PERTURBATIONS OF A ROTATING BLACK HOLE

Thesis by

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ABSTRACT

Decoupled, separable equations describing perturbations of a Kerr black hole are derived. These equations can be used to study black-hole processes involving scalar, electromagnetic, neutrino or gravitational fields. A number of astrophysical applications are made: Misner's idea that gravitational synchrotron radiation might explain Weber's observations is shown to be untenable; rotating black holes are shown to be stable against small perturbations; energy amplification by "superradiant scattering" of waves off a rotating black hole is computed; the "spin down" (loss of angular momentum) of a rotating black hole caused by a stationary non-axisymmetric perturbation is calculated.

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PART I

INTRODUCTION

This dissertation treats interactions of a rotating black hole by means of perturbation techniques. The motivation for this approach is that the full nonlinear system of equations describing black-hole interactions (e.g., the coupled Einstein-Maxwell system) is too complicated for an exact solution; however, a perturbation solution is often a good approximation for many physical situations.

The solution describing an unperturbed black hole will be taken to be the Kerr (1963) solution of the Einstein field equations. The Kerr metric describes the exterior gravitational field of a rotating black hole of mass M and angular momentum aM ($0 \leq a \leq M$; we use units with $c = G = 1$). There is good reason to believe that the Kerr black hole is the unique final state of the gravitational collapse of a sufficiently massive star (for discussion and references to the literature see Misner, Thorne and Wheeler 1973). Einstein's equations in fact allow the possibility that the black hole is charged; I shall adopt the viewpoint that any charged astrophysical object would quickly have its charge neutralized and so will consider only uncharged black holes.

Four types of perturbations will be treated: scalar, electromagnetic, neutrino and gravitational. Although no massless scalar field is known in nature, such a field is often used as a model for other interactions because of its mathematical simplicity.

Previous perturbation treatments were confined to Schwarzschild (non-rotating) black holes, which are much simpler to analyze--and also much less interesting! The background Schwarzschild metric is static and spherically symmetric, so the time and angular dependence can easily be separated out of the equations. It turns out that the resulting coupled radial equations can then be decoupled. (The scalar equation is of course already decoupled since there is only one field component.) Examples are the Regge-Wheeler (1957) and Zerilli (1970) equations, describing odd and even parity gravitational perturbations respectively. These approaches all used conventional tensor analysis, and the gravitational case in particular involved considerable algebraic complexity.

In the Kerr case, the background metric is much more complicated than the Schwarzschild metric, and an attempt to find convenient perturbation equations along the above lines seems doomed from the start. In addition, the replacement of spherical symmetry by axial symmetry means that a separation into spherical harmonics is no longer possible; one expects to end up with partial differential equations in r and θ instead of ordinary differential equations in r.

An alternative to the tensor approach is provided by the Newman-Penrose (1962) formalism, hereafter called "NP". This formalism was used to treat Schwarzschild perturbations by Price (1972), and his results were extended by Bardeen and Press (1973). Here again it was possible to obtain decoupled equations governing the perturbations.

The Schwarzschild and Kerr metrics are very similar from the NP point of view. (Technically, they are both Petrov Type D.) This similarity leads one to suspect that it may be possible to derive decoupled Kerr perturbation equations using the NP formalism. Moreover, Carter (1968) showed that the scalar wave equation in the Kerr background is completely separable into ordinary differential equations, even though there is no known group-theoretical reason for this.

The most important result of this thesis is the construction of decoupled, separable equations for Kerr perturbations. These equations allow one to treat a number of interesting black-hole problems in a convenient way.

The plan of the thesis is as follows: Part II consists of background material to the papers comprising the rest of the thesis. A general outline of the perturbation approach to black-hole problems is given. Also included is a short introduction to the Newman-Penrose formalism. Part III consists of two papers in which a scalar field is used to investigate properties of rotating black holes. The first paper discusses Misner's idea (1972) that gravitational synchrotron radiation might explain Weber's observations. It is shown that synchrotron radiation can be produced only by particles moving in astrophysically unreasonable orbits, orbits with "specific energy at infinity" large compared to 1. The Penrose process for extracting energy from a rotating black hole is discussed and also found to be astrophysically unlikely. The formalism of "Locally Non-Rotating Observers" is described and shown to be a powerful tool for analyzing physical processes near rotating black holes. The second paper in Part III discusses Misner's idea of "superradiant scattering", the wave analog of the Penrose process, which involves particles. The conditions for superradiant scattering to occur are much less stringent than those for the Penrose process, making it a phenomenon of possible astrophysical importance. The magnitude of the effect is found to be at most 0.3% energy amplification for scalar waves. The paper contains an idealized calculation of the "floating orbit" effect, where a particle in orbit around a rotating black hole can radiate energy to infinity without spiralling into the hole. The energy loss to infinity is balanced by superradiant scattering of the energy going down the hole.

Part IV contains the derivations of the separable, decoupled perturbation equations. The equations can all be written in a unified way, the master equation depending only on a parameter s describing the "spin weight" of the field ($s = 0$ for scalar, $s = \pm 1/2$ for neutrino, $s = \pm 1$ for electromagnetic and $s = \pm 2$ for gravitational). Included in this section are all the formulae necessary for doing an actual calculation, such as formulae for the energy flux and polarization of radiation at infinity.

Part V applies the perturbation equations to the important question of the stability of rotating black holes. It is shown numerically that rotating black holes are in fact stable against small perturbations; if they were not, one would have been forced to dismiss them as astrophysically interesting objects.

Part VI describes results from work still in progress. One interesting application of the perturbation equations is to the "spin down" effect which occurs when a rotating black hole is subjected to a stationary, non-axisymmetric perturbation. The magnitude of the effect is computed in the electromagnetic and gravitational cases. Also in Part VI is a discussion of the superradiance effect and the results of some computations of its magnitude in the electromagnetic and gravitational cases.

The Appendix contains some unrelated work on viscous Maclaurin spheroids.

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PART II

BACKGROUND MATERIAL

(a) The Perturbation Problem for Black Holes

When treating black-hole interactions we generally consider a gravitational field, which we denote symbolically by g, and a non-gravitational field ϕ produced by a source J (e.g., ϕ may be the electromagnetic field tensor $F_{\mu\nu}$ and J the electromagnetic 4-current J_μ). Einstein's equations are symbolically

$$\mathscr{D}(g) = T(g,\phi^2) \tag{2.1}$$

where $\mathscr{D}$ is a complicated nonlinear differential operator and T is the stress-energy tensor of the field ϕ. Note that T depends quadratically on ϕ. For gravitational perturbations (e.g., when ϕ is the mass of an infalling particle), this is the complete set of equations. For non-gravitational perturbations, we have in addition a set of field equations for ϕ (e.g., Maxwell's equations in curved spacetime); symbolically,

$$\mathscr{L}(g,\phi) = J \tag{2.2}$$

where $\mathscr{L}$ is a linear differential operator on ϕ in the cases we shall treat. In the perturbation approach, g is regarded as the sum of two pieces, g^A and g^B, say. The quantity g^A describes the unperturbed Kerr solution, while g^B is the perturbation due to the field ϕ^B. The quantity ϕ^A vanishes because the unperturbed hole possesses only a gravitational field.

Consider non-gravitational perturbations (no g^B present except that generated by ϕ^B). If we linearize equations (2.1) in g^B, we obtain the equations describing the perturbation of the gravitational field:

$$\mathscr{D}'(g^A,g^B) = T[g^A,(\phi^B)^2] \quad . \tag{2.3}$$

Here $\mathscr{D}'$ is some other differential operator, linear in g^B . If ϕ^B is a small non-gravitational perturbation, we see that g^B is a second-order small quantity; hence, in equation (2.2) we can use the unperturbed gravitational field g^A :

$$\mathscr{L}(g^A,\phi^B) = J^B \quad . \tag{2.4}$$

This equation describes the perturbation ϕ^B propagating in the unperturbed background geometry g^A , and is sometimes called the "test-field approximation". It is a good approximation whenever the density of energy-momentum due to ϕ^B is much smaller than $1/M^2$, a typical tidal gravitational force in the unperturbed solution. We see that, provided the questions we seek to answer do not require knowing g^B, it is relatively easy to treat non-gravitational perturbations: one simply writes down the equations for the field $\phi = \phi^B$ propagating in the unperturbed Kerr geometry. Note that this simplification would not occur if we were treating electromagnetic perturbations of a charged black hole. In that case there would be an unperturbed electromagnetic field ϕ^A . Since T is quadratic in $\phi = \phi^A + \phi^B$, T would be first-order in ϕ^B and would give rise to a first-order change in g .

Gravitational perturbations, on the other hand, by definition involve g^B and require an explicit linearization of the Einstein field equations. One is able to discuss with this approximation phenomena such as the interactions of a small mass m with a black hole ($m << M$), including the emission of gravitational waves, or one can look for vacuum solutions of the field equations "close" to the black-hole solution. This latter aspect is related to the stability problem for black holes (see Part V).

(b) The Newman-Penrose Formalism

The NP formalism arises naturally by introducing spinor calculus into general relativity, and by then building a tetrad calculus that mimics the spinor calculus. Since most readers are probably more familiar with tetrads than spinors, I shall briefly describe the NP formalism in the language of tetrads, without following the spinor route; a good description of the spinor approach has been given by Pirani (1964). More details on tetrad calculus can be found in Misner, Thorne and Wheeler (1973).

A tetrad is a set $\{\vec{e}_\alpha\}$ of four basis vectors chosen at each point of spacetime. The tetrad is normalized:

$$\vec{e}_\alpha \cdot \vec{e}_\beta = \eta_{\alpha\beta} \tag{2.5}$$

where $\eta_{\alpha\beta}$ is a matrix of <u>constants</u> with signature -2. A familiar example is an orthonormal tetrad, when $\eta_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1)$; we do not assume $\eta_{\alpha\beta}$ has this special form here. It is sometimes convenient to regard the tetrad as a linear combination of coordinate basis vectors:

$$\vec{e}_\alpha = L^{\mu'}{}_\alpha \, \vec{e}_{\mu'} \tag{2.6}$$

where $\vec{e}_{\mu'} = \partial/\partial x^{\mu'}$ in some coordinate system. (Throughout this section, unprimed indices refer to tetrad indices and primes refer to coordinate indices.) The components of the metric tensor in the coordinate system are

$$g_{\mu'\nu'} = \vec{e}_{\mu'} \cdot \vec{e}_{\nu'} = (L^\alpha{}_{\mu'} \vec{e}_\alpha) \cdot (L^\beta{}_{\nu'} \vec{e}_\beta) = L^\alpha{}_{\mu'} \eta_{\alpha\beta} L^\beta{}_{\nu'} \tag{2.7}$$

where $L^{\alpha}{}_{\mu'}$ is the inverse matrix of $L^{\mu'}{}_{\alpha}$. We thus see that (with $\eta_{\alpha\beta}$ chosen once and for all) specifying the tetrad everywhere, which is equivalent to specifying $L^{\mu'}{}_{\alpha}$ everywhere, is equivalent to specifying $g_{\mu'\nu'}$ everywhere. It is $g_{\mu'\nu'}$ which plays the dominant role in ordinary tensor analysis.

Note that equation (2.5) is invariant under Lorentz transformations of the tetrad:

$$\vec{e}_{\bar{\alpha}} = A^{\alpha}{}_{\bar{\alpha}}\vec{e}_{\alpha} \quad , \tag{2.8}$$

where

$$\eta_{\alpha\beta} \equiv \eta_{\bar{\alpha}\bar{\beta}} = A^{\alpha}{}_{\bar{\alpha}}\,\eta_{\alpha\beta}A^{\beta}{}_{\bar{\beta}} \quad . \tag{2.9}$$

Since the Lorentz group has six parameters, there are six degrees of freedom at each point in spacetime in choosing tetrads equivalent to the same $g_{\mu'\nu'}$. One can alternatively think of the tetrad as fixed (e.g., defined by some symmetry property of spacetime), while making the usual coordinate transformations of tensor analysis on $g_{\mu'\nu'}$. For this reason, the tetrad components of a tensor are often called "scalars" in the literature; a better name would be "coordinate scalars".

If the tetrad vectors are regarded as differential operators, they do not in general commute:

$$[\vec{e}_{\alpha},\vec{e}_{\beta}] = c_{\alpha\beta}{}^{\mu}\,\vec{e}_{\mu} \quad . \tag{2.10}$$

For a coordinate basis, the commutation coefficients $c_{\alpha'\beta'}{}^{\mu'}$ are zero since partial derivatives do commute. The commutation coefficients are related to the connection coefficients $\Gamma^{\mu}{}_{\beta\alpha}$ of the basis $\vec{e}_\mu$, which define covariant derivatives:

$$\nabla_\alpha \vec{e}_\beta = \Gamma^\mu{}_{\beta\alpha} \vec{e}_\mu \quad , \tag{2.11}$$

$$c_{\alpha\beta\mu} = \Gamma_{\mu\beta\alpha} - \Gamma_{\mu\alpha\beta} \quad . \tag{2.12}$$

These connection coefficients, which are antisymmetric in the first two indices, are sometimes called "Ricci rotation coefficients".

The Riemann tensor in a non-coordinate basis is given by

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} + \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta} - \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma} - \Gamma^\alpha{}_{\beta\mu}\, c_{\gamma\delta}{}^\mu \quad . \tag{2.13}$$

The Ricci tensor is defined by

$$R_{\alpha\beta} = R^\mu{}_{\alpha\beta\mu} \quad . \tag{2.14}$$

(This has the opposite sign to Misner, Thorne and Wheeler and is chosen to agree with Newman and Penrose.)

The Riemann tensor can be decomposed into the Weyl tensor, the Ricci tensor and the scalar curvature:

$$R^{\alpha\beta}{}_{\gamma\delta} = C^{\alpha\beta}{}_{\gamma\delta} - 2\delta^{[\alpha}{}_{[\gamma}\, R^{\beta]}{}_{\delta]} + \delta^{[\alpha}{}_{[\gamma}\, \delta^{\beta]}{}_{\delta]}\, R \quad . \tag{2.15}$$

The Bianchi identities are

$$R^\alpha{}_{\beta[\gamma\delta;\varepsilon]} = 0 \quad . \tag{2.16}$$

The Einstein field equations with these conventions are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi \, T_{\mu\nu} \quad . \tag{2.17}$$

So far we have described a general tetrad calculus. The NP version dispenses with the summation convention and gives every quantity its own name. The basis vectors are null vectors called $\vec{\ell}$, $\vec{n}$, $\vec{m}$ and $\vec{m}^*$. (An asterisk or a bar will be used to denote complex conjugation. The vectors $\vec{\ell}$ and $\vec{n}$ are real, while $\vec{m}$ is complex with real and imaginary parts each spacelike.) The tetrad satisfies

$$\vec{\ell}\cdot\vec{n} = 1 \ , \quad \vec{m}\cdot\vec{m}^* = -1 \ , \quad \text{all other dot products zero,} \tag{2.18}$$

i.e.,

$$\eta_{\alpha\beta} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad . \tag{2.19}$$

For example, from Cartesian basis vectors in special relativity one can construct an NP tetrad by setting

$$\vec{\ell} = \vec{e}_t + \vec{e}_z \ , \quad \vec{n} = (\vec{e}_t - \vec{e}_z)/2 \ , \quad \vec{m} = (\vec{e}_x + i\vec{e}_y)/\sqrt{2} \quad . \tag{2.20}$$

NP tetrads do not strictly speaking fall into the class of tetrads described earlier because they introduce complex quantities. However, the restrictions that $\vec{\ell}$ and $\vec{n}$ be real and that the real and imaginary parts of $\vec{m}$ be spacelike ensure that no problems arise if we manipulate NP tetrads in the same way as real tetrads of signature -2 .

Regarded as differential operators, the tetrad vectors are denoted by D, Δ, δ and $\bar{\delta}$ respectively (i.e., the intrinsic derivative along the $\vec{\ell}$ direction is D, etc.) The connection coefficients are denoted by twelve complex quantities, called "spin coefficients":

$$\kappa \equiv \Gamma_{m\ell\ell}, \quad \tau \equiv \Gamma_{m\ell n}, \quad \sigma \equiv \Gamma_{m\ell m}, \quad \rho \equiv \Gamma_{m\ell\bar{m}},$$

$$\pi \equiv \Gamma_{n\bar{m}\ell}, \quad \nu \equiv \Gamma_{n\bar{m}n}, \quad \lambda \equiv \Gamma_{n\bar{m}\bar{m}}, \quad \mu \equiv \Gamma_{n\bar{m}m}$$

$$\varepsilon + \bar{\varepsilon} \equiv \Gamma_{n\ell\ell}, \quad \varepsilon - \bar{\varepsilon} \equiv \Gamma_{m\bar{m}\ell}, \quad \gamma + \bar{\gamma} \equiv \Gamma_{n\ell n},$$

$$\gamma - \bar{\gamma} \equiv \Gamma_{m\bar{m}n}, \quad \bar{\alpha} + \beta \equiv \Gamma_{n\ell m}, \quad \beta - \bar{\alpha} \equiv \Gamma_{m\bar{m}m} \quad . \tag{2.21}$$

The commutation relations (2.10) in this notation are:

$$\begin{aligned}
(\Delta D - D\Delta)\varphi &= [(\gamma + \bar{\gamma})D + (\epsilon + \bar{\epsilon})\Delta \\
&\quad - (\tau + \bar{\pi})\bar{\delta} - (\bar{\tau} + \pi)\delta]\varphi \\
(\delta D - D\delta)\varphi &= [(\bar{\alpha} + \beta - \bar{\pi})D + \kappa\Delta \\
&\quad - \sigma\bar{\delta} - (\bar{\rho} + \epsilon - \bar{\epsilon})\,\delta]\varphi \\
(\delta\Delta - \Delta\delta)\varphi &= [-\bar{\nu}D + (\tau - \bar{\alpha} - \beta)\Delta \\
&\quad + \bar{\lambda}\bar{\delta} + (\mu - \gamma + \bar{\gamma})\delta]\varphi \\
(\bar{\delta}\delta - \delta\bar{\delta})\varphi &= [(\bar{\mu} - \mu)D + (\bar{\rho} - \rho)\Delta \\
&\quad - (\bar{\alpha} - \beta)\bar{\delta} - (\bar{\beta} - \alpha)\delta]\varphi.
\end{aligned} \tag{2.22}$$

The Weyl tensor is represented by five complex quantities:

$$\Psi_0 \equiv -C_{\ell m\ell m}, \quad \Psi_1 \equiv -C_{\ell n\ell m}, \quad \Psi_2 \equiv -C_{\ell m\bar{m}n},$$

$$\Psi_3 \equiv -C_{\ell n\bar{m}n}, \qquad \Psi_4 \equiv -C_{n\bar{m}n\bar{m}} \quad . \tag{2.23}$$

The scalar curvature is denoted by

$$\Lambda \equiv R/24 \quad . \tag{2.24}$$

The remaining nine degrees of freedom in the Ricci tensor are represented by three real quantities

$$\Phi_{00} \equiv -\frac{1}{2} R_{\ell\ell} , \qquad \Phi_{11} \equiv -\frac{1}{4}(R_{\ell n} + R_{m\bar{m}}) , \qquad \Phi_{22} \equiv -\frac{1}{2} R_{nn} \tag{2.25a}$$

and three complex quantities

$$\Phi_{01} \equiv -\frac{1}{2} R_{\ell m} , \qquad \Phi_{12} \quad -\frac{1}{2} R_{nm} , \qquad \Phi_{02} \equiv -\frac{1}{2} R_{mm} \quad . \tag{2.25b}$$

Equation (2.13) for the Riemann tensor becomes, using the decomposition (2.15), a set of eighteen complex equations:

$$
\begin{aligned}
D\rho - \bar{\delta}\kappa &= (\rho^2 + \sigma\bar{\sigma}) + (\epsilon + \bar{\epsilon})\rho - \bar{\kappa}\tau - \kappa(3\alpha + \bar{\beta} - \pi) + \Phi_{00} \\
D\sigma - \delta\kappa &= (\rho + \bar{\rho})\sigma + (3\epsilon - \bar{\epsilon})\sigma - (\tau - \bar{\pi} + \bar{\alpha} + 3\beta)\kappa + \Psi_0 \\
D\tau - \Delta\kappa &= (\tau + \bar{\pi})\rho + (\bar{\tau} + \pi)\sigma + (\epsilon - \bar{\epsilon})\tau - (3\gamma + \bar{\gamma})\kappa + \Psi_1 + \Phi_{01} \\
D\alpha - \bar{\delta}\epsilon &= (\rho + \bar{\epsilon} - 2\epsilon)\alpha + \beta\bar{\sigma} - \bar{\beta}\epsilon - \kappa\lambda - \bar{\kappa}\gamma + (\epsilon + \rho)\pi + \Phi_{10} \\
D\beta - \delta\epsilon &= (\alpha + \pi)\sigma + (\bar{\rho} - \bar{\epsilon})\beta - (\mu + \gamma)\kappa - (\bar{\alpha} - \bar{\pi})\epsilon + \Psi_1 \\
D\gamma - \Delta\epsilon &= (\tau + \bar{\pi})\alpha + (\bar{\tau} + \pi)\beta - (\epsilon + \bar{\epsilon})\gamma - (\gamma + \bar{\gamma})\epsilon + \tau\pi - \nu\kappa + \Psi_2 - \Lambda + \Phi_{11} \\
D\lambda - \bar{\delta}\pi &= (\rho\lambda + \bar{\sigma}\mu) + \pi^2 + (\alpha - \bar{\beta})\pi - \nu\bar{\kappa} - (3\epsilon - \bar{\epsilon})\lambda + \Phi_{20} \\
D\mu - \delta\pi &= (\bar{\rho}\mu + \sigma\lambda) + \pi\bar{\pi} - (\epsilon + \bar{\epsilon})\mu - \pi(\bar{\alpha} - \beta) - \nu\kappa + \Psi_2 + 2\Lambda \\
D\nu - \Delta\pi &= (\pi + \bar{\tau})\mu + (\bar{\pi} + \tau)\lambda + (\gamma - \bar{\gamma})\pi - (3\epsilon + \bar{\epsilon})\nu + \Psi_3 + \Phi_{21} \\
\Delta\lambda - \bar{\delta}\nu &= -(\mu + \bar{\mu})\lambda - (3\gamma - \bar{\gamma})\lambda + (3\alpha + \bar{\beta} + \pi - \bar{\tau})\nu - \Psi_4 \\
\delta\rho - \bar{\delta}\sigma &= \rho(\bar{\alpha} + \beta) - \sigma(3\alpha - \bar{\beta}) + (\rho - \bar{\rho})\tau + (\mu - \bar{\mu})\kappa - \Psi_1 + \Phi_{01} \\
\delta\alpha - \bar{\delta}\beta &= (\mu\rho - \lambda\sigma) + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \gamma(\rho - \bar{\rho}) + \epsilon(\mu - \bar{\mu}) - \Psi_2 + \Lambda + \Phi_{11} \\
\delta\lambda - \bar{\delta}\mu &= (\rho - \bar{\rho})\nu + (\mu - \bar{\mu})\pi + \mu(\alpha + \bar{\beta}) + \lambda(\bar{\alpha} - 3\beta) - \Psi_3 + \Phi_{21} \\
\delta\nu - \Delta\mu &= (\mu^2 + \lambda\bar{\lambda}) + (\gamma + \bar{\gamma})\mu - \bar{\nu}\pi + [\tau - 3\beta - \bar{\alpha}]\nu + \Phi_{22}
\end{aligned}
$$

$$
\begin{aligned}
\delta\gamma - \Delta\beta &= (\tau - \bar{\alpha} - \beta)\gamma + \mu\tau - \sigma\nu - \epsilon\bar{\nu} - \beta(\gamma - \bar{\gamma} - \mu) + \alpha\bar{\lambda} + \Phi_{12} \\
\delta\tau - \Delta\sigma &= (\mu\sigma + \bar{\lambda}\rho) + (\tau + \beta - \bar{\alpha})\tau - (3\gamma - \bar{\gamma})\sigma - \kappa\bar{\nu} + \Phi_{02} \\
\Delta\rho - \bar{\delta}\tau &= -(\rho\bar{\mu} + \sigma\lambda) + (\bar{\beta} - \alpha - \bar{\tau})\tau + (\gamma + \bar{\gamma})\rho + \nu\kappa - \Psi_2 - 2\Lambda \\
\Delta\alpha - \bar{\delta}\gamma &= (\rho + \epsilon)\nu - (\tau + \beta)\lambda + (\bar{\gamma} - \bar{\mu})\alpha + (\bar{\beta} - \bar{\tau})\gamma - \Psi_3 \, .
\end{aligned}
\tag{2.26}
$$

The Bianchi identities in this notation have been given by Pirani (1964):

$$
\begin{aligned}
&\bar{\delta}\Psi_0 - D\Psi_1 + D\Phi_{01} - \delta\Phi_{00} = (4\alpha - \pi)\Psi_0 - 2(2\rho + \epsilon)\Psi_1 + 3\kappa\Psi_2 + \\
&\qquad + (\bar{\pi} - 2\bar{\alpha} - 2\beta)\Phi_{00} + 2(\epsilon + \bar{\rho})\Phi_{01} + 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \bar{\kappa}\Phi_{02} \\
&\Delta\Psi_0 - \delta\Psi_1 + D\Phi_{02} - \delta\Phi_{01} = (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2 - \bar{\lambda}\Phi_{00} + \\
&\qquad + 2(\bar{\pi} - \beta)\Phi_{01} + 2\sigma\Phi_{11} + (2\epsilon - 2\bar{\epsilon} + \bar{\rho})\Phi_{02} - 2\kappa\Phi_{12} \\
&3(\bar{\delta}\Psi_1 - D\Psi_2) + 2(D\Phi_{11} - \delta\Phi_{10}) + \bar{\delta}\Phi_{01} - \Delta\Phi_{00} = 3\lambda\Psi_0 - 9\rho\Psi_2 + \\
&\quad + 6(\alpha - \pi)\Psi_1 + 6\kappa\Psi_3 + (\bar{\mu} - 2\mu - 2\gamma - 2\bar{\gamma})\Phi_{00} + (2\alpha + 2\pi + 2\bar{\tau})\Phi_{01} \\
&\quad + 2(\tau - 2\bar{\alpha} + \bar{\pi})\Phi_{10} + 2(2\bar{\rho} - \rho)\Phi_{11} + 2\sigma\Phi_{20} - \bar{\sigma}\Phi_{02} - 2\bar{\kappa}\Phi_{12} - 2\kappa\Phi_{21} \\
&3(\Delta\Psi_1 - \delta\Psi_2) + 2(D\Phi_{12} - \delta\Phi_{11}) + (\bar{\delta}\Phi_{02} - \Delta\Phi_{01}) = 3\nu\Psi_0 + 6(\gamma - \mu)\Psi_1 \\
&\quad - 9\tau\Psi_2 + 6\sigma\Psi_3 - \bar{\nu}\Phi_{00} + 2(\bar{\mu} - \mu - \gamma)\Phi_{01} - 2\bar{\lambda}\Phi_{10} + 2(\tau + 2\bar{\pi})\Phi_{11} \\
&\quad + (2\alpha + 2\pi + \bar{\tau} - 2\bar{\beta})\Phi_{02} + (2\bar{\rho} - 2\rho - 4\bar{\epsilon})\Phi_{12} + 2\sigma\Phi_{21} - 2\kappa\Phi_{22} \\
&3(\bar{\delta}\Psi_2 - D\Psi_3) + D\Phi_{21} - \delta\Phi_{20} + 2(\bar{\delta}\Phi_{11} - \Delta\Phi_{10}) = 6\lambda\Psi_1 - 9\pi\Psi_2 \\
&\quad + 6(\epsilon - \rho)\Psi_3 + 3\kappa\Psi_4 - 2\nu\Phi_{00} + 2\lambda\Phi_{01} + 2(\bar{\mu} - \mu - 2\bar{\gamma})\Phi_{10} \\
&\quad + (2\pi + 4\bar{\tau})\Phi_{11} + (2\beta + 2\tau + \bar{\pi} - 2\bar{\alpha})\Phi_{20} - 2\bar{\sigma}\Phi_{12} + 2(\bar{\rho} - \rho - \epsilon)\Phi_{21} - \bar{\kappa}\Phi_{22} \\
&3(\Delta\Psi_2 - \delta\Psi_3) + D\Phi_{22} - \delta\Phi_{21} + 2(\bar{\delta}\Phi_{12} - \Delta\Phi_{11}) = 6\nu\Psi_1 - 9\mu\Psi_2 \\
&\quad + 6(\beta - \tau)\Psi_3 + 3\sigma\Psi_4 - 2\nu\Phi_{01} - 2\bar{\nu}\Phi_{10} + 2(2\bar{\mu} - \mu)\Phi_{11} + 2\lambda\Phi_{02} - \bar{\lambda}\Phi_{20} \\
&\quad + 2(\pi + \bar{\tau} - 2\bar{\beta})\Phi_{12} + 2(\beta + \tau + \bar{\pi})\Phi_{21} + (\bar{\rho} - 2\epsilon - 2\bar{\epsilon} - 2\rho)\Phi_{22} \\
&\bar{\delta}\Psi_3 - D\Psi_4 + \bar{\delta}\Phi_{21} - \Delta\Phi_{20} = 3\lambda\Psi_2 - 2(\alpha + 2\pi)\Psi_3 + (4\epsilon - \rho)\Psi_4 - 2\nu\Phi_{10} \\
&\quad + 2\lambda\Phi_{11} + (2\gamma - 2\bar{\gamma} + \bar{\mu})\Phi_{20} + 2(\bar{\tau} - \alpha)\Phi_{21} - \bar{\sigma}\Phi_{22} \\
&\Delta\Psi_3 - \delta\Psi_4 + \bar{\delta}\Phi_{22} - \Delta\Phi_{21} = 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 + (4\beta - \tau)\Psi_4 \\
&\quad - 2\nu\Phi_{11} - \bar{\nu}\Phi_{20} + 2\lambda\Phi_{12} + 2(\gamma + \bar{\mu})\Phi_{21} + (\bar{\tau} - 2\bar{\beta} - 2\alpha)\Phi_{22} \\
&D\Phi_{11} - \delta\Phi_{10} - \bar{\delta}\Phi_{01} + \Delta\Phi_{00} + 3D\Lambda = (2\gamma - \mu + 2\bar{\gamma} - \bar{\mu})\Phi_{00} + (\pi - 2\alpha - 2\bar{\tau})\Phi_{01} \\
&\quad + (\bar{\pi} - 2\bar{\alpha} - 2\tau)\Phi_{10} + 2(\rho + \bar{\rho})\Phi_{11} + \bar{\sigma}\Phi_{02} + \sigma\Phi_{20} - \bar{\kappa}\Phi_{12} - \kappa\Phi_{21}
\end{aligned}
$$

$$
\begin{aligned}
&D\Phi_{12} - \delta\Phi_{11} - \bar{\delta}\Phi_{02} + \Delta\Phi_{01} + 3\,\delta\Lambda = (2\gamma - \mu - 2\bar{\mu})\Phi_{01} + \bar{\nu}\Phi_{00} - \bar{\lambda}\Phi_{10} \\
&\quad + 2(\bar{\pi} - \tau)\Phi_{11} + (\pi + 2\bar{\beta} - 2\alpha - \bar{\tau})\Phi_{02} + (2\rho + \bar{\rho} - 2\bar{\epsilon})\Phi_{12} + \sigma\Phi_{21} - \kappa\Phi_{22} \\
&D\Phi_{22} - \delta\Phi_{21} - \bar{\delta}\Phi_{12} + \Delta\Phi_{11} + 3\,\Delta\Lambda = \nu\Phi_{01} + \bar{\nu}\Phi_{10} - 2(\mu + \bar{\mu})\Phi_{11} - \lambda\Phi_{02} \\
&\quad - \bar{\lambda}\Phi_{20} + (2\pi - \bar{\tau} + 2\bar{\beta})\Phi_{12} + (2\beta - \tau + 2\bar{\pi})\Phi_{21} + (\rho + \bar{\rho} - 2\epsilon - 2\bar{\epsilon})\Phi_{22}
\end{aligned}
\tag{2.27}
$$

One way in which the formalism has been used is to find exact and approximate solutions of the vacuum Einstein equations. The reason for the success in this case is that the Bianchi identities (2.27) simplify considerably in vacuum, where all the Φ's are zero.

Maxwell's equations in NP formalism are written in terms of three complex quantities representing the electromagnetic field tensor:

$$
\Phi_0 \equiv F_{\ell m}\ , \qquad \Phi_1 \equiv \frac{1}{2}(F_{\ell n} + F_{\bar{m}m})\ , \qquad \Phi_2 \equiv F_{\bar{m}n}\ . \tag{2.28}
$$

They reduce to four complex equations:

$$
\begin{aligned}
D\Phi_1 - \bar{\delta}\Phi_0 &= (\pi - 2\alpha)\Phi_0 + 2\rho\Phi_1 - \kappa\Phi_2 + 2\pi J_\ell \\
D\Phi_2 - \bar{\delta}\Phi_1 &= -\lambda\Phi_0 + 2\pi\Phi_1 + (\rho - 2\varepsilon)\Phi_2 + 2\pi J_{\bar{m}} \\
\delta\Phi_1 - \Delta\Phi_0 &= (\mu - 2\gamma)\Phi_0 + 2\tau\Phi_1 - \sigma\Phi_2 + 2\pi J_m \\
\delta\Phi_2 - \Delta\Phi_1 &= -\nu\Phi_0 + 2\mu\Phi_1 + (\tau - 2\beta)\Phi_2 + 2\pi J_n\ .
\end{aligned}
\tag{2.29}
$$

Here $J_\ell = J_\mu \ell^\mu$ etc., where J^μ is the electromagnetic 4-current.

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PART III

SCALAR-FIELD CALCULATIONS OF ROTATING BLACK-HOLE PHENOMENA

(a) Rotating Black Holes: Locally Nonrotating Frames, Energy Extraction, and Scalar Synchrotron Radiation (Paper I; collaboration with J. M. Bardeen and W. H. Press, published in Ap. J. **178**, 347 [1972]).

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ROTATING BLACK HOLES: LOCALLY NONROTATING FRAMES, ENERGY EXTRACTION, AND SCALAR SYNCHROTRON RADIATION*

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ABSTRACT

This paper outlines and applies a technique for analyzing physical processes around rotating black holes. The technique is based on the orthonormal frames of "locally nonrotating observers." As one application of the technique, it is shown that the extraction of the rotational energy of a black hole, although possible in principle (e.g., the "Penrose-Christodoulou" process), is unlikely in any astrophysically plausible context. As another application, it is shown that, in order to emit "scalar synchrotron radiation," a particle must be highly relativistic as seen in the locally non-rotating frame—and can therefore not move along an astrophysically reasonable orbit. The paper includes a number of useful formulae for particle orbits in the Kerr metric, many of which have not been published previously.

I. INTRODUCTION

Although there is as yet no certain observational identification of a black hole, the study of the properties of black holes and their interactions with surrounding matter is astrophysically important. Black-hole astrophysics is important for the following reasons. (i) At least some stars of mass $\geqslant 2\ M_\odot$ probably fail to shed sufficient matter, when they die, to become white dwarfs or neutron stars, and instead collapse to form black holes. (ii) At least one irregularly pulsating X-ray source, Cygnus X-1, has been identified with a binary system which has a massive, invisible component; this might well be a black hole emitting X-rays as it accretes matter from its companion (for observations, see, e.g., Schreier *et al.* 1971 and Wade and Hjellming 1972). (iii) A black hole of 10^4–$10^8\ M_\odot$ might lie at the center of the Galaxy and be responsible for radio and infrared phenomena observed there (Lynden-Bell and Rees 1971). (iv) Gravitational waves seem to have been detected coming from the direction of the galactic center with such intensity (Weber 1971 and references cited therein) that black-hole processes are the least unreasonable source. We are faced with a double mystery: first, puzzling observations; second, a poor theoretical understanding of what processes *might* occur near a black hole. Both sides of the mystery call for further theoretical work.

Most interactions of a black hole with its surroundings can be treated accurately by perturbation techniques, where the dynamics of matter, electromagnetic and gravitational waves takes place in the fixed background geometry generated by the

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hole. (Notable exceptions are the interactions of two or more black holes, or of black holes with neutron stars of comparable mass, and the highly nonspherical collapse of a star to form a black hole; currently there are no adequate techniques for treating such processes.) Most previous perturbation analyses have dealt with nonrotating (Schwarzschild) black holes. The static nature of the Schwarzschild metric and its spherical symmetry vastly simplify most problems. The orbits of particles can be described easily, and the theory of electromagnetic (Price 1972*a*) and gravitational (Zerilli 1971; Price 1972*b*) perturbations is well developed. A number of interesting model applications have begun to appear in the literature (Davis *et al.* 1971, 1972; Press 1971; Misner 1972*a*; Misner *et al.* 1972).

However, black holes in nature are likely to be highly rotating (Bardeen 1970*a*), and must therefore be described by the Kerr (1963) metric, rather than the Schwarzschild metric. Phenomena in the vicinity of a rotating black hole are considerably more complicated than in the nonrotating case. The metric is only stationary, not static, and only axisymmetric, not spherically symmetric. A complete description of particle orbits is rather complex (e.g., de Felice 1968; Carter 1968*a*). The equations governing electromagnetic and gravitational perturbations have only recently been separated into ordinary differential equations (Teukolsky 1972). The scalar wave equation has been known to be separable for some time, and has therefore been heavily relied on for qualitative perturbation results, even though there are no known classical scalar fields in nature.

A further difficulty is the complexity of coordinate systems for describing processes near a Kerr hole. Boyer-Lindquist (1967) coordinates are the natural generalization of Schwarzschild curvature coordinates and are the best for many purposes, but sufficiently close to the hole—in the "ergosphere"—they are somewhat unphysical. Example: Physical observers cannot remain "at rest" (r, θ, φ = constant).

In this paper we outline a method for treating physical processes in the Kerr geometry which has proved extremely fruitful in our research. The method, previously used by one of us for a different application (Bardeen 1970*b*), replaces coordinate frames by orthonormal tetrads (i.e., nonholonomic frames) which are carried by "locally nonrotating observers." In essence, the nonrotating observers are chosen to cancel out, as much as possible, the "frame-dragging" effects of the hole's rotation. They "rotate with the black hole" in such a way that physical processes as analyzed in their frame are far more transparent than in any coordinate frame. The method of locally nonrotating frames (LNRF), and the nature of the Kerr geometry as seen from the LNRF, are described in § III.

In § II, as a foundation for the LNRF description, we review properties of the Kerr metric and formulae for its particle orbits. While many of these results are known to those working in the field, many have not appeared in the literature; also we have used computer-assisted algebraic techniques, and other methods, to find equivalent formulae much simpler than many in the literature. These should prove useful to other investigators.

In § IV we apply the formalism of locally nonrotating frames to the question of synchrotron radiation (here, scalar synchrotron radiation) from particles in orbits near a black hole. (See Teukolsky 1972 for a proof that electromagnetic and gravitational synchrotron radiation are qualitatively the same as the scalar case.) This type of mechanism has been proposed by Misner (1972*a*) as a possible explanation for the intensity of Weber's observed radiation: a narrow synchrotron cone beamed in the galactic plane. We find that substantial beaming is possible only for particles in unstable, highly energetic orbits—orbits much more energetic than mere infall from infinity can produce. It is theoretically possible to extract energy from the rotating black hole itself (Penrose 1969; Christodoulou 1970). The LNRF methods give a clear picture of this energy extraction process, and make the process seem astrophysically implausible. In particular, it seems unlikely that such extraction could realistically accelerate matter into a synchrotron-radiating orbit. These results make us pessimistic about the applicability of Misner's interesting synchrotron concept to any realistic astrophysical model.

In future papers, we will make use of methods described here to analyze more detailed and realistic processes near a rotating black hole.

II. BASIC PROPERTIES OF THE KERR METRIC AND ITS ORBITS

We choose units with $G = c = 1$. In Boyer-Lindquist coordinates the metric is

$$ds^2 = -(1 - 2Mr/\Sigma)dt^2 - (4Mar\sin^2\theta/\Sigma)dtd\varphi + (\Sigma/\Delta)dr^2 + \Sigma d\theta^2 + (r^2 + a^2 + 2Ma^2r\sin^2\theta/\Sigma)\sin^2\theta d\varphi^2, \qquad (2.1)$$

or, in contravariant form (matrix inverse),

$$\left(\frac{\partial}{\partial s}\right)^2 = -\frac{A}{\Sigma\Delta}\left(\frac{\partial}{\partial t}\right)^2 - \frac{4Mar}{\Sigma\Delta}\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial\varphi}\right) + \frac{\Delta}{\Sigma}\left(\frac{\partial}{\partial r}\right)^2 + \frac{1}{\Sigma}\left(\frac{\partial}{\partial\theta}\right)^2 + \frac{\Delta - a^2\sin^2\theta}{\Sigma\Delta\sin^2\theta}\left(\frac{\partial}{\partial\varphi}\right)^2. \qquad (2.2)$$

Here M is the mass of the black hole, a is its angular momentum per unit mass ($0 \leq a \leq M$), and the functions Δ, Σ, A are defined by

$$\begin{aligned} \Delta &\equiv r^2 - 2Mr + a^2\,, \\ \Sigma &\equiv r^2 + a^2\cos^2\theta\,, \\ A &\equiv (r^2 + a^2)^2 - a^2\Delta\sin^2\theta\,. \end{aligned} \qquad (2.3)$$

For $a = 0$, equations (2.1) and (2.2) reduce to the Schwarzschild solution in curvature coordinates.

It will be useful to express the metric (2.1) in the standard form valid for any stationary, axisymmetric, asymptotically flat spacetime—vacuum or nonvacuum—

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\varphi - \omega dt)^2 + e^{2\mu_1}dr^2 + e^{2\mu_2}d\theta^2\,. \qquad (2.4)$$

This standard metric becomes Kerr if

$$\begin{aligned} e^{2\nu} &= \Sigma\Delta/A\,, \qquad e^{2\psi} = \sin^2\theta A/\Sigma\,, \\ e^{2\mu_1} &= \Sigma/\Delta\,, \qquad e^{2\mu_2} = \Sigma\,, \qquad \omega = 2Mar/A\,. \end{aligned} \qquad (2.5)$$

The event horizon ("one-way membrane") is located at the outer root of the equation $\Delta = 0$,

$$r = r_+ \equiv M + (M^2 - a^2)^{1/2} \qquad (2.6)$$

for all θ, φ. Over the range $0 \leq a \leq M$, r_+ varies from $2M$ to M. The static limit (outer boundary of the ergosphere) is at the outer root of $(\Sigma - 2Mr) = 0$,

$$r = r_0 \equiv M + (M^2 - a^2\cos^2\theta)^{1/2}\,. \qquad (2.7)$$

A physical observer—i.e., one who follows a timelike world line—must be dragged in the positive φ direction if he is inside the static limit. Observers inside the static limit, i.e., in the ergosphere, have access to the "negative energy trajectories" which extract energy from the black hole (see § III).

The general orbits of particles (or photons) in the Kerr geometry are described by three constants of motion (Carter 1968*a*). In terms of the covariant Boyer-Lindquist components of the particle's 4-momentum at some instant, these conserved quantities are

$$E = -p_t = \text{total energy}\,,$$
$$L = p_\varphi = \text{component of angular momentum parallel to symmetry axis}\,,$$
$$Q = p_\theta{}^2 + \cos^2\theta[a^2(\mu^2 - p_t{}^2) + p_\varphi{}^2/\sin^2\theta]\,. \tag{2.8}$$

Here μ is the rest mass of the particle ($\mu = 0$ for photons), which is a trivial fourth constant of the motion. Note that $Q = 0$ is a necessary and sufficient condition for motion initially in the equatorial plane to remain in the equatorial plane for all time. Any orbit which crosses the equatorial plane has $Q > 0$. When $a = 0$, $Q + p_\varphi{}^2$ is the square of the total angular momentum. By solving equation (2.8) for the p_μ's and thence the p^μ's, one obtains equations governing the orbital trajectory,

$$\Sigma\frac{dr}{d\lambda} = \pm(V_r)^{1/2}\,, \tag{2.9a}$$

$$\Sigma\frac{d\theta}{d\lambda} = \pm(V_\theta)^{1/2}\,, \tag{2.9b}$$

$$\Sigma\frac{d\varphi}{d\lambda} = -(aE - L/\sin^2\theta) + aT/\Delta\,, \tag{2.9c}$$

$$\Sigma\frac{dt}{d\lambda} = -a(aE\sin^2\theta - L) + (r^2 + a^2)T/\Delta\,. \tag{2.9d}$$

Here λ is related to the particle's proper time by $\lambda = \tau/\mu$, and is an affine parameter in the case $\mu \to 0$, and

$$T \equiv E(r^2 + a^2) - La\,,$$
$$V_r \equiv T^2 - \Delta[\mu^2 r^2 + (L - aE)^2 + Q]\,,$$
$$V_\theta \equiv Q - \cos^2\theta[a^2(\mu^2 - E^2) + L^2/\sin^2\theta]\,. \tag{2.10}$$

Without loss of generality one is free to take $\mu = 1$ for particles and $\mu = 0$ for photons, in equations (2.8), (2.9), (2.10). (For particles this merely renormalizes E, L, and Q to a "per unit rest mass" basis.) V_r and V_θ are "effective potentials" governing particle motions in r and θ. Notice that V_r is a function of r only, V_θ is a function of θ only, and consequently equations (2.9a) and (2.9b) form a decoupled pair. Also, it is not difficult to show (Wilkins 1972) that if $E/\mu < 1$ the orbit is bound (does not reach $r = \infty$), while all orbits with $E/\mu > 1$ are unbound except for a "measure-zero" set of unstable orbits.

The single most important class of orbits are the circular orbits in the equatorial plane. For a circular orbit at some radius r, $dr/d\lambda$ must vanish both instantaneously and at all subsequent times (orbit at a perpetual turning point). Equation (2.9a) then gives the conditions

$$V_r(r) = 0\,, \qquad V_r{}'(r) = 0\,. \tag{2.11}$$

These equations can be solved simultaneously for E and L to give

$$E/\mu = \frac{r^{3/2} - 2Mr^{1/2} \pm aM^{1/2}}{r^{3/4}(r^{3/2} - 3Mr^{1/2} \pm 2aM^{1/2})^{1/2}}\,, \tag{2.12}$$

$$L/\mu = \frac{\pm M^{1/2}(r^2 \mp 2aM^{1/2}r^{1/2} + a^2)}{r^{3/4}(r^{3/2} - 3Mr^{1/2} \pm 2aM^{1/2})^{1/2}}\,. \tag{2.13}$$

In these and all subsequent formulae, the upper sign refers to direct orbits (i.e., corotating with $L > 0$), while the lower sign refers to retrograde orbits (counter-rotating with $L < 0$). For an extreme-rotating black hole, $a = M$, equations (2.12) and (2.13) simplify somewhat,

$$E/\mu = \frac{r \pm M^{1/2}r^{1/2} - M}{r^{3/4}(r^{1/2} \pm 2M^{1/2})^{1/2}}, \qquad \text{for } a = M; \tag{2.14}$$

$$L/\mu = \frac{\pm M(r^{3/2} \pm M^{1/2}r + Mr^{1/2} \mp M^{3/2})}{r^{3/4}(r^{1/2} \pm 2M^{1/2})^{1/2}}, \qquad \text{for } a = M. \tag{2.15}$$

The coordinate angular velocity of a circular orbit is

$$\Omega \equiv d\varphi/dt = \pm M^{1/2}/(r^{3/2} \pm aM^{1/2}). \tag{2.16}$$

Circular orbits do not exist for all values of r. The denominator of equations (2.12) and (2.13) is real only if

$$r^{3/2} - 3Mr^{1/2} \pm 2aM^{1/2} \geq 0. \tag{2.17}$$

The limiting case of equality gives an orbit with infinite energy per unit rest mass, i.e., a photon orbit. This photon orbit is the innermost boundary of the circular orbits for particles; it occurs at the root of (2.17),

$$r = r_{\rm ph} \equiv 2M\{1 + \cos[\tfrac{2}{3}\cos^{-1}(\mp a/M)]\}. \tag{2.18}$$

For $a = 0$, $r_{\rm ph} = 3M$, while for $a = M$, $r_{\rm ph} = M$ (direct) or $4M$ (retrograde).

For $r > r_{\rm ph}$ not all circular orbits are bound. An unbound circular orbit is one with $E/\mu > 1$. Given an infinitesimal outward perturbation, a particle in such an orbit will escape to infinity on an asymptotically hyperbolic trajectory. The unbound circular orbits are circular in geometry but hyperbolic in energetics, and they are all unstable. Bound circular orbits exist for $r > r_{\rm mb}$, where $r_{\rm mb}$ is the radius of the marginally bound ("parabolic") circular orbit with $E/\mu = 1$,

$$r_{\rm mb} = 2M \mp a + 2M^{1/2}(M \mp a)^{1/2}. \tag{2.19}$$

Note also that $r_{\rm mb}$ is the minimum perihelion of all parabolic ($E/\mu = 1$) orbits. In astrophysical problems, particle infall from infinity is very nearly parabolic, since the velocities of matter at infinity satisfy $v \ll c$. Any parabolic trajectory which penetrates to $r < r_{\rm mb}$ must plunge directly into the black hole. For $a = 0$, $r_{\rm mb} = 4M$; for $a = M$, $r_{\rm mb} = M$ (direct) or $5.83M$ (retrograde).

Even the bound circular orbits are not all stable. Stability requires that $V_r''(r) \leq 0$, which yields the three equivalent conditions

$$1 - (E/\mu)^2 \geq \tfrac{2}{3}(M/r),$$

$$r^2 - 6Mr \pm 8aM^{1/2}r^{1/2} - 3a^2 \geq 0,$$

or

$$r \geq r_{\rm ms}, \tag{2.20}$$

where $r_{\rm ms}$ is the radius of the marginally stable orbit,

$$\begin{aligned} r_{\rm ms} &= M\{3 + Z_2 \mp [(3 - Z_1)(3 + Z_1 + 2Z_2)]^{1/2}\}, \\ Z_1 &\equiv 1 + (1 - a^2/M^2)^{1/3}[(1 + a/M)^{1/3} + (1 - a/M)^{1/3}], \\ Z_2 &\equiv (3a^2/M^2 + Z_1{}^2)^{1/2}. \end{aligned} \tag{2.21}$$

For $a = 0$, $r_{\rm ms} = 6M$; for $a = M$, $r_{\rm ms} = M$ (direct) or $9M$ (retrograde). Figure 1

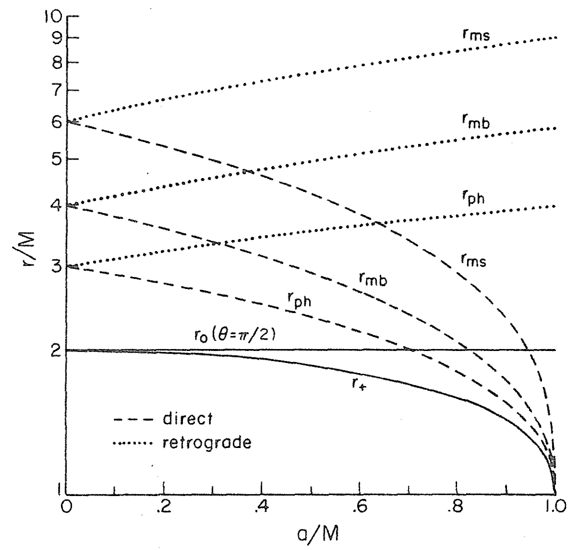

FIG. 1.—Radii of circular, equatorial orbits around a rotating black hole of mass M, as functions of the hole's specific angular momentum a. Dashed and dotted curves (for direct and retrograde orbits) plot the Boyer-Lindquist coordinate radius of the innermost stable (ms), innermost bound (mb), and photon (ph) orbits. Solid curves indicate the event horizon (r_+) and the equatorial boundary of the ergosphere (r_0).

shows the radii r_+, $r_0(\theta = \pi/2)$, r_{ph}, r_{mb}, and r_{ms} as functions of a for direct and retrograde orbits.

For $a = M, r_+ = r_{ph} = r_{mb} = r_{ms} = M$, and it appears that the photon, marginally bound, and marginally stable orbits are coincident with the horizon. Appearances are deceptive! The horizon is a null hypersurface, and no timelike curves can lie in it. The confusion is due to the subtle nature of the Boyer-Lindquist coordinates at $r = M$ for $a = M$. In fact the orbits at r_{ph}, r_{mb}, and r_{ms} are all outside the horizon and all distinct. Figure 2 illustrates the nature of the problem; it shows schematically the equatorial plane embedded in a Euclidean 3-space, for $a/M = 0.9, 0.99, 0.999$, and 1. In the limit $a \to M$ the orbits at r_{ph}, r_{mb}, and r_{ms} remain separated in proper radial distance, but the entire section of the manifold $r \leq r_{ms}$ becomes singularly projected into the Boyer-Lindquist coordinate location $r = M$. In the limit $a \to M$, the proper radial distance between r_{ms} and r_{mb} goes to infinity, as does that between r_{ms} and r_0. The proper distance between r_{mb} and r_{ph} remains finite and nonzero, as does that between r_{ph} and r_+. (The infinities are not physically important; an infalling particle follows a timelike curve, while the infinite distances are in a spacelike direction.)

For astrophysical applications with a very close to M (see Bardeen 1970*a*), one often needs to know explicitly the limiting behavior of r_+, r_{ph}, r_{mb}, and r_{ms}. Let $a = M(1 - \delta)$; then

$$r_+ \approx M[1 + (2\delta)^{1/2}], \qquad r_{ph} \approx M[1 + 2(\tfrac{2}{3}\delta)^{1/2}],$$
$$r_{mb} \approx M[1 + 2\delta^{1/2}], \qquad r_{ms} \approx M[1 + (4\delta)^{1/3}]. \tag{2.22}$$

Using these formulae, one finds that the proper radial distance between r_+ and r_{ph} becomes $\frac{1}{2}M \ln 3$, that between r_{ph} and r_{mb} becomes $M \ln [(1 + 2^{1/2})/3^{1/2}]$, and that between r_{mb} and r_{ms} becomes $M \ln [2^{7/6}(2^{1/2} - 1)\delta^{-1/6}]$ in the limit $\delta \to 0$.

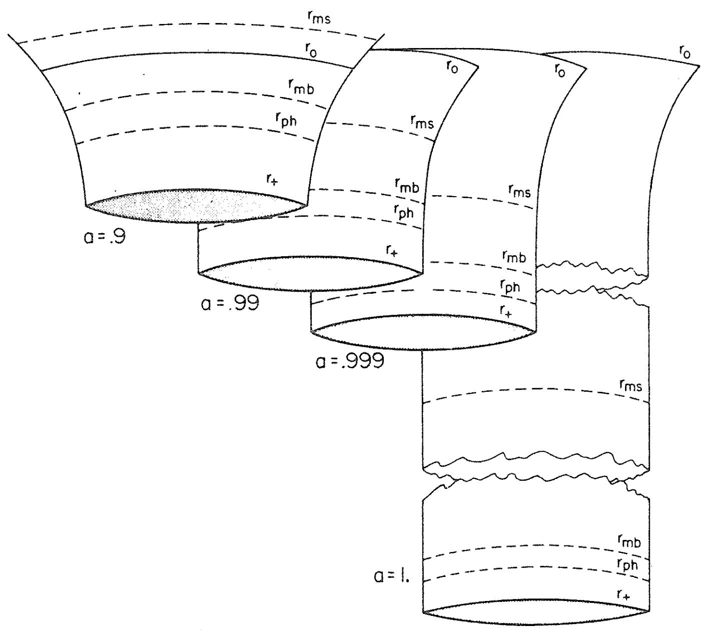

FIG. 2.—Embedding diagrams of the "plane" $\theta = \pi/2$, t = constant, for rotating black holes with near-maximum angular momentum. Here a denotes the hole's angular momentum in units of M. The Boyer-Lindquist radial coordinate r determines only the circumference of the "tube." When $a \to M$, the orbits at r_{ms}, r_{mb}, and r_{ph} all have the same circumference and coordinate radius, although—as the embedding diagram shows clearly—they are in fact distinct.

The orbits at $r = M$ are distinct energetically as well as geometrically. By taking appropriate limits of equations (2.12) and (2.13), one obtains

$$E/\mu \to 3^{-1/2}, \quad L/\mu \to 2M/3^{1/2} \quad \text{at } r = r_{\rm ms} \text{ as } a \to M,$$
$$E/\mu \to 1, \quad L/\mu \to 2M \quad \text{at } r = r_{\rm mb} \text{ as } a \to m,$$
$$E/\mu \to \infty, \quad L/\mu \to 2ME/\mu \quad \text{at } r = r_{\rm ph} \text{ as } a \to M. \tag{2.23}$$

A clearer picture of the relations among these various orbits, and among general orbits in the equatorial plane, will emerge in our consideration of locally nonrotating frames.

III. LOCALLY NONROTATING FRAMES

For any stationary, axisymmetric, asymptotically flat spacetime (for which the metric can always be written in the standard form of eq. [2.4]), it is useful to introduce a set of local observers who, in some sense, "rotate with the geometry" (Bardeen 1970*b*). Each observer carries an orthonormal tetrad of 4-vectors, his locally Minkowskian coordinate basis vectors. Rather than describe physical quantities (vectors, tensors, etc.) by their coordinate components at each point, one describes them by their projections on the orthonormal tetrad, i.e., their physically measured components in the local observer's frame. The desideratum governing the choice of observers is that physical processes described in their frames appear "simple." Physics is *not* simple in the Boyer-Lindquist coordinate frames because (i) the dragging of inertial frames becomes so severe that the t coordinate basis vector $(\partial/\partial t)$ goes spacelike at the static limit r_0, and (ii) the metric is nondiagonal, so raising and lowering tensor indices typically introduces algebraic complexity.

For metrics in the standard form (2.4), there is a uniquely sensible choice of observers and tetrads: the locally nonrotating frames (LNRF) for which the observers' world lines are r = constant, θ = constant, $\varphi = \omega t$ + constant. Here $\omega = -g_{\varphi t}/g_{\varphi\varphi}$ is the function appearing in equation (2.5). The orthonormal tetrad carried by such an observer (the set of LNRF basis vectors) at the point t, r, θ, φ is given by

$$
\begin{aligned}
e_{(t)} &= e^{-\nu}\left(\frac{\partial}{\partial t} + \omega\frac{\partial}{\partial\varphi}\right) = \left(\frac{A}{\Sigma\Delta}\right)^{1/2}\frac{\partial}{\partial t} + \frac{2Mar}{(A\Sigma\Delta)^{1/2}}\frac{\partial}{\partial\varphi}\,,\\
e_{(r)} &= e^{-\mu_1}\frac{\partial}{\partial r} = \left(\frac{\Delta}{\Sigma}\right)^{1/2}\frac{\partial}{\partial r}\,,\\
e_{(\theta)} &= e^{-\mu_2}\frac{\partial}{\partial\theta} = \frac{1}{\Sigma^{1/2}}\frac{\partial}{\partial\theta}\,,\\
e_{(\varphi)} &= e^{-\psi}\frac{\partial}{\partial\varphi} = \left(\frac{\Sigma}{A}\right)^{1/2}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\,.
\end{aligned}
\tag{3.1}
$$

Here the first expression for each basic vector is valid for any spacetime with the standard metric (2.4); the second expression specializes to the Kerr metric. The corresponding basis of one-forms (or covariant basis vectors) is

$$
\begin{aligned}
e^{(t)} &= e^{\nu}\boldsymbol{d}t = (\Sigma\Delta/A)^{1/2}\boldsymbol{d}t\,,\\
e^{(r)} &= e^{\mu_1}\boldsymbol{d}r = (\Sigma/\Delta)^{1/2}\boldsymbol{d}r\,,\\
e^{(\theta)} &= e^{\mu_2}\boldsymbol{d}\theta = \Sigma^{1/2}\boldsymbol{d}\theta\,,\\
e^{(\varphi)} &= -\omega e^{\psi}\boldsymbol{d}t + e^{\psi}\boldsymbol{d}\varphi = -\frac{2Mar\sin\theta}{(\Sigma A)^{1/2}}\boldsymbol{d}t + \left(\frac{A}{\Sigma}\right)^{1/2}\sin\theta\boldsymbol{d}\varphi\,.
\end{aligned}
\tag{3.2}
$$

From equations (3.1) and (3.2) one reads off directly the Boyer-Lindquist components $e^{\mu}{}_{(i)}$ and $e_{\mu}{}^{(i)}$ of the LNRF basis vectors, since

$$
e_{(i)} = e^{\mu}{}_{(i)}\frac{\partial}{\partial x^{\mu}} \qquad \text{and} \qquad e^{(i)} = e_{\mu}{}^{(i)}\boldsymbol{d}x^{\mu}\,.
\tag{3.3}
$$

As matrices $\|e^{\mu}{}_{(i)}\|$ and $\|e_{\mu}{}^{(i)}\|$, these components transform one back and forth between the LNRF frame and the Boyer-Lindquist coordinate frame. For example, the standard transformation law for components of a tensor reads

$$
J_{(a)(b)} = e^{\mu}{}_{(a)}e^{\nu}{}_{(b)}J_{\mu\nu}\,, \qquad J_{\mu\nu} = e_{\mu}{}^{(a)}e_{\nu}{}^{(b)}J_{(a)(b)}\,.
\tag{3.4}
$$

The rotation one-forms, which allow one to read off the connection coefficients $\Gamma_{(a)(b)(i)}$ by $\boldsymbol{\omega}_{(a)(b)} = \Gamma_{(a)(b)(i)}e^{(i)}$, are given by

$$
\begin{aligned}
\boldsymbol{\omega}_{(t)(r)} &= -\nu_{,r}\exp(-\mu_1)e^{(t)} - \tfrac{1}{2}\omega_{,r}\exp(\psi-\nu-\mu_1)e^{(\varphi)}\,,\\
\boldsymbol{\omega}_{(t)(\theta)} &= -\nu_{,\theta}\exp(-\mu_2)e^{(t)} - \tfrac{1}{2}\omega_{,\theta}\exp(\psi-\nu-\mu_2)e^{(\varphi)}\,,\\
\boldsymbol{\omega}_{(t)(\varphi)} &= -\tfrac{1}{2}\omega_{,r}\exp(\psi-\nu-\mu_1)e^{(r)} - \tfrac{1}{2}\omega_{,\theta}\exp(\psi-\nu-\mu_2)e^{(\theta)}\,,\\
\boldsymbol{\omega}_{(r)(\theta)} &= \mu_{1,\theta}\exp(-\mu_2)e^{(r)} - \mu_{2,r}\exp(-\mu_1)e^{(\theta)}\,,\\
\boldsymbol{\omega}_{(r)(\varphi)} &= -\psi_{,r}\exp(-\mu_1)e^{(\varphi)} + \tfrac{1}{2}\omega_{,r}\exp(\psi-\nu-\mu_1)e^{(t)}\,,\\
\boldsymbol{\omega}_{(\theta)(\varphi)} &= -\psi_{,\theta}\exp(-\mu_2)e^{(\varphi)} + \tfrac{1}{2}\omega_{,\theta}\exp(\psi-\nu-\mu_2)e^{(t)}\,.
\end{aligned}
\tag{3.5}
$$

Here a comma denotes partial differentiation. (Note that $\boldsymbol{\omega}_{(a)(b)} = -\boldsymbol{\omega}_{(b)(a)}$.)

One indication of the simplicity of the LNRF is the simplicity of the Kerr geometry's Riemann tensor when expressed in LNRF components. Define the four quantities

$$Q_1 \equiv Mr(r^2 - 3a^2\cos^2\theta)/\Sigma^3\,, \qquad Q_2 \equiv Ma\cos\theta(3r^2 - a^2\cos^2\theta)/\Sigma^3\,,$$
$$S \equiv 3a\sin\theta\Delta^{1/2}(r^2 + a^2)/A\,, \qquad z = \Delta a^2\sin^2\theta/(r^2 + a^2)^2\,. \tag{3.6}$$

(The quantities Δ, Σ, A are defined by eq. [2.3].) Then one obtains

$$\begin{aligned}
R_{(t)(\varphi)(t)(\varphi)} &= -R_{(r)(\theta)(r)(\theta)} = Q_1\,,\\
R_{(t)(\varphi)(r)(\theta)} &= -Q_2\,,\\
R_{(t)(r)(t)(r)} &= -R_{(\varphi)(\theta)(\varphi)(\theta)} = -Q_1\frac{2+z}{1-z}\,,\\
R_{(t)(r)(t)(\theta)} &= R_{(\varphi)(r)(\varphi)(\theta)} = SQ_2\,,\\
R_{(t)(r)(\varphi)(r)} &= -R_{(t)(\theta)(\varphi)(\theta)} = SQ_1\,,\\
R_{(t)(r)(\varphi)(\theta)} &= -Q_2\frac{2+z}{1-z}\,,\\
R_{(t)(\theta)(t)(\theta)} &= -R_{(\varphi)(r)(\varphi)(r)} = Q_1\frac{1+2z}{1-z}\,,\\
R_{(t)(\theta)(\varphi)(r)} &= -Q_2\frac{1+2z}{1-z}\,.
\end{aligned} \tag{3.7}$$

The other nonzero components follow directly from the symmetries of the Riemann tensor. Notice that Q_2 vanishes in the equatorial plane; also, that the dependence on z is always quite weak since

$$0 \le z \le 0.043$$

for all r, θ, a of interest ($r_+ \le r < \infty, 0 \le \theta \le \pi, 0 \le a \le M$).

At any instant in time, the local frame of *any* physical observer differs from the LNRF at the observer's location by a Lorentz transformation. One need only know the velocity of an observer relative to the LNRF, and the transformation formulae of special relativity, to obtain the Riemann tensor (or, similarly, any other physical quantity) in an arbitrary frame.

To use the LNRF in the analysis of processes in Kerr orbits, we must investigate the nature of the Kerr orbits as seen from the LNRF, i.e., their distribution in velocity space. In general, the 4-velocity $\boldsymbol{u}$ has the LNRF components

$$u^{(a)} = u^\mu e_\mu{}^{(a)}\,, \tag{3.8}$$

where the u^μ come from equation (2.9), and the $e_\mu{}^{(a)}$ from equation (3.2). The 3-velocity relative to the LNRF has components

$$\mathscr{V}^{(j)} = \frac{u^\mu e_\mu{}^{(j)}}{u^\nu e_\nu{}^{(t)}}\,, \qquad j = r, \theta, \varphi\,. \tag{3.9}$$

In particular, note that

$$\mathscr{V}^{(\varphi)} = e^{\psi - \nu}(\Omega - \omega)\,, \tag{3.10}$$

where $\Omega = u^\varphi/u^t$ as before. In the special case of circular, equatorial orbits, $\mathscr{V}^{(\varphi)}$ is the only nonvanishing velocity component, and is given by

$$\mathscr{V}^{(\varphi)} = \frac{\pm M^{1/2}(r^2 \mp 2aM^{1/2}r^{1/2} + a^2)}{\Delta^{1/2}(r^{3/2} \pm aM^{1/2})}\,. \tag{3.11a}$$

In the case $a = M$, equation (3.11a) further reduces to

$$\mathscr{V}^{(\varphi)} = \frac{\pm M^{1/2}(r^{3/2} \pm M^{1/2}r + Mr^{1/2} \mp M^{3/2})}{(r^{1/2} \pm M^{1/2})(r^{3/2} \pm M^{3/2})} . \tag{3.11b}$$

Corresponding to $\mathscr{V}^{(\varphi)}$, the quantity $\gamma \equiv (1 - [\mathscr{V}^{(\varphi)}]^2)^{-1/2}$ is given by

$$\gamma = \frac{\Delta^{1/2}(r^{3/2} \pm aM^{1/2})}{r^{1/4}(r^{3/2} - 3Mr^{1/2} \pm 2aM^{1/2})^{1/2}(r^3 + a^2r + 2Ma^2)^{1/2}} ; \tag{3.12a}$$

or for $a = M$,

$$\gamma = \frac{(r^{3/2} \pm M^{3/2})(r^{1/2} \pm M^{1/2})}{r^{1/4}(r^{1/2} \pm 2M^{1/2})^{1/2}(r^3 + M^2r + 2M^3)^{1/2}} . \tag{3.12b}$$

For all a, $\mathscr{V}^{(\varphi)}$ increases (but not monotonically!) from zero at $r = \infty$ to 1 (the speed of light) at the circular photon orbit $r = r_{ph}$. Another interesting point is that $\mathscr{V}^{(\varphi)}(r = r_{ms})$, the velocity of the most tightly bound circular orbit, goes to $\frac{1}{2}$ (not 1!) in the limit $a \to M$. The point once again is that for $a = M$, the marginally stable orbit and the photon orbit are distinct. The marginally bound orbit, also distinct, has $\mathscr{V}^{(\varphi)}(r = r_{mb}) \to 2^{-1/2}$ for $a \to M$. In fact, *all* stable, bound orbits around a rotating black hole—except "plunge" orbits irrevocably approaching the horizon—have $|\mathscr{V}|$ substantially bounded away from 1. Consequently, a Lorentz transformation from an LNRF to a stable, bound orbital frame never brings in factors greater than order unity.

We now consider noncircular orbits in the equatorial plane ($Q = 0$; E, L arbitrary). For each possible orbit, and at every radius r, we ask an LNRF observer to measure the velocity of the orbit at the instant that it passes him. The velocity is represented by a point in the ($\mathscr{V}^{(r)}$, $\mathscr{V}^{(\varphi)}$)-plane, somewhere inside the speed-of-light circle $[\mathscr{V}^{(r)}]^2 + [\mathscr{V}^{(\varphi)}]^2 = 1$. Thus, certain regions of the two-dimensional velocity space at radius r correspond to bound, stable orbits; other regions to hyperbolic orbits which escape to infinity; other regions to "plunge" orbits which go down the hole. Figure 3 shows a typical sequence of velocity-space diagrams corresponding to $a = 0.95M$ ($a = M$ would be similar, but would collapse several different interesting radii tc $r = M$). The following types of orbits are delineated in figure 3: bound stable orbits which exist for $r > r_{mb} \approx 1.94M$ (direct) or 8.86 (retrograde), denoted (B); plunge orbits originating at infinity, i.e., with $E/\mu \geq 1$, denoted (P); escape orbits which are the time reverse of (P) orbits, denoted (E) [since nothing can come out of the hole, some physical process near the hole is necessary to inject a particle into an (E) trajectory]; hyperbolic orbits which originate at infinity, and are scattered back to infinity by the hole (H); captured plunge orbits, i.e., plunge orbits with $E/\mu < 1$, denoted (C). Points on the border between regions (H) and (P) of velocity space correspond to unstable orbits, and the intersection of such a border with the line $\mathscr{V}^{(r)} = 0$ marks an unstable, unbound circular orbit.

Figure 3 also indicates the region of "negative energy states" first exploited by Penrose (1969). In the LNRF, a particle's 4-momentum has the flat-space form

$$\boldsymbol{p} = \mu(\gamma, \gamma\mathscr{V}) , \qquad \gamma = (1 - \mathscr{V}\cdot\mathscr{V})^{-1/2} \tag{3.13}$$

and its conserved total energy (dot product of 4-momentum with the time-coordinate Killing vector) is

$$\begin{aligned} E &= -\boldsymbol{p}\cdot(\partial/\partial t) = -p_t = -p^{(a)}e_{t(a)} \\ &= \mu\gamma(e^{\nu} + \omega e^{\psi}\mathscr{V}^{(\varphi)}) , \end{aligned} \tag{3.14}$$

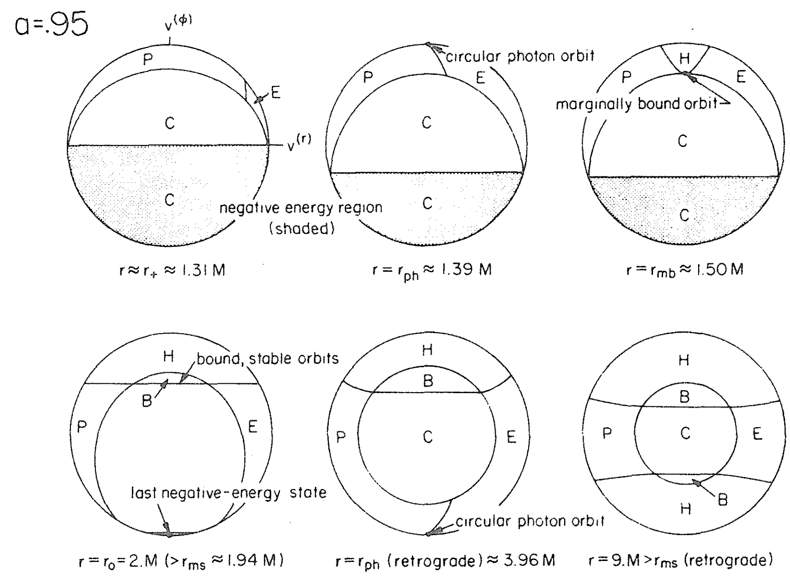

FIG. 3.—Distribution in velocity space of equatorial orbits passing through various radii r, around a rotating black hole with $a = 0.95M$. Each circle is the "space" of equatorial, ordinary velocities [d(proper distance)/d(proper time)] as measured in the proper reference frame of a locally nonrotating observer. The velocity circles are labeled by the radius r of the observer. The center of each circle is zero velocity; the edge is the speed of light; the $\mathcal{V}^{(r)}$ direction corresponds to radial velocities, the $\mathcal{V}^{(\varphi)}$ direction to tangential velocities. A particle which passes the observer with its velocity in an E-region will escape to infinity. Similarly, P denotes plunge trajectories from infinity into the hole; C denotes "captured" plunges which could not have come from infinity; H denotes "hyperbolic" orbits from infinity and to infinity; B denotes bound, stable orbits which neither plunge nor escape. The shaded regions are the "negative energy" orbits (see text for details). Diagrams for other values of a (the hole's specific angular momentum) are qualitatively similar.

so that in the equatorial plane,

$$E = \mu\gamma A^{-1/2}(r\Delta^{1/2} + 2Ma\mathcal{V}^{(\varphi)}) . \tag{3.15}$$

E is negative for

$$\mathcal{V}^{(\varphi)} < -\frac{r(r^2 - 2Mr + a^2)^{1/2}}{2Ma} , \tag{3.16}$$

that is, below a horizontal line in the velocity plane. Outside the ergosphere this line fails to intersect the velocity-space circle, and there are no negative energy states. At the event horizon the line is $\mathcal{V}^{(\varphi)} = 0$. Negative energy trajectories are always captured plunges (C).

In the Penrose energy-extraction process, a body breaks up into two or more fragments; if any fragments are injected into negative energy orbits, the sum of the total energy of the remaining fragments is greater than the total energy of the original body, since E is an additive conserved quantity. The extra energy comes from the rotational energy of the black hole (see Christodoulou 1970). Wheeler (1970) and others (see, e.g., Mashoon 1972) have speculated on the possibility that some natural astrophysical process, for example the breakup of a star by the tidal gravitational forces of the black hole, could result in the extraction of energy from the hole via the Penrose process. In the LNRF picture (fig. 3), the negative energy states and the (B) orbits are always separated by a substantial velocity, even for $a \approx M$ and $r \approx M$. Thus, if a star is taken initially on a bound, stable orbit in the equatorial plane, there can be no energy extraction from its breakup unless hydrodynamical boosts of $\sim\frac{1}{2}$ the speed of light occur. Similar results hold if the initial orbit is taken to be a plunge orbit of any reasonable sort, i.e., one no more bound than the *most* bound (B) orbit.

Appendix A proves the general theorem which the LNRF picture makes plausible: If two trajectories differ in energy per unit rest mass by an amount of order unity, then their locally measured relative velocities differ by a substantial fraction of the speed of light. This result holds everywhere outside the event horizon (and even inside it, for that matter). The most bound plunge orbit that is astrophysically plausible has $E/\mu = 3^{-1/2}$ (minimum energy of a plunge orbit which results from the decay of a bound, stable orbit around any rotating black hole). Such an orbit is bounded away from the negative energy states by $|\mathscr{V}| \geq 0.5c$. Thus, energy extraction cannot be achieved unless hydrodynamical forces or superstrong radiation reactions can accelerate fragments to more than this speed during the infall. On dimensional grounds, such boosts seem to be excluded: Suppose a self-gravitating object of mass m and radius r falls into a black hole of mass M. The criterion for Roche breakup at radius R is dimensionally

$$M/R^3 \sim m/r^3 \,. \tag{3.17}$$

After breakup, the object experiences tidal accelerations of magnitude $\sim r(M/R^3) \sim r(m/r^3)$ for a period of time $\sim(R^3/M)^{1/2} \sim (r^3/m)^{1/2}$, so the characteristic velocity of breakup is dimensionally

$$\mathscr{V} \sim (m/r)^{1/2} \,, \tag{3.18}$$

which is $\ll 1$ for any infalling object except highly bound neutron stars. Since equation (3.17) can be rewritten as

$$R/M = (r/m)(m/M)^{2/3} < 1 \qquad \text{if } m \ll M \text{ and } r \lesssim 10m \,, \tag{3.19}$$

for a neutron star falling into a substantially more massive black hole, the Roche limit is inside the event horizon. There will be no observable breakup at all.

As for the superstrong radiation reactions, we can only note that all calculations to date (e.g., Davis *et al.* 1971, 1972) show that energies radiated from plunge trajectories are typically

$$E_{\text{rad}} \sim m(m/M) \ll m \,, \tag{3.20}$$

so that reaction boosts are of the order of

$$\mathscr{V} \sim (m/M)^{1/2} \ll 1 \,. \tag{3.21}$$

In the next section we consider the scalar wave equation in the Kerr background and find no evidence of any breakdown in the estimate (3.20) for astrophysically plausible processes.

IV. THE SCALAR WAVE EQUATION AND SCALAR SYNCHROTRON RADIATION

The equation governing a scalar test field Φ in the Kerr background is

$$\Box\Phi = (-g)^{-1/2}[(-g)^{1/2}g^{\mu\nu}\Phi_{,\mu}]_{,\nu} = 4\pi T \,, \tag{4.1}$$

where T is the density of scalar charge per proper volume as measured in the rest frame of the charge and $g = \det(g_{\mu\nu})$. A comma denotes partial (not covariant) differentiation. In Boyer-Lindquist coordinates $(-g)^{1/2} = \Sigma \sin\theta$, and the metric $g^{\mu\nu}$ is given by equation (2.2); equation (4.1) becomes

$$\left\{\frac{\partial}{\partial r}\Delta\frac{\partial}{\partial r} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \left(\frac{1}{\sin^2\theta} - \frac{a^2}{\Delta}\right)\frac{\partial^2}{\partial\varphi^2} - \frac{4Mar}{\Delta}\frac{\partial^2}{\partial\varphi\partial t} - \left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2}{\partial t^2}\right\}\Phi = 4\pi\Sigma T\,. \quad (4.2)$$

Carter (1968*b*) first demonstrated the separability of equation (4.1), and the explicit separation of equation (4.2) has been given by Brill *et al.* (1972). The solutions have the form

$$\Phi = \sum_{lm}\int d\omega[R_{lm\omega}(r)S^m{}_l(-a^2\omega^2, \cos\theta)e^{im\varphi}e^{-i\omega t}]\,. \quad (4.3)$$

Here $S^m{}_l(-a^2\omega^2, \cos\theta)$ is the standard oblate spheroidal harmonic satisfying

$$\left(\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} + \lambda_{ml} + a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta}\right)S^m{}_l = 0\,, \quad (4.4)$$

where λ_{ml} is the eigenvalue of $S^m{}_l$. We write $S^m{}_l(\theta)$ for $S^m{}_l(-a^2\omega^2, \cos\theta)$ and take the normalization

$$\int_{-1}^{+1} d(\cos\theta)\int_0^{2\pi} d\varphi|S^m{}_l(\theta)e^{im\varphi}|^2 = 1\,. \quad (4.5)$$

Substituting equations (4.3)–(4.5) into (4.2), one finds that the radial function $R_{lm\omega}$ satisfies

$$\left[\frac{d}{dr}\Delta\frac{d}{dr} + \frac{a^2m^2 - 4Marm\omega + (r^2+a^2)^2\omega^2}{\Delta} - \lambda_{ml} - a^2\omega^2\right]R_{lm\omega}(r) = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_{-1}^{1} d(\cos\theta)\int_0^{2\pi} d\varphi[e^{i\omega t}e^{-im\varphi}S^m{}_l(\theta)(4\pi\Sigma T)]\,. \quad (4.6)$$

Although T is a scalar charge density, not a tensor gravitational source, one often seeks insight into gravitational-wave processes by taking T to be the trace of the stress-energy tensor; i.e., one sets the fictitious scalar charge of a point particle equal in magnitude to its mass μ. If the particle follows a world line $z^\mu(\tau)$, one has

$$T = \frac{\mu}{u^t}(-g)^{-1/2}\delta^3[x^i - z^i(\tau)] \qquad \text{for } i = r, \theta, \varphi\,, \quad (4.7)$$

where $u^t = dt/d\tau$. For a particle in an equatorial, circular orbit of radius r_p, with angular velocity $d\varphi/dt = \Omega$, this becomes

$$4\pi\Sigma T = \sum_{lm}(4\pi\mu/u^t)\delta(r - r_p)S^m{}_l(\theta)S^m{}_l(0)e^{-im\Omega t}e^{im\varphi}\,. \quad (4.8)$$

Thus, the Fourier-transformed source (right-hand side of eq. [4.6]) has nonvanishing ω-components only for $\omega = m\Omega$, $m = 0, \pm 1, \pm 2, \ldots$. Further, if by convention we take the real part of Φ to be the physical field, then we can restrict attention to $\omega \geq 0$ without loss of generality, so that only positive m's contribute if $\Omega > 0$, and negative if $\Omega < 0$. With this convention, the sum in equation (4.8) ranges from $m = 0$ to $m = \text{sgn}(\Omega)\infty$, and a factor 2 must be inserted on the right-hand side of (4.8) for $m \neq 0$.

Equation (4.6) can be simplified to an effective-potential equation by the introduction of a new coordinate r^* such that

$$dr^*/dr = r^2/\Delta\,. \tag{4.9}$$

Explicitly,

$$r^* = r + M \ln \Delta + \frac{(2M^2 - a^2)}{2(M^2 - a^2)^{1/2}} \ln \left(\frac{r - r_+}{r - r_-}\right); \tag{4.10a}$$

or for $a = M$,

$$r^* = r + 2M \ln (r - M) - M^2/(r - M)\,. \tag{4.10b}$$

[Recall that $r_\pm = M \pm (M^2 - a^2)^{1/2}$.] If we put

$$\psi = rR_{lm\omega}\,, \tag{4.11a}$$

then equations (4.6) and (4.8) become

$$\frac{d^2\psi}{dr^{*2}} + W(r)\psi = \frac{8\pi\mu}{r_p u^t} S^m{}_l(0)\delta(r^* - r_p{}^*)\,, \tag{4.11b}$$

where $S^m{}_l(0) = S^m{}_l(-a^2m^2\Omega^2, 0)$ and $W(r)$ is the effective potential

$$W(r) = m^2\left[\frac{(r^2 + a^2)\Omega - a}{r^2}\right]^2 - \frac{\Delta}{r^4}\left[\lambda_{ml} - 2a\Omega m^2 + a^2\Omega^2 m^2 + \frac{2(Mr - a^2)}{r^2}\right]. \tag{4.12}$$

Our boundary conditions for equation (4.11) agree with those of Misner (1972*b*), and we will not discuss them here, except for a brief summary in Appendix B. Misner and others use a slightly different r^* coordinate, $r^*{}_n$ defined by

$$dr^*{}_n/dr = (r^2 + a^2)/\Delta$$

instead of equation (4.9). This $r^*{}_n$ has the conceptual advantage that $t \pm r^*{}_n$ are null coordinates, but the practical disadvantage that it makes equation (4.12) and subsequent equations somewhat more complicated.

Locally nonrotating frames give insight into the physical content of the separated wave equation (4.11). We eliminate Ω in favor of $\mathscr{V}$, the LNRF linear velocity of the orbiting particle as measured in a LNRF (in previous sections $\mathscr{V}$ was denoted $\mathscr{V}^{(\varphi)}$). It is useful to define a function $\mathscr{V}(r)$, the linear velocity of the frame rigidly rotating with angular velocity Ω,

$$\mathscr{V}(r) = \frac{1}{r\Delta^{1/2}}\left[(r^3 + a^2r + 2Ma^2)\Omega - 2Ma\right]. \tag{4.13}$$

Thus, the particle's velocity is $\mathscr{V} = \mathscr{V}(r_p)$. Then equation (4.12) takes the simple form

$$W(r) = \frac{-\Delta}{r^4}\left[\lambda_{ml} - m^2\left(1 - \frac{1 - \mathscr{V}(r)^2}{1 + a^2/r^2 + 2Ma^2/r^3}\right) + \frac{2(Mr - a^2)}{r^2}\right]. \tag{4.14}$$

It is shown in Appendix C that $\lambda_{ml} \geq m^2$ for all physical cases. Since $\mathscr{V}(r_p) < 1$ and $Mr > a^2$ outside the horizon, $W(r_p) = O(m^2) < 0$. Thus, in the WKB limit of large barrier (large m), the field dies out exponentially as one moves radially away from the particle. Since $\mathscr{V}(r) \approx r\Omega \to \infty$ for large r, $W(r)$ becomes positive at some point $r_1 > r_p$, and traveling waves propagate from there to infinity. Similarly, $W(r)$ becomes positive at some point r_2, $r_+ < r_2 < r_p$, so traveling waves exist for $r < r_2$.

We are now in a position to discuss the interesting question of "beamed radiation" which was first raised by Misner (1972*a*). Two prerequisites for beamed radiation (i.e., radiation emitted into a solid angle much smaller than 4π) are (i) that the source itself contain high multipoles ($l, m \gg 1$) and (ii) that the field coupled to these multipoles radiate to infinity in a relatively unimpeded manner. For a point source, condition (i) is satisfied, so (ii) becomes the essential condition to check. The WKB barrier factor which separates the source from its wave zone is

$$\exp\left[-\int_{r_2}^{r_1} [-W(r)]^{1/2} dr^*\right] \equiv \exp(-B). \tag{4.15}$$

The question is: with $l \gg 1$, can B be made small? We will see below that the *most* favorable case (the case of smallest B) is $m = l$. Appendix C derives the result

$$l^2 \leq \lambda_{ll} \leq l(l+1); \tag{4.16}$$

so the effective potential (4.14) for $m = l$ is, with fractional errors of $O(1/l)$,

$$W(r) \simeq -\frac{l^2}{r^4}\Delta\frac{1-\mathscr{V}(r)^2}{1+a^2/r^2+2Ma^2/r^3} \qquad \text{for } l = m \gg 1. \tag{4.17}$$

The corresponding barrier-penetration factor for $m = l$ is

$$B \simeq l\int_{r_2}^{r_1} \frac{(1-\mathscr{V}(r)^2)^{1/2}}{(1+a^2/r^2+2Ma^2/r^3)^{1/2}(r^2-2Mr+a^2)^{1/2}}\,dr. \tag{4.18}$$

This barrier factor can be cast in a simple form by noticing that in the Kerr geometry, the proper circumferential radius R_c and proper radial distance R_p are given by

$$R_c = e^{\psi} = (r^2+a^2+2Ma^2/r)^{1/2}, \qquad dR_p/dr = (r^2/\Delta)^{1/2}. \tag{4.19}$$

So

$$B \simeq l\int_{r=r_2}^{r=r_1} R_c^{-1}(1-\mathscr{V}(r)^2)^{1/2} dR_p. \tag{4.20}$$

Large values of l will contribute to the radiation field only if the integral in (4.20) is $\ll 1$. This requires two conditions: First,

$$1-\mathscr{V}(r_p)^2 \ll 1; \tag{4.21}$$

i.e., the particle orbit must be highly relativistic *as seen in the LNRF*. Second, $|\mathscr{V}(r)|$ must increase monotonically as r increases from r_p to r_1. [If it decreases initially, then B cannot be made arbitrarily small even as $\mathscr{V}(r_p) \to 1$.] The fact that in the Kerr geometry, by contrast to flat space, the function $|\mathscr{V}(r)|$ can decrease with increasing r is closely related to the existence of circular photon orbits. At radius r the direction cosine relative to the φ-direction in the LNRF for a photon trajectory with energy E and axial angular momentum L is

$$\frac{p^{(\varphi)}}{p^{(t)}} = \frac{1}{(E/L-\omega)e^{\psi-\nu}} = \frac{1}{\mathscr{V}(r)}. \tag{4.22}$$

The quantity $\mathscr{V}(r)$ here is identical with that of equation (4.13) if $\Omega \doteq E/L$. Since $|\mathscr{V}(r)|$ increases outward at $r = r_1$, with $|\mathscr{V}(r_1)| = 1$, the photon trajectory that is tangential there is at an inner turning point. Conversely, since $|\mathscr{V}(r)|$ decreases outward at $r = r_2$, the tangential photon trajectory is at an outer turning point. The photon orbit is circular at $r = r_1 = r_2$, if $|\mathscr{V}(r)|$ is independent of r to first order near $r = r_1$. Thus if r_p is inside the circular photon orbit, high multipoles will not radiate to infinity even for $|\mathscr{V}| \to 1$. (In physical terms this is because, inside r_{ph}, the radiation is beamed "down the hole.")

There are no nonplunge geodesic orbits inside r_{ph} in any case; but our results are equally valid for accelerated circular trajectories inside r_{ph}, and for radiation emitted at pericenter by noncircular orbits and by accelerated trajectories in general. We can prove that no bound orbit satisfies $|\mathscr{V}| \to 1$ outside of r_{ph} as follows:

$$1 \geq E/\mu = -e^{t}{}_{(a)}u^{(a)} = (1 - \mathscr{V}^2)^{-1/2}(e^{\nu} + \omega e^{\psi}\mathscr{V}^{(\phi)})\,. \tag{4.23}$$

Since the last term in parentheses has no root outside r_{ph}, $\mathscr{V}$ is bounded away from 1. Our conclusion is that high multipole radiation is suppressed exponentially with increasing l for all astrophysically relevant equatorial orbits. There is no reason to believe that nonequatorial or noncircular orbits would be any more favorable than our arbitrarily accelerated circular trajectories.

Of course, there can be some *finite* beaming in the radiation by multipoles below the exponential cutoff. The characteristic l of the cutoff is that l for which $B \approx 1$. The most interesting cases are $r_p = r_{\text{ms}}$ and $r_p = r_{\text{mb}}$ in the limit $a \to M$, $l = m \gg 1$. In these cases equation (4.20) can be evaluated in terms of elementary functions with the results

$$B \approx 0.120l\,, \qquad r \to r_{\text{ms}}\,, \tag{4.24a}$$

or $l_{\text{cutoff}} \simeq 8$; and

$$B \approx 0.078l\,, \qquad r \to r_{\text{mb}}\,, \tag{4.24b}$$

or $l_{\text{cutoff}} \simeq 12$. In other cases, equation (4.20) (or [4.14] if $m < l$) can be integrated numerically. Figure 4 shows representative results with $a = M$ for various ratios m/l, for various geodesic circular orbits and circular accelerated trajectories chosen to be tangent to marginally bound ("parabolic") orbits at pericenter. One sees that $l = m$ is the case most favorable to propagation, and that the analytic results (4.24) correspond to the most favorable orbits. We have obtained similar results for various values of a, $0 \leq a \leq M$; the case $a = M$ is the most favorable to high multipoles.

Momentarily setting aside the question of astrophysical plausibility, it is interesting to see just how $l_{\text{cutoff}} \to \infty$ as $\mathscr{V} \to 1$. Choose the origin for proper radial distance to be $R_p = 0$ at $r = r_p$, and expand $\mathscr{V}(r)$ in a Taylor series

$$\mathscr{V}(r) = \mathscr{V}\left[1 + \alpha\frac{R_p}{R_c} + \tfrac{1}{2}\beta\left(\frac{R_p}{R_c}\right)^2 + O\left(\frac{R_p}{R_c}\right)^3\right]. \tag{4.25}$$

Thus

$$[1 - \mathscr{V}(r)^2]^{1/2} \approx (1 - \mathscr{V}^2)^{1/2}\left(1 - 2\alpha\gamma^2\frac{R_p}{R_c} - \beta\gamma^2\frac{R_p{}^2}{R_c{}^2}\right)^{1/2} \tag{4.26}$$

when $\gamma = (1 - \mathscr{V}^2)^{-1/2} \gg 1$. The first-order term in equation (4.26) is sufficient to represent $(1 - \mathscr{V}^2)^{1/2}$ accurately over the whole range of integration of equation (4.20) if $\alpha\gamma \gg 1$. For most accelerated (nongeodesic) trajectories α is nonzero in the limit $\gamma \to \infty$ and one obtains

$$B \simeq \frac{l}{2\alpha\gamma^3}\int_0^1 (1 - x)^{1/2}dx = \frac{l}{3\alpha\gamma^3}\,, \tag{4.27}$$

or $l_{\text{cutoff}} \simeq 3\alpha\gamma^3$. For geodesic orbits with $\gamma \gg 1$ (orbits just outside the circular photon orbit), $\alpha\gamma \ll 1$ and the second-order term is large relative to the first-order term over almost all of the range of integration. Therefore, in the latter case,

$$B \simeq \frac{l}{\beta^{1/2}\gamma^2}\int_0^1 (1 - x^2)^{1/2}dx = \frac{\pi}{4}\frac{l}{\beta^{1/2}\gamma^2} \tag{4.28}$$

and $l_{\text{cutoff}} \simeq 4\pi^{-1}\beta^{1/2}\gamma^2$. In other words, there is a qualitative difference between

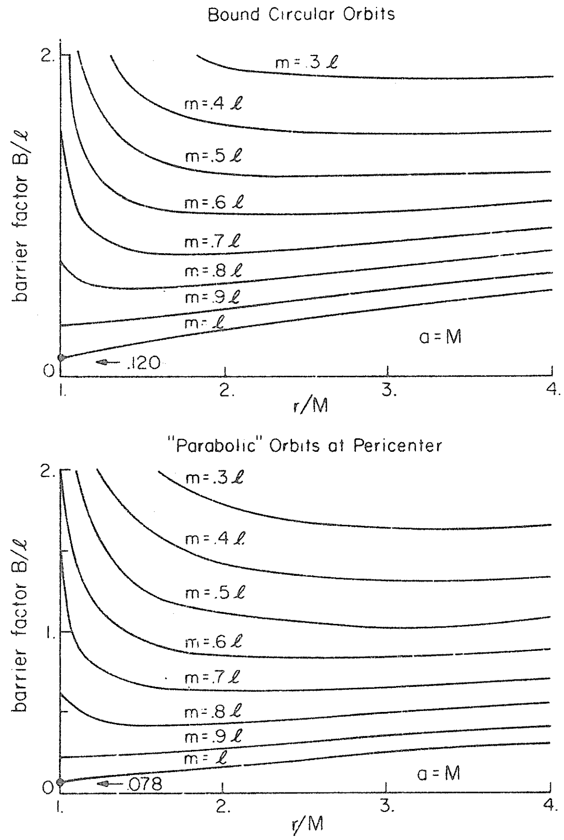

FIG. 4.—The "barrier factor" B for (scalar) synchrotron-type radiation from stable orbits around an extreme-rotating black hole. In the WKB approximation (valid for high multipoles, $l \gg 1$), the power radiated in a given l, m multipole is proportional to the exponential cutoff $\exp(-2B)$. Since B/l is seen to be bounded away from zero, modes of high l are always suppressed. The upper graph applies to stable, circular, geodesic orbits. The lower graph is computed for accelerated circular trajectories which are tangent to (and have the velocity of) marginally bound "parabolic" orbits at pericenter. We exclude extreme unbound orbits on the grounds of astrophysical implausibility (see text for details).

geodesic orbits and accelerated trajectories with the same LNRF velocity: the accelerated trajectories are more efficient sources of high-multipole radiation. In the Schwarzschild metric $\beta = 1$ at the circular photon orbit, while in the extreme ($a \simeq M$) Kerr metric $\beta = 12$ at the direct circular photon orbit ($r_p \simeq r_{\text{ph}} \simeq M[1 + 2(\frac{2}{3}\delta)^{1/2}]$) and $\beta = \frac{75}{64}$ at the retrograde circular photon orbit ($r_p \simeq r_{\text{ph}} = 4M$).

The locally nonrotating frame can also be used to interpret the radiation in the wave zone, $r > r_1$. As measured by an observer at rest in the LNRF at radius r the scalar field oscillates with a proper frequency

$$\tilde{\Omega} = (\Omega - \omega)e^{-\nu} . \tag{4.29}$$

A photon with energy E and axial angular momentum L has a locally measured energy (frequency) in the LNRF

$$p^{(t)} = e^{-\nu}(E - \omega L) . \tag{4.30}$$

In particular, the frequency of a photon emitted tangent to the velocity-of-light circle at $r = r_1$, for which $E/L = \Omega$, changes in the same way with radius as the frequency of the scalar synchrotron radiation.

V. CONCLUSION

Physical processes near a rotating black hole often reveal their underlying nature most clearly when they are examined in the locally nonrotating frames. In the case of rotational energy extraction, the LNRF picture points out the severe hydrodynamical constraints: energy extraction requires boosts of $\sim 0.5c$ in "short" hydrodynamical times. In the case of synchrotron radiation, the LNRF picture indicates that such beamed radiation is possible only from astrophysically implausible (unbound, unstable) orbits. The simplicity of the Riemann tensor in the LNRF picture points toward a number of future hydrodynamical applications. The physics of rotating black holes is sufficiently rich and varied as to require a variety of techniques, among which the LNRF picture is, we think, an important one.

We thank Paul Chrzanowski, Charles W. Misner, and Larry Smarr for valuable discussions and for making unpublished work available to us. We thank Kip S. Thorne for advice on preparing the manuscript.

APPENDIX A

BOUNDS ON ENERGIES AND RELATIVE VELOCITIES OF PARTICLE ORBITS

Consider a particle of rest mass μ and conserved energy $E = -\boldsymbol{p}\cdot\boldsymbol{\xi}_t$, where $\boldsymbol{p}$ is the 4-momentum and $\boldsymbol{\xi}_t$ is the time Killing vector. Not all values of E/μ are possible for trajectories through a given point in spacetime. For example, particles at radial infinity must have $E/\mu \geq 1$. We first ask, what is the bound on E/μ for a general point?

Pick an orthonormal frame at the point. The 4-velocity of a particle has components $\boldsymbol{u} = (\gamma, \gamma\boldsymbol{v})$ with $\boldsymbol{v}$ a 3-vector and $\gamma = (1 - v^2)^{-1/2}$; the time Killing vector has components $\boldsymbol{\xi}_t = (\xi_0, \boldsymbol{\xi})$, with $\boldsymbol{\xi}$ a 3-vector. The particle's ratio of energy to rest mass is given by

$$E/\mu = -\boldsymbol{u}\cdot\boldsymbol{\xi}_t = \gamma(\xi_0 - \boldsymbol{v}\cdot\boldsymbol{\xi}), \tag{A1}$$

where the dot denotes the scalar product in the local Euclidean 3-space. Evidently, a necessary (but not a sufficient) condition for an extremum (hence a bound) on E/μ is

$$\boldsymbol{v}\cdot\boldsymbol{\xi} = \pm v\xi, \tag{A2}$$

where $v = |\boldsymbol{v}|$, $\xi = |\boldsymbol{\xi}|$. Now we distinguish two cases: If $\boldsymbol{\xi}_t$ is spacelike (e.g., in the ergosphere of the Kerr geometry), then we have $\xi_0 < \xi$; and inspection of equation (A1) shows that all values of E/μ are possible,

$$-\infty < E/\mu < +\infty \qquad \text{for } \boldsymbol{\xi}_t \text{ spacelike}. \tag{A3}$$

The infinite limits correspond to $v \to 1$ with the two signs of equation (A2). If, instead, $\boldsymbol{\xi}_t$ is timelike (e.g., at radial infinity), so that $\xi_0 > \xi$, then the right-hand side of equation (A1) is always positive, and there is a nontrivial lower bound on E/μ. Rewriting equation (A1) and using equation (A2) with the upper sign, we obtain

$$(\xi^2 + E^2/\mu^2)v^2 - 2\xi\xi_0 v + (\xi_0{}^2 - E^2/\mu^2) = 0. \tag{A4}$$

The extremum in E/μ is obtained by setting the discriminant of this equation, a quadratic in v, equal to zero; this gives

$$0 = (E/\mu)^2[(E/\mu)^2 - \xi^2 + \xi_0{}^2] . \tag{A5}$$

The root $E/\mu = 0$ is spurious, and the lower bound on E/μ is

$$(E/\mu)^2 = \xi_0{}^2 - \xi^2 = -\boldsymbol{\xi}_t \cdot \boldsymbol{\xi}_t . \tag{A6}$$

We see that the allowed range of E/μ at a point depends only on the norm of the time Killing vector at that point,

$$(-\boldsymbol{\xi}_t \cdot \boldsymbol{\xi}_t)^{1/2} \leq E/\mu < +\infty \qquad \text{for } \boldsymbol{\xi}_t \text{ timelike} . \tag{A7}$$

Finally note that the two cases (A3) and (A7) imply as a general condition,

$$(E/\mu)^2 + \boldsymbol{\xi}_t \cdot \boldsymbol{\xi}_t > 0 . \tag{A8}$$

Now turn to a different problem: If two orbits through a point have different ratios of energy to rest mass, E_1/μ_1 and E_2/μ_2, they have different 4-velocities and therefore a nonzero relative 3-velocity, $|w|$ (velocity of one particle seen by an observer comoving with the other particle). What is a bound on $|w|$?

At the point of interest, choose the orthonormal frame which gives the orbits equal and opposite 3-velocities $\boldsymbol{v}$, so that the tangent 4-velocities have components

$$\boldsymbol{u}_1 = (\gamma, -\gamma \boldsymbol{v}) , \qquad \boldsymbol{u}_2 = (\gamma, +\gamma \boldsymbol{v}) . \tag{A9}$$

The magnitude v of $\boldsymbol{v}$ is related to the relative velocity $|w|$ by the velocity addition formula,

$$|w| = 2v/(1 + v^2) . \tag{A10}$$

By analogy with equation (A1) we have

$$E_1/\mu_1 = \gamma\xi_0 + \gamma \boldsymbol{v} \cdot \boldsymbol{\xi} , \qquad E_2/\mu_2 = \gamma\xi_0 - \gamma \boldsymbol{v} \cdot \boldsymbol{\xi} . \tag{A11}$$

Defining an angle η by $\boldsymbol{v} \cdot \boldsymbol{\xi} \equiv v\xi \cos\eta$, and solving equation (A11) for $\xi_0{}^2$ and ξ^2, we obtain

$$\xi_0{}^2 = (E_1/\mu_1 + E_2/\mu_2)^2/(4\gamma^2) , \tag{A12a}$$

$$\xi^2 = (E_1/\mu_1 - E_2/\mu_2)^2/(4\gamma^2 v^2 \cos^2\eta) . \tag{A12b}$$

Subtraction of (A12b) from (A12a) yields

$$\begin{aligned}(E_1/\mu_1 - E_2/\mu_2)^2 &= [(E_1/\mu_1 + E_2/\mu_2)^2 + 4\gamma^2 \boldsymbol{\xi}_t \cdot \boldsymbol{\xi}_t] v^2 \cos^2\eta \\ &\leq [(E_1/\mu_1 + E_2/\mu_2)^2 + 4\gamma^2 \boldsymbol{\xi}_t \cdot \boldsymbol{\xi}_t] v^2 .\end{aligned} \tag{A13}$$

This inequality can be solved for v; the result is

$$v^2 \geq \left[\frac{E_1/\mu_1 - E_2/\mu_2}{(E_1{}^2/\mu_1{}^2 + \boldsymbol{\xi}_t \cdot \boldsymbol{\xi}_t)^{1/2} + (E_2{}^2/\mu_2{}^2 + \boldsymbol{\xi}_t \cdot \boldsymbol{\xi}_t)^{1/2}}\right]^2 . \tag{A14}$$

By equation (A8), the quantities appearing inside the square roots are guaranteed to be positive.

To apply equation (A14) to the question of energy extraction in the Kerr geometry, we note that for all θ, φ, and $r \geq r_+$, $|\boldsymbol{\xi}_t \cdot \boldsymbol{\xi}_t| \leq 1$. If we take $E_1/\mu_1 = 3^{-1/2}$ (the minimum energy of a plunge orbit which can result from the decay of a bound, stable orbit around any rotating black hole) and $E_2 = 0$ (the boundary of the negative energy region), we obtain

$$v \geq 2 - 3^{1/2}; \tag{A15a}$$

or by equation (A10),

$$|w| \geq \tfrac{1}{2}. \tag{A15b}$$

Hence, this class of all physically plausible plunge orbits is always separated from the negative energy region by at least half the speed of light. To achieve energy extraction, hydrodynamical forces or superstrong radiation reactions would have to accelerate particle fragments to more than half the speed of light in the "short" characteristic time of the plunge (see eq. [3.17] and the discussion following it).

APPENDIX B

BOUNDARY CONDITIONS FOR EQUATION (4.11)

At $r^* \to +\infty$ the asymptotic solutions are

$$\psi = e^{-i\omega t}e^{+im\varphi}S^m{}_l(\theta)e^{\pm ik_+r^*}, \tag{B1}$$

where $k_+ = [W(r^* = +\infty)]^{1/2} = \omega$ (positive square root). By convention we may take ω as positive (see discussion following eq. [4.8]), so the correct solution, corresponding to outgoing waves, is the upper sign.

On the horizon, $r^* \to -\infty$, the discussion is not quite so simple. The asymptotic solutions are

$$\psi = e^{-i\omega t}e^{+im\varphi}S^m{}_l(\theta)e^{\pm ik_-r^*}, \tag{B2}$$

with $k_- = [W(r^* = -\infty)]^{1/2}$ (positive square root). Again by convention $\omega > 0$. The correct boundary condition is *not* that the wave appear ingoing in the coordinate frame (i.e., *not* necessarily the lower sign in eq. [B2]). Rather, the wave must be physically ingoing in the frame of a physical observer. Since all physical observers are related by Lorentz transformations, they will all agree on the boundary condition, and we can calculate with any convenient observer. Take an observer at constant r just outside the horizon. Since he is within the ergosphere, he is dragged in the positive φ direction with some angular velocity $d\varphi/dt = \Omega_d > 0$. This observer sees the local t, r dependence of ψ (eq. [B2]) as

$$\psi \sim e^{-i(\omega - m\Omega_d)t}e^{\pm ik_-r^*}. \tag{B3}$$

Hence, for physically ingoing waves one must choose the sign ($\pm ik_-$) opposite to the sign of $(\omega - m\Omega_d)$. On the horizon $\Omega_d \to \omega_+ = a/(2Mr_+)$ for all observers. Hence the correct sign in equation (B2) is

if $m < 0$, lower sign $(-)$;

if $m > 0$, lower sign $(-)$ if $\omega > m\omega_+$, upper sign $(+)$ if $0 < \omega < m\omega_+$.

In the last case the waves are apparently *outgoing* in the coordinate picture, and in fact they extract rotational energy from the rotating black hole, even though they are physically ingoing in the local frame of any physical observer. This kind of wave is generated by a particle in any direct, stable circular orbit for $a = M$, and also holds for small a if the particle orbit is sufficiently far out.

However, even for $a \approx M$ the highly relativistic orbits at $r \sim r_{\rm pb}$ cannot extract energy from the black hole. When $\delta = (1 - a/M) \ll 1$ and $r = M[1 + 2(\frac{4}{3}\delta)^{1/2}]$, then

$$\Omega - \omega_+ \approx \frac{1}{2M}(1 - 3^{1/2}/2)(2\delta)^{1/2} > 0\,, \tag{B4}$$

so the particle loses energy to the black hole.

For an alternative and more rigorous discussion, the reader is referred to Misner (1972*b*).

APPENDIX C

BOUNDS ON EIGENVALUES OF SPHEROIDAL HARMONICS

Define the following differential operator L on the closed interval $[-1, 1]$:

$$L = -\frac{d}{dx}(1 - x^2)\frac{d}{dx} + g(x)\,, \tag{C1}$$

where

$$g(x) = |c^2|(1 - x^2) + \frac{m^2}{1 - x^2} > 0\,. \tag{C2}$$

Then the oblate spheroidal harmonics $S^m{}_l(c^2, x)$, where $c^2 < 0$, are eigenfunctions of L which are regular at $x = \pm 1$:

$$LS^m{}_l = \alpha_{ml}S^m{}_l\,. \tag{C3}$$

Here m is fixed and $l = m, m + 1, \ldots$. In the text, we use $x = \cos\theta$, $c^2 = -a^2\omega^2$, and $\lambda_{ml} = \alpha_{ml} - |c^2|$. We use α_{ml} in this Appendix to make L a positive operator so that various theorems are directly applicable. In this Appendix, all functions u on which L acts will be normalized as follows:

$$\int_{-1}^{1} u^2 dx = 1\,. \tag{C4}$$

[This differs from the normalization of $S^m{}_l$ in the rest of the paper by a factor of 2π, and from the normalization used by Flammer (1957): $S^m{}_l(\text{here}) = N_{mn}{}^{-1/2}S_{mn}(\text{Flammer})$. Flammer (1957) tabulates the conventions used by various authors.]

Let u be a trial function for equation (C3). As Friedman (1956) shows, an upper bound ρ for the lowest eigenvalue, α_{mm}, is given by

$$\rho = \int_{-1}^{1} uLu dx\,, \tag{C5}$$

while a lower bound is

$$\rho - \left[\int_{-1}^{1} (Lu)^2 dx - \rho^2\right]^{1/2}. \tag{C6}$$

Taking as a trial function the associated Legendre function $u = P^m{}_l$ and using the identity

$$xP^m{}_l(x) = \frac{l - m + 1}{2l + 1}P^m{}_{l+1}(x) + \frac{l + m}{2l + 1}P^m{}_{l-1}(x) \tag{C7}$$

to perform the integrals, we find

$$l(l+1) - \frac{|c|^2}{2l+3}\left[1 + 2\left(\frac{l+1}{2l+5}\right)^{1/2}\right] \leq \lambda_{ll} \leq l(l+1) - \frac{|c|^2}{2l+3} \cdot \tag{C8}$$

The right-hand side of this inequality gives the upper bound quoted in § IV,

$$\lambda_{ll} \leq l(l+1) . \tag{C9}$$

Since $|c|^2 = a^2m^2\Omega^2$, and $\mathscr{V}^2 \leq 1$ implies $a^2\Omega^2 \leq \frac{1}{4}$, the left-hand side of inequality (C8) gives

$$\lambda_{mm} \geq m^2 . \tag{C10}$$

The eigenvalues of the Sturm-Liouville operator (C1) increase monotonically with l, hence inequality (C10) gives

$$\lambda_{ml} \geq m^2 . \tag{C11}$$

Inequalities (C10) and (C11) are the lower bounds used in § IV.

The upper bound (C9) holds for $l \neq m$ as well, since from the theory of Sturm-Liouville equations (e.g., Courant and Hilbert 1953), if we increase $g(x)$ to a new function $g'(x)$, then the new eigenvalues $\alpha'_{ml} = \lambda'_{ml} + |c|^2$ are all greater than the old ones. Choose

$$g'(x) = |c|^2 + \frac{m^2}{1-x^2} \cdot \tag{C12}$$

Then

$$\lambda_{ml} \leq \lambda'_{ml} = l(l+1) . \tag{C13}$$

An alternative lower bound can be derived by choosing

$$g'(x) = \frac{m^2}{1-x^2} \leq g(x) . \tag{C14}$$

Then

$$\lambda_{ml} \geq \lambda'_{ml} = l(l+1) - |c|^2 \geq l(l+1) - \tfrac{1}{4}m^2 . \tag{C15}$$

This inequality is stronger than the bound (C11) when $m^2 \leq \frac{4}{5}l(l+1)$.

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(b) Floating Orbits, Superradiant Scattering and the Black-Hole Bomb (Paper II; collaboration with W. H. Press, published in Nature 238, 211 [1972]).

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Floating Orbits, Superradiant Scattering and the Black-hole Bomb

Penrose[1] and Christodoulou[2] have shown how, in principle, rotational energy can be extracted from a black hole by orbiting and fissioning particles. Recently, Misner[3] has pointed out that waves can also extract rotational energy ("superradiant scattering" in which an impinging wave is amplified as it scatters off a rotating hole). As one application of superradiant scattering, Misner has suggested the possible existence of "floating orbits", that is, orbits in which a particle radiatively extracts energy from the hole at the same rate as it radiates energy to infinity; thereby it experiences zero net radiation reaction.

Here we point out a second application of superradiant scattering which we call the "black-hole bomb". We also present the chief results of quantitative analyses of superradiant scattering, floating orbits, and the black-hole bomb, for the case of scalar waves. Quantitative calculations are restricted to the scalar case because the scalar wave equation is separable[4] in the Kerr gravitational field of a black hole, whereas the gravitational wave equation appears not to be.

We expect the gravitational case to resemble the scalar case qualitatively if not quantitatively.

The scalar wave equation

$$\Box\Phi = 4\pi T \tag{1}$$

(T a scalar charge density) separates in Boyer–Lindquist coordinates (S. A. T., unpublished) by writing

$$\Phi = e^{-i\omega t}\, e^{im\varphi}\, S^m{}_l\,(\theta)\; \psi(r)/r \tag{2}$$

with $S^m{}_l$ an oblate spheroidal harmonic (D. R. Brill and colleagues, unpublished). We define a new radial coordinate r^* by $dr^*/dr = r^2/(r^2 - 2Mr + a^2)$, so that the equation for the radial function takes the form

$$d^2\psi/dr^{*2} - W(r^*)\psi = (\text{source term}) \tag{3}$$

The mass of the black hole is M and its angular momentum is aM. The effective potential $W(r^*)$ is negative at infinity and near the event horizon $r = r_+$, so travelling waves exist in those regions. In between, $W(r^*)$ is positive, that is, it becomes a potential barrier (J. M. Bardeen, W. H. P. and S. A. T., unpublished).

At infinity the asymptotic solutions for ψ are $\exp[-i\omega(t \pm r^*)]$ corresponding to ingoing ("+") and outgoing ("−") waves. By convention we set $G = c = 1$; also we take the real part of Φ as the physical field, which permits the convention $\omega \geq 0$ without loss of generality. On the horizon the asymptotic solutions are $\exp[-i(\omega t \pm kr^*)]$ where $k = [-W(r^* = -\infty)]^{\frac{1}{2}}$. The correct boundary condition on the horizon is not that the waves appear ingoing in the coordinate frame, but rather that the wave be physically ingoing in the frames of all physical observers, who are dragged around the hole by its rotation. If $m > 0$ and $0 < \omega < m\omega_{\text{horizon}}$, where $\omega_{\text{horizon}} \equiv$(angular velocity of "dragging" at the horizon)$= (a/2Mr_{\text{horizon}})$, this physically ingoing condition corresponds to a "coordinate outgoing" wave, $\exp[-i(\omega t - kr^*)]$ (C. M. Misner, unpublished).

We now consider a wave which is incident on the black hole. Normally, a part of the wave's energy reflects off the potential barrier $W(r^*)$, while the rest leaks through and is lost down the hole, so that the outgoing wave is weaker than the ingoing wave. If, however, m and ω are in the anomalous range $0 < \omega < m\omega_{\text{horizon}}$, the wave on the inside of the barrier is coordinate outgoing, it reinforces the reflected wave on the outside of the barrier, and there is thus more outgoing wave energy than ingoing. The extra energy comes from the rotational energy of the black hole. The amount of amplification is never very large, because there is always a potential barrier separating the travelling-wave regions.

Fig. 1 shows the results of our numerical integrations of equation (3) for the most favourable case, a maximally rotating black hole with $a=M$. The maximal amplification is a few tenths of a per cent in energy and occurs for low modes [$l=m\sim\varsigma(1)$] and for wave frequencies $\omega\sim(0.8$ to $1.0)$ $m\omega_{\mathrm{horizon}}$. For a maximally rotating hole of mass M,

$$\omega_{\mathrm{horizon}}\approx 10^5 \text{ rad/s } (M_\odot/M) \qquad (4)$$

By itself, a few tenths of a per cent is unimpressive; but any amplification mechanism admits improvement by positive feedback. To illustrate, in a rather speculative vein, we propose the "black-hole bomb" (closely related to a recent suggestion of Zel'dovich[6]): locate a rotating black hole and construct a spherical mirror around it. The mirror must reflect low-frequency radio waves (equation (4); and we now make the transition from scalar to electromagnetic fields) with reflectivity $\gtrsim 99.8\%$, so that in one reflexion and subsequent superradiant scattering there is a net amplification. The system is then

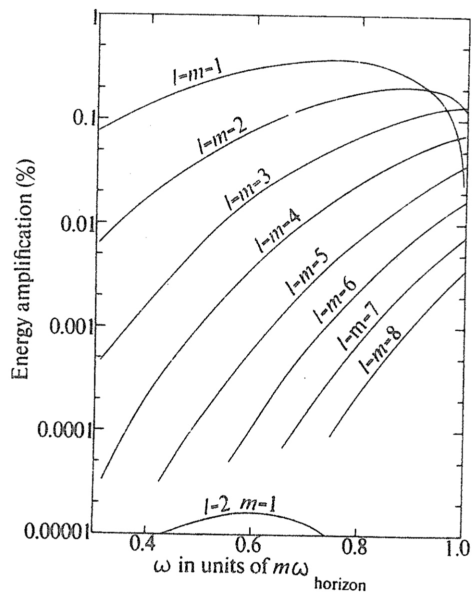

Fig. 1 Superradiant scattering of scalar radiation by maximally-rotating black hole. Radiation modes with axial eigenvalue $m>0$ and angular frequency $\omega<m\omega_{\mathrm{horizon}}$ are amplified by the hole, not absorbed by it. The fractional wave energy added by the hole is here shown as a function of wave frequency for the most favourable modes.

unstable against a number of exponentially growing electromagnetic modes which will be initiated by random "seed fields" (thermal noise). Because a typical amplification in one reflexion is $\sim 10^{-3}$, the *e*-folding time is roughly $\tau \sim 10^3 L/c$ where L is the radius of the mirror. For ease of construction, the mirror should not be too close to the hole; for a hole of $M_\odot$ an appropriate choice might be $L \sim 10^3$ km, so that $\tau \sim 3$ s. As the mode grows, electromagnetic pressure on the mirror increases until the mirror explodes, releasing the trapped electromagnetic energy in a time $\sim L/c$. This is the black-hole bomb. Alternatively a port hole in the mirror can be periodically opened, and the resultant radio flux rectified and used as a source of electric power.

Others may care to speculate on the possibility that nature provides her own mirror. The amplified wave frequencies are far below the plasma frequency of the interstellar medium, so that waves would reflect off the boundary of an evacuated cavity surrounding the hole; we are tempted to invoke radiation pressure to maintain the evacuation.

We turn now to "floating orbits". If a particle is in a stable circular orbit around the hole, it generates radiation both outward to infinity as a source term in equation (3), and also (physically) inward into the hole. If the particle is in a direct orbit, co-rotating with the hole, it generates only modes with $m > 0$. For holes with $a > 0.359\ M$, radiation from all direct, stable, circular orbits satisfied the anomalous boundary conditions $0 < \omega < m\omega_{\text{horizon}}$ for all l, m. The radiation energy balance of a particle in such an orbit has two parts: the power radiated to infinity, and the power extracted from (not deposited into) the rotating hole. The question of floating depends on the detailed numerical balance of these contributions. If at some radius more energy is extracted than is radiated, the particle will gradually spiral outward until the energy credits and debits are in balance. At this "floating" radius, the particle gradually radiates away the black hole's rotational energy; as a decreases, the radius of the lowest stable orbit moves outward with respect to the floating radius. When the two are equal, floating ceases, and the particle plunges into the hole.

The results of our calculations for scalar radiation are as follows: a system whose dominant radiation is in $m=1$ modes can float around any black hole with $a \gtrsim 0.985\ M$; for an $a=m$ hole, the floating radius is at $r \approx 1.4\ M$. A system whose dominant modes are $m=2$ can float for $a \gtrsim 0.9995$; for $a=M$ the floating radius is $r \approx 1.16\ M$. Systems radiating substantially in $m \geq 3$ modes cannot float at any radius for any $a \leq M$. In the particular case of a point particle in a circular equatorial orbit, it is not difficult to calculate the relative coupling of the source to various modes, and exhibit the energy "balance sheet". Table 1 shows this for the most favourable of all scalar-wave cases: an $a=M$ hole, with the particle very near the lowest stable orbit, $x \equiv r-M \ll M$. Although the first two modes give a net credit, the particle couples too strongly to non-floating modes and there is no net floating. If the particle were smeared out in azimuthal angle φ, these higher modes would be suppressed and floating would occur.

For the physical case of gravitational (not scalar) radiation, there is no $l=m=1$ radiation, and the numerical details for higher modes will be different. We do not know if there will be floating for the $l=m=2$ mode (or higher modes for that matter); our scalar results suggest only that gravitational floating is not implausible, and might conceivably enter into the dynamics of material processes near rotating black holes. (Some recent unpublished work by D. M. Chitre and R. H. Price, and by M. Davis and colleagues, suggests that source coupling to high, presumably non-floating, modes is weaker for gravitational than for scalar fields.)

Finally, we mention a curious aspect of our numerical results: in Fig. 1, the amplification factor with $l=m>1$ does not go to zero as $\omega \to m\omega_{\text{horizon}}$. Because the amplification factor is negative for $\omega > m\omega_{\text{horizon}}$ it must be a discontinuous function of frequency at $m\omega_{\text{horizon}}$. This discontinuity is an artefact of taking $a=M$. For a slightly lower value, we expect

Table 1 Energy Balance for Scalar Radiation from a Particle in Close Circular Orbit

Mode (l, m)	Power from hole* (credit)	Power lost to infinity* (debit)	Net energy loss (or gain)
1, 1	(0.074)	≈ 0	(0.074)
2, 2	(0.081)	0.032	(0.049)
3, 3	(0.062)	0.070	0.008
4, 4	(0.042)	0.089	0.047
5, 5	(0.027)	0.091	0.064
6, 6	(0.016)	0.087	0.071
7, 7	(0.010)	0.081	0.071
8, 8	(0.006)	0.073	0.067
All modes with $l>m$	≈ 0	≈ 0	≈ 0
Total for modes $l \leq 8$	(0.318)	0.523	0.205

* In units of $(c^5/G)\,(\mu/M)^2\,x$ where $x=(c^2r/GM-1) \ll 1$ and μ is the particle's scalar charge. It is an artefact of the scalar case that power$\to 0$ as $x \to 0$.

the discontinuity to disappear, with the curves of Fig. 1 showing a sharp turnover very near $\omega/m\omega_{\text{horizon}}=1$.

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PART IV

KERR PERTURBATION EQUATIONS

(a) Rotating Black Holes: Separable Wave Equations for Gravitational and Electromagnetic Perturbations (Paper III; published in Phys. Rev. Letters **29**, 1114 [1972]).

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Rotating Black Holes: Separable Wave Equations for Gravitational and Electromagnetic Perturbations*

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Separable wave equations with source terms are presented for electromagnetic and gravitational perturbations of an uncharged, rotating black hole. These equations describe the radiative field completely, and also part of the nonradiative field. Nontrivial, source-free, stationary perturbations are shown not to exist. The barrier integral governing synchrotron radiation from particles in circular orbits is shown to be the same as for scalar radiation. Future applications (stability of rotating black holes, "spin-down," superradiant scattering, floating orbits) are outlined.

It is generally accepted that the gravitational collapse of a massive rotating star can produce a rotating black hole. Moreover, black holes may play important roles in a number of astrophysical phenomena: (i) One or more black holes near the center of the Galaxy might be the origin of Weber's[1] gravitational-wave events; (ii) a massive black hole at the center of the Galaxy has been postulated[2] to explain radio and infrared phenomena there; (iii) the x-ray source Cyg-XI —and also 2U-0900-40—is likely to be a black hole in close orbit around a B-type supergiant star, with the x rays emitted by gas flowing from star to hole.[3]

These developments create an urgent need for two types of black-hole calculations: first, cal-

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culations on matters of principle, such as whether rotating black holes are unstable against spontaneous loss of rotational energy and angular momentum; second, realistic astrophysical calculations of black-hole processes. Both types of calculations are amenable to perturbation techniques, where the dynamics of matter, electromagnetism, and gravitational waves takes place in the fixed background geometry of the black hole. Until now, calculations of electromagnetic and gravitational perturbations were restricted to the nonrotating (Schwarzschild) case, where the static, spherically symmetric background geometry guarantees a complete separation of variables in the perturbation equations. In the rotating case, the background geometry is described by the Kerr[4] metric, and the electromagnetic[5] and gravitational[6] perturbation equations have appeared to be inseparable. However, the scalar wave equation $\Box\Phi = 4\pi T$ is separable[7,8] in Boyer-Lindquist[9] coordinates, and has been used to make qualitative predictions about the electromagnetic and gravitational cases.

In this Letter, we present separable equations for the radiative parts of electromagnetic and gravitational perturbations. The equations are consolidated into a single master equation, Eq. (5) below, with a "spin-weight parameter" s specifying the type of field under study (scalar, electromagnetic, gravitational) and the particular radiative part of the field involved. The derivations, which are similar to earlier approaches,[10,11,5,6] use the Newman-Penrose[12] formalism and will be published elsewhere. Sufficient information is given below to enable calculations to be performed.

In Boyer-Lindquist coordinates with $c = G = 1$, the Kerr metric is

$$ds^2 = (1 - 2Mr/\Sigma)dt^2 + (4Mar\sin^2(\theta)/\Sigma)dt\,d\varphi - (\Sigma/\Delta)dr^2 - \Sigma d\theta^2 - \sin^2(\theta)(r^2 + a^2 + 2Ma^2 r\sin^2(\theta)/\Sigma)d\varphi^2. \quad (1)$$

Here M is the mass of the black hole, aM its angular momentum, $\Sigma = r^2 + a^2\cos^2\theta$, and $\Delta = r^2 - 2Mr + a^2$. We introduce Kinnersley's[13] null tetrad, which has $[t, r, \theta, \varphi]$ components

$$l^\mu = [(r^2 + a^2)/\Delta, 1, 0, a/\Delta], \quad n^\mu = [r^2 + a^2, -\Delta, 0, a]/2\Sigma, \quad m^\mu = [ia\sin\theta, 0, 1, i/\sin\theta]/2^{1/2}(r + ia\cos\theta). \quad (2)$$

The electromagnetic field is characterized by the Newman-Penrose components

$$\varphi_0 = F_{\mu\nu} l^\mu m^\nu, \quad \varphi_1 = \tfrac{1}{2} F_{\mu\nu}(l^\mu n^\nu + m^{*\mu} m^\nu), \quad \varphi_2 = F_{\mu\nu} m^{*\mu} n^\nu, \quad (3)$$

where $F_{\mu\nu}$ is the electromagnetic field tensor and an asterisk denotes complex conjugation. The components φ_0 and φ_2 are the "ingoing and outgoing radiative parts" of the field. Maxwell's equations in the fixed Kerr background with four-current J^μ lead directly to Eq. (5) below for φ_0 and φ_2.

Gravitational radiation is described by perturbations in the Weyl tensor $C_{\alpha\beta\gamma\delta}$ (which is equal to the Riemann tensor in vacuum). The "ingoing and outgoing radiative parts" of the Weyl tensor are

$$\psi_0 = -C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta, \quad \psi_4 = -C_{\alpha\beta\gamma\delta} n^\alpha m^{*\beta} n^\gamma m^{*\delta}. \quad (4)$$

These quantities are invariant under gauge transformations and infinitesimal tetrad rotations. First-order perturbations of the Newman-Penrose version of the Einstein field equations, with material stress-energy tensor $T_{\mu\nu}$, lead to Eq. (5) below for ψ_0 and ψ_4.

The master perturbation equation is

$$\left[\frac{(r^2 + a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\,\partial\varphi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\varphi^2} - \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)$$
$$- 2s\left[\frac{a(r - M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\varphi} - 2s\left[\frac{M(r^2 - a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t} + (s^2\cot^2\theta - s)\psi = 4\pi\Sigma T. \quad (5)$$

Table I specifies the field quantities ψ which satisfy this equation, the corresponding values of the spin-weight parameter s, and the source terms T. The source terms are written in Newman-Penrose notation.

Equation (5) can be separated by writing

$$\psi = e^{-i\omega t} e^{im\varphi} S(\theta) R(r). \quad (6)$$

[Note that Eq. (5) is also separable in Kerr[14] coordinates or any other coordinate system related to

TABLE I. Field quantities ψ, spin-weight s, and source terms T for Eq. (5).

ψ	s	Source Term T
Φ	0	$\Box\Phi = 4\pi T$ (T is the scalar charge density)
φ_0	1	$(\delta - 2\tau)J_\ell - (D - 2\rho - \rho^*)J_m$
$\rho^{-2}\varphi_2$	-1	$(\Delta + \mu)\rho^{-2} J_{m^*} - (\delta^* + \pi - \tau^*)\rho^{-2} J_n$
ψ_0	2	$2\left\{(\delta - 2\beta - 4\tau)[(\delta - \pi^*)T_{\ell\ell} - (D - 2\rho^*)T_{\ell m}] + (D - 4\rho - \rho^*)[(D - \rho^*)T_{mm} - (\delta - 2\beta + 2\pi^*)T_{\ell m}]\right\}$
$\rho^{-4}\psi_4$	-2	$2\left\{(\Delta + 2\gamma + \mu)[(\Delta - 2\mu - \mu^*)\rho^{-4} T_{m^*m^*} - (\delta^* - 2\pi - 2\beta^* - 2\tau^*)\rho^{-4} T_{nm^*}]\right.$
		$\left. + (\delta^* + 3\pi - 2\beta^* - \tau^*)[(\delta^* - 2\pi - \tau^*)\rho^{-4} T_{nn} - (\Delta + 2\gamma + 2\mu^* - 4\mu)\rho^{-4} T_{nm^*}]\right\}$
		Notation: $D = \ell^\mu \partial/\partial x^\mu$, $\Delta^a = n^\mu \partial/\partial x^\mu$, $\delta = m^\mu \partial/\partial x^\mu$,
		$\rho = -1/(r - ia\cos\theta)$, $\beta = -\rho^* \cot\theta/2\sqrt{2}$, $\pi = i a\rho^2 \sin\theta/\sqrt{2}$,
		$\tau = -i a\rho\rho^* \sin\theta/\sqrt{2}$, $\mu = \rho^2\rho^*(r^2 - 2Mr + a^2)/2$, $\gamma = \mu + \rho\rho^*(r - M)/2$
		$J_n = J_\mu n^\mu$, $T_{nn} = T_{\mu\nu}n^\mu n^\nu$, etc.

[a] The operator Δ in this table is not to be confused with $\Delta = r^2 - 2Mr + a^2$ used in the body of the paper.

Boyer-Lindquist by $\bar{t} = t + f_1(r) + f_2(\theta)$, $\bar{\varphi} = \varphi + g_1(r) + g_2(\theta)$, $\bar{r} = h(r)$, $\bar{\theta} = j(\theta)$.] In vacuum ($T = 0$) we obtain

$$\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{dR}{dr}\right) + \{[(r^2 + a^2)^2\omega^2 - 4aMr\omega m + a^2m^2 + 2ia(r - M)ms - 2iM(r^2 - a^2)\omega s]\Delta^{-1} + 2ir\omega s - A - a^2\omega^2\}R = 0, \tag{7}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) + \left(a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta} - 2a\omega s\cos\theta - \frac{2ms\cos\theta}{\sin^2\theta} - s^2\cot^2\theta + s + A\right)S = 0. \tag{8}$$

Equation (8), together with boundary conditions of regularity at $\theta = 0$ and π, constitutes an eigenvalue problem for the separation constant A. When $s = 0$, the eigenfunctions ${}_sS^m{}_l$ are the oblate spheroidal wave functions[15] $S^m{}_l(-a^2\omega^2, \cos\theta)$. When $a\omega = 0$, the eigenfunctions are the spin-weighted spherical harmonics[16] ${}_sY^m{}_l(\theta, \varphi) = {}_sS^m{}_l(\theta)e^{im\varphi}$, and $A = (l - s)(l + s + 1)$. When sources are present ($T \neq 0$), we can use the eigenfunctions of Eq. (8) to separate Eq. (5) by expanding $T = \sum G(r)_sS^m{}_l(\theta)e^{im\varphi}e^{-i\omega t}$.

The asymptotic solutions of Eq. (7) at $r = \infty$ are $e^{i\omega r}/r^{(2s+1)}$ (outgoing waves) and $e^{-i\omega r}/r$ (ingoing waves). This corresponds to

$$\Phi, \varphi_2, \psi_4 \sim e^{i\omega r}/r, \quad \varphi_0 \sim e^{i\omega r}/r^3, \quad \psi_0 \sim e^{i\omega r}/r^5 \text{ (outgoing)};$$

$$\Phi, \varphi_0, \psi_0 \sim e^{-i\omega r}/r, \quad \varphi_2 \sim e^{-i\omega r}/r^3, \quad \psi_4 \sim e^{-i\omega r}/r^5 \text{ (ingoing)}.$$

One identifies Φ, φ_2, and ψ_4 ($s \leq 0$) as the outgoing radiative parts of the fields because for outgoing waves they alone die out as $1/r$; all other Newman-Penrose quantities die out more rapidly. For the analogous reason one identifies Φ, φ_0, and ψ_0 $(s \geq 0)$ as the ingoing radiative parts.

The event horizon is at $r = r_+$, the larger root of $\Delta = 0$. The solutions at the horizon are e^{ikr^*} and $e^{-ikr^*}\Delta^{-s}$, where $k = \omega - ma/2Mr_+$, $dr^*/dr = (r^2+a^2)/\Delta$, and $r^* \to -\infty$. The factor Δ^{-s} is not a physical singularity; it arises because the Newman-Penrose field quantities are projections on the tetrad (2), which goes singular at $\Delta = 0$. (See Ref. 10 for a discussion of this point in the Schwarzschild case.) The correct boundary condition at the horizon is *not* that the waves be ingoing in these coordinates[17,18]; rather, they must be ingoing as seen by all physical observers—who are "dragged" with angular velocity $d\varphi/dt = a/2Mr_+$ near the horizon. The correct ingoing boundary condition then turns out to be $R \sim e^{-ikr^*}\Delta^{-s}$.

Maxwell's equations are sufficiently simple that, when one has solved Eq. (5) for φ_2 say, one can find the complete electromagnetic field, up to an undetermined Coulomb term, by integrating (nonseparable) Pfaffian equations for φ_1 and φ_0 (see Ref. 5). Then the stress-energy tensor is given by

$$4\pi T_{\mu\nu} = \{\varphi_0\varphi_0^* n_\mu n_\nu + 2\varphi_1\varphi_1^*[l_{(\mu}n_{\nu)} + m_{(\mu}m_{\nu)}^*] + \varphi_2\varphi_2^* l_\mu l_\nu - 4\varphi_0^*\varphi_1 n_{(\mu}m_{\nu)} - 4\varphi_1^*\varphi_2 l_{(\mu}m_{\nu)} + 2\varphi_2\varphi_0^* m_\mu m_\nu\} + \text{c.c.}, \tag{9}$$

where round brackets on subscripts denote symmetrization. The total energy flux per unit solid angle for outgoing waves at infinity can be found from φ_2 alone:

$$\frac{d^2E}{dt\,d\Omega} = \lim_{r\to\infty} \frac{r^2}{2\pi}|\varphi_2|^2. \tag{10}$$

The squares of the real and imaginary parts of φ_2 give the amount of energy in the two linear polarization states along the directions $\hat{e}_{\hat\theta}$ and $\hat{e}_{\hat\varphi}$, respectively.

For gravitational waves with $\psi_4 \propto e^{-i\omega t}/r$,

$$\frac{d^2E}{dt\,d\Omega} = \lim_{r\to\infty} \frac{r^2}{4\pi\omega^2}|\psi_4|^2. \tag{11}$$

The real and imaginary parts of ψ_4 describe the linear polarization states along the directions $\hat{e}_{\hat\theta}$ and $\hat{e}_{\hat\varphi}$, and $\hat{e}_{\hat\theta} \pm \hat{e}_{\hat\varphi}$, respectively. It is likely, but not yet proved, that one can calculate the complete perturbed Riemann tensor $(\psi_0, \psi_1, \psi_2, \psi_3, \psi_4)$ from a knowledge of ψ_0 or ψ_4 alone, except for undertermined monopole and dipole terms.

A number of expected results can easily be verified by using Eq. (7). Misner[19] has proposed that a synchrotronlike mechanism might be responsible for Weber's observations. He has suggested that matter spiraling into a massive rotating black hole at the center of the Galaxy might have its radiation beamed into the galactic plane in high harmonics of the orbital frequency, instead of being generated isotropically. Using the scalar-field analog, it has been found[20,18] that the beaming is cut down by a barrier integral which is never small for astrophysically reasonable sources. If we take the large-m limit of Eq. (7) (with $\omega = m \times$ angular velocity of source), we obtain exactly the same barrier integral for the scalar, electromagnetic, and gravitational cases at dominant order (m^2). Polarization effects, which differ from one type of field to another, enter only at the next smallest order (order m). Thus, the barrier-integral conclusions of the scalar analysis are valid for the electromagnetic and gravitational cases.

If $\omega = 0$, Eq. (7) can be solved in terms of hypergeometric functions. There are no solutions which are well-behaved at infinity and at the event horizon. Maxwell's equations then imply (Ref. 5) that $\varphi_1 = Q\rho^2$, which corresponds to adding a constant charge Q to the black hole. This verifies Ipser's argument[21] that the only stationary electromagnetic perturbation of a rotating black hole corresponds to adding charge. If we had found a solution ψ_4 with $\omega = 0$, this would have implied the existence of a new stationary black-hole solution, which presumably would have violated the combined theorems of Carter[22] and Hawking.[23]

The "spin down" (loss of angular momentum) of a rotating black hole caused by a stationary, nonaxisymmetric perturbation has been discussed by Press[24] (scalar perturbation) and Hawking and Hartle[25] (gravitational perturbation with $a \ll M$). It turns out that this problem can be solved from a knowledge of φ_0 or ψ_0 produced by the perturbation. Details of this calculation will be published elsewhere.

Misner[17] has recently shown that by "superradiant scattering," waves can extract rotational energy from a black hole. The boundary condition $R \sim e^{-ikr^*}\Delta^{-s}$ implies that as waves with $k\omega < 0$ go down a black hole, they carry negative "energy at infinity" with themselves. If the radiation to infinity of an orbiting particle were balanced by such negative-energy flow down the hole, the particle would be in a "floating orbit." Preliminary calculations[26] with the scalar equation suggest that the details of super-radiant scattering and floating orbits may depend strongly on the spin of the field. The calculations for electromagnetic and gravitational radiation are now underway.

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(b) Perturbations of a Rotating Black Hole. I. Fundamental Equations for Gravitational, Electromagnetic and Neutrino-Field Perturbations (Paper IV; to be published in Ap. J., October 1973).

PERTURBATIONS OF A ROTATING BLACK HOLE: I. FUNDAMENTAL EQUATIONS FOR GRAVITATIONAL, ELECTROMAGNETIC, AND NEUTRINO-FIELD PERTURBATIONS*

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ABSTRACT

This paper derives linear equations that describe dynamical gravitational, electromagnetic, and neutrino-field perturbations of a rotating black hole. The equations decouple into a single gravitational equation, a single electromagnetic equation, and a single neutrino equation. Each of these equations is completely separable into ordinary differential equations. The paper lays the mathematical groundwork for later papers in this series, which will deal with astrophysical applications: stability of the hole, tidal friction effects, super-radiant scattering of electromagnetic waves, and gravitational-wave processes.

I. INTRODUCTION

This is the first in a series of papers which will deal with dynamical processes near a rotating black hole. The underlying mathematical technique throughout the series is to linearize the Einstein or Maxwell-Einstein equations around a known stationary black-hole solution, in this case the Kerr (1963) metric. This technique goes beyond previous work in which a rotating black hole has been treated as a fixed geometrical background for physical processes: the linearized equations give the hole the full dynamical freedom of small perturbations, including the possibility of gravitational and electromagnetic waves, secular changes in its mass and angular momentum, interaction with accreting test matter or distant massive objects, and so on.

The fundamental perturbation equations which will be used throughout the series are derived in this paper; in form, the equations are separable partial differential equations whose independent variables are certain decoupled components of the Weyl or Riemann tensor, or of the electromagnetic field tensor. Some of the applications to be treated in subsequent papers make direct use of only these decoupled components. Other applications require that one consider <u>all</u> components of the electromagnetic or gravitational field. Here, a concentration of attention on only the decoupled components would not <u>a priori</u> seem to be justified. However, for both gravitational and electromagnetic perturbations, it can be proved that the decoupled components contain complete information about all non-trivial features of the full perturbing field; this completeness will be discussed in a subsequent paper. For the electromagnetic case the result is due to Fackerell and Ipser (1972); for the gravitational case it is due to Wald (1973).

How does one obtain linearized perturbation equations, say for gravitational perturbations? A straightforward way is to start with the Einstein equations for a metric $g_{\mu\nu}$, and to let $g_{\mu\nu} = g_{\mu\nu}{}^{A} + h_{\mu\nu}{}^{B}$, where the superscripts A and B denote background and perturbation quantities respectively. The field equations are then expanded to first order in $h_{\mu\nu}{}^{B}$, yielding a set of linear equations for the perturbations.

This method has already been applied to the Schwarzschild metric [Regge and Wheeler (1957); Vishveshwara (1970); Zerilli (1970)]. In this case, the background metric is static and spherically symmetric, so the time and angular dependence can easily be separated out of the equations. The resulting coupled radial equations can then be reduced to two decoupled equations, one governing odd-parity perturbations (Regge and Wheeler 1957) and the other governing even-parity perturbations (Zerilli 1970).

Even in the Schwarzschild case, this procedure involves considerable algebraic complexity. In the Kerr case, where the background metric is much more complicated, nobody has carried out a similar program. Moreover, the replacement of spherical symmetry by axial symmetry means that a separation into spherical harmonics is no longer possible; one expects to end up with partial differential equations in r and Θ instead of ordinary differential equations in r.

Fortunately, there is an alternative approach to the problem. This is provided by the Newman-Penrose (NP) formalism. We shall use the notation of Newman and Penrose (1962), and equations from that paper will be cited as NP 2.1 and so forth. The NP formalism arises naturally from the introduction of spinor calculus into general relativity. It can also be regarded as a special type of tetrad calculus. Four null vectors, conventionally called $\underset{\sim}{\ell}$, $\underset{\sim}{n}$, $\underset{\sim}{m}$, and $\underset{\sim}{m}^*$, are chosen at every point of spacetime. (An asterisk denotes complex conjugation. The vectors $\underset{\sim}{\ell}$ and $\underset{\sim}{n}$ are real.) All tensors are projected onto the null tetrad. The full set of NP equations is a system of coupled first-order differential equations linking the tetrad, the spin coefficients (essentially Ricci rotation coefficients), the Weyl tensor, the Ricci tensor, and the scalar curvature. To do perturbation theory in this formalism, one specifies the perturbed geometry by $\underset{\sim}{\ell} = \underset{\sim}{\ell}^A + \underset{\sim}{\ell}^B$, $\underset{\sim}{n} = \underset{\sim}{n}^A + \underset{\sim}{n}^B$, etc. All the NP quantities can then be written in this form: $\psi_2 = \psi_2{}^A + \psi_2{}^B$, $D = D^A + D^B$, etc. The complete set of perturbation equations is obtained from the NP equations by keeping B terms only to first order.

In the Schwarzschild case, this program has been carried out by Price (1972), and extended by Bardeen and Press (1973). The most important result of this approach is a decoupled equation for each of two components of the Weyl tensor, $\psi_0{}^B$ and $\psi_4{}^B$. As mentioned, it turns out that each of these quantities alone contains complete information about all non-trivial perturbations.

The Schwarzschild and Kerr metrics are very similar from the NP point of view. This similarity allows us, in this paper, to derive decoupled Kerr-metric equations for $\psi_0{}^B$ and $\psi_4{}^B$. Moreover, we shall demonstrate the unexpected result that these equations, like those for Schwarzschild, are separable.

Some of the results in this paper have been reported without proof in a short letter (Teukolsky 1972). The purpose of this paper is to present the results in greater detail, with full derivations, and in a form which will lay the foundation for the applications to be discussed in subsequent papers of this series.

The plan of the paper is as follows: In §II the decoupled gravitational perturbation equations are derived using the NP formalism. Section III derives decoupled equations for electromagnetic test fields. In §IV the equations are separated and written as a single master equation. Section V discusses the physical boundary conditions associated with the equations, and how to calculate the energy flux and polarization of gravitational and electromagnetic waves. Section VI previews applications of the equations to astrophysical problems. Appendix B treats the neutrino equation in the Kerr background. Readers unfamiliar with the NP formalism may skip §II and §III and treat the first few equations of §IV as definitions of the NP quantities in terms of more familiar tensor quantities.

For reference, we give the definitions of the NP quantities on which attention will be focussed in this paper. The electromagnetic field is characterized by the three complex quantities

$$\phi_0 = F_{\mu\nu}\ell^\mu m^\nu , \qquad \phi_1 = \frac{1}{2} F_{\mu\nu}(\ell^\mu n^\nu + m^{*\mu} m^\nu) , \qquad \phi_2 = F_{\mu\nu} m^{*\mu} n^\nu , \tag{1.1}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor. Equivalently,

$$F_{\mu\nu} = 2\left[\phi_1(n_{[\mu}\ell_{\nu]} + m_{[\mu}\overset{*}{m}_{\nu]}) + \phi_2\, \ell_{[\mu}m_{\nu]} + \phi_0\, \overset{*}{m}_{[\mu}n_{\nu]}\right] + \text{c.c.} , \tag{1.2}$$

where square brackets on subscripts denote antisymmetrization, and where "c.c." denotes "complex conjugate of the preceding terms." The gravitational quantities of interest will be

$$\psi_0 = - C_{\alpha\beta\gamma\delta}\ell^\alpha m^\beta \ell^\gamma m^\delta , \qquad \psi_4 = - C_{\alpha\beta\gamma\delta} n^\alpha m^{*\beta} n^\gamma m^{*\delta} , \tag{1.3}$$

where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor, which is equal to the Riemann tensor in vacuum.

II. DECOUPLED GRAVITATIONAL EQUATIONS

The derivation in this section applies to any Type D vacuum background metric. (The Schwarzschild and Kerr solutions are both of this type.) Choose the $\underset{\sim}{\ell}$ and $\underset{\sim}{n}$ vectors of the unperturbed tetrad along the repeated principal null directions of the Weyl tensor. Then

$$\psi_0{}^A = \psi_1{}^A = \psi_3{}^A = \psi_4{}^A = 0$$

$$\kappa^A = \sigma^A = \nu^A = \lambda^A = 0 \quad . \tag{2.1}$$

Now consider the following three non-vacuum NP equations, taken from Pirani (1964):

$$(\delta^* - 4\alpha + \pi)\,\psi_0 - (D - 4\rho - 2\epsilon)\,\psi_1 - 3\kappa\,\psi_2 = (\delta + \pi^* - 2\alpha^* - 2\beta)\,\Phi_{00}$$
$$- (D - 2\epsilon - 2\rho^*)\,\Phi_{01} + 2\sigma\,\Phi_{10} - 2\kappa\,\Phi_{11} - \kappa^*\,\Phi_{02} \tag{2.2}$$

$$(\Delta - 4\gamma + \mu)\,\psi_0 - (\delta - 4\tau - 2\beta)\,\psi_1 - 3\sigma\,\psi_2 = (\delta + 2\pi^* - 2\beta)\,\Phi_{01}$$
$$- (D - 2\epsilon + 2\epsilon^* - \rho^*)\,\Phi_{02} - \lambda^*\,\Phi_{00} + 2\sigma\,\Phi_{11} - 2\kappa\,\Phi_{12} \tag{2.3}$$

$$(D - \rho - \rho^* - 3\epsilon + \epsilon^*)\,\sigma - (\delta - \tau + \pi^* - \alpha^* - 3\beta)\,\kappa - \psi_0 = 0\ . \tag{2.4}$$

The Ricci tensor terms on the right-hand sides of equations (2.2) and (2.3) are given by the Einstein field equations:

$$\Phi_{00} \equiv -\frac{1}{2} R_{\mu\nu}\ell^\mu \ell^\nu = 4\pi\, T_{\mu\nu}\ell^\mu\ell^\nu \equiv 4\pi\, T_{\ell\ell} \tag{2.5}$$

and so on, where $R_{\mu\nu}$ is the Ricci tensor and $T_{\mu\nu}$ the stress-energy tensor.

Since $\psi_0{}^A$, $\psi_1{}^A$, σ^A, κ^A, and all the Φ^A vanish, the perturbation equations corresponding to equations (2.2)-(2.4) are:

$$(\delta^* - 4\alpha + \pi)^A \psi_0{}^B - (D - 4\rho - 2\epsilon)^A \psi_1{}^B - 3\kappa^B \psi_2{}^A = 4\pi[(\delta + \pi^* - 2\alpha^* - 2\beta)^A T_{\ell\ell}{}^B - (D - 2\epsilon - 2\rho^*)^A T_{\ell m}{}^B] \tag{2.6}$$

$$(\Delta - 4\gamma + \mu)^A \psi_0{}^B - (\delta - 4\tau - 2\beta)^A \psi_1{}^B - 3\sigma^B \psi_2{}^A = 4\pi[(\delta + 2\pi^* - 2\beta)^A T_{\ell m}{}^B - (D - 2\epsilon + 2\epsilon^* - \rho^*)^A T_{mm}{}^B] \tag{2.7}$$

$$(D - \rho - \rho^* - 3\epsilon + \epsilon^*)^A \sigma^B - (\delta - \tau + \pi^* - \alpha^* - 3\beta)^A \kappa^B - \psi_0{}^B = 0 \ . \tag{2.8}$$

To simplify the notation, the labels A will now be dropped from all unperturbed quantities.

The background ψ_2 satisfies

$$D\,\psi_2 = 3\rho\,\psi_2 \,, \qquad \delta\,\psi_2 = 3\tau\,\psi_2 \quad . \tag{2.9}$$

Hence equation (2.8) gives

$$(D - 3\epsilon + \epsilon^* - 4\rho - \rho^*)\,\psi_2\,\sigma^B - (\delta + \pi^* - \alpha^* - 3\beta - 4\tau)\,\psi_2\,\kappa^B - \psi_0{}^B\,\psi_2 = 0 \ . \tag{2.10}$$

The key step in the derivation is to eliminate $\psi_1{}^B$ from equations (2.6) and (2.7). This is most easily effected by using the following commutation relation:

$$\Big(D - (p + 1)\,\epsilon + \epsilon^* + q\,\rho - \rho^*\Big)(\delta - p\beta + q\tau) - \Big(\delta - (p + 1)\,\beta - \alpha^* + \pi^* + q\tau\Big)(D - p\epsilon + q\rho) = 0 \ , \tag{2.11}$$

where p and q are any two constants. This relation holds in any Type D metric, where equations (2.1) hold, and can be proved using equations (NP 4.4), (NP 4.2c), (NP 4.2e), and (NP 4.2k).

Operate with $(D - 3\epsilon + \epsilon^* - 4\rho - \rho^*)$ on equation (2.7) and with $(\delta + \pi^* - \alpha^* - 3\beta - 4\tau)$ on equation (2.6) and subtract one equation from the other. The terms in $\psi_1^{\ B}$ then vanish by equation (2.11) with p = 2 and q = -4. The combination of σ^B and κ^B remaining is exactly that in equation (2.10), and so both of these quantities can be eliminated in favor of $\psi_2\,\psi_0^{\ B}$. The resulting equation is:

$$\Big[(D - 3\epsilon + \epsilon^* - 4\rho - \rho^*)(\Delta - 4\gamma + \mu) - (\delta + \pi^* - \alpha^* - 3\beta - 4\tau)(\delta^* + \pi - 4\alpha) - 3\,\psi_2\Big]\psi_0^{\ B} = 4\pi\, T_0 \quad , \tag{2.12}$$

where

$$T_0 = (\delta + \pi^* - \alpha^* - 3\beta - 4\tau)\Big[(D - 2\epsilon - 2\rho^*)\ T_{\ell m}^{\ B} - (\delta + \pi^* - 2\alpha^* - 2\beta)\ T_{\ell\ell}^{\ B}\Big] + (D - 3\epsilon + \epsilon^* - 4\rho - \rho^*)\Big[(\delta + 2\pi^* - 2\beta)\ T_{\ell m}^{\ B} - (D - 2\epsilon + 2\epsilon^* - \rho^*)\ T_{mm}^{\ B}\Big] . \tag{2.13}$$

This is the decoupled equation for $\psi_0^{\ B}$. The full set of NP equations is invariant under the interchange $\underset{\sim}{\ell} \leftrightarrow \underset{\sim}{n}$, $\underset{\sim}{m} \leftrightarrow \underset{\sim}{m}^*$ (Geroch <u>et al</u>. 1972). This symmetry is not destroyed by the choice of $\underset{\sim}{\ell}$ and $\underset{\sim}{n}$ which gave equations (2.1). We can therefore derive an equation for $\psi_4^{\ B}$ by applying this transformation to equations (2.12) and (2.13):

$$\Big[(\Delta + 3\gamma - \gamma^* + 4\mu + \mu^*)(D + 4\epsilon - \rho) - (\delta^* - \tau^* + \beta^* + 3\alpha + 4\pi)(\delta - \tau + 4\beta) - 3\psi_2\Big]\psi_4^{\ B} = 4\pi\ T_4 \quad , \tag{2.14}$$

where

$$T_4 = (\Delta + 3\gamma - \gamma^* + 4\mu + \mu^*)\Big[(\delta^* - 2\tau^* + 2\alpha)\ T_{nm^*} - (\Delta + 2\gamma - 2\gamma^* + \mu^*)\ T_{m^*m^*}\Big] + (\delta^* - \tau^* + \beta^* + 3\alpha + 4\pi)\Big[(\Delta + 2\gamma + 2\mu^*)\ T_{nm^*} - (\delta^* - \tau^* + 2\beta^* + 2\alpha)\ T_{nn}\Big] . \tag{2.15}$$

For those readers familiar with the Geroch-Held-Penrose (1972) version of the NP formalism, the derivation in this section is even simpler in that formalism. An equivalent derivation in that formalism has been given by

Stewart (1972).

Appendix A proves that $\psi_0{}^B$ and $\psi_4{}^B$ are invariant under gauge transformations and infinitesimal tetrad rotations, and are therefore completely measurable physical quantities.

III. DECOUPLED ELECTROMAGNETIC EQUATIONS

Many realistic problems involving electromagnetic interactions near uncharged black holes can be treated in the "test field" approximation. Since the amplitude of the electromagnetic stress-energy is second order in the electromagnetic field, the change in the background geometry caused by the electromagnetic perturbation is also second order. Thus in Maxwell's equations this change in the geometry can be neglected to first order.

When equations (2.1) are satisfied, Maxwell's equations are

$$(D - 2\rho)\,\phi_1 - (\delta^* + \pi - 2\alpha)\,\phi_0 = 2\pi\, J_\ell \tag{3.1}$$

$$(\delta - 2\tau)\,\phi_1 - (\Delta + \mu - 2\gamma)\,\phi_0 = 2\pi\, J_m \tag{3.2}$$

$$(D - \rho + 2\epsilon)\,\phi_2 - (\delta^* + 2\pi)\,\phi_1 = 2\pi\, J_{m^*} \tag{3.3}$$

$$(\delta - \tau + 2\beta)\,\phi_2 - (\Delta + 2\mu)\,\phi_1 = 2\pi\, J_n \quad , \tag{3.4}$$

where the ϕ's are the first-order test fields and $J_\ell = J_\mu \ell^\mu$, etc. with J_μ the 4-current density.

Operate on equation (3.1) with $(\delta - \beta - \alpha^* - 2\tau + \pi^*)$ and on equation (3.2) with $(D - \epsilon + \epsilon^* - 2\rho - \rho^*)$ and subtract one equation from the other. The identity (2.11) with $p = 0$ and $q = -2$ shows that the terms in ϕ_1 disappear, leaving a decoupled equation for ϕ_0:

$$\Big[(D - \epsilon + \epsilon^* - 2\rho - \rho^*)(\Delta + \mu - 2\gamma) - (\delta - \beta - \alpha^* - 2\tau + \pi^*)(\delta^* + \pi - 2\alpha)\Big]\phi_0 = 2\pi\, J_0 \quad , \tag{3.5}$$

$$J_0 = (\delta - \beta - \alpha^* - 2\tau + \pi^*)\ J_\ell - (D - \epsilon + \epsilon^* - 2\rho - \rho^*)\ J_m \quad . \tag{3.6}$$

By interchanging $\underset{\sim}{\ell}$ and $\underset{\sim}{n}$, and $\underset{\sim}{m}$ and $\underset{\sim}{m}^*$, we obtain the equation for ϕ_2 [which is also derivable directly from eqs. (3.3) and (3.4)]:

$$\Big[(\Delta + \gamma - \gamma^* + 2\mu + \mu^*)(D - \rho + 2\epsilon) - (\delta^* + \alpha + \beta^* + 2\pi - \tau^*)(\delta - \tau + 2\beta)\Big]\phi_2 = 2\pi\ J_2 \quad , \tag{3.7}$$

$$J_2 = (\Delta + \gamma - \gamma^* + 2\mu + \mu^*)\ J_{m^*} - (\delta^* + \alpha + \beta^* + 2\pi - \tau^*)\ J_n \quad . \tag{3.8}$$

Fackerell and Ipser (1972) derived an analogous decoupled equation for ϕ_1, but this equation does not appear to be separable in the Kerr case.

IV. SEPARATION OF THE EQUATIONS

The next step is to write out the equations in a particular coordinate system. In Boyer-Lindquist (1967) coordinates, and in units such that $c = G = 1$, the Kerr metric is

$$ds^2 = (1 - 2Mr/\Sigma)\ dt^2 + (4Mar\ \sin^2(\Theta)/\Sigma)dtd\varphi - (\Sigma/\Delta)\ dr^2 - \Sigma d\Theta^2 - \sin^2(\Theta)(r^2 + a^2 + 2Ma^2 r\ \sin^2(\Theta)/\Sigma)\ d\varphi^2 \quad . \tag{4.1}$$

Here M is the mass of the black hole, aM its angular momentum, $\Sigma = r^2 + a^2\cos^2\Theta$, and[1] $\Delta = r^2 - 2Mr + a^2$. When $a = 0$, the metric reduces to the Schwarzschild metric, a non-rotating black hole.

[1] In §§II and III, Δ denoted the NP operator $n^\mu\ \partial/\partial x^\mu$. In the remainder of the paper Δ is used in its other conventional sense, to denote the function $r^2 - 2Mr + a^2$.

Any NP tetrad must satisfy the following orthogonality relations:

$$\underset{\sim}{\ell} \cdot \underset{\sim}{n} = 1 \,, \quad \underset{\sim}{m} \cdot \underset{\sim}{m}^* = -1 \,, \quad \text{all other dot products zero,} \tag{4.2}$$

so the metric is

$$g^{\mu\nu} = \ell^\mu n^\nu + n^\mu \ell^\nu - m^\mu m^{*\nu} - m^{*\mu} m^\nu \quad . \tag{4.3}$$

The relations (4.2) are preserved under the 6-parameter group of Lorentz transformations at each point of spacetime. A convenient decomposition of these six degrees of freedom is described in Appendix A. Choosing the directions of $\underset{\sim}{\ell}$ and $\underset{\sim}{n}$ so that equations (2.1) hold uses up four degrees of freedom (eqs. A1 and A2). We choose to follow Kinnersley (1969) and use up the remaining freedom by making a "null rotation" (eq. A3) to set the spin coefficient $\epsilon = 0$. The resulting tetrad has $[t, r, \theta, \varphi]$ components:

$$\ell^\mu = [(r^2 + a^2)/\Delta, \ 1, \ 0, \ a/\Delta] \,, \qquad n^\mu = [r^2 + a^2, \ -\Delta, \ 0, \ a]/2\Sigma \,,$$
$$m^\mu = [ia \sin\theta, \ 0, \ 1, \ i/\sin\theta]/2^{1/2}(r + ia\cos\theta) \quad . \tag{4.4}$$

The non-vanishing spin coefficients are

$$\rho = -1/(r - ia\cos\theta), \quad \beta = -\rho^* \cot\theta/2\sqrt{2}, \quad \pi = ia\rho^2 \sin\theta/\sqrt{2} \,,$$
$$\tau = -ia\rho\rho^* \sin\theta/\sqrt{2}, \quad \mu = \rho^2\rho^* \Delta/2, \quad \gamma = \mu + \rho\rho^*(r - M)/2, \quad \alpha = \pi - \beta^* \,, \tag{4.5}$$

while

$$\Psi_2 = M\rho^3 \quad . \tag{4.6}$$

We use these expressions, and the fact that[1] $D = \ell^\mu \partial/\partial x^\mu$, $\Delta = n^\mu \partial/\partial x^\mu$, and $\delta = m^\mu \partial/\partial x^\mu$, to write equations (2.12), (2.14), (3.5), and (3.7) as a single master equation — valid equally well for a test scalar field in the Kerr background ($s = 0$, not derived here), a test neutrino field ($s = \pm 1/2$, derived in Appendix B), a test electromagnetic field ($s = \pm 1$, derived in §III), or a gravitational perturbation ($s = \pm 2$, derived in §II):

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2 \sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\varphi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\varphi^2} - \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right)$$

$$- \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\varphi} - 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t}$$

$$+ (s^2\cot^2\theta - s)\,\psi = 4\pi\Sigma T \quad . \qquad (4.7)$$

Here $\underline{s}$ is a parameter called the "spin weight" of the field. Table 1 specifies the field quantities ψ which satisfy this equation, the corresponding values of s, and the source terms T.

Consider first the vacuum case ($T = 0$). Then the master equation (4.7) can be separated by writing

$$\psi = e^{-i\omega t}\, e^{im\varphi}\, S(\theta)\, R(r) \quad . \qquad (4.8)$$

The equations for R and S are

$$\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{dR}{dr}\right) + \left(\frac{K^2 - 2is(r-M)K}{\Delta} + 4is\omega r - \lambda\right) R = 0 \quad , \qquad (4.9)$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) + \left(a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta} - 2a\omega s\cos\theta - \frac{2ms\cos\theta}{\sin^2\theta}\right.$$
$$\left. - s^2\cot^2\theta + s + A\right) S = 0 \quad , \qquad (4.10)$$

where $K \equiv (r^2 + a^2)\,\omega - am$ and $\lambda \equiv A + a^2\omega^2 - 2am\omega$. Equation (4.10), together with boundary conditions of regularity at $\theta = 0$ and π, constitutes a Sturm-Liouville eigenvalue problem for the separation constant $A = {}_sA^m_\ell(a\omega)$. For fixed s, m, and $a\omega$, we label the eigenvalues by ℓ. The smallest eigenvalue has $\ell = \max(|m|, |s|)$. From Sturm-Liouville theory, the eigenfunctions ${}_sS^m{}_\ell$ are complete and orthogonal on $0 \leq \theta \leq \pi$ for each m, s, and $a\omega$. When $s = 0$, the eigenfunctions are the spheroidal wave functions $S^m{}_\ell(-a^2\omega^2, \cos\theta)$ (cf. Flammer 1957). When $a\omega = 0$, the eigenfunctions are the spin-weighted spherical harmonics ${}_sY^m{}_\ell = {}_sS^m{}_\ell(\theta)e^{im\varphi}$, and $A = (\ell - s)(\ell + s + 1)$ (cf. Goldberg et al. 1967). In the general case, we shall refer to the eigenfunctions as "spin-weighted spheroidal harmonics." The numerical calculation of these functions and the corresponding eigenvaules is described in Paper II of this series.

When sources are present ($T \neq 0$), we can use the eigenfunctions of equation (4.10) to separate equation (4.7) by expanding

$$4\pi\Sigma T = \int d\omega \sum_{\ell,m} G(r)\ {}_sS^m{}_\ell(\theta)\ e^{im\varphi}\ e^{-i\omega t} \quad ,$$

$$\psi = \int d\omega \sum_{\ell,m} R(r)\ {}_sS^m{}_\ell(\theta)\ e^{im\varphi}\ e^{-i\omega t} \quad . \tag{4.11}$$

Then $R(r)$ satisfies equation (4.9) with $G(r)$ as source term on the right-hand side.

Equation (4.7) is also separable in Kerr coordinates (cf. eq. 5.7), or any other coordinates related to Boyer-Lindquist by $\bar{t} = t + f_1(r) + f_2(\theta)$, $\bar{\varphi} = \varphi + g_1(r) + g_2(\theta)$, $\bar{r} = h(r)$, $\bar{\theta} = j(\theta)$.

The reason for the factors ρ^{-2} and ρ^{-4} in front of ϕ_2 and $\psi_4{}^B$ to achieve separable equations (cf. Table 1) is related to the null rotation used to set $\epsilon = 0$. Had we made some other choice, there would in general be different factors in front of each of ϕ_0, ϕ_2, $\psi_0{}^B$, and $\psi_4{}^B$, but the master perturbation equation (4.7) would be left unchanged. (See Appendix A for the transformation properties of these quantities under null rotations.)

V. BOUNDARY CONDITIONS, ENERGY, AND POLARIZATION

To discuss the boundary conditions for the separated radial equation (4.9), it is useful to make the transformation

$$Y = \Delta^{s/2}(r^2 + a^2)^{1/2} R , \quad dr^*/dr = (r^2 + a^2)/\Delta \quad . \tag{5.1}$$

Then

$$Y_{,r^*r^*} + \left\{ [K^2 - 2is(r - M) K + \Delta(4ir\omega s - \lambda)]/(r^2 + a^2)^2 - G^2 - G_{,r^*} \right\} Y = 0, \tag{5.2}$$

where $G = s(r - M)/(r^2 + a^2) + r\Delta/(r^2 + a^2)^2$ and a comma denotes partial differentiation. At $r \to \infty$ ($r^* \to \infty$), equation (5.2) becomes

$$Y_{,r^*r^*} + (\omega^2 + 2i\omega s/r) Y \approx 0 \quad , \tag{5.3}$$

with asymptotic solutions $Y \sim r^{\pm s} e^{\mp i\omega r^*}$, i.e., $R \sim e^{-i\omega r^*}/r$ and $e^{i\omega r^*}/r^{(2s+1)}$.

This corresponds to

$$\Phi, \phi_2, \psi_4^B \sim e^{i\omega r^*}/r , \quad \phi_0 \sim e^{i\omega r^*}/r^3 , \quad \psi_0^B \sim e^{i\omega r^*}/r^5 \text{ (outgoing waves);}$$

$$\Phi, \phi_0, \psi_0^B \sim e^{-i\omega r^*}/r , \quad \phi_2 \sim e^{-i\omega r^*}/r^3 , \quad \psi_4^B \sim e^{-i\omega r^*}/r^5 \text{ (ingoing waves).} \tag{5.4}$$

The different power-law fall-offs are dictated by the "peeling theorem" (cf. Newman and Penrose 1962). They necessitate special care in numerical integration of the equations to avoid losing the small solution in the roundoff error of the large solution. Such an integration is described in Paper II.

The event horizon is at $r = r_+$ ($r^* \to -\infty$), the larger root of $\Delta = 0$. Near the event horizon the transformed radial equation (5.2) becomes

$$Y_{,r^*r^*} + [k^2 - 2is(r_+ - M) k/(2Mr_+) - s^2(r_+ - M)^2/(2Mr_+)^2] Y \approx 0 \quad , \tag{5.5}$$

where $k = \omega - m\omega_+$, $\omega_+ = a/(2Mr_+)$. The asymptotic solutions are

$$Y \sim e^{\pm i[k - is(r_+ - M)/(2Mr_+)]r^*} \sim \Delta^{\pm s/2}\, e^{\pm ikr^*} \;,$$

$$\text{i.e.,}\; R \sim e^{ikr^*} \;\text{or}\; R \sim \Delta^{-s}\, e^{-ikr^*} \;. \qquad (5.6)$$

The correct boundary condition at the horizon[2] can be formulated in a number of equivalent ways. For example, way (i): require that a physically well-behaved observer at the horizon see non-special fields. (Non-special means neither singular nor identically zero.) Equivalently, way (ii): demand that the radial group velocity of a wave packet, as measured by a physically well-behaved observer, be negative (i.e., signals can travel into the hole, but cannot come out).

Every physical observer with 4-velocity $\underset{\sim}{u}$ has associated with him an orthonormal tetrad, his local rest-frame with basis vectors $\{\underset{\sim}{e}_{\hat{t}} = \underset{\sim}{u}, \underset{\sim}{e}_{\hat{r}}, \underset{\sim}{e}_{\hat{\theta}}, \underset{\sim}{e}_{\hat{\varphi}}\}$. Corresponding to this is a null tetrad: $\underset{\sim}{\ell} = \underset{\sim}{e}_{\hat{t}} - \underset{\sim}{e}_{\hat{r}}$, $\underset{\sim}{n} = (\underset{\sim}{e}_{\hat{t}} + \underset{\sim}{e}_{\hat{r}})/2$, $\underset{\sim}{m} = (\underset{\sim}{e}_{\hat{\theta}} + i\underset{\sim}{e}_{\hat{\varphi}})/2^{1/2}$. Conversely, given a non-singular null tetrad, there is a corresponding physical observer. Thus condition (i) can be reformulated as: NP field quantities on the horizon should be non-special for non-singular null tetrads.

To examine the tetrad (4.4) on the horizon, we cannot use Boyer-Lindquist coordinates since they themselves are singular on the horizon. Hence, we transform to Kerr "ingoing" coordinates (cf. Misner, Thorne, and Wheeler 1973):

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[2] We discuss here only the future horizon; the past horizon need not even exist if the black hole was formed by collapse.

$$dv = dt + dr^*$$

$$d\tilde{\varphi} = d\varphi + a(r^2 + a^2)^{-1}\, dr^* \quad . \tag{5.7}$$

The tetrad (4.4) is still singular at $\Delta = 0$ when expressed in these well-behaved coordinates, but if we perform a null rotation with $\Lambda = \Delta/2(r^2 + a^2)$ (cf. Appendix A), the resulting tetrad has $[v,\ r,\ \Theta,\ \tilde{\varphi}]$ components

$$\ell^\mu = [\,1,\ \Delta/2(r^2 + a^2),\ 0,\ a/(r^2 + a^2)\,]\ , \quad n^\mu = [\,0,\ -(r^2 + a^2)/\Sigma,\ 0,\ 0\,]$$

$$m^\mu = [\,ia\ \sin\Theta,\ 0,\ 1,\ i/\sin\Theta\,]/2^{1/2}(r + ia\ \cos\Theta)\ , \tag{5.8}$$

which show that it is well-behaved at $\Delta = 0$. Under this null rotation, the NP quantities of interest transform as follows (cf. Appendix A):

$$\psi \to \psi^{\text{New}} = [\Delta/2(r^2 + a^2)]^s\ \psi \quad . \tag{5.9}$$

On the horizon, the asymptotic solutions (5.6) have the forms $\psi^{\text{New}} \sim e^{-i\omega t}\, e^{im\varphi}\, e^{-ikr^*}$ and $e^{-i\omega t}\, e^{im\varphi}\, e^{ikr^*}\, \Delta^s$. Clearly the first solution is the non-special one, as can be seen by writing it in the form $e^{-i\omega v}\, e^{im\tilde{\varphi}}$. The correct boundary condition is therefore

$$R \sim \Delta^{-s}\, e^{-ikr^*} \quad . \tag{5.10}$$

The group and phase velocities of this solution are

$$v_{\text{group}} = -\,dk/d\omega = -\,1\ , \quad v_{\text{phase}} = -\,k/\omega = -\,1 + m\omega_+/\omega\ . \tag{5.11}$$

The group velocity agrees with condition (ii) above. Note that if $m\omega_+/\omega > 1$, then v_{phase} is positive. It turns out that the energy flow down the hole, while always inward as seen locally, is determined by v_{phase} for an observer at infinity. If $m\omega_+/\omega > 1$, energy flows out of the hole and the corresponding scattering wave mode is amplified, or "super-radiantly scattered" [cf. Press and Teukolsky (1972), Misner (1972), and Zeldovich (1972)]. A detailed discussion of electromagnetic and gravitational super-radiance, including numerical values, will be given in a later paper in this series.

Turn now to the problem of extracting information from solutions of the perturbation equations. For scalar and electromagnetic fields, there is a well-defined energy-momentum tensor at every point of spacetime:

$$4\pi\, T_{\mu\nu}^{(\text{scalar})} = \Phi_{;\mu}\Phi_{;\nu} - \frac{1}{2} g_{\mu\nu}\Phi_{;\alpha}\Phi^{;\alpha}$$

$$\begin{aligned} 4\pi T_{\mu\nu}^{(\text{em})} = \Big\{ & \phi_0\phi_0{}^* n_\mu n_\nu + 2\phi_1\phi_1{}^*[\ell_{(\mu}n_{\nu)} + m_{(\mu}m^*{}_{\nu)}] + \phi_2\phi_2{}^*\ell_\mu \ell_\nu \\ & - 4\phi_0{}^*\phi_1 n_{(\mu}m_{\nu)} - 4\phi_1{}^*\phi_2\ell_{(\mu}m_{\nu)} + 2\phi_2\phi_0{}^* m_\mu m_\nu \Big\} + \text{c.c.} \quad , \qquad (5.12) \end{aligned}$$

where round brackets on subscripts denote symmetrization. Note that when one has solved equation (4.7) for ϕ_2, say, ϕ_1 and ϕ_0 can be found from equations (3.1)-(3.4) which are then integrable Pfaffian equations in r and θ (cf. Fackerell and Ipser 1972). The only arbitrariness in the solution is the freedom to add $Q\rho^2$ to ϕ_1, which corresponds to adding a constant charge Q to the hole.

Often one is interested only in the energy carried off by outgoing waves at infinity. Using equations (5.4) and (5.12), we find that the total energy flux per unit solid angle can be found from ϕ_2 alone:

$$\frac{d^2E}{dt\,d\Omega} = \lim_{r\to\infty} r^2\, T^r{}_t = \lim_{r\to\infty} \frac{r^2}{2\pi}\, |\phi_2|^2 \quad . \qquad (5.13)$$

For outgoing waves at infinity, the components of the electric and magnetic fields satisfy $E_{\hat\theta} = B_{\hat\phi}$, $E_{\hat\phi} = -B_{\hat\theta}$, so from equation (1.1) we find $\phi_2 \propto E_{\hat\theta} - iE_{\hat\phi}$. Thus the squares of the real and imaginary parts of ϕ_2 are proportional to the amounts of energy in the two linear polarization states along the directions $\underset{\sim}{e}_{\hat{\theta}}$ and $\underset{\sim}{e}_{\hat{\varphi}}$ respectively.

For gravitational waves, one could in principle proceed as follows: Having solved equation (4.7) for $\psi_4{}^B$, say, solve the complete set of (non-separable) NP equations for the perturbations in the metric. Then use the Isaacson (1968) stress-energy tensor to determine the energy-momentum flux at any point in spacetime. Unfortunately the equations are so complicated that this is an impractical task. One can, however, find the energy flux in the two most important cases: at infinity and on the horizon.

At infinity, one can use the standard equations of linearized theory (cf. Misner, Thorne, and Wheeler 1973) to find the energy flux. For outgoing waves with frequency ω,

$$\psi_4{}^B = -\left(R^B{}_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} - i\, R^B{}_{\hat{t}\hat{\theta}\hat{t}\hat{\varphi}}\right) = -\,\omega^2\left(h^B{}_{\hat{\theta}\hat{\theta}} - i\, h^B{}_{\hat{\theta}\hat{\varphi}}\right)/2 \quad .$$

Therefore

$$\frac{d^2E^{(\text{out})}}{dt\,d\Omega} = \lim_{r\to\infty} \frac{r^2\omega^2}{16\pi}\left[\left(h^B{}_{\hat{\theta}\hat{\theta}}\right)^2 + \left(h^B{}_{\hat{\theta}\hat{\varphi}}\right)^2\right] = \lim_{r\to\infty} \frac{r^2}{4\pi\omega^2}\,|\psi_4{}^B|^2 \quad . \tag{5.14}$$

The squares of the real and imaginary parts of $\psi_4{}^B$ are proportional to the amounts of energy in the linear polarization states along $\underset{\sim}{e}_{\hat{\theta}}$ and $\underset{\sim}{e}_{\hat{\varphi}}$, and $\underset{\sim}{e}_{\hat{\theta}} \pm \underset{\sim}{e}_{\hat{\varphi}}$ respectively. Similar results hold for $\psi_0{}^B$ and ingoing waves:

$$\frac{d^2E^{(\text{in})}}{dt\,d\Omega} = \lim_{r\to\infty} \frac{r^2}{64\pi\omega^2}\,|\psi_0{}^B|^2 \quad . \tag{5.15}$$

The extra factor of 1/16 comes from the 1/2 in the definition of $\underset{\sim}{n}$ as opposed to $\underset{\sim}{\ell}$.

Some problems require one to be able to find the ingoing energy at infinity from $\psi_4{}^B$ (or the outgoing energy from $\psi_0{}^B$). The method for doing this will be given in Paper II.

To calculate the gravitational wave energy flux on the horizon, one can use the results of Hartle and Hawking (1972). From $\psi_0^{\,B}$ on the horizon one can find the shear σ^B of the horizon. The shear gives the rate of change of the area of the horizon, dA/dt. The quantity dA/dt contains two terms: dM/dt and da/dt. (See Hartle and Hawking 1972 for details.) In our case, $da/dt = m(\omega M)^{-1}\, dM/dt$, thus enabling us to find both dM/dt and da/dt from $\psi_0^{\,B}$ on the horizon.

For a stationary, nonaxisymmetric perturbation ($\omega = 0$, $dM/dt = 0$, $m \neq 0$, $da/dt \neq 0$), the radial wave equation (4.9) can be solved in terms of hypergeometric functions. This enables one to calculate the spin-down (loss of angular momentum) of a rotating black hole caused by such a perturbation. [See analyses by Press (1972) (scalar perturbation) and Hartle (1973) (gravitational perturbation with $a \ll M$).] The calculation for arbitrary a will be published in a later paper in this series.

VI. DISCUSSION

The important result presented in this paper is that there exists a tractable method of treating perturbations of a rotating black hole. One has to solve a relatively simple ordinary differential equation, the radial wave equation (4.9), subject to boundary conditions described in §V. The solution lends itself to direct physical interpretation, and can be related at infinity to the energy flux of gravitational or electromagnetic waves. A subsequent paper in this series will discuss and apply to this work the stronger result [due to Fackerell and Ipser (1972) for the electromagnetic case, and due to Wald (1973) for the gravitational case] that the solution of equation (4.9) in fact determines all non-trivial details of the full perturbation, at all radii outside the horizon.

Later papers in this series will deal primarily with applications of the equations in astrophysical contexts, including the dynamical stability of the Kerr metric (Paper II), the super-radiant scattering of electromagnetic and gravitational waves by an astrophysical black hole, the spin-down of an arbitrarily rotating hole which is perturbed non-axisymmetrically by a distant massive object, and calculations of the gravitational waves emitted by accretion processes.

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APPENDIX A

The 6-parameter group of homogeneous Lorentz transformations, which preserves the tetrad orthogonality relations (4.2), can be decomposed into three abelian subgroups (Janis and Newman 1965):

$$\begin{aligned}
\text{(i)}\quad & \underset{\sim}{\ell} \to \underset{\sim}{\ell} \\
& \underset{\sim}{m} \to \underset{\sim}{m} + d\underset{\sim}{\ell} \\
& \underset{\sim}{n} \to \underset{\sim}{n} + d\underset{\sim}{m}^* + d^*\underset{\sim}{m} + dd^*\underset{\sim}{\ell} \quad ,
\end{aligned} \tag{A1}$$

$$\begin{aligned}
\text{(ii)}\quad & \underset{\sim}{n} \to \underset{\sim}{n} \\
& \underset{\sim}{m} \to \underset{\sim}{m} + e\underset{\sim}{n} \\
& \underset{\sim}{\ell} \to \underset{\sim}{\ell} + e\underset{\sim}{m}^* + e^*\underset{\sim}{m} + ee^*\underset{\sim}{n} \quad ,
\end{aligned} \tag{A2}$$

$$\begin{aligned}
\text{(iii)}\quad & \underset{\sim}{\ell} \to \Lambda\underset{\sim}{\ell} \\
& \underset{\sim}{n} \to \Lambda^{-1}\underset{\sim}{n} \\
& \underset{\sim}{m} \to e^{i\Theta}\underset{\sim}{m} \quad ,
\end{aligned} \tag{A3}$$

where d and e are complex numbers and Λ and Θ are real. Under transformations of type (i),

$$\begin{aligned}
& \Psi_0 \to \Psi_0 \,, \quad \Psi_4 \to \Psi_4 + 4d^*\,\Psi_3 + 6d^{*2}\,\Psi_2 + 4d^{*3}\,\Psi_1 + d^{*4}\,\Psi_0 \quad , \\
& \phi_0 \to \phi_0 \,, \quad \phi_2 \to \phi_2 + 2d^*\,\phi_1 + d^{*2}\,\phi_0 \quad .
\end{aligned} \tag{A4}$$

For type (ii),

$$\begin{aligned}
& \Psi_0 \to \Psi_0 + 4e\Psi_1 + 6e^2\,\Psi_2 + 4e^3\,\Psi_3 + e^4\,\Psi_4 \,, \quad \Psi_4 \to \Psi_4 \quad , \\
& \phi_0 \to \phi_0 + 2e\phi_1 + e^2\phi_2 \,, \quad \phi_2 \to \phi_2 \quad .
\end{aligned} \tag{A5}$$

For type (iii),

$$\psi_0 \to \Lambda^2 e^{2i\Theta} \psi_0 , \quad \psi_4 \to \Lambda^{-2} e^{-2i\Theta} \psi_4 ,$$

$$\phi_0 \to \Lambda e^{i\Theta} \phi_0 , \quad \phi_2 \to \Lambda^{-1} e^{-i\Theta} \phi_2 . \tag{A6}$$

The above relations can be used to prove that $\psi_0{}^B$ and $\psi_4{}^B$ are invariant under infinitesimal tetrad transformations; for, suppose d, e, $\Lambda - 1$, and Θ are infinitesimal. Then

$$\psi_0{}^B \to \psi_0{}^B , \quad \psi_4{}^B \to \psi_4{}^B + 4d^* \psi_3{}^A \qquad [\text{type (i)}] ,$$

$$\psi_0{}^B \to \psi_0{}^B + 4e\, \psi_1{}^A , \quad \psi_4{}^B \to \psi_4{}^B \qquad [\text{type (ii)}] , \tag{A7}$$

$$\psi_0{}^B \to \psi_0{}^B + 2[(\Lambda - 1) + i\Theta]\, \psi_0{}^A , \; \psi_4{}^B \to \psi_4{}^B - 2[(\Lambda - 1) + i\Theta]\psi_4{}^A \; [\text{type (iii)}].$$

Since $\psi_0{}^A = \psi_1{}^A = \psi_3{}^A = \psi_4{}^A = 0$, $\psi_0{}^B$ and $\psi_4{}^B$ are invariant.

The quantities $\psi_0{}^B$ and $\psi_4{}^B$ are also invariant under gauge transformations (i.e., infinitesimal changes of coordinates which leave the tetrad unchanged at each point of spacetime). Locally, these transformations are the inhomogeneous part of the Lorentz group:

$$x^\mu \to x^\mu + \xi^\mu \tag{A8}$$

where ξ^μ is infinitesimal. Since the ψ's are scalars, they change as a function of coordinate location by

$$\psi \to \psi - \psi_{,\mu}\, \xi^\mu . \tag{A9}$$

Therefore

$$\begin{aligned} \psi^B &\to \psi^B - \psi^A{}_{,\mu}\, \xi^\mu \\ &= \psi^B , \end{aligned} \tag{A10}$$

since $\psi_4^{\;B} = \psi_0^{\;B} = 0$.

APPENDIX B

In this Appendix we shall show that the neutrino equation, in two-component form, also leads to a separable wave equation. We shall not give any discussion of the source terms here, nor of the physical interpretation of the solutions. For these, the interested reader may refer to Hartle (1970), and Wainwright (1971) and references therein.

The sourceless neutrino equation (no coupling to electrons or muons) is

$$\nabla^{AA'} \Phi_A = 0 \quad . \tag{B1}$$

Where Φ_A is a two-component spinor. (Our notation follows Pirani 1964.) This equation can be written in NP form by letting χ_0 and χ_1 denote the components of Φ_A along the dyad legs $\underline{o}$ and $\underline{\iota}$ respectively. Then

$$(\delta^* - \alpha + \pi)\chi_0 = (D - \rho + \epsilon)\chi_1 \tag{B2}$$

$$(\Delta + \mu - \gamma)\chi_0 = (\delta + \beta - \tau)\chi_1 \quad . \tag{B3}$$

Now consider Φ_A as a test field on the Kerr background. Operate on equation (B3) with $(D + \epsilon^* - \rho - \rho^*)$ and on equation (B2) with $(\delta - \alpha^* - \tau + \pi^*)$ and subtract one equation from the other. The identity (2.11) with $p = -1$ and $q = -1$ shows that the terms in χ_1 disappear, leaving

$$[(D + \epsilon^* - \rho - \rho^*)(\Delta - \gamma + \mu) - (\delta - \alpha^* - \tau + \pi^*)(\delta^* - \alpha + \pi)]\chi_0 = 0 \ . \tag{B4}$$

The interchange $\underset{\sim}{\ell} \leftrightarrow \underset{\sim}{n}$, $\underset{\sim}{m} \leftrightarrow \underset{\sim}{m}^*$ gives

$$[(\Delta - \gamma^* + \mu + \mu^*)(D + \epsilon - \rho) - (\delta^* + \beta^* + \pi - \tau^*)(\delta + \beta - \tau)]\chi_1 = 0 \ . \tag{B5}$$

When written out in Boyer-Lindquist coordinates, these equations are of the same form as the master equation (4.7), with $\psi = \chi_0$ ($s = 1/2$) and $\psi = \rho^{-1} \chi_1$ ($s = -1/2$).

TABLE 1

FIELD QUANTITIES ψ, SPIN-WEIGHT s, AND SOURCE TERMS T FOR EQUATION (4.7)

ψ	s	T
Φ	0	$\Box\Phi = 4\pi T$
χ_0	$\frac{1}{2}$	See references in Appendix B
$\rho^{-1}\chi_1$	$-\frac{1}{2}$	
ϕ_0	1	J_0 (eq. 3.6)
$\rho^{-2}\phi_2$	-1	$\rho^{-2}J_2$ (eq. 3.8)
$\psi_0{}^B$	2	$2T_0$ (eq. 2.13)
$\rho^{-4}\psi_4{}^B$	-2	$2\rho^{-4}T_4$ (eq. 2.15)

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PART V

STABILITY OF ROTATING BLACK HOLES

(a) Perturbations of a Rotating Black Hole. II. Dynamical Stability of the Kerr Metric (Paper V; collaboration with W. H. Press, to be published in Ap. J., October 1973.

PERTURBATIONS OF A ROTATING BLACK HOLE
II. DYNAMICAL STABILITY OF THE KERR METRIC*

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ABSTRACT

If unstable, a rotating black hole would spontaneously radiate gravitational waves and evolve dynamically to some new (unknown) final state. This paper tests the dynamical stability of rotating holes by numerical integration of the separable perturbation equations for the Kerr metric. No instabilities are found in any of the dozen or so lowest angular modes tested, for any value of specific angular momentum $0 \leq a < M$. Even in the limit $a \to M$, the hole appears to be stable. These stability results add credibility to the use of the Kerr metric in detailed astrophysical models. A numerical technique for preserving accuracy on an asymptotically small solution to an O.D.E. in the presence of an asymptotically large one is described.

I. INTRODUCTION

The Schwarzschild (nonrotating) black hole has been shown by Vishveshwara (1970) to be dynamically stable against small perturbations; and more recent mathematical advances by Zerilli (1970) allow the proof of Schwarzshild stability to be put forth in about two sentences (see §II below). By contrast, almost nothing has been known about the stability of rotating black holes, described by the Kerr metric (of which Schwarzschild is a special case). The question of Kerr stability is interesting in principle of course, and it is also vitally important astrophysically: black holes formed from a collapsed star are likely to be highly rotating (Bardeen 1970; Thorne 1973). If a Kerr metric were unstable, it could not be the endpoint of a dynamical collapse, and the whole question of black-hole uniqueness (cf. Carter 1973) and inevitability would have to be reexamined in quite a new light.

What do we mean by stability? At time $t = 0$ (constant time hypersurfaces are well-defined because the metric is stationary) we imagine a Kerr black hole with a small gravitational perturbation, infinitesimal in amplitude. Physically, this perturbation may have been caused by a particle flying by or falling in; we take the perturbation as arbitrarily given, however, and do not consider its precise coupling to material sources. Since the perturbation is small, its time evolution is accurately treated by linearized perturbation equations, specifically the separable perturbation equations of the accompanying paper (Teukolsky 1973, hereafter cited as Paper I). If the perturbation radiates away down the hole and to infinity, remaining small and well-behaved all the while, and finally goes to zero, then the linearized equations will have described the whole process accurately, and we can conclude that the black hole is stable against small perturbations.

On the other hand, the linearized equations might predict that the perturbation grows in time without bound, and that the black hole never returns to its quiescent state. In this case, the black hole is unstable. The linearized equations fail when the perturbations become large, so we cannot determine what the hole is evolving to, but we can be certain that it does not return to its original configuration: the linearized equations <u>are</u> valid in a neighborhood of that configuration.

What would be possible final states of an unstable Kerr black hole (if any were unstable)? First, since Schwarzschild is stable, it is not difficult to show that all Kerr holes in some neighborhood of Schwarzschild ("slowly rotating" holes) are also stable (Press 1972). Thus, an unstable Kerr hole might radiate away mass and angular momentum in a burst of gravitational radiation, until it settles down to become a Kerr hole somewhere in the stable region.

Second, the hole might become highly dynamical for a finite period of time, and finally settle down to a new non-Kerr stationary, axisymmetric configuration with a horizon and with no naked singularities (a new type of hole). Carter (1971) has proved that such configurations must occur in two-parameter families (like the Kerr family), and that they must be disjoint from the Kerr family. No such disjoint families are presently known; it is generally suspected that none exist.

A third possibility would be that a sequence of nonaxisymmetric holes bifurcates from the Kerr sequence at some finite, specific angular momentum, and that some or all Kerr holes with greater specific angular momentum are unstable against migrating dynamically to the new sequence. This picture would be the analog of the situation for classical fluid ellipsoids (see Chandrasekhar 1969) where — when there is any dissipation — stability passes from the axisymmetric Maclaurin to the nonaxisymmetric Jacobi sequence at their point of bifurcation. The similarities between fluid ellipsoids and black holes, and the possibility of similar instabilities, have been emphasized by Smarr (1973). By a theorem of Hawking (1972), any nonaxisymmetric sequence must be dynamic in its own right, but this does not mean that it cannot be stable: a hole on the new sequence will have a definite trajectory of time evolution, and this time evolution can be stable against small perturbations.

A fourth possibility would be the least pleasant. The hypothesized unstable black hole might evolve to a naked singularity, visible or asymptotically visible from asymptotically flat infinity. One hopes that naked singularities will at some future time be ruled out by something more general than a case-by-case analysis (Penrose's "Cosmic Censorship Hypothesis"), but at present they are not.

We should note that the stability problem is also astrophysically important in connection with possible sources of gravitational waves. For a small mass m falling into a larger black hole of mass M, perturbation calculations (e.g., Davis et al. 1971, 1972) show that the rate of conversion of infalling mass to gravitational waves is never greater than

$$\frac{dm}{dt} \sim \left(\frac{m}{M}\right)^2$$

(units with $c = G = 1$). Thus, in the hole's characteristic time M the efficiency of mass conversion to a burst of waves is of order $m/M \ll 1$.

In realistic astrophysical situations the accretion of matter into a black hole will tend to increase its specific angular momentum (Bardeen 1970). If instability sets in at some point, there will be a last particle of mass $\underline{m}$ whose capture pushes the hole into the unstable region. To return to a stable configuration, the hole must emit a burst of energy $\sim \underline{m}$. Thus the particle — and all subsequent matter that accretes — is converted to gravitational waves with efficiency ~ 1. On the other hand, if the hole is not able to return to a stable configuration, then the initial burst of radiation emitted is even greater, perhaps even energy $\sim \underline{M}$.

Having raised these striking — if speculative — possibilities, we turn to the more mundane task of disproving them: We have tested the dynamic stability of the Kerr metric by searching for an onset of instability as we "spin up" the hole from the Schwarzschild configuration, known to be stable. Specifically, we have looked at the dozen or so lowest angular eigenvalues for vibrational modes of the hole, and have verified numerically that the corresponding vibrational frequencies do not cross the real axis of the complex frequency plane into the upper (unstable) half plane, for any specific angular momenta $\underline{a}$ in the black-hole range $0 \le \underline{a} < \underline{M}$. In other words, we find no configurations (not even the limit $\underline{a} \to \underline{M}$) which are marginally unstable at any real frequency, for any mode number $\ell \le 3$ (for all $|m| \le \ell$; ℓ and m are spin-weighted spheroidal harmonic indices). We have also "spot-checked" a few higher modes, and find that they seem to go smoothly to an asymptotic limit, and show no tendency toward marginal stability.

Strictly speaking, this paper describes an unsuccessful search for instabilities rather than an actual proof of stability. However, we suspect that the data presented here are in fact sufficient to support a rigorous mathematical proof: Since the underlying equation is linear and suitably analytic, it is almost certain that instabilities must necessarily set in by passage of some mode from the stable lower half of the frequency plane, through a real-frequency mode, into the unstable upper half-plane. Also, although our numerical approach can sample only a discrete number of possible frequencies and a's, it is probably possible to derive analytic bounds on the "smoothness" of what is computed. Finally, our numerical results are also very smooth as a function of mode number, and a matching of the numerical results to asymptotic solutions of large mode number is probably possible at very moderate values of ℓ.

Work which will strengthen the mathematical footing in this way is important; it may in fact lead to a purely analytic proof of stability. However, we feel that in the results here presented the essential evidence for stability is already clear.

In §II of this paper, we describe the formalism which relates the decoupled, separable perturbation equations for the Kerr metric to the stability problem. Section III and its related Appendices treat the actual numerical solution of the equations, and present the raw results. In integrating the separated radial equation we have used a technique which does not seem to have been described previously, a means of preserving accuracy on an asymptotically small solution in the presence of an asymptotically large one; this technique may be of some general use in unrelated problems. For the benefit of other workers on Kerr perturbation problems, we also describe our integration of the angular equations in some detail, and tabulate some of the eigenvalues. Section IV interprets the stability results and gives our conclusions.

II. USE OF THE DECOUPLED, SEPARATED WAVE EQUATION

The dynamical behavior of the Kerr black hole under small perturbations is described by a homogeneous wave equation in a single decoupled variable

$$\mathscr{L}\psi(t,r,\theta,\varphi) = 0 \tag{2.1}$$

Here t,r,θ,φ are Boyer-Lindquist (1967) coordinates for the Kerr metric

$$\begin{aligned} ds^2 &= (1-2Mr/\Sigma)dt^2 + [4Mar\,\sin^2(\theta)/\Sigma]dtd\varphi \\ &\quad - (\Sigma/\Delta)dr^2 - \Sigma d\theta^2 - \sin^2(\theta)[r^2+a^2+2Ma^2r\,\sin^2(\theta)/\Sigma]d\varphi^2, \end{aligned} \tag{2.2}$$

where $\Delta \equiv r^2 - 2Mr + a^2$, $\Sigma \equiv r^2 + a^2\cos^2\theta$, $\underline{M}$ is the mass of the hole, $\underline{a}$ is its specific angular momentum, $0 \le \underline{a} \le \underline{M}$. The hyperbolic, second-order, linear differential operator $\mathscr{L}$, as derived in Paper I, is

$$\begin{aligned} \mathscr{L} &= \left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2}{\partial t\partial\varphi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2}{\partial\varphi^2} \\ &\quad - \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial}{\partial\varphi} \\ &\quad - 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial}{\partial t} + (s^2\cot^2\theta - s). \end{aligned} \tag{2.3}$$

Since we are interested in gravitational perturbations, we will always take $s = \pm 2$. The precise definition of ψ in terms of components of the Riemann tensor is given in Paper I and in Teukolsky (1972). For the stability problem, one need only know that ψ grows without bound if and only if physically measurable quantities (e.g., the energy fluxes of gravitational waves at infinity) grow without bound; this point is discussed below.

We now consider an initial-value problem where ψ and $\psi_{,t}$ are specified on an initial hypersurface $t = 0$, and where $\mathcal{L}\psi = 0$ determines the subsequent evolution for $t > 0$. The function $\psi(t,r,\theta,\varphi)$ is assumed to satisfy boundary conditions of physical acceptability at radial infinity and on the horizon (see Paper I). A reasonable sufficient condition to insure this is to take ψ and $\psi_{,t}$ nonzero only in a finite range of r outside the horizon, at $t = 0$. By Fourier analysis in the complex plane we have

$$\psi(t,r,\theta,\varphi) = (2\pi)^{-\frac{1}{2}} \int_{-\infty+i\tau_o}^{+\infty+i\tau_o} \psi_\omega \, e^{-i\omega t} \, d\omega, \qquad t > 0 \tag{2.4}$$

where

$$\psi_\omega = \psi_\omega(r,\theta,\varphi) = (2\pi)^{-\frac{1}{2}} \int_0^\infty \psi \, e^{i\omega t} \, dt \tag{2.5}$$

and where τ_o is a positive real number such that $\exp(\tau_o t)$ is an upper bound on the growth of ψ at large times, i.e., faster than the fastest instability. (We will see below that such a bound exists.)

We now want to investigate the possibility of deforming the contour of integration in equation (2.4) into the lower half plane, by letting τ_o decrease and become negative. (Strictly speaking, convergence at infinity demands that the contour be left attached to the real axis at $\mathrm{Re}\ \omega = \pm\infty$, but it can be deformed to $\mathrm{Im}\ \omega = \tau_o < 0$ in any finite range $|\mathrm{Re}\ \omega| < B \to \infty$.) If ψ_ω, viewed as a function of ω, contains no poles (we will assume that branch cuts are ruled out by the form of $\mathcal{L}$) in the region above the contour, then equation (2.4) remains a valid "reconstruction" of the complete field ψ. If, however, the contour deformation crosses a pole, then the pole's residue must be included, and we obtain,

$$\psi(t,r,\theta,\varphi) = (2\pi)^{-\frac{1}{2}} \int_{-\infty+i\tau_o}^{+\infty+i\tau_o} \psi_\omega(r,\theta,\varphi)e^{-i\omega t}\, d\omega + \sum_j F_j(r,\theta,\varphi)e^{-i\omega_j t} \quad (2.6)$$

where the sum is over all poles above the contour, with frequencies ω_j and with residues F_j.

The next point is that the solution $\psi = F_j(r,\theta,\varphi)\exp(-i\omega_j t)$ associated with any single pole must by itself satisfy $\mathcal{L}\psi = 0$ with physically correct boundary conditions on the horizon and at infinity. Why? Because at late times $t \to \infty$ and at any fixed r, the contribution of the contour integral vanishes exponentially compared to the sum over the poles, while the total summed solution, by construction, satisfies the boundary condition at all times. Since the poles are a discrete set, their sum can satisfy the boundary condition only if each does separately.

Each ω_j represents a discrete frequency mode with the correct boundary conditions. At late times, any initial perturbation is asymptotically a superposition of these discrete modes; all other perturbations die away (i.e., radiate away to infinity or the horizon) faster than any exponential. [There is no contradiction with Price's (1972) power-law "tails"; his system was inhomogeneous with sources which became asymptotically static.] The discrete modes are what one calls the "vibrations" of the black hole, and have been seen numerically in previous work (Press 1971). Since the equation $\mathcal{L}\psi = 0$ is separable, and its angular eigenfunctions are complete, we can write

$$F_j(r,\theta,\varphi) = \sum_{\ell,m} S^m{}_\ell(\theta)\, e^{im\varphi}\, R_{\omega\ell m}(r) \;, \quad (2.7)$$

where S^m_ℓ is a regular function which satisfies the equation

$$\frac{1}{\sin\Theta}\frac{d}{d\Theta}\left(\sin\Theta\frac{dS}{d\Theta}\right)+\left(a^2\omega^2\cos^2\Theta-\frac{m^2}{\sin^2\Theta}-2a\omega s\cos\Theta-\frac{2ms\cos\Theta}{\sin^2\Theta}\right.$$
$$\left.-s^2\cot^2\Theta+E-s^2\right)S=0, \tag{2.8}$$

for eigenvalue E^m_ℓ (a "spin-weighted spheroidal harmonic"), and $R_{\omega\ell m}$ satisfies the separated radial equation

$$\Delta\frac{d^2R}{dr^2}+2(s+1)(r-M)\frac{dR}{dr}+\left\{\frac{K^2-2is(r-M)K}{\Delta}+4ir\omega s-\lambda\right\}R=0. \tag{2.9}$$

Here $K \equiv (r^2+a^2)\omega - am$ and $\lambda \equiv E - 2am\omega + a^2\omega^2 - s(s+1)$. The boundary conditions on $R_{\omega\ell m}$ at the horizon ($r \to r_+$) and at infinity ($r \to \infty$) are respectively,

$$R \sim \begin{cases} \Delta^{-s}\, e^{-ikr_*} & , \quad r \to r_+ \\ e^{i\omega r_*}/r^{(2s+1)} & , \quad r \to \infty \end{cases} \tag{2.10}$$

where $k \equiv \omega - ma/(2Mr_+)$, and r_* is defined by

$$dr_*/dr = (r^2+a^2)/\Delta \tag{2.11}$$

(see Paper I for derivation and details).

Equations (2.8), (2.9), and (2.10) determine a nonlinear eigenvalue problem for the frequencies ω_j of vibration, nonlinear because ω is tied rather intimately into the equations as a parameter — for example $E^m_\ell = E^m_\ell(a\omega)$. The stability problem is now very straightforward: if there are no solutions with ω_j in the upper half complex plane (for all angular modes ℓ and m) then the Fourier reconstruction of equation (2.4) is valid with $\tau_o \leq 0$ and there are no solutions which become unbounded in time; the hole is stable. If there are eigenfrequencies ω_j in the upper half plane, then each corresponds to a perturbation which is well behaved for all $r \geq r_+$ at time $t = 0$, but which grows exponentially in time; these are instabilities.

How can one determine whether there are solutions corresponding to instabilities? The direct way, in principle, is to examine the entire upper half complex ω-plane by "shooting": for each ω, start a solution $R_{\omega\ell m}$ with the correct asymptotic form on the horizon

$$R = \Delta^{-s} e^{-ikr_*}, \quad r \to r_+ ; \tag{2.12}$$

integrate this solution outward with the radial equation (2.9) to large values of r, and resolve it into the two asymptotic solutions

$$R = Z_{in}\, e^{-i\omega r_*}/r + Z_{out}\, e^{i\omega r_*}/r^{(2s+1)}, \quad r \to \infty . \tag{2.13}$$

The solutions we seek are zeros of Z_{in}, or equivalently poles of Z_{out}/Z_{in}, viewed as a function of ω in the complex plane. Note that by looking at Z_{out}/Z_{in} we cannot miss a zero of Z_{in} by Z_{out} going to zero at the same value of ω, for then R would be zero everywhere since it satisfies a linear second-order equation.

This paper reports a shooting search for poles of Z_{out}/Z_{in} where we have restricted ω to the real axis, and tested a large number of values of a in the range $0 \leq a < M$. There are good reasons for this restriction: Since the Schwarzschild case $a = 0$ is known to be stable, there are no poles in the upper half plane for this value. Since the linear differential operator $\mathcal{L}$ depends continuously (in fact analytically) on the parameter a, and since the boundary conditions (2.10) are also analytic in a, it is virtually certain (but strictly speaking, not yet rigorously proved!) that the eigenvalues ω_j vary continuously with a. Thus, a pole can migrate to the upper half plane only by crossing the real axis, and the restriction of our search to the real axis involves no loss of generality.

Going beyond the scope of this paper, it is worth noting that data on the real axis could rule out eigenvalues in the upper half plane even if the continuity assumption failed. It is not difficult to show that $Z_{in} = Z_{in}(\omega)$ is regular as $\omega \to \infty$ in the upper half-plane; a proof that $Z_{in}(\omega)$ is suitably meromorphic in the finite upper half plane would rigorously justify a "phase shift analysis" on the real-ω axis: the change in phase of Z_{in} over the interval $-\infty < \omega < +\infty$ would directly measure the number of eigenvalues in the upper half plane. This approach is discussed briefly in Appendix A. In that appendix we also show that $Z_{in} \to 1$ for large $|\omega|$; this places a bound on the growth of the fastest instability, and justifies the choice of a constant τ_o in equation (2.4). We postpone further discussion of the analyticity properties of the Kerr perturbation equations to a future paper.

Two loose ends can be tied up here. First, for the equation with spin-weight $s = -2$ (which for numerical reasons will turn out to be the actual case integrated, see §III), the ingoing piece of the solution $Z_{in} \exp(-i\omega r_*)/r$ is asymptotically small compared to the outgoing piece $Z_{out} \exp(i\omega r_*)r^3$. This is a direct consequence of the peeling theorems (see Newman and Penrose 1962): the Newman-Penrose field ψ_4 is "tailored" for outgoing waves, and suppresses ingoing waves by four powers of r. How do we know that the magnitude Z_{in} of this suppressed "ghost" of an incoming wave accurately reflects the physical magnitude of the wave (say, as an ingoing energy flux at infinity), so that a zero of Z_{in} is <u>necessary and</u>

sufficient for the correct physical boundary condition, as claimed? The answer is given in Appendix B, where we give the relation between ingoing and outgoing wave energy flux, and Z_{in} and Z_{out} as defined above. It turns out that the factor relating Z_{in} to ingoing flux is rather complicated and cannot be exhibited in closed form for $a \neq 0$; but we prove that it is positive definite, without zeros or poles. A subsequent paper will give numerical results for this factor and discuss applications to the problem of superradiant scattering (Misner 1972, Press and Teukolsky 1972) of gravitational waves.

Second, we can give a very easy proof that a Schwarzschild black hole is stable [modeled on the work of Vishveshwara (1970) and Zerilli (1970)]. For the Schwarzschild case one can pose the analogous eigenvalue problem to that determined by equations (2.8), (2.9), and (2.10). Here, however, one can use the Regge-Wheeler (1957) and Zerilli (1970) radial equations instead of the more complicated equation (2.9) of the Kerr background. The Regge-Wheeler and Zerilli equations are both of the form

$$\psi'' = (V - \omega^2)\,\psi, \tag{2.14}$$

where prime denotes d/dr_* and V is a real, positive function with no ω dependence. Thus the eigenvalue problem for ω^2 is linear, and self-adjoint. By self-adjointness, ω^2 must be real, so any instability must lie on the positive imaginary axis, $\omega = i\sigma$ say. Then the radial equations take the form

$$\psi'' = (\text{positive definite function})\,\psi \tag{2.15}$$

which manifestly has no solution that is regular at $r_* = \pm\,\infty$, q.e.d. Marginal instabilities, with ω^2 real and positive, are equally easy to rule out, using the fact that the Wronskian of a solution with its complex conjugate is conserved.

III. NUMERICAL SOLUTION OF THE EQUATIONS

a) Angular Eigenfunctions and Eigenvalues

The angular equation (2.8) can be written as an eigenvalue equation involving the sum of two operators

$$(\mathcal{H}_0 + \mathcal{H}_1)S = - ES, \tag{3.1}$$

where

$$\begin{aligned}\mathcal{H}_0 &\equiv \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - \left(\frac{m^2 + s^2 + 2ms\cos\theta}{\sin^2\theta}\right)\\ \mathcal{H}_1 &\equiv a^2\omega^2\cos^2\theta - 2a\omega s\cos\theta .\end{aligned} \tag{3.2}$$

The operator $\mathcal{H}_0$ has no dependence on $\underline{a}$ or ω; the equation

$$\mathcal{H}_0 S = - ES \tag{3.3}$$

has as its solutions the well-known spin-weighted spherical harmonics (Newman and Penrose 1966, Goldberg et al. 1967)

$$\left.\begin{aligned} S(\theta) &= {}_sY^m{}_\ell(\theta) \\ E &= \ell(\ell+1) \end{aligned}\right\} \quad \begin{aligned} &\ell = |s|,\ |s|+1,\ \ldots \\ &-\ell \le m \le +\ell. \end{aligned} \tag{3.4}$$

We can view $\mathcal{H}_1$ as a (noninfinitesimal) perturbation operator which takes us from the spherical ($a\omega = 0$) to the spheroidal ($a\omega \neq 0$) case. We will first consider the limiting case of $a\omega \ll 1$ and derive explicit perturbation solutions for ${}_sS^m{}_\ell(\theta, a\omega)$ and ${}_sE^m{}_\ell(a\omega)$; a numerical calculation then generalizes these solutions to arbitrary values of $a\omega$. In the present series of papers, we are primarily interested in the eigenvalues, which couple into the radial equation, and only secondarily interested in the details of the angular functions. Without loss of generality, we can restrict attention to cases with $s < 0$, $a\omega > 0$, since equation (3.2) admits the following symmetries:

$$\begin{aligned} {}_{-s}S^m{}_\ell(\Theta, a\omega) &= {}_sS^m{}_\ell(\pi - \Theta, a\omega) \\ {}_{-s}E^m{}_\ell(a\omega) &= {}_sE^m{}_\ell(a\omega) \end{aligned} \tag{3.5}$$

$$\begin{aligned} {}_sS^m{}_\ell(\Theta, -a\omega) &= {}_sS^{-m}{}_\ell(\pi - \Theta, a\omega) \\ {}_sE^m{}_\ell(-a\omega) &= {}_sE^{-m}{}_\ell(a\omega) \end{aligned} \tag{3.6}$$

For small $a\omega$, ordinary perturbation theory gives the results

$$\begin{aligned} {}_sE^m{}_\ell &= \ell(\ell+1) - \langle s\ell m|\mathscr{H}_1|s\ell m\rangle + \cdots \\ {}_sS^m{}_\ell &= {}_sY^m{}_\ell + \sum_{\ell' \neq \ell} \frac{\langle s\ell' m|\mathscr{H}_1|s\ell m\rangle}{\ell(\ell+1) - \ell'(\ell'+1)}\, {}_sY^m{}_{\ell'} \quad . \end{aligned} \tag{3.7}$$

Here

$$\langle s\ell' m|\mathscr{H}_1|s\ell m\rangle \equiv \int d\Omega\, {}_sY^m{}_{\ell'}{}^*\, {}_sY^m{}_\ell\, \mathscr{H}_1 \ . \tag{3.8}$$

The spin-weighted spherical harmonics are related to the rotation matrix elements of quantum mechanics (see Campbell and Morgan 1971), so standard formulae are available for the integrated product of three such functions.

The ones we need are

$$\langle s\ell'm|\cos^2\Theta|s\ell m\rangle = \frac{1}{3}\,\delta_{\ell\ell'} + \frac{2}{3}\left(\frac{2\ell+1}{2\ell'+1}\right)^{\frac{1}{2}} \langle \ell \;\; 2\,m\,0|\ell'\,m\rangle\langle \ell \;\; 2\,-s\,0|\ell'\,-s\rangle \tag{3.9a}$$

and

$$\langle s\ell'm|\cos\Theta|s\ell m\rangle = \left(\frac{2\ell+1}{2\ell'+1}\right)^{\frac{1}{2}} \langle \ell \;\, 1\,m\,0|\ell'm\rangle\langle \ell \;\, 1\,-s\,0|\ell'\,-s\rangle \tag{3.9b}$$

where $\langle j_1 j_2 m_1 m_2|JM\rangle$ is a Clebsch-Gordon coefficient. Equations (3.7), (3.8), (3.9) yield an explicit construction of the perturbation eigenfunctions and eigenvalues. For example,

$$ {}_sE^m{}_\ell(a\omega) = \begin{cases} \ell(\ell+1) - 2a\omega\,\dfrac{s^2 m}{\ell(\ell+1)} + \mathcal{O}\left[(a\omega)^2\right] , & s \neq 0 \\[2ex] \ell(\ell+1) + 2a^2\omega^2\left[\dfrac{m^2+\ell(\ell+1)-1}{(2\ell-1)(2\ell+3)}\right] & \\[2ex] \qquad\qquad + \mathcal{O}\left[(a\omega)^4\right] , & s = 0 \;. \end{cases} \tag{3.10}$$

For the case when $a\omega$ is not small, the perturbation method generalizes to a continuation method (cf. Wasserstrom 1972), giving differential equations in the independent variable $a\omega$ which can be integrated numerically to arbitrarily large values. If we choose the spherical functions as a representation, so that

$$ {}_sS^m{}_\ell(\Theta,a\omega) = \sum_{\ell'} {}_sA^m{}_{\ell\ell'}(a\omega)\; {}_sY^m{}_{\ell'}(\Theta) \tag{3.11}$$

then the continuation equations take the form

$$\frac{d\, A_{\ell\ell'}}{d(a\omega)} = \sum_{\alpha,\beta,\gamma \neq \ell} \frac{A_{\gamma\alpha}\, A_{\ell\beta}}{{}_sE^m{}_\ell - {}_sE^m{}_\gamma} \langle\alpha,\beta\rangle\, A_{\gamma\ell'} \tag{3.12a}$$

$$\frac{d\, E^m{}_\ell}{d(a\omega)} = \sum_{\alpha,\beta} A_{\ell\alpha}\, A_{\ell\beta}\, \langle\alpha,\beta\rangle \tag{3.12b}$$

where

$$\langle\alpha,\beta\rangle \equiv \langle s\,\alpha\, m | d\mathscr{V}_1/d(a\omega) | s\,\beta\, m\rangle$$

$$= \int d\Omega\; {}_sY^m{}_\alpha\; {}_sY^m{}_\beta\; (2\, a\omega \cos^2\Theta - 2\, s \cos\Theta) \tag{3.13}$$

which is evaluated using equations (3.9a) and (3.9b). Our numerical calculations integrate equations (3.12a) and (3.12b), truncated at $\alpha,\beta,\gamma = 20$ from $a\omega = 0$ to $a\omega = 3$, by 4th order Runge-Kutta integration. A step size of $\Delta(a\omega) = 0.25$ is sufficient for five place accuracy. The results are displayed in Appendix C as polynomial fits in the parameter $a\omega$ for the ranges $s = \pm 2$, $\ell \leq 6$, and $(a\omega) \lesssim 3$.

The continuation method described here works well for "mass producing" eigenvalues and eigenfunctions. Standard shooting methods are also useful if only a few functional values are desired. Finite difference methods should be used only with care, since the form of the functions is poorly suited to a uniform grid [see Keller (1968) for details on the standard methods].

b) Radial Equation

Except for the simpler case $\omega = 0$, the radial equation (2.9) has regular singular points at the two roots of Δ, and an irregular singular point at infinity; thus, it is not soluble in terms of hypergeometric or other standard functions, nor do its solutions admit a known integral representation. Numerical integration of the equation appears to be the only direct line of attack.

There are actually two radial equations, corresponding to the spin-weights $s = \pm 2$. The two equations contain the same information, so we need integrate only one. Which one? The choice is largely dictated by the boundary condition on the horizon. There, the regular and irregular radial solutions are respectively

$$R \sim \exp(-ikr_*)\Delta^{-s} \sim \exp\left\{-ikr_* - s\left[\frac{2(M^2 - a^2)^{\frac{1}{2}}}{r_+^{\,2} + a^2}\right] r_*\right\} \qquad \text{(regular)} \qquad (3.14a)$$

$$R \sim \exp(ikr_*) \qquad \text{(irregular)} \qquad (3.14b)$$

For $s = +2$ the regular solution is exponentially larger than the irregular solution near the horizon (large negative r_*). This is numerically very unpleasant: if we integrate the physically correct, regular solution out from the horizon, the integration will be unstable against bringing in an exponentially growing piece of the irregular solution. Likewise, if we integrate <u>in</u> to the horizon and attempt to impose boundary conditions there, we are faced with finding a zero of the exponentially small irregular solution, which is lost in numerical truncation.

The case $s = -2$ is pleasantly the reverse: integrating outward, any contamination of the irregular solution vanishes exponentially; or, integrating inward, the irregular solution is large, and can be zeroed definitively. On this basis, we choose to integrate the case $s = -2$ exclusively;

we always integrate outward from the horizon through many e-folds of equation (3.14a); thus, we need not start the solution on the horizon very carefully — we have adopted the conservative procedure of always starting two "random" linearly independent solutions and verifying that they yield identical final results.

There is a price exacted at infinity for this convenience at the horizon, however. For large r, the solution is as given in equation (2.13). To find instabilities, we seek zeros of Z_{in} and for $s = -2$, Z_{in} is the coefficient of a function which decreases faster by four powers of r than does the dominant term multiplying Z_{out}. (As before, integrating <u>in</u> from infinity is no better — the solution is then numerically unstable against contamination with the other solution.) Fortunately, the problem here is only a power law, not an exponential. We know of two ways to circumvent it:

First, it is not too difficult to integrate with sufficiently high precision to maintain significance in the small solution. This "brute force" method is not without pitfalls: If one resolves Z_{in} and Z_{out} on the basis of function and derivative at one large radius $r = R_0$, then the outgoing (large, Z_{out}) solution at R_0 must be known analytically to much higher accuracy than just its leading term in equation (2.9) — it must be known to the accuracy of the small solution associated with Z_{in}. This is not easy: a WKB solution is not sufficiently accurate, for example; and even adding further terms to an asymptotic series does not always work, since the asymptotic series is not convergent at fixed R_0. A satisfactory way to resolve the Z_{in} and Z_{out} components is to fit (e.g., least-squares fit) a numerical solution to the functional form of equation (2.13) over a range in r_* of at least π/ω, so that all relative phases of the oscillatory parts contribute.

A second method is the one which we have actually used for most of the results here presented. The method is based on a differential transformation which takes a new independent variable

$$\chi \equiv \mathscr{P}(r)\,\frac{dR}{dr} + \mathscr{Q}(r)R\ . \tag{3.15}$$

The new variable χ satisfies a second-order, linear O.D.E. of its own, which is the equation actually integrated. The functions $\mathscr{P}$ and $\mathscr{Q}$ are chosen to make the two independent solutions at $r \to \infty$ have the same asymptotic behavior. This method, including a scheme for choosing $\mathscr{P}$ and $\mathscr{Q}$, is described in detail in Appendix D; the method does not appear to be in the literature, and it may have some utility in other unrelated applications where the two solutions of an O.D.E. differ by a power law.

The use of this second method also makes the problem of resolving Z_{in} and Z_{out} much easier: since the two solutions are of comparable size, accurate results are obtained by using WKB solutions for large r, and comparing to function and derivative of the numerically integrated variable at a single large R_0. Further description is in Appendix D.

In the limit of zero frequency, $\omega \to 0$, the radial equation is soluble analytically in terms of hypergeometric functions. Matching the zero-frequency solution to asymptotic small-ω solutions at infinity gives a limiting expression for Z_{out}/Z_{in},

$$Z_{out}/Z_{in} = \frac{16(\omega M)^4}{(\ell-1)\ell(\ell+1)(\ell+2)}\left[1 + \mathscr{O}(a\omega)\right], \qquad a\omega \ll 1 \tag{3.16}$$

(The pole at $\omega = 0$ does not correspond to an instability, but rather to the sole zero in the factor relating Z_{in} to physical gravitational wave flux at infinity [see Appendix B].)

c) Numerical Results

It is convenient to plot our results in terms of a real function with the same physical poles as Z_{out}/Z_{in},

$$Z(\omega, a, \ell, m) \equiv \left| \frac{(\ell - 1)\ell(\ell+1)(\ell+2)}{16(M\omega)^4} \frac{Z_{out}}{Z_{in}} \right| \quad . \tag{3.17}$$

By equation (3.16), $Z \to 1$ as $\omega \to 0$. Complex conjugating the radial equation (2.9), and reversing the sign of m and ω, gives the symmetry

$$Z(\omega, a, \ell, -m) = Z(-\omega, a, \ell, m) \tag{3.18}$$

so we can restrict ourselves to non-negative m without loss of generality. When $m = 0$, we can further take $\omega \geq 0$.

Figures 1-3 plot Z as a function of ω and a for all modes with $\ell = 2$ and $\ell = 3$ and for a sample mode of $\ell = 4$. (Recall that $\ell = 0$, 1 modes do not exist, since $\ell \geq |s| = 2$.) Calculations for several higher modes have also been made, with similar results. The following features of the numerical results are worth noting: (i) for low ω, Z is a very nearly linear function of ω, with a value of order unity; as $|\omega|$ becomes larger, the potential barrier between the horizon and infinity is surmounted, and the ingoing wave on the horizon matches to a nearly pure ingoing wave at infinity — Z drops to zero exponentially. This fall-off is derived analytically in Appendix A. (ii) As one might expect intuitively, the effect of varying a is greatest for modes with $\ell = m$, $\omega > 0$ (roughly, wave packets in direct, equatorial orbits), and small for $m = 0$ and for $m = \ell$ with $\omega < 0$. (iii) Z is everywhere a very smooth function of a; not only is there no instability apparent, but it is difficult to find any particular value of a with distinguished behavior. (iv) There are no obvious qualitative differences between the $\ell = 2$ modes, the $\ell = 3$ modes, and higher modes, i.e., there is no particularly distinguished ℓ. (v) For $a \sim M$ and $\omega > 0$, the transition to an exponentially falling function occurs more or less at m times the rotational frequency of the horizon $\omega \approx a/(2Mr_+)$. This phenomenon is related to the superradiance effect and will be discussed in a later paper in this series. (vi) Z is everywhere finite in the limiting case $a \to M$ but a sharp corner appears in the double limit $a \to M$, $\omega \to \frac{1}{2}m/M$; this behavior has previously been noted for the scalar wave equation in the Kerr background (Press and Teukolsky 1972, Starobinsky 1972).

IV. DISCUSSION AND CONCLUSIONS

In the data of figures 1 and 2, there is no onset of instability via a real frequency mode for any angular mode with $\ell \leq 3$, at any value of a, $0 \leq a/M \leq 0.9999$. The same conclusion holds for a number of higher angular modes that we have spot-checked, e.g., $\ell = 4$, $m = \pm 4$ in figure 3. We cannot, of course, rule out the possibility of an instability in some mode of high ℓ that we have not examined. However, the lack of any qualitative differences between low discrete values of ℓ, and the fact that even these moderately low values seem to tend to a smooth asymptotic behavior as ℓ increases, seem to make the possibility unlikely.

As we have noted in §II, it is almost certain that an instability must set in through a real frequency mode; a rigorous proof would slightly extend known analyticity theorems so as to include our radial equation (2.9). In the present case, moreover, the smoothness in a of the families of curves shown in figures 1-3 argues directly for this analyticity, and against the sudden appearance of a pole in the upper half plane for some finite value of $\underline{a}$: in general such a pole would be expected to change the functional form of $|Z_{out}/Z_{in}|$ on the real-ω axis, as well as introducing a new phase shift of 2π in $\arg(Z_{out}/Z_{in})$.

Comparing the behavior of the data for $a/M = 0.99$, 0.999, and 0.9999, one sees no tendency towards instability even in the limit $a/M \to 1$, a value which Hawking (1973) has suggested as a possible candidate for a marginal instability. However, even this extrapolation is not necessary for astrophysical applications, since the natural upper limit to the spin-up of astrophysical black holes, $a/M \approx 0.9982$ (Thorne 1973), lies within the range of our calculations.

Our conclusion, that the Kerr metric is apparently dynamically stable, adds credibility to present and future detailed astrophysical models which include rotating black holes, e.g., models for variable X-ray sources in binary systems (Pringle and Rees 1973, Shakura and Sunyayev 1973, Novikov and Thorne 1973), and for infrared and radio emission at the galactic center (Lynden-Bell and Rees 1971); the conclusion also adds a bit of evidence to the conjecture (Carter 1973) that Kerr holes are the unique final states for collapsed isolated bodies; finally, the apparent stability also justifies the further use of the separable perturbation equations on the Kerr background for investigations of the dynamical behavior of rotating black holes in astrophysical contexts, e.g., tidal interactions and gravitational and electromagnetic wave processes, which subsequent papers in this series will investigate.

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APPENDIX A

ASYMPTOTIC FORMS OF Z_{OUT} AND Z_{IN} FOR $\omega \to \infty$

Let $\psi = \Delta^{s/2}(r^2 + a^2)^{\frac{1}{2}} R$. Then equation (2.9) becomes

$$\psi_{,r_* r_*} + V\psi = 0 \tag{A1}$$

where

$$V = \frac{K^2 - 2is(r - M)K + \Delta(4ir\omega s - \lambda)}{(r^2 + a^2)^2} - G^2 - G_{,r_*}$$

$$G = \frac{s(r - M)}{r^2 + a^2} + \frac{r\,\Delta}{(r^2 + a^2)^2} \quad . \tag{A2}$$

The conditions (2.12) and (2.13) become

$$\psi \to \Delta^{-s/2}\, e^{-ikr_*} \qquad , \; r \to r_+$$

$$\psi \to Z_{in}\, r^s\, e^{-i\omega r_*} + Z_{out}\, r^{-s}\, e^{-i\omega r_*} \qquad , \; r \to \infty \quad . \tag{A3}$$

Let ψ_-^L be a solution of equation (A1) satisfying $\psi_-^L \to \Delta^{-s/2}\, e^{-ikr_*}$ "on the left," at $r \to r_+$. Let $\psi_\pm^R$ be solutions satisfying $\psi_\pm^R \to r^{\mp s}\, e^{\pm i\omega r_*}$ "on the right," at $r \to \infty$. Since equation (A1) has only two linearly independent solutions, we have

$$\psi_-^L = Z_{in}\, \psi_-^R + Z_{out}\, \psi_+^R \; . \tag{A4}$$

Take the Wronskian of equation (A4) with ψ_+^R. Then

$$Z_{in} = W(\psi_-^L, \psi_+^R)/W(\psi_-^R, \psi_+^R)$$

$$= W(\psi_-^L, \psi_+^R)/2i\omega \; . \tag{A5}$$

Similarly

$$Z_{out} = - W(\psi_-^L, \psi_-^R)/2i\omega \quad . \tag{A6}$$

Now suppose $\phi_\pm$ satisfies

$$\phi_{\pm, r_* r_*} + (V + U_\pm)\phi_\pm = 0 \tag{A7}$$

$$\begin{aligned} \phi_\pm &\to \Delta^{\pm s/2} e^{\pm ikr_*} , \quad & r \to r_+ \\ \phi_\pm &\to r^{\mp s} e^{\pm i\omega r_*} , \quad & r \to \infty \ . \end{aligned} \tag{A8}$$

(We shall later choose ϕ to be "close" to ψ for $\omega \to \infty$, i.e., $|U| << |V|$ for $\omega \to \infty$.) Multiply equation (A1) by $\phi_\pm$, equation (A7) by ψ_-^L, and subtract. Integrate from $r_* = -\infty$ to $r_* = \infty$ and use $\phi_\pm \to \psi_\pm^R$ at $r_* = \infty$ and $\psi_-^L \to \phi_-$ at $r_* = -\infty$. Thus find

$$Z_{in} = \frac{1}{2i\omega}\left[W(\phi_-, \phi_+)_{-\infty} - \int_{-\infty}^{\infty} U_+ \phi_+ \psi_-^L \, dr_*\right] \tag{A9}$$

$$Z_{out} = \frac{1}{2i\omega}\int_{-\infty}^{\infty} U_- \phi_- \psi_-^L \, dr_* \tag{A10}$$

For $\omega \to \infty$, a good approximation for $\psi_\pm$ is the WKB solution $\psi_\pm \sim \exp(\pm i\int V^{\frac{1}{2}} dr_*)$. It can be shown by methods analogous to those of Erdélyi et al. (1955) that for $\omega \to \infty$,

$$\lambda \sim \begin{cases} \alpha a\omega + \mathcal{O}(1) \quad & , \quad \mathrm{Re}\ \omega \neq 0 \\ a^2\omega^2 + \beta a\omega + \mathcal{O}(1) \quad & , \quad \mathrm{Re}\ \omega = 0 \ , \end{cases} \tag{A11}$$

where α and β depend on ℓ, m, and s, and α is real. Let us consider first $\mathrm{Re}\ \omega \neq 0$. Then

$$V^{\frac{1}{2}} \sim H + \mathcal{O}(\omega^{-1}) \tag{A12}$$

where

$$H = \omega - \frac{am}{r^2 + a^2} - \frac{is(r - M)}{r^2 + a^2} + \frac{\Delta}{(r^2 + a^2)^2}\left(2irs - \frac{\alpha a}{2}\right) \quad . \tag{A13}$$

Choose

$$\begin{aligned}\phi_{\pm} &= \exp(\pm i \int H dr_*) \\ &= \Delta^{\pm s/2} (r^2 + a^2)^{\mp s} \exp\left[\pm i \int \left(\omega - \frac{am}{r^2 + a^2}\right) dr_*\right] \exp\left[\pm \frac{i\alpha}{4} \tan^{-1}\left(\frac{2a}{r}\right)\right] .\end{aligned} \tag{A14}$$

Then

$$U_{\pm} = H^2 \mp i\, H_{,r_*} - V \sim \mathcal{O}(1) \qquad \text{as } \omega \to \infty . \tag{A15}$$

Substituting ϕ_- for ψ_-^L in equation (A9), we find

$$\begin{aligned} Z_{in} &\sim \frac{1}{2i\omega}\left[2iH(r_+) - \int_{-\infty}^{\infty} U_+ \phi_+ \phi_- dr_*\right] \\ &= \frac{1}{2i\omega}\left[2i\left(\omega - \frac{am}{r_+^2 + a^2} - \frac{is(r_+ - M)}{r_+^2 + a^2}\right) - \int_{-\infty}^{\infty} U_+ dr_*\right] \\ &\to 1 \qquad \text{as } \omega \to \infty . \end{aligned}$$

Thus the wave is nearly perfectly transmitted for high frequencies. For $\mathrm{Re}\,\omega = 0$, a similar argument can be given using a slightly different H satisfying the relation (A12). Again we find $Z_{in} \to 1$ as $\omega \to \infty$. This result further supports the assertion that instabilities must necessarily set in via a zero of Z_{in} crossing the real-ω axis to the upper half plane: the possibility of a zero entering "from infinity" is excluded.

It is also interesting to show analytically that the exponential fall off in Z_{out}/Z_{in}, which is unmistakable in the numerical results for moderate ω, persists in the limit $\omega \to \pm\infty$ (ω real). This can be proved by replacing

ψ_-^L by ϕ_- in equation (A10):

$$Z_{out} \sim \frac{1}{2i\omega} \int_{-\infty}^{\infty} U_- \phi_-^2 \, dr_* = \frac{1}{2i\omega} \int_{r_+}^{\infty} U_- \phi_-^2 \frac{dr_*}{dr} \, dr. \qquad \text{(A16)}$$

The factor $e^{-2i\omega r_*}$ in ϕ_-^2 allows a treatment of this expression by the saddle-point method. The saddle points occur at $dr_*/dr = 0$, i.e., at $r^2 + a^2 = 0$. Using the explicit form of U_-, we see that these are also poles of U_-. Moreover, as $a \to 0$ the two saddle-points at $r = \pm ia$ coalesce. All these complications can be taken care of by Van der Waerden's (1951) treatment of the saddle-point method. The important result is that the dominant behavior of the integral is still determined by the value of $e^{-2i\omega r_*}$ at a saddle point; the complications give rise to different power-law dependences on ω when $a = 0$ or $a \gg 0$, with a transition region described by an incomplete gamma function. For ω positive, $r = -ia$ is the dominant saddle point, while for ω negative $r = +ia$ dominates. Taking care to define the branches of the logarithms in $r_*(r)$ consistently, we find

$$Z_{out} \approx e^{-2|\omega|B} \times \text{(weak dependence on } \omega)$$

where

$$B = 2\pi + a + \frac{\tan^{-1}[(1-a^2)^{\frac{1}{2}}/a]}{(1-a^2)^{\frac{1}{2}}} - \frac{\pi}{2}$$

$$\to \begin{cases} 2\pi, & a \to 0 \\ 2\pi + \left(2 - \frac{\pi}{2}\right), & a \to 1. \end{cases}$$

APPENDIX B

RELATION BETWEEN ψ_0 AND ψ_4 FOR INGOING WAVES

We wish to find the relation between ${\psi_4}^B$ and ${\psi_0}^B$ at infinity by using the asymptotic form of the NP equations. The notation in this Appendix is explained in Paper I, and we shall use the null tetrad equation (4.4) of that paper.

Since ${\psi_4}^B$ and ${\psi_0}^B$ are invariant under gauge transformations and infinitesimal tetrad rotations, we may make use of any convenient choice of gauge and tetrad. By infinitesimal transformations of types (i), (ii), and (iii) respectively (cf. Appendix A, Paper I), set $\lambda^B = \kappa^B = \epsilon^B = 0$. Set $D^B = 0$ by a gauge transformation. (Each of these transformations does not disturb the result of the previous ones.) For this choice of tetrad the necessary NP equations are:

NP 4.2a: $$D\rho^B = 2\rho\rho^B \tag{B1}$$

NP 4.2b: $$D\sigma^B = (\rho + \rho^*)\sigma^B + {\psi_0}^B \tag{B2}$$

NP 4.2d: $$D\alpha^B = \rho^B\alpha + \rho\alpha^B + \beta\sigma^{*B} + \rho\pi^B + \rho^B\pi \tag{B3}$$

NP 4.2e: $$D\beta^B = (\alpha + \pi)\sigma^B + \rho^*\beta^B + \rho^{*B}\beta^B + {\psi_1}^B \tag{B4}$$

NP 4.2g: $$-\delta^*\pi^B - \delta^{*B}\pi = \sigma^{*B}\mu + 2\pi\pi^B + (\alpha - \beta^*)^B\pi + (\alpha - \beta^*)\pi^B \tag{B5}$$

NP 4.4 : $$\delta^{*B}D - D\delta^{*B} = (\alpha + \beta^* - \pi)^B D - \sigma^{*B}\,\delta - \rho\delta^{*B} - \rho^B\delta^* \tag{B6}$$

NP 4.5 : $$(D - 4\rho){\psi_1}^B = (\delta^* + \pi - 4\alpha){\psi_0}^B \tag{B7}$$

NP 4.5 : $$(D - 3\rho){\psi_2}^B = (\delta^* + 2\pi - 2\alpha){\psi_1}^B + 3\rho^B\psi_2 \tag{B8}$$

NP 4.5 : $$(D - 2\rho)\psi_3{}^B = (\delta^* + 3\pi)\psi_2{}^B + (\delta^* + 3\pi)^B\psi_2 \tag{B9}$$

NP 4.5 : $$(D - \rho)\psi_4{}^B = (\delta^* + 4\pi + 2\alpha)\psi_3{}^B \tag{B10}$$

For ingoing waves at infinity, $\psi_0{}^B \sim S_0(\theta)\ e^{-i\omega(t+r_*)}\ e^{im\varphi}/r$. Let all the perturbation quantities in equations (B1)-(B10) be asymptotically

$$(\text{Quantity})^B \sim \rho^n\ S(\theta)\ e^{-i\omega(t+r_*)}\ e^{im\varphi}/r$$

where n is an integer to be determined along with the functions $S(\theta)$. Since the equations contain complex conjugate quantities, when we equate coefficients of $e^{-i\omega(t+r_*)}\ e^{im\varphi}$ we will get equations involving $S(\theta,\omega,m)$ and $S^*(\theta,-\omega,-m) \equiv S^\dagger(\theta,\omega,m)$. Because $\rho \sim 1/r$, we have

$$D\ (\text{Quantity})^B \sim -2i\omega\ (\text{Quantity})^B\ .$$

Thus, from equations (B1), (B2), (B7), (B4), and (B8) respectively, we find

$$\rho^B \sim 0,\ \sigma^B \sim \rho^0,\ \psi_1{}^B \sim \rho,\ \beta^B \sim \rho,\ \psi_2{}^B \sim \rho^2.$$

Let equation (B6) operate on ρ, and use the fact that, for this tetrad, $D\rho = \rho^2$, $\delta\rho = \rho\tau$, and $\delta^*\rho = -\rho\pi$. The result is

$$(-D + 3\rho)\ \delta^{*B}\rho = (\alpha + \beta^* - \pi)^B\ \rho^2 - \sigma^{*B}\ \rho\tau + \rho^B\rho\pi\ . \tag{B11}$$

Similarly, let equation (B6) operate on π, and use $D\pi = 2\rho\pi$, $\delta\pi = 2\pi(\tau+\beta)$, and $\delta^*\pi = -2\pi(\pi+\beta^*)$. Then find

$$2\pi\delta^{*B}\rho + (-D+3\rho)\delta^{*B}\pi = (\alpha+\beta^*-\pi)^B 2\rho\pi - 2\sigma^{*B}(\tau\pi+\pi\beta) + 2\rho^B(\pi^2+\pi\beta^*). \tag{B12}$$

Equations (B3), (B5), (B11), and (B12) give

$$\alpha^B \sim \rho,\ \pi^B \sim \rho^0,\ \delta^{*B}\rho \sim \rho^2,\ \delta^{*B}\pi \sim \rho^3.$$

Since $\psi_2 = M\rho^3$, $\delta^{*B}\psi_2 \sim \rho^4$ and so this latter term can be neglected in equation (B9) in comparison with $\pi^B\psi_2$. Equations (B2), (B5), (B7)-(B10) therefore determine the angular function S_0 in terms of S_4:

$$(\partial_\theta + m \operatorname{cosec} \theta - a\omega \sin \theta + 2 \cot \theta)S_0 = 2^{\frac{3}{2}} i\omega S_1 \tag{B13}$$

$$(\partial_\theta + m \operatorname{cosec} \theta - a\omega \sin \theta + \cot \theta)S_1 = 2^{\frac{3}{2}} i\omega S_2 \tag{B14}$$

$$(\partial_\theta + m \operatorname{cosec} \theta - a\omega \sin \theta)S_2 = 2^{\frac{3}{2}} i\omega S_3 + 2^{\frac{1}{2}} 3M S_\pi \tag{B15}$$

$$(\partial_\theta + m \operatorname{cosec} \theta - a\omega \sin \theta - \cot \theta)S_3 = 2^{\frac{3}{2}} i\omega S_4 \tag{B16}$$

$$(\partial_\theta + m \operatorname{cosec} \theta - a\omega \sin \theta - \cot \theta)S_\pi = - S_0{}^\dagger/(2^{\frac{3}{2}} i\omega) \tag{B17}$$

The equations for the $S^\dagger$ can be found by complex conjugating equations (B13)-(B17) and letting $\omega \to -\omega$, $m \to -m$.

Given an angular dependence of ψ_4, some particular $S_4(\theta)$, these ten equations (eqs. [B13]-[B17] and the corresponding equations for $S^\dagger$) explicitly determine a non-singular angular function for ψ_0, namely $S_0(\theta)$, as follows: It is not difficult to check that there are five regular solutions to the homogeneous (i.e., $S_4(\theta) = 0$) set at $\theta = 0$, and five more at $\theta = \pi$. Finding the linear combination of these homogeneous solutions which regularizes the inhomogeneous set at both $\theta = 0$ and π is equivalent to a well-posed set of ten linear equations in ten unknowns, and always has a unique solution.

When $a = 0$, these equations can be solved explicitly with the appropriate spin-weighted spherical harmonics for each S, since the differential operators reduce to "edth" operators (Goldberg et al. 1967). If $S_4(\theta)\, e^{im\varphi} = {}_{-2}Y_{\ell m}(\theta,\varphi)$, then we find $S_0(\theta)\, e^{im\varphi} = A\, {}_2Y_{\ell m}(\theta,\varphi)$, where

$$A = \frac{64\ \omega^4}{\ell(\ell+1)(\ell-1)(\ell+2) + 12\ iM\omega} \quad . \tag{B18}$$

Note that, if $\psi_4{}^B \sim \rho^4\ e^{im\varphi}\ e^{-i\omega t}\ S_4(\theta)\ (Z_{out}\ r^3\ e^{i\omega r_*} + Z_{in}\ e^{-i\omega r_*}/r)$ and $\psi_0{}^B \sim e^{im\varphi}\ e^{-i\omega t}\ S_0(\theta)\ Z_{in}\ e^{-i\omega r_*}/r$, then from equations (5.12) and (5.13) of Paper I, the ratio of outgoing to ingoing energy flux at infinity is

$$\frac{dE^{(out)}/dt}{dE^{(in)}/dt} = \frac{|Z_{out}|^2 \int |S_4|^2\ d(\cos\theta)}{|Z_{in}|^2 \int |S_0|^2\ d(\cos\theta)} \quad . \tag{B19}$$

APPENDIX C

ANGULAR EIGENVALUES FOR $s = \pm 2$

Table 1 gives the coefficients for polynomial approximations of the angular eigenvalues ${}_{\pm 2}E^{m}{}_{\ell}(a\omega)$ for $\ell \le 6$ and all values of m. The polynomial approximation is accurate to about five decimal places for $0 \le a\omega \le 3$; for $a\omega > 3$ (but not $a\omega \gg 3$) the coefficients can be used as an extrapolating polynomial. Negative values of $a\omega$ can be converted to positive values by the symmetry relation (3.6).

Note that the polynomial approximations given here are not the same as power series expansions around $a\omega = 0$. For example, there is not perfect symmetry between corresponding coefficients for m and -m; this is because the best interpolating polynomial for the range $0 \le a\omega \le 3$ is not identical to the best extrapolating polynomial for $-3 \le a\omega \le 0$.

All values in table 1 were computed by the continuation method described in §IIIa .

APPENDIX D

A METHOD FOR INTEGRATING EQUATION (2.9) AND SIMILAR EQUATIONS

We are given a linear, second order equation, say

$$\varphi'' = A\varphi' + B\varphi \tag{D1}$$

(prime denotes d/dr), such that two asymptotic solutions as $r \to \infty$ are of the form

$$\begin{aligned} \varphi_1 &\sim \exp(i\omega r_*)[1 + \mathcal{O}(1/r)] \\ \varphi_2 &\sim \left[\exp(-i\omega r_*)/r^n\right]\left[1 + \mathcal{O}(1/r)\right], \qquad n > 0 . \end{aligned} \tag{D2}$$

Here $r_*(r)$ can be any function such that $dr_*/dr \sim \mathcal{O}(1)$ at $r \to \infty$. The r^0 behavior of φ_1 at $r \to \infty$ involves no loss of generality, since a change of variable is allowed.

The problem is to integrate (D1) from some initial conditions at $r = r_0$ out to $r \to \infty$, without loss of significance to the small solution φ_2 in the presence of some large solution φ_1. Our method is as follows:

Let $S_1(r)$ be a function of r with the two properties, (i) $S_1(r) \neq 0$ for $r \geq r_0$, and (ii) for $r \to \infty$, S_1 and φ_1 agree asymptotically to order r^{-n}, i.e.,

$$\varphi_1 - S_1 = \mathcal{O}(1/r^{n-1}), \quad r \to \infty \quad . \tag{D3}$$

Now define a new variable χ by

$$\chi \equiv (\varphi/S_1)' . \tag{D4}$$

The two asymptotic solutions for χ at infinity are

$$\chi_1 = (\varphi_1/S_1)' = \left[1 + \mathcal{O}(1/r^{n-1})\right]' = \mathcal{O}(1/r^n)$$

$$\chi_2 = (\varphi_2/S_1)' = \left\{\exp(-2i\omega r_*)\ r^{-n}\left[1 + \mathcal{O}(1/r)\right]\right\}' \tag{D5}$$

$$= -2i\omega\ (dr_*/dr)\ r^{-n}\left[1 + \mathcal{O}(1/r)\right]$$

which manifestly decrease at the same asymptotic rate. Thus, a differential equation for χ can be integrated without loss of significance to either of the two solutions. Such an equation is obtainable by taking two successive derivatives of equation (D4) and using (D1) at each stage. The result is

$$\chi'' = \mathcal{A}\chi' + \mathcal{B}\chi \tag{D6}$$

where, if we define the intermediate quantities

$$\alpha \equiv A - \frac{2S_1'}{S_1}, \qquad \beta \equiv B + A\frac{S_1'}{S_1} - \frac{S_1''}{S_1}, \tag{D7}$$

$\mathcal{A}$ and $\mathcal{B}$ are given by

$$\begin{aligned} \mathcal{A} &\equiv \alpha + \beta'/\beta \\ \mathcal{B} &\equiv \alpha' + \beta - \alpha\beta'/\beta. \end{aligned} \tag{D8}$$

At any radius r, one can recover φ and φ' from χ and χ' by algebraic relations,

$$\begin{aligned} \varphi &= \frac{S_1}{\beta}(\chi' - \alpha\chi) \\ \varphi' &= \frac{S_1'}{\beta}\chi' + (S_1 - \frac{\alpha}{\beta}S_1')\chi. \end{aligned} \tag{D9}$$

Formally, the reverse relations are

$$\chi = \frac{1}{S_1}\varphi' - \frac{S_1'}{S_1^{\,2}}\varphi$$

$$\chi' = \frac{\alpha}{S_1}\varphi' + \left(\frac{\beta}{S_1} - \frac{\alpha S_1'}{S_1^{\,2}}\right)\varphi \qquad \text{(D10)}$$

but this transformation is ill-conditioned for large r; this is because the information in φ is very "delicately" encoded in the asymptotically small solution φ_2 while χ is not delicate.

The solution of a differential equation for χ instead of φ can be taken either as an analytic method (in which case α and β are worked out analytically for once and for all) or as a numerical method, where the numerical evaluation of α and β is built into the actual integrating program and done at each integration step. In this latter case (which is the way we have actually proceeded) special care is needed in the numerical evaluation of β, since the defining formula (D7) contains three terms whose sum is very nearly zero, for large r. This difficulty is easily handled without loss of significance, however, since β is simply an algebraic function of r, and is not subject to the additional truncation error of a differencing scheme as φ would be.

It is generally not difficult to find a function S_1 on which to base the method. For example, one can take the form

$$S_1 = \exp\left(i\omega r_* + \frac{C_1}{r} + \frac{C_2}{r^2} + \frac{C_3}{r^3} + \cdots\right) \qquad \text{(D11)}$$

and solve analytically for the first few C_i's by substitution into (D1). By construction S_1 is never zero, so property (i) above is satisfied.

Finally, the χ variable is well suited for directly resolving the components of the two independent solutions at large r; its two WKB solutions for $r \to \infty$ are

$$\chi = \exp\left[-\int_r^\infty dr\ (\tfrac{1}{2}\,\mathcal{a} \pm iV^{\frac{1}{2}})\right] \tag{D12}$$

where

$$V \equiv -\mathcal{B} + \tfrac{1}{2}\,\mathcal{a}' - \tfrac{1}{4}\,\mathcal{a}^2\ . \tag{D13}$$

For definiteness, here are the formulae which specifically apply the above method to equation (2.9) for s = -2 (a minor variation on the method is the factor r^3 left in φ_1):

Equation (D1):

$$\begin{aligned} \mathcal{A} &= 2(r-M)/\Delta \\ \mathcal{B} &= -\left[\frac{K^2 + 4i(r-M)K - (8ir\omega + \lambda)\Delta}{\Delta^2}\right] \\ \varphi &\equiv R \end{aligned} \tag{D14}$$

Equation (D11):

$$S_1 = \exp\left(i\omega r_* + 3\ \ell n\,\frac{r}{M} + \frac{C_1}{r} + \frac{C_2}{r^2}\right) \tag{D15}$$

where

$$\begin{aligned} C_1 &= -\ (\lambda + 2am\omega)/(2i\omega) \\ C_2 &= (6a^2\omega^2 + 4iam\omega^2 M - 6am\omega + 6i\omega M - \lambda)/(4\omega^2)\ . \end{aligned} \tag{D16}$$

If the fundamental solutions for φ, equation (D2) are normalized by

$$\begin{aligned} \varphi_1 &= r^3 \exp(i\omega r_*)\left[1 + \mathcal{O}(1/r)\right] \\ \varphi_2 &= r^{-1} \exp(-i\omega r_*)\left[1 + \mathcal{O}(1/r)\right] \end{aligned} \tag{D17}$$

then the corresponding solutions for χ are (eq. [D5])

$$\chi_1 = -3C_3 r^{-4}\left[1 + \mathcal{O}(1/r)\right]$$

$$\chi_2 = -2i\omega r^{-4}\exp(-2i\omega r_*)\left[1 + \mathcal{O}(1/r)\right] \tag{D18}$$

where

$$\begin{aligned} C_3 = -\Big\{ & 12\omega + 4ia^2\omega^2(m^2 + \lambda + 2am\omega + 3) \\ & + 8iam\omega(1 - 4\omega^2M^2 - 3i\omega M) \\ & - i(\lambda + 2am\omega)\left[\lambda + 2am\omega - 12i\omega M - 2\right]\Big\}/(24\omega^3). \end{aligned} \tag{D19}$$

TABLE 1

POLYNOMIAL APPROXIMATIONS FOR THE ANGULAR EIGENVALUES ${}_{\pm 2}E^{m}_{\ell}(a\omega)$ WITH $\ell \leq 6$ (SEE TEXT FOR DETAILS)

ℓ=2 E= 6. +						
m= 2	-2.66704	-.733301	-.063112	.0071519	.0000556	-.00006648
m= 1	-1.33324	-.577448	-.044922	-.0109315	.0038306	-.00034314
m= 0	0.00088	-.528430	.008472	-.0209644	.0025263	-.00000657
m=-1	1.33250	-.572316	.038860	-.0012912	-.0045311	.00056402
m=-2	2.66684	-.736281	.059979	.0027459	-.0022491	.00012955
ℓ=3 E= 12. +						
m= 3	-1.99983	-.501002	-.047865	-.0055100	.0017950	-.00012886
m= 2	-1.33281	-.410252	.007111	-.0134938	.0005287	.00012146
m= 1	-0.66698	-.349910	.006907	.0060328	-.0044674	.00046121
m= 0	-0.00078	-.329262	-.007370	.0126218	-.0020442	-.00005615
m=-1	0.66749	-.356215	-.003271	-.0054137	.0042397	-.00051531
m=-2	1.33319	-.406766	-.002443	-.0087279	.0020020	-.00019510
m=-3	2.00012	-.500668	.051347	-.0046268	-.0000742	.00000116
ℓ=4 E= 20. +						
m= 4	-1.59991	-.359008	-.033376	-.0040907	.0002589	.00003745
m= 3	-1.20023	-.358292	.001576	.0013031	-.0019482	.00018990
m= 2	-0.80017	-.359597	.015870	.0013607	-.0004809	-.00006020
m= 1	-0.39976	-.362199	.015394	-.0003395	.0008162	-.00013044
m= 0	-0.00009	-.360510	-.001067	.0032991	-.0004615	.00006777
m=-1	0.40003	-.361041	-.012565	.0014466	.0002530	-.00005840
m=-2	0.79994	-.360094	-.017876	.0010567	-.0000151	.00007306
m=-3	1.19992	-.359176	-.005328	-.0005678	.0001105	-.00003007
m=-4	1.60003	-.358728	.034592	-.0038102	.0001703	-.00001149
ℓ=5 E= 30. +						
m= 5	-1.33335	-.270527	-.023430	-.0022118	-.0002452	.00004840
m= 4	-1.06678	-.314780	-.000553	.0005883	-.0003756	-.00001574
m= 3	-0.79994	-.350509	.012217	.0000083	.0002382	-.00006680
m= 2	-0.53331	-.375132	.013208	.0007410	.0000150	.00000002
m= 1	-0.26668	-.389812	.007902	.0010792	-.0000942	.00001032
m= 0	0.00000	-.394892	.000033	.0008844	.0000053	-.00000827
m=-1	0.26665	-.389792	-.008281	.0011075	-.0000355	.00000915
m=-2	0.53336	-.375119	-.012738	.0007261	.0001401	-.00000736
m=-3	0.79996	-.349946	-.011916	.0010161	-.0001063	.00002469
m=-4	1.06665	-.315278	-.000740	-.0000760	-.0000756	-.00000416
m=-5	1.33334	-.270700	.023249	-.0025948	.0001484	-.00000914
ℓ=6 E= 42. +						
m= 6	-1.14288	-.212695	-.016325	-.0014776	-.0002005	.00002073
m= 5	-0.95237	-.276181	-.001386	-.0004355	.0001442	-.00004094
m= 4	-0.76188	-.328010	.006747	.0002154	.0001270	-.00002272
m= 3	-0.57143	-.368133	.009365	.0006675	-.0000201	.00000378
m= 2	-0.38095	-.396898	.008394	.0006334	-.0000398	.00000310
m= 1	-0.19048	-.414164	.004852	.0005355	-.0000276	-.00000057
m= 0	-0.00000	-.419891	-.000045	.0005375	-.0000169	.00000097
m=-1	0.19047	-.414155	-.004861	.0005496	.0000232	.00000030
m=-2	0.38095	-.396910	-.008393	.0006096	.0000424	.00000017
m=-3	0.57143	-.368142	-.009387	.0006528	.0000085	.00000278
m=-4	0.76189	-.327805	-.006629	.0005684	-.0000883	.00000988
m=-5	0.95238	-.276093	.001534	-.0002084	-.0000593	-.00000041
m=-6	1.14286	-.212837	.016084	-.0017368	.0001037	-.00000605
	$\times (a\omega)$	$\times (a\omega)^2$	$\times (a\omega)^3$	$\times (a\omega)^4$	$\times (a\omega)^5$	$\times (a\omega)^6$

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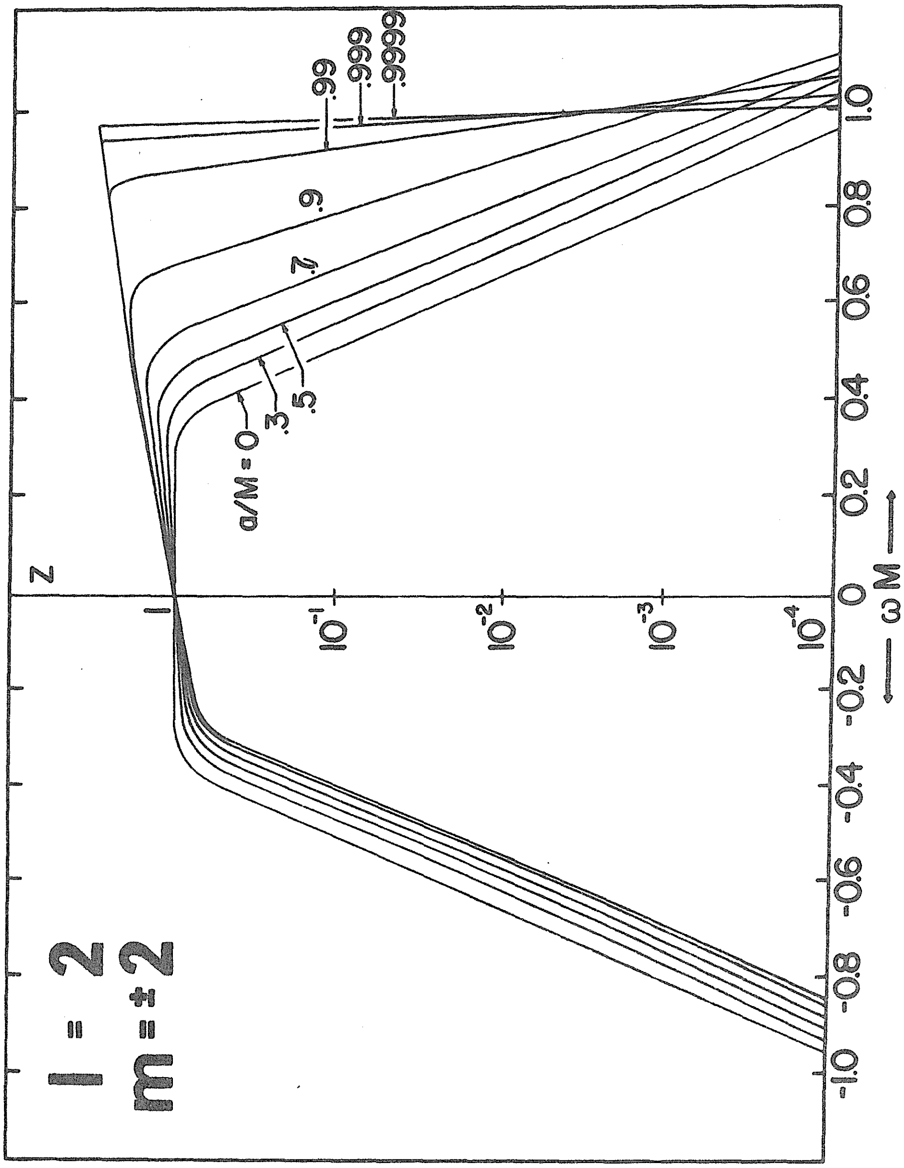
FIGURE CAPTIONS

Fig. 1a,b,c. Asymptotic behavior at radial infinity of a real-frequency perturbation which is regular on the event horizon. Except for normalizing factors (see text for details), Z is the ratio of outgoing-wave to ingoing-wave solution far from the black hole. An instability is a perturbation which can support an outgoing wave with <u>no</u> ingoing wave required; hence an onset of instability (as a/M increases from zero, the stable Schwarzschild hole) would correspond to a pole in Z. These figures show that there are no such poles for any $\ell = 2$ mode. The figures are drawn for m positive. The curves for corresponding negative m are obtained by reflection through $\omega M = 0$. There are no $\ell = 0$ or $\ell = 1$ gravitational modes in the formalism used in this paper.

Fig. 2a,b,c,d. Asymptotic behavior at radial infinity for $\ell = 3$ modes. As for the $\ell = 2$ modes, the function Z has no poles corresponding to an onset of instability through a real-frequency mode. It is almost certain that any instability must set in via a real frequency (see text); furthermore, the smoothness of the Z function in a/M argues directly that there is no sudden appearance of a pole in the upper-half complex plane. In this figure and fig. 1, note that there is not even a tendency toward marginal instability in the "unphysical" limit $a/M = 1$.

Fig. 3. A sample mode with $\ell = 4$, which is also stable. Several higher modes have also been spot-checked, and no tendencies towards instability have been found.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m = ±2
Z
a/M = 0
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.5
.7
.9
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.9999
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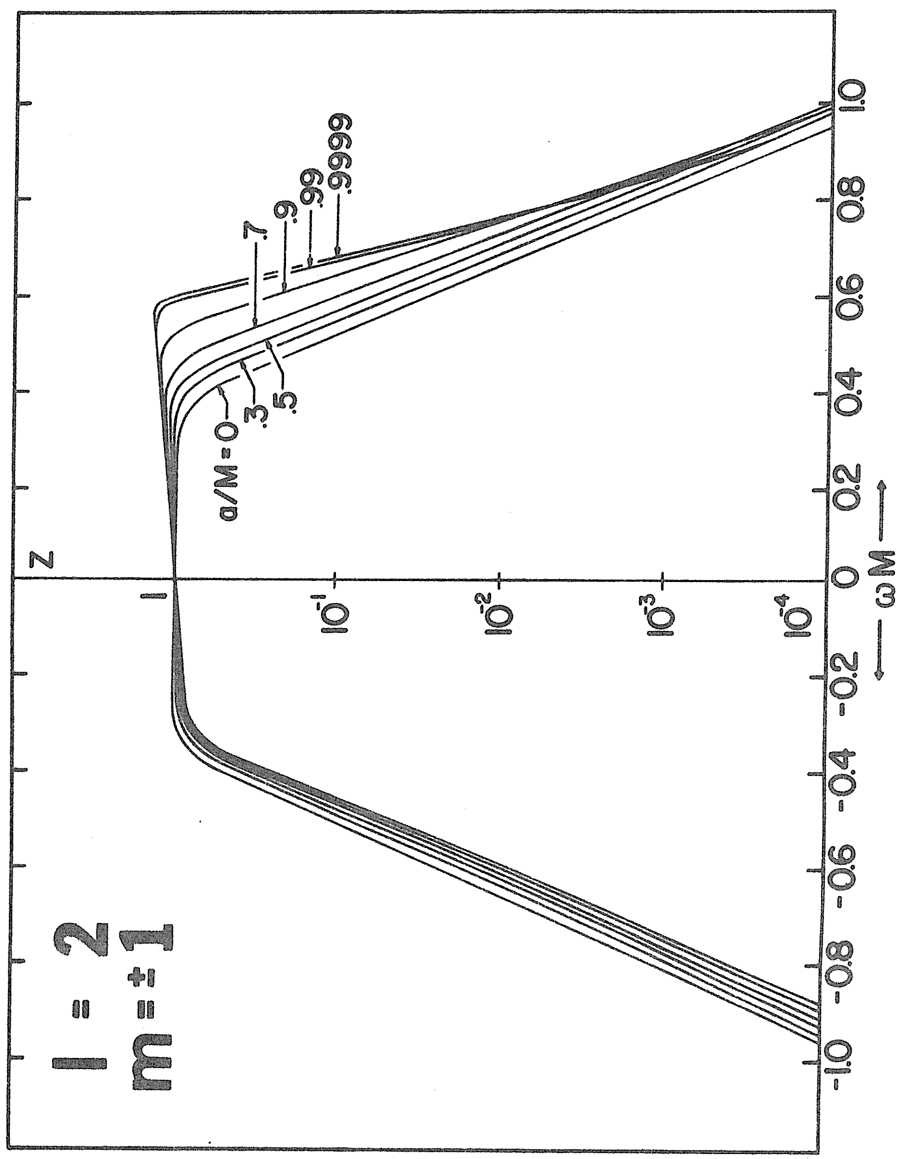
10^{-2}
10^{-3}
10^{-4}
-1.0
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1.0
ωM

Fig. 1a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a/M = 0
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.99
.9999
-1.0
-0.8
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-0.2
0
0.2
0.4
0.6
0.8
1.0
ωM

Fig. 1b

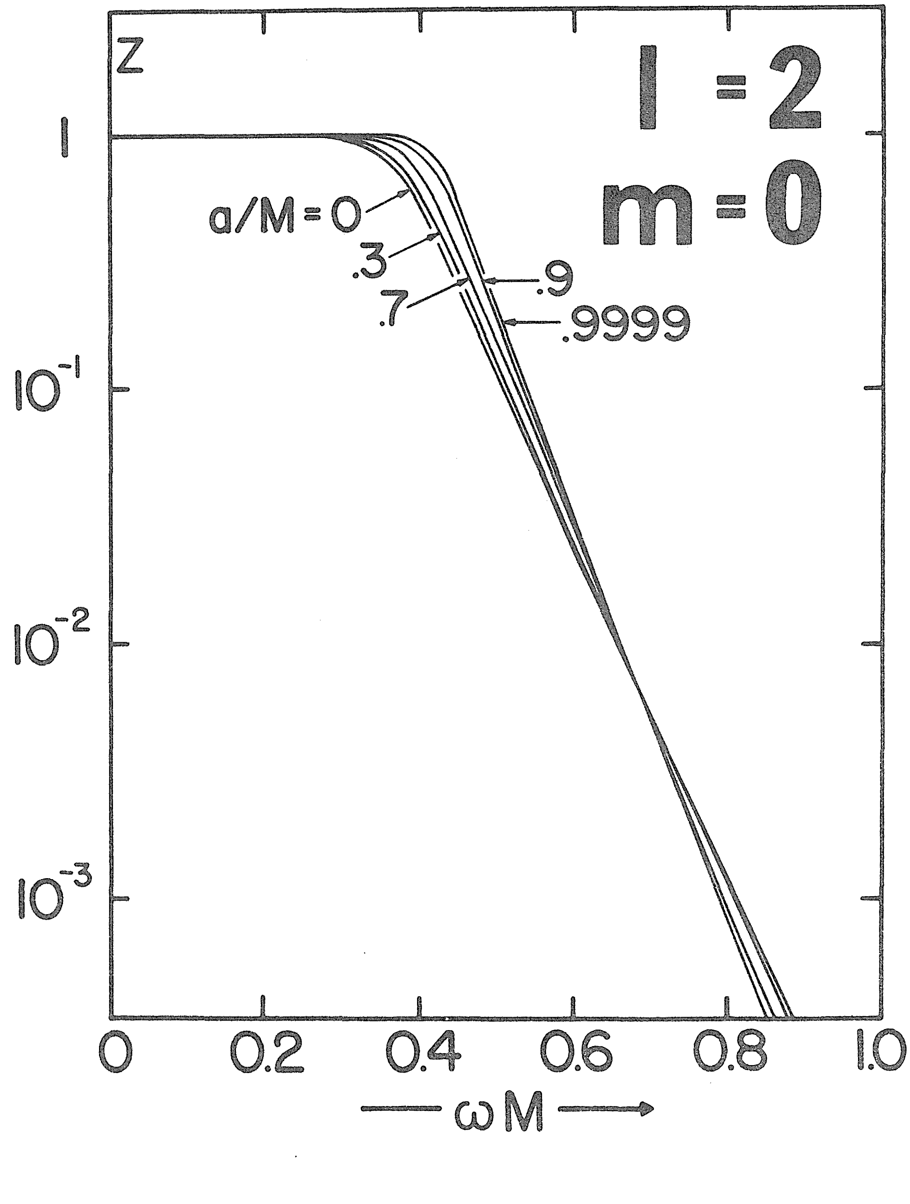
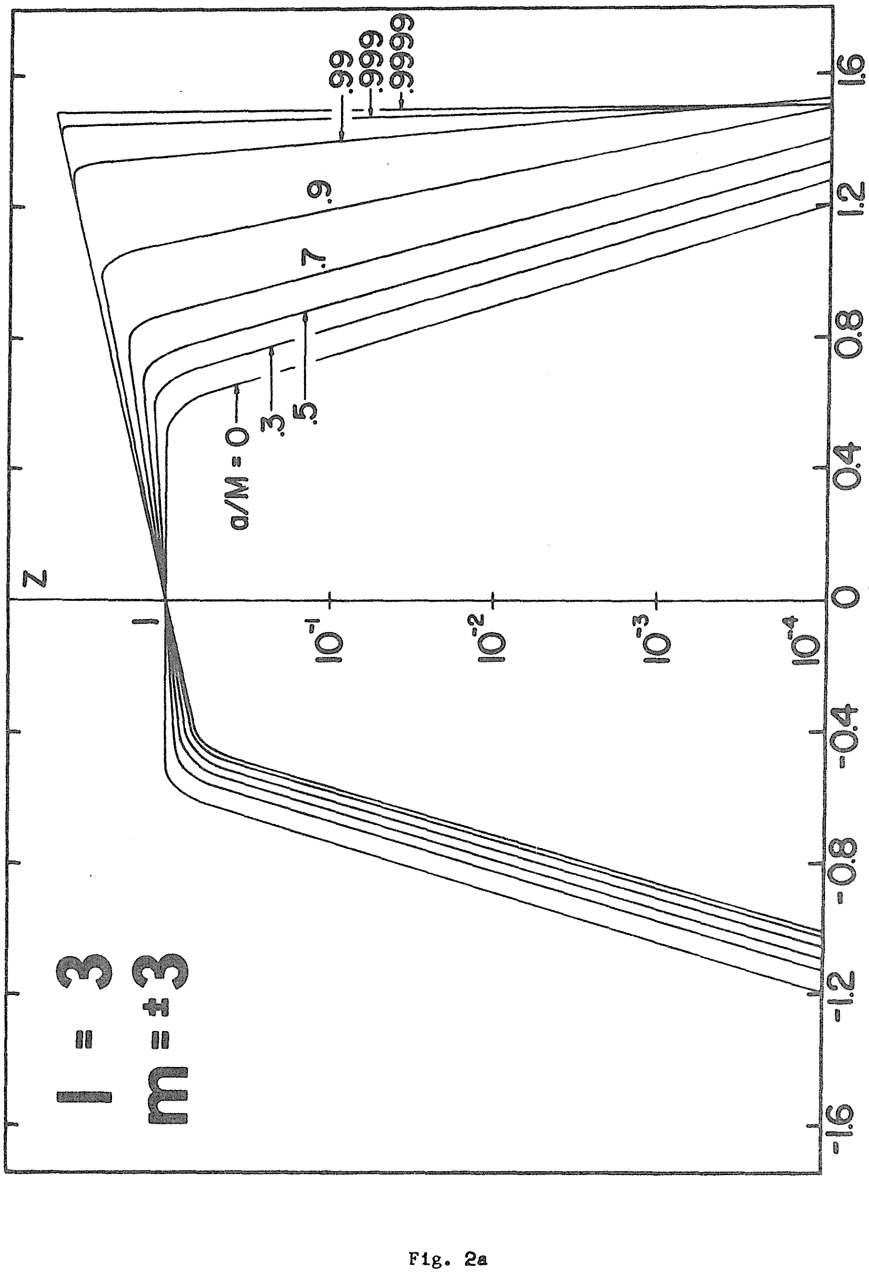


Fig. 1c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m = ±3
Z
a/M = 0
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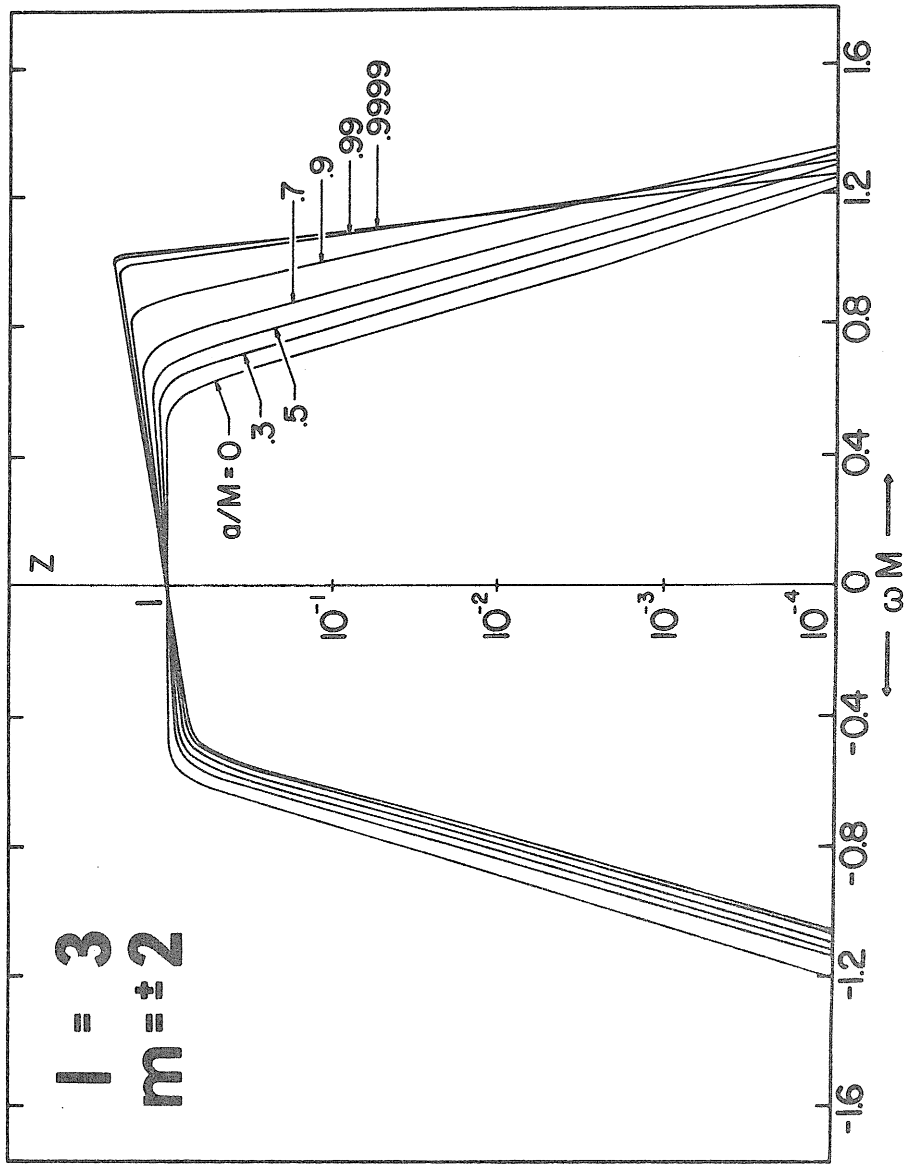
.5
.7
.9
.99
.999
.9999
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
-1.6
-1.2
-0.8
-0.4
0
0.4
0.8
1.2
1.6
ωM

Fig. 2a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m = ±2
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10^{-4}
a/M = 0
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.5
.7
.9
.99
.9999
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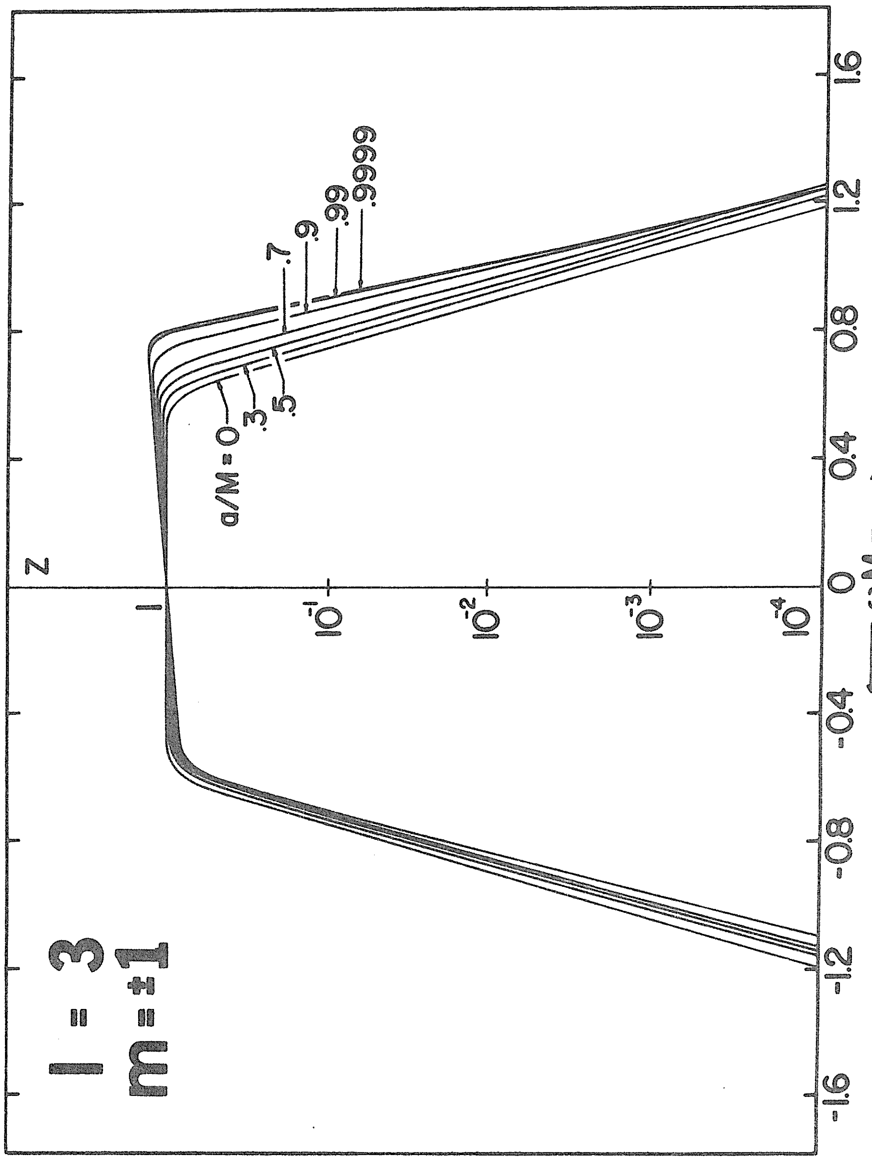
-0.8
-0.4
0
0.4
0.8
1.2
1.6
← ωM →

Fig. 2b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10⁻²
10⁻³
10⁻⁴
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0
0.4
0.8
1.2
1.6
ωM

Fig. 2c

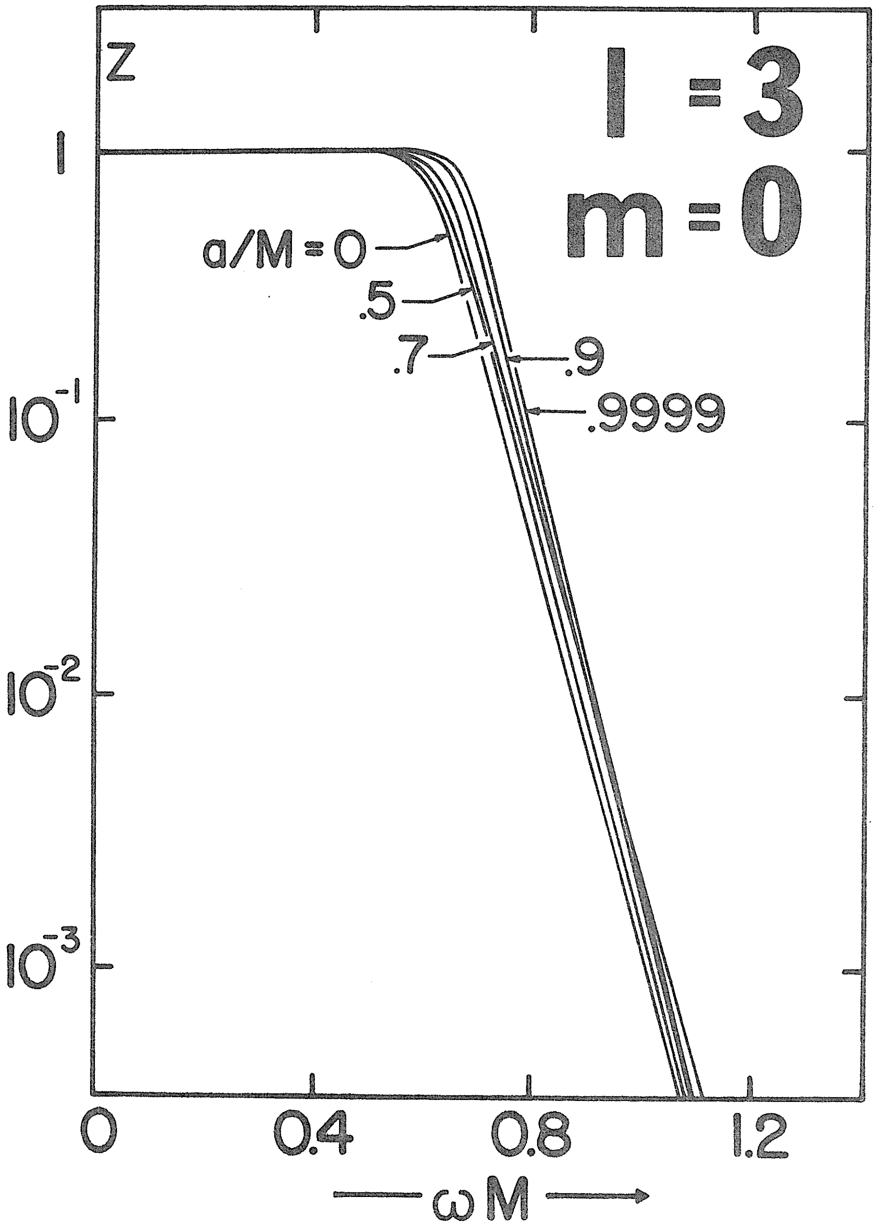
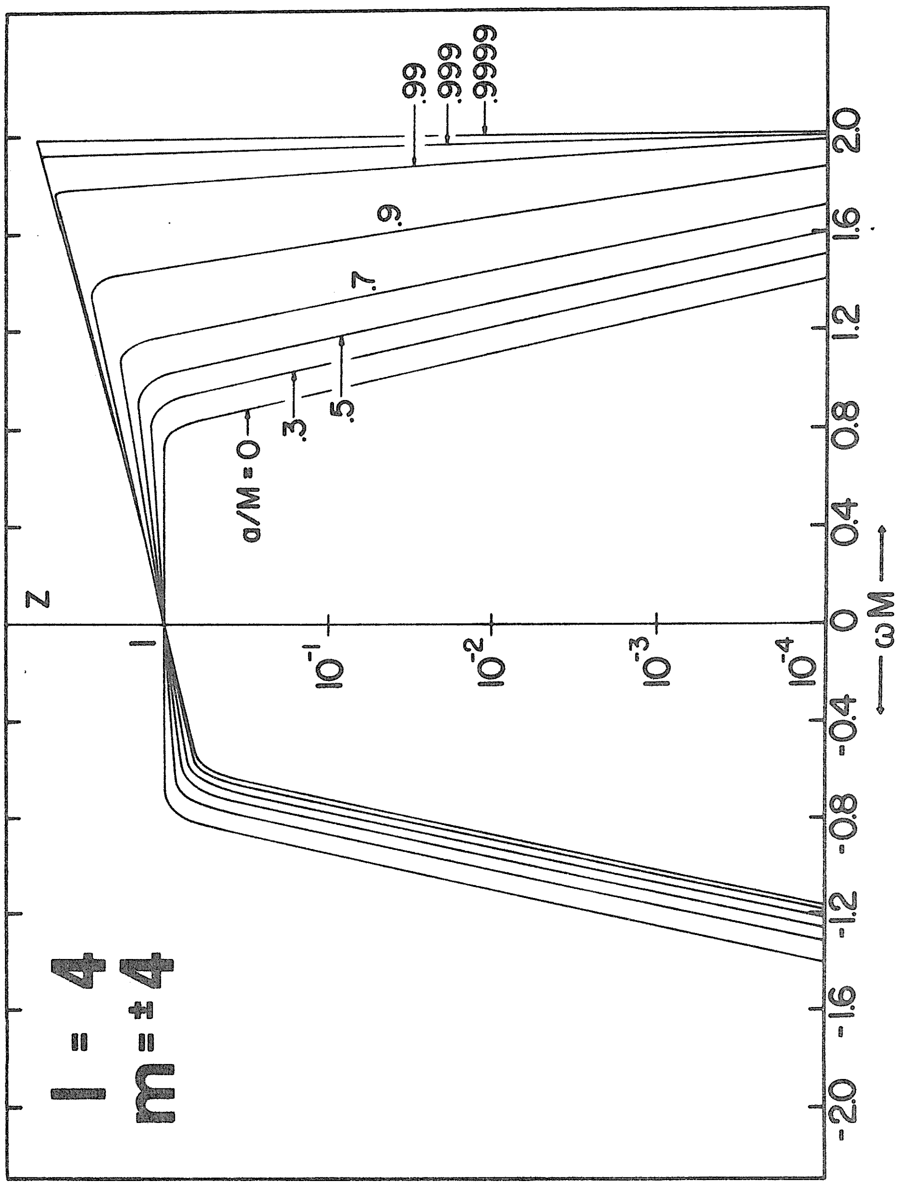


Fig. 2d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m = ±4
Z
a/M = 0
.3
.5
.7
.9
.99
.999
.9999
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
-2.0
-1.6
-1.2
-0.8
-0.4
0
0.4
0.8
1.2
1.6
2.0
ωM

Fig. 3

PART VI

FURTHER APPLICATIONS

(a) Spin Down of a Rotating Black Hole

(i) Introduction

Using powerful global methods, Hawking (1972) has proved an important theorem about black holes. The physical content of the theorem can be stated as follows: "A stationary black hole must be either static or axisymmetric." (Static means stationary plus time-reversal invariant and so excludes rotation). This means that if a rotating Kerr black hole is acted on by a stationary, non-axisymmetric perturbation, it must become non-stationary. Either it must lose its angular momentum and become a static Schwarzschild hole, or else it must find an axisymmetric orientation with respect to the perturbation (if such an orientation exists), or else it must do a combination of these possibilities, losing angular momentum and aligning itself axisymmetrically.

We can use the following picture to understand why the black hole evolves (cf. Ipser 1971). Inside the ergosphere there do not exist any physical observers who are stationary with respect to observers at infinity. Thus a physical observer inside the ergosphere sees a stationary perturbation as dynamical. Such an observer sees a local energy flux across the horizon. This energy cannot come from infinity, where things are stationary; rather, it must come from a decrease in the rotational energy of the hole. Thus, as seen from infinity, the hole loses angular momentum.

This picture is not fully convincing for purely gravitational perturbations, since there is no localizable energy or angular momentum flux. Also, one cannot in this case calculate the magnitude of the effect by integrating the appropriate components of a stress-energy tensor over the horizon. However, Hartle and Hawking (1972) have provided a way of calculating the gravitational spin down by finding the rate of change of the area of the black hole. As they have pointed out, the equations describing this bear a remarkable formal similarity to the equations describing the slowing down of a rotating planet by viscous dissipation in tides caused by an exterior moon. One can think of the perturbation as raising a tidal bulge in the surface of the black hole. As the black hole rotates under this bulge, its rotational energy is dissipated gravitationally.

Press (1972) carried out the first calculation of the spin down effect, using a scalar field perturbation. He was able to obtain both the change in the magnitude and the orientation of the hole by examining the torque acting back on the source of the perturbation. This torque is equal and opposite to the rate of change of the (vector) angular momentum of the hole. Hartle (1973) calculated the magnitude of the spin down (da/dt) for a gravitational perturbation in the limit $a \ll M$. In this section we shall calculate da/dt for scalar, electromagnetic and gravitational perturbations for arbitrary a. For definiteness we shall take the source of the perturbation to be a point particle and we shall be particularly interested in the case when the particle is far from the black hole so that only the lowest multipole of the perturbation is important in spinning down the hole.

(ii) The Perturbation Equations

The perturbations we are considering are governed by the equations of Paper IV. These equations were separated in Boyer-Lindquist coordinates. For stationary perturbations, each perturbation field can be expanded in the form

$$\psi = \sum_{\ell,m} R_{\ell m}(r)\, {}_sY_{\ell m}(\Omega) \quad . \tag{2.1}$$

Here $s = 0$, 1 or 2 according as to whether the perturbation is scalar, electromagnetic or gravitational respectively. The ${}_sY_{\ell m}(\Omega)$ are spin-weighted spherical harmonics (cf. Goldberg *et al*. 1967). The radial factor $R_{\ell m}$ satisfies the following equation:

$$\Delta^{-(s+1)} \frac{d}{dr}\left(\Delta^{s+1} \frac{dR}{dr}\right) + \left(\frac{a^2m^2 + 2iams(r-M)}{\Delta^2} - \frac{(\ell-s)(\ell+s+1)}{\Delta}\right)R$$

$$= \int d\Omega \; {}_sY^*_{\ell m}(\Omega) \; 4\pi\Sigma T/\Delta \quad . \tag{2.2}$$

where $\Delta = r^2 - 2Mr + a^2$ and $\Sigma = r^2 + a^2\cos^2\theta$. The source terms T are given in Table I of Paper IV.

The perturbation field ψ for electromagnetism is a certain tetrad component of the electromagnetic field tensor, while for gravity it is a certain tetrad component of the Weyl tensor. Equation (2.2) is valid for Kinnersley's null tetrad (see Paper IV for details).

Both Boyer-Lindquist coordinates and Kinnersley's tetrad are badly-behaved on the horizon, so at some stages of the calculation it is convenient to transform to Kerr "ingoing" coordinates,

$$dv = dt + dr^*$$

$$d\tilde{\phi} = d\phi + a/(r^2 + a^2)\, dr^* \qquad , \qquad (2.3)$$

and to the Hartle-Hawking null tetrad

$$\vec{\ell}^{HH} = \frac{\Delta}{2(r^2 + a^2)} \vec{\ell} \qquad , \qquad \vec{n}^{HH} = \frac{2(r^2 + a^2)}{\Delta} \vec{n} \qquad (2.4)$$

(see Section 5 of Paper IV for details). The 3-surface element of the horizon turns out to be

$$\begin{aligned} d^3\Sigma_\mu &= \ell_\mu^{HH}\, 2Mr_+ \sin\theta\, d\theta\, d\tilde{\phi}\, dv \\ &= \ell_\mu^{HH}\, 2Mr_+ \sin\theta\, d\theta\, d\phi\, dt \qquad , \end{aligned} \qquad (2.5)$$

where we have used the fact that the Jacobian $\partial(\tilde{\phi},v)/\partial(\phi,t) = 1$. We shall denote $2Mr_+ \sin\theta\, d\theta\, d\phi$ by dS .

(iii) Expressions for dJ/dt

Since the Kerr metric has an axial Killing vector

$$\vec{\xi}_{(\phi)} = \frac{\partial}{\partial\phi} = \frac{\partial}{\partial\tilde{\phi}} \qquad ,$$

there is a conserved angular momentum flux vector $T^\mu{}_\nu\, \xi^\nu{}_{(\phi)}$ associated with the stress-energy tensor $T^\mu{}_\nu$ of the perturbation. If there is a flux of angular momentum across the 2-surface element formed by the intersection of an element of the horizon with two surfaces of constant time v separated by dv , then the change in angular momentum of the hole is

$$dJ = - T_\mu{}^\nu \, \xi^\mu{}_{(\phi)} \, d^3\Sigma_\nu \quad . \tag{3.1}$$

Therefore

$$\frac{dJ}{dt} = - \int T_\mu{}^\nu \, \xi^\mu{}_{(\phi)} \, \ell_\nu{}^{HH} \, dS \quad . \tag{3.2}$$

But on the horizon

$$\vec{\ell}^{HH} = \vec{\xi}_{(t)} + \omega_+ \vec{\xi}_{(\phi)} \quad , \tag{3.3}$$

where $\vec{\xi}_{(t)} = \frac{\partial}{\partial t} = \frac{\partial}{\partial v}$

is the time Killing vector and $\omega_+ = a/(2Mr_+)$ is the angular velocity of the hole. Thus

$$\frac{dJ}{dt} = - \frac{1}{\omega_+} \int T_{\mu\nu} (\ell^{\mu(HH)} - \xi^\mu{}_{(t)}) \ell^{\nu(HH)} dS$$

$$= - \frac{1}{\omega_+} \int T_{\mu\nu} \, \ell^{\mu(HH)} \, \ell^{\nu(HH)} \, dS \tag{3.4}$$

since $T_{\mu\nu}\xi^\mu{}_{(t)}$ is the energy flux vector which is zero for stationary perturbations.

For a scalar perturbation,

$$T_{\mu\nu}\ell^{\mu(HH)}\ell^{\nu(HH)} = \frac{1}{4\pi} (\psi_{,\mu}\psi_{,\nu} - \frac{1}{2} g_{\mu\nu}\psi_{,\alpha}\psi^{,\alpha})\ell^{\mu(HH)}\ell^{\nu(HH)}$$

$$= \frac{(\omega_+)^2}{4\pi} \psi^*_{,\phi} \, \psi_{,\phi}$$

since $\psi = \psi^*$ for a scalar field. Thus

$$\frac{dJ^{(scalar)}}{dt} = -\frac{\omega_+}{4\pi}\int \psi^*_{,\phi}\,\psi_{,\phi}\,dS$$

$$= -\frac{a}{4\pi}\sum_{\ell,m} m^2|R_{\ell m}(r_+)|^2 \quad . \tag{3.5}$$

For an electromagnetic perturbation,

$$T^{\mu\nu}\,\ell_\mu^{\ HH}\,\ell_\nu^{\ HH} = \left(\frac{\Delta}{2(r^2+a^2)}\right)^2 T^{\mu\nu}\,\ell_\mu\ell_\nu$$

$$= \frac{1}{2\pi}\left(\frac{\Delta}{2(r^2+a^2)}\right)^2 |\phi_o|^2$$

where we have used equation (5.12) of Paper IV for the electromagnetic stress-energy tensor. Thus

$$\frac{dJ^{(em)}}{dt} = -\frac{1}{8\pi a}\lim_{r\to r_+}\sum_{\ell,m}|\Delta R_{\ell m}(r_+)|^2 \quad . \tag{3.6}$$

For gravity we use the results of Hartle and Hawking (1972). The area of the black hole is

$$A = 8\pi[M^2+(M^4-J^2)^{1/2}] \quad . \tag{3.7}$$

Thus, if $dM/dt = 0$,

$$\frac{dJ}{dt} = -\frac{(M^2-a^2)^{1/2}}{8\pi a}\frac{dA}{dt} \quad . \tag{3.8}$$

Hartle and Hawking show that

$$\frac{dA}{dt} = \frac{1}{\varepsilon}\int |\sigma^{HH}|^2\,dS \quad . \tag{3.9}$$

Here $\varepsilon = (M^2 - a^2)^{1/2}/(4Mr_+)$ is the unperturbed value of a spin coefficient in the Hartle-Hawking tetrad and σ^{HH} is the perturbation in the shear of the generators of the horizon. The quantity σ^{HH} is related to $\Psi_o{}^{HH}$, a component of the Weyl tensor, by

$$D\sigma^{HH} = 2\varepsilon\sigma^{HH} + \Psi_o{}^{HH} \quad . \tag{3.10}$$

Here the Newman-Penrose operator D for stationary perturbations is

$$D = \vec{\ell}^{HH} = \frac{\partial}{\partial v} + \omega_+ \frac{\partial}{\partial\tilde{\phi}} = \omega_+ \frac{\partial}{\partial\phi} \tag{3.11}$$

Therefore

$$\sigma^{HH} = \frac{1}{\omega_+} e^{2\varepsilon\phi/\omega_+} \int^{\phi} e^{-2\varepsilon\phi/\omega_+} \Psi_o{}^{HH} \, d\phi \quad . \tag{3.12}$$

But

$$\Psi_o{}^{HH} = \left(\frac{\Delta}{2(r^2 + a^2)}\right)^2 \Psi_o \quad , \tag{3.13}$$

where Ψ_o satisfies the separated equation (2.2); so

$$\sigma^{HH} = \left(\frac{\Delta}{2(r^2 + a^2)}\right)^2 \sum_{\ell,m} \frac{R_{\ell m}(r_+) \; {}_2Y_{\ell m}(\Omega)}{-2\varepsilon + im\omega_+} \quad . \tag{3.14}$$

Thus

$$\frac{dJ^{(grav)}}{dt} = - \frac{1}{64\pi a(M^2 - a^2)} \lim_{r \to r_+} \sum_{\ell,m} \left| \frac{\Delta^2 R_{\ell m}(r_+)}{1 - im\omega_+/2\varepsilon} \right|^2 \quad . \tag{3.15}$$

(iv) Source Terms

We shall take the source of the perturbation to be a point particle at fixed Boyer-Lindquist coordinate location $r = b$, $\theta = \theta_0$, $\phi = \phi_0$. The particle is held fixed by stresses which we shall henceforth ignore. For a scalar particle of "mass" μ, the source term is

$$T^{(\text{scalar})} = \mu \int \delta^4[x^\mu - z^\mu(\tau)]\, d\tau \quad , \tag{4.1}$$

where $z^\mu(\tau)$ is the world-line of the particle and τ is proper time along the world-line. Integrating over t gives

$$T^{(\text{scalar})} = \frac{\mu(g_{tt})^{1/2}}{b^2 + a^2\cos^2\theta_0}\, \delta(r-b)\, \delta(\Omega-\Omega_0) \quad , \tag{4.2}$$

where g_{tt} is a component of the metric tensor. Thus

$$\int d\Omega\, Y^*_{\ell m}(\Omega)\, 4\pi\Sigma T/\Delta = \frac{4\pi\mu(g_{tt})^{1/2}}{b^2 - 2Mb + a^2}\, Y^*_{\ell m}(\Omega_0)\delta(r-b) \ . \tag{4.3}$$

It is convenient to write the radial equation (2.2) in the form

$$\mathscr{L}(r)\, R_{\ell m}(r) = \mathscr{D}_{\ell m}(b)\, \delta(r-b) \quad . \tag{4.4}$$

Then

$$\mathscr{D}^{(\text{scalar})}_{\ell m} = \frac{4\pi\mu(g_{tt})^{1/2}}{b^2 - 2Mb + a^2}\, Y^*_{\ell m}(\Omega_0) \quad . \tag{4.5}$$

For a particle of charge e, the electromagnetic 4-current is

$$J = e \int \frac{dz^\mu}{d\tau}\, \delta^4[x^\mu - z^\mu(\tau)]\, d\tau \quad . \tag{4.6}$$

The only non-vanishing component is

$$J^t = \frac{e}{\Sigma}\,\delta(r-b)\,\delta(\Omega-\Omega_o) \quad . \tag{4.7}$$

From equation (3.6) of Paper IV, the electromagnetic source term is

$$\begin{aligned}\frac{4\pi\Sigma T^{(em)}}{\Delta} &= \frac{4\pi(r^2+a^2\cos^2\theta)}{\sqrt{2}(r+ia\cos\theta)\Delta}\Big(\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\phi}-ia\sin\theta\frac{\partial}{\partial r}\\ &\qquad -\frac{ia^2\sin\theta}{\Delta}\frac{\partial}{\partial\phi}\Big)\,J^t\\ &= \frac{4\pi e}{\sqrt{2}(b+ia\cos\theta_o)\Delta}\Big(\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\phi}-ia\sin\theta_o\frac{\partial}{\partial r}\\ &\qquad -\frac{ia^2\sin\theta_o}{\Delta}\frac{\partial}{\partial\phi}\Big)\,\delta(r-b)\delta(\Omega-\Omega_o) \quad . \end{aligned} \tag{4.8}$$

Here we have used the fact that $g(x)\partial_x\delta(x-y) = [g(y)\partial_x - g'(y)]\delta(x-y)$. Now $\partial_\theta + (i/\sin\theta)\partial\phi$ is the operator ð ("edth"; see Goldberg et al. 1967). We use the following relations:

$$\begin{aligned}\int d\Omega\,{}_1Y^*_{\ell m}(\Omega)\,ð\,\delta(\Omega-\Omega_o) &= \int d\Omega\,{}_1Y^*_{\ell m}(\Omega)\sum_{\ell',m'} Y^*_{\ell'm'}(\Omega_o)\,ð\,Y_{\ell'm'}(\Omega)\\ &= [\ell(\ell+1)]^{1/2}\,Y^*_{\ell m}(\Omega_o) \quad . \end{aligned} \tag{4.9a}$$

Similarly,

$$\int d\Omega\,{}_1Y^*_{\ell m}(\Omega)\partial_\phi\,\delta(\Omega-\Omega_o) = im\;{}_1Y^*_{\ell m}(\Omega_o) \quad , \tag{4.9b}$$

$$\int d\Omega\,{}_1Y^*_{\ell m}(\Omega)\,\delta(\Omega-\Omega_o) = {}_1Y^*_{\ell m}(\Omega_o) \quad . \tag{4.9c}$$

These enable us to write

$$\mathscr{D}^{(em)}_{\ell m} = \frac{4\pi e}{\sqrt{2}(b + ia \cos \theta_o)\Delta} ([\ell(\ell+1)]^{1/2} Y^*_{\ell m}(\Omega_o) + ia \sin \theta_o \; {}_1Y^*_{\ell m}(\Omega_o)\frac{\partial}{\partial b} + \frac{a^2 m \sin \theta_o}{b^2 - 2Mb + a^2} \; {}_1Y^*_{\ell m}(\Omega_o)) \tag{4.10}$$

The source term for a gravitational perturbation depends on the stress-energy tensor of the point-particle:

$$T^{\mu\nu} = \mu \int \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} \delta^4[x^\mu - z^\mu(\tau)] \, d\tau \quad . \tag{4.11}$$

The only non-vanishing component of $T^{\mu\nu}$ is

$$T^{tt} = \frac{\mu}{\Sigma(g_{tt})^{1/2}} \delta(r-b) \; \delta(\Omega-\Omega_o) \quad . \tag{4.12}$$

Using equation (2.13) of Paper IV, we find

$$\begin{aligned} 4\pi\Sigma T/\Delta = \frac{-4\pi\Sigma}{\Delta(r + ia \cos \theta)^2} [&(\frac{\partial}{\partial\theta} + \frac{i}{\sin \theta}\frac{\partial}{\partial\phi} - \cot \theta)(\frac{\partial}{\partial\theta} + \frac{i}{\sin \theta}\frac{\partial}{\partial\phi}) \\ &-2ia \sin \theta \; (\frac{\partial}{\partial\theta} + \frac{i}{\sin \theta}\frac{\partial}{\partial\phi})(\frac{\partial}{\partial r} + \frac{a}{\Delta}\frac{\partial}{\partial\phi}) \\ &-a^2 \sin^2\theta \; (\frac{\partial}{\partial r} + \frac{a}{\Delta}\frac{\partial}{\partial\phi})(\frac{\partial}{\partial r} + \frac{a}{\Delta}\frac{\partial}{\partial\phi})] \; T^{tt} \quad . \end{aligned} \tag{4.13}$$

In the limit $b \to \infty$, only the first term in square brackets is important. We have

$$4\pi\Sigma T/\Delta \to \frac{-4\pi\mu}{b^4} (\frac{\partial}{\partial\theta} + \frac{i}{\sin \theta}\frac{\partial}{\partial\phi} - \cot \theta)(\frac{\partial}{\partial\theta} + \frac{i}{\sin \theta}\frac{\partial}{\partial\phi}) \; \delta(r-b) \; \delta(\Omega-\Omega_o) \quad . \tag{4.14}$$

We use the relation

$$\int d\Omega \; {}_2Y^*_{\ell m}(\Omega) \; \eth\eth \; \delta(\Omega-\Omega_o) = [\ell(\ell+1)(\ell-1)(\ell+2)]^{1/2} \; Y^*_{\ell m}(\Omega_o) \tag{4.15}$$

to obtain

$$\mathscr{D}^{(\mathrm{grav})}_{\ell m} \rightarrow \frac{-4\pi\mu}{b^4} [\ell(\ell+1)(\ell-1)(\ell+2)]^{1/2} \; Y^*_{\ell m}(\Omega_o) \quad . \tag{4.16}$$

Note that $\mathscr{D}_{1m} = 0$; this is true even when the terms we have neglected are kept. This means that the lowest multipole of the source that contributes to the spin down effect is the quadrupole, in contrast with the dipole contribution for scalar and electromagnetic perturbations. This is as expected--gravitational spin down is a tidal effect.

(v) Solution of the Radial Equation

We wish to solve

$$\mathscr{L}(r) \; R_{\ell m}(r) = \mathscr{D}_{\ell m}(b) \; \delta(r-b) \tag{5.1}$$

subject to suitable boundary conditions. For $r \rightarrow \infty$, the asymptotic solutions of equation (5.1) are $R_{\ell m} \sim r^{\ell-s}$ or $r^{-(\ell+s+1)}$. The correct boundary condition is the falling solution $r^{-(\ell+s+1)}$. For $r \rightarrow r_+$, $R_{\ell m} \sim \Delta^{-s}\exp(im\omega_+ r^*)$ or $\exp(-im\omega_+ r^*)$. The first of these is the correct boundary condition; it represents an ingoing energy flux to a local observer (cf. Section 5 of Paper IV). Let $R^{\infty}{}_{\ell m}$ and $R^{+}{}_{\ell m}$ be solutions of the homogeneous equation

$$\mathscr{L}(r) \; R_{\ell m}(r) = 0 \tag{5.2}$$

such that

$$R^{\infty}{}_{\ell m} \sim r^{-(\ell+s+1)} \quad , \qquad (r \to \infty) \quad ,$$

$$R^{+}{}_{\ell m} \sim \Delta^{-s}\, e^{im\omega_+ r^*} \quad , \qquad (r \to r_+) \quad . \tag{5.3}$$

Then the solution of equation (5.1) satisfying the correct boundary conditions is

$$R_{\ell m}(r) = \begin{cases} \mathscr{D}_{\ell m}(b)\, \dfrac{R^{\infty}{}_{\ell m}(b)\; R^{+}{}_{\ell m}(r)}{W_{\ell m}} \;, & r \le b \\[2ex] \mathscr{D}_{\ell m}(b)\, \dfrac{R^{+}{}_{\ell m}(b)\; R^{\infty}{}_{\ell m}(r)}{W_{\ell m}} \;, & r \ge b \end{cases} \tag{5.4}$$

Here $W_{\ell m}$ is the Wronskian

$$W_{\ell m} = [R^{+}{}_{\ell m} \partial_r\, R^{\infty}{}_{\ell m} - R^{\infty}{}_{\ell m}\, \partial_r\, R^{+}{}_{\ell m}]_{r=b} \quad . \tag{5.5}$$

It is convenient to introduce dimensionless variables

$$x = \frac{r - r_+}{r_+ - r_-}$$

$$\gamma = \frac{iam}{r_+ - r_-} \quad . \tag{5.6}$$

Then two fundamental solutions of the homogeneous equation (5.2) are

$$y_1 = x^{-s+\gamma}(1+x)^{-s-\gamma}\, F(-\ell - s,\ \ell-s+1;\ 1-s+2\gamma;\ -x) \tag{5.7}$$

$$y_2 = x^{-\gamma}(1+x)^{\gamma}\, F(-\ell + s,\ \ell+s+1;\ 1+s-2\gamma;\ -x) \quad . \tag{5.8}$$

Here the hypergeometric functions F are polynomials which have the value 1 at the horizon $(x = 0)$. We can therefore take

$$R^{+}_{\ell m} = y_1 \quad . \tag{5.9}$$

An appropriate linear combination of y_1 and y_2 gives a solution with the correct behavior at infinity:

$$R^{\infty}_{\ell m} = \frac{\Gamma(2\ell+2)\ \ \Gamma(s-2\gamma)}{\Gamma(\ell+s+1)\ \Gamma(\ell+1-2\gamma)}\, y_1 + \frac{\Gamma(2\ell+2)\ \Gamma(2\gamma-s)}{\Gamma(\ell+1+2\gamma)\ \Gamma(\ell+1-s)}\, y_2$$

$$= x^{-(\ell+s+1)}(1+1/x)^{-s-\gamma}\ F(\ell+1-s,\ell+1-2\gamma;\ 2\ell+2;\ -1/x) \tag{5.10}$$

(cf. Erdelyi et al. 1953). With these normalizations, we find

$$W_{\ell m} = -\frac{\Gamma(2\ell+2)\ \Gamma(1-s+2\gamma)}{\Gamma(\ell-s+1)\ \Gamma(\ell+1+2\gamma)}\ \frac{(r_+ - r_-)^{2s+1}}{(b^2 - 2Mb + a^2)^{s+1}} \quad . \tag{5.11}$$

(vi) Magnitude of the Spin Down Effect

Combining equations (3.5), (5.4), (4.5) and (5.9)-(5.11), we have

$$\frac{dJ^{(\text{scalar})}}{dt} = -4\pi a\mu^2\left(1 - \frac{2Mb}{b^2 + a^2\cos^2\theta_o}\right)\sum_{\ell,m}\left|m\ Y_{\ell m}(\Omega_o)\ \frac{\ell!\ \Gamma(\ell+1+2\gamma)}{(2\ell+1)!\,\Gamma(1+2\gamma)}\right.$$

$$\times\ \left.\frac{(r_+ - r_-)^{\ell}}{(b - r_+)^{\ell+1}}\ F\!\left(\ell+1,\ell+1-2\gamma;\ 2\ell+2;\ -\frac{r_+ - r_-}{b - r_+}\right)\right|^2 . \tag{6.1}$$

For $b \to \infty$, the $\ell = 1$ term of the sum dominates. This term is

$$\frac{1}{9b^4}\sum_m m^2|Y_{1m}(\Omega_o)|^2\ (M^2 - a^2 + a^2m^2) \quad .$$

Using the formulae given in the Appendix to do the sum over m, we find

$$\frac{dJ^{(scalar)}}{dt} = - \frac{a\mu^2 M^2}{3b^4} \sin^2\theta_o \qquad (b \to \infty) \quad . \tag{6.2}$$

This agrees with the result of Press (1972), derived by a very different technique.

Using equations (3.6), (5.4), (4.10) and (5.9)-(5.11), we can write out an expression similar to equation (6.1) for $dJ^{(em)}/dt$. The general expression is rather cumbersome, so we shall restrict ourselves to the case $b \to \infty$, when only the $\ell = 1$ term of the sum is important. For $s = 1$ we have in the limit $b \to \infty$

$$\mathscr{D}_{1m} \to \frac{4\pi e}{b^3} Y^*_{1m}(\Omega_o)$$

$$R^{\infty}{}_{1m} \to \left(\frac{r_+ - r_-}{b}\right)^3$$

$$W_{1m} \to \frac{-3!}{2\gamma(2\gamma+1)} \frac{(r_+ - r_-)^3}{b^4} \quad .$$

Thus

$$\frac{dJ^{(em)}}{dt} = - \frac{2e^2 a M^2}{3b^4} \sin^2\theta_o \qquad (b \to \infty) \quad . \tag{6.3}$$

This is double the scalar value.

In the gravitational case we shall also only give explicit formulae for $b \to \infty$. For $s = 2$ and $\ell = 2$, equations (5.9)-(5.11) give

$$\mathscr{D}_{2m} \to \frac{-4\pi\mu}{b^4} (24)^{1/2} Y^*_{2m}(\Omega_o)$$

$$R^{\infty}{}_{2m} \to \left(\frac{r_+ - r_-}{b}\right)^5$$

$$W_{2m} \to \frac{-5!}{(2\gamma+2)(2\gamma+1)2\gamma(2\gamma-1)} \frac{(r_+ - r_-)^5}{b^6} .$$

Since $im\omega_+/2\varepsilon = 2\gamma$, equation (3.15) gives after some simplification

$$\frac{dJ^{(grav)}}{dt} = - \frac{8\mu^2 aM^4}{5b^6} \sin^2\theta_o \left(1 - \frac{3a^2}{4M^2} + \frac{15a^2}{4M^2} \sin^2\theta_o\right) . \qquad (6.4)$$

This is four times bigger than Hartle's result (1973) in the limit $a/M \to 0$, and presumably this is due to a slight error in either Hartle's or our calculations.

(vii) Discussion and Conclusions

Note that $dJ^{(scalar)}/dt$ is proportional to g_{tt} , which comes from the source term [cf. equation (4.5)]. The quantity $dJ^{(em)}/dt$ has no g_{tt} [cf. equation (4.10)], while $dJ^{(grav)}/dt$ is proportional to $1/g_{tt}$ [cf. equation (4.12)]. This is the same effect that is present in special relativity, where the apparent strength of a charge varies with its velocity v as $(1 - v^2)^{(1-s)/2}$, where s is the spin of the field [e.g., for mass, $m = m_o(1 - v^2)^{-1/2}$] . Here we see that as the source of the perturbation approaches the boundary of the ergosphere, where $g_{tt} \to 0$, the scalar effect disappears, the electromagnetic effect remains, while the gravitational effect becomes infinite. This is because the stationary charge is moving with a velocity approaching the speed of light as seen by a physical observer at the boundary of the ergosphere.

The above calculation does not enable us to compute the time-evolution of the hole's angular momentum: we need to know the rate of change of the orientation as well as that of the magnitude. Unfortunately, because there is only axial symmetry and not spherical symmetry, only the z-component of angular momentum density is well-defined everywhere, even in the strong field region near the black hole. Press (1972) was able to find dJ_x/dt and dJ_y/dt in the scalar case by examining the torque exerted by the perturbing field back on the source. Since the source is in a Newtonian region, one can use standard Newtonian formulae. The analogous calculation in the electromagnetic and gravitational cases appears to be considerably more complicated and is still in progress.

The most interesting question to be answered by the above analysis is: does the spin down effect limit the possibility of finding rotating black holes in nature? For example, imagine an isolated rotating black hole immersed in an interstellar magnetic field (cf. Press 1972). Equation (6.3) gives

$$\frac{dJ}{dt} \sim -\frac{J}{\tau} \quad , \tag{7.1}$$

where the time constant for angular momentum loss is

$$\tau \sim M^{-1} \times (\text{energy density in field})^{-1} \tag{7.2}$$

$$\sim 10^{37} \text{years} (M_{\odot}/M)(B/10^{-5}\text{gauss})^{-2} \quad . \tag{7.3}$$

This makes the effect completely negligible on astronomical time scales. Similarly, the gravitational effect has a time constant given by equation (6.4):

$$\tau \sim b^6/(\mu^2 M^3) \tag{7.4}$$

$$\sim (\text{perturbing Riemann tensor})^{-2} \times M^{-3} \quad . \tag{7.5}$$

This is completely negligible for an interstellar gravitational field. In the case of a binary star system, one of whose members is a black hole, equation (7.4) can be written (taking $\mu \sim M$)

$$\tau \sim \tau_{\text{Kepler}} \times (\frac{b}{M})^{9/2} \quad . \tag{7.6}$$

The time constant for decay of the binary system by gravitational radiation is, however,

$$\tau_{\text{grav.rad.}} \sim \tau_{\text{Kepler}} \times (\frac{b}{M})^{5/2} \quad . \tag{7.7}$$

Applying the results of our calculation for a stationary perturbation to a binary system is only valid when $b/M >> 1$, and in this case $\tau_{\text{spin down}} >> \tau_{\text{grav.rad.}}$, so once again the effect is negligible. The case of a close binary system $(b \sim M)$ requires a dynamical calculation using the equations of Paper IV.

APPENDIX: Certain Sums of Spherical Harmonics

In Section (vi) we needed to sum expressions of the form

$$I_n = \frac{4\pi}{2\ell+1} \sum_m m^{2n} |Y_{\ell m}(\theta,\phi)|^2 \quad . \tag{A.1}$$

Following Hartle (1973), we can manipulate this to give

$$I_n = (-1)^n \frac{4\pi}{2\ell+1} \sum_m \left[\frac{d^{2n}}{d\lambda^{2n}} Y^*_{\ell m}(\theta,\phi+\lambda)\ Y_{\ell m}(\theta,\phi)\right]_{\lambda=0}$$

$$= (-1)^n \frac{d^{2n}}{d\lambda^{2n}} P_\ell(\cos^2\theta + \sin^2\theta \cos\lambda)\Big|_{\lambda=0} \quad , \tag{A.2}$$

where we have used the addition theorem for spherical harmonics in the second line. Thus

$$I_1 = P'_\ell(1) \sin^2\theta = \frac{1}{2}\ell(\ell+1)\sin^2\theta \quad , \tag{A.3}$$

$$I_2 = P'_\ell(1)\sin^2\theta + 3P''_\ell(1)\sin^4\theta$$

$$= \frac{1}{2}\ell(\ell+1)\sin^2\theta\,[1 + \frac{3}{4}(\ell-1)(\ell+2)\sin^2\theta] \quad , \tag{A.4}$$

$$I_3 = P'_\ell(1)\sin^2\theta + 15P''_\ell(1)\sin^4\theta + 9P'''_\ell(1)\sin^6\theta$$

$$= \frac{1}{2}\ell(\ell+1)\sin^2\theta\,[1 + 15\sin^2\theta(\ell-1)(\ell+2)/4$$

$$+ 3\sin^4\theta(\ell-1)(\ell+2)(\ell-2)(\ell+3)/8] \quad . \tag{A.5}$$

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(b) Superradiant Scattering (Collaboration with W. H. Press and B. A. Zimmerman)

(i) Introduction

Rotating black holes are vast storehouses of rotational energy. The only known theoretical limit on the amount of energy that can be extracted in principle is provided by Hawking's (1972) theorem, that the surface area of a black hole cannot decrease. For an isolated black hole, this provides a limit to the amount of energy that can be extracted of 29% of the original mass, to be compared with less than 1% maximum efficiency for nuclear processes in stars. Are there plausible mechanisms for extracting this energy?

There are many rotating systems in physics which are unstable when their angular momentum exceeds a certain critical value. However, we have seen in Paper V that a rotating black hole is stable for all values of a . There is therefore no possibility of getting out large amounts of energy by dropping in a small particle to take the black hole into an unstable regime.

Historically, the first energy-extraction mechanism was suggested by Penrose (1969). A particle of rest mass m_0 and total energy E_0 travels from infinity through the ergosphere of a rotating black hole. Inside the ergosphere the particle splits into two particles, one of which goes down the hole and the other of which travels out to infinity with rest mass $m_1 < m_0$. However, if things are arranged just right, the particle can come out with a greater total energy E_1 than the initial energy E_0 --the extra energy comes from the rotational energy of the hole. Unfortunately, a detailed investigation of the conditions under which the Penrose process will work (see

Paper I) shows that the two particles in the ergosphere must separate with a locally measured velocity of at least $\frac{1}{2}$ c. This makes the process astrophysically unlikely.

A far more promising mechanism has been suggested by Misner (1972). (See also Zel'dovich 1971,1972). This is the wave analog of the Penrose process. A wave comes in from infinity and scatters off a rotating black hole. Part of the wave is absorbed and part is transmitted. Under the right conditions, more energy comes out than went in--the wave is "superradiantly scattered". As we shall see in the next section, the conditions under which superradiant scattering can occur are much less stringent than those for the Penrose process, making it a phenomenon of possible astrophysical importance.

Paper II described the results of a numerical calculation of the magnitude of the superradiance effect for scalar waves. It was found that the maximum amplification was 0.3% in energy. We report here results of numerical calculations for electromagnetic and gravitational waves.

(ii) Criterion for Superradiance

The conditions under which superradiance will occur can be derived by integrating the appropriate components of the stress-energy tensor of the waves over the horizon and determining criteria for the flux of "energy at infinity" to be outward, even though the flux of "local energy" is inward. This effect is possible because "energy at infinity" is defined with respect to the Killing vector which is timelike there, and this Killing vector is spacelike inside the ergosphere where the horizon resides. This method of computation does not work for gravitational waves because one has no stress-energy tensor for such waves valid near the horizon.

A much simpler way to find the conditions for superradiance, which works for gravitational waves as well as for scalar and electromagnetic waves, is to use the area theorem. (This approach has also been used by Bekenstein 1973.) The area of a Kerr black hole of mass M and angular momentum $J = aM$ is

$$A = 8\pi[M^2 + (M^4 - J^2)^{1/2}] \equiv 8\pi M r_+ , \tag{2.1}$$

where $r_+ = M + (M^2 - a^2)^{1/2}$. When a wave scatters off the hole, M , J and hence A change. Equation (2.1) gives

$$\begin{aligned}\frac{1}{8\pi}\,\delta A &= \left(2M + \frac{2M^3}{(M^4 - J^2)^{1/2}}\right)\delta M - \frac{J}{(M^4 - J^2)^{1/2}}\,\delta J \\ &= \frac{2M^2 r_+}{(M^4 - J^2)^{1/2}}\,\delta M - \frac{J}{(M^4 - J^2)^{1/2}}\,\delta J \quad .\end{aligned}$$

Hawking's theorem says $\delta A \geq 0$, so

$$2Mr_+\,\delta M \geq a\delta J \quad . \tag{2.2}$$

The Kerr metric is stationary and axisymmetric, so any wave can be decomposed into modes of the form

$$\psi = e^{-i\omega t}\,e^{im\phi}\,f(r,\theta) \quad . \tag{2.3}$$

For such a mode, the ratio of angular momentum to energy at infinity is m/ω ; thus

$$\frac{\delta J}{\delta M} = \frac{m}{\omega} \quad . \tag{2.4}$$

Equation (2.2) now gives

$$(2Mr_+ - am/\omega)\delta M \geq 0 \quad .$$

Superradiance occurs when $\delta M < 0$, i.e., when

$$0 < \frac{\omega}{m} < \omega_+ \quad , \tag{2.5}$$

where we have introduced the angular velocity of the black hole, $\omega_+ = a/(2Mr_+)$.

(iii) Computation of the Magnitude of Superradiance

The magnitude of the superradiance effect can be computed using the separable equations of Paper IV. A wave ψ (scalar, electromagnetic or gravitational) of amplitude Z_{in} is sent in from infinity, scatters off the hole, and comes out with amplitude Z_{out} . The ratio of energy flux out to energy flux in is not simply $|Z_{out}/Z_{in}|^2$, except in the scalar case. There is a correction factor because ψ does not treat ingoing and outgoing waves on an equal footing.

Let us consider the electromagnetic case, where we integrate numerically the separable equation for Φ_2 . This quantity behaves at large r as

$$\Phi_2 \sim e^{-i\omega t}\, e^{im\phi}\, S_2(\theta)(Z_{out}e^{i\omega r^*}/r + Z_{in}e^{-i\omega r^*}/r^3) \ . \qquad (3.1)$$

The other Newman-Penrose components of the electromagnetic field tensor at large r are

$$\Phi_1 \sim e^{-i\omega t}\, e^{im\phi}\, S_1(\theta)(Y_{out}e^{i\omega r^*}/r^2 + Y_{in}e^{-i\omega r^*}/r^2) \qquad (3.2)$$

$$\Phi_o \sim e^{-i\omega t}\, e^{im\phi}\, S_o(\theta)(X_{out}e^{i\omega rt}/r^3 + X_{in}e^{-i\omega r^*}/r) \ . \qquad (3.3)$$

The ratio of outgoing to ingoing energy flux at infinity can be found from the expression for the electromagnetic stress-energy tensor [Paper IV, equation (5.12)]. We find

$$\frac{dE^{(out)}/dt}{dE^{(in)}/dt} = \lim_{r \to \infty} \frac{4 \int |\Phi_2|^2\, r^2 d\Omega}{\int |\Phi_o|^2\, r^2 d\Omega}$$

$$= \frac{4|Z_{out}|^2 \int (S_2)^2\, d(\cos\theta)}{|X_{in}|^2 \int (S_o)^2\, d(\cos\theta)} \ . \qquad (3.4)$$

(The factor of 4 comes from the 1/2 in the definition of the tetrad vector $\vec{n}$ as opposed to $\vec{\ell}$.) We now use Maxwell's equations at large r [equations (3.1) and (3.3) of Paper IV] to relate $X_{in}S_o$ to $Z_{in}S_2$. The result is that we can normalize S_o such that

$$\left(\frac{\partial}{\partial\theta} + \frac{m}{\sin\theta} + \cot\theta - a\omega\sin\theta\right)S_o = S_1 \quad , \qquad (3.5a)$$

$$\left(\frac{\partial}{\partial\theta} + \frac{m}{\sin\theta} - a\omega\sin\theta\right)S_1 = S_2 \quad , \qquad (3.5b)$$

and then

$$X_{in} = -8\omega^2 Z_{in} \quad . \tag{3.6}$$

Thus we have a complete prescription for calculating both ingoing and outgoing energies from the solution for Φ_2 alone. The radial equation for Φ_2 is integrated from the horizon to infinity, with the boundary condition of ingoing local energy at the horizon (cf. Paper IV Section 5). At infinity we can read off Z_{out}/Z_{in} . The quantity $S_o(\theta)$ can be found from the known quantity $S_2(\theta)$ by integrating equations (3.5). Then equation (3.4) gives the ratio of the energies.

The gravitational calculation is similar to that described for electromagnetism above. The calculation of Z_{out}/Z_{in} is exactly the same as the calculation done in Paper V to investigate the stability of rotating black holes. The relevant equations for converting Z_{out}/Z_{in} to energy are given in Appendix B of Paper V. The energy correction factor was calculated by numerical integration of the equations in Appendix B of Paper V and some results are plotted in Figure 1.

(iv) Results and Discussion

While not all the relevant modes have yet been computed, the results so far are sufficient to draw several interesting conclusions. For the electromagnetic case, the superradiance effect is qualitatively similar to the scalar case, but somewhat larger. The maximum energy amplification is about 4%, and occurs for the lowest modes ($\ell = m = 1$, $\ell = m = 2$) when a approaches M . Modes with $\ell \neq m$

give negligible amplification, and the amplification decreases with increasing ℓ .

The gravitational effect is surprisingly larger than the scalar or electromagnetic cases. The maximum energy amplification is about 125% for the $\ell = m = 2$ mode in the limit $a \to M$. The $\ell = m = 3$ mode is around 20%, and other modes are much smaller.

The possible astrophysical significance of these results is not yet clear. For $a = M = 3M_{\odot}$, $\omega_{+} = 3 \times 10^{4}$Hz. The plasma frequency of the interstellar medium (1 electron per cm^{3}) is around 6×10^{4}Hz; however, accretion onto the black hole will produce a much higher electron density and so $\omega_{+} << \omega_{plasma}$ near the black hole. Perhaps this provides the "mirror" necessary for the "black hole bomb" effect (see Paper II). It is clear that more detailed astrophysical models are required of black-hole processes before we can say anything definitive about the importance of exotic effects such as superradiance.

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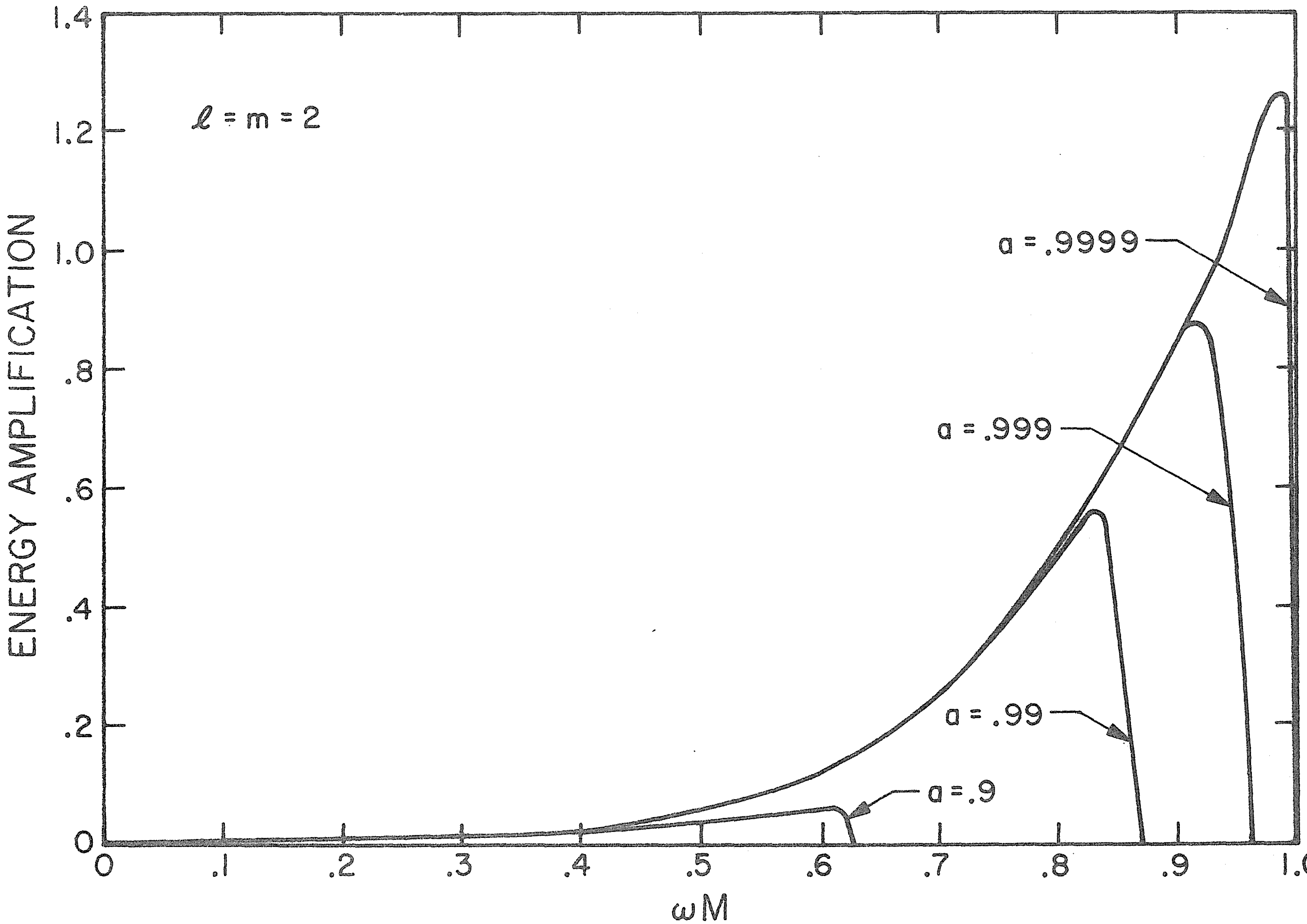
FIGURE 1

Plot of energy amplification as a function of frequency, for $\ell = m = 2$ and various values of a (a is in units of M). The vertical scale is $(|Z_{out}/Z_{in}|^2_{corrected} - 1)$, where "corrected" means converted to give energy rather than amplitude, as described in the text. The peak remains finite in the limit $a/M \to 1$.



ENERGY AMPLIFICATION
1.4
1.2
1.0
.8
.6
.4
.2
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
ωM
ℓ = m = 2
a = .9999
a = .999
a = .99
a = .9

PART VII

APPENDIX

(a) On the Evolution of the Secularly Unstable, Viscous MacLaurin Spheroids (Paper VI; collaboration with W. H. Press, published in Ap. J. 181, 513 [1972]).

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ON THE EVOLUTION OF THE SECULARLY UNSTABLE, VISCOUS MACLAURIN SPHEROIDS*

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ABSTRACT

Previous investigations, which are superficially contradictory, are here reconciled. With new numerical results, a consistent picture emerges: A secularly unstable, viscous Maclaurin spheroid slowly and monotonically deforms itself into a stable, Jacobi ellipsoid. The intermediate configurations are Riemann S-type ellipsoids. A misnomer from previous methods terms this monotonic evolution "anti-damped oscillation"; in actuality no physical fluid oscillation is involved. The evolutionary path is almost certainly independent of the details of the viscous force when the viscous force is sufficiently small.
Subject headings: hydrodynamics — rotation

I. DISCUSSION OF PREVIOUS INVESTIGATIONS

A mass M of viscous, incompressible (density ρ), self-gravitating fluid whose angular momentum L exceeds the value $0.2392G^{1/2}\rho^{-1/6}M^{5/3}$ admits two stationary equilibrium configurations, the Maclaurin spheroid and the Jacobi ellipsoid (see Chandrasekhar 1969 [hereafter cited as EFE], and references therein). If L does not exceed $0.4010G^{1/2}\rho^{-1/6}M^{5/3}$, both configurations are dynamically stable (i.e., absolutely stable against small perturbations in the limit of vanishing viscosity). Since, however, the Jacobi configuration has a lower total energy, it has long been presumed that any sort of dissipative viscosity will render the Maclaurin spheroid "secularly" unstable: given an initial perturbation, it should evolve gradually to the Jacobi ellipsoid of equal angular momentum.

Roberts and Stewartson (1963) first elucidated the details of this process by considering the effect of a uniform viscosity on the hydrodynamical equations linearized about a Maclaurin configuration. In this limit of small perturbations, they found one "anti-damped" oscillatory mode, and interpreted the growth of this mode as the initial motion away from the Maclaurin, and ("presumably") toward the Jacobi, solution. In this formulation, viscous stresses appear entirely in a boundary layer at the surface of the fluid mass, since the anti-damped mode has fluid velocities which are linear functions of coordinates—hence a viscous-stress tensor which vanishes at internal points.

Rosenkilde (1967) avoided boundary layer considerations by the use of the tensor virial theorem (see EFE §§ 11 and 37), and obtained identical results for the anti-damped oscillatory mode. Rosenkilde does not mention why the results of the two, very different, treatments must be identical. We do so here: The replacement of the exact hydrodynamical Navier-Stokes equations by equations of motion derived from the tensor virial theorem corresponds precisely to the imposition of fixed holonomic constraints on the mechanical system. These constraints require the system to be

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ellipsodial, with fluid velocities linear functions of coordinates ("Dirichlet," see EFE, p. 64), and reduce the system's degrees of freedom from an infinite number to (essentially) nine: three principal axes, three components each of total angular momentum and circulation.

In the case of zero viscosity, the constraints—though present—are "unused," i.e., the forces of constraint on any field element vanish identically. This is the reason that the Riemann-Lebovitz dynamical equations (EFE, p. 71) are indifferent to derivation by virial-theorem or Euler equation methods. When the viscosity is nonzero, however, the shear forces generated will not in general allow an ellipsoid with fluid velocities that are initially linear functions of coordinates to remain in such a state. The use of the tensor virial theorem in the viscous case instead of the Navier-Stokes equations corresponds to projecting the viscous forces into the allowed degrees of freedom, and utilizing constraint forces to counteract whatever piece is left over.

Why, now, do the results of Roberts and Stewartson—which use the full hydrodynamical equations—agree exactly with the results of the tensor virial approach? Because, in the former investigation, the mode of perturbation which is found to be anti-damped ($n = m = 2$) happens to lie precisely within the subspace of Dirichlet degrees of freedom. Thus, although a boundary-layer stress is carried into the hydrodynamical equations, the anti-damped term which appears in the equation of motion of the unstable mode is the same mode-projected term which is allowed in through the tensor-virial approach.

In physical terms, the second-order tensor-virial approach is justified when the projection of viscous forces into any non-Dirichlet degree of freedom is small compared to the "restoring force" of that degree of freedom. Thus, for small viscosity, the approach is justified as long as one is not too near a configuration with a neutral (or unstable) non-Dirichlet mode (in the terminology of EFE, mode of greater than second harmonic).

The works of Roberts and Stewartson, and of Rosenkilde, discuss only small perturbations of the Maclaurin spheroids, which are adequately treated by linearized hydrodynamical equations. Rossner (1967) treated a case of finite-amplitude oscillations of the Maclaurin spheroids, but only in the case of zero viscosity, so evolution to a Jacobi ellipsoid could not be observed. More recently Fujimoto (1971) has integrated numerically the nonlinear equations for finite-amplitude motions with a viscosity. Fujimoto's viscosity is different from all previous investigations. It is not a uniform bulk viscosity; rather it varies spatially throughout the ellipsoid in a very special way, which is postulated from the start (his eqs. [4] and [5]). This particular ad hoc choice of viscosity allows an ellipsoid with linear fluid velocities to remain as such, without the necessity of constraint forces. In other words, the tensor-virial equations are exactly equivalent to the Navier-Stokes equations for this choice of viscosity.

Fujimoto's numerical calculations (which are supported by our independent calculations; see below) show two effects present in the behavior of a perturbed, viscous Maclaurin spheroid. First, there is a smooth, secular monotonic change in the shape of the rotating configuration as it relaxes from spheroid to Jacobi ellipsoid; second, in the initial stages of the relaxation there is superposed a damped oscillatory motion (see Fujimoto 1971, fig. 2; or fig. 1 below). While his results are correct, Fujimoto erroneously states that they show behavior not predicted by the linearized treatment—even in the initial stage of relatively small oscillation.

In fact, however, there is full agreement between the numerical and linearized investigations. The point of confusion is that the anti-damped oscillations of the linearized treatments refer to Lagrangian displacements of fluid from a "fictional" unperturbed configuration; they are *not* overdamped oscillations in the physical shape of the ellipsoid. It is an artifact of the linearized mathematical formalism that a smooth relaxation in shape is represented as an anti-damped mode: If ξ is the Lagrangian displacement at position x, with time dependence $e^{-i\mathrm{Re}(\omega)t}$ removed, then the anti-damped mode is a toroidal mode (EFE, p. 88) with

$$\xi_1 = A(x_1 + ix_2)\,, \qquad \xi_2 = iA(x_1 + ix_2)\,, \qquad \xi_3 = 0\,. \tag{1}$$

Here the spheroid is assumed to rotate about the positive x_3 axis, and A is the instantaneous amplitude of the excitation $A \equiv A(t) = e^{\mathrm{Im}(\omega)t}$. It is straightforward to verify that for small ξ, (i) this displacement makes the configuration ellipsoidal, and (ii) the principal axes of the ellipsoid are $a_1(t) = a + A(t)$, $a_2(t) = a - A(t)$, where a denotes the unperturbed (equal) axes. Thus, for an anti-damped mode $\mathrm{Im}\,(\omega) > 0$, the axes diverge monotonically. The rapid oscillations $e^{-i\mathrm{Re}(\omega)t}$ represent only the rotation of the ellipsoid and its constant internal motions, not any change in its shape or "actual" fluid oscillations (cf. EFE, p. 82, eqs. [35] ff.). In Fujimoto's numerical solutions, the additional damped oscillations at early times occur because his initial perturbation is not a pure mode; it is a mixture of the two toroidal modes in the linearized theory; one of these damps out in time, leaving only the growing, "relaxation" mode. For a different initial perturbation, Fujimoto would have found only the monotonic secular piece.

In brief, the linearized small-perturbation analyses of Roberts and Stewartson and of Rosenkilde, and the nonlinear finite-amplitude analysis of Fujimoto, are all in agreement. A perturbed, secularly unstable, Maclaurin spheroid evolves, by monotonic relaxation in shape, into a Jacobi ellipsoid. A misnomer from the Lagrangian-displacement formalism terms this evolution "anti-damped oscillations."

II. NEW RESULTS

Our starting point is the Riemann-Lebovitz equation including viscous stresses (see EFE, § 37). This equation is derived by Rosenkilde (1967), but written explicitly only in its linearized form. In exact form it is

$$\begin{aligned}\frac{d^2A}{dt^2} + \frac{d}{dt}(A\Lambda - \Omega A) + \frac{dA}{dt}\Lambda - \Omega\frac{dA}{dt} + A\Lambda^2 + \Omega^2 A - 2\Omega A\Lambda \\ + 2\pi G\rho U A - \frac{2p_c}{\rho}A^{-1} = -\left(2\frac{dA}{dt}A^{-2} + A\Lambda A^{-2} - A^{-1}\Lambda\right)\frac{5}{V}\int_V \nu dV\,.\end{aligned} \tag{2}$$

Here A, Ω, Λ, U are 3×3 matrices representing the shape, angular velocity, internal motions, and gravitational potential of the ellipsoid (see EFE, chap. 4, for details). V is the ellipsoid's volume. Because the fluid is incompressible, the central pressure p_c is given by the algebraic relation

$$\begin{aligned}\frac{2p_c}{\rho} = \Bigg\{\mathrm{Tr}\left[\Lambda^2 + \Omega^2 - 2\Omega A\Lambda A^{-1} + \left(A^{-1}\frac{dA}{dt}\right)^2 + \frac{dA}{dt}A^{-3}\frac{10}{V}\int_V \nu dV\right] \\ + 4\pi G\rho\Bigg\}\Big/\mathrm{Tr}\,(A^{-2})\,.\end{aligned} \tag{3}$$

For nonzero coefficient of viscosity $\nu = \nu(x, t)$, the right-hand side of equation (2) gives explicitly the projection of the viscous shear tensor into the allowed Dirichlet modes. It is not necessary to assume that ν is small for equations (2) and (3) to be valid. Note that only the bulk average of ν over the volume of the ellipsoid enters. Thus *all* models for viscosity are equivalent, in this formulation, to a uniform bulk viscosity ν_{eff} which is at most a function of time. If ν is uniformly distributed, $\nu_{\text{eff}} = \nu$ and the right-hand side in equation (2) is just

$$\text{RHS} = -\left(2\frac{dA}{dt}A^{-2} + A\Lambda A^{-2} - A^{-1}\Lambda\right)5\nu_{\text{eff}}\,, \tag{4}$$

which corresponds to the viscosity of Rosenkilde. Fujimoto's viscosity $\nu(x, t)$ is

$$\nu = \nu_0 f(t)\left[1 - \left(\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2}\right)\right], \tag{5}$$

which gives an equivalent uniform viscosity

$$\nu_{\text{eff}} = \tfrac{2}{5}f(t)\nu_0\,. \tag{6}$$

The important point is that if an ellipsoidal configuration evolves quasi-statically—as is the case in slow relaxation (small ν) from Maclaurin to Jacobi (see below)—then $\nu_{\text{eff}}(t)$ also changes quasi-statically, and the evolutionary track from Maclaurin to Jacobi is independent of the precise model of viscosity taken.

What, then, is this evolutionary track? To answer this, we have numerically integrated the nine equations (2) with the right-hand side of equation (4), starting from various Maclaurin ellipsoids and with various initial perturbations. A typical time evolution is shown in figures 1 and 2. Figure 1 shows, on a highly expanded scale, how all modes of the ("random") perturbation damp out in time, except for the mode of quasi-static, secular deformation. Figure 2 shows the complete evolution from Maclaurin to Jacobi shape.

We find that the intermediate configurations, in the limit of small viscosity, are just the Riemann S-type ellipsoids described by Chandrasekhar (EFE, § 48). We have verified this in two ways: (i) direct comparison of the shape, circulation C, and angular momentum L to the known Riemann-S values, and (ii) numerical verification that the

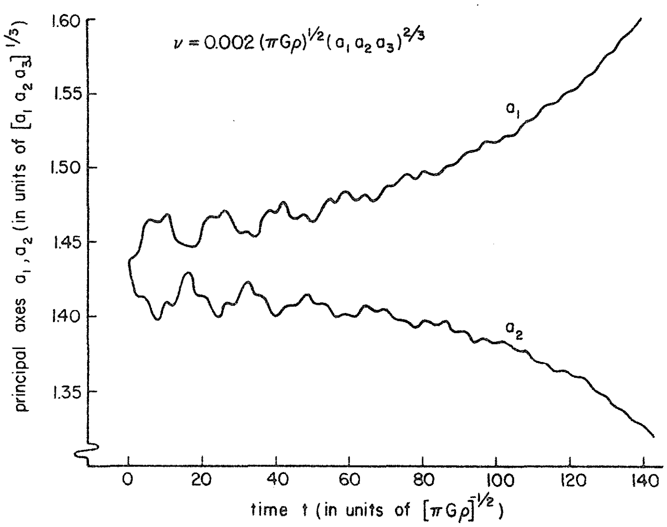

FIG. 1.—Principal axes of an initially perturbed, secularly unstable Maclaurin spheroid as a function of time. All modes of the perturbation except one are seen to damp out in time; the one growing mode corresponds to a monotonic, secular relaxation to a triaxial Jacobi ellipsoid. (The behavior of the third principal axis, with initial value 0.488, is qualitatively similar.) See fig. 2.

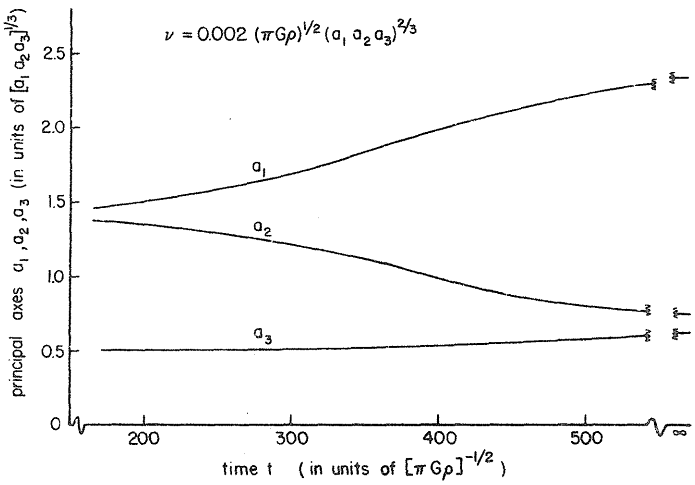

FIG. 2.—Subsequent behavior of the perturbed Maclaurin spheroid of fig. 1. Once the damped perturbations have died away, the system follows a unique evolutionary track through Riemann S-type configurations to the Jacobi sequence. Its rate of progress along the track depends only on the volume average of fluid viscosity, *not* on details of how the viscosity is distributed (see text for details).

intermediate states are themselves equilibrium configurations with L and C parallel, i.e., that there is no further evolution when the viscosity is suddenly "switched off." (Riemann-S ellipsoids are the unique equilibrium configurations for parallel L and C; see EFE, p. 133.) Departures from the small viscosity limit are also seen numerically: when a larger viscosity is switched off, the resultant configuration is a Riemann-S ellipsoid plus a small, stable oscillation. Evidently the oscillation occurs because the faster viscous evolutions are not quite quasi-static.

Concluding, we find that a secularly unstable, viscous Maclaurin spheroid evolves slowly, smoothly, and monotonically along a line of constant angular momentum through the Riemann-S plane, to its destination on the Jacobi sequence. *If* all configurations along this evolutionary path are stable with respect to small, non-Dirichlet perturbations ("third harmonics" and above), then one can be confident that this evolutionary path is independent of the details of the (small) viscosity.

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