ASYMPTOTIC EXPANSIONS FOR CHARACTERISTIC VALUES AND FUNCTIONS OF A SECOND ORDER ORDINARY LINEAR DIFFERENTIAL OPERATOR.

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ABSTRACT

Consider a second order ordinary linear differential operator on a real half-open interval $(0,b]$ $(b > 0)$ which contains no singular points. Suppose $x = 0$ is a singular point. The basic characteristic value problem is defined on this interval when suitable boundary conditions are adjoined at the endpoints. Two classes of perturbed characteristic value problems are defined on subintervals $[a,b]$, where a is a small positive number. It is proved under certain conditions on the basic problem that for each isolated characteristic value Λ of the basic problem, there is a characteristic value λ of the perturbed problem which is developable in an asymptotic expansion with leading term Λ, valid as $a \to 0$. Furthermore, the characteristic function corresponding to λ possesses an asymptotic expansion valid as $a \to 0$ uniformly in the interval $[a,b]$. These expansions are not asymptotic power series, but are asymptotic expansions of a more general type.

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1. Introduction.

We shall consider characteristic value problems for a second order ordinary linear differential operator defined on certain intervals of the real x-axis. We designate as the basic interval a half-open interval $(0,b]$ $(b > 0)$ which contains no singular points of the differential operator, and we suppose that the point $x = 0$ is a singular point of the operator. A self-adjoint operator over a suitable Hilbert space can then be defined by the application of appropriate boundary conditions (if needed) at the endpoints of the basic interval. We shall call this the basic operator and assume that there are isolated points in its spectrum.

Our aim is to investigate the spectrum of a self-adjoint differential operator defined on a subinterval $[a,b]$ of the basic interval, where a is a small positive number. Such an operator will be referred to as a perturbed operator. We shall give a perturbation procedure for estimating the difference between the characteristic values of a perturbed operator and those of the basic operator. Coddington [1]* has shown that for problems of this type, the spectral families belonging to a set of perturbed operators converge to the spectral family belonging to the basic operator as $a \to 0$. We shall establish more specific results for certain classes of problems when there are isolated points in the spectrum

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* Numbers in brackets are references to the books, reports, and articles which are listed at the end of the thesis. Pairs of numbers in parentheses are labels for statements in the text; thus (p.q) refers to statement q in section p.

of the basic operator: we shall obtain asymptotic expansions for the perturbed characteristic values and characteristic functions as $a \to 0$.

We shall consider two classes of problems, which roughly correspond to the cases when the basic operator possesses or does not possess linearly independent solutions whose ratio tends to zero as $a \to 0$. In the one class, an essentially arbitrary homogeneous boundary condition is imposed on solutions of the differential equation at the point $x = a$, and it turns out that the characteristic values of the perturbed operator, whose domain is restricted by this boundary condition, converge to those of the basic operator. In the other class, the homogeneous boundary condition at $x = a$ must have a special character in order for the characteristic values to converge to those of the basic operator. The first class includes problems of both the limit circle and the limit point type in Weyl's classification of singular points [7], [10]. On the other hand, the second class includes only problems of the limit circle type. Thus the criterion for distinguishing between our two classes of problems is not Weyl's criterion.

In the next two paragraphs we shall state certain known results in perturbation theory and indicate how our results differ from these. We are interested in problems where the perturbation arises from the domain of the operator, while in most existing theories it arises from the operator itself. In the latter, one considers a set of not necessarily bounded self-adjoint operators $\{A_\varepsilon\}$ defined on a certain Hilbert space, where $\varepsilon > 0$ is supposed small, such that A_ε converges in some sense to a self-adjoint operator A_0 as $\varepsilon \to 0$. The characteristic values λ_ε and the characteristic functions y_ε of A_ε are then developed in convergent or asymptotic power series in ε as $\varepsilon \to 0$. Specifically, the following result is true in analytic perturbation theory: if the resolvent R_ε corresponding to A_ε converges uniformly to the resolvent R_0 corresponding to A_0 and if R_ε is analytic in ε in some neighborhood of $\varepsilon = 0$, then for each isolated point Λ in the spectrum of A_0 (say of multiplicity one) there is a corresponding λ_ε in the spectrum of A_ε which converges to Λ as $\varepsilon \to 0$, and furthermore λ_ε is an analytic function of ε for small values of ε.

If the resolvent R_ε converges strongly but not uniformly, weaker results are obtained; in fact, the spectral family corresponding to A_ε then converges strongly to that corresponding to A_0. In asymptotic perturbation theory [5], the perturbed operator A_ε is defined formally by the relation $A_\varepsilon = A_0 + \varepsilon A^{(1)} + \varepsilon^2 A^{(2)} + \cdots$ where the operators on the right side are symmetric and semi-bounded from below. The right side is not in general self-adjoint even if A_0 and all the operators $A^{(j)}$ $(j = 1, 2, \ldots)$ are self-adjoint, but a distinguished self-adjoint extension of the right side can be obtained (the Friedrichs extension [5]). Then the spectral family corresponding to A_ε will converge strongly to that corresponding to A_0 as $\varepsilon \to 0$. In order to obtain asymptotic expansions of the characteristic values and characteristic functions belonging to A_ε, when A_0 has isolated points in its spectrum, one has to assume that the domain of A_ε is large enough to include the characteristic functions of A_0.

In the problems we wish to treat, such is not the case, and a different procedure is needed. Coddington [1] has discussed situations where the spectral family corresponding to a perturbed operator indeed converges, but we wish to obtain more: to give asymptotic expansions of the characteristic values and the characteristic functions of the perturbed operator as $a \to 0$, in the event that there exists at least one isolated point in the spectrum of the basic operator. Our general procedure is a familiar one in the theory of asymptotic solutions of differential equations: we compare the solutions of the perturbed problem with those of the basic problem by means of an integral equation of Volterra's type. Our asymptotic expansions will not be power series in the small parameter a as in analytic or asymptotic perturbation theory, but will be asymptotic expansions in a more general sense, and in fact these expansions will be convergent for sufficiently small values of a. The properties of general types of asymptotic expansions have been investigated by van der Corput [9].

In section 2 we shall give precise definitions of the basic and perturbed operators and their domains, make the appropriate assumptions for the two classes of problems under consideration, and set forth the asymptotic terminology to be used in the sequel. In section 3 we shall describe the machinery used for the comparison of the basic and the perturbed problems. This section includes the appropriate lemmas stating the existence of solutions of integral equations, and lemmas giving the asymptotic behavior of certain functions and estimates for other functions near the singular point. In section 4 we present the main theorem, which states the existence of asymptotic expansions for the characteristic values and the characteristic functions as $a \to 0$. Furthermore, we obtain explicit asymptotic forms, not depending upon integrals of characteristic functions of the basic problem, by replacing the first term in the above asymptotic expansion by an asymptotic form, and these asymptotic forms will be extremely useful in applications. Only in the trivial case that the exceptional point $x = 0$ is an ordinary point of the differential equation (so that it is not exceptional at all) will the results be convergent power series in the small parameter a. In section 5, we shall treat the case when the exceptional point is a regular singularity of the differential equation; then the asymptotic form referred to above will be dominated by certain elementary functions. For problems of the first class described earlier in this introduction, these functions will turn out to be simply powers of a, depending upon the exponents belonging to the regular singularity. In section 6, we shall treat the case when $x = 0$ is an irregular singular point of finite rank, and obtain asymptotic forms which are dominated by exponential functions multiplied by powers of a. In section 7, we shall weaken the assumption that $x = b$ is an ordinary point of the differential operator and permit it to be a regular singularity.

2. Formulation of the Problem.

This section contains a detailed formulation of the characteristic value problem under consideration, including all the definitions and assumptions needed in the sequel.

First, we give the asymptotic terminology to be used, of the type introduced by van der Corput [9]. It is to be observed that these definitions are not equivalent to analogous definitions used in Poincaré's classical theory of asymptotic series: the former make weaker demands than the latter on the functions under consideration. We shall be interested in obtaining asymptotic expansions of functions $f(\nu)$ for small values of the real variable ν.

Definition 1. A constant is a function which is independent of ν.

Definition 2. A function $f(\nu)$ is of the same order as $g(\nu)$, in symbols $f(\nu) = O[g(\nu)]$, if there exist positive constants ν_o and C such that $|f(\nu)| \leq C|g(\nu)|$ whenever $|\nu| \leq \nu_o$.

Definition 3. The order relation $f(\nu) = o[g(\nu)]$ as $\nu \to 0$ means that given any constant $\varepsilon > 0$, there exists a constant ν_o such that $|f(\nu)| \leq \varepsilon|g(\nu)|$ whenever $|\nu| \leq \nu_o$.

Definition 4. Suppose $f(\nu)$ depends upon an additional real variable x on an interval I. The order relation in definition 2 (or definition 3) is said to hold uniformly in x if ν_o and C (or ν_o) can be chosen independent of x for all $x \in I$.

<u>Definition 5</u>. The formal series $\sum_{j=1}^{\infty} f_j(\nu)$ is said to be an <u>asymptotic expansion</u> of a function $f(\nu)$, in symbols $f(\nu) \sim \sum f_j(\nu)$, if $f - f_1 - f_2 - \dots - f_h = O(\nu^h)$ for $h = 1, 2, \dots$.

<u>Definition 6</u>. A function f_1 is said to be an <u>asymptotic form</u> for f as $\nu \to 0$ if $f = f_1 + o(f_1)$ as $\nu \to 0$. The form is said to be valid <u>uniformly</u> in x if the order relation holds uniformly in x.

Suppose now that $\nu = \nu(a)$ is itself a function of a small positive variable a and that $\nu(a) = o(1)$ as $a \to 0$. The function $\nu(a)$ will be called a scale function, or simply a <u>scale</u>. Suppose that $\psi = \psi(a)$ is an additional function of a. Then we make the convention contained in the following definition.

<u>Definition 7</u>. The formal series $\sum_{j=1}^{\infty} f_j$ is said to be an asymptotic expansion of $f(\nu)$ with scale $\nu(a)$ as $a \to 0$ if there exists a function $\psi(a)$ such that $f - f_1 - f_2 - \dots - f_h = O(\psi\nu^h)$ for $h = 1, 2, \dots$.

If f has such an asymptotic expansion and if in addition $\psi\nu = o(f_1)$ as $a \to 0$, then $f = f_1 + o(f_1)$ as $a \to 0$.

Suppose now that f and f_j $(j = 1, 2, \dots)$ are functions of a and x, $x \in I_a$. The subscript a means that the x-interval can depend upon a. Let $\psi = \psi(a,x)$ be a function of a and x and let $\nu = \nu(a,x)$ be a function of a and x such that $\nu = o(1)$ as $a \to 0$ uniformly for $x \in I_a$.

<u>Definition 8</u>. The formal series $\sum_{j=1}^{\infty} f_j$ is said to be an asymptotic expansion of f with scale $\nu(a,x)$ as $a \to 0$, valid uniformly for $x \in I_a$ if there exists a function $\Psi(a,x)$ such that $f - f_1 - \cdots - f_h = O(\Psi \nu^h)$ uniformly for $x \in I_a$ $(h = 1, 2, \ldots)$.

We now proceed to formulate the characteristic value problems to be investigated in the sequel.

Differential operators of the type

$$(2.1) \quad L = [k(x)]^{-1} \left[- \frac{d^2}{dx^2} + q(x)\right] \qquad 0 < x \leq b$$

are under consideration, where $q(x)$ and $k(x)$ are real-valued, piecewise continuous functions on $(0,b]$ and $k(x)$ is positive-valued. Let $\mathfrak{h}$ denote the Hilbert space of all complex-valued functions $U(x)$ measurable on $(0,b)$ for which the integral $\int_0^b |U(x)|^2 k(x)\, dx$ exists in the sense of Lebesgue. The inner product in this space is defined by $(U,V) = \int_0^b U(x) \bar{V}(x) k(x)\, dx$ and the norm is defined by $||U|| = (U,U)^{1/2}$. We shall now proceed to define certain self-adjoint operators in $\mathfrak{h}$ which are associated with the formal operator L.

<u>Definition 9</u>. Suppose that the point $x = 0$ comes under the limit point case in Weyl's classification of singular points. Then we define the domain $\mathcal{D}_0$ to consist of those functions $Y \in \mathfrak{h}$ which satisfy the following conditions:

(a) Y and Y' are continuous and Y'' is piecewise continuous on $(0,b]$.

(b) $L\, Y \in \mathfrak{h}$.

(c) Y satisfies a homogeneous boundary condition at $x = b$ of the type

(2.2) $$\mathcal{B}_b[Y] \equiv \beta_o Y(b) + \beta_1 Y'(b) = 0,$$

where β_o, β_1 are real numbers, not both zero.

Let $\chi(x)$ be a non-identically zero solution of the equation $L\chi = i\chi$ such that

$$\lim_{x\to 0} [\bar{\chi}'(x)\chi(x) - \bar{\chi}(x)\chi'(x)] = 0 .$$

Such a function exists according to Weyl's theorem [10].

<u>Definition 10</u>. Suppose that the point $x = 0$ comes under the limit circle case in Weyl's classification. Then we define the domain $\mathcal{D}_o$ to consist of those functions $Y \varepsilon \mathfrak{h}$ which satisfy (a), (b), and (c) of definition 9 and in addition

(d) Y satisfies the following boundary condition at $x = 0$:

(2.3) $$\mathcal{B}_o[Y] \equiv \lim_{x\to 0} [\bar{\chi}'(x) Y(x) - \bar{\chi}(x) Y'(x)] = 0.$$

<u>Definition 11</u>. In either the limit point case or the limit circle case, we define the <u>basic operator</u> A_o in $\mathfrak{h}$ as follows: A_o has domain $\mathcal{D}_o$ and

$$A_o Y = LY \qquad Y \varepsilon \mathcal{D}_o .$$

It is known [7] that A_o on $\mathcal{D}_o$ is a self-adjoint operator in $\mathfrak{h}$.

We shall be interested only in the case that there exist isolated points in the spectrum of A_o. If the inverse operator $(A_o - i)^{-1}$ is completely continuous on the subspace of piecewise continuous functions of $\mathfrak{H}$ [7], then the spectrum will be entirely a point spectrum. However, we shall not in general assume the complete continuity of $(A_o - i)^{-1}$, but shall be content to assume that there exist isolated points in the spectrum.

Suppose there exists an isolated point Λ in the spectrum of A_o. Thus, if $Y_1 = Y_1(x) = Y_1(x, \Lambda)$ is the corresponding characteristic function,

(2.4) $$LY_1 = \Lambda Y_1 \qquad\qquad Y_1 \varepsilon \mathcal{D}_o .$$

All characteristic functions $Y(x)$ will be normalized so that

(2.5) $$||Y||^2 \equiv \int_o^b |Y(t)|^2 k(t)\, dt = 1 .$$

Let $Y_2 = Y_2(x) = Y_2(x, \Lambda)$ be a real-valued solution of the equation $LY = \Lambda Y$ such that Y_1 and Y_2 form a fundamental set of solutions. The Wronskian determinant of Y_1 and Y_2 is a constant which may be taken to be -1 without loss of generality,

(2.6) $$\mathcal{W}[Y_1, Y_2] = -1.$$

We shall now give formal definitions of the two classes of problems which will be under consideration in the sequel. These are the classes which have been discussed summarily in section 1. It is implicit that the assumptions of the previous paragraph apply to either class.

Definition 12. For a class 1 problem the point $x = 0$ is not an accumulation point of the zeros of Y_1 and*

$$(2.7)\qquad \frac{Y_1(x)}{Y_2(x)} \int_x^b |Y_1(t)\, Y_2(t)|\, k(t)\, dt = o(1) \quad \text{as} \quad x \to 0.$$

This implies that

$$(2.8)\qquad Y_1(x)/Y_2(x) = o(1) \quad \text{as} \quad x \to 0.$$

Definition 13. For a class 2 problem, there exists a pair of linearly independent solutions $U_1(x)$, $U_2(x)$ of the equation $LU = \Lambda U$ such that the point $x = 0$ is not an accumulation point of the zeros of these functions, the ratio $U_1(x)/U_2(x)$ is bounded and bounded away from zero in a neighborhood of $x = 0$, and

$$(2.9)\qquad \int_0^x U_1(t)\, U_2(t)\, k(t)\, dt = o(1) \quad \text{as} \quad x \to 0 .$$

Then for class 2 problems, the ratio $Y_1(x)/Y_2(x)$ is not supposed to be small for x near 0, and in fact the limit of this ratio as $x \to 0$ may not exist at all. It follows from the assumptions that both U_1 and U_2 belong to $\mathfrak{h}$, and hence class 2 problems are always of the limit circle type.

The perturbed characteristic value problems for class 1 and class 2 will now be defined.

Definition 14. The domain $\mathcal{D}_a$ for class 1 problems is the set of all functions $y \in \mathfrak{h}$ having the properties (b), (c) of definition 9 and

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* Since the zeros of a pair of linearly independent real solutions of a second order ordinary linear differential equation separate each other, it follows that the zeros of Y_2 as well do not accumulate at $x = 0$.

also the following three properties:

(e) y and y' are continuous on $(a,b]$ and y'' is piecewise continuous on $[a,b]$. Furthermore, y has a right derivative at $x = a$, which will be designated hereafter by $y'(a)$. Both y and y' are continuous from the right at $x = a$.

(f) $y \equiv 0$ on $(0,a)$.

(g) y satisfies a homogeneous boundary condition at the point $x = a$ of the type

$$(2.10) \qquad \mathcal{B}_a[y] \equiv \alpha_o(a)\, y(a) + \alpha_1(a)\, y'(a) = 0$$

where $\alpha_o(a)$, $\alpha_1(a)$ are real-valued functions, not both zero for any value of a, subject to the following condition: there exists a positive number a_o such that all the functions

$$(2.11) \qquad \frac{\alpha_o(a)\, Y_2(a)}{\mathcal{B}_a[Y_2]}\ ; \qquad \frac{\alpha_1(a)\, Y_2'(a)}{\mathcal{B}_a[Y_2]}\ ; \qquad \frac{\alpha_1(a)\, Y_2(a)\, Y_1'(a)}{Y_1(a)\, \mathcal{B}_a[Y_2]}$$

are bounded whenever $0 < a \leq a_o$.

Once the basic problem has been prescribed, this last condition is equivalent to the exclusion of certain special choices of the functions $\alpha_o(a)$, $\alpha_1(a)$ from consideration. This condition implies that the function

$$(2.12) \qquad \frac{Y_2(a)}{Y_1(a)}\ \frac{\mathcal{B}_a[Y_1]}{\mathcal{B}_a[Y_2]}$$

is bounded whenever $0 < a \leq a_o$. We shall now show that (2.11) will still be bounded if $Y_2(x)$ is replaced by any solution $Z_2(x)$ of $LY = \Lambda Y$ which is independent of a, and which forms with Y_1 a fundamental set of solutions of $LY = \Lambda Y$. There are constants c_1 and $c_2 \neq 0$, independent of a, such that $Z_2(x) = c_1 Y_1(x) + c_2 Y_2(x)$. Then

$$\frac{\alpha_o(a)\, Z_2(a)}{\mathcal{B}_a[Z_2]} = \frac{\alpha_o(a)\, Y_2(a)}{\mathcal{B}_a[Y_2]} \quad \frac{1 + c_1 Y_1(a)/c_2 Y_2(a)}{1 + c_1 \mathcal{B}_a[Y_1]/c_2 \mathcal{B}_a[Y_2]}$$

The first factor on the right side is bounded by (2.11), and the second factor tends to 1 as $a \to 0$ by (2.8), (2.12). Hence the boundedness of the first function (2.11) is independent of Y_2. Similar remarks apply to the second and third functions (2.11). This means that the dominant behavior of the excluded functions $\alpha_o(a)$, $\alpha_1(a)$ for a in the neighborhood of 0, as determined by (2.11), does not depend upon a special choice of the function Y_2, but only on the nature of the basic problem.

<u>Definition 15</u>. The domain $\mathcal{D}_a$ for class 2 problems is the set of all functions $y \in \mathcal{H}$ having the properties (b), (c) of definition 9, (e), (f) of definition 14, and also the property

$$(2.13) \qquad \mathcal{B}_a[y] \equiv \bar{\chi}'(a)\, y(a) - \bar{\chi}(a)\, y'(a) = 0$$

where the function $\chi(x)$ has been defined in connection with definition 10.

The boundary condition (2.13) has been chosen so that $\mathcal{B}_a[Y_1] = o(1)$ as $a \to 0$ for any function Y_1 which satisfies condition (2.3). The same purpose could be achieved by replacing $\chi(a)$ by any function with the asymptotic form $\chi(a)\,[1 + o(1)]$ as $a \to 0$ in (2.13). However, there is very little freedom for this function, and accordingly we refer to (2.13) as a distinguished boundary condition.

<u>Definition 16</u>. In either class 1 or class 2 problems, we define the <u>perturbed operator</u> A_a in $\mathfrak{h}$ as follows: A_a has domain $\mathcal{D}_a$ given by definition 14 or 15 and

$$A_a y = Ly \qquad\qquad y \varepsilon \mathcal{D}_a .$$

Then $\{A_a\}$ defines a set of self-adjoint operators on the space $\mathfrak{h}$ [7]. Coddington [1] has shown that under a slightly different definition of A_a, the spectral family of A_o is the limit of the spectral family of A_a as $a \to 0$. We shall proceed in a different way and show in a constructive fashion how to estimate the difference between the characteristic values of A_a and A_o for small values of a, and the difference between the corresponding characteristic functions.

The distinction between the boundary conditions (2.10) and (2.13), corresponding to class 1 and class 2 problems respectively, should be emphasized: the former is virtually arbitrary while the latter is determined in an essentially unique manner from the nature of the basic problem; the former may apply to either the limit point or the limit circle case while the latter is always a condition for the limit circle case. There are in fact problems which may be treated either way, so that class 1 and class 2 problems are not mutually exclusive.

3. <u>The Comparison Technique</u>.

The characteristic value problem under consideration is

$$(3.1)\qquad Ly = \lambda y \qquad y \in \mathcal{D}_a \qquad (0 < a < b) .$$

It has been assumed that there exists a solution Λ, Y_1 of the basic problem, that is

$$(3.2)\qquad LY_1 = \Lambda Y_1 \qquad Y_1 \in \mathcal{D}_0 .$$

It is our purpose now to connect the solutions of these two problems. Let Y_1, Y_2 be the fundamental set of solutions of the equation $LY = \Lambda Y$ which has been described in section 1, and let Y be any solution. Let y satisfy the Volterra-type integral equation

$$(3.3)\qquad y(x,\lambda) = Y(x) + (\Lambda - \lambda)\int_x^b G(x,t)\,k(t)\,y(t,\lambda)\,dt \qquad (a \leq x \leq b)$$

and let

$$y(x,\lambda) = 0 \qquad (0 < x < a) .$$

Here

$$(3.4)\qquad G(x,t) = Y_1(t)\,Y_2(x) - Y_1(x)\,Y_2(t).$$

Then the following lemmas are valid.

<u>Lemma 1</u>. For each fixed value of λ and a $(0 < a < b)$, there exists a unique solution $y(x,\lambda)$ of the integral equation (3.3) which is continuously differentiable and twice piecewise continuously differentiable on $(a,b]$, and which is continuously differentiable from the right at $x = a$. This function satisfies the differential equation (3.1), and if $Y(x)$ satisfies a homogeneous boundary condition at $x = b$, then $y(x,\lambda)$ satisfies the same condition.

This result follows from the Liouville-Neumann theorem on non-singular Volterra-type integral equations.

Lemma 2. Let Λ be an isolated point in the spectrum of the self-adjoint operator A_o, and let Y_1 be the corresponding characteristic function. Let $y(x, \lambda)$ be the solution of (3.3) when $Y = Y_1$. Suppose that λ satisfies the condition $\mathcal{B}_a[y(x, \lambda)] = 0$. Then $y(x, \lambda)$ is a characteristic function for A_a with corresponding characteristic value λ.

In the sequel we shall always choose $Y = Y_1$ in (3.3) so that lemma 2 can be applied.

Lemma 3. Let $\chi(x)$ be the function defined in connection with definition 10. Let Λ be an isolated point in the spectrum of A_o and let Y_1, Y_2 be the fundamental set of solutions of $LY = \Lambda Y$ used in (3.4). Then, for class 2 problems, there is a constant C such that

$$(3.5)\qquad \chi(x) = CY_1(x)\,[1 + o(1)] \qquad \text{as} \qquad x \to 0 .$$

Further, there exists a number a_o such that $\mathcal{B}_a[Y_2]$ is bounded away from zero whenever $a \leq a_o$.

Proof: Let $U_1(x), U_2(x)$ be the solutions of $LU = \Lambda U$ postulated in definition 13. Let $\phi_1(x), \phi_2(x)$ be defined by means of the integral equations

$$(3.6)\quad \phi_j(x) = U_j(x) + (\lambda - \Lambda)\int_o^x G(x,t)\,k(t)\,\phi_j(t)\,dt \qquad (j = 1,2)$$

where the function $G(x,t)$, given by (3.4), has the representation

$$G(x,t) = \Delta^{-1}[U_1(t)\,U_2(x) - U_1(x)\,U_2(t)]$$

in which Δ is a non-zero constant. Consider the sequences of successive approximations to the solutions of equations (3.6). The first approximations are for $j = 1$ and $j = 2$ respectively

$$U_1(x) + \frac{(i - \Lambda)}{\Delta} U_1(x) \int_0^x \left[\frac{U_2(x)}{U_1(x)} U_1^2(t) - U_1(t)\, U_2(t)\right] k(t)\, dt$$

$$U_2(x) + \frac{(i - \Lambda)}{\Delta} U_2(x) \int_0^x \left[U_1(t)\, U_2(t) - \frac{U_1(x)}{U_2(x)} U_2^2(t)\right] k(t)\, dt,$$

which are $U_1(x)\,[1 + o(1)]$, $U_2(x)\,[1 + o(1)]$ as $x \to 0$ because $U_1 \varepsilon \mathfrak{h}$ and $U_2 \varepsilon \mathfrak{h}$, and the assumptions of definition 13 apply. Then it follows by a standard procedure that for each j, equation (3.6) possesses a solution $\phi_j(x)$ with the property that

$$\phi_j(x) = U_j(x)\,[1 + o(1)] \quad \text{as} \quad x \to 0 \qquad (j = 1,2),$$

and furthermore ϕ_j satisfies the equation $L\phi = i\phi$. Let

$$\chi(x) = \gamma_1\, \phi_1(x) + \gamma_2\, \phi_2(x)$$

and

$$Y_1(x) = \delta_1\, U_1(x) + \delta_2\, U_2(x)\;.$$

Since $\bar{\chi}(x) = \chi(x)\,[1 + o(1)]$, the condition (2.3) leads to $\delta_1 : \delta_2 = \gamma_1 : \gamma_2$ and hence

$$\chi(x) = CY_1(x)\,[1 + o(1)] \quad \text{as} \quad x \to 0\;,$$

which is (3.5). Further, it follows from condition (2.13) that

$$\mathcal{B}_a[Y_2] = C\mathcal{W}[Y_1(a), Y_2(a)]\,[1 + o(1)]$$

$$= -C[1 + o(1)] \qquad \text{as} \quad a \to 0 ,$$

which is bounded away from zero for a sufficiently small. This completes the proof of lemma 3.

The solution of the integral equation (3.3) when $Y = Y_1$, for each fixed value of x and λ, is represented by the series

$$(3.7)\quad y(x, \lambda) = Y_1(x) + (\Lambda - \lambda)\, HY_1(x) + \cdots + (\Lambda - \lambda)^m\, H^m\, Y_1(x) + \cdots$$

where the operator H is defined by

$$(3.8)\qquad Hf(x) = \int_x^b G(x,t)\, k(t)\, f(t)\, dt$$

and H^m is the mth iterate of H $(m = 2, 3, \ldots)$. For each fixed a and each fixed λ, the series on the right of (3.7) is uniformly and absolutely convergent for $a \leq x \leq b$, and may be differentiated termwise.

Consider a characteristic function $Y_1 = Y_1(x, \Lambda)$ of the operator A_o for a class 1 or a class 2 problem. Let Y_1, Y_2 be the fundamental set of solutions of $LY = \Lambda Y$ described in section 2. We shall use the following notation:

$$(3.9)\qquad \hat{Y}(x) = \text{maximum}\ \left\{|Y_1(x)|,\ |Y_2(x)|\right\} \qquad (0 < x \leq b)\ .$$

For class 2 problems we define

$$(3.10)\qquad g(x) = \int_x^b \hat{Y}^2(t)\, k(t)\, dt \qquad (0 < x \leq b),$$

which is bounded since class 2 problems are always of the limit circle type.

For class 1 problems, we know from equation (2.6) and definition 12 that there exists a number x_o such that the ratio $Y_1(x)/Y_2(x)$ is a positive monotone increasing function of x whenever $0 < x \leq x_o$. Then we define

$$(3.11) \qquad g(x) = \int_x^b \hat{Y}^2(t)\, k(t)\, dt \qquad (x_o \leq x \leq b)$$

$$= g(x_o) + \int_x^{x_o} |Y_1(t)|\, \hat{Y}(t)\, k(t)\, dt \qquad (0 < x < x_o)$$

and further define

$$(3.12) \qquad \hat{Y}'(x) = \text{maximum} \left\{ |Y_1'(x)|,\ |Y_2'(x)| \right\} \qquad (x_o \leq x \leq b)$$

$$= \text{maximum} \left\{ \left| \frac{Y_2(x)}{Y_1(x)}\, Y_1'(x) \right|,\ |Y_2'(x)| \right\} \qquad (0 < x < x_o).$$

<u>Lemma 4</u>. The following asymptotic form is valid for class 1 problems:

$$(3.13) \qquad HY_1(x) = Y_2(x)\,[1 + o(1)] \quad \text{as} \quad x \to 0 .$$

Furthermore, there is a number C_1 independent of x and m such that

$$(3.14) \qquad |H^m\, Y_1(x)| \leq \frac{2^m}{(m-1)!}\, C_1 [g(x)]^{m-1}\, \hat{Y}(x)$$

$$(0 < x \leq b) \quad (m = 1, 2, \ldots).$$

<u>Proof</u>: It follows from equations (3.4) and (3.8) that

$$\frac{HY_1(x) - Y_2(x)}{Y_2(x)} = -\int_0^x Y_1^2(t)\, k(t)\, dt - \frac{Y_1(x)}{Y_2(x)} \int_x^b Y_1(t)\, Y_2(t)\, k(t)\, dt .$$

The first term on the right side is $o(1)$ as $x \to 0$ since $Y_1 \in \mathfrak{h}$, and the second term is $o(1)$ by assumption (2.7). This establishes the form (3.13).

The result (3.14) will be proved by mathematical induction on the positive integer m. First suppose that $x_o \leq x \leq b$. Then it follows from equations (3.4), (3.8), (3.9), and (3.11) that

$$|HY_1(x)| \leq \int_x^b \left\{ |Y_1^2(t)\, Y_2(x)| + |Y_1(t)\, Y_2(t)\, Y_1(x)| \right\} k(t)\, dt$$

$$\leq 2g(x)\, \hat{Y}(x) .$$

Since $g(x)$ is bounded above for $x_o \leq x \leq b$, the result (3.14) is true for $m = 1$. Under the hypothesis that it is true for m, it follows that

$$|H^{m+1}\, Y_1(x)| \leq \frac{2^m}{(m-1)!} C_1 \int_x^b \left\{ |Y_1(t)\, \hat{Y}(t)\, Y_2(x)| + \right.$$

$$\left. + |Y_2(t)\, \hat{Y}(t)\, Y_1(x)| \right\} [g(t)]^{m-1}\, k(t)\, dt$$

$$\leq \frac{2^{m+1}}{(m-1)!} C_1\, \hat{Y}(x) \int_x^b \hat{Y}^2(t)\, [g(t)]^{m-1}\, k(t)\, dt$$

$$= \frac{2^{m+1}}{m!} C_1 [g(x)]^m\, \hat{Y}(x) .$$

Hence it is true for $m+1$, and the induction is complete.

Suppose now that $0 < x < x_o$. Then

$$(3.15) \quad |HY_1(x)| \leq \int_x^{x_o} \left\{ |Y_1^2(t)\, Y_2(x)| + |Y_1(t)\, Y_2(t)\, Y_1(x)| \right\} k(t)\, dt +$$

$$+ \int_{x_o}^b \left\{ |Y_1^2(t)\, Y_2(x)| + |Y_1(t)\, Y_2(t)\, Y_1(x)| \right\} k(t)\, dt .$$

Since the ratio $Y_1(x)/Y_2(x)$ is a positive monotone increasing function on the interval $0 < x < x_o$, it follows that

$$|Y_2(t)\ Y_1(x)| \leq |Y_1(t)\ Y_2(x)| \ . \qquad (x \leq t \leq x_o) \tag{3.16}$$

Substitution of the inequality (3.16) into the inequality (3.15) then leads to

$$\begin{aligned}
|HY_1(x)| &\leq 2Y_2(x) \int_x^{x_o} Y_1^2(t)\ k(t)\ dt + 2\ \hat{Y}(x) \int_{x_o}^{b} \hat{Y}^2(t)\ k(t)\ dt \\
&\leq 2\ \hat{Y}(x) + 2\ \hat{Y}(x)\ g(x_o) \ .
\end{aligned}$$

Hence the result (3.14) is true for $m = 1$. It follows from the induction hypothesis and the inequality (3.16) that

$$\begin{aligned}
|H^{m+1}\ Y_1(x)| \leq{} & \frac{2^m}{(m-1)!}\ C_1 \int_x^{x_o} \left\{ |Y_1(t)\ \hat{Y}(t)\ Y_2(x)| + \right. \\
& \left. + |Y_2(t)\ \hat{Y}(t)\ Y_1(x)| \right\}\ [g(t)]^{m-1}\ k(t)\ dt \\
& + \frac{2^{m+1}}{(m-1)!}\ C_1\ \hat{Y}(x) \int_{x_o}^{b} \hat{Y}^2(t)\ [g(t)]^{m-1}\ k(t)\ dt \\
\leq{} & \frac{2^{m+1}}{(m-1)!}\ C_1 |Y_2(x)| \int_x^{x_o} |Y_1(t)|\ \hat{Y}(t)\ [g(t)]^{m-1}\ k(t)\ dt \\
& + \frac{2^{m+1}}{(m-1)!}\ C_1\ \hat{Y}(x) \int_{x_o}^{b} \hat{Y}^2(t)\ [g(t)]^{m-1}\ k(t)\ dt \\
\leq{} & \frac{2^{m+1}}{m!}\ C_1 [g(x)]^m\ \ \hat{Y}(x) \qquad\qquad (0 < x < x_o) .
\end{aligned}$$

This completes the inductive proof of the result (3.14).

Lemma 5. Let $\mathcal{B}_a[y] = 0$ be the boundary condition (2.10). Then the following asymptotic form is valid for class 1 problems:

$$\mathcal{B}_a[HY_1] = \mathcal{B}_a[Y_2]\,[1 + o(1)] \quad \text{as} \quad a \to 0 \,. \tag{3.17}$$

Furthermore, there is a number* C_1 independent of x and m such that

$$|[H^m Y_1(x)]'| \leq \frac{2^m}{(m-1)!} C_1[g(x)]^{m-1} \; \hat{Y}'(x) \tag{3.18}$$

$$(0 < x \leq b) \quad (m = 1,2, \ldots)$$

and

$$|\mathcal{B}_a[H^m Y_1]| \leq \frac{2^m}{(m-1)!} C_1[g(a)]^{m-1} \left\{ |\alpha_o(a)|\hat{Y}(a) + |\alpha_1(a)|\hat{Y}'(a) \right\} \tag{3.19}$$

$$(0 < a < b) \quad (m = 1,2, \ldots) \,.$$

Proof: In forming $\mathcal{B}_a[HY_1]$, we may differentiate HY_1 under the integral sign and obtain

$$\frac{\mathcal{B}_a[HY_1] - \mathcal{B}_a[Y_2]}{\mathcal{B}_a[Y_2]} \tag{3.20}$$

$$= - \int_o^a Y_1^2(t)\,k(t)\,dt - \frac{\mathcal{B}_a[Y_1]}{\mathcal{B}_a[Y_2]} \int_a^b Y_1(t)\,Y_2(t)\,k(t)\,dt$$

The first term on the right side is $o(1)$ as $a \to 0$. The second term may be decomposed into the factors

$$\frac{Y_1(a)}{Y_2(a)} \int_a^b Y_1(t)\,Y_2(t)\,k(t)\,dt, \qquad - \frac{Y_2(a)}{Y_1(a)} \frac{\mathcal{B}_a[Y_1]}{\mathcal{B}_a[Y_2]}$$

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* There is no loss of generality in assuming that the constants C_1 in lemmas 4, 5, and 6 are identical.

The first factor is $o(1)$ by hypothesis (2.7) and the second factor is bounded by hypothesis (2.12). These statements establish the asymptotic form (3.17). The proof of (3.18) is similar to that of (3.14) and will be omitted. The result (3.19) follows from (3.14) and (3.18).

Lemma 6. Let $\mathcal{B}_a[y] = 0$ be the boundary condition (2.13). Then the following asymptotic form is valid for class 2 problems

$$(3.21) \qquad \mathcal{B}_a[HY_1] = \mathcal{B}_a[Y_2]\;[1 + o(1)] \qquad \text{as} \qquad a \to 0$$

Furthermore, there is a number* C_1 independent of x and m such that

$$(3.22) \qquad |H^m\, Y_1(x)| \leq \frac{2^m}{(m-1)!}\; C_1[g(x)]^{m-1}\; \hat{Y}(x) \qquad (0 < x \leq b)$$

and

$$(3.23) \qquad |\mathcal{B}_a[H^m\, Y_1]| \leq \frac{2^m}{(m-1)!}\; C_1\; [g(a)]^{m-1}\; |\mathcal{B}_a[Y_2]| \qquad (0 < a \leq a_o)$$

Proof: Equation (3.20) holds as in lemma 5. The first integral on the right side is $o(1)$ as $a \to 0$ since $Y_1 \in \mathcal{Y}$, and the second integral tends to a finite limit as $a \to 0$ because class 2 problems are always of the limit circle type. On account of conditions (2.3) and (2.13) it follows that $\mathcal{B}_a[Y_1] = o(1)$ as $a \to 0$, and on account of lemma 3, $\mathcal{B}_a[Y_2]$ is bounded away from zero whenever $a \leq a_o$. Hence the right side of equation (3.20) is $o(1)$ as $a \to 0$. These considerations establish the asymptotic form (3.21).

The proof of the result (3.22) is similar to that of (3.14) on the interval $x_o \leq x \leq b$, and will be omitted. It follows from (3.22)

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* There is no loss of generality in assuming that the constants C_1 in lemmas 4,5, and 6 are identical.

that

$$|\mathcal{B}_a[H^m\, Y_1]| \leq \frac{2^{m-1}\, C_1}{(m-2)!} \int_a^b \left\{ |Y_1(t)\, \hat{Y}(t)\, \mathcal{B}_a[Y_2]| + \right.$$

$$\left. + |Y_2(t)\, \hat{Y}(t)\, \mathcal{B}_a[Y_1]| \right\} \quad [g(t)]^{m-2}\, k(t)\, dt$$

Since assumption (2.13) and lemma 3 show that $|\mathcal{B}_a[Y_1]| \leq |\mathcal{B}_a[Y_2]|$ whenever $a \leq a_o$, it follows that

$$|\mathcal{B}_a[H^m\, Y_1]| \leq \frac{2^m\, C_1}{(m-2)!}\, |\mathcal{B}_a[Y_2]| \int_a^b \hat{Y}^2(t)\, [g(t)]^{m-2}\, k(t)\, dt$$

$$= \frac{2^m\, C_1}{(m-1)!}\, [g(a)]^{m-1}\, |\mathcal{B}_a[Y_2]| \qquad (a \leq a_o)$$

Hence lemma 6 has been established.

Lemmas 5 and 6 can now be combined to give results which are valid for class 1 or class 2 problems. Let $g(x)$ be defined by equation (3.11) for class 1 and by equation (3.10) for class 2. Then

$$(3.24) \qquad \frac{\mathcal{B}_a[Y_1]}{\mathcal{B}_a[HY_1]} = \frac{\mathcal{B}_a[Y_1]}{\mathcal{B}_a[Y_2]}\, [1 + o(1)] \qquad \text{as} \quad a \to 0\,,$$

and

$$(3.25) \qquad \left| \frac{\mathcal{B}_a[H^m\, Y_1]}{\mathcal{B}_a[HY_1]} \right| \leq \frac{2^m\, C}{(m-1)!}\, [g(a)]^{m-1}$$

$$(0 < a \leq a_o) \quad (m = 1, 2, \ldots)$$

where C denotes an upper bound for

$$(3.26) \qquad C_1\, \frac{|\alpha_o(a)|\, \hat{Y}(a) + |\alpha_1(a)|\, \hat{Y}'(a)}{|\mathcal{B}_a[HY_1]|} \quad \text{or} \quad C_1 \left| \frac{\mathcal{B}_a[Y_2]}{\mathcal{B}_a[HY_1]} \right|$$

for class 1 or class 2 respectively. The first expression (3.26) is bounded by equation (3.17) and assumption (2.11) and the second expression is bounded by equation (3.21) and lemma 3.

4. Asymptotic expansions for characteristic values and characteristic functions.

The boundary condition $\mathcal{B}_a[y] = 0$, given by (2.10) for class 1 and (2.13) for class 2, is to be applied to the function $y(x, \lambda)$ with the series expansion (3.7). Considered as a function of the complex variable $\Lambda - \lambda$, the series is an entire function of $\Lambda - \lambda$ for each fixed value of a, and it represents $y(x, \lambda)$ for $a \leq x \leq b$. Termwise differentiation is valid since a standard argument shows that the derived series is uniformly convergent (see lemma 1). Thus

$$(4.1) \quad \mathcal{B}_a[y] = \mathcal{B}_a[Y_1] + (\Lambda - \lambda)\,\mathcal{B}_a[HY_1] + \cdots + (\Lambda - \lambda)^m\,\mathcal{B}_a[H^m Y_1] + \cdots$$

The series on the right is an entire function of $\Lambda - \lambda$.

Lemma 7. For each fixed value of $a \leq a_o$, the function $\mathcal{B}_a[y(x, \lambda)]$ possesses a unique zero $\lambda = \lambda(a)$ in the neighborhood of $\lambda = \Lambda$ which may be represented by the convergent series expansion

$$(4.2) \qquad \lambda(a) = \Lambda - \theta_o(a) \sum_{j=1}^{\infty} \mu_j(a), \qquad a \leq a_o$$

where the coefficients $\mu_j(a)$ are determined recursively by the formal relation

$$(4.3) \qquad 0 = w + \sum_{m=1}^{\infty} \theta_o^{m-1}\, \theta_m \Big(\sum_{j=1}^{\infty} \mu_j\, w^j\Big)^m,$$

where

$$(4.4) \qquad \theta_o(a) = \frac{\mathcal{B}_a[Y_1]}{\mathcal{B}_a[HY_1]}$$

and

$$(4.5) \qquad \theta_m(a) = \frac{\mathcal{B}_a[H^m\, Y_1]}{\mathcal{B}_a[HY_1]} \qquad (m = 1,2, \ldots) .$$

<u>Proof:</u> Let $\lambda - \Lambda = -s\ \theta_o(a)$. Then equation (4.1) may be rewritten in the form

$$(4.6) \qquad \frac{\mathcal{B}_a[y]}{\mathcal{B}_a[Y_1]} - 1 = F(s)$$

where

$$(4.7) \qquad F(s) = s + \Gamma_2\, s^2 + \cdots + \Gamma_m\, s^m + \cdots$$

and

$$(4.8) \qquad \Gamma_m = \Gamma_m(a) = \theta_o^{m-1}(a)\ \theta_m(a) \qquad (m = 1,2, \ldots).$$

It follows from the asymptotic form (3.24) and the inequality (3.25) that

$$(4.9) \qquad \begin{aligned} &\theta_o(a) = o(1) && \text{as} \quad a \to 0 , \\ &|\theta_m(a)| \leq 2C[2g(a)]^{m-1} && (0 < a \leq a_o) \quad (m = 1,2, \ldots). \end{aligned}$$

Then

$$(4.10) \qquad |\Gamma_m(a)| \leq 2C[2g(a)\ \theta_o(a)]^{m-1} \qquad (0 < a \leq a_o) \quad (m = 1,2, \ldots).$$

For class 1 problems, it follows from lemma 5 that

$$g(a)\ \theta_o(a) = \frac{\mathcal{B}_a[Y_1]}{\mathcal{B}_a[Y_2]} \int_a^b Y_1(t)\ \hat{Y}(t)\ k(t)\ dt[1 + o(1)] = o(1)$$

as $a \to 0$. For class 2 problems, $g(a)$ is bounded and hence $g(a)\ \theta_o(a) = o(1)$ as $a \to 0$.

For each value of a, $\Lambda - \lambda$ is a zero of $\mathcal{B}_a[y]$ if and only if s satisfies the equation $F(s) = -1$, since $\mathcal{B}_a[Y_1]$ exists and is different from zero. $F(s)$ represents an analytic function of the complex variable s for $|s| \leq R$, where R is arbitrarily large. Further, $F(s)$ has the properties that $F(0) = 0$ and $F'(0) \neq 0$. Then the equation $F(s) = -w$ possesses a unique solution $s = s(w)$, regular near $w = 0$, of the form

$$(4.11) \qquad s = \sum_{j=1}^{\infty} \mu_j w^j$$

according to Lagrange's theorem on the inversion of power series [2]. One obtains the coefficients μ_j by substituting (4.11) into the equation $F(s) = -w$, with the result

$$-w = F(\sum_{j=1}^{\infty} \mu_j w^j) = \sum_{m=1}^{\infty} \Gamma_m (\sum_{j=1}^{\infty} \mu_j w^j)^m ,$$

and equating the coefficients of successive powers of w.

The radius of convergence of the series on the right side of (4.11) is not less that $R^2|F'(0)|^2/6M$, where M is an upper bound for $F(s)$ in the region of regularity $|s| \leq R$. Since (4.7) and (4.10) show that $|F(s)| \leq 2|s|$ whenever $a \leq a_o$, such an upper bound is $M = 2R$. However, R is arbitrary and hence the series on the right side of (4.11) converges for all values of w. Let $w = 1$. Then it follows that the equation $F(s) = -1$ possesses a unique solution, which is represented by the convergent series $s = \sum_{j=1}^{\infty} \mu_j$. The substitution $\lambda = \Lambda - s\theta_o$ then establishes the result (4.2).

Let $g(x)$ and $\theta_o(a)$ be the functions defined by (3.10), (3.11) and (4.4) respectively. The next lemma yields an asymptotic expansion for the zero $\lambda(a)$ with scale $\nu(a) = g(a)\,\theta_o(a)$ as $a \to 0$ in the sense of definition 7.

Lemma 8. The series on the right side of (4.2) is an asymptotic expansion for the zero $\lambda(a)$ in lemma 7, with scale $g(a)\,\theta_o(a)$ as $a \to 0$. The function $\psi(a)$ in definition 7 is in this case $\theta_o(a)$.

Proof: The coefficients μ_j in lemma 7, as determined by (4.3), have the form

$$(4.12)\quad \mu_j = \delta_j - \sum \Phi \gamma_h \mu_{k_1} \mu_{k_2} \cdots \mu_{k_h} \qquad (j = 1,2, \dots),$$

where the summation extends over indices $h, k_1, k_2, \dots , k_h$ with the properties

$$h = 2, 3, \dots ;\ k_1 + k_2 + \dots + k_h = j,$$

where $\delta_1 = -1$, $\delta_j = 0$ for $j = 2,3, \dots$, and where the coefficients Φ depend upon the indices $h, k_1, k_2, \dots , k_h$ but not on ν . It follows from (4.10) that $\gamma_h = O(\nu^{h-1})$ $(h = 1,2, \dots)$. Clearly $\mu_1 = O(1)$. Under the assumption that $\mu_\ell = O(\nu^{\ell-1})$ for all $\ell \leq j-1$ it follows from (4.12) that

$$\begin{aligned} \mu_j &= O(\nu^{h-1+k_1+k_2+ \cdots k_h-h}) \\ &= O(\nu^{j-1}) \qquad (j = 2,3, \dots). \end{aligned}$$

Hence $\mu_j = O(\nu^{j-1})$ is valid for $j = 1,2, \dots$. by mathematical induction. Then it follows that

$$\lambda(a) - \Lambda + \theta_o(a)\,[\mu_1(a) + \cdots + \mu_j(a)] = O(\theta_o \nu^j)$$

$$(j = 1, 2, \ldots)$$

and hence the series on the right side of (4.2) is an asymptotic expansion for $\lambda(a)$ with scale $\nu(a)$ as $a \to 0$, in the sense of definition 7. The function $\psi(a)$ in that definition is in this case $\theta_o(a)$.

It follows in particular from lemma 8 that the following asymptotic forms are valid

$$(4.13) \qquad \begin{aligned} \lambda(a) &= \Lambda\,[1 + o(1)] && \text{as} \quad a \to 0 \\ \lambda(a) &= \Lambda + \theta_o(a)[1 + o(1)] && \text{as} \quad a \to 0\,. \end{aligned}$$

The proof of lemma 8 has been modeled after the proof of a general theorem of van der Corput [9], pp. 59-61.

<u>Theorem 1</u>. For each characteristic value Λ of the self-adjoint operator A_o, for class 1 or class 2 problems, there exists exactly one characteristic value $\lambda = \lambda(a)$ of A_a in the neighborhood of Λ such that the convergent asymptotic expansion (4.11) is valid. Also the corresponding characteristic function $y(x, \lambda)$ possesses the asymptotic expansion

$$(4.14) \qquad y[x, \lambda(a)] \sim Y_1(x, \Lambda) + \sum_{j=1}^{\infty} [\Lambda - \lambda(a)]^j\, H^j\, Y_1(x, \Lambda)$$

with scale $\nu(a,x) = g(x)\,\theta_o(a)$ as $a \to 0$, valid uniformly for $a \leq x \leq b$, in the sense of definition 8. The function $\psi(a,x)$ in this case is $\theta_o(a)\,\hat{Y}(x)$.

Proof: The zero $\lambda(a)$ described in lemmas 7 and 8, with the asymptotic expansion (4.11), satisfies the condition $\mathcal{B}_a[y(x, \lambda)] = 0$. Then all the conditions of lemma 2 are satisfied and it follows that $\lambda(a)$ is a characteristic value of A_a with corresponding characteristic function $y[x, \lambda(a)]$. If $\ell \neq \lambda$ were another characteristic value near Λ, then $\mathcal{B}_a[y(x, \ell)] \neq 0$ by the uniqueness of the zero, which contradicts $y(x, \ell) \in \mathcal{D}_a$.

In order to prove (4.14) we observe that

$$(4.15) \qquad y[x, \lambda(a)] - Y_1(x, \Lambda) = \sum_{j=1}^{\infty} [\Lambda - \lambda(a)]^j H^j Y_1(x, \Lambda).$$

Because of lemmas 4, 6, and 8 it follows that

$$|H^j Y_1(x, \Lambda)| \leq 2C_1[2g(x)]^{j-1} \hat{Y}(x) \qquad (a \leq x \leq b) \quad (j = 1,2, \ldots)$$

$$|\Lambda - \lambda(a)|^j \leq |2\theta_o(a)|^j \qquad (0 < a \leq a_o) \quad (j = 1,2, \ldots).$$

The terms after the mth in the series on the right of (4.15) can then be estimated as follows:

$$|(\Lambda - \lambda)^{m+1} H^{m+1} Y_1(x) + (\Lambda - \lambda)^{m+2} H^{m+2} Y_1(x) + \cdots |$$

$$\leq 4C_1|\theta_o(a) \hat{Y}(x)| \left\{|4g(x) \theta_o(a)|^m + |4g(x) \theta_o(a)|^{m+1} \cdots \right\}$$

$$\leq 8C_1|\theta_o(a) \hat{Y}(x)| \quad |4g(x) \theta_o(a)|^m$$

$$a \leq a_o, \qquad a \leq x \leq b, \qquad (m = 1,2, \ldots) .$$

Then (4.15) yields the result

$$(4.16) \qquad y[x, \lambda(a)] - Y(x,\Lambda) - \sum_{j=1}^{m} [\Lambda - \lambda(a)]^j H^j Y_1(x,\Lambda)$$

$$= 0 \left\{ \theta_o(a) \hat{Y}(x) [g(x) \theta_o(a)]^m \right\}$$

as $a \to 0$ uniformly for $a \leq x \leq b$ $(m = 1, 2, \ldots)$. Then, according to definition 8, the series on the right of (4.14) constitutes an asymptotic expansion of $y[x, \lambda(a)]$ with scale $g(x)\ \theta_o(a)$ as $a \to 0$, uniformly for $a \leq x \leq b$. This completes the proof of theorem 1.

<u>Corollary 1</u>. For class 1 or class 2 problems, the characteristic value $\lambda(a)$ of A_a has the asymptotic form

$$(4.17) \qquad \lambda(a) = \Lambda + \frac{\mathcal{B}_a[Y_1]}{\mathcal{B}_a[Y_2]} [1 + o(1)] \qquad \text{as} \qquad a \to 0.$$

<u>Proof</u>: This follows from (3.24), (4.4), and (4.13).

<u>Corollary 2</u>. Let $I[Y_1]$ be a closed subset of $(0,b]$ with the property that $Y_1(x,\Lambda) \neq 0$ whenever $x \in I[Y_1]$. Then the characteristic function $y(x,\lambda)$ has the asymptotic expansion

$$(4.18) \qquad y[x, \lambda(a)] = Y_1(x,\Lambda) + \sum_{j=1}^{m} [\Lambda - \lambda(a)]^j H^j Y_1(x,\Lambda) +$$

$$+ Y_1(x,\Lambda)\ O[\theta_o^{m+1}(a)]$$

with scale $\theta_o(a)$ as $a \to 0$, uniformly for $x \in I[Y_1]$.

Proof: Since $Y_1(x, \Lambda)$ is bounded away from zero and $\hat{Y}(x)$ is bounded whenever $x \in I[Y_1]$, it follows that $\hat{Y}(x)$ can be replaced by $Y_1(x, \Lambda)$ in the error term on the right side of (4.16). Furthermore, the function $g(x)$ in the error term can be replaced by a constant whenever $x \in I[Y_1]$.

It follows in particular from corollary 2 that

$$(4.19) \qquad y[x, \lambda(a)] = Y_1(x, \Lambda) \left\{1 + O[\theta_o(a)]\right\}$$

as $a \to 0$, uniformly for $x \in I[Y_1]$.

Corollary 3. The characteristic function $y(x, \lambda)$ has the following asymptotic form

$$(4.20) \qquad y[x, \lambda(a)] = Y_1(x, \Lambda) + \frac{\mathcal{B}_a[Y_1]}{\mathcal{B}_a[Y_2]} \; HY_1(x, \Lambda)[1 + o(1)]$$

as $a \to 0$, uniformly for $x \in I[Y_1]$.

Proof: This follows from (4.17) and (4.18).

Corollary 4. The characteristic function $y(x, \lambda)$ possesses a convergent series expansion, consisting of the series on the right side of (4.14), valid for $a \leq a_o$ uniformly for $a \leq x \leq b$. Furthermore, $y[x, \lambda(a)]$ has the property that

$$(4.21) \qquad ||y[x, \lambda(a)] - Y_1(x, \Lambda)|| = o(1) \qquad \text{as} \qquad a \to 0 .$$

Proof: The first statement follows from the proof of theorem 1. Since $y \equiv 0$ for $0 < x < a$ by definitions 14 and 15, it follows that

$$(4.22) \quad ||y[x, \lambda(a)] - Y_1(x, \Lambda)||^2$$

$$= \int_o^a Y_1^2(x, \Lambda)\, k(x)\, dx + \int_a^b |y[x, \lambda(a)] - Y(x, \Lambda)|^2\, k(x)\, dx.$$

The first term on the right side is $o(1)$ as $a \to 0$ since $Y_1 \in \mathfrak{h}$. It follows from (4.16) that there exist constants C and a_o such that

$$|y[x, \lambda(a)] - Y_1(x, \Lambda)| \leq C|\theta_o(a)|\ \hat{Y}(x, \Lambda)$$

whenever $a \leq a_o$, uniformly for $a \leq x \leq b$. On account of (2.8) and (3.9) there exists a number x_o such that $\hat{Y}(x, \Lambda) = Y_2(x, \Lambda)$ whenever $a \leq x \leq x_o$. Hence the second term on the right side of (4.22) does not exceed

$$(4.23) \qquad C^2|\theta_o(a)|^2 \int_a^{x_o} Y_2^2(x, \Lambda)\, k(x)\, dx +$$

$$+ C^2|\theta_o(a)|^2 \int_{x_o}^{b} \hat{Y}^2(x, \Lambda)\, k(x)\, dx$$

provided that $a \leq a_o$. The second term in (4.23) is $O(\theta_o^2) = o(1)$ as $a \to 0$. In the first term,

$$\theta_o(a) = O\left[\frac{Y_1(a)}{Y_2(a)}\right] , \qquad |Y_2(x)| \leq \left|\frac{Y_2(a)\ Y_1(x)}{Y_1(a)}\right|$$

whenever $a \leq a_o$, uniformly for $a \leq x \leq x_o$. Then the first term is dominated by

$$\frac{Y_1(a)}{Y_2(a)} \int_a^{x_o} Y_1(x)\ Y_2(x)\ k(x)\ dx,$$

which is $o(1)$ as $a \to 0$ by hypothesis (2.7). Hence the expression (4.23) is $o(1)$ as $a \to 0$. This completes the proof of (4.21).

5. Regular singularities.

The differential operator (2.1) is under consideration when $q(x)$ and $k(x)$ are real-valued, piecewise continuous functions of the real variable x, and $k(x)$ is positive-valued. The following specific assumptions will be made in this section

$$q(x) \sim q_{-2}\, x^{-2} + q_{-1}\, x^{-1} + \cdots \qquad \text{as } x \to 0$$
$$(q_{-1},\ q_{-2} \text{ not both zero})$$

(5.1)

$$k(x) \sim k_{-1}\, x^{-2+m} + k_{0}\, x^{-1+m} + \cdots \qquad \text{as } x \to 0$$
$$(m = 1,2, \ldots ; \quad k_{-1} > 0).$$

The symbols ρ_1 and ρ_2 will denote the roots of the quadratic equation

$$\rho(\rho - 1) - q_{-2} = 0. \tag{5.2}$$

Class 1 problems. It is assumed that $q_{-2} \geq -\frac{1}{4}$. If $q_{-2} > -\frac{1}{4}$, then the roots ρ_1 and ρ_2 are distinct ($\rho_1 > \rho_2$). The differential equation $LU = \Lambda U$ under consideration possesses linearly independent solutions $U_j(x)$ $(j = 1,2)$ with the asymptotic behavior

$$U_j(x) \sim x^{\rho_j} \qquad \text{as} \quad x \to 0 \quad (j = 1,2). \tag{5.3}$$

If in addition $q_{-2} \geq \frac{1}{4}(m^2 - 1) > -\frac{1}{4}$ $(m = 1,2, \ldots)$ it follows that $\rho_2 \leq \frac{1}{2}(1-m)$, so that U_2 is not in $\mathfrak{h}$. Hence this is the limit point case and no boundary condition is needed at the point $x = 0$. If $-\frac{1}{4} < q_{-2} < \frac{1}{4}(m^2 - 1)$, then $\rho_2 > \frac{1}{2}(1 - m)$ and hence both linearly independent solutions belong to $\mathfrak{h}$. This is the limit circle case and will not in general lead to a class 1 problem. However, it is known [7] that the function $\chi(x)$ in section 2 can be chosen so that the boundary condition (2.3) reduces to

(5.4) $$\mathcal{B}_o[Y] \equiv \lim_{x \to 0} x^{-\rho_2} Y(x) = 0 \,.$$

This condition requires that $Y(x) \sim Cx^{\rho}$ as $x \to 0$, with $\rho > \rho_2$ in order that $Y \varepsilon \mathcal{H}$. Since Y is supposed to satisfy the differential equation $LU = \Lambda U$ in order to be a characteristic function of A_o, it follows that $Y(x) \sim Cx^{\rho_1}$ as $x \to 0$. With the choice (5.4) for the boundary condition (2.3), any characteristic function Y_1 of A_o satisfies the hypothesis (2.7).

The boundary condition (5.4) ensures that the characteristic functions for the basic problem will have the same asymptotic behavior as the small solution U_1 in (5.3). If a more general boundary condition of the type (2.3) were to replace (5.4), a problem would arise which is in neither class 1 nor class 2.

If $q_{-2} = -\frac{1}{4}$, the roots of (5.2) are identical, $\rho_1 = \rho_2$. Then there are linearly independent solutions of $LU = \Lambda U$ with the asymptotic behavior

(5.5) $$U_1(x) \sim x^{1/2} \qquad U_2(x) \sim x^{1/2} \ln x \quad \text{as} \quad x \to 0.$$

The boundary condition (5.4) is replaced by

(5.6) $$\mathcal{B}_o[Y] \equiv \lim_{x \to 0} x^{-1/2} (\ln x)^{-1} Y(x) = 0.$$

Again this demands that $Y(x) \sim CU_1(x)$ as $x \to 0$, so that (2.7) is satisfied in this case also.

<u>Class 2 problems</u>. It is assumed that $q_{-2} < -\frac{1}{4}$. The solutions ρ_1 and ρ_2 of (5.2) are then of the form

$$(5.7)\qquad \rho_1 = \frac{1}{2} + i\omega , \qquad \rho_2 = \frac{1}{2} - i\omega$$

where ω is a positive number. The point $x = 0$ comes under the limit point case, and appropriate boundary conditions are known [7] to be

$$(5.8)\qquad \mathcal{B}_o[Y] \equiv \lim_{x\to 0} [\zeta'(x)\, Y(x) - \zeta(x)\, Y'(x)] = 0$$

where

$$(5.9)\qquad \zeta(x) = \frac{1}{2} i (x^{\rho_1} e^{i\theta} - x^{\rho_2} e^{-i\theta}) \qquad (0 \leq \theta < \pi).$$

These conditions are deduced from (2.3). Then the condition (2.9) is clearly satisfied for the solutions U_1, U_2 of $LU = \Lambda U$ with the asymptotic behavior $U_j(x) \sim x^{\rho_j}$ as $x \to 0$ $(j = 1,2)$.

In order to define the perturbed self-adjoint operator A_a, we shall now formulate the appropriate boundary conditions (2.10) for class 1 problems, or (2.13) for class 2 problems. We assume that

$$(5.10)\qquad \sigma = \lim_{a\to 0} \frac{a\, \alpha_o(a)}{\alpha_1(a)}$$

exists. The limit may be finite or ∞. For class 1 problems, the restriction (2.11) on the coefficients $\alpha_o(a)$ and $\alpha_1(a)$ will be satisfied provided that

$$(5.11)\qquad \sigma + \rho_2 \neq 0 .$$

This may be eased in the limit point case, but we shall not pursue this now. For class 1 problems, therefore, $\mathcal{D}_a$ is the domain of definition 14 with the condition (5.11) on the functions $\alpha_o(a)$, $\alpha_1(a)$. For class 2 problems, the distinguished boundary condition (2.13) is equivalent to

(5.12) $\qquad \mathcal{B}_a[y] \equiv \zeta'(a)\, y(a) - \zeta(a)\, y'(a) = 0,$

where the function $\zeta(a)$ is given by (5.9).

Because of the assumptions (5.1), it can be seen that the inverse operator $(A_o - i)^{-1}$ is completely continuous on the (dense) subspace of piecewise continuous functions of $\mathfrak{h}$. Then it follows from the result of von Neumann [7] that A_o on $\mathcal{D}_o$ has a denumerable set of characteristic values $\{\Lambda_n\}$ and a corresponding set of characteristic functions $\{Y_{1n}\}$ $(n = 1, 2, \ldots)$. The existence of these sets has also been established by McCrea and Newing [6]. The problem of finding characteristic values λ of A_a on $\mathcal{D}_a$ is the regular Sturm-Liouville problem, so there exists a denumerable set $\{\lambda_n\}$ and a corresponding set $\{y_n\}$. The following theorem can then be easily deduced from theorem 1.

<u>Theorem 2</u>. Let L be the differential operator (2.1) for class 1 or class 2 problems, under the assumption that the point $x = 0$ is a regular singularity and all points in the interval $(0,b]$ are ordinary points. Then the condition (2.7) is satisfied for class 1 problems and (2.9) is satisfied for class 2 problems. Suppose the functions $\alpha_o(a)$ and $\alpha_1(a)$ in (2.10) satisfy the condition (5.11). Then for every characteristic value Λ_n of the operator A_o there exists a unique characteristic value $\lambda_n(a)$ of A_a, which possesses the asymptotic expansion

(5.14) $\qquad \lambda_n(a) \sim \Lambda_n - \theta_o(a) \sum_{j=1}^{\infty} \mu_j(a)$

with scale $\theta_o(a)$ as $a \to 0$ $(n = 1,2, \ldots)$. The coefficients $\mu_j(a)$ are given by (4.3). Furthermore, the characteristic functions $y_n[x, \lambda_n(a)]$ have asymptotic expansions of the type (4.14) with scale $\theta_o(a)$ as $a \to 0$, uniformly for $a \leq x \leq b$.

<u>Corollary 5</u>. Let $U_{1n} = U_{1n}(x, \Lambda_n)$ be the solution of the equation $LY = \Lambda_n Y$ $(n = 1,2, \ldots)$ with the asymptotic behavior (5.3) or (5.5) according as $q_{-2} > -\frac{1}{4}$ or $q_{-2} = -\frac{1}{4}$. Let $\Omega_n = ||U_{1n}||$. Then for class 1 problems, the characteristic values $\lambda_n(a)$ of A_a have the following explicit asymptotic forms, valid as $a \to 0$ $(n = 1,2, \ldots)$ under the restrictions stated.

$$(5.16) \qquad \lambda_n(a) = \Lambda_n + \Omega_n^{-1}(\rho_1 - \rho_2)\, a^{\rho_1 - \rho_2}\,[1 + o(1)]$$

when $\sigma = \infty$, $q_{-2} > -\frac{1}{4}$

$$(5.17) \qquad \lambda_n(a) = \Lambda_n + \Omega_n^{-1}(\rho_1 - \rho_2)\, a^{\rho_1 - \rho_2} \left[\frac{\sigma + \rho_1}{\sigma + \rho_2} + o(1)\right]$$

when σ is finite, $\sigma + \rho_2 \neq 0$, $q_{-2} > -\frac{1}{4}$

$$(5.18) \qquad \lambda_n(a) = \Lambda_n - \Omega_n^{-1}(\ln a)^{-1}\,[1 + o(1)]$$

when $q_{-2} = -\frac{1}{4}$.

We shall now obtain the analogous corollary for class 2. Let $\mathcal{B}_a[y] = 0$ be the distinguished boundary condition (5.12). Then theorem 2 and lemma 6 give the asymptotic form

$$(5.19) \qquad \lambda_n(a) - \Lambda_n = \frac{\mathcal{B}_a[Y_{1n}]}{\mathcal{B}_a[Y_{2n}]}\,[1 + o(1)] \qquad \text{as} \quad a \to 0 .$$

To obtain a more explicit result, we need to investigate $\mathcal{B}_a[Y_{1n}]$ and $\mathcal{B}_a[Y_{2n}]$ as $a \to 0$. For class 2 problems the asymptotic behavior of the solutions $U_{jn}(x)$ $(j = 1,2;\ n = 1,2, \dots)$ of $LU = \Lambda_n U$ is

$$(5.20) \quad U_{jn}(x) \sim x^{1/2} e^{\pm i\omega \ln x} [1 + U_{jn}^{(1)} x + \cdots] \qquad \text{as} \quad x \to 0$$

$$(j = 1,2; \quad n = 1,2, \dots).$$

Then it follows from the boundary condition (5.8) that the characteristic functions have the form

$$(5.21) \quad Y_{1n}(x) = C_n \left\{ x^{1/2} \sin(\omega \ln x + \theta) [1 + B_n x + \cdots] + \right.$$

$$\left. + x^{1/2} \cos(\omega \ln x + \theta) [D_n x + \cdots] \right\}$$

$$0 < x \leq b, \qquad 0 \leq \theta < \pi \qquad (n = 1,2, \dots)$$

where C_n is a normalization constant determined so that $\|Y_{1n}\| = 1$, and B_n and D_n are constants. The second linearly independent solution of $LU = \Lambda_n U$, determined as in section 2, has the asymptotic behavior

$$(5.22) \qquad Y_{2n}(x) = C_n^{-1} \omega^{-1} x^{1/2} [\cos(\omega \ln x + \theta) + O(x)] \qquad \text{as} \quad x \to 0,$$

$$0 \leq \theta < \pi \quad (n = 1,2, \dots).$$

When (5.21), (5.22) as well as (5.9) are inserted into (5.12), the following results are obtained:

$$(5.23)\ \mathcal{B}_a[Y_{1n}] = C_n\, a[B_n \sin^2(\omega \ln a + \theta) - D_n \omega +$$

$$+ \frac{1}{2} D_n \sin(2\omega \ln a + 2\theta) + O(a)] \qquad \text{as} \quad a \to 0,$$

$$(5.24) \qquad \mathcal{B}_a[Y_{2n}] = - C_n^{-1}\, [1 + O(a)] \qquad \text{as} \quad a \to 0.$$

Substitution of (5.23) and (5.24) into (5.19) then leads to the following corollary.

<u>Corollary 6</u>. The characteristic values $\lambda_n(a)$ of A_a for class 2 problems have the asymptotic forms

$$(5.25) \qquad \lambda_n(a) = \Lambda_n - C_n^2\, a[B_n \sin^2(\omega \ln a + \theta) - D_n \quad +$$

$$+ \frac{1}{2} D_n \sin(2\omega \ln a + 2\theta) + o(1)] \qquad \text{as} \quad a \to 0$$

$$(0 \leq \theta < \pi) \quad (n = 1, 2, \ldots).$$

Success in obtaining a result like (5.25) depends in a crucial manner on a distinguished boundary condition being applied at the point $x = a$. The necessity of introducing such a boundary condition can be understood on the basis of Weyl's theorem [10], but the details of the discussion will be omitted here.

<u>Example 1</u>. The annular membrane with fixed outer edge. The characteristic value problem is to solve

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \lambda w = 0 \qquad (a^2 \leq r^2 = x^2 + y^2 \leq 1),$$

where $0 < a < 1$, under the boundary conditions

$$w = 0 \qquad \text{when} \quad r = 1,$$

$$\alpha_o w + \alpha_1 \frac{\partial w}{\partial r} = 0 \qquad \text{when} \quad r = a.$$

It is assumed that α_o and α_1 are independent of a and that they are not both zero. The characteristic functions have the form

$$w = r^{-1/2} u(r) e^{mi\theta} \qquad (m = 0, \pm 1, \ldots)$$

where r and θ are polar coordinates in the plane. The function $u(r)$ satisfies the equation

$$\frac{d^2u}{dr^2} + u \left[\lambda + \frac{1/4 - m^2}{r^2} \right] = 0$$

and the boundary conditions

$$u(1) = 0, \qquad \hat{\alpha}_o u(a) + \hat{\alpha}_1 u'(a) = 0$$

where

$$\hat{\alpha}_o = a^{-1/2} \alpha_o - \frac{1}{2} a^{-3/2} \alpha_1 ; \qquad \hat{\alpha}_1 = a^{-1/2} \alpha_1 .$$

The problem is in class 1, and corollary 5 gives the results

$$(5.26) \qquad \lambda_{n,m}(a) = \Lambda_{n,m} \pm \Omega^{-1}_{n,m} \; 2m \, a^{2m} [1 + o(1)]$$

$$\text{as } a \to 0 \qquad (n,m = 1,2, \ldots),$$

where the $+$ or $-$ sign is chosen according as $\alpha_1 = 0$ or $\alpha_1 \neq 0$, and where

$\Lambda_{n,m}$ satisfies $J_m(\Lambda^{1/2}_{n,m}) = 0$,

$$\Omega_{n,m} = \int_o^1 |Y_{n,m}(r)|^2 \, dr,$$

$$Y_{n,m} = C_{n,m} r^{1/2} J_m(r \Lambda^{1/2}_{n,m}),$$

and

$C_{n,m}$ is independent of r.

An explicit calculation yields [11], p. 381

$$\Omega_{n,m} = 2^{2m-1} [\Gamma(m+1)]^2 J^2_{m+1}(\Lambda^{1/2}_{n,m}) \Lambda^{-m}_{n,m}$$

$$(n,m = 1,2, \ldots).$$

Hence (5.26) becomes

$$(5.27) \qquad \lambda_{n,m}(a) = \Lambda_{n,m} \pm \frac{2m \Lambda^m_{n,m} \; a^{2m}}{2^{2m-1} [\Gamma(m+1)]^2 J^2_{m+1}(\Lambda^{1/2}_{n,m})} [1 + o(1)]$$

$$\text{as} \quad a \to 0 \qquad (n,m = 1,2, \ldots).$$

If $m = 0$, this is replaced by

$$(5.28) \qquad \lambda_{n,o}(a) = \Lambda_{n,o} \pm \frac{2}{J_1(\Lambda^{1/2}_{n,o})} \frac{1}{\ln a} [1 + o(1)]$$

$$\text{as} \quad a \to 0 \qquad (n = 1,2, \ldots),$$

where $\Lambda_{n,o}$ satisfies $J_o(\Lambda^{1/2}_{n,o}) = 0$.

Similar results can be obtained in the event that α_o and α_1 depend upon a.

6. Irregular singularities.

It will be assumed in this section that the differential operator (2.1) has an irregular singularity of finite rank p ($p = 1,2, \ldots$) at $x = 0$. For convenience the singularity will be transformed to the point at infinity, and the characteristic value problems will be considered on the interval $b \leq x < \infty$. The differential equation involved in these problems is

$$\frac{d^2U}{dx^2} + U[-q(x) + \Lambda k(x)] = 0 \qquad (b \leq x < \infty) \tag{6.1}$$

where $q(x)$ and $k(x)$ are piecewise continuous, real-valued functions on the interval $[b,\infty)$, and $k(x)$ is positive-valued. The following specific assumptions will be made*

$$q(x) \sim Q_{2p-2}\, x^{2p-2} + Q_{2p-3}\, x^{2p-3} + \cdots \qquad \text{as } x \to \infty \tag{6.2}$$

$$(Q_{2p-2} > 0)\ (p = 1,2, \ldots)$$

$$k(x) \sim K_m\, x^m + K_{m-1}\, x^{m-1} + \cdots \qquad \text{as } x \to \infty \tag{6.3}$$

$$(K_m > 0)\ (m = 2p-3,\ 2p-4, \ldots).$$

The positive integer p is the rank of the irregular singularity at $x = \infty$. Once p is prescribed, m assumes one of the values $2p-3$, $2p-4$, ... It is known [3], [7], that (6.1) possesses linearly independent solutions $U_j(x)$ $(j = 1,2)$, called the normal solutions, with the asymptotic behavior**

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* The case that the dominant term in $q(x)$ at $x = \infty$ has an odd exponent, or $Q_{2p-2} = 0$, will be treated later in this section.

** The convention will be made that $j = 1$ corresponds to the negative sign and $j = 2$ to the positive sign.

$$(6.4)\qquad U_j(x) \sim e^{\mp\omega(x)}\, x^{\gamma_j}\,(1+\sum_{h=1}^{\infty} U_{jh}\, x^{-h}) \qquad \text{as} \quad x \to \infty \qquad (j = 1,2)$$

where $\omega(x)$ is the polynomial

$$\omega(x) = \omega_p\, x^p + \omega_{p-1}\, x^{p-1} + \dots + \omega_1 x,$$

$$(6.5)\qquad \omega_p = p^{-1}\, Q_{2p-2}^{1/2} > 0 \qquad (p = 1,2, \dots).$$

The coefficients ω_ℓ ($\ell = p-1, p-2, \dots , 1$), γ_j, and U_{jh} ($j = 1,2$; $h = 1,2, \dots$) are determined recursively after formal substitution of (6.2), (6.3), and (6.5) into (6.1). Observe that ω_p is a positive number and ω_ℓ , γ_j, and U_{jh} are real when Λ is real. Further, $\gamma_1 + \gamma_2 = 1-p$ ($p = 1,2, \dots$).

In this section, $\mathfrak{h}$ denotes the Hilbert space of all complex-valued functions $U(x)$ such that $\int_b^\infty |U(x)|^2\, k(x)\, dx$ exists, with inner product defined analogously to that in section 2. The domain $\mathcal{D}_o$ consists of those functions Y belonging to $\mathfrak{h}$ which satisfy (a), (b), and (c) of definition 9, with the interval $(0,b]$ replaced by $[b,\infty)$. Definition 10 is void because the point $x = \infty$ always comes under the limit point case according to (6.4) and (6.5). Let a be a number in the interval $b < a < \infty$. Then $\mathcal{D}_a$ is defined by definition 6 with $[a,b]$ replaced by $[b,a]$. The operator A_o as well as A_a is a self-adjoint operator in the space $\mathfrak{h}$, but the inverse operator $(A_o - i)^{-1}$ is not necessarily completely continuous on the subspace of piecewise continuous functions of $\mathfrak{h}$ for arbitrary integers p and $m \leq 2p-3$.

It will be assumed that there exists an isolated point Λ in the spectrum of A_o. Let Y_1 be the corresponding characteristic function and let Y_2 be a real linearly independent solution of the equation $LU = \Lambda U$ such that $[Y_1, Y_2] = -1$. Let U_1, U_2 be the normal solutions of this equation and let

$$(6.6)\qquad \begin{aligned} \Delta &= \mathcal{W}[U_1, U_2] = 2Q_{2p-2}^{1/2}, \\ \Omega &= \|U_1\| = \int_b^\infty U_1^2(t)\, k(t)\, dt. \end{aligned}$$

It follows from (6.4) that the condition

$$(6.7)\qquad \frac{Y_1(x)}{Y_2(x)} \int_b^x Y_1(t)\, Y_2(t)\, k(t)\, dt = o(1) \qquad \text{as} \quad x \to \infty$$

is always satisfied under the assumptions of this section. Clearly (6.7) is equivalent to the assumption (2.7). Let $\alpha_o(a)$, $\alpha_1(a)$ be the coefficients occurring in the boundary condition (2.10). Suppose that

$$(6.8)\qquad \sigma = \lim_{a\to\infty} \frac{a^{1-p}\, \alpha_o(a)}{\alpha_1(a)} \qquad\qquad (p = 1, 2, \ldots)$$

exists. It may be a finite number or ∞. We shall assume that $\alpha_o(a)$, $\alpha_1(a)$ satisfy

$$(6.9)\qquad \sigma + p\,\omega_p \neq 0,$$

which is a sufficient condition for (2.11) to hold. Then the following result is established in the same way as theorem 1. The details of the proof will be omitted.

Theorem 3. Let L be the differential operator (2.1) defined on the interval $[b,\infty)$, where $x = \infty$ is an irregular singularity of finite rank and all other points are ordinary points. Under the assumptions of this section the problems will always be of class 1. For each isolated point Λ in the spectrum of A_o there exists exactly one point $\lambda(a)$ in the spectrum of A_a with the property that the convergent asymptotic expansion (4.2) is valid as $a \to \infty$. The corresponding characteristic function $y(x, \lambda)$ possesses a convergent asymptotic expansion of the type (4.14), valid as $a \to \infty$, uniformly for $b \leq x \leq a$.

Corollary 7. The characteristic value $\lambda(a)$ in theorem 3 has the following explicit asymptotic form:

$$\lambda(a) = \Lambda + \frac{\Delta}{\Omega} a^{\Upsilon_1 - \Upsilon_2} e^{-2\omega(a)} \left[\frac{\sigma - p\omega_p}{\sigma + p\omega_p} + o(1)\right] \tag{6.10}$$

$$\text{as} \quad a \to \infty, \qquad \sigma + p\,\omega_p \neq 0.$$

Results can still be obtained when $\sigma + p\omega_p = 0$ by modifying the treatment slightly. Suppose that $\sigma + p\,\omega_p = 0$ and that ω_{p-j} is the first non-vanishing coefficient in the series (6.5) after ω_p. In this event

$$\frac{\mathcal{B}_a[Y_1]}{\mathcal{B}_a[Y_2]} = a^{\Upsilon_1 - \Upsilon_2 + j} e^{-2\omega(a)} \left[\frac{\sigma - p\omega_p}{(p-j)\,\omega_{p-j}} + o(1)\right].$$

Then (6.10) is replaced by

$$\lambda(a) = \Lambda + \frac{\Delta}{\Omega} a^{\Upsilon_1 - \Upsilon_2 + j} e^{-2\omega(a)} \left[\frac{\sigma - p\omega_p}{(p-j)\,\omega_{p-j}} + o(1)\right] \quad \text{as} \quad a \to \infty \tag{6.11}$$

$$\sigma + p\omega_p = \omega_{p-1} = \cdots = \omega_{p-j-1} = 0, \quad \omega_{p-j} \neq 0.$$

Similar results can be written down in the event that $\sigma + p\,\omega_p = 0$ and $\omega_{p-j} = 0$ for each $j = 1,2, \dots , (p-1)$.

Consider the special case $p = 1$. Let $\omega = \omega_1$, $\Gamma = \Gamma_1 = -\Gamma_2$. Then (6.10) reduces to

$$(6.12) \qquad \lambda(a) = \Lambda + \frac{2\omega}{\Omega}\, a^{-2\Gamma} e^{-2\omega a}\left[\frac{\sigma - \omega}{\sigma + \omega} + o(1)\right] \quad \text{as} \quad a \to \infty,$$

$$\sigma + \omega \neq 0 .$$

For the case $p = 1$, it can be seen that the inverse operator $(A_o - i)^{-1}$ is completely continuous on the subspace of piecewise continuous functions of $\mathfrak{h}$. Then there is a denumerable set of characteristic values $\{\Lambda_n\}$ of A_o and a corresponding set $\{Y_{1n}\}$ which is complete $(n = 1,2, \dots)$. Then asymptotic forms like (6.12) are valid for each $\lambda_n(a)$ as $a \to \infty$ $(n = 1,2, \dots)$.

It has been assumed so far that $Q_{2p-2} > 0$ $(p = 1,2, \dots)$. If $Q_{2p-2} = 0$ and $Q_{2p-3} > 0$, then there do not exist normal solutions of equation (6.1). However, the problem can be reduced to the normal case by the transformation

$$\xi = x^{1/2}, \qquad V(\xi) = \xi^{-1/2}\, U(x)$$

of equation (6.1).

<u>Example 2</u>. Consider the characteristic value problem

$$(6.13) \qquad \frac{d^2u}{dx^2} + u(\lambda - x^2) = 0 \qquad (0 \leq x \leq a)$$

$$(6.14) \qquad u(0) = u(a) = 0 .$$

The corresponding basic problem is

$$\frac{d^2U}{dx^2} + U[\Lambda - x^2] = 0 \qquad (0 \leq x < \infty) \tag{6.15}$$

$$U(0) = 0 \qquad U \,\varepsilon\, \mathcal{G}. \tag{6.16}$$

Solutions of the latter are

$$Y_n(x, \Lambda_n) = B_n \, D_\nu \, (2^{1/2} x) \tag{6.17}$$

where $\nu = (\Lambda_n - 1)/2$, where B_n is a normalization constant to be determined, and where Λ_n satisfies the equation

$$D_\nu(0) = 0. \tag{6.18}$$

The function $D_\nu(2^{1/2} x)$ is a parabolic cylinder function of the first kind, with the asymptotic forms [11], p. 347

$$\begin{aligned} D_\nu(2^{1/2} x) &= 2^{\nu/2} x^\nu \, e^{-x^2/2} \, [1 + o(1)] && \text{as } x \to \infty \\ D_\nu(2^{1/2} x) &= \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} - \frac{1}{2}\nu)} \, 2^{\nu/2} + o(1) && \text{as } x \to 0 \end{aligned} \tag{6.19}$$

The condition (6.18) requires that

$$\nu = 2n + 1 \qquad (n = 0,1,2, \ldots), \tag{6.20}$$

or

$$\Lambda_n = 2\nu + 1 = 4n + 3 \qquad (n = 0,1,2, \ldots). \tag{6.21}$$

The normal solutions of (6.15) are

$$\begin{aligned} U_1 &\sim e^{-x^2/2} \, x^{2n+1} \\ U_2 &\sim e^{x^2/2} \, x^{-2n-2} \qquad \text{as } x \to \infty \quad (n = 0,1,2, \ldots). \end{aligned} \tag{6.22}$$

Comparison of (6.17), (6.18), (6.22) shows that

$$(6.23) \qquad B_n = 2^{-n-1/2} \qquad\qquad (n = 0,1,2, \ldots).$$

Then the normalized solutions of the basic problem are

$$(6.24) \qquad Y_n(x, \Lambda_n) = 2^{-n-1/2} D_{2n+1} (2^{1/2} x) \qquad (n = 0,1,2, \ldots)$$

where Λ_n is given by (6.21). Then (6.10) can be applied to the problem at hand, with the identification

$$\sigma = \infty , \qquad \Delta = 2$$

$$\gamma_1 = 2n + 1, \qquad \gamma_2 = -2n - 2$$

$$\omega (a) = \frac{1}{2} a^2$$

$$\Omega_n = 2^{-2n-1} \int_0^\infty D^2_{2n+1}(2^{1/2} x) \, dx = \pi^{1/2} 2^{-2n-2}(2n + 1)!$$

(see [11], p. 351), and the following asymptotic forms are obtained for the characteristic values:

$$(6.25) \quad \lambda_n(a) = (4n + 3) + \frac{2^{2n+3}}{\pi^{1/2}(2n+1)!} a^{4n+3} e^{-a^2} [1 + o(1)]$$

$$\text{as} \quad a \to \infty \quad (n = 0,1,2, \ldots).$$

Asymptotic forms for the characteristic values of the problem

$$\frac{d^2u}{dx^2} + u(\lambda - x^2) = 0 \qquad\qquad (0 \leq x \leq a)$$

$$u'(0) = u(a) = 0$$

can be obtained in a similar fashion. Combination of these forms with

(6.25) then leads to asymptotic forms for the characteristic values of the bounded quantum mechanical oscillator problem

$$(6.26) \qquad \frac{d^2u}{dx^2} + u(\lambda - x^2) = 0 \qquad\qquad (-a \leq x \leq a)$$

$$u(-a) = u(a) = 0,$$

as follows:

$$(6.27) \qquad \lambda_n(a) = (2n + 1) + \frac{2^{n+2}\, a^{2n+1}\, e^{-a^2}}{\pi^{1/2}\, n!}\,[1 + o(1)]$$

$$\text{as} \quad a \to \infty \quad (n = 0,1,2, \ldots).$$

7. <u>Equations with two singularities</u>.

So far it has been assumed that the fixed endpoint b of the interval under consideration is an ordinary point of the differential equation. We now weaken this restriction somewhat, and permit b to be a regular singularity.

Consider the differential operator L given by (2.1) on the interval $(0,b)$, where now $q(x)$ and $k(x)$ are piecewise continuous on $(0,b)$. It will be assumed that these functions have the asymptotic expansions

$$q(x) \sim p_{-2}(b-x)^{-2} + p_{-1}(b-x)^{-1} + \cdots \quad \text{as } x \to b,$$

(7.1)

$$k(x) \sim h_{-2}(b-x)^{-2+\ell} + h_{-1}(b-a)^{-1+\ell} + \cdots \quad \text{as } x \to b,$$

$$(h_{-2} > 0) \quad (\ell = 1, 2, \ldots).$$

Let η_1 and η_2 denote the roots of the quadratic equation

$$\eta(\eta - 1) - p_{-2} = 0. \tag{7.2}$$

Then the differential equation $LU = \Lambda U$ possesses linearly independent solutions $W_j(x)$ $(j = 1,2)$ with the asymptotic behavior

$$W_j(x) \sim (b-x)^{\eta_j} \quad \text{as } x \to b \quad (\eta_1 \neq \eta_2) \quad (j = 1,2) \tag{7.3}$$

$$W_1(x) \sim (b-x)^{1/2}, \qquad W_2(x) \sim (b-x)^{1/2} \ln(b-x)$$

$$\text{as } x \to b \quad (\eta_1 = \eta_2).$$

The same Hilbert space $\mathfrak{h}$ is used as in section 2. The boundary conditions at $x = a$ for both class 1 and class 2 problems are left unaltered. However, the boundary condition (2.2) is not appropriate when $x = b$ is a singular point of the differential equation: in the limit point case no condition is applied at $x = b$, and in the limit circle case (2.2) is replaced by the condition

$$(7.4) \qquad \mathcal{B}_b[Y] \equiv \lim_{x \to b} [\bar{\chi}_b'(x)\, Y(x) - \bar{\chi}_b(x)\, Y'(x)] = 0,$$

where the function $\chi_b(x)$ is defined analogously to $\chi(x)$ in section 2. Accordingly, the definitions 9, 10 and 14, 14 of $\mathcal{D}_o$ and $\mathcal{D}_a$ respectively are modified by the omission of (2.2) in the limit point case and the replacement of (2.2) by (7.4) in the limit circle case. Other definitions and assumptions are modified in an obvious way. The crucial assumption (2.7) for class 1 problems is left intact; this is reasonable because the contribution to the integral for values of t near b is small on account of the behavior (7.1) of $k(x)$ and (7.3) of $W_j(x)$ as $x \to b$. Then results corresponding to theorems 1, 2, and 3 and their corollaries can be obtained without substantial alteration of the proofs.

We shall omit the formal presentation of these modified results, and shall be content to treat an example of some importance in the case that the endpoint $x = 0$ as well as $x = b$ is a regular singularity.

Example 3. The bounded rigid rotator in quantum mechanics [8]. A mass particle is restricted to rotate at a constant distance from the origin in three-dimensional Euclidean space. For the bounded problem, the particle is excluded from entering a cone defined by an azimuthal angle a, and in solving the problem in quantum mechanics, one requires that the wave function vanish on the surface of the cone.

It turns out that the wave function must have the form

(7.5) $$\psi(\theta,\phi) = \sin^{-1/2}\theta\, y(\theta)\, e^{mi\phi} \qquad (m = 0, \pm 1, \dots)$$

where $y(\theta)$ satisfies the equation

(7.6) $$\frac{d^2y}{d\theta^2} + y\left[\frac{\frac{1}{4} - m^2}{\sin^2\theta} + \lambda\right] = 0 \qquad (0 < \theta \leq a)$$

and the conditions

(7.7) $$y(a) = 0 \qquad y \in \mathcal{L}^2(0,a).$$

The variables θ and ϕ are spherical polar angles ranging over the domain $0 \leq \phi \leq 2\pi$, $0 \leq \theta \leq a < \pi$. A simple linear transformation of the independent variable θ sends π into 0 and 0 into π. The problem at hand is then of the type discussed earlier in this section, and hence corollary 5 is applicable. The asymptotic form (5.16) becomes in the present notation

(7.8) $$\lambda_n(a) = \Lambda_n + \Omega_n^{-1}(\rho_1 - \rho_2)(\pi - a)^{\rho_1 - \rho_2}[1 + o(1)]$$

$$\text{as } a \to \pi- \qquad (\rho_1 > \rho_2)\ (n = 1, 2, \dots)$$

In this example, the functions $q(x)$ and $k(x)$ described by (5.1), (7.1) are replaced by

$$q(\theta) = (m^2 - \tfrac{1}{4})\sin^{-2}\theta, \qquad k(\theta) = 1 \qquad (0 < \theta < \pi)$$

and

$$q_{-2} = (m^2 - \tfrac{1}{4}) \geq -\tfrac{1}{4} \qquad (m = 0, \pm 1, \dots)$$

$$\rho_1 = \tfrac{1}{2} + m, \qquad \rho_2 = \tfrac{1}{2} - m.$$

It can be shown that the basic problem

$$(7.9) \quad \frac{d^2Y}{d\theta^2} + Y\Big[\frac{\frac{1}{4} - m^2}{\sin^2\theta} + \Lambda\Big] = 0 \qquad (0 < \theta < \pi)$$

$$Y \in \mathfrak{h} = \mathcal{L}^2(0,\pi)$$

has solutions

$$(7.10) \quad Y_1(\theta) \equiv Y_{1,n,m}(\theta) = C_{n,m} \sin^{1/2}\theta \, P_n^m(\cos\theta),$$

$$\Lambda \equiv \Lambda_{n,m} = (n + \tfrac{1}{2})^2$$

$$(n = 0,1,2, \ldots ; m = 0, \pm 1, \ldots , n)$$

where $P_n^m(z)$ is the associated Legendre function of degree n and order m, as defined on the cut $(-1,1)$ [4], and

$$C_{n,m} = (-1)^n \, 2^m \, m! \, \frac{(n-m)!}{(n+m)!}$$

The constant Ω_n in (7.8) is given by [11], p. 325

$$\Omega_n = \Omega_{n,m} = C_{n,m}^2 \int_0^\pi [P_n^m(\cos\theta)]^2 \sin\theta \, d\theta$$

or

$$\Omega_{n,m} = \frac{2^{2m+1} (m!)^2}{2n + 1} \; \frac{(n-m)!}{(n+m)!}$$

Hence (7.8) is rewritten

$$(7.11) \qquad \lambda_{n,m}(a) \sim (n + \tfrac{1}{2})^2 + \frac{m(2n + 1)}{2^{2m} (m!)^2} \; \frac{(n+m)!}{(n-m)!} \, (\pi - a)^{2m}$$

$$\text{as } a \to \pi, \quad (n = 0,1, \ldots ; m = 1,2, \ldots ,n).$$

Similarly, it is found that

$$(7.12)\qquad \lambda_{n,o}(a) \sim (n + \tfrac{1}{2})^2 - \frac{2n + 1}{2}\,[\ln(\pi - a)]^{-1}$$

$$\text{as} \quad a \to \pi \qquad (n = 0,1,2, \ldots).$$

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