Chapter 2

A Fast, High-Order Method in Two Dimensions: Theoretical Analysis

In this chapter, we present a theoretical analysis of the efficient, high-order method introduced in [13]. The high-order accuracy of this method derives from the concepts introduced in Section 1.5. However, this theoretical analysis is not simply an academic exercise, but rather puts to rest the considerable controversy generated by the claim of high-order accuracy. Additionally, as we will describe, this analysis reveals interesting and unexpected bounds on the convergence rates.

2.1 Approximate Integral Equation

As described in the introduction, the field scattered by an bounded inhomogeneity is given by the solution of the Lippmann-Schwinger integral equation (1.6). Although several existence and uniqueness results are known for this problem, the existing results do not address the full generality of the problem that we consider. Specifically, we assume that, given $m \in L^\infty$, (1.6) admits a unique solution $u \in L^\infty$. Certainly, this is the case for $m \in C^1$ as proven in [17, §8.1, 8.3]. Furthermore, it is well known that (1.6) admits a unique solution for constant inhomogeneities by consideration of the appropriate surface integral equation [16, pp. 100–101]; [39].

To obtain the numerical method, we construct an approximate integral equation. As mentioned in the introduction, we approximate the Green's function by a truncated Fourier series. We now prove that this approximate integral equation also admits a unique solution.

The related fact that the inverse operator for the approximate problem is uniformly bounded provides an essential part of our proof of high-order convergence.

After changing to polar coordinates, $x = ae^{i\phi}$ and $y = re^{i\theta}$, we obtain the Fourier representation of the Green's function by means of the addition theorem for the Hankel function [17, p. 67]

$$H_0^1(\kappa|ae^{i\phi} - re^{i\theta}|) = \sum_{\ell=-\infty}^{\infty} \mathcal{J}_\ell(a,r)e^{i\ell(\phi-\theta)},$$

where

$$\mathcal{J}_\ell(a,r) = H_\ell^1(\kappa \max(a,r))J_\ell(\kappa \min(a,r)). \tag{2.1}$$

This allows us to expand the integral operator, Ku, (see (1.6)) in a Fourier series

$$(Ku)(a,\phi) = -\frac{i\kappa^2}{4}\int H_0^1(\kappa|x-y|)m(y)u(y)dy = \sum_{\ell=-\infty}^{\infty}(K_\ell u)(a)e^{i\ell\phi},$$

where

$$(K_\ell u)(a) = -i\frac{\kappa^2}{4}\int_{R_0}^{R_1}\mathcal{J}_\ell(a,r)\left[\int_0^{2\pi} m(r,\theta)u(r,\theta)e^{-i\ell\theta}d\theta\right]rdr. \tag{2.2}$$

Here $R_0 \leq a \leq R_1$ defines an annular region that contains the support of m.

To obtain the approximate integral equation, we truncate this Fourier series. Furthermore, we also replace the incident field by its truncated Fourier series. Although not necessary for our results, this simplifies the presentation somewhat. We thereby obtain

$$v(a,\phi) = u^{i,M}(a,\phi) + (K^M v)(a,\phi), \tag{2.3}$$

where

$$u^{i,M}(a,\phi) = \sum_{\ell=-M}^{M} u_\ell^i(a)e^{i\ell\phi} \tag{2.4}$$

$$(K^M v)(a,\phi) = \sum_{\ell=-M}^{M}(K_\ell v)(a)e^{i\ell\phi}. \tag{2.5}$$

(Note: we will use superscript M throughout this text to denote the truncated Fourier series of order M of a given function, e.g., (2.4) and (2.5).)

Decomposing (2.3) into Fourier modes, we observe that

$$v_\ell(a) = \begin{cases} u_\ell^i(a) + (K_\ell v)(a), & \text{for } |\ell| \le M \\ 0, & \text{for } |\ell| > M. \end{cases}$$

Hence,

$$v(a,\phi) = v^M(a,\phi)$$

and solving (2.3) is equivalent to solving the following system of one-dimensional integral equations

$$v_\ell(a) - (K_\ell v^M)(a) = u_\ell^i(a),\ \ell = -M, \ldots, M. \tag{2.6}$$

To prove existence and uniqueness for this approximate integral equation, we make use of the following technical lemma.

Lemma 2.1. *There exists a constant $C > 0$ depending only on R_0, R_1 and κ such that*

$$\left\| \int_{R_0}^{R_1} |\mathcal{J}_\ell(a,r)|\, r dr \right\|_\infty \le \frac{C}{\max\{1, \ell^2\}},$$

where $\mathcal{J}_\ell(a,r)$ is defined in (2.1).

This lemma is proved in Appendix A.1 and allows us to prove the following lemma.

Lemma 2.2. *For any $m \in L^\infty$,*

$$\left\| K - K^M \right\|_\infty \to 0, \textit{ as } M \to \infty,$$

where the operator norm is the one induced by the L^∞-norm.

Proof. Let $u \in L^\infty$. Then,

$$\int_0^{2\pi} \left| m(r,\theta) u(r,\theta) e^{-i\ell\theta} \right| d\theta \le 2\pi \|m\|_\infty \|u\|_\infty.$$

Hence, for $M \geq 0$,

$$\begin{aligned} \left\|(K - K^M)u\right\|_\infty &\leq \frac{\pi\kappa^2}{2}\|m\|_\infty\|u\|_\infty \sum_{|\ell|>M} \left\|\int_{R_0}^{R_1} |\mathcal{J}_\ell(a,r)| r dr\right\|_\infty \\ &= \mathcal{O}\left(\sum_{|\ell|>M} \frac{1}{\ell^2}\right)\|u\|_\infty \\ &= \mathcal{O}\left(M^{-1}\right)\|u\|_\infty. \end{aligned}$$

Therefore,

$$\|K - K^M\|_\infty = \mathcal{O}\left(M^{-1}\right) \to 0,$$

as $M \to \infty$. □

This leads to the desired uniqueness and existence result.

Theorem 2.3. *Given $m \in L^\infty$, for M sufficiently large (2.3) admits a unique solution $v \in L^\infty$ for incident field u^i satisfying (1.1). Furthermore, for sufficiently large M, the operators $(I - K^M)^{-1}$ exist and are uniformly bounded.*

Proof. Since we have assumed that (1.6) admits a unique solution, $I - K$ has a bounded inverse. Since L^∞ is a Banach space, Lemma 2.2 and [38, Theorem 10.1, p. 142] imply that for all sufficiently large M the inverse operators $(I - K^M)^{-1}$ exist and are uniformly bounded. Therefore, (2.3) admits a unique solution $v \in L^\infty$, for M sufficiently large. □

2.2 Error Bounds

In summary, the approximate integral equation (2.3) is obtained by truncating the Fourier series of both the incident field u^i and the integral operator K at each radius to obtain an approximate solution v to (1.6), which itself is a truncated Fourier series. Roughly speaking, high-order accuracy is obtained because the integral operator Ku and the incident field u^i are smooth and periodic functions as a function of the angular variable. Therefore, their truncated Fourier series exhibit high-order convergence. In this section we derive bounds on the error in this approximation. Of course the full numerical implementation of the method introduces additional errors (e.g., quadrature errors), but here we consider only the accuracy of v, the solution of the approximate integral equation (2.3). All other errors are considered in our discussion of the numerical implementation of this method in Chapter 3.

2.2.1 Error in Approximated Fourier Modes

The error in the solution of the approximate integral equation (2.3) is given by

$$|u(x) - v^M(x)| \leq |u^M(x) - v^M(x)| + |u^T(x)|,$$

where u^T is the "tail" of the Fourier series of u,

$$u^T(a, \phi) = u(a, \phi) - u^M(a, \phi) = \sum_{|\ell|>M} u_\ell(a) e^{i\ell\phi}.$$

In this section, we derive a bound on the first of these two terms. Note that

$$\begin{aligned} u^M &= u^{i,M} + K^M u, \\ v^M &= u^{i,M} + K^M v^M. \end{aligned}$$

Hence, taking the difference of these equations we obtain

$$\begin{aligned} u^M - v^M &= K^M(u - v^M) \\ &= K^M(u^M - v^M) + K^M u^T. \end{aligned}$$

Theorem 2.3 ensures that $I - K^M$ is invertible for M sufficiently large and that the operators $(I - K^M)^{-1}$ are uniformly bounded by some $B > 0$, i.e., $\left\|(I - K^M)^{-1}\right\|_\infty \leq B$ for all sufficiently large M. Then

$$\varepsilon_M = \left\|u^M - v^M\right\|_\infty \leq B \left\|K^M u^T\right\|_\infty.$$

Hence, we seek a bound on $K^M u^T$.

We first note that

$$\left\|K^M u^T\right\|_\infty \leq \sum_{\ell=-M}^{M} \left\|K_\ell u^T\right\|_\infty.$$

Furthermore, note that

$$\int_0^{2\pi} m(r, \theta) u^T(r, \theta) e^{-i\ell\theta} d\theta = \sum_{|j|>M} m_{\ell-j}(r) u_j(r).$$

Then we have

$$\left\|K_\ell u^T\right\|_\infty \leq \frac{C}{\max\{1,\ell^2\}} \sum_{|j|>M} \|m_{\ell-j}\|_\infty \|u_j\|_\infty. \tag{2.7}$$

To bound this expression we use bounds on the Fourier coefficients of m and u. Let the space $C^{k,\alpha}_{pw}$ denote the space of functions whose k^{th} derivative is piecewise Hölder continuous. (Note: throughout this text we assume that $0 < \alpha < 1$.) We say that a function f is piecewise continuous on Ω if and only if there is a finite number of open, disjoint subsets of Ω, denoted by $\Omega_1, \ldots, \Omega_n$, such that $\bigcup_{i=1}^n \overline{\Omega_i} = \overline{\Omega}$ where $f|_{\Omega_i}$ can be extended as a continuous function to $\overline{\Omega_i}$ for each $i = 1, \ldots, n$. Given these definitions, the following lemma follows by repeated integration by parts.

Lemma 2.4. *If $g \in C^k([0,2\pi] \cap C^{k+2}_{pw}([0,2\pi])$, then the Fourier coefficients of g,*

$$\begin{aligned} |c_\ell| &= \left|\frac{1}{2\pi}\int_0^{2\pi} g(\theta)e^{-i\ell\theta}d\theta\right| \\ &\leq \frac{C}{\max\{1,|\ell|^{k+2}\}}, \end{aligned}$$

for some constant C.

If $g \in L^\infty([0,2\pi]) \cap C^1_{pw}([0,2\pi])$, then

$$|c_\ell| \leq \frac{C}{\max\{1,|\ell|\}}.$$

To bound the discrete convolution in (2.7) we need a relationship between the regularity of m and the regularity of u. Variations on results in [8, p. 223], [24, pp. 97, 102] and [28, pp. 53, 56] give us the following.

Theorem 2.5. *Let D be a bounded, open set. If $m \in L^\infty(D)$, then $u \in C^1(D)$. Furthermore, if $m \in C^{k,\alpha}(D)$, $k \geq 0$, then $u \in C^{k+2,\alpha}(D)$.*

We emphasize that D is an arbitrary bounded, open set. Hence, the theorem relates the *local* regularity of u to the *local* regularity of m.

Using these results, we obtain the necessary bounds on (2.7) by means of the following technical lemma, whose proof is contained in Appendix A.2.

Lemma 2.6. *Define the region $A = \{(a, \phi) : R_0 \le a \le R_1\}$. If $m \in C^{k,\alpha}(A) \cap C^{k+2,\alpha}_{pw}(A)$, $k \ge 0$, then there exists a constant $C > 0$ independent of ℓ and M such that for $\ell = 0, \dots, M$*

$$\left\|K_\ell u^T\right\|_\infty \le \frac{1}{M^{k+4}} \frac{4C}{\max\{1, \ell^2\}} \frac{1}{(M+1-\ell)^{k+1}},$$

Similarly, if $m \in L^\infty(A) \cap C^{1,\alpha}_{pw}(A)$, then

$$\left\|K_\ell u^T\right\|_\infty \le \frac{2C}{\max\{1, \ell^2\}} \left\{ \frac{1}{3M^3} + \frac{1}{M^2} \frac{1}{(M+1-\ell)} \right\}.$$

This leads us to a bound on the approximated Fourier modes.

Theorem 2.7. *If $m \in L^\infty(A) \cap C^{1,\alpha}_{pw}(A)$, then*

$$\varepsilon_M = \left\|u^M - v^M\right\| \le B \left\|K^M u^T\right\| = \mathcal{O}\left(\frac{1}{M^3}\right).$$

If $m \in C^{0,\alpha}(A) \cap C^{2,\alpha}_{pw}(A)$, then

$$\varepsilon_M = \mathcal{O}\left(\frac{1}{M^5}\right).$$

If $m \in C^{k,\alpha}(A) \cap C^{k+2,\alpha}_{pw}(A)$ for $k \ge 1$, then

$$\varepsilon_M = \mathcal{O}\left(\frac{1}{M^{k+6}}\right).$$

Proof. We use Lemma 2.6 to bound $\left\|K^M u^T\right\| \le \sum_{\ell=-M}^{M} \left\|K_\ell u^T\right\|_\infty$. For $\ell = 0$ and $m \in C^{k,\alpha}(A) \cap C^{k+2,\alpha}_{pw}(A)$, we obtain

$$\left\|K_0 u^T\right\|_\infty = \mathcal{O}\left(\frac{1}{M^{2k+5}}\right).$$

For $m \in L^\infty(A) \cap C^{1,\alpha}_{pw}(A)$, in turn, we have

$$\left\|K_0 u^T\right\|_\infty = \mathcal{O}\left(\frac{1}{M^3}\right).$$

For the remaining part of the sum, it is sufficient to bound sums of the following form

$$\sum_{\ell=1}^{M} \frac{1}{\ell^2} \frac{1}{(M+1-\ell)^p},$$

for $p = 1, 2, \ldots$. First consider $p \geq 2$.

$$\begin{aligned}
\sum_{\ell=1}^{M} \frac{1}{\ell^2} \frac{1}{(M+1-\ell)^p} &\leq \sum_{\ell=1}^{M} \frac{1}{\ell^2} \frac{1}{(M+1-\ell)^2} \\
&\leq 2 \sum_{\ell=1}^{\lceil \frac{M}{2} \rceil} \frac{1}{\ell^2} \frac{1}{(M+1-\ell)^2} \\
&= \mathcal{O}\left(\frac{1}{M^2}\right).
\end{aligned}$$

For $p = 1$, we write the summed quantity as partial fractions.

$$\begin{aligned}
\sum_{\ell=1}^{M} \frac{1}{\ell^2} \frac{1}{(M+1-\ell)} &= \frac{1}{(M+1)^2} \sum_{\ell=1}^{M} \frac{1}{\ell} + \frac{1}{M+1} \sum_{\ell=1}^{M} \frac{1}{\ell^2} + \frac{1}{(M+1)^2} \sum_{\ell=1}^{M} \frac{1}{(M+1-\ell)} \\
&= \mathcal{O}\left(\frac{1}{M}\right).
\end{aligned}$$

Combining these results, we arrive at the claims of the theorem. □

Remark 2.8. Of course, there are many other conditions on m that could be proposed for which the corresponding convergence rates could be determined. For instance, one might remove the requirement of Hölder continuity. In every case, the convergence rates are directly determined by the rate of decay of the Fourier coefficients of m and u. We do not attempt to provide a comprehensive listing of all possible regularity conditions and their corresponding convergence rates.

Remark 2.9. We have taken great care in the proof to obtain tight bounds; the resulting convergence rates depend on k in a particularly interesting way. Proceeding with less care, one might have expected a simpler dependence on k as follows.

$$\begin{aligned}
\varepsilon_M &\leq B \left\| K^M \right\|_\infty \left\| u^T \right\|_\infty \\
&= \mathcal{O}\left(\sum_{\ell > M} \frac{1}{\ell^{k+4}} \right) = \mathcal{O}\left(\frac{1}{M^{k+3}}\right).
\end{aligned}$$

This bound predicts second-order convergence for $m \in L^\infty(A) \cap C^{1,\alpha}_{pw}(A)$, third-order convergence for $m \in C^{0,\alpha}(A) \cap C^{2,\alpha}_{pw}(A)$, and fourth-order convergence for $m \in C^{1,\alpha}(A) \cap C^{3,\alpha}_{pw}(A)$. However, as we have proven, these simple bounds are not tight. In fact, the method exhibits *third-order* convergence for $m \in L^\infty(A) \cap C^{1,\alpha}_{pw}(A)$, *fifth-order* convergence for

$m \in C^{0,\alpha}(A) \cap C^{2,\alpha}_{pw}(A)$ and *seventh-order* convergence for $m \in C^{1,\alpha}(A) \cap C^{3,\alpha}_{pw}(A)$. This rather interesting and unexpected convergence behavior can be observed in the far field convergence rates of the computational examples in Section 5.1.1.

2.2.2 Total Error in the Interior and Exterior Fields

To this point, we have only computed the convergence rate in the approximated modes, i.e., the first M modes. Given these convergence rates, we can now easily estimate the total error. We make a distinction here between two types of error: the *interior field error* (the error on the domain of integration $A = \{(a, \phi) : R_0 \le a \le R_1\}$) and the *exterior field error* (the error outside of A). The interior field error is simply the difference between the true solution $u(x)$ and the solution $v^M(x)$ of (2.3) on A. Therefore, for $x \in A$

$$\begin{aligned} \left|u(x) - v^M(x)\right| &\le \left|u^M(x) - v^M(x)\right| + \left|u^T(x)\right| \\ &\le \varepsilon_M + \tau_M(|x|), \end{aligned}$$

where $\tau_M(|x|)$ is a bound on $\left|u^T(x)\right|$.

The important observation here is that τ_M depends on the decay rate of the Fourier modes of u at the radius $|x|$. Hence, even if m is discontinuous on A, if m is smooth as a function of angle on the circle with radius $|x|$ centered at the origin, then the Fourier modes of u at radius $|x|$ decay very rapidly. This result implies that, at a given point x, the method converges at a rate that depends on a combination of the regularity of m at the radius $|x|$ and the regularity of m in all of A. We state this result more precisely in the following corollary to Theorem 2.7.

Corollary 2.10 (Interior field error). *Let $x \in A$. Let S denote the circle with radius $|x|$ centered at the origin. Let $N(S)$ be a neighborhood of S in $\mathbb{R}^2$. If $m \in C^{p,\alpha}(N(S)) \cap C^{p+2,\alpha}_{pw}(N(S))$, then the interior field error is given by*

$$\left|u(x) - v^M(x)\right| \le \varepsilon_M + \tau_M(|x|),$$

where bounds on ε_M are given in Theorem 2.7 and

$$\left|u^T(x)\right| \le \tau_M(|x|) = \mathcal{O}\left(\frac{1}{M^{p+3}}\right).$$

This result holds with $p = -1$ for $m \in L^\infty(N(S)) \cap C^{1,\alpha}_{pw}(N(S))$.

Proof. We need only prove the bound on $u^T(x)$. By Theorem 2.5, $u \in C^{p+2,\alpha}(N(S)) \cap C^{p+4,\alpha}(N(S))$. Hence, by Lemma 2.4,

$$\begin{aligned} \left|u^T(x)\right| &\leq \tau_M(|x|) = \sum_{\ell > M} \frac{C}{\ell^{p+4}} \\ &= \mathcal{O}\left(\frac{1}{M^{p+3}}\right). \end{aligned}$$

The proof for $m \in L^\infty(N(S)) \cap C^{1,\alpha}_{pw}(N(S))$ is similar. □

Remark 2.11. Although the approximate Fourier modes v^M converge rapidly to u^M, the decay of u^T dominate the maximum interior field error. For example, in the case of $m \in L^\infty(A) \cap C^{1,\alpha}_{pw}(A)$, third-order convergence of v^M to u^M will be dominated by second-order decay of u^T in the maximum error. At the same time, if the interior field error is evaluated at a radius for which m is smooth, we will observe the more rapid convergence rate predicted by Theorem 2.7.

Before we can discuss convergence rates in the exterior field, we must describe how we extend our approximate solution v^M, which we have computed only on the interior of A, to the exterior field. Since the integration in (1.6) is performed only over the support of m, one can easily see that given the exact solution u on the boundary of A, the solution in the rest of $\mathbb{R}^2$ can be computed simply by an appropriate scaling of the Fourier modes of u. More precisely, define $A = \{(r,\theta) : R_0 \leq r \leq R_1\}$. Then,

$$u^s_\ell(r) = \begin{cases} \frac{J_\ell(\kappa r)}{J_\ell(\kappa R_0)} u^s_\ell(R_0), & \text{if } 0 \leq r < R_0 \\ \\ \frac{H^1_\ell(\kappa r)}{H^1_\ell(\kappa R_1)} u^s_\ell(R_1), & \text{if } r > R_1. \end{cases} \tag{2.8}$$

(Note that this result can also be obtained directly from the differential equation by means of separation of variables.) Similarly, to extend our approximate solution v^M to the exterior of A, we simply scale its Fourier modes in the same way.

Corollary 2.12 (Exterior field error). *Given $x \notin A$, extend the approximate solution*

v^M to the exterior of A as prescribed above. Then, the exterior field error is given by

$$\left|u(x) - v^M(x)\right| = \mathcal{O}\left(\varepsilon_M\right),$$

where $\varepsilon_M = \left\|u^M - v^M\right\|_\infty$ has bounds given by Theorem 2.7.

Proof. Denote the scaling factors for the given radius $r = |x|$ by $\beta_\ell(r)$. Assume that $r > R_1$. The proof for $0 \le r < R_0$ is similar. We have

$$\begin{aligned} \left|u(x) - v^M(x)\right| &\le \sum_{\ell=-M}^{M} |\beta_\ell(r)| \left|u_\ell^{s,M}(r) - v_\ell^{s,M}(r)\right| + \left|u^T(x)\right| \\ &\le \varepsilon_M \sum_{\ell=-M}^{M} |\beta_\ell(r)| + \left|u^T(x)\right|. \end{aligned}$$

As before, let S denote the circle of radius r about the origin. Since $r = |x| > R_1$, there exists a neighborhood $N(S)$ of S such that $m|_{N(S)} = 0$. Therefore, $u \in C^\infty(N(S))$ and $\left|u^T(x)\right| \le \frac{C}{M^p}$ for *any* integer $p > 0$. This implies that $\left|u^T(x)\right|$ is always dominated by ε_M.

To complete the proof, we use the asymptotic expressions for J_ℓ and Y_ℓ as found in [2, p. 365], i.e., for fixed z and as $\ell \to \infty$ through positive real values,

$$\begin{aligned} J_\ell(z) &\sim \frac{1}{\sqrt{2\pi\ell}} \left(\frac{ez}{2\ell}\right)^{\ell} \\ Y_\ell(z) &\sim -\sqrt{\frac{2}{\pi\ell}} \left(\frac{ez}{2\ell}\right)^{-\ell}. \end{aligned}$$

Therefore,

$$\begin{aligned} |\beta_\ell(r)|^2 &= \left|\frac{Y_\ell(\kappa r)}{Y_\ell(\kappa R_1)}\right|^2 \frac{1 + \left|\frac{J_\ell(\kappa r)}{Y_\ell(\kappa r)}\right|^2}{1 + \left|\frac{J_\ell(\kappa R_1)}{Y_\ell(\kappa R_1)}\right|^2} \\ &\sim \left(\frac{R_1}{r}\right)^{2\ell}. \end{aligned}$$

This implies that $|\beta_\ell(r)|$ is summable. Hence, $\sum_{\ell=-M}^{M} |\beta_\ell(r)|$ is bounded for all integers $M > 0$. We conclude that

$$\left|u(x) - v^M(x)\right| = \mathcal{O}\left(\varepsilon_M\right).$$

□

Remark 2.13. Note that $u \in C^\infty$ on the exterior of A and u may be much less regular on the interior of A (in general, $u \in C^1$ for a discontinuous scatterer). Hence, the decay of u^T on the exterior of A is superalgebraic, whereas u^T may decay as slowly as $\mathcal{O}(M^{-2})$ on the interior of A. This fact is responsible for the interesting result that the method converges more rapidly on the exterior of A than on the interior (where u^T may dominate ε_M).

This is particularly relevant in the evaluation of radar cross sections, an important measure in many applications. The evaluation of radar cross sections requires the computation of the *far field*. Although Corollary 2.12 does not directly address the error in the far field, we obtain an approximate far field by a scaling of the Fourier modes of v^M just as in the computation of the exterior field. As in [10, p. 6], we define the far field, u_∞, by the asymptotic representation of the scattered field as $r \to \infty$, i.e.,

$$u^s(r,\phi) = e^{i\left(\kappa r - \frac{\pi}{4}\right)}\sqrt{\frac{2}{\pi\kappa r}}\left[u_\infty(\phi) + \mathcal{O}(r^{-1})\right].$$

From (2.8) and the asymptotic expression for $H^1_\ell(z)$ for fixed ℓ as $z \to \infty$ [2, p. 364], we obtain the Fourier modes of u_∞ by a simple scaling of the Fourier modes of u^s.

$$(u_\infty)_\ell = \frac{u^s_\ell(R_1)}{i^\ell H^1_\ell(\kappa R_1)}.$$

If we define the approximate far field v_∞ by scaling the Fourier modes of $v^{s,M}$ in the same way, we can prove that

$$\|u_\infty - v_\infty\| = \mathcal{O}\left(\varepsilon_M\right).$$

The proof of this fact is nearly identical to the proof of Corollary 2.12.

The predicted convergence rates in both the interior field and the far field are verified through several computational examples in Section 5.1.1.

2.3 Computation of the Angular Integral

Thus we have shown that the method achieves high-order convergence even in the case of discontinuous scatterers. However, to this point, we have not discussed any methods for computing the required angular and radial integrals for each mode of the solution (2.6). Since this chapter primarily addresses the theoretical aspects of the method, we leave a discussion of a particular efficient, high-order radial integrator to the next chapter. On the other hand, the required Fourier coefficients of $m(r,\theta)v^M(r,\theta)$ can be computed efficiently and *exactly* (except for round-off error). Furthermore, the approach taken in computing the angular integrals was the primary source of controversy surrounding the method. Therefore, we briefly discuss the angular integration here before moving on to a discussion of the numerical implementation in the next chapter.

The required angular integrals are given by

$$I_\ell(r) = \int_0^{2\pi} m(r,\theta)v^M(r,\theta)e^{-i\ell\theta}d\theta, \tag{2.9}$$

where v^M solves the approximate integral equation (2.3). We can express this integral in terms of the Fourier coefficients of m and v, i.e.,

$$\begin{aligned} I_\ell(r) &= \int_0^{2\pi} \left(\sum_{j=-\infty}^{\infty} m_j(r)e^{ij\theta}\right)\left(\sum_{k=-M}^{M} v_k(r)e^{ik\theta}\right)e^{-i\ell\theta} \\ &= 2\pi \sum_{k=-M}^{M} m_{\ell-k}(r)v_k(r), \end{aligned} \tag{2.10}$$

where $\ell = -M, \ldots, M$.

Hence, we obtain a finite discrete convolution of Fourier coefficients of m and v at each radius. Since $|\ell| \le M$ and $|k| \le M$, we have $|\ell - k| \le 2M$. Thus, given the Fourier coefficients $m_\ell(r)$ for $|\ell| \le 2M$, we can compute the required angular integrals *exactly*. Furthermore, as is well known, such discrete convolutions may be evaluated (with no approximation) with the help of FFTs [45, pp. 531–537] yielding a complexity of $\mathcal{O}(M \log M)$ at each radial point. (As we will demonstrate in our numerical examples, Chapter 5, these Fourier coefficients can be computed quite easily for a wide range of scatterers.)

This method of computing the angular integrals has an interesting implication. Since the computation involves only modes m_ℓ, $|\ell| \le 2M$, replacing m with m^{2M} in the integral equation yields no additional error, i.e.,

$$I_\ell(r) = \int_0^{2\pi} m^{2M}(r,\theta)v^M(r,\theta)e^{-i\ell\theta}d\theta \tag{2.11}$$

Hence, in a sense, the truncation of the Fourier series of the integral operator *implies* an associated truncation of the Fourier series of the scatterer—as a result of the bandlimited nature of the solution v^M. Thus, surprisingly, the low-order approximation of a discontinuous scatterer at each radius by its truncated Fourier series yields *no more error* than our original, high-order truncation of the Fourier series of K. This illustrates the interesting *cancellation of errors* that underlies the power of this approach.

Note that this discrete convolution method of computing $I_\ell(r)$ for $\ell = -M, \ldots, M$ is equivalent to trapezoidal rule integration of (2.11) with a sufficiently large number of integration points N_θ. More precisely, it is not difficult to see that the trapezoidal rule with N_θ points integrates Fourier modes $e^{ik\theta}$ for $|k| < N_\theta$ exactly. Since the largest mode in the integrand of (2.11) is $2M + M + M = 4M$, if we choose $N_\theta > 4M$, the trapezoidal rule computes (2.11) exactly (assuming exact arithmetic) and the use of FFTs yields a complexity of $\mathcal{O}(M \log M)$. Algorithmically, this is entirely equivalent to computing the discrete convolution (2.10) using FFTs.