DEVELOPMENT OF SIMPLECTIC METHODS FOR THE METRICAL AND ELECTROMAGNETIC FIELDS

by

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ABSTRACT

The simplectic approach to discretization, as introduced by Regge, may be better suited to computer calculation than is the usual "finite difference" approach. This thesis describes a general simplectic formalism for coupled electromagnetic and metric fields including detailed discussion of nets for closed and open space-times. The rendering of this formalism into computer algorithms is then described, and indicative numerical results are reported.

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NOTATION AND GLOSSARY

The metric, $g_{\mu\nu}$, has signature $-+++$.

The summation convention applies unless it obviously does not.

$$R^{\mu}_{\nu\alpha\beta} = \Gamma^{\mu}_{\nu\alpha,\beta} + \Gamma^{\sigma}_{\nu\alpha}\,\Gamma^{\mu}_{\sigma\beta} - (\alpha \leftrightarrow \beta)$$

$$R = g^{\mu\mu} g^{\nu\nu}\, R_{\mu\nu\mu\nu} \equiv g^{\mu\alpha}\, g^{\nu\beta}\, R_{\mu\nu\alpha\beta}$$

$$S_g = -\frac{1}{2}\int R\, dV$$

$$S_e = -\frac{1}{4}\int F_{\mu\nu}\, F^{\mu\nu}\, dV$$

The contraction of two tensors is variously denoted; some examples:

$A \cdot B$, $\langle A, B \rangle$, $A(B)$, $A_{ij}B^{ij}$

If a,b,c are vectors, then

$a \wedge b = a \otimes b - b \otimes a$

$a \wedge b \wedge c$ has six terms each with coefficient ± 1, etc.

If E represents a (logical) condition then

$$(E) = \begin{cases} 1 & \text{if } E \text{ is true} \\ 0 & \text{if not.} \end{cases}$$

Thus, for example, $\sum_{n=1}^{\infty}$ means the same as $\sum_{n} (1 \le n < \infty)$.

$\sum$ = the net for space-time.

$\sum_m$ = set of all m-simplexes of $\sum$.

"vertices" and "legs" refer respectively to $\sum_o$ and $\sum_1$, while if n = dimension($\sum$), then "bones" and "cells" refer respectively to $\sum_{n-2}$ and $\sum_n$.

$[jk\ell]$: an element of $\sum_2$ with vertices $[j]$, $[k]$, $[\ell]$.

$\alpha|\beta$ (α and β are "incident") means for simplexes of α,β that α is a subsimplex of β or vice-versa.

$\mathfrak{S}_m(\alpha)$ (the m-star of α) $= \{\beta \in \sum_m : \alpha|\beta\}$

$\Theta(jk\cdots\ell) = ([j],[k],\cdots[\ell]$ are the vertices of some simplex in $\sum$)

thatch = simplectic analogue of a field

n = dimension of space under consideration (usually 4)

$\tilde{T}^{i\cdots j}_{k\cdots m}$ = the affine components of a tensor T .

$\tilde{\delta}^j_k = \delta^j_k - \frac{1}{n+1}$

Chapter I

INTRODUCTION

In order to prepare a problem in continuum physics for machine solution one almost always rewrites the basic partial differential equations (field equations) in discrete form. In other words, one samples the field quantities at finely spaced, selected points, and replaces derivatives by differences. The simplectic approach to "discretization" is altogether different from this "partial difference" scheme.

Rather than fill space-time with a grid of <u>points</u> one divides it into a <u>net</u> of simplicial cells. Instead of replacing derivatives by differences one seeks the simplectic <u>analogues</u> of the fundamental quantities and operations of the continuum theory. (The analogy is even so close as to be a singular instance of the continuum case.) The simplectic version of a field (I will call this a <u>thatch</u> for short) may be associated to any of the simplexes of the net, not necessarily just to points, with the tensor character of the thatch expressed by its mode of definition on the simplexes, rather than through many components.

As far as numerical calculations go, the simplectic approach can, when applicable, be expected to be more efficient both because it is more genetically related to the continuum case, and because, for that very reason, it makes sense even as a very crude approximation. It also provides a coordinate free way to express the solution and in general avoids the problems deriving from the need to work within a particular coordinate or "gauge" condition. What is probably most valuable, it is no harder to apply to complex topologies than to simple ones (see, e.g., Ref. [3]).

Even without these "practical" advantages simplectic methods would be of some interest for the insight they furnish into the corresponding continuum equations. And they might even offer a clue to possible discontinuous replacements for field theory that some people see as indicated by the "renormalization" and "quantum gravity" problems.

This thesis will describe a general simplectic formalism for coupled electromagnetic and (following Regge [1]) metric thatches, including detailed discussion of nets for "closed" and "open" space-time. The rendering of this formalism into computer algorithms is then described, and indicative numerical results are reported.

Chapter II

METRIC NETS

A. The Metric Thatch

One endows the net Σ with metrical character by assigning to each leg [ij] of the net a "length"--or rather, the square of a length-- ℓ_{ij}^2 .† Consider, then, a particular cell $\sigma \in \Sigma_4$ with vertices 0,1,2,3,4 (in other words, σ = [01234]). Just as the three edges of a triangle determine its internal geometry (it is "rigid"), the 10 leg lengths of σ determine its internal geometry. More formally, embed σ linearly in $\underset{\sim}{R}^4$. If under the embedding the vertex [i] corresponds to the point x_i , then we seek a (constant) metric $g_{\mu\nu}$ for $\underset{\sim}{R}^4$ such that for all i,j , $\ell_{ij}^2 = g_{\mu\nu}(x_i-x_j)^\mu\,(x_i-x_j)^\nu$. Since there are ten ℓ_{ij}^2 and ten independent components of $g_{\mu\nu}$, $g_{\mu\nu}$ must be uniquely determined. An explicit formula for it appears in Section IIIB.

It is not, however, enough that the ℓ_{ij}^2 define a metric $g_{\mu\nu}(\sigma)$ for the interior of σ . In order that σ can be a "piece" of space-time the metric must have the signature $-+++$. (This is the analogue of the triangle inequality in the Euclidean plane.) In numerical work one must check the signature at each stage.

Having defined a (flat) metric for the interior of each cell we can now "glue" these metrics together at the interface between any

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†See the summary of notation for definitions of Σ_4 , [ij] , etc.

two cells, in the obvious way. To be more precise one can introduce a coordinate system in terms of which $g_{\mu\nu}$ is constant throughout the two simplexes, σ, ρ and thus provide (the interior of) σ, ρ with a differentiable structure. Doing this for every pair of cells in Σ_4 we define a flat (pseudo-)Riemannian structure for all of the net except the boundaries of the interfaces between cells. At these latter points, the points of $\cup\Sigma_2$ (the set theoretic union of all 2-simplexes or "bones", which Regge calls the "skeleton") it may be impossible to find a coordinate system to cover smoothly all the cells which meet there. It is on these bones that the curvature is concentrated.

A two-dimensional example may clarify this. Any two of the four triangular faces of a tetrahedron join smoothly along their common edge. In fact, after removing the other faces, one could flatten them to lie in a plane without at all altering their intrinsic geometry. However, there is no coordinate patch covering a vertex and in which $g_{ij}(x)$ is a smooth function of position. The tetrahedron's intrinsic geometry is everywhere flat except at the four vertices (the "bones") where all the curvature is concentrated. In general the bones are of dimension 2 less than the manifold itself.

B. The "Defect" of a Bone

Consider the tetrahedron again. Near any particular vertex it is metrically like a cone and the deviation from flatness at the vertex can be characterized by the "defect angle", were one to cut and flatten the cone. (For a regular tetrahedron this angle is $2\pi - 3(\pi/3) = \pi$). It is easy to see that this characterization of the "defect" of a bone accords with the usual definition in terms of the non-integrability of parallel transport (see Figure 2.1).

In four dimensions the bones are 2-simplexes, but the notion of defect still applies. Since a net with metric thatch is flat everywhere but the bones, parallel transport around a loop has no effect unless the loop links some bone, and then the result depends only on which bones are linked with what orientation and in what order. In other words, it depends only on the homotopy class of the loop.

Think, now, of a single bone and a loop which circles it once. The space "surrounding" the bone comprises a "ring" of 4-simplexes (cells) whose mutual intersection is the bone itself. The loop begins in one of these, encounters the others in cyclical order, and returns finally to its point of departure. It is easy to see that a vector parallel to the bone remains unchanged throughout the whole process. Since parallel transport around any loop produces a Lorentz transformation, the "circulator" of the bone will be a Lorentz transformation which fixes the points of a 2-dimensional subspace (that of the bone)[†]. There are three cases depending on whether the bone is timelike, spacelike or null. For a timelike bone the most general circulator is rotation through an angle θ (If $t'=t$ and $z'=z$ then the most general Lorentz transformation is $x' = x \cos\theta - y \sin\theta$, $y' = y \cos\theta + x \sin\theta$ ($\theta = 0$ is <u>not</u> the same as $\theta = 2\pi$ though!)).

[†]Appendix B develops an explicit formula, in terms of affine coordinates (see below) for the circulator; it also confirms that b is invariant under "circulation".

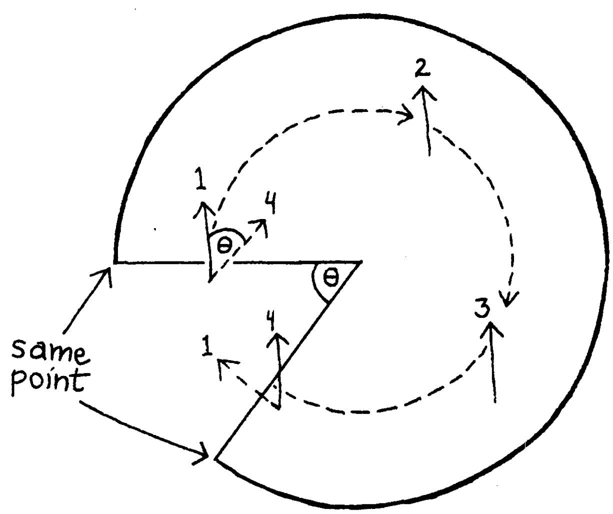


Figure 2.1 "Unrolled" cone illustrating the relation of the defect angle θ to the non-integrability of parallel transport around the bone (vertex).

For a space-like bone the most general circulator is a "boost" with parameter η (If $x'=x$, $y'=y$ then $t'=t \cosh \eta - z \sinh \eta$, $z' = z \cosh - t \sinh \eta$). And for a null bone the most general circulator is also characterized by a single parameter, which, however, is not dimensionless and can be fixed in magnitude only relative to the specific bone (see Appendix A).

C. The Action

As described above, a metric thatch induces in a net the structure of a (singular) Riemannian manifold. We show now that the continuum expression for the action $S = -\frac{1}{2}\int R\, dV$ makes sense for this manifold, and evaluate it in terms of a sum over the bones.

Since the manifold is flat everywhere outside the bones the only contribution to $R_{\mu\nu\alpha\beta}$ and a fortiori to R is from the neighborhood of a bone. But consider parallel transport around a loop linking some bone, which is a measure of $R_{\mu\nu\alpha\beta}$ there. Since the result is the same no matter where along the bone the loop links it, we see that the bone is homogeneous, and its contribution to S will be proportional to its area. Consider for example a timelike bone, infinitely extended, and find the action per unit area.

Let the bone be the t-z plane = $\{(txyz) \mid x=y=0\}$ and let it have defect θ . If $\theta = 0$ then, replacing the coordinates x,y by r,ϕ , one has for the metric tensor $g_{tt} = -1$, $g_{zz} = 1$, $g_{rr} = 1$, $g_{\phi\phi} = r^2$ with all others vanishing. We introduce the defect by deleting the "wedge" $2\pi - \theta \leq \phi < 2\pi$ from the space-time and "expand" ϕ to cover the remainder smoothly, with the result

$$g_{rr} = 1 \qquad g_{\phi\phi} = (1 - \frac{\theta}{2\pi})^2 \, r^2$$

This metric has a "cusp" at $r = 0$. In order to work with differentiable functions we will "smooth" the cusp temporarily. Thus we evaluate R for the metric

$$g_{rr} = 1 \qquad g_{\phi\phi} = e^{2\lambda(r)} \quad \text{with} \quad e^{2\lambda} = r^2 \qquad \text{for small } r$$

and

$$e^{2\lambda} = r^2(1 - \frac{\theta}{2\pi})^2 \qquad \text{for large } r .$$

The only non-vanishing Christoffel symbols are

$$\Gamma^r_{\phi\phi} = -\lambda' \, e^{2\lambda} \qquad \text{and} \qquad \Gamma^\phi_{\phi r} = \lambda' \qquad (\lambda' \equiv \frac{d\lambda}{dr})$$

Defining $R^\mu_{\nu\alpha\beta} \equiv \Gamma^\mu_{\nu\alpha,\beta} + \Gamma^{\bullet}_{\nu\alpha} \, \Gamma^{\ \mu}_{\bullet\beta} - (\alpha \leftrightarrow \beta)$

and $R \equiv g^{\mu\nu} \, g^{\alpha\beta} \, R_{\mu\alpha\nu\beta}$, one finds

$$R^\phi{}_{r\phi r} = \lambda'' + (\lambda')^2$$

$$R = 2(\lambda'' + (\lambda')^2)$$

$$\sqrt{-g} = \sqrt{-g_{tt} \, g_{zz} \, g_{rr} \, g_{\phi\phi}} = \sqrt{g_{rr} \, g_{\phi\phi}} = e^\lambda$$

Thus

$$R\sqrt{-g} = 2(\lambda'' + (\lambda')^2) \, e^\lambda = 2(e^\lambda)''$$

whence

$$-\frac{1}{2} \int\int R\sqrt{-g} \, dr \, d\phi = -2\pi \int_0^\infty (e^\lambda)'' \, dr = -2\pi (e^\lambda)' \Big]_0^\infty$$

But near $r = 0$, $e^\lambda = r \Rightarrow (e^\lambda)' = 1$, while near $r = \infty$,

$$e^{\lambda} = r(1 - \frac{\theta}{2\pi}) \Rightarrow (e^{\lambda})' = 1 - \frac{\theta}{2\pi}$$

Thus

$$-\frac{1}{2}\int\int R\sqrt{-g}\ dr\ d\phi = -2\pi\ [1 - \frac{\theta}{2\pi} - 1] = \theta$$

which is plainly independent of the degree of smoothing in the function $\lambda(r)$.

Extending the integration over the whole bone,

$$-\frac{1}{2}\iiiint R\sqrt{-g}\ dr\ d\phi\ dz\ dt = \theta\int\int dz\ dt = \theta A$$

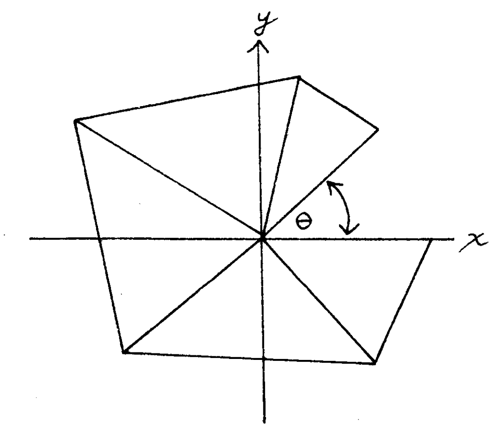
For a spacelike bone, one finds by a similar analysis $-\frac{1}{2}\int R\sqrt{-g}\ d^4x = \eta A$ in which η , the "boost parameter" is defined to be positive for a spacelike defect, which a little thought shows (Figure 2.2) to be equivalent to a timelike "infect" or "excess". For a null bone one must work with a three-dimensional metric, but finds without too much more trouble, that R and therefore S vanishes.

D. The Thatch Equations

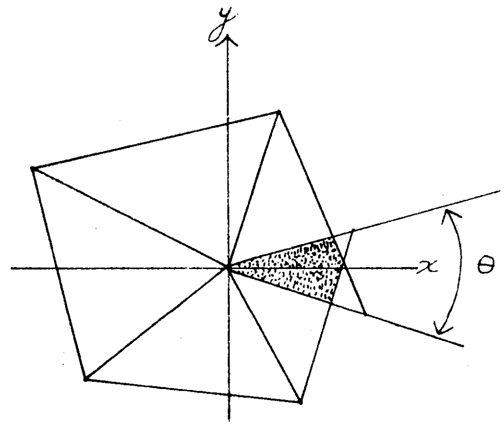
We have just found that each bone contributes to the action in an amount ηA when η stands for the defect (called "η" or "θ" above) and A is the area, considered as a positive number. Summing over all the bones we can write

$$S_g = \sum_{b\varepsilon\Sigma_2} \eta(b)\ A(b) \qquad (2.1)$$

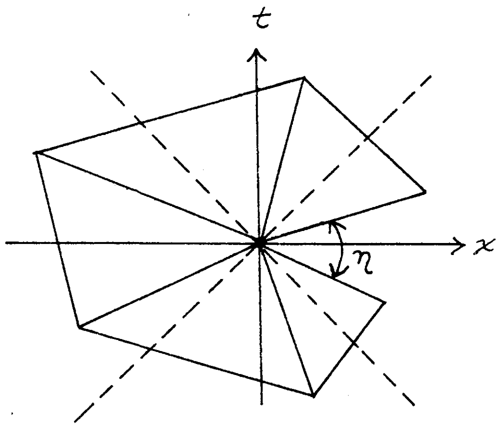
The "equations of motion" of the metrical thatch require that



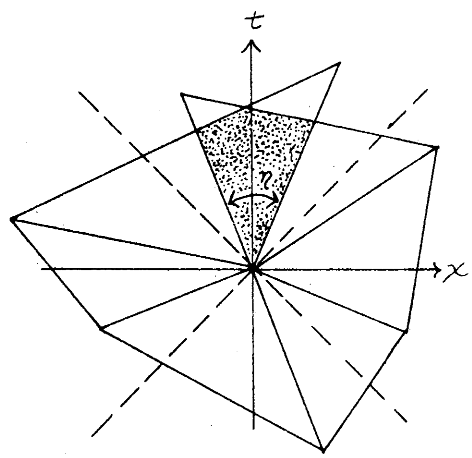
(a) timelike bone $S = \theta A$



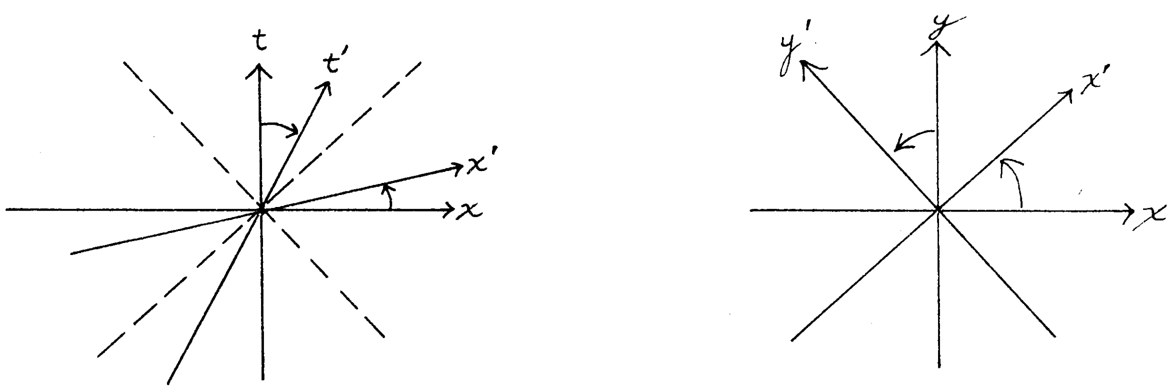
(b) timelike bone $S = -\theta A$



(c) spacelike bone $S = \eta A$



(d) spacelike bone $S = \eta A$



(e)

Figure 2.2 Typical examples of the action for a bone of area A . Cases (c) and (d) represent the <u>same</u> circulator for the reason indicated in (e).

$\delta S_g = 0$ for all variations of the thatch--in other words, for any variation of the squared leg lengths, ℓ_{ij}^2 . Carrying out the variation,

$$\delta S_g = \sum_b \eta(b)\ \delta A(b) + \sum_b \delta\eta(b)\ A(b)$$

But now, just as in the continuum theory, the second term vanishes identically (Appendix C) and we are left with

$$\delta S_g = \sum_b \eta(b)\ \delta A(b)$$

Let us express the variation with respect to a single element ℓ_{ij}^2 of the thatch. If $[ijk] \varepsilon \Sigma_2$ is a bone of the net then we will show in §IIIC that its area is

$$A(ijk) = \frac{1}{4}\sqrt{|[[i,j,k]]|}$$

in which, if we set $x = \ell_{ij}^2$, $y = \ell_{jk}^2$, $z = \ell_{ki}^2$, then

$$[[ijk]] \equiv x^2 + y^2 + z^2 - 2(xy + yz + zx)$$

Thus

$$\frac{\partial A}{\partial \ell_{ij}^2} = \frac{\partial A}{\partial x} = \frac{\pm 1}{16A}(\ell_{ij}^2 - \ell_{ik}^2 - \ell_{jk}^2)$$

where $\pm$ is the sign of $[[ijk]]$.[†] Calling this sign $\sigma(ijk)$ we get finally

$$G(ij) \equiv \frac{\partial S_g}{\partial \ell_{ij}^2} = \sum_{b\,\varepsilon\,\Sigma_2} \eta(b)\frac{\partial A(b)}{\partial \ell_{ij}^2} \qquad (2.2)$$

[†]The unlikely possibility that $A = 0$ is considered in detail in Appendix A.

$$G(ij) = \frac{1}{16} \sum_{k \varepsilon \Sigma_o} \Theta(ijk)\, \eta(ijk) \frac{\sigma(ijk)}{A(ijk)} (\ell^2_{ij} - \ell^2_{ik} - \ell^2_{jk}) \quad (2.3)$$

in which $\Theta(ijk)$ is 1 if i,j,k are the vertices of some 2-simplex (bone) and zero otherwise. The "empty space" thatch equations result from setting $G(ij) = 0$ for all legs $[ij] \varepsilon \Sigma_1$.

E. Examples in Two and Three Dimensions

In two dimensions the elements of $\sum_{n-2}$ (the bones) are 0-dimensional and curvature is concentrated entirely on the vertices. Since the "volume" of a point never changes, the variations (2.2) vanish identically, which implies that Sg is independent of the metric thatch ℓ^2_{ij} ! In fact it is well known that for a two-dimensional manifold (of signature ++) the integral of the curvature depends only on the topology of the manifold. This is the Gauss-Bonnet theorem which reads in simplectic terms

$$\sum_{x \varepsilon \Sigma_o} \theta(x) = 2\pi(V - E + F) \; ^{\dagger}$$

In three dimensions the bones coincide with the legs and (2.2) becomes

$$G(ij) = \eta(ij)$$

Then the variational equations, $G(ij) = 0$, require that all defects η vanish--the simplectic version of the well-known fact that Einstein's

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†See [1] for a proof of this from Euler's theorem.

equations have only trivial solutions in three dimensions.

F. The Thatch Equations with Source Term

If there are other terms in the action besides S_g then the variational equations will read

$$G(ij) = T(ij) \tag{2.4}$$

in which, of course, $-T(ij)$ results from varying these other terms with respect to $\ell^2(ij)$. Since $T(ij)$ must represent "matter" we can say that, simplectically, "energy-momentum is concentrated in the legs of the net", even though curvature is diffused throughout Σ_2 .

G. "Coordinate Invariance"

As pointed out in the introduction, the simplectic approach provides a coordinate-free method to specify a space-time geometry. Just for this reason, the well-known coordinate invariance of the continuum formulation finds no analogue here. It is not true, for example, that, corresponding to a given solution of $G(i,j) = 0$ there are an infinite number of others with the same boundary conditions†.

A soap film analogy may serve to clarify this. Aside(!) from the difference in dimension and in signature the "empty space" problem is very similar to that of approximating a minimal surface (soap-film)

†Except for the singular case of zero curvature.

by a polyhedron. Once the number and connectivity of the faces have been chosen, there will be a unique choice for the vertices which minimizes the total surface area (Figure 2.3). Thus, despite the coordinate ambiguity in an analytical solution, the thatch solution is unique.

Calculationally this is probably an advantage since it relieves one from the need to choose any "gauge condition" in order to define the time evolution problem. On the other hand, it leaves one with less freedom to adjust the net should the thatch begin to go singular during the course of a calculation. What one can adjust is the topological character of the net; in fact it is probably this freedom of topology† rather than any numerical freedom of the leg lengths which corresponds to the coordinate or "gauge" freedom in the continuum.

Unfortunately there is one geometry which does possess a full gauge freedom: flat space. Here, as is also clear from the soap-film analogy, the simplectic approximation to the metric is exact (all the defects vanish) and each vertex of Σ has a fourfold freedom to move without affecting the geometry. This means that the time evolution equations must become underdetermining in the flat space limit. In other words, both the attempt to produce very accurate solutions with fine nets and the treatment of asymptotically flat metrics can be expected to present extra (see Chapter IX) calculational difficulties.

†"Topology" here refers to the number and interconnection of the cells chosen to approximate a given manifold, not necessarily to the overall connectivity of the manifold itself.

Figure 2.3 A good and a bad way to approximate a curve with 4 segments. The one-dimensional representation falsifies the minimal character but is much easier to draw!

Chapter III

AFFINE COORDINATES AND SIMPLECTIC EXTERIOR CALCULUS

In working with tensors defined relative to an n-simplex it is convenient to use a system of coordinates which reflects the (n+1)-fold character of the vertices. The formalism described in this section simplifies many derivations--and it is essential to the expression of the simplectic analogue of Maxwell's equations.

A. Affine Coordinates

By considering a point P in the interior of an n-simplex as the centroid of $n+1$ masses t^j placed at the vertices $v_o \cdots v_n$ we can express it as an "affine sum"

$$P = \sum_{j=0}^{n} t^j v_j \Big/ \sum_{j=0}^{n} t^j$$

of the vertices v_j. Renormalizing the masses, we can write

$$P = \sum_{j=0}^{n} t^j v_j \tag{3.1}$$

in which

$$\sum_{j=0}^{n} t^j = 1 \tag{3.2}$$

and with all the $t^j \geq 0$. By relaxing this latter condition we can express any point in the affine space, S, of the simplex in the form (3.1), (3.2)[†]. We call $v_o, v_1, \cdots, v_n$ an affine point basis for S.

[†]An "affine space" is just a vector space in which no point is distinguished as the "origin".

A vector of an affine space is the "difference" of two points which we write as Q-P or $\overrightarrow{PQ}$. Let V denote the space of all vectors of S . If

$$P = \sum p^j v_j \quad , \quad Q = \sum q^j v_j$$

then we take for coordinates of $\overrightarrow{PQ}$ the differences $x^j \equiv q^j - p^j$. Then (3.2) implies:

$$\sum_{j=0}^{n} x^j = 0 \tag{3.3}$$

Another way to explain these coordinates is as components of $\overrightarrow{PQ}$ relative to the (redundant) "barycentric basis" comprising the n+1 vectors

$$\underset{\sim}{e}_i = v_i - \frac{1}{n+1} \sum_{k=0}^{n} v_k \tag{3.4}$$

A simple computation verifies this:

$$\sum_i x^i \underset{\sim}{e}_i = \sum_i x^i v_i - (\sum_i x^i)(\frac{1}{n+1} \sum_k v_k)$$

$$= \sum_i (q^i - p^i) v_i \qquad \text{(by (3.3))}$$

$$= \sum q^i v_i - \sum p^i v_i$$

$$= Q - P = \overrightarrow{PQ}$$

Corresponding to the basis $(\underset{\sim}{e}_j)$ for V , we introduce for the dual space V^* a basis $(\underset{\sim}{e}^i)$ defined by

$$\langle \underset{\sim}{e}^j, \underset{\sim}{e}_k \rangle = \tilde{\delta}^j_k \equiv \delta^j_k - \frac{1}{n+1} = \begin{cases} \frac{n}{n+1} & \text{if} \quad j = k \\ -\frac{1}{n+1} & \text{if} \quad j \neq k \end{cases} \tag{3.5}$$

Notice that

$$\sum_k \underset{\sim}{e}_k = 0 \quad , \quad \sum_k \underset{\sim}{e}^k = 0 \tag{3.6}$$

$$\sum_k \underset{\sim}{e}_k \otimes \underset{\sim}{e}^k = \underset{\sim}{1} \tag{3.7}$$

Let us check the last relation, for example, by applying its left hand side to the vector $a = \sum a^i \underset{\sim}{e}_i$. First, however, we introduce the self-evident lemma:

(3.8) Lemma: If for any quantities Q_j , $j=0,\cdots,n$, $\sum_j Q_j = 0$ then $Q_j = \tilde{\delta}^k_j \, Q_k$.

Continuing with the check,

$$\begin{aligned} \sum_k \underset{\sim}{e}_k \otimes \underset{\sim}{e}^k \cdot a &= \sum_k \underset{\sim}{e}_k (\underset{\sim}{e}^k \cdot \sum_i a^i \underset{\sim}{e}_i) \\ &= \sum_{k,j} \underset{\sim}{e}_k a^j (\underset{\sim}{e}^k \underset{\sim}{e}_j) \\ &= \sum_{k,j} \underset{\sim}{e}_k a^j \tilde{\delta}^k_j \\ &= \sum_j a^j \sum_k \tilde{\delta}^k_j \underset{\sim}{e}_k \\ &= \sum_j a^j \underset{\sim}{e}_j \qquad \text{(by the lemma and (3.6))} \\ &= a \end{aligned}$$

Q.E.D.

If T is any sort of tensor relative to the vector space V, we define its <u>affine components</u> $\tilde{T}^{j\cdots k}_{\ell\cdots m}$ by contracting it with the relevant product of basis vectors e_j, e^k. Then (3.7) guarantees the expansion:

$$T = \sum_{\substack{j\cdots k \\ \ell\cdots m}} \tilde{T}^{j\cdots k}_{\ell\cdots m} \; e_j \otimes \cdots \otimes e_k \otimes e^\ell \cdots \otimes e^m \tag{3.9}$$

from which follows, with the aid of (3.6),

$$\sum_j \tilde{T}^{j\cdots k}_{\ell\cdots m} = 0 \tag{3.10}$$

for any index j, up or down. This last result is the distinguishing feature of affine components and, together with (3.5) and the lemma, it guarantees that contraction works as usual by summing on the contracted indices.

Finally, we derive the affine components of two "special tensors". The "Kronecker delta tensor" δ has components formed as follows (in a slightly cumbersome notation):

$$\tilde{\delta}^j_k = \delta^\mu_\nu (e_k)^\nu \, (e^j)_\mu = (e_k)^\nu \, (e^j)_\nu = e_k \cdot e^j = \tilde{\delta}^j_k \quad ,$$

which shows the consistency of our earlier definition (3.5).

The other "special" tensor we will need is the epsilon symbol, which strictly speaking is not a tensor but a tensor density and thus defined <u>a priori</u> only up to an overall factor. We fix this factor by setting

$$\tilde{\varepsilon}^{1\,2\cdots n} = +1 \quad ,$$

from which it is easy to evaluate the other components using the antisymmetry of $\tilde{\varepsilon}^{i\cdots j}$ and the sum rule (3.10). Thus, for example,

$$\tilde{\varepsilon}^{213\cdots n} = -\tilde{\varepsilon}^{12\cdots n} = -1$$

and

$$\tilde{\varepsilon}^{023\cdots n} + \tilde{\varepsilon}^{123\cdots n} + \tilde{\varepsilon}^{223\cdots n} + \cdots + \tilde{\varepsilon}^{n23\cdots n} = 0$$

$$\tilde{\varepsilon}^{023\cdots n} + 1 + 0 + \cdots + 0 = 0$$

$$\tilde{\varepsilon}^{023\cdots n} = -1 \, .$$

Let $j_o j_1 \cdots j_n$ be any permutation of the indices $01\cdots n$. Then

$$\tilde{\varepsilon}^{j_1\cdots j_n} = \begin{cases} +1 & \text{if the permutation } j_o j_1 \cdots j_n \text{ is even} \\ -1 & \text{if the permutation } j_o j_1 \cdots j_n \text{ is odd} \end{cases} \qquad (3.11)$$

We note without proof that our definition is equivalent (for the contravariant ε) to

$$\varepsilon = \overrightarrow{v_o v_1} \wedge \overrightarrow{v_o v_2} \wedge \cdots \wedge \overrightarrow{v_o v_n}$$

A final subtlety needs mention. Let $n=3$ for definiteness. Then under the usual definitions

$$\varepsilon^{\sigma\mu\nu}\varepsilon_{\sigma\alpha\beta} = \delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\,\delta^{\nu}_{\alpha}$$

That the analogous formula apply to $\tilde{\varepsilon}^{ijk}$ and $\tilde{\varepsilon}_{i\ell m}$ requires, as is easily checked, that $\tilde{\varepsilon}^{123}\tilde{\varepsilon}_{123} = \frac{1}{4} = \frac{1}{n+1}$. Accordingly, we define the covariant ε with components of magnitude $(n+1)^{-1}$:

$$\tilde{\varepsilon}_{j_1\cdots j_n} = \frac{1}{n+1}\,\tilde{\varepsilon}^{j_1\cdots j_n} \tag{3.12}$$

With these definitions all the expected formulas obtain.

B. An Application

As a first application we calculate the metric tensor $\tilde{g}_{ij}$ from the edge-lengths of the simplex, as promised in Section IIA. Let ℓ^2_{ij} be the length squared of the edge joining v_i to v_j. Then, since, plainly, $\overrightarrow{v_i v_j} = \underset{\sim}{e}_j - \underset{\sim}{e}_i$

$$\begin{aligned}
\ell^2_{ij} &= \langle g, \overrightarrow{v_i v_j} \otimes \overrightarrow{v_i v_j} \rangle \\
&= \langle g, (\underset{\sim}{e}_j - \underset{\sim}{e}_i) \otimes (\underset{\sim}{e}_j - \underset{\sim}{e}_i)\rangle \\
&= \langle g, \underset{\sim}{e}_j \otimes \underset{\sim}{e}_j\rangle - 2\langle g, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j\rangle + \langle g, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_i\rangle \\
&= \tilde{g}_{jj} - 2\tilde{g}_{ij} + \tilde{g}_{ii} \\
&\equiv A_{ij}
\end{aligned}$$

By forming the combination $\tilde{\delta}^k_i \tilde{\delta}^\ell_j A_{k\ell}$ we can, in view of (3.8) and (3.10) as applied to $\tilde{\delta}$, recover $\tilde{g}_{ij}$:

$$\tilde{\delta}^k_i\, \tilde{\delta}^\ell_j\, \ell^2_{k\ell} = 0 - 2\tilde{g}_{ij} + 0$$

$$\tilde{g}_{ij} = -\frac{1}{2}\,\ell^2_{k\ell}\,\tilde{\delta}^k_i\,\tilde{\delta}^\ell_j \tag{3.13}$$

which says that $\tilde{g}_{ij}$ is just $-\frac{1}{2}\ell_{ij}^2$ "rendered affine" or "projected into the affine tensors".

C. Bordered Determinants and Volumes

In this section we fix some normalizations and derive a useful expression for the volume of a simplex.

Let the wedge product be defined in the usual way and normalized so that, for instance, the wedge product $a \wedge b \wedge c$ of three vectors consists of six terms each with coefficient ±1 . Then we take the product $a \wedge b$ to represent the parallelogram determined by a and b, and $\frac{1}{2} a \wedge b$ the triangle or 2-simplex spanned by them. In general, the normalized product

$$\omega = \frac{1}{m!} a_1 \wedge \cdots \wedge a_m \tag{3.14}$$

will represent the m-simplex spanned by vectors $a_1 \cdots a_m$. With these definitions the volume represented by any rank m totally antisymmetric tensor ω is

$$\text{volume}(\omega) = \sqrt{\frac{|\langle\omega,\omega\rangle|}{m!}} \equiv ||\,\omega\,|| \tag{3.15}$$

(The absolute value is needed because of the indefinite metric, i.e., the volume is defined to be a real number. Thus, for example, the bone [012] of some 4-simplex σ corresponds to the tensor

$$\begin{aligned}\omega &= \frac{1}{2!} \overrightarrow{v_o v_1} \wedge \overrightarrow{v_o v_2} \\ &= \frac{1}{2!} (\underset{\sim}{e}_1 - \underset{\sim}{e}_o) \wedge (\underset{\sim}{e}_2 - \underset{\sim}{e}_o)\end{aligned}$$

$$= \frac{1}{2!} (\underset{\sim}{e}_1 \wedge \underset{\sim}{e}_2 + \underset{\sim}{e}_2 \wedge \underset{\sim}{e}_o + \underset{\sim}{e}_o \wedge \underset{\sim}{e}_1)$$

$$= \frac{1}{2!} \tilde{\varepsilon}^{ij}_{(012)} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$$

where $\tilde{\varepsilon}^{ij}_{(012)}$ is of course formed from the indices 0,1,2 in the manner of (3.11). In general the m subsimplex with vertices $k_o \cdots k_m$ corresponds via (3.14) to the tensor with affine components.

$$\tilde{\omega}^{j_1 \cdots j_m} = \frac{1}{m!} \tilde{\varepsilon}^{j_1 \cdots j_m}_{(k_o \cdots k_m)} \tag{3.16}$$

According to (3.15) the volume, V, of this simplex is given by (assume $m=2=n$ for ease of notation)

$$\pm V^2 = \frac{\langle \omega, \omega \rangle}{m!} = \frac{\tilde{\omega}^{ij} \tilde{g}_{i\ell} \tilde{g}_{jk} \tilde{\omega}^{\ell k}}{m!}$$

But (3.13) and (3.8) imply a remarkable simplification:

$$\pm V^2 = (-\tfrac{1}{2})^m (m!)^{-1} \tilde{\omega}^{ij} \tilde{\omega}^{\ell k} \ell^2_{i\ell} \ell^2_{jk}$$

Or working out the general case, for ω of rank m :

$$\pm \mathrm{Vol}^2 = (-\tfrac{1}{2})^m \frac{1}{m!} \tilde{\omega}^{j_1 \cdots j_m} \tilde{\omega}^{k_1 \cdots k_m} \prod_{a=1}^{m} \ell^2_{j_a k_a} \tag{3.17}$$

To find the volume of any m-simplex of the net we can work within the m-dimensional affine space spanned by that simplex and (calling its vertices $0 \cdots m$) write

$$\tilde{\omega}^{j_1 \cdots j_m} = \frac{1}{m!} \tilde{\varepsilon}^{j_1 \cdots j_m}$$

Then

$$\pm \text{Vol}^2 = (-\tfrac{1}{2})^m \, (m!)^{-3} \, \tilde{\varepsilon}^{j_1 \cdots j_m} \, \tilde{\varepsilon}^{k_1 \cdots k_m} \, \ell^2_{j_1 k_1} \cdots \ell^2_{j_m k_m} \tag{3.18}$$

To facilitate numerical evaluation of such expressions we introduce the concept of a "bordered determinant" which has the form (with A representing any $m \times m$ matrix)

$$B(A) \equiv \begin{vmatrix} 0 & 1 & 1 \cdots 1 \\ 1 & & \\ 1 & \multicolumn{2}{c}{\begin{bmatrix} & \\ & A & \\ & \end{bmatrix}} \\ \vdots & & \\ 1 & & \end{vmatrix}$$

Then the expression

$$\tilde{\varepsilon}^{j_1 \cdots j_m} \, \tilde{\varepsilon}^{k_1 \cdots k_m} \, A_{j_1 k_1} \cdots A_{j_m k_m}$$

can be evaluated as $-m!\; B(A)$, the proof being left to the interested reader. Thus we get the expression for volume in terms of edges, as

$$\pm \text{Vol}^2 = -(-\tfrac{1}{2})^m \, (m!)^{-2} \begin{vmatrix} 0 & 1 & 1 \cdots 1 \\ 1 & & \\ 1 & & \ell^2_{ij} \\ \vdots & & \\ 1 & & \end{vmatrix} \tag{3.19}$$

a result which appears in [4].

For a triangle we find (setting $m=2$, $x = \ell^2_{01}$, $y = \ell^2_{02}$, $z = \ell^2_{12}$)

$$\pm A^2 = -(-\tfrac{1}{2})^2 \ (2!)^{-2} \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & x & y \\ 1 & x & 0 & z \\ 1 & y & z & 0 \end{vmatrix}$$

$$= -\frac{1}{16} [x^2 + y^2 + z^2 - 2(xy + yz + zx)]$$

as promised in Section IID.

D. Simplectic Exterior Calculus

Let Σ_o = set of all 0-simplexes (vertices) of the net,

Σ_1 = set of all oriented 1-simplexes (legs) of the net,

Σ_2 = set of all oriented 2-simplexes of the net,

etc.

and represent a typical oriented 2-simplex, e.g., as [xyz] where $x,y,z \in \Sigma_o$. Then we define†

a 0-form (scalar thatch) as a map $\phi : \Sigma_o \to \underset{\sim}{R}$;

a 1-form (co-vector thatch) as a map $A: \Sigma_1 \to \underset{\sim}{R}$ such that $A(xy) = -A(yx)$;

a 2-form as a map $F: \Sigma_2 \to \underset{\sim}{R}$ such that $F(xyz) = -F(yxz) = F(yzx)$, etc;

etc.

To understand these definitions one could think of $A(xy)$, e.g., as the line integral $\int_x^y A_\mu dx^\mu$ of some field A_μ along the leg [xy]. If, then, $F = dA$, then Stokes theorem becomes

† k-forms are also called k-cochains in combinatorial topology.

$$F(xyz) \leftrightarrow \int_{[xyz]} F_{\mu\nu} da^{\mu\nu} = \oint_{[xy]+[yz]+[zx]} A_\mu dx^\mu$$

$$\leftrightarrow \quad A(xy) + A(yz) + A(zx)$$

Generalizing this relation to arbitrary dimension we define the operator "d" from m- to (m+1)-forms as follows:

$$d\omega(k_o k_1 \cdots k_m) = \sum_{j=0}^{m} (-1)^j \; \omega(k_o \cdots \hat{k}_j \cdots k_m) \tag{3.20}$$

where the "hat" indicates omission. It is easy to check that

$$dd\omega = 0 \quad : \tag{3.21}$$

$$dd\omega(k_o \cdots k_{m+1}) = \sum_{j=0}^{m+1} d\omega \; (k_o \cdots \hat{k}_j \cdots k_{m+1})(-1)^j$$

$$= \sum_{j=0}^{m+1} (-1)^j \sum_{\ell=0}^{m+1} (-1)^\ell \; \mathrm{sgn}(\ell-j)$$

$$\times \; \omega(k_o \cdots \hat{k}_a \cdots \hat{k}_b \cdots k_{m+1})$$

(where $a = \min(j,\ell)$, $b = \max(j,\ell)$)

$$= \sum_{j,\ell=0}^{m+1} (-1)^{j+\ell} \; \mathrm{sgn}(\ell-j) \; \omega(k_o \cdots \hat{k}_a \cdots \hat{k}_b \cdots k_{m+1})$$

$$= \; 0$$

since $\mathrm{sgn}(\ell-j)$ alone in the penultimate expression is antisymmetric in the exchange of j with ℓ .

Q.E.D.

The main theorem of this section is in part a partial converse to (3.21).

Consider now a particular m-form ω and a particular simplex $\sigma \varepsilon \Sigma_4$ and ask whether ω "extends to the interior of σ", i.e., whether there is defined in the space of σ an m-form $\omega(\sigma)$ which agrees with ω on all the m-subsimplexes of σ. By definition $\omega(\sigma)$ "agrees with" ω on $[x_o \cdots x_m]$ iff

$$\langle\omega(\sigma),[x_o \cdots x_m]\rangle = \omega(x_o \cdots x_m) \tag{3.22}$$

If ω does extend to σ then we can form a simple expression (analogous to (3.13)) for the affine components of $\omega(\sigma)$:

From Section IIIC we know that, calling $\sigma = [01 \cdots n]$,

$$m![k_o \cdots k_m] = \overrightarrow{k_o k_1} \wedge \overrightarrow{k_o k_2} \wedge \cdots \wedge \overrightarrow{k_o k_m}$$

$$= (\underset{\sim}{e}_{k_1} - \underset{\sim}{e}_{k_o}) \wedge (\underset{\sim}{e}_{k_2} - \underset{\sim}{e}_{k_o}) \wedge \cdots \wedge (\underset{\sim}{e}_{k_m} - \underset{\sim}{e}_{k_o})$$

In the expansion of the right hand side only terms lacking or linear in $\underset{\sim}{e}_{k_o}$ survive since $\underset{\sim}{e}_{k_o} \wedge \underset{\sim}{e}_{k_o} = 0$, and one obtains, in a hopefully clear notation:

$$m![k_o \cdots k_m] = \underset{\sim}{e}_{k_1} \wedge \cdots \wedge \underset{\sim}{e}_{k_m} - \sum_{j=1}^{m} \underset{\sim}{e}_{k_1} \wedge \cdots \wedge \underset{(j)}{\underset{\sim}{e}_{k_o}} \cdots \wedge \underset{\sim}{e}_{k_m} \tag{3.23}$$

Because of (3.6) we can isolate $\underset{\sim}{e}_{k_1} \wedge \cdots \wedge \underset{\sim}{e}_{k_m}$ by summing on k_o :

$$m! \sum_{k_o=0}^{n} [k_o \cdots k_m] = (n+1)\, \underset{\sim}{e}_{k_1} \wedge \cdots \wedge \underset{\sim}{e}_{k_m}$$

$$\underset{\sim}{e}_{k_1} \wedge \cdots \wedge \underset{\sim}{e}_{k_m} = \frac{m!}{1+n} \sum_{k_o=0}^{n} [k_o k_1 \cdots k_m] \tag{3.24}$$

Applying $\omega(\sigma)$ to both sides

$$m! \; \tilde{\omega}(\sigma)_{k_1 \cdots k_m} = \frac{m!}{1+n} \sum_{k_o=0}^{n} \langle \omega(\sigma), \; [k_o \cdots k_m] \rangle$$

$$\tilde{\omega}(\sigma)_{k_1 \cdots k_m} = \frac{1}{1+n} \sum_{k_o=0}^{n} \omega(k_o k_1 \cdots k_m) \tag{3.25}$$

In order to study this condition more closely we make the definition (relative to the simplex σ)

$$S\omega(k_1 \cdots k_m) = \frac{1}{1+n} \sum_{k_o=0}^{n} \omega(k_o k_1 \cdots k_n) \tag{3.26}$$

so that (3.25) can be expressed in the droll form

$$\tilde{\omega}(\sigma)_{k_1 \cdots k_m} = S\omega(k_1 \cdots k_m) \tag{3.27}$$

It is easy to see that $S\omega$ is an (m-1)-form (on σ) when ω is an m-form, and that

$$S^2 = 0 \tag{3.28}$$

We can also verify the important relation (relative to σ as always)

$$Sd + dS = 1 \tag{3.29}$$

<u>Proof</u>:

$$dS\omega(k_o \cdots k_m) = \sum_{j=0}^{m} (-1)^j \; S\omega(k_o \cdots \hat{k}_j \cdots k_m) =$$

$$= \sum_{j=0}^{m} (-1)^j \frac{1}{1+n} \sum_{\ell=0}^{n} \omega(\ell k_o \cdots \hat{k}_j \cdots k_m)$$

$$= \frac{1}{1+n} \sum_{j=0}^{m} \sum_{\ell=0}^{n} \omega(k_o \cdots \underset{(j)}{\ell} \cdots k_m)$$

$$Sd\omega(k_o \cdots k_m) = \frac{1}{1+n} \sum_{\ell=0}^{n} d\omega(\ell k_o \cdots k_m)$$

$$= \frac{1}{1+n} \sum_{\ell=0}^{n} [\omega(k_o \cdots k_m) - \sum_{j=0}^{m} (-1)^j \omega(\ell k_o \cdots \hat{k}_j \cdots k_m)]$$

$$= \omega(k_o \cdots k_m) - \frac{1}{1+n} \sum_{\ell=0}^{n} \sum_{j=0}^{m} \omega(k_o \cdots \underset{(j)}{\ell} \cdots k_m)$$

Comparing the two expressions completes the proof.

Returning to the question whether ω extends to σ , we note that the formula for $\omega(\sigma)$ given in (3.27) or (3.25) will define a form in the space of σ whether or not ω extends to σ . If we call this form ρ then the condition that ρ agree with ω on σ (which is just that ω extend to σ) becomes

$$<\rho, [k_o \cdots k_m]> = \omega(k_o \cdots k_m)$$

But by (3.23)

$$<\rho, [k_o \cdots k_m]> = \frac{1}{m!} <\rho, \underset{\sim}{e}_{k_1} \wedge \cdots \wedge \underset{\sim}{e}_{k_m}> - \frac{1}{m!} \sum_{j=1}^{m} <\rho, \underset{\sim}{e}_{k_1} \wedge \cdots \underset{\sim}{e}_{k_o} \wedge \cdots \underset{\sim}{e}_{k_m}>$$

$$= \tilde{\rho}_{k_1 \cdots k_m} - \sum_{j=1}^{m} \tilde{\rho}_{k_1 \cdots \underset{(j)}{k_o} \cdots k_m}$$

$$= \frac{1}{1+n} \sum_{\ell=0}^{n} \omega(\ell k_1 \cdots k_m) - \sum_{j=1}^{m} \frac{1}{1+n} \sum_{\ell=0}^{n} \omega(\ell k_1 \cdots \underset{(j)}{k_o} \cdots k_m)$$

$$= \frac{1}{1+n} \sum_{\ell=0}^{n} \omega(\ell k_1 \cdots k_m) + \frac{1}{1+n} \sum_{j=1}^{m} \sum_{\ell=0}^{n} \omega(k_o k_1 \cdots \underset{(j)}{\ell} \cdots k_m)$$

$$= \frac{1}{1+n} \sum_{\ell=0}^{n} \sum_{j=0}^{m} \omega(k_o \cdots \underset{(j)}{\ell} \cdots k_m)$$

Comparing this with the proof of (3.29) furnishes the condition for ω to extend to σ in the form

$$dS\ \omega = \omega \qquad (3.30)$$

We can now prove the following fundamental theorem which has been the goal of this section:

(3.31) Theorem: Let ω be any form defined on a net including the simplex σ and set $\Omega = S\omega$ as defined in (3.26). Then the following three conditions are equivalent:

(1) $\omega = d\Omega$

(2) $d\omega = 0$

(3) ω extends to σ, the extension being furnished by (3.25).

Proof: We just saw that we can replace (3) by the condition

(3') $dS\omega = \omega$

we already know by (3.21)) that (1) $\Rightarrow$ (2), and (3') $\Rightarrow$ (1) is obvious. To complete the circle of implication we need only (2) $\Rightarrow$ (3') which follows immediately from (3.29) applied to ω.

Q.E.D.

By the way, Ω becomes unique through the condition $S\Omega = 0$, which follows from (3.28).

Chapter IV

THE ELECTROMAGNETIC THATCH

(All components in this chapter are affine components--but the tilde ($\tilde{}$) will usually be omitted.)

A. The Source Free Thatch Equations

In this section we assume a net Σ with fixed metric thatch ℓ^2_{ij} and the associated metric tensors $g(\sigma)$ for each $\sigma \varepsilon \Sigma_4$ (see Section IIIB).

The "vector potential" A is a 1-form on Σ, as defined in Section IIID, and $F = dA$ is the electromagnetic thatch. By the theorem of the previous chapter F extends in each cell $\sigma \varepsilon \Sigma_4$ to a tensor $F_{ij}(\sigma)$ given by (3.25). Calling $V(\sigma)$ the volume of σ, we take for the action

$$\begin{aligned} S_e &= -\frac{1}{4} \sum_{\sigma \varepsilon \Sigma_4} V(\sigma) \quad \langle F(\sigma), F(\sigma) \rangle_\sigma \\ &= -\frac{1}{4} \sum_\sigma V(\sigma)\ F^{ij}(\sigma)\ F_{ij}(\sigma) \\ &= -\frac{1}{4} \sum_\sigma V(\sigma)\ g(\sigma)^{ia}\ g(\sigma)^{jb}\ F(\sigma)_{ij}\ F(\sigma)_{ab} \end{aligned} \tag{4.1}$$

The thatch equations equate to zero the variation of S with respect to the thatch A:

$$\frac{\partial S}{\partial A(ij)} = 0 \qquad \text{for all legs } [ij] \ \varepsilon \ \textstyle\sum_1 \tag{4.2}$$

Well,

$$\delta S_e = -\frac{1}{2} \sum_{\sigma} V(\sigma)\ F^{ij}(\sigma)\ \delta F_{ij}(\sigma)$$

But because of (3.25)

$$\begin{aligned} F_{ij}(\sigma) &= \frac{1}{5} \sum_{k\epsilon\sigma} F(kij) \\ &= \frac{1}{5} \sum_{k} (A(ij) + A(jk) + A(ki)) \\ &= A(ij) + \frac{1}{5} \sum_{k} (A(jk) + A(ki)) \end{aligned} \tag{4.3}$$

or

$$\delta F_{ij}(\sigma) = \delta A(ij) + \frac{1}{5} \sum_{k\epsilon\sigma} (\delta A(jk) + \delta A(ki))$$

Substituting this into the expression for δS_e ,

$$\delta S_e = -\frac{1}{2} \sum_{\sigma} V(\sigma)\ F^{ij}(\sigma)\ [\delta A(ij) + \frac{1}{5} \sum_{k\epsilon\sigma} (\delta A(jk) + \delta A(ki))]$$

Because of the sum on i and j respectively, the second and third terms vanish, and there remains finally

$$\delta S_e = -\frac{1}{2} \sum_{\sigma} V(\sigma)\ F^{ij}(\sigma)\ \delta A(ij) \quad ^{\dagger} \tag{4.4}$$

---

† Thus δS_e is the same function of $\delta A(ij)$ that it is of δF_{ij} . In fact the reasoning leading to (4.4) reveals this general rule:

Whenever $\tilde{F}_{ij}(\sigma)$ occurs with both its indices contracted against affine indices, it can be replaced by $A(ij)$! The same applies to $\tilde{g}_{ij}(\sigma) \to -\frac{1}{2} \ell_{ij}^2$.

There is thus one thatch equation for each leg of the net:

$$\sum_{\sigma} (i,j|\sigma)\ V(\sigma)\ F^{ij}(\sigma) \doteq 0 \qquad (4.5)$$

where the symbol "$(i,j|\sigma)$" specifies that i and j are vertices of σ .

We can also express $F^{ij}(\sigma)$, and thereby the thatch equations directly in terms of A . From the footnote to (4.4)

$$F^{ij}(\sigma) = g^{ia}(\sigma)\ g^{jb}(\sigma)\ F_{ab} = g^{ia}(\sigma)\ g^{jb}(\sigma)\ A(ab)$$

$$F^{ij}(\sigma) = \frac{1}{2}\ h^{ijab}(\sigma)\ A(ab) \qquad (4.6)$$

where

$$h^{ijab} = g^{ia}\ g^{jb} - g^{ib}\ g^{ja}$$

B. The Equations with a Source--Charge Conservation

If there is prescribed a source, J , then the action has an additional term

$$S_i = \sum_{[ij]\,\varepsilon\,\Sigma_1} A(ij)\ J(ij) \qquad (4.7)$$

in which $J(ij)$ should be considered, not as a 1-form, but rather as a "vector density" or "current". In place of (4.4) stands (half of)

$$-\sum_{\sigma} V(\sigma)\ F^{ij}(\sigma)\ \delta A(ij) + J(ij)\ \delta A(ij)$$

so that the thatch equations become

$$\sum_{\sigma} V(\sigma)\ F^{ij}(\sigma) = J(ij) \tag{4.8}$$

The natural interpretation of J regards J(ij) as the charge flowing "along" leg [ij] of the net. It is as if $\sum_1$ were an electrical network, A the potential drop, and J the current. Then the conservation of charge (like one of Kirchhoff's laws) reads

$$\sum_{j} J(ij) = 0 \tag{4.9}$$

and follows from (4.8) because of the rule (3.10). To clarify notation we write the proof pedantically:

$$\sum_{j\,\varepsilon\,\Sigma_o} \Theta(i,j)\ J(ij)$$

$$= \sum_{j\,\varepsilon\,\Sigma_o} \Theta(i,j) \sum_{\sigma\,\varepsilon\,\Sigma_4} (i,j|\sigma)\ V(\sigma)\ F^{ij}(\sigma)$$

$$= \sum_{\sigma}\sum_{j} (i|\sigma)\ (j|\sigma)\ V(\sigma)\ F^{ij}(\sigma)$$

$$= \sum_{\sigma} (i|\sigma)\ V(\sigma) \sum_{j} (j|\sigma)\ F^{ij}(\sigma)$$

$$= \sum_{\sigma|i} V(\sigma) \sum_{j\,\varepsilon\,\sigma} F^{ij}(\sigma)$$

$$= 0$$

We can also cast the conservation law in an "integral" form as opposed to its "local" statement (4.9): Let $\Omega \subset \Sigma_o$ be all the vertices in some region of the net and form the two expressions

$$\sum_{i\in\Omega}\sum_{k\in\Sigma_o} J(ik) \qquad \text{and} \qquad \sum_{i,k\,\in\Omega} J(ik)$$

The first vanishes by the equation of conservation (4.9) and the second by the antisymmetry of J. Then

$$0 = \sum_{i\in\Omega}\sum_{k\in\Sigma_o} J(ik)$$

$$= \sum_{i\in\Omega}\Big(\sum_{k\in\Omega} + \sum_{k\notin\Omega}\Big) J(ik)$$

$$= 0 + \sum_{i\in\Omega}\sum_{k\notin\Omega} J(ik)$$

$$\sum_{i\in\Omega}\sum_{k\notin\Omega} J(ik) = 0 \tag{4.10}$$

In words: "The total charge leaving the region Ω vanishes."

C. Gauge Invariance

As usual $F = dA$ determines A only up to an addition of the form $d\theta$, for arbitrary 0-form θ. Since A does not occur explicitly in S_e we are free to require invariance under the "gauge transformation"

$$A \to A + d\theta \tag{4.11}$$

as long as the interaction term (4.7) is unaffected. But under (4.11) S_i acquires an additional term

$$\frac{1}{2}\sum_{i,j} d\theta(ij)\ J(ij)^{\dagger}$$

$$= \frac{1}{2}\sum [\theta(j) - \theta(i)]\ J(ij)$$

$$= -\sum_{i} \theta(i) \sum_{j} J(ij)$$

whence gauge invariance requires

$$\sum_{j} J(ij) = 0$$

since θ is arbitrary. This is exactly the familiar connection between gauge invariance and charge conservation.

Since the gauge freedom of A introduces a free number for each vertex of the net, one can remove this freedom by imposing one condition at each vertex. One which suggests itself is

$$\sum_{j} A(ij) = 0 \quad \text{at all} \quad i \in \Sigma_o \tag{4.12}$$

This looks something like the "Lorentz gauge" but it isn't, since A is a 1-form rather than a current.

D. Coupling to the Metric Thatch--The Energy-Momentum Tensor

Equation (4.5) already includes the effects of an arbitrary background metric. To find the reciprocal influence of the electromagnetic thatch on the metric we must evaluate

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† The factor $\frac{1}{2}$ avoids double counting of legs in (4.7).

$$T(ij) = -\frac{\partial S_e}{\partial \ell_{ij}^2} \tag{4.13}$$

Writing (4.1) in the form

$$\begin{aligned} S_e &= \sum_\sigma L(\sigma) \\ L(\sigma) &\equiv -\frac{1}{4} V(\sigma)\ g^{\mu\mu} g^{\nu\nu} F_{\mu\nu}(\sigma)\ F_{\mu\nu}(\sigma) \end{aligned} \tag{4.14}$$

and varying the metric $g(\sigma)$ interior to σ , one finds

$$\begin{aligned} 2\delta L &= -\frac{1}{2} \delta V <F,F> - V\ g^{\mu\mu} \delta g^{\nu\nu}\ F_{\mu\nu} F_{\mu\nu} \\ &= -\frac{1}{2} \delta V <F,F> + V\ g_{\mu\mu} \delta g_{\nu\nu}\ F^{\mu\nu} F^{\mu\nu} \end{aligned} \tag{4.15}$$

If we express this in affine components, then δV assumes a simple form which follows readily from the method of §IIIC:

$$\delta V = \frac{1}{2} V\ \tilde{g}^{ij}\ \delta \tilde{g}_{ij} \tag{4.16}$$

whence

$$\begin{aligned} 2\delta L &= V\ \tilde{g}_{ii}\ \delta \tilde{g}_{jj}\ \tilde{F}^{ij}\ \tilde{F}^{ij} - \frac{1}{4} V\ \tilde{F}^{ab}\ \tilde{F}_{ab}\ \tilde{g}^{jj}\ \delta \tilde{g}_{jj} \\ &= V\ \delta \tilde{g}_{jj}\ \{\tilde{F}^{ij} \tilde{g}_{ii} \tilde{F}^{ij} - \frac{1}{4} \tilde{F}^{ab} \tilde{F}_{ab} \tilde{g}^{jj}\} \\ &= V(\sigma)\ \delta \tilde{g}_{jk}(\sigma)\ \tilde{T}^{jk}(\sigma) \end{aligned} \tag{4.17}$$

in which

$$\tilde{T}^j{}_k(\sigma) \equiv \tilde{F}^{ja}(\sigma)\ \tilde{F}_{ka}(\sigma) - \frac{1}{4} \tilde{F}^{ab}(\sigma)\ \tilde{F}_{ab}(\sigma)\ \tilde{\delta}^j{}_k \tag{4.18}$$

is the well-known formation in terms of $\tilde{F}_{ij}, \tilde{g}_{ij}$. In view of the footnote to equation (4.4) this becomes

$$\delta L = -\frac{1}{4} \ell_{jk}^2 \, V \, \tilde{T}^{jk}$$

so that, finally

$$\begin{aligned} \delta S &= \sum_{\sigma} \delta L(\sigma) \\ &= -\frac{1}{4} \sum_{\sigma} \sum_{j,k} (j,k|\sigma) \, \ell_{jk}^2 \, V(\sigma) \, \tilde{T}^{jk}(\sigma) \\ &= -\frac{1}{4} \sum_{j,k} \ell_{jk}^2 \sum_{\sigma} (j,k|\sigma) \, V(\sigma) \, \tilde{T}^{jk}(\sigma) \\ &= - \sum_{[j,k] \in \Sigma_1} \ell_{jk}^2 \, T_e(jk) \end{aligned}$$

$$T_e(jk) = \frac{1}{2} \sum_{\sigma} (j,k|\sigma) \, V(\sigma) \, \tilde{T}^{jk}(\sigma) \tag{4.19}$$

The thatch equations (2.4) for coupled electromagnetic and metric thatches are thus given explicitly by (2.3) and (4.19).

Chapter V

NETS FOR OPEN AND CLOSED SPACE-TIMES

A. A Simplectic Net for $\underset{\sim}{R}^4$

The most natural path to arrive at a decomposition of $\underset{\sim}{R}^4$ into 4-simplexes is this: Cover or "tile" $\underset{\sim}{R}^4$ by rectangular regions of which at most 5 intersect at any point; the nerve of this covering (see [2]) will furnish the desired net (Figure 5.1). Having taken this path, however, it appears that the answer can be gained directly, and most clearly presented in affine coordinates, which moreover are perfectly suited to the symmetries of the net.

We will describe the net by specifying Σ_o and Σ_1 (that is, the "network" or "graph" formed by the legs). Then the following simple rule (which just expresses that Σ is a nerve) defines Σ_k for k=2,3,4:

Any k+1 vertices span a k-simplex of the net iff they are mutually joined by legs.

Σ_o comprises all the points of the lattice generated by the 5 vectors (e_j, $j=0,\cdots,4$) described in Section IIIA. More explicitly, it consists of

(i) all vectors with integral components (recall that affine vector components sum to zero)

(ii) vectors differing from those of (i) by one of the following 30 (=10 + 20) vectors:

$$\pm\frac{1}{5}(4 -1 -1 -1 -1)\ ,\ \pm\frac{1}{5}(-1\ 4 -1 -1 -1)\ ,\ \cdots\ ,\pm\frac{1}{5}(-1 -1 -1 -1\ 4)\ ,$$

$$\pm\frac{1}{5}(3\ 3 -2 -2 -2)\ ,\ \pm\frac{1}{5}(3 -2\ 3 -2 -2)\ ,\ \cdots\ ,\pm\frac{1}{5}(-2 -2 -2\ 3\ 3)$$

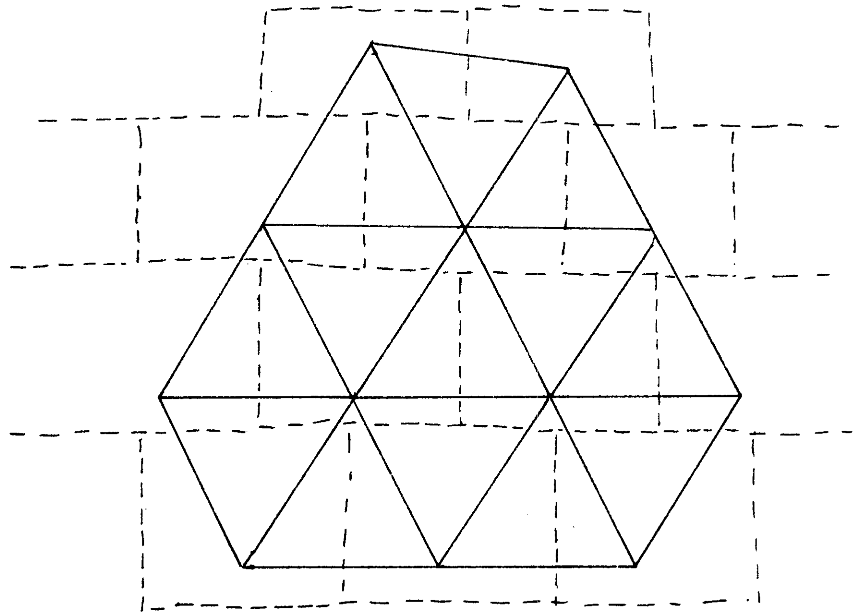

Figure 5.1 A familiar simplectic net for $\underset{\sim}{R}^2$ as the "nerve" of a tiling.

Finally, any pair of vertices which differ by a vector of the type (ii) (equivalently by a vector all of whose components are less than 1 in absolute value) determine a leg of the net.

This completes the definition of the net. In the rest of this section we verify that it is in fact a triangulation of $\underset{\sim}{R}^4$ and we expose some of its properties:

(5.1) All vertices of the net are equivalent. This follows from the definition of Σ , which is invariant under translation through any lattice vector.

(5.2) The "isotropic group" of all symmetries of Σ fixing the origin, comprises the 5! permutations of the coordinates with or without an overall sign change. It has therefore 240 elements.

(5.3) The cells (4 simplexes) of the net fill $\underset{\sim}{R}^4$ without gaps and without any overlap. In other words, the net really is a net. To prove this we note that if any flaw or overlap occurs, it must occur also in the neighborhood of some vertex. Then by (5.1) it is enough to look near the origin (00000). It is easy to see that the only cells that come near the origin are those related by one of the symmetries (5.2) to the cell σ_o with vertices

$$\tfrac{1}{5}(00000) \quad , \quad \tfrac{1}{5}(4-1-1-1-1) \quad , \quad \tfrac{1}{5}(3\,3\,-2-2-2)$$

$$\tfrac{1}{5}(2\,2\,2-3-3) \quad , \quad \tfrac{1}{5}(1111-4)$$

But this subset of the vectors of type (ii) (see above) is characterized by the ordering $x^o \geq x^1 \geq x^2 \geq x^3 \geq x^4$ for its coordinates. Furthermore any point interior to σ_o is a convex sum (with positive coefficients) of the vertices of σ_o and thus enjoys the same ordering. On the other hand any point in the neighborhood of the origin has some ordering of coordinates and will thus belong to that (or those if it is on a boundary) cell(s) whose vertices are those of σ_o with coordinates permuted to match that ordering. Since there are exactly 5! possible orderings and the same number of cells ("$-\sigma_o$" is the same as σ_o), the assertion is proved.

Finally we introduce some general definitions preparatory to listing some "incidence numbers" for the net.

(5.4) Definition: $\alpha|\beta$ (α and β are incident) iff α is a subsimplex of β or vice-versa.

(5.5) Definition: $\mathfrak{S}_m(\beta)$ = m-star of $\beta = \{\alpha \in \Sigma_m : \alpha|\beta\}$

(5.6) Definition: $I(m,k) = \text{card}\ \mathfrak{S}_m(\beta)$ = number of m-simplexes in the m-star of a k-simplex β. ($I(m,k)$ may have several values if there is more than one type of k-simplex in the net). Thus, for example,

$m \leq k \Rightarrow I(m,k) = \binom{1+k}{1+m}$, the binomial coefficient.

Here are some easily checked† incidence numbers of interest or relevance:

$I(1,0) = 30$, $I(2,1) = 8,14$

---

†In deriving such relations it is often convenient to characterize a simplex by its barycenter, for example σ_o above by the vector $\frac{1}{5}(2,1,0,-1,-2)$, which helps clarify the action of the symmetry group.

$I\ (4,0) = 120 \quad , \quad I\ (4,2) = 4,6$

Notice that legs and bones both come in two inequivalent types. On the other hand 0, 3, and 4-simplexes come in one type only.

B. A Simplectic Net for $S_3 \times \underset{\sim}{R}$

The spherical character of the net to be described is based on the 4-dimensional analogue of the octahedron, a "regular polyhedron" with four pairs of "antipodal" vertices (Figure 5.2). Each of these eight vertices implies, for the net, an event which recurs periodically (with period 4) at the same position in space and simultaneous to the antipodal event. The 4 pairs are staggered in phase by 0,1,2,3 respectively. A precise description follows.

Let the vertices be represented as $[t]$ or $[t^*]$ in which t is an integer. Then two vertices $[t_1]$ and $[t_2]$, or $[t_1^*]$ and $[t_2^*]$ determine a leg in Σ_1 iff $|t_1-t_2| \leq 4$, while $[t_1]$ and $[t_2^*]$ determine a leg iff $|t_1-t_2| \leq 3$. (Of course, $[t^*]$ is the vertex "antipodal" to $[t]$, and t is the "time".) As before we complete the description of Σ as the "nerve" determined by Σ_o and Σ_1 , so that, e.g., any 5 mutually connected vertices determine a cell of the net.

Let us show that Σ is topologically a 4-dimensional manifold without boundaries. I claim that this is equivalent to the following conditions:

(1) Every simplex belongs to at least one 4-simplex

(2) Every 3-simplex belongs to exactly two 4-simplexes.

(1) says that every point of Σ has a 4-dimensional "environment",

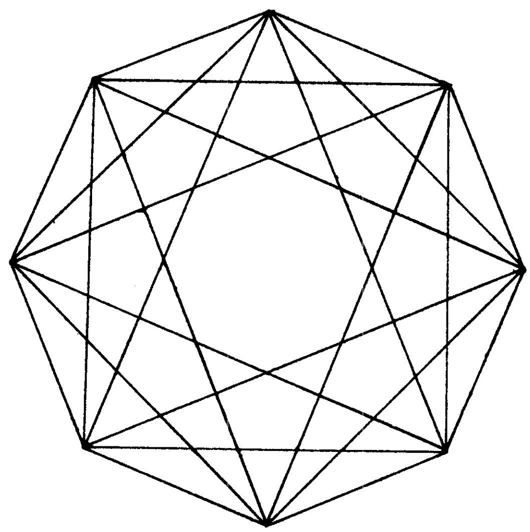

Figure 5.2 A regular "polyhedron" in 4 dimensions.

while (2) rules out any boundaries or "bifurcations". Detailed verification of (1) and (2) is an easy, if slightly tedious, matter.

Next we indicate why Σ is homeomorphic to $S_3 \times \underset{\sim}{R}$. In the first place we can embed it in $\underset{\sim}{R}^4 \times \underset{\sim}{R}$ by the scheme

$$[t](\text{resp.}[t^*]) \rightarrow (x,t)(\text{resp.}(-x,t))$$

in which x is the vector in $\underset{\sim}{R}^4$:

$$\begin{array}{lll} (1\ 0\ 0\ 0) & \text{if} & t = 1 \pmod 4 \\ (0\ 1\ 0\ 0) & \text{if} & t = 2 \pmod 4 \\ (0\ 0\ 1\ 0) & \text{if} & t = 3 \pmod 4 \\ (0\ 0\ 0\ 1) & \text{if} & t = 4 \pmod 4 \end{array}$$

Then it is easy to see that $\Sigma \subset \underset{\sim}{R}^4 \times \underset{\sim}{R}$ is a "cylinder" of the form $B \times \underset{\sim}{R}$ in which B is just the polyhedron pictured in Figure 5.2. Since B is thereby homeomorphic to the sphere S_3 , Σ is homeomorphic to $S_3 \times \underset{\sim}{R}$.

Here are some incidence numbers for this net:

$$I\ (1,0) = 14 \quad , \quad I\ (2,1) = 6,8,10$$

$$I\ (4,0) = 40 \quad , \quad I\ (4,2) = 4,6$$

Finally, we remark that this net for $S_3 \times \underset{\sim}{R}$ can only support a crude approximation to any particular 4-geometry or electromagnetic field. Unlike the net for $\underset{\sim}{R}^4$ which has no intrinsic scale and can be cast as finely as desired over any continuous space-time, this one cannot be refined without producing a topologically distinct net.

In the next chapter we examine the time-evolution and initial-value problems in terms of these nets.

Chapter VI

THE TOPOLOGICAL STRUCTURE OF THE THATCH EQUATIONS

a) General considerations

With both the metric and electromagnetic thatches, the basic data ($\ell^2(ij)$ and $A(ij)$) for the action principle involve one number for each leg of the net. Accordingly there is in each case a single variational equation associated to each leg. What other legs of the net are involved by such an equation?

In the latter case the action (4.1) is a sum of terms pertaining to a single cell of the net. The variational equation "of" a leg λ will therefore involve only legs of $\mathfrak{S}_4(\lambda)$, as can be verified from equation (4.5).

In the former case on the other hand, a single term in the action (2.1) pertains to all the cells of $\mathfrak{S}_4(b)$. We would thus expect the variational equation of the leg λ to involve all the legs of $\mathfrak{S}_4 \circ \mathfrak{S}_2 \circ \mathfrak{S}_4(\lambda)$. But because of the identity discussed in Appendix C the thatch equation (2.3) involves only $\mathfrak{S}_4(\lambda)$. This is no doubt the simplectic equivalent of Einstein's equations being only second-order despite that the Lagrangian R is already second-order in the metric tensor.

In both the cases just discussed then, the thatch equations have an identical structure. For the coupled electromagnetic and metric thatches everything is the same except both the number of variables and the number of equations doubles.

If two legs λ_1, λ_2 belong to $\mathfrak{S}_4(\lambda_o)$ for some λ_o then we will say they are "variationally related", or just "related" when the meaning is clear:

<u>Definition</u>: $\lambda, \mu \in \Sigma_1$ are variationally related iff

$$\lambda \in \mathfrak{S}_1 \circ \mathfrak{S}_4 \circ \mathfrak{S}_1 \circ \mathfrak{S}_4(\mu)$$

b) The time evolution problem

In this section we assume that everything is known up to a given "time" and consider the problem of carrying the solution forward a step. The next section will examine the problem of how to "begin" a solution.

Take first the case of the spatially closed net described in Section VB. (We discuss this in more detail because there are fewer simplexes to deal with--a closed space has less "space" than an open one!) Suppose known all thatch quantities pertaining to legs previous to $t = 3$ (We will say for short that "all legs previous to $t = 3$ are known".) and consider how to extend this knowledge to $t \leq 4$ by means of the thatch equations. In fact only legs lying wholly after $t = -3$ are variationally related to the unknown legs so it is enough to assume these known. We will call the subnet lying wholly between $t = \pm 3$ an "initial couche".

Consider for example the leg [04]. Since $\mathfrak{S}_1([4])$ includes six other legs lying prior to $t = 4$, the most we can really hope for is to find seven† equations in terms of which to solve for these seven "new" legs. Happily there are exactly 7 legs whose 1-stars include both new and couche legs. They are, as is easily checked, the 7 legs extending forward in time from [0]. To express the situation in more detail, we have seven new legs

[4,0], [4,1], [4,1*], [4,2], [4,2*], [4,3], [4,3*]

and for them the equations of the seven legs

[0,1], [0,1*], [0,2], [0,2*], [0,3], [0,3*], [0,4]

In other words, the seven "retarded" legs in $\mathfrak{S}_1([4])$ are determined by the equations of the seven "advanced" legs in $\mathfrak{S}_1([0])$. By symmetry the advanced legs of $\mathfrak{S}_1[0^*]$ will similarly determine the retarded legs of $\mathfrak{S}_1[4^*]$, and together these include all the unknown legs prior to $t=4$. Having thus advanced from $t=3$ to $t=4$, we can continue indefinitely, and we see that each step requires the solution of two sets of seven equations in seven unknowns (or 14 equations in 14 unknowns for coupled metric and electromagnetic thatches). Unfortunately the equations are nonlinear in the ℓ_{ij}^2.

Notice, by the way, that in this scheme all equations are utilized (as they must be since there is exactly one for each leg) so that a solution which begins consistent will remain so.

Turn now to the time evolution problem for the net of Section VA. We take the first affine coordinate as "time" and assume known all

† For definiteness we assume one thatch equation and one thatch quantity per leg.

legs prior to $t = 3/5$.

In the previous case it could be considered a convenience that the 14 new legs for $t \leq 4$ fell into two sets, each of which could be solved for separately. In this case, however, it is crucial that the equations fall into finite clusters in order to avoid solving for an infinite number of unknown legs subject to boundary conditions at spatial infinity, etc. etc. Fortunately the situation turns out to be completely analogous to the previous one with, e.g., the advanced legs of $\mathfrak{S}_1((00000))$ providing exactly enough equations to determine the retarded legs of $\mathfrak{S}_1(\frac{1}{5}(4\text{-}1\text{-}1\text{-}1\text{-}1))$. The only difference is that there are 15 legs in each cluster and an infinite number of clusters rather than only two.

c) The initial value problem

In contrast to the continuum case the initial value problem involves thatch equations of exactly the same type as does the time evolution problem. Where it differs is in its "topological" structure--in the relation of what is to be found to what is specified.

We begin again with the case of $S_3 \times \underset{\sim}{R}$. As pointed out in the last section, the problem is to specify consistently all the legs of the "initial couche" contained between $t = \pm 3$. Of the 66 couche legs there are 18 whose 4-stars lie entirely within the couche and therefore imply constraints on the initial value data. Specifically, they are, as is readily checked,

[-1 3] [-2 2] [-3 1]

[-1*3*] [-2*2*] [-3*1*]

[-1 2] [-1 2*] [-2 1] [-2 1*]

[-1*2*] [-1*2] [-2*1*] [-2*1]

[-1 1] [-1 1*]

[-1*1*] [-1*1]

The scheme which suggests itself is this: to specify freely all the couche legs except for the 18 listed above, and then to solve for the latter by using the 18 constraints which they themselves provide.

As far as $S_3 \times \underset{\sim}{R}$ is concerned then, beginning a solution involves the one-time solution of 18 equations in 18 unknowns, while continuing one begun involves the repeated solution of two sets of seven equations in seven unknowns.

One last point. In the electromagnetic case, gauge invariance introduces $2 \times 7 - 1 = 13$ extra degrees of freedom, which effectively cancel all but five of the constraint equations. But since the thatch equations are linear in A anyhow, this is not a momentous simplification.

In many respects the initial value problem for the net of Section VA (for $\underset{\sim}{R}^4$) is similar to that just discussed. On the other hand, the infinity of initial value data raises whole new problems which may or may not be severe. Only further theoretical investigation or experience with practical application will clarify some of these questions.

At any rate, the initial couche for this net may be taken as the subnet lying wholly between $t = \pm 3/5$. (Except for the conventional factor of $1/5$ this is just like the previous case.) Observing that a constraint leg is one whose equation involves only couche-legs (in other words, whose 4-star is in the couche) one can count without too much trouble, 72 couche legs for each vertex at $t = 0$ of which 17 imply constraints. We can therefore specify 55/72 of the couche legs and solve for the remainder in terms of those specified and of appropriate boundary conditions at "spatial infinity".

And it is easy to see what the boundary conditions should be. Assuming we pick the "constraint legs" as unspecified, and if we specify all others in a region Ω of the couche, then some of the constraint legs near the boundary of Ω will remain undetermined-- namely those whose 4-star extends outside Ω . To specify those in addition to the non-constraint legs in Ω is to impose boundary conditions at ∞.

Unfortunately there will be, in any practical case, so many initial value equations (almost 17 for each vertex at $t = 0$) that a direct solution is probably out of the question. Instead one would probably rely on a relaxation method, which, hopefully would be appropriate since the initial value equations ought, in some sense, to be "elliptic" in analogy with the continuum case. But this requires more study.

Alternatively, one might hope to begin somewhere at the "center" of the couche and proceed outward, specifying data until some leg (which must be still free!) becomes determined by those already specified. Assuming such a procedure is possible, there is the further requirement that it be stable in the sense of not leading to some sort of untenable behavior at spatial infinity. Again, these questions need further study.

Chapter VII

ENTR'ACTE: THE WAVE EQUATION IN TWO DIMENSIONS

The scalar wave equation in flat two-dimensional space-time offers a simple illustration of simplectic methods, especially as applied to linear theories.

If ϕ is the basic scalar thatch then, in analogy with the continuum theory, we choose for the action

$$S = \sum_{\sigma \varepsilon \Sigma_n} L(\sigma)\ V(\sigma) \tag{7.1}$$

where

$$\begin{aligned} L(\sigma) &= \frac{1}{2} <d\phi(\sigma), d\phi(\sigma)> \\ &= \frac{1}{2} \tilde{g}^{ij}(\sigma)\ d\tilde{\phi}_i(\sigma)\ d\tilde{\phi}_j(\sigma) \end{aligned} \tag{7.2}$$

Here, of course, $g(\sigma)$ and $d\phi(\sigma)$ are defined as in Sections IIIB and IIID, respectively. Thus

$$\begin{aligned} d\tilde{\phi}_i(\sigma) &= \frac{1}{1+n} \sum_{k \varepsilon \sigma} d\phi(ki) \\ &= \frac{1}{1+n} \sum_k \phi(i) - \phi(k) \end{aligned}$$

$$d\tilde{\phi}_i(\sigma) = \phi(i) - <\phi>_\sigma \tag{7.3}$$

where $<\phi>_\sigma$ is the average value of ϕ in the simplex ϕ. Then, since $\sum_j \tilde{g}^{ij} = 0$,

$$\tilde{g}^{ij}(\sigma)\ d\tilde{\phi}_j(\sigma) = \sum_{j \varepsilon \sigma} \tilde{g}^{ij}(\sigma)\ \phi(j)$$

and

$$L(\sigma) = \sum_{i,j \in \sigma} \frac{1}{2} \tilde{g}^{ij}(\sigma)\ \phi(i)\ \phi(j) \qquad (7.4)$$

whence

$$S = \frac{1}{2} \sum_{\substack{i,j \in \Sigma_o \\ \sigma \in \Sigma_n}} (i,j|\sigma)\ \tilde{g}^{ij}(\sigma)\ \phi(i)\ \phi(j)\ V(\sigma)^{\dagger} \qquad (7.5)$$

Varying $\phi(i)$:

$$\frac{\partial S}{\partial \phi(i)} = \sum_{j,\sigma} (i|\sigma)(j|\sigma) \quad \tilde{g}^{ij}(\sigma)\ \phi(j)\ \ V(\sigma)$$

$$= \sum_{\sigma} (i|\sigma)\ V(\sigma) \sum_{j \in \sigma} \tilde{g}^{ij}(\sigma)\ \phi(j) \qquad (7.6)$$

the vanishing of which constitutes the thatch equation for vertex i .

So far everything was general. We now specialize to various two-dimensional nets with flat metric. To evaluate $\tilde{g}^{ij}(\sigma)$ the following formula, which can be proved by the methods of Section IIIC, will prove very convenient:

$$\langle F(i),F(j)\rangle = \frac{\tilde{g}}{(n-1)!}\ \tilde{g}^{ij} = \pm(n)(n!)V^2\ \tilde{g}^{ij} \qquad (7.7)$$

Here everything relates to a particular n-simplex; $F(i)$ is the oriented face opposite to the vertex i , V the volume of the simplex, and

$$n!\ \tilde{g} = \tilde{\varepsilon}^{i\cdots j}\ \tilde{g}_{ia}\cdots\tilde{g}_{jb}\ \tilde{\varepsilon}^{a\cdots b} \qquad (7.8)$$

of course.

---

† "$(i,j|\sigma)$" is a "logical function" which = 1 when $i|\sigma$ and $j|\sigma$ and = 0 otherwise.

Work first with the net of Figure 7.1 (without the dotted line), and consider the equation of vertex A . Because all the cells have the same volume, V , equation (7.6) becomes

$$\sum_{\sigma}(i|\sigma) \sum_{j\varepsilon\sigma} \tilde{g}^{ij}(\sigma)\ \phi(j) = 0$$

or in view of (7.7),

$$\sum_{\sigma}(i|\sigma) \sum_{j\varepsilon\sigma} \langle F(i),F(j)\rangle\ \phi(j) = 0 \tag{7.9}$$

There are two types of cell in the net, of which α and β are exemplars. For α one finds from (7.7) (order: A B C)

$$\tilde{g}^{ij}(\alpha) = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} \tag{7.10}$$

and from this $\tilde{g}^{ij}(\beta)$ must be (order: A D C)

$$\tilde{g}^{ij}(\beta) = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \tag{7.11}$$

The equation of A is then

$$(\tilde{g}^{AA}(\alpha) + \tilde{g}^{AA}(\beta) + \tilde{g}^{AA}(\gamma) + \tilde{g}^{AA}(\delta) + \tilde{g}^{AA}(\varepsilon) + \tilde{g}^{AA}(\zeta))\ \phi(A)$$

$$+ (\tilde{g}^{AB}(\zeta) + \tilde{g}^{AB}(\alpha))\ \phi(B)$$

$$+ (\tilde{g}^{AC}(\alpha) + \tilde{g}^{AC}(\beta))\ \phi(C) \quad +$$

$$+ \cdots$$

$$+ (\tilde{g}^{AH}(\varepsilon) + \tilde{g}^{AH}(\zeta))\ \phi(H) = 0$$

or

$$(-1 + 1 + 0 - 1 + 1 + 0)\ \phi(A)$$

$$+ (1+1)\ \phi(B) + (0+0)\ \phi(C) + (-1-1)\ \phi(D)$$

$$+ \cdots + (-1-1)\ \phi(H) = 0$$

$$\phi(B) - \phi(D) + \phi(F) - \phi(H) = 0$$

$$\phi(B) + \phi(F) = \phi(H) + \phi(D) \tag{7.12}$$

which is exactly the equation used by the method of finite differences, in place of $\Box^2\phi = 0$ (in 2-dim.).

It is remarkable that $\phi(A)$, $\phi(C)$, $\phi(G)$ drop out of the equation completely. It is also odd that the vertices of the net fall into two variationally unrelated subsets, but there seems to be no way to set up a net which avoids this and still has basic equations of the type (7.12). The net indicated in Figure 7.2, for example, relates every point to every other, but through the typical equations

$$\phi(a) + 2\phi(h) + \phi(d) = \phi(b) + \phi(c) + \phi(e) + \phi(f)$$

which could be thought of as the sum of the two equations

$$\phi(a) + \phi(h) = \phi(f) + \phi(b) \quad \text{and} \quad \phi(d) + \phi(h) = \phi(c) + \phi(e)$$

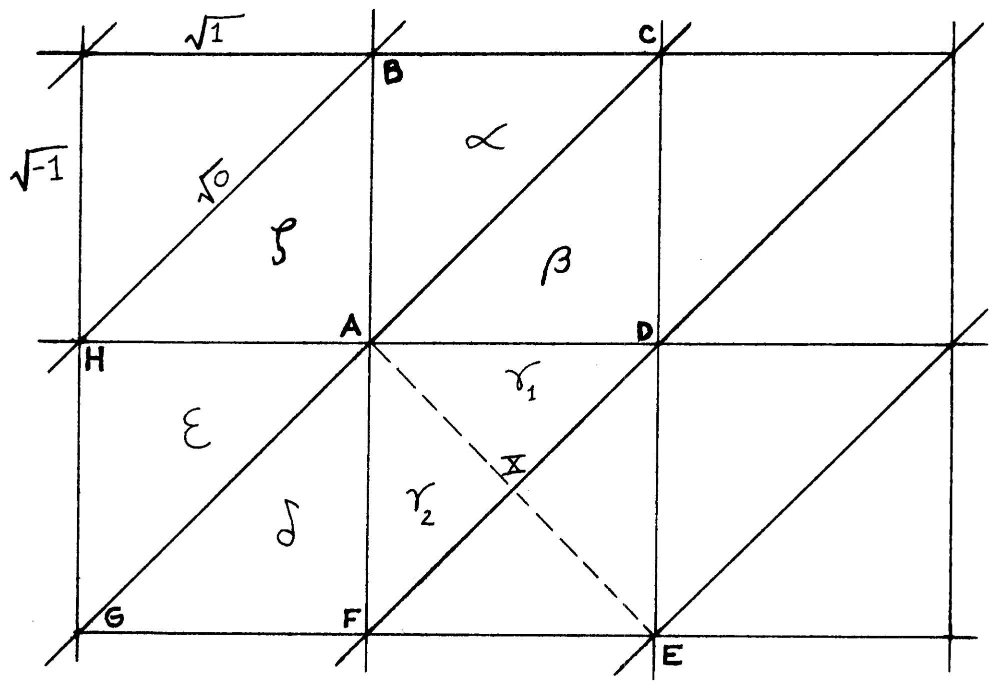

Figure 7.1 A rectangular net for a two-dimensional flat space-time. The diagonal lines are light-like.

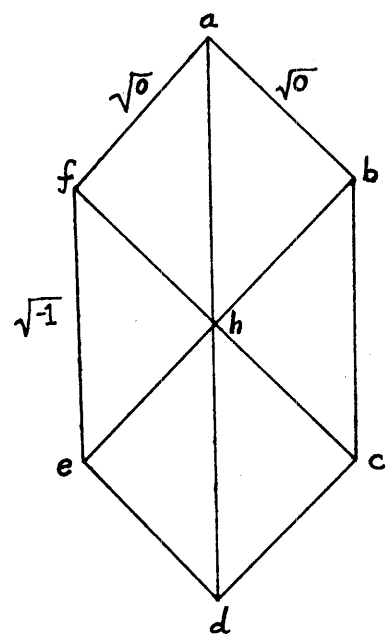

Figure 7.2 The analogue in two dimensions of the net of Section VA.

The most disconcerting phenomenon implied by (7.9) is that of the totally unrelated vertex as illustrated by Figure 7.3. The subnet pictured, which might be the refinement indicated by the dotted line in Figure 7.1, consists of four cells (triangles). According to (7.5) their contribution to the action is a sum of terms in $\phi(x)\ \phi(x)$, $\phi(x)\ \phi(A), \cdots, \phi(A)\ \phi(F), \cdots$. From (7.10) and (7.11) the coefficient of $\phi(x)\cdot\phi(x)$ is

$$\frac{1}{8} \sum_{j=1}^{4} \tilde{g}^{xx}(j) = \frac{1}{8}(-4+4-4+4) = 0$$

while that of $\phi(x)\ \phi(A)$, e.g., is

$$\frac{1}{4} \sum_{j=1}^{2} \tilde{g}^{xA}(j) = \frac{1}{4}(2-2) = 0$$

In other words $\phi(x)$ drops out of the action completely! In fact the expression for S is the same for both nets: the dotted line makes no difference.

Lest all these surprises give the impression that the simplectic approach is especially productive of anomalies, we should add that for any other than the 1-1 ratio of sides, the net of Figure 7.1 reproduces exactly the equation of the usual finite difference approximation. And, though we have stuck to flat space-time, the simplectic scheme comes into its own only with a curved background metric--which it handles with no extra trouble.

As a final example we take the two-dimensional potential equation. Using a "square" net with the topology pictured in Figure 7.1 one finds for vertex A , e.g., the equation

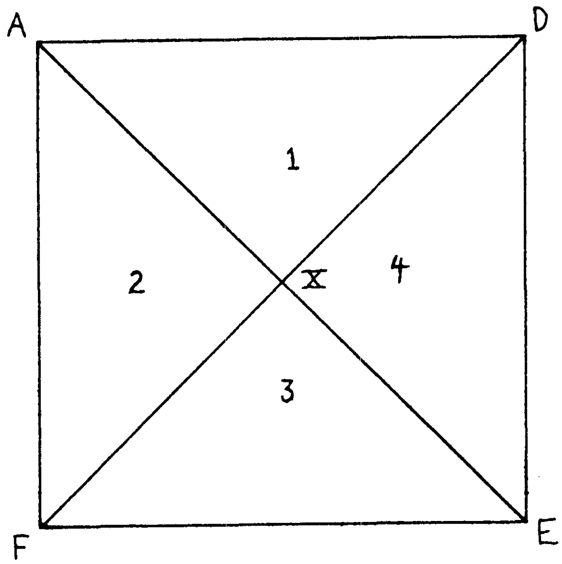

Figure 7.3 A refinement of the net of Figure 7.1.
$\phi(x)$ makes no contribution to the action.

$$4\phi(A) = \phi(H) + \phi(D) + \phi(B) + \phi(F)$$

which just says that $\phi(A)$ is the average of the neighboring ϕ values. And of course this is the well known characteristic feature of a solution of $\nabla^2\phi = 0$.

Chapter VIII

ALGORITHMS FOR NUMERICAL WORK

This chapter describes three key algorithms needed for numerical study of a metrical thatch.

A. To Find the Signature of a Cell

As discussed in Chapter II, any assignment of lengths to the edges of a simplex, σ , determines an internal geometry for σ . The algorithm of this section allows one to check whether that geometry has the correct signature of $-+++$.

To find the signature of any metric in an n-dimensional affine space amounts to finding n mutually orthogonal directions. If (v_j for $j=1$ to n) are vectors in such directions, then the signature is the number of signs of each type ($+-0$) among the n scalar products $\langle v_j, v_j \rangle$. One can thus find the overall signature by the inductive process of picking a direction, noting its sign and reducing everything to a subspace perpendicular to the direction chosen.

Applying this to an n-simplex, σ , let ℓ_{ij}^2 represent the metric thatch restricted to σ . In other words,

$$\ell_{ij}^2 = \langle [ij],[ij] \rangle$$

It is natural to choose as the direction to "factor out" that of one of the edges $[ij] \in \mathfrak{S}_1(\sigma)$. For numerical stability it will be best (see below) to pick the "largest" such edge, i.e., one with the greatest $|\ell_{ij}^2|$.

Suppose it is ℓ_{01}^2 . Then ℓ_{01}^2 contributes its sign to the total signature and the "inductive step" consists in reducing the remaining signature to that of an (n-1)-simplex, ρ , as follows.

The midpoint, P_1, of [01] will be the 0^{th} vertex of ρ , and the other n-1 vertices will be the remaining vertices, [2],$\cdots$ [n], of σ projected on the n-1 dimensional hyperplane passing through P_1 and perpendicular to [01]. The n-1 points $P_1 \cdots P_n$ define ρ as their convex hull.

The formula for the "reduced" $\hat{\ell}_{ij}^2$ in terms of the original ℓ_{ij}^2 is remarkably simple. If [01] is the factored leg, and

$$\Delta_j \equiv \ell_{oj}^2 - \ell_{1j}^2$$

then

$$\hat{\ell}_{ij}^2 = \ell_{ij}^2 - \frac{(\Delta_i - \Delta_j)}{4\ell_{01}^2} \quad ; \qquad i,j = 1,\cdots,n \tag{8.1}$$

Notice the division by ℓ_{01}^2 ; had we not picked ℓ_{01}^2 as the largest ℓ_{ij}^2 then (8.1) might be unstable against small changes (e.g., from roundoff error) in the ℓ_{ij}^2 . As defined, however, the algorithm is insensitive to such changes.

The $|\ell_{01}^2|$ at successive stages of the reduction process furnish a measure of how nearly singular σ is. The greater the ratio between the first and last of them, the more "squashed" is σ . Thus zeroes in the signature will usually appear as tiny $|\ell_{01}^2|$'s after being "filtered" through roundoff error, and the sign of relatively very small ℓ_{01}^2's is not reliable.

Finally, the product of the ℓ_{01}^2 turns out to be proportional to $\|\sigma\|^2$, the square of the volume of σ .

B. To Find the Defect of a Non-Null Bone

Let $b = [0\,1\,2]$ be a bone of the net and let the remaining vertices of $\mathfrak{S}_4(b)$ be numbered cyclically from 3 to k+2 . Then the cells,

$$\sigma_1 = [0\ 1\ 2\ 3\ 4]$$
$$\sigma_2 = [0\ 1\ 2\ 4\ 5]$$
$$\vdots$$
$$\sigma_k = [0\ 1\ 2\ k{+}2\ 3]$$

of $\mathfrak{S}_4(b)$ comprise a ring whose mutual intersection is b itself.

If we embed b in flat space-time and then successively σ_1 through σ_k , ranged about b just as they are in $\mathfrak{S}_4(b)$, then the ring will not in general "close"; the initial face of σ_1 and the final face of σ_k will not coincide (see Figure 8.1). As indicated in the figure we name the faces between successive cells in the ring as follows:

$$F_3 = [0\ 1\ 2\ 3]$$
$$F_4 = [0\ 1\ 2\ 4]$$
$$\vdots$$
$$F_{k+2} = [0\ 1\ 2\ k{+}2]$$
$$F_{k+3} = [0\ 1\ 2\ k{+}3]$$

Here [3] and [k+3] correspond to the same vertex--and F_3 and F_{k+3} to the same 3-simplex--of $\mathfrak{S}_4(b)$.

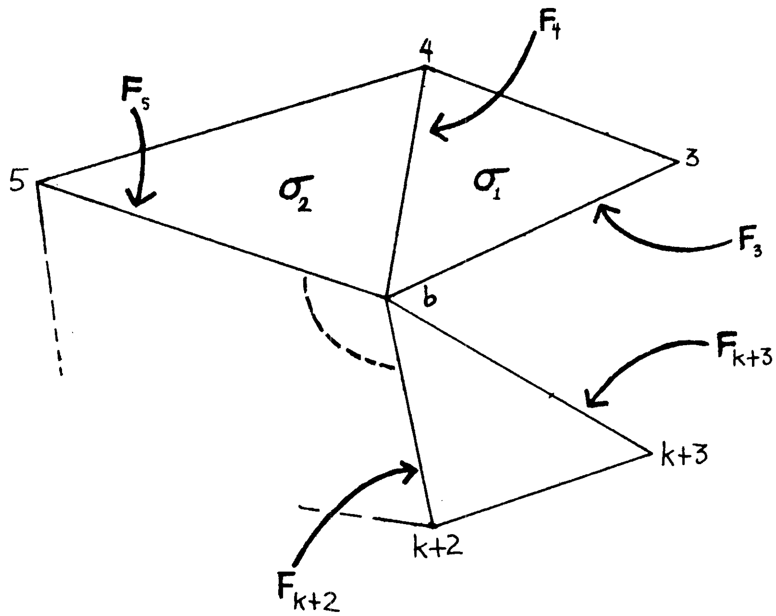

Figure 8.1 The numbering scheme for $\mathfrak{S}_4(b)$

The <u>defect</u> η of b measures the deviation of F_3 from F_{k+3} (see Sections IIB,C). If b is timelike ($\langle b,b\rangle < 0$) then η is just the missing solid angle θ . If b is spacelike ($\langle b,b\rangle > 0$) then η is just the "boost" parameter relating the two faces and is positive when either F_3 (and hence F_{3+k}) is spacelike and there is a gap, or F_3 is timelike and there is an overlap. (For brevity we ignore the possibility that F_3 is null.)

To calculate η it is convenient to work in the image space, pictured in Figure 8.1, in which b becomes a point. More formally, if B is the antisymmetric tensor representing the embedded bone (see Section IIIC) and (f_j for j=3 to k+3) the exterior tensors representing the tetrahedrons F_j then the image space M is a two-dimensional vector space with metric ++ or −+ according as B is timelike or spacelike respectively. In either case the image of b is the origin, and that of F_j a vector ξ_j in M , where the configuration of the ξ_j is defined by the equalities

$$\langle \xi_j,\xi_k\rangle = \langle f_j,f_k\rangle \qquad (k=j \text{ or } k=j+1) \qquad (8.2)$$

We will not give a formal proof that this image method furnishes the defect correctly, but one can understand why by thinking of M as the subspace orthogonal to B and ξ_j as defined by the relation

$$f_j = B \wedge \xi_j$$

Now we know $\langle f_j,f_j\rangle$ and $\langle f_{j+1},f_{j+1}\rangle$ from the internal geometry of F_j and F_{j+1} and $\langle f_{j+1},f_j\rangle$ from that of σ_{j-2} , so by

(8.2) we can image the serial embedding process described above once we solve the following problem: given α, $\langle\zeta,\alpha\rangle$, $\langle\zeta,\zeta\rangle$ for vectors α,ζ in two-space, to find ζ . The solution is easily attained and depends on the type of bone:

Timelike bone (++ image space):

$$\text{Let} \quad p = \langle\alpha,\zeta\rangle \ , \quad r^2 = \langle\zeta,\zeta\rangle\langle\alpha,\alpha\rangle - \langle\zeta,\alpha\rangle^2 \ ,$$

then

$$\underset{\sim}{\zeta} = \frac{1}{\langle\alpha,\alpha\rangle} \begin{bmatrix} p & -r \\ r & p \end{bmatrix} \cdot \underset{\sim}{\alpha}$$

Spacelike bone (−+ image space):

$$\text{Let} \quad p = \langle\alpha,\zeta\rangle \ (= -\alpha^o\zeta^o + \alpha^1\zeta^1) \ , \quad r^2 = \langle\zeta,\alpha\rangle^2 - \langle\zeta,\zeta\rangle\langle\alpha,\alpha\rangle$$

then

$$\underset{\sim}{\zeta} = \frac{1}{\langle\alpha,\alpha\rangle} \begin{bmatrix} p & r \\ r & p \end{bmatrix} \cdot \underset{\sim}{\alpha}$$

In each case ζ lies counterclockwise from α .

Finally, if we call ξ_3 and ξ_{k+3} respectively ξ_i and ξ_f , then for a timelike bone, η (called "θ" in Section IIB) is defined by the relations

$$\langle\xi_i,\xi_i\rangle \sin\eta = -(\xi_i \wedge \xi_f)^{12} \equiv -\xi_i^1\,\xi_f^2 + \xi_i^2\,\xi_f^1$$

$$\langle\xi_i,\xi_i\rangle \cos\eta = \langle\xi_i,\xi_f\rangle$$

(with -2π added for each extra circling of the bone), while for a spacelike bone there must be just 1 circling and then

$$<\xi_i,\xi_i> \sinh \eta = - (\xi_i \wedge \xi_f)^{12}$$

(Note that $<\xi_i,\xi_i> = <\xi_f,\xi_f>$ since both refer to the same tetrahedron, F_3 , of $\mathfrak{S}_4(b)$.)

C. To Solve a Nonlinear Algebraic System with Ill-Conditioned Jacobian

Almost every numerical algorithm for solving a nonlinear system of n equations in n unknowns is some variation of that scheme (called "Newton's method" in one dimension) in which one iteratively guesses the root of the function f on the basis of the current values of f and f'. In formulas:

$$f'(x) \cdot \Delta x = f(x) \tag{8.3}$$

$$x \to x - \Delta x \tag{8.4}$$

Unfortunately, when $n > 1$ this scheme is subject to instability. Suppose, for example, that $n = 2$ and the solution set is a curve ("degenerate" solution) as shown in Figure 8.2. Then at a point such as x_1 the Jacobian $f'(x_1)$ must be singular since motion along the solution curve leaves $f(x) = 0$. For a point such as x_o , which is very near to x_1 , $f'(x_o)$ will be almost singular (ill-conditioned) since $f'(x_o) \cdot \Delta x$ will be very small for Δx as depicted in the figure. The condition (8.3) will therefore not inhibit very decisively such a Δx and it may be that, rather than jumping to a point like x_1

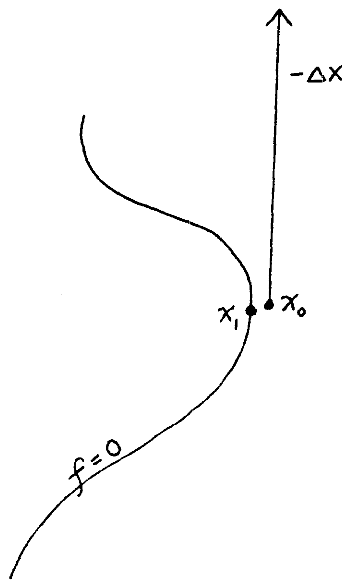

Figure 8.2   A possible instability

as one would wish, the iteration could jump clear away from the solution.

Let D be the determinant

$$\begin{vmatrix} \frac{\partial f_1}{\partial \theta} & \frac{\partial^2 f_1}{\partial n \partial \theta} \\ \frac{\partial f_2}{\partial \theta} & \frac{\partial^2 f_2}{\partial n \partial \theta} \end{vmatrix} \tag{8.5}$$

where θ parameterizes the curve $f = 0$ and $\frac{\partial}{\partial n}$ is the normal derivative. Then it turns out that, in the generic case $D \neq 0$, there is no problem as long as x_o is close enough to x_1 . But if $D = 0$ then the root x_1 can be absolutely unstable in the sense that no matter how close x_o is to it, the depicted behavior will occur.

Another difficulty with the undamped scheme is that it works well only when the initial guess is sufficiently near a root. Far from any root it has a tendency to jump about without ever "homing in" on a root. In this case the usual remedy is to proceed slowly in the direction of steepest descent of $||f||^2$, since any root is surely at a minimum of this function.

Because any flat-space solution of the thatch equations is degenerate (see Section IIG), any sufficiently fine net will involve thatch equations with an ill-conditioned Jacobian. And because we have no good way to guess the solution of these equations, a procedure such as that of Section VIB can expect to encounter one or both of the problems just described.

The solution (see [6]) is to replace the system (8.3),(8.4), which can be described as minimizing $||f||^2$ on the basis of the best linear approximation to f , by the minimization on the same basis of

$$||f||^2 + \varepsilon^2 ||\Delta x||^2 \tag{8.6}$$

Plainly this will discourage wild jumps like that shown in Figure 8.2. Moreover, when ε is large the direction of Δx will approach that of steepest descent, as solving (8.6) reveals (setting $y = f(x)$, $J = f'(x)$):

$$\delta ||y - J \cdot \Delta x||^2 + \varepsilon^2 \delta ||\Delta x||^2 = 0$$

$$-2 \langle J\delta\Delta x,\ y - J\cdot\Delta x\rangle + 2\varepsilon^2 \langle \delta\Delta x,\ \Delta x\rangle = 0$$

$$(J^*J + \varepsilon^2)\Delta x = J^*y \qquad \text{(since } \delta\Delta x \text{ is arbitrary)} \tag{8.7}$$

Thus, when ε is large, Δx will be in the steepest descent direction, J^*y .

To understand better the damping (8.6) requires the concept of singular value. Namely, every linear operator can assume the form

$$J = \sum_{k=1}^{n} |kb\rangle \sigma_k \langle ka| \quad , \quad \sigma_k \geq 0 \tag{8.8}$$

where $(|ka\rangle)$ and $(|kb\rangle)$ are orthonormal bases and the σ_k , which are unitary invariants of J , are its <u>singular values</u>. J is called "ill-conditioned" when its singular values are very disparate.

Now consider the instability pictured in Figure 8.2. J might be something like

$$J = |1\rangle \langle 1| + |2\rangle \, \delta \, \langle 2| \text{ with } \delta \ll 1$$

Then $J^{-1} = |1\rangle \langle 1| + |2\rangle \, \delta^{-1} \langle 2|$, whence

$$\Delta x = J^{-1}|y\rangle = |1\rangle \, y_1 + |2\rangle \, \delta^{-1} \, y_2 \qquad (8.9)$$

which will be bad unless $y_2 \lesssim y_1\delta$. A crude method to suppress the instability would be to replace δ^{-1} by 0 in (8.9). The prescription (8.6) replaces it instead by $\frac{\delta}{\delta^2 + \varepsilon^2}$, which has the same effect when $\varepsilon \gg \delta$ but allows a gradual transition to the undamped case.

From equation (8.8),

$$J^* = \sum |ka\rangle \, \sigma_k \, \langle kb|$$

Substituting this into (8.7) and applying $\langle ja|$ on the left,

$$\sum_{k,\ell} \langle ja|ka\rangle \, \sigma_k \, \langle kb|\ell b\rangle \, \sigma_\ell \langle \ell a|\Delta x\rangle + \varepsilon^2 \langle ja|\Delta x\rangle = \sum_k \langle ja|ka\rangle \, \sigma_k \langle kb|y\rangle$$

or by orthonormality

$$(\sigma_j^2 + \varepsilon^2) \, \langle ja|\Delta x\rangle = \sigma_j \, \langle jb|y\rangle$$

$$\Delta x = \sum_{k=1}^{n} \frac{|ka\rangle \, \sigma_k \, \langle kb|}{\sigma_k^2 + \varepsilon^2} \, |y\rangle \qquad (8.10)$$

Thus, as we just described $\varepsilon > 0$ functions to damp the action of each singular value by the factor $\sigma^2/(\sigma^2 + \varepsilon^2)$.

In practice one begins with a large value of ε , and lowers it past one singular value after another as long as f behaves sufficiently linearly at each step. When linearity fails, one initiates a binary search for the smallest acceptable ε . This procedure has proved successful in many cases where the undamped scheme was hopeless. The algorithm actually used differs in these ways (among others) from that charted in Figure 8.3:

(1) it has a provision for overriding the damping periodically under certain circumstances to avoid bogging down;

(2) it uses an inaccurate but convenient replacement for the singular values;

(3) for very small ε it reverts to (8.3) to reduce roundoff error;

(4) if $x - \Delta x$ would fall outside the domain of definition of f , it increases ε in order to decrease Δx ;

(5) it can keep Δx to within a specified size;

(6) it can stop.

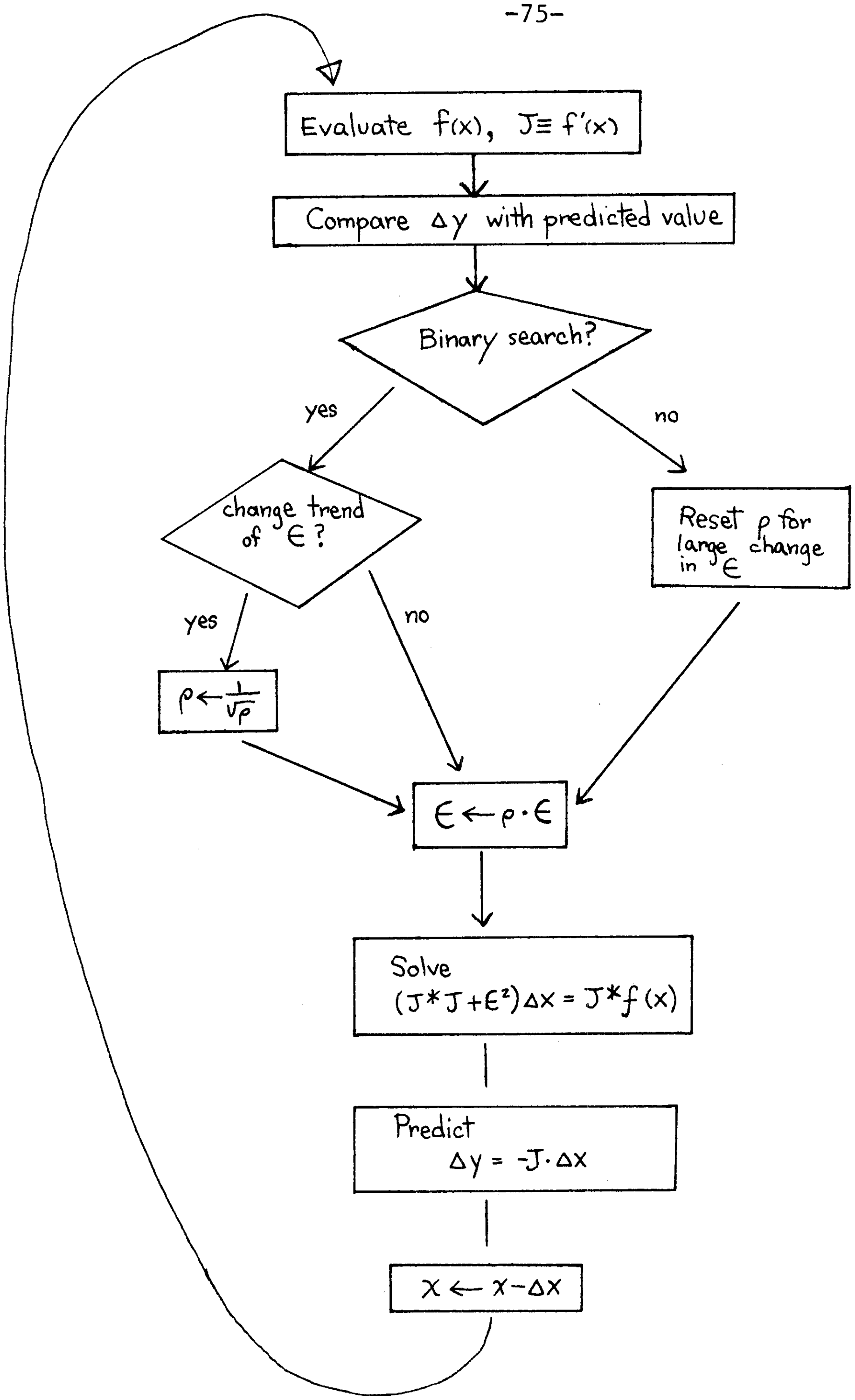

Figure 8.3 Highly simplified flow chart for the algorithm of Section VIIIC. The damping parameter, ε , decreases after a successful step, increases otherwise.

Chapter IX

INDICATIVE NUMERICAL RESULTS

Using the algorithms described in the last chapter I have investigated the scheme outlined in Chapter VI. Unfortunately the indications are that the combination of an IBM 370/55 with the language PL/1 is too slow to be practical. It is possible, however, that a faster machine and a more efficient compiler (FORTRAN on a CDC 7600) could provide sufficient economy to work out interesting applications.

A. Testing the Code

At first I set up the 15 equations in 15 unknowns germane to the time evolution problem of the purely metric mesh, as outlined in Chapter VI ("15-problem"). As one sees from (2.3) this involves only quantities pertaining to $\mathfrak{S}_4(x_o)$ for some vertex x_o . On the other hand, since, according to (2.2) $G(x_o x)$ is really

$$\frac{\partial S_g}{\partial \ell^2(x_o x)} \tag{9.1}$$

one has the symmetry relation

$$\frac{\partial G(x_o x)}{\partial \ell^2(x_o y)} = \frac{\partial G(x_o y)}{\partial \ell^2(x_o x)} \tag{9.2}$$

It is remarkable that $G(x_o x)$ can be considered (9.1) as the derivative of S_g even though S_g itself is not even defined by the legs involved in evaluating $G(x_o x)$ according to (2.3). The explanation is the identity (C1) used in deriving (2.3). Thus a verification of (9.2) serves at the same time to check the computer coding for errors and to confirm the correctness of the formalism in general.

Double precision computations in which the derivatives in (9.2) were evaluated directly as differences displayed the symmetry to an accuracy of 8 decimal places. Since reversing the relative sign of the defect for spacelike vs. timelike bones leaves no trace of symmetry, this also verifies the sign conventions described in Section IIe or Section Ca.

B. Conclusions from Calculations

Beginning with very crude approximations to known exact solutions, the routine LINEARG (§VIIIC) was able to produce one solution to the "15-problem" and two to the analogous, but easier "7-problem" based on the net of Section VB. In no case could the undamped Newton's method converge and even the damped scheme failed in a majority of cases. When it did converge it required between 36 and 200 iterations or about 40 to 600 seconds of computation.

The solutions found are notable, in the first place, for their lack of gauge-invariance; in agreement with the considerations of Section IIG they are unique (or at least discrete).

On the other hand, at none of the solutions was the Jacobian well-conditioned, the ratio of the greatest to the least singular value being about 25 in the best, and about 1000 in the worst case. This means first that the solution is hard to find, and second that it is unstable to small changes in the parameters. For let the equation be

$$F(x,z) = 0 \tag{9.3}$$

where x is sought, for parameters z . Then the Jacobian

$$J(x,z) \equiv \frac{\partial F}{\partial x} \tag{9.4}$$

and, under a variation of z

$$0 = \delta F = \frac{\partial F}{\partial x} \delta x + \frac{\partial F}{\partial z} \delta z$$

$$\delta x = -J^{-1} \circ \frac{\partial F}{\partial z} \delta z \tag{9.5}$$

If J is nearly singular, then J^{-1} in (9.5) will in general cause large changes in x even for relatively small δz .

It is possible that a more realistic choice for the parameter legs (for instance as part of a complete solution to the initial value problem) could remedy the conditioning problem. But it will require further theoretical or numerical work to assess this possibility, and to learn more in general about the solutions to the equations for the metric thatch†.

†Presumably, there can be no unusual problems with the equations for the electromagnetic thatch in a fixed background since these are linear.

Chapter X

FOR FURTHER INVESTIGATION

The next step in the development of simplectic methods would be to try to reproduce a known analytic solution. Of these the most suitable is probably the Taub space-time [7] which represents a closed empty universe and could be approximated on the net of Section VB.

Should such a trial succeed, the next step could be the Schwarzschild metric, for which the initial value problem is less straightforward, and the behavior of the thatch in the "throat" may or may not cause difficulty. Assuming any such troubles overcome, however, a situation such as the collision of two initially stationary black holes should present no further essential difficulties. One could take the black holes as the mouths of a single "wormhole" and realize that topology by deleting from the net of Section VA two space-time "cylinders" and identifying corresponding boundary simplexes.

The key to all such applications is the solution of the set of 7 or 15 equations which govern the time evolution problem (§VIB). One might study these in more detail in the hope of effecting an analytic simplification. Even an approximate simplification could be very useful in finding a good initial guess to the solution. Also, one might try to improve the equation-solving routine of Section VIIIC. In particular, it should be able to follow the boundary of the valid domain for the independent variables, rather than having to reduce the step size, possibly to nothing. (It is usually the signature condition which delimits the domain of valid leg lengths.)

More theoretical in character is the question of the hyperbolicity or "causality" of the thatch equations. Has this any simple meaning for a simplectic net? The question of the stability of the time evolution problem relates closely to this, as does that of clarifying the character of the continuum limit.

Similarly, are the initial value equations in some sense elliptical? If so will some relaxation method apply? In these areas especially it may be that the recent work of certain engineers will prove helpful; for they have been developing, under the title "finite element methods", an approach remarkably similar in many respects to that described in this thesis [5].

A number of questions not even touched on in previous chapters relate to the refinement, or other alterations, of the net. Is there a systematic way to refine the net in regions of special interest or rapid variation of the fields? Should one make any special alteration in the neighborhood of a singularity? Can one estimate discretization error or judge stability by refining a single cell and checking how well the new thatch agrees with the old one?

Another theoretical question concerns the identity proved in Appendix C. One may suspect that it is really the familiar relation

$$\int g^{\mu\nu}\, \delta R_{\mu\nu}\, dV = 0$$

in disguise. To make sense of such an assertion, however, one would have to extend the definition of the action (2.1) to cases where the connection (B.3) is not necessarily compatible with the metric thatch. Perhaps the method of Section IIC would apply.

Finally, there is the possibility of extending simplectic methods to forms of matter other than the electromagnetic field. An extension to matter with rest mass could have important astrophysical applications while, more speculatively, one to spinor thatches might be useful in theoretical investigations of quantum electrodynamics.

Appendix A: The Circulator of a Null Bone, and How It Enters into the Thatch Equations

Because a null bone is unlikely to arise in the course of an actual calculation we have relegated its discussion to an appendix. We discuss it here, not only for logical completeness, but also for the illumination shed by a "singular case" on a more familiar situation.

a) Parameterization of null rotations

As we have seen in Section IIB as well as in Appendix B, the most general circulator of a bone is a Lorentz transformation fixing some 2-dimensional subspace of space-time. When the bone is time (resp. space)-like such a transformation is a rotation (resp. boost) characterized by an invariant parameter called the angle (resp. rapidity). Similarly, the possible circulators of a null bone also comprise a one-parameter set. But, unlike the angle and the rapidity this parameter is not a Lorentz invariant (it is not "dimensionless"). Nonetheless, there is an invariant implied by the relation of the circulator to its specific bone, as we now show.

Let M be a 4-dimensional vector space, with a basis $\underset{\sim}{e}_1 \cdots \underset{\sim}{e}_4$ in terms of which the scalar product

$$\langle\xi,\eta\rangle = \xi^1\eta^1 + \xi^2\eta^2 - \xi^3\eta^4 - \xi^4\eta^3 \tag{A.1}$$

Thus $\underset{\sim}{e}_3$ is null and, together with $\underset{\sim}{e}_2$, spans a null 2-subspace, $\mathcal{B}$, of M. What is the most general Lorentz transformation fixing $\underset{\sim}{e}_2$ and $\underset{\sim}{e}_3$ (and hence $\mathcal{B}$)?

It is

$$\Lambda^{\pm}(\lambda) = \begin{bmatrix} \pm 1 & 0 & 0 & \lambda \\ 0 & 1 & 0 & 0 \\ \pm\lambda & 0 & 1 & \lambda^2/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(A.2)}$$

Since a circulator preserves orientation, we can ignore the minus signs and write simply $\Lambda(\lambda)$. Then

$$\Lambda(\lambda') \circ \Lambda(\lambda'') = \Lambda(\lambda' + \lambda'')$$

so that λ plays the role of an angle. Nevertheless, it is not an invariant because, for example, the Lorentz transformation

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \alpha & \\ & & & \alpha^{-1} \end{bmatrix} \qquad \text{(A.3)}$$

changes λ by a factor of α .

To examine this circumstance let $B^{\nu\beta}$ be any "surface tensor" in the fixed subspace $\mathcal{B}$ and set

$$V_{\mu\nu} = \sqrt{-g}\, \varepsilon_{\mu\nu\alpha\beta}\, B^{\alpha\beta} \qquad \text{(A.4)}$$

In particular, we can imagine that $\underset{\sim}{e}_2$ and $\underset{\sim}{e}_3$ span a bone b , that

$$B = \frac{1}{2!}\, \underset{\sim}{e}_2 \wedge \underset{\sim}{e}_3 \qquad \text{(A.5)}$$

and that $\Lambda(\lambda)$ is the circulator of b . Then we calculate successively

$$\sqrt{-g} = 1 \ , \quad B^{23} = -B^{32} = \frac{1}{2} \ , \quad V_{14} = -V_{41} = 1 \ ,$$

$$[V^{\mu}_{\nu}] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \ ,$$

or comparing with (A.2),

$$V = \frac{d}{d\lambda} \Lambda(\lambda) \Big|_{\lambda=0} \tag{A.6}$$

In other words, $V(=V^{\mu}_{\nu})$ is the infinitesimal generator for Λ , whence $\Lambda(\lambda) = e^{\lambda V}$. Noting that

$$V^3 \equiv V \circ V \circ V = 0 \tag{A.7}$$

we can express finally Λ in the form:

$$\Lambda(\lambda) = e^{\lambda V} = 1 + \lambda V + \frac{1}{2} \lambda^2 V^2 \tag{A.8}$$

Looking back over the previous paragraph shows that, relative to B and Λ , λ is invariantly defined by the relations (A.4) and (A.8). Furthermore (A.4) in the form

$$V^{\mu}_{\nu} = g^{\mu\sigma} \sqrt{-g}\ \varepsilon_{\sigma\nu\alpha\beta}\ B^{\alpha\beta} \tag{A.9}$$

shows that V has "dimensions" of L^2 , from which, with (A.8), λ must have "dimensions" of L^{-2}. The case of a non-null bone is precisely analogous except for one thing: one can normalize V through the requirement

$$\hat{V}^{\mu\nu}\hat{V}_{\mu\nu} = \pm 2$$

and define θ or η relative to this dimensionless tensor. In the present case, however, $V \cdot V = -4B \cdot B = 0$ and one is thrown back on (A.9) with its linear dependence on B.

b) An expression for the defect of a spacelike bone

Consider a pure boost Λ and some surface tensor $B^{\mu\nu}$ in the plane fixed by Λ. As before define

$$V^{\mu}{}_{\nu} = g^{\mu\mu} \sqrt{-g}\, \varepsilon_{\mu\nu\alpha\beta}\, B^{\alpha\beta} \qquad \text{(A.10)}$$

$$\Lambda = e^{\lambda V} \qquad \text{(A.11)}$$

Calculating as in the previous section one finds

$$\eta = \lambda \sqrt{2 \langle B,B \rangle} \qquad \text{(A.12)}$$

where η is the defect of Section IIC if Λ is the circulator corresponding to circulation in the sense indicated by $V^{\mu\nu}$.†

The significance of reexpressing η in terms of B, V, and the parameter λ is this: all these quantities will be continuous functions of the metric thatch. In fact (referring everything to some 4-simplex in $\mathfrak{S}_4(b)$) $\tilde{B}^{ij}$ is constant while $\tilde{\Lambda}^{i}{}_{j}$ is defined continuously in terms of the affine tensors $\tilde{g}_{ij}(\sigma)$ (see Appendix B) which in turn are continuous in the metric thatch ℓ^2_{ij}. Similarly for

† $\underset{\sim}{e} \wedge \underset{\sim}{f}$ "indicates" the sense $\underset{\sim}{e} \to \underset{\sim}{f}$.

$\tilde{V}^i{}_j$. Therefore by (A.11) λ is continuous in the ℓ^2_{ij} , and as b becomes null goes over into the null parameter λ of §Aa.

c) Contribution of a null bone to G(ij)

If $b = [ijk] \varepsilon \Sigma_2$, then from (2.3) its contribution to G(ij) is

$$\frac{\sigma(b)}{16} \frac{\eta(b)}{A(b)} (\ell^2_{ij} - \ell^2_{ik} - \ell^2_{jk}) \tag{A.13}$$

If b is null then A and (see Section IIC) η vanish, leading to 0/0. We can, however, evaluate (A.13) as a limit.

As before we refer everything to some cell in $\mathfrak{S}_4(b)$ and work with b spacelike. Then, in the first place, $\sigma(b) = -1$ and

$$A(b) = \sqrt{\frac{1}{2} \langle B,B \rangle}$$

Combining these with (A.12) furnishes for (A.13)

$$-\frac{1}{8} \lambda (\ell^2_{ij} - \ell^2_{ik} - \ell^2_{jk}) \tag{A.14}$$

with λ defined by (A.4) and (A.8). This is b's contribution to G(ij).

Appendix B: The Connection in Affine Coordinates

This appendix develops an expression in terms of the metric tensors $g(\sigma)$, $g(\rho)$ of adjoining cells σ,ρ for the affine mapping $\gamma(\sigma,\rho)$ implementing "parallel transport" from ρ into σ. More precisely, if $\tilde{v}^j(\rho)$ are the affine components of a vector in the space of ρ, then $\tilde{\gamma}^i_j(\sigma,\rho)\ \tilde{v}^i(\rho)$ can be considered as the components of the same vector relative to σ.

Let $\sigma = [01234]$ and $\rho = [51234]$, so that, e.g., $\sigma \cap \rho = [1234]$. Then if

$$P = \sum_{j=1}^{5} t^j(\rho)[j] \tag{B.1}$$

is a point (not a vector!) in ρ, we seek its affine coordinates $s^k(\sigma)$ relative to σ:

$$P = \sum_{k=0}^{4} s^k(\sigma)[k] \tag{B.2}$$

It is easy to see, by expressing [5] as an affine sum of $[0]\cdots[4]$, that s^k is related to t^j by a relation of the form

$$s^k = \gamma^k_j\ t^j \tag{B.3}$$

in which, because of (3.2),

$$\sum_k \gamma^k_j = 1 \tag{B.4}$$

In fact, if

$$[5] = \sum_{k=0}^{4} x^k[k] \qquad (\Sigma\, x^k = 1) \tag{B.5}$$

then γ is the matrix

$$\gamma(\sigma,\rho)^k{}_j = \begin{array}{c|ccccc} & 5 & 1 & 2 & 3 & 4 \\ \hline 0 & x^o & 0 & 0 & 0 & 0 \\ 1 & x^1 & 1 & 0 & 0 & 0 \\ 2 & x^2 & 0 & 1 & 0 & 0 \\ 3 & x^3 & 0 & 0 & 1 & 0 \\ 4 & x^4 & 0 & 0 & 0 & 1 \end{array} \qquad \text{(B.6)}$$

and that [0] and [5] lie on opposite sides of [1234] means x^o is negative:

$$x^o < 0 \qquad \text{(B.7)}$$

Now $\gamma(\sigma,\rho)^k{}_j$ represents an affine point transformation but can function equally well as a vector map[†] since a vector is just the difference of 2 points (see Section IIIA). From the characteristic that parallel transport preserves the metric, one deduces the crucial condition for γ :

$$\tilde{g}^{k\ell}(\sigma) = \gamma^k_i(\sigma,\rho)\gamma^\ell_j(\sigma,\rho)\ g^{ij}(\rho) \qquad \text{(B.8)}$$

which we express in terms of the inverse metric tensor only because this will produce the x^j with less effort.

(B.8) resolves into four typical equations:

$$\tilde{g}^{oo}(\sigma) = \tilde{\gamma}^o{}_i\tilde{\gamma}^o{}_j\tilde{g}^{ij}(\rho) = (x^o)^2\ \tilde{g}^{55}(\rho) \qquad \text{(B.9-1)}$$

$$g^{11}(\sigma) = (x^1)^2\tilde{g}^{55}(\rho) + 2x^1\ \tilde{g}^{15}(\rho) + \tilde{g}^{11}(\rho) \qquad \text{(B.9-2)}$$

[†]It is not, however, the affine components of a bitensor, as is $\tilde{\gamma}^k_j = \gamma^k_i\ \tilde{\delta}^i_j$, the preferred form of the linear transformation associated to γ .

$$\tilde{g}^{01}(\sigma) = x^o x^1 \, \tilde{g}^{55}(\rho) + x^o \, \tilde{g}^{51}(\rho) \qquad \text{(B.9-3)}$$

$$\tilde{g}^{12}(\sigma) = x^1 x^2 \, \tilde{g}^{55}(\rho) + x^1 \, \tilde{g}^{52}(\rho) + x^2 \tilde{g}^{51}(\rho) + \tilde{g}^{12}(\rho) \qquad \text{(B.9-4)}$$

Assuming $\tilde{g}^{55} \neq 0$ (which $\Rightarrow \tilde{g}^{oo} \neq 0$) it is easy to solve (B.9-1), (B.9-3) and (B.7) for x :

$$x^o = -\sqrt{\frac{\tilde{g}^{oo}(\sigma)}{\tilde{g}^{55}(\rho)}}$$

$$x^k = \frac{\frac{1}{x_o}\,\tilde{g}^{ok}(\sigma) - \tilde{g}^{5k}(\rho)}{\tilde{g}^{55}(\rho)} \, , \qquad k=1\cdots 4 \qquad \text{(B.10)}$$

If $\tilde{g}^{55} = 0$ then other, equally easily derived formulas obtain.

Now envision a ring of cells comprising the star, $\mathfrak{S}_4(b)$, of a bone b , and numbered from 1 to k . Then the composition

$$\Lambda(1) = \gamma(1,k) \circ \gamma(k,k-1) \circ \cdot \cdot \cdot \circ \gamma(3,2) \circ \gamma(2,1) \qquad \text{(B.11)}$$

is an affine transformation representing the circulator of b in the sense $1 \to 2 \to 3 \cdots \to 1$, and referred to cell #1. Because of (B.8) the matrix $g(1)$ of cell #1 is invariant under Λ . And obviously (see (B.6)) Λ leaves b, and therefore any vector parallel to b , unchanged. Considered as a linear transformation the circulator is thus a Lorentz transformation which leaves unaltered the vectors of some two-dimensional subspace.

Appendix C: Proof that $\sum A\,\delta\eta = 0$

In deriving the metrical thatch equations (Section IID) we used the identity

$$\sum_b A(b)\ \delta\eta(b) = 0 \tag{C.1}$$

where the sum is over all the bones of the net. Regge [1] has proved this for a positive definite metric; this appendix extends his proof to the signature $-+++$.

a) Space-time "trigonometry"

The algorithm described in Section VIIIB produces the defect η by "imaging" the faces F_i in a two-dimensional space of appropriate signature. But why not just add up the "angles" between the F_i and subtract the sum from 2π (not that this would be more efficient!)? For a timelike bone this prescription is easy to carry out, but for a spacelike bone ($-+$ image space) the definition of "angle" involves some subtlety, it being evident, for example, that θ cannot increase continuously from 0 to 2π during a complete circuit of the bone.

The basic property of angle is additivity on segments of the plane so that $\theta(x,y)$ in Fig. C.1 is independent of where z intervenes. But this additivity is tied up with the relation

$$\cos\,\theta(x,y) = \frac{x\cdot y}{|x|\cdot|y|} \tag{C.2}$$

which might, therefore, be able to define θ in general. However, whenever x or y is timelike there will be an ambiguity in the sign

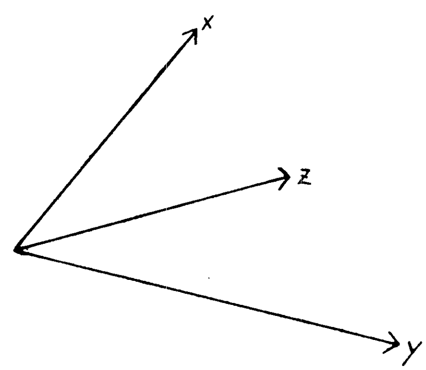

Figure C.1 $\theta(x,y) = \theta(x,z) + \theta(z,y)$

of the right hand side, not to mention that $\cos\theta$ itself determines θ (which is in general a complex number) only up to a sign. There are two consistent ways (complex conjugates of each other) to resolve these ambiguities and the following defines one of them.

If x, y are vectors, then define

$$|x| = \sqrt{\langle x, x\rangle}$$

$$|x \wedge y| = \sqrt{\frac{1}{2}\langle x \wedge y,\ x \wedge y\rangle} \qquad \text{(C.3)}$$

in which the root is by definition positive imaginary or positive real for negative or positive argument respectively. Then we determine θ from the formulae

$$\cos\theta = \frac{\langle x, y\rangle}{|x|\ |y|}$$

$$\sin\theta = \frac{|x \wedge y|}{|x|\ |y|} \qquad \text{(C.4)}$$

and complete the definition by stipulating

$$0 \leq \operatorname{Re}\theta \leq \pi \qquad \text{(C.5)}$$

Let us check, for example, the cosine of the additivity condition illustrated in Figure C.1. To insure that z is "between" x and y we can conveniently put (since the magnitude of z is irrelevant)

$$z = tx + (1-t)y \quad , \qquad t \geq 0 \qquad \text{(C.6)}$$

Then

$$\cos\,\theta(x,y) = \cos(\theta(x,z) + \theta(z,y))\ ?$$

$$\cos\,\theta(x,y) = \cos\,\theta(x,z)\cos\,\theta(z,y) - \sin\,\theta(x,z)\sin\,\theta(z,y)\ ?$$

$$\frac{<x,y>}{|x|\ |y|} = \frac{<x,z>\ <z,y>}{|x|\ |z|^2\ |y|} - \frac{|x\wedge z|\ |z\wedge y|}{|x|\ |z|^2\ |y|}\ ?$$

$$|z|^2 <x,y> = <x,z>\ <z,y> - |x\wedge z|\ |z\wedge y|\ ?$$

The equality is readily verified using (C.3) and (C.6).

From (C.4) it is easy to work out the path of θ in the complex plane during a complete circuit of a spacelike bone (Figure C.2). Taking differences shows that an angle within region I or III is positive pure imaginary, while one in II or IV is negative imaginary.

Finally, what is the formula for the action, S, in terms of defect angles as defined above? Let θ be the defect, B a tensor representing the bone, and

$$|B| = \sqrt{\frac{1}{2} <B,B>} \tag{C.7}$$

with the same convention that $|B| \sim +i$ or $+1$. In fact, if A is the (real) area of the bone as used in Section IID, then

$$||B|| = A$$

and a detailed check of the sign conventions as indicated, e.g., by Figure 2.2, reveals a simple formula for S valid in all cases:

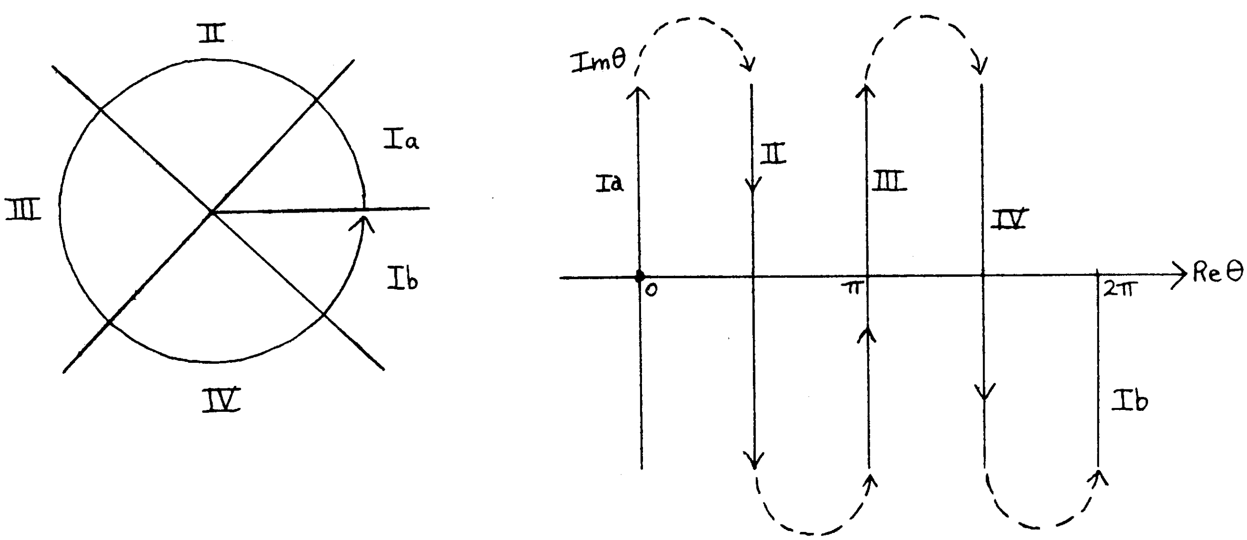

Figure C.2 The dotted lines are "at the point at infinity" in the complex θ-plane.

$$iA(b)\ \eta(b) = |B|\ \theta(b) \tag{C.8}$$

$$iS = \sum_b |B|\ \theta(b) \tag{C.9}$$

b) Derivation of the identity

If $\sigma \varepsilon \Sigma_4$ is a cell of the net then let $F(j)$ represent the face opposite vertex j , $B(jk)$ the bone opposite $[jk]$ (geometrically the intersection of F_j and F_k), and $\theta(jk)$ the contained angle. We will prove the

(C.10) Lemma: $$\sum_{k,j\,\varepsilon\,\sigma} |B(jk)|\ \delta\theta(jk) = 0$$

if δ denotes a variation in the length ℓ^2_{jk} of the legs of σ . Then the desired identity (C.1) will result from summing over all the cells of the net:

$$\sum_b |B(b)|\ \delta\theta(b) = -0 \quad ,$$

or by (C.8)

$$i \sum_b A(b)\ \delta\eta(b) = 0$$

In order to prove the lemma, it is most convenient to express everything in terms of $\tilde{g}^{ij}(\sigma)$. To that end we note that (C.4) defines the angle between two vectors whereas $\theta(jk)$ is in the first place an angle between two hyperplanes. If x,y are co-vectors corresponding to these hyperplanes, then it turns out that (C.4) applies to the complex conjugate θ^* , rather than to θ . But we can take for x,y the (co)basis vectors $\underset{\sim}{e}^i,\underset{\sim}{e}^j$ with a relative minus sign for proper orientation. Then

$$\cos\,\theta(i,j)^* = \frac{\langle \underset{\sim}{e}^i, -\underset{\sim}{e}^j\rangle}{|\underset{\sim}{e}^i|\;|\underset{\sim}{e}^j|}$$

$$\sin\,\theta(i,j)^* = \frac{|\underset{\sim}{e}^i \wedge \underset{\sim}{e}^j|}{|\underset{\sim}{e}_i|\;|\underset{\sim}{e}_j|} \quad ,$$

or, since

$$\langle \underset{\sim}{e}^i, \underset{\sim}{e}^j\rangle = \tilde{g}^{ij}$$

and

$$|\underset{\sim}{e}^i \wedge \underset{\sim}{e}^j|^2 = \langle \underset{\sim}{e}^i, \underset{\sim}{e}^i\rangle \; \langle \underset{\sim}{e}^j, \underset{\sim}{e}^j\rangle - \langle \underset{\sim}{e}^i, \underset{\sim}{e}^j\rangle^2$$

$$= \tilde{g}^{ii}\,\tilde{g}^{jj} - (\tilde{g}^{ij})^2 \equiv -\tilde{h}^{ijij}$$

$$\cos\,\theta^* = \frac{-\tilde{g}^{ij}}{\sqrt{\tilde{g}^{ii}}\sqrt{\tilde{g}^{jj}}} \quad , \quad \sin\,\theta^* = \frac{\sqrt{-\tilde{h}^{ijij}}}{\sqrt{\tilde{g}^{ii}}\sqrt{\tilde{g}^{jj}}} \tag{C.11}$$

Writing $\delta\theta^* = -\delta(\cos\,\theta)^*/\sin\,\theta^*$, evaluating, and simplifying:

$$\delta\theta(i,j) = \frac{i}{\sqrt{\tilde{h}^{ijij}}} \left\{\delta\tilde{g}^{ij} - \frac{1}{2}\left(\frac{\delta\tilde{g}^{ii}}{\tilde{g}^{ii}} + \frac{\delta\tilde{g}^{jj}}{\tilde{g}^{jj}}\right)\tilde{g}^{ij}\right\} \tag{C.12}$$

which expresses $\delta\theta$ in terms of $\delta\tilde{g}^{ab}$.

It is somewhat easier to express $|B(ij)|$ in these terms:

$$\tilde{B}(ij)^{k\ell} = \frac{1}{2}\,\tilde{\varepsilon}^{ijk\ell} \qquad \text{(see (3.16))}$$

where

$$|B|^2 = \frac{1}{2}\langle B,B\rangle = \frac{1}{2}\,\tilde{B}^{ab}\,\tilde{B}^{ab}\,\tilde{g}_{aa}\,\tilde{g}_{bb} =$$

$$= \frac{1}{8}\, \tilde{g}_{aa}\, \tilde{g}_{bb}\, \tilde{\varepsilon}^{ijab}\, \tilde{\varepsilon}^{ijab}$$

$$= \frac{1}{4}\, (-\tilde{g})\, \tilde{h}^{ijij}$$

$$|B| = \frac{\sqrt{-\tilde{g}}}{2} \sqrt{\tilde{h}^{ijij}} \tag{C.13}$$

Combining the last two results,

$$|B(ij)|\ \delta\theta(ij) = \frac{i}{2} \sqrt{-\tilde{g}}\ \left\{\delta\tilde{g}^{ij} - \frac{1}{2}\, \tilde{g}^{ij} \left(\frac{\delta\tilde{g}^{ii}}{\tilde{g}^{ii}} + \frac{\delta\tilde{g}^{jj}}{\tilde{g}^{jj}}\right)\right\}$$

the sum of which for all i,j vanishes in virtue of the rule (3.10). This proves the lemma.

Appendix D: The Character of the Continuum Limit

a) General considerations

All the thatch equations discussed above share this feature: the thatch represents a true field but one of a very simple (possibly singular) type. Thus $A(jk)$ corresponds to a piecewise linear electromagnetic potential, while $\ell^2(jk)$ defines a piecewise flat manifold. In terms of these fields one defines the action in the usual way; then the thatch equations just assert the stationarity of the action--but only for variations which maintain the correspondence of the field to some thatch. By using ever finer nets one allows for ever more delicate variations of the field so that, in the limit of an infinitely fine net, one expects the solution thatch to correspond exactly to the true field[†].

On the other hand, as we will see below in particular examples, it is in general false that the limit of a particular thatch equation is the correct field equation at that point. In other words, the discretization of an exact continuum solution will not produce a solution of the thatch equations, even in the continuum limit!

To understand this better remember that $A(jk)$, for example, corresponds to a piecewise linear field. At any given point this can agree with $A_\mu(x)$ only to terms of the first order in dx (precisely

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[†]Unfortunately the action is not positive definite (hyperbolic equation); so there will also be some requirement for overall stability. That is not discussed at all in this appendix.

those involved in the definition of the action!); it can reflect the second derivatives of A_μ only <u>on the average</u> over a small region. Thus one can expect A_μ and even $F_{\mu\nu}$, but not $\partial_\mu F_{\mu\nu}$ to become exact in the continuum limit. The field equations $\partial_\mu F_{\mu\nu} = 0$ can become exact only after averaging.

We can arrive at this conclusion again by a somewhat different argument. Let Ω be a region of spacetime and consider for simplicity the scalar thatch $\phi(j)$. In the continuum limit δS must vanish for any smooth variation $\delta\phi(x)$ of the field $\phi(x)$. In particular, it must vanish for the variation $\delta\phi$ = constant within Ω, $\delta\phi = 0$ outside. But for such a variation δS is just the sum

$$\delta\phi \cdot \sum_{j\in\Omega} \frac{\partial S}{\partial\phi(j)}$$

(The boundary terms are negligible if the net is sufficiently fine.) We conclude that even though the thatch equations, $\frac{\partial S}{\partial\phi(j)}$, may fail individually, their sum

$$\sum_{j\in\Omega} \frac{\partial S}{\partial\phi(j)} = 0$$

over any finite region Ω will be valid.

For the thatches A, ℓ^2 the same argument applies except that, in place of $\delta\phi$ = const. one must put a variation of the $\ell^2(ij)$ [respectively $\delta A(ij)$] which corresponds to $\delta g_{\mu\nu}$ = const. [resp. δA = const.]. There will be 10 [resp. 6] linearly independent such variations.

b) Illustration

The simplest example of these considerations is the flat scalar wave equation in two dimensions. We will examine the thatch equation (7.6) for various nets and show that while the continuum limit of (7.6) is always a homogeneous second order differential equation, it is sometimes the wrong one. As expected, however, an appropriate sum of these equations (over a "unit cell" of the lattice) always reduces to the correct equation in the continuum limit.

Let us write (7.6) for the vertex $[0] \varepsilon \Sigma_o$ in the form

$$\sum_k \mu(k)\ \phi(k) = 0 \tag{D.1}$$

where

$$\mu(k) \equiv \sum_\sigma (0, k \varepsilon \sigma)\ \tilde{g}^{0k}(\sigma)\ V(\sigma) \tag{D.2}$$

and k ranges over $\mathfrak{S}_o(\mathfrak{S}_1([0])$. If we expand $\phi(x)$ about $[0]$ (assuming flat space-time recall) then

$$\phi(j) = \phi(0) + \phi'(0) \cdot \overrightarrow{0j} + \frac{1}{2}\phi''(0) \cdot \overrightarrow{0j} \otimes \overrightarrow{0j} + \cdots \tag{D.3}$$

and (D.1) becomes

$$\phi(0) \sum_k \mu(k) + \phi'(0) \cdot \sum_k \mu(k)\ \overrightarrow{0k} + \frac{1}{2}\phi''(0) \cdot \sum_k \mu(k)\ \overrightarrow{0k} \otimes \overrightarrow{0k} + \cdots = 0 \tag{D.4}$$

Since one can prove in general (flat space) that

$$\sum_k \mu(k) = 0 \quad , \qquad \sum_k \mu(k)\ [k] = 0 \tag{D.5}$$

(D.4) becomes, to second order in $\overrightarrow{0k}$

$$\frac{1}{2}\,\phi''(0)\cdot\sum_k\ \mu(k)[k]\otimes[k] = 0 \tag{D.6}$$

(The notation of the second equation of (D.5) makes sense because of the first, just as that of (D.6) in turn makes sense because of (D.5).) In the continuum limit this has the form

$$a^{\mu\nu}\ \partial_\mu\partial_\nu\phi(0) = 0 \tag{D.7}$$

Unfortunately $a^{\mu\nu} \neq g^{\mu\nu}$ in general.

Consider for example the star shown in Figure D.1a. The corresponding equation (D.7) works out as

$$-\frac{1}{2}\,(1+q)\,\frac{\partial^2\phi}{\partial x^2} - \frac{1}{2}\,(1+\frac{1}{q})\,\frac{\partial^2\phi}{\partial y^2} = 0 \tag{D.8}$$

in which

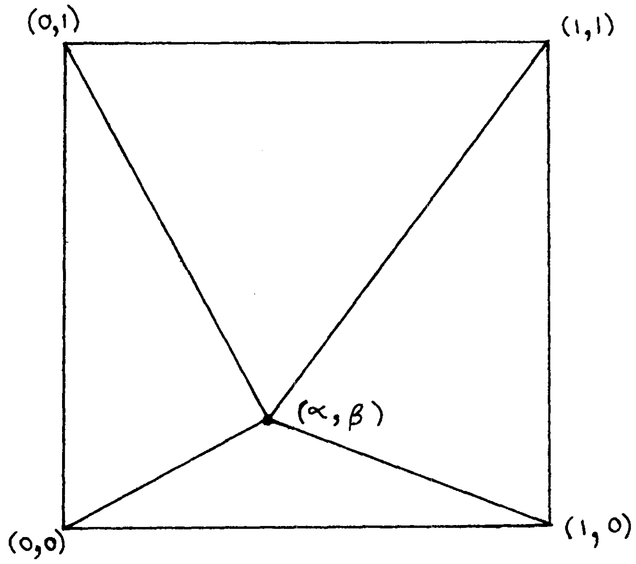
$$q = \frac{\alpha(1-\alpha)}{\beta(1-\beta)}$$

(For convenience we deal with a positive definite metric.) This differs from the correct $\nabla^2\phi = 0$ unless $q = 1$, i.e., unless (α,β) lies on one of the diagonals of the square.

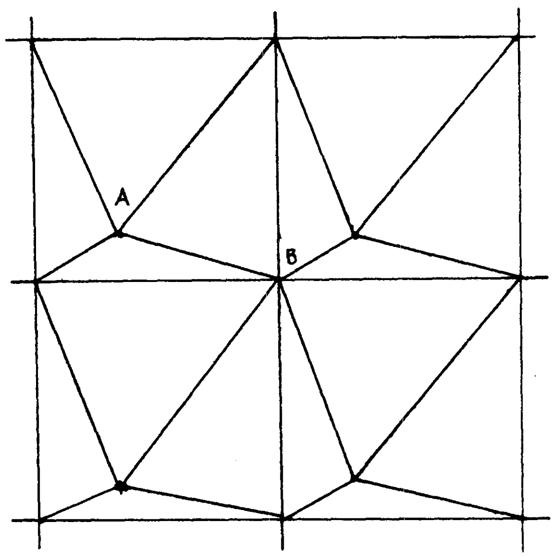
On the other hand, for a vertex such as B in Fig. D.1b, (D.7) becomes, with the same normalization,

$$-\frac{1}{2}\,(3-q)\,\frac{\partial^2\phi}{\partial x^2} - \frac{1}{2}(3-\frac{1}{q})\,\frac{\partial^2\phi}{\partial y^2} = 0 \tag{D.9}$$

Since on the average there are equal numbers of vertices of types A and B, the average thatch equation is the average of (D.8) and (D.9):



(a) A star in flat two-dimensional space-time. The vertices are labelled by their rectangular coordinates.



(b) A complete net made up of repetitions of this star.

Figure D.1

$$- \nabla^2 \phi = 0$$

which is the correct continuum equation.

Notice that, in forming the average thatch equation it was enough to consider one equation for each type of net vertex. In a net where all vertices were equivalent, each individual equation would already be completely typical. This explains why the nets of Chapter VII produced the correct continuum limit without any averaging process.

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