

ON THE BRST INVARIANCE  
OF STRING THEORY

Thesis by  
Soo-Jong Rey

In Partial Fulfillment of the Requirements  
for the Degree of  
Doctor of Philosophy

California Institute of Technology  
Pasadena, California.

1988  
(Submitted May 2, 1988)

## ACKNOWLEDGEMENTS

Many thanks go to Sumit R. Das (renormalization group flows...), Georgi Djorgovski (remember the Compton cooling diagram), Murray Gell-Mann (do you understand complexity?), Antal Jevicki ( $N_c$  is large anyhow), John M. Kosterlitz (melting two-dimensional ice), John P. Preskill (he understands QCD), and John H. Schwarz (staying on the leading Regge trajectory), from whom I learned an enormous amount of physics. My fellow graduate students at the fourth floor of Lauritsen at Caltech were the constant dynamos to my pursuit of physics: Theodore Allen (unconstraining all the constraints), Michael Douglas (who can divide G by H), Elias Kiritsis (never tired), Milan Mijic (let there be  $R^2!$ ), and Alexios Polychronakos (a man from  $2 + 1$  dimensional world). It was a special and fruitful moment to collaborate with some excellent physicists, S. R. Das, B. Grinstein (never makes a mistake), H. D. Politzer (with deep intuition), and M. B. Wise (watch out CP!): I express my thanks to them for such an opportunity. I am grateful to Theodore Allen and Roxanne Springer for their wonderful proofreadings.

It was my pleasure to have Mark B. Wise as a research supervisor. More than anything else, he exemplified the model theoretical physicist: hard working, full of ideas and great enthusiasm, and possessing enormously strong dedication to physics.

Most of all, deep love from my parents, brothers and sisters has always been beyond any possible literal description, and I wish to dedicate this thesis to them.

## ABSTRACT

Radiative corrections to string scattering amplitude generate, in general, BRST anomalies due to the massless particle tadpole, the on-shell external two-point function singularities, and the intermediate unphysical cut singularities. It is proved that they originate from the boundaries of moduli space. Unitarity, analyticity, and BRST invariance are restored only after we add appropriate local counterterms to the scattering amplitudes. We show that counterterms have physical interpretations as the Fischler-Susskind mechanism, the mass renormalization, and the contact interactions respectively.

# Table of Contents

<i>Acknowledgements</i>	iii
<i>Abstract</i>	iv
<i>Table of Contents</i>	v
<b>1 Introduction</b>	<b>1</b>
References	6
<b>2 The BRST Quantization of String Theory</b>	<b>8</b>
2.1 Introduction to the BRST Quantization	8
2.2 BRST Invariant String Path Integral	18
2.3 Scattering Amplitude on the Punctured Riemann Surface	26
2.4 Example - Calculation of the Dilaton Tadpoles	34
References	46
<b>3 BRST Anomalies in One Loop Scattering Amplitudes</b>	<b>49</b>
3.1 Degeneration of One Loop Scattering Amplitudes	49
3.2 Structure of BRST Anomalies	55
References	58
<b>4 One Loop Local Counterterms</b>	<b>59</b>
4.1 General Strategy for Local Counterterms	59
4.2 Fischler-Susskind Mechanism as Tadpole Counterterm	61
4.3 Mass Renormalization as On-Shell Counterterm	65
4.4 Contact Interaction as Intermediate Cut Counterterm	67
References	69
<b>5 Analysis of Higher Loop Scattering Amplitudes</b>	<b>71</b>
5.1 Scattering Amplitudes at Two Loops	71
5.2 Structure of Two Loop BRST Anomalies and Counterterms	75
5.3 Higher Loop Generalization	81
References	84

## 1. Introduction

Our current understanding of the experimentally verifiable fundamental interactions is well described by the gauge theory of quarks and leptons based on the local symmetry group  $SU_c(3) \times SU_w(2) \times U_y(1)$  [1]. However, inclusion of *quantum* gravity has remained an unsolved problem. Of course, gravitational quantum fluctuations are not significant until one reaches the Planck energy scale of  $10^{19}$  GeV. One may try to quantize the covariant action of Einstein's general relativity, but the major problem is its apparent lack of renormalizability and unitarity [2]. At the moment, we do not have any viable alternative candidates whose low energy approximation reduces to the standard Einstein gravity and at the same time circumvents both of the above problems.

The superstring theory [3] is a candidate for self-consistent, supersymmetric, finite, and unique quantum theory of gravity. Importantly enough, it also unifies both the spacetime geometry and the matter fields into string excitations. A realistic unified model must contain the chiral fermions we observe in the low energy world. However this is strongly constrained due to potential chiral anomalies of the gauge, gravitational, and supersymmetry invariances on the quantum level. Great progress has been made since the discovery of anomaly cancellation mechanism in type-I superstring theory [4]. There are at present several classes of anomaly-free models: type-I open and closed superstring theory with gauge group  $SO(32)$  [5], type-II closed string theory with no gauge particles and with chiral and nonchiral matter fermions [6], heterotic strings based upon gauge group of  $SO(32)$  or  $E_8 \times E_8$  [7], and nonsupersymmetric, tachyon free  $O(16) \times O(16)$  heterotic string theory [8]. Lower dimensional variants of these models grow in number very rapidly but we do not count them as independent theories.

One crucial step to the internal consistency of string theory is a proof of unitarity and finiteness. Without a consistent second-quantized formulation of string theory, however, one finds it difficult to prove the unitarity of string theory. One may argue the unitarity by appealing to the light-cone formalism of string theory a *la* Mandel-

stam [9]. Only physical degrees of freedom are kept and there are no time-like or longitudinal components propagating throughout the scattering process. However, this is not manifestly covariant, and its proof of hidden covariance is another complicated calculational check. One may try to prove the Lorentz invariance by showing the equivalence between the light-cone formalism and a certain covariant formalism [10]. Still, this should not be regarded as a complete proof of unitarity, finiteness, and covariance since the above *formal* proof misses certain factorization limits of the scattering amplitudes which actually turn out to be the central source of potential unitarity violation. It is therefore important to examine this issue carefully.

This thesis addresses the issue of unitarity of string theories. For concreteness, we mainly consider a closed orientable bosonic string theory in detail. Calculationally, this is the simplest model, yet the underlying physics is very similar to that of the open and closed bosonic string theory or the superstring theory. Unlike the standard point particle theory, the string theory doesn't have any concrete set of Feynman rules for generic string field theories. This is an especially acute problem for the closed (super) string field theory, and its construction has remained as an important unsolved problem so far [11]. Therefore, we content ourselves with the proof of the unitarity of the on-shell scattering amplitudes. The scattering amplitude may be defined as a path integral over both the intrinsic string trajectory and its embedding into the target spacetime [12]. The spacetime covariance is manifest only if we choose a conformal gauge for fixing the worldsheet diffeomorphism symmetry. The gauge fixing requires a subtraction of the gauge volume through the standard Faddeev-Popov procedure. The gauge fixed path integral is most suitably described by introducing the diffeomorphism ghost fields. Combined with the original gauge fixed action, this results in BRST symmetry whose existence is traced back to the original conformal symmetry [13].

This BRST formulation gives much useful information with which we study the unitarity of scattering amplitudes [14]. We introduce a reparametrization ghost system  $c^z, b_{zz}$  which is the basis of one-forms in tangent and cotangent space of moduli

space respectively. The Hilbert space of the resulting  $c = 0$  conformal field theory is

$$\mathcal{H}_{string} = \mathcal{H}[X^\mu, \mu = 1, \dots, 26] \otimes \mathcal{H}[c^z, b_{zz}] \otimes \mathcal{H}[c^{\bar{z}}, b_{\bar{z}\bar{z}}]$$

and an exterior derivative

$$Q_{\text{BRST}} = \oint \frac{dz}{2\pi i} j_z^{\text{BRST}}$$

provides the  $\mathcal{H}_{string}$  with a nice cohomological structure. The physical states are represented by

$$\{\text{Ker } Q_{\text{BRST}} / \text{Im } Q_{\text{BRST}}\}.$$

What one has to check is the decoupling of unphysical states in the scattering amplitude. For this purpose, one may calculate the scattering amplitudes involving states in  $\text{Im } Q_{\text{BRST}}$ . Unitarity is guaranteed only if these states decouple. What one has to bear in mind is that there is a ghost charge duality in tangent space, so the decoupling of  $\text{Im } Q_{\text{BRST}}$  automatically ensures the decoupling of  $(\text{Ker } Q_{\text{BRST}})^\perp$ . The formal proof of the decoupling proceeds via the integration contour deformation [14] of the BRST current. Since the physical states are nonsingular when  $j_{\text{BRST}}$  passes through, a complete deformation leaves  $\oint j_{\text{BRST}} \cdot 1 = 0$ . However, this has to be understood with care, since when we factorize the scattering amplitude, the world sheet configuration approaches the boundary of the moduli space. The above contour deformation argument implicitly assumes that the moduli space is compact. The factorization is a fundamental *local* property of the scattering amplitude, and the unitarity provides the amplitudes with factorizability. Violation of factorizability also means the loss of the analytic property of the scattering amplitudes, and eventually of the BRST invariance of the underlying two-dimensional field theory.

Upon careful examination of the BRST invariance of the closed bosonic string scattering amplitudes, we find potential BRST anomalies. The sources of BRST anomalies are: (1) massless state tadpoles; (2) on-shell external two-point amplitudes;

and (3) intermediate pole singularities of unphysical states at some momentum configurations. They all arise from the boundary of moduli space in a total derivative structure.

Once the BRST anomalies are present, we must add local counterterms to restore the unitarity and the analyticity, hence the two-dimensional conformal invariance. We find that (1) the Fischler-Susskind mechanism [15,16], (2) the vertex operator renormalization [17, 18], and (3) the contact interactions [19] are the appropriate counterterms. It is probable that these constitute the only possible forms of counterterms.

Extension of this proof to the superstring theory is straightforward. One has a supersymmetry on the worldsheet so that one has fermionic spacetime coordinates  $\psi^\mu$  in addition to the bosonic coordinates  $X^\mu$  introduced in eq. (1.1). Similarly, the path integral of string trajectory is extended to fermionic parametrization variables (worldsheet gravitino) [20]. Thus, after gauge fixing the super-diffeomorphism symmetry, one is left with the supersymmetric ghosts  $\beta_{z\theta}$  and  $\gamma^\theta$ . The total Hilbert space is an extension of eq. (1.1) to

$$\mathcal{H}_{\text{superstring}} = \mathcal{H}[X^\mu, \psi^\mu] \otimes \mathcal{H}[c^z, b_{zz} : \gamma^\theta, \beta_{z\theta}] \otimes \mathcal{H}[c^{\bar{z}}, b_{\bar{z}\bar{z}} : \gamma^{\bar{\theta}}, \beta_{\bar{z}\bar{\theta}}]$$

with a supersymmetric BRST charge  $Q_{BRST}$ . Worldsheet fermions may be given two inequivalent spin structures: Neveu-Schwarz [21] and Ramond [22] satisfying antiperiodic and periodic boundary conditions respectively. Spacetime supersymmetry is achieved only after a GSO spin structure projection. Therefore, the proof of unitarity and finiteness must be extended to incorporate both the worldsheet fermions and the spin structure summation. This is not a problem at all. Actual calculation in the lower orders of the string loop expansion shows a mathematical equivalence to the bosonic strings and the absence of tachyons. For superstring amplitudes, in particular, one wants to establish a *precise* statement of the nonrenormalization theorem [23] which exists once we have at least one spacetime supersymmetry.

The present thesis [24] is organized as follows. Chapter 2 is about BRST invariant formulation of the string theory. Section 2.1 briefly illustrates BRST quantization method in a simple point particle field theory. In sections 2.2 and 2.3, we formulate BRST invariant Polyakov scattering amplitudes in terms of the punctured Riemann surface  $\mathcal{R}_{g,N}$ . We illustrate this formalism with a simple calculation of a dilaton tadpole in section 2.4 [25]. Using this formalism, we find the BRST anomalies in the form of total derivatives of moduli space integration in chapter 3. The one loop investigation is complete, proving that the divergence sources are a tadpole amplitude, an on-shell two point amplitude, and intermediate state singularity. In chapter 4, we describe the appropriate local counterterms we must add in order for BRST invariance to be restored. We explain in detail what spacetime interpretation we can give to each type of the local counterterms. The analysis of chapter 3 and 4 are on one-loop level. We consider two-loop and higher-loop BRST anomalies and the counterterms in chapter 5, essentially proving that leading BRST anomalies all cancel out by appropriate choice of counterterms.

## References

1. For a review, see G. Ross, *Grand Unified Theories*, Benjamin Cummings, Reading, Massachuttes (1984).
2. G. 'tHooft and M. Veltman, *Ann. Inst. Henri Poincaré*, **20** , 69 (1974).
3. For an excellent review, see M.B. Green, J.H. Schwarz, and E. Witten, *Superstring Theory*, 2 vols. Cambridge University Press, Cambridge (1987).
4. M.B. Green and J.H. Schwarz, *Phys. Lett.* **B149**, 117 (1984).
5. M.B. Green and J. H. Schwarz, *Nucl. Phys.* **B255**, 93 (1985).
6. See J. H. Schwarz, *Superstring Theory*, *Phys. Rep.* **89**, 223 (1982).
7. D. J. Gross, J. A. Harvey, E. Martinec, and R. Rohm, *Nucl. Phys.* **B256**, 253 (1985).
8. L. Alvarez-Gaume, P. Ginsparg, G. Moore, and C. Vafa, *Phys. Lett.* **B 171**, 155 (1986):  
 L. Dixon and J. A. Harvey, *Nucl. Phys.* **B274**, 93 (1986).
9. S. Mandelstam, in *Unified String Theory* ed. by M.B. Green and D. Gross, World Scientific, Singapore (1986).
10. E. D'Hoker and S. Giddings, *Nucl. Phys.***B291** , 90 (1987).
11. Recently, progress has been made by A. Strominger. See ICTP lecture, *Lectures on Closed String Field Theory* (1987).
12. A. M. Polyakov, *Phys. Lett.***103B**, 207 (1981).
13. M. Kato and K. Ogawa, *Nucl. Phys.* **B212** , 443 (1983).
14. D. Friedan, E. Martinec, and S. Shenker, *Nucl. Phys.* **B271** , 93 (1986).
- 15 W. Fischler and L. Susskind, *Phys. Lett.* **B171** , 383 (1986) : *ibid.***B173** , 262 (1986).
16. S. R. Das and S.-J. Rey, *Phys. Lett.* **B186** , 328 (1987):

- C. G. Callan, C. Lovelace, C. Nappi, and S. Yost, Nucl. Phys. **B288** , 525 (1987): *ibid.* **B293** , 83 (1987).
17. S. Weinberg, talk at *American Physical Society Meeting* , Eugene, Oregon (1985).
  18. N. Seiberg, Phys. Lett. **B187** , 56 (1987):  
A. Sen, SLAC preprint SLAC-PUB-4383, (1987) unpublished.
  19. M. B. Green and N. Seiberg, Institute for Advanced Study preprint, IASSNS-HEP-87/38 (1987) unpublished.
  20. L. Brink, P. DiVecchia, and P. Howe, Phys. Lett. **B65** , 471 (1976):  
S. Deser and B. Zumino, Phys. Lett. **B65** , 369 (1976):  
A. M. Polyakov, Phys. Lett. **B103** , 211 (1981).
  21. A. Neveu and J. H. Schwarz, Nucl. Phys. **B31** , 86 (1971).
  22. P. Ramond, Phys. Rev. **D3** , 2415 (1971).
  23. E. Martinec, Phys. Lett. **B171** , 189 (1986).
  24. S.-J. Rey, Caltech Preprint, CALT-68-1468 (1988) To be published in Nucl. Phys. B.
  25. S.-J. Rey, Phys. Lett. **B203**, 393 (1988).

## 2. The BRST Quantization of String Theory

### 2.1 Introduction to BRST Quantization

In classical physics dynamics are completely specified by equations of motion and initial data which provides the equation of motion with causal structure. Dynamical variables and their conjugate momenta form a phase space into which physical solutions are mapped. However, in many cases, we are interested in situations with dynamical constraints. The simplest example is a particle moving on a surface of sphere of radius  $R$  when we use spherical polar coordinates  $r, \theta, \phi$  as dynamical variables (constraint is  $c(r) - (r - R) = 0$  in this case). The constraints restrict the dynamical variables and their conjugate momenta to lie in a physically acceptable subspace of the phase space. In principle one imposes constraints at every stage of solving equations of motion. In practice we find it more convenient to extend dynamical variables to include auxiliary ones, the so-called ‘method of Lagrange multipliers’ [1]. This enlarges the phase space so that it has a higher dimension. It is precisely based on this observation that a physical phase space manifold, complicated by constraints, can be described in simpler way by embedding it in a higher dimensional, unconstrained, phase space.

In quantum mechanics we need to quantize the dynamical variables. At the initial configuration this can be done once and for all with all of the conjugate pairs. However, constraints are now operator-valued and in general do not commute with the Hamiltonian. Therefore quantization cannot be consistently maintained at subsequent time given a quantization prescription at initial data spacelike hypersurface. As in the classical problem, we solve this problem by enlarging the quantum phase space including constraints as dynamical fields then truncating the Hilbert space to a subspace that obeys the operator-valued constraint equations. What one does is to make a canonical transformation of the Hamiltonian such that the constraints are dynamical variables and the truncate terms with conjugate momenta to the constraint out. The result is a simultaneously diagonalized Hamiltonian and quantum constraint

conditions [2].

It is always possible to formulate constrained systems as gauge theories [3]. By gauge transformation we mean time-dependent transformation of the dynamical variables of extended phase space. An action is invariant under the gauge transformation or, equivalently, the Lagrangian is changed by a total derivative. However, the Hamiltonian which is derived from the extended phase space is not gauge invariant. One can make a time-dependent gauge transformation that vanishes at the initial data hypersurface and this is a gauge-theoretic manifestation of quantization problem mentioned above. Because of these gauge transformations, the Hilbert space is even larger than the Hilbert space for the extended phase space described above. The physical projection is implemented by demanding gauge invariance of Hilbert space states. This is an analog of demanding that the operator-valued constraints be satisfied in the extended phase space description. In covariant quantum field theories such as Yang-Mills theory, we must have a perturbative scheme to calculate quantum Green functions and subsequently S-matrix elements. However, a perturbative scheme is impossible without fixing the gauge arbitrariness. Also one wants to keep the manifest spacetime covariance of the theory. Canonical quantization does not fit into this procedure and standard method one uses is an indefinite metric (Gupta-Bleuler) quantization prescription [4].

Beccei, Rouet, and Stora [5] and independently Tyutin [6] realized yet another scheme to fix the gauge invariance while maintaining a residual so-called BRST symmetry. BRST transformations correspond roughly to a fermionic gauge transformation. Therefore the BRST transformation must be nilpotent:  $\{Q_{BRST}, Q_{BRST}\} = 0$  where  $Q_{BRST}$  is the BRST charge operator. This also makes the Hilbert space of the BRST formulation even bigger than the gauge theoretic or extended phase space Hilbert spaces. Physical states are manifestly BRST invariant elements in the subspace  $\text{Ker } Q_{BRST}$ , annihilated by the BRST charge  $Q_{BRST}$ . However, due to the nilpotency of BRST charge algebra, there are trivially BRST invariant states in the subspace of  $\text{Im } Q_{BRST}$ . We therefore identify physical states as an equivalence class defined by a quotient space  $\text{Ker } Q_{BRST}/\text{Im } Q_{BRST}$ . Beccei, Rouet, Stora, and Tyutin

showed that both the perturbative renormalizability and gauge invariance of physical S-matrices are easier to derive in this formalism.

In this section we illustrate BRST quantization with the simplest gauge theory: Abelian gauge theory. First we briefly go through quantization in canonical and indefinite metric method and finally present BRST method.

Abelian gauge theory is a theory of massless vector boson field. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad (2.1.1)$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \quad (2.1.2)$$

The action is invariant under a local gauge transformation

$$\delta A_\mu = \partial_\mu \Lambda(x) \quad (2.1.3)$$

with arbitrary spacetime-dependent function  $\Lambda(x)$ . When coupled to matter, the gauge invariance implies a conservation of the gauge current. If we denote a matter current by  $j_\mu^M$ , a gauge invariant minimal coupling gives

$$\mathcal{L}_{tot.} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + j_\mu^M A^\mu. \quad (2.1.4)$$

Under the gauge transformation, eq. (2.1.4) changes by

$$\delta\mathcal{L}_{tot.} = j_\mu^M \partial^\mu \Lambda(x). \quad (2.1.5)$$

Upon integration by parts, we find that the action is invariant if and only if the matter current is conserved.

Canonical quantization of this theory is done by constructing a Hamiltonian and commutation relations between dynamical variables and their conjugate momenta (another method is by constructing a transition amplitude through path integration with the above Lagrangian which will be mainly used in the subsequent discussion of quantum string theory). Denoting

$$E_i = \partial_0 A_i - \partial_i A_0 \quad (2.1.6)$$

and

$$\epsilon_{ijk} B_k = \partial_i A_j - \partial_j A_i, \quad (2.1.7)$$

the conjugate momentum of  $\vec{A}$  is

$$\vec{\Pi} = \frac{\partial \mathcal{L}}{\partial \vec{A}_0} = \vec{E} \quad (2.1.8)$$

and  $A_0$  is not dynamical. By Legendre transformation, the Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}(\vec{\Pi}^2 + \vec{B}^2) - A_0 \nabla \cdot \vec{\Pi} \quad (2.1.9)$$

with the canonical commutation relation

$$[\Pi_i(\vec{x}, t), A_j(\vec{y}, t)] = -i\delta_{ij}\delta^{(3)}(\vec{x} - \vec{y}). \quad (2.1.10)$$

However, given an initial data of  $\vec{A}$  and  $\vec{\Pi}$ , we cannot consistently impose this canonical commutation relation at later time since a time-dependent gauge transformation modifies canonical variables. Under a time-dependent gauge transformation, the Hamiltonian changes into:

$$\mathcal{H} \rightarrow \mathcal{H}' = \mathcal{H} - \partial_0 \Lambda \nabla \cdot \vec{\Pi}. \quad (2.1.11)$$

We see that  $\nabla \cdot \vec{\Pi}$  is a generator of time-independent gauge transformation. This means that the invariance of the Hamiltonian under time-dependent gauge transformation is equivalent to the invariance of Hilbert space state under time-independent

gauge transformations. Therefore, restricting ourselves to gauge invariant states (or equivalently, local operators), we get a consistent quantization rule. That is, the physical Hilbert space is the subspace of the full Hilbert space which is annihilated by  $\nabla \cdot \vec{\Pi}$ .

All is well except that we lost manifest Lorentz covariance through the use of the canonical quantization procedure. Also this procedure, when applied to non-Abelian gauge theory, makes it impossible to develop a consistent perturbation expansion procedure. This is a serious problem, for example, if we need to prove renormalizability and unitarity of the quantum gauge theory. What one does instead is to extend the phase space to include the auxiliary variable  $A_0$  as a dynamical field. One conventionally adds a kinetic term for the  $A_0$  field to the Lagrangian:

$$\mathcal{L}_{g.f.} = -\frac{\alpha}{2}(\partial_\mu A^\mu)^2. \quad (2.1.12)$$

Another way to see this is through the path integral quantization. The partition function is defined by

$$\mathcal{Z} = \int DA_\mu \exp[-\int d^4x \frac{1}{4} F_{\mu\nu} F^{\mu\nu}] \quad (2.1.13)$$

but there is a flat direction of the integrand precisely along the gauge orbit. This makes the functional integration ill-defined, and we need to add an longitudinal damping term of the form in eq.(2.1.12). With the gauge-fixing term eq.(2.1.12) added to the Lagrangian, the equation of motion is modified to

$$[g_{\mu\nu}\partial^2 - (1 - \alpha)\partial_\mu\partial_\nu]A^\nu(x) = 0. \quad (2.1.14)$$

From this we find that the on-shell solution satisfies

$$\partial^2(\partial_\mu A^\mu)(x) = 0 \quad (2.1.15)$$

and the gauge fields can be consistently solved for once initial data for  $A^\mu$  and  $\partial_\mu A^\mu$

are given over a spacelike hypersurface. In Gupta-Bleuler indefinite metric quantization, the equal-time commutation relation is modified and postulated to be

$$[\Pi^\mu(\vec{x}, t), A^\nu(\vec{y}, t)] = -ig^{\mu\nu}\delta^{(3)}(\vec{x} - \vec{y}) \quad (2.1.16)$$

consistent with the general covariance and the conjugate momenta are defined by

$$\begin{aligned} \Pi_\mu &= \frac{\partial \mathcal{L}}{\partial A_{,0}^\mu} \\ &= F_{0\mu} + \alpha g_{0\mu} (\partial_\nu A^\nu). \end{aligned} \quad (2.1.17)$$

However, we immediately find that the gauge condition is incompatible with quantization since  $\partial_\mu A^\mu$  does not commute with  $A_0$  due to eq. (2.1.16). At best we demand its quantum average over its positive frequency part be satisfied in the linear physical Hilbert space:

$$\langle phys. | \partial_\mu A_{(+)}^\mu | phys. \rangle = 0. \quad (2.1.18)$$

One can factorize the physical Hilbert states into products of transverse states and longitudinal and scalar states:

$$|phys\rangle = |transverse\rangle \otimes |long. + scalar\rangle \quad (2.1.19)$$

where  $|long. + scalar\rangle$  satisfies eq.(2.1.18). This means that the physical states are defined by an equivalence class since we can add arbitrary states of purely longitudinal and scalar mode excitations without modifying the physical spectrum. Due to the indefiniteness of commutation relation, we see that any such excitations have zero norms once they satisfy eq. (2.1.18). Equivalently, one finds that  $\langle long. + scalar | A_\mu(x) | long. + scalar \rangle$  can always be written as a total derivative of some spacetime dependent function.

BRST quantization starts by promoting the gauge transformations to dynamics by introducing Grassmann-valued BRST ghost and antighost fields  $b$  and  $c$ , and a bosonic auxiliary variable  $\psi$ . From the gauge transformation of eq. (2.1.3), we replace

$$\delta A_\mu = \epsilon \partial_\mu c(x). \quad (2.1.20)$$

Under this BRST transformation, the gauge fixing term is transformed by

$$\delta \mathcal{L}_{g.f.} = -\alpha (\partial_\mu A^\mu) \epsilon \partial^2 c. \quad (2.1.21)$$

Therefore we add a new BRST ghost Lagrangian

$$\mathcal{L}_{gh.} = b \partial_\mu \partial^\mu c \quad (2.1.22)$$

and postulate that the antighost  $b$  changes under BRST transformation by

$$\delta b = \epsilon \alpha \partial_\mu A^\mu. \quad (2.1.23)$$

Once this is done, the total Lagrangian is invariant under BRST transformations. Obviously any gauge invariant operators are BRST invariant. However the converse is not true in general. To match the counting of degrees of freedom we introduce an auxiliary bosonic field  $\psi$ . Thus we write the gauge fixing term as

$$\mathcal{L}_{g.f.} = -\frac{\psi^2}{2\alpha} + \psi (\partial_\mu A^\mu) \quad (2.1.24)$$

and define

$$\delta b = \epsilon \psi, \quad \delta \psi = 0,$$

$$\delta c = 0. \quad (2.1.25)$$

The BRST Lagrangian, which is a sum of the gauge fixing term and the ghost kinetic

term, can be written as

$$\mathcal{L}_{BRST} = \delta[b(x)(\partial_\mu A^\mu - \frac{1}{2\alpha}\psi)], \quad (2.1.26)$$

or, using the BRST charge,

$$Q_{BRST} = \int d^3\vec{x} J_{BRST}^0,$$

$$J_{BRST}^0 = -ic\vec{\nabla} \cdot \vec{E} - i(\partial^0 c)\Pi_0, \quad (2.1.27)$$

as

$$\mathcal{L}_{BRST} = \{Q_{BRST}, b(\partial_\mu A^\mu - \frac{1}{2\alpha}\psi)\}. \quad (2.1.28)$$

A similar invariance exists for the anti-BRST transformation. Therefore, physical states satisfy

$$Q_{BRST}|phys.\rangle = 0 = \bar{Q}_{BRST}|phys.\rangle \quad (2.1.29)$$

or equivalently

$$\vec{\nabla} \cdot \vec{E}|phys.\rangle = 0$$

and

$$\Pi_0|phys.\rangle = \partial_\mu A_{(+)}^\mu|phys.\rangle = 0. \quad (2.1.30)$$

One also finds that the ghost and antighost form an indefinite metric Hilbert space. Let us see this by quantizing the ghost system. One can write the ghost

Lagrangian as

$$\mathcal{L}_{gh.} = -\partial_\mu b \partial^\mu c. \quad (2.1.31)$$

The conjugate momenta are

$$\Pi_b = \partial_0 c, \quad (2.1.32)$$

and

$$\pi_c = \partial_0 b, \quad (2.1.33)$$

so that the Hamiltonian is

$$\mathcal{H}_{gh.} = \Pi_c \cdot \Pi_b + \vec{\nabla} b \cdot \vec{\nabla} c. \quad (2.1.34)$$

Since the ghost and the antighost fields are independent of each other, we must satisfy

$$\{b, c\} = 0. \quad (2.1.35)$$

Differentiating with respect to time, we get

$$\{\Pi_c, c\} = -\{b, \Pi_b\} = -1,$$

$$\{\Pi_b, \Pi_c\} = 0. \quad (2.1.36)$$

Therefore one has an indefinite metric Hilbert space of the ghosts. Combined with the gauge field Hilbert space, the physical subspace consists of only positive norm states and hence has no ghost excitations. Combined with the BRST invariance condition eq. (2.1.29), one finds that this BRST invariant subspace is equivalent to the one derived from the Gupta-Bleuler quantization procedure.

One finally needs to identify the physical subspace with  $\text{Ker } Q_{BRST}/\text{Im } Q_{BRST}$ . Suppose one start with a state in the physical Hilbert space. The state is BRST invariant since it is constructed out of the ghost and antighost creation operators and also by an arbitrary gauge-invariant field creation operators. Acting on another BRST gauge-fixing operator of the form eq. (2.1.28), we get a state in the Hilbert subspace  $\text{Im } Q_{BRST}$ :

$$\{Q_{BRST}, \mathcal{O}\}|phys.\rangle = Q_{BRST}\mathcal{O}|phys.\rangle. \quad (2.1.37)$$

The norm of this state is equal to

$$\langle phys.|\mathcal{O}^+Q_{BRST}^2\mathcal{O}|phys.\rangle = \langle phys.|\mathcal{O}^+[Q_{BRST}, \{Q_{BRST}, \mathcal{O}\}]|phys.\rangle = 0, \quad (2.1.38)$$

and one finds that states in this subspace have zero norms, isomorphic to the zero norm states in Gupta-Bleuler formalism.

From this argument it follows that the equivalence class,  $\text{Ker } Q_{BRST}/\text{Im } Q_{BRST}$ , constructed by moding out the gauge direction from any subspace of the covariantly gauge-fixed theory, generates a theory isomorphic to the physical theory obtained from canonical or indefinite metric quantized physical Hilbert space defined by constraint.

## 2.2 BRST Invariant String Path Integral

In the quantum theory of a point particle with mass  $m$ , the wave-particle duality enables us to write a transition amplitude as a sum over the phase along all possible particle trajectories [7]. We introduce a one dimensional metric  $g(t)$  along the particle trajectory and write the transition amplitude  $\mathcal{Z}$  on the background of  $G_{\mu\nu}$  as a functional integration over  $g(t)$  as well as  $X^\mu$ :

$$\mathcal{Z} = \int \mathcal{D}X^\mu \cdot \mathcal{D}g \cdot \exp\left[\frac{i}{2} \int dt \sqrt{g}(m^2 - g^{-1}(t)X'^\mu(t)X'^\nu(t)G_{\mu\nu}(X))\right] \quad (2.2.1)$$

with the boundary conditions at the end points specified. This is a manifest Poincaré and reparametrization invariant, where the latter follows from the invariant length  $(ds)^2 = g(t) \cdot (dt)^2$  along the trajectory.

In a similar manner, the first quantized string transition amplitude may be written as a phase sum over all possible trajectories with the given boundary conditions [8]. The trajectory of the string is a two dimensional manifold, hereafter called the worldsheet, and the sum over the history naturally includes the sum over all possible shapes of the two dimensional worldsheet geometries. This is conveniently described by a map  $X^\mu(\sigma)$  from the two dimensional parameter space manifold of genus  $g$   $\mathcal{M}_g$ ,  $\sigma^a \in \mathcal{M}_g$  to the target spacetime. In the first quantization, the geometry of the target spacetime is given, and the string represents fluctuations propagating on this background. The dynamical determination of the background geometry requires the second quantized formulation such as the string field theory. Still, one can examine a consistent string propagation by demanding conformal invariance of the first quantized string theory as will be discussed at length later. The most generally covariant action is

$$\begin{aligned} S &= \sum_i \int_{\mathcal{M}} \lambda_i(X) \hat{\mathcal{O}}_{(d)}^i(\sigma) \epsilon^{d-2} \\ &= \int \left( \frac{d\sigma}{\epsilon} \right)^2 \sqrt{g} [\epsilon^2 T(X) + \frac{1}{T} g^{ab} \epsilon \partial_a X^\mu \epsilon \partial_b X^\nu G_{\mu\nu}(X) + \frac{1}{2\pi} \epsilon^2 R^{(2)} D(X) + \dots]. \end{aligned} \quad (2.2.2)$$

Here,  $\epsilon$  is a coordinate lattice cutoff to regularize the worldsheet short distance diver-

gence,  $T$  is the string tension, and the ‘geometric couplings’  $\lambda = T(X), G_{\mu\nu}(X), D(X)$  etc. are dimensionless spacetime functionals associated with covariant worldsheet operators  $\mathcal{O}_{(d)}^i = 1, g^{ab} \partial_a X^\mu \partial_b X^\nu, R^{(2)}$  etc. The action is invariant under local worldsheet transformations of the following types:

(1) Diffeomorphism:

$$\sigma^a \rightarrow \eta^a(\sigma), \quad g_{ab}(\sigma) d\sigma^a d\sigma^b = \tilde{g}_{ab}(\eta) d\eta^a d\eta^b$$

(2) Weyl transformation:

$$g_{ab}(\sigma) \rightarrow \Omega(\sigma) \cdot g_{ab}(\sigma), \quad \sigma^a \rightarrow \sigma^a. \quad (2.2.3)$$

Formally, the string path integral is [9]

$$\mathcal{Z} \equiv \sum_g \mathcal{Z}_g = \sum_g \int_{\mathcal{M}_g} \frac{\mathcal{D}g_{ab} \mathcal{D}X^\mu}{\text{Vol}(G)} \cdot \exp[-S(g_{ab}, X^\mu)]. \quad (2.2.4)$$

The functional measure  $\mathcal{D}g_{ab}$  and  $\mathcal{D}X^\mu$  are defined by the metric [10]:

$$\|\delta X^\mu\|^2 = \int d^2\sigma \sqrt{g} \delta X^\mu \delta X^\nu H_{\mu\nu}(X),$$

and

$$\|\delta g_{ab}\|^2 = \int d^2\sigma \sqrt{g} \delta g_{ac} \delta g_{bd} [(g^{ab} g^{cd} - \frac{1}{2} g^{ac} g^{bd}) \cdot A(X) + g^{ac} g^{bd} \cdot B(X)], \quad (2.2.5)$$

where  $H_{\mu\nu}(X)$ ,  $A(x)$ , and  $B(X)$  are some spacetime functions to be related to  $G_{\mu\nu}(X)$ ,  $T(X)$ , and  $D(X)$  in eq. (2.2.2) by a renormalization procedure of worldsheet action  $S$ . One notes that the measures are diffeomorphism invariant but not Weyl invariant. However, we find that a total Weyl anomaly is cancelled if we choose  $D=26$  or  $10$  for bosonic and supersymmetric strings respectively, thereby retaining the symmetries

of eq. (2.2.3). We will strictly set the spacetime dimensionality to these critical values from now on. Therefore, with  $G = G_{Dif f} \otimes G_{Weyl}$  in eq. (2.2.4) we choose a conformal gauge for gauge fixing:

$$g_{ab}(\sigma) = e^{\phi(\sigma)} \hat{g}_{ab}(m_i). \quad (2.2.6)$$

Here  $\hat{g}_{ab}(m_i)$  is a fixed reference metric which depends on moduli parameters  $m_i, i = 1, \dots, 3g - 3$  to describe a global structure of the worldsheet manifold  $\mathcal{M}_g$ . In each local coordinate patch,  $\hat{g}_{ab}(m_i)$  can be chosen as  $\delta_{ab}$  so that a complex structure is manifest. The nontrivial moduli dependence comes from the global patching of coordinates. Still, there is a residual conformal symmetry which is a combination of a diffeomorphism and a Weyl transformation satisfying

$$\partial_a \delta \eta_b + \partial_b \delta \eta_a = \hat{g}_{ab} \partial_c \delta \eta^c \equiv \delta \phi(\sigma) \hat{g}_{ab}.$$

We use complex coordinates  $z = \exp(t + i\sigma)$  compatible with the complex structure at each local coordinate patch with the reference metric  $\hat{g}_{ab}(m_i)$ . The most general coordinate transformation compatible with the complex structure turns out to be the conformal transformation

$$z \rightarrow w(z) \quad \text{and} \quad \bar{z} \rightarrow \bar{w}(\bar{z}).$$

A gauge slice of eq. (2.2.6) is defined over the Teichmuller space with complex coordinates  $m_i, i = 1, \dots, 3g - 3$ . The tangent space variation of the metric is written as

$$\delta g_{zz} = \partial_i g_{zz} \delta m^i + \partial_{\bar{i}} g_{zz} \delta m^{\bar{i}} + 2P_{1z} \delta \eta_z, \quad (2.2.7)$$

and

$$\delta g_{z\bar{z}} = \partial_i g_{z\bar{z}} \delta m^i + \partial_{\bar{i}} g_{z\bar{z}} \delta m^{\bar{i}} + \delta \phi g_{z\bar{z}}. \quad (2.2.8)$$

Here,  $\phi$  is a conformal factor,  $\eta_z$  a diffeomorphism vector, and  $P_j$  the differential

operator mapping  $(1,0)$  vector field into  $(-2,0)$  tensors:

$$\delta t_{zz} = (P_1 \delta \eta)_{zz} = \nabla_z \delta \eta_z$$

with an associated adjoint structure

$$(P_1^+ \delta t | \delta \eta) = (\delta t | P_1 \delta \eta).$$

One can further decompose  $\partial_i g_{zz}$  into zero modes  $h_{izz}$  of  $P_1$ , called holomorphic quadratic differentials, and deformation orthogonal to them

$$\partial_i g_{zz} = \nabla_z v_{iz} + M_i^j h_{jzz}, \quad (2.2.9)$$

and similarly for  $\partial_{\bar{i}} g_{z\bar{z}}$ . Using (2.2.8), eq.(2.2.5) can be written as

$$\begin{aligned} \|\delta g_{ab}\|^2 &= A(X) \int dz \wedge d\bar{z} g^{z\bar{z}} |[P_1(2\delta \eta_z + v_{iz} \delta m^i + v_{\bar{i}z} \delta m^{\bar{i}})]_{zz}|^2 \\ &\quad + 4A(X) \delta m^i \delta m^{\bar{j}} M_i^k M_{\bar{j}}^{\bar{l}} (h_{kzz} | h_{\bar{l}\bar{z}\bar{z}}) \\ &\quad + B(X) \int dz \wedge d\bar{z} g_{z\bar{z}} |\delta \phi + \delta m^i g^{z\bar{z}} \partial_i g_{z\bar{z}} + \delta m^{\bar{i}} g^{z\bar{z}} \partial_{\bar{i}} g_{z\bar{z}}|^2. \end{aligned} \quad (2.2.10)$$

From eq. (2.2.9), we calculate  $M_i^j$ :

$$M_i^j = (\mu_{iz}^{\bar{z}} | h_{\bar{k}\bar{z}\bar{z}}) \cdot (h_{\bar{k}\bar{z}\bar{z}} | h_{jzz})^{-1} \quad (2.2.11)$$

where the Beltrami differential,  $\mu_{iz}^{\bar{z}}$ , and the inner product are defined by

$$\begin{aligned} \mu_{iz}^{\bar{z}} &= \frac{1}{2} g^{z\bar{z}} \partial_i g_{z\bar{z}}, \\ (\mu_{iz}^{\bar{z}} | h_{jzz}) &= \int d^2 z \mu_{iz}^{\bar{z}} h_{jzz}. \end{aligned} \quad (2.2.12)$$

Therefore, the partition function eq. (2.2.3) can be written as

$$\mathcal{Z} = \int \prod_{i=1}^{3g-3} dm_i \wedge dm_{\bar{i}} \cdot \left| \frac{\det(\mu_{i\bar{z}}^z | h_{j\bar{z}\bar{z}})}{\det^{1/2}(h_{izz} | h_{j\bar{z}\bar{z}})} \right|^2 \cdot \mathcal{Z}_{\text{flat}}(m_i, m_{\bar{i}}). \quad (2.2.13)$$

Here,

$$\mathcal{Z}_{\text{flat}}(m_i, m_{\bar{i}}) = \int \text{Vol}^{-1}(CKV) \mathcal{D}X \mathcal{D}b \mathcal{D}c \mathcal{D}\bar{b} \mathcal{D}\bar{c} \exp[-S_{BRST}(X, b, c : \hat{g})] \quad (2.2.14)$$

is a conformal gauge-fixed partition function of spacetime coordinates and diffeomorphism ghost fields  $b_{zz}$  and  $c^z$  and

$$S_{BRST}(X, b, c : \hat{g}) = S(X : \hat{g}) + \int d^2z [b_{zz} \nabla_{\bar{z}} c^z + c.c.]. \quad (2.2.15)$$

This yields the two-point functions

$$\langle X^\mu(z) X^\nu(w) \rangle = -\ln |z - w|^2$$

and

$$\langle b_{zz}(z) c^w(w) \rangle = (z - w)^{-1}. \quad (2.2.16)$$

For a flat spacetime background, we have:

$$\begin{aligned} \mathcal{Z}_g &= \int \prod_{i=1}^{3g-3} dm_i \wedge dm_{\bar{i}} \left| \frac{\det(\mu_{i\bar{z}}^z | h_{j\bar{z}\bar{z}})}{\det^{1/2}(h_{izz} | h_{j\bar{z}\bar{z}})} \right|^2 \\ &\quad \cdot \frac{|\det' P_1^+ P_1|^{1/2}}{\text{Vol}(CKV)} \cdot \left| \frac{\det'(-g^{z\bar{z}} \partial_z \partial_{\bar{z}})}{\int dz \wedge d\bar{z}} \right|^{-D/2}. \end{aligned} \quad (2.2.17)$$

The action  $S_{BRST}$  in eq. (2.2.15) is BRST invariant. The BRST transformation

is deduced from the diffeomorphism transformation

$$\delta X^\mu = -\epsilon(c^z \partial_z + c^{\bar{z}} \partial_{\bar{z}}) X^\mu,$$

$$\delta c^z = -\epsilon c^z \partial_z c^z,$$

$$\delta b_{zz} = \epsilon [G_{\mu\nu}(X) \partial_z X^\mu \partial_z X^\nu + \nabla_z b_{zz} \cdot c^z + 2b_{zz} \cdot \nabla_z c^z], \quad (2.2.18)$$

where  $\epsilon$  is a constant Grassman c-number on the worldsheet.

The energy momentum tensor is defined by a response to the change of action  $S_{BRST}$  with respect to that of worldsheet metric. The  $X^\mu$  responds according to

$$T_{zz} = -\frac{1}{2} \partial_z X^\mu \partial_z X^\nu G_{\mu\nu}(X) + \dots,$$

$$T_{\bar{z}\bar{z}} = \bar{T}_{zz}. \quad (2.2.19)$$

With a flat spacetime, the energy momentum tensor satisfies an operator algebra:

$$T_{zz} T_{ww} = \frac{D}{2} (z-w)^{-4} + 2(z-w)^{-2} T_{ww} + (z-w)^{-1} \partial_w T_{ww}. \quad (2.2.20)$$

The first term is a manifestation of the conformal anomaly, and is related to the component  $T_{zz}$ . The energy momentum tensor associated with the ghosts is [11]:

$$T_{gh}(z) = c \nabla b + 2(\nabla c) b. \quad (2.2.21)$$

The operator product expansion is

$$T_{gh}(z) T_{gh}(w) = -13(z-w)^{-4} + 2(z-w)^{-2} T_{gh}(w) + (z-w)^{-1} \partial T_{gh}(w). \quad (2.2.22)$$

We find that the total conformal anomaly of eq.(2.2.20) and eq.(2.2.22) disappears only if the spacetime dimension is 26, and allows decoupling of the unphysical states by application of these Virasoro conditions.

This can be compactly described by the BRST Noether current [20]:

$$\begin{aligned} J_z^{BRST} &= \frac{1}{2}c^z(\delta b_{zz}/\delta\epsilon) \\ &= c^z[T_{zz}(X) + \frac{1}{2}T_{zz}(b_{zz}, c^z)] \end{aligned}$$

which satisfies the chiral current conservation law

$$\partial_{\bar{z}}J_z^{BRST} = 0, \quad \partial_zJ_{\bar{z}}^{BRST} = 0. \quad (2.2.23)$$

Using energy-momentum conservation, we can show that these are equivalent to

$$T_{z\bar{z}} = 0 = T_{\bar{z}z}, \quad (2.2.24)$$

which is the condition for conformal invariance.

In BRST formulation, there are nontrivial normalizable classical solutions of

$$\nabla_{\bar{z}}(\hat{g})b_{zz}^{(0)} = 0,$$

$$\nabla_{\bar{z}}(\hat{g})c_{(0)}^z = 0$$

in a given background metric  $\hat{g}_{ab}$ . Because of these zero modes, the ghost density currents  $j_z = c^z b_{zz}$  and  $j_{\bar{z}} = c^{\bar{z}} b_{\bar{z}\bar{z}}$  are anomalous [13]

$$\partial_{\bar{z}}j_z = \frac{3}{16\pi}\sqrt{g}R^{(2)}$$

and

$$\partial_zj_{\bar{z}} = \frac{3}{16\pi}\sqrt{g}R^{(2)}. \quad (2.2.25)$$

Integrating this, we get the relation of the zero modes to the topological invariant of manifold  $\mathcal{M}_g$  called Euler characteristic of the background worldsheet:

(number of c complex zero modes) – (number of b complex zero modes)

$$\begin{aligned}
&\equiv \int_{\mathcal{M}_g} [\partial_{\bar{z}} j_z + \partial_z j_{\bar{z}}] \\
&= \frac{3}{2} \cdot \chi_g \\
&= 3 \cdot (1 - g). \tag{2.2.26}
\end{aligned}$$

We must couple the dilaton background to the diffeomorphism ghost fields starting from the action eq.(2.2.2). This is achieved by a dilaton coupled rotation defined by the new BRST transformation [14]

$$\delta X^\mu = -\epsilon \cdot e^{4D(X)/3} (c^z \partial_z + c^{\bar{z}} \partial_{\bar{z}}) X^\mu,$$

$$\delta(e^{4D(X)/3} \cdot c^z) = -\epsilon e^{8D(X)/3} c^z \partial_z c^z,$$

$$\delta(e^{-4D(X)/3} \cdot b_{zz}) = \epsilon (G_{\mu\nu} \partial_z X^\mu \partial_z X^\nu + \frac{4}{3} \partial_z D(X) b_{zz} c^z + \partial_z b_{zz} \cdot c^z + 2b_{zz} \cdot \partial_z c^z). \tag{2.2.27}$$

New BRST invariant action is

$$S_{BRST} = \int d^2 z [G_{\mu\nu}(X) \partial_z X^\mu \partial_z X^\nu + b_{zz} \nabla_{\bar{z}} c^z + \frac{4}{3} \partial_{\bar{z}} D(X) b_{zz} c^z + c.c.] \tag{2.2.28}$$

and the BRST Noether current becomes

$$J_z^{BRST} = c^z G_{\mu\nu} \partial_z X^\mu \partial_z X^\nu + c^z b_{zz} \partial_z c^z - \frac{1}{2\pi} c^z \partial_z^2 D(X). \tag{2.2.29}$$

This BRST invariant dilaton coupling will be used later when we construct local perturbations of spacetime backgrounds.

### 2.3 Scattering Amplitude on the Punctured Riemann Surface

In the previous section, we defined the covariant, first quantization of quantum string through path integration and BRST invariance. A physically measurable quantity is a scattering amplitude directly compared to an experimental setup. We prepare with the ‘in’ and ‘out’ set of complete physical states at asymptotic past and future respectively. The unitary transformation of the prepared incoming particle Hilbert space states to the outgoing particle Hilbert space states corresponds to the scattering amplitude [15].

First, to each asymptotic physical state of quantum number  $A$  and momentum  $p$ , we associate a local worldsheet vertex operator, denoted by  $V_A(z : p)$  with its position  $z$  on the worldsheet. The vertex operator represents a small fluctuation around the given background of quantum number  $A$ :  $\delta T(X), \delta G_{\mu\nu}(X), \delta D(X)$  and so on. The partition function of eq. (2.2.3) is nothing but a generating functional of scattering amplitudes [16] Denoting a functional average of an operator  $\mathcal{O}$  with the action eq. (2.2.3) by  $\langle \mathcal{O} \rangle_{\text{background}}$ ,

$$\mathcal{Z}(T + \delta T, G_{\mu\nu} + \delta G_{\mu\nu}, D + \delta D, \dots) = \langle \exp[-\delta S] \rangle_{T, G_{\mu\nu}, D, \dots}$$

where

$$\delta S = \int \frac{d^D p}{(2\pi)^D} \int dz \wedge d\bar{z} [\delta \tilde{T}(p) \cdot V_T(z : p) + \delta \tilde{G}_{\mu\nu} \cdot V^{\mu\nu}(z : p) + \delta \tilde{D}(p) \cdot V_D(z : p)],$$

with

$$V_T(z : p) = \exp[ip \cdot X(z)],$$

$$V^{\mu\nu}(z : p) = \partial_z X^\mu \partial_{\bar{z}} X^\nu \exp[ip \cdot X(z)],$$

$$V_D(z : p) = R^{(2)} \exp[ip \cdot X(z)]. \tag{2.3.1}$$

However, these set of local operators are not of conformal dimension (1,1) in general. This is because there arise operator mixings of operators with the same naive dimensions. First, we prescribe a normal ordering to cure the Weyl anomaly. Then, new

invariant operators are

$$V_T(X) = \kappa_T \exp[ip \cdot X(z)],$$

$$V_G^{\mu\nu}(X) = \kappa_G : \partial_z X^\mu \partial_{\bar{z}} X^\nu : \exp[ip \cdot X(z)],$$

$$V_D(X : p) = \kappa_D [ : \partial_z X^\mu \partial_{\bar{z}} X_\mu : - \frac{1}{16\pi T} R^{(2)} ] \exp[ip \cdot X(z)], \quad (2.3.2)$$

for tachyon, graviton, and physical dilaton vertex operators respectively. We can generalize this procedure to arbitrary higher mass levels [17]. We explicitly displayed the vertex operator coupling constants  $\kappa_A$  to emphasize that they differ for different particle states. Their renormalized coupling constants must satisfy the unitarity relation that we will derive below (cf. eq. (2.3.8)).

A normal ordered vertex operator is equivalent to a string wave function defined on the rim of a small hole which is then shrunk to a point to get a local operator perturbation [18]. To see this, consider a surface where a vertex operator on a puncture is replaced by a small disc of radius  $\delta$  centered at the puncture position. Then the amplitude computed by an insertion of the vertex operator is the same as the one found by doing the path integral over the disc whose boundary is  $S_1$ . That is, the wave function equals the path integral of the vertex operator over the disc

$$\Psi[X(z)] = \int \mathcal{D}X \cdot V_A(z : p) \exp[-S(X)]$$

with generic vertex operator of a form

$$V_A(z : p) = \mathcal{O}(\partial X; \bar{\partial} X) \exp[ip \cdot X](z). \quad (2.3.3)$$

We may evaluate the wave function by using a change of coordinates

$$X^\mu(z) = \tilde{X}^\mu(z) + \xi^\mu(z)$$

where  $X^\mu$  satisfies the classical equation of motion, while  $\xi^\mu$  is a fluctuation satisfying

the Dirichlet boundary conditions at  $\partial D_1 = S_1$ . Then,

$$\Psi[X(z)] = \int \mathcal{D}\xi \cdot \mathcal{O}(\partial X + \partial\xi) \exp(ip \cdot \xi - \frac{1}{2} \int d^2\xi \partial\xi \bar{\partial}\xi) \exp(ip \cdot X - \frac{1}{2} \int_{\partial D} dn X \partial_n X). \quad (2.3.4)$$

If we normal order the original vertex operators the integration is trivial. So we get

$$\Psi[X(z)] = \mathcal{O}(\partial X; \bar{\partial} X) \exp(ip \cdot X) \quad (2.3.5)$$

modulo a renormalization of the vertex operator coupling constant. Note that the surface term drops out in the limit  $\delta$  goes to zero. This shows an equivalence between normal ordered vertex operators and string wave functions defined on the rim around the punctured disc.

Next, we examine what kinds of string loop expansions and vertex operator normalizations are consistent with unitarity.

An N-point scattering amplitude is defined by

$$iS_N = (2\pi)^D \delta^{(D)}(p_1 + p_2 + \cdots + p_N) \cdot \mathcal{A}_N(p_1, \cdots, p_N) / \left[ \prod_{i=1}^N (2\pi)^{(D-1)} \cdot 2p_i^0 \right]^{\frac{1}{2}} \quad (2.3.6)$$

where the reduced amplitude  $\mathcal{A}_N(p_1, \cdots, p_N)$  is

$$\mathcal{A}_N = \sum_g \mathcal{A}_{g,N}(p_1, \cdots, p_N),$$

$$\mathcal{A}_{g,N} = C_g \int_{\mathcal{R}_g} \mathcal{D}g_{ab} \mathcal{D}X^\mu \exp[iS(g, X)] \cdot \int \prod_{i=1}^N d^2\sigma_i V(\sigma_i, p_i). \quad (2.3.7)$$

Here,  $g$  sums over all genera with different weights  $C_g$ , and a sum over different topology is made at each given genus  $g$ . The normalization of vertex operators and the weighting factor  $C_g$  must be consistent with the unitarity of the invariant amplitude.

So consider a generic  $g$  loop,  $N$  external state invariant amplitude  $\mathcal{A}_{g,N}$ . Factorization into  $M$ -particle discontinuity transforms this into a product of two invariant amplitudes integrated over all allowed phase space of intermediate states. Certainly, they are diagrams with  $N_1, g_1$  and  $N_2, g_2$  respectively. The diagrammatics tell us that  $N = (N_1 - M) + (N_2 - M)$  and  $g_1 + g_2 + (M - 1) = g$ . Rearranging this to eliminate  $M$ , we get

$$N + 2g - 2 = (N_1 + 2g_1 - 2) + (N_2 + 2g_2 - 2)$$

We note that  $2g - 2$  is essentially the factor counting the string loop expansion coupling constant, which is accounted for by the vacuum expectation value of the dilaton field  $\langle D \rangle = \frac{1}{2} \log \kappa$  in eq. (2.2.2) [16]. Here  $\kappa$  is the renormalized string loop coupling constant. We also find that correctly normalized vertex operators are needed to satisfy unitarity. Now, consider a one-particle unitarity cut. The invariant amplitude must obey:

$$\mathcal{A}(p_1, \dots, p_N) = \sum_I \mathcal{A}(p_1, \dots, p_L, -p_I) \cdot \frac{-i}{(p_I^2 + M_I^2)} \cdot \mathcal{A}(p_I, p_{L+1}, \dots, p_N). \quad (2.3.8)$$

Here, the intermediate sum is over all possible on-shell states with masses  $M_I$ .

In this section, we reformulate the above scattering amplitude in a manifestly BRST invariant way. At  $g$ -th order in the string perturbation expansion, the world-sheet has  $g$  handles, and  $N$  punctures associated with the vertex operator insertions. The Riemann surface of such type is denoted by  $\mathcal{M}_{g,N}$ .

We evaluated the partition function of closed bosonic string with an exact normalization in the previous section. Then the Polyakov prescription for an  $N$ -point,  $g$ -loop scattering amplitude  $\mathcal{A}_g(1, \dots, N)$  for a closed bosonic string is defined by the

average of a product of vertex operators [17]:

$$\begin{aligned}
\mathcal{A}_g(1, \dots, N) = & \int_{\mathcal{M}_{g,0}} \prod_{i=1}^{3g-3} dm_i \wedge d\bar{m}_i \int \prod_{A=1}^N dz_A \wedge d\bar{z}_A \\
& \cdot \left| \frac{\det(\mu_i | h_j)}{\det^{1/2}(h_i | h_j)} \right|^2 \cdot \left| (\det' \nabla_z^+ \nabla_{\bar{z}})^{1/2} \right|^2 \\
& \cdot \left| \frac{\det'(-g^{z\bar{z}} \partial_z \partial_{\bar{z}})}{\int dz \wedge d\bar{z}} \right|^{-D/2} \cdot \left\langle \prod_{B=1}^N V_B(z_B) \right\rangle_X
\end{aligned} \tag{2.3.9}$$

Here,  $\mathcal{M}_{g,0}$  denotes the moduli space of genus- $g$  compact Riemann surfaces  $\mathcal{R}_{g,0}$  with local complex coordinates  $(m_i, \bar{m}_i)$  at each slice  $(i = 1, \dots, 3g - 3)$ . In tangent and cotangent space, we have associated one forms: Beltrami differentials  $\mu_{i\bar{z}}$  and quadratic holomorphic differentials  $h_{izz}$  respectively. The inner product between them is defined by

$$(\mu_i | h_j) = \int dz \wedge d\bar{z} \mu_{i\bar{z}}^z h_{jzz} \tag{2.3.10}$$

and

$$(h_i | h_j) = \int dz \wedge d\bar{z} g^{z\bar{z}} h_{i\bar{z}\bar{z}} h_{jzz}. \tag{2.3.11}$$

The vertex operators  $V_A(z_A), A = 1, \dots, N$  are conformal dimension (1,1) highest weight states, and they are averaged over  $X^\mu$  coordinates in eq. (2.3.9).

A manifestly BRST invariant description of the scattering amplitude is formed by considering a genus  $g$ ,  $N$ -punctured Riemann surface  $\mathcal{R}_{g,N}$ , and summing over all its conformally inequivalent configurations. The resulting moduli space  $\mathcal{M}_{g,N}$  [18] has a complex dimension  $(3g + N - 3)$ . This formulation treats the vertex operator insertion points  $z_A$  on an equal footing with  $m_i$ 's, and this way, we will find all BRST anomalies can be traced from the same source: the boundary of moduli space  $\mathcal{M}_{g,N}$ .

Given the vertex operators  $V_A(z_A)$  of conformal dimension (1,1), the BRST in-

variant vertex operators [17] are

$$W^{A\bar{A}} \equiv c^{z_A} c^{\bar{z}_A} V_A(z_A), \quad A = 1, \dots, N, \quad (2.3.12)$$

in the sense that

$$[Q_{\text{BRST}}, W^{A\bar{A}}] = 0 \quad (2.3.13)$$

identically. Here

$$Q_{\text{BRST}} = \oint \frac{dw}{2\pi i} j_w^{\text{BRST}}, \quad (2.3.14)$$

$$j_w^{\text{BRST}} = c^w T_{ww}(X^\mu) + b_{ww} c^w \partial_w c^w + \frac{3}{2} \partial_w^2 c^w.$$

In  $\mathcal{M}_{g,N}$ ,  $W^{A\bar{A}}$ 's are modular (1,1) forms with respect to the moduli coordinates  $z_A$ , and  $N$ -point correlation functions are  $(N, N)$  modular forms. We can define a Beltrami differential  $\mu_{A\bar{z}}^z$  dual to  $dz_A$  which induces a shift of  $z_A$  by unity. Locally, we can always find a vector field  $v_A^z$  defining the tangent direction in  $\mathcal{M}_{g,N}$ . It is

$$v_A^{zB} = \delta_A^B, \quad (2.3.15)$$

and

$$\mu_{A\bar{z}}^z = \nabla_{\bar{z}} v_A^z, \quad A = 1, \dots, N. \quad (2.3.16)$$

These distributional Beltrami differentials  $\mu_{A\bar{z}}^z$  are unique up to a  $\bar{z}$ -derivative of some vector field vanishing at every puncture.

Cotangent space conjugate one-forms are defined by scalar products

$$(\mu_A | b) \equiv \int dz \wedge d\bar{z} \mu_{A\bar{z}}^z b_{zz} = \oint \frac{dz}{2\pi i} v_A^z b_{zz}, \quad (2.3.17)$$

and together with conjugate one-forms associated with moduli variables  $m_i$  and  $\bar{m}_i$

$$(\mu_i | b) \equiv \int dz \wedge d\bar{z} \mu_{i\bar{z}}^z b_{zz}, \quad (2.3.18)$$

the volume form of the moduli space  $\mathcal{M}_{g,N}$  is written as

$$d\mathcal{V}(\mathcal{M}_{g,N}) = \prod_{i=1}^{3g-3} dm_i \wedge d\bar{m}_i |(\mu_i|b)|^2 \prod_{A=1}^N dz_A \wedge d\bar{z}_A |(\mu_A|b)|^2. \quad (2.3.19)$$

Therefore, we can rewrite eq. (2.3.9) as an integration over moduli of  $\mathcal{M}_{g,N}$  with combined  $X^\mu$  and  $(b_{zz}, c^z)$  conformal field theory of central charge  $c = 0$ :

$$\begin{aligned} \mathcal{A}_{g \geq 2}(1, \dots, N) &= \int_{\mathcal{M}_{g,N}} \left\langle d\mathcal{V}(\mathcal{M}_{g,N}) \prod_{A=1}^N W^{A\bar{A}}(z_A) \right\rangle_{X,b,c} \\ &= \int \prod_{i=1}^{3g-3} dm_i \wedge d\bar{m}_i \prod_{A=1}^N dz_A \wedge d\bar{z}_A \\ &\quad \cdot \left\langle \prod_{j=1}^{3g-3} |(\mu_j|b)|^2 \prod_{B=1}^N |(\mu_B|b)|^2 W^{B\bar{B}}(z_B) \right\rangle_{X,b,c} \end{aligned} \quad (2.3.20)$$

Taking  $c$ -zero modes for  $g = 0, 1$  cases, the same amplitudes can be expressed for all  $g$  and  $N$ . The equivalence between eq. (2.3.9) and (2.3.16) also explains the proper  $(b_{zz}, c^z)$  ghost insertion rules with respect to each puncture generated by conformal dimension (1,1) operator (i.e., marginal operator) perturbations, which will be used in chapter 3.

There seems to remain a puzzle. From eq. (2.2.2), we find that a dilaton field condensate  $\langle D(X) \rangle = \frac{1}{2} \log \kappa$  generates a multiplicative factor  $\kappa^{-\chi_g}$  to the  $g$ -loop string path integral in eq. (2.2.4). String coupling constant is part of string dynamics. This result is also consistent with the unitarity relation that we discussed following eq. (2.3.7). However, in the BRST gauge fixed formulation, it appears that we do not get a relation between the string coupling constant and the dilaton condensate. The dilaton turned out to interact with the ghosts through a derivative coupling as in eq. (2.2.28) and the action does not say anything whether the dilaton is condensated or not. The resolution is in the definition of path integral of eq. (2.2.14) or equivalently of eq. (2.3.20). Due to the ghost number anomaly relation eq. (2.2.25), the path

integration measure in eq. (2.2.14) must take into account of insertions of  $b_{zz}$  and  $c^z$  zero modes. This is explicitly shown in eq. (2.3.20) through  $|(\mu_j|b)|^2$  insertions. We observe that dilaton condensate scales the ghost and the antighost fields according to

$$b_{zz} \rightarrow e^{-\frac{4}{3}\langle D(X)\rangle} b_{zz}$$

and

$$c^z \rightarrow e^{\frac{4}{3}\langle D(X)\rangle} c^z. \tag{2.3.21}$$

But this means that the path integrals of eq. (2.2.14) and eq. (2.3.20) are multiplied by  $e^{-2\chi_g \langle D(X)\rangle}$  and correctly reproduce the coupling constant multiplicative factor  $\kappa^{-\chi_g}$ .

## 2.4 Example-Calculation of the Dilaton Tadpoles

In previous sections, we showed that the path integral over the punctured Riemann surface  $\mathcal{R}_{g,N}$  corresponds to a  $N$  point on-shell  $g$ -loop scattering amplitude. In this section, we confirm this by an explicit calculation of simple scattering amplitudes already known. The Riemann surface we are considering is a disc with one puncture and a real projective plane with one puncture. We construct them from the annulus and the Möbius strip by taking a shrinking hole limit.

This calculation is important in another respect too. In string theory, solutions of the classical equations of motion are equivalent to conformally invariant sigma models. Therefore, Gell-Mann–Low beta functions [19] of string condensate fields vanish. Actually, one has to minimize the quantum effective potential, that is,

$$\sum_{genus} \langle V_A \rangle_g = 0 \tag{2.4.1}$$

for all vertex operators denoted by subscript  $A$ , and the sum is over string quantum loops. Equation (2.4.1) is on-shell only for massless particles, since momentum conservation requires the vertex operator to carry zero momentum. Tree level approximation generates the beta functions for massless fields. In particular, the dilaton one-point function determines the effective action for massless fields [20].

The calculation goes schematically as follows: we first prescribe the one-point amplitude on the annulus ( $C_2$ ) and Möbius strip ( $M_2$ ) topologies. In the limit that the small hole shrinks to a punctual state, we get a zero momentum tadpole of scalar particles. We extract the logarithmic divergence and its residue to get a massless state amplitude by the standard LSZ reduction formula. Finally, we work out the amplitudes with equivalent operator insertions on  $D_2$  or  $RP_2$  topologies. The ghost contribution terms may be crucial in extracting consistent loop corrected beta-functions of a string propagating in background fields [21].

The world-sheet topology of interest can be described by a rectangle  $A_2$  with a

metric

$$(ds)^2 = (d\sigma_1)^2 + \lambda^2(d\sigma_2)^2 = dz \cdot d\bar{z}, \quad (0 \leq \lambda \leq \infty), \quad (2.4.2)$$

$$\sigma_1, \sigma_2 \in [0, 1], \quad z = \sigma_1 + i\lambda\sigma_2.$$

$A_2$  can be mapped into the cylinder  $C_2$  or the Möbius strip  $M_2$  through a mapping

$$w = \exp[2\pi iz]. \quad (2.4.3)$$

As  $\lambda \rightarrow \infty$ ,  $C_2$  and  $M_2$  become the disc  $D_2$  and the real projective plane  $RP_2$  respectively with a puncture at the center. The inner circles of  $C_2$  and  $M_2$  are parametrized by  $t$ , and the off-shell state is described by  $X^\mu(t)$ .

The string one-point off-shell amplitude can be written as:

$$G(l) = \int \frac{d\Sigma(t)}{\mathcal{V}(\text{Diff.}S^1)} \int \frac{Dg_{ab} \cdot DX^\mu}{\mathcal{V}(\text{Diff.}) \otimes \mathcal{V}(W)} \cdot e^{-S[g;X]} \quad (2.4.4)$$

and

$$\begin{aligned} S[g; X] = & \frac{T}{2} \int_M d^2\sigma \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X^\mu \\ & + \frac{\ln g}{8\pi} \left[ \int_M d^2\sigma \sqrt{g} R + 2 \int_{\partial M} dt K \right] \\ & + \frac{D}{8\pi} \int_{\partial M} dt K \end{aligned}$$

where  $K$  denotes the extrinsic curvature on the boundary.

The second term signifies a string loop expansion via topology, and the last term is a Weyl anomaly counterterm. The punctual state we are interested in is described

by a constant map [22]:

$$X_0^\mu(t) = \bar{X}^\mu, \quad X^\mu = X_0^\mu + \delta X^\mu. \quad (2.4.5)$$

With this prescription, the amplitude (2.4.4) can be written, after metric gauge fixing,

$$G(l) = \frac{1}{2} \int d\lambda \frac{\mu_{W-P}}{\mathcal{V}_{\text{ol}}(\text{CKV})} [\det P^+ P]^{1/2} [\det \Delta]^{-D/2} \exp[-S(X_0^\mu)]. \quad (2.4.6)$$

$\mu_{W-P}$  denotes the Weil-Petersson measure. The factor  $\frac{1}{2}$  is included to take care of the orientation reversing discrete transformation

$$\sigma_1 \rightarrow 1 - \sigma_1, \quad \sigma_2 \rightarrow \sigma_2. \quad (2.4.7)$$

We now briefly describe the boundary conditions. The diffeomorphism vector fields  $\delta\xi^a(\sigma)$  satisfy [23]:

$$\begin{aligned} g_{ab} n^a \delta\xi^b \Big|_{\partial M} &= 0, \\ g_{ab} (n^c t^a + n^a t^c) \nabla_c \delta\xi^b \Big|_{\partial M} &= 0. \end{aligned} \quad (2.4.8)$$

The vectors  $n^a$  and  $t^a$  denote the inward normal vector and the tangent vector, respectively, on the boundary  $\partial M$ . Therefore, we have

$$\delta\xi^1(\sigma^1, \sigma^2) = \delta\xi^1(\sigma^1 + 1, \sigma^2)$$

$$\partial_2 \delta\xi^1(\sigma^1, \sigma^2 = 0) = 0 = \partial_2 \delta\xi^1(\sigma^1, \sigma^2 = 1)$$

and

$$\delta\xi^2(\sigma^1, \sigma^2) = \delta\xi^2(\sigma^2 + 1, \sigma^2)$$

$$\delta\xi^2(\sigma^1, \sigma^2 = 0) = 0 = \delta\xi^2(\sigma^1, \sigma^2 = 1). \quad (2.4.9)$$

The Weil-Petersson measure, and the Faddeev-Popov ghost determinant can be

calculated for the metric (2.4.2)

$$\mu_{W-P} = \sqrt{\frac{2}{\lambda}}, \quad V(\text{CKV}) = \sqrt{\lambda},$$

and

$$[\det P^+ P]^{1/2} = \sqrt{2}\lambda e^{-\pi\lambda/3} \prod_{n \geq 1} (1 - e^{-4\pi n\lambda})^2. \quad (2.4.10)$$

For  $X^\mu$  field, the boundary condition is such that

$$\delta X^\mu(\sigma^1, \sigma^2) = \delta X^\mu(\sigma^1 + 1, \sigma^2),$$

$$\partial_2 \delta X^\mu(\sigma^1, \sigma^2 = 0) = 0,$$

and

$$\delta X^\mu(\sigma^1, \sigma^2 = 1) = 0. \quad (2.4.11)$$

Therefore the scalar Laplacian determinant is

$$\begin{aligned} \det \Delta &\equiv \det(-g^{ab} \partial_a \partial_b) \\ &= \prod_{\substack{n_2 \geq 0 \\ n_1 = Z}} \left[ 4\pi^2 n_1^2 + \frac{\pi^2}{\lambda^2} (n_2 + \frac{1}{2})^2 \right] \end{aligned}$$

By Sommerfeld-Watson evaluation, we get

$$[\det \Delta]^{-D/2} = \left[ 2e^{-\pi\lambda/3} \prod_{n \geq 1} (1 + e^{-4\pi n\lambda})^2 \right]^{-D/2}. \quad (2.4.12)$$

Finally, the solution  $X_0^\mu$  to the boundary condition (2.4.5) and  $\partial_2 X_0^\mu(\sigma^1, \sigma^2 = 0) = 0$

is simply

$$X_0^\mu(\sigma^1, \sigma^2) = \bar{X}^\mu, \quad (2.4.13)$$

and  $s[X_0^\mu] = 0$ . Therefore, substituting equations (2.4.9), (2.4.11), and (2.4.12) into eq. (2.4.6),

$$\begin{aligned} G(l) &= 2^{-D/2} \int_0^\infty d\lambda e^{4\pi\lambda} \prod_{n \geq 1} \left[ (1 - e^{-4\pi n\lambda})^2 (1 + e^{-4\pi n\lambda})^{-D} \right] \\ &\equiv G(\bar{X}^\mu). \end{aligned} \quad (2.4.14)$$

The momentum space one-point string amplitude is defined by

$$i(2\pi)^D \delta^{(D)}(p) G(p) = \int d^D \bar{X}^\mu \exp[ip\bar{X}] G(\bar{X}^\mu). \quad (2.4.15)$$

So,

$$\begin{aligned} i\tilde{G}(p=0) &= 2^{-D/2} \int_0^\infty d\lambda e^{4\pi\lambda} \prod_{n \geq 1} \left[ \left( \frac{1 - e^{-4\pi n\lambda}}{1 + e^{-4\pi n\lambda}} \right)^2 (1 + e^{-4\pi n\lambda})^{-(D-2)} \right] \\ &= \frac{2T}{2^{D/2}} \int_0^\infty \frac{d\lambda}{2T} \sum_{n=0}^\infty d_B(n, D) \exp \left[ \frac{\lambda}{2T} (p^2 - (n-1)8\pi T) \right] \Big|_{p^2=0} \\ &= \frac{2T}{2^{D/2}} \sum_{n=0}^\infty \frac{d_B(n, D)}{p^2 - (n-1)8\pi T} \Big|_{p^2=0} \end{aligned} \quad (2.4.16)$$

and  $d_B(n, D)$  denotes the “bare” degeneracy of the  $n$ th mass level. In particular,

$$\begin{aligned} d(n=0, D) &= 1, & \text{off-shell tachyon,} \\ d(n=1, D) &= -(D-2) - 4, & \text{on-shell dilaton.} \end{aligned} \quad (2.4.17)$$

When the punctual state limit is taken, we need to renormalize punctures so that the propagator takes the canonical form  $-i(p^2 + M^2)^{-1}$ . This “wave-function” renormalization can be determined by considering the off-shell propagator [21] in the punctual

state limit. The appropriate coordinate boundary conditions are:

$$\begin{aligned} X_0^\mu(\sigma^1, \sigma^2 = 0) &= \bar{X}_i^\mu, \\ X_0^\mu(\sigma^1, \sigma^2 = 1) &= \bar{X}_f^\mu. \end{aligned} \quad (2.4.18)$$

and

$$\delta X^\mu(\sigma^1, \sigma^2 = 0, 1) = 0.$$

The classical action becomes

$$S(X_0) = \frac{T}{2\lambda} (\bar{X}_f^\mu - \bar{X}_i^\mu)^2. \quad (2.4.19)$$

With these modifications, the propagator is

$$G(\bar{X}_i^\mu, \bar{X}_f^\mu) = 2^{-D/2} \int_0^\infty \frac{d\lambda}{\lambda^{D/2}} e^{-4\pi\lambda} e^{-s(X_0)} \prod_{n \geq 1} [1 - e^{-4\pi n\lambda}]^{-(D-2)} \quad (2.4.20)$$

The momentum space propagator is

$$i(2\pi)^D \delta^{(D)}(p_1 + p_2) \tilde{G}f(p_1, p_2) = \int \int d^D \bar{X}_i d^D \bar{X}_f \exp[i(p_1 \bar{X}_i + p_2 \bar{X}_f)] G(\bar{X}_i, \bar{X}_f). \quad (2.4.21)$$

This gives

$$\begin{aligned} & i\tilde{G}(p_1 = -p_2 = p) \\ &= \left(\frac{\pi}{T}\right)^{D/2} \int_0^\infty d\lambda \exp\left[-\frac{\lambda}{2T}(p^2 - 8\pi T)\right] \prod_{n \geq 1} [1 - e^{-4\pi n\lambda}]^{-(D-2)} \\ &= \sum_{n=0}^\infty Z_n(D) [p^2 - (n-1)8\pi T]^{-1}. \end{aligned} \quad (2.4.22)$$

and ‘‘wave-function’’ renormalizations  $Z_n(D)$  of interest are

$$\begin{aligned} Z_{n=0}(D) &= 2T(\pi/T)^{D/2} && \text{off-shell tachyon,} \\ Z_{n=1}(D) &= 2T(D-2)(\pi/T)^{D/2} && \text{on-shell dilaton.} \end{aligned} \quad (2.4.23)$$

Therefore, eqs. (2.4.15), (2.4.16), and (2.4.22) enable us to extract the punctual

dilaton tadpole amplitude

$$\begin{aligned} \tilde{D}(p=0) &\equiv +ip^2 \tilde{G}(p^2) \Big|_{p^2=0} (Z_{n=1})^{-1/2} \text{Tr}(1)_{N \times N} \\ &= -4\sqrt{3}T^7 \pi^{-13/2} \left( \frac{N}{2^{D/2}} \right) \left( 1 + \frac{4}{D-2} \right) \end{aligned} \quad (2.4.24)$$

where we include the Chan-Paton factor [24] for the gauge group  $\text{SO}(N)$  on the outer boundary of  $C_2$ .

Next, we evaluate the same amplitude for the Möbius strip. In terms of the coordinate  $(\sigma^1, \sigma^2)$  of eq. (2.3.2), it is identified with

$$(\sigma_1, \sigma_2) \simeq (\sigma_1 + 1, -\sigma_2 + 1) \text{ mod } 2 \quad (2.4.25)$$

Therefore, the boundary conditions for the diffeomorphism vector are

$$\delta\xi^1(\sigma_1 + 1, -\sigma_2 + 1) = \delta\xi^1(\sigma_1, \sigma_2),$$

$$\partial_2 \delta\xi^1(\sigma_1, \sigma^2 = 0) = 0 = \partial_2 \delta\xi^1(\sigma^1, \sigma^2 = 1),$$

and

$$\delta\xi^2(\sigma_1 + 1, -\sigma_2 + 1) = -\delta\xi^2(\sigma_1, \sigma_2),$$

$$\delta\xi^2(\sigma_1, \sigma_2 = 0) = 0 = \delta\xi^2(\sigma_1, \sigma_2 = 1). \quad (2.4.26)$$

Similarly, the boundary condition for  $X^\mu(\sigma)$  are

$$X_0^\mu(\sigma^1, \sigma^2) = \bar{X}^\mu,$$

$$\delta X^\mu(\sigma_1 + 1, -\sigma_2 + 1) = \delta X^\mu(\sigma_1, \sigma_2),$$

and

$$\delta X^\mu(\sigma_1, \sigma_2 = 0) = 0 = \delta X^\mu(\sigma^1, \sigma^2 = 1). \quad (2.4.27)$$

The evaluation of the determinants is straightforward, and the one-point string

amplitude reads

$$G'(\bar{X}^\mu) = \frac{1}{4} \int_0^\infty d\lambda e^{\pi\lambda} \prod_{n \geq 1} \left[ (1 - (-)^n e^{-\pi n\lambda})^2 (1 + (-)^n e^{-\pi n\lambda})^{-D} \right]. \quad (2.4.28)$$

The factor  $\frac{1}{4}$  again comes from the order of the mapping class group of the Möbius strip in unoriented open string theory. Therefore,

$$\begin{aligned} i\tilde{G}'(p=0) &= \frac{1}{4} \int_0^\infty d\lambda e^{\pi\lambda} \prod_{n \geq 1} \left[ \left( \frac{1 - (-)^n e^{-\pi n\lambda}}{1 + (-)^n e^{-\pi n\lambda}} \right)^2 (1 + (-)^n e^{-\pi n\lambda})^{-(D-2)} \right] \\ &= 2T \int_0^\infty \frac{d\lambda}{8T} \sum_{n=0}^\infty d'_B(n, D) \exp \left[ -\frac{\lambda}{8T} (p^2 - (n-1)8\pi T) \right]_{p^2=0} \\ &= 2T \sum_{n=0}^\infty \frac{d'_B(n, D)}{p^2 - (n-1)8\pi T} \end{aligned} \quad (2.4.29)$$

and again  $d'_B(b, D)$  denotes the “bare” degeneracy of the  $n$ th mass level. Also,

$$\begin{aligned} d'(n=0, D) &= 1 && \text{off-shell tachyon,} \\ d'(n=1, D) &= +(D-2) + 4 && \text{on-shell dilaton.} \end{aligned} \quad (2.4.30)$$

Renormalizing the wave-function of the puncture the dilaton tadpole amplitude of  $RP_2$  is

$$\begin{aligned} \tilde{D}'(p=0) &= ip^2 \tilde{G}'(p^2) \Big|_{p^2=0} (Z_{n=1})^{-1/2} \\ &= +4\sqrt{3}T^7 \pi^{-13/2} \left( 1 + \frac{4}{D-2} \right). \end{aligned} \quad (2.4.31)$$

Equations (2.4.23) and (2.4.31) are the results of the dilaton tadpole amplitudes. The first term of each equation denotes the previously calculated dilaton vertex operator tadpoles [25]. These tadpoles cancel with each other when the gauge group is chosen to be  $SO(2^{D/2})$ . The off-shell zero momentum tachyon tadpoles also have been calculated [26], and the off-shell prescription automatically ensures that the off-shell tachyon tadpole is finite.

The second terms of eq. (2.4.23) and (2.4.31) however were not found in the previous calculations in terms of vertex operators. Of course, they cancel each other if the above gauge group is chosen. Similar terms have been considered previously within the context of a two loop vacuum amplitude factorization [27]. This term looks obscure within the context of tadpole calculation using vertex operators. We want to interpret them arising from some conformal dimension (1,1) operator at zero momentum. For this purpose, we consider the BRST invariant action of a string moving on background condensates, considered by Banks *et al.*, [28]

$$S = \frac{T}{2} \int d^2z [\partial X^\mu \bar{\partial} X^\nu G_{\mu\nu}(X) + \{2b_{zz} \bar{\partial}_z c^z + \frac{8}{3} \bar{\partial}_z (\Phi(X) b_{zz} c^z) + c.c.\}] \quad (2.4.32)$$

and for simplicity, we consider only graviton and dilaton condensates. The one-point amplitude of massless particles in conventional vertex operator language corresponds to the linear approximation of eq. (2.4.32) in a flat background spacetime. The first term of eq. (2.4.32) generates, in particular, the trace part of  $\delta G_{\mu\nu}(x)$ , which is proportional to

$$\int d^2z \eta_{\mu\nu} \partial_z X^\mu \bar{\partial}_z X^\nu e^{ik \cdot X}, \quad k^2 = 0. \quad (2.4.33)$$

Similarly, the ghost couplings generate the vertex operator proportional to

$$\int d^2z [\bar{\partial}_z (b_{zz} c^z e^{ik \cdot X}) + c.c.], \quad k^2 = 0. \quad (2.4.34)$$

The right combination to give the dilaton vertex operator was derived in the previous section (also derived in ref. [29]) which is

$$-\frac{2T}{\sqrt{D-2}} \kappa \int d^2z \left( : \partial_z X^\mu \bar{\partial}_z X_\mu e^{ik \cdot X} - \left[ \frac{1}{12\pi T} \bar{\partial}_z (b_{zz} c^z e^{ik \cdot X}) + c.c. \right] : \right). \quad (2.4.35)$$

At zero momentum this reduces to

$$-\frac{2T}{\sqrt{D-2}} \kappa \int d^2z \left( : \partial_z X^\mu \bar{\partial}_z X_\mu - \frac{1}{12\pi T} [\bar{\partial}_z (b_{zz} c^z) + c.c.] : \right). \quad (2.4.36)$$

Comparing this with eqs. (2.4.23) and (2.4.31), we see that the second terms of

these equations are generated by the insertion of

$$\begin{aligned} & A \frac{4}{D-2} \frac{1}{2\pi T} \int d^2 z \{ \bar{\partial} z (b_{zz} c^z + c.c.) \} \\ & = A \frac{4}{D-2} \frac{1}{2\pi T} \frac{6}{8} \left[ \int_{M_0} d^2 z \sqrt{g} R^{(2)} + 2 \int_{\partial M_0} dt K \right] \end{aligned} \quad (2.4.37)$$

where  $A$  denotes the numerical coefficient of the first terms in eqs. (2.4.23) and (2.4.31) and we used the ghost current anomaly equation of eq. (2.2.23) generalized to a manifold with a boundary:

$$\int \bar{\partial}_z (b_{zz} c^z) \equiv \int \bar{\partial}_z j_z^{ghost} = \int_{M_0} \frac{3}{8} \sqrt{g} R^{(2)} + \frac{3}{4} \int_{\partial M_0} K. \quad (2.4.38)$$

In eq. (2.4.37), the manifold  $M_0$  to integrate over is clearly  $D_2$  or  $RP_2$ , since we have already taken the singular puncture limit and replaced the puncture by an equivalent vertex operator insertion. Therefore, eq. (2.4.37) becomes

$$\frac{A}{4T} \chi_E(M_0) = \frac{A}{4T} \quad \text{for } M_0 = D_2 \text{ or } RP_2$$

(full treatment of ghost current contributions could be done explicitly by including ghosts in Polyakov path integration). This argument can be shown explicitly through a careful treatment of vertex operators in the factorization of a (generically higher genus Riemann surface) bosonic string amplitude [30].

Suppose we have an  $N$ -point amplitude on  $D_2$  or  $RP_2$ . When all  $N$  moduli are near the origin, the  $D_2$  or  $RP_2$  degenerate into an  $N+1$ -point amplitude on a sphere  $S_2$  whose one extra operator insertion comes from a tadpole on  $D_2$  or  $RP_2$ . This degeneration could be understood as the following procedure. Draw an imaginary closed path around which pinching occurs. We have two surfaces by this procedure:  $D_2 \oplus C_2$  for the original  $D_2$  amplitude, and  $D_2 \oplus M_2$  for the original  $RP_2$  amplitude.

All the  $N$  vertex operators are inside  $D_2$ , which is a result of the dissecting process. We give a Dirichlet boundary condition on the imaginary closed path, and integrate over all possible reparametrizations on the path to make the whole amplitude Weyl-invariant. As the pinching is performed, the original amplitudes factorize into the following form:

$$\begin{aligned} & \langle V_1 \dots V_N \rangle_{D_2 \text{ or } RP_2} \\ & \sim \sum_{\{n\}} \langle V_1 \dots V_N \mathcal{O}(n) \rangle_{S_2} \left[ \frac{-i}{p^2 + M_n^2} \right]_{p^2=0} \langle \check{\mathcal{O}}(n) \rangle_{D_2 \text{ or } RP_2} \end{aligned} \quad (2.4.39)$$

where  $\{n\}$  runs over all conformal dimension (1,1) operator sets. This contains, in particular, the operator set of eq. (2.4.36), which generates a logarithmic divergence due to the zero momentum propagator of these massless states. The Fischler-Susskind mechanism was shown to provide a consistent background solution by cancelling these infinities [31]. Obviously, the world-sheet curvature or, equivalently, a ghost current insertion contributes to the Gell-Mann–Low  $\beta$ -function of the dilaton.

At this point, it is worthwhile to mention why only physical states appear as we take an on-shell limit both in the two point propagator eq. (2.4.22), and the one point tadpoles eq. (2.4.16) or eq. (2.4.29). Actually, this is a general feature of the open Dirichlet boundaries [21]. Recall that our off-shell amplitudes are defined to be invariant under the local worldsheet symmetries. This essentially guarantees the decoupling of the unphysical states in the on-shell limit, since any anomalous contribution to the worldsheet symmetries cannot appear and the longitudinal degrees of freedom give, at most, surface terms which vanish as the boundary runs off to infinity. Alternatively, a semi-off-shell amplitude with open Dirichlet boundaries can be reduced into an on-shell scattering amplitude without the boundaries. It is easy to see that the amplitude agrees with the dual scattering amplitude evaluated with additional ‘physical state’ vertex operators, corresponding to each Dirichlet boundary. Either argument proves that unphysical states such as the trace of the graviton never appear in the on-shell limit of the Dirichlet open boundary. (This relationship between

the local worldsheet symmetries and the spacetime symmetry was exploited in the second paper of reference [21])

## References

1. See, for example, A. L. Fetter and J. D. Walecka, *Theoretical Mechanics of Particles and Continua*, Mc-Graw-Hill, New York (1980).
2. L.D. Faddeev, *Theo. Math. Phys.* **1,3** (1969).
3. L.D. Faddeev, *Lecture in Methods in Field Theory*, Les Houches Summer School, France (1975).
4. C. Itzykson and B. Zuber, *Quantum Field Theory*, Chapter 9, McGraw-Hill, New York (1980).
5. C. Beccei, A. Rouet and R. Stora, *Phys. Lett.* **B52**, 344 (1974): *ibid.* *Comm. Math. Phys.* **42**, 127 (1975): *ibid.* *Ann. Phys.* **98**, 287 (1976).
6. I.V. Tyutin, *Gauge invariance in field theory and in statistical physics in the operator formulation*, in Russian, unpublished (1975).
7. R. Feynman and A. Hibbs, *Quantum Mechanics and Path Integral*, McGraw-Hill, New York (1965).
8. Y. Nambu, *Duality and Hadrodynamics*, unpublished lecture at the Copenhagen Summer symposium (1970).
9. A. M. Polyakov, *Phys. Lett.* **B103** , 207 (1981).
10. O. Alvarez, *Nucl. Phys.* **B216** , 129 (1983):  
G. Moore and P. Nelson, *Nucl. Phys.* **B266**, 58 (1986):  
J. Polchinski, *Comm. Math. Phys.* **104**, 37 (1986).
11. D. Friedan, talk at *The Copenhagen Workshop on String Models*, unpublished (1981).
12. D. Friedan, E. Martinec, and S. Shenker, *Nucl. Phys.* **B271**, 93 (1986).  
M. Kato and K. Ogawa, *Nucl. Phys.* **B212**, 443 (1983).
13. O. Alvarez, unpublished work (1985): See also ref. 10 and ref. 12.

14. T. Banks, D. Nemeschansky, and A. Sen, Nucl. Phys. **B277**, 67 (1986).
15. C. Itzykson and B. Zuber, *Quantum Field Theory*, Chapter 5, McGraw-Hill, New York (1980).
16. C. G. Callan, D. Friedan, E. Martinec, and M. Perry, Nucl. Phys. **B262**, 593 (1985):
  - A. Sen, Phys. Rev. **D32** , 2102 (1985):
  - B. Fridling and A. Jevicki, Phys. Lett. **B174**, 75 (1986).
17. S. Weinberg, Phys. Lett. **B156**, 309 (1985):
  - I. Ichinose and B. Sakita, Phys. Lett. **B175**, 423 (1986):
  - E. D'Hoker and S. Giddings, Nucl. Phys. **B291**, 90 (1987).
18. H. M. Farkas and I. Kra, *Riemann Surfaces* , Springer-Verlag, New York (1980):
  - R. Penner, Bull. Am. Math. Soc., **15**, 73 (1986).
19. M. Gell-Mann and F. E. Low, Phys. Rev. **84**, 350 (1951): *ibid.* **95**, 1300 (1954).
20. C. Lovelace, Nucl. Phys. **B273**, 413 (1986).
21. S. R. Das and S.-J. Rey, unpublished work (1987):
  - C. G. Callan, C. Lovelace, C. Nappi, and S. Yost, Nucl. Phys. **B293** , 83 (1987).
22. A. Cohen, G. Moore, P. Nelson, and J. Polchinski, Nucl. Phys. **B267**, 143 (1987): *ibid.* **B281**, 127 (1987).
24. M. B. Green, Nucl. Phys. **B103**, 333 (1976): *ibid.* **B124**, 461 (1977).
23. S.-J. Rey, Phys. Lett. **B203**, 393 (1988): See also ref. 10.
24. H.-M. Chan and S.-T. Tsou, Phys. Lett. **B28**, 485 (1969):
  - C. J. Goebel and B. Sakita, Phys. Rev. Lett. **22** , 257 (1969):

- Z. Koba and H. B. Nielsen, Nucl. Phys. **B10** , 633 (1969), *ibid.* **B12** , 517 (1969).
25. B. Grinstein and M. B. Wise, Phys. Rev. **D35**, 655 (1987):  
M. Douglas and B. Grinstein, Phys. Lett. **B183**, 52 (1987).
26. S. Weinberg, Phys. Lett. **B187**, 278 (1987).
27. G. Moore, Phys. Lett. **B176**, 369 (1987).
28. T. Banks, D. Nemeschansky, and A. Sen, Nucl. Phys. **B277**, 67 (1986).
29. J. Polchinski, Nucl. Phys. **B289**, 465 (1987).
26. J. Polchinski, Texas preprint (1988) unpublished.
27. W. Fischler and L. Susskind, Phys. Lett. **B171**, 383 (1986); *ibid.* **B173**, 262 (1986):  
S. R. Das and S.-J. Rey, Phys. Lett. **B186**, 328 (1987).

### 3 BRST Anomalies in One Loop Scattering Amplitudes

#### 3.1 Degeneration of One Loop Scattering Amplitudes

As alluded to in the previous section, potential BRST anomalies can arise when the world sheet Riemann surface  $\mathcal{R}_{g,N}$  becomes degenerate. In this section, we evaluate the BRST anomalies explicitly by examining all possible degeneration configurations.

Consider, for this purpose,  $(N+1)$ -point scattering amplitudes involving  $N$  physical vertex operators and one unphysical state, represented by  $\{\text{Im}Q_{\text{BRST}}\}$ , which is supposed to decouple from a physical scattering process [1]. In this chapter, we examine the lowest order radiative correction, namely a torus  $T_2$  diagram:

$$\begin{aligned} \mathcal{A}_{g=1}^0(1, \dots, N+1) &= C_{(g=1)} \int d^2\tau \int \prod_{A=1}^N d^2z_A \langle |(\mu_\tau|b)|^2 \\ &\cdot \prod_{B=1}^N |(\mu_B|b)|^2 W^{B\bar{B}} \cdot W_{N+1}^0(z_{N+1}) \rangle_{T_2}. \end{aligned} \quad (3.1.1)$$

Here,  $C_{(g=1)}$  denotes a combinatoric factor that satisfies the factorization condition eq. (2.2.8). We choose the  $(N+1)$ -st vertex operator\* to be a representative of  $\{\text{Im}Q_{\text{BRST}}\}$ :

$$\begin{aligned} W_{N+1}^0(z_{N+1}) &= \{Q_{\text{BRST}}, \bar{c}V_{N+1}^0(z_{N+1})\} \\ &= \oint_{w=z_{N+1}} \frac{dw}{2\pi i} j_w^{\text{BRST}} \bar{c}V_{N+1}^0(z_{N+1}). \end{aligned} \quad (3.1.2)$$

This will be useful later once we choose  $V_{n+1}^0(z_{N+1})$  to be a physical vertex operator.

---

\* We take  $V_{N+1}$  other than the dilaton vertex operator, in which case we must consider the holomorphic and anti-holomorphic pieces together [2].

One may also consider a longitudinal state

$$\begin{aligned} W_{N+1}^0(z_{N+1}) &= \{\bar{Q}_{BRST}, cV_{N+1}^0(z_{N+1})\} \\ &= - \oint_{\bar{w}=z_{N+1}} \frac{d\bar{w}}{2\pi i} j_{\bar{w}}^{BRST} cV_{N+1}^0(z_{N+1}) \end{aligned}$$

but the following argument is exactly same in this case also. We will unwrap the BRST current contour around  $V_{N+1}$  and move it to other regions [3] using the worldsheet analytic property of the integrand in eq. (3.1.1). Since

$$[Q_{BRST}, W^{B\bar{B}}] = 0, \quad B = 1, \dots, N, \quad (3.1.3)$$

the only nontrivial term comes from [4]:

$$\{Q_{BRST}, (\mu_A|b)\} = (\mu_A|T_{zz}). \quad (3.1.4)$$

This gives a total derivative with respect to the modulus  $z_A$ , which is the source of the BRST anomaly. Consider the degeneration limits of the amplitude  $\mathcal{A}_{g=1}(1, \dots, N+1)$  of eq. (3.1.1). These configurations are conformally equivalent to the ones achieved by the following transformation<sup>\*</sup> of the complex  $z_A$  moduli coordinates

$$z_A - z_{N+1} = tw_A \quad (A \geq k+1), \quad (3.1.5)$$

and

$$w_{k+1} = 1$$

Actually, we need a more careful treatment of the coordinate transformation, since the worldsheet needs a specification of local coordinates around the pinching point. The correct method is to use a Riemann normal coordinate expansion between  $z_A$  and  $z_{N+1}$ . This often gives an extra piece to the conformal anomaly in factorizations [14]. Still, the factorizations that we discuss are free of this complication and we use the coordinate transformation eq. (3.1.5).

---

\* In paper [5], the Susskind-Fischler mechanism in open and closed bosonic string theory was studied using this transformation.

Under this coordinate transformation, the integration measure in eq. (3.1.1) is transformed into:

$$(d^2\tau \prod_{A=1}^N d^2z_A) = (d^2\tau \prod_{A=1}^k d^2z_A) \cdot d^2t |t|^{2N-2k-2} \cdot \left( \prod_{B=k+2}^N d^2w_B \right), \quad (3.1.6)$$

since this measure is an invariant density under the coordinate transformation of  $\mathcal{M}_{g,N}$ . Also, using (3.1.4),

$$\left\langle \{Q_{\text{BRST}}, (\mu_A|b)\} \hat{\mathcal{O}}(z_A) \right\rangle = \left\langle (\mu_A|T_{zz}) \hat{\mathcal{O}}(z_A) \right\rangle = \frac{\partial}{\partial z_A} \left\langle \hat{\mathcal{O}}(z_A) \right\rangle$$

where  $\hat{\mathcal{O}}(z_A)$  is a (0,1) modular form with respect to  $z_A$ . Therefore, the  $A = k + 1$  derivative term becomes a total derivative in  $t$  modular integration. It is

$$\int \left( d^2\tau \prod_{A=1}^k d^2z_A \right) \left( \prod_{B=k+2}^N d^2w_B \right) \int d^2t \frac{\partial}{\partial t} \left[ |t|^{2N-2k-2} \right] \cdot \left\langle |(\mu_\tau|b)|^2 (\bar{\mu}_{k+1}|\bar{b}) \cdot \prod_{D=k+2}^N |(\mu_D|b)|^2 \cdot \prod_{E=1}^N W^{E\bar{E}} \cdot \bar{c}V_{N+1}^0(z_{N+1}) \right\rangle_{T_2} \quad (3.1.8)$$

There are other terms involving total derivatives in the  $w_A$ . However, it is easy to show that these terms do not generate any nonvanishing contributions at  $t = 0$ . The boundary of  $\text{Im } \tau \rightarrow \infty$ , we will discuss at the end of this section.

It is appropriate to recapitulate a heuristic argument why the BRST anomaly appears in the degeneration limit as described above. In the degeneration configuration, we try to deform the BRST current around the  $(N+1)$ -th puncture. Deformation generates new closed BRST current contours around the other punctures at  $k+1, \dots, N$ , and a node denoted as  $P$ . Contours around punctures vanish due to eq. (3.1.3). However the closed contour at the node  $P$  is singular in the following sense: the contour must contract to an ill-defined ‘‘point loop’’ if we try to pull the contour into the other side past the other node denoted as  $Q$ . Only after we can accomplish this the

BRST invariance is guaranteed. The singularity mentioned above as a surface term is an obstruction to the stripping off of the BRST current, and hence generates a BRST anomaly. Equivalently, by representing the nodes  $P$  and  $Q$  as states in the string Hilbert space of BRST cohomology ( as will be described in the following paragraph), unphysical state propagation is signaled by nonvanishing action of  $Q_{BRST}$  on the nodes.

The correlation function in eq. (3.1.8) is written as

$$\left\langle |(\mu_\tau|b)|^2 \prod_{C=1}^k |(\mu(z_C)|b)|^2 W^{C\bar{C}}(\bar{\mu}(z_{N+1}+t)|\bar{b}) \cdot W^{k+1, \bar{k}+1} \right. \\ \left. \cdot \prod_{D=k+2}^N |(\mu(z_{N+1}+\tau w_D)|b)|^2 W^{D\bar{D}} \cdot \bar{c}V_{N+1}^0(z_{N+1}) \right\rangle_{T_2} \quad (3.1.9)$$

We may insert a complete set of highest weight states, i.e., primary fields  $\{\phi_I\}$  of conformal dimension  $(h_I, \bar{h}_I)$  propagating on the collar between two necks  $P$  and  $Q$  [7]. We describe the collar as a cylinder  $C = D_t - \{z_{N+1}\}$  where  $D_t$  denotes a disc of radius  $tw_I$  around the point  $z_{N+1}$ . The length of the cylinder is  $\log|t|^{-1}$  as  $t \rightarrow 0$ . Then, eq. (3.1.9) becomes

$$\sum_I (tw_I)^{h_I} (\bar{t}w_I)^{\bar{h}_I} \left\langle \tilde{\phi}_I(z_I) |(\mu_\tau|b)|^2 \prod_{A=1}^k |(\mu_A|b)|^2 W^{A\bar{A}} \right\rangle_{T_2} \\ \cdot \left\langle (\bar{\mu}_t|\bar{b}) W^{k+1, \bar{k}+1}(t) \prod_{B=k+2}^N |(\mu(tw_B)|b)|^2 W^{B\bar{B}}(tw_B) \cdot \bar{c}V_{N+1}^0(0) \phi_I(tw_I) \right\rangle_{S_2}$$

after a uniform shift of  $w_A$  coordinates on  $S_2$  using  $SL(2, C)$  invariance.

The intermediate states  $\tilde{\phi}_I$  may be written as [3]

$$\tilde{\phi}_I(z) = c^z \bar{c}^{\bar{z}} V_I(z), \quad (h_I = \bar{h}_I). \quad (3.1.11)$$

Using the fact that  $\mu_A$  is a conjugate one form with respect to  $z_A$ , we find that

$$(\mu(tw)|b) \simeq \left(\frac{1}{t}\right)(\mu(w)|b) \quad \text{as } t \rightarrow 0.$$

Therefore, we have

$$\begin{aligned} & \sum_I \bar{t}^{-1} |t|^{\Delta_I - 2(N-k-1)} |w_I|^{\Delta_I} \cdot \left\langle |(\mu_\tau|b)|^2 c \bar{c} V_I(z_I) \cdot \prod_{A=1}^k |(\mu_A|b)|^2 W^{A\bar{A}} \right\rangle_{T_2} \\ & \cdot \left\langle \bar{c} V_{N+1}^0(0) c \bar{c} V_{k+1}(1) c \partial c \bar{c} V_I(w_I) \cdot \prod_{B=k+2}^N |(\mu_B|b)|^2 W^{B\bar{B}} \right\rangle_{S_2} \end{aligned} \quad (3.1.12)$$

where

$$\Delta_I = (p_I^2 + m_I^2), \quad p_I = \sum_{i=1}^k p_i, \quad m_I = \text{mass of } V_I.$$

From eq. (3.1.8) and (3.1.11), the moduli space boundary term is

$$\sum_I \mathcal{A}_{g=1}(1, \dots, k; I) \cdot \mathcal{D}_I(p_I) \cdot \mathcal{A}_{g=0}^0(I; k+1, \dots, N+1) \quad (3.1.13)$$

with

$$\mathcal{D}_I(p_I) = \int_{\text{unit disc}} d^2 t \frac{\partial}{\partial t} \left( \frac{1}{t} |t|^{\Delta_I} \right) |\alpha|^{\Delta_I}, \quad (3.1.14)$$

$$\mathcal{A}_{g=1}(1, \dots, k; I) = \int d^2 \tau \prod_{A=1}^k d^2 z_A \left\langle |(\mu_\tau|b)|^2 \prod_{B=1}^k |(\mu_B|b)|^2 W^{B\bar{B}} \cdot c \bar{c} V_I(z_I) \right\rangle_{T_2} \quad (3.1.15)$$

and

$$\mathcal{A}_{g=0}^0(I; k+1, \dots, N+1) =$$

$$\int \prod_{A=k+2}^N d^2 w_A \left\langle \bar{c}V_{N+1}^0(0) \cdot c\bar{c}V_{k+1}(1) \cdot c\partial c\bar{c}V_I(w_I) \prod_{B=k+2}^N |(\mu_B|b)|^2 W^{B\bar{B}} \right\rangle_{S_2}. \quad (3.1.16)$$

In this expression,  $\{I\}$ -summation means the sum over degenerate states allowed by the selection rule.  $\mathcal{A}_{g=1}$  is the scattering amplitude of  $1, \dots, k$  partial vertex operators, and one intermediate state of momentum  $p_I = \sum_{i=1}^k p_i$ . All the information on the boundary behavior of the moduli space is summarized in  $\mathcal{D}_I(p_I)$  of eq. (3.1.14).

Also, we note a choice-dependent factor  $\alpha$  in  $\mathcal{D}_I(p_I)$ , which is traced back to the “off-shell” states of  $V_I$ ’s for general momentum  $p_I$ ’s. This has been observed in the two-loop vacuum amplitude factorization [8], and in the context of the dilaton tadpole factorization of open bosonic strings [9]. This arises because the conformal invariance is broken, and the amplitude involving “each” separate  $V_I$  insertion explicitly depends upon the choice of the  $w_I$  position on the world sheet. However, the *whole* expression (3.1.13) is a conformally invariant statement.

Now, eq. (3.1.14) is

$$|\alpha|^\Delta \int_{\substack{\text{unit} \\ \text{disc}}} d^2 t \frac{\partial}{\partial t} \left( \frac{1}{t} |t|^\Delta \right) \equiv \lim_{\epsilon \rightarrow 0} |\alpha|^\Delta \int_{|t| \leq \epsilon} d^2 t \frac{\partial}{\partial t} \left( \frac{1}{t} |t|^\Delta \right) = \lim_{\epsilon \rightarrow 0} \left[ -\frac{1}{2} |\alpha|^\Delta |\epsilon|^\Delta \right] \quad (3.1.17)$$

by using analyticity away from  $t = 0$  and the Stoke’s theorem. Therefore, for  $\Delta \leq 0$ , we have a nonzero  $\mathcal{D}_I(p_I)$  and BRST invariance is lost.

One must not regard the BRST anomalies as arising from a short distance physics in spacetime. String theory has a fundamental scale parameter, the string tension, and it is impossible to squeeze a string down below this size. The BRST anomalies we encountered come from worldsheet short distance physics. Actually short distance singularity of worldsheet corresponds to a larger and larger spacetime domain fluctuation so it is rather in a spacetime infrared singularity limit that BRST symmetries become anomalous.

### 3.2 Structure of BRST Anomalies

We now examine the  $k = 0, 1$ , and  $k \leq 2$  cases separately for an explicit calculation. The BRST anomalies have distinctive structure for each case.

The case  $k = 0$  corresponds to a one-loop tadpole configuration. Eq. (3.1.13) can be written, using eq. (3.1.17) and  $p_I = 0$ , as

$$\begin{aligned}
& -\frac{1}{2} \lim_{\epsilon \rightarrow 0} |\alpha\epsilon|^{-2} \int d^2\tau \langle |(\mu_\tau|b)|^2 c\bar{c}V_T(z_I) \rangle_{T_2} \\
& \cdot \int \prod_{A=2}^N d^2w_A \left\langle \bar{c}V_{N+1}^0(0) \cdot c\bar{c}V_1(1) \cdot c\partial c\bar{c}V_T(w_I) \prod_{B=2}^N |(\mu_B|b)|^2 W^{B\bar{B}} \right\rangle_{S_2} \\
& -\frac{1}{2} \int d^2\tau \langle |(\mu_\tau|b)|^2 c\bar{c}V_D(z_I) \rangle_{T_2} \\
& \cdot \int \prod_{A=2}^N d^2w_A \left\langle \bar{c}V_{N+1}^0(0) \cdot c\bar{c}V_1(1) \cdot c\partial c\bar{c}V_D(\infty) \prod_{B=2}^N |(\mu_B|b)|^2 W^{B\bar{B}} \right\rangle_{S_2} .
\end{aligned} \tag{3.2.1}$$

The first term is the anomaly due to the off-shell tachyon, and the second is the one due to an on-shell graviton and a dilaton. The tachyon anomaly is a pathology of generic bosonic strings, which we want to eliminate by considering supersymmetric string theories instead. The on-shell graviton/dilaton BRST anomaly is a physical anomaly, in the sense that it has a spacetime local interpretation, and even for supersymmetric strings it is known that this tadpole can arise in certain circumstances [10]. Therefore, we mainly concentrate on the second term in eq. (3.2.1).

Next,  $k = 1$  generates an on-shell two-point function. We assume no degeneracies in the particle spectrum. Then only one state contributes in  $I$ -summation in

eq. (3.1.13). It is

$$-\frac{1}{2} \int d^2\tau d^2z \langle |(\mu_\tau|b)|^2 |(\mu_z|b)|^2 W^{z\bar{z}} \cdot c\bar{c}V_I(z_I) \rangle_{T_2}$$

$$\cdot \int \prod_{A=3}^N d^2w_A \left\langle \bar{c}V_{N+1}^0(0) \cdot c\bar{c}V_2(1) \cdot c\partial c\bar{c}V_I(\infty) \prod_{B=3}^N |(\mu_B|b)|^2 W^{B\bar{B}} \right\rangle_{S_2}, \quad (3.2.2)$$

a BRST anomaly for each vertex operator  $W^{z_A\bar{z}_A}$ . By our assumption,  $c\bar{c}V_I(z_I) \equiv W^{I\bar{I}}(z = z_I)$ , and the first correlation of eq. (3.2.2) is just the two point function on a torus.

Finally, we look at  $k \geq 2$  configurations. From eq. (3.1.17) and a discussion thereafter, we see that the BRST anomalies arise for all intermediate states for which  $\Delta_I \leq 0$ . So, eq. (3.1.14) is

$$\lim_{\epsilon \rightarrow 0} \sum_{\Delta_I \leq 0} \left[ -\frac{1}{2} |\alpha\epsilon|^{\Delta_I} \right]$$

$$\cdot \int d^2\tau \prod_{A=1}^k d^2z_A \left\langle |(\mu_\tau|b)|^2 \prod_{B=1}^k |(\mu_B|b)|^2 W^{B\bar{B}} c\bar{c}V_I(z_I) \right\rangle_{T_2}$$

$$\cdot \int \prod_{A=k+2}^N d^2w_A \left\langle \bar{c}V_{N+1}^0(0) \cdot c\bar{c}V_{k+1}(1) \cdot c\partial c\bar{c}V_I(w_I) \prod_{B=k+2}^N |(\mu_B|b)|^2 W^{B\bar{B}} \right\rangle_{S_2}. \quad (3.2.3)$$

In general, a sufficiently timelike configuration of intermediate momentum  $p_I$  will generate a large number of divergent BRST anomalous terms. The usual argument for getting around this is to start with a sufficiently spacelike configuration of  $p_I$  so that there is no BRST anomaly at all (i.e.,  $p_I^2 \rightarrow \infty$ ), then analytically continue to other momentum configurations. However, we need to have a prescription to calculate scattering amplitudes directly at physical momentum without a *priori* assuming analytic continuation of external momentum [11].

There is also a boundary of moduli space where  $\text{Im}\tau \rightarrow \infty$  [12]. This configuration corresponds to the scattering amplitude on  $S_2$  with  $N$  vertex operators plus two intermediate state punctures, connected together by an appropriate propagator. This limit shows an infinity due to the existence of the tachyon in the spectrum of the bosonic string. Therefore, this divergence has a fundamentally different origin from the ones we discuss in this paper. Actually, we will find that there are no *local* counterterms, which may cancel out this type of anomaly. This problem is another indication that bosonic string theory is internally inconsistent; of course it does not arise in the superstring theories with consistent spacetime backgrounds.

## References

1. D. Friedan, E. Martinec, and S. Shenker, Nucl. Phys. **B271**, 93 (1986).
2. S.-J. Rey, unpublished work (1987):
  - H. Terao and S. Uehara, Phys. Lett. **B188**, 198 (1987):
  - S. Penati and M. Pernici, Phys. Lett. **B197**, 33, (1988).
3. E. Martinec, Phys. Lett. **B189**, (1986): See also ref. 1.
4. T. Eguchi and H. Ooguri, Nucl. Phys. **B284**, 335 (1987).
5. S. R. Das and S.-J. Rey, Phys. Lett. **B186**, 328 (1987).
6. J. Polchinski, Texas preprint (1988) unpublished.
7. D. Friedan, Nucl. Phys. **B 289**, 543 (1987).
8. G. Moore, Phys. Lett. **B176**, 369 (1986).
9. S. R. Das and S.-J. Rey, unpublished work (1987).
10. M. Dine, N. Seiberg, and E. Witten, Nucl. Phys. **B289**, 589 (1987):
  - M. Dine, I. Ichinose, and N. Seiberg, Nucl. Phys. **B293**, 253 (1987):
  - J. Atick, L. Dixon, and A. Sen, Nucl. Phys. **B292**, 109 (1987).
11. M. B. Green and N. Seiberg, Institute for Advanced Study preprint, IASSNS-HEP-87/38 (1987) unpublished.
12. S.-J. Rey, Unpublished work (1986).

## 4 One Loop Local Counterterms

### 4.1 General Strategy for Local Counterterms

Once BRST anomalies show up, we may look for local (both on the worldsheet and in spacetime) counterterms that can restore the BRST invariance of the scattering amplitudes. In this section, we explain the idea behind the introduction of the counterterms necessary for those BRST anomalies we found in the previous chapter.

The BRST invariance was required to decouple unphysical states in any unitarity cut of the string loop diagrams. What we have found is that the BRST anomalies arise whenever the worldsheet becomes degenerate at the boundary of moduli space. This boundary is the boundary of moduli space between different values of  $g$  and  $N$ :  $\mathcal{M}_{g,N}$  and  $\mathcal{M}_{g',N'}$ . This suggests that we can introduce local counterterms to the scattering amplitude defined on  $\mathcal{R}_{g,N}$  as some scattering amplitudes defined on the manifold  $\mathcal{R}_{g',N'}$ , thereby approaching their common boundary of moduli space from opposite sides. The unique feature of string theory is that these counterterms admit interesting physical interpretations from the point particle field theory viewpoint [1].

One now wants to relate this to the string infinities cancellation mechanism [2]. String theory exhibits a better ultraviolet divergence behavior than point particle field theories. The cause of this is duality or, equivalently, the modular invariance of the scattering amplitudes [3]. The string theory introduces a built-in cutoff in the ultraviolet limit of the radiative processes, thereby eliminating the need for infinite renormalization. There is another source of the divergences, the infrared divergence. This arises because there are exceptional momentum subspaces in the scattering kinematics. In point particle field theory, we can always work away from the exceptional momentum kinematics. This is also guaranteed by the analyticity of the scattering amplitudes. In string theory, one must prove the analyticity of the scattering amplitudes in order to follow the same argument as in the point particle field theory. The spacetime infrared divergence is associated with the short distance behavior on the string worldsheet as we discussed at the end of section 3.1. Therefore, the classifica-

tion of the potential (infrared) infinities in scattering amplitudes is directly related to the classifications of BRST anomalies [1].

It is also related to potential chiral anomalies in superstring theories [6]: gauge, gravitational, local Lorentz, and supersymmetry anomalies. A realistic superstring must yield chiral fermions with the nontrivial local symmetry charge assignments as we observe in the low energy world. On the other hand this charge assignments are strongly constrained by condition that chiral anomalies be absent in order to have gauge invariances intact on the quantum level. The chiral anomalies mean that unphysical longitudinal components of the gauge fields *do* propagate in the scattering amplitudes. However, we saw that the unphysical longitudinal states of gauge fields are elements of  $\text{Im}Q_{BRST}$  as discussed in section 2.1 and their propagation should again arise at the boundaries of moduli space by the same argument as the BRST anomalies. This can be also seen by observing that the longitudinal components of gauge fields may be written as

$$p_\mu V^\mu(z : p) = p_\mu \partial X^\mu \mathcal{O}(\partial X, \bar{\partial} X) \exp(ip \cdot X).$$

This is a total derivative in the puncture moduli, and we need to take care of its worldsheet short distance singularities. All these are not accidental at all. Actually, the BRST anomalies, the gauge anomalies, and the infinities arising to the string scattering amplitudes are all generated by the unphysical states  $\text{Im}Q_{BRST}$  in the indefinite metric Hilbert space. The short distance limit on the worldsheet is where the Hilbert space structure gets distorted.

## 4.2 Fischler-Susskind Mechanism as Tadpole Counterterm

Once BRST anomalies show up, we may look for local (both on the world sheet and in space-time) counterterms that can restore the BRST invariance of the scattering amplitudes. In this section, we present the counterterms necessary for those BRST anomalies we found in chapter 3. As we will see, the procedure of adding local counterterms is more natural in a space-time point of view.

First we look at  $k = 0$ , the one-loop tadpole case. The second term of eq. (3.2.1) is due to the exchange of the graviton and the dilaton. Rewriting eq. (3.2.1),

$$-\frac{1}{2}\mathcal{T}_1(p=0) \cdot \int \prod_{A=2}^N d^2 w_A \left\langle \bar{c}V_{N+1}(0) \cdot c\bar{c}V_1(1) \cdot c\partial c\bar{c}V_D(w=\infty) \prod_{B=2}^N |(\mu_B|b)|^2 W^{B\bar{B}} \right\rangle_{S_2} \quad (4.2.1)$$

denoting the one-loop tadpole as

$$\mathcal{T}_1(p=0) = \int d^2 \tau \langle |(\mu_\tau|b)|^2 c\bar{c}V_D(z_1) \rangle_{T_2}. \quad (4.2.2)$$

We note that this has the structure of a string *tree* level amplitude with one local operator insertion at  $w = \infty$  (after BRST current contour deformation). Therefore, it is natural to look for a counterterm on string *tree* level with an explicitly BRST anomalous local operator insertion. Since BRST invariance amounts to restricting all mass-level states on their classical ones satisfying the equations of motion, we consider general states away from classical solutions. It is sufficient to consider only massless states, i.e., a graviton condensate  $G_{\mu\nu}(X)$  and a dilaton condensate  $D(X)$ . The action for a string propagating in this background [7] is written as

$$S_b = \int d^2 z \left[ \frac{1}{2} \partial_z X^\mu \partial_{\bar{z}} X^\nu G_{\mu\nu}(X) + \frac{4}{3} \partial_{\bar{z}} (D(X) b_{zz} c^z) + c.c. \right]. \quad (4.2.3)$$

Now, a weak field perturbative expansion amounts to a systematic insertion of conformal dimension (1,1) (i.e., marginal perturbation) local operators on the Riemann surface  $\mathcal{R}_{g,N+1}$  to make an amplitude on  $\mathcal{R}_{g,N+2}$ . The Feynman rules for ghost insertion were given in section 2.3. Eq. (4.2.1) tells us that a single insertion of marginal

perturbation for the action of eq. (4.2.3) can serve as the local counterterm we are looking for. So, with that insertion, we consider an amplitude

$$\begin{aligned} & \delta\mathcal{A}_{g=0}^0(I; 1, \dots, N+1) \\ &= \kappa^{-2} \int \prod_{A=2}^N d^2w_A \left\langle \{Q_{\text{BRST}}, \bar{c}V_{N+1}(0)\} c\bar{c}V_1(1) c\bar{c} \delta\mathcal{L}(w = \infty) \prod_{B=2}^N |(\mu_B|b)|^2 W^{B\bar{B}}(w_B) \right\rangle_{S_2} \end{aligned} \quad (4.2.4)$$

where

$$\delta\mathcal{L}(z) = \frac{1}{2} \partial X^\mu \bar{\partial} X^\nu [G_{\mu\nu}(X) - \eta_{\mu\nu}] + \frac{4}{3} \partial_{\bar{z}} [D(X) b_{zz} c^z] + c.c. \quad (4.2.5)$$

After the BRST contour deformation, this is written as

$$\begin{aligned} & \delta\mathcal{A}_{g=0}^0(I; 1, \dots, N+1) \\ &= \kappa^{-2} \int \prod_{A=2}^N d^2w_A \left\langle \bar{c}V_{N+1}(0) \cdot c\bar{c}V_1(1) \cdot [Q_{\text{BRST}}, c\bar{c} \delta\mathcal{L}(w = \infty)] \prod_{B=2}^N |(\mu_B|b)|^2 W^{B\bar{B}} \right\rangle_{S_2} \end{aligned} \quad (4.2.6)$$

Calculating the BRST charge commutator, we find

$$\begin{aligned} [Q_{\text{BRST}}, c\bar{c} \delta\mathcal{L}(z)] &= \oint_{w=z} [c^w \partial_w X^\mu \partial_w X_\mu(w) + b_{ww} c^w \partial_w c^w + \frac{3}{2} \partial_w^2 c^w] \\ &\cdot c^z \bar{c}^{\bar{z}} \left[ -\frac{1}{2} R_{\mu\alpha\nu\beta} \partial_z X^\mu \partial_{\bar{z}} X^\nu X^\alpha X^\beta + \frac{4}{3} \nabla_\mu \nabla_\nu \Phi X^\mu X^\nu \partial_{\bar{z}} (b_{zz} c^z) + c.c. + \dots \right] \\ &= c^z \partial_z c^z \bar{c}^{\bar{z}} \left[ -\frac{1}{2} \beta_{\mu\nu}^G \partial_z X^\mu \partial_{\bar{z}} X^\nu + \beta^\Phi \cdot \left( \frac{4}{3} \partial_{\bar{z}} (b_{zz} c^z) + c.c. \right) \right] + \dots \end{aligned} \quad (4.2.7)$$

with

$$\beta_{\mu\nu}^G = R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi + \dots,$$

and

$$\beta^\Phi = -\nabla^2\Phi - \frac{1}{4}R + \dots \quad (4.2.8)$$

Therefore,

$$\begin{aligned} & \delta\mathcal{A}_{g=0}^0(I; 1, \dots, N+1) \\ &= \kappa^{-2} \int \prod_{A=2}^N d^2w_A \langle \bar{c}V_{N+1}(0) c\bar{c}V_1(1) c\partial\bar{c}\{ -\frac{1}{2}\beta_{\mu\nu}^G \partial X^\mu \bar{\partial} X^\nu + \beta^\Phi (\frac{4}{3}\bar{\partial}(b_{zz}c^z) + c.c.) \}(\infty) \\ & \quad \cdot \prod_{B=2}^N |(\mu_B|b)|^2 W^{B\bar{B}} \rangle_{S_2}, \end{aligned} \quad (4.2.9)$$

and for the marginal operators

$$\partial X^\mu \bar{\partial} X^\nu \quad \text{and} \quad \bar{\partial}(b_{zz}c^z), \quad (4.2.10)$$

we have local counterterms with appropriate choices of  $\beta_{\mu\nu}^G$  and  $\beta^\Phi$ . However, with the ghost current anomaly equation (cf. eq. (2.2.23)):

$$\partial_{\bar{z}} \langle b_{zz}c^z \rangle = \frac{3}{8}\sqrt{g}R^{(2)}(z), \quad (4.2.11)$$

we find that the torus has no tadpole associated with the insertion of  $\partial_{\bar{z}}(b_{zz}c^z)$  or its complex conjugate operator. Therefore, we get

$$\begin{aligned} & \frac{1}{2}\kappa^{-2}\beta_{\mu\nu}^G \partial X^\mu \bar{\partial} X^\nu \\ &= -\frac{1}{2} \int d^2\tau \langle |(\mu_\tau|b)|^2 c\bar{c} \partial X^\mu \bar{\partial} X_\mu \rangle_{T_2} \cdot \partial_z X^\mu \partial_{\bar{z}} X^\nu \eta_{\mu\nu} \\ &= -\frac{1}{2} \mathcal{I}_1(\partial X^\mu \bar{\partial} X_\mu) \cdot \partial_z X^\mu \partial_{\bar{z}} X^\nu \eta_{\mu\nu} \end{aligned}$$

and

$$\beta^\Phi \cdot \frac{4}{3} \partial_{\bar{z}}(b_{zz}c^z) = 0, \quad (4.2.12)$$

in the lowest order of the string loop perturbation expansion. Therefore setting

$$\begin{aligned} \beta_{\mu\nu}^G &= R_{\mu\nu} + 2\nabla_\mu \nabla_\nu D(X) + \dots \\ &= \kappa^2 \mathcal{T}_1(X^\mu \partial X_\mu) G_{\mu\nu}(X) \end{aligned}$$

and

$$\beta^\Phi = 0, \quad (4.2.13)$$

we cancel a tadpole BRST anomaly in the lowest order of string loop and background expansions. This is the same counterterm as the one that Fischler-Susskind [8] introduced to cancel an infinity due to non-vanishing one loop dilaton tadpole. As we discussed in section 4.1, we have anticipated that same local counterterm is used to cancel both the finite tadpole BRST anomaly and the infinity due to dilaton tadpole.

### 4.3 Mass Renormalization as Mass-Shell Counterterm

The case  $k = 1$  is an on-shell two-point function. The BRST anomaly is

$$-\frac{1}{2}\Pi_1 \int \prod_{A=3}^N d^2 w_A \left\langle \bar{c}V_{N+1}^0(0) \cdot c\bar{c}V_2(1) \cdot c\partial c\bar{c}V_1(\infty) \prod_{B=3}^N |(\mu_B|b)|^2 W^{B\bar{B}} \right\rangle_{S_2}$$

where

$$\Pi_1 = \int d^2 \tau d^2 z \left\langle |(\mu_\tau|b)|^2 |(\mu_z|b)|^2 c\bar{c}V_1(z) c\bar{c}\tilde{V}_J(z_J) \right\rangle_{T_2}. \quad (4.3.1)$$

The correlation function on  $S_2$  is exactly the same amplitude with the  $N + 1$  vertex operators of the original set we had on the torus, eq. (3.2.1). Therefore, the only source of local counterterms would be a modification of the vertex operator  $V_1(z)$  in an explicitly BRST anomalous way. A general vertex operator is of the form

$$V(z; p) = D_N e^{ip \cdot X} \quad (4.3.2)$$

where  $D_N$  is a function of covariant derivatives on  $X^\mu$ , apart from polarization tensor, momentum  $p$ , and coupling constant dependences. The subscript  $N$  denotes the conformal dimension  $(N, N)$  of  $D_N$ , where  $N$  is the total number of covariant derivatives  $\nabla_z$ . Therefore,  $V(z; p)$  has a conformal dimension  $(N + p^2/2, N + p^2/2)$ .

$$T_{zz}V(w) = \left[ N + \frac{p^2}{2} \right] (z - w)^{-2}V(w) + (z - w)^{-1}\partial_w V(w) + \text{regular terms}. \quad (4.3.3)$$

The vertex operator  $W^{z\bar{z}} = c^z \bar{c}^{\bar{z}} V(z)$  is BRST invariant only if

$$N + \frac{p^2}{2} = 1, \quad (4.3.4)$$

i.e., if

$$[Q_{\text{BRST}}, W^{z\bar{z}}(p)] = \left( N + \frac{p^2}{2} - 1 \right) c^z \partial_z c^{\bar{z}} \bar{c}^{\bar{z}} V(z; p) = 0. \quad (4.3.5)$$

Therefore, the counterterm amounts to a shift of the mass-shell condition of eq. (4.3.4),

and we consider the local counterterm amplitude

$$\begin{aligned}
& \delta \mathcal{A}_{g=0}^0(1, \dots, N+1) \\
&= \kappa^{-2} \int \prod_{A=3}^N d^2 w_A \left\langle [Q_{\text{BRST}}, \bar{c}V_{N+1}(0)] \cdot c\bar{c}V_2(1) \cdot c\bar{c}V_1(\infty, p_1 + \delta p_1) \prod_{B=3}^N |(\mu_B|b)|^2 W^{B\bar{B}} \right\rangle_{S_2} \\
&= -\kappa^{-2} \left( \frac{\Delta p_1^2}{2} \right) \int \prod_{A=3}^N d^2 w_A \\
&\quad \cdot \left\langle \bar{c}V_{N+1}(0) \cdot c\bar{c}V_2(1) \cdot c\partial c\bar{c}V_1(\infty) \prod_{B=3}^N |(\mu_B|b)|^2 W^{B\bar{B}} \right\rangle_{S_2}
\end{aligned}$$

with

$$\Delta p_1^2 = (p_1 + \delta p_1)^2 - p_1^2. \quad (4.3.6)$$

Therefore, the BRST anomaly in eq. (3.3.1) can be cancelled by choosing  $\Delta p^2$  in eq. (4.3.6) by

$$\Delta p_1^2 = \kappa^2 \cdot \Pi_1(p_1^2 = -M_1^2), \quad (4.3.7)$$

and the shift in mass spectrum is given by two-point amplitudes [9]. In case of massless gauge particles, we do not expect any mass-shell shift since their gauge invariances protect them from getting any masses.

Analogous to the dual model of hadrons, the mass-shell renormalization must become smaller for higher massive level states. This follows from a string version of correspondence principle\*. Even in superstring theories we expect similar finite mass-shell renormalizations. This also suggests that by considering scattering amplitudes involving massless state vertex operators only we can avoid complications due to mass-shell renormalizations. From these massless state scattering amplitudes, massive state scattering amplitudes can be obtained by factorization in appropriate channels.

---

\* I thank the late R.P. Feynman for this argument.

#### 4.4 Contact Interaction as Intermediate Cut Counterterm

Finally, if  $k \geq 2$ , we have an anomaly due to the intermediate state singularity. It is

$$-\frac{1}{2} \sum_{\Delta_I < 0} \lim_{\epsilon \rightarrow 0} |\alpha|^{\Delta_I} |\epsilon|^{\Delta_I} \cdot \mathcal{A}_{g=1}(1, \dots, k; I) \cdot \mathcal{A}_{g=0}^0(I; k+1, \dots, N+1) \quad (4.4.1)$$

where  $\mathcal{A}_{g=1}$  and  $\mathcal{A}_{g=0}^0$  are given by eqs. (3.1.15) and (3.1.16) respectively. Note that  $\Delta_I$  in eq. (3.4.1) is defined by

$$\Delta_I = \left( \sum_{i=1}^k p_i \right)^2 + M_I^2 \quad (4.4.2)$$

and therefore depends upon the intermediate momentum configuration. One should recall that  $\mathcal{A}_{g=1}$  and  $\mathcal{A}_{g=0}^0$  are  $(k+1)$  and  $(N-k+1)$  point amplitudes respectively with one vertex operator (corresponding to the intermediate exchange state) *off-shell* in general. The local counterterm one must add is a contact interaction [10] amplitude:

$$\begin{aligned} & \delta_I \mathcal{A}_{g=0}^0(I, k+1, \dots, N+1) \\ &= \kappa^{-2} \int \prod_{A=k+2}^N d^2 w_A \langle \{ Q_{BRST}, \bar{c} V_{N+1}(0) \} c \bar{c} V_{k+1}(1) c \bar{c} V_I^c(w_I) \prod_{B=k+2}^N |(\mu_B | b)|^2 W^{B\bar{B}} \rangle_{S_2} \end{aligned} \quad (4.4.3)$$

where

$$V_I^c(w_I; p) = \left[ \frac{\kappa^2}{2} \int_{|s| \leq |\alpha \epsilon|} d^2 s \delta^{(2)}(s) |\alpha s|^{\Delta_I} \right] V_I(w_I; p). \quad (4.4.4)$$

Therefore, the complete counterterm is

$$\frac{1}{2} \sum_{\Delta_I} \delta_I \mathcal{A}_{g=0}^0 \cdot \mathcal{A}_{g=1} = \sum_{\Delta_I < 0} \lim_{\epsilon \rightarrow 0} |\alpha \epsilon|^{\Delta_I} \mathcal{A}_g(1, \dots, k; I)$$

$$\int \prod_{A=k+2}^N d^2 w_A \left\langle \bar{c}V_{N+1}(0) \cdot c\bar{c}V_{k+1}(1) \cdot c\partial c\bar{c}V_I(w_I; p) \prod_{B=k+2}^N |(\mu_B|b)|^2 W^{B\bar{B}} \right\rangle_{S_2}. \quad (4.4.5)$$

This cancels the BRST anomalies arising in eq. (4.4.1) to guarantee (perturbative) unitarity, and analyticity over all intermediate momentum configurations. The singular limit of eq. (4.4.4) is characteristic of the contact interaction, meaning a pointlike interaction in the boundary of moduli space. The contact interactions can be understood in terms of the principal value prescription of the worldsheet short distance singularity by subtracting out the delta function part:

$$(z - w)^{-1} = P.P.(z - w)^{-1} + i\pi\delta^{(2)}(z - w).$$

In terms of BRST formalism, the propagation of  $\text{Im}Q_{BRST}$  states is cancelled by that of  $(\text{Ker } Q_{BRST})^\perp$  states in the dual Hilbert space of ghost number [11]. This is very analogous to the point particle field theory cases. For example, in quantum electrodynamics, the Ward identity is satisfied only after including the contribution of the seagull term [12].

The reason why contact interactions arise is because particle spectrum of string excitations has its lower bound at the tachyon state. Once energy transfer across the degeneration point is below the tachyon mass threshold we cannot create any particles and must have a regular behavior of the string scattering amplitudes.

A brief digression is in order. A Ward identity could be derived also within the fully second quantized string field theory [13]. However, the derivation is with an assumption of no contact interaction contribution. Following the above description in the first quantized language, one must expect a similar contribution in the string field theoretic derivations also. Actually, in superstring field theory, both in light-cone gauge [10] and in covariant gauge [14], it was pointed out that the contact interactions are important to guarantee unitarity.

## References

1. S.-J. Rey, Caltech preprint, CALT-68-1468 (1988) To be published in Nucl. Phys. **B** .
2. M. B. Green and J. H. Schwarz, Phys. Lett. **B151** , 21 (1985):  
C. Burgess and T. Morris, Nucl. Phys. **B291** , 285 (1987):  
M. Douglas and B. Grinstein, Phys. Lett. **B183** , 52 (1987).
3. M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory*, vol. 2, Cambridge University Press, Cambridge (1987).
4. C. Lovelace, Nucl. Phys. **B273** , 413 (1986).
5. D. Gross, in *Unified String Theory* , ed. by M. B. Green and D. Gross, world Scientific, Singapore (1986).
6. M. B. Green and J. H. Schwarz, Phys. Lett. **B149** , 117 (1984): Nucl. Phys. **B255** , 93 (1985).
7. T. Banks, D. Nemeschansky, and A. Sen, Nucl. Phys. **B277** , 67 (1986).
8. W. Fischler and L. Susskind, Phys. Lett. **B171** , 383 (1986): *ibid.* **B173** , 262 (1986):  
S. R. Das and S.-J. Rey, Phys. Lett. **B186** , 328 (1987):  
C. G. Callan, C. Lovelace, C. Nappi, and S. Yost, Nucl. Phys. **B288** , 575 (1987).
9. S. Weinberg, talk at *American Physical Society Meeting* , Eugene, Oregon (1985):  
N. Seiberg, Phys. Lett. **B187** , 56 (1987):  
A. Sen, SLAC preprint SLAC-PUB-4383 (1987).
10. M. B. Green and N. Seiberg, Institute for Advanced Study preprint, IASSNS-HEP-87/38 (1987) Unpublished.

11. I. Frenkel, H. Garland, and G. Zuckerman, Proc. Natl. Acad. Sci. USA, **83** , 8442 (1986):  
M. D. Freeman and D. Olive, Phys. Lett. **B175** , 151 (1986).
12. See, for example, C. Itzykson and B. Zuber, *Quantum Field Theory* , McGraw-Hill, New York (1980).
13. S. P. de Alwis, M. Grisaru, and L. Mezincescu, Phys. Lett. **B197** , 96 (1987).
14. C. Wendt, SLAC preprint SLAC-PUB-4442 (1987) Unpublished.

## 5 Analysis of Higher Loop Scattering Amplitudes

### 5.1 Two Loop Scattering Amplitudes

The next step is to extend the previous analysis to higher orders in the radiative correction. It becomes vastly complicated, however, since we must deal with many possible sources of BRST anomalies together. For a consistency check it is necessary to do a calculation explicitly, at least to the next to leading order in string loop expansion. Fortunately, the degeneration process is a *local* statement on the worldsheet, and this essentially enables us to analyze BRST anomalies order by order in the string loop expansion.

The same spirit as the one loop order of sections 3 and 4 applies straightforwardly to multiloop diagrams. However, we have to identify all the overlapping divergences and the one-particle reducible diagrams beforehand. This requires an examination in the higher co-dimension factorization of scattering amplitudes around the boundary of moduli space  $\mathcal{M}_{g,N}$  for the punctured Riemann surfaces  $\mathcal{R}_{g,N}$ .

‘One particle reducible’ diagrams are those which contains string subdiagrams connected by a single particle state propagator. Analogously to ordinary field theory, one particle reducible diagrams are not counted for higher order corrections. This is ensured by the lower order local counterterms, and their various combinations.

This section is devoted to demonstrating the above assertion through an explicit calculation at two-loop order.

Consider a  $N$ -punctured Riemann surface  $\mathcal{R}_{2,N}$  of genus two. This defines  $N$ -point scattering amplitudes of two loop level:

$$\begin{aligned} & \mathcal{A}_{g=2}(1, \dots, N+1) \\ &= \kappa^2 \int_{\mathcal{F}_2^0} \prod_{i=1}^3 d^2 \tau_i \prod_{A=1}^{N+1} d^2 z_A \left\langle \prod_{i=1}^3 |(\mu_{\tau_i}|b)|^2 \prod_{B=1}^N |(\mu_B|b)|^2 W^{B\bar{B}} \right\rangle_{g=2} \end{aligned} \quad (5.1.1)$$

Here,  $\tau_i$ ,  $i = 1, 2, 3$  denotes the moduli variables of an unpunctured  $g = 2$  Riemann

surface  $\mathcal{R}_{2,0}$  related to the Siegel period matrix [1]:

$$\Omega = \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{pmatrix}, \quad (5.1.2)$$

and  $\mathcal{F}_2^0$  denotes the fundamental domain of the moduli space, which we may choose as [2]

$$\begin{aligned} (1) \quad & |\operatorname{Re}\tau_i| \leq \frac{1}{2}, \quad i = 1, 2, 3, \\ (2) \quad & 0 \leq |2\operatorname{Im}\tau_3| \leq \operatorname{Im}\tau_1 \leq \operatorname{Im}\tau_2 \\ (3) \quad & \text{For } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \operatorname{Sp}(4, \mathcal{Z}), \quad |\det(C\Omega + D)| \geq 1. \end{aligned} \quad (5.1.3)$$

However, we do not need any specific information other than this.

We again look at  $(N + 1)$  point amplitude involving a null state

$$W_0(z_{N+1}) = \{Q_{\text{BRST}}, \bar{c}V^0(z_{N+1})\} :$$

$$\mathcal{A}_{g=2}^0(1, \dots, N + 1)$$

$$= \kappa^2 \int_{\mathcal{F}_2^0} \prod_{i=1}^3 d^2\tau_i \prod_{A=1}^{N+1} d^2z_A \left\langle \prod_{i=1}^3 |(\mu_{\tau_i}|b)|^2 \prod_{B=1}^{N+1} |(\mu_B|b)|^2 W^{B\bar{B}} \right\rangle_{g=2}$$

$$= \kappa^2 \int d^2z_{N+1} \prod_{A=1}^N d^2z_A \prod_{i=1}^3 d^2\tau_i$$

$$\left\langle \prod_{i=1}^3 |(\mu_{\tau_i}|b)|^2 \prod_{B=1}^N |(\mu_B|b)|^2 W^{B\bar{B}} \cdot |(\mu_{N+1}|b)|^2 \{Q_{\text{BRST}}, \bar{c}V^0(z_{N+1})\} \right\rangle_{g=2} \quad (5.1.4)$$

One configuration with BRST anomalies is when all punctures approach one another,

described by the  $t \simeq 0$  boundary of

$$\left\{ \begin{array}{l} z_A = z_{N+1} + tw_A \quad (k+1 \leq A \leq N), \\ w_{k+1} = 1 \end{array} \right\} \quad (5.1.5)$$

Deforming the BRST current contour,

$$\begin{aligned} & \mathcal{A}_{g=2}^0(1, \dots, N+1) \\ &= \kappa^2 \int d^2t \frac{\partial}{\partial t} |t|^{2N-2k-2} \int \prod_{A=1}^k d^2z_A d^2z_{N+1} \int \prod_{B=k+1}^N d^2w_B \int \prod_{i=1}^3 d^2\tau_i \\ & \quad \left\langle \prod_{i=1}^3 |(\mu_{\tau_i}|b)|^2 \prod_{C=1}^k |(\mu_C|b)|^2 W^{c\bar{c}}(\bar{\mu}_{k+1}|\bar{b}) W^{k+1\overline{k+1}} \cdot \right. \\ & \quad \left. \prod_{D=k+2}^N |(\mu_D|b)|^2 W^{D\bar{D}} \cdot |(\mu_{N+1}|b)|^2 \bar{c}V_0(N+1) \right\rangle_{g=2} \end{aligned} \quad (5.1.6)$$

Inserting a complete basis set, we find that the correlation function becomes

$$\begin{aligned} & \sum_I \left\langle \prod_{i=1}^3 |(\mu_{\tau_i}|b)|^2 \prod_{C=1}^k |(\mu_C|b)|^2 W^{C\bar{C}} |\Phi_I\rangle \right\rangle_{g=2} \\ & \quad \left\langle \check{\Phi}_I | (\bar{\mu}_{k+1}|\bar{b}) W^{k+1\overline{k+1}} \prod_{D=k+2}^N |(\mu_D|b)|^2 W^{D\bar{D}} \cdot |(\mu_{N+1}|b)|^2 \bar{c}V_0(z_{N+1}) \right\rangle_{S_2} \end{aligned} \quad (5.1.7)$$

and we let

$$|\Phi_I\rangle = W^{II}(z_{N+2})$$

where  $z_{N+2} = z_{N+1} + tw_\infty$ . Therefore, the amplitude  $\mathcal{A}_{g=2}^0$  becomes

$$\kappa^2 \int d^2t \frac{\partial}{\partial t} |t|^{2N-2k-2} \int d^2z_{N+1} \prod_{A=1}^k d^2z_A \prod_{i=1}^3 d^2\tau_i \cdot \int \prod_{B=k+2}^N d^2w_B$$

$$\begin{aligned}
& \sum_I \left\langle \prod_{i=1}^3 |(\mu_{\tau_i}|b)|^2 \prod_{A=1}^k |(\mu_A|b)|^2 W^{A\bar{A}} \cdot |(\mu_{N+1}|b)|^2 W^{I\bar{I}}(z_{N+2}) \right\rangle_{g=2} \\
& \cdot \left( \frac{1}{t} \right) \left\langle c\partial c\bar{c}V_I(w_\infty) \cdot c\bar{c}V_1(t) \prod_{B=k+2}^N |(\mu_B(tw_B)|b)|^2 W^{B\bar{B}} \bar{c}V_0(0) \right\rangle_{S_2}. \quad (5.1.8)
\end{aligned}$$

This can be rewritten as

$$\begin{aligned}
& \kappa^2 \sum_I \int d^2t \frac{\partial}{\partial t} \left[ \left( \frac{1}{t} \right) |t|^{\Delta_I} \right] |\alpha|^{\Delta_I} \\
& \cdot \mathcal{A}_{g=2}(1, \dots, k; I) \cdot \mathcal{A}_{g=0}^0(I; k+1, \dots, N+1)
\end{aligned}$$

where

$$\begin{aligned}
& \mathcal{A}_{g=2} = \kappa^2 \int \prod_{i=1}^3 d^2\tau_i d^2z_{N+2} \prod_{A=1}^k d^2z_A \\
& \cdot \left\langle \prod_{i=1}^3 |(\mu_{\tau_i}|b)|^2 \prod_{A=1}^k |(\mu_A|b)|^2 W^{A\bar{A}} |(\mu_{N+1}|b)|^2 W^{I\bar{I}}(z_{N+2}) \right\rangle_{g=2}
\end{aligned}$$

and

$$\mathcal{A}_{g=0}^0 = \kappa^{-2} \int \prod_{A=k+2}^N d^2w_A \left\langle \bar{c}V_0(0) \cdot c\bar{c}V_1(1) \cdot c\partial c\bar{c}V_I(w_\infty) \prod_{B=k+2}^N |(\mu_B|b)|^2 W^{B\bar{B}} \right\rangle_{S_2} \quad (5.1.9)$$

after a change of variables from  $z_{N+1}$  to  $z_{N+2}$ . Now, part of the potential BRST anomaly comes from the second order processes of the one-loop anomalies in a form of ‘‘one particle reducible’’ diagrams [3]. However, at the same order, we have contributions coming from the lower order local counterterms. We will explicitly verify that these two sources cancel with each other.

## 5.2 Structure of Two-Loop Anomalies and Counterterms

First, consider the two loop tadpole. Its BRST anomaly comes from eq. (5.9) with  $k = 0$ :

$$\kappa^2 \int d^2 t \frac{\partial}{\partial t} \left( \frac{1}{t} \right) \mathcal{T}_2(p_I = 0) \cdot \mathcal{A}_{g=0}^0(p_I = 0; 1, \dots, N+1) \quad (5.2.1)$$

where

$$\mathcal{T}_2(p_I = 0) = \int \prod_{i=1}^3 d^2 \tau_i \cdot d^2 z_{N+2} \left\langle \prod_{i=1}^3 |(\mu_{\tau_i} | b)|^2 (\mu_{N+2} | b)|^2 W^{I\bar{I}}(z_{N+2}) \right\rangle_{g=2}. \quad (5.2.2)$$

From eq. (5.1.2), we see that  $\tau_3 \sim 0$  gives a one-particle reducible diagram

$$\begin{aligned} \mathcal{T}_2(p_I = 0) \sim \int_{\tau_3 \sim 0} \frac{d^2 \tau_3}{|\tau_3|^2} \int d^2 \tau_1 d^2 z_{N+2} \left\langle |(\mu_1 | b)|^2 |(\mu_{N+2} | b)|^2 W^{I\bar{I}}(z_{N+2}) c\bar{c}\tilde{V}(z_\infty) \right\rangle_{T_2} \\ \cdot \int d^2 \tau_2 \left\langle |(\mu_2 | b)|^2 c\bar{c}\tilde{V}(z_\infty) \right\rangle_{T_2}. \end{aligned} \quad (5.2.3)$$

Here we used the fact that

$$(\mu_3 | b)_{\tau_3 \rightarrow 0} \longrightarrow \frac{1}{\tau_3} b_0$$

and similarly,

$$(\bar{\mu}_3 | \bar{b})_{\tau_3 \rightarrow 0} \longrightarrow \frac{1}{\bar{\tau}_3} \bar{b}_0.$$

This one particle irreducible part shows a structure of

$$\mathcal{T}_2(p_I = 0) \sim -\ln |t| \cdot \mathcal{T}_1(p_I = 0) \cdot \Pi_1(p_I = 0). \quad (5.2.4)$$

Therefore, eq. (4.2.1) gives a one particle reducible BRST anomaly of

$$\frac{\kappa^2}{2} \ln |t| \cdot \mathcal{T}_1(p_I = 0) \cdot \Pi_1(p_I = 0) \cdot \mathcal{A}_{g=0}^0(p_I = 0; 1, \dots, N+1). \quad (5.2.5)$$

Now we examine the amplitudes coming from the lower order local counterterms.

They are

$$-\frac{1}{4} \left\langle \beta_{\mu\nu}^G \partial X^\mu \bar{\partial} X^\nu \right\rangle_{T_2} \cdot \ln |t| \cdot \Pi_1(p_I = 0) \cdot \mathcal{A}_{g=0}^0(p_I = 0; 1, \dots, N+1) \quad (5.2.6)$$

and

$$-\frac{1}{4} \mathcal{T}_1(p_I = 0) \cdot \ln |t| \cdot \delta M_1^2(p_I = 0) \cdot \mathcal{A}_{g=0}^0(p_I = 0; 1, \dots, N+1) \quad (5.2.7)$$

respectively. Therefore, we see that equations (5.2.5)-(5.2.6) give a final BRST anomaly

$$\frac{\kappa^2}{4} \mathcal{A}_{g=0}^0(p_I = 0; 1, \dots, N+1) \cdot \Pi_1(p_I = 0) \cdot \mathcal{T}'_1(p_I = 0), \quad (5.2.7)$$

where  $\mathcal{T}'_1(p_I = 0)$  denotes the one-loop tadpole less the massless particle state singularity. This can be cancelled by a higher order BRST local counterterm in eq. (4.2.12) and (4.2.13):

$$\begin{aligned} & \frac{1}{2} \kappa^{-2} \{ -\beta_{\mu\nu}^G \partial X^\mu \bar{\partial} X^\nu + \beta^\Phi \frac{8}{3} [\bar{\partial}(b_{zz}c^z) + c.c.] \} \\ & = -\frac{1}{2} \mathcal{T}_1(\eta_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu) \partial X^\mu \bar{\partial} X^\nu \eta_{\mu\nu} \\ & + \frac{\kappa^2}{4} \Pi_1 \cdot [ -\eta_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu + \frac{8}{3} \bar{\partial}(b_{zz}c^z) + c.c. ] \cdot \mathcal{T}_1(p_I = 0). \end{aligned} \quad (5.2.8)$$

Next, we consider the two loop on-shell two point amplitude:

$$\kappa^2 \int d^2 t \frac{\partial}{\partial t} \left( \frac{1}{t} \right) \Pi_2(p_1^2 = -M_1^2) \cdot \mathcal{A}_{g=0}^0(p_1^2 = -M_1^2; 2, \dots, N+1) \quad (5.2.9)$$

where

$$\begin{aligned} \Pi_2 = & \int \prod_{i=1}^3 d^2 \tau_i d^2 z_1 d^2 z_{N+2} \left\langle \prod_{i=1}^3 |(\mu_{\tau_i}|b)|^2 |(\mu_1|b)|^2 |(\mu_{N+2}|b)|^2 \right. \\ & \left. \cdot W^{1\bar{1}}(z_1) W^{I\bar{I}}(z_{N+2}) \right\rangle_{g=2}. \end{aligned} \quad (5.2.10)$$

Again, the degeneration around the trivial homology cycle gives

$$\begin{aligned}
\Pi_2 = & \int_{\tau_3 \simeq 0} \frac{d^2 \tau_3}{|\tau_3|^2} \left\{ \int d^2 \tau_1 d^2 z_{N+2} \left\langle |(\mu_{\tau_1}|b)|^2 \cdot |(\mu_{N+2}|b)|^2 c\bar{c}V(z_\infty) W^{I\bar{I}}(z_{N+2}) \right\rangle_{T_2} \right. \\
& \cdot \int d^2 \tau_2 d^2 z_1 \left\langle |(\mu_{\tau_2}|b)|^2 |(\mu_1|b)|^2 W^{1\bar{1}}(z_1) c\bar{c}V(z_\infty) \right\rangle_{T_2} \\
& + \int d^2 \tau_1 d^2 z_1 d^2 z_{N+2} \left\langle |(\mu_{\tau_1}|b)|^2 |(\mu_1|b)|^2 W^{1\bar{1}}(z_1) \right. \\
& \left. \cdot |(\mu_{N+2}|b)|^2 W^{I\bar{I}}(z_{N+2}) c\bar{c}V_0(z_\infty) \right\rangle \cdot \int d^2 \tau_2 \left\langle |(\mu_{\tau_2}|b)|^2 c\bar{c}V_0(z_\infty) \right\rangle. \quad (5.2.11)
\end{aligned}$$

The first term arises when the two vertex operators at  $z_1$  and  $z_{N+2}$  are in different tori after the pinching, while the second term comes from the case that two vertex operators are on the same torus. Therefore, the one particle reducible pieces are

$$\begin{aligned}
& -\frac{1}{2} \kappa^2 \ln |t| \{ [\Pi_1(p_1^2 = -M_1^2)]^2 + \mathcal{T}_1 \cdot \mathcal{A}_{g=1}(p_0; 1, I) \} \\
& \cdot \mathcal{A}_{g=0}^0(p_1^2 = -M_1^2; 2, \dots, N+1). \quad (5.2.12)
\end{aligned}$$

On the other hand, those terms coming from the lower order local counterterms are

$$\begin{aligned}
& 2\kappa^2 \int d^2 t \frac{\partial}{\partial t} \left( \frac{1}{t} \ln t\bar{t} \right) \left( \frac{\Delta p_1^2}{8\pi T} \right) \delta M_1^2 \cdot \mathcal{A}_{g=0}^0(p_1^2 = -M_1^2; 2, \dots, N+1) + \\
& + \frac{1}{2} \kappa^2 \ln |t| \cdot \mathcal{T}_1(\partial X^\mu \bar{\partial} X_\mu) \cdot \mathcal{A}_{g=1}(\delta \mathcal{L}; 1, I) \cdot \mathcal{A}_{g=0}^0(p_1^2 = -M_1^2; 2, \dots, N+1). \quad (5.2.13)
\end{aligned}$$

The first term comes from a renormalized vertex operator and the second term from the background field insertion. Again, eq. (5.2.13) cancels eq. (5.2.12) completely.

Finally, we are left with the intermediate state anomaly. It is

$$\begin{aligned} & \sum_{\Delta_I \leq 0} \kappa^2 \int d^2 t \frac{\partial}{\partial t} \left[ \frac{1}{t} |t|^{\Delta_I} \right] |\alpha|^{\Delta_I} \\ & \cdot \mathcal{A}_{g=2}(1, \dots, k; I) \cdot \mathcal{A}_{g=0}^0(I; k+1, \dots, N+1) \end{aligned} \quad (5.2.14)$$

with

$$\begin{aligned} \mathcal{A}_{g=2} &= \kappa^2 \int \prod_{i=1}^3 d^2 \tau_i d^2 z_{N+2} \prod_{A=1}^{k \geq 2} d^2 z_A \\ & \cdot \left\langle \prod_{i=1}^3 |(\mu_{\tau_i} | b)|^2 \prod_{A=1}^k |(\mu_A | b)|^2 W^{A\bar{A}} |(\mu_{N+2} | b)|^2 W^{I\bar{I}}(z_{N+2}) \right\rangle_{g=2} \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_{g=0}^0 &= \kappa^{-2} \int \prod_{A=k+2}^N d^2 w_A \left\langle \bar{c}V_0(0) c\bar{c}V_1(1) c\partial\bar{c}V_I(w_\infty) \right. \\ & \cdot \left. \prod_{B=k+2}^N |(\mu_B | b)|^2 W^{B\bar{B}} \right\rangle_{S_2} \end{aligned}$$

First consider one-particle reducible diagrams with one or two loop tadpoles. They have anomaly structures

$$\begin{aligned} & \kappa^2 \sum_{\Delta_I \leq 0} \int d^2 t \frac{\partial}{\partial t} \left[ \frac{1}{t} |t|^{\Delta_I} \right] |\alpha|^{\Delta_I} \\ & \cdot \mathcal{A}_{g=0}^0(I; k+1, \dots, N+1) \\ & \cdot \{ \mathcal{A}_{g=0}(I; 1, \dots, k; J) \cdot [-\ln |t| \cdot \mathcal{T}_{g=2}(p_J = 0)] \\ & + \mathcal{A}_{g=1}(I; 1, \dots, k; J) \cdot [-\ln |t| \cdot \mathcal{T}_{g=1}(p_J = 0)] \} \end{aligned} \quad (5.2.15)$$

However, inside the bracket, each term can be renormalized by adding local countert-

erms. That is, by adding local counterterms

$$\begin{aligned} & \kappa^2 \sum_{\Delta_I \leq 0} \int d^2 t \delta^{(2)}(t) |t|^{\Delta_I} |\alpha|^{\Delta_I} \mathcal{A}_{g=0}^0(I; k+1, \dots, N+1) \\ & \cdot \{ \mathcal{A}_{g=0}(I; 1, \dots, k; J) + \mathcal{A}_{g=1}(I; 1, \dots, k; J) \}_{1PI} \end{aligned} \quad (5.2.16)$$

to the contact interaction, we cancel completely all the anomalies in eq. (5.2.14). A similar analysis goes through for other configurations of puncture moduli variables. One thing to note in this analysis is that, for a general intermediate state configuration we have two possible degeneration combinatorics

$$\begin{aligned} & \kappa^2 \sum_{\Delta_I \leq 0} \int d^2 t \frac{\partial}{\partial t} \left[ \frac{1}{t} |t|^{\Delta_I} \right] |\alpha|^{\Delta_I} \\ & \cdot \mathcal{A}_{g=0}^0(I; k+1, \dots, N+1) \\ & \{ \mathcal{A}_{g=0}(I; l+1, \dots, k; J) \cdot \int d^2 t' |t'|^{\Delta_J-2} |\beta|^{\Delta_J} \cdot \mathcal{A}_{g=2}(J; 1, \dots, l) \\ & + \mathcal{A}_{g=2}(I; l+1, \dots, k; J) \cdot \int d^2 t' |t'|^{\Delta_J-2} |\beta|^{\Delta_J} \cdot \mathcal{A}_{g=0}(J; 1, \dots, l) \}. \end{aligned} \quad (5.2.17)$$

Of course, they receive separate local counterterms. We checked the cancellation for all possible degeneration limits, and confirmed the notion of one-particle irreducible diagrams in a systematic renormalization scheme.

There is an another degeneration limit which give one-particle reducible diagrams. It comes from the  $\tau_3 \sim 0$  boundary of the genus two moduli space with a base manifold

of genus one [2]. The BRST anomaly structure is

$$\kappa^2 \sum_I \int d^2t \frac{\partial}{\partial t} \left[ \frac{1}{t} |t|^{\Delta_I} \right] |\alpha|^{\Delta_I} \cdot \mathcal{A}_{g=1}^0(I; k+1, \dots, N+1) \cdot \mathcal{A}_{g=1}(1, \dots, k; I) \quad (5.2.18)$$

with

$$\mathcal{A}_{g=1} = \int d^2\tau \prod_{A=1}^k d^2z_A \left\langle |(\mu_\tau|b)|^2 \prod_{A=1}^k |(\mu_A|b)|^2 W^{A\bar{A}} c\bar{c}V_I(z_\infty) \right\rangle_{T_2}$$

and

$$\mathcal{A}_{g=1}^0 = \int d^2\tau \prod_{A=k+1}^{N+1} d^2z_A \left\langle |(\mu_\tau|b)|^2 |(\mu_{N+1}|b)|^2 \bar{c}V_0(0) \cdot c\partial c\bar{c}V_I(z_\infty) \prod_{B=k+1}^N |(\mu_B|b)|^2 W^{B\bar{B}} \right\rangle_{T_2}$$

Again, depending on whether  $k = 0, 1$ , or  $\geq 2$ , we have a tadpole, on-shell two-point amplitude, and intermediate state singularity BRST anomalies. The cancellation of the anomalies is exactly the same as in chapter 3, except now we are dealing with a genus one null amplitude  $\mathcal{A}_{g=1}^0$ .

### 5-3. Higher Loop Generalization

The one and two loop analysis of BRST anomalies and local counterterms revealed that only three types of anomaly are relevant: tadpole, on-shell two point amplitude, and intermediate unphysical cut. In this section, we give a general inductive argument to extend the proof of the BRST anomaly and its cancellation to any finite order of string loop perturbation theory ( In superstring theory, a complete argument has not been achieved yet, due to mathematical difficulties. See, for example, ref. [4]).

Given the one loop BRST anomalies of chapter 3, we introduced zero loop (tree level diagram) local counterterms to restore BRST invariance. This can be generalized trivially to the following situations. Suppose we start from a  $g + 1$  loop scattering amplitude involving a longitudinal state:

$$\mathcal{A}_{g+1}(1, \dots, N) = \int_{\mathcal{M}_{g+1, N}} \prod_{i=1}^{3(g+1)-3} dm_i \wedge dm_{\bar{i}} \prod_{A=1}^N dz_A \wedge d\bar{z}_A$$

$$\cdot \left\langle \prod_{j=1}^{3(g+1)-3} |(\mu_j|b)|^2 \prod_{B=1}^N |(\mu_B|b)|^2 W^{B\bar{B}}(z_B) \right\rangle_{X, b, c}. \quad (5.3.1)$$

We consider a boundary of moduli space such that this amplitude is degenerated into  $g$  loop scattering amplitude and a one loop scattering amplitude:

$$\mathcal{A}_{g+1}(1, \dots, N) \rightarrow \mathcal{A}_g(1, \dots, M, I) \mathcal{D}_I(p_I) \mathcal{A}_1(I, M + 1, \dots, N), \quad (5.3.2)$$

where  $\mathcal{D}_I(p_I)$  is given in eq. (3.1.14). If there is no vertex operator on the one loop amplitude part, this is the tadpole BRST anomaly. Then we introduce a local counterterm of  $g$  loop scattering amplitude through the Fischler-Susskind mechanism. Since the counterterm is local on the worldsheet, there is no special role of the base worldsheet manifold of genus  $g$  with vertex operators. Similarly, we can introduce local counterterms to the other BRST anomalies we classified in chapter 3. This completes an introduction of  $g$  loop level counterterms to  $g + 1$  loop BRST anomalies.

Next, consider a boundary of moduli space where  $g + 1$  loop degenerates into  $g - 1$  loop and 2 loop amplitudes. The analysis in the previous two sections shows that all subleading anomalies are automatically cancelled by one loop counterterms. After subtracting them, the ‘one-particle irreducible’ BRST anomaly can be again cancelled by a two-loop local counterterm by the Fischler-Susskind mechanism.

We now generalize inductively this process: suppose we degenerate a  $g$  loop scattering amplitude into all possible  $g_1$  loop and  $(g - g_1)$  loop scattering amplitudes. Regard  $g_1$  loop amplitude part as a base manifold, even though we have a reflection symmetry between two degenerated amplitudes of  $g_1$  loop and  $(g - g_1)$  loop:

$$\mathcal{A}_g(1, \dots, N) \rightarrow \sum_{g_1} \sum_M \mathcal{A}_{g-g_1}(1, \dots, M, I) \mathcal{D}_I(P_I) \mathcal{A}_{g_1}(I, M + 1, \dots, N). \quad (5.3.3)$$

We denote by  $\tilde{\mathcal{A}}(1, \dots, N)$  a scattering amplitude summed over all orders in string loop perturbation expansion:

$$\tilde{\mathcal{A}}(1, \dots, N) = \sum_{genus} \mathcal{A}_g(1, \dots, N). \quad (5.3.4)$$

Then, the sum over  $g$  in eq. (5.3.3) gives

$$\tilde{\mathcal{A}}(1, \dots, N) = \sum_M \tilde{\mathcal{A}}(1, \dots, M, I) \mathcal{D}_I(P_I) \sum_{g_1} \mathcal{A}_{g_1}(I, M + 1, \dots, N). \quad (5.3.5)$$

The structure of counterterms, which we denote as  $\mathcal{C}_g$  for  $g$  loop order counterterm in the string loop perturbation expansion, is as follows:

$$\mathcal{C}_g(1, \dots, N) = \sum_{g_1} \sum_M \mathcal{A}_{g-g_1}(1, \dots, M, I) \mathcal{C}_{g_1}(I, M + 1, \dots, N). \quad (5.3.6)$$

Again summing over  $g$ , we have

$$\tilde{\mathcal{C}}(1, \dots, N) = \sum_M \tilde{\mathcal{A}}(1, \dots, M, I) \sum_{g_1} \mathcal{D}_{g_1}(I, M + 1, \dots, N). \quad (5.3.7)$$

Therefore, adding eq. (5.3.5) to eq. (5.3.7), we have a ‘renormalized’ amplitude:

$$\begin{aligned} \tilde{\mathcal{A}}(1, \dots, N) + \tilde{\mathcal{C}}(1, \dots, N) &= \sum_M \tilde{\mathcal{A}}(1, \dots, M, I) \\ &\sum_{g_1} [\mathcal{A}_{g_1}(I, M+1, \dots, N) + \mathcal{C}_{g_1}(I, M+1, \dots, N)]. \end{aligned} \quad (5.3.8)$$

Note that we have introduced the local counterterms order by order in string loop perturbative expansion. Therefore, the sum over  $g_1$  gives a ‘renormalized’ scattering amplitude, which we denote with subscript ‘ren.’:

$$\tilde{\mathcal{A}}_{ren.}(1, \dots, N) = \sum_M \tilde{\mathcal{A}}(1, \dots, M, I) \tilde{\mathcal{A}}_{ren.}(I, M+1, \dots, N). \quad (5.3.9)$$

The above invariant renormalization structure is again independent of the number of external legs. This is especially true for zero-point amplitude and one-point amplitude. Now we can now proceed again with an inductive argument to show that the right hand side of eq. (5.3.9) must hold for any value of  $N$  and any decomposition  $M$ . Therefore, we find that the left hand side of eq. (5.3.9) is indeed ‘renormalized’ to give a BRST invariant amplitude to all orders of the string loop expansion. This completes an inductive argument to prove a perturbative restoration of BRST invariance to all finite orders in the string loop expansion.

A similar argument may be implemented in the superstring perturbation expansion to prove the BRST invariance, especially the finiteness of scattering amplitudes. The above bosonic scattering amplitude analysis is particularly relevant to the heterotic string theory with anomalous  $U(1)$  gauge groups since in this class of theories it is known that vacuum energy is induced at two-loop order due to nonvanishing one-loop Fayet-Iliopoulos D-term [5]. Also, this argument may be used to establish an exact (to all orders of string loop expansion) nonrenormalization theorem [6] to massless particle scattering amplitudes. These are beyond the scope of the present thesis, but under current investigation [7].

## References

1. H. M. Farkas and I. Kra, *Riemann Surfaces*, Springer-Verlag, Berlin (1980):  
E. Freitag, *Siegelsche Modulfunktionen*, Springer-Verlag, Berlin (1983).
2. G. Moore, Phys. Lett. **B176**, 369 (1986).
3. A. Sen, SLAC preprint SLAC-PUB-4383 (1987) Unpublished.
4. J. Atick, G. Moore, and A. Sen, SLAC preprint SLAC-PUB-4500 (1988) Unpublished.
5. M. Dine, N. Seiberg, and E. Witten, Nucl. Phys. **B289**, 589 (1987):  
M. Dine, I. Ichinose, and N. Seiberg, Nucl. Phys. **B293**, 253 (1987):  
J. Atick, L. Dixon, and A. Sen, Nucl. Phys. **B292**, 67 (1987).
6. E. Martinec, Phys. Lett. **B171**, 189 (1986).
7. S.-J. Rey, Work in progress.