delta\mathcal{E}_{\Delta^+} = \tfrac{1}{3}\left(2\delta\mathcal{E}_{\Delta^{++}} + \delta\mathcal{E}_{\Delta^-}\right), \tag{3.16}$$

$$\delta\mathcal{E}_{\Delta^0} = \tfrac{1}{3}\left(\delta\mathcal{E}_{\Delta^{++}} + 2\delta\mathcal{E}_{\Delta^-}\right), \tag{3.17}$$

$$\delta\mathcal{E}_{\Sigma^{*0}} = \tfrac{1}{2}\left(\delta\mathcal{E}_{\Sigma^{*+}} + \delta\mathcal{E}_{\Sigma^{*-}}\right). \tag{3.18}$$

Clearly, in neutron-proton symmetric nuclear matter the baryons within isomultiplets remain degenerate up to the splitting between helicity states.

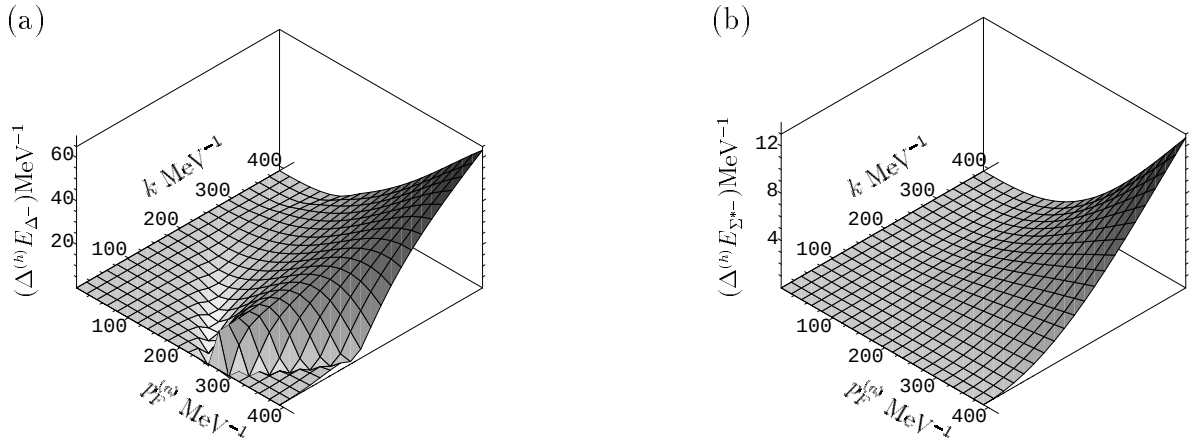


Figure 3.2: Leading-order helicity splitting of the decuplet baryon self-energy in nuclear matter $\Delta^{(h)}E = \delta E^{(1/2)} - \delta E^{(3/2)}$ as a function of neutron Fermi momentum $p_F^{(n)}$ and baryon momentum k, (a) for the Δ^- baryon and (b) for the Σ^{*-} baryon.

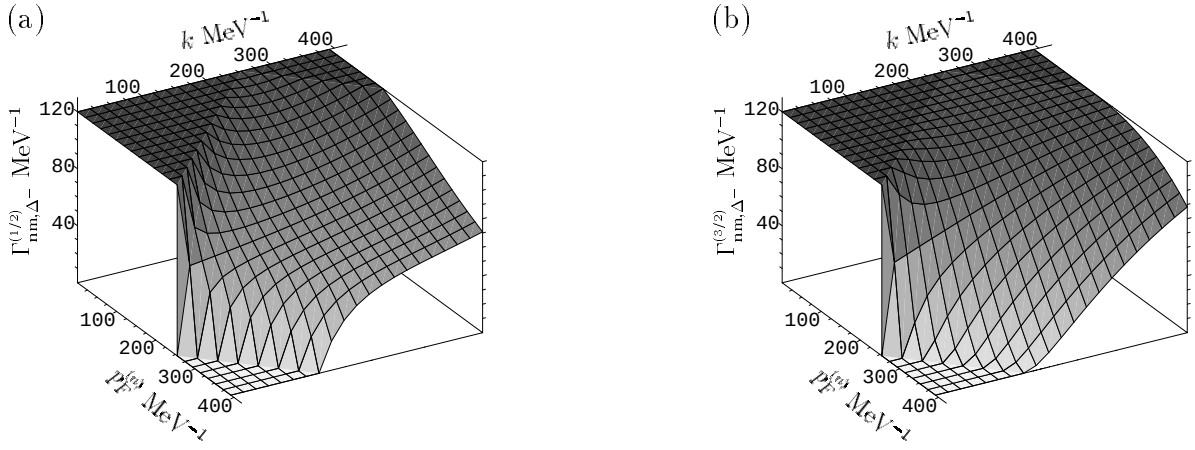


Figure 3.3: Leading-order Δ^- resonance widths in nuclear matter as functions of neutron Fermi momentum $p_F^{(n)}$ and baryon momentum k, (a) for helicities $\pm\frac{1}{2}$ and (b) for helicities $\pm\frac{3}{2}$.

Figure 3.4: (a) Infrared divergent baryon-meson box diagram and (b) overlapping meson-loop diagram obtained from the box diagram by contracting nucleon lines 2 and 3. The two-loop diagram in (b) avoids the infrared divergence of (a).

3.4 Discussion and Conclusion

3.4.1 Corrections at Higher-Order

Contrary to the discussion in our prior work, the leading corrections to our results are not from infrared divergent two-loop diagrams. We previously thought diagrams with overlapping meson loops would be afflicted with the infrared divergence that appears in baryon-meson box diagrams, see Figure 3.4. That the two-loop diagrams avoid the divergence is most easily seen in terms of time-ordered perturbation theory as discussed by Weinberg [30, 31], where field propagators and integrals over four-momenta are replaced by intermediate-state energies in the denominator and integrals over three-momenta. In the time-ordered approach, infrared divergences arise from intermediate states with 'small' energies of order Q^2/m_B; however, in loop diagrams contributing to the proper self-energy $\Sigma(k_0, \vec{k}\,)$ all intermediate states contain mesons with energies on the characteristic scale Q. Consequently the two-loop diagram in Figure 3.4(b) is governed by equation (2.28) and is suppressed relative to the leading-order result by $(Q/\Lambda_\chi)^2$ and not by $m_B Q/\Lambda_\chi^2 \sim Q/\Lambda_\chi$.

The leading-order results ($\sim Q^3$) presented in section 3.3 are determined from the difference $\Sigma_{\rm nm}(\Delta m, \vec{k}\,) - \Sigma_{\rm vac}(\Delta m, \vec{k}\,)$. A priori, we would expect corrections at order Q^4 from three sources: 1) the chiral expansion of the proper self-energies $\Sigma_{\rm nm}$ and $\Sigma_{\rm vac}$ based on the power counting of equation (2.28), 2) the expansion of $\Sigma_{\rm nm}$ about $\mathcal{E}_{\rm vac}$ in equation (3.5), and 3) the Q-expansion of $\mathcal{E}_{\rm vac}$ as $\mathcal{E}_{\rm vac} = \Delta m + k^2/2m_T + \mathcal{O}(Q^3)$. In the following two paragraphs we argue that the corrections from the first two expansions do

not occur until order Q^5. The only corrections of order Q^4 arise from the third expansion and are included by making the replacements

$$\begin{aligned} \mu &\longrightarrow \mu + \left(\frac{\Delta m}{\mu}\right)\frac{k^2}{2m_T} \\ \tilde{\mu} &\longrightarrow \tilde{\mu} - \left(\frac{\Delta m}{\tilde{\mu}}\right)\frac{k^2}{2m_T} \end{aligned} \tag{3.19}$$

in the leading-order results for the Δ and Σ^* isomultiplets. The results for the Ξ^* isomultiplet and the Ω^- baryon do not receive corrections at order Q^4.

The chiral expansion of the proper self-energies $\Sigma_{\rm nm}$ and $\Sigma_{\rm vac}$ would contribute at order Q^4 through one-loop diagrams obtained from Figure 3.1 by replacing a single vertex with one which satisfies $d + \frac{1}{2}n = 3$. The only relevant operators, invariant under rotations, P, and G_χ, are constructed from products of the form $(i\partial_0)TB^\dagger A_\mu$ for the meson-nucleon loop diagrams or $(i\partial_0)TB(TB)^\dagger$ for the contact diagrams. Partial integration and baryon field redefinitions can be used together to eliminate both classes of operators in favor of terms higher-order in the expansion of the effective Lagrangian. (Such field redefinitions are reviewed in Chapter 4.) Consequently, there are no corrections at order Q^4 from the chiral expansion. Order-Q^5 corrections arise from one-loop diagrams with a single vertex which satisfies $d + \frac{1}{2}n = 4$ and two-loop diagrams, such as in Figure 3.4(b).

The primary corrections from the expansion of $\Sigma_{\rm nm}$ about $\mathcal{E}_{\rm vac}$ are suppressed by powers of Q contributed by $\partial\Sigma_{\rm nm}/\partial k_0$ in equation (3.5). Because the Q-expansion of $\Sigma_{\rm nm}$ begins at order Q^2, e.g., through an insertion of the kinetic energy operator, we would expect the corrections to be suppressed by Q, or to contribute at order Q^4. However, all k_0-dependence of $\Sigma_{\rm nm}$ can be moved into loop diagrams which are order Q^3 or higher by suitable redefinitions of the decuplet baryon field. Thus, $\partial\Sigma_{\rm nm}/\partial k_0 \sim Q^2$ and the primary correction contributed by the expansion of $\Sigma_{\rm nm}$ about $\mathcal{E}_{\rm vac}$ is order Q^5.

Throughout this calculation we have made the assumption that the power counting in equation (2.28) applies. However, large scattering lengths for nucleon-nucleon or nucleon-

delta interactions will require a revised power counting scheme,[2] as seen in references [40, 41]. Using the revised power counting for the problem considered here is presently not tractable. How these effects may be taken into account remains to be seen; the work presented here is the best that can be done at the present time and uses the power counting of equation (2.28). We hope that the discussion of some technical details, particularly those presented in the next subsection, will be useful in further studies of the decuplet self-energy.

3.4.2 Δ Self-Energy Near Saturation Density

One important feature of the self-energy shifts of the Δ isomultiplet, not apparent in Figure 3.2(a), is a logarithmic divergence when $\vec{k} = 0$ at a 'threshold' Fermi momentum p_F^* (coincident with the discontinuity of the widths). The divergence arises due to degeneracy of the Δ baryon with a pion and a nucleon on the surface of the Fermi sea; for static baryons $p_F^* = \mu$, which is just the solution of $m_\Delta = m_N + \sqrt{m_\pi^2 + p_F^2}$. To the extent that the divergence complicates the analysis of the self-energy shifts, it is unfortunate that $p_F^* \simeq 255$ MeV is so close to the Fermi momentum associated with nuclear matter at saturation density, $p_F^{(sat)} \simeq 262$ MeV [53]. In contrast, the Σ^* isomultiplet self-energy shifts do not have a similar divergence because the $\Sigma^* \to NK$ decay is kinematically forbidden for any $p_F^{(n)}$ and $p_F^{(p)}$.

In the Δ^- self-energy shift, equation (3.11), the divergence manifests itself as a cancellation among quantities of order Q in the argument of the logarithm, i.e., when

$$\left(\sqrt{\mathcal{E}^2 - m_\pi^2} - p_F\right)\Big|_{\mathcal{E}=\Delta m} = \mu - p_F = 0. \tag{3.20}$$

The divergence is resolved by keeping higher-order terms in the argument of the logarithm, specifically the imaginary part in the expansion of $\mathcal{E}$ about Δm. When the imaginary part is kept, the logarithmic divergence is replaced by a logarithmic enhancement of

[2] We thank M. Wise for bringing this point to our attention.

the self-energy shift and the result is order $Q^3 \ln Q^2$. For Fermi momenta sufficiently below p_F^*, the cancellation is not significant and we expect equation (3.11) to provide a good description of the Δ^- self-energy shift. The effect of the cancellation becomes important when $\mu - p_F \sim Q^3$, or approximately for $p_F \gtrsim (0.7) p_F^* \simeq 180$ MeV.

The logarithmically divergent term can be traced to the proper self-energy in nuclear matter $\Sigma_{\rm nm}(\mathcal{E}, 0)$. At the threshold Fermi momentum p_F^*, $\Sigma_{\rm nm}$ is singular when $\mathcal{E} = \Delta m$ and the expansion in equation (3.5) fails. To calculate the self-energy $\mathcal{E}_{\rm nm}$ in the vicinity of p_F^*, we evaluate $\Sigma_{\rm nm}$ at $\mathcal{E}_{\rm nm}$ and expand $\Sigma_{\rm vac}(\mathcal{E}_{\rm vac}, 0)$ about $\mathcal{E}_{\rm nm}$ ($\Sigma_{\rm vac}$ is smooth in this region),

$$\begin{aligned} \mathcal{E}_{\rm nm} &= \mathcal{E}_{\rm vac} + \Sigma_{\rm nm}(\mathcal{E}_{\rm nm}, 0) - \Sigma_{\rm vac}(\mathcal{E}_{\rm nm}, 0) \\ &\quad + (\mathcal{E}_{\rm nm} - \mathcal{E}_{\rm vac}) \frac{\partial}{\partial k_0} \Sigma_{\rm vac}(k_0, 0) \Big|_{k_0 = \mathcal{E}_{\rm nm}} + \cdots \end{aligned} \tag{3.21}$$

As in equation (3.5), all the terms of order Q^3 appear on the first line and the second line may be neglected at leading order.

In equation (3.21) we pay a price to determine the self-energy $\mathcal{E}_{\rm nm}$ near the threshold Fermi momentum; $\mathcal{E}_{\rm nm}$ is the solution of a transcendental equation and must be found numerically. Figure 3.5 presents a numerical calculation of the self-energy shift δE_{Δ^-} compared with the divergent behavior of equation (3.11). Both curves are plotted as functions of $p_F^{(n)}$ with $\vec{k} = 0$ and we assume the coefficients of the contact terms are $d_1 = d_2 = 0$. Because the contributions of the contact diagrams are effectively omitted, the results plotted in the figure are not model-independent in the sense of effective field theory. Until the values of the coefficients d_1 and d_2 are determined, there is little motivation for further numerical study of equation (3.21).

Having resolved the logarithmic divergence in the real part of the Δ^- self-energy shift, next we consider the behavior of the resonance width $\Gamma_{{\rm nm},\Delta^-}$ near p_F^*. The basic features of the plots in Figure 3.3 are easily understood in terms of the nucleon momentum in $\Delta \to N\pi$ decay. When the momentum of the Δ^- baryon is $\vec{k}$, then up to recoil correc-

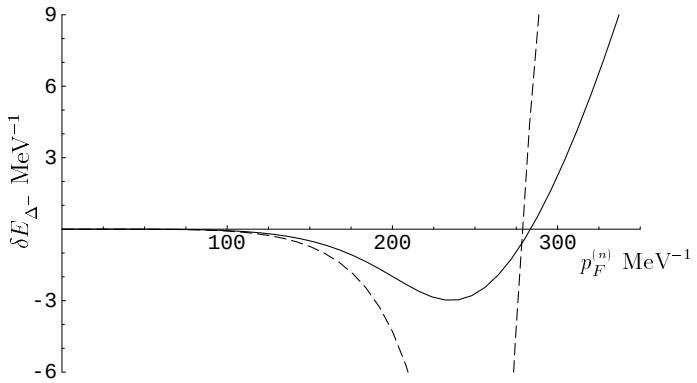


Figure 3.5: Self-energy of the Δ^- baryon at rest in nuclear matter, omitting contact diagrams ($d_1 = d_2 = 0$). The solid curve is the numerical solution of equation (3.21); the dashed curve is a plot of equation (3.11).

tions the nucleon and pion momenta in the final state are respectively $\vec{k} + \vec{\mu}$ and $-\vec{\mu}$, where $|\vec{\mu}\,|$ equals the mass parameter μ. Naively, when $|\vec{k}\,| + \mu \leq p_F$ all the nucleon final states are occupied by nucleons in the Fermi sea; the Δ^- baryon becomes stable against $\Delta \to N\pi$ decay and the width in nuclear matter vanishes. When $p_F \leq \left||\vec{k}\,| - \mu\right|$ all the nucleon states are available for decay and the width in nuclear matter is the same as the free-space width $\Gamma_{\rm vac}$. For values of p_F between those two cases, the Fermi sea partially obscures the shell of nucleon momenta and $0 < \Gamma_{\rm nm,\Delta^-} < \Gamma_{\rm vac}$. When $\vec{k} = 0$ the width transitions abruptly from $\Gamma_{\rm vac}$ (for $p_F < \mu$) to zero (for $p_F > \mu$) creating the discontinuities in Figure 3.3.

The pitfall in the simple analysis of the previous paragraph is determining the momentum transfered to the intermediate-state nucleon $\vec{\mu}$ in terms of the free-space energy of the Δ baryon. More rigorous consideration of equation (3.4) shows that the Δ resonance width is non-zero for any p_F, although it may decrease sharply near p_F^*. Setting $\vec{k} = 0$, $\mathcal{E}_{\rm nm}$ is the solution of

$$\begin{aligned} \Delta m &= \mathcal{E} - \Sigma_{\rm nm}(\mathcal{E}) \qquad (3.22) \\ &= \frac{\mathcal{C}^2(m_\pi^2 - \mathcal{E}^2)^{\frac{3}{2}}}{24\pi^2 F_0^2} \left\{ \arccos\left(\frac{-\mathcal{E}}{m_\pi}\right) - 2\arctan\left(\frac{p_F}{\sqrt{m_\pi^2 - \mathcal{E}^2}}\right) \right\} + f(\mathcal{E}, p_F), \end{aligned}$$

which is analytically continued to $\mathcal{E} > m_\pi$ by assigning m_π^2 an infinitesimal imaginary part, $m_\pi^2 \to m_\pi^2 - i\epsilon$. The last term, $f(\mathcal{E}, p_F)$, is an analytic function of $\mathcal{E}$ below the threshold for $N\pi\pi$ production from a virtual Δ baryon and does not influence our reasoning. The only real-valued solutions of equation (3.22) satisfy either $\mathcal{E} < m_\pi$ or $p_F > \sqrt{\mathcal{E}^2 - m_\pi^2}$; however, as p_F is decreased with Δm fixed, none of these solutions flow continuously to the physical, complex-valued solution for small p_F. Starting with $\mathcal{E}_{\rm vac}$ at $p_F = 0$, as p_F is increased the physical solution bypasses the region of real-valued solutions by passing onto a different sheet of the Riemann surface.

The main result of this subsection is the logarithmic divergence of $\delta E_{\rm nm,\Delta^-}$ and the discontinuity of $\Gamma_{\rm nm,\Delta^-}$, seen in section 3.3, can be removed by changing how the evaluation of $\Sigma_{\rm nm}$ is handled. Quantitative predictions for $\mathcal{E}_{\rm nm}$ near p_F^*, by numeric solution of equation (3.21) or (3.22), require values for the coefficients d_1 and d_2. Independent of d_1 and d_2, however, we conclude that the resonance width of the Δ isomultiplet is non-zero for any p_F; the decay channel $\Delta \to N\pi$ is never fully blocked by the presence of the nuclear medium.

3.4.3 Conclusion

To complete the description of the self-energy shifts of the spin-$\frac{3}{2}$ decuplet baryons in nuclear matter, the coefficients d_i of the contact terms must still be determined. In the absence of low-energy octet-decuplet scattering data, one way to estimate the values of the coefficients may be through appealing to a larger symmetry group, in this case the approximate SU(6) spin-flavor symmetry. Because the spin-$\frac{1}{2}$ octet and spin-$\frac{3}{2}$ decuplet baryons form a single 56-dimensional representation of spin-flavor SU(6), the SU(6)-invariant Lagrangian determines the octet-decuplet coefficients d_1–d_8, the Savage-Wise coefficients c_1–c_6 for octet-octet interactions [45], and 14 coefficients for decuplet-decuplet interactions. The SU(6)-invariant Lagrangian contains only two four-baryon contact terms of dimension six, with coefficients a and b as defined by Kaplan and Savage [54]. In terms

of a and b, the octet-decuplet coefficients in equation (3.7) are

$$\begin{aligned} d_1 &= 2a + \tfrac{5}{9}b, & d_5 &= \tfrac{1}{9}b, \\ d_2 &= -\tfrac{5}{9}b, & d_6 &= -\tfrac{1}{3}b, \\ d_3 &= -\tfrac{5}{9}b, & d_7 &= -\tfrac{1}{9}b, \\ d_4 &= -\tfrac{2}{9}b, & d_8 &= -\tfrac{2}{9}b. \end{aligned} \tag{3.23}$$

Unfortunately, neither a nor b is reliably known; therefore, for the phenomenologically-interesting Δ isomultiplet, we have exchanged unknowns d_1 and d_2 for unknowns a and b. What we gain from spin-flavor SU(6) is relations between the octet-octet coefficients c_i and the octet-decuplet coefficients d_i. A determination of a subset of the coefficients c_i from low-energy octet-octet scattering would permit an estimate of the desired octet-decuplet coefficients d_i.

In summary, we have calculated the leading-order shift in the self-energy of the spin-$\frac{3}{2}$ decuplet baryons in nuclear matter. Our work differs in two ways from earlier calculations of the Δ isomultiplet self-energies in nuclear matter; we use chiral SU(3) symmetry to extend the calculation to include Σ^*, Ξ^*, and Ω^-baryons and find new momentum-independent contributions from four-baryon operators in the effective Lagrangian. We have identified quantities independent of the coefficients d_i, the helicity-splitting of the Δ and Σ^* self-energy shifts and the resonance width of the Δ isomultiplet, which are presented in Figures 3.2 and 3.3. In section 3.4 we discuss the origin and resolution of a logarithmic divergence of the Δ self-energy near nuclear saturation density, which is particularly relevant for future work when the coefficients d_1 and d_2 are known. The two major short-comings of our results are that the coefficients d_i of the four-baryon operators have not yet been determined and possible effects of the large nucleon-nucleon scattering lengths have not been included.

Chapter 4

Heavy Kaon/Eta Effective Theory

In this chapter an effective field theory for treating kaon and eta interactions in a non-relativistic framework is developed. The relevant effective Lagrangians are derived to an order sufficient for one-loop calculations of $\pi\eta$, πK, and KK scattering processes. Coefficients in the effective Lagrangians at low orders are determined from a matching calculation with SU(3) χPT and used to predict KK scattering phase shifts.

In section 4.1 we present the motivation for developing the non-relativistic theory, describe the three key ideas which form the foundation, and outline the overall program to be followed. Sections 4.2 and 4.3 establish the elements and principles from which the effective Lagrangian is developed then detail the construction of the first several orders of the Lagrangian. The coefficients of the effective Lagrangian in the lowest orders of the chiral expansion are determined from a matching calculation in section 4.4. A prediction of the s-wave KK scattering phase shift and brief concluding remarks on the outlook for the heavy kaon/eta effective theory are contained in section 4.5.

4.1 Prospectus

For the range of momenta $Q \lesssim m_\eta$ the rate of convergence of the chiral expansion in SU(3) chiral perturbation theory is limited by approximately the ratio of mass scales $(m_\eta/\Lambda_\chi)^2 \sim 0.3$. For processes involving only pions with momenta $Q \lesssim m_\pi$ chiral perturabation theory based on a smaller $\mathrm{SU(2)_L} \times \mathrm{SU(2)_R}$ chiral symmetry provides an alternative description. One advantage of using SU(2) χPT for low-energy pion interactions is a better rate of convergence; generalizing the power counting arguments in section 2.2 the expansion parameter is approximately $(m_\pi/m_K)^2 \sim 0.1$. However, compared to SU(3) χPT, a

drawback of SU(2) χPT is the great reduction in the variety of processes to which the theory can be applied with only a slight reduction in the number of coefficients to be determined in the Lagrangian. Today, SU(2) χPT and SU(3) χPT are considered complementary descriptions of pion interactions at low-energy and the relations between the coefficients in the two order-Q^4 effective Lagrangians are known [20].

Motivated by the comparison above, we seek an effective field theory tailored for low-energy interactions ($Q \lesssim m_\pi$) of pions with kaons or eta mesons to exploit the relatively small ratio m_π/m_K. One key ingredient of the effective field theory, as an expansion in $1/m_K$, is a non-relativistic treatment of the kaon and eta degrees of freedom. Because the momentum scale dictates a relativistic treatment of the pion field, we cannot build the $G_\chi = \mathrm{SU}(3)_\mathrm{L} \times \mathrm{SU}(3)_\mathrm{R}$ chiral symmetry of QCD into the effective Lagrangian explicitly; flavor rotations in $\mathrm{SU}(3)_\mathrm{V}$ would map relativistic pions into non-relativistic kaons and vice versa. However, we are able to include the smaller $G'_\chi = \mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ chiral symmetry of QCD and a description of the pion degrees of freedom based on SU(2) χPT becomes the second key ingredient of the effective field theory.

The non-relativistic field theory naturally divides into a number of n-body sectors distinguished by strangeness and the number of heavy fields n in the initial state. Sectors of the theory in which the number of heavy fields is not conserved present a problem for the momentum expansion; the annihilation of a heavy particle introduces pions with 'hard' momenta ($\sim m_{K,\eta}$) in intermediate states and the power counting scheme breaks down. This problem afflicts sectors involving either $K\overline{K}$ pairs or the eta meson, which is subject to pion conversion as in $\eta K \to \pi K$. The exception to the general rule proscribing the eta meson is for a single eta meson interacting with any number of pions; in that case the eta meson is protected by G-parity [55]. (When isospin-violating effects are included, the decays $\eta \to 3\pi$ create new problems in the single-eta sector.) We focus on applying the effective field theory to the unaffected one- and two-body sectors, i.e., to $\pi\eta$, πK, KK, and the sectors related by charge conjugation.

When constructing the Lagrangian for the heavy kaon/eta effective theory, we must

introduce coefficients, or low-energy constants, which are not determined by the imposed symmetries. Even at low orders in the chiral expansion, several low-energy constants will appear in each of the independent sectors we consider. Due to the scarcity of experimental data in some sectors, the standard method for extracting values for the low-energy constants is impractical. The alternative is to pursue a matching calculation onto a better-known theory at higher energy to determine the low-energy constants in terms of known parameters of the high-energy theory. In the regime of non-perturbative QCD, SU(3) χPT is the natural choice for the 'high-energy' theory. An additional benefit of matching onto SU(3) χPT is that the full G_χ chiral symmetry is implicitly restored to the effective Lagrangian through constraints on the low-energy constants. Because the matching calculation enhances the predictive power of the heavy kaon/eta effective theory, we consider it the third (and final) key ingredient of the theory. The essence of heavy kaon/eta effective theory is a reorganization of the chiral expansion in SU(3) χPT, keeping order-by-order only the terms relevant to the low-energy regime, which should result in improved convergence.

The steps in the program we follow are to establish the effects of symmetry transformations on the fields of the theory, detail the rules for constructing a minimal effective Lagranian, build the Lagrangian for the sectors of interest to order Q^4, perform the matching calculation onto SU(3) χPT, extract values for the low-energy constants of the heavy kaon/eta effective theory, apply the results to scattering problems near threshold, and finally look for further problems to which the theory can be profitably applied. For the πK sector, this program has been started independently by Roessl [56]; however, we find he has omitted terms from the effective Lagrangian at higher order ($\geq Q^3$). Several topics relevant to developing the effective field theory are addressed in the discussion: field redefinitions (or use of equations of motion); reparameterization invariance constraints on the effective lagrangian; reconciling relativistic and non-relativistic treatments in the matching calculations; and consequences of differences between the power counting schemes of SU(3) χPT and of the heavy kaon/eta effective theory.

To conclude a section titled "Prospectus" by considering future applications for the heavy kaon/eta effective theory seems appropriate, so we briefly skip ahead to the final step in the program. While the effective Lagrangians written out for the πK and KK sectors include isospin-violating terms for generality, the values of the associated low-energy constants have not been determined in the matching calculations. Extending the matching to determine the low-energy constants for isospin violation and electromagnetic interactions would be straight forward and suitable for studying processes such as $\pi K \to \gamma\pi K$ and $\gamma K \to \pi\pi K$ near threshold. In another application, baryon isomultiplets from the spin-$\frac{1}{2}$ octet and spin-$\frac{3}{2}$ decuplet can be coupled to the heavy kaon effective theory with the coefficients in the Lagrangian matched to the G_χ-invariant theory in sections 2.3, 2.4. Unfortunately, in many interesting kaon-baryon systems, strangeness-exchange reactions do not conserve the number of heavy particles, e.g., $N\overline{K} \to \Sigma\pi$ [57], resulting in a breakdown of the power counting scheme; however, the NK and ΔK sectors are protected by strangeness conservation and are compatible with the heavy-kaon framework. On a more speculative note, phenomenological models inspired by and incorporating elements of the heavy kaon/eta effective theory may be developed for the problematic sectors like $K\overline{K}$ [58]. For example, similar work has been done on $\pi^+\pi^-$ atoms [59], and by integrating out π^0 pairs and allowing for a non-hermitian effective Lagrangian [60]. Finally, if experimental data for the threshold energy region become available in abundance, extracting the low-energy constants of the heavy kaon/eta effective theory from the data may provide insights into SU(3) χPT. Knowing values for the low-energy constants would permit verification of the constraints on the low-energy constants imposed by the matching calculation and would provide a means of testing the SU(3) chiral expansion.

4.2 Elements and Principles

The Lagrangian of heavy kaon/eta effective theory is divided into sectors by the heavy fields present,

$$\mathcal{L} = \mathcal{L}^{\pi\pi} + \mathcal{L}^{\pi K} + \mathcal{L}^{\pi\overline{K}} + \mathcal{L}^{\pi\eta} + \mathcal{L}^{\pi KK} + \cdots \tag{4.1}$$

and the Lagrangian for each sector is subdivided as $\mathcal{L}^X = \sum_j \mathcal{L}^X_j$ in a chiral expansion. All the purely pionic interactions are derived from $\mathcal{L}^{\pi\pi}$, where $j = 2, 4, \ldots$, which is identically the Lagrangian of SU(2) χPT. The effective Lagrangians for interactions of a single heavy field with any number of pions are characterized by the chiral expansion $j = 1, 2, 3, \ldots$; the expansion of $\mathcal{L}^{\pi KK}$ is similar except that the expansion starts with $j = 0$ due to heavy-field contact terms. In this section we briefly establish our notation and identify the principles used to construct the effective Lagrangians.

The building blocks of the heavy kaon/eta effective theory are introduced in direct analogy to SU(3) χPT. The three pseudo-Goldstone bosons of SU(2) χPT are introduced in the exponential representation; the most elementary constituents are $u = e^{i\Phi/2F}$ and $\chi = 2B\mathcal{M}_q$ where

$$\Phi = \pi_a \tau_a = \begin{pmatrix} \pi^0 & \sqrt{2}\,\pi^+ \\ \sqrt{2}\,\pi^- & -\pi^0 \end{pmatrix}, \qquad \mathcal{M}_q = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}, \tag{4.2}$$

and the parameters F and B are non-trivially related to the SU(3) χPT parameters F_0 and B_0. (The relations between the parameters were obtained by Gasser and Leutwyler [20]; we review them in section 4.4.) In the purely pionic sector, the effective Lagrangian is written in terms of the fields $U = u^2$ and χ as in section 2.2. Because the kaon fields are treated as matter fields which transform in the fundamental representation of $\mathrm{SU(2)_V}$, for coupling pions to kaons we adopt the notation used in section 2.3 and reintroduce

$$V_\mu = \tfrac{1}{2}(u^\dagger \partial_\mu u + u \partial_\mu u^\dagger), \tag{4.3}$$

$$A_\mu = \tfrac{i}{2}(u^\dagger \partial_\mu u - u \partial_\mu u^\dagger), \tag{4.4}$$

$$\chi_\pm \;=\; u^\dagger \chi u^\dagger \pm u \chi^\dagger u. \tag{4.5}$$

Throughout this chapter this notation is reserved for the SU(2) versions of the fields. The heavy fields are the η meson and the kaons, which form two $\mathrm{SU(2)_V}$ doublets

$$K = \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}, \qquad K_c = \begin{pmatrix} \overline{K}^0 \\ -K^- \end{pmatrix}. \tag{4.6}$$

As non-relativistic fields, the creation and annihilation operators are treated separately, thus we distinguish $K^\dagger$ from K_c.

The symmetries which we explicitly build into the effective Lagrangian are G'_χ chiral symmetry, parity (P), charge conjugation (C), and reparameterization invariance. The implementation of reparameterization invariance is discussed separately in the next subsection. The transformations of the fields under G'_χ, P, C, and hermitian conjugation (†) are collected for reference in Table 4.1 along with the power of Q, i.e., the chiral index, associated with each field. Under each of the transformations listed, a covariant derivative of a field $D_\mu O$ transforms in the same way as O provided, in the case of parity, the Lorentz index of the derivative is also raised or lowered. The number of low-energy constants required is restricted sector-by-sector by imposing the constraints of hermiticity and invariance under G'_χ and P on the effective Lagrangian. Within sectors of non-zero strangeness charge conjugation does not restrict the form of the effective Lagrangian, but determines $\mathcal{L}_{\pi\overline{K}}$ from $\mathcal{L}_{\pi K}$ and $\mathcal{L}_{\pi\overline{K}\overline{K}}$ from $\mathcal{L}_{\pi KK}$.

4.2.1 Reparameterization Invariance

By treating the kaon and eta mesons as non-relativistic (heavy) fields we have performed an implicit rephasing, as in section 2.4, compared to the relativistic counterparts K_r and η_r,

$$K(x) \sim \sqrt{2M}\, e^{iMv\cdot x} K_r(x), \qquad \eta(x) \sim \sqrt{2M'}\, e^{iM'v\cdot x} K_r(x). \tag{4.7}$$

field	G'_χ	P	C	†	index
U	$RUL^\dagger$	$U^\dagger$	U^T		0
χ	$R\chi L^\dagger$	$\chi^\dagger$	χ^T		2
V_μ	$HV_\mu H^\dagger - (\partial_\mu H)H^\dagger$	$+V^\mu$	$-V_\mu{}^T$	$-V_\mu$	1
A_μ	$HA_\mu H^\dagger$	$-A^\mu$	$A_\mu{}^T$	A_μ	1
χ_+	$H\chi_+ H^\dagger$	$+\chi_+$	$\chi_+{}^T$	χ_+	2
χ_-	$H\chi_- H^\dagger$	$-\chi_-$	$\chi_-{}^T$	$-\chi_-$	2
K	HK	$-K$	$K_c{}^T$	$K^\dagger$	0
K_c	$K_c H^\dagger$	$-K_c$	K^T	$K_c^\dagger$	0
η	η	$-\eta$	η	$\eta^\dagger$	0

Table 4.1: Transformations of fields and under G'_χ chiral symmetry, parity P, charge conjugation C, and hermitian conjugation †, and the chiral index associated with each field.

Reparameterization invariance (RPI) [61, 62, 63, 64] is a consequence of arbitrariness in the choice of the velocity v^μ used in the rephasing. The choice of a different velocity in the relations between the relativistic and heavy fields is compensated by a shift in the residual momentum k^μ of the heavy field. Lagrangians written in terms of heavy fields rephased with different velocities must still give the same result for physical S-matrix elements, which implies invariance of the Lagrangian under shifts of v^μ. A consequence of RPI symmetry in the effective Lagrangian is that coefficients of terms at different orders in the Q-expansion are related [62] and by building reparameterization invariance into an effective Lagrangian we reduce the number of low-energy constants to be determined. In this subsection we use the kaon as a representative example; all of the arguments apply equally well to the case of the eta meson.

To construct a RPI Lagrangian it is necessary and sufficient, as shown by Luke and Manohar [62], that the rephasing velocity and derivatives of the kaon fields appear only in the following combinations:

$$\begin{aligned} \mathcal{V}_\mu K &= (-iMv_\mu + D_\mu)K, \\ \mathcal{V}_\mu K^\dagger &= (+iMv_\mu + D_\mu)K^\dagger, \end{aligned} \tag{4.8}$$

where we will refer to the operator $\mathcal{V}_\mu$ as the RPI-covariant derivative. $\mathcal{V}_\mu$ is a valid derivative in the sense that it satisfies the chain rule,

$$D_\mu(K^\dagger OK) = (\mathcal{V}_\mu K^\dagger)OK + K^\dagger(D_\mu O)K + K^\dagger O(\mathcal{V}_\mu K), \tag{4.9}$$

which permits the use of partial integration and total derivative arguments with respect to $\mathcal{V}_\mu$. Only after building the effective Lagrangian in terms of the RPI-covariant derivative and expanding each term in powers of the mass M do we make any particular choice of frame, i.e., explicitly setting $v_\mu = (1,0,0,0)$.

Superficially, developing the effective Lagrangian in terms of the RPI-covariant derivative presents a problem for power counting. An infinite tower of possible terms, such as $K^\dagger K$, $K^\dagger \mathcal{V}_\mu \mathcal{V}^\mu K$, $K^\dagger \mathcal{V}_\mu \mathcal{V}_\nu \mathcal{V}^\mu \mathcal{V}^\nu K$, ..., will contribute to each order of the chiral expansion of the Lagrangian. On closer inspection, the fact that RPI guarantees a particular relationship between coefficients of terms in the effective Lagrangian assures that the effect of adding another term with more factors of $\mathcal{V}_\mu$ can always be compensated by a redefinition of the coefficients already present. As an explicit example we consider the effect of adding the term $\delta\mathcal{L} = b\, K^\dagger \ell^\mu \mathcal{V}_\mu \mathcal{V}_\nu \mathcal{V}^\nu K$, where ℓ^μ represents an unspecified combination of light degrees of freedom, to the interaction Lagrangian $\mathcal{L}_{\rm int} = a\, K^\dagger \ell^\mu \mathcal{V}_\mu K$. Expanding the combined result gives

$$\begin{aligned} \mathcal{L}_{\rm int} + \delta\mathcal{L} \;\; = \;\; & -iM(a - bM^2)\, K^\dagger (v\cdot\ell) K + (a - bM^2)\, K^\dagger \ell^\mu D_\mu K \\ & - 2bM^2\, K^\dagger (v\cdot\ell)(v\cdot D K) - ibM\, K^\dagger (v\cdot\ell) D_\mu D^\mu K + \cdots , \end{aligned} \tag{4.10}$$

where we have used $v_\mu v^\mu = 1$. A subsequent change of a, $a \to a + bM^2$, absorbs the contribution of $\delta\mathcal{L}$ to the terms already present in $\mathcal{L}_{\rm int}$. The addition of another term, $\delta\mathcal{L}' = c\, \mathcal{V}_\lambda K^\dagger \ell^\mu \mathcal{V}_\mu \mathcal{V}_\nu \mathcal{V}^\nu \mathcal{V}^\lambda K$, would be followed by a change of a and b to restore the form of the terms present before the addition of $\delta\mathcal{L}'$. Generally, RPI makes the possibility of such redefinitions certain to all orders in the expansion of the effective Lagrangian and in every sector of the theory.

The procedure just proposed is an iteration of adding new terms to the effective Lagrangian then redefining the coefficients of the terms already present. An equivalent and more practical procedure is to subtract the contribution to terms already present in the effective Lagrangian from each new term when it is added, for instance

$$\begin{aligned} & \mathcal{V}^\mu K^\dagger \mathcal{V}_\mu \mathcal{V}_\nu \mathcal{V}^\nu K \\ & \longrightarrow \quad \mathcal{V}^\mu K^\dagger \mathcal{V}_\mu \mathcal{V}_\nu \mathcal{V}^\nu K - M^2 \left(K^\dagger \mathcal{V}_\mu \mathcal{V}^\mu K + M^2 K^\dagger K \right) \\ & \qquad + M^2 \left(\mathcal{V}_\mu K^\dagger \mathcal{V}^\mu K - M^2 K^\dagger K \right) + M^4 K^\dagger K. \end{aligned} \tag{4.11}$$

RPI-covariant derivatives contracted with Lorentz indices on light degrees of freedom are never dropped. We implicitly make such subtractions from every term when we include the term in the effective Lagrangian.

Having resolved the issue of infinitely many contributions to each order of the chiral expansion of the effective Lagrangian, we turn now to determining the proper power of Q to associate with the leading contribution of a term in RPI form. Occurrences of RPI-covariant derivatives are divided into two classes, those contracted with another RPI-covariant derivative and those contracted into the light degrees of freedom. In the expansion of a contracted pair of RPI-covariant derivatives, the contribution from $(\pm iMv)^2$ is cancelled by the subtracted terms; the leading contribution to the power counting comes from $v\cdot D$ so each contracted pair contributes Q^1. The leading-order contribution from expanding a RPI-covariant derivative contracted with the light degrees of freedom is unaffected by the subtraction; the RPI-covariant derivative is counted as one power of v_μ or order Q^0. To determine the effective Lagrangian to order Q^D, in terms where the operators for the light degrees of freedom contribute d powers of Q and carry n Lorentz indices, we must consider some terms with as many as

$$N = n + 2(D - d) \tag{4.12}$$

added RPI-covariant derivatives. Toward the upper limit of the necessary number of

RPI derivatives, some relief comes from the fact that total derivative arguments will allow replacing some terms with alternates where a derivative is moved onto the light degrees of freedom and the alternate terms may be neglected to the order we are working.

In addition to the ambiguity in the choice of the rephasing velocity v_μ, we have freedom to choose the mass M used in the rephasing relation, equation (4.7). This freedom was exploited in equation (2.26) to avoid awkward x^μ-dependent phases in the heavy baryon Lagrangian. The only constraint on the choice of M is that any residual mass δM is small ($\lesssim Q$) to prevent failure of the power counting scheme. In the context of heavy kaon/eta theory, we perform the rephasing of the kaon and eta fields with the masses $\bar{M}_K$ and $\bar{M}_\eta$ respectively, where $\bar{M}_K$ and $\bar{M}_\eta$ are defined as the meson masses in the SU(2) chiral limit $m_{u,d} \to 0$ and m_s finite. With this choice, the residual masses of the K and η mesons at tree-level are generated through interactions with the field χ_+ and are included perturbatively. Because the residual masses vanish in the SU(2) chiral limit, by definition, the effective Lagrangian does not include any explicit residual mass term $-\delta M\, K^\dagger K$; in this sense our choice for the rephasing mass is the 'natural' choice.

4.2.2 Use of Field Redefinitions

Redefinitions of the fields of the theory can be used in two ways to potentially improve or simplify the effective Lagrangian. The first class of field redefinitions eliminates terms from the effective Lagrangian which are proportional to a classical equation of motion; this class of field redefinitions, combined with total derivative arguments, is a powerful method for reducing the number of unknown coefficients required. The technique is well-documented in the literature [65, 66, 67, 68] for simplifications of effective Lagrangians and we have nothing to add aside from identifying the general form of the terms we eliminate by this method. Terms with an explicit factor of $D_\mu A^\mu$, $\mathcal{V}_\mu \mathcal{V}^\mu K$, $\mathcal{V}_\mu \mathcal{V}^\mu \eta$, or the hermitian conjugates can be combined with contributions from other terms in the effective Lagrangian to form complete equation of motion terms and are subsequently eliminated.

The second class of field redefinitions allows the elimination of all time-like derivatives

of the heavy fields from the interaction Lagrangian [31, 35, 69]. The principle is closely related to the elimination of equation of motion terms, and if all equation of motion terms have already been eliminated by redefinitions of the first class then further redefinitions from the second class will permit a re-expression of the effective Lagrangian but not the further elimination of any low-energy constants present. Instead, the motive for considering field redefinitions in the second class is to simplify the application of the effective Lagrangian by replacing time-like derivatives of the heavy fields with spatial derivatives. Spatial derivatives permit a more transparent power counting for Q in loop diagrams than is possible when time-like derivatives occur in vertices.

Considering the free field theory of a (complex) heavy scalar is sufficient to illustrate the general method and highlight conclusions relevant to how the matching calculation between relativistic and non-relativistic theories is performed. The free-field Lagrangian for the heavy scalar, after rephasing as in equation (4.7), becomes

$$\mathcal{L}_{\phi_0} = \phi_0^\dagger \left[iv\cdot\partial + \frac{\nabla_\perp^2 + (iv\cdot\partial)^2}{2m} \right] \phi_0 \tag{4.13}$$

where the spatial derivative $\vec{\nabla}_\perp$ is defined by $\partial_\mu \partial^\mu = (v\cdot\partial)^2 - \nabla_\perp^2$. After substituting a field redefinition $\phi_0 = \phi_1 + \delta\phi$, we find the well-known result that $\delta\phi = \frac{-1}{4m} iv\cdot\partial\phi$ cancels the time-like derivatives from terms at order $1/m$ and gives

$$\mathcal{L}_{\phi_1} \;=\; \phi_1^\dagger \left[iv\cdot\partial + \frac{\nabla_\perp^2}{2m} - \frac{3(iv\cdot\partial)^3}{16m^2} - \frac{(iv\cdot\partial)\nabla_\perp^2}{4m^2} + \mathcal{O}(1/m^3) \right] \phi_1. \tag{4.14}$$

(A slightly different approach is taken in references [35, 69].) Substituting a second field redefinition $\phi_1 = \phi_2 + \delta\phi'$, dependence on time-like derivatives is canceled in $\mathcal{L}_{\phi_2}$ through order $1/m^2$ by $\delta\phi' = (4\nabla_\perp^2 + 3(iv\cdot\partial)^2)\phi/32m^2$. Repeating this process, time-like derivatives of ϕ can be eliminated to any arbitrary order in an expansion in terms of (∂/m).

Our original contribution to this discussion is the observation that the infinite series of field redefinitions can be formally summed and a relatively simple expression for the field redefinition $\phi_0 = \mathcal{F}[\tilde{\phi}]$ can be obtained. The approach is based on substituting a

series representation for $\mathcal{F}[\tilde{\phi}]$, and deriving recursion relations from the constraint that all time derivatives cancel from $\mathcal{L}_{\tilde{\phi}}$. Here we present a brief plausibility argument for the inverse solution $\tilde{\phi} = \mathcal{F}^{-1}[\phi_0]$; the actual recursion relations generated are presented in Appendix B with a short description of their solution. The first step is to complete the square for the time-like derivative appearing in equation (4.13);

$$\begin{aligned} \mathcal{L}_{\phi_0} &= \tfrac{1}{2m}\phi_0^\dagger \left[(iv\cdot\partial + m)^2 - m^2 + \nabla_\perp^2 \right] \phi_0 \\ &= \tfrac{1}{2m}\phi_0^\dagger \left[iv\cdot\partial + m - \sqrt{m^2 - \nabla_\perp^2} \right] \left[iv\cdot\partial + m + \sqrt{m^2 - \nabla_\perp^2} \right] \phi_0. \end{aligned} \tag{4.15}$$

Second, observing $(\sqrt{m^2 - \nabla_\perp^2} - m)$ is the relativistic kinetic energy operator $\widehat{K}$, we recognize the first expression in brackets is the kernel of the desired Lagrangian for $\tilde{\phi}$, $\mathcal{L}_{\tilde{\phi}} = \tilde{\phi}^\dagger (iv\cdot\partial - \widehat{K})\tilde{\phi}$. The second term in brackets is $(iv\cdot\partial + \widehat{K} + 2m)$ which we divide between ϕ_0 and $\phi_0^\dagger$ and finally use to identify $\tilde{\phi}$;

$$\begin{aligned} \mathcal{L}_{\phi_0} &= \left[\sqrt{\frac{-iv\cdot\partial + \widehat{K} + 2m}{2m}}\ \phi_0^\dagger \right] \left(iv\cdot\partial - \widehat{K} \right) \left[\sqrt{\frac{iv\cdot\partial + \widehat{K} + 2m}{2m}}\ \phi_0 \right] \\ &\longrightarrow \quad \tilde{\phi} = \sqrt{(iv\cdot\partial + \widehat{K} + 2m)/2m}\ \ \phi_0 = \mathcal{F}^{-1}[\phi_0]. \end{aligned} \tag{4.16}$$

The explicit form for the redefinition resolves some differences between the perturbative field theories of the heavy fields ϕ_0 and $\tilde{\phi}$. Developing the field propagator from the complete free-field Lagrangian in equation (4.13) results in two poles in the propagator, at $iv\cdot\partial = \pm(m + \widehat{K})$. Only one pole is relevant for the non-relativistic treatment and the second pole is eliminated differently in the ϕ_0 and $\tilde{\phi}$ theories. The second pole is completely absent from the Lagrangian $\mathcal{L}_{\tilde{\phi}}$; the disappearance of the second pole is traced to a singularity of the field redefinition $\mathcal{F}[\tilde{\phi}]$ which causes $\tilde{\phi}$ to vanish at the location of the second pole. In the ϕ_0 theory, the pole is removed by treating the operator $(iv\cdot\partial)^2/2m$ in $\mathcal{L}_{\phi_0}$ as a perturbative correction to the ϕ_0 two-point function; the second pole does not appear at any finite order. The second difference between the two non-relativistic theories is the normalization of one-particle states relative to the normalization of the rel-

ativistic one-particle state [33, 35], which becomes relevant when we pursue the matching calculation between the non-relativistic heavy kaon/eta theory and fully relativistic χPT. We determine the relative normalization by evaluating $\tilde{\phi} = \mathcal{F}^{-1}[\phi_0]$ at the location of the relevant pole, $(iv\cdot\partial) = \widehat{K} = E - m$, which gives $\sqrt{2m}\,\tilde{\phi} = \sqrt{2E}\,\phi_0$. Which factor is included in matching calculations between relativistic and non-relativistic theories depends on whether relativistic corrections to the propagator are included as insertions of $(iv\cdot\partial)^2/2m$ in the ϕ_0 theory or as insertions of $\nabla_\perp^4/8m^3 + \nabla_\perp^6/16m^5 + \cdots$ in the $\tilde{\phi}$ theory.

As stated above, the motivation for eliminating time-like derivatives of heavy fields is to simplify the power counting of loop diagrams involving both light- and heavy-field propagators. However, the field redefinitions required to eliminate time-like derivatives are not reparameterization invariant. We consider the question open as to whether RPI can be reformulated as a principle of the redefined $\tilde{\phi}$ scalar field theory. Without explicit RPI symmetry of the $\tilde{\phi}$ effective Lagrangian, the relations between coefficients of different orders in the Q-expansion are not guaranteed. As a consequence, we forego using further non-RPI field redefinitions and treat the heavy kaon and eta fields in analogy with the ϕ_0 scalar field theory described in this subsection.

4.3 The Effective Lagrangians

The approach we take to developing the effective Lagrangians is based on the work of Fearing and Scherer [25] extending $\mathcal{L}_{\chi\mathrm{PT}}$ to order Q^6. We begin by addressing two points where we use a different approach, specifically when performing the $\mathrm{SU}(2)_\mathrm{V}$ contractions and when using total derivative arguments to eliminate terms. After discussing those issues we give a brief summary of the overall procedure then present the explicit construction of the effective Lagrangians $\mathcal{L}^{\pi K}$, $\mathcal{L}^{\pi\eta}$, and $\mathcal{L}^{\pi KK}$.

Given a product of field operators, Fearing and Scherer form G_χ-invariant contractions by taking traces of all possible permutations of the matrix fields. (Roessl applies the

same method for $\mathcal{L}^{\pi K}$ by introducing a matrix field $KK^\dagger$ for the kaons [56].) Then trace relations, such as

$$\begin{aligned} &\mathrm{Tr}[ABC] + \mathrm{Tr}[ACB] \\ &= \mathrm{Tr}[AB]\mathrm{Tr}[C] + \mathrm{Tr}[CA]\mathrm{Tr}[B] + \mathrm{Tr}[BC]\mathrm{Tr}[A] - \mathrm{Tr}[A]\mathrm{Tr}[B]\mathrm{Tr}[C] \end{aligned} \tag{4.17}$$

for 2×2 matrices, can be used to eliminate some of the resulting terms in favor of others in the set. For the relatively simple case of $\mathrm{SU}(2)_\mathrm{V}$, we prefer to form a minimal set of contractions by appealing to the algebra for addition of angular momenta. We prefer the second method because 1) it permits a clean separation between isospin-conserving and isospin-violating terms, 2) the connection between generated terms and physical processes is more meaningful and direct, and 3) the method is more familiar.

A general matrix field O transforms under $\mathrm{SU}(2)_\mathrm{V}$ as $\mathbf{0} \oplus \mathbf{1}$, where the $\mathbf{0}$ component is identified as $\mathrm{Tr}[O]$. To separate out the $\mathbf{1}$ component of a matrix field, we define the notation

$$\widehat{O} = O - \tfrac{1}{2}\mathrm{Tr}[O]. \tag{4.18}$$

Since $\mathrm{Tr}[A_\mu]$ and $\mathrm{Tr}[V_\mu]$ both vanish, these fields transform in just the $\mathbf{1}$ representation. This notation is particularly helpful when applied to $\chi_\pm$; we find that $\mathrm{Tr}[\chi_+]$ and $\hat{\chi}_-$ are proportional to $\hat{m}$, but $\mathrm{Tr}[\chi_-]$ and $\hat{\chi}_+$ are proportional to $(m_u - m_d)$ and give isospin-violating terms.

As a simple illustration of the method, the product of fields $KK^\dagger A_\mu \chi_-$ transforms as $\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} \otimes \mathbf{1} \otimes (\mathbf{0} \oplus \mathbf{1})$ which contains three singlet representations; the resulting set of contracted forms is $\mathrm{Tr}[\hat{\chi}_- A_\mu]K^\dagger K$, $\mathrm{Tr}[\chi_-]K^\dagger A_\mu K$, and $K^\dagger[\hat{\chi}_-, A_\mu]K$. We choose to take $K^\dagger[\hat{\chi}_-, A_\mu]K$ instead of the equally suitable form $K^\dagger \hat{\chi}_- A_\mu K$ because the commutator projects out just the $\mathbf{1}$ component of the product $(\hat{\chi}_- \times A_\mu)$. Also, we can distribute any covariant derivatives which are present after performing all $\mathrm{SU}(2)_\mathrm{V}$ contractions since they do not change the $\mathrm{SU}(2)_\mathrm{V}$ transformation of the field on which they operate. (However, antisymmetric expressions such as $[A_\mu, A^\mu]$ must be kept until derivatives have been

distributed across the fields.)

The second point to discuss is how to select terms which are to be eliminated by adding total derivatives to the effective Lagrangian. Clearly, the set of selected terms should contain as few terms proportional to an equation of motion (EoM terms) as possible, since such terms will be eliminated anyway by subsequent field redefinitions. Fearing and Scherer [25] take an approach driven by intuition, and apply rules such as *all terms where more than half of the derivatives act on a single field may be eliminated,* and use Lorentz index exchange arguments to preserve as many EoM terms as possible from the remaining set. Because our power counting scheme permits many more derivatives at any order than appear in $\mathcal{L}_{\chi\mathrm{PT}}$, the intuitive approach is vulnerable to questions of whether every possible total derivative was used to eliminate a term and whether the set of terms retained contains the most EoM terms possible.

We present an algorithm which reduces the use of total derivative arguments to linear algebra and assures that no EoM terms are eliminated at the expense of keeping another term (and low-energy constant) in the effective Lagrangian. We view the set of m terms prior to any eliminations as the basis of a m-dimensional vector space Ω, and the set of coefficients of those terms in the effective Lagrangian represents an arbitrary vector in Ω. Also, each total derivative, when expanded by the chain rule, is represented by a vector in Ω; we define ω to be the n-dimensional sub-space of Ω spanned by the complete set of total derivatives. The key ideas are that the vector representing the most general effective Lagrangian is arbitrary up to the addition of any vector from ω and any set of $m-n$ terms which spans the complementary sub-space Ω/ω is sufficient for the most general effective Lagrangian.

We construct a matrix P from the column vectors of a complete set of total derivatives, not necessarily linearly independent. A complete set of total derivatives for terms containing k derivatives can always be generated by explicitly taking the derivatives of each possible term with $k-1$ derivatives. For any matrix O we define the operation NS$[O]$ which returns a matrix whose columns span the (right) null space of O. Then, trivially, the

columns of the $(m-n)\times m$ matrix $\mathrm{NS}[P^T]$ span the desired sub-space Ω/ω. A particular choice of $m-n$ terms from the original set also spans Ω/ω if the matrix R constructed from the column vectors representing the terms satisfies the relation $\det[R^T\cdot\mathrm{NS}[P^T]]\neq 0$. Thus we can guarantee a complete set of total derivatives and test any particular choice of terms for suitability as a basis of Ω/ω. The remaining hurdle is to establish a means for identifying which choices of terms contain the maximum number of EoM terms; because the number of acceptable sets grows combinatorially with the number of derivatives and fields in each term, exhaustive testing of the possible sets is impractical.

By definition the columns of $\mathrm{NS}[O]$ represent the linear combinations of the columns of O which vanish. The key observation is if we construct a matrix Q from the rows of $\mathrm{NS}[P^T]$ which correspond to the EoM terms, then the columns of $\mathrm{NS}[Q]$ represent all independent linear combinations of the columns of $\mathrm{NS}[P^T]$ with vanishing projection onto the EoM terms. The result is the columns of the matrix $\mathrm{NS}[P^T]\cdot\mathrm{NS}[Q]$ span the sub-space of Ω/ω which is orthogonal to all of the EoM terms, and the number of columns is exactly the minimum number of non-EoM terms which must be included in the effective Lagrangian.

One considerable advantage of the algorithm just described is that the computations, from developing the complete sets of terms and total derivatives to selecting a single optimal basis of terms to include in the effective Lagrangian, are easily programmed with software for symbolic mathematics. We also note that for very large sets of terms, the problem can be broken into two smaller problems by dividing the full set of terms considered $\{t_j\}_{1\leq j\leq m}$ into disjoint sets $\{t_j+t_j^\dagger\}$ and $\{i(t_j-t_j^\dagger)\}$ and applying the algorithm to each smaller set. Finally, if the full set of terms forms a hierarchy of 'better' terms and 'lesser' terms (beyond the distinction between EoM and non-EoM terms), the steps for removing a maximal set of EoM terms from the basis of Ω/ω can be repeated for each level in the hierarchy to assure the minimum number of terms must be selected from each of the successive 'lesser' sets of terms.

In summary, we list the steps of the procedure, based on Fearing and Scherer [25],

which we apply to construct the minimal effective Lagrangians $\mathcal{L}^{\pi K}$ and $\mathcal{L}^{\pi KK}$ order by order in Q.

1. Identify all simple (uncontracted) products of fields and derivatives consistent with parity and of the proper power of Q.

2. For each product, form a minimal set of forms contracted over $\mathrm{SU(2)_V}$ indices in analogy to addition of angular momenta.

3. For each contracted form, distribute the derivatives over the fields and contract Lorentz indices all possible ways, keeping in mind that derivatives of $\chi_\pm$ are unnecessary and $D_\mu A_\nu$ and all multiple derivatives are implicitly symmetrized.

4. For sets of terms with many derivatives, the power counting may indicate that some contractions of Lorentz indices yield terms higher order in Q than the order to which we are working; drop all such terms.

5. Replace each pair of terms t and $t^\dagger$ with the hermitian combinations $(t + t^\dagger)$ and $i(t - t^\dagger)$.

6. For each set of terms, apply the total derivative algorithm to select a basis set containing the most EoM terms possible.

7. Eliminate EoM terms, proportional to either $D_\mu A^\mu$, $\mathcal{V}_\mu \mathcal{V}^\mu K$, or $\mathcal{V}_\mu \mathcal{V}^\mu K^\dagger$ through use of implicit field redefinitions.

8. Expand RPI-covariant derivatives of the kaon fields as $\mathcal{V}_\mu = \pm i \bar{M}_K v_\mu + D_\mu$ then set $v_\mu = (1, 0, 0, 0)$.

In practice, after identifying a product of fields in step (1.), we work out all the terms generated by adding covariant derivatives to the product of fields up to the highest relevant order in Q before moving on to the next product of fields.

4.3.1 The πK and $\pi\eta$ Sectors

In the interest of transparency, so that the process and results can be independently verified and because some knowledge of intermediate results is necessary to extend the Lagrangian to higher order, we present details on some of the steps in the construction of the effective Lagrangian. After the discussion related to each product of fields, we list the RPI forms of the terms which are kept, together with the coefficient associated with each. The particular coefficient we assign to a given term depends on the leading power of Q contributed by the term after the RPI-covariant derivatives are expanded.

The only product of fields permitted at order Q^0 is $K^\dagger K$. Because covariant derivatives can be freely moved from one field to the other, sophisticated total derivative arguments are unnecessary regardless of the number of derivatives distributed on the kaon fields; in each order of the Q-expansion, only a single term of this form needs to be retained. For no derivatives or two derivatives, the terms we keep are $K^\dagger K$ and $\mathcal{V}_\mu K^\dagger \mathcal{V}^\mu K$. By definition, the coefficient of the mass term is the square of the mass used in the heavy field rephasing, i.e., the mass of the kaon in the chiral limit $m_{u,d} \to 0$. For four or more derivatives, we can always move all derivatives except a single contracted pair off the kaon field leaving only the EoM term which is subsequently eliminated. The only combination of terms generated from the product $K^\dagger K$ (at any order) is

$$\mathcal{V}_\mu K^\dagger \mathcal{V}^\mu K - \bar{M}_K^2 (K^\dagger K).$$

No product of fields at order Q^1 is consistent with RPI and parity invariance. At order Q^2 two products of field operators must be taken into account, $KK^\dagger\chi_+$ and $KK^\dagger A_\mu A_\nu$.

For the field product $KK^\dagger\chi_+$ we use the contracted forms

$$\mathrm{Tr}[\chi_+]K^\dagger K \quad (c_3^{\pi K}), \qquad K^\dagger \hat{\chi}_+ K \quad (c_4^{\pi K}).$$

Because χ_+ is effectively 'transparent' to covariant derivatives (since $D_\mu \chi_+$ is equivalent to alternate terms we consider elsewhere), we can freely move covariant derivatives between

the kaon fields just as with the product $K^\dagger K$. Consequently, for any number of derivatives distributed on this form, we can always reduce the set of terms to a single EoM term proportional to $\mathcal{V}_\mu \mathcal{V}^\mu K$.

For the field product $KK^\dagger A_\mu A_\nu$ we choose the contracted forms $\mathrm{Tr}[A_\mu A_\nu]K^\dagger K$ and $K^\dagger[A_\mu, A_\nu]K$, and up to order Q^4 in each case we can distribute up to six derivatives over the fields. First we consider the form $\mathrm{Tr}[A_\mu A_\nu]K^\dagger K$; adding two derivatives, we must keep a total of seven terms of which five may be EoM terms; adding four derivatives, we must keep a total of nine terms at order Q^4 of which eight may be EoM terms; adding six derivatives, we must keep a total of four terms at order Q^4 of which three may be EoM terms. The terms we select after all eliminations are

$$\begin{gathered}
\mathrm{Tr}[A_\mu A^\mu]K^\dagger K \quad (c_2^{\pi K}), \\
\mathrm{Tr}[A^\mu A^\nu]\mathcal{V}_\mu K^\dagger \mathcal{V}_\nu K \quad (c_1^{\pi K}), \\
\mathrm{Tr}[D_\mu A_\nu D^\mu A^\nu]K^\dagger K \quad (e_3^{\pi K}), \\
\mathrm{Tr}[A^\mu D^\nu A^\lambda]\mathcal{V}_\lambda K^\dagger \mathcal{V}_\mu \mathcal{V}_\nu K + \mathrm{h.c.} \quad (e_2^{\pi K}), \\
\mathrm{Tr}[D^\mu A^\nu D^\lambda A^\kappa]\mathcal{V}_\mu \mathcal{V}_\nu K^\dagger \mathcal{V}_\lambda \mathcal{V}_\kappa K \quad (e_1^{\pi K}).
\end{gathered}$$

The second form $K^\dagger[A_\mu, A_\nu]K$ vanishes unless derivatives are added; adding two derivatives, we must keep three terms of which two may be EoM terms; adding four derivatives, we must keep six terms of which five may be EoM terms; adding six derivatives, we only need to keep two EoM terms. The terms we select after all eliminations are

$$\begin{gathered}
\mathcal{V}_\mu K^\dagger [A^\mu, A^\nu]\mathcal{V}_\nu K \quad (d_2^{\pi K}), \\
\mathcal{V}_\mu K^\dagger [A^\mu, D^\nu A^\lambda]\mathcal{V}_\nu \mathcal{V}_\lambda K + \mathrm{h.c.} \quad (d_1^{\pi K}).
\end{gathered}$$

The only product of fields consistent with RPI and parity invariance at order Q^3 is $KK^\dagger A_\mu \chi_-$. The product results in three contracted forms, $\mathrm{Tr}[\hat{\chi}_- A_\mu]K^\dagger K$, $K^\dagger[\hat{\chi}_-, A_\mu]K$, and $\mathrm{Tr}[\chi_-]K^\dagger A_\mu K$, which up to order Q^4 can be combined with either one or three

derivatives. Because all three of the contracted forms are a product of the same four distinguishable fields (no exchange symmetry), the total derivative arguments are exactly the same in each case; with one derivative we must keep two terms of which one may be an EoM term; with three derivatives we may keep only two EoM terms. The three non-EoM terms which we elect to keep are

$$\begin{aligned}
&\mathrm{Tr}[\hat{\chi}_- A^\mu] K^\dagger \mathcal{V}_\mu K + \text{h.c.} \quad (d_3^{\pi K}),\\
&i K^\dagger [\hat{\chi}_-, A^\mu] \mathcal{V}_\mu K + \text{h.c.} \quad (d_4^{\pi K}),\\
&\mathrm{Tr}[\chi_-] K^\dagger A^\mu \mathcal{V}_\mu K + \text{h.c.} \quad (d_5^{\pi K}).
\end{aligned}$$

At order Q^4 four products of field operators are taken into account: $KK^\dagger\chi_+\chi_+$, $KK^\dagger\chi_-\chi_-$, $KK^\dagger A_\mu A_\nu \chi_+$, and $KK^\dagger A_\mu A_\nu A_\lambda A_\kappa$. The first two products will only appear without any additional derivatives, and are both contracted the same way under $\mathrm{SU}(2)_\mathrm{V}$. When coupling the representations of the two matrix fields, only the symmetric combinations $[(\mathbf{0}\oplus\mathbf{1})\otimes(\mathbf{0}\oplus\mathbf{1})]_\mathbf{S} = \mathbf{0}\oplus\mathbf{0}\oplus\mathbf{1_S}\oplus\mathbf{2}$ will appear resulting in three $\mathrm{SU}(2)_\mathrm{V}$ contractions for each product. The six terms generated in this fashion are

$$\begin{aligned}
&\mathrm{Tr}[\chi_+]^2 K^\dagger K \quad (e_{12}^{\pi K}), \qquad &&\mathrm{Tr}[\hat{\chi}_-\hat{\chi}_-] K^\dagger K \quad (e_{13}^{\pi K}),\\
&\mathrm{Tr}[\chi_+] K^\dagger \hat{\chi}_+ K \quad (e_{18}^{\pi K}), \qquad &&\mathrm{Tr}[\chi_-] K^\dagger \hat{\chi}_- K \quad (e_{19}^{\pi K}),\\
&\mathrm{Tr}[\hat{\chi}_+\hat{\chi}_+] K^\dagger K \quad (e_{20}^{\pi K}), \qquad &&\mathrm{Tr}[\chi_-]^2 K^\dagger K \quad (e_{21}^{\pi K}).
\end{aligned}$$

The product of fields $KK^\dagger A_\mu A_\nu \chi_+$ permits six distinct $\mathrm{SU}(2)_\mathrm{V}$ contractions and under the exchange $A_\mu \leftrightarrow A_\nu$ three are symmetric and three are antisymmetric. For the six contracted forms, the power counting permits the addition of two derivatives. The three antisymmetric forms are $\mathrm{Tr}[\hat{\chi}_+ A_\mu A_\nu] K^\dagger K$, $\mathrm{Tr}[\chi_+] K^\dagger [A_\mu, A_\nu] K$, and $K^\dagger [\hat{\chi}_+, [A_\mu, A_\nu]] K$ which all vanish if no derivatives are included. In each of the three cases, when two derivatives are included, the symmetry allows only a single term of order Q^4 which can be eliminated by total derivative arguments. The three symmetric contracted forms are $\mathrm{Tr}[\chi_+]\mathrm{Tr}[A_\mu A_\nu] K^\dagger K$, $\mathrm{Tr}[A_\mu A_\nu] K^\dagger \hat{\chi}_+ K$, and $K^\dagger (A_\mu \hat{\chi}_+ A_\nu + A_\nu \hat{\chi}_+ A_\mu) K$. With the addi-

tion of two derivatives to each of the symmetric forms, we are required to keep only a single non-EoM term. The three symmetric forms result in a total of six terms in the effective Lagrangian at order Q^4,

$$\begin{aligned}
&\mathrm{Tr}[\chi_+]\mathrm{Tr}[A_\mu A^\mu]K^\dagger K \quad (e_{11}^{\pi K}), && \mathrm{Tr}[\chi_+]\mathrm{Tr}[A^\mu A^\nu]\mathcal{V}_\mu K^\dagger \mathcal{V}_\nu K \quad (e_{10}^{\pi K}),\\
&\mathrm{Tr}[A_\mu A^\mu]K^\dagger \hat{\chi}_+ K \quad (e_{17}^{\pi K}), && \mathrm{Tr}[A^\mu A^\nu]\mathcal{V}_\mu K^\dagger \hat{\chi}_+ \mathcal{V}_\nu K \quad (e_{16}^{\pi K}),\\
&K^\dagger A_\mu \hat{\chi}_+ A^\mu K \quad (e_{15}^{\pi K}), && \mathcal{V}_\mu K^\dagger (A^\mu \hat{\chi}_+ A^\nu + A^\nu \hat{\chi}_+ A^\mu)\mathcal{V}_\nu K \quad (e_{14}^{\pi K}).
\end{aligned}$$

The final product of fields, $KK^\dagger A_\mu A_\nu A_\lambda A_\kappa$, transforms as $\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$ which contains nine singlet representations. Three of the nine $\mathrm{SU(2)_V}$ contractions correspond to $\mathrm{Tr}[A_\mu A_\nu]\mathrm{Tr}[A_\lambda A_\kappa]K^\dagger K$ under distinct permutations of Lorentz indices; the remaining six contracted forms are distinct permutations among the Lorentz indices of $\mathrm{Tr}[A_\mu A_\nu]K^\dagger [A_\lambda, A_\kappa]K$. In each of the two cases, the power counting permits distributing up to four derivatives on the fields, but the power counting also prohibits any EoM terms in the set of terms which can contribute at order Q^4. First we consider the second, commutator-type form. The contributions of order Q^4 from this term vanish if either no derivatives or four derivatives are distributed on the fields; in the case of adding two derivatives, a single term of order Q^4 must be kept. The term we choose to keep is

$$i\mathrm{Tr}[A_\mu A^\nu]K^\dagger [A^\mu, A^\lambda]\mathcal{V}_\nu \mathcal{V}_\lambda K + \mathrm{h.c.} \quad (e_9^{\pi K}).$$

For the double-trace contracted form, there are two ways to contract the Lorentz indices without adding derivatives of the fields. In addition, with two derivatives on the fields we must keep two terms at order Q^4; with four derivatives we must keep a single term at order Q^4. The net contribution to the effective Lagrangian is five terms; we select the following terms for this set:

$$\begin{aligned}
&\mathrm{Tr}[A_\mu A^\mu]\mathrm{Tr}[A_\nu A^\nu]K^\dagger K \quad (e_7^{\pi K}),\\
&\mathrm{Tr}[A_\mu A_\nu]\mathrm{Tr}[A^\mu A^\nu]K^\dagger K \quad (e_8^{\pi K}),
\end{aligned}$$

$$\mathrm{Tr}[A_\mu A^\mu]\mathrm{Tr}[A^\nu A^\lambda]\mathcal{V}_\nu K^\dagger \mathcal{V}_\lambda K \quad (e_5^{\pi K}),$$
$$\mathrm{Tr}[A_\mu A^\nu]\mathrm{Tr}[A^\mu A^\lambda]\mathcal{V}_\nu K^\dagger \mathcal{V}_\lambda K \quad (e_6^{\pi K}),$$
$$\mathrm{Tr}[A^\mu A^\nu]\mathrm{Tr}[A^\lambda A^\kappa]\mathcal{V}_\mu \mathcal{V}_\nu K^\dagger \mathcal{V}_\lambda \mathcal{V}_\kappa K \quad (e_4^{\pi K}).$$

We collect the terms of $\mathcal{L}^{\pi K}$ listed above and present the complete results for the chiral expansion $\mathcal{L}^{\pi K} = \sum_j \mathcal{L}_j^{\pi K}$ to order Q^4. We expand the RPI-covariant derivatives to separate the terms which contribute with different powers of Q. We also explicitly symmetrize the multiple derivatives and derivatives of A_μ wherever necessary and absorb some constant factors into the coefficients where convenient. Finally, all of the isospin-violating terms are collected together at the end of each equation, offset with parentheses.

$$\mathcal{L}_1^{\pi K} = iK^\dagger D_0 K \tag{4.19}$$

$$\begin{aligned}\bar{M}_K \mathcal{L}_2^{\pi K} &= \tfrac{1}{2} D_\mu K^\dagger D^\mu K + c_1^{\pi K}\mathrm{Tr}[A^0 A^0]K^\dagger K + c_2^{\pi K}\mathrm{Tr}[A_\mu A^\mu]K^\dagger K \\ &\quad + c_3^{\pi K}\mathrm{Tr}[\chi_+]K^\dagger K \Big(+ c_4^{\pi K} K^\dagger \hat{\chi}_+ K\Big)\end{aligned} \tag{4.20}$$

$$\begin{aligned}\bar{M}_K^2 \mathcal{L}_3^{\pi K} &= ic_1^{\pi K}\mathrm{Tr}[A^0 A^\mu](K^\dagger D_\mu K - \mathrm{h.c.}) + id_1^{\pi K} K^\dagger [A^0, D_0 A^0] K \\ &\quad + id_2^{\pi K}(K^\dagger [A^0, A^\mu] D_\mu K + \mathrm{h.c.}) + id_3^{\pi K}\mathrm{Tr}[\hat{\chi}_- A^0]K^\dagger K \\ &\quad + d_4^{\pi K} K^\dagger [\hat{\chi}_-, A^0] K \Big(+ id_5^{\pi K}\mathrm{Tr}[\chi_-]K^\dagger A^0 K\Big)\end{aligned} \tag{4.21}$$

$$\begin{aligned}\bar{M}_K^3 \mathcal{L}_4^{\pi K} &= c_1^{\pi K}\mathrm{Tr}[A^\mu A^\nu] D_\mu K^\dagger D_\nu K \\ &\quad - \tfrac{1}{2} d_1^{\pi K}\Big\{ K^\dagger([A^\mu, D_0 A^0] + [A^0, D_0 A^\mu] + [A^0, D^\mu A^0]) D_\mu K - \mathrm{h.c.}\Big\} \\ &\quad + d_2^{\pi K} D_\mu K^\dagger [A^\mu, A^\nu] D_\nu K - \tfrac{1}{2} d_3^{\pi K}\mathrm{Tr}[\hat{\chi}_- A^\mu](K^\dagger D_\mu K - \mathrm{h.c.}) \\ &\quad + \tfrac{i}{2} d_4^{\pi K}(K^\dagger [\hat{\chi}_-, A^\mu] D_\mu K - \mathrm{h.c.}) + e_1^{\pi K}\mathrm{Tr}[D_0 A^0 D_0 A^0]K^\dagger K \\ &\quad + e_2^{\pi K}\mathrm{Tr}[A^\mu D_0 A^0](K^\dagger D_\mu K + \mathrm{h.c.}) + e_4^{\pi K}\mathrm{Tr}[A^0 A^0]\mathrm{Tr}[A^0 A^0]K^\dagger K \\ &\quad + e_3^{\pi K}\mathrm{Tr}[D_\mu A_\nu (D^\mu A^\nu + D^\nu A^\mu)]K^\dagger K + e_5^{\pi K}\mathrm{Tr}[A^0 A^0]\mathrm{Tr}[A_\mu A^\mu]K^\dagger K \\ &\quad + e_6^{\pi K}\mathrm{Tr}[A^0 A_\mu]\mathrm{Tr}[A^0 A^\mu]K^\dagger K + e_7^{\pi K}\mathrm{Tr}[A_\mu A^\mu]\mathrm{Tr}[A_\nu A^\nu]K^\dagger K \\ &\quad + e_8^{\pi K}\mathrm{Tr}[A_\mu A_\nu]\mathrm{Tr}[A^\mu A^\nu]K^\dagger K + ie_9^{\pi K}\mathrm{Tr}[A^0 A_\mu]K^\dagger [A^0, A^\mu] K \\ &\quad + e_{10}^{\pi K}\mathrm{Tr}[\chi_+]\mathrm{Tr}[A^0 A^0]K^\dagger K + e_{11}^{\pi K}\mathrm{Tr}[\chi_+]\mathrm{Tr}[A_\mu A^\mu]K^\dagger K\end{aligned} \tag{4.22}$$

$$
\begin{aligned}
&+ e_{12}^{\pi K}\mathrm{Tr}[\chi_+]^2 K^\dagger K + e_{13}^{\pi K}\mathrm{Tr}[\hat\chi_-\hat\chi_-]K^\dagger K \\
&\Big(- \tfrac{1}{2} d_5^{\pi K}\mathrm{Tr}[\chi_-](K^\dagger A^\mu D_\mu K - \text{.h.c}) \\
&\quad + e_{14}^{\pi K} K^\dagger A^0 \hat\chi_+ A^0 K + e_{15}^{\pi K} K^\dagger A_\mu \hat\chi_+ A^\mu K + e_{16}^{\pi K}\mathrm{Tr}[A^0 A^0]K^\dagger \hat\chi_+ K \\
&\quad + e_{17}^{\pi K}\mathrm{Tr}[A_\mu A^\mu]K^\dagger \hat\chi_+ K + e_{18}^{\pi K}\mathrm{Tr}[\chi_+]K^\dagger \hat\chi_+ K + e_{19}^{\pi K}\mathrm{Tr}[\chi_-]K^\dagger \hat\chi_- K \\
&\quad + e_{20}^{\pi K}\mathrm{Tr}[\hat\chi_+\hat\chi_+]K^\dagger K + e_{21}^{\pi K}\mathrm{Tr}[\chi_-]^2 K^\dagger K \Big)
\end{aligned}
$$

This procedure does not need to be repeated for the $\pi\eta$ sector. The construction of $\mathcal{L}^{\pi\eta}$ is identical to the construction of $\mathcal{L}^{\pi K}$ with three exceptions: the eta transforms as a singlet under $\mathrm{SU(2)_V}$; all isospin-violating terms must be omitted as discussed in section 4.1; and the effective Lagrangian within the $\pi\eta$ sector is invariant under charge conjugation. From the results for $\mathcal{L}^{\pi K}$ we construct $\mathcal{L}^{\pi\eta}$ directly by simply dropping all terms which violate isospin conservation or are not proportional to the general form $(D^m K^\dagger D^n K)$, replacing $\bar M_K$ by $\bar M_\eta$ and each occurrence of $(D^m K^\dagger D^n K)$ by $(\partial^m \eta^\dagger \partial^n \eta)$, and finally checking the remaining terms for charge conjugation invariance.

$$
\begin{aligned}
\mathcal{L}_1^{\pi\eta} &= i\eta^\dagger \partial_0 \eta && (4.23)\\
\bar M_\eta \mathcal{L}_2^{\pi\eta} &= \tfrac{1}{2}\partial_\mu \eta^\dagger \partial^\mu \eta + c_1^{\pi\eta}\mathrm{Tr}[A^0 A^0]\eta^\dagger\eta && (4.24)\\
&\quad + c_2^{\pi\eta}\mathrm{Tr}[A_\mu A^\mu]\eta^\dagger\eta + c_3^{\pi\eta}\mathrm{Tr}[\chi_+]\eta^\dagger\eta \\
\bar M_\eta^2 \mathcal{L}_3^{\pi\eta} &= i c_1^{\pi\eta}\mathrm{Tr}[A^0 A^\mu](\eta^\dagger\partial_\mu\eta - \text{h.c.}) + d^{\pi\eta}\mathrm{Tr}[\hat\chi_- A^0]\eta^\dagger\eta && (4.25)\\
\bar M_\eta^3 \mathcal{L}_4^{\pi\eta} &= c_1^{\pi\eta}\mathrm{Tr}[A^\mu A^\nu]\partial_\mu\eta^\dagger\partial_\nu\eta - \tfrac{1}{2} d^{\pi\eta}\mathrm{Tr}[\hat\chi_- A^\mu](\eta^\dagger\partial_\mu\eta - \text{h.c.}) && (4.26)\\
&\quad + e_1^{\pi\eta}\mathrm{Tr}[D_0 A^0 D_0 A^0]\eta^\dagger\eta + e_2^{\pi\eta}\mathrm{Tr}[A^\mu D_0 A^0](\eta^\dagger\partial_\mu\eta + \text{h.c.}) \\
&\quad + e_3^{\pi\eta}\mathrm{Tr}[D_\mu A_\nu (D^\mu A^\nu + D^\nu A^\mu)]\eta^\dagger\eta + e_4^{\pi\eta}\mathrm{Tr}[A^0 A^0]\mathrm{Tr}[A^0 A^0]\eta^\dagger\eta \\
&\quad + e_5^{\pi\eta}\mathrm{Tr}[A^0 A^0]\mathrm{Tr}[A_\mu A^\mu]\eta^\dagger\eta + e_6^{\pi\eta}\mathrm{Tr}[A^0 A_\mu]\mathrm{Tr}[A^0 A^\mu]\eta^\dagger\eta \\
&\quad + e_7^{\pi\eta}\mathrm{Tr}[A_\mu A^\mu]\mathrm{Tr}[A_\nu A^\nu]\eta^\dagger\eta + e_8^{\pi\eta}\mathrm{Tr}[A_\mu A_\nu]\mathrm{Tr}[A^\mu A^\nu]\eta^\dagger\eta \\
&\quad + e_{10}^{\pi\eta}\mathrm{Tr}[\chi_+]\mathrm{Tr}[A^0 A^0]\eta^\dagger\eta + e_{11}^{\pi\eta}\mathrm{Tr}[\chi_+]\mathrm{Tr}[A_\mu A^\mu]\eta^\dagger\eta \\
&\quad + e_{12}^{\pi\eta}\mathrm{Tr}[\chi_+]^2\eta^\dagger\eta + e_{13}^{\pi\eta}\mathrm{Tr}[\hat\chi_-\hat\chi_-]\eta^\dagger\eta
\end{aligned}
$$

4.3.2 The πKK Sector

In the πKK sector, the full expression for the effective Lagrangian to order Q^4 is not necessary to treat $2 \leftrightarrow 2$ scattering processes until very high order. For instance terms involving four factors of A_μ only appear with at least two closed pion loops and will be suppressed by at least a factor of Q^6. We develop the full effective Lagrangian to order Q^2; at orders Q^3 and Q^4 we consider only those terms which give rise to four-kaon contact terms. To distinguish contact terms from pion interactions, we note that an expansion of quantities in the number of pion fields gives $A_\mu \simeq -\partial_\mu \pi/2F$, $\hat{\chi}_- \simeq -4i\hat{m}B\pi/F$, and $\mathrm{Tr}[\chi_-] \simeq -4i(m_u - m_d)B\pi^0/F$.

Also, we choose to couple the kaon annihilation operators together as $K\epsilon K$ and $K\epsilon\tau_j K$, where $\epsilon = i\tau_2$, before contracting with the other fields present for the advantage of greater symmetry under exchanges, antisymmetric and symmetric respectively. Coupling four kaon operators together in this way gives six representations under $\mathrm{SU(2)_V}$. The three representations involving an antisymmetric contraction of fields are simple products of the two forms above and transform as $\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{1}$; the remaining three combinations of fields are $(K\epsilon\tau_j K)^\dagger(K\epsilon\tau_j K)$ (a $\mathbf{0}$), $i\epsilon_{jkl}(K\epsilon\tau_j K)^\dagger(K\epsilon\tau_k K)$ (a $\mathbf{1}$), and $\big((K\epsilon\tau_j K)^\dagger(K\epsilon\tau_k K) + (K\epsilon\tau_k K)^\dagger(K\epsilon\tau_j K)\big)$ (a $\mathbf{2}$), where use of ϵ_{jkl}, the three-dimensional Levi-Civita symbol, is always clear from context. For brevity of notation we define the combinations

$$(K_1\ K_2|K_3\ K_4) = (K_1\epsilon\tau_j K_2)^\dagger(K_3\epsilon\tau_j K_4), \tag{4.27}$$

$$(K_1\ K_2|O|K_3\ K_4) = i\epsilon_{jkl}(K_1\epsilon\tau_j K_2)^\dagger \mathrm{Tr}[\tau_k O](K_3\epsilon\tau_l K_4) \tag{4.28}$$

where K_j represent general fields in the $\frac{\mathbf{1}}{\mathbf{2}}$ representation of $\mathrm{SU(2)_V}$ and O is any $\mathrm{SU(2)_V}$ matrix operator.

Without any factors of A_μ or $\chi_\pm$ there are two contracted forms of four kaon field operators, $(K\epsilon K)^\dagger(K\epsilon K)$ and $(K\ K|K\ K)$. To determine all contact operators up to order Q^4 requires distributing up to eight RPI-covariant derivatives on each form. (Many

with several derivatives contribute only at much higher order in tree diagrams; however, we include them for completeness regardless.) The algorithm for making optimal use of total derivative arguments serves us very well in this endeavor; we need to deal with as many as 221 candidate terms restricted by 179 linearly independent total derivatives. We consider the symmetric contracted form first and count the terms left after total derivative arguments; one term with no derivatives is necessary; adding two derivatives, we must keep three terms of which two are EoM terms; adding four derivatives, we must keep nine terms of which seven are EoM terms; adding six derivatives, 20 terms are required of which 18 are EoM terms; finally adding eight derivatives, 42 terms are required of which 39 are EoM terms. After all redundant terms are eliminated, we select the following set of terms:

$$
\begin{gathered}
(K\ K|K\ K)\ \ (a^{\pi KK}), \qquad (K\ \mathcal{V}_\mu K|K\ \mathcal{V}^\mu K)\ \ (b^{\pi KK}),\\
(\mathcal{V}_\mu K\ \mathcal{V}^\mu K|\mathcal{V}_\nu K\ \mathcal{V}^\nu K)\ \ (c_1^{\pi KK}), \qquad (\mathcal{V}_\mu K\ \mathcal{V}_\nu K|\mathcal{V}^\mu K\ \mathcal{V}^\nu K)\ \ (c_2^{\pi KK}),\\
(\mathcal{V}_\mu K\ \mathcal{V}_\nu \mathcal{V}_\lambda K|\mathcal{V}^\mu K\ \mathcal{V}^\nu \mathcal{V}^\lambda K)\ \ (d_1^{\pi KK}), \qquad (\mathcal{V}_\nu K\ \mathcal{V}_\mu \mathcal{V}_\lambda K|\mathcal{V}^\lambda K\ \mathcal{V}^\mu \mathcal{V}^\nu K)\ \ (d_2^{\pi KK}),\\
(\mathcal{V}_\mu \mathcal{V}_\nu K\ \mathcal{V}_\lambda \mathcal{V}_\kappa K|\mathcal{V}^\mu \mathcal{V}^\nu K\ \mathcal{V}^\lambda \mathcal{V}^\kappa K)\ \ (e_1^{\pi KK}),\\
(\mathcal{V}_\mu \mathcal{V}_\lambda K\ \mathcal{V}_\nu \mathcal{V}^\lambda K|\mathcal{V}_\kappa \mathcal{V}^\mu K\ \mathcal{V}^\kappa \mathcal{V}^\nu K)\ \ (e_2^{\pi KK}),\\
(\mathcal{V}_\mu \mathcal{V}_\nu K\ \mathcal{V}^\mu \mathcal{V}^\nu K|\mathcal{V}_\lambda \mathcal{V}_\kappa K\ \mathcal{V}^\lambda \mathcal{V}^\kappa K)\ \ (e_3^{\pi KK}).
\end{gathered}
$$

For the antisymmetric contracted form the term with no derivatives vanishes; adding two derivatives, we must keep a single non-EoM term; adding four derivatives, we must keep four terms of which three are EoM terms; adding six derivatives, twelve terms must be retained of which ten are EoM terms; adding eight derivatives, 28 terms must be retained of which 26 are EoM terms. After all redundant terms are eliminated, we select the following set of terms:

$$
\begin{gathered}
(K\epsilon\mathcal{V}_\mu K)^\dagger(K\epsilon\mathcal{V}^\mu K)\ \ (c_3^{\pi KK}),\\
(K\epsilon\mathcal{V}_\mu K)^\dagger(\mathcal{V}_\nu K\epsilon\mathcal{V}^\mu\mathcal{V}^\nu) + \text{h.c.}\ \ (d_4^{\pi KK}),
\end{gathered}
$$

$$(\mathcal{V}_\nu K\epsilon\mathcal{V}_\mu\mathcal{V}_\lambda K)^\dagger(\mathcal{V}^\lambda K\epsilon\mathcal{V}^\mu\mathcal{V}^\nu K)\ \ (d_3^{\pi KK}),$$
$$(\mathcal{V}_\nu K\epsilon\mathcal{V}_\mu\mathcal{V}^\nu K)^\dagger(\mathcal{V}_\lambda K\epsilon\mathcal{V}^\mu\mathcal{V}^\lambda K)\ \ (e_5^{\pi KK}),$$
$$(\mathcal{V}_\mu\mathcal{V}_\nu K\epsilon\mathcal{V}_\lambda\mathcal{V}_\kappa K)^\dagger(\mathcal{V}^\mu\mathcal{V}^\nu K\epsilon\mathcal{V}^\lambda\mathcal{V}^\kappa K)\ \ (e_4^{\pi KK}),$$
$$(\mathcal{V}_\nu K\epsilon\mathcal{V}_\mu\mathcal{V}^\nu K)^\dagger(\mathcal{V}_\lambda\mathcal{V}_\kappa K\epsilon\mathcal{V}^\mu\mathcal{V}^\lambda\mathcal{V}^\kappa K).$$

The final term in the above list does not contribute until order Q^5, so we do not list an associated coefficient for that term.

The next product of fields we consider is $KKK^\dagger K^\dagger\chi_+$ which results in five contracted forms and each form must be considered with up to four derivatives. Because χ_+ is transparent to covariant derivatives, three of the contracted forms yield terms analogous to a product of fields $KKK^\dagger K^\dagger$. The contracted forms $\mathrm{Tr}[\chi_+](K\ K|K\ K)$ and $(K\ K|\chi_+|K\ K)$ give four terms each after all redundant terms are eliminated in analogy to $(K\ K|K\ K)$;

$$\mathrm{Tr}[\chi_+](K\ K|K\ K)\ \ (c_{10}^{\pi KK}),\qquad (K\ K|\chi_+|K\ K)\ \ (c_{11}^{\pi KK}),$$
$$\mathrm{Tr}[\chi_+](K\,\mathcal{V}_\mu K|K\,\mathcal{V}^\mu K)\ \ (d_5^{\pi KK}),\qquad (K\,\mathcal{V}_\mu K|\chi_+|K\,\mathcal{V}^\mu K)\ \ (d_6^{\pi KK}),$$
$$\mathrm{Tr}[\chi_+](\mathcal{V}_\mu K\,\mathcal{V}^\mu K|\mathcal{V}_\nu K\,\mathcal{V}^\nu K)\ \ (e_6^{\pi KK}),\qquad (\mathcal{V}_\mu K\,\mathcal{V}^\mu K|\chi_+|\mathcal{V}_\nu K\,\mathcal{V}^\nu K)\ \ (e_{11}^{\pi KK}),$$
$$\mathrm{Tr}[\chi_+](\mathcal{V}_\mu K\,\mathcal{V}_\nu K|\mathcal{V}^\mu K\,\mathcal{V}^\nu K)\ \ (e_7^{\pi KK}),\qquad (\mathcal{V}_\mu K\,\mathcal{V}_\nu K|\chi_+|\mathcal{V}^\mu K\,\mathcal{V}^\nu K)\ \ (e_{12}^{\pi KK}).$$

The contracted form $\mathrm{Tr}[\chi_+](K\epsilon K)^\dagger(K\epsilon K)$ gives two terms after all redundant terms are eliminated in analogy to $(K\epsilon K)^\dagger(K\epsilon K)$;

$$\mathrm{Tr}[\chi_+](K\epsilon\mathcal{V}_\mu K)^\dagger(K\epsilon\mathcal{V}^\mu K)\ \ (e_8^{\pi KK}),$$
$$\mathrm{Tr}[\chi_+](K\epsilon\mathcal{V}_\mu K)^\dagger(\mathcal{V}_\nu K\epsilon\mathcal{V}^\mu\mathcal{V}^\nu K).$$

Again, the final term in the list contributes only beyond order Q^4. The final two contracted forms for the product $KKK^\dagger K^\dagger\chi_+$ are the hermitian conjugate pair $(K\epsilon K)^\dagger(K\epsilon\hat{\chi}_+K)$ and $(K\epsilon\hat{\chi}_+K)^\dagger(K\epsilon K)$. Each of the forms vanishes without derivatives; adding two derivatives, in each case only one EoM term is required; adding four derivatives, four EoM terms are required. These final two contracted pairs do not result in any necessary terms in the

effective Lagrangian.

The only remaining terms we will consider at order Q^4 are derived from the field product $KKK^\dagger K^\dagger \chi_+\chi_+$ without any derivatives. Considering the $\mathrm{SU(2)_V}$ couplings gives eight distinct contractions, but four vanish due to antisymmetry. The remaining four terms contributing to the effective Lagrangian at order Q^4 are

$$\begin{aligned} &\mathrm{Tr}[\chi_+]^2(K\,K|K\,K)\ \ (e_9^{\pi KK}), \qquad &&\mathrm{Tr}[\chi_+](K\,K|\hat{\chi}_+|K\,K)\ \ (e_{10}^{\pi KK}),\\ &\mathrm{Tr}[\hat{\chi}_+\hat{\chi}_+](K\,K|K\,K)\ \ (e_{13}^{\pi KK}), \qquad &&(K\epsilon\hat{\chi}_+K)^\dagger(K\epsilon\hat{\chi}_+K)\ \ (e_{14}^{\pi KK}). \end{aligned}$$

The final product of fields in the πKK sector is $KKK^\dagger K^\dagger A_\mu A_\nu$ and to derive the Lagrangian to order Q^2 we only need to consider adding up to two derivatives. The $\mathrm{SU(2)_V}$ contractions yield six independent forms, two of which are related by hermitian conjugation. Without adding any derivatives to the product of fields, four of the contracted forms vanish due to an antisymmetry under an exchange of fields. Adding two derivatives to each of these four contracted forms gives terms which only appear at higher order than Q^2; we list the terms derived with the power of Q at which it would first contribute. From the hermitian-conjugate pair of contracted forms we get two EoM terms and

$$\begin{aligned} &(K\epsilon\mathcal{V}_\mu K)^\dagger(K\epsilon[A^\mu,A^\nu]\mathcal{V}_\nu K)\ \ (\sim Q^3,\ \text{twice}),\\ &(K\epsilon\mathcal{V}_\mu K)^\dagger(K\epsilon[A_\nu,D^\mu A^\nu]K)\ \ (\sim Q^4,\ \text{twice}); \end{aligned}$$

each form contributing both as $(O+O^\dagger)$ and as $i(O-O^\dagger)$. The other two contracted forms with a pair of fields antisymmetric under exchange give two EoM terms and the higher-order terms

$$\begin{aligned} &(K\,\mathcal{V}_\mu K|A^\mu A^\nu|K\,\mathcal{V}_\nu K)\ \ (\sim Q^3),\\ &\mathrm{Tr}[A_\mu A^\mu](K\epsilon\mathcal{V}_\nu K)^\dagger(K\epsilon\mathcal{V}^\nu K)\ \ (\sim Q^4),\\ &\mathrm{Tr}[A^\mu A^\nu](K\epsilon\mathcal{V}_\mu K)^\dagger(K\epsilon\mathcal{V}_\nu K)\ \ (\sim Q^4). \end{aligned}$$

The two terms without any derivatives which do contribute to $\mathcal{L}_2^{\pi KK}$ are

$$\mathrm{Tr}[A_\mu A^\mu](K\ K|K\ K)\ \ (c_6^{\pi KK}), \qquad (K\epsilon A_\mu K)^\dagger(K\epsilon A^\mu K)\ \ (c_9^{\pi KK}).$$

When two derivatives are distributed on either form, we must retain eleven terms of which five may be EoM terms. Of the twelve terms appearing in the effective Lagrangian, the only four contributing to $\mathcal{L}_2^{\pi KK}$ are

$$\begin{gathered}
\mathrm{Tr}[A^\mu A^\nu](K\,\mathcal{V}_\mu K|K\,\mathcal{V}_\nu K)\ \ (c_4^{\pi KK}),\\
\mathrm{Tr}[A^\mu A^\nu](K\ K|\mathcal{V}_\mu K\,\mathcal{V}_\nu K)+\mathrm{h.c.}\ \ (c_5^{\pi KK}),\\
(K\epsilon A^\mu K)^\dagger(\mathcal{V}_\mu K\epsilon A^\nu\mathcal{V}_\nu K)+\mathrm{h.c.}\ \ (c_7^{\pi KK}),\\
(K\epsilon A^\mu\mathcal{V}_\mu K)^\dagger(K\epsilon A^\nu\mathcal{V}_\nu K)+(K\epsilon A^\mu\mathcal{V}_\nu K)^\dagger(K\epsilon A^\nu\mathcal{V}_\mu K)\ \ (c_8^{\pi KK}).
\end{gathered}$$

From the product of fields $KKK^\dagger K^\dagger A_\mu A_\nu$, the remaining eight terms derived are

$$\begin{gathered}
i\mathrm{Tr}[A^\mu A^\nu](K\ K|\mathcal{V}_\mu K\,\mathcal{V}_\nu K)+\mathrm{h.c.}\ \ (\sim Q^3),\\
i(K\epsilon A^\mu K)^\dagger(\mathcal{V}_\mu K\epsilon A^\nu\mathcal{V}_\nu K)+\mathrm{h.c.}\ \ (\sim Q^3),\\
\mathrm{Tr}[A_\mu A^\mu](K\ K|\mathcal{V}_\nu K\,\mathcal{V}^\nu K)\ \ (\sim Q^3,\ \text{twice}),\\
(K\epsilon A_\mu K)^\dagger(\mathcal{V}_\nu K\epsilon A^\mu\mathcal{V}^\nu K)\ \ (\sim Q^3,\text{twice}),\\
\mathrm{Tr}[D_\mu A_\nu D^\mu A^\nu](K\ K|K\ K)\ \ (\sim Q^4),\\
(K\epsilon D_\mu A_\nu K)^\dagger(K\epsilon D^\mu A^\nu K)\ \ (\sim Q^4).
\end{gathered}$$

We collect the terms listed above and present the results for $\mathcal{L}_0^{\pi KK}$ through $\mathcal{L}_3^{\pi KK}$. The contact terms of $\mathcal{L}_4^{\pi KK}$ are easily derived from the enumerated terms above; we do not present the fully-expanded form of $\mathcal{L}_4^{\pi KK}$.

$$\bar{M}_K^2\mathcal{L}_0^{\pi KK} = a^{\pi KK}(K\ K|K\ K) \tag{4.29}$$

$$\bar{M}_K^3\mathcal{L}_1^{\pi KK} = ib^{\pi KK}\{(K\ K|K\ D_0K)-\mathrm{h.c.}\} \tag{4.30}$$

$$
\begin{aligned}
\bar{M}_K^4 \mathcal{L}_2^{\pi KK} &= b^{\pi KK}(K\ D_\mu K|K\ D^\mu K) + c_3^{\pi KK}(K\epsilon D_\mu K)^\dagger(K\epsilon D^\mu K) \qquad (4.31)\\
&\quad + (c_1^{\pi KK} + 2c_2^{\pi KK})(K\ D_0 K|K\ D_0 K)\\
&\quad - c_2^{\pi KK}\{(K\ K|D_0 K\ D_0 K) + \text{h.c.}\}\\
&\quad + (c_4^{\pi KK} + c_5^{\pi KK})\text{Tr}[A_0 A_0](K\ K|K\ K) + c_6^{\pi KK}\text{Tr}[A_\mu A^\mu](K\ K|K\ K)\\
&\quad + (c_7^{\pi KK} + c_8^{\pi KK})(K\epsilon A_0 K)^\dagger(K\epsilon A_0 K) + c_9^{\pi KK}(K\epsilon A_\mu K)^\dagger(K\epsilon A^\mu K)\\
&\quad + c_{10}^{\pi KK}\text{Tr}[\chi_+](K\ K|K\ K)\Big(+ c_{11}^{\pi KK}(K\ K|\hat{\chi}_+|K\ K)\Big)\\
\bar{M}_K^5 \mathcal{L}_3^{\pi KK} &= \tfrac{i}{2}c_1^{\pi KK}\{(K\ D_0 K|D_\mu K\ D^\mu K) - \text{h.c.}\} \qquad (4.32)\\
&\quad + 2ic_2^{\pi KK}\{(K\ D_\mu K|D_0 K\ D^\mu K) - \text{h.c.}\}\\
&\quad + i(d_1^{\pi KK} + d_2^{\pi KK})\Big\{\ (K\ K|D_0 K\ D_0 D_0 K) - (K\ D_0 K|K\ D_0 D_0 K)\\
&\qquad\qquad - 2(K\ D_0 K|D_0 K\ D_0 K) - \text{h.c.}\ \Big\}\\
&\quad + id_3^{\pi KK}\{(K\epsilon D_0 K)^\dagger(K\epsilon D_0 D_0 K) - \text{h.c.}\}\\
&\quad + id_4^{\pi KK}\Big\{\ (K\epsilon D_\mu K)^\dagger(K\epsilon D_0 D^\mu K) + (K\epsilon D_\mu K)^\dagger(K\epsilon D^\mu D_0 K)\\
&\qquad\qquad + 2(K\epsilon D_\mu K)^\dagger(D_0 K\epsilon D^\mu K) - \text{h.c.}\ \Big\}\\
&\quad + id_5^{\pi KK}\text{Tr}[\chi_+]\{(K\ K|K\ D_0 K) - \text{h.c.}\}\\
&\quad \Big(+ id_6^{\pi KK}\{(K\ K|\hat{\chi}_+|K\ D_0 K) - \text{h.c.}\}\Big)\\
&\quad + \text{other terms involving } A_\mu
\end{aligned}
$$

Having worked out the forms of the effective Lagrangians, we turn in the next section to a matching calculation to determine the coefficients which appear in terms of the parameters of the 'high-energy' theory, SU(3) χPT.

4.4 Matching Calculations

A topic which needs to be addressed before setting up the matching calculations between the heavy kaon/eta theory and SU(3) chiral perturbation theory is the difference between the expansions of the two theories. The heavy kaon/eta effective theory is an

expansion in Q/M, where Q represents the generic scale of m_π and external momenta $|\vec{p}\,|$, and M represents the scale of heavy masses $\bar{M}_{K,\eta}$. In contrast, SU(3) χPT is an expansion in $(Q/\Lambda)^2$ where Q represents the scale of external momenta and the full set of pseudo-Goldstone boson masses. To reconcile the two effective field theories in the low-energy regime $|\vec{p}\,| \lesssim m_\pi$, we must recognize three distinct mass scales Q, M, and Λ which represent respectively external momenta and m_π, $\bar{M}_K$ and $\bar{M}_\eta$, and the chiral symmetry breaking scale.

Distinguishing the scales in χPT is trivial. We compute scattering amplitudes to a given order in $1/\Lambda$, then in each order separate the expression by powers of $m_s B_0 \sim M^2$. In contrast, the role of Λ in the heavy kaon/eta effective theory is buried in the coefficients of the effective theory, just as the coefficients of χPT are implicitly unknown functions of $\Lambda_{\rm QCD}$ and the heavy quark masses [13]. From within the heavy kaon/eta theory there is no way to determine the dependence of the coefficients on Λ via scaling arguments as in section 2.2 because the dependence is obscured by the intermediate scale M. The matching calculation provides a direct way to establish the relationship between the coefficients and the chiral symmetry breaking scale.

We match the theories by equating on-shell scattering amplitudes and the locations and residues of poles in the heavy field propagators of the two theories. (Off the mass-shell, the scattering amplitudes depend, in an unphysical way, on how the meson fields are defined and matching would require including all of the EoM terms eliminated by field redefinitions.) We choose to match amplitudes with a relativistic normalization; so a factor of $\sqrt{2\bar{M}_{K,\eta}}$ is included for each external heavy-particle state in the heavy kaon/eta theory, consistent with the discussion in subsection 4.2.2. For the amplitudes calculated in χPT we perform a non-relativistic expansion, making the replacement $p^0 \to (\bar{M}_{K,\eta} + k^0)$. The expansions for an arbitrary amplitude $\mathcal{A}$ of mass-dimension d, in χPT and in heavy kaon/eta theory respectively, are (symbolically)

$$\mathcal{A}_{\chi\rm PT} \;=\; M^d \sum_{j,k} \bar{\alpha}_{2j,k} \frac{Q^{2j} M^{2(k-j)}}{\Lambda^{2k}} = M^d \sum_j \frac{Q^{2j}}{M^{2j}} \sum_k \bar{\alpha}_{2j,k} \frac{M^{2k}}{\Lambda^{2k}}, \tag{4.33}$$

$$\mathcal{A}_{NR} \;=\; M^d \sum_j \alpha_j \frac{Q^j}{M^j}, \tag{4.34}$$

where the $(1/\Lambda)$-expansion of χPT has been reorganized as an expansion in M/Λ within an expansion in Q/M. Equating the expansions in powers of Q determines the combinations of terms $\alpha_j Q^j = \sum_k \bar{\alpha}_{j,k} Q^j (M^2/\Lambda^2)^k$ for even j and $\alpha_j Q^j = 0$ for odd j. Solving for the coefficients in the heavy kaon/eta effective theory, we find that matching gives the coefficients as an expansion in powers of $(M/\Lambda)^2$. This behavior is seen explicitly in the results for matching F and B to the parameters of SU(3) χPT by comparing amplitudes for pion interactions. The matching relations were derived by Gasser and Leutwyler [20];

$$F \;=\; F_0 \left\{ 1 - \frac{\bar{M}_K^2}{32\pi^2 F_0^2} \ln \frac{\bar{M}_K^2}{\mu^2} + 8 L_4^r \frac{\bar{M}_K^2}{F_0^2} + \mathcal{O}(M^4/\Lambda^4) \right\}, \tag{4.35}$$

$$B \;=\; B_0 \left\{ 1 - \frac{\bar{M}_\eta^2}{96\pi^2 F_0^2} \ln \frac{\bar{M}_\eta^2}{\mu^2} - 16(L_4^r - 2L_6^r) \frac{\bar{M}_K^2}{F_0^2} + \mathcal{O}(M^4/\Lambda^4) \right\}. \tag{4.36}$$

The leading-order results from χPT determine coefficents of the effective Lagrangians $\mathcal{L}^{(X)}_{j\leq 3}$ to leading order in $(M/\Lambda)^2$ (with the exception of coefficients for terms with time-like derivatives). Next-to-leading order results from χPT determine the same coefficients at the next order in $(M/\Lambda)^2$ and the leading order of a further set of coefficients from the heavy kaon/eta theory.

We present the results for the leading order matching calculation then apply the results to make a prediction for KK scattering phase shifts in the next section. The parameters of $\mathcal{L}^{\pi K}$ are, up to corrections of order $(M/\Lambda)^2$,

$$\bar{M}_K^2 = m_s B_0, \qquad c_1^{\pi K} = 0, \qquad c_2^{\pi K} = -1/4, \qquad c_3^{\pi K} = -1/16, \tag{4.37}$$

and $d_1^{\pi K} = d_2^{\pi K} = d_4^{\pi K} = 0$. The parameters appearing in $\mathcal{L}^{\pi\eta}$ are, up to corrections of order $(M/\Lambda)^2$,

$$\bar{M}_\eta^2 = \tfrac{4}{3} m_s B_0, \qquad c_1^{\pi\eta} = c_2^{\pi\eta} = 0, \qquad c_3^{\pi\eta} = -1/16. \tag{4.38}$$

The coefficients $d_3^{\pi K}$ and $d^{\pi\eta}$ are not determined until matching at next-to-leading order in χPT. For $\mathcal{L}^{\pi KK}$ we only perform the matching for the coefficients of the four-kaon contact terms and find the leading order results are

$$a^{\pi KK} = \frac{-m_s B_0}{16F_0^2}, \qquad b^{\pi KK} = \frac{-m_s B_0}{8F_0^2}, \qquad c_3^{\pi KK} = 0, \qquad c_{10}^{\pi KK} = \frac{m_s B_0}{128F_0^2}, \tag{4.39}$$

with corrections of order $(M/\Lambda)^4$.

In effect, the heavy kaon/eta effective theory becomes a dual expansion in Q/M and $(M/\Lambda)^2$ through the matching calculation. Applying the theory to physical processes requires establishing a relative weight to the two expansion parameters. The relative weight dictates to what order of the expansion in powers of $(M/\Lambda)^2$ one must work for a consistent result to a particular order in Q/M. Equivalently we introduce a common expansion parameter ξ and assign Q/M and M/Λ each a characteristic power of ξ. The ratio of masses $m_\pi/M_K \simeq 0.3$ and the typical size of SU(3) symmetry breaking effects [20] suggest assigning the ratios of mass scales Q/M and $(M/\Lambda)^2$ equal powers of the parameter ξ, so we take $\xi \sim Q/M \sim (M/\Lambda)^2 \sim 0.3$.

4.5 Application: KK Scattering

From the effective Lagrangians $\mathcal{L}^{\pi K}$ and $\mathcal{L}^{\pi KK}$ we calculate the KK scattering amplitudes in the isospin I=0,1 channels to order Q^2. Using the results of the matching calculations of section 4.4, we make a prediction for the KK s-wave scattering phase shift in the I=1 channel. We plot a comparison of the leading order results of the heavy kaon/eta theory and SU(3) χPT and comment on the usefulness of this approach.

Initially we calculate the KK scattering amplitude as an expansion in powers of Q/M, treating all coefficients as intrinsically $\mathcal{O}(\xi^0)$. The leading-order contribution to the scattering amplitude is order Q^0 and arises from the tree diagram with an $a^{\pi KK}$-vertex. Because time derivatives on external legs contribute Q^2, the effective Lagrangian results in no contribution from tree diagrams at order Q. At order Q^2 we include tree diagrams with

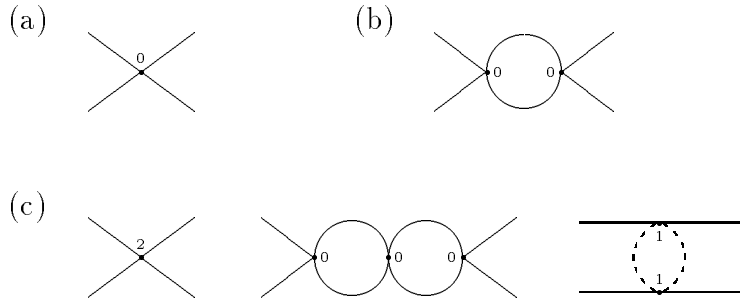


Figure 4.1: Feynman diagrams contributing to the KK scattering amplitude in the heavy kaon/eta effective theory; (a) order Q^0 tree diagram, (b) order Q kaon-bubble diagram, (c) order Q^2 tree, double-bubble, and pion exchange diagrams.

a vertex from $\mathcal{L}_1^{\pi KK}$ or $\mathcal{L}_2^{\pi KK}$ and which give one of the coefficients $b^{\pi KK}$, $c_3^{\pi KK}$, or $c_{10}^{\pi KK}$. Because the two-body sector admits nearly-infrared-divergent behavior, the appropriate power counting for loop diagrams is given in equation (2.31). The power counting shows that kaon-bubble diagrams are suppressed by a single power of Q, so to order Q^2 we take double- and single-bubble diagrams with only the $a^{\pi KK}$-vertex included. Finally, one-pion loop diagrams are suppressed by only the powers of Q generated at the vertices, which means double pion exchange is included with vertices from the order-Q effective Lagrangian $\mathcal{L}_1^{\pi K}$. The set of relevant diagrams is illustrated in Figure 4.1.

Integrations over loop energies dq_0 are performed by contour integration and the resulting momentum integrals $d\vec{q}$ are evaluated in dimensional regularization. The scattering amplitudes are calculated in the center-of-mass frame and the order-Q^2 equation of motion, $2\bar{M}_K E_k = k^2 - 16 c_3^{\pi K} \hat{m} B$, gives the kaon energy of the external states. The results for $\mathcal{A}_{NR}^{(\mathrm{I})}$ are

$$i\mathcal{A}_{NR}^{(0)}(k) = 32 i c_3^{\pi KK} \frac{k^2 \cos\theta}{\bar{M}_K^2}, \tag{4.40}$$

$$\begin{aligned} i\mathcal{A}_{NR}^{(1)}(k) &= 32 i a^{\pi KK} \left\{ 1 + i a^{\pi KK} \sqrt{\frac{2E_k}{\pi^2 \bar{M}_K}} - \frac{(a^{\pi KK})^2 2E_k}{\pi^2 \bar{M}_K} \right\} \\ &\quad + 64 i b^{\pi KK} \frac{E_k}{\bar{M}_K} + 256 i c_{10}^{\pi KK} \frac{\hat{m} B}{\bar{M}_K^2} - i \frac{m_\pi^2 \bar{M}_K^2}{16\pi^2 F^4} \ln \frac{m_\pi^2}{\mu^2}. \end{aligned} \tag{4.41}$$

From the matching calculation we determined $c_3^{\pi KK} \simeq 0$; consequently, the (I=0) p-wave scattering amplitude and phase shift vanish to the order we are working. Including the ξ-scaling of the coefficients, the leading contribution to $\mathcal{A}_{NR}^{(1)}$ comes from the order-Q^0 tree diagram and counts as order ξ. The next terms in the phenomenological expansion in ξ, of equation (4.41), are the order-Q kaon-bubble diagram and the order-Q^2 tree diagram. To work consistently at this order would require extending the matching of the coefficient in the leading-order result, $a^{\pi KK}$, by two orders in $(M/\Lambda)^2$.

For comparison, the result for the scattering amplitude in χPT is

$$i\mathcal{A}_{\chi\mathrm{PT}}^{(1)} = \frac{-1}{3F_0^2}\left[8M_K^2 + 12k^2 - 2(m_s + \hat{m})B_0\right]. \tag{4.42}$$

The leading-order results for the s-wave phase shift in both the heavy kaon/eta theory and SU(3) χPT are presented together in Figure 4.2. The s-wave KK scattering length determined from the phase shift plotted in the figure is

$$a^{(1)} = 0.45 \times 10^{-13}\ \mathrm{cm}, \tag{4.43}$$

which is consistent with a repulsive K-K interaction potential. In both cases the leading corrections are suppressed by $\sim(M/\Lambda)^2 \sim 0.3$. In light of the fact that the heavy kaon/eta theory is determined directly from the scattering amplitudes of χPT, the agreement of the two expansions to within the expected 30% corrections is not surprising.

The calculation of the s-wave phase shift in both χPT and the heavy kaon/eta theory has illustrated a fundamental point relevant to the utility of the matching calculation. The motivation for developing the heavy kaon/eta theory was to achieve a better expansion in the threshold regime by virtue of the better ratio of scales $(Q/M)^2 \sim .1$ versus $(M/\Lambda)^2 \sim .3$. In effect, the price for the improved convergence was an increase in the number of low-energy constants, i.e., coefficients in the Lagrangian, to be determined. The initial proposal was made to determine the unknowns of the low-energy theory by performing a matching calculation onto a theory with fewer unknown parameters, SU(3) χPT.

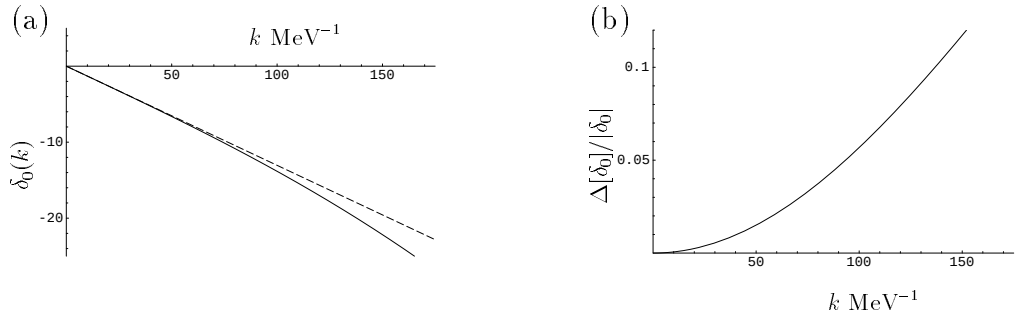


Figure 4.2: (a) Leading-order calculations of the KK s-wave (I=1) scattering phase shift $\delta_0(k)$ in degrees as a function of center-of-mass momentum k; the dashed line is heavy kaon/eta theory, the solid line is SU(3) χPT. (b) Relative error between the results $|\delta_0^{NR} - \delta_0^{\chi\mathrm{PT}}|/|\delta_0^{\chi\mathrm{PT}}|$.

However, by matching onto χPT we guarantee that the heavy kaon/eta theory is limited to converging no better than the theory to which it is matched. Indeed, we find that the matching calculation explicitly reintroduces the mass-scale ratio $(M/\Lambda)^2$ that we attempted to circumvent. The matching calculation appears to be both the savior and the Achilles' heel of the heavy kaon/eta effective theory.

For a wide variety of processes, including KK scattering, the heavy kaon/eta effective theory does not provide a computational advantage over χPT, and the additional work required to carry out the matching calculation cannot be justified. However, the outlook for the heavy kaon/eta theory or for the use of matching calculations is not entirely bleak. The approach is useful is two senses that were suggested in section 4.1. The first is that independent of the matching calculation, when data are available for a direct empirical determination of the low-energy constants, then taking advantage of the better rate of convergence in the non-relativistic theory will be possible. For a short-term perspective, the available data on πK scattering [70, 71] may be useful in this way.

The second way in which the non-relativistic effective theory may be useful is calculating quantities for which the relativistic formalism of χPT is awkward. An example where a non-relativistic treatment has proven beneficial is calculating the properties of bound-state systems [9, 32, 72, 73]. In applications of this sort, reviving the matching calculation

approach has merit because the motive for using the non-relativistic effective theory is not solely to improve the convergence of the expansion. Since bound state formation always involves summing an infinite number of Feynman diagrams, the power counting considered here cannot be directly applicable for these problems. Two cases should be considered regarding bound states in the heavy kaon/eta effective theory. First, the effective field theory cannot be applied to deeply bound systems because they lie outside the range of the momentum expansion. Second, for shallow bound states, the power counting scheme needs to reflect the fine-tuning which is implicit in the associated large scattering length. In systems with a shallow bound state, the matching calculation will not provide an estimate of coefficients of the terms responsible for binding the system, but may permit the determination of coefficients which contribute perturbatively to properties of the bound state. The interesting possibilities for strong-interaction bound states, $K\overline{K}$ and $\overline{K}N$, both suffer kaon annihilation and are not suitable candidates. Another possibility is to calculate strong-interaction perturbations to a Coulomb bound state with a kaon, for instance $\pi^- K^+$ or $K^-\Sigma^+$, or kaon electromagnetic form-factors via $e^- K^+$.

In conclusion, we have presented the foundations and laid out the principles for an effective field theory to describe the interactions of pions with non-relativistic kaons and eta mesons. The effective Lagrangians for few-body sectors were constructed explicitly for the first several orders in a chiral expansion. Much work remains to be done both in applying the theory to systems of interest and in generalizing the theory to include electromagnetic interactions, couplings to baryons, and the study of isospin violation.

Appendix A

Symbolic Expansion of $\mathcal{L}_{\chi\mathrm{PT}}$

This appendix presents *Mathematica*[1] routines [74] for the symbolic expansion of the SU(3) chiral lagrangian in terms of the embedded chiral SU(2) fields in the sincere hope that they will prove useful to others. The source code is divided into two files, `lagrangian.math` on page 72 which specifies the sequence of steps in expanding and saving the desired parts of the lagrangian and `definitions.math` on pages 73–74 which defines the functions called to perform the expansion and simplify the results. Currently, the chiral Lagrangian $\mathcal{L}_2$ in equation (2.8) is expanded out to terms including six boson fields; $\mathcal{L}_4$ in equation (2.10) is expanded to four boson fields. The source code can be applied to higher-order parts of the chiral Lagrangian or to terms involving more bosons by adding transformation rules to the `trace` routine in `definitions.math` at line 42.

The routines are of very limited use without a brief description of the symbols appearing in the input and output files. In addition to the obvious parameters `F`, `B`, and `L1`–`L8` defined as in section 2.2, the quark masses are included as

$$\texttt{mhat} \to \hat{m} = \tfrac{1}{2}(m_u + m_d), \qquad \texttt{mdiff} \to (m_u - m_d), \qquad \texttt{ms} \to m_s.$$

The correspondences for the boson fields are

$$\texttt{pion} \to \pi_a \tau_a, \qquad \texttt{kaon} \to K, \qquad \texttt{kbar} \to \overline{K}, \qquad \texttt{eta} \to \eta.$$

Each occurrence of a boson field is multiplied by a tag, `boson`, which simplifies expanding the lagrangian and separating the lagrangian into parts by the number of bosons

[1] *Mathematica* is a registered trademark of Wolfram Research, Inc.

appearing. Also, two matrices are given symbolic names in the input files; `one` refers to the 2×2 identity matrix and `tau3` refers to the Pauli matrix $\tau_3 = \sigma_z$, which appears in isospin violating terms. Finally, the following notation appears in the output:

- derivatives, `d[A,x]` $\to \partial_x A$,
  `d[pion,mu]` $\to (\partial_\mu \pi_a)\tau_a$, `d[kaon,nu]` $\to \partial_\nu K$
- isovector products, `v[A,B]` $\to \frac{1}{2}\mathrm{Tr}[AB]$,
  `v[pion,pion]` $\to \frac{1}{2}\pi_a\pi_b\mathrm{Tr}[\tau_a\tau_b] = \pi \cdot \pi$, `v[d[pion,mu],tau3]` $\to \partial_\mu \pi^0$
- isospinor contractions, `m[A,B,C]` $\to A_\alpha B_{\alpha\beta} C_\beta$,
  `m[kbar,d[kaon,mu]]` $\to \overline{K}\partial_\mu K$, `m[kbar,pion,kaon]` $\to \pi \cdot (\overline{K}\tau K)$
- commutators, `c[A,B]` $\to [A,B]$,
  `v[c[pion,d[pion,mu]],tau3]` $\to \pi_a(\partial_\mu \pi_b)(2i\epsilon_{ab3}) = 2i\,(\pi \times \partial_\mu \pi)\cdot\tau_3$,
  `m[kbar,c[d[pion,mu],d[pion,nu]],kaon]` $\to 2i\,(\partial_\mu \pi \times \partial_\nu \pi)\cdot(\overline{K}\tau K)$

The output is written to the file `lagrangian.save` and breaks the lagrangian into pieces of manageable size. The different parts of the lagrangian are named `L`XyZ, where X is the chiral order as in $\mathcal{L}_2$ versus $\mathcal{L}_4$, Z is the number of boson fields appearing in the terms, and y is a letter which indicates the number of factors of $\mathcal{M}_q$ occurring in place of derivatives. A few select lines of `lagrangian.save` are given below to illustrate.

```
L2a2 = d[eta, mu]^2/2 + m[d[kbar, mu], d[kaon, mu]] +
     v[d[pion, mu], d[pion, mu]]/2

L2b2 = eta^2*(-(B*mhat)/3 - (2*B*ms)/3) + (-(B*mhat) - B*ms)*m[kbar, kaon] -
     B*mhat*v[pion, pion]
```

The translation of the lines above is

$$\begin{aligned}
\mathcal{L}_{2a2} &= \tfrac{1}{2}\,\partial_\mu \pi \cdot \partial^\mu \pi + \partial_\mu \overline{K}\partial^\mu K + \tfrac{1}{2}\,\partial_\mu \eta \partial^\mu \eta, \\
\mathcal{L}_{2b2} &= -B_0 \hat{m}\,\pi \cdot \pi - B_0(m_s + \hat{m})\,\overline{K}K - \tfrac{1}{3}B_0(2m_s + \hat{m})\,\eta^2.
\end{aligned}$$

```
(******************** lagrangian.math ************************)
(* symbolically expands the SU(3) ChPT lagrangian in terms of the      *)
(*   embedded SU(2) fields, out to 6 bosons for L2 and 4 bosons for L4 *)
<< "definitions.math";  timetag["Session started:"];

mdiff=0;  (* drops all isospin violating terms, remove line to keep them *)
X=Xd=2*B*{{mhat*one+mdiff*tau3/2,0},{0,ms}};
P=boson*Sqrt[2]*{{pion/Sqrt[2]+eta*one/Sqrt[6],kaon},{kbar,-2*eta/Sqrt[6]}};
U=matrixseries[Exp[(I/F)*#]&,P,{boson,6}];    Um=derivative[U,mu];
Ud=matrixseries[Exp[(-I/F)*#]&,P,{boson,6}];  Udm=derivative[Ud,mu];
timetag["Defined the basics:"];

(* Calculate L2a *)
UmUdm=product[dot,Um,Udm,{boson,6}];  L2a=trace[UmUdm]*(F/2)^2;
{L2a2,L2a4,L2a6}=Map[reduce[Coefficient[L2a,boson,#]]&,{2,4,6}];
L2aN=Expand[reduce[L2a/.boson->1]-(L2a2+L2a4+L2a6)];
Save["lagrangian.save",L2a2,L2a4,L2a6,L2aN];
{Um,Udm,UmUdm}=Map[truncate[#,{boson,4}]&,{Um,Udm,UmUdm}];
Clear[L2a,L2a2,L2a4,L2a6,L2aN];  timetag["Calculated L2a:"];
(* Calculate L2b *)
XUd=dot[X,Ud];  UXd=dot[U,Xd];  L2b=trace[XUd+UXd]*(F/2)^2;
{L2b0,L2b2,L2b4,L2b6}=Map[reduce[Coefficient[L2b,boson,#]]&,{0,2,4,6}];
L2bN=Expand[reduce[L2b/.boson->1]-(L2b0+L2b2+L2b4+L2b6)];
Save["lagrangian.save",L2b0,L2b2,L2b4,L2b6,L2bN];
{XUd,UXd}=Map[truncate[#,{boson,4}]&,{XUd,UXd}];
Clear[U,Ud,L2b,L2b0,L2b2,L2b4,L2b6,L2bN];  timetag["Calculated L2b:"];
(* Calculate L4a *)   TUmUdm=trace[UmUdm];
TUmUdn=trace[product[dot,Um,(Udm/.mu->nu),{boson,4}]];
L4a=(L1*product[Times,TUmUdm,(TUmUdm/.mu->nu),{boson,4}]
    +L2*product[Times,TUmUdn,TUmUdn,{boson,4}]
    +L3*trace[product[dot,UmUdm,(UmUdm/.mu->nu),{boson,4}]] );
L4a4=reduce[Coefficient[L4a,boson,4]]; L4aN=Expand[reduce[L4a/.boson->1]-L4a4];
Save["lagrangian.save",L4a4,L4aN];
Clear[Um,Udm,TUmUdn,L4a,L4a4,L4aN];  timetag["Calculated L4a:"];
(* Calculate L4b *)   TpXU=trace[XUd+UXd];
L4b=(L4*product[Times,TUmUdm,TpXU,{boson,4}]
    +L5*trace[product[dot,UmUdm,(XUd+UXd),{boson,4}]] );
{L4b2,L4b4}=Map[reduce[Coefficient[L4b,boson,#]]&,{2,4}];
L4bN=Expand[reduce[L4b/.boson->1]-(L4b2+L4b4)];
Save["lagrangian.save",L4b2,L4b4,L4bN];
Clear[UmUdm,TUmUdm,L4b,L4b2,L4b4,L4bN];  timetag["Calculated L4b:"];
(* Calculate L4c *)   TmXU=trace[XUd-UXd];
TpXUXU=trace[product[dot,XUd,XUd,{boson,4}]+product[dot,UXd,UXd,{boson,4}] ];
L4c=(L6*product[Times,TpXU,TpXU,{boson,4}]
    +L7*product[Times,TmXU,TmXU,{boson,4}]+L8*TpXUXU );
{L4c0,L4c2,L4c4} = Map[reduce[Coefficient[L4c,boson,#]]&,{0,2,4}];
L4cN = Expand[reduce[L4c/.boson->1]-(L4c0+L4c2+L4c4)];
Save["lagrangian.save",L4c0,L4c2,L4c4,L4cN];  timetag["Calculated L4c:"];
(******************   end of file   ************************)
```

```
(************************ definitions.math ************************)
(* defines functions called by "lagrangian.math" *)

(* time-stamp printing function *)
timetag[msg_]:=(Print[];  timenew=SessionTime[];
    If[untimed, (timeorigin=timeold=timenew; untimed=False); ];
    Print[ StringTake[msg<>spaces,20],PaddedForm[(timenew-timeold),{9,3}],
        PaddedForm[(timenew-timeorigin),{9,3}],PaddedForm[TimeUsed[],{9,3}] ];
    timeold=timenew );
untimed=True;  spaces="                        ";

(* catagories of symbols which are used in simplifying rules *)
fields={pion,kaon,kbar,eta};  ident={{one,0},{0,1}};
structs={pion,kaon,kbar,one,tau3};  bras={kbar};  kets={kaon};

(* functions called directly from "lagrangian.math" *)
truncate[exp_,xpn_]:=Expand[Normal[Series[exp,{First[xpn],0,Last[xpn]}]]];
derivative[exp_,var_]:=(D[exp,var,NonConstants->fields]
    //. Dot[___,0,___]->0 //. Literal[D][f_,v_,___]->d[f,v] );
dot[x_,y_]:=(distribute[{
    {Dot[x[[1,1]],y[[1,1]]]+Dot[x[[1,2]],y[[2,1]]],
        Dot[x[[1,1]],y[[1,2]]]+x[[1,2]]*y[[2,2]]},
    {Dot[x[[2,1]],y[[1,1]]]+x[[2,2]]*y[[2,1]],
        Dot[x[[2,1]],y[[1,2]]]+x[[2,2]]*y[[2,2]]} }]);
product[mult_,x_,y_,xpn_]:=(Module[{i,j,coeff,result=0},(
    coeff=Coefficient[#1,First[xpn],#2]&;
    For[i=0,i<=Last[xpn],i++, For[j=0,j<=i,j++,
        result+=mult[coeff[x,j],coeff[y,(i-j)]]*First[xpn]^i; ];];
    Return[Expand[result]] );]);
matrixseries[func_,matr_,xpn_]:=(Module[{x,i=0,m={truncate[matr,xpn]}},(
    While[Union[Flatten[Last[m]]]!={0},
        (AppendTo[m,product[dot,Last[m],First[m],xpn]]; i++); ];
    Return[ReleaseHold[Expand[ truncate[func[x],{x,0,i}]
        +Limit[func[x],x->0]*(Hold[ident]-1) ] /. x^n_.:>Hold[m[[n]]] ]]);]);
trace[exp_]:=(Expand[ExpandAll[tr[exp[[1,1]]]+tr[exp[[2,2]]]]
    //.{tr[x_?(FreeQ[#,any[structs]]&)]->x,
        tr[x:(_Plus|_Times)]:>Map[tr,x], tr[x_^n_Integer]->tr[x]^n }
    /. tr[x_Dot]:>isoscalars[Apply[List,x]]
    //.{tr[one]->2, tr[tau3|pion|d[pion,_]]->0, tr[x_,y_]->2*v[x,y],
        tr[x_,a__,x_]->v[x,x]*tr[a], tr[a___,x_,x_,b___]->v[x,x]*tr[a,b],
        m[a__,x_,x_,b__]->v[x,x]*m[a,b] }
    /. {tr[x_,y_,z_]->v[c[x,y],z], tr[x_,y_,x_,y_,z_]->2*v[x,y]*v[c[x,y],z],
        tr[x_,y_,z_,w_]->2*v[x,y]*v[z,w]-2*v[x,z]*v[y,w]+2*v[x,w]*v[y,z],
        m[a_,x_,y_,b_]->v[x,y]*m[a,b]+m[a,c[x,y],b]/2,
        m[a_,x_,y_,z_,b_]->v[c[x,y],z]*m[a,b]/2
            +v[x,y]*m[a,z,b]-v[x,z]*m[a,y,b]+v[y,z]*m[a,x,b],
        m[a_,x_,y_,x_,y_,b_]->v[x,y]*m[a,c[x,y],b]
            +2*v[x,y]*v[x,y]*m[a,b]-v[x,x]*v[y,y]*m[a,b] }
    /. {v[tau3,tau3]->1, m[_,c[x_,x_],_]->0,
        v[c[x_,y_],z_]:>0/;((x===y)||(x===z)||(y===z)) }]);
reduce[exp_]:=(Module[{terms,forms,freeq=FreeQ[#,any[fields]]&},(
```

```
    terms=Map[commonform,Flatten[{
        Replace[ExpandAll[exp],x_Plus:>Apply[List,x]] }]];
    forms=Union[DeleteCases[terms,_?freeq,{1}] //. _?freeq*x_->x];
    Return[Collect[Apply[Plus,terms],forms]] );]);

(* functions used internally to simplify and reduce expressions *)
any[list_]:=Apply[Alternatives,list];
indexed[form_]:=(MatchQ[form,( any[structs]|d[any[structs],__]
    |Dot[_?(FreeQ[#,any[bras]]&),__]|Dot[__,_?(FreeQ[#,any[kets]]&)] )]);
distribute[exp_]:=(ExpandAll[exp] //. Dot[___,0,___]->0
    //.{Dot[w___,x_Plus,y___]:>Distribute[Dot[w,x,y],Plus],
        Dot[w___,a_*(x_?indexed),y___]->a*Dot[w,x,y],
        Dot[w___,one,y___]->Dot[w,y] });
isoscalars[list_]:=(Module[{olist,break=Map[First,Position[list,any[bras]]]},(
    If[break=={},Return[Apply[tr,list]];];
    olist=RotateLeft[list,First[break]-1];
    break=Append[break-First[break]+1,Length[olist]+1];
    Return[Product[ Apply[m,Take[olist,{break[[i]],break[[i+1]]-1}]],
        {i,1,Length[break]-1} ]]);]);
commonform[exp_]:=(Module[{vars,swaps,array},(
    array={1,exp} //. List[a_,b_?(FreeQ[#,any[fields]]&)*x_]->{a*b,x}
    //.{{a__,x_*y_,z___}->{a,x,y,z}, {a__,x_^n_Integer,y___}->{a,x^(n-1),x,y}};
    array=Apply[Union[Flatten[Outer[Times,##]]]&,Map[Union, Map[List,array]
    /. {{m[a_,c[x_,y_],b_]}->{m[a,c[x,y],b],-m[a,c[y,x],b]},
        {v[c[x_,y_],z_]}->{v[c[x,y],z],-v[c[y,x],z],
            v[c[y,z],x],-v[c[z,y],x],v[c[z,x],y],-v[c[x,z],y]},
        {v[x_,y_]}->{v[x,y],v[y,x]} }
    /. {m[a_,c[tau3,x_],b_]->-m[a,c[x,tau3],b], v[tau3,x_]->v[x,tau3],
        (v[c[tau3,x_],y_]|v[c[y_,tau3],x_])->v[c[x,y],tau3] }]];
    vars=Union[Cases[exp,(d[_,x_]->x),Infinity]];
    swaps=Map[Thread[Rule[vars,#]]&,Permutations[vars],{1}];
    Return[First[Union[Flatten[Map[(#/.swaps)&,array]]]]]; ]);
(************************    end of file    ************************)
```

Appendix B

Recursion Relations for $\phi \to \mathcal{F}[\tilde{\phi}]$

We start with the free-field Lagrangian for a (complex) heavy scalar field,

$$\mathcal{L}_\phi = \phi^\dagger \left[iv\cdot\partial + \frac{\nabla_\perp^2 + (iv\cdot\partial)^2}{2m} \right] \phi$$

where v^μ is the scalar-field velocity and $\partial_\mu \partial^\mu = (v\cdot\partial)^2 - \nabla_\perp^2$. We seek a redefinition of the scalar field $\phi = \mathcal{F}[\tilde{\phi}]$ which eliminates secondary time-like derivatives of the heavy scalar in favor of higher powers of the space-like derivative $\nabla_\perp^2$,

$$\mathcal{L}_{\tilde{\phi}} = \tilde{\phi}^\dagger \left[iv\cdot\partial + m \sum_{j=1}^{\infty} g_j \left(\frac{\nabla_\perp^2}{m^2} \right)^j \right] \tilde{\phi}.$$

The general field redefinition we consider here is

$$\mathcal{F}[\tilde{\phi}] = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} f_k^j \left(\frac{iv\cdot\partial}{m} \right)^j \left(\frac{\nabla_\perp^2}{m^2} \right)^k \tilde{\phi},$$

and constraints on the form of $\mathcal{L}_{\tilde{\phi}}$ give recursion relations for the coefficients f_k^j. Using integration by parts, we find the recursion relations are

$$\begin{aligned} (f_0^0)^2 = 1, \qquad f_0^1 = -\tfrac{1}{4} f_0^0, \qquad f_1^0 = -\tfrac{1}{2} f_0^1, \\ -f_0^0 f_0^{j+1} \;=\; \tfrac{1}{4} f_0^0 f_0^j + \tfrac{1}{4} \sum_{p=0}^{j-1} \left(f_0^p f_0^{j-p} + 2 f_0^{p+1} f_0^{j-p} \right), \end{aligned}$$

$$
\begin{aligned}
-f_0^0 f_{k+1}^j &= -\tfrac{1}{2} f_0^j f_{k+1}^0 + \tfrac{1}{4} \sum_{p=0}^{j-1} \sum_{q=0}^{k} \left(f_q^p f_{k-q+1}^{j-p-1} + 2 f_q^{p+1} f_{k-q+1}^{j-p-1} + f_q^{p+1} f_{k-q}^{j-p} \right) \\
&\quad + \tfrac{1}{4} \sum_{p=0}^{j-1} \left(f_{k+1}^p f_0^{j-p-1} + 2 f_{k+1}^p f_0^{j-p} \right) + \tfrac{1}{2} \sum_{q=0}^{k} \left(f_q^0 f_{k-q}^{j+1} + f_{q+1}^0 f_{k-q}^j \right), \\
-f_0^0 f_{k+1}^0 &= \tfrac{1}{4} (f_0^0 f_k^1 + f_0^1 f_k^0) + \tfrac{1}{4} \sum_{q=0}^{k-1} \left(f_q^0 f_{k-q}^1 + f_q^1 f_{k-q}^0 + 2 f_{q+1}^0 f_{k-q}^0 \right), \\
g_{k+1} &= \tfrac{1}{2} \sum_{q=0}^{k} f_q^0 f_{k-q}^0,
\end{aligned}
$$

and determine the coefficients in the sequence $f_0^0, f_0^1, f_1^0, f_0^2, f_1^1, f_2^0, f_0^3, \ldots$

Solving for $\mathcal{F}[\tilde{\phi}]$ is simplified by recognizing that the coefficients g_j must reproduce the kinetic energy and well-known relativistic corrections; $\mathcal{L}_{\tilde{\phi}} = \tilde{\phi}^\dagger (iv\cdot\partial - \hat{K}) \tilde{\phi}$ where

$$
\begin{aligned}
\hat{K} &= \sqrt{m^2 - \nabla_\perp^2} - m \\
&= -\frac{\nabla_\perp^2}{2m} - \frac{\nabla_\perp^4}{8m^3} - \frac{\nabla_\perp^6}{16m^5} - \cdots
\end{aligned}
$$

We postulate that the field redefinition is a function of one of the combinations $(iv\cdot\partial \pm \nabla_\perp^2)$ or $(iv\cdot\partial \pm \hat{K})$ only, then verify that only the form $(iv\cdot\partial + \hat{K})$ satisfies the recursion relations. We construct the solution through trial and error, guided by comparing expansions of the trial function with the recursion relations above, and find

$$
\mathcal{F}[\tilde{\phi}] = \pm \left[1 + \frac{iv\cdot\partial + \hat{K}}{2m} \right]^{-\frac{1}{2}} \tilde{\phi}.
$$

Inspired by hindsight we present a formal, and much shorter, derivation of this result in Chapter 4.

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