RESPONSE OF MECHANICAL SYSTEMS

TO

RANDOM EXCITATION

Thesis by

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In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1960

ACKNOWLEDGMENT

The author is indebted to Dr. T. K. Caughey who directed this research. His many helpful suggestions and constant encouragement made this work possible.

A portion of this research was completed while the author held fellowships from the Sunray Mid-Continent Oil Company and the Curtiss-Wright Corporation.

Thanks is also due Mr. Robert Oliver, President of AER, Inc., for permitting the author to use the results of some work he completed there.

ABSTRACT

The Fourier Series and Fokker-Planck Methods, the available techniques for solving vibration problems when the exciting force is a stochastic process, are reviewed and several detailed examples are given. In particular a two-degree-of-freedom system is considered which is excited by a non-stationary input and which possesses a general type of viscous damping.

Several typical engineering problems involving stochastic processes are considered. In the case of fatigue it is shown that a criterion for fatigue failure in multi-degree-of-freedom systems may be established using Miner's cumulative damage hypothesis and the number of zero crossings per second.

In the earthquake problem it is shown that when certain inequalities involving the natural frequencies of the building are valid, cross-product terms may be neglected in computing mean square displacements.

Two problems involving beams are considered. In one case it is demonstrated that a convergent expression for the mean square bending moment may be obtained for a Bernoulli-Euler beam excited by white noise, provided a finite cutoff frequency is used. In the other case involving random end motion, a one-term approximation to the mean square bending moment may be obtained, when the correlation time is not too small.

The isolation problem is considered and the concept of the "white spectrum fragility curve" is established as a criterion for adequate isolation.

Finally the motion of a single-degree-of-freedom system over a rough surface is considered. It is shown that for an exponential type of auto-correlation the mean square displacement is finite for unaccelerated motion and diverges when the system is accelerated.

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Roman Letters

$A_i^{(r)}$	Maximum displacement of i^{th} mass in r^{th} vibrational mode [ft]
$\bar{A}(z)$	Function in Fokker-Planck Equation (see eq. (133))
$\mathcal{A}$	Characteristic acceleration of system (see Appendix I, Sect. 4.2)
a	Acceleration $\left[\frac{ft}{(sec)^2}\right]$
$B(z)$	Function in Fokker-Planck Equation (see eq. (134))
$\bar{B}$	Constants (see eqs. (214) and (365))
$\mathcal{B}$	Characteristic length (see Appendix I, Sect. 4.2)
b	Constant (see eq. (105))
$\mathcal{b}$	Constant (see eq. (267))
C	Generalized damping coefficient
$\bar{C}$	Constant (see eq. (566))
c	Damping Coefficient $\left[\frac{lb_f\text{-}sec}{ft}\right]$
$\mathcal{C}_i$	Integral (see eq. (I-76))
$\bar{\mathcal{C}}$	Function defined by eq. (119)
D	Spectral density of power spectrum

$d, \bar{d}$	Coefficients in a Fourier Series (eq. (75))
$\mathcal{D}$	Dissipation function
d	Absolute damping coefficient $\left[\frac{lb_f\text{-}sec}{ft}\right]$
E	Product of modulus of elasticity and moment of inertia for a beam
$\mathcal{E}$	Random function (see Part II, Sect. 2.3)
e	Voltage
F	Forcing function $\left[lb_f\right]$
$\mathcal{F}$	Fatigue damage factor (see Part III, Sect. 1.1)
g	Function used in Laplace's Method (see eq. (I-47))
H	Indicial admittance of a simple harmonic oscillator
h	Response of a simple harmonic oscillator to a unit impulse
$\mathcal{h}$	Variable defined in eq. (259)
$\mathcal{I}$	General integral
$\bar{I}$	Variable defined in eq. (259)
J	Determinant of a Jacobian matrix
K	Generalized Spring constant
k	Spring Constant $\left[\frac{lb_f}{ft}\right]$

TABLE OF NOTATION (CONTD.)

L	Parameter defined in eqs. (I-74) through (I-81)
$\mathcal{L}$	Characteristic length $[ft]$ (see Appendix I, Sect. 4.3)
ℓ	Function used in Laplace's Method (see eq. (I-47))
$\bar{\ell}$	Constant (see eq. (246))
M	Generalized mass
$\bar{M}$	Bending moment $[lb_f - ft]$
$\mathfrak{m}$	Dimensionless displacement (see Appendix I, Sect. 4.2)
m	Mass $\left[\frac{lb_f - (sec)^2}{ft}\right]$
η	Number of stress reversals at fixed stress level needed to cause failure
n	Number of stress reversals at fixed stress level
P	Probability distribution
$\mathcal{P}$	Power
q	Generalized coordinate
$\mathfrak{q}$	Arbitrary function (see eq. (I-5))
R	Ratio of force to mass $\left[\frac{ft}{(sec)^2}\right]$
R_s	Electrical resistance
r	Variable of integration defined by eq. (I-43)

TABLE OF NOTATION (CONTD.)

$\mathcal{S}_i$	Integral (see eq. (I-76))
$\mathcal{S}$	Stress Amplitude $\left[\frac{lb_f}{(in)^2}\right]$
$\bar{\mathcal{S}}$	Function defined by eq. (120)
s	Variable of integration defined by eq. (I-44)
T	Period of vibration [sec]
$\bar{T}$	Kinetic energy of a vibrating system
$\mathcal{T}$	Dimensionless time parameter (see Appendix I, Sect. 4.2)
t	Time [sec]
U	Linear combination of random variables (see eq. (104))
$\mathcal{U}$	Dimensionless variable of integration in Laplace's Method (see Sect. 4.2)
u	Function used in Laplace's Method (see eq. (I-50))
V	Potential energy of a vibrating system
$\bar{V}$	Linear combination of random variables (see eq. (104))
$\mathcal{V}$	Dimensionless variable of integration in Laplace's Method (see Appendix I, Sect. 4.2)
v	Velocity $\left[\frac{ft}{sec}\right]$
W	General probability distribution
$\mathcal{W}$	Total energy of a system

TABLE OF NOTATION (CONTD.)

W	Work of viscous friction forces for a vibrating system
X	Random variable (see eq. (100))
$\mathcal{X}$	Expected number of zeros per second (see eq. (257))
x	Displacement of a mass [ft]
Y	Random variable (see eq. (100))
y_1	Absolute displacement of mass [ft]
Z	Impedance of a simple harmonic oscillator
z	Variable in symmetric form of Fokker-Planck Equation
$\mathfrak{z}$	Relative displacement of mass [ft]

Greek Letters

α	Eigenvalue of a differential equation (see eq. (4))
β	$= \alpha\phi$ (see eq. (38))
$\bar{\beta}$	$= 2\rho\omega_0$ (see eq. (139))
Γ	Gamma Function
γ	Parameter in autocorrelation (see eq. (I-22))
Δ	Fourier Transform parameters
δ	Dirac Delta Function

TABLE OF NOTATION (CONTD.)

$\bar{\epsilon}$	Coefficients in a partial differential equation (see eq. (146))
ζ	Function of time in complex normal mode analysis (see eq. (46))
η	Parameter used in Laplace's Method (see eq. (I-47))
κ	Function used in Laplace's Method (see Sketch I-5)
Λ	Remainder in asymptotic series (see eqs. (169) and (170))
λ	Coefficients in a partial differential equation (see eq. (146))
μ	First moment of a probability distribution
$\bar{\mu}$	Constant (see eq. (241))
Ξ	Correction term (see eq. (I-77))
ξ	Normal coordinate
Π	Power spectrum
ρ	Ratio of damping coefficient to critical damping
σ^2	Second moment of a probability distribution (see eq. (98))
τ	Time parameter [sec]
$\bar{\tau}$	Correlation coefficient
Υ	Integrals defined by eqs. (329) through (332)

TABLE OF NOTATION (CONTD.)

ϕ	Mode shape for continuous systems (see eq. (37))
Ψ	Eigenfunction of a differential equation (see eqs. (218) through (221))
ψ	Autocorrelation function
$\Omega, \bar{\Omega}$	Functions defined by eq. (318)
ω	Frequency of vibration $\left[\frac{rad}{sec}\right]$

Matrix	Elements Refer to:
$\left[C \right]$	Generalized Damping Coefficients
$\left\{ F \right\}$	Components of force (see eq. (3))
$\left\{ f \right\}$	Components of force (see Part II, Sect. 1.21)
$\left[g \right]$	Second moments of a probability distribution
$\left\{ \mathcal{H} \right\}$	Random variables
$\left[I \right]$	Constants (see eq. (5))
$\left[\mathcal{J} \right]$	Constants (see eq. (103))
$\left[K \right]$	Generalized spring constants
$\left[k \right]$	Spring constants
$\left[M \right]$	Generalized masses
$\left[m \right]$	Masses

TABLE OF NOTATION (CONTD.)

Matrix	Elements Refer to:
$[Q]$	Maximum displacements of a mass for a particular mode (see eq. (400))
$[\mathcal{R}]$	Parameters of a vibrating system (see eq. (3))
$[r]$	Damping coefficients
$[\mho]$	Parameters of a vibrating system (see eq. (5))
$\{X\}$	Displacements of masses (see eq. (3))
$\{y\}$	Displacements of masses (see eq. (185))
$[Z]$	Impedance of a particular mode
$[\Delta]$	Linear combinations of second moments of a probability distribution (see eq. (112))
$[\Sigma]$	Second moments of a probability distribution (see eq. (111))
$\{\Phi\}$	Components of an eigenvector (see eq. (4))
$\{\phi\}$	Components of an eigenvector (see Part II, Sect. 1.211)

Subscript	
AV	Average value
Cr	Critical value
IN	Input

TABLE OF NOTATION (CONTD.)

OUT	Output
Rc	Remainder term in approximation of an integral containing cosine
RS	Remainder term in approximation of an integral containing sine
To	Total
T	Truncated function
ij	Element in ( ) row and ( ) column of a square matrix
r,s	Mode of vibration
ϵ	Equivalent
o	Natural frequency of vibration
a	Constant acceleration
v	Constant velocity

Superscript

$(r),(s)$	Mode of vibration

Special Mathematical Symbols

$(\tilde{\ })$ or $\mathcal{F}\{\ \}$	Fourier Transform
$\langle\ \ \rangle$	Ensemble Average

TABLE OF NOTATION (CONTD.)

$(\dot{\ }), (\ddot{\ })$	Differentiation with respect to time variable
$(\ ')$, $(\ '')$	Differentiation with respect to space variable
$(^{I}), (^{II})$	Differentiation with respect to a difference of space variable (see eq. (I-21))
$[\ \ ]$	Square Matrix
$[\ \ ]^{T}$	Transpose of a square matrix
$[\ \ ]^{-1}$	Inverse of a square matrix
$\{\ \ \}$	Column matrix
$\|\ \ \|$	Determinant of a square matrix
$O(\ )$	Order of
$(^{+})$	Distinguishes variables of integration (see eq. (160))
$(^{*})$	Complex conjugate
δ_{nm}	Kronecker's delta ($=0, n \neq m$; $= 1, n = m$)

PART I

INTRODUCTION AND SUMMARY

The analysis of mechanical systems subjected to random exciting forces is quite naturally dependent upon the theory of probability. Fortunately, for much of the work, only rudimentary concepts from this field are necessary, and in many cases successful treatments of random processes have been made on the basis of a purely intuitive approach. In random processes, unlike the problems where the exciting forces are a definite function of time, we are forced to deal with statistical parameters, that is, quantities which can be expressed as expectations of certain random variables of interest. These parameters are, in general, directly observable either through experiment or the analyses of statistical data but they do not completely characterize the probability distributions of the random variates. The various moments, for example the mean and variance, convey a limited amount of information such as the average value and spread of the distribution about the mean but they do not uniquely define the distribution. The determination of the probability distribution is the most general quantity we can compute in a problem involving random processes, but unfortunately it is only in special circumstances that it can be found. The variance or mean square is usually computed since this is the quantity, which in general, is most easily observed. In vibration problems, it is often a straightforward calculation to find the mean square values although the details may become quite tedious.

In the following sections a brief historical review of the development of the mathematical theory of random processes and a summary of the main body of the report are given.

1.0 Historical Background

In 1827 the English botanist, Robert Brown, noticed that small particles suspended in a fluid performed peculiarly erratic movements. This was, perhaps, one of the first problems involving a random phenomenon to be recognized as such. It soon became apparent that this so-called Brownian Motion was an outward manifestation of the molecular motion postulated by the kinetic theory of matter. It was not until 1905, however, that a satisfactory theory was advanced by Einstein for the case of the free particle.

Einstein treated $X(t)$, the X coordinate of the particle at time t as a chance variable and found that the probability distribution of $\{X(t)-X(0)\}$ was Gaussian with mean zero and variance $b|t|$ where b is a positive constant which can be determined from the properties of the particles and fluid. This result was a consequence of his showing that $P(X_0|X,t)$ was the fundamental solution to a partial differential equation of the diffusion type, where $P(X_0|X,t)$ is the probability that at time t the particle will be between X and $X+dX$ if at time $t=0$ it was at X_0.

The theory was soon generalized by the Polish physicist, Smoluchowski. A natural extension of Einstein's work arose when outside forces were considered, and it was Smoluchowski who showed how the Einstein equation is modified in this case. Contributions of major importance were also made by Fokker, Planck, Burger, Furth, Ornstein, Uhlenbeck, Chandrasekhar, Kramers, and others resulting in the general partial differential equation satisfied by $P(x_0/x,t)$ known as the Fokker-Planck equation. The purely mathematical aspects of the theory were analyzed by Weiner, Khintchine, Kolomgoroff, Feller Lèvy, Doob, and others. One conceptual difficulty of the theory lay in the fact that it does not appear possible to derive the diffusion equation in a rigorous fashion, unless one assumes that the change in the particle position ΔX in time Δt is independent of the fact that at $t=0$ the particle was at X_0 with velocity V_0. For this reason Smoluchowski suggested treating the problem as a discrete random walk. The theories of both Einstein and Smoluchowski are valid only for large t and in the case of an elastically-bound particle only for the overdamped condition.

In 1919 Ornstein used a different approach in solving for the probability distribution, which consisted of computing all mean values for the random variable by integrating the equation of motion. If all the moments are known, the probability distribution is defined. At best we can only approximate the distribution in general using this procedure.

Another approach is the Fourier Series method in which the random variable is expanded in a Fourier Series in time. The coefficients are no longer constants but vary in a random fashion. Weiner in 1930 and Khintchine in 1934 discovered the fundamental theorem of this method which relates the power spectrum to the correlation function by a Fourier Cosine Transform. Rice has systematically applied this method to a large number of problems concerned with electronic circuit elements.

With the advent of the guided missile and its associated problems in random vibrations a great number of papers have been written in which the responses of discrete and continuous systems to random excitation are treated. The methods of analysis used generally follow those mentioned.

To summarize then, there are three general techniques for handling problems involving random processes. The first, in which the Fokker-Planck equation is used, is best suited to problems where the power spectrum of the random force is constant. Otherwise a partial differential equation with variable coefficients results which is generally extremely difficult to solve. When the input power spectrum is constant the output is always Gaussianly distributed. No restrictions are placed on the probability distribution of the input in this method.

In Ornstein's method the input need not have a flat power spectrum; but unless its probability distribution is known to be Gaussian, we can only approximate the output distribution function by computing a large number of moments. When the input is Gaussian, the output is Gaussian, and we need only calculate the first and second moments to completely define the probability distribution.

The Fourier Series method, like Ornstein's, need not restrict the power spectrum of the input to a constant value; but unless the input is Gaussianly distributed, we have no information about the distribution function of the output.

For the Gaussian random process in which the input has a Gaussian probability distribution and a flat power spectrum, the Fokker-Planck and Fourier Series methods are generally used.

2.0 Summary of Thesis

The main body of the thesis is divided into four parts, Part II, Part III, Part IV, and Appendix I. In Part II one aspect of linear, damped multi-degree-of-freedom systems is considered, that being the existence of normal modes. The most general form of damping for which Rayleigh normal modes exist is given and then a more general type of damping is considered for which only complex normal modes exist. In this latter case a method developed by K. Foss for determining the coordinates which uncouple the equations of motion of damped, linear, dynamic systems is outlined in some detail.

The next topic treated is that of systems subjected to random exciting forces. A brief review of the general theory is presented which describes the Fourier Series and Fokker-Planck methods of handling the Gaussian random process. In addition the Fokker-Planck equation and the Weiner-Khintchine theorem are derived in detail. An extension of the Weiner-Khintchine theorem to the case of non-stationary processes due to D. G. Lampard is also reviewed. The method of computing mean square values is then outlined.

Although a large portion of the work in Part II is a summary of the known techniques for handling stochastic processes, several of the sections are thought to present new material. In these sections several problems are solved in detail. The well known result for the response of a simple harmonic oscillator excited by white noise is given. This problem is extended to finding the time dependent solution for the mean square displacement of the simple harmonic oscillator excited by a random forcing function possessing an arbitrary power spectrum. This problem is also the simplest type of non-stationary process since the output is non-stationary in the sense that it is transient, building up exponentially with time to a steady state value.

A two-degree-of-freedom system possessing general type damping and excited by a simple non-stationary random force is then treated in detail illustrating the method of Foss. Continuous systems, represented by a vibrating beam, are also briefly considered.

Part III, Part IV, and Appendix I represent the major portion of the original work. Part III treats several typical engineering problems involving stochastic processes. First of all the problem of fatigue failure is considered. For a single-degree-of-freedom system the output due to a random input is a curve whose frequency is equal to that of the system and whose amplitude is random in time. Using Miner's cumulative damage hypothesis, a criterion for fatigue failure can be obtained. This work done by J. W. Miles, is briefly reviewed. When multi-degree-of-freedom systems are considered, the situation is far more complex since the output contains many frequencies. It is shown that by using Miner's hypothesis and the number of zero crossings per second as a measure of the number of stress reversals per unit time a criterion for fatigue failure can be obtained.

Next the earthquake problem, considered as the response of a tall shear structure to random ground motion, is discussed. For a building which possesses Rayleigh normal modes it is shown that the cross-product terms may be neglected when computing mean square quantities provided the damping is small and certain inequalities involving the natural frequencies of the building are valid. The power spectrum of the ground motion is assumed to be peaked which is typical of earthquakes.

Two problems involving Bernoulli-Euler beams are treated next. The first problem is that of a beam subjected to a random transverse loading. A. C. Eringen has shown that if the random force is considered to have a white power spectrum over the entire frequency range, the mean square bending moment is infinite although the mean square displacement is finite. By using the more elaborate Timoshenko beam theory, the mean square bending moment can be shown to converge. The white noise excitation considered is not physically realizable since it implies infinite energy input. If the power spectrum is cut off at some finite frequency, it is shown that a convergent result for the mean square bending moment can be obtained even for the relatively simple Bernoulli-Euler beam theory. The problem of a beam with random end motion is also considered. It is shown that for an exponential type of autocorrelation for the end motion a one-term approximation to the mean square bending moment can be obtained, provided the correlation time is small compared to the lowest natural frequency of the beam.

In Part IV the problem of vibration isolation under random excitation is considered. The major problem is the determination of a criterion which can be used to ascertain whether isolation is adequate. Two different situations may arise. One in which an item will fail if some maximum value of acceleration, velocity, or displacement is exceeded and the other when the item fails due to fatigue failure. When a maximum value of some parameter is the critical quantity and the input is Gaussian, we may design the isolation system to make the mean square value of the critical parameter as small as practicable, since the probability of exceeding a given value is directly related to the mean square or variance. When fatigue failure is important, the concept of the "white spectrum fragility curve" may be used.

In this method a system is tested by subjecting it to white noise whose spectral density is varied until failure within a prescribed time interval results. The power spectrum of this white noise excitation is called the "white spectrum fragility curve." The isolation system is then designed so that the power spectrum of the parameter of interest at the point where the given element is mounted does not exceed the "white spectrum fragility curve" at any frequency. The expressions for mean square values of acceleration and displacement, and the response power spectra are determined for two-degree-of-freedom systems possessing Rayleigh normal modes or complex normal modes.

In Appendix I the motion of a single-degree-of-freedom system over a rough surface is considered. The profile of the surface is represented by a stationary random process. The equations of motion are derived and the expressions for the mean square values of displacement and velocity are developed. It is shown that unless the acceleration of the system is zero, the input is non-stationary. The quantities appearing in the autocorrelation of the forces are then related to the known properties of the surface.

A simple surface shape is then considered and the integral expressions for the mean square values are derived for the cases where the system moves with constant velocity and constant acceleration.

Two particular mean square values are considered, those for the absolute displacement of the mass when the system moves with constant velocity and with constant acceleration. It is not possible to integrate the expressions directly and approximate methods must be used. When the system velocity is constant, the input is stationary and we may find the power spectrum of the exciting forces quite readily. For small damping, the major contribution to the mean square displacement is for input frequencies near the natural frequency of the system. The power spectrum is therefore evaluated at this frequency and then considered as a flat power spectrum. The integration can then be carried out quite easily.

When the system acceleration is constant, the input is non-stationary, and the previous method cannot be applied very readily. Instead we use Laplace's method to reduce the double integral to a single integral. Small damping is considered and an approximate solution is obtained.

A numerical example is then considered. It is shown that when the system moves with constant velocity, the mean square displacement is finite; but when the system is accelerated, the expression for the mean square displacement diverges.

PART II

LINEAR MULTI-DEGREE-OF-FREEDOM SYSTEMS

1.0 Uncoupling the Equations of Motion

In many engineering problems the forces which excite a mechanical system are not always harmonic with fixed frequency and amplitude. Quite often the input is random in nature and the output displays no orderly pattern. In such cases as these, instantaneous values and phase of the parameters of interest are meaningless and the problem must be treated statistically. Before proceeding with the general problem of the response of mechanical systems to random excitation, a brief review of the problem of separating the equations of motion for linear, damped systems will be given.

1.1 CLASSICAL NORMAL MODES

The theory of undamped, linear systems is well understood and it has been known for some time that such systems possess normal modes. When damping is introduced, however, this property does not exist in general and the more comprehensive treatment of K. Foss is required. Lord Rayleigh showed that classical normal modes exist in damped systems if the damping matrix is a linear combination of the inertia and stiffness matrices. More recently Dr. T. K. Caughey has shown that a necessary and sufficient condition for the existence of classical normal modes in such systems is that the damping matrix be diagonalized by the same transformation which uncouples the equations of motion of the undamped system. A sufficient condition on the damping matrix is shown to be

$$\left[m\right]^{-\frac{1}{2}}\left[c\right]\left[m\right]^{-\frac{1}{2}} = \sum_{n=1}^{\infty}\sum_{\ell=0}^{N-1} a_{n\ell}\left(\left[m\right]^{-\frac{1}{2}}\left[k\right]\left[m\right]^{-\frac{1}{2}}\right)^{\ell/n} \tag{1}$$

If the terms $n=1$, $\ell=0$, and $\ell=1$ are taken in eq. (1), Rayleigh's criterion is obtained.

1.2 COORDINATES WHICH UNCOUPLE THE EQUATIONS OF MOTION OF LINEAR, DAMPED SYSTEMS

In a recent paper, K. A. Foss has shown how to obtain the coordinates which will uncouple the equations of motion of linear damped systems with general damping. The following material summarizes the results of interest from that paper and includes a more detailed analysis of the continuous system.

1.21 Discrete Systems

The equations of motion of a multi-degree-of-freedom system can be written as

$$\left[m_{ij}\right]\left\{\ddot{q}_j\right\} + \left[r_{ij}\right]\left\{\dot{q}_j\right\} + \left[k_{ij}\right]\left\{q_j\right\} = \left\{f_i(t)\right\}, \tag{2}$$

where

$[\;\;]$ denotes a square matrix,

and

$\left\{ \quad \right\}$ denotes a column matrix.

We have seen that unless $[r_{ij}]$ is proportional to $[m_{ij}]$ or $[k_{ij}]$ or is a linear combination of the two, velocity coupling exists. Equation (2) may be rewritten in the form

$$[R]\{\dot{x}\} + [K]\{x\} = \{F(t)\}\ , \tag{3}$$

where

$$\{X\} = \begin{Bmatrix} \{\dot{q}\} \\ \{q\} \end{Bmatrix},$$

$$\{F(t)\} = \begin{Bmatrix} \{0\} \\ \{f(t)\} \end{Bmatrix},$$

$$[R] = \begin{bmatrix} [0] & [m] \\ [m] & [r] \end{bmatrix},$$

and

$$[K] = \begin{bmatrix} -[m] & [0] \\ [0] & [k] \end{bmatrix}.$$

Subscripts have been omitted for simplicity.

1.211. HOMOGENEOUS SOLUTION. To obtain the homogeneous solution of eq. (3) we assume that $\{X(t)\} = e^{\alpha t}\{\Phi\}$ where α is an eigenvalue of eq. (3) and $\{\Phi\}$ is the corresponding eigenvector. In general, both

α and $\{\Phi\}$ will be complex. Substituting this value of $\{x(t)\}$ into the relation (3) yields

$$\alpha [R]\{\Phi\} + [K]\{\Phi\} = \{0\} . \tag{4}$$

We may rewrite eq. (4) as

$$[U]\{\Phi\} = \frac{1}{\alpha}\{\Phi\} , \tag{5}$$

where

$$[U] = -[K]^{-1}[R] = \begin{bmatrix} [0] & [I] \\ -[c][m] & -[c][r] \end{bmatrix} ,$$

$$[I] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix}$$, the unit matrix ,

$$[c] = [k]^{-1}$$, a set of influence coefficients,

and

$[\ ]^{-1}$ denotes the inverse of a matrix.

For a system with N degrees of freedom eq. (5) will yield $2N$ eigenvalues α_n and eigenvectors $\{\Phi^n\} = \begin{Bmatrix} \alpha_n \{\phi^n\} \\ \{\phi^n\} \end{Bmatrix}$.

For a stable system α_n must be real and negative or complex with a negative real part. The complex roots appear as complex conjugate pairs with corresponding complex conjugate modal columns.

If $[m]$, $[k]$ and $[r]$ are symmetric, $[\mathcal{R}]$ and $[K]$ are symmetric and eq. (4) may be written as

$$\alpha_n [\mathcal{R}]\{\Phi^n\} + [K]\{\Phi^n\} = \{0\}, \tag{6}$$

or

$$\alpha_m \{\Phi^m\}^T[\mathcal{R}] + \{\Phi^m\}^T[K] = \{0\}, \tag{7}$$

where

$[\ ]^T$ or $\{\ \}^T$ denotes a transposed matrix. Premultiply eq. (6) by $\{\Phi^m\}^T$ and postmultiply eq. (7) by $\{\Phi^n\}$ giving

$$\alpha_n \{\Phi^m\}^T[\mathcal{R}]\{\Phi^n\} + \{\Phi^m\}^T[K]\{\Phi^n\} = 0, \tag{8}$$

and

$$\alpha_m \{\Phi^m\}^T[\mathcal{R}]\{\Phi^n\} + \{\Phi^m\}^T[K]\{\Phi^n\} = 0. \tag{9}$$

Subtracting eq. (9) from eq. (8) yields

$$(\alpha_n - \alpha_m)\{\Phi^m\}^T[\mathcal{R}]\{\Phi^n\} = 0. \tag{10}$$

If $\alpha_n \neq \alpha_m$ we have the orthogonality relations

$$\{\Phi^m\}^T[\mathcal{R}]\{\Phi^n\} = 0 \qquad m \neq n, \tag{11}$$

and

$$\{\Phi^m\}^T[K]\{\Phi^n\} = 0 \qquad m \neq n. \tag{12}$$

Equations (11) and (12) are analogous to the orthogonality conditions of undamped systems. The analog of no dynamic or static coupling here would be for the off diagonal terms of $[\mathcal{R}]$ or $[K]$ to be zero.

1.212 NON-HOMOGENEOUS SOLUTION. To obtain the non-homogeneous solution of eq. (3) we expand $\{X(t)\}$ in a modal series, that is

$$\{X(t)\} = \sum_{n=1}^{N} \{\Phi^n\} \xi_n(t) \quad , \tag{13}$$

where the functions $\xi_n(t)$ are to be solved for. Substituting eq. (13) into eq. (3) yields

$$\sum_{n=1}^{N} [\mathcal{R}]\{\Phi^n\}\dot{\xi}_n + \sum_{n=1}^{N} [K]\{\Phi^n\}\xi_n = \{F(t)\}. \tag{14}$$

Premultiplying by $\{\Phi^m\}^T$ yields

$$\sum_{n=1}^{N} \{\Phi^m\}^T[\mathcal{R}]\{\Phi^n\}\dot{\xi}_n + \sum_{n=1}^{N}\{\Phi^m\}^T[K]\{\Phi^n\}\xi_n = \{\Phi^m\}^T\{F(t)\}. \tag{15}$$

Applying the orthogonality relations (11) and (12) reduces eq. (15) to

$$\mathcal{R}_n \dot{\xi}_n - \alpha_n \mathcal{R}_n \xi_n = F_n(t) , \tag{16}$$

where

$$\mathcal{R}_n = \{\Phi^n\}^T [\mathcal{R}] \{\Phi^n\} = 2\alpha_n \{\phi^n\}^T [m] \{\phi^n\} + \{\phi^n\}^T [r] \{\phi^n\} , \tag{17}$$

and

$$F_n(t) = \{\Phi^n\}^T \{F(t)\} = \{\phi^n\}^T \{f(t)\} . \tag{18}$$

We see that the coordinates ξ_n lead to uncoupled equations of motion. It is of interest to note the analogy between these results and those for undamped systems. Here $\mathcal{R}_n$ is analogous to the generalized mass $M_r = \sum_{j=1}^{N} m_j A_j^{(r)2}$ and $F_n(t)$ is analogous to the generalized force $G_r = \sum_{j=1}^{N} F_j A_j^{(r)}$.

It can be shown that the exciting forces of undamped systems may be expanded in a series

$$F_i = \sum_{r=1}^{N} f^{(r)} m_i A_i^{(r)} . \tag{19}$$

Trying a similar procedure we assume that we may write

$$\{F(t)\} = \sum_{n=1}^{N} \lambda_n [\mathcal{R}] \{\Phi^n\} . \tag{20}$$

Premultiply eq. (20) by $\{\Phi^m\}^T$ to give

$$\{\Phi^m\}^T \{F(t)\} = \sum_{n=1}^{N} \lambda_n \{\Phi^m\}^T [\mathcal{R}] \{\Phi^n\} . \tag{21}$$

Using the orthogonality relation eq. (11) reduces eq. (21) to

$$\left\{\Phi^n\right\}^T \left\{F(t)\right\} = \lambda_n \left\{\Phi^n\right\}^T \left[\mathcal{R}\right] \left\{\Phi^n\right\} = \lambda_n \mathcal{R}_n . \tag{22}$$

Since $\left\{\Phi^n\right\}^T \left\{F(t)\right\} = F_n(t)$ by eq. (18) we have

$$\lambda_n = \frac{F_n(t)}{\mathcal{R}_n} , \tag{23}$$

and hence

$$\left\{F(t)\right\} = \sum_{n=1}^{N} \frac{F_n(t)}{\mathcal{R}_n} \left[\mathcal{R}\right] \left\{\Phi^n\right\} . \tag{24}$$

From eq. (24) we see

$$\sum_{n=1}^{N} \frac{F_n(t)}{\mathcal{R}_n} \Big(\alpha_n \left[m\right] + \left[r\right] \Big) \left\{\phi^n\right\} = \left\{f(t)\right\}, \tag{25}$$

and

$$\sum_{n=1}^{N} \frac{F_n(t)}{\mathcal{R}_n} \left\{\phi^n\right\} = 0 . \tag{26}$$

We must now solve eq. (16) for ξ_n . Using the impulse method we see that for a unit impulse eq. (16) becomes

$$\mathcal{R}_n \dot{\xi}_{nf} - \alpha_n \mathcal{R}_n \xi_{nf} = \delta(t) . \tag{27}$$

Hence

$$\xi_{nf}(t) = \frac{1}{\mathcal{R}_n} e^{a_n t} , \tag{28}$$

or

$$\xi_{nf}(t-\tau) = \frac{1}{\mathcal{R}_n} e^{a_n (t-\tau)} , \tag{29}$$

and

$$\xi_n(t) = \int_0^t \xi_{nf}\,(t-\tau)\,F_n(\tau)\,d\tau\,, \tag{30}$$

or

$$\xi_n(t) = \frac{1}{\mathcal{R}_n}\int_0^t e^{a_n(t-\tau)}\,F_n(\tau)\,d\tau. \tag{31}$$

Finally

$$\left\{x(t)\right\} = \sum_{n=1}^{N}\left\{\Phi^n\right\}\xi_n(t) = \sum_{n=1}^{N}\left\{\Phi_n\right\}\frac{1}{\mathcal{R}_n}\int_0^t e^{a_n(t-\tau)}F_n(\tau)\,d\tau\,. \tag{32}$$

From the definitions of $\left\{x(t)\right\}$ and $\left\{\Phi^n\right\}$ we see

$$\left\{q_i\right\} = \sum_{n=1}^{N}\frac{1}{\mathcal{R}_n}\left\{\phi_i^n\right\}\int_0^t e^{a_n(t-\tau)}F_n(\tau)\,d\tau, \tag{33}$$

and

$$\left\{\dot{q}_i\right\} = \sum_{n=1}^{N}\frac{a_n}{\mathcal{R}_n}\left\{\phi_i^n\right\}\int_0^t e^{a_n(t-\tau)}F_n(\tau)\,d\tau\,. \tag{34}$$

1.22 Continuous Systems

To derive the analogous relations for continuous systems we take as a specific example the damped motion of simple beams. The equation of motion is

$$m(y)\ddot{x}\,(y,t) + r(y)\dot{x}(y,t) + \int_{y_0}^{y_1} k(y,\eta)\,x(\eta,t)\,d\eta = f(y,t)\,, \tag{35}$$

where

$k(y,\eta)$ is a stiffness influence function.

In general, it is quite difficult to determine $k(y,\eta)$ and it will be shown later how the relations are modified if we use

$$(E x''(y,t))'' \qquad \text{for} \qquad \int_{y_0}^{y_1} k(y,\eta)\, x(\eta,t)\, d\eta \,,$$

where

$(\;')$ denotes $\dfrac{\partial}{\partial y}$.

If we substitute $\xi(y,t)$ for $\dot{x}(y,t)$ in eq. (35) we obtain

$$\left.\begin{aligned} & m(y)\dot{x}(y,t) - m(y)\xi(y,t) = 0 \,, \\ \text{and} \\ & m(y)\dot{\xi}(y,t) + r(y)\dot{x}(y,t) + \int_{y_0}^{y_1} k(y,\eta)x(\eta,t)\, d\eta = f(y,t) \,. \end{aligned}\right\} \qquad (36)$$

1.221 HOMOGENEOUS SOLUTION. To obtain the homogeneous solution of eq. (36) we assume a solution of the form

$$\left.\begin{aligned} & x(y,t) = \phi(y)e^{at} \,, \\ \text{and} \\ & \xi(y,t) = \beta(y)e^{at} \,. \end{aligned}\right\} \qquad (37)$$

Substituting relations (37) into (36) we have

$$\left.\begin{aligned} & am(y)\,\phi(y) - m(y)\,\beta(y) = 0 \,, \\ \text{and} \\ & am(y)\beta(y) + ar(y)\,\phi(y) + \int_{y_0}^{y_1} k(y,\eta)\,\phi(\eta)\, d\eta = 0 \,. \end{aligned}\right\} \qquad (38)$$

The solutions of eqs. (38) yield an infinite number of eigenvalues a_n and eigenfunctions $\beta_n(y) = a_n\,\phi_n(y)$. For the n^{th} mode eqs. (38)

become

$$\alpha_n m(y)\, \phi_n(y) - m(y)\, \beta_n(y) = 0 \quad , \tag{39}$$

and

$$\alpha_n m(y)\, \beta_n(y) + \alpha_n r(y)\phi_n(y) + \int_{y_0}^{y_1} k(y,\eta)\, \phi_n(y)\, d\eta = 0. \tag{40}$$

Multiplying eq. (39) by $\beta_m(y)$ and eq. (40) by $\phi_m(y)$, adding and integrating over y from y_0 to y_1 we have

$$\alpha_n \int_{y_0}^{y_1} \Big(m(y)\phi_n(y)\beta_m(y) + r(y)\, \phi_n(y)\, \phi_m(y) + m(y)\beta_n(y)\phi_m(y) \Big) dy$$

$$+ \int_{y_0}^{y_1} \Big(\int_{y_0}^{y_1} k(y,\eta)\, \phi_n(\eta)\phi_m(y)\, d\eta - m(y)\beta_n(y)\beta_m(y) \Big) dy = 0 . \tag{41}$$

Interchanging indices in eq. (41) we get

$$\alpha_m \int_{y_0}^{y_1} \Big(m(y)\, \phi_m(y)\beta_n(y) + r(y)\phi_m(y)\, \phi_n(y) + m(y)\beta_m(y)\, \phi_n(y) \Big) dy$$

$$+ \int_{y_0}^{y_1} \Big(\int_{y_0}^{y_1} k(y,\eta)\, \phi_m(\eta)\, \phi_n(y)\, d\eta - m(y)\beta_m(y)\, \beta_n(y) \Big) dy = 0 . \tag{42}$$

Subtracting eq. (42) from eq. (41) and noting that $k(y,\eta)$ is a symmetric kernal we have

$$\left(\alpha_n - \alpha_m\right)\int_{y_0}^{y_1}\left(m(y)\phi_n(y)\beta_m(y) + r(y)\phi_n(y)\,\phi_m(y) + m(y)\beta_n(y)\,\phi_m(y)\right)dy = 0. \quad (43)$$

The orthogonality relations are hence

$$\int_{y_0}^{y_1}\left(m(y)\,\phi_n(y)\,\beta_m(y) + r(y)\,\phi_n(y)\,\phi_m(y) + m(y)\beta_n(y)\,\phi_m(y)\right)dy = 0, \qquad m \neq n, \quad (44)$$

and

$$\int_{y_0}^{y_1}\left(\int_{y_0}^{y_1} k(y,\eta)\,\phi_n(\eta)\,\phi_m(y)\,d\eta - m(y)\,\beta_n(y)\beta_m(y)\right)dy = 0, \qquad m \neq n. \quad (45)$$

1.222 NON-HOMOGENEOUS SOLUTION

To obtain the non-homogeneous solutions of eqs. (36) we expand $\chi(y,t)$ and $\xi(y,t)$ as a series of eigenfunctions, that is

$$\left.\begin{aligned} \chi(y,t) &= \sum_{n=1}^{\infty} \phi_n(y)\,\zeta_n(t), \\ &\text{and} \\ \xi(y,t) &= \sum_{n=1}^{\infty} \beta_n(y)\,\zeta_n(t). \end{aligned}\right\} \quad (46)$$

Using eqs. (46) in eqs. (36) we get

$$m(y)\sum_{n=1}^{\infty}\phi_n(y)\dot{\zeta}_n(t) - m(y)\sum_{n=1}^{\infty}\beta_n(y)\zeta_n(t) = 0, \tag{47}$$

and

$$m(y)\sum_{n=1}^{\infty}\beta_n(y)\dot{\zeta}_n(t) + r(y)\sum_{n=1}^{\infty}\phi_n(y)\dot{\zeta}_n(t) + \int_{y_0}^{y_1} k(y,\eta)\sum_{n=1}^{\infty}\phi_n(\eta)\zeta_n(t)\,d\eta = f(y,t). \tag{48}$$

Multiplying eq. (47) by $\beta_m(y)$, eq. (48) by $\phi_m(y)$, adding and integrating over y from y_0 to y_1 results in

$$\sum_{n=1}^{\infty}\dot{\zeta}_n(t)\int_{y_0}^{y_1}\Big(m(y)\phi_n(y)\beta_m(y) + r(y)\phi_n(y)\phi_m(y) + m(y)\phi_m(y)\beta_n(y)\Big)dy$$

$$+\sum_{n=1}^{\infty}\zeta_n(t)\int_{y_0}^{y_1}\Big(\int_{y_0}^{y_1}k(y,\eta)\phi_n(\eta)\phi_m(y)\,d\eta - m(y)\beta_n(y)\beta_m(y)\Big)dy = \int_{y_0}^{y_1}f(y,t)\phi_m(y)\,dy. \tag{49}$$

Using the orthogonality relations (44) and (45) and calling

$$\int_{y_0}^{y_1}f(y,t)\,\phi_n(y)\,dy = F_n \quad ,$$

$$\int_{y_0}^{y_1}\Big(2m(y)\phi_n(y)\beta_n(y) + r(y)\phi_n^2(y)\Big)dy = M_n ,$$

and

$$\int_{y_0}^{y_1}\Big(\int_{y_0}^{y_1}k(y,\eta)\,\phi_n(\eta)\phi_m(y)\,d\eta - m(y)\beta_n^2(y)\Big)dy = K_n ,$$

eq. (49) becomes

$$M_n \dot{\zeta}_n + K_n \zeta_n = F_n \ . \tag{50}$$

We see, therefore, that the coordinates ζ_n uncouple the equations of motion. Since $m(y)$, $r(y)$, $k(y,\eta)$, $\phi(y)$, $\beta(y)$, and $f(y,t)$ are all known, M_n , K_n , and F_n can be evaluated explicitly.

Since $\beta_n(y) = \alpha_n \phi_n(y)$ we may write

$$-\alpha_n M_n = \int_{y_0}^{y_1} \phi_n(y) \left(-\alpha_n^2 m(y) \phi_n(y) - \alpha_n r(y) \phi_n(y) - \alpha_n^2 m(y) \phi_n(y) \right) dy \ . \tag{51}$$

We also see that a solution of the homogeneous form of eq. (35) is $\chi(y,t) = \phi_n(y) e^{\alpha_n t}$. Substituting this into eq. (35) and dividing by $e^{\alpha_n t}$ we have

$$\alpha_n^2 m(y) \phi_n(y) + \alpha_n r(y) \phi_n(y) + \int_{y_0}^{y_1} k(y,\eta) \phi_n(\eta) \, d\eta = 0 \ ,$$

or

$$-\alpha_n^2 m(y) \phi_n(y) = \alpha_n r(y) \phi_n(y) + \int_{y_0}^{y_1} k(y,\eta) \phi_n(\eta) d\eta \ . \tag{52}$$

Using eq. (52) in eq. (51) we get

$$-\alpha_n M_n = \int_{y_0}^{y_1} \phi_n(y) \left(\int_{y_0}^{y_1} k(y,\eta) \phi_n(\eta) d\eta - \alpha_n^2 m(y) \phi_n(y) \right) dy \ . \tag{53}$$

From the definition of K_n we see $K_n = -\alpha_n M_n$ and eq. (50) becomes

$$M_n \dot{\zeta}_n - \alpha_n M_n \zeta_n = F_n \quad . \tag{54}$$

This is similar to eq. (16) and solving eq. (54) in the same manner we have

$$\zeta_n(t) = \frac{1}{M_n} \int_0^t e^{\alpha_n (t-\tau)} F_n(\tau)\, d\tau \quad . \tag{55}$$

From eq. (46) we have

$$x(y,t) = \sum_{n=1}^{\infty} \frac{\phi_n(y)}{M_n} \int_0^t e^{\alpha_n (t-\tau)} F_n(\tau)\, d\tau . \tag{56}$$

We could have written eq. (35) as

$$m(y)\ddot{x}(y,t) + r(y)\dot{x}(y,t) + \left[E x''(y,t)\right]'' = f(y,t) , \tag{57}$$

where

$(\dot{\ })$ denotes $\frac{\partial}{\partial t}$,

and

$(\ ')$ denotes $\frac{\partial}{\partial y}$.

If each end of the simple beam is free, simply supported, or built-in, then integrating by parts four times shows that

$$\int_{y_0}^{y_1} \left[E\,\phi_n''(y)\right]''\phi_m(y)\,dy \quad = \quad \int_{y_0}^{y_1} \left[E\,\phi_m''(y)\right]''\,\phi_n(y)\,dy\;. \tag{58}$$

Hence proceeding as before we find that the orthogonality relations become in this case

$$\int_{y_0}^{y_1} \Big(m(y)\,\phi_n(y)\,\beta_m(y) \;+ m(y)\,\beta_m(y)\,\phi_m(y) \;+\; r(y)\,\phi_n(y)\,\phi_m(y)\Big)\,dy = 0, \quad m \neq n, \tag{59}$$

and

$$\int_{y_0}^{y_1} \Big(\left[E\,\phi_n''(y)\right]''\phi_m(y) \;-\; m(y)\,\beta_m(y)\,\beta_n(y)\Big)\,dy \;=\; 0\,, \qquad m \neq n, \tag{60}$$

where K_n is modified to

$$\int_{y_0}^{y_1} \Big(\left[E\,\phi_n''(y)\right]''\phi_n(y) \;-\; m(y)\,\beta_n^2(y)\Big)\,dy \;=\; K_n\;. \tag{61}$$

The remaining equations are the same.

2.0 Response to Random Exciting Forces

When the forces which excite a mechanical system are random in nature, the methods for obtaining the response due to periodic excitation are not directly applicable. The external forces are no longer periodic and do not go to zero for large values of time so that $\int_{-\infty}^{\infty} \left| R(t) \right| dt$ does not converge.

2.1 GENERAL THEORY

Roughly speaking, what is meant by a random excitation is one in which the forcing function does not depend in a completely definite way on the independent variable time as in a causal process. On the contrary one gets in different observations different functions of time so that it is only the probability distributions that are directly observable. The following set of probability distributions will completely define a random function

$W_1(y,t)\,dy$	=	probability of finding y in the range y to $y+dy$ at time t.
$W_2(y_1t_1, y_2t_2)\,dy_1\,dy_2$	=	joint probability of finding y in the range y_1 to y_1+dy_1 at time t_1 and in the range y_2 to y_2+dy_2 at time t_2.
$W_3(y_1t_1, y_2t_2, y_3t_3)\,dy_1\,dy_2\,dy_3$	=	joint probability of finding y in the range y_1 to y_1+dy_1 at time t_1, in the range y_2 to y_2+dy_2 at time t_2 and in the range y_3 to y_3+dy_3 at time t_3.

and so on. The higher probability distributions W_n, $n = 4, 5, 6$ --- are defined in a similar fashion. Each W_n must satisfy the following conditions

(i) $W_n \geq 0$,

(ii) W_n is symmetric in y_1t_1, y_2t_2, --- y_nt_n,

(iii) $W_k = \int_{n-k} \text{------} \int W_n \, dy_{k+1} \text{ ---- } dy_n$.

Condition (iii) is just the equation for determining a marginal distribution.

The function W_n can be used as a means of classifying a random function. The simplest case is that of a purely random function. This means that the value of y at some time t_1 does not depend upon, or is not correlated with the value of y at any other time t. The probability distribution $W_1(yt)\,dy$ completely describes the function in this case since higher distributions are found by the following equation

$$W_n(y_1t_1,\ y_2t_2,\ ---,\ y_nt_n) = W_1(y_1t_1)\ W_1(y_2t_2)\ ---\ W_1(y_nt_n)\,. \tag{62}$$

The next most complicated case is where the probability distribution W_2 completely describes the function. This is the so-called Markoff Process. To define a Markoff Process more precisely we introduce the idea of the conditional probability. We define $P_2(y_1|y_2,t)dy_2$ to mean the probability that for a given y_1 we find y in the range y_2 to $y_2 + dy_2$ at a time t later. We find P_2 by the relation

$$W_2(y_1t_1,\ y_2t_2) = W_1(y_1t_1)P_2(y_1|y_2t)\,. \tag{63}$$

Equation (63) is analogous to the joint probability of two dependent events. In this case, we would have

$$P(AB) = P(A)P_A(B)\quad , \tag{64}$$

where

$P(AB)$ = probability of events A and B occurring,

$P(A)$ = probability that event A occurs,

and

$P_A(B)$ = probability that event B occurs given that event A has occurred.

Then

$P(AB)$ is the analog of W_2 ,

$P(A)$ is the analog of W_1 ,

and

$P_A(B)$ is the analog of P_2 .

The function P_2 must satisfy the conditions

(i) $P_2(y_1 | y_2 t) \geq 0$,

(ii) $\int dy_2 P_2(y_1 | y_2 t) = 1$,

and

(iii) $W_1(y_2 t_2) = \int W_1(y_1 t_1) P_2(y_1 | y_2 t)\, dy_1$.

We can now define the Markoff Process to mean that the conditional probability that y lies in the interval y_n to $y_n + dy_n$ at time t_n given that y is in the interval y_1 to $y_1 + dy_1$ at t_1 , y_2 to $y_2 + dy_2$ at t_2 , ---- y_{n-1} to $y_{n-1} + dy_{n-1}$ at t_{n-1} , depends only upon the values of y at t_n and t_{n-1} . That is for a Markoff Process

$$P_n(y_1 t_1, y_2 t_2, ---, y_{n-1} t_{n-1} | y_n t_n) = P_2(y_{n-1} t_{n-1} | y_n t_n) . \tag{65}$$

It is now possible to derive W_3 , W_4 , etc. from W_2 and eq. (63). For example

$$W_3(y_1 t_1, y_2 t_2, y_3 t_3) = W_2(y_1 t_1, y_2 t_2) P_2(y_2 t_2 | y_3 t_3) = \frac{W_2(y_1 t_1, y_2 t_2) W_2(y_2 t_2, y_3 t_3)}{W_1(y_2 t_2)} \tag{66}$$

It is clear that

$$W_4(y_1 t_1, y_2 t_2, y_3 t_3, y_4 t_4) = W_3(y_1 t_1, y_2 t_2, y_3 t_3) P_2(y_3 t_3 | y_4 t_4) , \tag{67}$$

and from eqs. (66) and (63) we see that eq. (67) becomes

$$W_4(y_1t_1, y_2t_2, y_3t_3, y_4t_4) = \frac{W_2(y_1t_1, y_2t_2)W_2(y_2t_2, y_3t_3)}{W_1(y_2t_2)} \cdot \frac{W_2(y_3t_3, y_4t_4)}{W_1(y_3t_3)} . \qquad (68)$$

We may determine W_5 , W_6 - - - W_n in a similar fashion.

In addition to the previously-mentioned conditions on P_2 it must also satisfy the condition

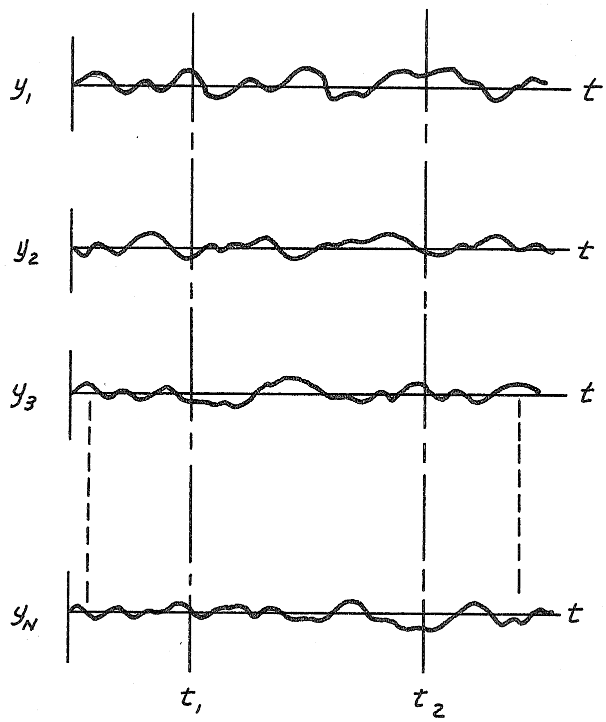
$$P_2(y_1|y_2t) = \int dy P_2(y_1|yt_1)P_2(y|y_2t-t_1) , \qquad 0 \leq t_1 \leq t . \qquad (69)$$

Equation (69) is called the Smoluchowski Equation.

The next step would be to consider processes that are completely described by W_3 , W_4 , W_5 , etc. Physically there are few examples studied which involve these higher order processes. Sometimes when a process is not a Markoff Process we can find another variable z which combined with y makes the process Markoffian. The variable z may be $\frac{dy}{dt}$ or another coordinate. In this case, the Smoluchowski Equation (eq. (69)) becomes

$$P_2(y_1z_1|y_2z_2t) = \iint dy\, dz\, P_2(y_1z_1|yzt_1) P_2(yz|y_2z_2t-t_1) . \qquad (70)$$

A further classification of the random process results when we consider whether it is stationary or non-stationary. Although precise mathematical definitions exist describing stationary and non-stationary processes, it is not necessary for our purposes to consider them and we will use a much simpler and perhaps intuitive concept. Assume for example that we have a large number of identical systems each one acted upon by the same type of random process. If we record some parameter of interest y for such systems over the same interval of time, we would have an ensemble of records as shown below.



ENSEMBLE OF RECORDS FOR PARAMETER y

SKETCH 1

Assume that we compute the average value of y^n at times t_1 and t_2. We will say that the process is stationary if

$$\langle y^n(t_1)\rangle = \langle y^n(t_2)\rangle \quad , \tag{71}$$

and non-stationary if

$$\langle y^n(t_1)\rangle \neq \langle y^n(t_2)\rangle \quad , \tag{72}$$

where

$$\langle y^n(t_1)\rangle = \frac{y_1^n(t_1) + y_2^n(t_1) + ---- + y_N^n(t_1)}{N}. \tag{73}$$

Roughly speaking, the process is stationary if the ensemble of functions $y_1, y_2, ------ y_N$ is invariant under translations in time.

Another important concept is the Ergodic Hypothesis. If we have a random process in which the functions of the ensemble are similar in nature, a kind of statistical homogeneity exists. This is desirable since then the statistical properties of the process can be deduced by considering any one function of the ensemble at various times or considering the various functions at a single time. Essentially, what we mean when we say a random process has the ergodic property is that time averages are equivalent to ensemble averages. Determining whether an ensemble of functions has the ergodic property or not is in general a very difficult problem.

One problem in particular has received a great deal of attention, that being the Gaussian Random Process. In this case we say that the process is purely random, stationary and possesses the ergodic property. In addition, the distribution function of the process is Gaussian. By purely random we mean that the process is completely described if we know $W_1(yt)$. We assume that W_1 is a Gaussian distribution which is a reasonable assumption. From the central limit theorem, we know that if some process produces an effect y with mean μ and variance σ^2 a large number of samples will have $\langle y\rangle$ normally or Gaussianly distributed. The stationary property insures that the underlying mechanism causing the fluctuations is not changing with time so that

W_1 is independent of time and hence $W_1(yt)$ becomes $W_1(y)$. The ergodic property, of course, insures the equivalence of time and ensemble averages so that we may say

$$\lim_{M\to\infty} \frac{\sum_{m=1}^{M} F_m(t)}{M} = \lim_{T\to\infty} \frac{1}{T}\int_0^T F(t)\,dt \quad . \tag{74}$$

Hence, we need only one record to obtain the required statistical information provided we average over intervals of length T where T is much longer than the greatest period appearing in the frequency spectrum.

In general, there are two methods of dealing with the Gaussian Random Process, these being the Fourier Series method and the Fokker-Planck method. In the Fourier Series method we consider the actual fluctuations in time of the parameter of interest. This parameter is developed in a Fourier Series in time but now the coefficients of the terms of the series are no longer constants but random variables. An important theorem in this method is the Weiner-Khintchine relation which connects the autocorrelation and power spectrum of a stationary process by means of a Fourier Cosine Transform. In the Fokker-Planck method we note that for an ensemble of systems we can consider the variations which occur as similar to a diffusion process. The distribution function of the random variables of the system is seen to satisfy a partial differential equation of the diffusion type.

For more complicated processes it is possible to obtain mean square values of the parameter of interest, (i.e., $\langle y^2(t) \rangle$), by using a modified form of the impulse method but this gives no information about the probability distributions which one obtains by using the Fourier Series or Fokker-Planck methods. The following material summarizes the Fourier Series and Fokker-Planck methods of treating Gaussian Random Processes.

2.2 FOURIER SERIES METHOD

We will begin this section by demonstrating some results which will be needed later, namely the relationship which exists between the power spectrum and autocorrelation and the form of the multi-dimensional Gaussian distribution.

2.21 Weiner-Khintchine Theorem

To begin with, assume we apply a periodic voltage e_{in} to a linear, time-invariant circuit which acts as a filter with a frequency response function $H(i\omega)$. We may represent the input voltage e_{in} and the resulting output voltage by Fourier Series. If we place a resistance R_s across the output terminals, and assume the circuit is sharply selective, we can show that the average power dissipated is

$$\mathcal{P}_{AV} = \frac{1}{2R_s} \sum_{n=m}^{\ell} d_n^2 \,, \tag{75}$$

where d_n are the coefficients of the Fourier Series for e_{in} .

In order to avoid concerning ourselves with factors of proportionality, we generalize our definition of power. We say that for a real-valued function of time $X(t)$ the instantaneous power associated with $X(t)$ is $X^2(t)$ and that the total energy $\mathscr{H}_{to}$ is

$$\mathscr{H}_{to} = \int_{-\infty}^{\infty} \mathcal{P}(t)\, dt = \int_{-\infty}^{\infty} X^2(t) dt , \tag{76}$$

and the average power is,

$$\mathcal{P}_{AV} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X^2(t)\, dt . \tag{77}$$

When the total energy of $X(t)$ is infinite we still assume that it has a finite average power which means that the limit of eq. (77) exists. The integrated square of $X(t)$, however, is infinite and we cannot define a Fourier Transform for $X(t)$ since the integral for $\tilde{X}(\omega)$ will in general not converge. By using the truncated functions $X_T(t)$ and defining the power spectral density of $X(t)$ as

$$\frac{\mathscr{H}_T(\omega)}{2T} = 2 \left| \frac{\tilde{X}_T(\omega)}{\sqrt{2T}} \right|^2 = \frac{|\tilde{X}_T(\omega)|^2}{T} , \tag{78}$$

we are able to show that

$$\int_0^{\infty} \Pi(\omega)\, d\omega = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X^2(t)\, dt . \tag{79}$$

It can also be seen that $\Pi(\omega)$ is real-valued, non-negative, an even function of ω, and invariant under a translation in time.

Now assume that $X(t)$ is a function such that $\tilde{X}_T(\omega)$ and $\Pi(\omega)$ exist and that $X(t)$ represents a stationary process having the ergodic property. By defining the autocorrelation function $\psi(\tau)$ as

$$\psi(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)\,X(t+\tau)\,dt\,, \tag{80}$$

and computing its Fourier transform we may show that

$$\Pi(\omega) = \frac{2}{\pi}\int_{0}^{\infty} \psi(\tau)\cos\omega\tau\,d\tau\,, \tag{81}$$

and

$$\psi(\tau) = \int_{0}^{\infty} \Pi(\omega)\cos\omega\tau\,d\omega. \tag{82}$$

Equations (81) and (82) are the Weiner-Khintchine relations connecting the autocorrelation and power spectrum by means of a Fourier Cosine transform. An important case is that of so-called "white noise" where all frequencies are present in equal amounts and the power spectrum is "flat" or constant. In this case eqs. (81)and (82) reduce to

$$\left.\begin{aligned} \Pi(\omega) &= \frac{4D}{2\pi}\,, \\ &\text{and} \\ \psi(\tau) &= 2D\,\delta(\tau)\,. \end{aligned}\right\} \tag{83}$$

For a non-stationary process D. G. Lampard has generalized the relations (83). He considers a pair of non-stationary functions $X_1(\tau)$ and $X_2(\tau)$ defined in the range $-\infty \leq \tau \leq t$ and which are zero for $\tau > t$. The joint energy of the component of frequency f of X_1 and X_2 in this order is

$$\mathcal{W}_{12}(t,f) = \tilde{X}_1(t,f)\,\tilde{X}_2^*(t,f) = \int_{-\infty}^{t}\int_{-\infty}^{t} X_1(t_1)X_2(t_2)e^{i2\pi f(t_2-t_1)}dt_1\,dt_2\,. \tag{84}$$

The cross correlation function $\psi_{12}(t,\tau)$ of X_1 and X_2 in this order is defined as

$$\psi_{12}(t,\tau') = \langle X_1(t)X_2(t-\tau')\rangle\,, \tag{85}$$

where the brackets denote the average value of the product is to be taken. In the non-stationary case, ensemble averages rather than time averages must be taken. Taking an ensemble average of both sides of eq. (84) and using eq. (85) yields

$$\langle \mathcal{W}_{12}(t,f)\rangle = \int_{-\infty}^{t}\int_{-\infty}^{t} \psi_{12}(t_1, t_1-t_2)e^{i2\pi f(t_2-t_1)}dt_1\,dt_2\,. \tag{86}$$

If we define $\tau = t_2 - t_1$ eq. (86) may be written as

$$\langle \mathcal{W}_{12}(t,f)\rangle = \int_{-\infty}^{t}\int_{0}^{\infty} \psi_{12}(t_2-\tau,-\tau)\,e^{i2\pi f\tau}dt_2\,d\tau + \int_{-\infty}^{t}\int_{-\infty}^{0} \psi_{12}(t_1,-\tau)\,e^{-i2\pi f\tau}\,dt_1\,d\tau$$

$$= \int_{-\infty}^{t}\int_{0}^{\infty}\left\{\psi_{12}(t_1-\tau,-\tau)e^{-i2\pi f\tau} + \psi_{12}(t_1,\tau)\,e^{i2\pi f\tau}\right\}dt_1\,d\tau\,. \tag{87}$$

Since $\psi_{12}(t_1-\tau,-\tau) = \psi_{21}(t_1,\tau)$, eq. (87) becomes

$$\langle \mathcal{W}_{12}(t,f)\rangle = \int_{-\infty}^{t}\int_{0}^{\infty}\left\{\psi_{12}(t_1,\tau)e^{-i2\pi f\tau} + \psi_{21}(\tau_1,\tau)e^{i2\pi f\tau}\right\}dt_1 d\tau. \quad (88)$$

Defining the cross power spectrum as

$$\Pi_{12}(t,f) = 2\frac{\partial}{\partial t}\left\{\langle \mathcal{W}_{12}(t,f)\rangle_{AV}\right\}, \quad (89)$$

we find

$$\Pi_{12}(t,f) = 2\int_{0}^{\infty}\left\{\psi_{12}(t,\tau)e^{-i2\pi f\tau} + \psi_{21}(t,\tau)e^{i2\pi f\tau}\right\}d\tau. \quad (90)$$

For the case where $x_1(t) = x_2(t)$ we have

$$\Pi(t,f) = 4\int_{0}^{\infty}\psi(t,\tau)\cos 2\pi f\tau\, d\tau, \quad (91)$$

which is analogous to eq. (81) of the stationary case.

To get the inverse of the relation given by eq. (91) multiply $\Pi_{12}(t,f)$ by $e^{i2\pi f\bar{\rho}}$, $\Pi_{21}(t,f)$ by $e^{-i2\pi f\bar{\rho}}$, integrate over f from 0 to ∞, and add the resulting expressions. This yields

$$\int_0^\infty \left\{ \Pi_{12}(t,f)\, e^{i2\pi f\bar{\rho}} + \Pi_{21}(t,f)\, e^{-i2\pi f\bar{\rho}} \right\} df$$

$$= 4\int_0^\infty\int_0^\infty \left\{ \psi_{12}(t,\tau)\cos 2\pi f(\bar{\rho}-\tau) + \psi_{21}(t,\tau)\cos 2\pi f(\bar{\rho}+\tau) \right\} d\tau\, df. \qquad (92)$$

Integrating the right-hand side of eq. (92) we find

$$\psi_{12}(t,\bar{\rho}) = \frac{1}{2}\int_0^\infty \left\{ \Pi_{12}(t,f)\, e^{i2\pi f\bar{\rho}} + \Pi_{21}(t,f)\, e^{-i2\pi f\bar{\rho}} \right\} df. \qquad (93)$$

This is the inverse of eq. (90). If $X_1(t) = X_2(t)$, eq. (93) becomes

$$\psi(t,\tau) = \int_0^\infty \Pi(t,f)\cos 2\pi f\tau\, df. \qquad (94)$$

Equation (94) is analogous to eq. (82) for the stationary case.

2.22 Multi-Dimensional Gaussian Distributions

If y_1 , y_2 , --- y_n are random variables with means zero, we say that they are normally distributed in n dimensions if their distribution function $P(y_1$, y_2 , --- $y_n)$ is of the form

$$P(y_1, y_2, ---, y_n) = \frac{1}{(2\pi)^{n/2}\sqrt{|\mathcal{B}|}} \exp\left\{ -\frac{1}{2} \left| \mathcal{Y}^T \mathcal{B}^{-1} \mathcal{Y} \right| \right\}, \qquad (95)$$

where

$[\mathcal{G}]$ is the matrix of the second moments σ_{ij}^2 ;

$\sigma_{ij}^2 = \langle y_i y_j \rangle$, i,j , $= 1, 2, 3, -----, n$;

$[\mathcal{G}]^{-1}$ is the inverse of the matrix $[\mathcal{G}]$;

$|\mathcal{G}|$ is the determinant of the matrix $[\mathcal{G}]$;

$\{\mathcal{Y}\}$ is a column matrix with elements y_1 , y_2 , ---- y_n ;

and $\{\mathcal{Y}\}^T$ is the transpose of the column matrix $\{\mathcal{Y}\}$.

Hence

$$\left| \mathcal{Y}^T \mathcal{G}^{-1} \mathcal{Y} \right| = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\mathcal{G}_{ij}}{|\mathcal{G}|} y_i y_j , \tag{96}$$

with $\mathcal{G}_{ij}$ the cofactor of σ_{ij}^2 in the matrix $[\mathcal{G}]$. The marginal distributions $P(y_1, y_2, --- y_r)$, where $r \leq n$, are found in the following way

$$P(y_1, y_2, --- y_r) = \int_{-\infty}^{\infty} -- \int_{-\infty}^{\infty} P(y_1, y_2, ---, y_n) dy_{r+1} --- dy_n . \tag{97}$$

If the y_i are independent

$$\sigma_{ij}^2 = \langle y_i y_j \rangle = \langle y_i \rangle \langle y_j \rangle = 0. \tag{98}$$

Hence $[\mathcal{G}]$ is a diagonal matrix and

$$P(y_1, y_2, ---, y_n) = P(y_1) P(y_2) --- P(y_n) , \tag{99}$$

with each $P(y_i)$ being a one-dimensional Gaussian distribution with mean zero and variance σ_{ii}^2 .

For the special case of two dimensions

$$[\mathcal{G}] = \begin{bmatrix} \sigma_{11}^2 & \sigma_{12}^2 \\ \sigma_{21}^2 & \sigma_{22}^2 \end{bmatrix} ,$$

$$[\mathcal{G}]^{-1} = \frac{1}{|\mathcal{G}|} \cdot \begin{bmatrix} \sigma_{22}^2 & -\sigma_{12}^2 \\ -\sigma_{21}^2 & \sigma_{11}^2 \end{bmatrix} ,$$

$$|\mathcal{G}| = \sigma_{11}^2 \sigma_{22}^2 - \sigma_{12}^2 \sigma_{21}^2 ,$$

$$\{\mathcal{X}\} = \begin{Bmatrix} X \\ Y \end{Bmatrix} ,$$

$$\{\mathcal{X}\}^T = \{X\,Y\} ,$$

and hence

$$|\mathcal{X}^T \mathcal{G}^{-1} \mathcal{X}| = \frac{1}{|\mathcal{G}|}\left(\sigma_{22}^2 X^2 - 2\sigma_{12}^2 XY + \sigma_{11}^2 Y^2\right). \tag{100}$$

If we define

$$\sigma_{11}^2 = \sigma_X^2 ,$$

$$\sigma_{22}^2 = \sigma_y^2 ,$$

and

$$\sigma_{12}^2 = \bar{\tau}\sigma_X \sigma_y ;$$

then eq. (95) becomes

$$P(X,Y) = \frac{1}{2\pi\sqrt{1-\bar{\tau}^2}\,\sigma_X \sigma_y} \exp\left\{\frac{-1}{2(1-\bar{\tau}^2)}\left(\frac{X^2}{\sigma_X^2} - \frac{2\bar{\tau}XY}{\sigma_X \sigma_y} + \frac{Y^2}{\sigma_y^2}\right)\right\} . \tag{101}$$

If X and Y are independent then $\bar{\tau}$, the correlation coefficient, is zero and we have

$$P(X,Y) = P(X)P(Y) = \left(\frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{X^2}{2\sigma_X^2}}\right)\left(\frac{1}{\sqrt{2\pi}\,\sigma_y} e^{-\frac{Y^2}{2\sigma_y^2}}\right) . \tag{102}$$

In general we may say that if y_1, y_2 - - -, y_n are n independent, normally distributed random variables with means zero and variances σ_i^2 and Z_1, Z_2 - - -, Z_m are m linear combinations of the y_i, then the Z_i have an m dimensional Gaussian distribution given by

$$P(z_1, z_2, ---, z_m) = \frac{1}{(2\pi)^{m/2}\sqrt{|\Delta|}} \exp\left\{-\frac{1}{2}\left| z^T \Delta^{-1} z \right|\right\} , \tag{103}$$

where

$$[\Delta] = [g][\Sigma][\Sigma]^T[g]^T ,$$

$$[\Sigma] = [\delta_{ij}\ \sigma_{ij}] ,$$

$$\delta_{ij} = \begin{cases} 0 , & i \neq j \\ 1 , & i = j \end{cases} ,$$

$$\sigma_{ij}^2 = \text{second moments} ,$$

and

$[g] = [g_{ij}]$ a non-singular matrix of rank $m \leq n$ whose elements g_{ij} are constants.

For the special case of two dimensions, $n = m = 2$, and

$$P(X) = \frac{1}{\sqrt{2\pi}\ \sigma_x} e^{-\frac{X^2}{2\sigma_x^2}} ,$$

and

$$P(Y) = \frac{1}{\sqrt{2\pi}\ \sigma_y} e^{-\frac{Y^2}{2\sigma_y^2}} .$$

Let $Z_1 = U$ and $Z_2 = V$ where

$$\left.\begin{aligned} U &= g_{11} X + g_{12} Y , \\ \text{and} \quad V &= g_{21} X + g_{22} Y . \end{aligned}\right\} \tag{104}$$

We may solve eqs. (104) for X and Y getting

$$\left.\begin{aligned} X &= b_{11} U + b_{12} \overline{V} , \\ \text{and} \qquad & \\ Y &= b_{21} U + b_{22} \overline{V} . \end{aligned}\right\} \tag{105}$$

Now $P(U\overline{V}) = J\left(\frac{X,Y}{U,\overline{V}}\right) P(XY)$ where $J\left(\frac{X,Y}{U,\overline{V}}\right)$ is the Jacobian of the transformation. We see that

$$J\left(\frac{X,Y}{U,\overline{V}}\right) = \left| g^{-1} \right| , \tag{106}$$

and

$$[g]^{-1} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \frac{1}{|g|} \begin{bmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{bmatrix} . \tag{107}$$

Hence

$$P(U\overline{V}) = \frac{1}{2\pi |g| \sigma_x \sigma_y} \exp\left\{ -\frac{1}{2} \left(\frac{X^2}{\sigma_x^2} + \frac{Y^2}{\sigma_y^2} \right) \right\} . \tag{108}$$

Since

$$\left.\begin{aligned} X^2 &= b_{11}^2 U^2 + 2 b_{11} b_{12} U\overline{V} + b_{12}^2 \overline{V}^2 , \\ \text{and} \qquad & \\ Y^2 &= b_{21}^2 U^2 + 2 b_{21} b_{22} U\overline{V} + b_{22}^2 \overline{V}^2 , \end{aligned}\right\} \tag{109}$$

eq. (108) becomes

$$P(U\overline{V}) = \frac{1}{2\pi |g| \sigma_x \sigma_y} \exp\left\{ -\frac{1}{2} \left[\left(\frac{b_{11}^2}{\sigma_x^2} + \frac{b_{21}^2}{\sigma_y^2} \right) U^2 \right.\right.$$

$$\left.\left. + 2 \left(\frac{b_{11} b_{12}}{\sigma_x^2} + \frac{b_{21} b_{22}}{\sigma_y^2} \right) U\overline{V} + \left(\frac{b_{12}^2}{\sigma_x^2} + \frac{b_{22}^2}{\sigma_y^2} \right) \overline{V}^2 \right] \right\} . \tag{110}$$

Now

$$[\Sigma] = [\Sigma]^T = \begin{bmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{bmatrix} , \tag{111}$$

$$[\Delta]^{-1} = [g^T]^{-1} [\Sigma^T]^{-1} [\Sigma]^{-1} [g]^{-1} = \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} \sigma_x^{-2} & 0 \\ 0 & \sigma_y^{-2} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} , \tag{112}$$

and

$$z = \begin{Bmatrix} U \\ \bar{V} \end{Bmatrix} . \tag{113}$$

Hence

$$-\frac{1}{2}\left| z^T \Delta^{-1} z \right| = -\frac{1}{2}\left\{ \left(\frac{b_{11}^2}{\sigma_x^2} + \frac{b_{21}^2}{\sigma_y^2} \right) U^2 + 2\left(\frac{b_{11} b_{12}}{\sigma_x^2} + \frac{b_{21} b_{22}}{\sigma_y^2} \right) U\bar{V} + \left(\frac{b_{12}^2}{\sigma_x^2} + \frac{b_{22}^2}{\sigma_y^2} \right) \bar{V}^2 \right\} . \tag{114}$$

In addition

$$|\Delta| = |g|^2 \, |\Sigma|^2 = |g|^2 \sigma_x^2 \sigma_y^2 , \tag{115}$$

so that

$$\sqrt{|\Delta|} = |g| \, \sigma_x \sigma_y . \tag{116}$$

Using eqs. (113) and (115) in eq. (110) reduces it to the standard form

$$P(U\bar{V}) = \frac{1}{2\pi\sqrt{|\Delta|}} \exp\left\{ -\frac{1}{2} \left| z^T \Delta^{-1} z \right| \right\} . \tag{117}$$

This does not prove relation (103) but does verify it for the two-dimensional case.

2.23 Relation Between Power Spectrum and Mean Square of Fourier Coefficients

Before presenting the treatment of the Gaussian, random process, one more relation is needed. Let $X(t)$ be a stationary, Gaussian, random process. In some interval $0 < t < T$ we may represent $X(t)$ by its Fourier Series

$$X(t) = \sum_{n=1}^{\infty} (d_n \cos n\omega t + \bar{d}_n \sin n\omega t), \tag{118}$$

where d_n and $\bar{d}_n$ are determined in the usual way. If we define the functions $\bar{\mathcal{C}}_n(t)$ and $\bar{\mathcal{S}}_n(t)$ as

$$\bar{\mathcal{C}}_n(t) = \begin{cases} \frac{2}{T}\cos n\omega t, & 0 \le t \le T, \\ 0, & \text{otherwise}, \end{cases} \tag{119}$$

and

$$\bar{\mathcal{S}}_n(t) = \begin{cases} -\frac{2}{T}\sin n\omega t, & 0 \le t \le T, \\ 0, & \text{otherwise}, \end{cases} \tag{120}$$

then we have

$$\left. \begin{aligned} d_n &= \int_0^T \bar{\mathcal{C}}_n(T-\tau)\, X(\tau)\, d\tau, \\ \text{and} \quad \bar{d}_n &= \int_0^T \bar{\mathcal{S}}_n(T-\tau)\, X(\tau)\, d\tau . \end{aligned} \right\} \tag{121}$$

The coefficients d_n and $\bar{d}_n$ are assumed to be normally distributed and it can be shown that for large T

$$\langle \bar{d}_k^2 \rangle = \langle d_k^2 \rangle \doteq \frac{2\pi}{T} \Pi(k\omega) \quad . \tag{122}$$

2.24 Solution for Gaussian Random Process

With these rather lengthy preliminaries completed, we may now proceed with the Fourier Series method of treating the Gaussian Random Process.

Consider $y(t)$ to be a random, stationary function repeated periodically with period T, where T is large. We can develop $y(t)$ in a Fourier Series

$$y(t) = \sum_{k=1}^{\infty} \left(d_k \cos 2\pi f_k t + \bar{d}_k \sin 2\pi f_k t \right) , \tag{123}$$

where

$$f_k = \frac{k}{T} \quad .$$

We assume $\langle y(t) \rangle = 0$, hence we have no constant term. In addition, we assume that d_k and $\bar{d}_k$ are random variables, Gaussianly distributed with average value zero, and that they are independent of each other. The distribution function for the d_k and $\bar{d}_k$ then becomes, using eqs. (99) and (102),

$$P(d_1, d_2, ---, \bar{d}_1, \bar{d}_2, ---) = \prod_{k=1}^{\infty} \frac{1}{\sqrt{2\pi}\, \sigma_k} \exp\left\{ -\frac{(d_k^2 + \bar{d}_k^2)}{2\sigma_k^2} \right\} , \tag{124}$$

where

$$\sigma_k^2 = \langle d_k^2 \rangle = \langle \bar{d}_k^2 \rangle .$$

We may now derive from eq. (123) the quantities $\langle y^2(t) \rangle$, $\langle \dot{y}^2(t) \rangle$, and $\langle y(t)\dot{y}(t) \rangle$. We see that

$$\langle y^2(t) \rangle = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \Big\{ \langle d_k d_\ell \rangle \cos 2\pi f_k t \cos 2\pi f_\ell t + \langle \bar{d}_k \bar{d}_\ell \rangle \sin 2\pi f_k t \sin 2\pi f_\ell t$$

$$+ \langle d_k \bar{d}_\ell \rangle \cos 2\pi f_k t \sin 2\pi f_\ell t + \langle d_\ell \bar{d}_k \rangle \cos 2\pi f_\ell t \sin 2\pi f_k t \Big\} . \qquad (125)$$

Since the d_k and $\bar{d}_k$ are independent having means zero eq. (125) becomes

$$\langle y^2(t) \rangle = \sum_{k=1}^{\infty} \Big\{ \langle d_k^2 \rangle \cos^2 2\pi f_k t + \langle \bar{d}_k^2 \rangle \sin^2 2\pi f_k t \Big\} . \qquad (126)$$

Using eqs. (122) we have

$$\langle y^2(t) \rangle = \frac{1}{T}\sum_{k=1}^{\infty} \Pi(f_k) \doteq \int_0^{\infty} \Pi(f)\, df . \qquad (127)$$

Since

$$\dot{y}(t) = \sum_{k=1}^{\infty} \Big\{ -2\pi f_k d_k \sin 2\pi f_k t + 2\pi f_k \bar{d}_k \cos 2\pi f_k t \Big\} , \qquad (128)$$

proceeding as before we find

$$\langle \dot{y}^2(t) \rangle = \frac{4\pi^2}{T}\sum_{k=1}^{\infty} f_k^2\, \Pi(f_k) \doteq 4\pi^2 \int_0^{\infty} f^2\, \Pi(f)\, df ; \qquad (129)$$

and

$$\langle y(t)\dot{y}(t) \rangle = \sum_{k=1} 2\pi f_k \sin 2\pi f_k t \cos 2\pi f_k t \Big\{ -\langle d_k^2 \rangle + \langle \bar{d}_k^2 \rangle \Big\} = 0 . \qquad (130)$$

If $y(t)$ is considered as the output of a linear system with a Gaussian, random input, then it is known that $y(t)$ is Gaussian also. Hence when $\langle y(t)\rangle = 0$ all that need be computed is $\langle y^2(t)\rangle$ for this completely defines the probability distribution in this case.

2.3 FOKKER-PLANCK METHOD

In this method we solve directly for the output probability distribution since it is seen to satisfy a partial differential equation of the diffusion type.

2.31 General Theory

Let us begin with the Smoluchowski Equation (eq. (69)) written in the form

$$P(x|y, t+\Delta t) = \int dz\, P(x|z, t) P(z|y, \Delta t) \quad . \tag{131}$$

Equation (131) tacitly assumes that the random process is a Markoff Process. In addition, the moments of the change in the space coordinate in a small time Δt are given by

$$\sigma^n(z, \Delta t) = \int dy\, (y-z)^n P(z|y, \Delta t) \; . \tag{132}$$

It is assumed that as $\Delta t \to 0$ only the first two moments become proportional to Δt so that

$$\bar{A}(z) = \lim_{\Delta t \to 0} \frac{\sigma^1(z, \Delta t)}{\Delta t} , \tag{133}$$

and

$$B(z) = \lim_{\Delta t \to 0} \frac{\sigma^2(z, \Delta t)}{\Delta t} , \tag{134}$$

both exist. All $\sigma^n(z,\Delta t)$, where $n \geq 3$, become proportional to $(\Delta t)^m$ where $m \geq 2$. In the case of a linear, single-degree-of-freedom oscillator, the values of $\bar{A}(z)$ and $B(z)$ can be computed from the equation of motion which is referred to as the Langevin Equation.

Consider now the following integral

$$\mathcal{J} = \int dy\, \mathcal{E}(y) \frac{\partial P(x|y,t)}{\partial t} , \tag{135}$$

and let $\mathcal{E}(y)$ go to zero as $y \to \pm\infty$ sufficiently fast so that the integral exists. With the use of eq. (131) and the Taylor Series expansion of $\mathcal{E}(y)$ about Z it is possible to show that

$$\int dy\, \mathcal{E}(y) \left\{ \frac{\partial P}{\partial t} + \frac{\partial (\bar{A}P)}{\partial y} - \frac{1}{2} \frac{\partial^2 (BP)}{\partial y^2} \right\} = 0 . \tag{136}$$

Since eq. (136) holds for any $\mathcal{E}(y)$ we have

$$\frac{\partial P}{\partial t} = -\frac{\partial (\bar{A}P)}{\partial y} + \frac{1}{2} \frac{\partial^2 (BP)}{\partial y^2} . \tag{137}$$

In N dimensions eq. (137) becomes

$$\frac{\partial P}{\partial t} = -\sum_{i=1}^{N} \frac{\partial (\bar{A}_i P)}{\partial y_i} + \frac{1}{2} \sum_{k=1}^{N} \sum_{\ell=1}^{N} \frac{\partial^2 (B_{k\ell} P)}{\partial y_k\, \partial y_\ell} \tag{138}$$

Equations (137) and (138) are the one-dimensional and N-dimensional Fokker-Planck equations. We must compute the $\bar{A}_i$ and $B_{k\ell}$ and solve the equations for the probability distribution function P .

2.32 Solution for Linear, Damped, Single-Degree-of-Freedom System

The equation of motion in this case is

$$\frac{d^2y}{dt^2} + \bar{\beta}\frac{dy}{dt} + \omega_o^2 y = F(t) \quad , \tag{139}$$

where

$$\bar{\beta} = 2\rho\omega_o \; .$$

We will assume that $F(t)$ is purely random and Gaussian so that

$$\left.\begin{aligned} \langle F(t) \rangle &= 0 \quad , \\ \langle F(t_1) F(t_2) \rangle &= 2D\,\delta(t_1 - t_2) \, . \end{aligned}\right\} \tag{140}$$

Equation (139) may be rewritten as

$$\left.\begin{aligned} \frac{dy}{dt} &= p \quad , \\ \frac{dp}{dt} + (\bar{\beta}p + \omega_o^2 y) &= F(t) . \end{aligned}\right\} \tag{141}$$

It is clear from eqs. (132), (133), (134), and (140) that

$$\left.\begin{aligned}
\bar{A}_1 &= \lim_{\Delta t \to 0} \frac{\langle \Delta y \rangle}{\Delta t} = p \,, \\
\bar{A}_2 &= \lim_{\Delta t \to 0} \frac{\langle \Delta p \rangle}{\Delta t} = -(\bar{\beta} p + \omega_0^2 y) \,, \\
B_{11} &= \lim_{\Delta t \to 0} \frac{\langle (\Delta y)^2 \rangle}{\Delta t} = 0 \,, \\
B_{12} &= B_{21} = \lim_{\Delta t \to 0} \frac{\langle \Delta y \, \Delta p \rangle}{\Delta t} = 0 \,, \\
B_{22} &= \lim_{\Delta t \to 0} \frac{\langle (\Delta p)^2 \rangle}{\Delta t} = 2D \,,
\end{aligned}\right\} \tag{142}$$

and hence eq. (138) becomes

$$\frac{\partial P}{\partial t} = -p \frac{\partial P}{\partial y} + \frac{\partial}{\partial p}\left\{(\bar{\beta} p + \omega_0^2 y) P\right\} + D \frac{\partial^2 P}{\partial p} \,. \tag{143}$$

Equation (143) is to be solved for P with the initial condition

$$P(y \mid p, 0) = \delta(y - y_0)\delta(p - p_0) \,. \tag{144}$$

Equation (144) is merely the statement that at $t = 0$ we know for certain that $y = y_0$ and $p = \dot{y} = p_0$. By defining

$$z_1 = p + f y \,,$$

and

$$z_2 = p + g y \,,$$

where

$$f = \frac{\bar{\beta}}{2} + i\omega_1 \,,$$

$$g = \frac{\bar{\beta}}{2} - i\omega_1 \,,$$

and

$$\omega_1 = \omega_0\sqrt{1-\rho^2} ,$$

it can be shown that eq. (143) reduces to

$$\frac{\partial P}{\partial t} = g\frac{\partial}{\partial z_1}(z_1 P) + f\frac{\partial}{\partial z_2}(z_2 P) + D\left(\frac{\partial}{\partial z_1}+\frac{\partial}{\partial z_2}\right)^2 P . \tag{145}$$

Now consider the solution of

$$\frac{\partial P}{\partial t} = -\sum_{i=1}^{N}\lambda_i\frac{\partial}{\partial y_i}(y_i P) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\bar{\epsilon}_{ij}\frac{\partial^2 P}{\partial y_i\,\partial y_j} , \tag{146}$$

with the initial condition

$$P(y_1, y_2, ---, y_N, 0) = \delta(y_1-y_{10})\,\delta(y_2-y_{20}) ---- \delta(y_N-y_{N0}) . \tag{147}$$

This partial differential equation can be solved by the Fourier Transform method and we find

$$\tilde{P}(\Delta_1, \Delta_2, ---, \Delta_N, t) = exp\left\{-i\sum_{j=1}^{N}\Delta_j y_{jo} e^{\lambda_j t} + \frac{1}{2}\sum_{k=1}^{N}\sum_{j=1}^{N}\bar{\epsilon}_{kj}\frac{\Delta_k\Delta_j}{\lambda_k+\lambda_j}\left[1-e^{(\lambda_k+\lambda_j)t}\right]\right\}. \tag{148}$$

This can be shown to be the Fourier Transform of an N -dimensional Gaussian distribution with means

$$\langle y_i \rangle = y_{io}\, e^{\lambda_i t} , \tag{149}$$

and variances

$$\langle (y_i - \bar{y}_i)(y_j - \bar{y}_j) \rangle = -\frac{\sigma_{ij}}{\lambda_i + \lambda_j}\left\{1 - e^{(\lambda_i+\lambda_j)t}\right\} , \tag{150}$$

where

$$\bar{y}_i = \langle y_i \rangle .$$

Now eq. (145) is a special case of eq. (146), hence the solution of eq. (145) is a two-dimensional Gaussian distribution in the variables z_1 and z_2 where

$$\left.\begin{aligned}
\langle z_1 \rangle &= z_{10}\, e^{-g t} , \\
\langle z_2 \rangle &= z_{20}\, e^{-f t} , \\
\langle (z_1 - \bar{z}_1)^2 \rangle &= \frac{D}{g}(1 - e^{-2 g t}) , \\
\langle (z_2 - \bar{z}_2)^2 \rangle &= \frac{D}{f}(1 - e^{-2 f t}) , \\
\text{and} \quad \langle (z_1 - \bar{z}_1)(z_2 - \bar{z}_2) \rangle &= \frac{2D}{f+g}(1 - e^{-(f+g)t}) .
\end{aligned}\right\} \tag{151}$$

z_{10} and z_{20} are the initial values of z_1 and z_2 corresponding to the initial values of y and p, that is

$$\left.\begin{aligned}
z_{10} &= p_0 + f y_0 , \\
\text{and} \quad z_{20} &= p_0 + g y_0 .
\end{aligned}\right\} \tag{152}$$

Hence, the solution of eq. (143) with the initial condition eq. (147) is, in terms of the variables z_1 and z_2, given by

$$P(z_1, z_2) = \frac{1}{2\pi\sqrt{1 - \bar{\tau}^2}\,\sigma_1 \sigma_2} \exp\left\{ - \frac{1}{2(1 - \bar{\tau}^2)} \left(\frac{z_1^2}{\sigma_1^2} - \frac{2\bar{\tau}}{\sigma_1 \sigma_2} z_1 z_2 + \frac{z_2^2}{\sigma_2^2} \right) \right\} , \tag{153}$$

where

$$\sigma_1^2 = \langle (z_1 - \bar{z}_1)^2 \rangle \quad ,$$

$$\sigma_2^2 = \langle (z_2 - \bar{z}_2)^2 \rangle \quad ,$$

and

$$\sigma_1 \sigma_2 \bar{\tau} = \langle (z_1 - \bar{z}_1)(z_2 - \bar{z}_2) \rangle \, .$$

Using eqs. (151), (152), and the relations

$$z_1 = p + f y \quad ,$$

$$z_2 = p + g y \quad ,$$

$$f = \frac{\bar{\beta}}{2} + i\omega_1 \quad ,$$

and

$$g = \frac{\bar{\beta}}{2} - i\omega_1 \quad ,$$

equation (153) can be written in terms of y , P , $\bar{\beta}$, ω_0 , and ω_1 . We must, however, compute $\langle P \rangle$, $\langle y \rangle$, $\langle (p - \bar{p})^2 \rangle$, $\langle (y - \bar{y})^2 \rangle$ and $\langle (p - \bar{p})(y - \bar{y}) \rangle$. After some tedious algebra we find,

$$\langle p \rangle = \frac{p_0}{\omega_1} e^{-\frac{\bar{\beta}}{2}t} \left\{ \omega_1 \cos \omega_1 t - \frac{\bar{\beta}}{2} \sin \omega_1 t \right\} - \frac{\omega_0^2}{\omega_1^2} y_0 e^{-\frac{\bar{\beta}}{2}t} \sin \omega_1 t \quad , \tag{154}$$

$$\langle y \rangle = \frac{p_0}{\omega_1} e^{-\frac{\bar{\beta}}{2}t} \sin \omega_1 t + \frac{y_0}{\omega_1} e^{-\frac{\bar{\beta}}{2}t} \left(\omega_1 \cos \omega_1 t + \frac{\bar{\beta}}{2} \sin \omega_1 t \right) \quad , \tag{155}$$

$$\langle (p - \bar{p})^2 \rangle = \frac{D}{\bar{\beta}} \left\{ 1 - \frac{e^{-\bar{\beta}t}}{\omega_1^2} \left(\omega_1^2 + \frac{\bar{\beta}^2}{2} \sin^2 \omega_1 t - \bar{\beta}\omega_1 \sin \omega_1 t \cos \omega_1 t \right) \right\} \quad , \tag{156}$$

$$\langle \omega_o^2 (y-\bar{y})^2 \rangle = \frac{D}{\bar{\beta}} \left\{ 1 - \frac{e^{-\bar{\beta}t}}{\omega_1^2} \left(\omega_1^2 + \frac{\bar{\beta}^2}{2} \sin^2 \omega_1 t + \bar{\beta}\omega_1 \sin \omega_1 t \cos \omega_1 t \right) \right\} , \tag{157}$$

and

$$\langle \omega_o (p-\bar{p})(y-\bar{y}) \rangle = \frac{D\omega_o}{\omega_1^2} e^{-\bar{\beta}t} \sin^2 \omega_1 t \quad . \tag{158}$$

With the results of eqs. (154) through (158) we have the solution for the joint probability distribution of the variables y and $\dot{y} = p$ for a single-degree-of-freedom, linear oscillator by the Fokker-Planck method.

We now have two methods for solving random vibration problems, namely the Fourier Series method and the Fokker-Planck method. The limitations of each method should be noted, however. In the Fourier Series method we are unable to solve for the probability distribution, unless we postulate that the input is Gaussianly distributed; for then the output is also Gaussian. The power spectrum on the other hand may be any appropriate function. In the Fokker-Planck method the input has a flat power spectrum but not necessarily a Gaussian distribution function. We saw that for this case the output will always be Gaussianly distributed regardless of the input probability distribution.

When we deal with more general problems the probability distributions are very difficult to determine. If we concern ourselves only with mean square averages, we may compute them quite readily by the impulse method discussed previously. The next section deals with this matter.

2.4 CALCULATION OF MEAN SQUARE VALUES BY THE IMPULSE METHOD

When the input to a system does not have a Gaussian distribution function or a flat power spectrum it is in general extremely difficult, if not impossible, to find the probability distribution for the output. If we limit ourselves then to finding average values for the variables of interest we may compute these quantities by using the much more elementary concepts discussed in Sect. 1.21. We will begin by computing the mean square displacement for a single-degree-of-freedom, linear, damped oscillator when the exciting force has a "white" or "flat" power spectrum and when it has a more general power spectrum.

2.41 Single-Degree-of-Freedom System

It is well known that the response of the simple harmonic oscillator is

$$x(t) = \int_0^t h(t-\tau)R(\tau)\,d\tau \,, \tag{159}$$

where $h(t-\tau)$ is the response to a unit impulse. We write $x^2(t)$ in the form

$$x^2(t) = \int_0^t\int_0^t h(t-\tau)\,h(t-\tau^+)\,R(\tau)\,R(\tau^+)\,d\tau\,d\tau^+ \;. \tag{160}$$

Taking the average of both sides of eq. (160) leads to

$$\langle x^2(t)\rangle = \int_0^t\int_0^t h(t-\tau)\,h(t-\tau^+)\,\langle R(\tau)R(\tau^+)\rangle\;d\tau\,d\tau^+ \;. \tag{161}$$

Equation (161) is the expression for the mean square displacement of a single-degree-of-freedom oscillator.

2.411 WHITE NOISE EXCITATION. Now $\langle R(\tau)R(\tau^+)\rangle$ is just the autocorrelation of the exciting force and for the case of "white" noise we see from eq. (83) that

$$\langle R(\tau)R(\tau^+)\rangle = 2D\,\delta(\tau^+-\tau) \quad . \tag{162}$$

Using eq. (162) in eq. (161) we find

$$\langle x^2(t)\rangle = 2D\int_0^t \left\{h(t-\tau)\right\}^2 d\tau \, . \tag{163}$$

Using the expression for $h(t-\tau)$ in eq. (163) yields

$$\langle x^2(t)\rangle = \frac{2D}{\omega_1^2}\int_0^t e^{-2\rho\omega_0(t-\tau)}\sin^2\omega_1(t-\tau)\,d\tau \, . \tag{164}$$

Performing the integration, rearranging terms, and recalling the definitions of $\bar{\beta}$ and ω_1 we find that

$$\langle \omega_0^2 x^2(t)\rangle = \frac{D}{\bar{\beta}}\left\{1 - \frac{e^{-\bar{\beta}t}}{\omega_1^2}\left(\omega_1^2 + \frac{\bar{\beta}^2}{2}\sin^2\omega_1 t + \bar{\beta}\omega_1\sin\omega_1 t\cos\omega_1 t\right)\right\} \, . \tag{165}$$

Equation (165) is, of course, the same as eq. (157) found by the more elaborate Fokker-Planck method. We may use the same procedure as was used to determine the result of eq. (165) to compute other averages such as $\langle x(t)\,\dot{x}(t)\rangle$ or $\langle \dot{x}^2(t)\rangle$.

2.412 GENERAL POWER SPECTRUM. Let us now consider the case where the exciting force no longer has a flat power spectrum but where it has some general shape, say $\Pi(\omega)$. We begin as before and write the mean square displacement as

$$\langle x^2(t)\rangle = \int_0^t \int_0^t h(t-\tau)\, h(t-\tau^+) \langle R(\tau) R(\tau^+)\rangle \, d\tau\, d\tau^+ \,. \tag{161}$$

We now use eq. (82) which relates the autocorrelation to the power spectrum and find

$$\langle R(\tau)R(\tau^+)\rangle = \psi(\tau-\tau^+) = \int_0^\infty \Pi(\omega) \cos\omega(\tau-\tau^+)\, d\omega \,. \tag{166}$$

Substituting eq. (166) into eq. (161) results in

$$\langle x^2(t)\rangle = \int_0^\infty \Pi(\omega) d\omega \int_0^t \int_0^t h(t-\tau)\, h(t-\tau^+) \cos\omega(\tau-\tau^+) d\tau\, d\tau^+ \,. \tag{167}$$

Using the expression for $h(t-\tau)$, eq. (167) becomes

$$\langle x^2(t)\rangle = \int_0^\infty \frac{\Pi(\omega)}{\omega_1^2} d\omega \int_0^t \int_0^t e^{-p\omega_0(t-\tau)} e^{-p\omega_0(t-\tau^+)} \sin\omega_1(t-\tau) \sin\omega_1(t-\tau^+) \cos\omega(\tau-\tau^+)\, d\tau\, d\tau^+ . \tag{168}$$

If we use the exponential expression for $\{\sin\omega_1(t-\tau)\sin\omega_1(t-\tau^+)\cos\omega(\tau-\tau^+)\}$ and perform the integration we find

$$\langle x^2(t)\rangle = \int_0^\infty \frac{\Pi(\omega)}{|Z(\omega)|^2}\, d\omega \Bigg[1 + e^{-\bar{\beta} t}\Bigg\{ 1 + \frac{\bar{\beta}}{\omega_1} \sin\omega_1 t \cos\omega_1 t - \left(2\cos\omega_1 t + \frac{\bar{\beta}}{\omega_1}\sin\omega_1 t \right)\cos\omega t$$

$$- \frac{2\omega}{\omega_1}\sin\omega_1 t \sin\omega t + \frac{\bar{\beta}^2 - 4\omega_1^2 + 4\omega^2}{4\omega_1^2}\sin^2\omega_1 t \Bigg\}\Bigg] . \qquad (169)$$

Of course, when $\Pi(\omega) = \frac{4D}{2\pi}$ we have the case of the flat spectrum and eq. (169) reduces to eq. (165) as expected.

The impulse method allows the average values to be computed in a straight-forward manner even though the integrations may be quite tedious. The method, however, has the disadvantage in that it yields no information about the probability distributions of the variables.

In the next section we shall consider multi-degree-of-freedom systems, excited by a very simple type of non-stationary exciting force.

2.42 Multi-Degree-of-Freedom System

We will consider a multi-degree-of-freedom system with a general type of damping so that real normal modes do not exist and use must be made of Foss' method described in Sect. 1.2.

2.421 DISCRETE SYSTEMS - NON-STATIONARY EXCITING FORCE. From eq. (33) we see that we can write for the response of the i^{th} mass

$$\{q_i\} = \sum_{n=1}^{N} \frac{1}{Q_n}\{\phi_i^n\}\int_0^t e^{a_n(t-\tau)} F_n(\tau)\, d\tau . \qquad (33)$$

The transpose of eq. (33) is

$$\left\{ q_i \right\}^T = \sum_{m=1}^{N} \frac{1}{\mathcal{R}_m} \left\{ \phi_i^m \right\}^T \int_0^t e^{\alpha_m (t-\tau^+)} F_m(\tau^+)\, d\tau^+ \,. \tag{170}$$

We write q_i^2 in the form

$$(q_i^2) = \left\{ q_i \right\}^T \left\{ q_i \right\} = \sum_{n=1}^{N} \sum_{m=1}^{N} \frac{1}{\mathcal{R}_n \mathcal{R}_m} \left\{ \phi_i^m \right\} \left\{ \phi_i^n \right\} \int_0^t \int_0^t e^{\alpha_n (t-\tau)} e^{\alpha_m (t-\tau^+)} F_n(\tau) F_m(\tau^+)\, d\tau\, d\tau^+ \tag{171}$$

Using eq. (18) we see

$$F_n(\tau) = \left\{ \phi^n \right\}^T \left\{ f(\tau) \right\} = \sum_{i=1}^{N} \phi_i^n f_i(\tau) \,, \tag{172}$$

and

$$F_m(\tau^+) = \left\{ \phi^m \right\}^T \left\{ f(\tau^+) \right\} = \sum_{j=1}^{N} \phi_j^m f_j(\tau^+) \,. \tag{173}$$

Using eqs. (172) and (173) in eq. (171) and taking the average of both sides yields

$$\langle q_i^2 \rangle = \sum_{n=1}^{N} \sum_{m=1}^{N} \frac{1}{\mathcal{R}_m \mathcal{R}_n} \left\{ \phi_i^m \right\}^T \left\{ \phi_i^n \right\} \int_0^t \int_0^t e^{\alpha_n (t-\tau)} e^{\alpha_m (t-\tau^+)} \left[\sum_{i=1}^{N} \sum_{j=1}^{N} \phi_i^n \phi_j^m \langle f_i(\tau) f_j(\tau^+) \rangle \right] d\tau\, d\tau^+ . \tag{174}$$

Note that $f_i(\tau)$ is the force acting on the i^{th} mass at time τ .
We see that

$\langle f_i(\tau) f_j(\tau^+) \rangle$ is the cross-correlation of the forces for $i \neq j$,

and

$$\langle f_i(\tau) f_i(\tau^+) \rangle$$ is the autocorrelation of the forces.

Let us consider a very simple type of non-stationary forcing function defined by the following relations

$$\langle f_i(\tau) f_j(\tau^+) \rangle = 0, \qquad i \neq j, \tag{175}$$

and

$$\langle f_i(\tau) f_j(\tau^+) \rangle = 2D\left(1 - e^{-\lambda\tau}\right)^{\frac{1}{2}}\left(1 - e^{-\lambda\tau^+}\right)^{\frac{1}{2}} \delta\left(\tau - \tau^+\right), \qquad i = j . \tag{176}$$

Equations (175) and (176) mean that there is no correlation between the forces acting upon different masses and that the autocorrelations of the forces represent random processes that at time $t=0$ begin to build up exponentially to random stationary processes. They may thus be thought of as transient non-stationary processes. Using eqs. (176) and (175) in eq. (174) yields

$$\langle q_i^2 \rangle = \sum_{n=1}^{N}\sum_{m=1}^{N} \frac{2D}{Q_n Q_m} \left\{\phi_i^m\right\}^T \left\{\phi_i^n\right\} \left\{\phi^m\right\}^T \left\{\phi^n\right\} e^{(\alpha_n+\alpha_m)t} \int_0^t e^{-(\alpha_n+\alpha_m)\tau}(1 - e^{-\lambda\tau})\, d\tau, \tag{177}$$

where we have written

$$\left\{\phi^m\right\}^T \left\{\phi^n\right\} \quad \text{for} \quad \sum_{i=1}^{N} \phi_i^m \phi_i^n .$$

Integrating eq. (177) we have

$$\langle q_i^2 \rangle = \sum_{n=1}^{N}\sum_{m=1}^{N} \frac{2D}{Q_n Q_m} \left\{\phi_i^m\right\}^T \left\{\phi_i^n\right\} \left\{\phi^m\right\}^T \left\{\phi^n\right\} \left[\frac{(\alpha_n+\alpha_m)(e^{-\lambda t} - 1) + \lambda(e^{(\alpha_n+\alpha_m)t} - 1)}{(\alpha_n+\alpha_m)^2 + \lambda(\alpha_n+\alpha_m)} \right]. \tag{178}$$

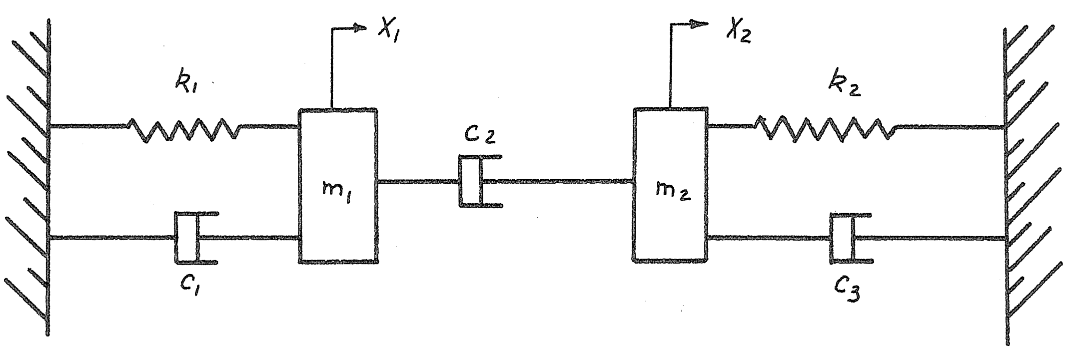
Using the definition of $\left\{\dot{q}_i\right\}$ from eq. (34) and following the same procedure we find

$$\langle \dot{q}_i^2 \rangle = \sum_{n=1}^{N}\sum_{m=1}^{N} \frac{2D\alpha_n\alpha_m}{Q_n Q_m}\left\{\phi_i^m\right\}^T\left\{\phi_i^n\right\}\left\{\phi^m\right\}^T\left\{\phi^n\right\}\left[\frac{(\alpha_n+\alpha_m)(e^{-\lambda t}-1)+\lambda(e^{(\alpha_n+\alpha_m)t}-1)}{(\alpha_n+\alpha_m)^2+\lambda(\alpha_n+\alpha_m)}\right], \quad (179)$$

and

$$\langle q_i\dot{q}_i \rangle = \sum_{n=1}^{N}\sum_{m=1}^{N} \frac{2D\alpha_n}{Q_n Q_m}\left\{\phi_i^m\right\}^T\left\{\phi_i^n\right\}\left\{\phi^m\right\}^T\left\{\phi^n\right\}\left[\frac{(\alpha_n+\alpha_m)(e^{-\lambda t}-1)+\lambda(e^{(\alpha_n+\alpha_m)t}-1)}{(\alpha_n+\alpha_m)^2+\lambda(\alpha_n+\alpha_m)}\right]. \quad (180)$$

To clarify this method, let us consider the two-degree-of-freedom system shown in Sketch 2.



TWO-DEGREE-OF-FREEDOM SYSTEM

SKETCH 2

The system parameters have the following numerical values

$$m_1 = 20 \frac{lb\text{-}sec^2}{ft} ,$$

$$m_2 = 30 \frac{lb\text{-}sec^2}{ft} ,$$

$$k_1 = 400 \frac{lb}{ft} ,$$

$$k_2 = 300 \frac{lb}{ft} ,$$

$$c_1 = 1 \frac{lb\text{-}sec}{ft} ,$$

$$c_2 = 3 \frac{lb\text{-}sec}{ft} ,$$

and

$$c_3 = 5 \frac{lb\text{-}sec}{ft} .$$

The kinetic energy $\mathcal{T}$ is

$$\mathcal{T} = \frac{1}{2}\left(m_1 \dot{x}_1^2 + m_2 \dot{x}_2^2\right) , \tag{181}$$

the potential energy V is

$$V = \frac{1}{2}\left(k_1 x_1^2 + k_2 x_2^2\right) , \tag{182}$$

and the dissipation function $\mathcal{D}$ is

$$\mathcal{D} = \frac{1}{2}\left(c_1 \dot{x}_1^2 + c_2(\dot{x}_1 - \dot{x}_2)^2 + c_3 \dot{x}_2^2\right) . \tag{183}$$

Using the expressions for T, V, and $\mathcal{D}$ in Lagrange's Equations yields

$$\left.\begin{aligned} m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + k_1 x_1 &= 0 , \\ m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_1 + c_3)\dot{x}_2 + k_2 x_2 &= 0 . \end{aligned}\right\} \tag{184}$$

In matrix notation, eq. (184) becomes

$$[m]\{\ddot{x}\} + [r]\{\dot{x}\} + [k]\{x\} = 0 , \tag{185}$$

where

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} ,$$

$$[r] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} ,$$

$$[k] = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} ,$$

$$\{\ddot{x}\} = \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} ,$$

$$\{\dot{x}\} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} ,$$

and

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} .$$

Defining $[\mathcal{R}]$, $[K]$, and $\{y\}$ as

$$[\mathcal{R}] = \begin{bmatrix} [0] & [m] \\ [m] & [r] \end{bmatrix} ,$$

$$[K] = \begin{bmatrix} -[m] & [0] \\ [0] & [k] \end{bmatrix} ,$$

and

$$\{y\} = \begin{Bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} \\ \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \end{Bmatrix} ,$$

eq. (185) becomes

$$[\mathcal{R}]\{\dot{y}\} + [k]\{y\} = 0 . \tag{186}$$

Letting $\{y\} = e^{\alpha t}\{\Phi\}$ eq. (186) becomes

$$\alpha[\mathcal{R}]\{\Phi\} + [K]\{\Phi\} = 0 . \tag{187}$$

Equation (187) may be written as

$$[U]\{\Phi\} = \frac{1}{\alpha}\{\Phi\} , \tag{188}$$

where

$$[U] = -[K]^{-1}[R] = \begin{bmatrix} [0] & [I] \\ -[c][m] & -[c][r] \end{bmatrix} , \qquad (189)$$

and

$$[c] = [k]^{-1} = \begin{bmatrix} \frac{1}{k_1} & 0 \\ 0 & \frac{1}{k_2} \end{bmatrix} . \qquad (190)$$

Hence, we find that

$$-[c]\,[m] = \begin{bmatrix} -\frac{m_1}{k_1} & 0 \\ 0 & -\frac{m_2}{k_2} \end{bmatrix} , \qquad (191)$$

and

$$-[c]\,[r] = \begin{bmatrix} -\frac{c_1 + c_2}{k_1} & \frac{c_2}{k_2} \\ \frac{c_2}{k_2} & -\frac{c_2 + c_3}{k_2} \end{bmatrix} . \qquad (192)$$

From eq. (188) we see that if non-trivial solutions for $\{\Phi\}$ are to exist then the following condition must hold

$$[U] - \frac{1}{\alpha}[I] = 0 \quad . \qquad (193)$$

Using eqs. (189), (191), and (192) in eq. (193) yields

$$\begin{bmatrix} -\frac{1}{\alpha} & 0 & 1 & 0 \\ 0 & -\frac{1}{\alpha} & 0 & 1 \\ -\frac{m_1}{k_1} & 0 & -\frac{c_1+c_2}{k_1}-\frac{1}{\alpha} & \frac{c_2}{k_2} \\ 0 & -\frac{m_2}{k_2} & \frac{c_2}{k_2} & -\frac{c_2+c_3}{k_2}-\frac{1}{\alpha} \end{bmatrix} = 0. \tag{194}$$

Expanding this matrix in cofactors of the fourth row yields

$$\left(\frac{m_1}{k_1}\alpha^2 + \frac{c_1+c_2}{k_1}\alpha + 1\right)\left(\frac{m_2}{k_2}\alpha^2 + \frac{c_2+c_3}{k_2}\alpha + 1\right) - \frac{c_2^2}{k_1 k_2}\alpha^2 = 0. \tag{195}$$

Substituting the numerical values for the parameters yields

$$(10\alpha^2 + 2\alpha + 200)(30\alpha^2 + 8\alpha + 900) - 4.5\alpha^2 = 0 ,$$

or

$$\alpha^4 + \frac{1.4}{3}\alpha^3 + \frac{90.115}{3}\alpha^2 + \frac{22}{3}\alpha + 200 = 0 . \tag{196}$$

Using the Test-function Method for determining the roots of an algebraic equation we find that the four roots of eq. (196) are

$$
\left.\begin{aligned}
\alpha_1 &= -0.0973 + 4.42i \, , \\
\alpha_2 &= -0.0973 - 4.42i \, , \\
\alpha_3 &= -0.1372 + 3.24i \, , \\
\text{and} \qquad \alpha_4 &= -0.1372 - 3.24i \, .
\end{aligned}\right\} \tag{197}
$$

These are the eigenvalues of eq. (186). To find the eigenvectors we make use of the fact that they are proportional to the cofactors of any row of the matrix of equation (194). We thus have for the fourth row

$$
\left\{ \Phi^n \right\} = \left\{ \begin{matrix} \left\{ \begin{matrix} -\dfrac{c_2}{\alpha_n k_1} \\[2ex] -\dfrac{1}{\alpha_n^2} - \dfrac{c_1 + c_2}{\alpha_n k_1} - \dfrac{m_1}{k_1} \end{matrix} \right\} \\[6ex] \left\{ \begin{matrix} -\dfrac{c_2}{\alpha_n^2 k_1} \\[2ex] -\dfrac{1}{\alpha_n^3} - \dfrac{c_1 + c_2}{\alpha_n^2 k_1} - \dfrac{m_1}{\alpha_n k_1} \end{matrix} \right\} \end{matrix} \right\} = \left\{ \begin{matrix} \alpha_n \left\{ \begin{matrix} -\dfrac{c_2}{\alpha_n^2 k_1} \\[2ex] -\dfrac{1}{\alpha_n^3} - \dfrac{c_1 + c_2}{\alpha_n^2 k_1} - \dfrac{m_1}{\alpha_n k_1} \end{matrix} \right\} \\[6ex] \left\{ \begin{matrix} -\dfrac{c_2}{\alpha_n^2 k_1} \\[2ex] -\dfrac{1}{\alpha_n^3} - \dfrac{c_1 + c_2}{\alpha_n^2 k_1} - \dfrac{m_1}{\alpha_n k_1} \end{matrix} \right\} \end{matrix} \right\} . \tag{198}
$$

Hence we see that the form of $\left\{ \Phi^n \right\}$ is

$$\left\{\Phi^n\right\} = \left\{\begin{array}{c} \alpha_n \left\{\phi^n\right\} \\ \left\{\phi^n\right\} \end{array}\right\} , \tag{199}$$

as was indicated in Sect. 1.2.

The four eigenvectors $\left\{\Phi^n\right\}$ are found by using the values of α_n from eqs. (197) in eq. (198). Substituting the numerical values into eq. (198) we have

$$\left\{\Phi^1\right\} = \left\{\begin{array}{c} (-0.10 + 4.42i) \left\{\begin{array}{c} 3.83 \times 10^{-4} - 1.69 \times 10^{-5} i \\ -3.00 \times 10^{-6} - 2.62 \times 10^{-4} i \end{array}\right\} \\ \left\{\begin{array}{c} 3.83 \times 10^{-4} - 1.69 \times 10^{-5} i \\ -3.00 \times 10^{-6} - 2.62 \times 10^{-4} i \end{array}\right\} \end{array}\right\} , \tag{200}$$

$\left\{\Phi^2\right\}$ is the complex conjugate of $\left\{\Phi^1\right\}$,

$$\left\{\Phi^3\right\} = \left\{\begin{array}{c} (-0.14 + 3.24i) \left\{\begin{array}{c} 7.10 \times 10^{-4} - 6.20 \times 10^{-5} i \\ -2.11 \times 10^{-3} - 1.38 \times 10^{-2} i \end{array}\right\} \\ \left\{\begin{array}{c} 7.10 \times 10^{-4} - 6.20 \times 10^{-5} i \\ -2.11 \times 10^{-3} - 1.38 \times 10^{-2} i \end{array}\right\} \end{array}\right\} , \tag{201}$$

and

$\left\{\Phi^4\right\}$ is the complex conjugate of $\left\{\Phi^3\right\}$.

We may now compute the values of $\mathcal{R}_n$. From eq. (17) we see that

$$\mathcal{R}_n = 2\alpha_n \left\{ \phi^n \right\}^T \left[m \right] \left\{ \phi^n \right\} + \left\{ \phi^n \right\}^T \left[r \right] \left\{ \phi^n \right\} , \tag{17}$$

where

$\left\{ \phi^n \right\}$ is defined by comparing eqs. (198) and (199).

Hence

$$\left\{ \phi^n \right\} = \left\{ \begin{array}{c} -\dfrac{c_2}{\alpha_n^2 k_1} \\ \\ -\dfrac{1}{\alpha_n^3} - \dfrac{c_1 + c_2}{\alpha_n^2 k_1} - \dfrac{m_1}{\alpha_n k_1} \end{array} \right\} . \tag{202}$$

Substituting numerical values into eq. (17) we find the four values of $\mathcal{R}_n$ to be

$$\left.\begin{aligned} \mathcal{R}_1 &= (179 + 829i) \times 10^{-8} , \\ \mathcal{R}_2 &= (179 - 829i) \times 10^{-8} , \\ \mathcal{R}_3 &= (-1,120,000 - 3,580,000i) \times 10^{-8} , \\ \text{and} \quad \mathcal{R}_4 &= (-1,120,000 + 3,580,000i) \times 10^{-8} . \end{aligned}\right\} \tag{203}$$

Consider the external exciting forces to be described by eqs. (175) and (176). Then for coordinate X_2 we have

$$\langle x_2^2 \rangle = \sum_{n=1}^{N} \sum_{m=1}^{N} \frac{2D}{\mathcal{R}_n \mathcal{R}_m} \left\{ \phi_2^m \right\}^T \left\{ \phi_2^n \right\} \left\{ \phi^m \right\}^T \left\{ \phi^n \right\} \left[\frac{(\alpha_n + \alpha_m)(e^{-\lambda t} - 1) + \lambda (e^{(\alpha_n + \alpha_m)t} - 1)}{(\alpha_n + \alpha_m)^2 + \lambda (\alpha_n + \alpha_m)} \right] , \tag{204}$$

where

$$\phi_2^n \quad = \quad -\frac{1}{\alpha_n^3} - \frac{C_1 + C_2}{\alpha_n^2\, k_1} - \frac{m_1}{\alpha_n k_1} \quad . \tag{205}$$

For the case where $\lambda = 3$ using eqs. (200), (201), (202), and (203) in eq. (204) we can solve for $\langle x_2^2 \rangle$. The algebra involved is tedious but the final result may be written as

$$\frac{10^6}{D}\langle x_2^2\rangle \quad = \quad 977 + 44e^{-3t} - 592\,e^{-0.195t} - 431\,e^{-0.274t}$$

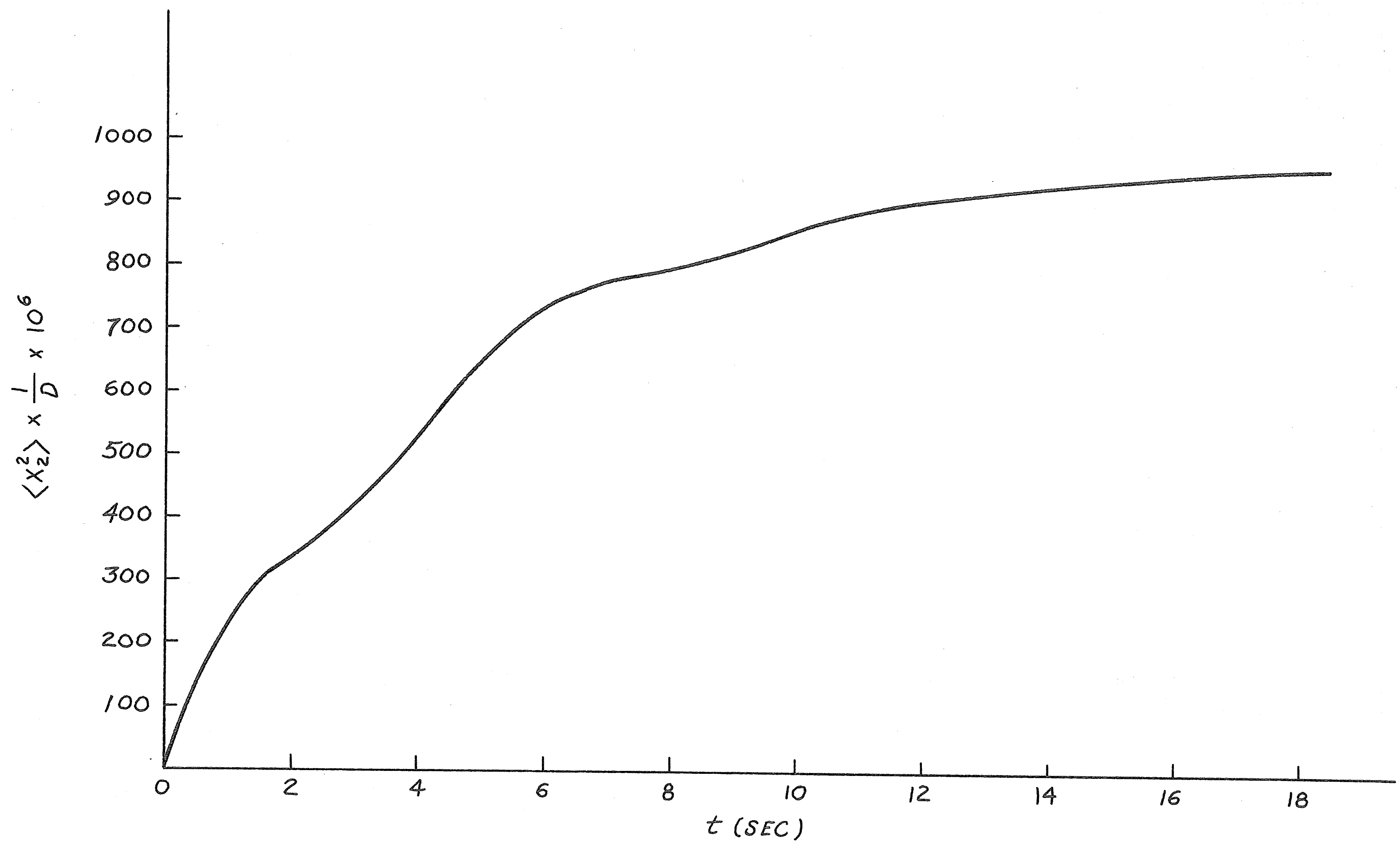
$$+ 2.31\,e^{-0.195t}\sin(8.84t + 5.20) + 2.56\,e^{-0.274t}\sin(6.48t + 3.45)$$

$$+ 4.08\,e^{-0.235t}\sin(7.66t + 4.27) + 92.1\,e^{-0.235}\sin(1.18\,t + 0.48) . \tag{206}$$

It can be seen that $\langle x_2^2\rangle$ approaches the value $977D \times 10^{-6}$ as $t \to \infty$. The buildup to this value is exponential with sinusoidal fluctuations superimposed upon it. These fluctuations are damped out after approximately 15 seconds. Hence, as the random exciting forces build up to their peak values the response of the system builds up to a steady state value and the initial transient fluctuations are eventually damped out. A curve showing $\langle x_2^2\rangle$ as a function of time is shown in Fig. 1.

2.422 CONTINUOUS SYSTEMS. As a specific example, we take the case of the simple beam. We see from eq. (56) that the response is

$$x(y,t) \quad = \quad \sum_{n=1}^{\infty} \frac{\phi_n(y)}{M_n} \int_0^t e^{\alpha_n(t-\tau)} F_n(\tau)\, d\tau \,, \tag{56}$$



MEAN SQUARE DISPLACEMENT OF MASS m_2 vs. TIME FOR A NON-STATIONARY INPUT

FIGURE I

where M_n and F_n are defined in Sect. 1.222 and $\phi_n(y)$ and α_n are eigenfunctions and eigenvalues respectively. We write $x^2(y,t)$ in the form

$$x^2(y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\phi_n(y)\,\phi_m(y)}{M_n M_m} \int_0^t\int_0^t e^{\alpha_n(t-\tau)} e^{\alpha_m(t-\tau^+)} F_n(\tau) F_m(\tau^+)\, d\tau\, d\tau^+ . \qquad (207)$$

Using the definitions of F_n and taking the average of both sides of eq. (207) we have

$$\langle x^2(y,t)\rangle = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\phi_n(y)\,\phi_m(y)}{M_n M_m} \int_0^t\int_0^t e^{\alpha_n(t-\tau)} e^{\alpha_m(t-\tau^+)} \left[\int_{y_0}^{y_1}\int_{y_0}^{y_1} \phi_n(y)\phi_m(y^+)\langle f(y,\tau) f(y^+,\tau^+)\rangle\, dy\, dy^+\right] d\tau\, d\tau^+ . \qquad (208)$$

We must consider two types of correlations for the forces; cross correlations for $y \neq y^+$, and autocorrelations for $y = y^+$. Other mean square averages may be found by noting from eq. (56) that

$$\dot{x}(y,t) = \sum_{n=1}^{\infty} \frac{\alpha_n \phi_n(y)}{M_n} \int_0^t e^{\alpha_n(t-\tau)} F_n(\tau)\, d\tau . \qquad (209)$$

We then have

$$\langle \dot{x}^2(y,t)\rangle = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\alpha_n\alpha_m\phi_n(y)\phi_m(y)}{M_n M_m} \int_0^t\int_0^t e^{\alpha_n(t-\tau)} e^{\alpha_m(t-\tau^+)} \left[\int_{y_0}^{y_1}\int_{y_0}^{y_1} \phi_n(y)\phi_m(y^+)\langle f(y,\tau) f(y^+,\tau^+)\rangle\, dy\, dy^+\right] d\tau\, d\tau^+ , \qquad (210)$$

and

$$\langle x(y,t)\,\dot{x}(y,t)\rangle = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\alpha_m\phi_n(y)\phi_m(y)}{M_n M_m} \int_0^t\int_0^t e^{\alpha_n(t-\tau)} e^{\alpha_m(t-\tau^+)} \left[\int_{y_0}^{y_1}\int_{y_0}^{y_1} \phi_n(y)\phi_m(y^+)\langle f(y,\tau) f(y^+,\tau^+)\rangle\, dy\, dy^+\right] d\tau\, d\tau^+ . \qquad (211)$$

If we have the necessary correlation functions for the forces, we then need only the values for the α_n and $\phi_n(y)$ to evaluate the averages given by eqs. (208), (210), and (211). In order to determine these quantities we substitute $X(y,t) = \phi(y) e^{\alpha t}$ into the homogeneous form of eq. (57). After dividing by $e^{\alpha t}$ and letting $E = U(y)$ we have the following fourth order, ordinary differential equation

$$U(y)\,\phi^{IV}(y) + 2U'(y)\,\phi'''(y) + U''(y)\,\phi''(y) + (\alpha^2 m(y) + \alpha r(y))\,\phi(y) = 0, \tag{212}$$

where

$$(\;') = \frac{d}{dy} ,$$

$$(\;'') = \frac{d^2}{dy^2} ,$$

$$(\;''') = \frac{d^3}{dy^3} ,$$

and

$$(\;^{IV}) = \frac{d^4}{dy^4} .$$

To be specific, let

$$\left.\begin{aligned} U(y) &= U_0 + U_1 y + U_2 y^2 , \\ m(y) &= m_0 + m_1 y , \\ \text{and} \quad r(y) &= r_0 + r_1 y + r_2 y^2 . \end{aligned}\right\} \tag{213}$$

Assuming a series solution of the form

$$\phi(y) = y^s \sum_{n=0}^{\infty} \bar{B}_n y^n , \tag{214}$$

and substituting it into eq. (212) we have, after collecting terms in like powers of y, the following expression

$$\left\{U_0 s(s-1)(s-2)(s-3)\overline{B}_0\right\}y^{s-4} + \left\{U_0(s+1)(s)(s-1)(s-2)\overline{B}_1 + U_1(s)(s-1)^2(s-2)\overline{B}_0\right\}y^{s-3}$$

$$+\left\{U_0(s+2)(s+1)(s)(s-1)\overline{B}_2 + U_1(s+1)(s)^2(s-1)\overline{B}_1 + U_2(s)^2(s-1)^2\overline{B}_0\right\}y^{s-2}$$

$$+\left\{U_0(s+3)(s+2)(s+1)(s)\overline{B}_3 + U_1(s+2)(s+1)^2(s)\overline{B}_2 + U_2(s+1)^2(s)^2\overline{B}_1\right\}y^{s-1}$$

$$+\left\{U_0(s+4)(s+3)(s+2)(s+1)\overline{B}_4 + U_1(s+3)(s+2)^2(s+1)\overline{B}_3 + U_2(s+2)^2(s+1)^2\overline{B}_2 + (\alpha^2 m_0 + \alpha r_0)\overline{B}_0\right\}y^{s}$$

$$+\left\{U_0(s+5)(s+4)(s+3)(s+2)\overline{B}_5 + U_1(s+4)(s+3)^2(s+2)\overline{B}_4 + U_2(s+3)^2(s+2)^2\overline{B}_3 + (\alpha^2 m_0 + \alpha r_0)\overline{B}_1 + (\alpha^2 m_1 + \alpha r_1)\overline{B}_0\right\}y^{s+1}$$

$$+\sum_{n=6}^{\infty}\left\{U_0(s+n)(s+n-1)(s+n-2)(s+n-3)\overline{B}_n + U_1(s+n-1)(s+n-2)^2(s+n-3)\overline{B}_{n-1} + U_2(s+n-2)^2(s+n-3)^2\overline{B}_{n-2}\right.$$

$$\left. + (\alpha^2 m_0 + \alpha r_0)\overline{B}_{n-4} + (\alpha^2 m_1 + \alpha r_1)\overline{B}_{n-5} + \alpha r_2 \overline{B}_{n-6}\right\}y^{s+n-4} = 0 \quad . \tag{215}$$

Since the coefficient of each power of y must be zero we see that

(i) $s = 0, 1, 2,$ or 3 ,

(ii) $\overline{B}_0$ is arbitrary ,

and

(iii) $\overline{B}_1$, $\overline{B}_2$, $\overline{B}_3$, $\overline{B}_4$, and $\overline{B}_5$ may be written in terms of $\overline{B}_0$.

The recursion relation, given by the summation term, can then be used to express all other $\overline{B}_n$ in terms of $\overline{B}_0$. For each value of s we get an infinite series and the general solution for $\phi(y)$ will be a linear combination of these different series. Hence, from eq. (214) letting $s = 0, 1, 2,$ and 3 respectively we have

$$\phi(y) = \sum_{n=0}^{\infty} \bar{B}_n y^n + \sum_{n=0}^{\infty} \bar{B}_n y^{n+1} + \sum_{n=0}^{\infty} \bar{B}_n y^{n+2} + \sum_{n=0}^{\infty} \bar{B}_n y^{n+3} . \qquad (216)$$

In each series all coefficients can be written in terms of one arbitrary constant so that we may write eq. (216) as

$$\phi(y) = \bar{B}_1 \Psi_1(y) + \bar{B}_2 \Psi_2(y) + \bar{B}_3 \Psi_3(y) + \bar{B}_4 \Psi_4(y) . \qquad (217)$$

From eqs. (213) we see that

$$\Psi_1(y) = 1 + \frac{U_1}{U_0} y - \frac{(\alpha^2 m_0 + \alpha r_0)}{24 U_0} y^4 + \left\{ \frac{2U_1(\alpha^2 m_0 + \alpha r_0)}{120 U_0^2} - \frac{(\alpha^2 m_1 + \alpha r_1)}{120 U_0} \right\} y^5 + \cdots, \qquad (218)$$

$$\Psi_2(y) = y - \frac{(\alpha^2 m_0 + \alpha r_0)}{120 U_0} y^5 + \left\{ \frac{2U_1(\alpha^2 m_0 + \alpha r_0)}{360 U_0^2} - \frac{(\alpha^2 m_1 + \alpha r_1)}{360 U_0} \right\} y^6 + \cdots, \qquad (219)$$

$$\begin{aligned} \Psi_3(y) = {} & y^2 - \frac{U_1}{3U_0} y^3 + \frac{1}{6}\left\{ \frac{U_1^2}{U_0^2} - \frac{U_2}{U_0} \right\} y^4 + \frac{1}{10}\left\{ \frac{2U_1 U_2}{U_0^2} - \frac{U_1^3}{U_0^3} \right\} y^5 \\ & + \left\{ \frac{1}{15}\left(\frac{U_1^4}{U_0^4} - \frac{3U_1^2 U_2}{U_0^3} + \frac{U_2^2}{U_0^2} \right) - \frac{(\alpha^2 m_0 + \alpha r_0)}{360 U_0} \right\} y^6 \\ & + \left\{ \frac{1}{21}\left(-\frac{U_1^5}{U_0^5} + \frac{4U_1^3 U_2}{U_0^4} - \frac{3U_1 U_2^2}{U_0^3} \right) + \frac{2U_1(\alpha^2 m_0 + \alpha r_0)}{840 U_0^2} - \frac{(\alpha^2 m_1 + \alpha r_1)}{840 U_0^3} \right\} y^7 + \cdots, \end{aligned} \qquad (220)$$

and

$$\Psi_4(y) = y^3 - \frac{U_1}{2U_0}y^4 + \frac{3}{10}\left\{\frac{U_1^2}{U_0^2} - \frac{U_2}{U_0}\right\}y^5 + \frac{1}{5}\left\{\frac{2U_1U_2}{U_0^2} - \frac{U_1^3}{U_0^3}\right\}y^6$$

$$+\left\{\frac{1}{7}\left(\frac{U_1^4}{U_0^4} - \frac{3U_1^2U_2}{U_0^3} + \frac{U_2^2}{U_0^2}\right) - \frac{(\alpha^2 m_0 + \alpha r_0)}{840U_0}\right\}y^7 \tag{221}$$

$$+\left\{\frac{3}{28}\left(-\frac{U_1^5}{U_0^5} + \frac{4U_1^3U_2}{U_0^4} - \frac{3U_1U_2^2}{U_0^3}\right) + \frac{2U_1(\alpha^2 m_0 + \alpha r_0)}{1680U_0^2} - \frac{(\alpha^2 m_1 + \alpha r_1)}{1680U_0}\right\}y^8 + \cdots .$$

By applying the boundary conditions at $y = 0$ we eliminate two of the functions $\Psi_n(y)$. For example, if at $y = 0$ the beam is simply supported, we know $\phi(0) = \phi''(0) = 0$. Using $\phi(0) = 0$ we see that for $y=0$ only $\Psi_1(0) \neq 0$ hence its coefficient $\overline{B}_1$ must be zero. Using $\phi''(0) = 0$ we see that only $\Psi_3''(0) \neq 0$ hence its coefficient $\overline{B}_3$ must be zero. Thus, we are left with

$$\phi(y) = \overline{B}_2\Psi_2(y) + \overline{B}_4\Psi_4(y) . \tag{222}$$

Now assume that at $y = L$ the beam is also simply supported then we have $\phi(L) = \phi''(L) = 0$ and using eq. (222) we get two equations

$$\left.\begin{aligned} \overline{B}_2\Psi_2(L) + \overline{B}_4\Psi_4(L) &= 0 , \\ \overline{B}_2\Psi_2''(L) + \overline{B}_4\Psi_4''(L) &= 0 . \end{aligned}\right\} \tag{223}$$

In order for eqs. (223) to be compatible, the determinant of the matrix of coefficients of $\overline{B}_2$ and $\overline{B}_4$ must be zero so that

$$\Psi_2(L)\Psi_4''(L) - \Psi_2''(L)\Psi_4(L) = 0 . \tag{224}$$

Depending upon the number of terms used in the series expressions for $\Psi_2(y)$ and $\Psi_4(y$, eq. (224) will yield an equation in α of order n . This then gives approximations to the first n eigenvalues α_n . Using each α_n in eq. (222) yields approximations to the first n eigenfunctions $\phi_n(y)$.

If at $y=0$ the beam is built in, $\phi(0) = \phi'(0) = 0$ and

$$\phi(y) = \bar{B}_3 \Psi_3(y) + \bar{B}_4 \Psi_4(y). \tag{225}$$

If at $y=0$ the beam is free, $\phi''(0) = \phi'''(0) = 0$ and

$$\phi(y) = \bar{B}_1 \Psi_1(y) + \bar{B}_2 \Psi_2(y) . \tag{226}$$

This method is very tedious and if more than the first few eigenvalues and eigenfunctions are desired, it becomes impractical to use this procedure. An initial condition is required to determine the arbitrary constants $\bar{B}_1$, $\bar{B}_2$, $\bar{B}_3$, and $\bar{B}_4$.

To solve the problem of forced vibrations we make use of eq. (208) and the definition of M_n given in Sect. 1.222. The accuracy of the forced vibration solution of course depends upon the number of eigenvalues and eigenfunctions that have been determined from eq. (224).

2.5 NON-STATIONARY, GAUSSIAN, EXCITING FORCE

We will conclude Part II by showing that if the input to a linear system is non-stationary but has a Gaussian distribution function then the output will also be Gaussianly distributed. For this case the problem is completely solved if we are able to determine the mean square values of the parameters of interest.

We first consider some properties of a Gaussian frequency function. If a quantity X has a Gaussian distribution with mean zero, we may write

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}} , \tag{227}$$

where

$$\sigma^2 = \langle x^2 \rangle ,$$

$$\mu = \langle x \rangle = 0 ,$$

and

$f(x) =$ probability that X lies between X and $X + dX$.

We define the $2n^{th}$ moment of the distribution by $\langle x^{2n} \rangle$ where

$$\langle x^{2n} \rangle = \int_{-\infty}^{\infty} x^{2n} f(x)\, dx = \int_{-\infty}^{\infty} x^{2n} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}} dx . \tag{228}$$

Letting $\frac{x}{\sqrt{2}\,\sigma} = \xi$ we reduce eq. (228) to

$$\langle x^{2n} \rangle = \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \xi^{2n} e^{-\xi^2} d\xi . \tag{229}$$

The integral of eq. (229) is in standard form and hence we have

$$\langle x^{2n} \rangle = 1 \cdot 3 \cdot 5 \cdot \text{----} \cdot (2n-1)(\sigma^2)^n = 1 \cdot 3 \cdot 5 . \text{-----} \cdot (2n-1) \langle x^2 \rangle^n . \tag{230}$$

This is a characteristic result for a Gaussian distribution. It is clear that all odd moments, that is $\langle x^{2n+1} \rangle$, are zero since the integral appearing in eq. (229) would then have an odd integrand which makes the integral zero. We conclude that if the even moments of a distribution are related as in eq. (230), and if all odd moments are zero, then the distribution is Gaussian.

Let us now consider the simple one-degree-of-freedom system governed by the differential equation

$$\ddot{y} + 2\rho\omega_o\dot{y} + \omega_o^2 y = F(t) \quad . \tag{231}$$

Assume that $F(t)$ is purely random and stationary so that

$$\left.\begin{aligned} &\langle F(t)\rangle = 0 \quad , \\ \text{and} \\ &\langle F(t_1)F(t_2)\rangle = 2D\,\delta(t_1 - t_2) \quad . \end{aligned}\right\} \tag{232}$$

In addition, assume that $F(t)$ is Gaussian and denote this by writing

$$\left.\begin{aligned} &\langle F(t_1)F(t_2) ---- F(t_{2n+1})\rangle = 0 \quad , \\ \text{and} \\ &\langle F(t_1)F(t_2) --- F(t_{2n})\rangle = \sum_{ALL\ PAIRS} \langle F(t_i)F(t_j)\rangle\,\langle F(t_k)F(t_\ell)\rangle \cdots \quad . \end{aligned}\right\} \tag{233}$$

To show that this defines a Gaussian distribution we consider how many different pairs we can form from $2n$ items. This first pair can be chosen in $2n(2n-1)$ ways but since the order is irrelevant, that is the pair ij is considered to be the same as the pair ji, we divide by the number of ways we may order 2 items which is just $2!$ or 2. The next pair can be chosen in $\frac{(2n-2)(2n-3)}{2!}$ ways and so on. The total number of ways of forming n pairs is then just $\frac{(2n)(2n-1)}{2!}$ $\times\frac{(2n-2)(2n-3)}{2!}\times ---- \times\frac{(2)(1)}{2!}$. The order of these n pairs, however, is irrelevant, since a set of pairs ij ; $k\ell$ is considered the same as $k\ell$; ij , hence we must divide the above expression by the number of ways we can order n objects which is just $n!$. Hence, the total number of different ways that n pairs may be formed from $2n$ objects is just $\frac{(2n)(2n-1)}{2!\,n!} \cdot \frac{(2n-2)(2n-3)}{2!} \cdot \text{----} \cdot \frac{(2)(1)}{2!} = \frac{1}{n!} \cdot (n)(2n-1) \cdot (n-1)(2n-3) \cdot \text{----} \cdot (1)(1)$. We note that $(n)(n-1)(n-2) \cdot \text{-----} \cdot (1) = n!$ so that

$$S_{t_0} = (2n-1)(2n-3)(2n-5) \cdot \text{---} \cdot (3)(1) = 1 \cdot 3 \cdot 5 \cdot 7 \cdot \text{----} \cdot (2n-1), \qquad (234)$$

where S_{t_0} is the number of different ways that n pairs may be formed from $2n$ objects. For example if $2n = 4$ we may form the following different pairs

$$\langle 12 \rangle \langle 34 \rangle ,$$

$$\langle 13 \rangle \langle 24 \rangle ,$$

and

$$\langle 14 \rangle \langle 23 \rangle .$$

We say that such a pair as $\langle 34 \rangle \langle 21 \rangle$ is the same as $\langle 21 \rangle \langle 43 \rangle$, $\langle 12 \rangle \langle 34 \rangle$, etc. From eq. (234) we see that with $2n = 4$ we should be able to form three pairs, which agrees with the above result.

Now for the equation of motion, eq. (231), we have for the $2n^{th}$ moment

$$\langle y^{2n} \rangle = \underbrace{\int_0^t \text{-----} \int_0^t}_{2n} h(t-\tau_1)h(t-\tau_2) \text{---} h(t-\tau_{2n}) \langle F(\tau_1)F(\tau_2) \text{---} F(\tau_{2n}) \rangle d\tau_1 \text{---} d\tau_{2n} . \qquad (235)$$

From eq. (235) we see that

$$\langle F(\tau_1)F(\tau_2) \text{-----} F(\tau_{2n}) \rangle = \sum_{\text{ALL PAIRS}} \langle F(\tau_i)F(\tau_j) \rangle \langle F(\tau_k)F(\tau_\ell) \rangle \cdot \text{---} , \qquad (236)$$

and as was shown above, there are $1\cdot3\cdot5\cdot\text{----}\cdot(2n-1)$ such sets. Hence, the $2n$ integrals become

$$\langle y^{2n}\rangle = 1\cdot3\cdot5\cdot\text{----}\cdot(2n-1)\int_0^t\int_0^t h(t-\tau_i)h(t-\tau_j)\langle F(\tau_i)F(\tau_j)\rangle d\tau_i d\tau_j \int_0^t\int_0^t h(t-\tau_k)h(t-\tau_\ell)\langle F(\tau_k)F(\tau_\ell)\rangle d\tau_k d\tau_\ell\cdot\text{---}. \quad (237)$$

Now each of the double integrals above is just $\langle y^2\rangle$ and there are n of them. We can arrange the integrals in this fashion since all τ_i and τ_j are dummy variables so that

$$h(t-\tau_i) \equiv h(t-\tau_j),$$

$$\langle F(\tau_i)F(\tau_j)\rangle \equiv \langle F(\tau_k)F(\tau_\ell)\rangle, \text{ etc.}$$

Hence

$$\langle y^{2n}\rangle = 1\cdot3\cdot5\cdot\text{----}\cdot(2n-1)\langle y^2\rangle^n . \quad (238)$$

Therefore the output y is Gaussianly distributed since it has the Gaussian property given by eq. (230).

We may now assume that $F(t)$ is random, Gaussian, and non-stationary so that

$$\left.\begin{aligned}
&\langle F(t)\rangle = 0 ,\\
&\langle F(t_1)F(t_2)\rangle = \psi(t_1, t_2-t_1) ,\\
&\langle F(t_1)F(t_2)\text{----}F(t_{2n-1})\rangle = 0 ,\\
&\text{and}\\
&\langle F(t_1)F(t_2)\text{----}F(t_{2n})\rangle = \sum_{\text{ALL PAIRS}}\langle F(t_i)F(t_j)\rangle\langle F(t_k)F(t_\ell)\rangle\cdot\text{----}.
\end{aligned}\right\} \quad (239)$$

Proceeding as before, we write for the $2n^{th}$ moment of the output

$$\langle y^{2n} \rangle = 1 \cdot 3 \cdot 5 \cdot \text{----} \cdot (2n-1) \int_0^t \int_0^t h(t-\tau_i) h(t-\tau_j) \psi(t, \tau_j - \tau_i)\, d\tau_i\, d\tau_j$$

$$\cdot \int_0^t \int_0^t h(t-\tau_k) h(t-\tau_\ell) \psi(t, \tau_\ell - \tau_k) d\tau_k\, d\tau_\ell \cdot \text{----} \quad . \tag{240}$$

Again each integral is just $\langle y^2 \rangle$ and since there are n of them we have the result given by eq. (224).

There is, however, a very important difference between the stationary and non-stationary cases. From eq. (240) we see that when the input is stationary, the mean square of the output approaches a constant value, independent of the time t, as the upper limits of the integrals approach infinity. In the non-stationary case, the autocorrelation of the forces is a function of time and so the mean square of the output will always depend upon t regardless of how large t becomes.

PART III

EXAMPLES OF MECHANICAL SYSTEMS EXCITED BY RANDOM INPUTS

Having developed the necessary relations for treating systems excited by random forces, some typical problems will now be considered.

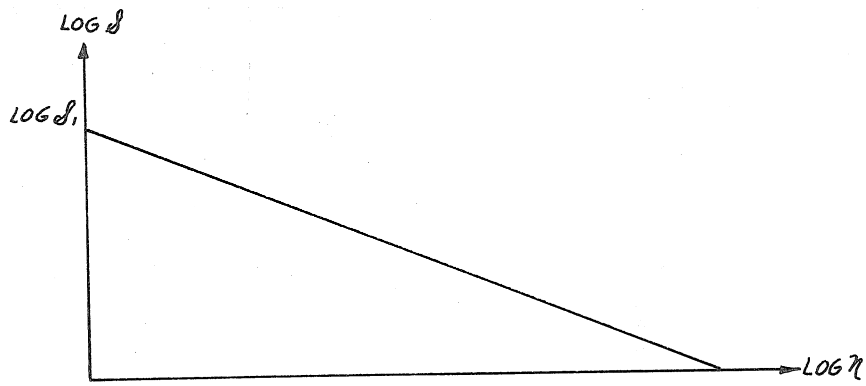
1.0 Fatigue Under Random Loading

The problem of determining the stresses due to random loading is considered here as well as the prediction of fatigue failure. The first section deals with an analysis made by John W. Miles on a single-degree-of-freedom system. The last section deals with multi-degree-of-freedom systems.

1.1 LIGHTLY DAMPED, SINGLE-DEGREE-OF-FREEDOM OSCILLATOR

There does not exist at present a comprehensive theory to describe fatigue. Little analysis has been done, the majority of the work being of an experimental nature. Since it does not seem possible to predict fatigue life any closer than within a factor of two at present, the relatively simple concept of cumulative fatigue proposed by M. A. Miner appears to be adequate for determining fatigue life.

The cumulative damage hypothesis is briefly described in the following. We make use of an $(\mathcal{S}-\eta)$ plot where η is the number of cycles of complete stress reversal of fixed amplitude $\mathcal{S}$ which is required to cause failure. A typical curve is shown in Sketch 3.



TYPICAL $\mathcal{S}$-η CURVE

SKETCH 3

It can be seen that the functional relationship existing between η and $\mathcal{S}$ is of the form

$$\eta(\mathcal{S}) = \left(\frac{\mathcal{S}_1}{\mathcal{S}}\right)^{\mu} , \tag{241}$$

where $\mathcal{S}_1$ is the stress at which eq. (241) predicts failure will occur in one cycle. In some cases an endurance limit exists such that below some particular stress, an infinite number of stress reversals is required for failure. We will not consider this case in the following work.

We must now establish some criterion to determine the amount of damage done by a number of stress reversals less than that which causes failure. Miner's Rule assumes that the damage accumulates linearly, hence if the system undergoes η_i stress reversals at a stress level $\mathcal{S}_i$ the partial damage is given by

$$\mathcal{F}_i = \frac{n_i}{\eta_i} , \tag{242}$$

where $\eta_i = \eta(\mathcal{S}_i)$ is the number of stress reversals at stress level $\mathcal{S}_i$ which will cause failure.

The cumulative damage is given by

$$\mathcal{F} = \sum_{i=1}^{N} \mathcal{F}_i = \sum_{i=1}^{N} \frac{n_i}{\eta_i} . \tag{243}$$

Substituting eq. (241) in eq. (243) we have

$$\mathcal{F} = \sum_{i=1}^{N} n_i \left(\frac{\mathcal{S}_i}{\mathcal{S}_1}\right)^{\bar{\mu}} . \tag{244}$$

Fatigue failure occurs when $\mathcal{F} = 1$. From eq. (244) we may define an equivalent stress which produces the same fatigue damage as n_1 cycles at stress $\mathcal{S}_1$, n_2 cycles at stress $\mathcal{S}_2$, etc. Calling this equivalent stress $\mathcal{S}_\epsilon$ we have

$$\mathcal{S}_\epsilon = \left\{ \sum_{i=1}^{N} n_i \mathcal{S}_i^{\bar{\mu}} \Big/ \sum_{i=1}^{N} n_i \right\}^{\frac{1}{\bar{\mu}}} . \tag{245}$$

In order to account for uncertainties in the data and the hypotheses we may rewrite eq. (245) as

$$\mathcal{S}_\epsilon = \left\{ \sum_{i=1}^{N} n_i \mathcal{S}_i^{\bar{\mu}} \Big/ \sum_{i=1}^{N} n_i \right\}^{\frac{1}{\bar{e}\bar{\mu}}} , \tag{246}$$

where $\bar{e}$ is a constant with a conservative upper limit of about 2.

In order to compute $\mathcal{S}_\epsilon$ we note that for small values of damping the single-degree-of-freedom oscillator acts like a narrow band filter and passes only those frequencies in the neighborhood of its natural frequency ω_0. Hence, it passes frequencies in the range $\omega_0 \pm \Delta\omega_0$. From the phenomenon of beats we see that the system oscillates at the mean frequency ω_0 and that the amplitude envelope exhibits a random fluctuation whose frequency is of the order of $2\Delta\omega_0$. The probability distribution of the amplitude can be shown to be a Rayleigh distribution whose analytic form is

$$P\,dy = \frac{y}{\psi(0)} e^{-\frac{y^2}{2\psi(0)}} dy\,, \tag{247}$$

where $P\,dy$ is the probability that the amplitude $y(t)$ lies between y and $y+dy$ and $\psi(0)$ is the mean square value of $y(t)$.

In order to obtain the probable number of cycles of loading having an amplitude in the range $\mathcal{S}$ to $\mathcal{S}+d\mathcal{S}$, we merely multiply the total number of cycles at frequency $\omega_0/2\pi$ by the probability distribution $P\,d\mathcal{S}$ given by eq. (247). We then find that the equivalent stress given by eq. (246) becomes

$$\mathcal{S}_\epsilon = \left\{ \int_0^\infty \mathcal{S}^{\bar{\ell}\bar{\mu}} P\,d\mathcal{S} \Big/ \int_0^\infty P\,d\mathcal{S} \right\}^{\frac{1}{\bar{\ell}\bar{\mu}}}. \tag{248}$$

Substituting eq. (247) in eq. (248) we have

$$\mathcal{S}_\epsilon = \left\{ \int_0^\infty \mathcal{S}^{\bar{\ell}\bar{\mu}+1} e^{-\frac{\mathcal{S}^2}{2\psi(0)}} d\mathcal{S} \Big/ \int_0^\infty \mathcal{S} e^{-\frac{\mathcal{S}^2}{2\psi(0)}} d\mathcal{S} \right\}^{\frac{1}{\bar{\ell}\bar{\mu}}}. \tag{249}$$

The Gamma Function $\Gamma(z+1)$ is defined by

$$\Gamma(z+1) = 2^{-z}\int_0^\infty \zeta^{2z+1} e^{-\frac{\zeta^2}{2}} d\zeta \; . \tag{250}$$

Hence, using eq. (250) in eq. (249) we have

$$\mathcal{S}_\epsilon = \left\{\Gamma\left(\frac{\bar{\ell}\bar{\mu}}{2}+1\right)\right\}^{\frac{1}{\bar{\ell}\bar{\mu}}}\left\{2\psi(0)\right\}^{\frac{1}{2}} \; . \tag{251}$$

It is clear that $\psi = \langle \mathcal{S}^2 \rangle$ the mean square stress. For large values of z the Gamma Function may be approximated by Sterling's formula as

$$\Gamma(z+1) = \sqrt{2\pi}\, e^{-z} z^{z+\frac{1}{2}}, \qquad z \gg 1 \; . \tag{252}$$

We thus find that eq. (251) becomes

$$\mathcal{S}_\epsilon = (\pi\bar{\ell}\bar{\mu})^{\frac{1}{2\bar{\ell}\bar{\mu}}}\left(\frac{\bar{\ell}\bar{\mu}\langle \mathcal{S}^2\rangle}{e}\right)^{\frac{1}{2}} \doteq \left(\frac{\bar{\ell}\bar{\mu}\langle \mathcal{S}^2\rangle}{e}\right)^{\frac{1}{2}} , \qquad \bar{\mu} \gg 1. \tag{253}$$

We know $\psi(0) = \int_0^\infty \Pi_{OUT.}(\omega)\,d\omega$ in general and for a lightly damped system $\Pi_{OUT.}(\omega) = \Pi_{IN.}(\omega)/|Z(\omega)|^2$ where $\Pi_{IN.}(\omega)$ is the power spectrum of the forcing function and $Z(\omega)$ is the impedance of the vibrating system. Using these relations we find

$$\psi = \langle \mathcal{S}^2 \rangle = \left(\frac{\pi}{4\rho}\right)\left\{\frac{\omega_0 \Pi_{IN.}(\omega)}{F_0^2}\right\}\mathcal{S}_0^2 \; , \tag{254}$$

where

$\mathcal{S}_0$ is the stress that would be produced by the root mean square of the force F_0,

and

$$\rho = \frac{c}{c_r} = \frac{c}{2\sqrt{km}} .$$

From eqs. (253) and (254) we find

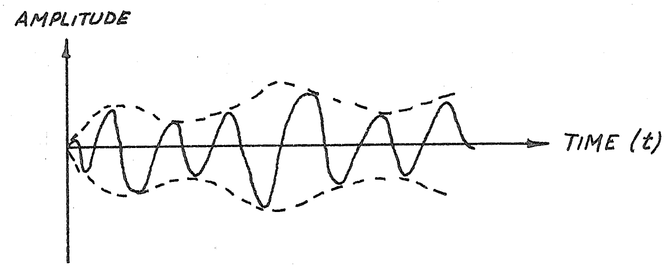
$$\frac{\mathcal{S}_\epsilon}{\mathcal{S}_0} = \left\{ \frac{\bar{\ell}\,\pi\bar{\mu}}{4e\rho} \cdot \frac{\omega_0 \Pi_{IN.}(\omega)}{F_0^2} \right\}^{\frac{1}{2}} . \tag{255}$$

Hence, we may compute $\mathcal{S}_\epsilon$ from eq. (255) and using this value for the stress we find from the $(\mathcal{S}-\eta)$ curve the number of stress reversals which cause failure. Since the system will vibrate at approximately $\omega_0/2\pi$ cycles per second, the fatigue life T_f is given by

$$T_f = \frac{2\pi\eta(\mathcal{S}_\epsilon)}{\omega_0} . \tag{256}$$

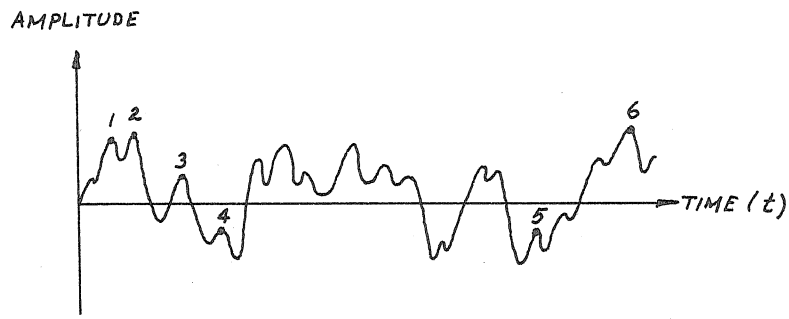
1.2 LIGHTLY DAMPED, MULTI-DEGREE-OF-FREEDOM SYSTEMS

When we deal with multi-degree-of-freedom systems the problem is more complicated since the output is composed of many frequencies. We no longer have the relatively simple case of a single harmonic with an amplitude which varies randomly with time. In Sketches 4 and 5 a comparison between the outputs of single and multi-degree-of-freedom systemsis shown.



OUTPUT OF SINGLE-DEGREE-OF-FREEDOM SYSTEM

SKETCH 4



OUTPUT OF MULTI-DEGREE-OF-FREEDOM SYSTEM

SKETCH 5

Since fatigue appears to depend upon the level of stress and the number of stress reversals, it is necessary to determine some measure of the number of stress reversals per unit time and the equivalent stress level. It is clear that by using the lowest frequency of the system as a measure of the number of stress reversals per unit time we underestimate this parameter. Similarly, we overestimate it by using the highest frequency of the system, in fact, in continuous systems this frequency would be infinite and, hence, this method would predict instantaneous failure. Referring to Sketch 5, denoting the output of a multi-degree-of-freedom system, it seems reasonable to assume that a stress reversal of the type (1-2) will not contribute as much to fatigue failure as will a stress reversal of type (3-4). It is the higher frequencies which cause the motion of type (1-2), hence, the high frequency components of the motion do not appear to be as important as the low frequency components. By using the number of zero crossings per second as a measure of the number of stress reversals we are weighting the lower frequencies more heavily than the higher ones and to some extent do account for the reversals of type (1-2). It has been shown by S. O. Rice that the expected number of zeros per second can be computed from the relation

$$\mathcal{K} = \frac{1}{\pi}\left\{-\frac{(d^2\psi/d\tau^2)_{\tau=0}}{\psi(0)}\right\}^{\frac{1}{2}}, \tag{257}$$

where

$\mathcal{K}$ is the expected number of zeros per second,

and

$\psi(\tau)$ is the autocorrelation of the random function.

The frequency will be set equal to $\mathcal{K}/2$ and so we use as the frequency of vibration

$$\omega_0 = \left\{-\frac{(d^2\psi/d\tau^2)_{\tau=0}}{\psi(0)}\right\}^{\frac{1}{2}}. \tag{258}$$

Using the expected number of zeros as a measure of the number of stress reversals we must now find some means of obtaining an equivalent stress level. Rice has also developed a relation for the distribution of the maxima of a random function which is given as

$$P d\bar{I} = \frac{d\bar{I}}{3\sqrt{2\pi\psi(0)}}\left[2e^{-\frac{9h^2}{8}} + \left(\frac{5\pi}{2}\right)^{\frac{1}{2}} h e^{-\frac{h^2}{2}}\left\{1 + \operatorname{erf}\left(\frac{5}{8}\right)^{\frac{1}{2}} h\right\}\right] , \tag{259}$$

where

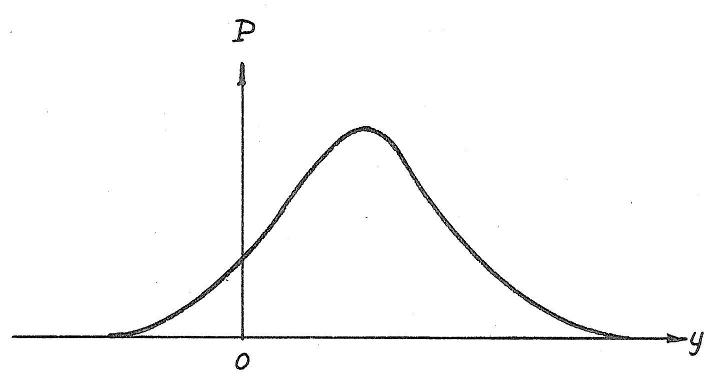
$$h = \frac{\bar{I}}{\sqrt{\psi(0)}} ,$$

$\psi(\tau)$ is the autocorrelation of the random function,

and

$P d\bar{I}$ = probability that the amplitude lies between $\bar{I}$ and $\bar{I}+d\bar{I}$.

A sketch of this relation is shown in Sketch 6.



APPROXIMATE PLOT OF EQ. (259)

SKETCH 6

The negative tail of this curve is due to maxima like point (5) in Sketch 5, which will not contribute much to fatigue failure. It is the maxima like points (6) in Sketch 5 that are the more important ones for fatigue calculations. At high stress levels, that is at large h, the probability distribution approaches the Rayleigh distribution given by eq. (247), that is

$$P\,d\bar{I} \doteq \frac{\sqrt{5}\,h}{3\sqrt{\psi(0)}}\,e^{-\frac{h^2}{2}}\,d\bar{I}\,, \qquad h \text{ large} \tag{260}$$

Hence, by assuming it is the stress reversals at high stress levels that are important, we may approximate the probability distribution given by eq. (259) by the Rayleigh distribution of eq. (260). We may then proceed according to the method of Miles using one half of the expected number of zeros per second as the frequency of the random function and the Rayleigh distribution as the probability distribution of maxima of the function.

In order to compute the expected number of zeros per second, from eq. (257), we must first determine the autocorrelation $\psi(\Delta\tau)$. Let us assume that we are interested in the relative motion between the i^{th} and j^{th} masses of an N-degree-of-freedom system. The autocorrelation of interest would then be

$$\psi(\Delta\tau) = \left\langle \left\{x_i(\tau) - x_j(\tau)\right\}\left\{x_i(\tau^+) - x_j(\tau^+)\right\}\right\rangle. \tag{261}$$

Equation (261) represents the average value of the product of the relative displacements of the i^{th} and j^{th} masses as a function of the time difference $\Delta\tau$.

We can express the coordinates x_i and x_j in terms of the generalized coordinates q_r and the normal mode shapes $A_i^{(r)}$. Hence, we may write

$$x_i(\tau) = \sum_{r=1}^{N} A_i^{(r)} q_r(\tau) , \tag{262}$$

and

$$\left\{x_i(\tau) - x_j(\tau)\right\}\left\{x_i(\tau^+) - x_j(\tau^+)\right\} = \sum_{r=1}^{N}\sum_{s=1}^{N}\left\{A_i^{(r)} - A_j^{(r)}\right\}\left\{A_i^{(s)} - A_j^{(s)}\right\} q_r(\tau) q_s(\tau^+) . \tag{263}$$

We may write q_r in integral form as

$$q_r(t) = \frac{1}{\omega_{r_1}} \int_0^t R_r(\tau)\, e^{-\beta_r \omega_{r_0}(t-\tau)} \sin \omega_{r_1}(t-\tau)\, d\tau . \tag{264}$$

Hence, substituting eq. (264) into eq. (263) and using eq. (261) we find

$$\psi(\Delta\tau) = \sum_{r=1}^{N}\sum_{s=1}^{N}\left\{A_i^{(r)} - A_j^{(r)}\right\}\left\{A_i^{(s)} - A_j^{(s)}\right\} \frac{1}{\omega_{r_1}\omega_{s_1}} \int_0^t\int_0^t e^{-\beta_r \omega_{r_0}(t-\tau)} e^{-\beta_s \omega_{s_0}(t-\tau^+)} \sin\omega_{r_1}(t-\tau) \sin\omega_{s_1}(t-\tau^+) \langle R_i R_j \rangle\, d\tau\, d\tau_j^+ \tag{265}$$

where R_i is the generalized force for the i^{th} mode. We may rewrite the autocorrelation $\langle R_i R_j \rangle$ as

$$\langle R_i R_j \rangle = \sum_{u=1}^{N}\sum_{v=1}^{N} \langle F_u F_v \rangle \frac{A_u^{(i)} A_v^{(j)}}{M_i M_j} , \tag{266}$$

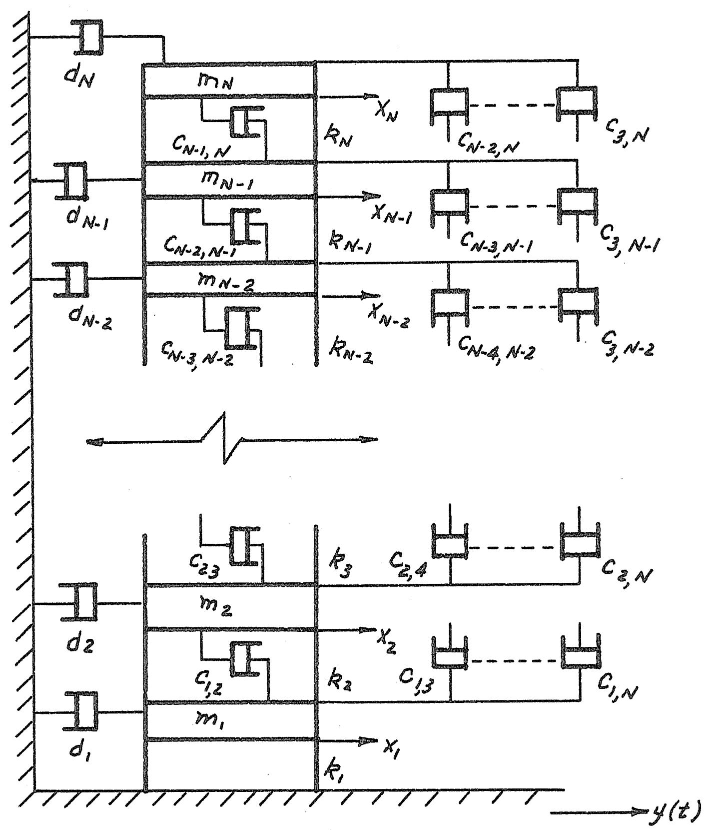
where F_u is the force on the u^{th} mass and M_i is the generalized mass of the i^{th} mode.

By means of eq. (265) we may compute $\psi(0)$ and $\left[d^2\psi/d(\Delta\tau)^2\right]_{\Delta\tau=0}$ and, hence, find the expected number of zeros per second from eq. (257).

2.0 Earthquake Problem

A tall building subjected to ground motion represents a special type of multi-degree-of-freedom system since the power spectra of the exciting forces acting on each of the masses is the same except possibly for a numerical constant.

Consider the system shown in Sketch 7.



MULTI-DEGREE-OF-FREEDOM SYSTEM

SKETCH 7

Assume that some random support motion $y(t)$ exists so that an inertial force of $- m_j \ddot{y}(t)$ acts upon each mass. It will be assumed that the power spectrum of the support acceleration $\ddot{y}(t)$ is known and that it is of the form

$$\Pi_s(\omega) = \ell\,\omega^2 e^{-\rho\omega^2} , \tag{267}$$

where ℓ and ρ are constants. This is a typical form for the power spectra of earthquakes. It is clear that the power spectrum given by eq. (267) peaks at $\omega = \sqrt{\frac{1}{\rho}}$.

Dynamic measurements which have been made indicate that the typical building has a constant damping ratio for each normal mode. Since phase measurements are usually not made, however, it is questionable whether it is classical normal modes which are being measured or the more general complex modes described in Foss' method. If absolute damping and relative damping between adjacent floors is considered, it does not appear possible to get a constant damping ratio for each normal mode. If relative damping between all floors is considered, however, it is possible to get a constant damping ratio for each mode.

Let us consider this latter case. The equations of motion can then be represented as a set of N ordinary, linear differential equations of the form

$$M_j \ddot{q}_j + C_j \dot{q}_j + K_j q_j = Q_j , \tag{268}$$

where

$$Q_j = \sum_{r=1}^{N} F_r A_r^{(j)} ,$$

and

$$F_r = -m_r \ddot{y}(t) .$$

Equation (268) may be rewritten as

$$\ddot{q}_j + 2\omega_{jo} \rho_j \dot{q}_j + \omega_{jo}^2 q_j = \ddot{y}(t) R_j , \tag{269}$$

where

$$\omega_{jo} = \sqrt{\frac{K_j}{M_j}} ,$$

$$\rho_j = \frac{C_j}{2\sqrt{M_j K_j}} ,$$

and

$$R_j = -\sum_{r=1}^{N} m_r A_r^{(j)} \Big/ M_j = \text{constant.}$$

We may write for the response of the j^{th} mass

$$x_j = \sum_{r=1}^{N} A_j^{(r)} q_r , \tag{270}$$

and hence

$$\langle x_j^2 \rangle = \sum_{r=1}^{N} \sum_{s=1}^{N} A_j^{(r)} A_j^{(s)} \langle q_r q_s \rangle . \tag{271}$$

Calling $Z_r(\omega)$ the impedance of the r^{th} mode and using the definitions of R_j and the power spectrum we have

$$\langle q_r^2 \rangle = \frac{R_r^2}{2\pi} \int_0^\infty \frac{\Pi_s(\omega)}{|Z_r(\omega)|^2}\, d\omega\ , \tag{272}$$

and

$$\langle q_r q_s \rangle = \frac{R_r R_s}{2\pi} \int_0^\infty \frac{\Pi_s(\omega)}{Z_r Z_s^*}\, d\omega\ . \tag{273}$$

Equation (272) may be evaluated easily by using the approximation of Sect. 3.1 in Appendix I. Evaluating the power spectrum at $\omega = \omega_{ro}$ and assuming it to be a flat spectrum with that spectral density we have

$$\langle q_r^2 \rangle \doteq \frac{R_r^2 (b\, \omega_{ro}^2 e^{-p\omega_{ro}^2})}{2\pi} \int_0^\infty \frac{d\omega}{|Z_r(\omega)|^2}\ . \tag{274}$$

Recalling the definition of $|Z_r(\omega)|^2$ we see that eq. (274) becomes

$$\langle q_r^2 \rangle \doteq \frac{R_r^2 (b\, e^{-p\omega_{ro}^2})}{8\,\omega_{ro}\,\rho_r}\ . \tag{275}$$

In eq. (271) we see that terms involving $\langle q_r q_s \rangle$ and $\langle q_s q_r \rangle$ appear and so we have

$$A_j^{(r)} A_j^{(s)} \left\{ \langle q_r q_s \rangle + \langle q_s q_r \rangle \right\} = \frac{A_j^{(r)} A_j^{(s)} R_r R_s}{2\pi} \int_0^\infty \left\{ \frac{\Pi_s(\omega)}{Z_r(\omega) Z_s^*(\omega)} + \frac{\Pi_s(\omega)}{Z_s(\omega) Z_r^*(\omega)} \right\} d\omega$$

$$= \frac{A_j^{(r)} A_j^{(s)} R_r R_s}{2\pi} \int_0^\infty \frac{\left\{ Z_s(\omega) Z_r^*(\omega) + Z_r(\omega) Z_s^*(\omega) \right\} \Pi_s(\omega)}{|Z_r(\omega)|^2 \, |Z_s(\omega)|^2}\, d\omega. \tag{276}$$

If the value of the integrand in eq. (276) is plotted as a function of ω it will be seen that the resulting curve has two peaks, one near ω_{r_o} and the other near ω_{s_o}. For small values of damping these peaks will be very narrow and the major contributions to the integral of eq. (276) will be near ω_{r_o} and ω_{s_o}. If we consider these peaks to be well separated and take the case where $\omega_{r_o} < \omega_{s_o}$ then the contribution at ω_{r_o} will be much greater than at ω_{s_o}. We may then evaluate the power spectrum of the ground acceleration at $\omega = \omega_{r_o}$ and consider it to be a white spectrum of spectral density $b\, \omega_{r_o}^2\, e^{-\rho\omega_{r_o}^2}$. The resulting integral can then be evaluated by means of contour integration in the complex plane. It is seen that the result is of order ρ so that the cross-product terms are negligible compared to those of the square terms defined by eq. (275). The conditions under which these approximations may be made are

$$e^{-\rho(\omega_{s_o}^2 - \omega_{r_o}^2)} \gg 1 \quad , \tag{277}$$

$$16\rho^2\left(\frac{\omega_{r_o}}{\omega_{s_o}}\right)^4 \ll 1 \quad , \tag{278}$$

and

$$16\rho^2\left(\frac{\omega_{r_o}}{\omega_{s_o}}\right)^3 e^{\rho(\omega_{s_o}^2 - \omega_{r_o}^2)} \ll 1 \quad , \tag{279}$$

where

$$\omega_{r_o} < \omega_{s_o} \quad .$$

If the condition of eq. (277) is satisfied the resonance peak at ω_{so} is negligible compared to that at ω_{ro}. Equations (278) and (279) are the conditions under which the cross-product terms are negligible compared to the square terms.

Equations (277) through (279) are usually satisfied if the fundamental period of the structure is less than 1.5 seconds. When this is not the case, both resonance peaks are important and a different approximation is needed. The integrand is evaluated at each resonance peak and the result multiplied by the band width $2\rho_i\omega_{io}$. The sum of the areas of the resulting rectangles is then approximately equal to the value of the integral in eq. (276). The condition under which the cross-product terms may be neglected is then

$$\frac{8\rho^2}{\pi}\left(\frac{\omega_{ro}}{\omega_{so}}\right)^2 \ll 1 \qquad \omega_{ro} < \omega_{so} . \tag{280}$$

If either of the above mentioned conditions hold the cross-product terms of eq. (276) may be neglected and we have

$$\langle x_j^2 \rangle = \sum_{r=1}^{N} A_j^{(r)^2} \langle q_r^2 \rangle . \tag{281}$$

Using eq. (275) the mean square displacement may then be written as

$$\langle x_j^2 \rangle \doteq \sum_{r=1}^{N} \frac{A_j^{(r)^2} R_r^2 \not{f} e^{-\rho\omega_{ro}^2}}{8\omega_{ro}\rho_r} . \tag{282}$$

If the ground acceleration has a Gaussian probability distribution, the displacements will also be Gaussianly distributed and the result of eq. (282) is sufficient to completely solve the problem.

3.0 Response of Beams to Random Loads

Since the beam represents a fundamental structural member, it is natural to consider the effect of a known transverse force distribution, random in time, upon such a member. The first known work on the subject was done by Houdijk and Ornstein in 1927. They treated the problems of the Brownian motion of strings and cantilever beams. In 1931 Van Lear and Uhlenbeck discussed the same problems. More recently Eringen and Samuels have considered the response of beams and plates to random loads. They found that a knowledge of the cross-correlation functions of the loads enabled them to compute the cross-correlation functions of the displacements and stresses, hence the corresponding mean square values. In the following sections several typical problems are considered.

3.1 BEAM SUBJECTED TO RANDOM LOADING

Perhaps the most elementary mathematical formulation of the vibrating beam is that given by the Bernoulli-Euler theory which considers only lateral inertia and bending deflections. The equation of motion for such a beam is given as

$$E \frac{\partial^4 y}{\partial x^4} + m \frac{\partial^2 y}{\partial t^2} = F(x,t) \; . \tag{283}$$

This equation tacitly assumes that plane sections remain plane in bending, bending slopes are small, the line of centroids is a straight line, the principal axes of the cross sections form two principal planes, and the loading is applied in one of these planes.

If one of the components of $F(x, t)$ has a frequency which is very close to one of the natural frequencies of the beam, the response of the beam in that mode will be very large and in the limit will diverge. Since any physically realizable beam contains some damping, it is necessary to introduce into eq. (283) a term which will simulate this mechanism. One of the simplest ways to accomplish this is to add to the left-hand side of eq. (283) a term of the form $\bar{\beta} \frac{\partial y}{\partial t}$. The resulting equation yields useful results when the first few modes of vibration are the important ones, but it suffers from the disadvantage that the fraction of critical damping in each mode decreases with mode order so that the higher modes of vibration are almost undamped, a fact not in agreement with experimental evidence.

When the forcing function is considered as being random in time, a broad band of frequencies is involved; hence unless some damping mechanism is considered, the expressions for mean square displacements and stresses will surely diverge. It seems natural, therefore, to consider first the problem of a Bernoulli-Euler beam with viscous external damping subjected to random loading. Eringen considered this problem and chose as the forcing function a stochastic load distributed along the beam with infinite correlation at $x = \xi$. The problem may be stated mathematically as

$$E\,\frac{\partial^4 y}{\partial x^4} + \bar{\beta}\,\frac{\partial y}{\partial t} + m\,\frac{\partial^2 y}{\partial t^2} = F(x,t)\ , \qquad (284)$$

where

$$\langle F(x,t)\,F(\xi,t^+)\rangle = D\ \delta(x-\xi)\,\delta(t-t^+)\ . \qquad (285)$$

Using a Fourier integral expression for the transverse displacement y, and expanding the transformed variable in a series of appropriate eigenfunctions, Eringen showed that the series for the mean square displacement converged but that the series for the mean square bending moment diverged. Admittedly a forcing function of the type given by eq. (285) is a severe test of the convergence of series type solutions. It appears that the infinite mean square bending stresses are caused by the severeness of the assumptions of the Bernoulli-Euler beam theory and the fact that eq. (285) implies an infinite energy input. Eringen and Samuels have recently shown that by using the Timoshenko beam equation, which includes the effect of shear and rotary inertia, and using a linear, viscous damping term, the mean square displacements and bending stresses remain finite when the beam is subjected to a forcing function of the type given by eq. (285).

Since the forcing function that has been considered represents an infinite energy input, it appears reasonable to employ a more realistic function which possesses a clipped white spectrum whose cutoff frequency is ω_c . In this case all normal modes of the beam with frequencies of vibration much greater than ω_c will not contribute any appreciable amount to the mean square displacements or bending stresses and a convergent series results for both quantities.

Let us consider the equation of motion of the Bernoulli-Euler beam with viscous external damping to be given by

$$E \frac{\partial^4 y}{\partial x^4} + m\beta \frac{\partial y}{\partial t} + m \frac{\partial^2 y}{\partial t^2} = F(x,t) , \tag{286}$$

where $F(x, t)$ is random, stationary and Gaussian with its time dependent part having a power spectrum of the form

$$\left.\begin{aligned} \Pi(\omega) &= \frac{4D}{2\pi} , & 0 \leq \omega \leq \omega_c , \\ \Pi(\omega) &= 0 , & \omega > \omega_c . \end{aligned}\right\} \tag{287}$$

We assume a solution of the form

$$y = \sum_{n=1}^{\infty} \zeta_n(t)\, \phi_n(x) , \tag{288}$$

where the functions $\zeta_n(t)$ are to be solved for and the functions $\phi_n(x)$ satisfy the differential equation

$$E \frac{d^4\phi_n}{d x^4} - m\omega_n^2 \phi_n = 0 , \tag{289}$$

along with the appropriate boundary conditions for the beam. If the ends of the beam are free, fixed, or simply supported, the functions defined by eq. (289) are orthogonal and may be normalized so that

$$\int_0^L \phi_n(x)\, \phi_m(x)\, dx = \delta_{mn} , \tag{290}$$

where δ_{mn} is Kronecker's delta. If we substitute eq. (288) into eq. (286) and make use of eq. (289), we have

$$\sum_{n=1}^{\infty}\left(m\phi_n\ddot{\zeta}_n + m\bar{\beta}\phi_n\dot{\zeta}_n + m\omega_n^2\phi_n\zeta_n\right) = F(x,t). \tag{291}$$

Multiplying both sides of eq. (291) by ϕ_m, integrating over x from 0 to L, and using the orthogonality relation of eq. (290) results in

$$\ddot{\zeta}_n + \bar{\beta}\dot{\zeta}_n + \omega_n^2\zeta_n = f_n(t) \quad , \tag{292}$$

where

$$f_n(t) = \frac{1}{m}\int_0^L F(x,t)\,\phi_n(x)\,dx \quad . \tag{293}$$

The solution of eq. (292) may be written as

$$\zeta_n = \int_0^t h_n(t-\tau)\,f_n(\tau)\,d\tau \quad , \tag{294}$$

where the function $h_n(t-\tau)$ is the response to a unit impulse. The mean square displacement for the beam is given as

$$\langle y^2\rangle = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\phi_n\phi_m\langle\zeta_n\zeta_m\rangle \quad , \tag{295}$$

where

$$\langle\zeta_n\zeta_m\rangle = \int_0^t\int_0^t h_n(t-\tau)\,h_m(t-\tau^+)\langle f_n(\tau)f_m(\tau^+)\rangle\,d\tau\,d\tau^+ \quad , \tag{296}$$

and

$$\langle f_n(\tau)f_m(\tau^+)\rangle = \frac{1}{m^2}\int_0^L\int_0^L \phi_n(\xi)\,\phi_m(\eta)\langle F(\xi,\tau)F(\eta,\tau^+)\rangle\,d\xi\,d\eta \quad . \tag{297}$$

Let us take the form of the correlation function for $F(x, t)$ to be

$$\langle F(x,t)\ F(x^+, t^+)\rangle = \delta(x-x^+)\,\psi(t-t^+), \tag{298}$$

where $\psi(t-t^+)$ is as yet not defined. Substituting eq. (298) into eq. (297) results in

$$\langle f_n(\tau)\, f_m(\tau^+)\rangle = \frac{1}{m^2}\,\psi(\tau-\tau^+)\,\delta_{mn}\ . \tag{299}$$

Using eq. (299) in eq. (296) we have

$$\langle \zeta_n \zeta_m\rangle = \frac{\delta_{mn}}{m^2}\int_0^t\int_0^t h_n(t-\tau)\,h_m(t-\tau^+)\,\psi(\tau-\tau^+)\,d\tau\, d\tau^+\ . \tag{300}$$

Since the relation between the power spectrum and autocorrelation is

$$\psi(\tau-\tau^+) = \int_0^\infty \Pi(\omega)\cos\omega(\tau-\tau^+)\,d\omega\ , \tag{301}$$

we can show that eq. (300) may be written as

$$\langle \zeta_n^2\rangle = \frac{1}{m^2}\int_0^\infty \frac{\Pi(\omega)\,d\omega}{|Z_n(\omega)|^2}\Bigg\{1 + e^{-\bar{\beta}t}\Bigg(1 + \frac{\bar{\beta}}{\omega_{Dn}}\sin\omega_{Dn}t\cos\omega_{Dn}t$$

$$-\left(2\cos\omega_{Dn}t + \frac{\bar{\beta}}{\omega_{Dn}}\sin\omega_{Dn}t\right)\cos\omega t - \frac{2\omega}{\omega_{Dn}}\sin\omega_{Dn}t\,\sin\omega t$$

$$+\frac{\bar{\beta}^2 - 4\omega_{Dn}^2 + 4\omega^2}{4\omega_{Dn}^2}\sin^2\omega_{Dn}t\Bigg)\Bigg\}\ . \tag{302}$$

Substituting eq. (302) into eq. (295) would yield the general time dependent solution for the mean square displacement of the beam. Since transient effects in statistical computations do not have a great deal of value in engineering systems we will consider the steady state solution only. Hence for large values of time eq. (302) reduces to

$$\langle \zeta_n^2 \rangle = \frac{1}{m^2} \int_0^\infty \frac{\Pi(\omega)\, d\omega}{|Z_n(\omega)|^2} \; . \tag{303}$$

Using eq. (287) in eq. (303) we have

$$\langle \zeta_n^2 \rangle = \frac{2D}{\pi m^2} \int_0^{\omega_c} \frac{d\omega}{|Z_n(\omega)|^2} \; . \tag{304}$$

This integral can be transformed into a tabulated integral, the result of the integration being

$$\langle \zeta_n^2 \rangle = \frac{D}{2\pi m^2 \omega_n^2 \omega_{Dn}} \left\{ \frac{1}{2} \log \left[\frac{\omega_c^2 + \omega_n^2 + 2\omega_c \omega_n (1-\rho_n^2)^{\frac{1}{2}}}{\omega_c^2 + \omega_n^2 - 2\omega_c \omega_n (1-\rho_n^2)^{\frac{1}{2}}} \right] \right.$$

$$\left. + \frac{(1-\rho_n^2)^{\frac{1}{2}}}{\rho_n} \arctan \frac{2\omega_c \omega_n \rho_n}{\omega_n^2 - \omega_c^2} \right\} \; . \tag{305}$$

If $\rho_n << 1$ then $\omega_n \doteq \omega_{on}$ and $1-\rho_n^2 \doteq 1$. When $\omega_c << \omega_N$, where ω_N is the N^{th} natural frequency of the beam, eq. (305) becomes

$$\langle \zeta_N^2 \rangle \doteq \frac{D}{2\pi m^2 \omega_N^3} \left\{ \log \frac{\omega_N}{\omega_N} + \frac{1}{\rho_N} \arctan \frac{2\omega_c \rho_N}{\omega_N} \right\} \doteq 0 \ . \tag{306}$$

Hence we may neglect terms in the series of eq. (295) for $n > N$ where N is such that $\omega_c << \omega_N$ so that eq. (306) is approximately correct. Since $\omega_n = n^2 \omega_1$ we find that for the case of small damping the mean square displacement becomes

$$\langle y^2 \rangle = \frac{D}{2\pi m^2 \omega_1^3} \sum_{n=1}^{N} \frac{1}{n^6} \left\{ \log \frac{\omega_c + \omega_n}{\omega_c - \omega_n} + \frac{1}{\rho_n} \arctan \frac{2\omega_c \omega_n \rho_n}{\omega_n^2 - \omega_c^2} \right\} \phi_n^2 \ . \tag{307}$$

Since this is a finite series, the displacements are finite. The bending moment is given by

$$\bar{M} = E \frac{\partial^2 y}{\partial x^2} = E \sum_{n=1}^{\infty} \zeta_n \frac{d^2 \phi_n}{d x^2} \ , \tag{308}$$

so that the mean square bending moment is

$$\langle \bar{M}^2 \rangle = E^2 \sum_{n=1}^{N} \langle \zeta_n^2 \rangle \left(\frac{d^2 \phi_n}{d x^2} \right)^2 \ . \tag{309}$$

For the simply supported beam $\dfrac{d^2\phi_n}{dx^2} = -\dfrac{n^2\pi^2}{L^2}\phi_n$ and eq. (309) becomes

$$\langle \bar{M}^2 \rangle = \frac{D E^2 \pi^3}{2m^2\omega_1^3 L^4} \sum_{n=1}^{N} \frac{1}{n^2} \left\{ \log \frac{\omega_c+\omega_n}{\omega_c-\omega_n} + \frac{1}{\rho_n} \arctan \frac{2\omega_c\omega_n\rho_n}{\omega_n^2-\omega_c^2} \right\} \phi_n^2 \quad . \tag{310}$$

Again this is a convergent expression because of the finite number of terms.

Hence by assuming that the forcing function acting on a Bernoulli-Euler beam with viscous external damping possesses a clipped white power spectrum, it is shown that the series expressions for the mean square displacements and stresses are finite. Useful results can be obtained, therefore, without using the more complicated Timoshenko beam equations.

3.2 BEAM WITH RANDOM END MOTION

Consider a Bernoulli-Euler beam with viscous external damping and simply supported ends and let one end of the beam, say $x = L$, be given some time dependent motion $f(t)$ where $f(t)$ is random, stationary, and Gaussian. The equation of motion is

$$E\frac{\partial^4 y}{\partial x^4} + m\bar{\beta}\frac{\partial y}{\partial t} + m\frac{\partial^2 y}{\partial t^2} = 0 \quad , \tag{311}$$

and the boundary conditions are

$$\left.\begin{aligned} y(0,t) = \frac{\partial^2 y(0,t)}{\partial x^2} &= 0 \quad , \\ y(L,t) &= f(t) \quad , \\ \text{and} \quad \frac{\partial^2 y(L,t)}{\partial x^2} &= 0 \quad . \end{aligned}\right\} \tag{312}$$

The inhomogeneous boundary condition can be eliminated by letting $y = y_0 + y_1$ where $y_0 = u(x) f(t)$ and $u(x)$ satisfies the homogeneous beam equation

$$E \frac{d^4 u}{d x^4} = 0 \quad , \tag{313}$$

and the boundary conditions

$$\left.\begin{aligned} u(0) = \frac{d^2 u(0)}{d x^2} &= 0 \quad , \\ u(L) &= 1 \quad , \\ \text{and} \quad \frac{d^2 u(L)}{d x^2} &= 0 \quad . \end{aligned}\right\} \tag{314}$$

Using the above transformation in eq. (311) and making use of eq. (313) we have

$$E\frac{\partial^4 y_1}{\partial x^4} + m\bar{\beta}\frac{\partial y_1}{\partial t} + m\frac{\partial^2 y_1}{\partial t^2} = -m\bar{\beta}u\frac{\partial f}{\partial t} - mu\frac{\partial^2 f}{\partial t^2} \quad , \tag{315}$$

where y_1 satisfies the homogeneous boundary conditions for a simply supported beam.

We assume a solution of the form

$$y_1 = \sum_{n=1}^{\infty} \phi_n \zeta_n(t) \quad , \tag{316}$$

where the functions $\zeta_n(t)$ are to be solved for and ϕ_n satisfies eqs. (289) and (290) and the boundary conditions for a simply supported beam. Substituting eq. (316) into eq. (315) we find

$$\ddot{\zeta}_n + \bar{\beta}\dot{\zeta}_n + \omega_n^2 \zeta_n = \Omega_n \ddot{f}(t) + \bar{\Omega}_n \dot{f}(t) \quad , \tag{317}$$

where

$$\left.\begin{aligned} \Omega_n &= -\int_0^L u\,\phi_n\,dx \quad , \\ \text{and} \qquad \bar{\Omega}_n &= -\bar{\beta}\int_0^L u\,\phi_n\,dx \quad . \end{aligned}\right\} \tag{318}$$

In order to compute mean square quantities we must determine $\langle \zeta_n \zeta_m \rangle$. Proceeding as in Sect. 3.1 we find

$$\langle \zeta_n \zeta_m \rangle = \int_0^t\!\!\int_0^t h_n(t-\tau)h_m(t-\tau^+)\left\langle\left\{\Omega_n\ddot{f}(\tau)+\bar{\Omega}_n\dot{f}(\tau)\right\}\left\{\Omega_m\ddot{f}(\tau^+)+\bar{\Omega}_m\dot{f}(\tau^+)\right\}\right\rangle d\tau\, d\tau^+ \quad . \tag{319}$$

From eq. (318) we see that $\Omega_n \bar{\Omega}_m = \Omega_m \bar{\Omega}_n$, hence eq. (319) may be written as

$$\langle \zeta_n \zeta_m \rangle = \int_0^t \int_0^t h_n(t-\tau) h_m(t-\tau^+) \Big\{ \Omega_n \Omega_m \langle \ddot{f}(\tau) \ddot{f}(\tau^+) \rangle + \bar{\Omega}_n \bar{\Omega}_m \langle \dot{f}(\tau) \dot{f}(\tau^+) \rangle$$

$$+ \Omega_n \bar{\Omega}_m \Big(\langle \dot{f}(\tau) \ddot{f}(\tau^+) \rangle + \langle \ddot{f}(\tau) \dot{f}(\tau^+) \rangle \Big) \Big\} \, d\tau \, d\tau^+ \quad . \tag{320}$$

In Appendix I it is shown that the various correlations of derivatives of a function can be related to derivatives of the auto-correlation of the function. It can thus be shown that

$$\langle \dot{f}(\tau) \dot{f}(\tau^+) \rangle = \psi''(\tau-\tau^+) \quad , \tag{321}$$

$$\langle \dot{f}(\tau) \ddot{f}(\tau^+) \rangle = -\psi'''(\tau-\tau^+) \quad , \tag{322}$$

$$\langle \ddot{f}(\tau) \dot{f}(\tau^+) \rangle = \psi'''(\tau-\tau^+) \quad , \tag{323}$$

and

$$\langle \ddot{f}(\tau) \ddot{f}(\tau^+) \rangle = \psi^{IV}(\tau-\tau^+) \quad , \tag{324}$$

where

$$(\ ') = \frac{d}{d(\tau-\tau^+)} \quad ,$$

$$(\ '') = \frac{d^2}{d(\tau-\tau^+)^2} \quad , \text{ etc.}$$

Using eqs. (321) through (324) the expression for $\langle \zeta_n \zeta_m \rangle$ can be written as

$$\langle \zeta_n \zeta_m \rangle = \int_0^t \int_0^t h_n(t-\tau) h_m(t-\tau^+) \left\{ \Omega_n \Omega_m \psi^{iv}(\tau-\tau^+) - \bar{\Omega}_n \bar{\Omega}_m \psi''(\tau-\tau^+) \right\} d\tau \, d\tau^+ . \tag{325}$$

Since the autocorrelation must be differentiable at least four times, it appears that one of the simplest forms for $\psi(\tau - \tau^+)$ is

$$\psi(\tau-\tau^+) = \psi_0 \, e^{-\gamma(\tau-\tau^+)^2} . \tag{326}$$

Using this form for the autocorrelation eq. (325) becomes for the case $n = m$

$$\langle \zeta_n^2 \rangle = \frac{\psi_0 \, e^{-2\rho_n \omega_n t}}{2 \omega_{nD}^2} \int_0^t \int_0^t e^{\rho_n \omega_n (\tau+\tau^+)} \left\{ \cos \omega_{nD}(\tau^+ - \tau) - \cos \omega_{nD}(2t - \{\tau + \tau^+\}) \right\}$$

$$\cdot \left\{ \Omega_n^2 (12\gamma^2 - 48\gamma^3(\tau^+ - \tau)^2 + 16\gamma^4 (\tau^+ - \tau)^4) - \bar{\Omega}_n^2 (4\gamma^2(\tau^+ - \tau)^2 - 2\gamma) \right\} e^{-\gamma(\tau^+ - \tau)^2} d\tau \, d\tau^+ . \tag{327}$$

Letting $r = \tau^+ + \tau$ and $s = \tau^+ - \tau$ we can simplify eq. (327) and write it as

$$\langle \zeta_n^2 \rangle = \frac{\psi_0 e^{-2\beta_n \omega_n t}}{2\omega_{nD}^2} \int_{s=0}^{t} \left\{ \Omega_n^2 (12\gamma^2 - 48\gamma^3 s^2 + 16\gamma^4 s^4) - \bar{\Omega}_n^2 (4\gamma^2 s^2 - 2\gamma) \right\} e^{-\gamma^2 s^2} ds$$

$$\cdot \int_{r=s}^{-s+2t} e^{-\beta_n \omega_n r} \left\{ \cos \omega_{nD} s - \cos \omega_{nD} (2t - r) \right\} dr. \qquad (328)$$

The integration over the variable r is easily carried out but the resulting single integral contains terms of the form

$$I_{nc} = \int_{y_1}^{y_2} y^n e^{-y^2} \cos ay \, dy \quad , \qquad (329)$$

or

$$I_{ns} = \int_{y_1}^{y_2} y^n e^{-y^2} \sin ay \, dy \quad . \qquad (330)$$

It is possible to integrate by parts successively and reduce all such integrals to expressions involving only I_{oc} and I_{os} where

$$I_{oc} = \int_{y_1}^{y_2} e^{-y^2} \cos ay \, dy = \frac{\pi^{\frac{1}{2}}}{4e^{a^2/4}} \left[\operatorname{erf}\left(y - \frac{ia}{2}\right) + \operatorname{erf}\left(y + \frac{ia}{2}\right) \right]_{y_1}^{y_2} \quad , \qquad (331)$$

and

$$I_{os} = \int_{y_1}^{y_2} e^{-y^2} \sin ay \, dy = \frac{\pi^{\frac{1}{2}}}{4ie^{a^2/4}} \left[\operatorname{erf}\left(y - \frac{ia}{2}\right) - \operatorname{erf}\left(y + \frac{ia}{2}\right) \right]_{y_1}^{y_2} \quad . \qquad (332)$$

This results in an extremely lengthy expression and since the transient solution is not of great interest we will concern ourselves with the steady state solution.

The steady state problem is easier to handle by power spectrum methods, since the autocorrelation of the forcing function is stationary. Using the relationship between the autocorrelation and power spectrum given by eq. (81) we find that the power spectrum of the forcing function for the case $n = m$ is

$$\Pi_{n=m}(\omega) = \psi_0 \sqrt{\frac{1}{\gamma\pi}}\, \omega^2 e^{-\frac{\omega^2}{4\gamma}} \left(\Omega_n^2 \omega^2 + \bar{\Omega}_n^2\right) \quad . \tag{333}$$

For the cross product terms, that is $n \neq m$, the power spectrum becomes

$$\Pi_{n\neq m}(\omega) = \psi_0 \sqrt{\frac{1}{\gamma\pi}}\, \omega^2 e^{-\frac{\omega^2}{4\gamma}} \left(\Omega_n \Omega_m \omega^2 + \bar{\Omega}_n \bar{\Omega}_m\right) \quad . \tag{334}$$

We then have the relations

$$\langle \zeta_n^2 \rangle = \int_0^\infty \frac{\Pi_{n=m}(\omega)\, d\omega}{|Z_n|^2} \quad , \tag{335}$$

and

$$\langle \zeta_n \zeta_m \rangle = \int_0^\infty \frac{\Pi_{n\neq m}(\omega)\, d\omega}{Z_n Z_m^*} \quad . \tag{336}$$

In Sect. 2.0 it was shown that the cross-product terms are of order ρ^2 compared to the square terms. For small damping, therefore, we need only consider the terms given by eq. (335) and we may neglect those given by eq. (336).

If it is desired to compute the mean square bending moment, we have

$$\bar{M} = E\frac{\partial^2 y}{\partial x^2} = E\left(\frac{\partial^2 y_0}{\partial x^2} + \frac{\partial^2 y_1}{\partial x^2}\right) = E\left(f(t)\frac{d^2 u}{dx^2} + \sum_{n=1}^{\infty}\zeta_n(t)\frac{d^2\phi_n}{dx^2}\right). \tag{337}$$

Solving eqs. (313) and (314) for u we find $u = \frac{x}{L}$ so that $\frac{d^2 u}{dx^2} = 0$. We then have for the case of small damping

$$\bar{M} = E\sum_{n=1}^{\infty}\zeta_n(t)\frac{d^2\phi_n}{dx^2}, \tag{338}$$

and

$$\langle \bar{M}^2\rangle \doteq E^2\sum_{n=1}^{\infty}\langle\zeta_n^2\rangle\left(\frac{d^2\phi_n}{dx^2}\right)^2. \tag{339}$$

For the simply supported beam $\phi_n = \sqrt{\frac{2}{L}}\,\sin\frac{n\pi x}{L}$ so that

$$\frac{d^2\phi_n}{dx^2} = -\frac{n^2\pi^2}{L^2}\sqrt{\frac{2}{L}}\,\sin\frac{n\pi x}{L}. \tag{340}$$

The expression for the mean square bending moment thus becomes

$$\langle \bar{M}^2\rangle \doteq \frac{2E^2\pi^4}{L^5}\,\psi_0\sqrt{\frac{1}{\gamma\pi}}\sum_{n=1}^{\infty}n^4\sin^2\frac{n\pi x}{L}\int_0^{\infty}\frac{\omega^2 e^{-\frac{\omega^2}{4\gamma}}\left(\Omega_n^2\omega^2+\bar{\Omega}_n^2\right)d\omega}{|Z_n|^2}. \tag{341}$$

For small damping we may approximate the integral appearing in eq. (341) by evaluating the power spectrum at ω_n and assuming that the power spectrum is flat with spectral density $\Pi(\omega_n)$. We thus have

$$\int_0^\infty \frac{\omega^2 e^{-\frac{\omega^2}{4\delta}} (\Omega_n^2 \omega^2 + \bar{\Omega}_n^2)\, d\omega}{|Z_n|^2} \doteq \omega_n^2 e^{-\frac{\omega_n^2}{4\delta}} (\Omega_n^2 \omega_n^2 + \bar{\Omega}_n^2) \int_0^\infty \frac{d\omega}{|Z_n|^2}$$

$$= \omega_n^2 e^{-\frac{\omega_n^2}{4\delta}} (\Omega_n^2 \omega_n^2 + \bar{\Omega}_n^2) \left(\frac{\pi}{2\bar{\beta}\omega_n^2}\right) . \quad (342)$$

For the simply supported beam we may use eq. (318) to solve for Ω_n^2 and $\bar{\Omega}_n^2$ and find

$$\Omega_n^2 = \frac{2L}{n^2 \pi^2} , \quad (343)$$

and

$$\bar{\Omega}_n^2 \quad \frac{2\bar{\beta}^2 L}{n^2 \pi^2} . \quad (344)$$

Substituting eqs. (342) through (344) into eq. (341) we have

$$\langle \bar{M}^2 \rangle \doteq \frac{2E^2 \pi^3}{L^4 \bar{\beta}} \Psi_0 \sqrt{\frac{1}{\gamma \pi}} \sum_{n=1}^\infty n^2 \sin^2 \frac{n\pi x}{L} (\omega_n^2 + \bar{\beta}^2) e^{-\frac{\omega_n^2}{4\delta}} . \quad (345)$$

Since the natural frequencies for the beam are related by the equation

$$\omega_n = n^2 \omega_1 \quad , \tag{346}$$

we see that eq. (345) becomes

$$\langle \bar{M}^2 \rangle \doteq \frac{2E^2\pi^3}{L^4 \bar{\beta}} \psi_0 \sqrt{\frac{1}{\gamma\pi}} \sum_{n=1}^{\infty} n^2 \sin^2 \frac{n\pi x}{L} (n^4\omega_1^2 + \bar{\beta}^2)\, e^{-\frac{n^4\omega_1^2}{4\gamma}} \quad . \tag{347}$$

If the damping is small then for beams with fundamental frequencies of the order of unity $n^4\omega_1^2 \gg \bar{\beta}^2$ and eq. (347) becomes

$$\langle \bar{M}^2 \rangle \doteq \frac{2E^2\pi^3\omega_1^2}{L^4 \bar{\beta}} \psi_0 \sqrt{\frac{1}{\gamma\pi}} \sum_{n=1}^{\infty} n^6 \sin^2 \frac{n\pi x}{L}\, e^{-\frac{n^4\omega_1^2}{4\gamma}} \quad . \tag{348}$$

If the autocorrelation of the end motion is not peaked very sharply, then γ is small. In this case the factor $e^{-\frac{n^4\omega_1^2}{4\gamma}}$ in eq. (348) will cause very rapid convergence of the series. For example, if $\omega_1 = 2$ and $\gamma = 0.25$ we see that the term for $n = 2$ is much smaller than that for $n = 1$ since

$$\frac{\left(n^6 e^{-\frac{n^4\omega_1^2}{4\gamma}}\right)_{n=2}}{\left(n^6 e^{-\frac{n^4\omega_1^2}{4\gamma}}\right)_{n=1}} = 64 e^{-60} = 4.7 \times 10^{-25} . \tag{349}$$

We may thus approximate the mean square bending moment by means of the first term in eq. (348) which yields

$$\langle \bar{M}^2 \rangle \doteq \frac{2E^2\pi^3\omega_1^2}{L^4\bar{\beta}} \psi_0 \sqrt{\frac{1}{\gamma\pi}} \sin^2\frac{\pi x}{L}\, e^{-\frac{\omega_1^2}{4\gamma}} . \tag{350}$$

The maximum occurs when $\sin^2\frac{\pi x}{L} = 1$, that is $x = \frac{L}{2}$, in which case we have

$$\langle \bar{M}^2 \rangle_{MAX.} \doteq \frac{2E^2\pi^3\omega_1^2}{L^4\bar{\beta}} \psi_0 \sqrt{\frac{1}{\gamma\pi}}\, e^{-\frac{\omega_1^2}{4\gamma}} . \tag{351}$$

When the autocorrelation for the end motion is a very sharply peaked function γ is large and the convergence of the series for the mean square bending moment is poor. This agrees with the result which would be expected if the autocorrelation for $f(t)$ were a delta function. In this latter case $\gamma = \infty$ and the expression for the mean square bending moment given by eq. (348) would yield a divergent result.

When the end conditions for the beam are such that the expression for u is not constant or a linear function of x so that $\frac{d^2u}{dx^2} \neq 0$ the expression for the mean square bending moment is not as simple as that given by eq. (339). In this case we find

$$\langle \bar{M}^2 \rangle \doteq E^2 \Bigg\{ \left(\frac{d^2u}{dx^2}\right)^2 \langle f(\tau) f(\tau^+) \rangle + \sum_{n=1}^{\infty} \left(\frac{d^2\phi_n}{dx^2}\right)^2 \langle \zeta_n(\tau) \zeta_n(\tau^+) \rangle$$

$$+ 2 \sum_{n=1}^{\infty} \frac{d^2u}{dx^2} \frac{d^2\phi_n}{dx^2} \langle f(\tau) \zeta_n(\tau^+) \rangle \Bigg\} , \tag{352}$$

and two additional terms must be evaluated. This adds no new difficulty and the previous methods may be used to compute $\langle \bar{M}^2 \rangle$.

Other attempts to formulate a theory of beam vibrations with viscous damping based upon the Bernoulli-Euler equation led K. Sezawa to use a damping term which was proportional to the time rate of extensional strain. The simplest law, which introduces this type of internal damping into a elastic solid, is Stokes' law, which assumes that stress is a linear function of strain and strain rate. In the case of the beam

$$\mathcal{S} = \left(A + B\frac{\partial}{\partial t}\right)\epsilon \quad , \tag{353}$$

where A and B are constants. The beam equation becomes

$$E\left(\frac{\partial^4 y}{\partial x^4} + \bar{\beta}\frac{\partial^5 y}{\partial x^4 \partial t}\right) + m\frac{\partial^2 y}{\partial t^2} = F(x, t) \quad . \tag{354}$$

It is also necessary to modify the boundary conditions so that for a simply supported end we have

$$y = 0 \quad , \tag{355}$$

and

$$E\left(\frac{\partial^2 y}{\partial x^2} + \bar{\beta}\frac{\partial^3 y}{\partial x^2 \partial t}\right) = 0 \quad . \tag{356}$$

For a built-in end we have

$$y = 0 \quad , \tag{357}$$

and

$$\frac{\partial y}{\partial x} = 0 \quad . \tag{358}$$

For a free end we have

$$E\left(\frac{\partial^2 y}{\partial x^2} + \bar{\beta}\,\frac{\partial^3 y}{\partial x^2 \partial t}\right) = 0 \quad , \tag{359}$$

and

$$E\left(\frac{\partial^3 y}{\partial x^3} + \bar{\beta}\,\frac{\partial^4 y}{\partial x^3 \partial t}\right) = 0 \quad . \tag{360}$$

Assuming a solution of the form given by eq. (288) we have

$$\ddot{\zeta}_n + \bar{\beta}\,\omega_n^2\,\dot{\zeta}_n + \omega_n^2\,\zeta_n = f_n(t) \quad . \tag{361}$$

This is similar to eq. (292) except that the coefficient of the damping term is now $\bar{\beta}\omega_n^2$ instead of $\bar{\beta}$. We may use the results of the previous analysis if we replace $\bar{\beta}$ with $\bar{\beta}\omega_n^2$. For the case of random end motion this improves the convergence of the series. Equation (348) becomes

$$\langle \bar{M}^2 \rangle = \frac{2E^2\pi^3}{L^4\bar{\beta}}\,\Psi_0\sqrt{\frac{1}{\gamma\pi}}\sum_{\pi=1}^{\infty} n^4 \, SIN^2\,\frac{n\pi x}{L}\; e^{-\frac{n^4\omega_1^2}{4\gamma}} \quad . \tag{362}$$

This theory is not realistic, however, since this form of damping results in the eventual suppression of the higher modes of vibration which is certainly not in agreement with experimental evidence.

Part IV of this thesis was written as a supplement to NAVORD Report 7010. The equation and sketch numbers correspond to those in the Navy report but Part IV is a unit in itself and may be read without reference to that document.

PART IV

VIBRATION ISOLATION UNDER RANDOM EXCITATION

The problem of vibration isolation is certainly not a new one and the theory has been developed to a high degree in the past. Most of the work, however, has been done for systems excited by forces having single frequency components or by forces that can be described by the first few terms of a Fourier series. Little work has been done for the case of random inputs.

1.0 General Discussion of the Problem

The principle of isolation has essentially two aspects, isolation of the motion and isolation of the force. Isolation of the motion usually applies to reduction of stresses or deflections of components whose supports experience motion from some source while isolation of the forces involves reducing the transmission of forces to the supporting structure. For the case of single frequency excitation the criterion for good isolation can usually be established quite readily. If failure occurs when a particular value of amplitude, velocity, or acceleration is exceeded, the design problem is clear. If fatigue is the criterion for failure, the component may be subjected to a shake test at the frequency of interest and the number of cycles at some particular value of amplitude, velocity, or acceleration for which failure occurs can be determined. When more than one frequency is involved, it is often the case that only the fundamental and the first harmonic are important; again some similar test may be performed to determine how damage accumulates for each frequency and amplitude.

Clearly as the number of frequency components increase, the problem of determining analytically or experimentally what portion of the total damage causing failure results from each complete cycle at each frequency becomes extremely complex.

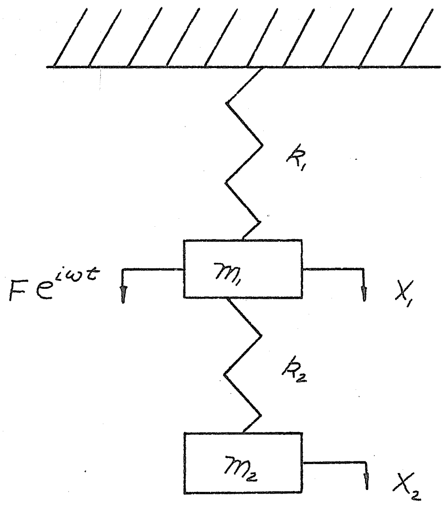
When the input is a random function, the isolation problem becomes more difficult. It is now no longer possible to design so that a particular value of displacement, velocity, or acceleration is never exceeded, for we can only describe the various quantities statistically. Theoretically any value of a parameter is possible however unlikely it may be when its amplitude is described by a probability distribution. Hence all that can be done in this case is to design so that the probability of exceeding a certain value of the parameter of interest is small.

When fatigue is the criterion for failure, it is not at all clear at present how damage accumulates at the various amplitudes and frequencies. Since in the random case a wide band of frequencies is involved, there appears to be little that can be done in a quantitative sense.

In the following sections an attempt is made to determine some criterion for proper isolation which may be used for the case of random inputs and to develop some of the expressions for the response of multi-degree-of-freedom systems for the case of random support motion.

1.1 INPUT WITH DISCRETE FREQUENCY

For the simple, undamped harmonic oscillator excited by a force having a single frequency component we may decrease the motion of the primary mass by merely attaching a secondary spring-mass system to it as shown in the following sketch.



UNDAMPED TWO-DEGREE-OF-FREEDOM SYSTEM

SKETCH 18

The equations of motion are

$$m_1 \ddot{X}_1 + (k_1 + k_2) X_1 - k_2 X_2 = F e^{i\omega t} , \tag{563}$$

and

$$m_2 \ddot{X}_2 + k_2 X_2 - k_2 X_1 = 0 . \tag{564}$$

Assuming solutions of the form

$$X_1 = \bar{B} e^{i\omega t} , \tag{565}$$

and

$$X_2 = \bar{C} e^{i\omega t} , \tag{566}$$

we find

$$-m_1 \omega^2 \bar{B} + (k_1 + k_2) \bar{B} - k_2 \bar{C} = F , \tag{567}$$

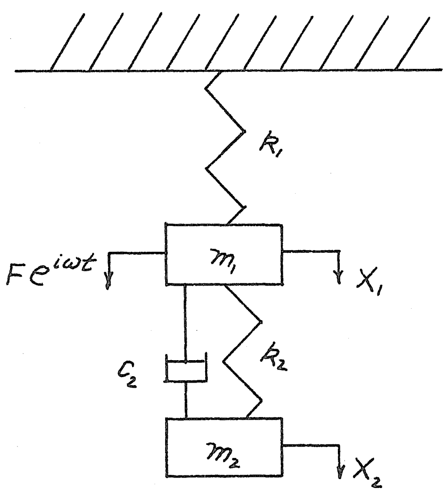
and

$$-m_2\omega^2\bar{C} + k_2\bar{C} - k_2\bar{B} = 0 \quad . \tag{568}$$

It is clear from eq. (568) that if $\frac{k_2}{m_2} = \omega^2$ then $\bar{B} = 0$ and the motion of the primary mass is eliminated completely. This result is true only when the auxiliary system is tuned to the driving frequency. For other driving frequencies the primary mass will vibrate and, depending upon the driving frequency, its amplitude may be many times greater with the auxiliary mass than without it.

1.2 INPUT WITH MANY FREQUENCIES

When we deal with inputs which have many frequency components, it is not possible to decrease the motion of the primary mass to zero; instead we can only minimize it over the entire range of frequency by adding damping to the system in addition to the auxiliary mass. Consider the damped vibration absorber shown below.



DAMPED TWO-DEGREE-OF-FREEDOM SYSTEM

SKETCH 19

The equations of motion are

$$m_1 \ddot{X}_1 + k_1 X_1 + k_1 (X_1 - X_2) + c_2 (\dot{X}_1 - \dot{X}_2) = F e^{i\omega t} \quad , \tag{569}$$

and

$$m_2 \ddot{X}_2 + k_2 (X_2 - X_1) + c_2 (\dot{X}_2 - \dot{X}_1) = 0 \ . \tag{570}$$

Again we assume harmonic solutions for X_1 and X_2 and after performing some algebraic manipulations we find

$$\frac{X_1^2}{F^2} = \frac{(k_2 - m_2\omega^2)^2 + \omega^2 c_2^2}{(-m_1\omega^2 + k_1)(-m_2\omega^2 + k_2) - m_2\omega^2 k_2^2 + \omega^2 c_2^2 (-m_1\omega^2 + k_1 - m_2\omega^2)} \ . \tag{571}$$

By defining

$$c_{cr} = 2 m_2 \sqrt{k_1/m_1} \quad ,$$

$$j = \omega \Big/ \sqrt{k_1/m_1} \quad ,$$

$$f = \sqrt{k_2/m_2} \Big/ \sqrt{k_1/m_1} \quad ,$$

$$X_{st} = F/k_1 \quad ,$$

and

$$r = m_2/m_1 \quad ,$$

we find

$$\frac{X_1}{X_{st}} = \sqrt{\frac{\left(2\frac{c}{c_{cr}} j\right)^2 + (j^2 - f^2)^2}{\left(2\frac{c}{c_{cr}} j\right)^2 (j^2 - 1 + r j^2)^2 + \left[r f^2 j^2 - (j^2 - 1)(j^2 - f^2) \right]^2}} \ . \tag{572}$$

We see that when $c_2 = 0$ there are two driving frequencies for which the amplitude of m_1 becomes infinite. Also if $c_2 = \infty$ the masses are essentially clamped together and there is one driving frequency at which the amplitude becomes infinite. It is reasonable to expect that some intermediate value of damping will lead to a maximum dissipation of energy and hence the smallest resonant amplitude for the mass .

It is possible to show that for any given set of system parameters a series of curves of X_1/X_{st} vs f all pass through two given points. Hence the best value for c_2 is the one that gives a horizontal tangent through the highest of these points. In addition by varying f we can change the relative heights of these points. Hence by making the points occur at the same height through the proper choice of the parameter f and choosing c_2 so that a horizontal tangent occurs at one of the points, we have optimized the absorber. The algebra involved in determining the proper value of f and the resulting value for X_1/X_{st} is quite tedious. The results, however, are

$$f = \frac{1}{1+h} , \tag{573}$$

and

$$\frac{X_1}{X_{st}} = \sqrt{1 + \frac{2}{h}} . \tag{574}$$

This problem is treated in considerable detail in <u>Mechanical Vibrations</u> by J. P. Den Hartog.

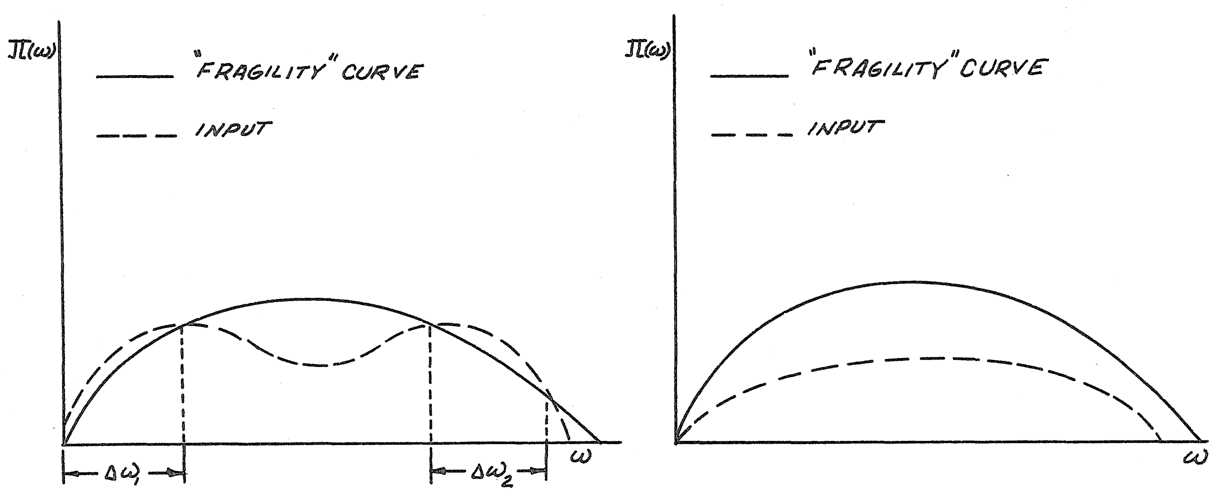
The preceding results were mentioned only to illustrate the manner in which simple systems can be treated to improve vibrational environments.

It is obvious that for many degrees of freedom the problem becomes extremely difficult to handle analytically since the number of free parameters that must be dealt with becomes very large.

1.3 RANDOM INPUTS

When the exciting force is a random function a broad band of frequencies is involved. Minimization of the amplitude is meaningless since the Fourier components of the force are themselves random variables with some probability distribution. Minimizing the mean square value is also not a valid criterion for good isolation, unless the statistics of the input are Gaussian and failure occurs when a critical value of some parameter is exceeded. For example, assume a component is located at a particular point in the system and assume it is known that this piece of equipment will fail if subjected to an acceleration of more than 12 g for any frequency. Furthermore, suppose a calculation for the mean square acceleration at this point yields $\langle \ddot{X}_i^2 \rangle = 4$ g . If the statistics are Gaussian, we know that the probability of exceeding an acceleration of 12 g at this point is only 0.0027. It is difficult to give any strict interpretation to this number, but it may be considered as meaning that if 10,000 identical systems were subjected to this vibrational environment roughly 27 of them would fail. If $\langle \ddot{X}_i^2 \rangle$ were decreased to 3 g, then only 6 out of 100,000 would fail on the average. It is of course a matter of judgment as to when the isolation is acceptable for this particular case.

When fatigue is the cause of failure, the problem becomes more complicated. It is not known quantitatively, for example, how to add the fatigue damage caused by a complete stress reversal at some particular frequency to that of some other frequency. If failure occurred after n_1 cycles at frequency ω_1 and amplitude δ_1, or after n_2 cycles at frequency ω_2 and amplitude δ_2, it is not clear how to determine at what time failure would occur if both frequencies acted simultaneously. When a wide band of frequencies is involved, it would be impractical to even attempt to gather the required experimental data. Ideally what would be desired is to have a "fragility" curve available for each component. This would essentially be the power spectrum of the vibrational environment that would cause failure within some prescribed period of time. If the power spectrum of the actual environment exceeded that of the "fragility" curve at any point, then the component would most likely fail before the minimum time of required operation elapsed. The sketches shown below clarify the idea.



a - Poor Isolation

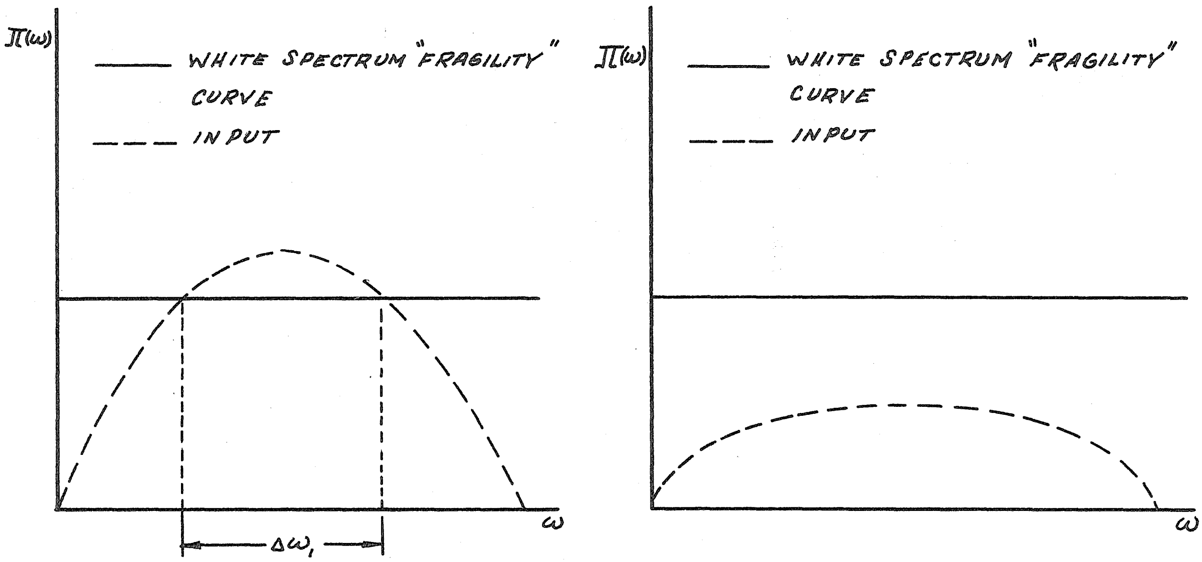
b - Good Isolation

EXAMPLES OF VIBRATION ISOLATION

SKETCH 20

Sketch 20(a) shows an example of poor vibration isolation since in the two ranges $\Delta\omega_1$ and $\Delta\omega_2$ the input power spectrum lies above the "fragility" curve, hence failure would very likely occur before the minimum time required. Sketch 20(b) shows an example of good isolation since the power spectrum of the input lies below the "fragility" curve at all frequencies. This is of course just one possible criterion to use and certainly there may be others that would be more appropriate for some particular conditions. Unfortunately these ideal "fragility" curves would be extremely difficult if not impossible to construct. Suppose the component is required to operate satisfactorily for some time t_f. The component might be tested at some frequency ω_1 at amplitude δ_1 and say failure occurred after time $t_1 > t_f$. The amplitude δ_1 could be increased and the test repeated until the time of failure was equal to or less than t_f for some amplitude δ_{1f}. This test could be repeated at other frequencies ω_i to determine other δ_{if}. A curve of δ_{if} vs ω_i could then be plotted. From this we may compute the power spectrum by means of eq. (189). This would not be the "fragility" curve, however, since in an actual process all the various frequencies act simultaneously, not independently as in the test just described.

This difficulty may be circumvented by subjecting the component to a test where the input is a random function with a white power spectrum having a cutoff frequency larger than any that would be encountered in the actual case. The level of this white spectrum would be increased in successive tests until failure occurred in some time less than t_f. The criterion for good isolation would be that the maximum of the power spectrum of the vibration environment be less than that of the white spectrum "fragility" curve. This is illustrated in the sketch shown below.



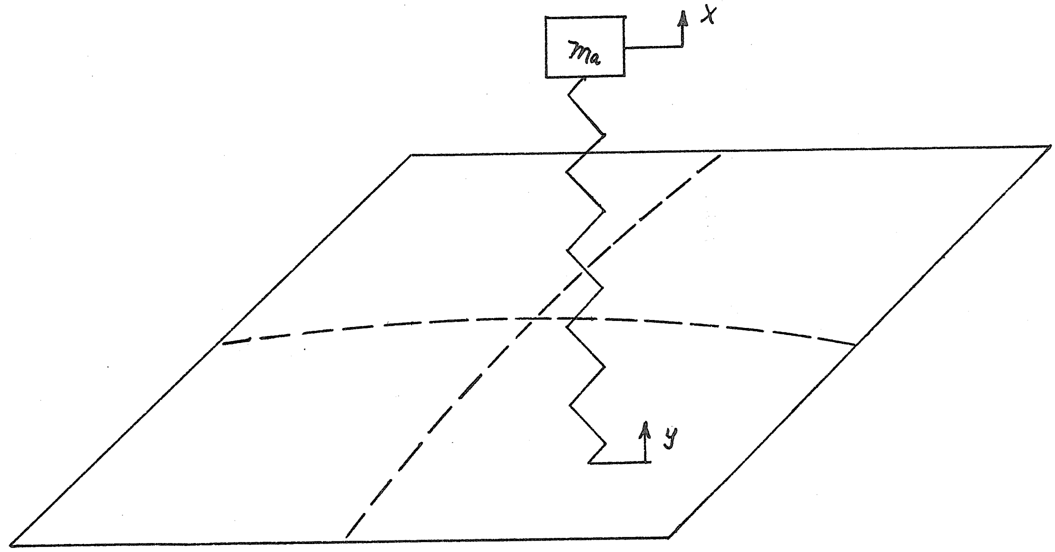
a - Poor Isolation b - Good Isolation

EXAMPLES OF VIBRATION ISOLATION

SKETCH 21

Sketch 21(a) shows an example of poor isolation since the input power spectrum exceeds the white spectrum "fragility" curve in the region $\Delta\omega_1$ while in Sketch 21(b) the isolation is satisfactory.

W. T. Thomson of the Ramo-Wooldridge Corp. has done some work on the problem of vibration isolation under random excitation in which he shows that components attached to a randomly vibrating structure may be subjected to a vibration environment which is substantially less severe than the environment of the primary structure at the attachment point. His analysis proceeds as follows. Consider, for example, the system shown below.



EXAMPLES OF COMPONENT ISOLATION

SKETCH 22

If the system is assumed to have structural damping, we may write for the displacement of the attachment point

$$y(\xi,\eta,t) = \sum_{n=1}^{N} q_n(t)\, \phi_n(\xi,\eta) \quad . \tag{575}$$

The generalized coordinate $q_n(t)$ is found by solving the second order differential equation

$$\ddot{q}_n + (1+ic)\,\omega_n^2\, q_n = \frac{1}{M_n}\int\!\!\int F(\xi,\eta,t)\,\phi_n(\xi,\eta)\, d\xi\, d\eta$$

$$- \frac{m_a \ddot{X}\, \phi_n(a,b)}{M_n} \; , \tag{576}$$

where

ϕ_n = normal mode shape of the primary structure,

M_n = generalized mass of the primary structure,

and

ω_n = normal mode frequency.

The equation of motion for the auxiliary mass m_a is given as

$$\ddot{X} + (1 + iC_a)\omega_a^2 (X - y(a,b)) = 0 \quad . \tag{577}$$

Solving eqs. (576) and (577) for a harmonic input we find

$$X = \frac{\displaystyle\sum_{n=1}^{N} \frac{\phi_n(a,b)\iint F(\xi,\eta)\,\phi_n(\xi,\eta)\,d\xi\,d\eta}{\omega_n^2 M_n \left(1 - \left(\frac{\omega}{\omega_n}\right)^2 + ic\right)} \quad e^{i\omega t}}{1 - \left(\frac{\omega}{\omega_a}\right)^2 + iC_a - \displaystyle\sum_{n=1}^{N} \frac{m_a}{M_n}\phi_n^2(a,b) \frac{\left(\frac{\omega}{\omega_n}\right)^2}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + ic}}$$

$$= H(i\omega)\, e^{i\omega t} \quad , \tag{578}$$

where the definition of the admittance $H(i\omega)$ is clear from eq. (578).

If we consider a random input which can be written as

$$F(x,y,t) = F_t(t)\, F_s(x,y) \quad , \tag{579}$$

and restrict ourselves to the case of two masses and two springs, we find

$$H(i\omega) = \frac{\displaystyle\sum_{n=1}^{N} A_n \frac{\left(\frac{\omega}{\omega_n}\right)^2}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + ic}}{1 - \left(\frac{\omega}{\omega_a}\right)^2 + iC_a - \displaystyle\sum_{n=1}^{N} N_n \frac{\left(\frac{\omega}{\omega_n}\right)^2}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + ic}} \quad , \tag{580}$$

where

$$A_n = \frac{\phi_n(a,b)}{M_n} \iint F(\xi,\eta)\, \phi_n(\xi,\eta)\, d\xi\, d\eta$$

and

$$N_n = \frac{m_a}{M_n} \phi_n^2(a,b)$$

For the case where $m_a \to 0$ we find

$$H_o(i\omega) = \sum_{n=1}^{N} \frac{A_n \left(\frac{\omega}{\omega_n}\right)^2}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + iC} \quad , \tag{581}$$

and for the case where the secondary mass is rigidly attached

$$H_\infty(i\omega) = \frac{\displaystyle\sum_{n=1}^{N} A_n \frac{\left(\frac{\omega}{\omega_n}\right)^2}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + iC}}{1 - \displaystyle\sum_{n=1}^{N} N_n \frac{\left(\frac{\omega}{\omega_n}\right)^2}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + iC}} \quad . \tag{582}$$

The zeros of the denominator of eq. (582) are readily approximated since near the first zero $\left(\frac{\omega}{\omega_1}\right)^2 \gg \left(\frac{\omega}{\omega_2}\right)^2 \gg \left(\frac{\omega}{\omega_3}\right)^2 \gg \text{-------------} \gg \left(\frac{\omega}{\omega_N}\right)^2$ so that by solving the following equation we can determine $\left(\frac{\omega}{\omega_1}\right)^2$ quite accurately.

$$1 - \frac{N_1 \left(\frac{\omega}{\omega_1}\right)^2}{1 - \left(\frac{\omega}{\omega_1}\right)^2} = 0 \quad , \tag{583}$$

or

$$\left(\frac{\omega}{\omega_1}\right)^2 = \frac{1}{1 + N_1} \quad . \tag{584}$$

The reduction in the first resonant frequency is then

$$1 - \left(\frac{\omega}{\omega_1}\right)^2 = \frac{N_1}{1 + N_1} \tag{585}$$

In a similar fashion we find

$$\left(\frac{\omega}{\omega_n}\right) = \frac{1 + \sum_{i=1}^{N-1} N_i}{1 + \sum_{i=1}^{N} N_i} \tag{586}$$

and

$$1 - \left(\frac{\omega}{\omega_n}\right) = \frac{N_n}{1 + \sum_{i=1}^{N} N_i} \quad .$$

Considering white noise excitation we may compare the response of the structure at the point of attachment with and without the added mass. The acceleration response is given as

$$\int_0^{\infty} \left| H(i\omega) \right|^2 d\omega = \frac{\pi}{2} \sum (\text{peak value}) \cdot (\text{band width at half power level}). \tag{587}$$

Hence when $m_2 = 0$ we have

$$\int_0^\infty H_0(i\omega)\,d\omega = \frac{\pi}{2C}\left(A_1^2\omega_1 + A_2^2\omega_2 + A_3^2\omega_3 + \text{----------}\right) , \tag{588}$$

and when the secondary mass is rigidly attached

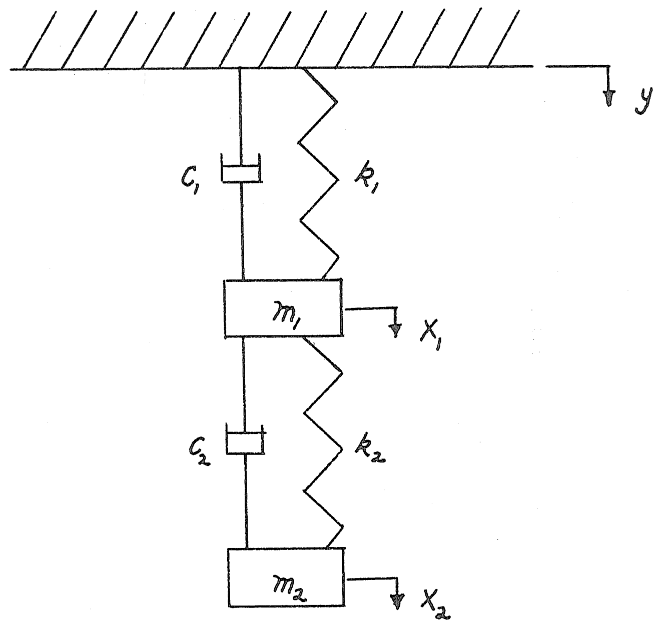
$$\int_0^\infty H_\infty(i\omega)\,d\omega = \frac{\pi}{2C}\left(\frac{A_1^2\omega_1}{(1+N_1)^{\frac{3}{2}}} + \frac{A_2^2\omega_2(1+N_1)^{\frac{1}{2}}}{(1+N_1+N_2)^{\frac{3}{2}}} \text{-----}\right) . \tag{589}$$

Since N_n is positive the acceleration response at the point of attachment may be substantially reduced by the addition of a small mass. Hence assuming that the vibrational environment experienced by a component attached to a vibrating structure is that of the structure alone may be seriously in error.

In the following sections multi-degree-of-freedom systems will be considered for the case where the support undergoes random motion.

2.0 Normal Mode Analysis

Consider the two-degree-of-freedom system with a moving support shown below.



TWO-DEGREE-OF-FREEDOM SYSTEM WITH MOVING SUPPORT

SKETCH 23

The equations of motion are

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 + (k_1 + k_2)x_1 - c_2\dot{x}_2 - k_2 x_2 = c_1\dot{y} + k_1 y \; , \tag{590}$$

and

$$m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2 x_2 - c_1\dot{x}_1 - k_1 x_1 = 0 \; . \tag{591}$$

If we transform eqs. (590) and (591) by means of the relations

$$\xi = x_1 - y \; , \tag{592}$$

and

$$\zeta = x_2 - y \; , \tag{593}$$

we find

$$m_1\ddot{\xi} + (c_1 + c_2)\dot{\xi} + (k_1 + k_2)\xi - c_2\dot{\zeta} - k_2\zeta = -m_1\ddot{y} \; , \tag{594}$$

and

$$m_2\ddot{\bar{\xi}} + c_2\dot{\bar{\xi}} + k_2\bar{\xi} - c_1\dot{\bar{\xi}} - k_1\bar{\xi} = -m_2\ddot{y} \quad . \tag{595}$$

Hence we have changed our problem from one where the support is moving with no forces acting on the masses to one in which the support is stationary with inertial forces $-m_i\ddot{y}$ acting on the masses. Equations (594) and (595) are usually easier to solve and once the relative motion is known it is simply a matter of using eqs. (592) and (593) to find the absolute motion.

If we consider an N-degree-of-freedom system for which normal modes exist and write the equations of motion in terms of the relative coordinates, the result in matrix notation is

$$\left[m\right]\left\{\ddot{x}\right\} + \left[c\right]\left\{\dot{x}\right\} + \left[k\right]\left\{x\right\} = -\left[m\right]\left\{\ddot{y}\right\} \quad . \tag{596}$$

In order to insure the existence of real eigenvalues and orthogonal eigenvectors, we make the transformation of coordinates

$$\left\{\bar{\xi}\right\} = \left[m\right]^{\frac{1}{2}}\left\{x\right\} \quad , \tag{597}$$

or

$$\left\{x\right\} = \left[m\right]^{-\frac{1}{2}}\left\{\bar{\xi}\right\} \quad . \tag{598}$$

Since $\left[m\right]$ is a diagonal matrix, we know that $\left[m\right]^{n} = \left[m^{n}\right]$. Hence substituting eq. (597) into eq. (598) and premultiplying by $\left[m\right]^{-\frac{1}{2}}$ we have

$$\{\ddot{\bar{\zeta}}\} + [m]^{-\frac{1}{2}}[c][m]^{-\frac{1}{2}}\{\dot{\bar{\zeta}}\} + [m]^{-\frac{1}{2}}[k][m]^{-\frac{1}{2}}\{\bar{\zeta}\} = -[m]^{\frac{1}{2}}\{\ddot{y}\} \quad . \tag{599}$$

Since $[m]^{-\frac{1}{2}}[k][m]^{-\frac{1}{2}}$ is a symmetric matrix it possesses real eigenvalues and orthogonal eigenvectors. We make the following transformation

$$\{\bar{\zeta}\} = [Q]\{\xi\} \quad , \tag{600}$$

where $[Q]$ is composed of columns which are the normalized eigenvectors of the matrix $[m]^{-\frac{1}{2}}[k][m]^{-\frac{1}{2}}$. We know from matrix theory that $[Q]^T = [Q]^{-1}$ hence

$$[Q]^{-1}[Q] = [Q]^T[Q] = [I] \tag{601}$$

It is also known that

$$[Q]^T[m]^{-\frac{1}{2}}[k][m]^{-\frac{1}{2}}[Q] = [\omega^2] = \begin{bmatrix} \omega_1^2 & 0 & \cdots & 0 \\ 0 & \omega_2^2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \omega_N^2 \end{bmatrix} \quad , \tag{602}$$

where ω_i is the natural undamped frequency of the i^{th} normal mode. Substituting eq. (600) into eq. (599), premultiplying by $[Q]^T$ and noting that if normal modes are to exist $[c] = \bar{\alpha}[k]$ we find

$$\{\ddot{\xi}\} + \bar{\alpha}\left[\omega^2\right]\{\dot{\xi}\} + \left[\omega^2\right]\{\xi\} = -\left[Q\right]^T\left[m\right]^{\frac{1}{2}}\{\ddot{y}\} \quad . \tag{603}$$

Equation (603) represents a system of uncoupled equations which in two dimensions would have the form

$$\ddot{\xi}_1 + \bar{\alpha}\omega_1^2\dot{\xi}_1 + \omega_1^2\xi_1 = -\left(\sqrt{m_1}\,A_1^{(1)} + \sqrt{m_2}\,A_2^{(1)}\right)\ddot{y} \quad , \tag{604}$$

and

$$\ddot{\xi}_2 + \bar{\alpha}\omega_2^2\dot{\xi}_2 + \omega_2^2\xi_2 = -\left(\sqrt{m_1}\,A_1^{(2)} + \sqrt{m_2}\,A_2^{(2)}\right)\ddot{y} \quad , \tag{605}$$

where

$$A_1^{(1)} = \frac{k_2/m_2 - \omega_1^2}{\sqrt{\bar{\Delta}_1}} \quad ,$$

$$A_1^{(2)} = \frac{k_2/m_2 - \omega_2^2}{\sqrt{\bar{\Delta}_2}} \quad ,$$

$$A_2^{(1)} = \frac{k_2}{\sqrt{m_1 m_2 \bar{\Delta}_1}} \quad ,$$

$$A_2^{(2)} = \frac{k_2}{\sqrt{m_1 m_2 \bar{\Delta}_2}} \quad ,$$

$$\bar{\Delta}_1 = \left(k_2/m_2 - \omega_1^2\right) + \frac{k_2^2}{m_1 m_2} \quad ,$$

and

$$\bar{\Delta}_2 = \left(k_2/m_2 - \omega_2^2\right) + \frac{k_2^2}{m_1 m_2} \quad .$$

We may determine the solutions of eqs. (604) and (605) in several ways but two particular forms will be of interest. They are the impulse method and the impedance method.

2.1 IMPULSE METHOD

For the impulse method the solution of the uncoupled equations of motion may be written in the form

$$\xi_i = \int_{-\infty}^{t} h_i(t-\tau) f_i(\tau)\, d\tau \quad , \tag{606}$$

where

$$f_i(\tau) = \left(- \sum_{j=1}^{N} \sqrt{m_j}\, A_j^{(i)} \right) \ddot{y}(\tau) \quad ,$$

$$h_i(t-\tau) = \frac{1}{\omega_{iD}}\, e^{-\rho_i \omega_i (t-\tau)} \sin \omega_{iD}(t-\tau) \quad ,$$

$$\omega_{iD}^2 = \omega_i^2 (1-\rho_i^2) \quad ,$$

and

$$\rho_i = \frac{\alpha}{2}\, \omega_i \quad .$$

Hence in matrix notation the solution of eq. (603) is

$$\left\{\xi\right\} = - \int_{-\infty}^{t} \left[h(t-\tau)\right] \left[Q\right]^{T} \left[m\right]^{\frac{1}{2}} \left\{\ddot{y}(\tau)\right\} d\tau \quad , \tag{607}$$

where

$$\left[h(t-\tau)\right] = \begin{bmatrix} h_1(t-\tau) & 0 & \cdots & 0 \\ 0 & h_2(t-\tau) & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & h_N(t-\tau) \end{bmatrix} .$$

From eq. (607) we find

$$\left\{\dot{\xi}\right\} = - \int_{-\infty}^{t} \left[\dot{h}(t-\tau)\right] \left[Q\right]^{T} \left[m\right]^{\frac{1}{2}} \left\{\ddot{y}(\tau)\right\} d\tau \quad , \tag{608}$$

and

$$\left\{\ddot{\xi}\right\} = - \int_{-\infty}^{t} \left[\ddot{h}(t-\tau)\right] \left[Q\right]^{T} \left[m\right]^{\frac{1}{2}} \left\{\ddot{y}(\tau)\right\} d\tau \quad - \left[Q\right]^{T} \left[m\right]^{\frac{1}{2}} \left\{\ddot{y}(t)\right\} . \tag{609}$$

Using eqs. (600) and (597) and noting that $\left\{\ddot{x}\right\} + \left\{\ddot{y}\right\} = \left\{\ddot{x}_{abs}\right\}$ we see that eq. (609) becomes

$$\left\{\ddot{x}_{abs}\right\} = - \int_{-\infty}^{t} \left[m\right]^{-\frac{1}{2}} \left[Q\right] \left[\ddot{h}(t-\tau)\right] \left[Q\right]^{T} \left[m\right]^{\frac{1}{2}} \left\{\ddot{y}(\tau)\right\} d\tau \quad . \tag{610}$$

For a two-degree-of-freedom system this becomes

$$\ddot{x}_{1abs} = -\int_{-\infty}^{t} \frac{\ddot{h}_1(t-\tau)\bar{A}^2 + \ddot{h}_2(t-\tau)\bar{N}^2 + \sqrt{m_2/m_1}\,\bar{A}\bar{N}\left(\ddot{h}_1(t-\tau) - \ddot{h}_2(t-\tau)\right)}{\bar{A}^2 + \bar{N}^2}\,\ddot{y}(\tau)\,d\tau \quad , \qquad (611)$$

and

$$\ddot{x}_{2abs} = -\int_{-\infty}^{t} \frac{\ddot{h}_2(t-\tau)\bar{A}^2 + \ddot{h}_1(t-\tau)\bar{N}^2 + \sqrt{m_1/m_2}\,\bar{A}\bar{N}\left(\ddot{h}_1(t-\tau) - \ddot{h}_2(t-\tau)\right)}{\bar{A}^2 + \bar{N}^2}\,\ddot{y}(\tau)\,d\tau \quad , \qquad (612)$$

where

$$\bar{A} = \frac{k_2}{m_2} - \omega_1^2 \quad ,$$

and

$$\bar{N} = \frac{k_2}{\sqrt{m_1 m_2}} \quad .$$

It can be seen that only the expression for the absolute acceleration yields a simple expression as in eq. (610). The expressions for velocity and displacement have an additional term and are of the form

$$\{\dot{x}_{abs}\} = -\int_{-\infty}^{t} [m]^{-\frac{1}{2}}[Q][h(t-\tau)][Q]^T[m]^{\frac{1}{2}}\{\ddot{y}(\tau)\}\,d\tau + \{\dot{y}(t)\} \quad , \qquad (613)$$

and

$$\{x_{abs}\} = -\int_{-\infty}^{t} [m]^{-\frac{1}{2}}[Q][h(t-\tau)][Q]^T[m]^{\frac{1}{2}}\{\ddot{y}(\tau)\}\,d\tau + \{y(t)\} \quad . \qquad (614)$$

2.2 IMPEDANCE METHOD

We may rewrite eq. (603) in a slightly different form as

$$\left\{\ddot{\xi}\right\} + \left[2\rho\omega\right]\left\{\dot{\xi}\right\} + \left[\omega^2\right]\left\{\xi\right\} = -\left[Q\right]^T\left[m\right]^{\frac{1}{2}}\left\{\ddot{y}\right\} \quad , \tag{615}$$

where

$$\left[2\rho\omega\right] = \begin{bmatrix} 2\rho_1\omega_1 & 0 & \cdots & 0 \\ 0 & 2\rho_2\omega_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 2\rho_N\omega_N \end{bmatrix}$$

Defining the Fourier transform as

$$\tilde{\xi}(\Delta) = \int_{-\infty}^{\infty} \xi(t)\, e^{-i\Delta t}\, dt \quad , \tag{616}$$

and the inverse Fourier transform as

$$\xi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\xi}(\Delta)\, e^{i\Delta t}\, d\Delta \quad , \tag{617}$$

we find that the Fourier transform of eq. (613) is

$$\left[Z\right]\left\{\tilde{\xi}\right\} = -\left[Q\right]^T\left[m\right]^{\frac{1}{2}}\left\{\tilde{\ddot{y}}\right\} \quad , \tag{618}$$

where

$$[Z] = \begin{bmatrix} -\Delta^2+\omega_1^2+2i\rho_1\omega_1\Delta & 0 & \cdots & 0 \\ 0 & -\Delta^2+\omega_2^2+2i\rho_2\omega_2\Delta & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & -\Delta^2+\omega_N^2+2i\rho_N\omega_N\Delta \end{bmatrix} .$$

Hence

$$\{\tilde{\xi}\} = - [Z]^{-1}[Q]^{T}[m]^{\frac{1}{2}}\{\tilde{\ddot{y}}\} ,$$

and

$$\{\xi\} = -\frac{1}{2\pi}\int_{-\infty}^{\infty}[Z]^{-1}[Q]^{T}[m]^{\frac{1}{2}}\{\tilde{\ddot{y}}\}\, e^{i\Delta t}\, d\Delta . \qquad (619)$$

Again using eqs. (597) and (600) we have

$$\{X_{abs}\} = -\frac{1}{2\pi}\int_{-\infty}^{\infty}[m]^{-\frac{1}{2}}[Q][Z]^{-1}[Q]^{T}[m]^{\frac{1}{2}}\{\tilde{\ddot{y}}\}\, e^{i\Delta t} d\Delta + \{y(t)\} , \qquad (620)$$

$$\{\dot{X}_{abs}\} = -\frac{i}{2\pi}\int_{-\infty}^{\infty}[m]^{-\frac{1}{2}}[Q][Z]^{-1}[Q]^{T}[m]^{\frac{1}{2}}\{\tilde{\ddot{y}}\}\,\Delta\, e^{i\Delta t} d\Delta + \{\dot{y}(t)\} , \qquad (621)$$

and

$$\{\ddot{X}_{abs}\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}[m]^{-\frac{1}{2}}[Q][Z]^{-1}[Q]^{T}[m]^{\frac{1}{2}}\{\ddot{y}\}\Delta^{2}e^{i\Delta t}d\Delta + \{\ddot{y}(t)\} . \qquad (622)$$

In this form of the solution none of the expressions for the absolute motion have the simple form of eq. (610). It is clear that the relative motion is found by omitting the last term in eqs. (620), (621), and (622).

3.0 Method of Foss

When normal modes in the Rayleigh sense do not exist but the damping is still proportional to the velocity, we may make use of Foss' method to uncouple the equations of motion.

3.1 IMPULSE METHOD

We rewrite eq. (103) as

$$[\Phi]^{T}[\mathcal{R}][\Phi]\{\dot{\xi}\} + [\Phi]^{T}[K][\Phi]\{\xi\} = [\Phi]^{T}\{F\} , \qquad (623)$$

where $[\Phi]^{T}[\mathcal{R}][\Phi]$ and $[\Phi]^{T}[K][\Phi]$ are diagonal matrices. This can easily be seen from the orthogonality relations of eqs. (98) and (99). Equation (623) represents a set of uncoupled first order differential equations, and we may write the solution for each ξ_n as

$$\xi_n = \int_0^t h_n(t-\tau)\, F_n(\tau)\, d\tau \quad , \tag{624}$$

where

$$h_n(t-\tau) = \frac{1}{\mathcal{R}_n}\, e^{\alpha_n (t-\tau)} \quad ,$$

$$\mathcal{R}_n = \left\{\Phi^n\right\}^T \left[\mathcal{R}\right] \left\{\Phi^n\right\} = 2\alpha_n \left\{\phi^n\right\}^T \left[m\right] \left\{\phi^n\right\} + \left\{\phi^n\right\}^T \left[r\right] \left\{\phi^n\right\} \quad ,$$

and

$$F_n = \left\{\Phi^n\right\}^T \left\{F\right\} = \left\{\phi^n\right\}^T \left\{f\right\} \quad .$$

Hence we may write

$$\left\{\xi\right\} = \int_0^t \left[h(t-\tau)\right] \left[\Phi\right]^T \left\{F(\tau)\right\} d\tau \quad , \tag{625}$$

where

$$\left[h(t-\tau)\right] = \begin{bmatrix} h_1(t-\tau) & 0 & \cdots & 0 \\ 0 & h_2(t-\tau) & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & h_{2N}(t-\tau) \end{bmatrix} \quad .$$

From eq. (100) we see that

$$\left\{x\right\} = \left[\Phi\right] \left\{\xi\right\} \quad . \tag{626}$$

Hence using eqs. (625) and (626) we find

$$\{X\} = \int_0^t [\Phi][h(t-\tau)][\Phi]^T\{F(\tau)\}\,d\tau \quad , \tag{627}$$

and

$$\{\dot{X}\} = \int_0^t [\Phi][\dot{h}(t-\tau)][\Phi]^T\{F(\tau)\}\,d\tau + [\Phi][h(0)][\Phi]^T\{F(t)\} \quad , \tag{628}$$

where

$$[h(0)] = \begin{bmatrix} \frac{1}{R_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{R_2} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \frac{1}{R_{2N}} \end{bmatrix} .$$

Expanding the last term of eq. (628) and using the relationship of eq. (112) we find

$$[\Phi][h(0)][\Phi]^T\{F(t)\} = \begin{Bmatrix} \sum_{n=1}^{2N} \frac{\alpha_n F_n}{R_n}\{\phi^n\} \\ \{0\} \end{Bmatrix} . \tag{629}$$

In a similar fashion we find

$$[\Phi][\dot{h}(t-\tau)][\Phi]^T\{F(\tau)\} = \begin{Bmatrix} \sum_{n=1}^{2N} \frac{\alpha_n^2 F_n}{R_n} e^{\alpha_n(t-\tau)} \{\phi^n\} \\ \sum_{n=1}^{2N} \frac{\alpha_n F_n}{R_n} e^{\alpha_n(t-\tau)} \{\phi^n\} \end{Bmatrix} . \tag{630}$$

Hence from eqs. (90) and (627) through (630) we have

$$\begin{Bmatrix} \{\ddot{q}\} \\ \{\dot{q}\} \end{Bmatrix} = \int_0^t d\tau \begin{Bmatrix} \sum_{n=1}^{2N} \frac{\alpha_n^2 F_n}{R_n} e^{\alpha_n(t-\tau)} \{\phi^n\} \\ \sum_{n=1}^{2N} \frac{\alpha_n F_n}{R_n} e^{\alpha_n(t-\tau)} \{\phi^n\} \end{Bmatrix} + \begin{Bmatrix} \sum_{n=1}^{2N} \frac{\alpha_n F_n}{R_n} \{\phi^n\} \\ \{0\} \end{Bmatrix} . \qquad (631)$$

If we multiply the expression for $\{\ddot{q}\}$ by $[m]$, use eq. (111), and note that $\sum_{n=1}^{2N} \frac{F_n}{R_n} [r] \{\phi^n\} = 0$ and $\{f\} = -[m]\{\ddot{y}\}$ we have

$$\{\ddot{q}_{abs}\} = \{\ddot{q}\} + \{\ddot{y}\} = \int_0^t \sum_{n=1}^{2N} \frac{\alpha_n^2 F_n}{R_n} e^{\alpha_n(t-\tau)} \{\phi^n\} d\tau . \qquad (632)$$

Here again we see that the absolute acceleration has a relatively simple expression. For the displacement and velocity we find

$$\{\dot{q}_{abs}\} = \int_0^t \sum_{n=1}^{2N} \frac{\alpha_n F_n}{R_n} e^{\alpha_n(t-\tau)} \{\phi^n\} d\tau + \{\dot{y}(t)\} , \qquad (633)$$

and

$$\{q_{abs}\} = \int_0^t \sum_{n=1}^{2N} \frac{F_n}{R_n} e^{\alpha_n(t-\tau)} \{\phi^n\} d\tau + \{y(t)\} . \tag{634}$$

3.2 IMPEDANCE METHOD

From eq. (623) we see that we have a series of uncoupled equations of the form

$$R_n \dot{\xi}_n - \alpha_n R_n \xi_n = F_n . \tag{635}$$

Taking the Fourier transform we have

$$(i\Delta - \alpha_n) R_n \tilde{\xi}_n = \tilde{F}_n . \tag{636}$$

In matrix notation this becomes

$$\left[(i\Delta - \alpha_n) R_n \right] \{\tilde{\xi}\} = [\Phi]^T \{\tilde{F}\} , \tag{637}$$

where

$$\left[(i\Delta - \alpha_n) R_n \right] = \begin{bmatrix} (i\Delta - \alpha_1) R_1 & 0 & \cdots & 0 \\ 0 & (i\Delta - \alpha_2) R_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & (i\Delta - \alpha_{2N}) R_{2N} \end{bmatrix} .$$

Hence

$$\left\{\tilde{\xi}\right\} = \left[(i\Delta-\alpha)\mathcal{R}\right]^{-1}\left[\Phi\right]^{T}\left\{\tilde{F}\right\} , \tag{638}$$

$$\left\{\xi\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[(i\Delta-\alpha)\mathcal{R}\right]^{-1}\left[\Phi\right]^{T}\left\{\tilde{F}\right\} e^{i\Delta t}\, d\Delta , \tag{639}$$

$$\left\{X\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\Phi\right]\left[(i\Delta-\alpha)\mathcal{R}\right]^{-1}\left[\Phi\right]^{T}\left\{\tilde{F}\right\} e^{i\Delta t}\, d\Delta , \tag{640}$$

and

$$\left\{\dot{X}\right\} = \frac{i}{2\pi}\int_{-\infty}^{\infty}\left[\Phi\right]\left[(i\Delta-\alpha)\mathcal{R}\right]^{-1}\left[\Phi\right]^{T}\left\{\tilde{F}\right\} \Delta\, e^{i\Delta t}\, d\Delta . \tag{641}$$

Expanding the quantity $\left[\Phi\right]\left[(i\Delta-\alpha)\mathcal{R}\right]^{-1}\left[\Phi\right]^{T}\left\{\tilde{F}\right\}$

we find

$$\left[\Phi\right]\left[(i\Delta-\alpha)\mathcal{R}\right]^{-1}\left[\Phi\right]^{T}\left\{\tilde{F}\right\} = \left\{\begin{array}{c} \displaystyle\sum_{n=1}^{2N}\frac{\alpha_n \tilde{F}_n}{\mathcal{R}_n(i\Delta-\alpha_n)}\left\{\phi^n\right\} \\ \\ \displaystyle\sum_{n=1}^{2N}\frac{\tilde{F}_n}{\mathcal{R}_n(i\Delta-\alpha_n)}\left\{\phi^n\right\} \end{array}\right\} \tag{642}$$

Hence we have

$$\left\{\ddot{q}_{abs}\right\} = \frac{i}{2\pi}\int_{-\infty}^{\infty}\sum_{n=1}^{2N}\frac{\alpha_n\,\tilde{F}_n}{R_n\,(i\Delta-\alpha_n)}\left\{\phi^n\right\}\Delta e^{i\Delta t}\,d\Delta \;+\; \left\{\ddot{y}(t)\right\}, \qquad (643)$$

$$\left\{\dot{q}_{abs}\right\} = \frac{i}{2\pi}\int_{-\infty}^{\infty}\sum_{n=1}^{2N}\frac{\tilde{F}_n}{R_n\,(i\Delta-\alpha_n)}\left\{\phi^n\right\}\Delta e^{i\Delta t}\,d\Delta \;+\; \left\{\dot{y}(t)\right\}, \qquad (644)$$

and

$$\left\{q_{abs}\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=1}^{2N}\frac{\tilde{F}_n}{R_n\,(i\Delta-\alpha_n)}\left\{\phi^n\right\} e^{i\Delta t}\,d\Delta \;+\; \left\{y(t)\right\}. \qquad (645)$$

If $F_n(t)$ is a reasonable function, the integrals of eqs. (643), (644), and (645) can be evaluated by contour integration.

These are the expressions for the various quantities of interest. They are quite complex; and if N is large, the computations are extremely tedious. If we consider the simple case where failure occurs when a particular value of some parameter is exceeded, and the statistics are Gaussian, then we need only compute the first and second moments for this describes the probability distribution completely. We can then easily predict the probability of exceeding the critical value for failure.

In this case we want the mean square value to be as small as practicable since the probability of exceeding some predetermined value of the parameter of interest is directly related to the second moment or mean squared value.

When a fatigue phenomenon is the failure criterion, we may use the concept of the white spectrum "fragility" curve to determine whether the system is well isolated. In this case the power spectrum of the response is required and we must make use of some of the expressions found for the impedance solution. Adjusting the values of the many variables that enter into the problem so as to keep the value of the response power spectrum below the white spectrum "fragility" curve for all frequencies, appears to be a hopeless analytical task when the number of degrees of freedom is large.

In the next section several expressions for mean squared values are determined which are useful for the case of Gaussian statistics with failure occurring when some critical value is exceeded.

4.0 Mean Squared Values for Multi-Degree-of-Freedom Systems

To simplify the computational work we will consider a simple two-degree-of-freedom system and determine the mean square acceleration and displacement. A simple coordinate transformation can be made so that the mean value is zero and hence only the second moment need be considered.

4.1 MEAN SQUARE DISPLACEMENT

From eq. (614) we see that

$$\{X\} = -\int_{-\infty}^{t} [m]^{-\frac{1}{2}} [Q] [h(t-\tau)] [Q]^{T} [m]^{\frac{1}{2}} \{\ddot{y}(\tau)\}\, d\tau \quad . \tag{646}$$

Performing the indicated matrix multiplication we find for the two-degree-of-freedom system that

$$X_1 = -\int_{-\infty}^{t} \frac{\bar{A}^2 h_1(t-\tau) + \bar{N}^2 h_2(t-\tau) + \sqrt{m_2/m_1}\,\bar{A}\bar{N}\,\left[h_1(t-\tau) - h_2(t-\tau)\right]}{\bar{A}^2 + \bar{N}^2}\,\ddot{y}(\tau)\, d\tau \ , \tag{647}$$

and

$$X_2 = -\int_{-\infty}^{t} \frac{\bar{A}^2 h_2(t-\tau) + \bar{N}^2 h_1(t-\tau) + \sqrt{m_1/m_2}\,\bar{A}\bar{N}\,\left[h_1(t-\tau) - h_2(t-\tau)\right]}{\bar{A}^2 + \bar{N}^2}\,\ddot{y}(\tau)\, d\tau \ , \tag{648}$$

where the various quantities are defined as before. If we consider m_2 to represent an instrument package mounted on a rocket sled, then $\langle X_2^2 \rangle$ may be of interest. We would then have for the case of a "white" acceleration power spectrum

$$\langle X_2^2 \rangle = 2D \int_{-\infty}^{t} \left[\frac{\left(\bar{N}^2 + \sqrt{m_1/m_2}\,\bar{A}\bar{N}\right)^2 h_1^2(t-\tau) + 2\left(\bar{N}^2 + \sqrt{m_1/m_2}\,\bar{A}\bar{N}\right)\left(\bar{A}^2 - \sqrt{m_1/m_2}\,\bar{A}\bar{N}\right) h_1(t-\tau)\, h_2(t-\tau)}{\left(\bar{A}^2 + \bar{N}^2\right)^2} \right.$$

$$\left. + \frac{\left(\bar{A}^2 - \sqrt{m_1/m_2}\,\bar{A}\bar{N}\right)^2 h_2^2(t-\tau)}{\left(\bar{A}^2 + \bar{N}^2\right)^2} \right] d\tau \quad . \tag{649}$$

The integrations are straightforward but lengthy and so only the result will be given. We find that

$$\langle x_2^2 \rangle = \frac{2D}{\left(\frac{k_1+k_2}{m_1}+\frac{k_2}{m_2}\right)^2-\frac{4k_1k_2}{m_1m_2}}\left\{\frac{\rho_2\omega_2^7+\rho_1\omega_1^7}{4\rho_1\rho_2\omega_1^3\omega_2^3}-\frac{4\left(\rho_1\omega_1^3\omega_2^2+\rho_2\omega_1^2\omega_2^3\right)}{(\omega_1^2-\omega_2^2)^2+4\omega_1\omega_2(\rho_1\omega_1+\rho_2\omega_2)(\rho_1\omega_2+\rho_2\omega_1)}\right\}, \tag{650}$$

where

$$\omega_1^2 = \frac{1}{2}\left(\frac{k_1+k_2}{m_1}+\frac{k_2}{m_2}\right)-\sqrt{\frac{1}{4}\left(\frac{k_1+k_2}{m_1}+\frac{k_2}{m_2}\right)^2-\frac{k_1k_2}{m_1m_2}}\,, \tag{651}$$

and

$$\omega_2^2 = \frac{1}{2}\left(\frac{k_1+k_2}{m_1}+\frac{k_2}{m_2}\right)+\sqrt{\frac{1}{4}\left(\frac{k_1+k_2}{m_1}+\frac{k_2}{m_2}\right)^2-\frac{k_1k_2}{m_1m_2}}\,. \tag{652}$$

If the primary system parameters m_1, k_1 and c_1 are fixed, then we would be left with three variables m_2, k_2 and c_2 which could be changed so as to decrease $\langle x_2^2 \rangle$. It is clear that even in this special case the process would be quite tedious to carry out analytically. It would be simpler to set up an electric analog of the system and vary the parameters until a satisfactory system is found.

4.2 MEAN SQUARE ACCELERATION

Proceeding as in Section 4.1, we find that

$$\langle \ddot{X}_2^2 \rangle = 2D \int_{-\infty}^{t} \frac{\left[\left(\bar{N}^2 + \sqrt{m_1/m_2}\,\bar{A}\bar{N}\right)^2 \ddot{h}_1^2(t-\tau) + 2\left(\bar{N}^2 + \sqrt{m_1/m_2}\,\bar{A}\bar{N}\right)\left(\bar{A}^2 - \sqrt{m_1/m_2}\,\bar{A}\bar{N}\right)\ddot{h}_1(t-\tau)\ddot{h}_2(t-\tau)\right.}{\left(\bar{A}^2 + \bar{N}^2\right)^2}$$

$$\left. + \frac{\left(\bar{A}^2 - \sqrt{m_1/m_2}\,\bar{A}\bar{N}\right)^2 \ddot{h}_2^2(t-\tau)}{\left(\bar{A}^2 + \bar{N}^2\right)^2}\right] d\tau \quad . \quad (653)$$

Carrying out the integrations the result is

$$\langle \ddot{X}_2^2 \rangle = \frac{2D}{\left(\frac{k_1+k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - \frac{4k_1k_2}{m_1m_2}} \left[-\left(\frac{k_2}{m_2} - \omega_2^2\right)\left(\frac{k_2(m_1+m_2)}{m_2^2} + \frac{m_2\omega_2^2 - m_1\omega_1^2}{m_2}\right)\left(\frac{\omega_1(1+4\rho_1^2}{4\rho_1}\right)\right.$$

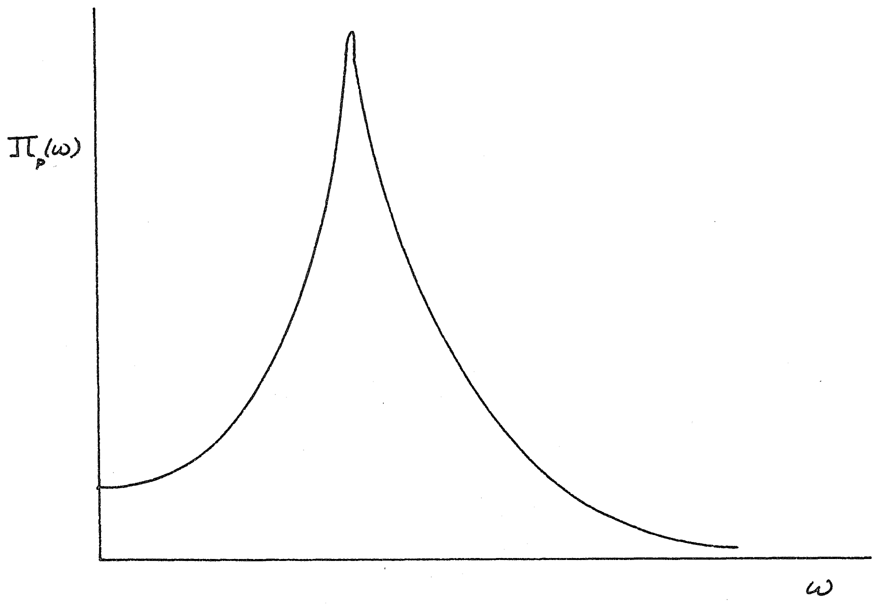
$$\left. -\left(\frac{k_2}{m_2} - \omega_1^2\right)\left(\frac{k_2(m_1+m_2)}{m_2^2} + \frac{m_2\omega_1^2 - m_1\omega_2^2}{m_2}\right)\left(\frac{\omega_2(1+4\rho_2^2)}{4\rho_2}\right) - \frac{2k_1k_2}{m_1m_2}\left(\frac{2\rho_1\omega_1\omega_2^2(\omega_2^2 + 4\rho_2^2\omega_1^2) + 2\rho_2\omega_1^2\omega_2(\omega_1^2 + 4\rho_1^2\omega_2^2)}{(\omega_1^2 + \omega_2^2 + 2\rho_1\rho_2\omega_1\omega_2)^2 - 4\omega_1^2\omega_2^2(1-\rho_1^2)(1-\rho_2^2)}\right)\right] . \quad (654)$$

If we consider the case where the parameters of the primary system are fixed and those of the secondary system may be varied, we may easily deduce some qualitative points of interest.

By neglecting the interaction between the primary and secondary systems, which is a reasonable assumption if $m_1 >> m_2$, we may consider the secondary system to be on a vibrating platform represented by the primary mass. The power spectrum of the motion of the primary mass can be found by means of the relation

$$\Pi_p(\omega) = \frac{\Pi_{IN}(\omega)}{\left| Z(\omega) \right|^2} , \tag{655}$$

where $Z(\omega)$ is the impedance of the primary system. A typical curve for $\Pi_p(\omega)$ is shown below.



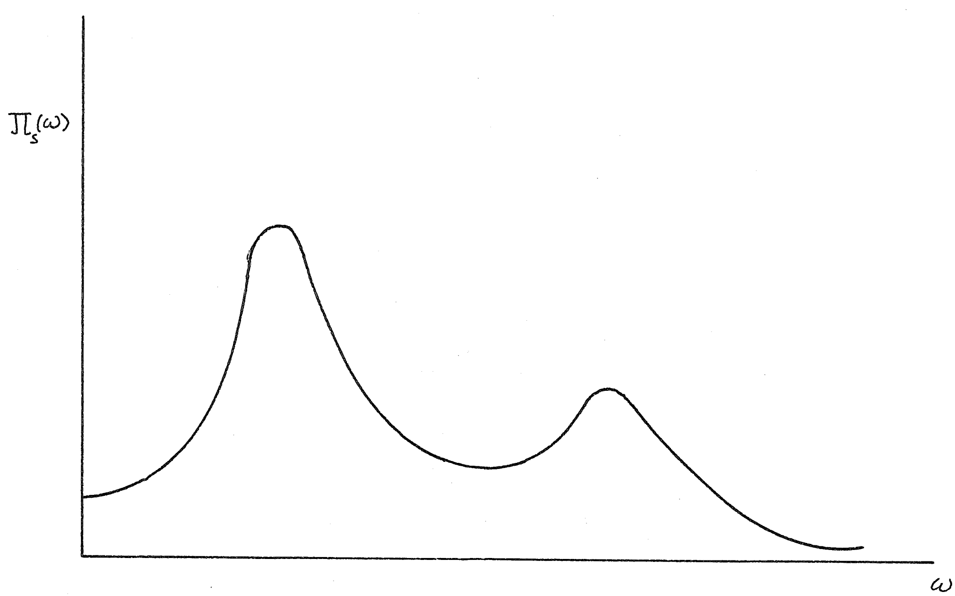
POWER SPECTRUM OF DISPLACEMENT OF PRIMARY MASS

SKETCH 24

The power spectrum for the secondary system motion would also be similar to that shown in the previous sketch. When the interaction is neglected, then the ratio of the acceleration of m_2 to that of the base is proportional to the product of the power spectra, that is

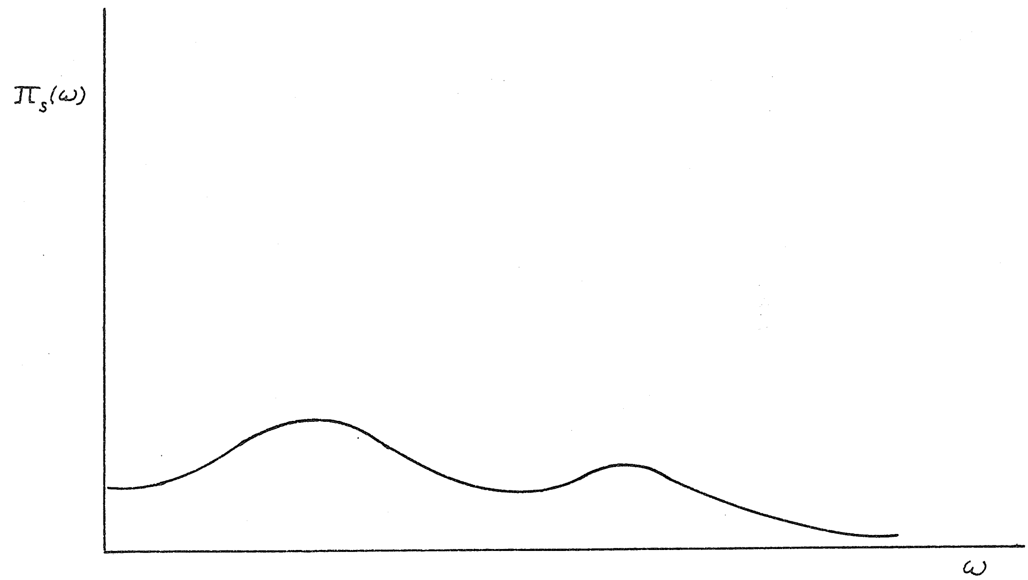
$$\frac{\ddot{X}_2}{\ddot{y}} = \Pi_p(\omega)\,\Pi_s(\omega) \tag{655}$$

If the resonant frequency of the primary structure is less than that of the secondary system, the primary system resonant peak is reproduced almost unchanged; and it is little effected by changes in the secondary system. This may lead to large values for $\ddot{X}_2$. By making the resonant frequency of the secondary system appreciably less than that of the primary system, we can decrease this resonant peak by making changes in the parameters of the secondary system. These results are shown in Sketches 25 and 26.



POWER SPECTRUM OF DISPLACEMENT OF SECONDARY MASS ($\omega_p << \omega_s$)

SKETCH 25



POWER SPECTRUM OF DISPLACEMENT OF SECONDARY MASS ($\omega_p >> \omega_s$)

SKETCH 26

Hence for the case of a two-degree-of-freedom system with fixed primary parameters isolation of the secondary system is best carried out if the natural frequency of the secondary system is made considerably less than that of the primary system, that is $\omega_p >> \omega_s$.

5.0 Response Power Spectra for Multi-Degree-of-Freedom Systems

To make use of the white spectrum "fragility" curve concept we must compute the power spectrum of the response for a multi-degree-of-freedom system. Using the definition of the power spectrum from eq. (189) along with the results of sections 2.2 and 3.2, we may compute the required quantities.

5.1 SYSTEMS POSSESSING NORMAL MODES

We will consider systems which possess normal modes and compute the power spectrum of the displacement response. From eq. (618) we see that

$$\{\tilde{\xi}\} = -[Z]^{-1}[Q]^{T}[m]^{\frac{1}{2}}\{\ddot{\tilde{y}}\} . \tag{656}$$

Hence

$$\{\tilde{x}\} = -[m]^{-\frac{1}{2}}[Q][Z]^{-1}[Q]^{T}[m]^{\frac{1}{2}}\{\ddot{\tilde{y}}\} . \tag{657}$$

Performing the indicated matrix multiplication, we have

$$\{\tilde{x}\} = -\left\{\begin{array}{c} \sum_{i=1}^{N}\sum_{j=1}^{N} A_1^{(i)} A_j^{(i)} \dfrac{\sqrt{m_j}\;\ddot{\tilde{y}}}{Z_i\;\sqrt{m_1}} \\ \vdots \\ \sum_{i=1}^{N}\sum_{j=1}^{N} A_N^{(i)} A_j^{(i)} \dfrac{\sqrt{m_j}\;\ddot{\tilde{y}}}{Z_i\;\sqrt{m_N}} \end{array}\right\} . \tag{658}$$

Hence

$$\tilde{x}_k = -\sum_{i=1}^{N}\sum_{j=1}^{N} A_k^{(i)} A_j^{(i)} \frac{\sqrt{m_j}\;\ddot{\tilde{y}}}{Z_i\sqrt{m_k}} \tag{659}$$

Since all terms in eq. (659) are real except Z_i and $\ddot{\tilde{y}}$, we find

$$\tilde{x}_k^{*} = -\sum_{\ell=1}^{N}\sum_{n=1}^{N} A_k^{(\ell)} A_n^{(\ell)} \frac{\sqrt{m_n}\;\ddot{\tilde{y}}^{*}}{Z_\ell^{*}\sqrt{m_k}} \tag{660}$$

From the definition of the power spectrum, we find

$$\Pi_k(\omega) = \lim_{T\to\infty} \frac{1}{T} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{\ell=1}^{N} \sum_{n=1}^{N} \frac{A_k^{(i)} A_j^{(i)} A_k^{(\ell)} A_n^{(\ell)} \sqrt{m_j \; m_n}}{m_k \, Z_i \, Z_\ell^*} \ddot{y}_T \, \ddot{y}_T^* \quad . \tag{661}$$

Hence

$$\Pi_k(\omega) = \sum_{i=1}^{N} \sum_{\ell=1}^{N} \frac{\Pi_{IN}(\omega)}{Z_i \, Z_\ell^*} \sum_{j=1}^{N} \sum_{n=1}^{N} A_k^{(i)} A_j^{(i)} A_k^{(\ell)} A_n^{(\ell)} \frac{\sqrt{m_j \cdot m_n}}{m_k} \quad . \tag{662}$$

The criterion for acceptable isolation would then be

$$\Pi_k(\omega) < \Pi_f \qquad \text{for all } \omega \quad , \tag{663}$$

where Π_f is the white spectrum "fragility" curve and $\Pi_k(\omega)$ is the power spectrum of the displacement response of the k^{th} mass. When the number of degrees of freedom exceeds three, there is usually no explicit expression for the eigenvectors or eigenvalues so that satisfying the condition given by eq. (663) when $N > 3$ and the input power spectrum $\Pi_{IN}(\omega)$ is some general function would involve tedious numerical trial and error solutions. The problem can be somewhat simplified by assuming that the input power spectrum is a white spectrum whose spectral density is equal to the maximum spectral density of the actual input spectrum $\Pi_{IN}(\omega_{MAX.})$. This is certainly a conservative estimate. We may then determine whether the isolation is acceptable by evaluating $\Pi_k(\omega) / \Pi_{IN}(\omega_{MAX.})$ at the various resonant frequencies ω_i .

If the largest of these values is less than $\Pi_f / \Pi_{IN}(\omega_{MAX.})$ then the system is adequately isolated.

For example, in the case of a two-degree-of-freedom system where the response of the second mass is of interest eq. (662) would become:

$$\Pi_2(\omega) = \Pi_{IN}(\omega) \frac{A_2^{(1)} A_2^{(2)}}{m_2} \left(A_1^{(1)}\sqrt{m_1} + A_2^{(1)}\sqrt{m_2}\right)\left(A_1^{(2)}\sqrt{m_1} + A_2^{(2)}\sqrt{m_2}\right)$$

$$\cdot \left\{ \frac{A_2^{(1)}\left(A_1^{(1)}\sqrt{m_1} + A_2^{(1)}\sqrt{m_2}\right)}{A_2^{(2)}\left(A_1^{(2)}\sqrt{m_1} + A_2^{(2)}\sqrt{m_2}\right) Z_1 Z_1^*} + \frac{A_2^{(2)}\left(A_1^{(2)}\sqrt{m_1} + A_2^{(2)}\sqrt{m_2}\right)}{A_2^{(1)}\left(A_1^{(1)}\sqrt{m_1} + A_2^{(1)}\sqrt{m_2}\right) Z_2 Z_2^*} + \frac{Z_1^* Z_2 + Z_1 Z_2^*}{Z_1 Z_1^* Z_2 Z_2^*} \right\} . \quad (664)$$

We know that the impedance of the k^{th} mode is given by

$$Z_k = \omega_k^2 - \omega^2 + 2i\rho_k\omega_k\omega \quad . \quad (665)$$

Using eq. (665) we find

$$Z_1 Z_1^* = (\omega_1^2 - \omega^2)^2 + 4\rho_1^2\omega_1^2\omega^2 \quad , \quad (666)$$

and

$$Z_2 Z_2^* = (\omega_2^2 - \omega^2)^2 + 4\rho_2^2\omega_2^2\omega^2 \quad . \quad (667)$$

Substituting eqs. (666) and (667) into eq. (664), using the expressions for the $A_i^{(r)}$ and assuming that the resonance at ω_1 is larger than at

ω_2 , we find

$$\frac{\Pi_2(\omega_1)}{\Pi_{IN}(\omega_{MAX})} = \left(\frac{k_2}{m_1 m_2}\right)^2 \frac{1}{\bar{\Delta}_1 \bar{\Delta}_2}\left[k_2 + m_1\left(\frac{k_2}{m_2} - \omega_1^2\right)\right]\left[k_2 + m_1\left(\frac{k_2}{m_2} - \omega_2^2\right)\right]$$

$$\cdot \left\{ \frac{\left[k_2 + m_1\left(\frac{k_2}{m_2} - \omega_1^2\right)\right]\bar{\Delta}_2}{\left[k_2 + m_1\left(\frac{k_2}{m_2} - \omega_2^2\right)\right]\bar{\Delta}_1 (4\rho_1^2\omega_1^4)} + \frac{\left[k_2 + m_1\left(\frac{k_2}{m_2} - \omega_2^2\right)\right]\bar{\Delta}_1}{\left[k_2 + m_1\left(\frac{k_2}{m_2} - \omega_1^2\right)\right]\bar{\Delta}_2\left[(\omega_2^2 - \omega_1^2) + 4\rho_2^2\omega_2^2\omega_1^2\right]} + \frac{2\rho_2\omega_2}{\rho_1\omega_1\left[(\omega_2^2 - \omega_1^2)^2 + 4\rho_2^2\omega_2^2\omega_1^2\right]} \right\}, \quad (668)$$

where the $\bar{\Delta}_i$ are defined after eq. (605). If this expression is less than $\Pi_f / \Pi_{IN}(\omega_{MAX})$ we would have acceptable isolation. If this is not the case, then the parameters m_2, c_2, and k_2 must be varied until this condition is met.

To find the velocity and acceleration power spectra we would merely multiply eq. (662) by ω^2 or ω^4 respectively and proceed as before. The corresponding expressions for the absolute quantities are more complicated because cross-product terms occur due to the present of the various time derivatives of the base motion.

5.2 SYSTEMS WITH GENERAL VISCOUS DAMPING

From eq. (645) and the definition of the Fourier transform we find that

$$\tilde{q}_k = \sum_{n=1}^{2N} \frac{\tilde{F}_n \phi_k^n}{R_n (i\Delta - \alpha_n)} , \quad (669)$$

and

$$\tilde{q}_k^* = \sum_{m=1}^{2N} \frac{F_m^* \, \phi_k^{m*}}{R_m^*(i\Delta - \alpha_m)^*} \quad . \tag{670}$$

Hence the power spectrum of the displacement of the k^{th} mass is given by

$$\Pi_k(\omega) = \lim_{T\to\infty} \frac{1}{T} \sum_{n=1}^{2N} \sum_{m=1}^{2N} \frac{\phi_k^n \, \phi_k^{m*} \, \tilde{F}_n \, \tilde{F}_m^*}{R_n \, R_m^* \, (i\Delta - \alpha_n)(i\Delta - \alpha_m)^*} \quad . \tag{671}$$

From eq. (105) we see that

$$\tilde{F}_n = \{\phi^n\}^T \{\tilde{f}\} = \{\phi^n\}^T [m] \{\ddot{\tilde{y}}\} = \sum_{i=1}^{2N} \phi_i^n \, m_i \, \ddot{\tilde{y}} \quad . \tag{672}$$

Hence eq. (671) becomes

$$\Pi_k(\omega) = \sum_{n=1}^{2N} \sum_{m=1}^{2N} \sum_{i=1}^{2N} \sum_{j=1}^{2N} \frac{\phi_k^n \, \phi_k^{m*} \, \phi_i^n \, \phi_j^{n*} \, m_i \, m_j}{R_n \, R_m^* (i\Delta - \alpha_n)(i\Delta - \alpha_m)^*} \, \Pi_{IN}(\omega) \quad . \tag{673}$$

This is the formal expression for the power spectrum of the k^{th} mass. There is little point in expanding the expression even for simple two-degree-of-freedom systems since the evaluation of the various α_i, R_i, and ϕ_i^j is tedious.

6.0 Summary of Results

From the previous results it is evident that the problem of isolation under random inputs is quite complex even for very simple systems. One of the first difficulties arising is the establishment of the criterion for adequate isolation. If a failure occurs when certain values of displacement, velocity, or acceleration are exceeded, then the system must be designed so that the probability of this occurrence is very small. When the input statistics are Gaussian, the problem then reduces to that of making the mean square value of interest as small as practicable, since the probability of exceeding a given value decreases as the second moment decreases. If the input statistics are not Gaussian, then little is known about the output probability distribution and no simple relation exists between the second moment and the probability of exceeding certain values of the parameter of interest.

When a fatigue phenomenon is the criterion for failure, one means of determining whether isolation is adequate is to construct white spectrum "fragility" curves for the components under study. These are essentially the white power spectra which cause failure within a prescribed period of time. The system must then be designed so that the power spectrum of the vibration environment at the point of interest is at all frequencies less than the white spectrum "fragility" curve. To simplify the computational work, the input power spectrum can be replaced by a white spectrum whose spectral density is that of the maximum of the input spectrum. We then need only evaluate $\Pi_k(\omega) / \Pi_{IN}(\omega_{MAX.})$ at the various resonant frequencies. If the largest of these values is below the value of the white spectrum "fragility" curve, then isolation is adequate.

The solution of the isolation problem by analytic means appears to be quite tedious and can probably best be handled by analog techniques using the criterion discussed above.

PART V

CONCLUSIONS

It is now possible to arrive at certain conclusions as a result of the analyses carried out in the main body of the thesis. These will be discussed in the next section.

1.0 Conclusions

There are essentially two ways of treating problems in random vibrations, the Fokker-Planck and Fourier Series methods, and each is restricted to certain types of problems.

The Fokker-Planck method determines the probability distribution of the random variable as the fundamental solution of a partial differential equation of the diffusion type. The procedure is best suited to inputs possessing a flat power spectrum in which case the output is always Gaussianly distributed. When the input power spectrum is not flat, the corresponding Fokker-Planck equation generally cannot be solved. The input distribution function need not be Gaussian in this method.

In the Fourier Series method the variable of interest is expanded in a Fourier Series in which the coefficients are random variables. The Weiner-Khintchine theorem relating the power spectrum to the autocorrelation is of fundamental importance to the method. The major restriction here is that the input must have a Gaussian distribution, for then at any instant of time the Fourier Series represents an N-dimensional Gaussian distribution and the output distribution function is uniquely determined. If the input is not Gaussian, the method can only approximate the output distribution function by the laborious technique of computing a large number of the moments of the distribution. In this method the power spectrum need not be flat.

Unfortunately in most practical problems we do not deal with Gaussian inputs or flat power spectra. It is then generally not possible to determine the output probability distribution and we must be content with finding several mean square values of interest. This is a straightforward procedure in most cases if use is made of the inpulse method. Of course, this yields no information about the distribution functions unless we are dealing with Gaussian inputs.

The two-degree-of-freedom system analyzed in Part II indicates that the output of a damped, linear system, excited by a non-stationary input which exponentially approaches a stationary input, will also approach a steady state value for the mean square displacement in the same exponential manner as the force except that sinusoidal variations will be superimposed upon it. The more quickly the force reaches a steady value and the more damping the system possesses, the more quickly the output reaches equilibrium.

In Part III we see that a criterion for fatigue failure in multi-degree-of-freedom systems can be obtained by assuming that the stress reversals at high stress levels contribute most to fatigue failure and by considering the number of zero crossings per second as a measure of the frequency of stress reversal. Miles' method for computing the mean square stress level can then be used in conjunction with Miner's cumulative damage hypothesis to predict the time for failure.

For a shear building possessing normal modes and subjected to random ground motion the cross-product terms may be neglected in computing mean square values, provided the difference in the square of any two natural frequencies is small compared to the square of the frequency at which the power spectrum of the ground acceleration peaks, and provided the natural frequencies are well separated.

The divergent result for the bending moment of a Bernoulli-Euler beam subjected to transverse random loading possessing a white power spectrum can be eliminated by using a finite cutoff frequency for the spectrum instead of using the more elaborate Timoshenko beam equation.

The transverse displacement of a beam subjected to random end motion may be expanded in a series of normal modes and a one-term approximation to the mean square bending moment obtained, provided the correlation time of the random end motion is large compared to the fundamental frequency of the beam.

In the problem of vibration isolation under random excitation, two criteria for determining the adequacy of the vibration mounting may be used depending upon whether the system to be isolated will fail because some parameter, say acceleration, velocity, or displacement, exceeds a critical value or because the system fails in fatigue. In the first case if the input is Gaussian, the criterion is to make the mean square value of the critical parameter as small as is practical since the probability of exceeding a given value is proportional to the root mean square. When fatigue is the cause of failure, the concept of the "white spectrum fragility curve" may be used. This curve is found by subjecting the particular piece of equipment in question to a vibration test in which the exciting force is random with a white power spectrum. The spectral density of the excitation parameter of interest, say acceleration, velocity, or displacement, is increased in successive tests until the time for failure reaches the minimum required time of operation. The corresponding white spectrum is the "white spectrum fragility curve." The criterion for proper isolation is that the power spectrum of the critical parameter at the point where the equipment is located must not exceed the "white spectrum fragility curve" at any frequency.

For the problem of the motion of a single-degree-of-freedom system over a rough surface the most important parameter is the autocorrelation of the surface profile. This quantity must be known before any numerical results can be obtained. All of the quantities appearing in the autocorrelation of the input forces can be related to derivatives of the autocorrelation of the surface profile. The exciting forces are seen to be non-stationary unless the acceleration of the system is zero. Mean square values for the displacement and velocity of the mass were developed for a particular surface profile. If the distribution function of the surface height is Gaussian, the output of the system is Gaussian and the mean square values will determine the output probability distribution uniquely. The double integrals for the various mean square values generally cannot be integrated in closed form so approximate techniques are required. Machine calculations yield a quick method for small values of time and one particular integral has been evaluated by means of a digital computer. For large values of time machine computations are impractical, since the integrand behaves like a delta function; and it must be evaluated at a large number of points if numerical accuracy is to be attained. This results in slow convergence and a great deal of machine time. Laplace's method is quite applicable here and an approximate solution has been obtained which agrees quite well with the available machine calculations. In addition it becomes increasingly more accurate for larger values of time.

For the particular autocorrelation chosen for the surface profile it is seen that for constant system velocity the mean square displacement of the mass is bounded and approaches zero as the velocity becomes very large. If the system moves with constant acceleration the mean square displacement of the mass grows without bound as time increases. These results are, of course, a consequence of the particular autocorrelation chosen for the surface profile and cannot be construed as a general result.

It can be seen that the term causing the solution to diverge for the case of constant system acceleration results from the fact that the damping force is proportional to the relative velocity between the system mass and the surface. If some form of damping existed so that the damping forces were proportional to the absolute velocity of the mass, for example air damping, then this term would not appear and the solution would converge.

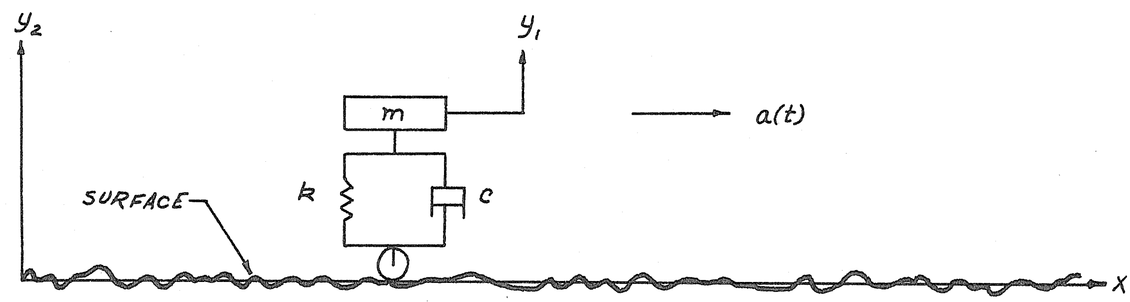
APPENDIX I

Motion Over a Rough Surface

An interesting problem arises when the motion of a mechanical system over a rough surface is considered. To simplify the analysis the system shall be represented by a linear, damped, single-degree-of-freedom oscillator and motion of the system over the surface shall be restricted to one dimension.

1.0 Development of Equations for the Mean Square Responses

Consider a single-degree-of-freedom system accelerating over a rough surface as shown below



SINGLE-DEGREE-OF-FREEDOM SYSTEM
ACCELERATING OVER A ROUGH SURFACE

SKETCH I-1

The equation of motion for the system is

$$m\ddot{y}_1 + c(\dot{y}_1 - \dot{y}_2) + k(y_1 - y_2) = 0 \; . \tag{I-1}$$

We may wish to determine the tension or compression in the spring in which case the important variable is $z = (y_1 - y_2)$ and eq. (I-1) becomes

$$\ddot{z} + 2\rho\omega_0 \dot{z} + \omega_0^2 z = -\ddot{y}_2 \quad . \tag{I-2}$$

If we wish to find the absolute displacement of the mass, the important variable is y_1 and eq. (I-1) becomes

$$\ddot{y}_1 + 2\rho\omega_0 \dot{y}_1 + \omega_0^2 y_1 = 2\rho\omega_0 \dot{y}_2 + \omega_0^2 y_2 \quad . \tag{I-3}$$

We must now find expressions for $\ddot{y}_2$, $\dot{y}_2$, and y_2 in terms of the quantities which are known for the surface. If we consider the surface to be a stationary random process, we will assume that we know the autocorrelation

$$\psi(\Delta X) = \langle y_2(X)\, y_2(X^+) \rangle = \langle y_2(X)\, y_2(X+\Delta X) \rangle \quad , \tag{I-4}$$

where

$$\left.\begin{aligned} y_2 &= g_i(X) \;, \\ X &= g_j(t) \;, \\ \Delta X &= X^+ - X \;. \end{aligned}\right\} \tag{I-5}$$

and

Equations (I-5) express the fact that the surface height y_2 is a function of position X ; and since the system is accelerating along the surface, its position X will depend upon the time t .

It is easy to show that

$$\dot{y}_2 = V(t)\, y_2' \quad , \tag{I-6}$$

where

$$(\;') = \frac{d}{dX} \quad ,$$

$$(\;'') = \frac{d^2}{dX^2} \quad ,$$

and

$V(t)$ is the velocity of the system at time t ,

and that

$$\ddot{y}_2 = a(t)y_2' + V^2(t)y_2'' \quad , \tag{I-7}$$

where

$a(t)$ is the acceleration of the system at time t .

Substituting eqs. (I-6) and (I-7) into the equations of motion, (eqs. (I-2) and (I-3)), we see that they become

$$\ddot{z} + 2\rho\omega_0\dot{z} + \omega_0^2 z = -\left(a(t)y_2' + V^2(t)y_2''\right), \tag{I-8}$$

and

$$\ddot{y}_1 + 2\rho\omega_0\dot{y}_1 + \omega_0^2 y_1 = 2\rho\omega_0\left(V(t)\,y_2'\right) + \omega_0^2 y_2 \quad . \tag{I-9}$$

We will now develop the mean square expressions for the case where the parameter of interest is z , the relative motion of the mass.

1.1 RELATIVE MOTION OF THE MASS

From the results of eq. (161) we see that the mean square displacement $\langle z^2(t)\rangle$, and mean square velocity $\langle \dot{z}^2(t)\rangle$ may be expressed as

$$\langle z^2(t)\rangle = \int_0^t\int_0^t h(t-\tau)h(t-\tau^+)\,\langle R(\tau)R(\tau^+)\rangle\; d\tau\, d\tau^+ , \tag{I-10}$$

and

$$\langle \dot{z}^2(t)\rangle = \int_0^t\int_0^t \dot{h}(t-\tau)\,\dot{h}(t-\tau^+)\,\langle R(\tau)R(\tau^+)\rangle\, d\tau\; d\tau^+ , \tag{I-11}$$

where

$$R(\tau) = -\ddot{y}_2(\tau) = -\left(a(\tau)y_2'(x) + V^2(\tau)y_2''(x)\right) .$$

We then see that

$$R(\tau)R(\tau^+) = \left(-a(\tau)y_2'(x) - V^2(\tau)y_2''(x)\right)\left(-a(\tau^+)y_2'(x^+) - V^2(\tau^+)y_2''(x^+)\right)$$

$$= \left(a(\tau)y_2'(x)\,a(\tau^+)y_2'(x^+) + V^2(\tau)y_2''(x)V^2(\tau^+)y_2''(x^+)\right. \tag{I-12}$$

$$\left. + a(\tau)y_2'(x)V^2(\tau^+)y_2''(x^+) + V^2(\tau)y_2''(x)a(\tau^+)y_2'(x^+)\right) .$$

We next take an ensemble average of both sides of eq. (I-12) and note that since $a(\tau)$ and $V(\tau)$ are deterministic functions of τ they will be constants in the ensemble average. Let us also use the notation that the first term in the brackets $\langle \quad \rangle$ will refer to a function of x and the second to a function of x^+. In addition, the first term outside the bracket $\langle \quad \rangle$ refers to a function of τ and the second to a function of τ^+. Hence $aV^2\langle y_2' y_2''\rangle$ would denote $a(\tau)V^2(\tau^+)\langle y_2'(x)y_2''(x^+)\rangle$. With this notation, eq. (I-12) becomes

$$\langle R(\tau)R(\tau^+)\rangle = aa\langle y_2'y_2'\rangle + V^2V^2\langle y_2''y_2''\rangle + aV^2\langle y_2'y_2''\rangle + V^2a\langle y_2''y_2'\rangle . \tag{I-13}$$

We must now relate

$\langle y_2'y_2'\rangle$, $\langle y_2''y_2''\rangle$, $\langle y_2'y_2''\rangle$, and $\langle y_2''y_2'\rangle$ to $\langle y_2 y_2\rangle$.

Since the surface is assumed to be a stationary process, averages are not affected by translations in the variable and we may write

$$\langle y_2 y_2\rangle = \langle y_2(x)y_2(x^+)\rangle = \langle y_2(x)y_2(x+\Delta x)\rangle = \langle y_2(x-\Delta x)y_2(x)\rangle = \psi(\Delta x). \tag{I-14}$$

We will also consider each product $y_2(x)\,y_2(x+\Delta x)$ and $y_2(x-\Delta x)\,y_2(x)$ to be a function of Δx only so that we may write

$$\frac{d}{d(\Delta x)}\langle y_2(x)y_2(x^+)\rangle = \left\langle \frac{d}{d(\Delta x)}\left(y_2(x)y_2(x^+)\right)\right\rangle$$

$$= \left\langle \frac{dx^+}{d(\Delta x)}\,\frac{d}{dx^+}\left(y_2(x)y_2(x^+)\right)\right\rangle = \langle y_2(x)y_2'(x^+)\rangle\ , \qquad \text{(I-15)}$$

and

$$\frac{d}{d(\Delta x)}\langle y_2(x)y_2(x^+)\rangle = \left\langle \frac{dx}{d(\Delta x)}\,\frac{d}{dx}\left(y_2(x)y_2(x^+)\right)\right\rangle = -\langle y_2'(x)y_2(x^+)\rangle\ . \qquad \text{(I-16)}$$

Hence, omitting the variables x^+ and x and using the notation discussed previously we have

$$\frac{d}{d(\Delta x)}\langle y_2y_2\rangle = \langle y_2y_2'\rangle = -\langle y_2'y_2\rangle\ . \qquad \text{(I-17)}$$

In a similar way we find

$$\frac{d^2}{d(\Delta x)^2}\langle y_2y_2\rangle = \langle y_2y_2''\rangle = \langle y_2''y_2\rangle = -\langle y_2'y_2'\rangle\ , \qquad \text{(I-18)}$$

$$\frac{d^3}{d(\Delta x)^3}\langle y_2y_2\rangle = \langle y_2y_2'''\rangle = -\langle y_2'''y_2\rangle = \langle y_2''y_2'\rangle = -\langle y_2'y_2''\rangle\ , \qquad \text{(I-19)}$$

and

$$\frac{d^4}{d(\Delta x)^4}\langle y_2y_2\rangle = \langle y_2y_2^{IV}\rangle = \langle y_2^{IV}y_2\rangle = -\langle y_2'y_2'''\rangle = -\langle y_2'''y_2'\rangle = \langle y_2''y_2''\rangle\ . \qquad \text{(I-20)}$$

Equations (I-17) through (I-20) relate the various correlations appearing in eq. (I-13) to derivatives of the autocorrelation of the surface with respect to ΔX, and these are the required relations.

We may now write eq. (I-13) as

$$\langle R(\tau)R(\tau^{+})\rangle = -aa\psi^{II} + V^2V^2\psi^{IV} - aV^2\psi^{III} + V^2a\psi^{III}, \tag{I-21}$$

where

$$\psi^{I} = \frac{d\psi}{d(\Delta x)},$$

$$\psi^{II} = \frac{d^2\psi}{d(\Delta x)^2} \quad \text{etc.}$$

In order to be able to treat a specific example, we must now consider the form of $\psi(\Delta X)$. Since the autocorrelation is to be differentiated several times, we must obviously choose $\psi(\Delta X)$ so that it is at least four times differentiable. For the purely random process we would have $\psi(\Delta X) = \delta(\Delta X)$ which immediately leads to difficulty when we try to differentiate. The next simplest process is the Markoff process in which case $\psi(\Delta X) = \psi_{(0)}\, e^{-\beta|\Delta x|}$ where $\psi_{(0)}$ and β are constants. This function has a cusp at $\Delta X = 0$ and hence its derivative is undefined there. It appears that one of the simplest processes we can investigate then is one whose autocorrelation is of the form

$$\psi(\Delta X) = \psi_{(0)}\, e^{-\gamma^2(\Delta X)^2}, \tag{I-22}$$

where

$\psi_{(0)} = \langle y_2^2\rangle$ the mean square value of the surface height,

and

$$\gamma = \text{constant} .$$

Using eq. (I-22) we see

$$\psi^{I} = -2\gamma^2 (\Delta X)\, \psi \qquad , \tag{I-23}$$

$$\psi^{II} = \left(-2\gamma^2 + 4\gamma^4 (\Delta X)^2\right) \psi , \tag{I-24}$$

$$\psi^{III} = \left(12\gamma^4 (\Delta X) - 8\gamma^6 (\Delta X)^3\right) \psi \quad , \tag{I-25}$$

and

$$\psi^{IV} = \left(16\gamma^8 (\Delta X)^4 - 48\gamma^6 (\Delta X)^2 + 12\gamma^4\right) \psi \quad . \tag{I-26}$$

1.11 Constant Velocity

When the system moves with uniform velocity, the acceleration is zero and the autocorrelation of the forces becomes

$$\langle R(\tau)\, R(\tau^+)\rangle = V^4\, \psi^{IV} \quad , \tag{I-27}$$

and the parameter ΔX is given by

$$\Delta X = X^+ - X = V(\tau^+ - \tau) \quad . \tag{I-28}$$

Substituting eqs. (I-26) and (I-27) into eqs. (I-10) and (I-11) and using the expressions for $h(t-\tau)$, and $h^{\circ}(t-\tau)$ we find

$$\langle z^2(t)\rangle = \frac{\psi_{(0)}\, e^{-2\rho\omega_0 t}}{2\omega_1^2} \int_0^t \int_0^t e^{\rho\omega_0(\tau+\tau^+)}\, e^{-\gamma^2 V^2 (\tau^+-\tau)^2} \left[\cos\omega_1(\tau^+-\tau) - \cos\omega_1(2t-\tau^+-\tau)\right]$$

$$\cdot \left[16\gamma^8 V^8 (\tau^+-\tau)^4 - 48\gamma^6 V^6 (\tau^+-\tau)^2 + 12\gamma^4 V^4\right] d\tau\, d\tau^+, \tag{I-29}$$

and

$$\langle \dot{z}^2(t) \rangle = \psi_{(0)} e^{-2\rho\omega_0 t} \int_0^t \int_0^t e^{\rho\omega_0(\tau+\tau^+)} e^{-\gamma^2 V^2 (\tau^+-\tau)^2} \left[\frac{1}{2} \left(\frac{\omega_0}{\omega_1}\right)^2 \cos\omega_1 (\tau^+-\tau) \right. \tag{I-30}$$

$$\left. + \left(\frac{\omega_1^2 - \rho^2\omega_0^2}{2\omega_1^2}\right) \cos\omega_1(2t-\tau-\tau^+) - \left(\frac{\rho\omega_0}{\omega_1}\right) \sin\omega_1(2t-\tau-\tau^+) \right]$$

$$\cdot \left[16\gamma^8 V^8 (\tau^+-\tau)^4 - 48\gamma^6 V^6 (\tau^+-\tau)^2 + 12\gamma^4 V^4 \right] d\tau\, d\tau^+ .$$

Equations (I-29) and (I-30) give the mean square values for the relative displacement, and velocity of the mass of a single-degree-of-freedom system moving with uniform velocity along a surface whose shape has an autocorrelation given by eq. (I-22).

1.12 Constant Acceleration

When the system moves with constant acceleration and starts from rest at position $x=0$, the autocorrelation of the forces is

$$\langle R(\tau) R(\tau^+) \rangle = -a^2 \psi^{II} + a^4 \tau^2 \tau^{+2} \psi^{IV} - a^3(\tau^{+2} - \tau^2)\psi^{III} , \tag{I-31}$$

and the parameter Δx is given by

$$\Delta x = x^+ - x = \frac{a}{2}(\tau^{+2} - \tau^2) . \tag{I-32}$$

Proceeding as in Sect. 1.11 we find that the mean square values of the relative displacement and velocity of the mass are given for the case of constant acceleration of the system by

$$\langle z^2(t)\rangle = \frac{\Psi_{(0)}\, e^{-2\rho\omega_0 t}}{2\omega_1^2}\int_0^t\int_0^t e^{\rho\omega_0(\tau+\tau^+)}\, e^{-\frac{\gamma^2 a^2}{4}(\tau^+ + \tau)^2(\tau^+ - \tau)^2}$$

$$\cdot\Big\{2\gamma^2 a^2 - \gamma^4 a^4(\tau^+ + \tau)^2(\tau^+ - \tau)^2\left[\gamma^2 a^2(\tau^+ + \tau)^2(\tau^+ - \tau)^2 - 7\right] + \tau^{+2}\tau^2\gamma^4 a^4\Big[\gamma^4 a^4(\tau^+ + \tau)^4(\tau^+ - \tau)^4$$

$$-12\gamma^2 a^2(\tau^+ + \tau)^2(\tau^+ - \tau)^2 + 12\Big]\Big\}\Big\{\cos\omega_1(\tau^+ - \tau) - \cos\omega_1(2t - \tau - \tau^+)\Big\}\, d\tau\, d\tau^+ \quad , \qquad \text{(I-33)}$$

and

$$\langle \dot z^2(t)\rangle = \Psi_{(0)}\, e^{-2\rho\omega_0 t}\int_0^t\int_0^t e^{\rho\omega_0(\tau+\tau^+)}\, e^{-\frac{\gamma^2 a^2}{4}(\tau^+ + \tau)^2(\tau^+ - \tau)^2}$$

$$\Big\{2\gamma^2 a^2 - \gamma^4 a^4(\tau^+ + \tau)^2(\tau^+ - \tau)^2\left[\gamma^2 a^2(\tau^+ + \tau)^2(\tau^+ - \tau)^2 - 7\right] + \tau^{+2}\tau^2\gamma^4 a^4\Big[\gamma^4 a^4(\tau^+ + \tau)^4(\tau^+ - \tau)^4$$

$$-12\gamma^2 a^2(\tau^+ + \tau)^2(\tau^+ - \tau)^2 + 12\Big]\Big\}\Big\{\frac{1}{2}\left(\frac{\omega_0}{\omega_1}\right)^2\cos\omega_1(\tau^+ - \tau)$$

$$+\left(\frac{\omega_1^2 - \rho^2\omega_0^2}{2\omega_1^2}\right)\cos\omega_1(2t - \tau - \tau^+) - \left(\frac{\rho\omega_0}{\omega_1}\right)\sin\omega_1(2t - \tau - \tau^+)\Big\}\, d\tau\, d\tau^+ \quad . \qquad \text{(I-34)}$$

1.2 ABSOLUTE MOTION OF THE MASS

When the parameter of interest is y_1, the absolute motion of the mass, the expressions for mean square values are analogous to eqs. (I-10) and (I-11) with z replaced by y_1 and the forcing function $R(\tau)$ given by

$$R(\tau) = 2\rho\omega_0\dot y_2 + \omega_0^2 y_2 = 2\rho\omega_0 V(\tau)\, y_2'(x) + \omega_0^2 y_2(x) \;. \qquad \text{(I-35)}$$

Following the procedure used in establishing eq. (I-21), we find that we may write the autocorrelation of the forces in this case as

$$\langle R(\tau)R(\tau^+)\rangle = -4\rho^2\omega_0^2 v(\tau)v(\tau^+)\psi^{II} + \omega_0^4\psi + 2\rho\omega_0^3 v(\tau^+)\psi^{I} - 2\rho\omega_0^3 v(\tau)\psi^{I} \quad . \qquad \text{(I-36)}$$

1.21 Constant Velocity

When the system velocity is constant, ΔX is given by eq. (I-28), and the autocorrelation becomes

$$\langle R(\tau)R(\tau^+)\rangle = -4\rho^2\omega_0^2 v^2\psi^{II} + \omega_0^4\psi \quad . \qquad \text{(I-37)}$$

Forming the mean square values as in Sect. 1.1, we find

$$\langle y_1^2(t)\rangle = \frac{\psi_{(0)}\,\omega_0^2}{2\omega_1^2} e^{-2\rho\omega_0 t}\int_0^t\int_0^t e^{\rho\omega_0(\tau+\tau^+)} e^{-\gamma^2 v^2(\tau^+-\tau)^2}\left\{\omega_0^2 + 8\rho^2\gamma^2 v^2 - 16\rho^2\gamma^4 v^4(\tau^+-\tau)^2\right\}$$

$$\cdot\left\{\cos\omega_1(\tau^+-\tau) - \cos\omega_1(2t-\tau-\tau^+)\right\} d\tau\, d\tau^+ \quad , \qquad \text{(I-38)}$$

and

$$\langle \dot{y}_1^2(t)\rangle = \psi_{(0)}\,\omega_0^2 e^{-2\rho\omega_0 t}\int_0^t\int_0^t e^{\rho\omega_0(\tau+\tau^+)} e^{-\gamma^2 v^2(\tau^+-\tau)^2}\left\{\omega_0^2 + 8\rho^2\gamma^2 v^2 - 16\rho^2\gamma^4 v^4(\tau^+-\tau)^2\right\}$$

$$\cdot\left\{\frac{1}{2}\left(\frac{\omega_0}{\omega_1}\right)^2\cos\omega_1(\tau^+-\tau) + \left(\frac{\omega_1^2-\rho^2\omega_0^2}{2\omega_1^2}\right)\cos\omega_1(2t-\tau-\tau^+) - \left(\frac{\rho\omega_0}{\omega_1}\right)\sin\omega_1(2t-\tau-\tau^+)\right\}d\tau\, d\tau^+. \qquad \text{(I-39)}$$

1.22 Constant Acceleration

In this case ΔX is given by eq. (I-32) and the autocorrelation is given by

$$\langle R(\tau)\,R(\tau^+)\rangle = -4\rho^2\omega_0^2 a^2 \tau\tau^+ \psi^{II} + \omega_0^4 \psi + 2\rho\omega_0^3 a(\tau^+-\tau)\,\psi^{I}. \tag{I-40}$$

The mean square values then become

$$\langle y_1^2(t)\rangle = \frac{\psi_{(0)}\,\omega_0^2}{2\omega_1^2} e^{-2\rho\omega_0 t} \int_0^t \int_0^t e^{\rho\omega_0(\tau+\tau^+)} e^{-\frac{\gamma^2 a^2}{4}(\tau^+-\tau)^2(\tau^++\tau)^2} \left\{\cos\omega_1(\tau^+-\tau) - \cos\omega_1(2t-\tau-\tau^+)\right\}$$

$$\cdot \left\{-4\rho^2 a^2 \tau\tau^+ \gamma^2\left[-2+\gamma^2 a^2(\tau^+-\tau)^2(\tau^++\tau)^2\right] + \omega_0^2 - 2\rho\omega_0 a^2\gamma^2(\tau^+-\tau)^2(\tau^++\tau)\right\} d\tau\, d\tau^+, \tag{I-41}$$

and

$$\langle \dot{y}_1^2(t)\rangle = \psi_{(0)}\,\omega_0^2\, e^{-2\rho\omega_0 t} \int_0^t \int_0^t e^{\rho\omega_0(\tau+\tau^+)} e^{-\frac{\gamma^2 a^2}{4}(\tau^+-\tau)^2(\tau^++\tau)^2}$$

$$\cdot \left\{\frac{1}{2}\left(\frac{\omega_0}{\omega_1}\right)^2 \cos\omega_1(\tau^+-\tau) + \left(\frac{\omega_1^2-\rho^2\omega_0^2}{2\omega_1^2}\right)\cos\omega_1(2t-\tau-\tau^+) - \left(\frac{\rho\omega_0}{\omega_1}\right)\sin\omega_1(2t-\tau-\tau^+)\right\}$$

$$\cdot \left\{-4\rho^2 a^2 \tau\tau^+ \gamma^2\left[-2+\gamma^2 a^2(\tau^+-\tau)^2(\tau^++\tau)^2\right] + \omega_0^2 - 2\rho\omega_0 a^2\gamma^2(\tau^+-\tau)^2(\tau^++\tau)\right\} d\tau\, d\tau^+. \tag{I-42}$$

2.0 Approximate Solutions

The double-integral expressions for the mean square values of displacement, velocity, and acceleration developed in the previous section may be reduced to forms involving only single integrals.

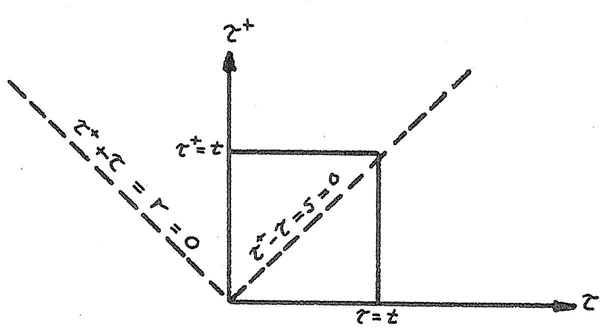
The integrands, however, will then involve error functions of real and imaginary arguments and hence it is not possible to express the solution with a finite number of elementary or tabulated functions. It becomes necessary to evaluate the integrals by approximate or numerical methods.

As a specific example, consider eqs. (I-41) and (I-42) which give the mean square displacements of the mass when the system moves with constant acceleration and constant velocity, respectively. It is immediately seen that if we retain the variables τ and τ^+ we are faced with the problem of trying to integrate such terms as $(\tau^4 e^{-\tau^4} \cos \omega_1 \tau)$. It is convenient to use the new variables r and s defined as

$$r = \tau^+ + \tau \quad , \tag{I-43}$$

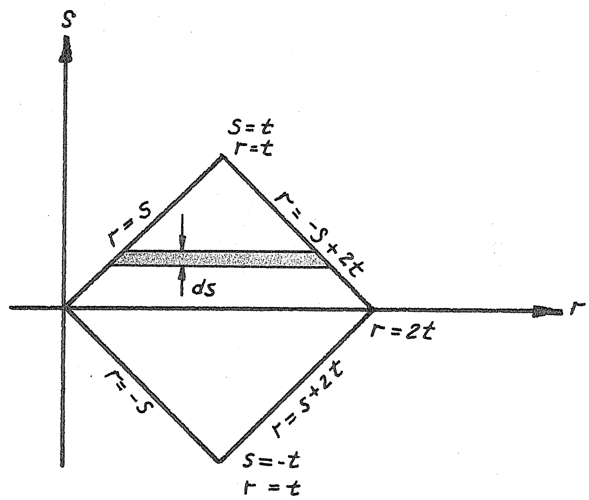
$$s = \tau^+ - \tau \quad . \tag{I-44}$$

Note that the Jacobian of this transformation is 1/2. Sketches I-2 and I-3 show the areas of integration in the τ, τ^+ plane and in the r, s plane.



AREA OF INTEGRATION IN τ, τ^+ PLANE

<u>SKETCH I-2</u>



AREA OF INTEGRATION IN r, s PLANE

SKETCH I-3

Substituting eqs. (I-43) and (I-44) into eqs. (I-41) and (I-38) and rearranging terms we find that the expressions for the mean square displacements become

$$\langle y_1^2(t)\rangle_a = \frac{\psi_{(0)}\,\omega_0^2 e^{-2p\omega_0 t}}{2\omega_1^2}\int_{s=0}^{t}\int_{r=s}^{-s+2t} e^{-\frac{\gamma^2 a^2}{4} r^2 s^2 + p\omega_0 r}\left\{\cos\omega_1 s - \cos\omega_1(2t-r)\right\}$$

$$\cdot\left\{(\omega_0 - p\gamma^2 a^2 r s^2)^2 + 2\gamma^2 a^2 p^2 (r^2 - s^2) - p^2\gamma^4 a^4 r^4 s^2\right\} dr\,ds\,, \qquad \text{(I-45)}$$

and

$$\langle y_1^2(t)\rangle_v = \frac{\psi_{(0)}\,\omega_0^2 e^{-2p\omega_0 t}}{2\omega_1^2}\int_{s=0}^{t}\int_{r=s}^{-s+2t} e^{-\gamma^2 v^2 s^2 + p\omega_0 r}\left\{\cos\omega_1 s - \cos\omega_1(2t-r)\right\}$$

$$\cdot\left\{\omega_0^2 + 8p^2\gamma^2 v^2 - 16p^2\gamma^4 v^4 s^2\right\} dr\,ds\,, \qquad \text{(I-46)}$$

where

$\langle\ \rangle_a$ refers to average values when system has constant acceleration,

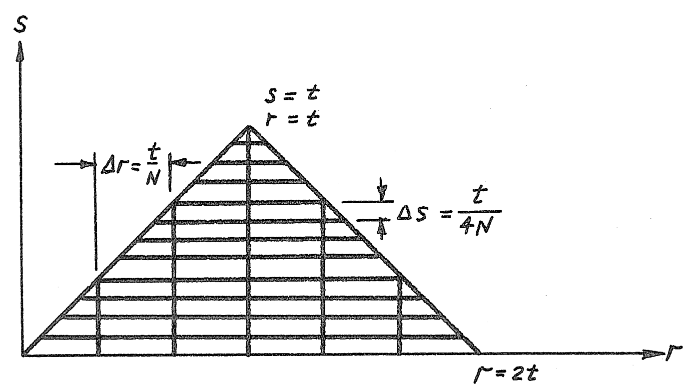
and

$\langle\ \rangle_v$ refers to average values when system has constant velocity.

The limits of integration in eqs. (I-45) and (I-46) may be verified by referring to Sketch I-3 and noting that the integrands in these expressions are symmetric about the r axis.

2.1 NUMERICAL METHODS

Equations (I-45) and (I-46) may be evaluated by numerical techniques but then the only practical way to perform the calculations is by means of an automatic computer. One method of doing the integrations numerically is to divide the area of integration into convenient rectangles as shown in Sketch I-4.



DIVISION OF AREA OF INTEGRATION INTO RECTANGLES

SKETCH I-4

For the case where the system is traveling with large values of velocity or acceleration, we see that the exponential terms in the integrands of eqs. (I-45) and (I-46) cause the functions to peak sharply about the r axis. For this reason the interval of integration along the s axis should be divided more finely than the interval along the r axis (say $\Delta s = \frac{t}{4N}$ and $\Delta r = \frac{t}{N}$).

The integration is performed by evaluating the integrand at $2N$ values of r for a fixed value of s and using Simpson's Rule to find the area under this curve. The process is repeated for the $4N$ values of s yielding $4N$ such areas. Simpson's Rule is again applied to these $4N$ areas resulting in a volume, the numerical value of which is the mean square displacement.

Calculations carried out on an automatic computer have shown that it is not practical to use this numerical technique for large values of time t . Because of this limitation other approximate techniques must be used when large values of t are considered. The form of the integrands of eqs. (I-45) and (I-46) are such that Laplace's Method may be used. A brief discussion of this method is given in the next section.

2.2 LAPLACE'S METHOD

If in eqs. (I-45) and (I-46) the products $\gamma^2 a^2$ and $\gamma^2 v^2$ are large corresponding to large system accelerations and velocities, one of the integrations in each equation is of the form

$$\mathcal{G}(\eta) = \int_{\zeta_1}^{\zeta_2} e^{\eta \ell(\zeta)} g(\zeta)\, d\zeta \quad , \qquad \text{(I-47)}$$

where

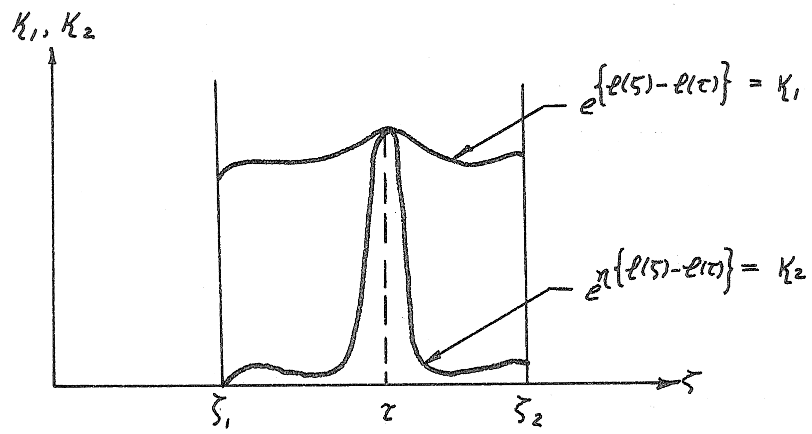
η is a large positive parameter,

$\ell(\zeta)$ is a real function of ζ, and

$g(\zeta)$ is a real or complex function of ζ.

We will assume that $g(\zeta)$ is continous in the interval $\zeta_1 \zeta_2$ and that $\ell(\zeta)$ is continuously differentiable in that interval.

Now assume that $\ell(\zeta)$ has a finite number of maxima in the interval $\zeta_1 \zeta_2$ and let the greatest of these maxima occur at $\zeta = \tau$. Then the function $e^{\{\ell(\zeta) - \ell(\tau)\}}$ is unity at $\zeta = \tau$ and less than unity everywhere else in the interval $\zeta_1 \zeta_2$. Since η is assumed to be large, the function $e^{\eta\{\ell(\zeta) - \ell(\tau)\}}$ has an appreciable value only near $\zeta = \tau$ and is quite small elsewhere as shown in Sketch I-5.



EXAMPLE OF FUNCTIONS K_1 AND K_2 USED IN LAPLACE'S METHOD

SKETCH I-5

It is clear that the main contribution to $g(\eta)$ comes from the vicinity of $\zeta = \tau$. For simplicity assume $\ell(\zeta)$ has only one maximum in the interval $\zeta_1 \zeta_2$ and introduce a new variable u defined as

$$\left.\begin{aligned} u &= \left\{\ell(\tau) - \ell(\zeta)\right\}^{\frac{1}{2}}, \qquad \zeta > \tau, \\ \text{and} \qquad u &= -\left\{\ell(\tau) - \ell(\zeta)\right\}^{\frac{1}{2}}, \qquad \zeta < \tau. \end{aligned}\right\} \tag{I-48}$$

Using this new variable in eq. (I-47) yields

$$\mathcal{J}(\eta) \doteq e^{\eta \ell(\tau)} \int_{\Omega_1}^{\Omega_2} e^{-\eta u^2} g(\zeta) \frac{d\zeta}{du}\, du \quad , \tag{I-49}$$

where

$$\Omega_1 = -\left\{\ell(\tau) - \ell(\zeta_1)\right\}^{\frac{1}{2}} \quad ,$$

and

$$\Omega_2 = \left\{\ell(\tau) - \ell(\zeta_2)\right\}^{\frac{1}{2}} \quad .$$

Since $e^{-\eta u^2}$ is small except when u^2 is near zero, we may replace $g(\zeta)\dfrac{d\zeta}{du}$ by its value at $u = 0$ or equivalently $\zeta = \tau$. Hence we replace $g(\zeta)$ by $g(\tau)$. From eq. (I-48) we see $u^2 = \ell(\tau) - \ell(\zeta)$ and using the Taylor Series representation for $\ell(\zeta)$ we have

$$u^2 = \ell(\tau) - \ell(\tau) - (\zeta - \tau)\left(\frac{d\ell(\zeta)}{d\zeta}\right)_{\zeta=t} - \frac{(\zeta-\tau)^2}{2!}\left(\frac{d^2\ell(\zeta)}{d\zeta^2}\right)_{\zeta=\tau} - \cdots \quad . \tag{I-50}$$

Since $\zeta = \tau$ is a maximum, $\left(\dfrac{d\ell(\zeta)}{d\zeta}\right)_{\zeta=\tau} = 0$ and we have

$$u^2 \doteq -\frac{1}{2}(\zeta-\tau)^2\left(\frac{d^2\ell(\zeta)}{d\zeta^2}\right)_{\zeta=\tau} , \tag{I-51}$$

and

$$\frac{d\zeta}{du} \doteq \left\{-2\Big/\left(\frac{d^2\ell(\zeta)}{d\zeta^2}\right)_{\zeta=\tau}\right\}^{\frac{1}{2}} . \qquad \text{(I-52)}$$

Using eq. (I-52) in eq. (I-49) and letting $\sqrt{\eta}\,u = \nu$ we have

$$\mathcal{g}(\eta) \doteq g(\tau)\, e^{\eta\ell(\tau)} \left\{-2\Big/\eta\left(\frac{d^2\ell(\zeta)}{d\zeta^2}\right)_{\zeta=\tau}\right\}^{\frac{1}{2}} \int_{\sqrt{\eta}\,\Omega_1}^{\sqrt{\eta}\,\Omega_2} e^{-\nu^2}\, d\nu . \qquad \text{(I-53)}$$

If $g(\tau) = 0$, we may use the first non-vanishing term of the Taylor Series expansion for $g(\tau)$. For example, if $\left(\frac{d^2 g(\zeta)}{d\zeta^2}\right)_{\zeta=\tau}$ were the first non-zero term, eq. (I-53) would become

$$\mathcal{g}(\eta) \doteq \frac{\sqrt{\pi}}{4}\left(\frac{d^2 g(\zeta)}{d\zeta^2}\right)_{\zeta=\tau} e^{\eta\ell(\tau)} \left\{2\Big/\eta\left|\left(\frac{d^2\ell(\zeta)}{d\zeta^2}\right)_{\zeta=\tau}\right|\right\}^{\frac{3}{2}}, \quad \eta \to \infty . \qquad \text{(I-54)}$$

If

$$\left(\frac{d\ell(\zeta)}{d\zeta}\right)_{\zeta=\tau} = \cdots = \left(\frac{d^{2m-1}\ell(\zeta)}{d\zeta^{2m-1}}\right)_{\zeta=\tau} = 0, \quad \text{and} \quad \left(\frac{d^{2m}\ell(\zeta)}{d\zeta^{2m}}\right)_{\zeta=\tau} < 0 ,$$

for m = 1, 2, ---, then $\zeta=\tau$ is called the critical point of order $2m-1$. In this case we introduce the new variable $\overline{u}$ defined by

$$\overline{u} = \ell(\tau) - \ell(\zeta) \doteq \frac{1}{(2m)!}(\zeta-\tau)^{2m}\left(\frac{d^{2m}\ell(\zeta)}{d\zeta^{2m}}\right)_{\zeta=\tau} . \qquad \text{(I-55)}$$

We then find that eq. (I-49) becomes

$$\mathcal{J}(\eta) \doteq \frac{1}{m}\Gamma\left(\frac{1}{2m}\right)g(\tau)\,e^{\eta\ell(\tau)}\left\{2m!\Big/\eta\left|\left(\frac{d^{2m}\ell(\zeta)}{d\zeta^{2m}}\right)_{\zeta=\tau}\right|\right\}^{\frac{1}{2m}}, \quad \eta\to\infty \,. \qquad \text{(I-56)}$$

It is clear that eq. (I-53) is just a special case of eq. (I-56).

This method will be used to reduce eq. (I-45) to a single integral expression. Of course, we could do this by integrating over one of the variables directly but then there would be far more terms in the integrand. It is clear that we will still be left with error functions in the integrand and further approximations will be necessary. In the case of constant system velocity a simpler approach may be used and this will be discussed in the next section.

3.0 Solution for Constant Velocity

When the system moves with constant velocity, it is easier to determine the mean square displacement by making use of the power spectrum concept. By referring to eqs. (I-22), (I-24), (I-28), and (I-37), it is seen that for the case of constant system velocity the autocorrelation of the input is stationary, since it is only a function of the variable $s = (\tau^{+} - \tau)$; and we may use eq. (81) to determine the power spectrum. It is now possible to compute the mean square displacement of the mass from a knowledge of the input power spectrum and the impedance of the system.

3.1 POWER SPECTRUM METHOD

Assume that we know the power spectrum of the output of a single-degree-of-freedom system. From eq. (82) we see that by setting $\tau = 0$ we have

$$\psi_{OUT.}(0) = \int_0^\infty \Pi_{OUT.}(\omega)\, d\omega \quad , \tag{I-57}$$

where the subscript $OUT.$ refers to the output. From eq. (80), however, we see that when $\tau = 0$ we have

$$\psi_{OUT.}(0) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t)\, dt = \langle x^2(t) \rangle \quad . \tag{I-58}$$

Combining eqs. (I-57) and (I-58) and using y_1 for x we have

$$\langle y_1^2(t) \rangle_V = \int_0^\infty \Pi_{OUT.}(\omega) d\omega \quad . \tag{I-59}$$

It is clear then that by knowing the power spectrum of the output we can determine the mean square value of the output by means of eq. (I-59). In our case the input power spectrum is known, but it can be shown that

$$\Pi_{OUT.}(\omega) = \frac{\Pi_{IN.}(\omega)}{|Z(i\omega)|^2} \quad . \tag{I-60}$$

Hence eq. (I-58) becomes

$$\langle y_1^2(t) \rangle_V = \int_0^\infty \frac{\Pi_{IN.}(\omega)}{|Z(i\omega)|^2}\, d\omega \quad . \tag{I-61}$$

It is now necessary to compute the power spectrum of the input. Combining eqs. (81), (I-22), (I-24), (I-28), and (I-37) we have

$$\Pi_{IN.}(\omega) = \frac{2}{\pi}\psi_{(0)}\int_0^\infty e^{-\gamma^2 v^2 s^2}\left\{\omega_0^4 + 8\rho^2\omega_0^2\gamma^2 v^2 - 16\rho^2\omega_0^2 v^2\gamma^2 s^2\right\}\cos\omega s\,ds. \qquad \text{(I-62)}$$

Performing the integration and combining terms result in

$$\Pi_{IN.}(\omega) = \frac{\psi_{(0)}}{\sqrt{\pi}\,\gamma v}\, e^{-\frac{\omega^2}{4\gamma^2 v^2}}\left\{\omega_0^4 + 4\rho^2\omega_0^2\omega^2\right\}, \qquad \text{(I-63)}$$

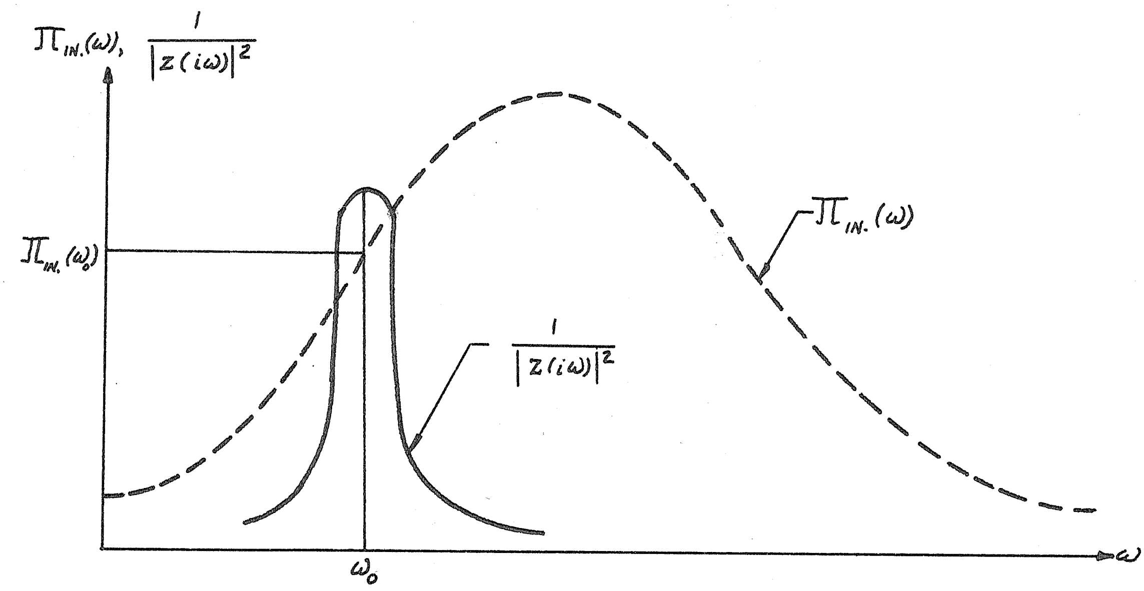
which is the power spectrum of the input for a constant system velocity. Substituting eq. (I-63) into eq. (I-61) leads to rather complicated integrals but in the case of small damping an approximate solution is easily obtained.

3.2 FLAT POWER SPECTRUM APPROXIMATION

For a damped, single-degree-of-freedom system we know that

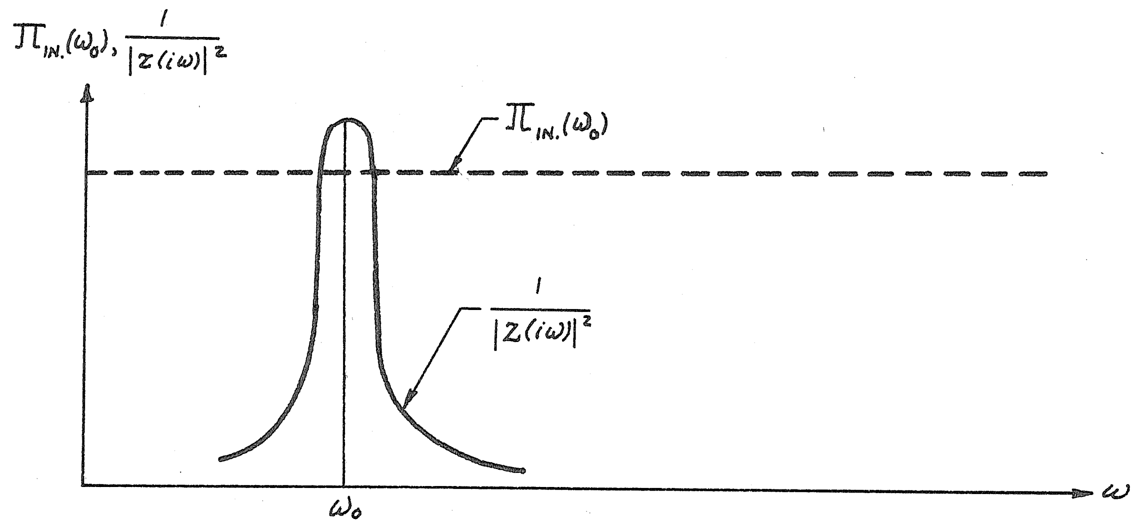
$$|Z(i\omega)|^2 = (\omega_0^2-\omega^2)^2 + (2\rho\omega_0\omega)^2. \qquad \text{(I-64)}$$

For lightly damped systems $\frac{1}{|Z(i\omega)|^2}$ is a function which peaks sharply at $\omega=\omega_0$ and hence the major contribution to the integral of eq. (I-61) is in the vicinity of ω_0. We may then approximate the mean square displacement in this case by evaluating the input power spectrum at $\omega=\omega_0$ and consider it to be a flat power spectrum with spectral density $\Pi_{IN.}(\omega_0)$. Sketches I-6 and I-7 give a pictorial representation of this approximation.



EXACT PROBLEM FOR RESPONSE $\langle y_1^2(t) \rangle_V$

SKETCH I-6



APPROXIMATE PROBLEM FOR RESPONSE $\langle y_1^2(t) \rangle_V$

SKETCH I-7

Combining eqs. (I-61), (I-63), and (I-64) and using $\Pi_{IN.}(\omega_o)$ for $\Pi_{IN.}(\omega)$ we have

$$\langle y_1^2(t)\rangle_V \doteq \frac{\Psi_{(0)}\,\omega_o^4}{\sqrt{\pi}\,\gamma V}\, e^{-\frac{\omega_o^2}{4\gamma^2 V^2}} \left\{1 + 4\rho^2\right\} \int_0^\infty \frac{d\omega}{(\omega_o^2-\omega^2)^2 + (2\rho\omega_o\omega)^2} . \qquad \text{(I-65)}$$

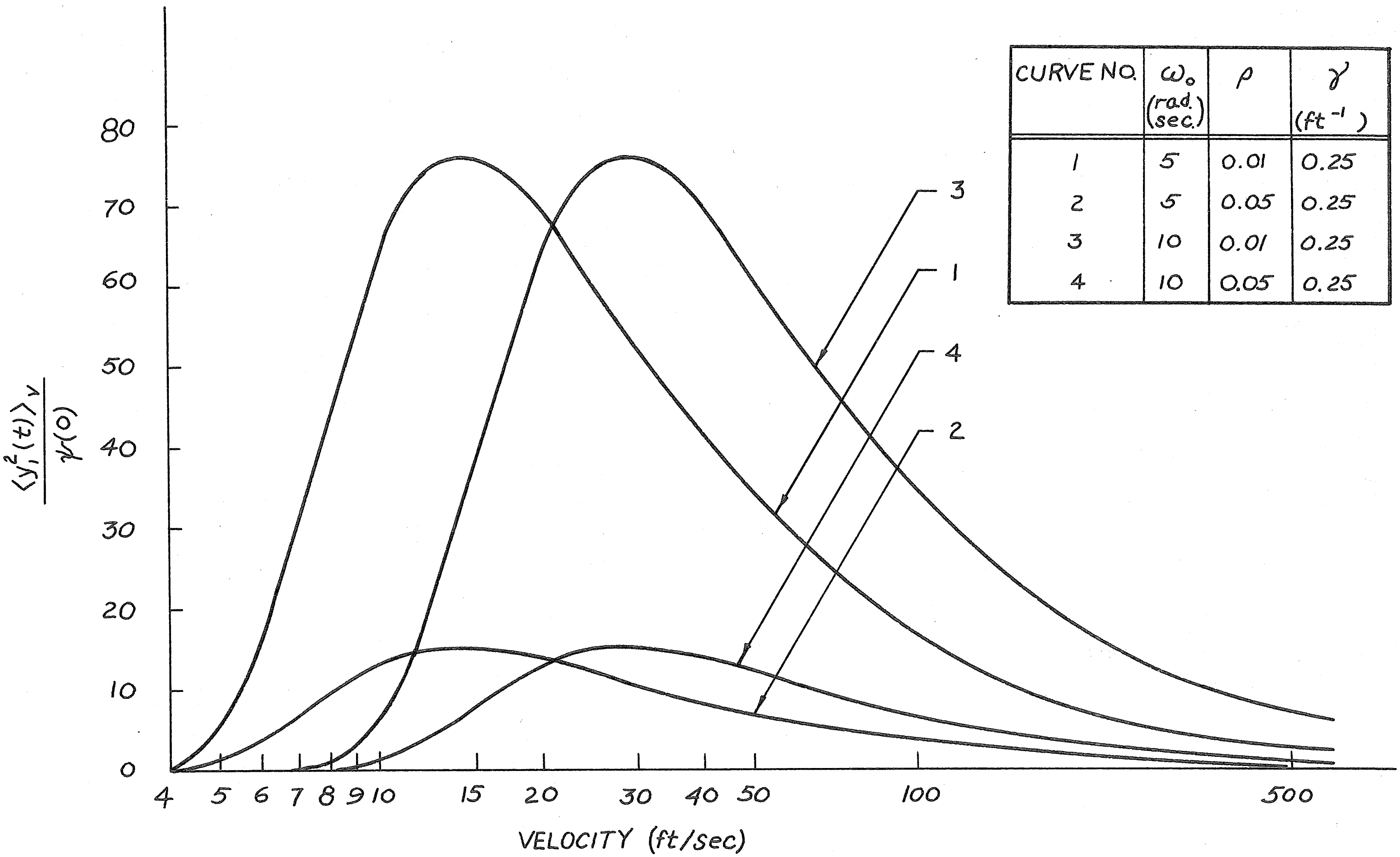
Carrying out the integration and combining terms yield

$$\langle y_1^2(t)\rangle_V \doteq \frac{\sqrt{\pi}\,\Psi_{(0)}\,\omega_o}{2\gamma V\rho}\, e^{-\frac{\omega_o^2}{4\gamma^2 V^2}} \left\{1 + 4\rho^2\right\} . \qquad \text{(I-66)}$$

Equation (I-66) is the approximate steady-state solution for the mean square displacement of the mass when the system moves with constant velocity. It is of interest to note that as the velocity becomes very large the mean square displacement approaches zero. For $V = 0$ we see that $\langle y_1^2(t)\rangle_V = 0$ as is expected. It must be remembered that these results depend upon the autocorrelation chosen for the surface and so each case must be analyzed separately. It is quite possible that other autocorrelations would result in expressions for the mean square displacement that would diverge. The solution of the problem, therefore, depends upon knowing the surface shape and its autocorrelation. Figure 2 is a plot of eq. (I-66) for several different systems.

4.0 Solution for Constant Acceleration

When the system moves with constant acceleration the autocorrelation of the input is non-stationary, since it is a function of both the variables $r = (\tau^+ + \tau)$ and $s = (\tau^+ - \tau)$; and the problem of



CURVE NO.	ω_o (rad./sec.)	ρ	γ (ft^{-1})
1	5	0.01	0.25
2	5	0.05	0.25
3	10	0.01	0.25
4	10	0.05	0.25

STEADY STATE MEAN SQUARE DISPLACEMENT OF MASS vs. SYSTEM VELOCITY WHEN SYSTEM IS UNACCELERATED

FIGURE 2

determining the corresponding power spectrum is quite difficult. It is simpler to apply Laplace's Method to eq. (I-45) which reduces it to a single integral and then by considering lightly damped systems we are able to obtain approximate solutions for the mean square displacement.

4.1 APPLICATION OF LAPLACE'S METHOD

Since Laplace's Method results in error integrals it is convenient to have symmetric limits on these integrals. We will thus apply the method to the integration involving the variable S since the integrand is symmetric with respect to the r axis. It is then necessary to rearrange the limits of integration and we have, symbolically,

$$\int_{s=-t}^{t}\int_{r=s}^{-s+2t} dr\,ds = \int_{r=0}^{t}\int_{s=-r}^{r} ds\,dr + \int_{r=t}^{2t}\int_{s=r-2t}^{-r+2t} ds\,dr \; . \tag{I-67}$$

Applying Laplace's Method to eq. (I-45) it can be shown that the mean square displacement may be approximated as

$$\langle y_1^2(t)\rangle_a \doteq \frac{\Psi_{(0)}\,\omega_0^2\sqrt{\pi}}{2\omega_1^2\gamma a}\, e^{-\epsilon p\omega_0 t}\left\{ \int_0^t \frac{e^{p\omega_0 r}}{r}\, erf\!\left(\frac{\gamma a}{2} r^2\right)\left[\omega_0^2 + 2\gamma^2 a^2 r^2\right]\left[1-\cos\omega_1(2t-r)\right] dr \right.$$

$$\left. + \int_t^{2t} \frac{e^{p\omega_0 r}}{r}\, erf\!\left[\frac{\gamma a}{2} r(2t-r)\right]\left[\omega_0^2 + 2\gamma^2 a^2 p^2 r^2\right]\left[1-\cos\omega_1(2t-r)\right] dr \right\}. \tag{I-68}$$

If we approximate the error function by a series of straight lines, then all but two of the integrals of eq. (I-68) are given in closed form. We may obtain an asymptotic expression for these integrals by successive integrations by parts. This method leads to

$$\int_{-\infty}^{\xi} \frac{e^{br}}{r} \cos \omega_1 r \, dr = e^{b\xi} \sum_{n=1}^{N} \frac{(n-1)! \cos(\omega_1 \xi - n\Theta)}{\xi^n \left\{\sqrt{b^2 + \omega_1^2}\right\}^n} + \Lambda_{RC} , \qquad \text{(I-69)}$$

and

$$\int_{-\infty}^{\xi} \frac{e^{br}}{r} \sin \omega_1 r \, dr = e^{b\xi} \sum_{n=1}^{N} \frac{(n-1)! \sin(\omega_1 \xi - n\Theta)}{\xi^n \left\{\sqrt{b^2 + \omega_1^2}\right\}^n} + \Lambda_{RS} , \qquad \text{(I-70)}$$

where

$$\Theta = \arc\tan\left(\frac{\omega_1}{a}\right) ,$$

and

Λ_{RC} and Λ_{RS} are remainders given by integral expressions having finite series for coefficients.

The terms of the series in eqs. (I-69) and (I-70) at first decrease. They then increase and eventually tend to infinity. We can obtain the best approximation to the integrals, using this method, if we sum the terms up to but not including the smallest term. There is little to be gained in evaluating Λ_{RC} or Λ_{RS} since in using Laplace's Method we have already introduced an error which is difficult to evaluate.

With these results it is then possible to approximate the integral of eq. (I-68) but it is evident that numerical computations are extremely tedious. If an automatic computer were available there would be little point in evaluating the integral of eq. (I-68) since the direct evaluation of the integral eq. (I-45) should offer no more difficulty.

It is clear then that we must make some further approximations if we wish to obtain a solution from which numerical results are readily available.

4.2. FURTHER APPROXIMATIONS

We shall begin by putting the integral of eq. (I-68) into dimensionless form. Define the following dimensionless parameters as

$$\langle m \rangle = \frac{\langle y_1^2(t) \rangle_a}{\psi(0)} \quad ,$$

$$\mathcal{T} = t\,\omega_0 \quad ,$$

$$\mathcal{U} = r\,\omega_0 \quad ,$$

$$\mathcal{V} = s\,\omega_0 \quad ,$$

$$\mathcal{A} = \frac{a}{\omega_0^2 \mathcal{L}} \quad ,$$

and

$$\mathcal{B} = \gamma \mathcal{L} \quad ,$$

where

$\mathcal{L}$ is a characteristic length associated with the surface.

Using these parameters in eq. (I-68) we have

$$\langle m \rangle \doteq \frac{\sqrt{\pi}}{2\mathcal{A}\mathcal{B}} e^{-2\rho\mathcal{T}} \left\{ \int_0^{\mathcal{T}} \frac{e^{\rho u}}{u} \operatorname{erf}\left(\frac{\mathcal{A}\mathcal{B}u^2}{2}\right)\left(1 + 2\mathcal{A}^2\mathcal{B}^2\rho^2 u^2\right)\left(1 - \cos\left[2\mathcal{T} - u\right]\right) du \right.$$

$$\left. + \int_{\mathcal{T}}^{2\mathcal{T}} \frac{e^{\rho u}}{u} \operatorname{erf}\left[\frac{\mathcal{A}\mathcal{B}u(2\mathcal{T}-u)}{2}\right]\left(1 + 2\mathcal{A}^2\mathcal{B}^2\rho^2 u^2\right)\left(1 - \cos\left[2\mathcal{T} - u\right]\right) du \right\} . \quad \text{(I-71)}$$

Equation (I-71) is, of course, more general than eq. (I-68) since the former satisfies all systems having the same values of $\mathcal{A}$, $\mathcal{B}$, and ρ , whereas the latter must be evaluated each time one of the parameters is changed.

For lightly damped systems $\rho \ll 1$ and we may approximate $e^{\rho u}$ by $1 + \rho u$. For small values of u we may neglect the term $2\mathcal{A}^2\mathcal{B}^2\rho^2 u^2$. Finally we approximate the error function by the relations

$$\left.\begin{aligned} \operatorname{erf}\zeta &\doteq \zeta , && 0 \leq \zeta \leq 1 , \\ \operatorname{erf}\zeta &\doteq 1 , && \zeta \geq 1 . \end{aligned}\right\} \quad \text{(I-72)}$$

It is clear that the solution will be different for various ranges of $\mathcal{T}$. It will depend upon which of the relations of eq. (I-72) is valid, whether $e^{\rho u} \doteq (1 + \rho u)$ is a good approximation and whether the term $2\mathcal{A}^2\mathcal{B}^2\rho^2 u^2$ may be neglected.

For a given system and for values of $\mathcal{J}$ such that $\frac{\mathcal{A}\mathcal{B}\mathcal{J}^2}{2}$ is no larger than unity we may approximate eq. (I-71) by

$$\langle m \rangle \doteq \frac{\sqrt{\pi}}{2\mathcal{A}\mathcal{B}} e^{-2\rho\mathcal{J}} \left\{ \int_0^{\mathcal{J}} \frac{(1+\rho u)}{u} \left(\frac{\mathcal{A}\mathcal{B}u^2}{2}\right)\left(1 - \cos\left[2\mathcal{J}-u\right]\right) du \right.$$

$$\left. + \int_{\mathcal{J}}^{2\mathcal{J}} \frac{(1+\rho u)}{u} \left(\frac{\mathcal{A}\mathcal{B}u\left[2\mathcal{J}-u\right]}{2}\right)\left(1 - \cos\left[2\mathcal{J}-u\right]\right) du \right\} . \qquad \text{(I-73)}$$

Carrying out the integration and simplifying we have

$$\langle m \rangle \doteq \frac{\sqrt{\pi}}{4} e^{-\rho\mathcal{J}} \left\{ 1 + 2\rho\mathcal{J} + \mathcal{J}^2 + \rho\mathcal{J}^3 - 4\rho\sin\mathcal{J} - 2(1+\rho\mathcal{J})\cos\mathcal{J} \right.$$

$$\left. + 2\rho\sin 2\mathcal{J} + \cos 2\mathcal{J} \right\}, \qquad 0 \leq \mathcal{J} \leq L_1, \qquad \text{(I-74)}$$

where

$$\frac{\mathcal{A}\mathcal{B}L_1^2}{2} \leq 1 .$$

When $\mathcal{J} > L_1$ we approximate the error function by the second relation of eq. (I-72) and proceed as before. Now, however, the first integral of eq. (I-71) must be evaluated over two ranges of the variable u. The first over the interval $0, L_1$ with $\operatorname{erf}\zeta \doteq \zeta$ and the second over the interval $L_1, \mathcal{J}$ with $\operatorname{erf}\zeta \doteq 1$. For values of u near $2\mathcal{J}$ the argument of the error function in the second integral of eq. (I-71) is small and hence $\operatorname{erf}\zeta \neq 1$ but we will neglect this. Hence for $\mathcal{J} > L_1$ we have

$$\langle m \rangle \doteq \frac{\sqrt{\pi}}{2\mathcal{A}\mathcal{B}} e^{-2\rho\mathcal{J}} \left\{ \int_0^{L_1} \frac{(1+\rho u)}{u} \left(\frac{\mathcal{A}\mathcal{B}u^2}{2}\right)\left(1 - \cos\left[2\mathcal{J}-u\right]\right) du + \int_{L_1}^{2\mathcal{J}} \frac{(1+\rho u)}{u}\left(1 - \cos\left[2\mathcal{J}-u\right]\right) du . \right. \qquad \text{(I-75)}$$

All of the integrals are elementary or tabulated functions and so eq. (I-75) becomes

$$\langle \mathcal{M} \rangle \doteq \frac{\sqrt{\pi}}{4} e^{-2\rho\mathcal{T}} \left\{ \frac{\mathcal{U}^2}{2} + \frac{\rho\mathcal{U}^3}{3} + (\mathcal{U} + \rho\mathcal{U}^2 - 2\rho)\sin(2\mathcal{T} - \mathcal{U}) - (1 + 2\rho\mathcal{U})\cos(2\mathcal{T} - \mathcal{U}) \right\}_0^{L_1}$$

$$+ \frac{\sqrt{\pi}}{2\mathcal{A}\mathcal{B}} e^{-2\rho\mathcal{T}} \left\{ \ln \mathcal{U} + \rho\mathcal{U} - C_i(\mathcal{U}) \cos 2\mathcal{T} - \mathcal{S}_i(\mathcal{U}) \sin 2\mathcal{T} + \rho \sin(2\mathcal{T} - \mathcal{U}) \right\}_{L_1}^{2\mathcal{T}}, \quad L_1 \leq \mathcal{T} \leq L_2 \qquad \text{(I-76)}$$

where

$$C_i(\mathcal{U}) = -\int_{\mathcal{U}}^{\infty} \frac{\cos\zeta}{\zeta} d\zeta ,$$

$$\mathcal{S}_i(\mathcal{U}) = \int_0^{\mathcal{U}} \frac{\sin\zeta}{\zeta} d\zeta ,$$

and

$$2\mathcal{A}^2\mathcal{B}^2\rho^2 L_2^2 \ll 1 .$$

Both of the functions $C_i(\mathcal{U})$ and $\mathcal{S}_i(\mathcal{U})$ are tabulated in <u>Tables of Functions</u> by Jahnke and Emde.

When $\mathcal{T} > L_2$ the term $2\mathcal{A}^2\mathcal{B}^2\rho^2\mathcal{U}^2$ becomes significant. For these larger values of $\mathcal{T}$ the approximation $e^{\rho\mathcal{U}} \doteq 1 + \rho\mathcal{U}$ is not very good and we retain the exponential term instead. Hence when $\mathcal{T} > L_2$ we must add a term to eq. (I-76) given by

$$\Xi = \frac{\sqrt{\pi}}{2\mathcal{A}\mathcal{B}} e^{-2\rho\mathcal{T}} \int_{L_2}^{2\mathcal{T}} \frac{e^{\rho\mathcal{U}}}{\mathcal{U}} (2\mathcal{A}^2\mathcal{B}^2\rho^2\mathcal{U}^2) \left\{ 1 - \cos(2\mathcal{T} - \mathcal{U}) \right\} d\mathcal{U} . \qquad \text{(I-77)}$$

Carrying out the integration and combining terms we have

$$\Xi = \sqrt{\pi}\,\mathcal{A}\mathcal{B}\left\{\frac{\rho^2(\rho^2-1)}{(\rho^2+1)^2} - \frac{2\mathcal{J}\rho^3}{\rho^2+1} + 2\mathcal{J}\rho - 1\right\}$$

$$+\sqrt{\pi}\,\mathcal{A}\mathcal{B}\,e^{\rho(L_2-2\mathcal{J})}\left\{1 - L_2\rho + \frac{\rho^2(L_2\sqrt{\rho^2+1}-1)}{(\rho^2+1)^{3/2}}\left[\cos 2\mathcal{J}(\rho\cos L_2 + \sin L_2)\right.\right.$$

$$\left.\left. + \sin 2\mathcal{J}(\rho \sin L_2 - \cos L_2)\right]\right\}. \qquad \text{(I-78)}$$

For small values of ρ eq. (I-78) may be approximated by

$$\Xi \doteq \sqrt{\pi}\,\mathcal{A}\mathcal{B}\left\{2\mathcal{J}\rho - 1 + (1 - L_2\rho)\,e^{\rho(L_2-2\mathcal{J})}\right\} . \qquad \text{(I-79)}$$

We see from eq. (I-79) that for large values of $\mathcal{J}$ the mean square response becomes proportional to $\mathcal{J}$ and hence diverges. Also since eq. (I-76) contains the factor $e^{-2\rho\mathcal{J}}$ this part of the response becomes negligible for large $\mathcal{J}$ and so we may approximate the response in this case by

$$\langle m \rangle \doteq 2\sqrt{\pi}\,\mathcal{A}\mathcal{B}\rho\mathcal{J}, \qquad \mathcal{J} > L_3 , \qquad \text{(I-80)}$$

where

$$2\rho L_3 \leq -\ln 0.1$$

The relation for L_3 was arrived at by requiring $e^{-2\rho L_3} \leq 0.1$.

The mean square displacement of the mass when the system is moving with constant acceleration is therefore given approximately by a set of relations each of which is valid over a particular range of the time parameter $\mathcal{J}$. We group these relations below for convenience.

$$\langle m \rangle \doteq \frac{\sqrt{\pi}}{4} e^{-2\rho\mathcal{T}} \left\{ 1 + 2\rho\mathcal{T} + \mathcal{T}^2 + \rho\mathcal{T}^3 - 4\rho \sin\mathcal{T} - 2(1+\rho\mathcal{T})\cos\mathcal{T} + 2\rho\sin 2\mathcal{T} + \cos 2\mathcal{T} \right\}, \quad 0 \leq \mathcal{T} \leq L_1, \qquad \text{(I-74)}$$

$$\langle m \rangle \doteq \frac{\sqrt{\pi}}{4} e^{-2\rho\mathcal{T}} \left\{ \frac{u^2}{2} + \frac{\rho u^3}{3} + (u + \rho u^2 - 2\rho)\sin(2\mathcal{T} - u) - (1 + 2\rho u)\cos(2\mathcal{T} - u) \right\}_0^{L_1}$$

$$+ \frac{\sqrt{\pi}}{2\mathcal{A}\mathcal{B}} e^{-2\rho\mathcal{T}} \left\{ \ln u + \rho u - C_i(u)\cos 2\mathcal{T} - S_i(u)\sin 2\mathcal{T} + \rho\sin(2\mathcal{T} - u) \right\}_{L_1}^{2\mathcal{T}}, \quad L_1 \leq \mathcal{T} \leq L_2, \qquad \text{(I-76)}$$

and

$$\langle m \rangle \doteq \frac{\sqrt{\pi}}{4} e^{-2\rho\mathcal{T}} \left\{ \frac{u^2}{2} + \frac{\rho u^3}{3} + (u + \rho u^3 + 2\rho)\sin(2\mathcal{T} - u) - (1 + 2\rho u)\cos(2\mathcal{T} - u) \right\}_0^{L_1}$$

$$+ \frac{\sqrt{\pi}}{2\mathcal{A}\mathcal{B}} e^{-2\rho\mathcal{T}} \left\{ \ln u + \rho u - C_i(u)\cos 2\mathcal{T} - S_i(u)\sin 2\mathcal{T} + \rho\sin(2\mathcal{T} - u) \right\}_{L_1}^{2\mathcal{T}}$$

$$+ \sqrt{\pi}\,\mathcal{A}\mathcal{B} \left\{ 2\mathcal{T}\rho - 1 + (1 - L_2\rho) e^{\rho(L_2 - 2\mathcal{T})} \right\}, \qquad L_1 \leq \mathcal{T} \leq L_3, \qquad \text{(I-81)}$$

and

$$\langle m \rangle \doteq 2\sqrt{\pi}\,\mathcal{A}\mathcal{B}\rho\mathcal{T}, \qquad \mathcal{T} > L_3, \qquad \text{(I-80)}$$

where the L_i are defined by the inequalities

$$L_1^2 \leq \frac{2}{\mathcal{A}\mathcal{B}},$$

$$L_2^2 \leq \frac{0.1}{2\mathcal{A}^2\mathcal{B}^2\rho^2},$$

and

$$L_3 \leq \frac{-\ln 0.1}{2\rho}.$$

These relations are, of course, arbitrary and should serve only as a guide.

The comments made at the end of Sect. 3.2 apply equally well here. The divergent result for the mean square displacement obtained here is the result of the particular autocorrelation that was chosen. A bounded solution may result for some other surface shape and autocorrelation.

4.3 SOLUTION FOR A PARTICULAR SYSTEM

A particular system will now be considered to indicate how the mean square displacement behaves as a function of the time parameter $\mathcal{T}$. Let the parameters of the system have the following values

$$a = 400 \text{ ft}/(\text{sec})^2 ,$$
$$\gamma = \tfrac{1}{4}(\text{ft})^{-1} ,$$
$$\mathcal{L} = 16 \text{ ft} ,$$
$$\rho = 0.01 ,$$

and

$$\omega_0 = 5 \text{ rad}/\text{sec}.$$

We see then from the definitions of the dimensionless parameters and the L_i that

$$\mathcal{A} = 1 ,$$
$$\mathcal{B} = 4 ,$$
$$L_1 = \frac{1}{\sqrt{2}} ,$$
$$L_2 = 5.6 ,$$

and

$$L_3 = 115 .$$

Using these numerical values in eqs. (I-74), (I-76), (I-80) and (I-81) we have

$$\langle m \rangle \doteq \frac{\sqrt{\pi}}{4} e^{-0.02\mathcal{T}} \Big\{ 1 + 0.02\mathcal{T} + \mathcal{T}^2 + 0.01\mathcal{T}^3 - 0.04 \sin \mathcal{T}$$

$$- (2 + 0.02\mathcal{T})\cos\mathcal{T} + 0.02 \sin 2\mathcal{T} + \cos 2\mathcal{T} \Big\}, \quad 0 \leq \mathcal{T} \leq 0.707, \qquad \text{(I-82)}$$

$$\langle m \rangle \doteq \frac{\sqrt{\pi}}{4} e^{-0.02\mathcal{T}} \Big\{ 0.25 + \cos 2\mathcal{T} - \cos(2\mathcal{T} - 0.707) + 0.707 \sin (2\mathcal{T} - 0.707) \Big\}$$

$$+ \frac{\sqrt{\pi}}{8} e^{-0.02\mathcal{T}} \Big\{ 0.347 + \ln 2\mathcal{T} + 0.02\mathcal{T} - \big(C_i(2\mathcal{T}) - 0.107\big) \cos 2\mathcal{T}$$

$$- \big(S_i(2\mathcal{T}) - 0.687\big) \sin 2\mathcal{T} \Big\}, \qquad 0.707 \leq \mathcal{T} \leq 5.6, \qquad \text{(I-83)}$$

$$\langle m \rangle \doteq \frac{\sqrt{\pi}}{4} e^{-0.02\mathcal{T}} \Big\{ 0.25 + \cos 2\mathcal{T} - \cos(2\mathcal{T} - 0.707) + 0.707 \sin (2\mathcal{T} - 0.707) \Big\}$$

$$+ \frac{\sqrt{\pi}}{8} e^{-0.02\mathcal{T}} \Big\{ 0.347 + \ln 2\mathcal{T} + 0.02\mathcal{T} - \big(C_i(2\mathcal{T}) - 0.107\big)\cos 2\mathcal{T} - \big(S_i(2\mathcal{T}) - 0.687\big) \sin 2\mathcal{T} \Big\}$$

$$+ \Big\{ 0.142\mathcal{T} - 7.1 + 6.73 e^{.01(5.6 - 2\mathcal{T})} \Big\}, \qquad 0.707 \leq \mathcal{T} \leq 115,$$

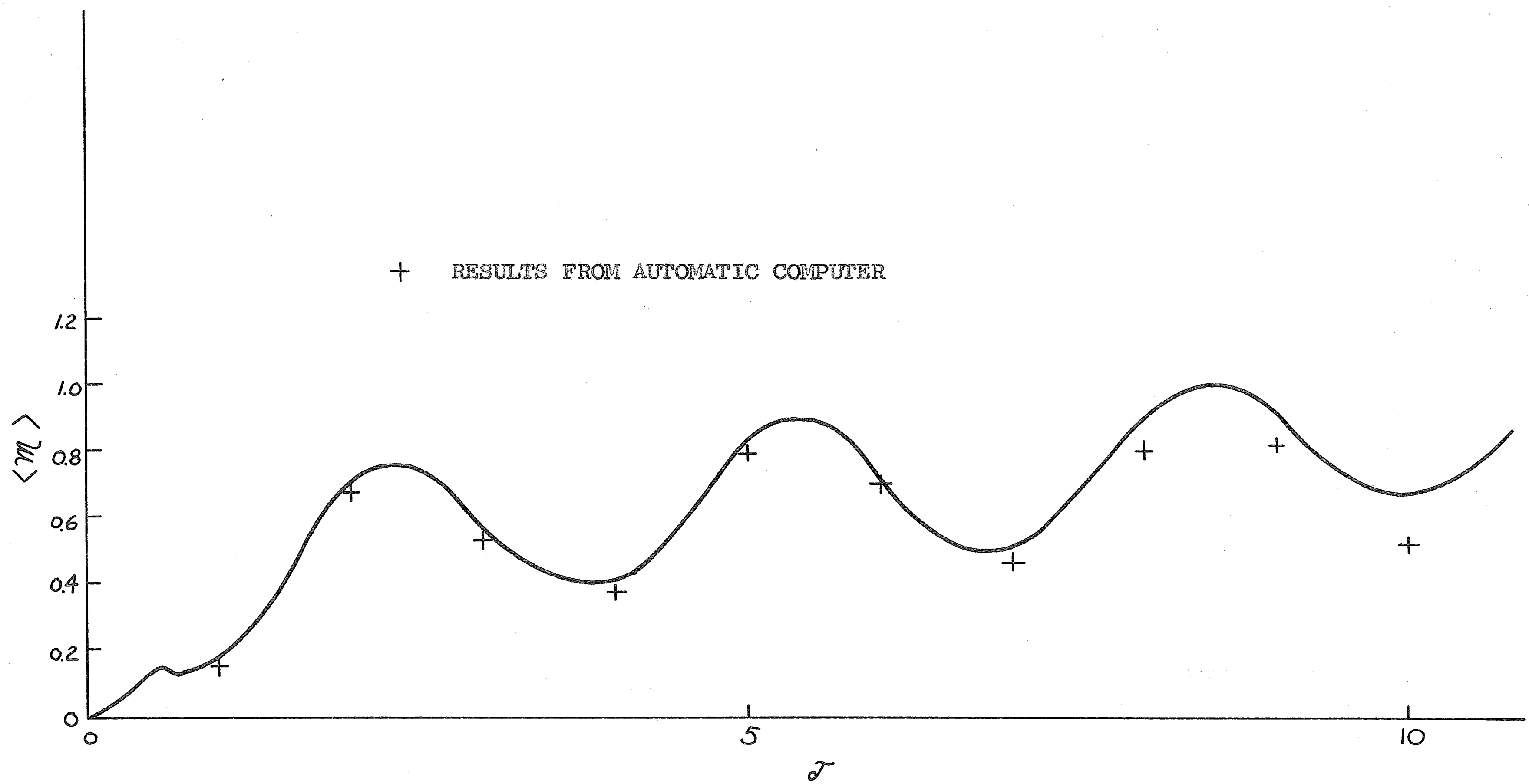
$$\text{(I-84)}$$

and

$$\langle m \rangle \doteq 0.08\sqrt{\pi}\,\mathcal{T}, \qquad \mathcal{T} > 115. \qquad \text{(I-85)}$$

Equations (I-82) through (I-85) were evaluated for values of $\mathcal{T}$ up to 12. These results as well as the results of some of the machine calculations are plotted in Fig. 3. It can be seen that the two methods agree quite well in the interval $0 \leq \mathcal{T} \leq 10$. For values of $\mathcal{T}$ greater than 10 it is not practical to use machine calculations, since convergence is quite slow; and in addition the approximate method becomes increasingly more accurate. These results follow from the fact that for large values of $\mathcal{T}$ the term $e^{-\frac{\gamma^2 a^2}{4} r^2 s^2} \equiv e^{-\frac{\mathcal{A}^2 \mathcal{B}^2 u^2 v^2}{4}}$ appearing in the integrand of eq. (I-45) behaves like a Delta function. It has a value of unity when $v = 0$ and is extremely small when $v \neq 0$. For example when $t = 10$ and $\omega_0 = 5$, the value of u at the midpoint of its range is 50 and hence $e^{-\frac{\mathcal{A}^2 \mathcal{B}^2 u^2 v^2}{4}} = e^{-10{,}000 v^2}$. It is clear then that even for values of v near zero this term makes the integrand quite small. The function is therefore very sharply peaked for large values of u; and in order for the numerical method to be accurate, an extremely fine division of the v axis is required. The major contribution to the double integral of eq. (I-45) is thus along the u axis which is exactly what is assumed in Laplace's Method.

We may treat all of the double integrals for the mean square values of displacement and velocity of the mass by Laplace's Method and then make the appropriate simplifications to arrive at results analogous to those of this section. This method, however, gives no information about the probability distribution of the variables unless the input is Gaussianly distributed.



MEAN SQUARE DISPLACEMENT OF MASS vs. TIME FOR CONSTANT SYSTEM ACCELERATION

FIGURE 3

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