WAVE PROPAGATION IN A CONTINUOUSLY STRATIFIED FLUID

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ABSTRACT

The problem consists of finding the pressure response to a pressure source in an acoustical system of a stratified fluid overlying a rigid surface; a uniform gravitational field of acceleration g is directed normal to the rigid surface. The stratification is to be understood as implying that $\nabla\Omega^{-1}$, the gradient of the velocity of sound, and $\underset{\sim}{\mu}$, the negative logarithmic gradient of the density, are directed parallel or antiparallel to the gravitational field. The magnitudes of the quantities involved are assumed to be appropriate to the atmosphere of the earth.

The first problem treated assumes Ω and μ are constant, a situation which would obtain in an isothermal atmosphere. The pressure response exhibits 1) an appreciable phase shift upon reflection from the rigid surface at large angles of incidence, 2) a surface wave, and 3) appreciable distortion of the pulse shape.

The second problem neglects g and μ. The solution given is asymptotically (high frequency) valid only in regions of the fluid reached by least time rays which have not been refracted through the horizontal. This solution indicates that the time average energy flux propagates along rays which differ from the least time rays by an amount dependent upon the frequency, i.e. such an atmosphere exhibits angular dispersion. Asymptotic expressions are derived for the magnitude of the time average energy flux and for the pressure pulse distortion.

Finally, variations in μ and Ω are admitted simultaneously with the presence of a gravitational field. The solution is qualitatively the same as that of the preceding problem.

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I. INTRODUCTION

The problem to be considered here may be stated as follows: An atmosphere consisting of a perfect fluid is bounded below by a rigid plane surface. At a distance h above this surface is located a point source. The source disturbance is considered as a perturbation of a state of static equilibrium. It will be assumed that the perturbation (propagation of an acoustic wave) is a quasistatic adiabatic process, that the fluid is acted upon by a constant gravitational force directed everywhere normal to the plane rigid surface, and that no other forces act upon the fluid. It is required to find the pressure response.

The symmetry of the problem suggests employing cylindrical coordinates r, z, and φ oriented so that the plane $z = 0$ coincides with the rigid surface and located so that the source lies on the z-axis at z = h. Inasmuch as there is no preferred azimuth one may assume that there will be no dependence upon φ.

Several special cases will be treated before finally arriving at a result applicable to the atmosphere of the earth.

II. DIFFERENTIAL SYSTEM

2.1. First Order Perturbation Equations

Bergmann (1946, pp.329-33) has derived the differential equations governing the first order perturbation of sound propagation in a heterogeneous nonviscous fluid. The results of this paper will be used with the following modification in notation (the associated parenthetical symbol being that used by Bergmann for the same quantity): P (p_o) is the static (unperturbed) pressure; p (p_1) is the sound pressure; ρ (ρ_o) is the static density; σ (ρ_1) is the variation in density caused by the sound wave; $\underset{\sim}{u}$ ($\underset{\sim}{u}_1$) is the fluid velocity, k (k_o) is the adiabatic bulk modulus of the fluid; and V (V) is the potential of the force of gravity. P, ρ, k, and V, the zero order terms, are assumed to be independent of time. p, σ, and $\underset{\sim}{u}$, the first order terms, are in general functions of time.

It will be assumed that the time dependence is separable and is given by $e^{i\omega t}$; thus

$p(z,r,t) = p(z,r) \; e^{i\omega t}$, $\underset{\sim}{u}(z,r,t) = \underset{\sim}{u}(z,r) \; e^{i\omega t}$, etc.

Unless written out specifically as $p(z,r,t)$ the symbol p will denote $p(z,r)$; a similar convention holds for $\underset{\sim}{u}$ and σ.

In this new notation Bergmann's equations (28) and (29) become

$$\sigma = \frac{\omega^2 \Omega^2 p + \underset{\sim}{G} \cdot \nabla p}{\omega^2 + \rho^{-1} \underset{\sim}{G} \cdot \nabla P} , \qquad (1)$$

$$\underset{\sim}{G} = -\rho^{-1} \nabla \rho + \kappa^{-1} \nabla P , \qquad (2)$$

$$\nabla^2 p + \omega^2 \sigma - \rho^{-1} \nabla P \cdot \nabla \sigma = 0, \tag{3}$$

where $\Omega^2 = \rho/\kappa$. (4)

Notice that $\underset{\sim}{G}$ has been defined as the negative of that used by Bergmann. The quantity Ω is the reciprocal of the velocity of sound and is known as the wave slowness.

Bergmann's equation (7) may be written as

$$\underset{\sim}{u} = i\rho^{-1}\omega^{-1}(\nabla p + \sigma \nabla V). \tag{5}$$

The assumption that the force of gravity is constant and perpendicular to the rigid surface z = 0 may be stated by the equation

$$\nabla V = g\, \underset{\sim}{e}_z, \tag{6}$$

where $\underset{\sim}{e}_z$ is the unit vector in the positive z direction and g the (constant) acceleration of gravity. This relation may be combined with Bergmann's equation (5) to yield

$$\nabla P = -\rho g\, \underset{\sim}{e}_z, \tag{7}$$

This implies P = P(z) and $\rho = \rho(z)$. It will be assumed that $\Omega = \Omega(z)$. Then from (2)

$$\underset{\sim}{G} = (\mu - g\Omega^2)\, \underset{\sim}{e}_z, \tag{8}$$

where $\mu = -\rho^{-1} \dfrac{d\rho}{dz}$. (9)

These relations permit the simplification of (1), (3), and (5) to

$$\sigma = \lambda^{-2}\left(\omega^2 \Omega^2 p + G \frac{\partial p}{\partial z}\right), \tag{10}$$

$$\nabla^2 p + \omega^2 \sigma + g \frac{\partial \sigma}{\partial z} = 0, \tag{11}$$

$$\underset{\sim}{u} = i\rho^{-1}\omega^{-1}(\nabla p + \sigma g\, \underset{\sim}{e}_z) \tag{12}$$

where $\lambda = (\omega^2 - gG)^{1/2}$. (13)

Bergmann has identified the term

$$\underset{\sim}{J} = p\underset{\sim}{u} \tag{14}$$

from equation (11) of his paper as the energy flux.

In order to express the boundary condition at $z = 0$ it is necessary to have an expression for the vertical component of the fluid velocity. This may be obtained from (10) and (12); whence

$$W = \underset{\sim}{u} \cdot \underset{\sim}{e}_z = i\,\omega\rho^{-1}\lambda^{-2}\left(\frac{\partial p}{\partial Z} + g\Omega^2 p\right). \tag{15}$$

Substitution of (10) into (11) leads to

$$\lambda^2 \nabla^2 p + gG\frac{\partial^2 p}{\partial Z^2} + \omega^2\left(\mu + \frac{gG'}{\lambda^2}\right)\frac{\partial p}{\partial Z} + \omega^2\left[\omega^2\Omega^2 + g\,\Omega^{2\prime} + \lambda^{-2}g^2\Omega^2 G'\right]p = 0, \tag{16}$$

where the prime denotes differentiation with respect to the argument (Z in this case).

The boundary condition appropriate to the rigid surface $z = 0$ is the vanishing of the vertical component of the fluid velocity. From (15) it is seen that a sufficient condition that w vanish at $z = 0$ is that

$$\left[\frac{\partial p}{\partial Z} + g\,\Omega^2 p\right]_{z=0} = 0. \tag{17}$$

2.2. Separation of Variables

Following the procedures common to this type of problem, one may now seek elementary solutions of the differential equation (16). It will be assumed that an elementary solution may be represented by the product

$$p(z,r) = \chi(z,h)\,y(z)\,J_0(\omega v r), \tag{18}$$

where $$\chi(z,h) = \left[\frac{\rho(z)}{\rho(h)}\right]^{1/2} \frac{\lambda(z)}{\lambda(h)} . \tag{19}$$

$J_0(x)$ is the ordinary Bessel function, and ωv is the separation constant. $y(z)$ is, of course, also a function of v.

By direct operation one may show that

$$\frac{\partial p}{\partial z} = \left[-\tfrac{1}{2}(\nu + \lambda^{-2} g G')y + y'\right]\chi\, J_0(\omega v r),$$

$$\frac{\partial^2 p}{\partial z^2} = \Big[y'' - (\nu + \lambda^{-2} g G')y' + \Big(\frac{\nu^2}{4} - \frac{\nu'}{2} + \frac{\nu g G' - gG''}{2\lambda^2} - \frac{g^2 G'^2}{4\lambda^2}\Big) y \Big] \chi(z,h)\, J_0(\omega v r),$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) = -\omega^2 v^2 y\, \chi(z,h)\, J_0(\omega v r) .$$

Substitution of these relations into (16) and (17) yields

$$y'' + q(z,\lambda)\, y = 0, \tag{20}$$

$$\left[y' + (m - \tfrac{1}{2}\lambda^{-2} g G')y\right]_{z=0} = 0, \tag{21}$$

where $$q(z,\lambda) = -\lambda^2\left[v^2 - \Omega^2(z)\right] - m^2 + m' + \lambda^{-2}(gG'm - \tfrac{1}{2} g G'') - \tfrac{3}{4} g^2 G'^2 \lambda^{-4}, \tag{22}$$

$$m(z) = g\Omega^2 - \tfrac{1}{2}\nu = \left(\frac{\nu^2}{4} - gG\Omega^2\right)^{1/2}. \tag{23}$$

It may be mentioned that a similar problem could be considered which models the ocean rather than the atmosphere. In this case the z-axis would be directed vertically downward and the boundary at $z = 0$ would be a free surface. The differential equation (20) remains valid; the boundary condition, however, must be replaced by

$$\left[y' + \left(-\frac{\omega^2}{g} + \frac{\nu}{2} - \frac{gG'}{2\lambda^2}\right) y\right]_{z=0} = 0 \tag{24}$$

Because of the reversal in the direction of the z-axis it will be found that g, G, and μ are intrinsically negative.

III. ISOTHERMAL ATMOSPHERE

Equations (20) and (21) admit tremendous simplification if one assumes that both Ω and μ are constant. It is possible in this case to study the effects upon wave propagation of gravity and density gradient isolated from velocity variation. As a matter of fact it will be possible to isolate the effects of density variation alone by making g approach zero.

It is of some interest to note that Ω and μ are constant in a perfect gas at constant temperature (isothermal atmosphere). For a temperature of 15°C. the numerical values of some of the constants in the MKS system of units are as follows:

$$\begin{aligned} \Omega^2 &= 8.65 \times 10^{-6} \qquad & gG &= 3.33 \times 10^{-4} \\ m &= 2.54 \times 10^{-5} \qquad & \mu &= 1.19 \times 10^{-4} . \end{aligned} \tag{25}$$

These values will be used whenever either numerical or order of magnitude calculation is required.

In this problem it will be possible to synthesize a pulse from the steady state solution.

3.1. Proposed Solution

For constant Ω and μ equations (20) and (21) reduce to

$$y'' - \lambda^2 c^2 y = 0, \tag{26}$$

$$(y' + my)_{z=0} = 0, \tag{27}$$

$$\text{where} \quad c = (v^2 - \Gamma^2)^{\frac{1}{2}}, \tag{28}$$

$$\Gamma = (\Omega^2 - \lambda^{-2} m^2)^{\frac{1}{2}}. \tag{29}$$

Inasmuch as the ultimate synthesis of a pulse will require dealing with all ω, it would be well to adopt a convention now as to the phase of the radicals having branch points in the ω-plane. Without exception the phase of these radicals will be specified as zero for large real ω; the required cuts, all of which are finite in length, in the ω-plane may be drawn along the real axis. It will be assumed that ω has an arbitrarily small imaginary part.

The complete solution for an isothermal atmosphere with $\mathcal{R}\omega > 0$ (where $\mathcal{R}\omega$ indicates the real part of ω) is given by

$$p(z,r) = +\sum_{k=1}^{3} p_k(z,r), \tag{30}$$

where

$$p_1(z,r) = \chi(z,h)\int_{0+i0}^{\infty+i0} J_0(\omega v r)\, e^{-\lambda c|z-h|}\, c^{-1}\, v\,dv, \tag{31}$$

$$p_2(z,r) = \chi(z,h)\int_{0+i0}^{\infty+i0} J_0(\omega v r)\, e^{-\lambda c(z+h)}\, c^{-1}\, v\,dv, \tag{32}$$

$$p_3(z,r) = 2m\,\chi(z,h)\int_{0+i0}^{\infty+i0} J_0(\omega v r)\, e^{-\lambda c(z+h)}\, \frac{v\,dv}{(\lambda c-m)c}\,. \tag{33}$$

The limits of the integrals indicate the path of integration lies slightly above the real axis. The phase of c is chosen to be zero for large real v.

The solution for $\mathcal{R}\omega < 0$ may be obtained from (30) by 1) changing the sign from plus to minus, 2) replacing λ by $-\lambda$, and 3) modifying the path of integration so that it lies below the real v axis.

Notice that the sum of the integrands of the three terms of (30) depends upon z and r as

$$\chi(z,h)\, J_0(\omega v r)\left[e^{-\lambda c|z-h|} + \frac{\lambda c+m}{\lambda c-m}\, e^{-\lambda c(z+h)}\right].$$

Because the bracketed quantity is a solution of the system (26) and (27), this equation is identical with (18). Thus, insofar as the operations of differentiation and $\lim_{z\to 0}$ commute with the integration over v, equation (30) satisfies the differential equation and also the boundary condition at $z = 0$. It is easy to show that in the vicinity of $z = 0$ both differentiation with respect to either z or r and the process $\lim_{z\to 0}$ commute with the integration over v by virtue of the uniform convergence provided by $e^{-\lambda ch}$, $h \neq 0$. The fact that the integrand satisfies the differential equation (16) implies that the differential operator of that equation commutes with the integration. The same argument with suitable changes of sign may be applied to the case $\mathcal{R}\,\omega < 0$.

It will be shown in the next section that (30) contains a proper source representation.

3.2. Source and Reflected Waves

If one introduces the notation

$$Z_1 = |z - h|, \quad Z_2 = (z + h), \tag{34}$$

the integrals of (31) and (32) may be treated simultaneously in the form

$$p_j(z,r) = \chi(z,h) \int_{0+i0}^{\infty+i0} J_0(\omega v r)\, e^{-\lambda c Z_j}\, c^{-1}\, v\, dv\,. \tag{35}$$

This integral may be identified with the classical source representation (Watson, 1944, p. 416). If one replaces Watson's a, b, t, and y respectively by λZ_j, ωr, v, and Γ, it is found that

$$p_j(z,r) = \frac{\chi(z,h)}{R_j\, \nu_j} \exp\left\{ i\left(\omega t - \Gamma R_j \nu_j\right)\right\}, \quad (j = 1,2), \tag{36}$$

where $R_j = (r^2 + Z_j^2)^{\frac{1}{2}}$, (37)

$$\nu_j = \left(\omega^2 - g\, G\, Z_j^2 / R_j^2\right)^{1/2}. \tag{38}$$

The extension of Watson's formula to complex λ where λ has a positive real part is accomplished by analytic continuation.

In the case $\omega < 0$ the expression for $p_j(z,r)$ may be obtained from the negative of (35) by changing the sign of λ and replacing the path of integration by a path below the real v axis. The same integral identity (Watson, 1944, p. 416) may be used to reduce the integral to equation (36). The fact that (36) is valid for $\omega \lessgtr 0$ is due to the choice of phase of ν_j.

It may be observed that for values of $\omega^2 > gG + \frac{m^2}{\Omega^2}$ equation (36) represents a wave propagating outward from the point $R_j = 0$. For smaller values of ω^2 no particular interpretation need be attached; it suffices that (36) is in a form suited to the synthesis of a pulse.

Inasmuch as μ is a constant, equation (9) implies

$$\rho(z) = \rho(h)\ e^{-\mu(z-h)} \qquad (39)$$

The fact that λ is a constant implies through (19) and (39) that

$$\chi(z,h) = e^{-\frac{\mu}{2}(z-h)}. \qquad (40)$$

3.3. Diffracted Wave

The term $p_3(z,r)$ is most conveniently treated by contour integration on the complex v-plane, a process which requires that the singularities on the v-plane be known. It will suffice to consider $\omega > (gG + \frac{m^2}{\Omega^2})^{1/2}$; the results may be extended to smaller values of ω by analytic continuation.

The radical c contributes branch points at $v = \pm\Gamma$. In order to ensure single valuedness one may cut the v-plane vertically downward from each of these branch points (fig. 1). The factor $(\lambda c - m)^{-1}$ contributes poles at $v = \pm\Omega$. It is not difficult to show that the pole $v = +\Omega$ lies on the upper Riemann sheet, i.e., the sheet upon which c is real positive as v approaches infinity, if $m > 0$, as would be the case in an atmosphere of a perfect gas at a constant temperature subject to gravity. For $m < 0$, as would be the case if $g = 0$, the pole lies on the lower Riemann sheet.

Fortunately, the degenerate case m = 0 leads to no complication since p_3 is identically zero. The following treatment assumes m > 0; the modifications for m < 0 are not difficult and will be given later.

For r = 0 and sufficiently large Z_2, (33) may be evaluated by the method of steepest descents (Jeffreys, 1950, p. 501). The saddle point is the root of

$$\frac{dc}{dv} = 2v\,(v^2-\Gamma^2)^{-1/2} = 0 \ .$$

The only root is, of course, v = 0. The path of steepest descents is given by

$$\Im c = +\Gamma \ .$$

This equation may be converted to the form

$$x = +y\,\Gamma\,(\Gamma^2-y^2)^{-1/2} \ , \quad x > 0 \ .$$

where v = x + iy. The upper curve in figure 2 is a sketch of this function.

The path of integration deforms continuously into the path of steepest descents. This deformation is completely independent of the existence of the pole; thus, the result will be equally valid for any m.

On the path of steepest descents the phase of c is such that the variable part of c is always pure real and positive, the imaginary part being a constant $i\Gamma$. It follows that the exponential of equation (33) is a decreasing function of v. Then by the application of Watson's lemma it is found that

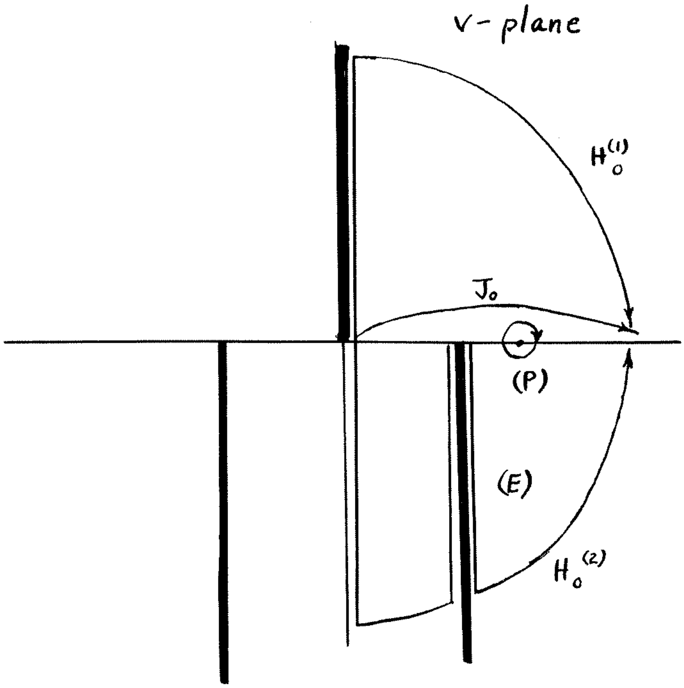

Figure 1

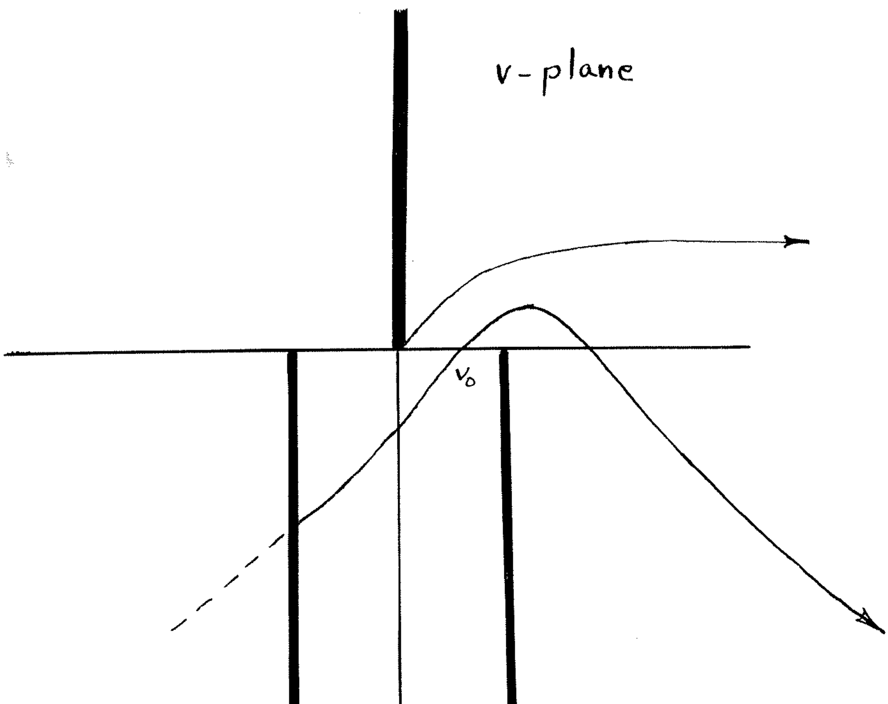

Figure 2

$$p_3(z,0) \sim \frac{2m\chi(z,h)}{(i\lambda\Gamma - m)\lambda Z_2} \quad e^{-i\lambda(z+h)\Gamma}. \tag{41}$$

On the other hand when $r \neq 0$ the integral of (33) may be transformed to a more tractable integral by a method which may be referred to as the Sommerfeld transformation (see Pekeris (1946, p. 305) for a more detailed exposition of the method). Briefly, the transformation proceeds as follows: Replace $2J_0(\omega vr)$ by the sum of the Hankel functions $(H_0^{(1)}(\omega vr) + H_0^{(2)}(\omega vr))$. The introduction of the Hankel functions requires a cut along the positive imaginary axis (fig. 1). The integral of (33) may be split into two integrals, the first containing $H_0^{(1)}$ and the second $H_0^{(2)}$. The former integral may be deformed continuously into an integral along the right side of the positive imaginary v-axis plus an integral along an arc of infinite radius connecting $v = i\infty$ to $v = +\infty$ (fig. 1). The latter integral may be deformed into a path along the negative imaginary axis, a branch line integral (E), a circuit (P) about the pole $v = +\Omega$ in the negative trigonometric sense, and an arc of infinite radius terminating at $v = +\infty$ (figure 1). Since

$$H_0^{(1)}(-iy) = -H_0^{(2)}(iy),$$

it can be shown that the two integrals along the imaginary axis cancel exactly. Furthermore, the integrals on the infinite arcs are zero. Then

$$p_3(z,r) = m\chi(z,h)\int_{(E)+(P)} H_0^{(2)}(\omega vr)\, e^{-\lambda c(z+h)} \frac{v\,dv}{(\lambda c - m)c} \quad .$$

For sufficiently large r the Hankel function may be replaced by its asymptotic representation, whence

$$p_3(z,r) \sim \left(\tfrac{1}{2}\pi\omega r\right)^{-\frac{1}{2}} m\chi(z,h)\, e^{i\pi/4}\int_{(P)+(E)} e^{-r\zeta^{\phi}(v)} \frac{v^{\frac{1}{2}}dv}{(\lambda c - m)c}, \tag{42}$$

where $\xi(v) = r^{-1}(i\omega v r + \lambda c Z_2)$.

This integral may be asymptotically evaluated for large r by steepest descents (Jeffreys, 1950, p. 503). The saddle point is at $v = v_0$, where

$$v_0 = \frac{\omega r}{\nu_2 R_2} \Gamma . \tag{43}$$

The path of steepest descents is given by

$$\mathcal{I}\, \xi(v) = \Gamma \nu_2 \frac{R_2}{r} , \tag{44}$$

where $\mathcal{I}$ denotes the imaginary part of the subsequent quantity. It is not difficult to show that the path of steepest descents cuts the real axis at v_0 and Γ^2/v_0; this path also asymptotically (large $|v|$) approaches

$$\omega x \pm \lambda y \frac{Z_2}{r} = 0 ,$$

where $v = x + iy$ and the upper sign applies to the right half v-plane and the lower sign to the left half. Generally the path will be of the form shown in figure 2 (page 12). In this case the standard evaluation gives

$$p_3(z,r) \sim \frac{2m\, \chi(z,h)}{\nu_2 R_2 (i\frac{\lambda^2}{\nu_2}\frac{Z_2}{R_2}\Gamma - m)}\, e^{-i\nu_2 R_2 \Gamma} . \tag{45}$$

For Z_2/R_2 sufficiently small, say $Z_2/R_2 < \frac{m}{\omega\Omega}$, the second crossing of the real axis lies between the pole and the branch point. It is then necessary to add to (45) the pole contribution, which is found by the method of residues to be

$$p_3^{(P)}(z,r) \sim \frac{2m}{\lambda}\, \chi(z,h) \left(\frac{2\pi}{\omega\Omega r}\right)^{1/2} \exp\left\{-m(z+h) - i(\omega\Omega r + \pi/4)\right\} . \tag{46}$$

Even the sum of (45) and (46) is a rather poor approximation for Z_2/R_2 very small since the saddle point is close to a pole. In order to obtain a better approximation it is best to start from (42). The pole contribution being given by (46), only the branch line integral (E) need be evaluated. In this integral it is desirable to make the change of variable from v to $u = iv$. With this change of variable the branch line integral of (42) becomes

$$p_3^{(E)}(z,r) \sim -i\left(\tfrac{1}{2}\pi\omega r\right)^{-\frac{1}{2}} m\,\chi(z,h)\,e^{i\pi/4}\int_{(E')} e^{-\omega r u}\,P(u)\,du\,, \tag{47}$$

where

$$p(u) = -e^{+i\lambda(z+h)(u^2+\Gamma^2)^{\frac{1}{2}}}\,u^{\frac{1}{2}}\,(u^2+\Gamma^2)^{-\frac{1}{2}}\left[\lambda\,(u^2+\Gamma^2)^{\frac{1}{2}} - i\,m\right]^{-1} e^{-i\pi/4}. \tag{48}$$

The u-plane, including cuts, poles, and the path of integration (E'), may be obtained from figure 1 by a 90° counterclockwise rotation. For consistency the phase of $(u^2 + \Gamma^2)^{\frac{1}{2}}$ must be chosen as $\pi/2$ at $u = i\,\infty$.

The purpose of this change of variable is to reduce the integral to one treated by van der Waerden (1951, p. 40). Van der Waerden has shown that the asymptotic evaluation of the integral of (47) is given by

$$\begin{aligned} -i\int_{(E)} e^{-\omega r u}\,P(u)\,du \sim {} & +i\,e^{-i\omega r\Gamma + i\pi/4}\,m^{-1}\left(\frac{2\pi}{\omega r}\right)^{\frac{1}{2}} \\ & + e^{-m(z+h) - i\,\omega r\Gamma - i\pi/4}\,\frac{K}{2\lambda(\Omega v)^{\frac{1}{2}}}\,, \end{aligned} \tag{49}$$

where $\quad v = \Omega - \Gamma\,,$ $\hfill (50)$

$$K = 2\left(\frac{\pi}{\omega r}\right)^{\frac{1}{2}} - 2\pi\,e^{-i\omega r v}\,(-iv)^{\frac{1}{2}}\,\operatorname{erfc}\,(-i\omega r v)^{\frac{1}{2}}, \tag{51}$$

$$\operatorname{erfc} z = 2\pi^{-\frac{1}{2}}\int_z^{\infty} e^{-t^2}\,dt = 1 - \operatorname{erf} z. \tag{52}$$

Notice that the definition of the complementary error function (52) differs by a factor 2 from that used by van der Waerden.

Rosser (1948, p. 15) has shown that

$$\operatorname{erfc}\,(-i\omega r v)^{\frac{1}{2}} = 1 - 2^{\frac{1}{2}}e^{-i\pi/4}\left[C\left([2\omega r v/\pi]^{\frac{1}{2}}\right) + i\,S\left([2\omega r v/\pi]^{\frac{1}{2}}\right)\right]\,, \tag{53}$$

where C(x) and S (x) are the Fresnel integrals

$$C(x) + iS(x) = \int_0^x \cos\frac{\pi x^2}{2}\,dx + i\int_0^x \sin\frac{\pi x^2}{2}\,dx = \int_0^x e^{i\frac{\pi x^2}{2}}dx. \tag{54}$$

Substituting (49) into (47)

$$p_3^{(E)}(z,r) \sim -\frac{2}{\omega r}\chi(z,h)\,e^{-i\omega r\Gamma} + \frac{m}{\lambda}(2\pi\Omega v\omega r)^{-1/2}$$

$$\chi(z,h)\,e^{-m(z+h)-i\omega r\Gamma}\,K. \tag{55}$$

The foregoing discussion applies to the case $m > 0$. For $m < 0$ the pole at $v = \Omega$ lies on the lower sheet and cannot contribute directly to the solution. However, the evaluation of the branch line integral leading to (55) remains valid. Therefore, the pole exerts an influence on the solution through the term K in (55).

The evaluation of $p_3(z,r)$ for $\Re\omega < 0$ may be carried out by a straightforward modification of the above procedure, the principal change being that the v-plane is cut in an opposite manner from that just employed. It is found that the equations (41), (45), (46), and (55) are valid for all ω.

3.4. Total Response for Steady State

First consider the case where $R_2 \gg 1$ and $Z_2/R_2 > 10^{-2}$. Since equation (45), which was derived for large r, reduces in the limit as $r \to 0$ to (41), which was derived for r = 0, it may be presumed that (45) is valid for small as well as large r. The total response is given by the sum of the three terms of (30). From (36) and (45)

$$p(z,r,t) \sim \chi(z,h)\Big\{R_1^{-1}\nu_1^{-1}e^{i(\omega t-\Gamma R_1\nu_1)} + R_2^{-1}\nu_2^{-1}e^{i(\omega t-\Gamma R_2\nu_2)}$$

$$\Big[1 + 2m\Big(i\frac{\lambda^2}{\nu_2}\frac{Z_2}{R_2}\Gamma - m\Big)^{-1}\Big]\Big\}. \tag{56}$$

For $\omega \gtrsim 1$ the bracketed quantity may be written to a very good approximation as

$$[1 + \frac{2m}{\Omega}(i\omega\, Z_2/R_2 - \frac{m}{\Omega})^{-1}] . \tag{57}$$

Perhaps the greatest interest centers on the case where $Z_2 \ll R_2$. In this case it is appropriate to neglect terms of order Z_2^2/R_2^2. Then (36) implies

$$p_1(z,r) + p_2(z,r) \sim 2\, \frac{\chi(z,h)}{\omega r}\, e^{-i\Gamma\omega r} .$$

The sum of this and (55) is

$$p^{(K)}(z,r) \sim m\lambda^{-1}(2\pi\Omega v\omega r)^{-\frac{1}{2}}\, \chi(z,h)\, e^{-m(z+h)-i\omega r\Gamma}\, K . \tag{58}$$

For $m < 0$ this represents the total response; for $m > 0$ there is in addition the pole contribution (46).

An estimate of the importance of (58) in comparison to (46) may be obtained rather easily from the asymptotic expansion of K given by van der Waerden (1951, p. 42):

$$K \sim \left(\frac{\pi}{\omega r}\right)^{\frac{1}{2}} \frac{i}{\omega r v}\left(1 - \frac{3i}{2\omega r v} + \cdots\right).$$

For large $\omega r v$, (58) may be written as

$$p^{(K)}(z,r) \sim [\frac{m}{2\lambda}\, \chi(z,h)\left(\frac{2}{\Omega\omega r}\right)^{\frac{1}{2}} e^{-m(z+h)}\, e^{-i\omega r\Gamma}]$$
$$i(\omega r v)^{-\frac{3}{2}}\left(1 - \frac{3i}{2\omega r v} + \cdots\right) . \tag{59}$$

The bracketed quantity is equal to about $p_3^{(P)}(z,r)/10$ (see (46)). It appears that for sufficiently large r, say r large enough that $\omega r v > 10$, $p^{(K)}(z,r)$ will be less than 1% of $p_3^{(P)}(z,r)$. Suppose $\omega^2 > 10^{-3}$; then

$$v \sim \frac{1}{2} \frac{m^2}{\lambda^2 \Omega} \left(1 + \frac{1}{4} \frac{m^2}{\lambda^2 \Omega^2} + \cdots \right) .$$

The conditions that $\omega r v > 10$ and $\omega^2 > 10^{-3}$ imply

$$r > 20\,\omega \frac{\Omega}{m^2} . \tag{60}$$

Using the values (25), one sees that $r > 10^5 \omega$ km. It is apparent then that at any reasonable distance, say less than 1000 km., the term in K must be retained in spite of its asymptotic behavior as $r^{-3/2}$.

$p^{(K)}(z,r)$ may be conveniently written as the sum of two terms

$$p^{(B)}(z,r) \sim \frac{\chi(z,h)\, m}{\lambda \omega r} \left(\frac{2}{\Omega v}\right)^{1/2} e^{-m(z+h) - i\omega r \Omega} , \tag{61}$$

$$p^{(S)}(z,r) \sim -\frac{m}{\lambda} \chi(z,h) \left(\frac{2\pi}{\omega r \Omega}\right)^{1/2} e^{-m(z+h) - i\omega r \Omega - i\pi/4} \operatorname{erfc}\left(-i\omega r v\right)^{1/2} . \tag{62}$$

The error function may, of course, be replaced by (53).

The results of this section may be summarized by observing that the pressure response for $R_2 \gg 1$ is given by:

equation (56) if $Z_2/R_2 > 10^{-2}$,

equation (61) and (62) if $Z_2/R_2 < 10^{-2}$,

equation (59) if $Z_2/R_2 < 10^{-2}$ and $r > 10^5$;

in the last two cases equation (46) must be added if $m > 0$.

3.5. Physical Interpretation of Steady State

It is of some interest to consider first the simple case of the propagation of a pressure wave from a point source in an infinite medium, i.e., to remove temporarily the complications due to the presence of a rigid boundary. The solution to this problem is given by (36) with j set equal to unity. For $\omega^2 > gG + \frac{m^2}{\Omega^2}$ the surfaces of constant phase are given by

$$\omega t = \Gamma R_1 \nu_1 .$$

The surfaces of constant phase are prolate spheroids of ellipticity (ratio of the difference of the semiaxes to the semimajor axis) gG/ω^2. The orthogonal trajectories to these surfaces have the same direction at any point as the vector

$$\nabla \Gamma R_1 \nu_1 = \Gamma \nu_1 \left(\underset{\sim}{e}_R + \frac{gG}{2\nu^2} \sin 2\theta \; \underset{\sim}{e}_\theta \right),$$

where the notation refers to a system of sperical coordinates the origin of which lies at the source and the angle θ being measured from the z-axis to the radius vector, $\underset{\sim}{e}_R$ and $\underset{\sim}{e}_\theta$ respectively being the unit vectors in the radial and θ directions. These trajectories are not rays, however, since the influence of gravity has destroyed the isotropy.

The energy flux computed from equations (12), (14), and (36) is

$$\underset{\sim}{J} = \omega \Gamma \left[2 R_1^2 \nu_1^3 \rho(h) \right]^{-1} \underset{\sim}{e}_R ,$$

where terms which average to zero over a period $(2\pi/\omega)$ have been neglected. The energy propagates radially outward from the source, but the source does not radiate energy symmetrically in all directions, the most energy being radiated vertically and the least horizontally.

Now consider the influence of the rigid surface upon the pressure response. Suppose r is large, but $Z_2/r > 10^{-2}$. The pressure response is then given by (56). It is easily seen that the first term represents the direct wave from the source, and the second term the wave reflected at the rigid surface. The bracketed quantity which modifies the latter may be interpreted as a reflection coefficient. The reflection coefficient is a complex number of modulus unity; the phase, however, depends upon both ω and Z_2/R_2. Therefore, there is a phase shift upon reflection from a rigid surface which varies with the angle of incidence and also with ω.

Finally, one must consider the case where R_2 is large and $Z_2/R_2 \ll 1$. The pressure response for this case is given by the sum of (61) and (62) plus, if $m > 0$, (46). The r^{-1} dependence of (61) identifies it as a body wave. For $\omega^2 > 10^{-3}$ and neglecting terms in Z_2^2/R_2^2, (61) reduces to

$$p^{(B)}(z,r,t) \sim \frac{\chi(z,h)}{r\omega} e^{-m(z+h)} e^{i\omega(t-r\Gamma)}$$

$$+ \frac{\chi(z,h)}{r\omega} e^{-m(z+h)} e^{i\omega(t-r\Gamma)} \left(1 - \frac{1}{8}\frac{m^2}{\lambda^2\Omega^2} + \cdots\right).$$

In this form $p^{(B)}(z,r,t)$ may be recognized as the sum of the direct and reflected waves.

In so far as $\omega r v$ is small, equation (62) has the characteristics of a surface wave, i.e., the amplitude depends upon the inverse square root of r and is exponentially attenuated with increasing z. As r increases and $\omega r v$ becomes so large that the complementary error function may be replaced by its asymptotic representation, it is found that $p^{(S)}(z,r)$ approaches $-p^{(B)}(z,r)$. In this case it is convenient to revert to equation (58) and its asymptotic expansion. The behavior of $p^{(S)}(z,r,t)$ is that of a surface wave at moderate distances and of a body wave at distances of the order of 10^5 km. (see (60)). In view of this behavior $p^{(S)}(z,r,t)$ may be referred to as a psuedo-surface wave.

For the case $m > 0$ there is in addition the pole contribution (46). As one might expect this wave has all the attributes of a pressure surface wave and may be referred to as such. The associated particle motion is completely horizontal. The phase velocity is the Laplace velocity of sound Ω^{-1}. The free wave which corresponds to this surface wave is the "generalized sound wave" of Bjerknes (1933, p. 335).

The pressure response at the surface for $m > 0$ may be described as follows: At a moderately large distance the response is made up of the sum of a surface wave, a psuedo-surface wave, and a body wave. The first two add to an amplitude of about half that of the pure surface wave alone; the total surface wave is much smaller than the body wave. As r increases, the psuedo-surface wave behaves more as a body wave, ultimately canceling the true body wave and leaving only the pure surface wave. It appears as if the energy of the body wave is transferred to the surface wave through the intermediary of the psuedo-surface wave.

The same end result must, of course, be obtained from the steepest descents evaluation, i.e., the sum of (56) and (46). For very small Z_2/R_2 the reflection coefficient (the bracketed quantity) in (56) approaches -1. Close to the surface the reflected wave, having undergone a 180° phase shift upon reflection, interferes destructively with the direct wave. The body wave is virtually cancelled, leaving only the surface wave (46).

It is of some interest to note that the surface wave, the only wave propagating as a purely elastic wave, depends upon the influence of gravity at the surface ($m > 0$ implies $g > \frac{\mu}{2\Omega^2}$) for its excitation. In this connection it may also be remarked that no pure gravity waves are excited although a free gravity wave exists (Bjerknes, 1933, p. 333). The boundary condition at a free surface (24) would introduce a pole leading to these gravity waves.

The solution for pressure response is exponentially attenuated with increasing z by the factor $\chi(z,h)$ (see (40)). Quite the contrary is the case for the fluid velocity. From (12) and (56) it is seen that the velocity amplitude of the body waves depends upon

$$\rho^{-1}(z)\,\chi(z,h) = e^{\frac{\mu}{2}(z-h)};$$

for the surface waves, equations (12) and (47) imply a dependence of

$$\rho^{-1}(z)\,\chi(z,h)\,e^{-mz} \propto e^{+Gz}.$$

The fact that the velocity amplitude can become arbitrarily large contradicts the basic assumption of the linearized equations employed. This is not as serious as it appears. The steady state solution is merely a mathematical tool for generating a pulse solution. In the pulse solution the disturbance is confined to a finite region of space at any given time. The initial amplitude may then be chosen so that the amplitude remains sufficiently small up to any preassigned time; the pulse solution will then remain valid at least up to that time.

There are some advantages, however, to supposing the atmosphere to be bounded by a perfect absorber at a level $z = H$ which may be selected sufficiently remote from the source and the rigid surface. The condition to be imposed at this level is that all propagating waves crossing it pass from the atmosphere into the absorber and not in the reverse direction. This overcomes two objections: 1) that the differential system is not valid for very low densities, and 2) that the steady state solution remain bounded. The concept of an absorber lying above a certain level does, in a rough way, take account of the fact that absorption, not allowed for in the equations used here, plays an increasingly important role as z increases. In any case a pulse solution synthesized from such solutions will not be affected by the introduction of this artificial boundary until such a time as the pulse front first reaches the level $z = H$.

3.6. Synthesis of a Pulse

A pulse may be synthesized by a linear integral operation upon the steady solution. One such operation which is convenient here is

$$\bar{p}(z,r,t) = (2\pi i)^{-1}\int_{-\infty-i0}^{+\infty-i0} F(\omega)\, p(z,r,t)\, d\omega . \tag{63}$$

$F(\omega)$ may be chosen as a convenient function of ω subject to the restriction that the integral must exist.

As an example of the procedure to be employed, a pulse solution may be generated from (36). Let $F(\omega) = 1$; then

$$\bar{p}_j(z,r,t) = \frac{\chi(z,h)}{R_j}\left[\frac{1}{2\pi i}\int_{-i\infty+0}^{+i\infty+0} e^{st}\, f_j(s)\, ds\right], \tag{64}$$

where the variable of integration has been changed from ω to $s = i\omega$. The quantity $f_j(s)$ appearing in (64) is defined as follows

$$f_j(s) = \tilde{\nu}_j^{-1}\, e^{-\tilde{r}\tilde{\nu}_j R_j} = \left(\tilde{\nu}_j^{-1}\, e^{-s\Omega R_j}\right) e^{-R_j(\tilde{r}\tilde{\nu}_j - s\Omega)} . \tag{65}$$

The tilde ($\sim$) indicates that in the quantity so modified, ω is to be replaced by $-is$, and the phase of this new quantity taken to be 0 for very large real s; thus,

$$\tilde{\nu}_j = (s^2 + gGZ_j^2/R_j^2)^{1/2} = i\,\nu_j . \tag{66}$$

The advantage of this transformation is that the bracketed quantity in (64) may be identified as the complex inversion formula (Widder, 1946, p. 66) applied to the Laplace transform $f_j(s)$. Then, (64) may be written as

$$\bar{p}_j(z,r,t) = \frac{\chi(z,h)}{R_j}\,\mathcal{L}^{-1}\left\{f_j(s)\right\}, \tag{67}$$

where the symbol $\mathcal{L}^{-1}\{\ \}$ indicates the inverse Laplace transform of the quantity enclosed in braces.

The first term on the right of (65) is a well known transform:

$$\mathcal{L}^{-1}\left\{\tilde{\nu}_j^{-1} e^{-s\Omega R_j}\right\} = 1(\tau_j)\ J_0\left[(gG)^{\frac{1}{2}} \frac{Z_j}{R_j} \tau_j\right], \tag{68}$$

where τ_j is the retarded time

$$\tau_j = (t - \Omega R_j), \tag{69}$$

and 1 (τ) is the unit step function

$$1(\tau) = \begin{cases} 0 & \tau < 0 \\ 1 & \tau > 0 \end{cases}. \tag{70}$$

The second exponential on the right of (65), being analytic at infinity, may be developed in a power series of inverse powers of s. Widder (1946, p. 94) has shown that such a series may be inverted term by term to form an entire series in t.

$$\mathcal{L}^{-1}\left\{e^{-R_j(\tilde{\Gamma}\tilde{\nu}_j - s\Omega)}\right\} = \delta(t) - C_1 1(t) + C_2 t + \cdots, \tag{71}$$

where δ (t) is the Dirac delta function. It is important to observe that the coefficients C_1, C_2, ... are all of order R_j and must approach zero if R_j becomes arbitrarily small.

Finally, the transform of $f_j(s)$ itself may be formed by taking the convolution (classical resultant) of (68) and (71) (Widder, 1946, p. 91). Thus,

$$\mathcal{L}^{-1}\{f_j(s)\} = 1(\tau_j)\left[J_0\left(\sqrt{gG}\ \frac{Z_j}{R_j} \tau\right) - B_j \tau_j + \frac{1}{4} B_j^2 \tau_j^2 + \cdots\right], \tag{72}$$

where $\quad B_j = \frac{1}{2} R_j\, \Omega\, (gGZ_j^2/R_j^2 + m^2/\Omega^2)$. (73)

From (67) one finds

$$\bar{p}_j(z,r,t) = \frac{\chi(z,h)}{R_j}\ 1(\tau_j)\left[J_0\left(\sqrt{gG}\ \frac{Z_j}{R_j} \tau_j\right) - B_j \tau_j + \frac{1}{4} B_j^2 \tau_j^2 + \cdots\ \right]. \tag{74}$$

The source is found from (74) by neglecting all terms enclosed in brackets except the first.

Application of the operation (63) to (56), (46), (61), and (62) leads to

$$\bar{p}\,(z,r,t) = \frac{\chi(z,h)}{R_1}\,1(\tau_1)\left[J_0\left(\sqrt{gG}\,\frac{Z_1}{R_1}\tau_1\right) - B_1\tau_1 + \cdots\right]$$

$$+\frac{\chi(z,h)}{R_2}\,1(\tau_2)\left[J_0\left(\sqrt{gG}\,\frac{Z_2}{R_2}\tau_2\right) - \left(B_2 - \frac{2mR_2}{\Omega Z_2}\right)\tau_2 + \cdots\right], \tag{75}$$

$$\bar{p}^{(P)}(z,r,t) = 4m\,\chi(z,h)\,e^{-m(z+h)}\,1(\tau)\left(\frac{2\tau}{\Omega r}\right)^{1/2}\left[1 - \frac{2}{15}gG\tau^2 + \cdots\right], \tag{76}$$

$$\bar{p}^{(B)}(z,r,t) \sim \frac{2}{r}\,\chi(z,h)\,e^{-m(z+h)}\,1(\tau)\left[1 - \frac{1}{2}r\Omega\frac{m^2}{\Omega^2}\tau + \cdots\right], \tag{77}$$

$$\bar{p}^{(S)}(z,r,t) \sim -2m\,\chi(z,h)\,e^{-m(z+h)}\,1(\tau)\left(\frac{2\tau}{\Omega r}\right)^{1/2}$$

$$\left[1 - \left(\frac{r\Omega}{2}\,\frac{m^2}{\Omega^2}\tau\right)^{1/2} + \frac{1}{6}\left(\frac{r\Omega}{2}\,\frac{m^2}{\Omega^2}\tau\right)^{3/2} + \cdots\right], \tag{78}$$

where $\tau = (t - r\Omega)$.

3.7. Special Cases

$m = 0$. Certainly the simplest special case is $m = 0$ ($g\Omega^2 = \mu/2$). The diffracted wave vanishes identically and the total response is given by the sum of $\bar{p}_1$ and $\bar{p}_2$. But for $m = 0$

$$f_j\,(s) = \tilde{\nu}_j^{-1}\,e^{-\Omega R_j \tilde{\nu}_j}.$$

This is a known transform (Bateman, 1954, p. 248). The pressure response is given exactly by

$$\bar{p}\,(z,r,t) = \chi(z,h)\,\frac{1(\tau_1)}{R_1}\,J_0\left[\sqrt{gG}\,\frac{Z_1}{R_1}\left(\tau_1^2 + 2R_1\Omega\tau_1\right)^{1/2}\right]$$

$$+\chi(z,h)\,\frac{1(\tau_2)}{R_2}\,J_0\left[\sqrt{gG}\,\frac{Z_2}{R_2}\left(\tau_2^2 + 2R_2\Omega\tau_2\right)^{1/2}\right].$$

The source behavior is included in the first term as $R_1 \to 0$.

$g = 0$. This case is of some interest in that it isolates the effects of density variation. Some simplification results if the operation (63) is performed with $F(\omega) = \omega\left(\omega^2 - \frac{\nu^2}{4\Omega^2}\right)^{-1/2}$. Then, operating upon (36) one finds

$$\bar{p}_j(z,r,t) = \frac{1(\tau_j)}{R_j}\,\chi(z,h)\;J_0\left[\frac{\nu}{2\Omega}\left(\tau_j^2 + 2R_1\Omega\tau\right)^{1/2}\right].$$

If the same operation is applied to (61) and (62) it is found that both (77) and (78) remain valid. There is, of course, no direct pole contribution.

3.8. Comments on the Pulse Solution

Admittedly there is no interest in making detailed calculations on pulse distortion due to propagation in an isothermal atmosphere. However, there is some interest in the magnitude of the distortion introduced by gravity and density gradients. It will be found that in certain regions of the atmosphere the effects of gravity, density variation, and the variation of the velocity of sound are all comparable. Thus, an estimate of the pulse distortion may be obtained from the results in an isothermal atmosphere.

The behavior of the source is given by the limit as $R_1 \to 0$ of (74) with j set equal to 1. Thus, the source pressure is given by

$$\bar{p}_s(z,r,t) = \frac{1(\tau_1)}{R_1}\,J_0\left(\sqrt{gG}\,\frac{Z_1}{R_1}\,\tau_1\right).$$

It is apparent that the source pressure is not spherically symmetric but depends upon the ratio Z_1/R_1. However, if interest is restricted to the first several seconds of the pulse, the pressure input closely approximates a spherically symmetric pressure source having a unit step function time dependence; in fact, for the first eleven seconds ($\tau_1 \leq 11$) the source pressure differs from spherical symmetry by less than 1%.

Suppose h is so large that at the time t under consideration, $\tau_2 < 0$; the boundary at $z = 0$ can have no effect upon the solution under these circumstances. The pressure response is most conveniently given by the first term of (75). In order to find the magnitude of the distortion of the pulse induced by gravity and density variation, consider the case $R_1 = 200$ km. Then, from (73) it is seen that $.02 \leq B_1 \leq .12$, the exact value depending upon Z_1/R_1. Substituting into the first term of (75), one sees that a square pulse of two seconds duration may be distorted at the rear end by from 4% for $Z_1/R_1 = 0$ to 24% for $Z_1/R_1 = 1$. The effects at moderately large distances upon moderately long pulses may be appreciable.

If one includes the effects of the boundary it is seen from (75) that there is added a reflected pulse, i.e., a pulse which appears to come from an image source located a distance h below the rigid surface. In general, this pulse will be more distorted than the source pulse. In the case where $Z_2 \ll R_2$ it is no longer convenient to divide the response into a direct and reflected wave; the natural division appears to be a surface wave and a body wave. It is of interest to find at what distance the surface wave attains the same amplitude as the body wave. From (76) and (77) it is seen that the amplitudes of the surface and body waves are approximately equal at $r = 570/\tau$ km. For example, suppose the source input is a square pulse of two seconds duration; the surface wave will attain an amplitude equal to that of the body wave at a distance of 280 km. If one includes the psuedo-surface wave as part of the surface wave it is found that equal amplitudes are attained at a distance $r = 2300/\tau$ km., eg., for the example cited before, r must be 1150 km. before equal amplitudes are attained.

The special cases require little discussion. Their chief interest lies in the fact that a part of the solution or even all of the solution may be obtained in an elementary closed form.

IV. SOLUTION FOR AN ATMOSPHERE IN WHICH $\Omega = \Omega(z)$

In the preceding section it was possible to obtain some estimate of the effects of gravity and density variation upon wave propagation. Actually the principal complications of the problem of sound propagation in a stratified fluid are due to the fact that the wave slowness is not a constant. For definiteness a velocity variation of the form found in the atmosphere of the earth will be considered. Figure 3 shows the general form of this function; there are two maxima, $z = z_1$ and z_3, and one minimum, $z = z_2$. Typical values of z_1, z_2, and z_3 respectively might be 15, 55, and 90 km.

In order to study the velocity variation as simply as possible one may assume that $g = \mu = \Omega'(0)$ and also that the source is at the surface. In section V a general solution not restricted by these assumptions will be given.

It will be convenient in this problem to use the concept of an artificial absorbing boundary which was introduced in section 3.5. For the methods which will be employed in this section it is required that there exist a true minimum of the wave slowness below the absorber and that the value of Ω at this minimum be less that $\Omega(H)$. In figure 3 the minimum z_2 will serve this purpose provided H is near z_3. Subject to this assumption it will be shown at the end of this section that the waves crossing the level $z = H$ have an upward component in their direction of propagation.

Inasmuch as constant density and zero gravity imply $\chi(z,h) = 1$, the form of the elementary solution (18) is simplified somewhat. One may seek a solution of the form

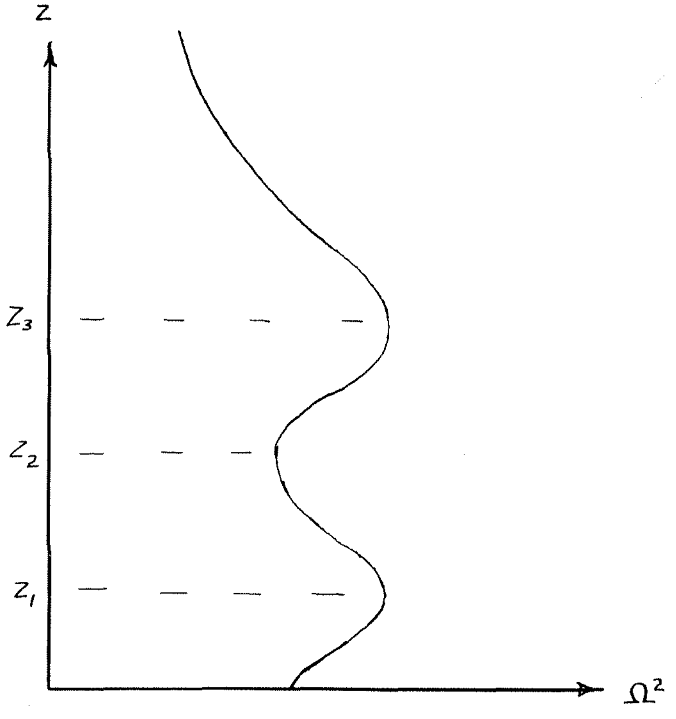

Figure 3

$$p(z,r) = \int_D J_0(\omega v r)\, y(z)\, v\, dv \tag{79}$$

where the path D lies wholly in the first quadrant of the v-plane and joins the origin to positive real infinity. The differential system (20) and (21) becomes

$$y'' - \omega^2 a^2 y = 0, \tag{80}$$

$$y'(0) = 0, \tag{81}$$

where $a^2 = v^2 - \Omega^2(z)$. (82)

Numerical values appropriate to the earth's atmosphere will be used wherever order of magnitude calculations are required. The Rocket Panel (1952, p. 1027) has given a tabulation which suffices for order of magnitude computation.

4.1. The Green's Function

A tool which will be required in the general solution of the differential system (80) and (81) is the appropriate Green's function. This function may be defined as a solution of the differential system (Ince, 1926, pp. 254-7)

$$\left.\begin{aligned} L[\mathcal{G}(z|\zeta)] &= 0, \\ \lim_{v\to\infty} \mathcal{G}(z|\zeta) &= 0, \end{aligned}\right\} \tag{83}$$

$$\left[\frac{\partial}{\partial z}\mathcal{G}(z|\zeta)\right]_{z=0} = 0, \tag{84}$$

$$\left.\begin{aligned} \mathcal{G}(\zeta+|\zeta) - \mathcal{G}(\zeta-|\zeta) &= 0, \\ \left[\frac{\partial}{\partial z}\mathcal{G}(z|\zeta)\right]_{z=\zeta+} - \left[\frac{\partial}{\partial z}\mathcal{G}(z|\zeta)\right]_{z=\zeta-} &= 1, \end{aligned}\right\} \tag{85}$$

where

$$L[y] = y'' - [\omega^2 a^2(z) + M(z,v)]\, y, \tag{86}$$

$$M(z,v) = 3/4 \quad (a'/a)^2 - \tfrac{1}{2} \quad (a''/a) = a^{\frac{1}{2}} \frac{\partial^2}{\partial z^2} (a^{-\frac{1}{2}}). \tag{81}$$

Define

$$E(\mp {}^{z}_{\zeta}, v) = \exp\left\{ \mp \omega \int_{\zeta}^{z} a(x)\, dx \right\}. \tag{88}$$

The phase of a(x) is chosen as zero for very large real v.

Direct substitution shows that

$$\mathcal{G}(z|\zeta) = -\frac{1}{2\omega} \left[a(z)\, a(\zeta)\right]^{-\frac{1}{2}} \left[E(\mp {}^{z}_{\zeta}, v) + E(-{}^{z}_{0}, v)\, E(-{}^{\zeta}_{0}, v) \right], \tag{89}$$

where the $\mp$ ambiguity is resolved by choosing the sign to agree with that of (ζ - z). In substituting into (84), one should keep in mind the assumption that $\Omega'(0) = 0$.

4.2. Source Representation

The values which the separation variable may assume lie on the path D, i.e., v must lie in the first quadrant of the v-plane. For such values of v the real part of a(z) cannot vanish. Erdélyi (1956, pp. 83-4) has shown that the asymptotic solution of (80) is given by a linear combination of

$$y_{\pm}(z) = a^{-\frac{1}{2}}(z)\, E(\pm {}^{z}_{0}, v),$$

provided that $\Re a(z)$ does not vanish.

From such solutions one can form a function which represents a source at z = h and which also satisfies (81), namely

$$y_0(z) = \left[a(z)\, a(h)\right]^{-\frac{1}{2}} \left[E(\mp {}^{z}_{h}, v) + E(-{}^{z}_{0}, v)\, E(-{}^{h}_{0}, v) \right].$$

In the limit as $h \to 0$

$$y_0 = 2\left[a(z)\, a(0)\right]^{-\frac{1}{2}}\, E(-{}^{z}_{0}, v). \tag{90}$$

Unfortunately, in this form it is not apparent that (81) is satisfied. However, upon substitution into the form of (79) one finds

$$p_o(z,r) = 2\int_0 J_o(\omega v r)\left[a(z)\,a(0)\right]^{-\frac{1}{2}} E(-{}^{z}_{0},v)\; v\,dv\,. \tag{91}$$

In the limit as $z \rightarrow 0$ this integral may be identified with that of (35). Then, for sufficiently small z

$$p_o(z,r) \rightarrow \frac{2}{\omega R}\, e^{-i\omega\Omega R}, \tag{92}$$

$$\text{where } R = (z^2 + r^2)^{\frac{1}{2}}. \tag{93}$$

In the form (92) it is easy to show that the normal component of the fluid velocity vanishes at z = 0, i.e., since g = 0, (92) satisfies the boundary condition (17).

4.3. Solution by Successive Approximations

In order to arrive at an exact solution of the problem a method of successive approximations will be employed. For this purpose set

$$y(z) = V(z) + y_o(z). \tag{94}$$

Then,

$$L\left[V(z)\right] = -M(z,v)\left[V(z) + y_o(z)\right], \tag{95}$$

$$V'(0) = 0. \tag{96}$$

The differential system (95) and (96) may be solved by a method of successive approximations due to Liouville (Ince, 1926, pp. 263-4). Briefly, the method consists of solving the inhomogeneous differential system

$$L\left[V_n(z)\right] = -M(z,v)\left[V_{n-1}(z) + y_o(z)\right],$$

$$V_n'(0) = 0,$$

where $V_o(z) \equiv 0$.

Ince (1926, p. 256) has given the solution of this system as

$$V_n(z) = -\int_0^H \mathcal{G}(z,\zeta)\, M(\zeta,v)\left[V_{n-1}(\zeta) + y_0(\zeta)\right] d\zeta, \tag{97}$$

where it will be recalled that H is the height of the absorbing layer in the atmosphere. The solution of the system (95) and (96) is the limit of the sequence $\{V_n(z)\}$, provided such a limit exists.

The method may be formulated as a series solution if one defines

$$y_n(z) = V_n(z) - V_{n-1}(z), \quad (n = 1,2, \ldots).$$

This equation implies

$$V_n(z) = \sum_{k=1}^{n} y_k(z).$$

From (97)

$$y_n(z) = -\int_0^H \mathcal{G}(z|\zeta)\, M(\zeta,v)\, y_{n-1}(\zeta)\, d\zeta, \quad (n = 1,2,\ldots). \tag{98}$$

It remains to establish the uniform convergence of the series $\sum_{n=1}^{\infty} y_n(z)$ and $\sum_{n=1}^{\infty} y_n'(z)$. On the path D, $|M|$ has an upper bound $\hat{M}$, and $|a|$ has a lower bound $\check{a}(v)$. If Y_n is the upper bound of $|y_n(z)|$, it is a consequence of (98) that

$$|y_n(z)| \leqslant Y_n \leqslant (\hat{M}H/2\omega\check{a})\; Y_{n-1}. \tag{99}$$

For sufficiently large ω, (99) implies the convergence of $\sum_{n=1}^{\infty} Y_n$, which in turn implies the uniform convergence of $\sum_{n=1}^{\infty} y_n(z)$.

Differentiation of (98) leads to

$$y_n'(z) = -\int_0^H \frac{\partial \mathcal{G}}{\partial z} M(\zeta,v)\, y_{n-1}(\zeta)\, d\zeta .$$

Inasmuch as all the terms in the integrand are bounded, there exists an A independent of n such that

$$|y_n'(z)| \leqslant AY_{n-1}.$$

Uniform convergence of $\sum_{n=1}^{\infty} y_n'(z)$ follows from this.

It might be mentioned that uniform convergence of both of these series may be established by virtue of the exponentials contained in $\mathcal{G}(z \mid \zeta)$ independent of the finiteness of H. However, this generalization is not required.

Substitution of the solution $y(z) = y_o(z) + \sum_{n=1}^{\infty} y_n(z)$ into (79) leads to

$$p(z,r) = \int_D J_0(\omega v r)\, y_0(z)\, v\, dv + \int_D J_0(\omega v r) \sum_{n=1}^{\infty} y_n(z)\, v\, dv . \tag{100}$$

Since the path of integration extends into the complex v-plane, the singularities of the integrand of equation (100) must be investigated. It is shown in appendix A that the singularities of $E(\frac{z}{h}, v)$ are branch points at $v = \pm\Omega(z), \pm\Omega(h)$, and also at $v = \pm\Omega(z_i)$ where $\Omega(z_i)$ is any maximum or minimum of $\Omega(x)$, $h < x < z$. The remaining singularities may be identified by inspection. The v-plane may be made single valued by drawing cuts from the branch points vertically downward in the right half plane and vertically upward in the left half plane. All singularities are on the real axis and in the intervals defined by $\check{\Omega}(0,H) \leq v^2 \leq \hat{\Omega}(0,H)$ where $\check{\Omega}(0,H)$ and $\hat{\Omega}(0,H)$ are respectively the smallest and largest values of $\Omega(x)$, $0 \leq x \leq H$. The cut v-plane is sketched in figure 4.

The number of branch points, and therefore the number of cuts, depends upon the value of z. For example consider the wave slowness versus altitude function shown in figure 3. It is seen that for $z < z_1$ there will be two pairs of branch points, for $z_1 < z < z_2$ there will be three pairs, for $z_2 < z < z_3$ there will be four pairs, and for $z_3 < z$ there will be five pairs.

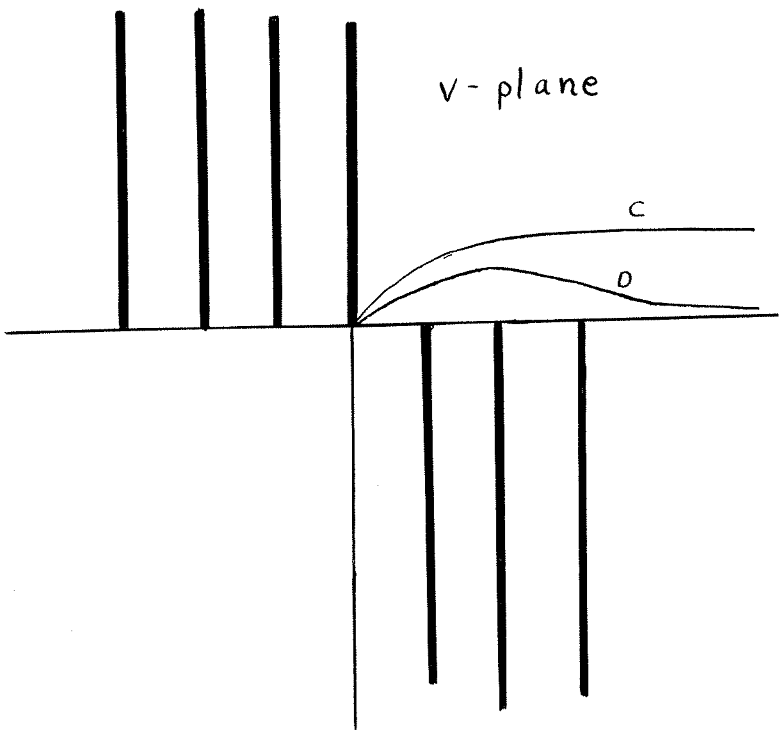

Figure 4

For convenience the path D may be thought of as consisting of two parts. The first part D_1 is an arc in the first quadrant connecting the origin to a point on the real axis to the right of $\widehat{\Omega}(0,H)$; the second part D_2 lies on the real axis joining the right terminus of D_1 to positive real infinity. Consider the last integral in (100). Along the path D_1 the integrand is bounded for any finite ωr; on the path D_2 the integrand is bounded and for sufficiently large v behaves as v^{-3}. It is a consequence of these facts that the integral converges absolutely; this absolute convergence permits the interchange of the order of integration and summation. Equation (100) becomes

$$p(z,r) = \sum_{n=0}^{\infty} p_n(z,r), \tag{101}$$

where

$$p_n(z,r) = \int_D J_0(\omega v r)\, y_n(z)\; v\, dv \tag{102}$$

Although (101) is a valid solution of the problem, its usefulness is restricted to regions of z and r such that the first few terms completely dominate the solution. Inasmuch as y(z) has been developed as a convergent asymptotic series, it is logical to investigate under what conditions is the asymptotic property transferred to the series (101). One should notice that on the section of the path D which is a finite distance above the real axis, the Bessel function will behave for large ωr as $e^{-i\omega vr}$ where v has a positive imaginary part. If succeeding terms are to be neglected on the basis of the largeness of ω it is necessary that there be an exponential quantity common to all terms of (101) which will dominate $e^{-i\omega vr}$ for sufficiently large ω. A term that will serve this purpose in a limited region of the r, z plane is $E(-\frac{z}{O},v)$, which may be factored from each of the terms $y_n(z)$. If on D the real part of $(ivr + \int_0^z a(x)dx)$ is greater than or equal to zero, (101) is an asymptotic series in ω. A somewhat less stringent condition will be derived in section 4.5. In any case, it is apparent that for r = 0 the series of (101) is asymptotic in ω for all z.

The initial term of the series of (101) has been written out explicitly in (91). The second term may be conveniently divided into two terms on the basis of the two terms of $\mathcal{G}(z \mid \zeta)$. Thus,

$$p_1(z,r) = p_{11}(z,r) + p_{12}(z,r), \tag{103}$$

where

$$p_{11}(z,r) = \omega^{-1} \int_D \frac{J_0(\omega v r)\, v\, dv}{[a(z)\, a(0)]^{1/2}} \int_0^H a^{-1}(\zeta)\, M(\zeta,v)\, E(\mp{}^{z}_{\zeta}, v)\, E(-{}^{\zeta}_{0}, v)\, d\zeta, \tag{104}$$

$$p_{12}(z,r) = \omega^{-1} \int_D \frac{J_0(\omega v r)\, v\, dv}{[a(z)\, a(0)]^{1/2}} \int_0^H a^{-1}(\zeta)\, M(\zeta,v)\, E^2(-{}^{\zeta}_{0}, v)\, E(-{}^{z}_{0}, v)\, d\zeta. \tag{105}$$

4.4. Evaluation on z-Axis

As was mentioned in section 4.3 the series (101) is both convergent and asymptotic for r = 0. Therefore, an asymptotic expression for p(z,r) may be obtained from the asymptotic evaluation of the sum of equations (91), (104), and (105).

For r = 0, (91) becomes

$$p_0(z,0) = z \int_D a^{-1/2}(z)\, a^{-1/2}(0)\, e^{-\omega z f_0(v,z)}\, v\, dv, \tag{106}$$

where $f_0(v,z) = z^{-1} \int_0^z a(x)dx.$ (107)

Equation (106) is in a form convenient for the application of the method of steepest descents (Jeffreys, 1950, pp. 501-6) in the form of Watson's lemma.

The saddle point is given by the root v of the equation

$$\frac{\partial}{\partial v} f_0(v,z) = v\, z^{-1} \int_0^z a^{-1}(x)\, dx = 0.$$

An obvious solution is $v = 0$.

The path of steepest descents C_o is given by

$$\mathscr{I} f_o(v,z) = \mathscr{I} f_o(0,z) = z^{-1}\int_0^z \Omega(x)\,dx,$$

where $\mathscr{I}$ denotes the imaginary part of the subsequent quantity. It is not difficult to show that C_o must 1) leave the origin of the complex v-plane at a 45° slope, 2) asymptotically (large $|v|$) approach

$$\mathscr{I}\, v \sim z^{-1}\int_0^z \Omega(x)\,dx,$$

and 3) not cross the real axis in the right half plane. C_o is sketched in figure 4.

The path D readily deforms into C_o; therefore, (106) may be written

$$p_o(z,0) = 2\,e^{-\omega z f_o(0,z)}\int_{C_o} a^{-\frac{1}{2}}(z)\, a^{-\frac{1}{2}}(0)\, e^{-\omega z \xi}\, v\,dv,$$

where $\xi = f_o(v,z) - f_o(0,z)$

is a real positive quantity on the path C_o.

This integral may be conveniently evaluated by Watson's lemma provided it is possible to develop v^2 as a power series in ξ. This may be accomplished by expanding the definition of ξ in a power series in v and then inverting the series. Next one changes the variable of integration from v to ξ and then applies Watson's lemma; whence

$$p_o(z,0) \sim \frac{2\,e^{-\omega\int_0^z \Omega(x)\,dx}}{[\Omega(z)\,\Omega(0)]^{\frac{1}{2}}\,\omega\int_0^z \Omega^{-1}(x)\,dx}\left(1 + \frac{i\,b_o(z)}{\omega z}\right) + O(\omega^{-3}), \tag{109}$$

where

$$b_o(z) = z^2\left[\int_0^z \Omega^{-1}(x)\,dx\right]^{-2}\left\{\frac{1}{2}\left[\Omega^{-2}(z) + \Omega^{-2}(0)\right] z^{-1}\int_0^z \Omega^{-1}(x)\,dx - z^{-1}\int_0^z \Omega^{-3}(x)\,dx\right\}. \tag{110}$$

Turning now to equations (104) and (105), one notices that the order of integration may be reversed; thus,

$$p_{1j}(z,r) = \omega^{-1}\int_0^H d\zeta \int_0 J_0(\omega v r) \quad \frac{M(\zeta)\, e^{-\omega z f_{1j}}}{a(\zeta)[a(z)\, a(0)]^{+1/2}} \; v dv\,, \tag{111}$$

where

$$f_{1j}(v,\zeta) = z^{-1} \begin{cases} \pm\int_\zeta^z a(x)\,dx + \int_0^\zeta a(x)\,dx & (j = 1) \\ \int_0^z a(x)\,dx + 2\int_0^\zeta a(x)\,dx & (j = 2) \end{cases} \quad . \tag{112}$$

For r = 0 the inner integral of equation (111) is in the same form as the integral of equation (106) and may be evaluated by the method of steepest descents. The saddle point is found to be at v = 0 and the path of steepest descents essentially the same as C_0. Straightforward evaluation leads to

$$p_{1j}(z,r) \sim \omega^{-2}\int_0^H \frac{-i\, M(\zeta,0)}{\Omega(\zeta)} \; \frac{e^{-\omega z f_{1j}(0)}}{[\Omega(z)\,\Omega(0)]^{1/2} \; z\,|f_{1j}''(0)|} \; d\zeta\,. \tag{113}$$

Since

$$\omega z f_{12}(0) = i\omega\left[\int_0^z \Omega(z)\,dx + 2\int_0^\zeta \Omega(x)\,dx\right]\,,$$

equation (113) with j = 2 is in a form suitable for evaluation by the method of stationary phase (Jeffreys, 1950, p. 505). There is no point of stationary phase within the interval of integration; thus, p_{12}, being of order ω^{-3}, may be neglected.

For the case j = 1, equation (113) may be broken into two integrals on the basis of the interval of integration, the first integral being over the interval $0<\zeta<z$ and the second over the interval $z<\zeta<H$. The method of stationary phase may be applied to the latter to prove that it is of order

ω^{-3}. Therefore,

$$p_1(z,r) \sim \frac{2e^{-\omega z\int_0^z \Omega(x)dx}}{[\Omega(z)\Omega(0)]^{\frac{1}{2}}\,\omega\int_0^z \Omega^{-1}(x)dx} \int_0^z -\frac{i\,M(\zeta,0)}{2\omega\Omega(\zeta)}d\zeta + O(\omega^{-3}). \tag{114}$$

The asymptotic behavior of the pressure response is given by the sum of (109) and (114), whence

$$p(z,r) \sim \frac{2e^{-i\omega\int_0^z \Omega(x)dx}}{[\Omega(z)\,\Omega(0)]^{\frac{1}{2}}\,\omega\int_0^z \Omega^{-1}(x)\,dx}\left\{1+\frac{i}{\omega}\left[\frac{b_0(z)}{z}-\int_0^z \frac{M(\zeta,0)}{2\Omega(\zeta)}d\zeta\right]\right\} + O(\omega^{-3}).$$

Identifying the quantity in braces as the first two terms of an exponential expansion, one may write the above equation as

$$p(z,0,t) \sim 2\left[\Omega^{\frac{1}{2}}(z)\,\Omega^{\frac{1}{2}}(0)\,\omega\int_0^z \Omega^{-1}(x)dx\right]^{-1} \exp i\omega\left\{t+\frac{b_0(z)}{\omega^2 z} - \int_0^z\left[\Omega(\zeta)+\frac{M(\zeta,0)}{2\omega^2\Omega(\zeta)}\right]d\zeta\right\} + O(\omega^{-3}). \tag{115}$$

For sufficiently large z the term in b_0 may be neglected.

4.5. Asymptotic Evaluation $r \neq 0$

Equation (91) may be reduced by means of the Sommerfeld transformation (see section 3.3) to the form

$$p_0(z,r) = \int_B H_0^{(2)}(\omega v r)\,[a(z)\,a(0)]^{-\frac{1}{2}}\, E(-{}^{z}_{0},v)\, v\,dv\,. \tag{116}$$

The path of integration B joins $v = -i\infty$ to $v = \infty$, avoiding the singularities and staying on the upper Riemann sheet of the v-plane. This path is sketched in figure 5. For sufficiently large r the Hankel function may be replaced by the first two terms of its asymptotic expansion; thus

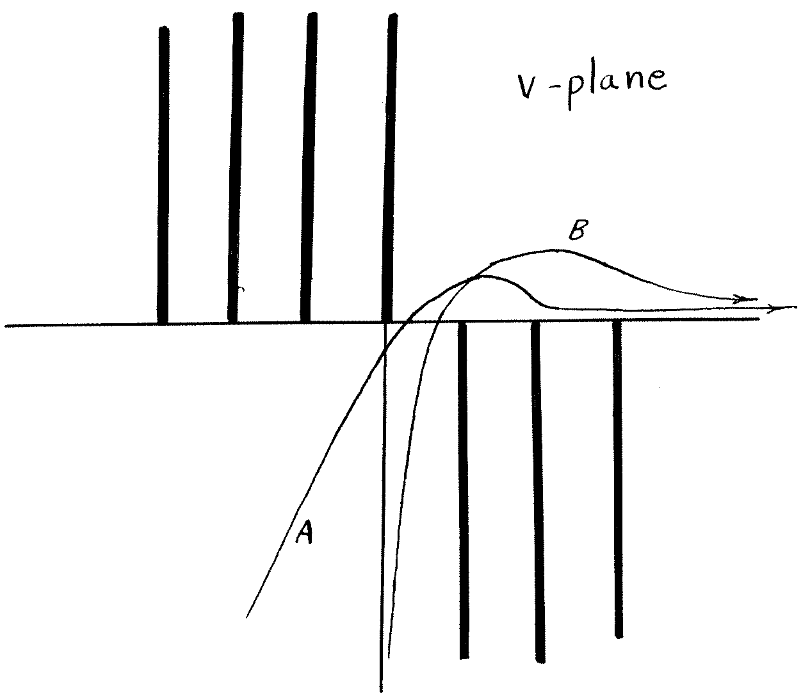

Figure 5

$$p_0(z,r) \sim \frac{e^{i\pi/4}}{(\frac{1}{2}\pi\omega r)^{1/2}} \int_B e^{-\omega r u_0(v)} \frac{v^{1/2}}{[a(z)\,a(0)]^{1/2}} \left[1 + (8\omega v r)^{-1}\right] dv, \tag{117}$$

where $u_0(v) = iv + r^{-1}\int_0^z a(x)\,dx.$ (118)

The form of equation (117) suggests evaluation by steepest descents. The saddle points, being the zeros of $u_0'(v)$, are the roots of

$$r = v_0 \int_0^z \alpha^{-1}(x, v_0)\,dx, \tag{119}$$

where $\alpha(x,v) = -i\,a(x) = [\Omega^2(x) - v^2]^{1/2}.$ (120)

It will be supposed that the ratio r/z is sufficiently small that a root $v = v_0$ exists on the real axis in the interval $0 \leq v_0 \leq \breve{\Omega}(0,z)$. v_0 is the only root of (119) existing on the axes of the v-plane on the upper Riemann sheet.

It remains to consider the possibility of the existence of saddle points elsewhere in the v-plane. The major portion of the v-plane may be excluded on the basis of the condition implied in (119), namely

$$\mathcal{R}\left[v\int_0^z a^{-1}(x)\,dx\right] = \int_0^z |a(x)|^{-2}\,\mathcal{R}(v a^*(x))\,dx = 0,$$

a^* being the complex conjugate of a. To the right of the cuts in both the upper half and lower half v-plane, $\mathcal{R}\,va^*(x)$ has a single sign for all x, $0 < x < z$; thus, the integral cannot vanish. A similar statement applies to the region to the left of the cuts in both upper and lower half planes. If other saddle points exist they must lie in the strips bounded by the cuts and the real axis.

The path of steepest descents through $v = v_0$ is given by

$$\mathcal{J}\,u_0(v) = |u_0(v_0)| \quad .$$

If $v = x + iy$ then asymptotically the path of steepest descents must approach

$$rx \pm zy = v_o r + \int_0^z \alpha(x, v_o)\,dx,$$

the upper sign being valid in the right half plane and the lower sign in the left half plane. The fact that in the interval $0 \leq v < \breve{\Omega}(0,z)$ the maximum value of $\mathfrak{R}u_o$ is attained at the saddle point implies that the path of steepest descents can cross the real axis to the left of $v = \breve{\Omega}(0,z)$ only at $v = v_o$. The path of steepest descents begins in the third quadrant, crosses the real axis at $v = v_o$, arches up into the first quadrant, and then turns downward to intersect the real axis at a point to the right of $v = \breve{\Omega}(0,x)$. Consider the path A_o which follows the path of steepest descent from the point at infinity, through the saddle point, to a point a finite distance above the second crossing of the real axis; from this point A_o parallels the real axis to positive infinity. The path A_o is sketched in figure 5. On the section of A_o which parallels the real axis, the integrand of (117) is exponentially attenuated by the real part of $e^{-\omega r u_o}$; consequently, the integral over this path becomes arbitrarily small as r increases. The remainder of A_o is a path of steepest descents.

It is easily seen that the path B deforms continuously into A_o. Moreover, the asymptotic evaluation of the integral of (117) over the path A_o may be accomplished by the application of the method of steepest descents at the saddle point $v = v_o$; whence

$$p_o(z,r) \sim \frac{2}{\omega} \mathfrak{X}(r,z,v_o)\, e^{-i\omega W_o}\left(1 + \frac{i\,b_1(z,r,v_o)}{\omega r}\right) + O(\omega^{-3}), \tag{121}$$

where

$$\mathfrak{X}(r,z,v_o) = \frac{v_o^{1/2}}{r}\left[\alpha(z,v_o)\,\alpha(0,v_o)\,|u_o''(v_o)|\right]^{-1/2}, \tag{122}$$

$$W_o(r,z,v_o) = v_o r + \int_0^z \alpha(x,v_o)\,dx, \tag{123}$$

$$|u_0''(v_0)| = r^{-1}\left[\frac{\partial}{\partial v}\left(v\int_0^z \alpha^{-1}(x,v)\,dx\right)\right]_{v=v_0} = r^{-1}\,\frac{\partial r(v_0,z)}{\partial v_0}. \tag{124}$$

The expression on the extreme right of (124) is derived from the quantity to its left by considering (119) as a definition of $r = r(v_0,z)$. The term in $b_1(z,r,v_0)$ is derived and defined in appendix B; it is included here only for completeness since it will be neglected later on the basis of the largeness of r.

The second term of the series (101) must also be evaluated. Equation (111) gives this term in its most convenient form. By applying the Sommerfeld transformation to the inner integral and substituting for the Hankel function its asymptotic representation, equation (111) may be transformed to

$$p_{1j}(z,r) \sim \omega^{-1} e^{i\pi/4}(2\pi\omega r)^{-1/2}\int_0^H d\zeta\int_B e^{-\omega r u_{1j}}\, M(\zeta,v)$$
$$a^{-1}(\zeta)\, v^{1/2}\,[a(z)\,a(0)]^{-1/2}\,dv, \tag{125}$$

where $u_{1j} = iv + z\,f_{1j}(v)/r$.

The inner integral may be evaluated by steepest descents for at least a limited region of the r,z plane. Saddle points are defined as the roots of

$$r = \begin{cases} \pm v\int_\zeta^z \alpha^{-1}(x,v)\,dx + v\int_0^\zeta \alpha^{-1}(x,v)\,dx & (j=1) \\[2ex] +v\int_0^z \alpha^{-1}(x,v)\,dx + 2v\int_0^\zeta \alpha^{-1}(x,v)\,dx & (j=2) \end{cases} \tag{126}$$

It will be supposed that r and z are such that for all ζ, $0 \leq \zeta \leq H$, (126) admits a positive real root $v = v_{1j}(\zeta)$ less than the smaller of $\breve{\Omega}(0,z)$ and $\breve{\Omega}(0,\zeta)$.

The path of integration B may be deformed continuously into a path

A_{1j} which, being analogous to A_o, consists of a portion of the path of steepest descents and a line parallel to the real axis. An argument similar to that employed previously shows that for sufficiently large ωr the asymptotic expression for the inner integral depends only upon the integral over the neighborhood of the saddle point $v = v_{1j}(\zeta)$. This contribution is found by steepest descents to be

$$p_{1j}(z,r) \sim \frac{1}{\omega^2 r}\int_0^H \frac{-i M(\zeta, v_{1j})}{\alpha(\zeta, v_{1j})} v_{1j}^{1/2}\left[\alpha(z,v_{1j})\,\alpha(0,v_{1j})\,|u_{1j}''(v_{1j})|\right]^{1/2} e^{-\omega r u_{1j}}\, d\zeta . \tag{127}$$

But

$$\omega r u_{1j} = i\omega \begin{cases} v_{11} r \pm \int_\zeta^z \alpha(x,v_{11})dx + \int_0^\zeta \alpha(x,v_{11})dx & (j = 1) \\ v_{12} r + \int_0^z \alpha(x,v_{12})dx + 2\int_0^\zeta \alpha(x,v_{12})dx & (j = 2) \end{cases} .$$

For j = 2, equation (127) is in a form suitable for evaluation by the method of stationary phase. It is easily shown in the case j = 2 that the point of stationary phase $v_{12} = \Omega(\zeta)$ does not lie in the interval of integration. The integral must be of order ω^{-1} and p_{12}, being of order ω^{-3}, may be neglected.

For j = 1 it is convenient to split the integral of equation (127) into two parts, the first part covering the interval $0 < \zeta < z$ and the second, $z < \zeta < H$. The argument applied to the case j = 2 shows that the second of these integrals is also negligible. However, for j = 1 and $\zeta < z$, v_{11} becomes identical with v_o (see equation (119)). In this case the exponential is no longer a function of ζ, which, of course, eliminates the possibility of evaluation by the method of stationary phase. Neglecting terms of order ω^{-3}, one finds

$$p_1(z,r) \sim \frac{2}{\omega} \mathfrak{X}(r,z,v_0)\, e^{-i\omega W_0} \int_0^z \frac{-i\, M(\zeta,v_0)}{2\omega\alpha(\zeta,v_0)}\, d\zeta . \tag{128}$$

In deriving (121) and (128), one has supposed that r and z are such that there exists for all ζ, $0 \leq \zeta \leq H$, a positive real number v_3 (ζ) such that

$$\left. \begin{aligned} r &= \pm v_3 \int_\zeta^z \alpha^{-1}(x,v)\,dx + v_3 \int_0^\zeta \alpha^{-1}(x,v)\,dx , \\ 0 &\leq v_3 \leq \begin{cases} \breve{\Omega}(0,z) & z > \zeta \\ \breve{\Omega}(0,\zeta) & z < \zeta \end{cases} . \end{aligned} \right\} \tag{129}$$

It may now be shown that (129) is a sufficient condition that $p(z,r) \sim p_0(z,r) + p_1(z,r)$.

From equation (100)

$$p(z,r) - \sum_{n=0}^{1} p_n(z,r) = \int_D J_0(\omega v r) \sum_{n=2}^{\infty} y_n(z)\, v\, dv .$$

Replacing $y_n(z)$ by (98), one finds

$$p(z,r) - \sum_{n=0}^{1} p_n(z,r) = -\int_D J_0(\omega v r)\, v\, dv \int_0^H \mathcal{G}(z|\zeta)\, M(\zeta,v) \sum_{n=2}^{\infty} y_{n-1}(\zeta)\, d\zeta .$$

For $n \geq 2$ it may be shown by induction that

$$y_{n-1}(\zeta) = \omega^{-1} e^{-\omega \int_0^\zeta a(x)dx} F_n(\zeta,v)$$

where $F_n(\zeta, v)$ contains only exponentials of the form $\exp\left\{-\omega \int_{\zeta_1}^{\zeta_2} a(x)dx\right\}$, $\zeta_2 > \zeta_1$. The proof of section 4.3 may be used to prove that the series $F = \sum_{n=2}^{\infty} F_n$ converges for sufficiently large ω. Thus equation (100) becomes

$$p(z,r) - \sum_{n=0}^{1} p_n(z,r) = -\omega^{-1} \int_D J_0(\omega v r)\, v\, dv \int_0^H \mathcal{G}(z|\zeta)\, M(\zeta,v)\, e^{-\omega \int_0^\zeta a(x)dx} F(\zeta,v)\, d\zeta ,$$

$$p(z,r) - \sum_{n=0}^{1} p_n(z,r) = -\omega^{-1}\int_0^H d\zeta \int_D J_0(\omega v r)\, e^{-\omega\int_0^\zeta a(x)dx}\, F(\zeta,v)$$

$$M(\zeta,v)\, \mathcal{G}(z|\zeta)\, v dv .$$

The interchange of order of integration is justified provided (129) is satisfied, since a path D has already been described on which $\left| J_0(\omega v r) \exp\left\{-\omega\int_0^\zeta a(x)dx\right\}\right|$ is bounded. The inner integral may be transformed by the Sommerfeld transformation and evaluated by steepest descents. The existence of a root v_3 of (129) guarantees that there will exist a saddle point on the positive real axis to the left of the singularities. The integral is thus shown to be of order ω^{-2}(an $\omega^{-1}$is included in $\mathcal{G}(z|\zeta)$, an $\omega^{-1/2}$ is introduced when the Hankel function is replaced by its asymptotic representation, and an additional $\omega^{-\frac{1}{2}}$ is contributed by steepest descents evaluation). Therefore, the existence of a root of (129) is a sufficient condition that

$$p(z,r) - \sum_{n=0}^{1} p_n(z,r) = O(\omega^{-3}),$$

or, what is the same thing,

$$p(z,r) \sim \sum_{n=0}^{1} p_n(z,r). \tag{130}$$

Substituting (121) and (128), one may write (130) as

$$p(z,r) \sim \frac{2}{\omega}\, \Sigma(r,z,v_0)\, e^{-i\omega W_0}\left[1 + \frac{i\, b_1(z,r,v_0)}{\omega r} - \frac{i}{2\omega}\int_0^z \frac{M(\zeta,v_0)}{\alpha(\zeta,v_0)}d\zeta\right] + O(\omega^{-3}).$$

Finally, one may identify the bracketed term as the first two terms of the expansion of an exponential. Then

$$p(z,r,t) \sim \frac{2}{\omega}\, \Sigma(r,z,v_0)\, e^{i\omega(t-W)} + O(\omega^{-3}), \tag{131}$$

where

$$W = W_0 - \frac{b_1(z,r,v_0)}{\omega^2 r} + \frac{1}{2\omega^2}\int_0^z \frac{M(\zeta,v_0)}{\alpha(\zeta,v_0)}\,d\zeta . \tag{132}$$

For sufficiently large r one may neglect the term in b_1 (z,r,v_0); then

$$W(r,z,v_0) \sim v_0 r + \int_0^z \left[\alpha(x,v_0) + \frac{M(\zeta,v_0)}{2\omega^2\alpha(\zeta,v_0)}\right] d\zeta . \tag{133}$$

It should be recalled that a sufficient condition that (131) be valid is that a root $v_3(\zeta)$ exists which satisfies (129) for all ζ, $0 \leq \zeta \leq H$.

4.6 Physical Interpretation

Geometrical acoustics is the high frequency limit of the solution of the wave equation. Specifically, it is derived (appendix C) by assuming a solution of the form (131) will satisfy the differential equation (16). It appears likely then that geometrical acoustics will provide a convenient framework upon which to hang the interpretation of the results of the preceding section.

It is shown in appendix C that the system of surfaces W_0 = constant is the eikonal. The orthogenal trajectories to this system are the rays. It follows that the unit vector tangent to the ray at any point is given by

$$\underset{\sim}{e}_p = \nabla W_0 / |\nabla W_0| = \Omega^{-1}(z)\left[v_0 \underset{\sim}{e}_r + \alpha(z,v_0) \underset{\sim}{e}_z\right] \tag{134}$$

The attitude to be adopted in this investigation is that the receiver is located at the point (z,r). Reference to a ray should always be understood to mean the ray passing through the point (z,r) (it will be shown that within the region of validity of the asymptotic approximation (131) only one ray passes through each point). It is convenient to define a quantity $\vartheta(\zeta)$ as the complement of the angle between the upward vertical and the ray tangent at altitude ζ. From (134) it is seen that $v_0 = \Omega(z) \cos\vartheta(z)$. For any given ray, v_0 is a constant; therefore,

$$v_0 = \Omega(z) \cos\vartheta(z) = \Omega(\zeta) \cos\vartheta(\zeta), \tag{135}$$

which is, of course, Snell's law. It follows that

$$\alpha(\zeta, v_0) = \left[\Omega^2(\zeta) - v_0^2\right]^{\frac{1}{2}} = \Omega(\zeta) \sin\vartheta(\zeta). \tag{136}$$

The ray theory given above, being valid only for very large frequencies, may be referred to as the zero order ray theory. A more precise theory will be formulated on the basis of (131).

Equation (129), which provides a sufficient condition for the validity of (131) is susceptible of a zero order ray theory interpretation. Inasmuch as v_3 is less than the smaller of $\breve{\Omega}(0,z)$ and $\breve{\Omega}(0,\zeta)$, it is possible to define a real variable $\psi(z,r,\zeta)$ by $v_3 = \Omega(0)\cos\psi$. Suppose $z > \zeta$; then ψ is no longer dependent upon ζ. Equation (129) requires that a ψ exist such that

$$r = \Omega(0) \cos\psi \int_0^z \left[\Omega^2(x) - \Omega^2(0)\cos^2\psi\right]^{-\frac{1}{2}} dx,$$

$$\Omega(0) \cos\psi < \breve{\Omega}(0,z).$$

Suppose one identifies $\psi = \vartheta(0)$. The first of these equations becomes the equation of the zero order ray leaving the source at an angle $\vartheta(0)$ and always having a slope $\tan\vartheta(z) > 0$. The second equation is a necessary condition that such a ray can reach the point (z,r) (see appendix C, equations (186) and (187)). If one supposes $z < \zeta$, (129) requires that there be no ψ such that

$$r = \Omega(\zeta)\left\{\int_0^\zeta \left[\Omega^2(x) - \Omega^2(\zeta)\right]^{-\frac{1}{2}} dx + \int_z^\zeta \left[\Omega^2(x) - \Omega^2(\zeta)\right]^{-\frac{1}{2}} dx\right\}.$$

This equation may be recognized as that of the zero order ray path which has had a positive slope from $z = 0$ to ζ and then a negative slope from

ζ to z. Such a ray, i.e., one which has reversed its vertical direction of propagation, may be referred to as a "completely refracted" ray. Therefore, a sufficient condition that (131) be a valid asymptotic approximation is that the point (z,r) lie on a zero order ray which has maintained a positive slope and that that point does not lie on a "completely refracted" ray.

A second order ray theory may be deduced from equation (131). In that equation the surfaces of constant phase are given by W = constant. Inasmuch as isotropy obtains, the orthogonal trajectories to these surfaces are the rays. From (133)

$$W \sim W_0 + \int_0^z \frac{M(\zeta,v_0)}{2\omega^2 \alpha(\zeta,v_0)}\, d\zeta .$$

By direct operation

$$\nabla W \sim \left[\Omega(z) + \frac{M(z,v_0)}{2\omega^2 \Omega(z)}\right] \underset{\sim}{e}_p + \frac{1}{2\omega^2}\left[N(z,v_0)\,\Omega(z) - \frac{M(z,v_0)}{\Omega(z)} \operatorname{ctn} \vartheta(z)\right] \underset{\sim}{e}_n , \tag{137}$$

where

$$N(z,v_0) = \left[\alpha(z,v_0) \frac{\partial r(v_0,z)}{\partial v_0}\right]^{-1} \int_0^z \frac{\partial}{\partial v_0}\left(\frac{M(\zeta,v_0)}{\alpha(\zeta,v_0)}\right) d\zeta , \tag{138}$$

$$\underset{\sim}{e}_n = \underset{\sim}{e}_r \sin\vartheta(z) - \underset{\sim}{e}_z \cos\vartheta(z). \tag{139}$$

It is apparent from (134) that $\underset{\sim}{e}_n$ is the unit principal normal to the zero order ray at altitude z. The unit tangent vector to the second order ray at any point is

$$\underset{\sim}{e}_p^{(2)} = \nabla W / |\nabla W| \sim \underset{\sim}{e}_p + \frac{1}{2\omega^2}\left[N(z,v_0) - \frac{M(z,v_0)}{\Omega^2(z)} \operatorname{ctn} \vartheta(z)\right] \underset{\sim}{e}_n . \tag{140}$$

The second order ray path is, therefore, frequency dependent.

It is of some interest to have an estimate of the order of magnitude of the bracketed quantity in (140). From (87), (135), and (136) one finds

$$M(z,v_0) = M_1(z)\sin^{-4}\vartheta(z) - M_2(z)\sin^{-2}\vartheta(z), \tag{141}$$

$$\frac{\partial}{\partial v_0}\frac{M(\zeta,v_0)}{\alpha(\zeta,v_0)} = \frac{\cos\vartheta(\zeta)}{\sin^3\vartheta(\zeta)}\left[5M_1(\zeta)\sin^{-4}\vartheta(\zeta) - 3M_2(\zeta)\sin^{-2}\vartheta(\zeta)\right], \tag{142}$$

where $M_1(z) = (5/16)(\Omega^{2\prime}(z)/\Omega^2(z))^2$, $M_2(z) = \frac{1}{4}\Omega^{2\prime\prime}(z)/\Omega^2(z)$. (143)

For the lower 70 km. of the earth's atmosphere

$$\left.\begin{aligned} &M_1 \doteq M_2 \doteq 10^{-10}\ (\text{meters})^{-2},\\ &\Omega^{-2}(z) \doteq 10^{5}\ (\text{meters/sec.})^{2},\\ &M/\Omega^2(z) \doteq 10^{-5}(|\sin^{-4}\vartheta(z)| + |\sin^{-2}\vartheta(z)|)\ (\text{sec.})^{-2}. \end{aligned}\right\} \tag{144}$$

where the symbol $\doteq$ is to be read "is the same order of magnitude as".

In view of these magnitudes one may say that the deviation of the second order ray from the zero order ray is likely to be of importance only for small ϑ, say less than 10°.

Consider a point (z,r) lying near the zero order limiting ray (the boundary of the shadow zone corresponding to $v_0 = \Omega(0)$), e.g., the point A in figure 6. On this ray $\vartheta(\zeta)$ is small near $\zeta = 0$; from equations (138) and (142) it is seen that this implies a large positive $N(z,v_0)$. It follows from (140) that the second order rays are directed somewhat into the zero order shadow zone, the deviation from the zero order ray being greater the lower the frequency.

At a point near the apex of a ray, e.g., B in figure 6, $\vartheta(z)$ is small; this implies large N and M $\text{ctn}\,\vartheta/\Omega^2$. Moreover, if B is sufficiently close to the limiting ray then the former

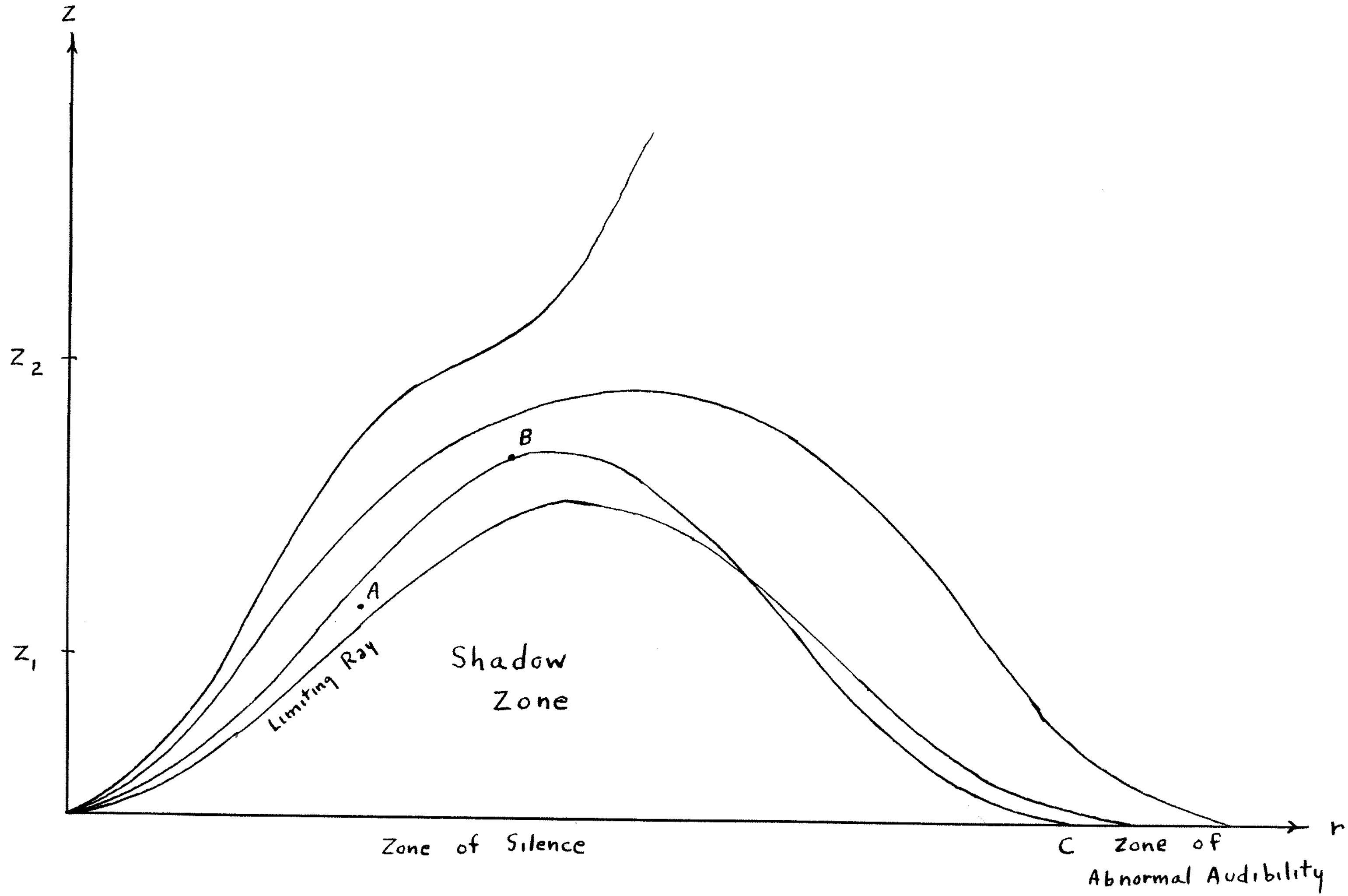
z
z_2
z_1
r
B
A
Limiting Ray
Shadow Zone
Zone of Silence
C
Zone of Abnormal Audibility

Figure 6

dominates the latter. It follows from (140) that the second order ray will have a smaller slope than the zero order ray; in fact, for sufficiently large ω and small $\vartheta(z)$, the second order ray may be horizontal or even sloping downward. It is, of course, impossible to trace the second order ray beyond, or even all the way to, the limiting ray on the basis of the theory presented here.

A natural question which arises here concerns the possibility of assigning to the wave a phase velocity which is not dependent upon r such that the second order ray may be traced by Snell's law; in other words, does there exist a $v = v(z,\omega)$ such that

$$v(z,\omega)\, \underset{\sim}{e}_r \cdot \underset{\sim}{e}_p^{(2)} = v(z,\omega) \quad \cos\,(\underset{\sim}{e}_r, \underset{\sim}{e}_p^{(2)}) = \text{constant}.$$

From (140) it may be shown that

$$\underset{\sim}{e}_r \cdot \underset{\sim}{e}_p = \Omega^{-1}(z) \left\{ v_o + \tfrac{1}{2}\omega^{-2}\left[\alpha(z,v_o)N(z,v_o) \; - \; v_oM(z,v_o)/\,\Omega^2(z)\right] + O(\omega^{-3})\right\}.$$

This expression, being a function of r, rules out the possibility that a velocity of the hypothesized form may exist.

It is of great importance to state specifically the limitations of the second order ray theory. At any point of the region of z and r in which (131) is a valid asymptotic solution it is possible to choose ω so large that (131) differs from the true pressure response by less than any preassigned quantity. Unfortunately, there is no easy way of determining just how large ω must be to obtain the desired accuracy. Generally, however, it may be presumed that if the second order term is small then the sum of the higher order terms will be negligible. Thus, the second order ray theory given here can at best give only small differences from the zero order ray theory.

It is of some interest to consider the energy flux. For this purpose it will be advantageous to write (131) in the form

$$p(z,r,t) = 2\,\omega^{-1} X \exp\left\{ i\omega t - i\omega W + S/\omega^2 + O(\omega^{-3}) \right\},$$

where S, a function which will be determined in section V, is independent of ω. Since g = 0, (12) implies

$$\underset{\sim}{u}(z,r,t) = \frac{2X}{\rho\omega}\, e^{i\omega t - i\omega W + \omega^{-2}S} \left\{ \nabla W + i\frac{\nabla X}{\omega X} + O(\omega^{-3}) \right\}$$

From (14) it is seen that the energy flux is the product of the two preceding equations. The complex phase representation is, of course, unsatisfactory for such a product; it is, therefore, necessary to form the product from only the real parts of each expression. Moreover, the energy flux is of interest only insofar as it represents propagating energy; thus it will suffice to compute only the time average energy flux $\underset{\sim}{\mathcal{J}}$ which is equal to the energy flux $\underset{\sim}{J}$ less terms sinusoidal in time, e.g., $\sin\omega(t - W)\cos\omega(t - W)$. Taking account of these considerations, one finds

$$\underset{\sim}{\mathcal{J}}(r,z) = \frac{2X^2}{\rho\omega^2}\, e^{2\omega^{-2}\Re S} \left\{ \nabla W + O(\omega^{-3}) \right\}$$

From (137) and (140) $$\underset{\sim}{\mathcal{J}}(r,z) = \frac{2X^2\Omega(z)}{\rho\omega^2}\, e^{-\omega^{-2}\left(2\Re S + \frac{M(z,v_0)}{2\Omega^2}\right)} \underset{\sim}{e}_p^{(2)} + O(\omega^{-5}). \qquad (145)$$

The fact that the time average energy flux flows along the second order rays gives a significance to these rays. The discussion of this equation in its complete form must be deferred until the function S is found in section V. However, some significance may be attached to (145) by considering only terms of order ω^{-2}. Then

$$\underset{\sim}{\mathcal{J}}(z,r) = \frac{2X^2\Omega(z)}{\rho\omega^2}\, \underset{\sim}{e}_p + O(\omega^{-4}).$$

Using (122), (124), (135), and (136), one may write this as

$$\underset{\sim}{\mathcal{J}}(z,r) = \frac{2\Omega(0)}{\rho\omega^2 r \sin\vartheta(z)}\left[\frac{\partial r(v_0,z)}{\partial(\sin\vartheta(0))}\right]^{-1}\underset{\sim}{e}_p\,. \tag{146}$$

For the interpretation of (146) it is desirable to know the time average rate of energy emission E from the source. Computation of E requires a knowledge of $\underset{\sim}{\mathcal{J}}$ in the immediate vicinity of the source. This may be computed from (12), (14), and (92). Integrating the $\underset{\sim}{\mathcal{J}}$ so found over an arbitrarily small sphere enclosing the source, one finds

$$E = 8\pi\Omega(0)/(\rho\omega^2) + O(\omega^{-4})\,. \tag{147}$$

With this information one may calculate the energy flux at the point (z,r) from the simple assumption that energy flows unimpeded in tubes bounded by the zero order rays. In appendix C, where this has been done, it is found that one arrives at equation (146). This provides the physical significance of X, namely it is the factor which takes account of the geometric divergence of the wave energy. In a homogeneous medium $X = R^{-1}$ as may be shown directly.

In the foregoing paragraphs attention has been devoted to the case $r \neq 0$. It will now be shown that (131) includes (115) as a special case provided that R is sufficiently large; specifically, neglecting $b_0(z)/(\omega^2 z)$ in (115) and $b_1(z,r,v_0)/(\omega^2 r)$ in (131), one may show that the limit of (131) as $r \to 0$ is (115). In the special case $r = 0$, $v_0 \to 0$, $\alpha(x,v_0) \to \Omega(x)$.

From (124)

$$|u_0''(v_0)| \to r^{-1}\int_0^z \Omega(x)\,dx\,.$$

From (119)

$$r/v_0 \to \int_0^z \Omega(x)\,dx\,.$$

From (122)

$$X \to [\Omega(z)\,\Omega(0)]^{-1/2}\,[\int_0^z \Omega(x)\,dx]^{-1}. \tag{148}$$

From (133)

$$W(r,z,v_o) \to \int_0^z [\Omega(\zeta) + \frac{M(\zeta,0)}{2\omega^2\Omega(\zeta)}]\,d\zeta. \tag{149}$$

Substituting (148) and (149) into (131), one finds

$$p(z,0) = \frac{2}{\omega}\,[\Omega^{1/2}(z)\,\Omega^{1/2}(0)\int_0^z \Omega(x)\,dx]^{-1}$$

$$\exp i\omega\{t - \int_0^z [\Omega(\zeta) + \frac{M(\zeta,0)}{2\omega^2\Omega(\zeta)}]\,d\zeta\}. \tag{150}$$

Comparison of (115) and (150) shows that they differ only by a term of order $(\omega^{-2}z^{-2})$, a term which is negligible for sufficiently large z.

Although equation (150) is a very special case, nevertheless, it is of some interest because only along the z-axis do all frequencies travel identical ray paths. Thus, in this case one may speak of group velocity. From (150) the phase velocity is found to be

$$v \sim \Omega^{-1}(z)\,[1 - \frac{M(z,0)}{2\omega^2\Omega^2(z)}] + O(\omega^{-4}).$$

The group velocity

$$U = v + \frac{\omega}{v}\frac{dv}{d(\omega/v)} \sim \Omega^{-1}(z)\,[1 + \frac{M(z,0)}{2\omega^2\Omega^2(z)}] + O(\omega^{-4}).$$

In the lowest 40 km. of the atmosphere $M(z,0) > 0$; from 40 to 70 km. altitude $M(z,0) < 0$. Thus a wave packet would undergo anomalous dispersion $(U > v)$ in the former region and normal dispersion in the latter.

Finally, it is desirable to obtain some information on the initial behavior of an acoustic pulse. The operation and methods of section 3.6 are applicable. The operation (63) will be employed with $F(\omega)$ set equal to unity.

First, the source behavior should be specified. Applying the operation (63) to (92), one finds by the procedures of section 3.6 that the source pressure is given by

$$\bar{p}_s(z,r,t) \rightarrow 2R^{-1}\ 1(t - \Omega R)\ . \tag{151}$$

The problem of finding the pressure response at some point (z,r) which is distant from the source is, unfortunately, not so straightforward. The difficulty arises from the fact that (131) is an asymptotic solution requiring not only large r, but also large ω, for its validity. Thus, the operation (63) applied to (131) yields an asymptotic approximation to the Laplace transform of the pressure response at (z,r), namely,

$$\mathcal{L}\{\bar{p}(z,r,t)\} = 2X\ e^{-sW_0}\left[\frac{1}{s} + \frac{1}{2s^2}\int_0^z \frac{M(\zeta,v_0)}{\alpha(\zeta,v_0)}\,d\zeta + O(s^{-3})\right].$$

The inverse transform of e^{-sW_0} is known to be the Dirac delta function $\delta(t - W_0)$. The bracketed quantity is not known to be analytic at infinity; consequently, the justification for term by term inversion used in section 3.6 is no longer applicable. Karamata's theorem (Widder, 1946, p. 197) may be used provided one is willing to make certain plausible assumptions about the pressure response. Specifically, it must be assumed that the pressure response is bounded for all t and maintains the same sign for at least a finite interval of time after zero, i.e., the function is not infinitely oscillating at the time origin. Subject to these assumptions the bracketed quantity may be inverted term by term. The convolution of the two inverse transforms gives an asymptotic ($t \rightarrow 0$) expression for the pressure response

$$\bar{p}(z,r,t) = 2X\ 1(t - W_0)\left[1 + \tfrac{1}{2}(t-W_0)\int_0^z M(\zeta,v_0)/\alpha(\zeta,v_0)\,d\zeta + O((t-W_0)^2)\right]. \tag{152}$$

W_0 is the time required for a particle to travel from the source to the point (z,r) along the zero order ray path, the particle having a velocity $\Omega^{-1}(x)$ appropriate to its instantaneous altitude x.

Since $M(\zeta,v_0)/\alpha(\zeta,v_0) > 0$ in the lowest 40 km. of the earth's atmosphere there is a tendency for the amplitude of the pulse to increase slightly after onset. At points (z,r) near either the limiting ray ($v_0 = \Omega(0)$) or near the apex of a zero order ray, the coefficient of the term in (152) which is linear in time becomes quite large; at such points the pulse is significantly distorted even in the first few seconds after onset. It is interesting that the distortion is such as to increase the amplitude of the pulse.

Finally it must be established that the waves crossing the level $z = H$ have an upward component in their direction of propagation. For $z = H$ there is always a root of (129). This may be seen by observing that in the vicinity of the minimum z_2

$$\lim_{v_3 \to \Omega(z_2)} \int_{z_2-\epsilon}^{z_2+\epsilon} \alpha^{-1}(x,v_3)\,dx \to \infty .$$

Thus, for any r one may choose v_3 sufficiently close to $\Omega(z_2)$ so that (129) is satisfied. It follows that the plane $z = H$ is included in the region in which the solution of this section is valid. But the solution (131) represents an upward propagating wave. Therefore the boundary condition at $z = H$ is satisfied.

V. GENERAL ATMOSPHERE

The purpose of this section is to generalize the results of the preceding section in order to arrive at equations applicable to the idealized atmosphere of the earth. For this purpose the restriction imposed in section IV that $\mu = h = \Omega'(0) = g = 0$ will be removed. The concept of an absorber at altitude H will be retained.

Wherever order of magnitude calculations are required numbers appropriate to the physical properties of the earth's atmosphere will be employed.

5.1. Asymptotic Solution

It will be assumed that a solution of the differential system (16) and (17) may be formed from the superposition of the elementary solutions (18); then

$$p(z,r) = \chi(z,h) \int_D J_o(\omega v r)\, y(z)\, v\, dv. \tag{153}$$

y(z) must, of course, satisfy (20) and (21). By expanding $q(z,\lambda)$ in a power series of inverse powers of ω, one may write equation (20) as

$$y''(z) + \sum_{n=0}^{\infty} q_n(z) = 0, \tag{154}$$

where
$$\left.\begin{aligned} &q_{2n+1}(z) = 0 \quad (n = 0, 1, 2, \ldots), \\ &q_o(z,v) = -a^2(z), \\ &q_2(z,v) = -m^2 + m' + a^2(z) gG, \\ &\ldots \qquad \ldots \end{aligned}\right\} \tag{155}$$

The asymptotic solution of (154) is a linear combination of the two solutions Y_j given by Erdélyi (1956, pp. 83-4),

$$Y_j(z) = a^{-\frac{1}{2}}(z) \exp\left\{\pm\omega \int_0^z \left[a(\zeta) - \omega^2 Q(\zeta,v)/a(\zeta)\right] d\zeta \right. \\ \left. + Q(z,v)/(2a^2(z)\,\omega^2) + O(\omega^{-3})\right\}, \tag{156}$$

the upper sign being chosen for $j = 1$ and the lower sign for $j = 2$. $Q(\zeta, v)$ is defined by

$$Q(\zeta,v) = \tfrac{1}{2}\left[M(\zeta,v) + q_2(\zeta,v)\right] . \tag{157}$$

From such solutions one may construct a function which asymptotically satisfies (21), namely

$$y(z) = \left[a(z)a(h)\right]^{-\frac{1}{2}} \left\{ E(\mp {}^{z}_{h}, v)\, K_1(z,v) + B(z)\, E(-{}^{z}_{0}, v) E(-{}^{h}_{0}, v) K_2(z,v) \right\} , \tag{158}$$

where

$$K_1(z,v) = \exp\left\{\pm \omega^{-1}\int_h^z Q(\zeta,v)/a(\zeta)\; d\zeta + \omega^{-2} S(z,v) + O(\omega^{-3})\right\}, \tag{159}$$

$$K_2(z,v) = \exp\left\{\omega^{-1}\int_0^z Q(\zeta,v)/a(\zeta)\; d\zeta + \omega^{-1}\int_0^h Q(\zeta,v)/a(\zeta)\; d\zeta + \omega^{-2} S(z,v) + O(\omega^{-3})\right\}, \tag{160}$$

$$B(v) = 1 - \omega^{-1}\left[a'(0)/a^2(0) - 2m(0)/a(0)\right] + \tfrac{1}{2}\omega^{-2}\left[a'(0)/a^2(0) - 2m(0)/a(0)\right]^2 + O(\omega^{-3}), \tag{161}$$

$$S(z,v) = \tfrac{1}{2}\left[Q(z,v)/a^2(z) - Q(h,v)/a^2(h)\right]. \tag{162}$$

The $\pm$ ambiguity in (158) and (159) is resolved by choosing the upper sign if $z > h$ and the lower if $z < h$.

Substitution of (158) into (153) leads to

$$p(z,r) = p_1(z,r) + p_2(z,r), \tag{163}$$

$$p_1(z,r) = \chi(z,h)\int_D J_0(\omega v r)\; E(\mp {}^{z}_{h}, v)\; \frac{K_1}{[a(z)\,a(h)]^{1/2}}\; v\,dv , \tag{164}$$

$$p_2(z,r) = \chi(z,h)\int_D J_0(\omega v r)\; B(v)\; E(-{}^{z}_{0}, v)\; E(-{}^{h}_{0}, v)\; \frac{K_2}{[a(z)a(h)]^{1/2}}\; v\,dv . \tag{165}$$

As $z \to h$ and $r \to 0$

$$p_1(z,r) \to \int_D J_0(\omega v r)\; a^{-1}(h)\; e^{-\lambda a(h)|z-h|}\; v\,dv \quad + O(\omega^{-4}) + O(z-h)$$

Notice that $\exp(\frac{1}{2}\omega^{-1} gGa|z-h|)$ has been retained from the K_1 term since it is an increasing function of v; it is this extra term which accounts for the dependence of the exponent upon λ rather than ω. In complete analogy with the transformation from (35) to (36), one may employ Watson's integral identity to show that as $z \to h$ and $r \to 0$

$$p_1(z,r) \to R_1^{-1}\, \nu_1^{-1} \exp(-\Omega R_1\, \nu_1), \tag{166}$$

where R_1 and ν_1 were defined in (37) and (38). This is an acceptable source representation.

It is worth pausing a moment to compare the procedure above with that employed in section IV. In the latter section a convergent series solution was found for $y(z)$; in the present section one has taken advantage of an asymptotic solution by Erdélyi. One may show the two methods to be equivalent by observing that the method used by Erdélyi to establish the asymptotic nature of (156) is essentially the method of Liouville.

One may evaluate (164) asymptotically by first applying the Sommerfeld transformation and then replacing the Hankel function by its asymptotic representation. Thus

$$p_1(z,r) \sim \frac{\chi(z,h)}{(2\pi\omega r)^{1/2}}\, e^{i\pi/4} \int_B e^{-\omega r u_1}\, \frac{v^{1/2} K_1(z,v)}{[a(z)\, a(h)]^{+1/2}}\, dv, \tag{167}$$

where $$u_1(v) = iv \pm r^{-1} \int_h^z a(x)\, dx \; . \tag{168}$$

For sufficiently large ω, $K_1(z,v)$ is a slowly varying function; consequently, (167) may be evaluated by steepest descents in a manner completely analogous to that employed in reducing (117) to (121). Neglecting terms in $\omega^2 r$, one finds that

$$p_1(z,r) \sim \frac{\chi(z,h)}{\omega r}\, v_1^{1/2} \left[\alpha(z,v_1)\, \alpha(h,v_1)\, |u_1''(v_1)|\right]^{-1/2} K_1(z,v_1)$$

$$\exp\left\{-i\omega\left[v_1 r \pm \int_h^z \alpha(\zeta,v_1)\, d\zeta\right]\right\} + O(\omega^{-4}), \tag{169}$$

where

$$|u_1''(v_1)| = \frac{\partial}{\partial v_1}\left[\pm v_1 \int_h^z \alpha^{-1}(x,v_1)\ dx\right] = \frac{\partial r(v_1,z)}{\partial v_1}, \tag{170}$$

v_1 being defined by

$$r = \pm\ v_1 \int_h^z \alpha^{-1}(x,v_1)\ dx, \qquad v_1 < \breve{\Omega}(h,z). \tag{171}$$

The ambiguity of sign in each case is resolved by choosing the sign of $(z-h)$. The partial derivative on the right hand side of (170) is made meaningful by considering (171) as the definition of $r = r(v_1,z)$.

A similar treatment applied to (165) leads to

$$p_2(z,r) \sim \frac{\chi(z,h)}{\omega r} v_2^{\frac{1}{2}}\left[\alpha(z,v_2)\alpha(h,v_2)|u_2''(v_2)|\right]^{-\frac{1}{2}} B(v_2)\ K_2(z,v_2)$$
$$\exp\left\{-i\omega\left[v_2 r + \int_0^z \alpha(\zeta,v_2)\,d\zeta + \int_0^h \alpha(\zeta,v_2)\,d\zeta\right\} + O(\omega^{-4}), \tag{172}$$

where

$$|u_2''(v_2)| = \frac{\partial}{\partial v_2}\left[v_2\int_0^z \alpha^{-1}(x,v_2)\,dx + v_2\int_0^h \alpha^{-1}(x,v_2)\,dx\right] = \frac{\partial r(v_2,z)}{\partial v_2}, \tag{173}$$

v_2 being defined by

$$r = v_2\int_0^z \alpha^{-1}(x,v_2)\,dx + v_2\int_0^h \alpha^{-1}(x,v_2)\,dx, \qquad v_2 < \breve{\Omega}(0,z),\ \breve{\Omega}(0,h). \tag{174}$$

It is, of course, necessary to specify the region of z and r in which (169) and (172) are valid asymptotic approximations; in other words, for what values of z and r can one neglect terms on the basis of the largeness of ω? This question has already been discussed in the paragraph following equation (102); it was shown in that paragraph that a sufficient condition is the existence of an exponential factor in $y(z)$ that is capable of asymptotically dominating $J_0(\omega vr)$ on the path D. In (158) the only exponentials which are increasing functions of ω are the terms of the form

$E(-\frac{z}{0}, v)$; these, then, must be the terms upon which the validity of the asymptotic approximations depends. But these terms are independent of μ, g, and $\Omega'(0)$. Thus, it appears that the region of validity of the asymptotic approximation depends upon the zero order terms in the approximation of $y(z)$. In these terms the approximations of the present section and section IV differ only in that h is zero in the latter. It may be presumed upon this basis that the region of r and z for which the asymptotic approximations are valid is given by the same criterion as that found in section IV, namely the point (z,r) must lie on a zero order ray but must not lie on a " completely refracted" zero order ray.

5.2 Physical Interpretation

By neglecting terms of order ω^{-2}, one can see that p_1 and p_2 respectively reduce to representations of the direct wave and the wave reflected at the rigid surface. Presumably, this interpretation can be extended to the more complete approximations of p_1 and p_2; thus, (169) will be referred to as the direct wave and (172) as the reflected wave.

Equation (131) may be profitably compared to (169). Many of the differences between these equations are merely a consequence of the fact that $h = 0$ in the former equation. For instance, the factor 2 in (131) expresses the fact that the reflected wave reinforces the direct wave immediately since the source lies on the reflecting surface. Moreover, u_0 (equation (118)) differs from u_1 (equation (168)) only in that $h = 0$ in the former; a similar statement applies to the pair v_0 (equation (119)) and v_1 (equation (171)). A very real difference, however, is the factor $\chi(z,h)$ in (169). From (19)

$$\chi(z,h) = e^{-\frac{1}{2}\int_h^z \mu(x)dx}\left\{\left[1 - \frac{g}{2\omega^2}\left(G(z) - G(0)\right)\right] + O(\omega^{-4})\right\}. \tag{175}$$

Inasmuch as $gG \doteq 10^{-4}$ sec.$^{-2}$ the bracketed quantity in (175) differs from unity by an amount which will generally be unimportant. Quite the contrary is the case for the exponential. Since $\mu \doteq 10^{-4}$ it is obvious that for values of z of the order of say 10 km. the exponential of (175) is one of the most important factors determining the pressure amplitude. The factor $\chi(z,h)$ provides an exponential decrease in pressure amplitude as z increases; from (12) it may be seen that the velocity amplitude increases with altitude exponentially. The product of the two amplitudes does not depend exponentially upon z as must be the case, the product being proportional to the energy flux (14).

It remains to compare the complex phase factors of the two equations (169) and (131). The phase factor of (169) is given by

$$E(\mp {}^{z}_{h}, v)\, F_1(z,v) = \exp\left\{-i\omega v_1 r \mp i\omega \int_h^z \left[\alpha(\zeta, v_1) + \frac{1}{\omega^2}\, \frac{Q(\zeta,v_1)}{\alpha(\zeta,v_1)}\right] d\zeta \right.$$
$$\left. + \omega^{-2} S(z, v_1) + O(\omega^{-3})\right\}.$$

and that of (131) by

$$\exp\left\{-i\omega v_0 r - i\omega \int_0^z \left[\alpha(\zeta, v_0) + \frac{1}{\omega^2}\, \frac{M(\zeta,v_0)}{\alpha(\zeta,v_0)}\right] d\zeta + O(\omega^{-2})\right\}.$$

It has already been mentioned that if $h = 0$, $v_1 = v_0$. Thus, neglecting differences arising solely from the fact that h differs from zero in the former of these two phase factors, one may see that the phase factors differ in two significant ways: 1) the phase factor corresponding to (169) has been carried to a higher order of approximation by including the term S, and 2) to the order of terms in ω^{-1} the phase factor corresponding to (169) may be formed from that corresponding to (131) by replacing $\frac{1}{2}M(\zeta, v_0)$ by $Q(\zeta, v_1)$.

The second difference will be considered first. From (157) it is seen that $\frac{1}{2}M(\zeta,v_o)$ differs from $Q(\zeta,v_o)$ by $\frac{1}{2}q_2(\zeta,v_o)$; from (155)

$$\frac{1}{2}\ q_2(\zeta,v_o) = -m^2(\zeta)+m'(\zeta)-\alpha^2(\zeta,v_o)gG(\zeta) \doteq 10^{-10} \quad (\text{meters})^{-2}.$$

What is of interest is whether the term $q_2(\zeta,v_o)$ is likely to be of importance in comparison with $M(\zeta,v_o)$. In order to discuss this question one must resort to the zero order ray theory of appendix C. The derivation there is valid even if μ and g differ from zero. Consequently, the quantities $\alpha(\zeta,v_1)$ and v_1 respectively may be identified with $\Omega(\zeta)$ $\sin\vartheta(\zeta)$ and $\Omega(z)\cos\vartheta(z)$ in the same manner as the corresponding identification in equations (135) and (136). The magnitude of $M(\zeta,v_o)$ is then given by (144). It appears that as long as $\vartheta(\zeta)$ is nowhere on the ray less than, say 30°, q_2 and M are of comparable magnitude; however, as $\vartheta(\zeta)$ becomes smaller $M(\zeta,v_o)$ completely dominates $q_2(\zeta,v_o)$. Now, it was shown in section IV that it is only in regions reached by rays which have at some point been nearly horizontal that the term in $M(\zeta,v_o)$ is of any real importance. It follows then that the term q_2 is likely to be completely negligible.

The first difference between the phase factors cited concerns the factor S. It is one of the advantages of the method employed in this section that the higher order approximation S/ω^2 could be obtained so easily. It is this factor which was required in equation (145).

So far attention has been devoted solely to the direct wave. The reflected wave (172) differs from the direct wave (169) in two important respects: 1) it appears to originate from an image source at $z = -h$, $r = 0$, and 2) its amplitude is modified by a reflection coefficient $B(v_2)$.

The first difference is, of course, expected; the second difference merits some discussion. Equation (161) may be written as

$$B(v_2) = \exp\ (ib/\omega) + O(\omega^{-3}) \tag{176}$$

where

$$b(v_2) = 2\left[\frac{\Omega^{2\prime}(0)}{4\alpha^3(0,v_2)} - \frac{m(0)}{\alpha(0,v_2)}\right]$$

The quantities $\alpha(0,v_2)$ and v_2 respectively may be identified as $\Omega(0)$ $\sin\vartheta(0)$ and $\Omega(\zeta)\cos\vartheta(\zeta)$ where ϑ now refers to a ray starting from the image source and traveling to the point (z,r). It may be shown that $b \doteq 10^{-2}(\frac{1}{2}\sin^{-3}\vartheta(0) + 2\sin^{-1}\vartheta(0))$.

$\vartheta(0)$ is the complement of the angle of incidence. It appears that for an angle of incidence greater than, say 60°, the phase shift of (176) becomes important.

5.3. Conclusions

The purpose of this section will be to generalize the discussion of section 4.6. It will be convenient to choose h to be arbitrarily small. Then the sum of (169) and (172) may be written as

$$p(z,r) = (1 + B(v_0))\,\frac{\chi(z,0)}{\omega}\,I\,\exp\left\{-i\omega W + \omega^{-2}S(z,v_0) + O(\omega^{-3})\right\}, \tag{177}$$

where

$$W \sim W_0 + \omega^{-2}\int_0^z Q(\zeta,v_0)/\alpha(\zeta,v_0)\,d\zeta. \tag{178}$$

Notice (178) reduces to (133) for g and μ both zero; thus, (178) is the generalization of the definition (133).

From (176)

$$(1 + B(v_2)) = 2\exp\left\{\frac{ib}{2\omega} - \frac{b^2}{8\omega^2} + O(\omega^{-3})\right\}.$$

This equation may be employed to rewrite (177) as

$$p(z,r) = \frac{2\chi(z,0)}{\omega}\, X \quad \exp\Big\{-i\omega\,\left(W - b/2\omega^2\right) + \omega^{-2}\left(S' - b^2/8\right) + O(\omega^{-3})\Big\}. \tag{179}$$

Inasmuch as the influence of gravity has destroyed the isotropy, the orthogonal trajectories of the surfaces of constant phase are not necessarily rays. As was pointed out in section 4.6 the physical significance of the rays is that the ray tangents at each point give the direction of the time average energy flux at that point. Thus, the ray direction may be determined from the time average energy flux $\underset{\sim}{\mathcal{J}}$.

It may be shown by the same methods employed in section 4.6 that

$$\underset{\sim}{\mathcal{J}}(z,r) = \frac{2X^2\chi^2(z,0)}{\rho\,\omega^2}\; e^{-\omega^{-2}(2S - b^2/4)}\Big\{\nabla\left(W - b/2\omega^2\right) + \frac{gG}{\omega^2}\frac{\partial W_0}{\partial z}\,\underset{\sim}{e}_z\Big\} + O(\omega^{-5}).$$

By direct operation this expression may be reduced to

$$\underset{\sim}{\mathcal{J}}(z,r) = \frac{2X^2\chi^2(z,0)\,\Omega(z)}{\rho\,\omega^2}\; e^{-\Lambda/\omega^2}\,\underset{\sim}{e}_P^{(2)} + O(\omega^{-5}), \tag{180}$$

where

$$\Lambda = b^2/4 - 2S' - \left(\frac{Q(z,v_0)}{\alpha(z,v_0)} + \alpha(z,v_0)\,g\,G(z)\right)\frac{\sin\vartheta(z)}{\Omega(z)}, \tag{181}$$

and

$$\underset{\sim}{e}_P^{(2)} \sim \underset{\sim}{e}_P + \omega^{-2}\Big\{\Big[\int_0^z \frac{\partial}{\partial v_0}\left(\frac{Q(\zeta,v_0)}{\alpha(\zeta,v_0)}\right)d\zeta - \frac{1}{2}\frac{\partial b(v_0)}{\partial v_0}\Big]\Big[\alpha(z,v_0)\frac{\partial r}{\partial v_0}\Big]^{-1} - \Big[\frac{Q(z,v_0)}{\alpha(z,v_0)} + g\,G(z)\,\alpha(z,v_0)\Big]\frac{\cos\vartheta(z)}{\Omega(z)}\Big\}\underset{\sim}{e}_n. \tag{182}$$

The definition of the second order ray tangent unit vector in (182) is the generalization of the definition (140).

The exact behavior of the second order ray would be very difficult to ascertain; nevertheless, a great deal of information may be gained from (182) by inspection. The second order term in (182) is generally very small; however, this term becomes of major importance if, on the ray path under consideration, the angle $\vartheta(\zeta)$ (and, thus, $\alpha(\zeta, v_0)$) becomes moderately small. Therefore, rays which have a steep slope throughout their length will differ only trivially from the zero order ray. On the other hand, any ray some section of which is nearly horizontal will differ appreciably from the zero order ray. If the horizontal section is somewhere other than in the neighborhood of the source, it is not difficult to show that the important terms of (182) are the same as those of (140). The same result holds even if the horizontal section is in the neighborhood of the origin provided $\vartheta(0)$ is sufficiently small. It follows that the time average energy flux either near the apex of a zero order ray or near the limiting ray may be resolved into two components one of which is parallel to the zero order ray and the other normal to the zero order ray, the latter being directed into the shadow zone. The results are, therefore, qualitatively the same as those obtained in section 4.6; in fact, in the limit of small ϑ they are the same as those of 4.6.

The quantity Λ in (181) may be rewritten as

$$\Lambda = \frac{Q(z,v_0)}{\alpha^2(z,v_0)} \cos^2\vartheta(z) + \left[\frac{b^2(v_0)}{4} - \frac{Q(0,v_0)}{\alpha^2(0,v_0)} \right] - g\,G(z)\sin^2\vartheta(z). \tag{183}$$

At any point either sufficiently near the limiting ray or sufficiently near the apex of a zero order ray $\Lambda > 0$. It follows from (180) that the energy flux at such points is somewhat less than might be expected from the zero order ray theory. It should be noted that the apparent exponential dependence upon Λ in (180) is deceptive. It would be better, perhaps, to write

$$e^{-\Lambda/\omega^2} = 1 - \Lambda/\omega^2 + O(\omega^{-3}),$$

since it is only to this accuracy that the approximation is valid.

The principal interest in the foregoing analysis relates to the evidence of dispersion in the atmosphere. This dispersion manifests itself principally by a separation in the direction of propagation of the time average energy flux of the various frequencies. This phenomenon is then related to what is called angular dispersion in optics (e.g., the angular dispersion of light by a prism). The deviation of the second order ray from the zero order ray, of course, accounts for the leakage of energy into the shadow zone.

There is, perhaps, a more interesting aspect of this angular dispersion. Consider a point near the apex of a zero order ray, e.g., B in figure 5. In order that the second order ray theory be applicable it is necessary that the second order quantities in the vector $\underset{\sim}{e}_p^{(2)}$ (equation (182)) be small, i.e., the direction of $\underset{\sim}{e}_p^{(2)}$ may differ from $\underset{\sim}{e}_p$ by only a small amount. However, if $\vartheta(z)$ is sufficiently small then the second order quantity (the $\underset{\sim}{e}_n$ component of $\underset{\sim}{e}_p^{(2)}$) may be greater than the vertical component of $\underset{\sim}{e}_p^{(2)}$. In such a case the second order ray has reached its apex and at the point (z,r) has a downward vertical component. Therefore, for sufficiently high frequencies at least, the path of the time average energy flux reaches its apex at a smaller r than the corresponding zero order ray, the value of r at which the apex is reached increasing as ω increases. Unfortunately, it is not possible to trace the second order ray beyond, or even quite up to, the limiting ray. The conclusion of this theory must be that angular dispersion is likely to be of importance at points reached by rays which have been "completely refracted" or even almost "completely refracted".

Cox (1949, pp. 13-16) first suggested that angular dispersion might be of some importance in the atmosphere on the basis of observations made during the Helgoland blast.

These observations showed a predominately low frequency pressure response at the inner edge of the first zone of abnormal audibility (point C in figure 6); at points farther from the source the spectrum of the response shifted toward higher frequencies. The theory proposed by Cox to account for angular dispersion is not satisfactory (Cox, 1949, p. 501). It is interesting to note that the observed phenomenon is what would be predicted by the extrapolation of the theory presented in this paper.

It is hardly surprising that the atmosphere should behave as a dispersive medium. Although very short wave lengths may propagate locally practically undisturbed by the stratification, longer waves will not. Probably an even more serious hindrance to the propagation of long waves is the curvature of the wave front induced by the velocity variation of the atmosphere. Although the causes of dispersion in the atmosphere are obvious, nevertheless, it is difficult to predict what the effects of these causes will be. The present theory would indicate the effects are principally 1) low frequency waves are more sharply refracted than high frequency waves, 2) pulse shape is distorted, and 3) the reflection coefficient of even a rigid surface is considerably modified. If the receiver should lie on a zero order ray some portion of which is nearly horizontal, then these effects are greatly magnified.

Appendix A

Special attention must be devoted to the identification of the singularities $E(-\frac{z}{h},v)$. Since this function is an exponential it is apparent that its singularities are the same as those of the exponent $\int_h^z (v^2 - \Omega^2(x))^{\frac{1}{2}}dx$. Moreover, the singularities of the integral must lie on the locus of the singularities of the integrand, i.e., on the line segments defined by $\check{\Omega}^2(h,z) \leq v^2 \leq \hat{\Omega}^2(h,z)$ where $\hat{\Omega}(h,z)$ and $\check{\Omega}(h,z)$ are respectively the largest and smallest values of $\Omega(z)$ in the closed interval (h,z).

The nature and location of the singularities of this integral may be investigated qualitatively by considering only real v. For $v > \hat{\Omega}(h,z)$ the integral is pure real since the integrand is pure real by choice of the phase of $a(x)$. For v slightly less than $\hat{\Omega}(h,z)$ the integral has a small imaginary part of ambiguous sign; this portion corresponds to integration over those values of $a(x)$ in which $v^2 < \Omega^2(x)$. The sign of this imaginary portion is $\pm$ according to whether v has a $\pm$ imaginary part. Thus, the point $v = \hat{\Omega}(h,z)$ is a branch point. The property which this point possesses that identifies it as a branch point is that it is a local extreme value of $\Omega(x)$. The other values of x which yield local extreme values of Ω^2 are h, z, and any of the maxima or minima z_1, z_2, z_3, ... of $\Omega(x)$ which lie in the closed interval (h,z); these values of x will be denoted by x_i. By an argument similar to that employed for $v = \hat{\Omega}(h,z)$ one can show that $v = \Omega(x_i)$ is a branch point.

A more precise investigation may be formulated as follows: Using the definition of x_i given in the paragraph above

$$\int_h^z a(x)\, dx = \int_h^{x_1} a(x)\, dx + \dots + \int_{x_i}^z a(x)\, dx.$$

It will suffice to consider only one of these integrals, say the last. Suppose $v = \Omega(X)$, $\Omega'(X) \neq 0$, where $x_i < X < z$.

Define $s(x) = (v - \Omega(x))^{\frac{1}{2}}$. Then x is a unique analytic function of s^2 in the neighborhood of $s = 0$ (Jeffreys, 1950, p. 380). Therefore, $a(x)$ may be expanded as power series in s^2 with a finite radius of convergence. Let $X \pm \delta$ lie within this region of convergence. Then

$$\int_{x_i}^{z} a(x)\ dx = \int_{x_i}^{X-\delta} a(x)dx + \int_{X+\delta}^{z} a(x)dx + \int_{s(X-\delta)}^{s(X+\delta)} (b_1 s^2 + b_2 s^4 + \ldots.)ds$$

$$= \int_{x_i}^{X-\delta} a(x)\ dx + \int_{X+\delta}^{z} a(x)dx + \Big[\tfrac{1}{3} b_1 (v - \Omega(x))^{\frac{3}{2}} + \ldots \Big]_{X-\delta}^{X+\delta}$$

In this form it is apparent that for small variations in v (which implies variations in X) all terms are uniquely defined. Thus points on the real axis other than $v = \Omega(x_i)$ are regular.

Appendix B

Equation (117) may be reduced to the form of Watson's lemma (Jeffreys, 1950, p. 501) by deforming the path of integration into the path of steepest descents and changing the variable from v to ξ where

$$\xi = u_o(v) - u_o(v_0).$$

All of the terms of the integrand with the exception of the exponential may be developed in a power series in ξ. The resultant series may then be integrated term by term to yield the desired asymptotic series.

The most convenient way of accomplishing the expansion in terms of ξ is to develop ξ as a power series in $\gamma = (v^2 - v_o^2)$; this series may then be inverted to obtain γ as a power series in ξ. Thus if

$$i\xi = C_1 \gamma^2 - C_2 \gamma^3 + C_3 \gamma^4 + \ldots$$

then

$$\gamma = \pm \left(\frac{i\xi}{c_1}\right)^{1/2}\left[1 \pm \frac{1}{2} c_2 \left(\frac{i\xi}{c_1}\right)^{1/2} + \left(\frac{5}{8} c_2^2 - \frac{c_3}{2}\right)\frac{i\xi}{c_1} + \cdots\right].$$

$v^{\frac{1}{2}}(a(z)a(0))^{-\frac{1}{2}}\frac{dv}{d\xi}$ may be expanded in a power series in γ; then by means of the equation above, this last series may be converted to a power series in ξ. Application of Watson's lemma leads to equation (121) in which

$$b_1(z,r,v_0) = \Big\{\left(\tfrac{3}{2} c_2 - \tfrac{1}{4} v_0^{-2}\right)\left[\alpha^{-2}(z,v_0) + \alpha^{-2}(0,v_0)\right]$$

$$+ \tfrac{1}{8}\left[5\alpha^{-4}(z,v_0) + 2\alpha^{-2}(z,v_0)\alpha^{-2}(0,v_0) + 5\alpha^{-4}(0,v_0)\right]$$

$$+ 6\left(\tfrac{5}{4} c_2^2 - c_3\right) - \tfrac{3}{2}\frac{c_2}{v_0^2} + \tfrac{5}{8} v_0^{-4}\Big\}\frac{1}{8c_1} + \frac{1}{8v_0}, \tag{183}$$

where

$$c_1 = \tfrac{1}{8}\left(v_0^{-3} + r^{-1}\int_0^z \alpha^{-3}(x,v_0)\,dx\right) = \frac{|u_0''(v_0)|}{v_0^2},$$

$$c_2 = \frac{1}{16c_1}\left(v_0^{-5} - r^{-1}\int_0^z \alpha^{-5}(x,v_0)\,dx\right),$$

$$c_3 = \frac{5}{128c_1}\left(v_0^{-7} + r^{-1}\int_0^z \alpha^{-7}(x,v_0)\,dx\right).$$

Appendix C

A zero order ray theory may be derived from equation (16) by the method of the eikonal (Sommerfeld, 1954, page 207). One assumes that a solution of (16) exists in the form

$$p(z,r) = A(z,r)\exp(-i\omega W_0)$$

Substitution of this expression into (16) leads to

$$(\nabla W_0)^2 = \Omega^2(z) + O(\omega^{-1})$$

An asymptotic solution of this equation is given by (123). It is shown in section 4.6 that from the eikonal W_0 = constant one may derive the ray equation (Snell's law)

$$\Omega(z)\cos\vartheta(z) = \text{constant}, \tag{184}$$

$\vartheta(z)$ being the complement of the angle between the vertical and the ray tangent at the altitude z.

In order to derive the equation of the ray path it is more convenient to write (184) in the form

$$\operatorname{ctn}^2 \vartheta(z) = \left(\frac{dr}{dz}\right)^2 = \frac{\Omega^2(h)\cos^2\vartheta(h)}{\Omega^2(z) - \Omega^2(h)\cos^2\vartheta(h)} . \tag{185}$$

An integral of (185) is

$$r = \pm\, \Omega(h)\cos\vartheta(h) \int_h^z \left[\Omega^2(x) - \Omega^2(h)\cos^2\vartheta(h)\right]^{-\frac{1}{2}} dx . \tag{186}$$

It should be noticed that (185) requires

$$\Omega(z) \geq \pm\Omega(h)\cos\vartheta(h), \tag{187}$$

in order that all quantities in (185) be real. Suppose a ray starts from the source at an angle $\vartheta(h)$ and that there exists an altitude $z = \zeta_0$ such that

$$\Omega(\zeta_0) = \Omega(h)\cos\vartheta(h). \tag{188}$$

Then (185) implies that this ray can never pass above $z = \zeta_0$; it may, however, pass continuously from a positive to a negative slope at this altitude since a slope of either sign is an equally valid root of (185). In this case the equation of the ray path becomes

$$r = \Omega(\zeta_0)\left\{\int_h^{\zeta_0}\left[\Omega^2(x) - \Omega^2(\zeta_0)\right]^{-\frac{1}{2}} dx + \int_z^{\zeta_0}\left[\Omega^2(x) - \Omega^2(\zeta_0)\right]^{-\frac{1}{2}} dx\right\}, \tag{189}$$

the first quantity in the braces being an integral of the positive root of (185) and the second quantity an integral of the negative root. Other integrals of this form are possible if there is a reflector below the source; however, the ray paths of equations (186) and (188) will suffice for the purposes of this paper.

For sufficiently high frequencies it is generally supposed that energy propagates unimpeded along the zero order rays. On the basis of this assumption one may arrive at an asymptotic expression for the energy flux at any point not lying in the shadow zone.

Consider first the energy flux along the z-axis due to a source which emits energy at a time average rate E. It will be supposed that the emission is uniform in all directions. Let dE be the time average energy flow into a vertical tube which is bounded by the bundle of rays leaving the source at such an angle that the ray tangents make an angle ϵ with the vertical, ϵ being arbitrarily small; then $dE = \lim_{R \to 0} \frac{\pi(\epsilon R)^2}{4\pi R^2} E = \frac{1}{4} E \epsilon^2$.

The radius of the tube at any altitude z is given by (186),

$$r = \Omega(h)\, \epsilon \int_h^z \Omega^{-1}(x)\, dx \quad + O(\epsilon^3).$$

The rate of energy emitted by the source is given by (147). The time average energy flux $\underset{\sim}{g}$ at altitude z is given by $\frac{dE}{dS}$ where dS is the differential cross sectional area of the tube (πr^2). Thus,

$$\underset{\sim}{g}(z,0) = \frac{2}{\rho(h)\Omega(h)\omega^2} \left[\int_h^z \Omega^{-1}(x)\, dx \right]^{-2}. \tag{190}$$

Haskell (1951, p. 157) has found for the more general case $r \neq 0$ that the time average energy flux is given by

$$|g(z,r)| = \frac{2\Omega(h) \sin^{-1}\vartheta(z)}{\rho(h)\, \omega^2\, r} \left[\frac{\partial\, r(v_0,z)}{\partial(\sin\vartheta(h))} \right]^{-1}. \tag{191}$$

GLOSSARY

SYMBOL	PAGE
a	31
A_o	44
$b(v)$	67
b_1	74
c	7
C_o	39
D	31,36
$\underset{\sim}{e}_n$	51
$\underset{\sim}{e}_p$	49
$\underset{\sim}{e}_p^{(2)}$	51,68
$\underset{\sim}{e}_r$	49
$\underset{\sim}{e}_z$	3
$E(\mp\frac{z}{h},v)$	32
f_o	38
f_{1j}	40
g	3
$\underset{\sim}{G}$	2
$\mathfrak{J}(z\|\zeta)$	31
h	1
H	22
$\mathcal{H}$	14
$\underset{\sim}{J}$	4
$\underset{\sim}{\mathcal{J}}$	55
k	2
K	15
K_1	61
K_2	61
$\mathcal{L}^{-1}\{\ \}$	23
$L[\ ]$	31
m	5
M	22
M_1	52
M_2	52
N	51
P	2
p	2
$\mathfrak{p}$	4
p_n	37
q	5
q_n	60
Q	61
r	1
R	33
R_j	9
$\mathcal{R}$	8
S	61
$\underset{\sim}{u}$	2
u_o	43
u_1	62
u_{1j}	45
v_o	43
v_1	63

SYMBOL	PAGE
v_{1j}	45
v_2	63
V	2
w	4
W	49,67
W_o	44
X	44
y_o	32
y_n	34
z	1
Z_j	9
α	43
Γ	47
$\delta(t)$	24
$\theta(\zeta)$	49
λ	4
Λ	68
μ	3
∂	9
ρ	2
σ	2
τ_j	24
υ	15
ϑ	1
χ	5

SYMBOL	PAGE
ω	2
Ω	3
$\breve{\Omega}(0,H)$	35
$\hat{\Omega}(0,H)$	35
$'$	4
$1(t)$	24
$*$	43
$\doteq$	52

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