# STATISTICAL MECHANICS OF TWO-DIMENSIONAL EULER EQUATIONS AND JUPITER'S GREAT RED SPOT

Thesis by

Jonathan Miller

In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1991

(Submitted October 17, 1990)

## ACKNOWLEDGEMENTS

The origin of this work was a suggestion by M.C. Cross, who contributed essential ideas throughout its execution. P.B. Weichman is responsible for the clean exposition in chapter 5, the introduction of the Kac-Hubbard-Stratonovitch transformation, and the neat characterization of the energy/enstrophy theory.

We are grateful to Phil Marcus of the Department of Mechanical Engineering, University of California at Berkeley, for discussions and for making his articles available to us prior to publication. Phil Marcus and Changhoon Lee kindly provided the dynamical simulation data described in Appendix B.

The Shell Foundation provided partial support.

## ABSTRACT

For the first time, we construct a statistical mechanics for the two-dimensional Euler fluid which respects all conservation laws. We derive mean-field equations for the equilibrium, and show that they are exact. Our methods ought to apply to a wide variety of Hamiltonian systems possessing an infinite family of Casimirs. We illustrate our theory by a comparison to numerical simulations of Jupiter's Great Red Spot.

# TABLE OF CONTENTS

## FIGURE CAPTIONS

Fig. 1 Comparison of a direct solution (smooth curve) of the differential equation (5-65) to a Monte Carlo calculation (jagged curve) for the same values of the conserved quantities. The domain is the unit disc with rigid boundaries.

$$g(\sigma) = 0.265\delta(\sigma - 1) + 0.735\delta(\sigma)$$
energy: $0.0108\pi$; angular momentum: $0.0988\pi$

Fig. 2 Simulation I: initial conditions. The grey levels denote evenly spaced potential vorticity values ranging from $-1.0$ (lightest) to 1.0 (darkest).

$$\vec{u} = \hat{u} + \vec{u}_p; \qquad \hat{u} = (0, \beta r^2/3); \qquad \beta = 1; \qquad \Gamma_i = 0$$
$$\omega_p = \nabla \times \vec{u}_p = (1/2)\{\tanh 10(r - 11/15) - \tanh 10(r - 14/15)\}$$
$$R_o = 4/3; \qquad R_i = 1/3$$

Fig. 3 Simulation I: long-time vorticity profile from dynamical simulation.

Fig. 4 Simulation I: statistical equilibrium profile from Monte Carlo calculation.

Fig. 5 Simulation II: initial conditions. The grey levels denote evenly spaced potential vorticity values, extending from $-1.0$ (lightest) to 1.0 (darkest).

$$\vec{u} = \hat{u} + \vec{u}_p; \qquad \hat{u} = (0, \beta r^2/3); \qquad \beta = 1; \qquad \Gamma_i = 0$$
$$\omega_p = \nabla \times \vec{u}_p = f(r, \theta, \theta_0) - f(r, \theta, \theta_0 + \pi)$$
$$f(r, \theta, \hat{\theta}) = (1/2)\{\tanh 0.857 \left(\eta(r, \theta, \hat{\theta}) + 1\right) - \tanh 0.857 \left(\eta(r, \theta, \hat{\theta}) - 1\right)\}$$
$$\eta(r, \theta, \hat{\theta}) = \left[\frac{(5/6)(\theta - \hat{\theta})}{0.35}\right]^2 + \left[\frac{r - 5/6}{0.175}\right]^2$$
$$R_o = 4/3; \qquad R_i = 1/3$$

Fig. 6 Simulation II: long-time vorticity profile from dynamical simulation.

Fig. 7 Simulation II: statistical equilibrium profile from Monte Carlo calculation.

# Chapter 1

## Introduction

The study discussed in this thesis originates from M.C. Cross' suggestion that the long-time evolution of the two-dimensional fluid might be described by a statistical equilibrium theory. That notion derives in turn from some numerical calculations by Marcus (1988) on a model for the Great Red Spot of Jupiter.

We begin by describing the model (Marcus, 1988; see also Ingersoll and Cuong, 1981). Jupiter for our purposes comprises a two-dimensional annulus, with rigid boundaries. The winds of Jupiter are realized by applying to the annulus an external potential, which is equivalent to some combination of a background shear, potential vorticity, and Coriolis force. The external potential possesses an orientation, which breaks the equivalence of positive and negative vorticity.

The initial conditions for the calculation are determined completely by the initial vorticity field. A typical initial vorticity field involves two spots of vorticity of opposite sign, placed on opposite sides of the annulus.

We may then follow the time evolution of the fluid. The spot of vorticity with opposite orientation to the external potential breaks down and disperses, whereas the spot of vorticity with the same orientation as the external potential remains largely intact. At long times, only the vortex sharing the sense of the external potential persists.

Marcus claims that the persisting vortex possesses a number of characteristics that bear qualitative comparison to Jupiter's Great Red Spot. In particular, the shape and vorticity profile of the persistent vortex are consistent with those of the Spot. Of course, the model used in the simulation is idealized; one could, if one wished, include the effects of the free surface, bottom topography, planetary curvature, etc. Nevertheless, the simulation suggests that the basic physics of the phenomenon may well be encapsulated by Marcus' simplified model.

Our goal in the present work is to ask whether we can predict and explain the long-time evolution of flows such as the one introduced above without resort to dynamics. There are two reasons for wishing to accomplish such a task. One is simply pragmatic: dynamical simulations demand extensive computation and incur substantial error at long times. An independent method of calculating the asymptotic evolution of a flow provides a check on the dynamical calculation and might plausibly replace it. The other reason is that when dynamics are unimportant, we may understand the basic physics through simpler, static considerations.

The treatment we describe here comprises the most naive application of statistical mechanics to the two-dimensional Euler fluid. The justification for expecting this kind of approach to work is merely historical.

After we remind the reader of some basic properties of Euler flow in chapter 2, we argue in chapter 3 that earlier attempts at a comprehensive statistical mechanics for the two-dimensional inviscid fluid have been unsuccessful. These failures originate primarily in the mistreatment of the infinite family of conserved quantities of Euler flow. Chapter 4 lays out the computation that needs to be done, in a manner that highlights the role of the conserved quantities. In chapter 5, we present two different methods for deriving the mean-field equations that equilibrium configurations of the fluid must satisfy, and we

prove the "dressed vorticity corollary." We derive the Lynden-Bell theory of star clusters, which ante-dates our own statistical mechanics. We discuss the physical content of these theories in chapter 6.

## Chapter 2

## Primer on the ideal fluid in two dimensions

In this section we collect a few standard facts about Euler flow in two dimensions, to which we shall refer repeatedly in the ensuing text. A transparent derivation of the properties of Euler flow may be found in Chorin and Marsden (1979).

*Notation*

$\vec{u}(\vec{r})$ denotes the two-dimensional velocity field; $\omega(\vec{r})$ denotes the (scalar) vorticity field, defined as

$$\omega = \nabla \times \vec{u} = \partial_x u_y - \partial_y u_x \tag{2-1}$$

We label by $\Omega$ the region containing the fluid; $p$ is the pressure, $\rho$ the density.

*Euler equation*

Following Landau and Lifshitz (1980a) we observe that in a coordinate system co-moving with an (infinitesimal) fluid element $\varrho$, the force on the fluid element is given by

$$-\oint_{\partial\varrho} p\, d\vec{S} = -\int_{\varrho} \nabla p\, dV \tag{2-2}$$

so that Newton's equation reads

$$\rho D\vec{u}/Dt = -\nabla p \tag{3-3}$$

where $D/Dt$ denotes a covariant (convective, material) derivative. In a stationary coordinate system, the covariant derivative may be rewritten to obtain Euler's equation

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\,\vec{u} = -\frac{1}{\rho}\nabla p \tag{2-4}$$

When we require the density $\rho$ to be constant, the equation for mass conservation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{u} = 0 \tag{2-5}$$

entails the incompressibility condition:

$$\nabla \cdot \vec{u} = 0 \tag{2-6}$$

which will be taken for granted in the remainder of our work.

Taking the curl of Euler's equation (2-4) and using the incompressibility condition, we readily obtain in two dimensions

$$D\omega/Dt = \partial\omega/\partial t + (\vec{u} \cdot \nabla)\omega = 0 \tag{2-7}$$

That is, the convective derivative of the vorticity vanishes.

*Stream function*

The vanishing of $\nabla \cdot \vec{u}$ in two dimensions entails the existence of a scalar function (the stream function) $\psi(\vec{r})$ such that

$$\vec{u} = \vec{\nabla} \times \psi \equiv (\partial_y \psi, -\partial_x \psi) \tag{2-8}$$

and consequently $\omega(\vec{r}) = -\nabla^2 \psi(\vec{r})$.

*Conserved quantities*

For any closed path $\partial\eta(t)$ moving with the fluid in time $t$, and whose interior $\eta$ is contained in the fluid, the circulation

$$\Gamma = \oint_{\partial\eta(t)} \vec{u} \cdot d\vec{\ell} = \int_{\eta(t)} \omega \; d^2\vec{r} \tag{2-9}$$

is conserved by Euler flow. In fact, in two dimensions, given *any* function $f(\omega)$ of the vorticity field, the quantity $\int_{\eta(t)} f(\omega(\vec{r}))\; d^2\vec{r}$ is conserved by the flow:

$$\frac{\partial}{\partial t}\int_{\eta(t)} d^2\vec{r}\; f(\omega(\vec{r})) = \int_{\eta(t)} d^2\vec{r}\; f'(\omega)\{\frac{\partial\omega}{\partial t} + \vec{u}\cdot\nabla\omega\} = 0 \tag{2-10}$$

If we take $f(\omega) = \omega^2$, the corresponding integral invariant is called the "enstrophy," a term apparently originating in Leith (1968).

*Boundary conditions*

Ordinarily, we shall assume rigid boundaries: that is, $\vec{u}$ is assumed to be tangent to the boundary $\partial\Omega$ of a bounded planar region $\Omega$ containing the fluid. When $\Omega$ possesses $n$ holes, we may define the quantities $\Gamma_n$ to be the circulations around each of these holes. We observe (Holm et al., 1985) that together $\omega(\vec{r})$ and $\Gamma_1, \ldots, \Gamma_n$ uniquely determine the velocity field $\vec{u}$.

# Chapter 3

## Review of statistical mechanics of the two-dimensional fluid

Our discussion throughout will be confined to the two-dimensional Newtonian fluid. The Navier-Stokes equations for an incompressible fluid

$$\begin{aligned} \rho(\partial\vec{u}/\partial t + \vec{u}\cdot\nabla\vec{u}) &= -\nabla p + \nu\nabla^2\vec{u} \\ \nabla\cdot\vec{u} &= 0 \end{aligned} \tag{3–1}$$

are thought to be a valid description of a viscous fluid in a variety of physical regimes. We shall not discuss when and why the two-dimensional fluid is a good physical model for fluids in a three-dimensional world. We refer the reader to the text by Pedlosky (1987) for a justification of the two-dimensional model. We remark only that a variety of laboratory and geophysical flows display essentially two-dimensional behavior, a nearly invariant prerequisite for which is rotation of the plane containing the fluid along the perpendicular axis. The well known Taylor-Proudman column (see, e.g., Tritton, 1988) graphically illustrates this kind of phenomenon in the laboratory; planetary atmospheres provide a geophysical setting. The elimination of the third dimension entails several consecutive approximations, and one must maintain careful control of the time and length scales that one wishes to describe.

Similar attention to time and length scales is needed to justify the additional approximation of discarding the viscous term in the Navier-Stokes equation to derive the Euler

equations, which in two dimensions take the form (2-7):

$$\partial\omega/\partial t = -\vec{u}\cdot\nabla\omega$$
$$\nabla\cdot\vec{u} = 0 \qquad (3\text{–}2)$$

Later on, we shall give an informal argument that the physics of certain flows may permit us to neglect the effect of viscosity on length scales of interest. For the bulk of our work, we shall simply assume that the physics is described by the *inviscid* Euler equations in two dimensions, (3-2).

Onsager (1949) first attempted the construction of a statistical mechanics for the two-dimensional fluid. Since subsequent papers refer to this one with hardly an exception, we discuss the ideas contained in his paper in detail.

Onsager wants to explain the generic occurrence of large isolated vortices in unsteady flow. He begins by restricting himself to a subset of Euler flows: those that can be described by a system of point vortices. The point vortex system consists of a linear superposition of point sources of vorticity:

$$\int d^2\vec{r}\,\omega(\vec{r}) = \sum_i \omega_i\delta(\vec{r}-\vec{r_i}) \qquad (3\text{–}3)$$

Writing $\nabla^2\psi(\vec{r}) = -\omega_0\delta(\vec{r}-\vec{r_0})$ for a single point vortex with charge $\omega_0$ located at $\vec{r_0}$, we obtain $\psi(\vec{r}) = \omega_0\mathcal{G}(\vec{r},\vec{r_0})$. Here $\mathcal{G}(\vec{r},\vec{r_0})$ is the Green's function for the Laplacian on the region containing the fluid, and its form depends on the boundary conditions. Obviously, $\mathcal{G}(\vec{r},\vec{r_0})$ corresponds to the potential at $\vec{r}$ due to a vortex at $\vec{r_0}$. For example, free boundary conditions yield a logarithmic potential $\mathcal{G}(\vec{r},\vec{r_0} = -\frac{1}{2\pi}\log(|\vec{r}-\vec{r_0}|/R)$ where $R$ is an arbitrary constant with dimensions of length.

The potential of a point vortex is finite at any finite distance from the source. *There exists no* a priori *bound on how closely two point vortices may approach each other.*

It had been observed much earlier that the equations of motion of a superposition of $N$ point vortices can be derived from a Hamiltonian (Kirchhoff, 1877). It is precisely this feature that leads Onsager to restrict himself to the special case of point vortices. Without a Hamiltonian, it is not obvious how to construct a statistical mechanics; once one has a Hamiltonian, no choice in that construction remains.

The Hamiltonian is:

$$\mathcal{H} = -\frac{1}{2}\sum_{i \neq j} \omega_i \omega_j \mathcal{G}(\vec{r}_i, \vec{r}_j) \tag{3-4}$$

and the equations of motion are derived by regarding the two components of the position vector of each vortex as conjugate variables:

$$\omega_i \frac{d\vec{r}_i}{dt} = \vec{\nabla}_i \times \mathcal{H} \tag{3-5}$$

Notice that phase space coincides with configuration space. Using Onsager's notation, we write the volume of phase space near some configuration:

$$d\Omega = dx_1 dy_1 \ \ldots \ dx_N dy_N \tag{3-6}$$

If we demand that the fluid be contained in some compact region of space, we see that $\int d\Omega$, the *total* volume of phase space, is finite.

The boundedness of phase space has a peculiar consequence. Ordinarily, we expect that, as a function of energy, the volume of phase space available to a system, $\Omega(E)$, increases exponentially (Landau and Lifshitz, 1980b). But here, the finite phase volume precludes this alternative, and the possibility exists that the volume of phase space available to the system could *decrease* as we increase the energy. Put another way, if we take the entropy at some system energy to be given by $S = \log \Omega(E)$, then, above some energy $E_m$, the quantity $dS/dE$, which is formally equal to the temperature $T$, may be *negative*. In this regime, the energy is so high that vortices with the same sign cluster together,

and vortices with opposite signs repel. The equilibrium consists of two isolated clusters of vortices.

Addition of the usual kinetic momentum term $p^2/2m$ to the Hamiltonian (3-4) would eliminate the negative temperature regime. The total phase volume would no longer be finite, and the momentum degrees of freedom could accommodate any energy. Microscopic degrees of freedom for example, are usually described by a Hamiltonian with a kinetic momentum, (with the exception of spin systems) and can not take on negative temperature. These remarks emphasize a basic feature of Onsager's theory for macroscopic vortices in fluids: the fluid system at negative temperature can not be in equilibrium with molecular degrees of freedom. Negative temperatures are higher than any positive temperature and a negative temperature system, on being placed in thermal contact with an ordinary positive temperature heat bath, will dump all its energy into the bath.

While this analysis does encompass the essential physics of the problem, Onsager remarks that it can not apply quantitatively to many flows of physical interest. Flows such as those in the model for the Red Spot discussed earlier involve *continuous* distributions of vorticity. But how to approximate a continuous distribution of vorticity by means of point vortices is not obvious, and in fact, generally not possible.

One way to apprehend the difficulty is to look at quantities conserved by Euler flow. Consider, for example, the integral of any finite power of the vorticity. For a vorticity field consisting of point sources, these integrals involve powers of delta functions. Yet in typical physical models the vorticity field has perfectly well-defined moments, to which we ought to be able to assign any consistent set of values.

A direct approach to the statistical mechanics of a continuous vorticity field is taken by Lee (1952). In his statistical field theory, the only conserved quantity is the energy.

Rewriting the Euler equations in Fourier space, he obtains a Liouville theorem:

$$\frac{\delta\dot{\omega}_{\vec{k}}}{\delta\omega_{\vec{k}}} = 0 \tag{3-7}$$

implying incompressible flow in the spectrally decomposed phase space. Lee's derivation applies in any dimension; we give here an argument valid in two dimensions only. Using the stream function $\psi$, we write

$$\vec{u}(\vec{r}) = i\sum_{\vec{l}} (l_y, -l_x)\psi_{\vec{l}} e^{i\vec{l}\cdot\vec{r}}; \qquad \omega(\vec{r}) = \sum_{\vec{m}} m^2 \psi_{\vec{m}} e^{i\vec{m}\cdot\vec{r}}; \tag{3-8}$$

In using the stream function we have incorporated the incompressibility condition of (2-6); now we Fourier transform the equation of motion in (3-2) to obtain

$$\dot{\omega}_{\vec{m}} = -\sum_{\vec{q}} (\vec{q}\times\vec{m})\omega_{\vec{q}}\psi_{\vec{m}-\vec{q}} \tag{3-9}$$

Differentiating with respect to $\omega_k$, we find

$$\frac{\delta\dot{\omega}_{\vec{m}}}{\delta\omega_{\vec{k}}} = 2(\vec{k}\times\vec{m})\psi_{\vec{m}-\vec{k}} \tag{3-10}$$

Setting $\vec{k} = \vec{m}$ yields our result. The Liouville equation plays an important role in chapter four, where it determines the measure on our phase space.

At finite temperature, imposing only energy conservation yields an energy spectrum of the form $E(k) \propto k$, which is ultraviolet divergent. Lee imposes an ultraviolet momentum cutoff to obtain a finite energy. Lee remarks that in two dimensions conservation laws preclude ergodic behavior, and in consequence he restricts himself to three dimensions.

Following Lee's work, an analogous problem involving an infinite set of constants of motion was stated and solved by Lynden-Bell (1967) in the context of stellar dynamics. As we shall discuss later, his statement of the problem apparently incorporates physics inappropriate to clusters of stars, and so his solution is of questionable relevance to the

systems he was trying to understand; however, when his reasoning is applied to the two-dimensional inviscid fluid, it is exactly correct. It seems that no one was aware of this parallel until E. Ott pointed out to us the formal similarity of our equilibrium equations to Lynden-Bell's, well after our own work was completed.

With this aside to maintain historical continuity, we postpone a detailed explication of the Lynden-Bell theory, and proceed to describe Kraichnan's method. Kraichnan (1967) constructs a statistical mechanics by keeping as his conserved quantities not only the energy, but also the enstrophy. He truncates the Fourier representation of the Euler equations to obtain a *truncated* dynamics which explicity conserves these quantities, and for which a Liouville theorem holds in the remaining finite number of spectral variables (Kraichnan, 1975; Kraichnan and Montgomery, 1980). Since both the energy and the enstrophy are quadratic in the stream function, this field theory is Gaussian. By the standard procedure he obtains an equilibrium spectrum of the form:

$$E(k) = 1/(\beta k^2 + \alpha)$$
$$\Omega(k) = k^2/(\beta k^2 + \alpha) \tag{3–11}$$

where $\alpha$ and $\beta$ are inverse temperatures conjugate to the energy and the enstrophy respectively. As in the theory conserving exclusively energy, we need a large momentum cutoff at any finite temperature. For some reason we refer to the theory that conserves only energy and enstrophy as the energy/enstrophy theory.

Kraichnan justifies the elimination of constants of motion other than the energy and enstrophy on grounds that the cut off dynamics conserve quadratic constants of motion. Agreement with point vortex theory is also deduced (we shall show below that this correspondence may only be justified in a very limited regime). The reader will find additional supporting arguments of an informal nature in an article by Kraichnan (1975). The truncated dynamics, when used properly, is claimed to be faithful to both inviscid and viscous fluids in statistical equilibrium.

Within the context of the 1967 theory, the energy/enstrophy theory contains three parameters: a momentum cutoff, a temperature conjugate to the energy, and a temperature conjugate to the enstrophy. Each of these quantities is to be finite. If we take the limit of infinite momentum cutoff with the remaining two parameters finite, we deduce easily from (3-11) that the energy diverges logarithmically, and the enstrophy quadratically.

A dynamical simulation by Deem and Zabusky (1971) elicited a spurt of publications in which the predictions of the energy/enstrophy theory are compared to results of dynamical simulation. The comparison is generally quite favorable. We mention two examples. Fox and Orszag (1973) test by means of numerical simulation an explanation of the Deem and Zabusky results in terms of the energy/enstrophy theory. They find general agreement with the theory, except for substantial deviations at low and high wavenumbers, which they attribute either to insufficient relaxation time or to the effect of additional constants of the motion. The computational technology of fluid simulation has evolved in the last twenty years, and the extent to which numerical simulation by present standards would validate these results is unclear. Basdevant and Sadourny (1975) confirm the energy/enstrophy theory for the *truncated* dynamics by means of an Arakawa code, which explicitly conserves energy and enstrophy. These latter authors emphasize that the energy/enstrophy theory ought only to apply to the truncated dynamics and not to the full inviscid dynamics.

In all applications of the energy/enstrophy theory to this point the momentum cutoff is finite. Kraichnan (1975) and Basdevant and Sadourny (1975) point out that by choosing the following scaling for the energy and enstrophy temperatures one can obtain finite values for the energy and enstrophy in the limit of infinite momentum cutoff $k_m$:

$$\begin{aligned} \beta &\to \pi k_m^2/2\Omega \\ \alpha/\beta &\to -k_0^2 + k_m^2 \exp(-k_m^2 E/\Omega) \end{aligned} \tag{3-12}$$

where $k_0$ is an infrared momentum cutoff determined by the finite system size. Note that this scaling is only possible at negative values of the inverse energy temperature, $\alpha$.

The need to scale the temperatures as one increases the number of degrees of freedom also emerges in the point vortex theory. In the mid-seventies a large number of authors derived and studied mean-field equations for the the equilibrium configuration of a system of point vortices (Kida, 1975; Katz and Lynden-Bell, 1978; Lundgren and Pointin, 1977a,1977b; Montgomery et al., 1974, 1979; Ostriker, 1964; Pointin and Lundgren, 1976; Stodolkiewicz, 1963). The most readable of these papers are those of Lundgren and Pointin (1977a, 1977b). They derive two limits of the point vortex system as $N$, the number of vortices, diverges. The two limits are distinguished by the scaling of the energy with $N$. In the "low energy limit," the energy scales as the number of vortices, whereas in the "high energy limit," the energy scales as the *square* of the number of vortices. For ease of discussion in what follows, we consider only systems with vanishing total vorticity, except where otherwise noted.

The low energy limit can apply only at positive temperatures, for reasons that emerge below, and entails an extensive thermodynamic scaling: for non-trivial results we require the density to be constant in the limit of large $N$, with energy proportional to $N$. This regime corresponds to the neutral two-dimensional coulomb gas, to which the standard thermodynamic prescription may be applied.

The high energy limit obtains at negative temperatures for neutral systems (for non-neutral systems the high energy limit is also correct at positive temperatures and the low energy limit is trivial). For the high energy limit to be valid, the area of the system must be *finite* in the absence of additional constraints on the phase space, such as angular momentum (Lundgren and Pointin, 1977a). In the high energy limit, the area of the system remains constant and the density of vortices diverges as $N$, with energy proportional to

$N^2$. Furthermore, because the high energy limit is not extensive, *the temperature must be scaled* as $1/N$ to yield a non-trivial equilibrium. One can derive a mean-field equation for the spatial distribution of vorticity as $N \to \infty$:

$$\omega(\vec{r}) = -\vec{\nabla}^2\psi = \{\exp -\psi/\bar{T}\}\{\int_\Omega d^2\vec{r}\ \exp -\psi/\bar{T}\}^{-1} \tag{3–13}$$

where $\bar{T}$ denotes the *scaled* temperature. This equation has been deduced by a number of authors, using a variety of approximations. Lundgren and Pointin (1977b) claim to derive it rigorously, but a truncation of the frequency spectrum at the very beginning of their argument depends on an unproven assumption that two particular limits commute. On the other hand, we expect that (3-13) is nevertheless valid.

Using numerical methods, many authors went on to compare the long-time dynamics of point vortices with the predictions of this statistical mechanical theory of point vortices. They obtain good agreement between the long-time spatial distribution of vortices in dynamical simulations and the equilibrium vortex density profiles calculated from equation (3-13). We draw attention in particular to Kida (1975), Lundgren and Pointin (1976, 1977a), and Aref and Siggia (1980). The latter authors study long-time dynamics of point vortices in a shear layer.

In calculating the charge density response function in the low energy limit, Lundgren and Pointin (1977b) use an approximation introduced by Edwards (1957) in a calculation of the properties of a neutral plasma at (finite) positive temperature. Edwards approximates the Jacobian in a change of variables from particle to collective coordinates, and is emulated by Edwards and Taylor (1974), Seyler et al. (1974, 1975, 1976), Taylor et al. (1971, 1972, 1973) and others in treatments of the 2-d plasma. The approximation, which they call the random phase approximation [RPA], is equivalent to that of Pines and Bohm (1952). RPA is a well-understood *perturbative* tool which has been used in a variety of physical

contexts (Kadanoff and Baym, 1962). In the particular case of particles with a Coulomb interaction, RPA is only valid in a *screened* phase, where the number of particles within a Debye screening length, $\lambda_D = (2\pi e^2 n/T)^{-\frac{1}{2}}$, of one another is large ($e$ the charge of a particle; $n$ the particle density). Under these conditions, charge fluctuations around the mean charge density are small, justifying a perturbative treatment. A linear-response calculation of the density-density correlation yields (Pines and Bohm, 1952)

$$|\rho_k|^2 = nk^2/(k^2 + \lambda_D^2) \tag{3-14}$$

which is to be compared to the enstrophy spectrum of the Kraichnan theory, equation (3-11). The similarity of equations (3-11) and (3-14) has led several authors to conclude that the 2-d Coulomb gas possesses the same energy spectrum as the energy-enstrophy theory of the Euler fluid (Benfatto et al., 1987; Cook and Taylor, 1972; Kraichnan, 1975; the low energy limit of Pointin and Lundgren 1976; and Seyler, 1976); however, this identification can in fact be made only in the limited regime in which RPA is applicable. In particular, whenever the fluid undergoes *macroscopic* charge separation in equilibrium, RPA breaks down, and a non-perturbative calculation is required. In the limit of infinite charge density, the energy spectrum obtained from (3-11) applies only to a homogeneous vorticity field with no structure on any finite length scale.

We next discuss two mathematical papers that describe rigorous results on the statistical mechanics of point vortices and Euler flow. Fröhlich and Ruelle (1982) prove that extensive negative temperature states do not exist in the point vortex gas; Benfatto et al. (1987) claim that the statistical mechanics of an Euler fluid is given by the energy/enstrophy theory. We will argue that each of these results, while correct, applies only in regimes with trivial structure at finite length scales. In both cases, their conclusions depend on using a temperature that is not appropriately scaled with the number

of degrees of freedom. Their arguments and conclusions can be completely understood within the context of the earlier studies we have sketched above.

Fröhlich and Ruelle treat a neutral collection of point vortices on a torus. They wish to prove that there can be no negative temperatures in the thermodynamic limit of Onsager's point vortex gas. By thermodynamic limit, they mean the limit in which the volume of the fluid diverges, while the density and energy per vortex converge to finite values; the temperature, of course, remains finite. Whereas Onsager argues that for a *finite* collection of vortices there exists an energy at which $dS/dE$ changes sign from positive to negative, Fröhlich and Ruelle show that this energy itself diverges as the number of point vortices goes to infinity. Consequently, a regime where $S$ decreases as a function of $E$ is unattainable.

This outcome may be attributed in a simple way to screening. First, let us observe with Fröhlich and Ruelle that the absence of a scale in the logarithmic potential permits us to rescale the volume of the fluid to a finite value as $N \to \infty$. So we may just as well talk about a limit in which the density diverges, but the energy per vortex remains finite. We understand the physics of this system very well at positive temperatures: charge fluctuations will be *screened*, and the long-range potential becomes effectively short-range. At infinite charge density, the screening length vanishes and the interaction becomes irrelevant. Consequently, the thermodynamic limit yields a vanishing mean charge density and trivial correlations. In fact the sign of the temperature makes no difference here: there is simply not enough energy per particle to yield charge fluctuations on length scales of order of the system size.

To see this, think for a moment of $N$ positive charges contained in a domain $\Omega$ and distributed uniformly in a blob of finite radius. Now, multiply the number of charges by some factor $\eta$, and ask what would be the energy of a blob with precisely the same

relative charge distribution. It is not difficult to see that the *long range* of the logarithmic interaction entails that the energy scales by a factor $\eta^2$. If we were not to rescale the energy, then the $\eta N$ charges would distribute themselves more evenly over $\Omega$: the normalized amplitude of the blob would decrease. *In any mean-field description, the normalized amplitude of a charge inhomogeneity will remain constant only if the energy is scaled by the square of the number of charges.*

On the other hand, if we scale the energy in the manner suggested by the high energy limit, that is as $N^2$, then a new feature emerges. While at positive temperatures screening dominates since opposite signs attract, at negative temperatures where opposite signs repel, the screening of the long-range potential is suppressed. Macroscopic charge inhomogeneities persist (and are in fact unavoidable) in the limit of large $N$. As discussed earlier, the scaling of the temperature as $1/N$ is corequisite to the scaling of energy with $N^2$. Fröhlich and Ruelle eliminate any possibility for negative temperatures and macroscopic charge inhomogeneities in the thermodynamic limit by their most basic assumption of the scaling that constitutes a thermodynamic limit.

Benfatto et al. also predicate their argument on an unnecessary restriction of the scalings they permit in the thermodynamic limit. These authors once again treat neutral flow on a torus, represented by $N$ blobs of vorticity with radius $\varepsilon$ and charge $\pm\sqrt{\sigma}$. Their Hamiltonian describes a gas of particles interacting with a logarithmic potential which is smoothed over a length of order $\varepsilon$, so as to remove the short-distance logarithmic divergence. They construct a canonical partition function with inverse temperature $\alpha$, where $\alpha$ is a *finite* quantity that is *not* scaled with the number of particles. As $N \to \infty$ at constant $V$, the radius and charge squared per vortex scale as $\varepsilon = N^{-\delta}$ and $\sigma = (2\pi)^2/\beta N$ respectively. The quantity $\delta$ must be chosen correctly from the interval $(0, \beta/2\pi\alpha)$. Benfatto

et al.'s principal result is that for large $N$, this system may be described by the energy/enstrophy partition function, with energy temperature $\alpha$ and enstrophy temperature $\beta$.

Benfatto et al. assume that both $\alpha$ and $\beta$ possess finite and positive values. For $\alpha$ finite, we expect the vortices to be uncorrelated in the infinite density limit. For example, in the case of periodic boundary conditions, it has been shown by Lundgren and Pointin that the uncorrelated vortices will have an energy (contributed by the image charges) diverging as $\log N$, where we have corrected for the scaling of the charge used by Benfatto et al. As we have observed earlier, the energy/enstrophy partition function yields a logarithmically diverging energy for this parameter range.

In summary, Benfatto et al. have shown that the low energy limit coincides with the energy/enstrophy theory for a neutral vortex gas at positive temperature in the dense limit. This regime is homogeneous, and has no interesting structure at scales on order of system size. Its behavior on scales large compared to the screening length should be given simply by a theory that incorporates the energy and particle number as conserved quantities. For a gas of $\pm 1$ charges, the sum square charge is sufficient to determine the latter. Higher order couplings are expected to be irrelevant on these scales. Neither Fröhlich and Ruelle nor Benfatto et al. have anything to say about regimes in which the fluid has non-trivial structure on length scales larger than the screening length.

We now assume that the proper way to use these statistical methods is to take the limit of an infinite number of degrees of freedom, and scale the temperature(s) in such a way as to extract the desired energy (and enstrophy). Following Carnevale and Frederiksen (1987) we review what is entailed by this procedure. As observed above, we can choose temperatures $\alpha$ and $\beta$ so that the energy/enstrophy theory yields any consistent *total* energy $E$ and enstrophy $\Omega$. Values of $\Omega$ are in fact constrained by a choice of value for $E$.

An elementary variational argument reveals that the minimum value of $\Omega$ corresponds to $\Omega_{min}(E) = {k_0}^2 E$, where $|k_0|$ is the system's smallest wavenumber.

A body of literature investigates the properties of flows possessing the minimum value of $\Omega$ consistent with a given $E$. The rationale is a "minimum enstrophy principle" based on crude arguments about the action of viscosity (Bretherton and Haidvogel, 1976). We shall not discuss "minimum enstrophy vortices" (Leith, 1984) in any detail here; however, Carnevale and Frederiksen point out that these same flows emerge from the energy/enstrophy theory. Equation (3-11) determines the distribution of energy and enstrophy among the Fourier modes. In the limit described by equation (3-12), the distribution has a very simple form: all the energy resides in the smallest wavenumber $k_0$; the enstrophy in the smallest wavenumber is exactly $\Omega_{min}(E) = {k_0}^2 E$; whatever enstrophy is left over sits at *infinite* wavenumber. This partitioning of the enstrophy carries a sweeping implication: *the energy is sufficient to determine the macroscopic flow completely*. The enstrophy exceeding $\Omega_{min}(E)$ disappears completely from the flow at infinite time, since it drains to infinitesimal spatial scales, at infinite wavenumber. Consequently, *all long-time flows possessing the same energy must be identical*. The dependence of the energy/enstrophy theory on two parameters is illusory.

There is nothing unsound about a one parameter theory; we shall see that the one parameter theory, which we derive using our formalism at the end of chapter five, comprises a crude approximation to the full statistical mechanics of the two-dimensional inviscid fluid.

We have not discussed several relevant issues, for the details of which we refer the reader to the literature: (1) Dynamical properties of point vortices. (2) Applications of statistical mechanics to geophysical flows. We expect our work to supersede much of this literature. (3) Turbulence. We do not concede the relevance of equilibrium statistical

mechanics to turbulent cascades; however, following Onsager, we do hope that statistical mechanics may yield a description of turbulent flow on length scales of order of the system size. (4) Dynamical simulation of two-dimensional fluids. A thorough understanding of numerical computation of inviscid fluid flow, especially with regard to its faithfulness to the conservation laws, may be necessary in order to generate simulations that confirm the predictions of statistical mechanics.

# Chapter 4

## Statistical mechanics

Our construction of a statistical mechanical formalism for the two-dimensional inviscid fluid consists of little more than the "cookbook" application of completely standard ideas from statistical thermodynamics. Landau and Lifschitz (1980b) review these ideas.

In short, we calculate the expectation value of a quantity $Q$ by the rule: *average $Q$ over all regions of phase space sharing given fixed values of the conserved quantities.* Underlying this rule is the basic assumption of statistical mechanics, the ergodic hypothesis, which postulates that our averaging assigns to each element of phase space an identical weight.

We dissect this rule into a few discrete steps:

(1) Define phase space variables so that the dynamics preserves arbitrary phase space volumes: phase space flow must be *incompressible*. Since the ergodic hypothesis ascribes weights in proportion to phase space volume, the phase space flow preserves relative probabilities (weights) in the chosen variables. Equivalently, we need to choose variables for which a Liouville theorem holds.

(2) Determine those functions of the phase variables $C$ which are conserved by the dynamics. For a typical particle Hamiltonian, the only such quantities are energy and

total density. Anything we wish to calculate will be parameterized by these quantities, so we shall fix them to particular values, $\hat{C}$.

(3) Given a quantity $Q$ whose expectation in statistical equilibrium you want, calculate its average over all of the phase space in which the conserved quantities take their chosen values. In this manner, we arrive at $\langle Q \rangle_{\hat{C}}$, the expectation value of $Q$ in statistical equilibrium.

The recipe we have given is a *microcanonical* formulation. We might just as well employ a canonical or grand canonical approach; in the case of interest here it is simplest to think within a microcanonical framework, but one obtains identical results, no matter which of the three ensembles one chooses. For purposes of calculation, we shall find the canonical point of view most useful.

Statistical mechanics is most often applied to dynamics that can be derived, at least in some limit, from a Hamiltonian. The reason is that (canonical) Hamiltonian dynamics manifestly preserves phase space volumes. The Liouville theorem follows trivially, almost tautologically, from the Hamiltonian equations of motion:

$$\frac{\partial \dot{x}_i}{\partial x_i} + \frac{\partial \dot{p}_i}{\partial p_i} = \frac{\partial}{\partial x_i}\frac{\partial H}{\partial p_i} - \frac{\partial}{\partial p_i}\frac{\partial H}{\partial x_i} \equiv 0 \qquad (4\text{–}1)$$

In a closed box, the usual many-particle Hamiltonian $H(\vec{x_1}, \vec{p_1} \ldots \vec{x_N}, \vec{p_N})$ with particle coordinates $\vec{x_i}$ and momenta $\vec{p_i}$ possesses one conserved quantity, the energy, which coincides with the scalar value of the (manifestly) conserved quantity $H$. (We do not allow particles to enter or leave the box). Given a value of the energy $E_0$, we may calculate the expectation value of a quantity $Q(\vec{x_1}, \vec{p_1} \ldots \vec{x_N}, \vec{p_N})$ by integrating over all values of $\vec{x_i}, \vec{p_i}, i = 1 \ldots N$ on the manifold determined by the constraint $E_0 = H(\vec{x_1}, \vec{p_1} \ldots \vec{x_N}, \vec{p_N})$.

We follow an analogous recipe for the two-dimensional inviscid fluid:

1) We choose the phase variables $\omega(\vec{r})$, for which a Liouville theorem holds. Our phase space consists of scalar fields defined on the region containing the fluid, $\Omega$.

2) We set the values of quantities conserved by Euler flow. These quantities comprise the energy, and some infinite set of scalars which determines the vorticity distribution. We are now restricted to the scalar fields on $\Omega$ which share the fixed values for these quantities.

3) We integrate over all fields on the manifold determined by the constraints given in 2).

The main hurdle we face is in setting up the calculation of 3) in such a way that we can carry it out. Before we do so, we shall discuss steps 1) and 2) in greater detail.

*Hamiltonian formulation of inviscid fluid flow*

Just as in the discrete particle case, a Hamiltonian formalism may be used to set up our approach to statistical mechanics. In fact, any explicit use of the Hamiltonian properties for our system seems arguably superfluous. Lynden-Bell (1967) sets up the problem for the collisionless Boltzmann equation without any explicit recognition that his equations of motion could be recast in Hamiltonian form. The feature of the equations of motion, which in both cases leads immediately to statistical mechanics, is that they give rise to a phase space flow that is incompressible. That is, one may derive directly from the equations of motion a Liouville theorem. We don't mean to suggest that the Hamiltonian, as an energy, doesn't play an important role in the eventual result. For example, the Liouville theorem for particle Hamiltonians describes incompressible phase space flow, yet we would not bother to look at the statistical mechanics of phase space flow, since there the advected scalar, in contrast to the vorticity field in two-dimensional fluid flow, is *passive*.

On the other hand, the Hamiltonian nature of the system deserves some emphasis for two reasons: first, it underlies the well-known, indeed classic, Hamiltonian dynamics of point vortices, in which Onsager's statistical mechanics originated; second, it falls under the rubric of the "non-canonical Hamiltonian formalism" into which a variety of physical systems have been collected (see, e.g., Holm et al., 1985), each of which is amenable to the statistical mechanical program which we are constructing in this work. These systems, among which are examples of classical and relativistic dynamics in both two and three space dimensions, each possess at least one infinite family of conservation laws. On these grounds we have chosen merely to outline very briefly the basic features of the Hamiltonian formalism for ideal fluid mechanics in two dimensions. A substantial body of literature exists to which you may refer for details.

Underlying the Hamiltonian description is a symmetry of the physics: the equations of motion in the form (2-3) are invariant under volume-preserving coordinate reparameterizations. This invariance, a kind of Newtonian covariance principle known also as "particle relabelling symmetry," can be viewed as a gauge symmetry, and through Noether's theorem gives rise to the infinite set of conservation laws of two-dimensional Euler flow (Salmon, 1982). These same coordinate transformations, which constitute the group of volume-preserving diffeomorphisms, $\mathrm{Diff}_V(\Omega)$, form the configuration space for fluid motions. Starting from an arbitrary reference coordinate frame, an element of $\mathrm{Diff}_V(\Omega)$ carries particles in the reference frame to a new configuration which may be labeled by this group element. Any motion of the fluid is described by the action of a group element on a configuration; Arnol'd (1978) describes a variational method to derive the motion of the fluid, which of course must correspond to the Euler equations.

Rather than discuss Arnol'd's construction, we shall follow the (equivalent) route of Marsden and co-workers (see, e.g., Holm et al., 1985). (We ignore any technicalities

involving boundary conditions, etc.) For arbitrary functions $F$ and $G$ of $\omega$ we define a Poisson bracket:

$$\{F,G\}(\omega) = \int_\Omega d^2\vec{r}\,\omega(\vec{r})\left\{\frac{\delta F}{\delta\omega},\frac{\delta G}{\delta\omega}\right\}_{xy} \tag{4-2}$$

where $\{f,g\}_{xy} = (\partial_x f)(\partial_y g) - (\partial_x g)(partial_y f)$, the usual (canonical) Poisson bracket in the variables $x$ and $y$. The Hamiltonian is the kinetic energy of the fluid, $\mathcal{H} = \int d^2\vec{r}\,\vec{u}^2 = -\int d^2\vec{r}\,\psi\omega$, the Poisson bracket of which we take with the field $\omega$, as usual:

$$\partial\omega/\partial t = \{\omega,\mathcal{H}(\omega)\} \tag{4-3}$$

which by way of an easy calculation yields Euler's equations.

In particular, for any integral of the form $C_f = \int d^2\vec{r}\,f(\omega)$, the Poisson bracket of $C_f$ with any functional of $\omega$ yields zero, implying that these integrals would be conserved quantities for any Hamiltonian system possessing this Poisson bracket. Such quantities are labeled "distinguished functionals," by Olver (1986), who argues that because of their trivial character with respect to the Poisson bracket, they *do not suffice to make the Hamiltonian system integrable.*

Not surprisingly, the Hamiltonian description of point vortices can be viewed as a special case of this more general formulation, as discussed by Marsden and Weinstein (1983).

*Liouville theorem*

Because $\omega$ is evidently not a canonical coordinate, we have no conjugate variable with which to write a Liouville equation in the form (4-1). Nevertheless, as we proved in the historical introduction, we are easily able to write a Liouville equation in the Fourier components of $\omega$. Equation (3-7) demonstrates precisely the incompressibility of phase flow in the space of $\omega_k$s. Now we observe that the transformation to real space variables

is trivial: it is merely a linear superposition of $\omega_k$s, with coefficients that do not depend on the vorticity field in any way. Consequently, we shall obtain a factor in the partition function, corresponding to the determinant of the transformation between real space and Fourier variables. Since the Fourier transform is an orthogonal transformation, we may ignore the trivial constant which we obtain from this determinant.

The conserved quantities make no difference to the particular volume form we use in the phase space, provided that we restrict ourselves in the functional integration to the manifold determined by these same constraints. For example, a microcanonical partition function for a particle Hamiltonian would entail the integral

$$Z(E) = \int \prod_i dp_i dq_i \; \delta(H(\vec{x_1}, \vec{p_1} \ldots \vec{x_N}, \vec{p_N}) - E) \tag{4–5}$$

with phase volume form $\prod_i dp_i dq_i$. This is the measure on phase space conserved by the Liouville flow, *not* the measure on the surface to which the single conserved quantity, the Hamiltonian, constrains the integration. This surface may be complicated, depending on the derivative of $H$ perpendicular to the energy hypersurface. In a parallel manner, we expect a microcanonical partition function for the Euler equations to take the form (up to some overall factor accounting for the change of variables from real space to Fourier space)

$$\mathcal{Z}(E) = \int \prod_i d\omega_i \; \delta(\mathcal{H}(\omega) - E) \prod_j \delta(\mathcal{C}_j(\omega) - \hat{\mathcal{C}}_j) \tag{4–6}$$

where $\prod_j$ denotes a product over all independent conserved functions of the vorticity.

It is this partition function (4-6), whose properties we need to compute. We carry out the necessary computations, on the canonical form of (4-6), in the next chapter.

## Chapter 5

## Derivation of the mean-field equations

In this section we present two derivations of the mean-field equations. Very similar to that of Lynden-Bell (1967), the first derivation (Miller, 1990) is combinatorial and relies on the weakness of the divergence in the Coulomb-type interaction between vortices at small distances. On very short length scales the vortices may be treated as an ideal gas of hard-core particles, whereas on large length scales the interaction energetics are determined entirely by the averaged Coulomb potential of small patches over which the ideal gas degrees of freedom have been integrated out. This separation of length scales is the key to the exactness of mean-field theory.

A second and more formal derivation, due to P.B. Weichman and presumably equivalent to the first, may be comforting to those familiar with the standard model mappings and manipulations used in critical phenomena. The formal device is the Kac-Hubbard-Stratanovitch transformation which allows one to convert the long-range Coulomb interaction into a purely local square-gradient interaction. The exactness of mean-field theory is a consequence of the standard wisdom in critical phenomena that interactions of sufficiently long range give rise to mean-field critical behavior. More precisely, if the interaction in $d$ dimensions decays less rapidly than $r^{-3d/2}$, mean-field critical behavior will result. The Coulomb interaction $r^{2-d}$, ($\log(r)$ in $d = 2$), clearly satisfies this condition, and decays so slowly (in fact growing in $d = 2$) that mean-field theory holds over the *entire phase diagram.*

Both derivations generalize to higher-dimensional models, Lynden-Bell's being an example. As will emerge later, Debye-Hückel theory of the three-dimensional plasma is a special case of our theory, the limit of a dense *point* charge distribution (we are indebted to Daniel Arovas for showing us this relation.)

### 5.1 Definitions and conservation laws

To begin with, we write the kinetic energy of the Euler fluid in terms of the vorticity field:

$$\begin{aligned} \mathcal{H} &= 1/2 \int d^2\vec{r}\, |\vec{u}(\vec{r})|^2 = -1/2 \int d^2\vec{r}\, \psi(\vec{r}) \nabla^2 \psi(\vec{r}) \\ &= 1/2 \int d^2\vec{r} \int d^2\vec{r}'\, \omega(\vec{r}) \mathcal{G}(\vec{r},\vec{r}') \omega(\vec{r}') \end{aligned} \tag{5-1}$$

where $\psi(\vec{r})$ is the (scalar) stream function, $\vec{u}(\vec{r}) = \nabla \times \psi(\vec{r})$ is the velocity field, and $\omega(\vec{r}) = \nabla \times \vec{u}(\vec{r}) = -\nabla^2 \psi(\vec{r})$ is the vorticity field. The Green's function $\mathcal{G}(\vec{r},\vec{r}')$ satisfies $-\nabla^2 \mathcal{G}(\vec{r},\vec{r}') = \delta(\vec{r}-\vec{r}')$, is symmetric in its arguments, and in each variable satisfies the same boundary conditions that $\psi(\vec{r})$ does. For *free* boundary conditions $\mathcal{G}(\vec{r},\vec{r}') = -\frac{1}{2\pi} \log |\frac{\vec{r}-\vec{r}'}{R_0}|$, where $R_0$ is a reference separation. For an arbitrary vorticity configuration, we have

$$\psi(\vec{r}) = \int d^2\vec{r}'\, \mathcal{G}(\vec{r},\vec{r}') \omega(\vec{r}') \tag{5-2}$$

the boundary conditions on $\mathcal{G}(\vec{r},\vec{r}')$ ensuring the corresponding ones on $\psi(\vec{r})$. For periodic boundary conditions, we have to be more careful. We require a uniform, compensating background vorticity in order to define $\mathcal{G}(\vec{r},\vec{r}')$. Equivalently, we demand that the system be neutral: $\int d^2\vec{r}\, \omega(\vec{r}) = 0$. The equation for $\mathcal{G}(\vec{r},\vec{r}')$ becomes $-\nabla^2 \mathcal{G}(\vec{r},\vec{r}') = \delta(\vec{r}-\vec{r}') - \frac{1}{V}$. The $\frac{1}{V}$ term ensures that the left-hand side integrates to zero, and removes the zeroth order Fourier component from the $\delta$-function. Neutrality ensures that (5-2) still holds.

Next we discuss the conserved quantities of Euler flow. Incompressibility, $\nabla \cdot \vec{u} = 0$, has been incorporated through the existence of the stream function. The energy was already displayed in equation (5-1). We suppose the fluid confined to a bounded domain,

so that the net momentum vanishes. When the region possesses azimuthal symmetry, the angular momentum will be conserved. Only the component perpendicular to the flow plane is non-vanishing, and is given by

$$\begin{aligned} L &\equiv \int_\Omega d^2\vec{r}\, \vec{r} \times \vec{u}(\vec{r}) \\ &= -1/2 \int_\Omega d^2\vec{r}\, r^2 \omega(\vec{r}) + 1/2 \int_{\partial\Omega} r^2 \vec{u}(\vec{r}) \cdot d\vec{\ell} \end{aligned} \tag{5-3}$$

Angular momentum will be conserved only if the domain has cylindrical symmetry. In that case the last term has the value $R^2\Gamma/2$ where $R$ is the radius of the system, and $\Gamma = \int_{r=R} \vec{u} \cdot d\vec{\ell} = \int d^2\vec{r}\, \omega(\vec{r})$ is the circulation, which is also a conserved quantity, as we show below. For an annulus, with inner radius $r = R_i$ and outer radius $r = R_o$, things are a bit more complicated. The last term becomes $R_o^2\Gamma_o - R_i^2\Gamma_i$, and the difference $\Delta\Gamma = \Gamma_o - \Gamma_i$ arises from a static source of vorticity located inside $r = R_i$, which enforces the boundary conditions at the inner radius. Irrespective of angular momentum considerations, the circulation around any impenetrable boundary is a conserved quantity.

One may also consider a boundary that *comoves* with the fluid, locally adjusting its shape with the flow so that no fluid breaches it. Incompressibility entails that any such boundary encloses a constant area. It follows from the vanishing of the convective derivative of the vorticity, $D\omega/Dt = 0$, that the circulation around the moving boundary is also constant in time, leading to an infinite set of conservation laws. The most convenient way to parameterize these conservation laws is, for each real number $\sigma$, to define the quantity $\Omega_\sigma$, the region of fluid on which $\omega \leq \sigma$. The area of this region is conserved by the flow, and the normalized area

$$G(\sigma) = V(\Omega_\sigma)/V = \int \frac{d^2\vec{r}}{V}\, \eta(\sigma - \omega(\vec{r})) \tag{5-4}$$

is a constant of motion. Here $\eta(\sigma)$ is the step function: $\eta(\sigma) = 1$ for $\sigma \geq 0$, and $\eta(\sigma) = 0$ for $\sigma < 0$. Perhaps more convenient is the derivative

$$g(\sigma) = \frac{dG}{d\sigma} = \int \frac{d^2\vec{r}}{V}\, \delta\big(\sigma - \omega(\vec{r})\big) \tag{5-5}$$

We may interpret $g(\sigma)d\sigma$ as the fractional area on which $\sigma \leq \omega \leq \sigma + d\sigma$.

Now the question arises: does the single variable $\sigma$ suffice to parameterize all relevant conservation laws? At first glance, it might seem that a single variable is inadequate. As is well-known, a region $\Omega_0$ which is connected at time 0 remains connected for all time (see, e.g., Arnold (1978)). In principle, two distinct blobs of vorticity cannot evolve into a single blob, or vice versa, even though we can easily envision cases where both configurations possess the same function $G(\sigma)$. Loosely speaking, a full set of conservation laws ought to specify the number of distinct disconnected domains into which each set $\{\omega(\vec{r}) = \sigma\}$ is divided, the topologies of these domains (simply connected, annular, figure eight, etc.), and the fractional areas of the domains, which are determined by $g(\sigma)$ and also known as "area integrals." The flow will preserve each of these properties.

It should be observed that conservation of connectivity differs from conservation of area integrals in a basic way. Up to sets of measure zero, the topological constraints do not preclude the vorticity field from taking any particular spatial configuration we wish to choose. That is, the set of configurations accessible to the topologically constrained fluid is dense in the set of configurations available to the *unconstrained* fluid. A ring of vorticity can evolve into two widely separated blobs, provided the topology is preserved by a filament of vorticity. Since we generally confine ourselves in this discussion to vorticity fields with a finite maximum vorticity, a fine thread of vorticity can contribute very little to the dynamics, especially if the dynamics are in fact dominated by structures on much larger scales. In contrast, the area integrals *do* affect the accessible vorticity configurations,

essentially by rendering the vorticity field itself incompressible. When we alter the value of an area integral, we find that new vorticity fields arise which cannot necessarily be duplicated by the action of any volume-preserving diffeomorphism on the original flow. Of course, through this alteration vorticity fields also arise which *can* be duplicated by an area-preserving transform. Our statistical average will range over all accessible vorticity fields, subject to the conservation laws. Consequently, we neglect the topological laws, which entail no restriction on the vorticity field, but we take into account explicitly the area integrals to the extent that they in fact constrain the vorticity field.

This basic assumption of our work can be justified by several lines of physical argument. (1) We are interested exclusively in long-time properties of the flow. In the infinite-time limit the fluid will be kneaded on all scales: infinitely folded and stretched. A single connected blob of vorticity can give birth to two (apparently) separated blobs, provided that the two blobs are in fact joined by a thread of vorticity of infinitesimal width. The finite maximum of the vorticity suggests that the thread can have no dynamical effect. (2) Topological conservation laws of this type exist in dynamical systems without constraining ergodicity. For example, phase space flow for any Hamiltonian system may be viewed as flow of an incompressible fluid of probability density (Reichl, 1980). The volume of any closed hypersurface in phase space is preserved by the flow, although its shape may change. Nevertheless, phase space flows can be *mixing*: the average of the probability density becomes *uniform* in phase space. The topology of the particular closed hypersurface does not matter. Mixing can be thought of as a source of macroscopic irreversibility (Reichl, 1980) is accompanied by a loss of information in the evolution from microscopic to coarse-grained vorticity distribution, which we discuss later. (3) *Point vortex systems are subject to these same topological constraints*. For point vortex systems the area integrals are singular, and their only effect is to set the relative numbers of vortices carrying given charges. The area integrals no longer prevent adjacent charges from approaching each

other arbitrarily closely, so that the point vortex gas is compressible; nevertheless, point vortices yield a divergence-free velocity field, and any curves we draw in the fluid must maintain their connectivities. But the ergodicity of a system of point vortices has been amply demonstrated by a number of authors (Kida, 1975; Lundgren and Pointin, 1976, 1977a; Aref and Siggia, 1981). These authors find that dynamical simulation of systems of point vortices confirms the predictions of statistical equilibrium theory. Kraichnan (1967) has also argued that the topological constraints have no effect on the flow at long times.

In contrast to conservation of connectivity, the conservation of $g(\sigma)$ affects the fluid motion in a fundamental way. The infinite folding and stretching of the fluid at long times leads to the *apparent* non-conservation of $g(\sigma)$; nevertheless, the equilibria depend quantitatively on the $g(\sigma)$ from which we started. We shall find that in general the initial and infinite time $g(\sigma)$ are not in one-to-one correspondence, as demonstrated by the "dressed vorticity corollary," which we discuss later, and their relation to one another is subtle.

### 5.2 Mean-field equations: combinatorial derivation

With the above discussion in mind, we proceed to derive the equilibrium statistics under the assumption that, at most, $H$, $L$, and $g(\sigma)$ need to be explicitly accounted for. For counting purposes we discretize space into a lattice with spacing $a$, and perform calculations with fixed $a$. We shall eventually take the limit $a \to 0$. Convergence is more or less ensured by the softness of the Coulomb interaction at short distances. More explicitly, we define the coarse-grained vorticity field

$$\omega_i = \int_{box_i} \frac{d^2\vec{r}}{a^2}\, \omega(\vec{r}) \tag{5-6}$$

where $box_i$ denotes the lattice square centered at $\vec{r}_i$. The lattice clearly breaks (local) Galilean invariance, so an associated discrete dynamics can probably not be realized. We

are investigating discretizations that preserve Galilean invariance, but we expect them to yield the same equilibrium statistics. The discrete static energy is perfectly well-defined:

$$\mathcal{H}_a = \frac{1}{2}a^4 \sum_{i \neq j} \mathcal{G}_{ij}\omega_i\omega_j \tag{5-7}$$

The matrix $\mathcal{G}_{ij}$ is the inverse of the discrete Laplacian:

$$-\sum_{n.n.\ \gamma} (\mathcal{G}_{i+\gamma,j} - \mathcal{G}_{i,j}) = \delta_{ij} \tag{5-8}$$

We need to choose a scale $R_0$ which sets the value of $\mathcal{G}_{0,0}$. A standard choice is to take $R_0 \propto a$, which corresponds to $\mathcal{G}_{0,0} = 0$, but we shall later require that $R_0$ be of order system size, independent of $a$ as $a \to 0$ (see the discussion following (5-46)). Appropriate boundary conditions should be imposed.

The canonical statistical configuration average is also well-defined and consists of a sum over all permutations of the $N = V/a^2$ boxes, in which we regard boxes possessing the same vorticity $\omega$ as indistinguishable). The permutations are weighted by the Gibbs factor $e^{-\beta_a \mathcal{H}_a}$, where $\beta_a = 1/T_a$ sets the average energy. We shall see later that $T_a$ must scale with $a$. This prescription obviously respects conservation of the function $g(\sigma) = N_\sigma/N$, which merely counts $N_\sigma$, the number of squares with vorticity $\sigma$.

In order to carry out the configuration sum, we take advantage of the separation of scales alluded to earlier. We need to alter the discretization slightly: we discretize the field $\omega(\vec{r})$ on a *finer* scale than the matrix $\mathcal{G}(\vec{r}, \vec{r}')$. In other words, we satisfy (5-8), but allow the indices $i, j$ to run over a lattice with spacing $\ell >> a$. We define

$$\mathcal{H}_{\ell,a} = \frac{1}{2}a^4 \sum_{\substack{i,j \\ \alpha,\beta}} \mathcal{G}_{ij}\omega_{i\alpha}\omega_{j\beta} \tag{5-9}$$

where the Greek indices run over the $(\ell/a)^2$ $a$-cells within each $\ell$-cell, which is labeled by Latin indices. In what follows we shall consider the limit $a \to 0$ for fixed $\ell$, then take

$\ell \to 0$ in the end. Since $\mathcal{G}_{ij}$ is well-behaved for small $|i-j|$, this procedure should yield the correct continuum limit, so long as the bulk correlations are much larger than $\ell$.

Equation (5-9) is constructed in such a way that permutations of the $\omega$s within a given $\ell$-cell do not affect the energy. We may perform the sum over this restricted set of permutations explicitly. In order to control the limiting procedure properly, the allowed values of the vorticity must be discretized as well. A convenient way to do this is to partition the *range* $[0,1]$ of $G(\sigma)$ into $p_\ell$ uniform intervals of width $\Delta_\ell = 1/p_\ell$ which will vanish when $\ell \to 0$. For small $\Delta_\ell$ we have partitioned the $\sigma$-axis into intervals $\sigma_k \leq \sigma < \sigma_k + \Delta\sigma_k$, where $G(\sigma_k) = k\Delta_\ell$, and $g(\sigma_k)\Delta\sigma_k \approx \Delta_\ell$ with $k = 0, 1 \ldots p_\ell$. Strictly speaking we must assume here that $g(\sigma)$ has compact support; otherwise $\Delta\sigma_0$ and $\Delta\sigma_{p_\ell}$ could be infinite (i.e., $G^{-1}(0) = -\infty$ and/or $G^{-1}(1) = +\infty$). If $g(\sigma)$ decays to zero sufficiently rapidly, we could relax this condition, and allow the support of $g(\sigma)$ to diverge at the same time that $\ell \to 0$. We shall have more to say about what happens when $g(\sigma)$ does not decay sufficiently rapidly later - this condition obtains for *point* vortices.

We now introduce the following notation. Let $N = V/a^2$ be the number of $a$-cells and $M = V/\ell^2 \equiv N/n$ be the number of $\ell$-cells. Let $\nu_i(\sigma_k)$ be the number of $a$-cells in $\ell$-cell $i$ with vorticity $\sigma_k$, and let $\nu^T(\sigma_k) = \sum_i \nu_i(\sigma_k)$ be the total number of $a$-cells with vorticity $\sigma_k$. Note that $\sum_k \nu_i(\sigma_k) = n$, $\sum_k \nu^T(\sigma_k) = N$ and that

$$\nu^T(\sigma_k)/N = \Delta_\ell \approx g(\sigma_k)\Delta\sigma_k \tag{5-10}$$

The last inequality holds generally, even when the intervals $\Delta_\ell$ are not independent of $k$. The total vorticity in $\ell$-cell $i$ is

$$\bar{\omega}_i = \frac{1}{n}\sum_k \sigma_k \nu_i(\sigma_k) = \frac{1}{n}\sum_\alpha \omega_{i\alpha} \tag{5-11}$$

The energy (5-9) remains constant so long as the $\bar{\omega}_i$ are fixed throughout. We now examine the combinatorial factor that results from distributing vorticity in all possible ways for given values of $\{\nu_i(\sigma_k)\}$. This factor is given by:

$$W_\ell\{\nu_i(\sigma_k)\} = \prod_i \frac{n!}{\prod_k \nu_i(\sigma_k)!} \tag{5-12}$$

which represents the number of ways of distributing each $\nu_i(\sigma_k)$ among the $n$ $a$-cells in $\ell$-cell $i$. The total partition function is

$$\mathcal{Z}[g] = \tilde{tr}^{\{\nu_i(\sigma_k)\}}\left[W_\ell\{\nu_i(\sigma_k)\}e^{-\beta_a \mathcal{H}_{\ell,a}\{\nu_i(\sigma_k)\}}\right] \tag{5-13}$$

where the trace is over all distinct ways of assigning the $\{\nu_i(\sigma_k)\}$ with fixed $\{\nu^T(\sigma_k)\}$ given by (5-10).

We must be careful to normalize the trace correctly. The Liouville theorem implies that the correct configuration sum should have uniform measure in the space of fields $\omega(\vec{r})$, or, on a lattice, in the space of the $\omega_{i\alpha}$. Before discretizing $g(\sigma)$, the trace is defined by

$$tr^{\{\omega_{i\alpha}\}} \equiv \prod_{i,\alpha} \int_{-\infty}^{\infty} \frac{d\omega_{i\alpha}}{q_0} \tag{5-14}$$

where $q_0$ is a reference vorticity needed to make $\mathcal{Z}$ dimensionless. Upon discretizing $g(\sigma)$ this becomes

$$tr^{\{\omega_{i\alpha}\}} = \prod_{i,\alpha} \sum_{k=1}^{p} \left(\frac{\Delta\sigma_k}{q_0}\right) = \prod_{i,\alpha} \sum_{\omega_{i\alpha}} \prod_k \left(\frac{\Delta\sigma_k}{q_0}\right)^{\nu^T(\sigma_k)} \tag{5-15}$$

where the sum is now over the discrete values $\omega_{i\alpha} \in \{\sigma_k\}_{k=1}^{p}$. It is understood that appropriate $\delta$-functions should be introduced in (5-14) and (5-15) to preserve the conservation laws. In (5-15) they take the form

$$\Delta(\nu, g) \equiv \prod_k \delta_{\nu^T(\sigma_k), \sum_i \nu_i(\sigma_k)} \prod_i \delta_{n, \sum_k \nu_i(\sigma_k)} \tag{5-16}$$

where we remind you that $\nu_i(\sigma_k) = \sum_\alpha \delta_{\omega_{i\alpha},\sigma_k}$. Our final expression for the partition function then becomes

$$\mathcal{Z}(g) = tr^{\{\nu_i(\sigma_k)\}}[\prod_k \left(\frac{\Delta\sigma_k}{q_0}\right)^{\nu^T(\sigma_k)} W_\ell\{\nu_i(\sigma_k)\}\Delta(\nu,g)e^{-\beta\mathcal{H}}] \tag{5–17}$$

in which the trace is now a *free* sum over all $0 \le \nu_i(\sigma_k) \le n$. It is easy to check that when $\mathcal{H} \equiv 0$, (5-17) yields the expected hard-core ideal gas result

$$\mathcal{Z}(g, \mathcal{H} \equiv 0) = \frac{N!}{\prod_k [\nu_T(\sigma_k)! \left(\frac{\Delta\sigma_k}{q_0}\right)^{-\nu_T(\sigma_k)}]} \tag{5–18}$$

Using Stirling's formula $m! \approx \sqrt{2\pi m}(m/e)^m$, which is valid for large $m$, we may further evaluate (5-17). Define the $\ell$-scale entropy by

$$S_\ell = \log[\prod_k \left(\frac{\Delta\sigma_k}{q_0}\right)^{\nu^T(\sigma_k)} W_\ell\{\nu_i(\sigma_k)\}] \tag{5–19}$$

We obtain

$$\begin{aligned} S_\ell = &\sum_k \nu_T(\sigma_k) \log\left(\frac{\Delta\sigma_k}{q_0}\right) + N\log(n) \\ &- \sum_{i,k} \nu_i(\sigma_k)\log[\nu_i(\sigma_k)] + O[\log(N)] \end{aligned} \tag{5–20}$$

On defining $n_i(\sigma_k)\Delta\sigma_k = \nu_i(\sigma_k)/n$, and dropping terms of $O[\log(N)]$, (5-20) becomes

$$\begin{aligned} S_\ell &= -N \sum_{i,k} \frac{\Delta\sigma_k}{M}\, n_i(\sigma_k)\log[q_0 n_i(\sigma_k)] \\ &\to -N \int d^2\vec{r} \int d\sigma\; n(\vec{r},\sigma)\log[q_0 n(\vec{r},\sigma)], \text{ as } n \to \infty \end{aligned} \tag{5–21}$$

Similarly, the logarithm of equation (5-18) has the continuum limit

$$S(\mathcal{H} \equiv 0) \to -N \int d\sigma\; g(\sigma)\log[q_0 g(\sigma)] \tag{5–22}$$

where we have used equation (5-10). The field $n(\vec{r},\sigma) = \int\limits_{\substack{\ell \text{ cell} \\ \text{at } \vec{r}}} \frac{d^2\vec{r}}{V}\, \delta(\sigma - \omega(\vec{r}))$ , which represents the fine-grained number density of vorticity $\sigma$ at $\vec{r}$, obeys the two constraints

$$\begin{cases} \int d\sigma\; n(\vec{r},\sigma) = 1 & \text{incompressibility} \\ \int \frac{d^2\vec{r}}{V}\; n(\vec{r},\sigma) = g(\sigma) & \text{conservation laws} \end{cases} \tag{5–23}$$

The coarse-grained vorticity at $\vec{r}$ is

$$\bar{\omega}(\vec{r}) = \int d\sigma \; \sigma n(\vec{r}, \sigma) \tag{5–24}$$

so that the Hamiltonian reads

$$\mathcal{H} \to \frac{1}{2} \int d^2\vec{r} \int d^2\vec{r}' \int d\sigma \int d\sigma' \; \sigma n(\vec{r}, \sigma) \; \sigma' n(\vec{r}', \sigma') \; \mathcal{G}(\vec{r}, \vec{r}'), \tag{5–25}$$

as $n \to \infty$. We now make the key observation which leads to the scaling of the temperature with $a$, and to mean-field theory: the coarse-grained entropy $S_\ell$ diverges as $VN = V^2/a^2$ and is extensive relative to $a$ in the sense that it is proportional to the number of $a$-lattice sites. Extensivity ordinarily refers to scaling with the volume $V = V(\Omega)$, which is independent of $a$. The energy (5-25) is finite as $a \to 0$, scaling only with $V(\Omega)$, but as we shall see shortly, *not* linearly. Clearly, if $\beta_a$ is independent of $a$ in (5-13), the entropy will completely dominate the energy, and (5-22) will ensue for any $\mathcal{H}$. In order to have competition between energy and entropy, *both* terms must be of the same order. This condition entails the scaling

$$\beta_a = \bar{\beta}/a^2 \Rightarrow T_a = \bar{T}a^2 \tag{5–25}$$

where $\bar{\beta}$ remains fixed as $a \to 0$. The combination $\beta_a \mathcal{H}_{\ell,a}$ then scales as $V^p/a^2$, where we must now address the value of the exponent $p$.

### 5.3 Negative temperatures

The *neutral* lattice Coulomb gas at positive temperature $T_{coul}$ described by the Hamiltonian $\mathcal{H}_a$ in (5-7), but *without* the $a^4$ prefactor,

$$\mathcal{H}_{coul} \equiv \frac{1}{2} \sum_{i,j} \mathcal{G}_{ij} \omega_i \omega_j; \qquad \sum_i \omega_i = 0, \tag{5–26}$$

is well understood. The average energy scales linearly with $N$, the number of lattice sites. The low-temperature phase consists of tightly bound, oppositely charged dipole

pairs (we consider the two-species case $\omega_i = \pm q$ or 0 for simplicity). There is a transition at finite temperature $T_c$ (the Kosterlitz-Thouless transition in two dimensions) to a neutral plasma phase, in which the opposite charges are decoupled, which persists for arbitrarily high temperature. In both phases, correlations decay *on the scale of the lattice spacing* $a$, (as power laws for $T < T_c$ and exponentially for $T > T_c$). The origin of the decay of correlations is screening: opposite charges attract and interfere with one another, yielding an effective interaction whose range is *finite* on the scale of the lattice spacing $a$. It is clear that any coarse-grained $\ell$-cell will be effectively *neutral* when $\ell >> a$, and that charges in different $\ell$-cells will be *uncorrelated*. These features violate the assumptions upon which we base our argument that (5-9) will yield the correct continuum limit, and suggest that the positive temperature neutral system is not properly described by (5-9).

There is a simpler way to understand the failure of (5-9). Since (5-26) yields an energy which scales linearly with $N$, the equivalent hydrodynamic energy (5-9) will scale as $Na^4 = Va^2$, and *vanish* with $a$. Equivalently, we have the correspondence $\beta_{coul} = \beta_a a^4 = \beta a^2$ so that $\bar{T} = T_{coul} a^2$ vanishes with $a$ at fixed $T_{coul}$. Intuitively, a positive hydrodynamic energy requires macroscopic flows, on the scale of the system size, while screening implies flows only on the microscale, invisible on any macroscopic scale. As is apparent from (5-24), hydrodynamic flows require macroscopic non-neutral regions of vorticity. In a neutral system, charge can segregate macroscopically only if, by some means, like charges happen to *attract* rather than repel. Homophilic charges correspond to reversing the sign of the energy (5-9) or (5-26), or equivalently of the temperature (5-25), and naturally lead us to examine *negative temperature* states of $\mathcal{H}_{\ell,a}$. As we remarked in the historical review, negative temperatures are inaccessible to systems consisting of real particles, since (the kinetic part of) the energy is not bounded from above. Crudely, we need to ensure that $\int d\varepsilon\ \rho(\varepsilon) \exp -\varepsilon/T$ be finite, where $\rho(\varepsilon)$ denotes the density of configurations with energy $\varepsilon$. This condition will be satisfied when the energy is bounded from above and below, or

when the configuration space is sufficiently constrained to yield at least exponential decay in $\rho(\varepsilon)$.

Since inviscid Euler dynamics is an approximation that neglects coupling between molecular and hydrodynamic degrees of freedom, it is perfectly consistent for the degrees of freedom at these respective length scales to be out of equilibrium with each other: the former at positive temperature; the latter at negative temperature. Viscosity, an agent of diffusive transport, would provide a route for energy to drain from the hydrodynamic degrees of freedom and excite molecular degrees of freedom, yielding in the long-time limit a warmer fluid, devoid of macroscopic motion. (More generally, since viscosity preserves the total vorticity depending on boundary and initial conditions, a rigidly rotating fluid might result).

In order to demystify the idea of negative temperature, which originated in this context with Onsager, we may view it simply as a mathematical device for setting the energy. In the microcanonical ensemble, thermodynamics is inferred from appropriate averages over the phase space hypersurface at a given energy $E$. By the standard argument, we allow $E$ to fluctuate, while we fix only the average energy using the Lagrange multiplier $\beta$ to obtain the canonical ensemble, both descriptions coinciding in the thermodynamic limit. If the density of configurations $\rho(\varepsilon)$ decays rapidly enough, there is no reason for $\beta$ to be positive. Only by recognizing that any real system will be in contact with positive-$\beta$ molecular degrees of freedom do we restrict our attention to positive-$\beta$ equilibria. If contacts between the system of interest and the positive-$\beta$ bath are weak, there may arise a timescale much longer than intra-system equilibration times, but much shorter than the inter-system equilibration time through the contacts, over which negative-$\beta$ equilibria are a valid thermodynamic description. This separation of time scales occurs in certain paramagnetic systems of nuclear moments in a crystal, where negative (spin-) temperature

states have been experimentally demonstrated (see Landau and Lifshitz (1980), chapter 6).

The presence of a separation of timescales may not be easy to establish, especially as different parts of our system may equilibrate more rapidly through the contacts than others. It is clear, for example, that viscosity ought to first equilibrate regions with large second derivatives in the velocity field, so that small-scale structures such as filaments of vorticity will be rapidly smoothed out, whereas large-scale structures will have longer lifetimes. We might hope that high-Reynolds number turbulence, in which the dimensionless parameter reflects the dominance of convective processes over the diffusive effects of viscosity, might permit such a separation of timescales, particularly when an additional length scale (such as system size) exists to set the scale for coherent structures.

The reader may find it useful to think about positive and negative temperatures in the context of nearest-neighbor ferromagnetic spin models. There, negative temperature equilibria are the usual positive temperature equilibria of the corresponding nearest-neighbor *anti*-ferromagnet. Statistics of the *high-energy* states of a ferromagnet are simply those of the low-energy states of the antiferromagnet. In general, a sure sign of the existence of negative temperature states is an energy that remains finite when $T \to +\infty$. This energy is the *unweighted* average of the energy over all states, and the system can never achieve energies larger than this particular energy unless $T$ becomes negative. We observe that the thermodynamics are continuous through $T = \pm\infty$, ($\beta = 0$), *not* $T = 0$. The latter corresponds to the ground state for $T \to 0^+$, and to the most excited state for $T \to 0^-$.

In contrast to the ferromagnet, where interactions are short-ranged, the negative temperature states of the Coulomb gas do not have energies that scale linearly in system size. This property returns us to the question of the value of the exponent $p$ in eq. (5-26). An easy calculation reveals that rescaling the number of lattice sites by a factor $\alpha$

yields an energy rescaling factor $\alpha^2$ (modulo $\log(\alpha)$ corrections that are unimportant in this argument). In a general dimension $d$, the energy rescaling factor is $\alpha^{2/d+1}$. So long as decreasing the coarseness of the mesh in (5-9) does not affect the macroscopic vorticity distribution, (an assumption basically equivalent to convergence as $a \to 0$), the energy $\mathcal{H}_{\ell,a}$ will scale as $N^2a^4 = V^2$. The combination $\beta_a\mathcal{H}_{\ell,a}$ scales as $V^2/a^2$, i.e., $p = 2$. This value corresponds to the scaling for $S_\ell$, confirming that we have made the correct choice for $\beta_a$.

Returning now to (5-21), (5-25) and (5-13), the partition function reads

$$\mathcal{Z}(g) = \lim_{N\to\infty} tr^n \exp -N\Big\{\int\frac{d^2\vec{r}}{V}\int d\sigma\; n(\vec{r},\sigma)\log[q_0 n(\vec{r},\sigma)] + (\bar{\beta}/2V)\int d^2\vec{r}\int d^2\vec{r}'\;\bar{\omega}(\vec{r})\bar{\omega}(\vec{r}')\mathcal{G}(\vec{r},\vec{r}')\Big\} \tag{5-27}$$

where the trace is over all fields $n(\vec{r},\sigma)$ subject to (5-23). It is now clear why mean-field theory is valid: in the limit $N \to \infty$ the trace will pick out the minimum of the quantity in braces. The average of the field $n(\vec{r},\sigma)$, which we denote by $n_0(\vec{r},\sigma)$, is determined by minimizing the functional

$$-\frac{1}{\beta_a V}\log(\mathcal{Z}) = \mathcal{F}[n] = \frac{1}{\bar{\beta}}\int\frac{d^2\vec{r}}{V}\int d\sigma\; n(\vec{r},\sigma)\log[q_0 n(\vec{r},\sigma)] + (1/2V)\int d^2\vec{r}\int d^2\vec{r}'\;\bar{\omega}(\vec{r})\bar{\omega}(\vec{r}')\mathcal{G}(\vec{r},\vec{r}') \tag{5-28}$$

subject to (5-23). In order to derive a differential equation from (5-28) we use Lagrange multipliers to enforce the constraints. Define

$$\mathcal{G}[n,\mu,\lambda] = \mathcal{F}[n] - \int d\sigma\;\mu(\sigma)\int\frac{d^2\vec{r}}{V}\;n(\vec{r},\sigma) - \int\frac{d^2\vec{r}}{V}\;\lambda(\vec{r})\int d\sigma\;n(\vec{r},\sigma) \tag{5-29}$$

where the constraints (5-23) determine $\mu(\sigma)$ and $\lambda(\vec{r})$. The required differential equation is obtained by differentiating (5-29) freely with respect to $n(\vec{r},\sigma)$ and setting it equal to

zero:

$$\frac{\delta \mathcal{F}}{\delta n(\vec{r},\sigma)} = 0 = \log[q_0 n_0(\vec{r},\sigma)] + 1 + \bar{\beta}\int d^2\vec{r}'\ \sigma\omega_0(\vec{r}')\mathcal{G}(\vec{r},\vec{r}') - \bar{\beta}[\mu(\sigma)+\lambda(\vec{r})] \tag{5–30}$$

where $\omega_0(\vec{r}) =< \omega(\vec{r}) >= \int d\sigma\ \sigma n_0(\vec{r},\sigma)$ is the equilibrium vorticity field. Define the equilibrium *stream function*

$$\psi_0(\vec{r}) = \int d^2\vec{r}'\ \mathcal{G}(\vec{r},\vec{r}')\omega_0(\vec{r}') \tag{5–31}$$

which, since (5-31) is a linear relationship, is just the average $< \psi(\vec{r}) >$. We then have

$$q_0 n_0(\vec{r},\sigma) = \exp\{-1 + \bar{\beta}\mu(\sigma) + \bar{\beta}\lambda(\vec{r}) - \bar{\beta}\sigma\psi_0(\vec{r})\} \tag{5–32}$$

The function $\lambda(\vec{r})$ may be immediately eliminated by integrating both sides over $\sigma$:

$$\exp\{-\lambda(\vec{r})\bar{\beta}+1\} = \int d\sigma\ \exp -\bar{\beta}[\sigma\psi_0(\vec{r}) - \mu(\sigma)] \tag{5–33}$$

so that

$$n_0(\vec{r},\sigma) = \frac{\exp -\bar{\beta}[\sigma\psi_0(\vec{r}) - \mu(\sigma)]}{\int d\sigma' \exp -\bar{\beta}[\sigma'\psi_0(\vec{r}) - \mu(\sigma')]} \tag{5–34}$$

A differential equation for $\psi_0(\vec{r})$ may be obtained by multiplying both sides of (5-32) by $\sigma$ and integrating:

$$\omega_0(\vec{r}) = -\nabla^2\psi_0(\vec{r}) = \frac{\int \sigma d\sigma\ \exp -\bar{\beta}[\sigma\psi_0(\vec{r}) - \mu(\sigma)]}{\int d\sigma \exp -\bar{\beta}[\sigma\psi_0(\vec{r}) - \mu(\sigma)]} \tag{5–35}$$

and the function $\mu(\sigma)$ is determined by integrating (5-34) over all space:

$$g(\sigma) = \int \frac{d^2\vec{r}}{V}\ \frac{\exp -\bar{\beta}[\sigma\psi_0(\vec{r}) - \mu(\sigma)]}{\int d\sigma \exp -\bar{\beta}[\sigma\psi_0(\vec{r}) - \mu(\sigma)]} \tag{5–36}$$

Equations (5-34),(5-35) and (5-36) constitute a complete equilibrium description of the system. $\bar{\beta}$ determines the energy $\frac{1}{2}\int d^2\vec{r}\,\omega_0(\vec{r})\psi_0(\vec{r})$ and may take on any real value. The above equations are implicit, nonlinear, and contain in principle a continuous infinity of

free parameters $\mu(\sigma)$. When there are only a few species of charge, they may be simplified considerably; however, we postpone discussion of specific cases.

### 5.4 Mean-field theory: Kac-Hubbard-Stratanovitch transformation

We now turn to the second derivation of the mean-field equations. This time we introduce appropriate Lagrange multipliers into the statistical mechanics from the outset. Unlike the multipliers introduced in (5-29), which constitute merely an exact method for solving constrained differential equations, statistical Lagrange multipliers allow us to remove constraints on the partition sum directly. In the mean-field limit the two constructions coincide because only a single state contributes to the partition sum.

In standard fashion, we introduce into the Hamiltonian a Lagrange multiplier $\mu_i$ for each conserved quantity, $Q_i[\omega]$:

$$\bar{\mathcal{H}}[\omega] = \beta\left[\mathcal{H}[\omega] - \sum_i \mu_i Q_i[\omega]\right] \tag{5–37}$$

The partition function consists of an unconstrained trace: $\mathcal{Z} = tr^{\omega} \exp \bar{\mathcal{H}}$. The value of a conserved quantity may be obtained from the appropriate free energy derivative

$$q_i \equiv \frac{1}{V} < Q_i[\omega] >= -\partial F/\partial \mu_i \tag{5–38}$$

where $F = -(\beta V)^{-1} \log(\mathcal{Z})$. Conservation of $g(\sigma)$ leads to a term

$$\mathcal{H}_\mu = -\int d\sigma\ \mu(\sigma) \int \frac{d^2\vec{r}}{V}\ \delta(\sigma - \omega(\vec{r})) = -\int \frac{d^2\vec{r}}{V}\ \mu(\omega(\vec{r})) \tag{5–39}$$

with the function $\mu(\sigma)$ to be determined from the functional derivative

$$g(\sigma) = -\delta F/\delta \mu(\sigma) \tag{5–40}$$

An alternative formulation of this term is obtained by associating, to each $n = 1, 2, 3, \ldots$ a Lagrange multiplier $\mu_n$ with the corresponding power of the vorticity $\Omega_n = \int d^2\vec{r}\, \omega^n(\vec{r})$:

$$\mathcal{H}_\mu = -\sum_n \mu_n \Omega_n \tag{5–41}$$

We observe that the $\mu_n$ are the Taylor coefficients of $\mu(\sigma)$

$$\mu(\sigma) = \sum_{n=1}^{\infty} \mu_n \sigma^n \tag{5-42}$$

and corresponding to (5-40) we have

$$\frac{1}{V} < \Omega_n >= \int d\sigma \; \sigma^n g(\sigma) = -\partial F/\partial \mu_n \tag{5-43}$$

so that $< \Omega_n >$ are the moments of $g(\sigma)$.

Angular momentum conservation is incorporated by means of a term

$$\mathcal{H}_h = -\int d^2\vec{r}\; h(\vec{r})\omega(\vec{r}) \tag{5-44}$$

where $h(\vec{r}) = -\frac{1}{2}\alpha r^2 + h_{ext}(\vec{r})$, and $h_{ext}(\vec{r})$ contains any external fields that might couple linearly to the vorticity field. For example, the Coriolis force in the $\beta$-plane approximation yields $h_{ext}(\vec{r}) = \beta r^3$. The derivative $-\partial F/\partial\alpha$ yields the first term in (5-3). To obtain the contribution to $L$ from the circulation, observe that $R_o^2\Gamma_o - R_i^2\Gamma_i = R_o^2\Omega_1 + (R_o^2 - R_i^2)\Gamma_i$ where $\Omega_1$ is calculated from (5-40) or (5-43) and $\Gamma_i$ is enforced by an imaginary point vortex at the origin, of strength $\Gamma_1$. The latter appears as an additional external field $h_1(\vec{r}) = \Gamma_1 \mathcal{G}(\vec{0},\vec{r})$ in (5-44), which may be seen most simply by separating out a static contribution, $\Gamma_1\delta(\vec{r})$, to the vorticity field in (5-1), and dropping the self-energy term proportional to $\Gamma_1^2$.

The complete Hamiltonian now reads

$$\begin{aligned}\bar{\mathcal{H}} = \frac{1}{2}\beta \int d^2\vec{r} \int d^2\vec{r}' \; \mathcal{G}(\vec{r},\vec{r}')\omega(\vec{r})\omega(\vec{r}') \\ - \beta \int \frac{d^2\vec{r}}{V}\; \mu(\omega(\vec{r})) - \beta \int d^2\vec{r}\; h(\vec{r})\omega(\vec{r})\end{aligned} \tag{5-45}$$

and $\hat{z} = tr^{\omega}\left[\exp -\bar{\mathcal{H}}\right]$ is now a *free* trace over all vorticity configurations $\omega(\vec{r})$. Equation (5-45) is the Hamiltonian for a continuum, continuous spin Ising model, with exchange constants $\mathcal{G}(\vec{r},\vec{r}')$, spin weighting factor $\mu(\sigma)$, and external magnetic field $h(\vec{r})$. More

typically, for applications to critical phenomena, $\mathcal{G}(\vec{r},\vec{r}')$ is short-ranged (e.g., $\mathcal{G}(\vec{r},\vec{r}') = -\delta(\vec{r}-\vec{r}')\nabla^2$), and $\mu(\sigma)$ is a low-order polynomial, $\beta\mu(\sigma) = \frac{1}{2}r\sigma^2 + u\sigma^4$ (the $\phi^4$ model). We are not restricted to analytic forms for $\mu(\sigma)$. The usual Ising discrete measure arises from the choice $\exp\beta\mu(\sigma) = \frac{1}{2}\left[\delta(\sigma-1)+\delta(\sigma+1)\right]$; whenever the function $g(\sigma)$ consists of a series of $\delta$-functions (a finite number of charge species), so does $\exp\beta\mu(\sigma)$.

We now introduce the Kac-Hubbard-Stratonovitch transformation. The idea is the following. All of the terms in $\bar{\mathcal{H}}$, except for the first, are *local*. We may convert the first term into one that is purely local at the cost of introducing a new field $\psi(\vec{r})$. The basic formula we use is

$$\exp\frac{1}{2}\sum_{i,j}A_{ij}s_is_j = \frac{1}{\mathcal{N}}\prod_i\int_{-\infty}^{\infty}d\psi_i\ \exp\left\{-\frac{1}{2}\sum_{i,j}(A^{-1})_{ij}\psi_i\psi_j - \sum_i\psi_is_i\right\} \qquad (5\text{–}46)$$

where $A$ is any positive definite matrix, $\{s_i\}$ any set of real or complex numbers, and $\mathcal{N} = \det(2\pi A)^{\frac{1}{2}}$ is a normalization factor. If $A$ is *negative* definite the same formula holds with $\psi_i$ replaced by $i\tilde{\psi}_i$ in the exponent on the right-hand side. We shall apply (5-46) with $A_{ij} = -\beta^{-1}\mathcal{G}_{ij}$ and $s_i = \beta\omega_i a^2$ (see (5-7)).

Since $\mathcal{G}_{ij}$ is the inverse of the negative Laplacian operator $-\Delta_{ij}$, $\mathcal{G}$ will be positive definite except, perhaps, for the eigenvalue corresponding to the *constant* eigenfunction. If conducting boundary conditions are used, the $\delta$-function $\delta_{ij}$ has no constant component, so $-\Delta_{ij}$ is positive definite and invertible and this problem does not arise. For periodic boundary conditions we must be more careful. In that case, the constant component, $\mathcal{G}_0$ of $\mathcal{G}_{ij}$ is arbitrary, and we choose it to be positive. This choice is equivalent to demanding that $\mathcal{H}[\omega]$ be positive for a *uniform* vorticity field $\omega(\vec{r}) \equiv$ constant $\neq 0$, which may be ensured by requiring the reference separation, $R_0$, to be larger than the system size so that $-\frac{1}{2\pi}\log|\frac{\vec{r}-\vec{r}'}{R_0}|$ is always positive. The constant $\mathcal{G}_0$ does not affect the physics, since it only sets a reference energy, which corresponds to an additive term $\frac{1}{2}\mathcal{G}_0\Omega_1^2$ and vanishes for periodic boundary conditions where $\Omega_1 = 0$.

With proper choice of $\mathcal{G}$, we have the representation for the discretized Coulomb energy

$$\exp -\frac{1}{2}\beta_a a^4 \sum_{i,j} \mathcal{G}_{ij}\omega_i\omega_j = tr^{\psi}\left[\exp\left\{+\frac{1}{2}\beta_a \sum_{i,j}(\mathcal{G}^{-1})_{ij}\psi_i\psi_j - \bar{\beta}\sum_i \psi_i\omega_i\right\}\right] \tag{5-47}$$

where $tr^{\psi} \equiv \mathcal{N}^{-1}\prod_i \int_{-\infty}^{\infty} d\psi_i$, $\mathcal{N} = \det(2\pi\beta_a^{-1}\mathcal{G})^{\frac{1}{2}}$, and $\bar{\beta} = \beta_a a^2$ as before. We have incorporated a rescaling of the temperature with lattice spacing. (5-47) is valid for $\beta < 0$. For $\beta_a > 0$, $i\tilde{\psi}_i$ must be used in place of $\psi_i$ in the exponent. By construction

$$(\mathcal{G}^{-1})_{ij} = -\sum_{n.n.\ \delta}(\delta_{i+\delta,j} - \delta_{ij}) + (\mathcal{G}_0 N)^{-1} \tag{5-48}$$

where the constant term is present for periodic boundary conditions (we shall see shortly that it is canceled).

Now define the Laplace transform

$$\exp \bar{\beta}W(\tau) = \int_{-\infty}^{\infty} d\sigma\ \exp -\bar{\beta}[\sigma\tau - \mu(\sigma)] \tag{5-49}$$

in which the variable $\tau$ may be complex. The partition function may be written

$$\mathcal{Z} = tr^{\psi}\exp\left\{\frac{1}{2}\beta_a\sum_{i,j}(\mathcal{G}^{-1})_{ij}\psi_i\psi_j + \bar{\beta}\sum_i W[\psi_i - h_i]\right\} \tag{5-50}$$

In the continuum limit we have $\frac{1}{a^2}(\mathcal{G}^{-1})_{ij} \to -\nabla^2\delta(\vec{r}-\vec{r}') + (\mathcal{G}_0 V)^{-1}$, yielding

$$\mathcal{Z} = tr^{\psi}\exp -\beta_a V\mathcal{F}[\psi] \tag{5-51}$$

where

$$\mathcal{F}[\psi] = -\int \frac{d^2\vec{r}}{V}\left\{\frac{1}{2}\psi(\vec{r})(-\nabla^2)\psi(\vec{r}) + W[\psi(\vec{r}) - h(\vec{r})]\right\} - \frac{1}{2\mathcal{G}_0}\left[\int \frac{d^2\vec{r}}{V}\ \psi(\vec{r})\right]^2 \tag{5-52}$$

$\bar{\beta} = \beta_a a^2$ has again emerged as the natural temperature variable. Mean-field theory results as the coefficient $\beta V = \bar{\beta}N$ in (5-51) diverges as $a \to 0$. The saddle point of (5-52)

determines the free energy, here a maximum, since $\bar{\beta} < 0$. For $\bar{\beta} > 0$ we replace $\psi(\vec{r})$ by $i\tilde{\psi}(\vec{r})$ and seek the minimum; however, as is common in steepest descent calculations, we shall always find the saddle point to be at *real* values of $i\tilde{\psi}(\vec{r})$. We shall write the free energy in the form (5-52) and remember to seek the appropriate extremum depending on the sign of $\bar{\beta}$.

We proceed to derive the mean-field equations from (5-52). The extremum, $\psi_0(\vec{r})$, is given by the equation

$$\begin{aligned}\frac{\delta\mathcal{F}[\psi]}{\delta\psi(\vec{r})} = 0 = -\nabla^2\psi_0(\vec{r}) + \frac{1}{\mathcal{G}_0}\int\frac{d^2\vec{r}}{V}\,\psi_0(\vec{r}) \\ - \frac{\int_{-\infty}^{\infty}\sigma d\sigma\ \exp -\bar{\beta}[\sigma(\psi_0(\vec{r}) - h(\vec{r})) - \mu(\sigma)]}{\int_{-\infty}^{\infty} d\sigma \exp -\bar{\beta}[\sigma(\psi_0(\vec{r}) - h(\vec{r})) - \mu(\sigma)]}\end{aligned} \tag{5–53}$$

and the averaged vorticity field is then given by

$$\omega_0(\vec{r}) \equiv < \omega(\vec{r}) > = -\frac{\delta\mathcal{F}[\psi]}{\delta h(\vec{r})} = -\nabla^2\psi_0(\vec{r}) + \frac{1}{\mathcal{G}_0}\int\frac{d^2\vec{r}}{V}\,\psi_0(\vec{r}) \tag{5–54}$$

(5-54) implies that $\psi_0(\vec{r}) = \int d^2\vec{r}'\ \mathcal{G}(\vec{r},\vec{r}')\omega_0(\vec{r}')$, which yields $\frac{1}{\mathcal{G}_0}\int\frac{d^2\vec{r}}{V}\ \psi_0(\vec{r}) = \Omega_1/V$. $\Omega_1 = \int d^2\vec{r}\ \omega_0(\vec{r})$ vanishes whenever the $\frac{1}{\mathcal{G}_0}$ term is present as we remarked following (5-46). Evidently, we may discard the $1/\mathcal{G}_0$ term. Equations (5-53) and (5-54) reduce to the previous result (5-35), with the exception that here we have added a field $h(\vec{r})$, which could have been trivially included in (5-35). Finally, the constraint equation is

$$g(\sigma) = -\frac{\delta\mathcal{F}[\psi]}{\delta\mu(\sigma)} = \int\frac{d^2\vec{r}}{V}\frac{\exp -\bar{\beta}[\sigma\big(\psi_0(\vec{r}) - h(\vec{r})\big) - \mu(\sigma)]}{\int_{-\infty}^{\infty} d\sigma'\ \exp -\bar{\beta}[\sigma'\big(\psi_0(\vec{r}) - h(\vec{r})\big) - \mu(\sigma')]} \tag{5–55}$$

which is (5-36). An equation for angular momentum conservation can be similarly obtained by differentiating the $\alpha$-dependence of $h(\vec{r})$ (see (5-44)):

$$< L > = -\frac{1}{2}\int d^2\vec{r}\ r^2\omega_0(\vec{r}) + R_o^2\Omega_1 + (R_o^2 - R_i^2)\Gamma_i \tag{5–56}$$

This derivation, although requiring the introduction of the $a$-lattice, made no use of the $\ell$-lattice. The results of both derivations coincide, verifying the irrelevance of interactions between vortices below the microscopic scale $\ell$.

It is amusing to see the averaged flow potential, $\psi_0(\vec{r})$, emerge as the Kac-Hubbard transformed vorticity variable. The origin of $\psi_0(\vec{r})$ lies in the form of the energy, which may be written $\frac{1}{2}\int d^2\vec{r}\,\omega(\vec{r})\psi(\vec{r})$, similar in form to the coupling term in (5-47), and in the relation $\omega = -\nabla^2\psi$ which minimizes the exponent in (5-47) for given $\psi$: mean-field theory enforces this relation *on average*. It is also interesting to see, from the first derivation of the mean-field equations, the interpretation of the integrand of the spatial integral in (5-55) as the microscale vorticity distribution function $n_0(\vec{r},\sigma)$ (see eq. (5-34) and (5-58)).

### 5.5 Dressed vorticity corollary

A key observation is that, except for $\Omega_1$, the vorticity conservation laws are all *violated* on the macroscopic scale. If we introduce the *dressed* distribution function

$$g_d(\sigma) = \int \frac{d^2\vec{r}}{V}\,\delta(\sigma - \omega_0(\vec{r})) \tag{5–57}$$

then except under very special circumstances, we will find that $g_d(\sigma) \neq g(\sigma)$. Since $g_d(\sigma)$ will be the distribution function observed on *any finite* length scale, we see that it is (experimentally) *impossible* to infer $g(\sigma)$ from the equilibrium state alone. At first glance, this loss of microscopic information would apparently make a theoretical prediction impossible in the absence of any knowledge of the initial conditions: information about initial conditions is absent for geophysical flows like the ones on Jupiter.

Fortunately, knowing only $g_d(\sigma)$, we may make some partial predictions based on the following *dressed vorticity corollary* (Miller, 1990): the averaged vorticity field, $\omega_0(\vec{r})$, is the *maximum energy solution* (corresponding to $\bar{T} \to 0^-$ or $\bar{\beta} \to -\infty$) of the mean field equations with constraint function $g_d(\sigma)$. For the maximum energy solution, $g_d(\sigma) = g(\sigma)$: the constraint function is unrenormalized.

To verify this claim, we consider the generalization of (5-34) to include $h(\vec{r})$:

$$n_0(\vec{r},\sigma) = \frac{\exp -\bar{\beta}\sigma[\psi(\vec{r}) - h(\vec{r})] + \bar{\beta}\mu(\sigma)}{\int_{-\infty}^{\infty} d\sigma' \ \exp -\bar{\beta}\sigma'[\psi(\vec{r}) - h(\vec{r})] + \bar{\beta}\mu(\sigma')} \tag{5-58}$$

where the constraints (5-23) hold. As $\bar{\beta} \to -\infty$, we expect that for each $\vec{r}$, $n_0(\vec{r},\sigma)$ becomes peaked in $\sigma$ around the maximum of the exponent $\sigma[\psi(\vec{r}) - h(\vec{r})] - \mu(\sigma)$. That is,

$$\lim_{\bar{\beta}\to-\infty} n_0(\vec{r},\sigma) = \delta(\sigma - \omega_\infty(\vec{r})) \equiv n_\infty(\vec{r},\sigma) \tag{5-59}$$

where $\omega_\infty(\vec{r})$ satisfies

$$\frac{d\mu}{d\sigma}(\omega_\infty(\vec{r})) = \psi_\infty(\vec{r}) - h(\vec{r}) \tag{5-60}$$

(for $\bar{\beta} \to +\infty$ we would look for the *minimum* rather than the maximum). We find that

$$\omega_\infty(\vec{r}) = -\nabla^2\psi_\infty(\vec{r}) = \int d\sigma \ \sigma n_\infty(\vec{r},\sigma) \tag{5-61}$$

and by assumption

$$g_d(\sigma) \equiv \int \frac{d^2\vec{r}}{V} \ n_\infty(\vec{r},\sigma) \tag{5-62}$$

which establishes that $\omega_\infty(\vec{r})$ has the same constraint function as $\omega_0(\vec{r})$.

To prove that $\omega_\infty(\vec{r}) = \omega_0(\vec{r})$ we argue as follows. It is easy to see that $\omega_\infty(\vec{r})$ is the result of maximizing the energy (5-45) alone, subject to the constraint $g_d(\sigma)$, while $\omega_0(\vec{r})$ is the result of maximizing the sum of the entropy and energy (5-28) or (5-29), with constraint $g(\sigma)$. We claim that $\omega_0(\vec{r})$ yields the same energy as $\omega_\infty(\vec{r})$; that is, the maximum possible energy subject to the coarse-grained constraint $g_d(\sigma)$. Suppose $\omega_0(\vec{r})$ were to yield a *smaller* energy. Since $\omega_0(\vec{r})$ and $\omega_\infty(\vec{r})$ have the same constraint function $g_d(\sigma)$, they are related by an area-preserving diffeomorphism: there exists some function $\vec{M} : V \to V$ with unit Jacobian such that $\omega_\infty(\vec{r}) = \omega_0\big(\vec{M}(\vec{r})\big)$. We define $\tilde{n}_0(\vec{r},\sigma) = n_0\big(\vec{M}(\vec{r}),\sigma\big)$, which yields the same entropy and satisfies the same constraints (5-23) as $n_0(\vec{r},\sigma)$, but has first moment $\int d\sigma \ \sigma\tilde{n}_0(\vec{r},\sigma) = \omega_\infty(\vec{r})$. Intuitively, we view

$\vec{M}(\vec{r})$ as a reshuffling of the $\ell$-cells in the argument leading to (5-21), in which we keep the $a$-cell microstructure within each $\ell$-cell fixed. But since $\omega_\infty(\vec{r})$ has a higher energy than $\omega_0(\vec{r})$, it is clear that $\tilde{n}_0(\vec{r},\sigma)$ has a higher *free* energy (5-28), than $n_0(\vec{r},\sigma)$. This deduction contradicts our assumption that $n_0(\vec{r},\sigma)$ gave the free energy maximum. Provided $\omega_\infty(\vec{r})$ is a unique energy maximum, we may conclude that $\omega_0(\vec{r}) = \omega_\infty(\vec{r})$, which establishes the corollary.

### 5.6 Single charge fluid and point vortices

We illustrate the formalism with a simple example. We consider the case of two charge species, $\sigma = 0$ and $\sigma = q$, on fractional areas of $1-\alpha$ and $\alpha$ respectively:

$$g(\sigma) = \alpha\delta(\sigma - q) + (1-\alpha)\delta(\sigma), \qquad 0 \le \alpha \le 1 \tag{5–63}$$

We require just a single (relative) chemical potential, $\mu_q$:

$$\exp\bar{\beta}\mu(\sigma) = \exp\{\bar{\beta}\mu_q\}\delta(\sigma - q) + \delta(\sigma) \tag{5–64}$$

which, from (5-53), yields the mean-field equation

$$\begin{aligned}\omega_0(\vec{r}) &= -\nabla^2\psi_0(\vec{r}) \\ &= q\Big\{1 + \exp -\bar{\beta}\big[q(\psi_0(\vec{r}) - h(\vec{r})) - \mu_q\big]\Big\}^{-1}\end{aligned} \tag{5–65}$$

with the single unknown $\mu_q$ determined from

$$\begin{aligned}\alpha &= \int \frac{d^2\vec{r}}{qV}\,\omega_0(\vec{r}) \\ &= \int \frac{d^2\vec{r}}{V}\,\Big\{1 + \exp -\bar{\beta}\big[q(\psi_0(\vec{r}) - h(\vec{r})) - \mu_q\big]\Big\}^{-1}\end{aligned} \tag{5–66}$$

We may simplify further to the *point vortex limit* which we *define* as the limit in which the fractional area $\alpha \to 0$, but the total charge $\alpha q \equiv Q$ remains *fixed*. In the limit $\alpha \to 0$ it is clear that we may neglect the 1 in the denominator on the right-hand sides of (5-65,66)

(we require $\exp -\bar{\beta}[q(\psi_0 - h) - \mu_q] \gg 1$ in order to ensure that the integrand of (5-66) be small). We obtain

$$\exp -\bar{\beta}\mu_q = \alpha / \int \frac{d^2\vec{r}}{V} \exp \bar{\beta} q \big(\psi_0(\vec{r}) - h(\vec{r})\big) \tag{5-67}$$

and hence

$$\begin{aligned} \omega_0(\vec{r}) &= -\nabla^2 \psi_0(\vec{r}) \\ &= Q \frac{\exp \hat{\beta}[\psi_0(\vec{r}) - h(\vec{r})]}{\int \frac{d^2\vec{r}}{V} \exp \hat{\beta}[\psi_0(\vec{r}) - h(\vec{r})]} \end{aligned} \tag{5-68}$$

where $\hat{\beta} = \bar{\beta} q$ is a new rescaled temperature variable, which should remain fixed as $q \to \infty$ in order that a well-defined limiting profile $\omega_0(\vec{r})$ exists. This requires that $\bar{T} = \hat{T} Q \alpha^{-1}$ diverge as $\alpha \to 0$. The reason is that point charges tend to collapse unless the temperature is very high: a hard core no longer keeps them apart. Note that we have taken the $a \to 0$ limit before the $\alpha \to 0$ limit, so that the mean-field equations are valid at each step. We might instead consider a simultaneous limit, in which case we have the appropriately rescaled temperature

$$\hat{T} = T / a^2 q \tag{5-69}$$

We might envision maintaining $a^2 q$, the charge per $a$-cell, fixed as $a \to 0$, yielding a finite number $N = Q/a^2 q$ of point charges in this limit, with $\hat{T} \propto T$. This scaling was treated rigorously by Fröhlich and Ruelle (1982), in the limit $V \to \infty$, with $n = N/V$ fixed. They obtain a fluctuating thermodynamics with positive temperature equilibria only. It thus emerges that in order for mean-field theory to hold, we must allow $a^2 q \to 0$, and $N \to \infty$, at fixed $\hat{T}$, yielding an infinite density of particles each with infinitesimal charge. We obtain the "point charge" limit described by (5-68).

### 5.7 Kraichnan energy/enstrophy theory

Another example worth discussing, because it arises so often in the literature, is the energy/enstrophy theory (Kraichnan, 1975). Here we make the choice

$$\mu(\sigma) = \tfrac{1}{2}\mu_2 \sigma^2 \tag{5-70}$$

We find (for $\bar{\beta}\mu_2 < 0$),

$$W(\tau) = -\tau^2/2\mu_2 + \frac{1}{2\bar{\beta}} \log(-2\pi/\bar{\beta}\mu_2) \tag{5-71}$$

and hence

$$\mathcal{F}[\psi] = -\frac{1}{2} \int \frac{d^2\vec{r}}{V} \left\{ |\nabla\psi(\vec{r})|^2 - \frac{1}{\mu_2} \left(\psi(\vec{r}) - h(\vec{r})\right)^2 \right\} \tag{5-72}$$

where we have dropped the trivial constant in (5-71). Let us restrict ourselves to $h \equiv 0$ for simplicity. For $\bar{\beta} > 0$ and $\mu_2 < 0$, we find $\psi \equiv 0$ as the only consistent minimum (recall that in this case $\psi$ ought to be replaced by $i\psi$). For $\bar{\beta} < 0$ and $\mu_2 > 0$, we seek the maximum, and the problem is well-defined only if $\frac{1}{\mu_2} \leq \lambda_0$, where $\lambda_0 \approx \frac{1}{L^2}$ is the smallest eigenvalue of the negative Laplacian consistent with the boundary conditions. The mean-field equation for $\psi_0$,

$$\nabla^2\psi_0(\vec{r}) + \frac{1}{\mu_2}\psi_0(\vec{r}) = 0 \tag{5-73}$$

is basically an eigenvalue problem for $\mu_2$, which is evidently not a free variable. $\mu_2$ can take on only the degenerate value $\mu_2 = 1/\lambda_0$, and consequently $\psi_0(\vec{r}) = A\psi_{\lambda_0}(\vec{r})$, the normalized eigenfunction associated with $\lambda_0$, multiplied by an arbitrary amplitude. The order parameter is then $\omega_0(\vec{r}) = (\lambda_0 A)\psi_0(\vec{r})$, and the energy is $E_0 = \lambda_0 A^2$ which is determined by the single free parameter $A$:

$$\psi_0(\vec{r}) = \sqrt{E_0/\lambda_0}\psi_{\lambda_0}(\vec{r}); \qquad \omega_0(\vec{r}) = \sqrt{E_0\lambda_0}\psi_{\lambda_0}(\vec{r}) \tag{5-74}$$

For a square box of side $L$, with $\psi_0$ vanishing on the boundary, we have

$$\psi_{\lambda_0} = (2/L) \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi y}{L}\right); \qquad 0 \leq x, y \leq L \tag{5-75}$$

with $\lambda_0 = 2\pi^2/L^2$. This stream function yields a blob-like shape centered on $(L/2, L/2)$. The constraint function $g(\sigma)$ is given by

$$g(\sigma) = \int \frac{d^2\vec{r}}{V} \frac{\exp \frac{1}{2}\bar{\beta}\mu_2[\sigma - \omega_0(\vec{r})]^2}{\sqrt{2\pi/(-\bar{\beta}\mu_2)}} \tag{5-76}$$

a non-trivial distribution of vorticity (observe that $\int d\sigma\ g(\sigma) = 1$, as we expect). By introducing the dressed distribution $g_d(\sigma)$, this expression can be rewritten in the form

$$g(\sigma) = \int_{-\infty}^{\infty} d\sigma'\ g_d(\sigma') \frac{\exp \frac{1}{2}\bar{\beta}\mu_2[\sigma - \sigma']^2}{\sqrt{2\pi/(-\bar{\beta}\mu_2)}} \tag{5-77}$$

which may be inverted to yield

$$\begin{aligned} g_d(\sigma) &\equiv \int \frac{d^2\vec{r}}{V}\ \delta\big(\sigma - \omega_0(\vec{r})\big) \\ &= \int_{-\infty}^{\infty} d\sigma'\ g(\sigma') \frac{\exp \frac{1}{2}\bar{\beta}\mu_2[\sigma - \sigma']^2}{\sqrt{2\pi/(-\bar{\beta}\mu_2)}} \end{aligned} \tag{5-78}$$

Note that $g_d(\sigma)$ has compact support while $g(\sigma)$ does not.

### 5.8 Lynden-Bell theory

To conclude this chapter, we connect our work to that of Lynden-Bell (1967) and to the Debye-Hückel theory of electron systems. We must first agree to give up our interpretation of the theory as describing Euler's equation, which can not be simply reduced to Coulomb form in higher dimensions. The derivation of the mean-field equations is valid in any dimension, so long as $\mathcal{G}(\vec{r}, \vec{r}')$ is replaced by the appropriate higher-dimensional Coulomb interaction between point charges, and the temperature is scaled as $\bar{\beta} = \beta_a a^d$ (see (5-5)). The mean-field equations derived here may then be used to describe equilibria of three-dimensional Coulomb systems with continuous charge density, characterized by $g(\sigma)$. In particular, the point charge limit, described by (5-68), turns out to correspond to the Debye-Hückel theory for electrolytes (Debye and Hückel, 1923). In its quantum generalization, Thomas-Fermi theory, Boltzmann factors are replaced by free-electron Fermi functions.

The derivation is very simple. The local charge density $n(\vec{r})$ is assumed to be governed by the local electric potential $\phi(\vec{r})$ via the Boltzmann distribution:

$$n(\vec{r}) = \exp \beta[e\phi(\vec{r}) - \mu] \tag{5-79}$$

with the self-consistent relation

$$n(\vec{r}) = -\nabla^2 \phi(\vec{r}) \tag{5–80}$$

($e$ denotes the electron charge). Here $\phi(\vec{r})$ includes both externally applied fields, such as charged impurities, and those fields originating from the induced electron density. The chemical potential $\mu$ is eliminated in favor of the average density

$$n_0 = \int \frac{d^2\vec{r}}{V}\, n(\vec{r}) \Rightarrow \exp \beta\mu = n_0 / \int \frac{d^2\vec{r}}{V}\, \exp \beta e\phi(\vec{r}) \tag{5–81}$$

which upon substitution yields the analogue of (5-68). The validity of this self-consistent theory requires that $\phi(\vec{r})$ vary slowly on the scale of the average separation between charges, and becomes exact in the limit of infinite density (compare with the discussion below (2-69)).

The theory developed by Lynden-Bell is based on the collisionless Boltzmann equation for a gravitating mass distribution. We examine the distribution function $f(\vec{r}, \vec{p}, t)$, which denotes the probability density at time $t$ for finding a particle at $\vec{r}$ with momentum $\vec{p}$. The time evolution of $f$ is assumed to be given by the Boltzmann equation

$$\frac{Df}{Dt} \equiv \dot{\vec{r}} \cdot \frac{\partial f}{\partial \vec{r}} + \dot{\vec{p}} \cdot \frac{\partial f}{\partial \vec{p}} + \frac{\partial f}{\partial t} = 0 \tag{5–82}$$

where now the convective derivative is with respect to a $2d$-dimensional space. We have the relations

$$\begin{aligned} \dot{\vec{r}} &= \vec{p} \\ \dot{\vec{p}} &= -\nabla \phi(\vec{r}, t) \\ \phi(\vec{r}, t) &= -\int d^{2d}\tau'\, \mathcal{G}(\vec{r}, \vec{r}') f(\vec{r}', \vec{p}', t) \end{aligned} \tag{5–83}$$

where $d^{2d}\tau' \equiv d^d\vec{r}' d^d\vec{p}'$, $\phi(\vec{r}, t)$ is the gravitational potential at $\vec{r}$ due to the mass distribution $f$, and $\mathcal{G}(\vec{r}, \vec{r}')$ is the appropriate $d$-dimensional Coulomb interaction (like charges

attract at positive temperatures in gravitational systems). The total energy (kinetic plus potential) is given by

$$E = \int d^d\vec{r}d^d\vec{p}\ \tfrac{1}{2}p^2 f(\vec{r},\vec{p}) - \tfrac{1}{2}\int d^{2d}\tau d^{2d}\tau'\ f(\vec{r},\vec{p})\mathcal{G}(\vec{r},\vec{r}')f(\vec{r}',\vec{p}') \tag{5-84}$$

with the constraint function

$$g(\sigma) = \int d^{2d}\tau\ \delta\big(\sigma - f(\vec{r},\vec{p})\big) \tag{5-85}$$

Since $\vec{p}$ is an unbounded variable $\int g(\sigma)d\sigma$ is unbounded (the divergence is at small $\sigma$); however, we do have the normalization

$$\int d\sigma\ \sigma g(\sigma) = \int d^{2d}\tau\ f(\vec{r},\vec{p}) = 1 \tag{5-86}$$

We are therefore lead to consider the Boltzmann factor $\exp -\beta\mathcal{H}$ with, in an obvious shorthand,

$$\mathcal{H} = -\tfrac{1}{2}\int d^{2d}\tau d^{2d}\tau'\ fGf - \int d^{2d}\tau\ hf - \int d^{2d}\tau\ \mu[f] \tag{5-87}$$

where $h(\vec{p},\vec{r})$ contains the $\frac{1}{2}p^2$ term in (5-84), and any other "external fields" we might wish to add. The Kac-Hubbard-Stratonovitch transformation to the new field $\psi(\vec{r},\vec{p})$ is now straightforward. Since the Coulomb interaction $\mathcal{G}$ does not depend on momentum, we use the identity (5-46) with $\omega(\vec{r}) \equiv \int d^d\vec{p}\ f(\vec{r},\vec{p})$. With the definitions $\bar{\beta} = \beta a^{2d}$ and

$$\exp \bar{\beta}W(\tau) = \int_{-\infty}^{\infty} d\sigma\ \exp -\bar{\beta}[\sigma\tau - \mu(\sigma)] \tag{5-88}$$

as before, the functional to be minimized becomes

$$\mathcal{F}[\psi(\vec{r})] = \frac{1}{2}\int d^d\vec{r}\ |\nabla_r\psi(\vec{r})|^2 - \frac{1}{2}\int d^{2d}\tau\ W[\psi(\vec{r}) - h(\vec{r},\vec{p})] \tag{5-89}$$

We may define

$$n_0(\vec{r},\vec{p},\sigma) = \frac{\exp -\bar{\beta}\big[\sigma\big(\psi(\vec{r},\vec{p}) - h(\vec{r},\vec{p})\big) - \mu(\sigma)\big]}{\int_{-\infty}^{\infty} d\sigma'\ \exp -\bar{\beta}\big[\sigma'\big(\psi(\vec{r},\vec{p}) - h(\vec{r},\vec{p})\big) - \mu(\sigma')\big]} \tag{5-91}$$

which we interpret as the coarse-grained equilibrium distribution function for charge species $\sigma$ at the coarse-grained phase space point $(\vec{r},\vec{p})$. The resulting mean-field equations can be written

$$-\nabla^2\psi_0(\vec{r}) = -\int d^d\vec{p}\int d\sigma\ \sigma n_0(\vec{r},\vec{p},\sigma) \tag{5-92}$$

and the full equilibrium distribution is given by

$$f_0(\vec{r},\vec{p}) = \int d\sigma\ \sigma n_0(\vec{r},\vec{p},\sigma) \tag{5-93}$$

with constraint function

$$g(\sigma) = \int d^{2d}\tau\ n_0(\vec{r},\vec{p},\sigma) \tag{5-94}$$

These expressions correspond to the mean-field equation of Lynden-Bell (1967).

# Chapter 6

## Summary and conclusions

We now turn to a discussion of the physical content of the theories we have derived in the last few chapters. It is instructive to start with an account of the physics of the Lynden-Bell model. As we have remarked earlier, the origin of the connection between Lynden-Bell's statistical mechanics and our own is the fundamental equation of motion:

$$\frac{Df}{Dt} \equiv \dot{\vec{r}} \cdot \frac{\partial f}{\partial \vec{r}} + \dot{\vec{p}} \cdot \frac{\partial f}{\partial \vec{p}} + \frac{\partial f}{\partial t} = 0 \qquad (5\text{–}82)$$

Obviously, we recover the Euler equations if we throw out explicit dependence on the momentum in (5-82). An additional feature common to both the Euler fluid and stellar clusters is the existence of a scalar field, the evolution of which is described by the equation of motion, and which interacts through a long-range potential. The Coulomb potential governs the interaction of the vorticity in the fluid, and the mass density in the gravitational system. In both cases, the equation of the motion says that the *covariant* derivative of the scalar field vanishes, giving rise to the infinite family of Casimirs.

The collisionless Boltzmann equation describes the motion of point masses interacting by a gravitational potential, *provided we do not permit the masses to collide*. This motion is Hamiltonian: $\vec{r}$ and $\vec{p}$ constitute canonical coordinates; the introduction of collisions destroys the Hamiltonian character unless details of the interaction of nearby particles are added. Collisions, by permitting the merger of particles and inelastic scattering, further

introduce source and sink terms into the Boltzmann equation, giving rise to a covariant derivative that no longer vanishes: the evolution of the mass density becomes *dissipative*. The collisionless Boltzmann equation represents an idealization of the motion of stars in which we view the discrete collection as a *continuous medium*, described by a continuous mass density $f(\vec{r},\vec{p},t)$. This idealization is only consistent to the extent that we may neglect collisions for the time scales in which we are interested.

In fact, the collisionless regime is of considerable interest to astrophysicists (Binney and Tremaine, 1987; Spitzer, 1987). In studying the formation of, say, an elliptical galaxy from some ancient distribution of stars, they estimate the "collision time," which sets the time scales for which we may view the dynamics as described by a collisionless Boltzmann equation. Lynden-Bell asked the question: can statistical equilibrium theory for the *collisionless* dynamics yield the observed structure of stellar clusters? That is: (1) is it possible that the dynamics determining the mass and velocity distribution in a galaxy occur within the time for which collisions between stars can be neglected, and (2) if so, then do the collisionless degrees of freedom of the system equilibrate in this process. Antonov (1962) had already argued that, in a physically significant regime, an isothermal system of gravitating *point* masses within an isolating spherical shell has a negative specific heat, and collapses catastrophically. If statistical mechanics was to have any bearing on the problem, equilibration of *all* degrees of freedom had to be excluded.

Lynden-Bell (1967) tried to answer his question by constructing a statistical mechanics for the collisionless Boltzmann equation, obtaining generalizations of our mean-field equations; however, a lengthy and convoluted chain of results has lead many workers, including Lynden-Bell, to conclude that the simple answer is *no*. (Less simple answers exist, and not surprisingly, are more, well, complicated. See Tremaine et al. (1986)). It emerges that in an infinite volume, which most people seem to believe constitutes an appropriate

physical boundary condition for galactic evolution, *the mean-field equations have no solution*. The star cluster can always achieve greater entropy by expelling a negligible mass to large distances, and increasing its density in a hot central core. Conservation of the mass distribution $g(\sigma)$ can't prevent this "violent relaxation," since the density can *increase* in the spatial dimensions, accompanied by a compensating *decrease* in the velocity dimensions. Astrophysicists must then explain how features of purely dynamical origin prevent complete equilibration, since observed galaxies evolve well into the collisional regime, and have evidently not collapsed.

Within a *finite* volume, it turns out that, at least for points, one encounters negative heat capacities and collapse under what are considered appropriate physical boundary conditions. Lynden-Bell and Wood (1968) have claimed that finite cores rescue the system from these pathologies. We know of no effort to determine the extent to which equilibrium statistical mechanics is useful under these circumstances, which, in any event, may not be physically significant. We feel that the statistical mechanics of self-gravitating systems is in an unsatisfactory state of affairs, not necessarily because of error, but because it has not been stated in the clarity and completeness we have come to expect in most applications of statistical mechanics (for a partial review, see Padmanabhan (1990)).

Our reason for discussing the Lynden-Bell theory in such detail is that we wish to contrast the Euler fluid with stellar mechanics. As Katz and Lynden-Bell (1978) have demonstrated (without reference to the inviscid fluid), in a finite volume the two-dimensional, attractive, single charge species point vortex gas does not display the negative specific heat that is characteristic of the stellar thermodynamics. Nor is the single species finite-core gas subject to collapse in infinite volume, since here we have only spatial, and not velocity, degrees of freedom. In fact, in two dimensions, we see no reason why solutions to

the mean-field equations should not exist for any consistent set of values of the conserved quantities.

But the differences run even deeper. In fluid mechanics, it has long been observed that *in certain physical regimes* the Euler equations, equations of non-dissipative motion for a continuous medium, yield a (surprisingly?) correct description of the long-time behavior of real, viscous fluids. That is, the Euler equations don't seem to be themselves aware of when they're not supposed to be valid any more. Marcus, for example, has successfully modeled a variety of non-trivial long-time laboratory flows by means of *inviscid* dynamics (Marcus, 1988, 1990, unpublished; Sommeria et al., 1988, 1989). Onsager (1949) proposed that statistical mechanics of inviscid flow could explain *turbulent* phenomena. Aref and Siggia (1981) claim that *inviscid* statistical mechanics models the turbulent shear layer. In the case of fluids, it seems that the collisionless approximation can be extraordinarily effective, for reasons that we do not entirely understand.

We should like to informally propose a mechanism that explains how *inviscid* statistical mechanics could usefully describe the long-time behavior of certain stationary *viscous* flows. Our notions rely on a *separation of time scales*. Without wishing to assign any blame, we attribute some part of our thinking to Aref and Siggia (1981).

We emphasize first that our statistical mechanics is meant only to be a *macroscopic* description of fluid properties. That is, we have nothing to say about correlations on scales small compared to the *finite* system size. We don't concede, for example, any but the most qualitative connection between the equilibrium properties that we hope to predict, and *turbulence*. The turbulent cascade, which we expect to dominate the small scales, constitutes a strongly forced and dissipated stationary state, and there is simply no reason to expect the turbulent scales to be in detailed balance. In an infinite volume, we would expect any forcing and dissipation to lead to the $k^{-5/3}$ cascade to small wavenumber

and the $k^{-3}$ cascade to large wavenumber, as suggested by Kraichnan (1967), or to the trivial long-wavelength properties suggested by Forster et al. (1977).

We maintain that the finite size of the fluid container introduces a new scale into the problem. It allows us to envision a situation in which wavenumbers of the order of inverse system size are *in equilibrium* with each other, while much larger wavenumbers are out of equilibrium, their behavior being dominated by the energy cascades. Whereas small-scale correlations would be determined by properties of the forcing, correlations on larger scales might not be affected by small-scale forcing. Central to this picture is the notion that the time scale for equilibration of long-wavelength modes be small compared to the characteristic time scale for the forcing and dissipation of these modes by communication with the short wavelengths. The self-consistency of our picture at long wavelengths is supported by the dressed vorticity corollary, which shows that, in equilibrium, we may ignore short-wavelength properties and focus on averages of the vorticity field over some scale determined by the system size. Any localized turbulence or viscosity-mediated diffusion that acts to smear the vorticity on this scale has no effect on the macroscopic flow.

A small-scale forcing of the kind we envision here may be a feature of some geophysical flows, where small-scale atmospheric storms drive the large-scale dynamics. In addition, it may be appropriate for certain laboratory flows, such as Swinney's spots (Sommeria et al. (1988); Marcus, personal communication).

As the above remarks no doubt convey, we do not have an adequate understanding of the applicability of a statistical mechanical theory such as ours to fluid flows in general. In particular, the entire theory is based upon a presumption of ergodicity. We expect that there are many situations, such as steady laminar flow, for which this presumption is most certainly wrong. On the other hand, even small amounts of turbulence or externally induced noise could serve to open up all significant parts of phase space, so that flow

becomes, effectively, ergodic. For example, Marcus (1990) finds that an axisymmetric ring of vorticity is a steady flow in his dynamical simulation; however, when subject to an asymmetric perturbation ten orders of magnitude lower in relative vorticity, the flow destabilizes and evolves to a single blob. In other simulations, he finds that steady flows consisting of a number of separated blobs can be further mixed if he throws in filaments of vorticity, causing the blobs to merge and shed vorticity. Presumably, noise introduced by small-scale turbulence or other processes could play an analogous role in paring down the class of stable flows.

We have emphasized earlier that a number of authors have numerically demonstrated ergodic behavior for point vortices. An intrinsic property of point vortex methods is that they preserve their (singular) values for the area integrals exactly. Their behavior may suggest that for a continuous vorticity field, a simulation faithful to the ideal Euler fluid would also be ergodic. (It is also possible that the apparently ergodic behavior of point vortices merely reflects cumulative numerical error). It could be that the smoothing effects of dissipation stabilize flows that are not true long-time stationary states of the Euler equations. We have seen that when we take the envelope of a finite temperature equilibrium flow, we obtain a zero temperature flow. Thus, viscosity might act locally to reduce the temperature, creating by its local smoothing effect stationary flows that are not in global equilibrium. Finally, Marcus (1990) draws a distinction between "filamentous" and "non-filamentous" flows, defined by the presence or absence of small-scale filaments of vorticity. He finds that non-filamentous flows remember their initial conditions, in contrast to filamentous flows, in which only conserved quantities seem to matter. Perhaps any truly dissipationless flow ought to be filamentous and consequently ergodic, since viscosity would not be present to remove the small-scale filamentation. We would hope that our remarks apply also to high-Reynolds number flows, where filamentation could occur on many more length scales than can be accommodated in any present-day numerical simulation. (These

small-scale filaments should be distinguished from the microscopic filamentation discussed below.)

The objection might be raised that, in fact, the fluid never achieves a stationary state. For any finite time, the fluid is still evolving at some non-zero length scale that vanishes only at infinite time. Consequently, the fluid motion can not be ergodic. We believe that this objection is of a formal nature. We would characterize the non-stationary behavior in a different manner: at any finite time, the fluid is still relaxing to an asymptotic equilibrium that will have structure at all length scales. It is the asymptotic state that maximizes the entropy and represents the long-time vorticity configuration that follows from the assumption of ergodicity.

The consistency of this characterization depends upon the irrelevance of short length-scale structures to the macroscopic vorticity configuration at long times. We would argue that Marcus (1990) has observed just this irrelevance in his dynamical simulations. By careful control of the short-wavelength cutoff, he demonstrates that the long-time macroscopic vorticity configuration is achieved *before* the enstrophy has reached the smallest scales of his numerics. If he increases his numerical resolution, Marcus obtains the same long-wavelength vorticity configuration, even though the simulation may now be continued to longer times in which structure at still smaller scales has emerged.

The long-range potential governing the interaction of vorticity is an additional factor suggesting that small-scale structures ought not to affect the macroscopic dynamics of a fluid near macroscopic equilibrium. Because of the long range, we may calculate the dynamics contributed by a small-scale structure to other structures at long distances by using the *average* of the vorticity over some small area. A similar feature emerges in the statistical mechanics: *in equilibrium, it is only the average of the vorticity field over small*

*length scales that affects the macroscopic structures.* Indeed, this is precisely what leads to the dressed vorticity corollary.

These remarks also pertain to the divergence of integrals of the gradient of the vorticity in our statistical equilibria. In any statistical mechanical system, one can define quantities that are finite for some class of initial conditions, but that diverge in statistical equilibrium. The defect lies not in statistical mechanics, but rather in the choice of quantities at which to look. For the Euler fluid, we would argue that the diverging integral vorticity gradients, which may be viewed as reflecting the length of the boundary between regions on which the microscopic vorticity field takes different values, are completely irrelevant to the macroscopic long-time flow. Our statistical formulation is consistent with many possible microscopic shapes for the vorticity field. It provides no useful information about them, and in turn, they don't matter to the macroscopic equilibrium at all.

Further difficulties arise in trying to test the theory by current numerical methods. Dynamical simulation of inviscid fluids is by no means a well-understood subject: in fact, appreciation of the constraining properties of the conservation laws has been sorely missing in this field. Even the very best studies seem to assume that the macroscopic enstrophy ought to be conserved at long times, whereas we have argued that it should *not* be conserved. Nor has anyone checked his simulation to ensure that he has controlled the vorticity distribution at the *microscopic* level. For example, let us suppose our initial conditions involve both positive and negative vorticity, and that the predicted equilibrium based on the microscopic vorticity distribution consists of spatially separated regions of positive and negative vorticity. We expect that, in evolving from initial to long-time state, regions of positive and negative vorticity might be at times well-mixed, with some of the vorticity again separating out asymptotically. Yet because of finite resolution, the numerical method could easily *cancel* the positive with negative vorticity in the well-mixed

intermediate stages, yielding a long-time vorticity distribution *consistent* with the initial conditions, but incorrect nevertheless. We would be unable to distinguish the failure of statistical mechanics from the failure of the simulation. (Notice that point vortex methods are not subject to this objection, since the methods explicitly conserve particle number and charge). In view of these realities, our recent preliminary comparison of predicted equilibrium flows with Marcus' dynamical simulations yields remarkable agreement (see Appendix B).

Certainly, possibilities exist for a numerical method that correctly incorporates the conservation laws. String theorists have uncovered a truncated dynamics that seems to converge, as the number of Fourier modes increases, to Euler dynamics (Fairlie and Zachos, 1989; Hoppe, 1989). As the number of modes increases, so does the number of conservation laws. (This literature was related to us by A. Rouhi, who independently discovered the truncated dynamics.) Efforts are underway to design a computational method based on the truncated dynamics (Rouhi and Meiron, personal communication). Perhaps this method will eventually confirm our equilibrium predictions.

We close by pointing out that there exist a variety of dynamical systems sharing the essential features of the Euler fluid that allowed us to construct a statistical mechanics. We refer the reader to Holm et al. (1985) for a partial listing of systems with an infinite family of Casimirs. One particularly interesting example might be the "meteorological primitive equations," describing three-dimensional rotating, stratified, compressible flow of an ideal gas (Shepherd, 1990). Possibly of interest to astrophysicists would be the relativistic generalization of the Boltzmann equation.

## References

V.A. Antonov, Vest. leningr. gos. Univ. **7**, 135 (1962).

H. Aref and E.D. Siggia, Vortex dynamics of the two-dimensional turbulent shear layer, J. Fluid Mech. **100**, 705 (1980).

V.I. Arnold, *Mathematical methods of classical mechanics* (New York: Springer-Verlag, 1978).

D. Bai and A. Brandt, Local mesh refinement multilevel techniques, Siam J. Scie. **8**, 109 (1967).

C. Basdevant and R. Sadourny, Ergodic properties of inviscid truncated models of two-dimensional incompressible flows, J. Fluid Mech. **69**, 673 (1975).

J.T. Beale and A. Majda, High order accurate vortex methods with explicit velocity kernals, J. Comp. Phys. **58**, 188 (1985).

G. Benfatto, P. Picco, and M. Pulvirente, On the invariant measures for the two-dimensional Euler flow, J. Stat. Phys. **46**, 729 (1987).

J. Binney and S. Tremaine, *Galactic dynamics* (Princeton: Princeton University Press, 1987).

F.P. Bretherton and D.B. Haidvogel, Two-dimensional turbulence above topography, J. Fluid Mech. **78**, 129 (1976).

G.F. Carnevale and J.S. Frederiksen, Nonlinear stability and statistical mechanics of flow over topography, J. Fluid. Mech. **175**, 157 (1987).

G.F. Carnevale and G.K. Vallis, Pseudo-advective relaxation to stable states of inviscid two-dimensional fluids, J. Fluid Mech. **213**, 549 (1990).

A.J. Chorin and J.E. Marsden, *A Mathematical introduction to fluid mechanics*, (New York: Springer-Verlag, c1979).

J.P. Christiansen, Numerical simulation of hydrodynamics by the method of point vortices, J. Comp. Phys. **13**, 363 (1973).

I. Cook and J.B. Taylor, Stationary states of two-dimensional turbulence, Phys. Rev. Lett. **28**, 82 (1972).

M. Creutz, Microcanonical Monte Carlo simulation, Phys. Rev. Lett. **50**, 1411 (1983).

P. Debye and E. Hückel, Phys. Z. **24**, 185 (1923); 305 (1923).

G.S. Deem and N.J. Zabusky, Ergodic boundary in numerical simulations of two-dimensional turbulence, Phys. Rev. Lett. **27**, 396 (1971).

S.F. Edwards, A variational calculation of the equilibrium properties of a classical plasma , 119 (1957).

S.F. Edwards and J.B. Taylor, Negative temperature states of two-dimensional plasmas and vortex fluids, Proc. R. Soc. Lond. A. **336**, 257 (1974).

D.B. Fairlie and C.K. Zachos, Infinite-dimensional algebras, sine brackets, and SU($\infty$), Phys. Lett. B **224**, 101 (1989).

D. Forster, D.R. Nelson, and M.J. Stephen, Large-distance and long-time properties of a randomly stirred fluid, Phys. Rev. A **16**, 732 (1977).

D.G. Fox and S.A. Orszag, Inviscid dynamics of two-dimensional turbulence, Phys. Fluids **16**, 169 (1973).

J. Fröhlich and D. Ruelle, Statistical mechanics of vortices in an inviscid two-dimensional fluid, Comm. Math. Phys. **87**, 1 (1982).

A. Griffa and R. Salmon, Wind-driven ocean circulation and equilibrium statistical mechanics, J. Marine Res. **47**, 457 (1989).

G. Holloway, Eddies, waves, circulation and mixing: statistical geofluid mechanics, Ann. Rev. Fluid Mech. **18**, 91 (1986).

D. Holm, Hamiltonian formulation of the baroclinic quasigeostrophic fluid equations, Phys. Fluids **29**, 7 (1986).

D. Holm, J.E. Marsden, T. Ratiu and A. Weinstein, Nonlinear stability of fluid and plasma equilibria, Phys. Rep. C **123**, 1 (1985).

J. Hoppe, Diffeomorphism groups, quantization, and SU($\infty$), Int. J. Mod. Phys. A **4**, 5235 (1989).

A.P. Ingersoll and P.G. Cuong, Numerical model of long-lived Jovian vortices, J. Atmos. Sci. **38**, 2067 (1981).

G. Joyce and D. Montgomery, Negative temperature states for the two-dimensional guiding-centre plasma, J. Plasma Physics **10**, 107 (1973).

J. Katz and D. Lynden-Bell, The gravothermal instability in two dimensions, Mon. Not. R. astr. Soc. **184**, 709 (1978).

S. Kida, Statistics of the systems of line vortices, J. Phys. Soc. Japan **39**, 1395 (1975).

G. Kirchhoff, *Lectures on Mathematical Physics, Mechanics*, (Leipzig: B.G. Teubner, 1877).

R.H. Kraichnan, Inertial ranges in two-dimensional turbulence, Phys. Fluids **10**, 1417 (1967).

R.H. Kraichnan, Statistical dynamics of two-dimensional flow, J. Fluid Mech. **67**, 155 (1975).

R.H. Kraichnan and D. Montgomery, Rep. Prog. Phys. **43**, 547 (1980).

L.D. Landau and E.M. Lifshitz, *Fluid mechanics*, (New York: Pergamon, 1980a).

L.D. Landau and E.M. Lifshitz, *Statistical physics*, v.1 (New York: Pergamon, 1980b).

T.D. Lee, On some statistical properties of hydrodynamical and magneto- hydrodynamical fields, Q. Appl. Math. **10**, 69 (1952).

C.E. Leith, Diffusion approximation for two-dimensional turbulence, Phys. Fluids **11**, 671 (1968).

C.E. Leith, Minimum enstrophy vortices, Phys. Fluids **27**, 1388 (1984).

A. Leonard, Vortex methods for flow simulation, J. Comput. Phys. **37**, 289 (1980).

C.C. Lin, On the motion of vortices in two dimensions - I. Existence of the Kirchhoff-Routh function, Proc. N.A.S. **27**, 570 (1941).

C.C. Lin, On the motion of vortices in two dimensions - II. Some further investigations on the Kirchhoff-Routh function, Proc. N.A.S. **27**, 575 (1941).

T.S. Lundgren and Y.B. Pointin, Statistical mechanics of two-dimensional vortices, J. Stat. Phys. **17**, 323 (1977a).

T.S. Lundgren and Y.B. Pointin, Non-gaussian probability distribution for a vortex fluid, Phys. Fluids **20**, 356 (1977b).

D. Lynden-Bell, Statistical mechanics of violent relaxation in stellar sytems, Mon. Not. R. astr. Soc. **136**, 101 (1967).

D. Lynden-Bell and R. Wood, The gravo-thermal catastrophe in isothermal spheres and the onset of red-giant structure for stellar systems, Mon. Not. R. astr. Soc. **138**, 495 (1968).

P.S. Marcus, Numerical simulation of Jupiter's Great Red Spot, Nature **331**, 693 (1988); Vortex dynamics in a shearing zonal flow, J. Fluid Mech. **215**, 393 (1990).

J. Marsden, A group theoretic approach to the equations of plasma physics, Canad. Math. Bull. **25**, 129 (1982).

J. Marsden and A. Weinstein, Coadjoint orbits, vortices, and Clebsch variables for incompressible fluids, Physica **7D**, 305 (1983).

N. Metropolis, A. Rosenbluth, M. Rosenbluth, A. Teller, and E. Teller, J. Chem. Phys. **21**, 1087 (1953).

J. Miller, Statistical mechanics of Euler equations in two dimensions, submitted to Phys. Rev. Let., October 1989.

J. Miller, Statistical mechanics, field theory, and Jupiter's Red Spot, abstract B28 1, A.P.S. March meeting (1990).

D. Montgomery, Two-dimensional vortex motion and "negative temperatures," Phys. Lett. **39A**, 7 (1972).

D. Montgomery and G. Joyce, Statistical mechanics of "negative temperature" states, Phys. Fluids **17**, 1139 (1974).

D. Montgomery, L. Turner, and G. Vahala, Most probable states in magnetohydrodynamics, J. Plasma Physics **21**, 239 (1979).

D. Montgomery and G. Vahala, Two-dimensional magnetohydrodynamic turbulence: cylindrical, non-dissipative model, J. Plasma Phys. **21**, 71 (1979).

P.J. Olver, *Applications of Lie groups to differential equations* (New York: Springer-Verlag, 1986).

L. Onsager, Statistical hydrodynamics, Nuovo Cimento Suppl. **6**, 279 (1949).

J. Ostriker, Astrophys. J. **140**, 1056 (1964).

T. Padmanabhan, Statistical mechanics of gravitating systems, Phys. Rep. **188**, 285 (1990).

J. Pedlosky, *Geophysical fluid dynamics* (New York: Springer-Verlag, 1987).

D. Pines and D. Bohm, A collective description of electron interactions: II. Collective *vs* individual particle aspects of the interactions, Phys. Rev. **85**, 338 (1952).

Y.B. Pointin and T.S. Lundgren, Statistical mechanics of two-dimensional vortices in a bounded container, Phys. Fluids **19**, 1459 (1976).

P.G. Saffman and G.R. Baker, Vortex interactions, Ann. Rev. Fluid. Mech. **11**, 95 (1979);

R. Salmon, Hamiltonian fluid mechanics, Ann. Rev. Fluid Mech. **20**, 225 (1988).

W. Saslaw, *Gravitational physics of stellar and galactic systems* (Cambridge: Cambridge University Press, 1985).

T.G. Shepherd, A general method for finding extremal states of Hamiltonian dynamical systems, with applications to perfect fluids, J. Fluid Mech. **213**, 573 (1990).

C.E. Seyler, Partition function for a two-dimensional plasma in the random-phase approximation, Phys. Rev. Lett. **32**, 515 (1974).

C.E. Seyler, Thermodynamics of two-dimensional plasmas or discrete line vortex fluids, Phys. Fluids **19**, 1336 (1976).

C.E. Seyler, Y. Salu, D. Montgomery, and G. Knorr, Two-dimensional turbulence in inviscid fluids or guiding center plasmas, Phys. Fluids **18**, 803 (1975).

R.A. Smith, Phase-transition behavior in a negative-temperature guiding-center plasma, Phys. Rev. Lett. **63**, 1479 (1989).

R.A. Smith and T.M. O'Neil, Nonaxisymmetric thermal equilibria of a cylindrically bounded guiding center plasma or discrete vortex system, submitted to Phys. Fluids B, (1990).

J. Sommeria, S.D. Meyers, and H.L. Swinney, Laboratory simulation of Jupiter's Great Red Spot, Nature **331**, 689 (1988).

J. Sommeria, S.D. Meyers, and H.L. Swinney, Laboratory model of a planetary eastward jet, Nature **337**, 58 (1989).

L. Spitzer, *Dynamical evolution of globular clusters*, (Princeton: Princeton University Press, 1987).

J.S. Stodolkiewicz, Acta. Astr. **13**, 30 (1963).

J.B. Taylor, Negative temperatures in two-dimensional vortex motion, Phys. Lett. **40A**, 1 (1972).

J.B. Taylor and B. McNamara, Plasma diffusion in two dimensions, Phys. Fluids **14**, 1492 (1971).

J.B. Taylor and W. B. Thompson, Fluctuations in guiding center plasma in two dimensions, Phys. Fluids **16**, 111 (1973).

S. Tremaine, M. Henon, and D. Lynden-Bell, H-functions and mixing in violent relaxation, Mon. Not. R. astr. Soc. **219**, 285 (1986).

D.J. Tritton, *Physical fluid dynamics*, $2^{nd}$ Ed. (New York: Oxford University Press, 1988).

A. Weinstein, Hamiltonian structure for drift waves and geostrophic flow, Phys. Fluids **26**, 388 (1983).

V.E. Zakharov, Zh. Eksp. Teor. Fiz. **60**, 1714 (1971).

# Appendix A

## Statistical mechanics of Euler equations in two dimensions

Jonathan Miller

Submitted to *Physical Review Letters* in October 1989

accepted October 1990

Kirchhoff[1] observed that the equations of motion for point vortices in a two-dimensional, inviscid, incomprehensible[2] fluid can be derived from a Hamiltonian:

$$\mathcal{H} = -\sum_{i \neq j} \omega_i \omega_j \log |\vec{r_i} - \vec{r_j}|; \qquad \omega_i \frac{d\vec{r_i}}{dt} = \vec{\nabla}_i \times \mathcal{H} \tag{1}$$

The conjugate variables are the coordinates of the $i$th vortex $x_i, y_i$ and we use the notation

$$\vec{u} = \vec{\nabla} \times \psi = (\partial_y \psi, -\partial_x \psi); \qquad \omega = \vec{\nabla} \times \vec{u} = -\vec{\nabla}^2 \psi \tag{2}$$

where $\omega$ is the (scalar) vorticity field, $\vec{u}$ is the velocity field, and $\psi$ is the stream function. A substantial body of work is based on the premise that the properties of this system, the point vortex gas, have implications for the flow described by the Euler equations[3]. In particular, given a Hamiltonian, it is natural to ask about equilibrium properties, using the methods of statistical mechanics.

The evolution of large-scale coherent structures (or blobs) is an oft-noted feature of two-dimensional fluid flow. The notion that blobs might be a simple equilibrium phenomenon was suggested by Onsager[4]. He pointed out that in bounded regions and at high energies the vortex gas, with Hamiltonian (1) and all $|\omega_i| = \omega_0$ gives rise to clusters

of vortices of the same sign. Onsager argued that the bounded phase space implies that above a certain energy the number of states available to the gas decreases as a function of energy, giving rise, at least for a finite number of vortices, to "negative temperature" states.

While interest in this system has surfaced on many occasions since Onsager's proposal[5], unresolved problems remain. For example, questions have been raised as to whether negative temperatures and blobs persist in the thermodynamic limit[6]. Onsager himself was uncertain how the statistics of point vortices applied to the more familiar situation in which initial conditions specify a continuous distribution of vorticity. A related issue concerns the proper treatment of the infinite number of integrals of motion in two-dimensional Euler flow: $\int_\Omega d^2\vec{r}\,\omega^n(\vec{r})$ for integer $n$, where $\Omega$ is the region containing the fluid. These quantities are conserved since

$$\frac{d}{dt}\int_\Omega d^2\vec{r}\,\omega^n(\vec{r}) = \int_\Omega d^2\vec{r}\,n\omega^{n-1}(\vec{r})(\frac{\partial\omega}{\partial t} + \vec{u}\cdot\vec{\nabla}\omega) = 0. \tag{3}$$

For $n = 2$, this integral is known as the enstrophy.

The point vortex model represents a singular case of Euler flow, since constants of motion with $n > 1$ involve powers of delta functions. A natural way to go about eliminating this defect is to write down a partition function incorporating the constraints as is usual in statistical mechanics :

$$\int \mathcal{D}\psi \exp \int_\Omega d^2\vec{r}\,\{-\frac{1}{T}(\vec{\nabla}\psi)^2 + \sum_n \alpha_n\omega^n(\vec{r})\} \tag{4}$$

where the constants $\alpha_n, n \geq 1$ and $1/T$ are Lagrange multipliers. In taking this approach, Kraichnan discarded all constants of motion except for the energy and the enstrophy[7]; however, integrals of other powers of the vorticity cannot be neglected in the study of long-wavelength properties of Euler flow in a compact domain.

In this paper we construct a theory of statistical equilibrium for the two-dimensional Euler fluid which incorporates all constants of motion. The equilibria generically feature blobs. We find that in a fluid evolving from some initial condition to statistical equilibrium, only the energy and integrals linear in the vorticity appear to be conserved. All other constants, including the enstrophy, are found to be altered. This situation reflects the fact that averages of such quantities over a finite area need not coincide with their unaveraged values.

A symmetry of the Euler equations enables us to include all constants of motion in our statistical mechanics. The invariance of physical quantities under smooth area-preserving coordinate reparameterizations (the group of area-preserving diffeomorphisms) leads, in two dimensions, to local conservation of vorticity[8]. Integrals over $\Omega$ of any smooth function of the vorticity are conserved by the flow; these quantities are the Casimirs of the theory[9]. Equivalently, we may say that the vorticity distribution function, $G(\omega)$, which yields the measure of the subset of $\Omega$ on which the vorticity takes on a value less than $\omega$, is preserved by the flow.

The preceding assertions follow from the Hamiltonian formulation of the Euler equations described by a number of authors[8,9,10,11]. Our equilibrium statistical mechanics will be obtained by averaging over all configurations of the fluid which share the same $G(\omega)$ and energy[12], with a weighting arising naturally from the Hamiltonian[13]. As in many applications of statistical mechanics, we can not rigorously justify our assumption of ergodicity[14].

We now sketch the construction of our theory. For ease of presentation we consider only the very simplest case: two-dimensional Euler flow in a disc $\Omega$ of radius 1. We impose free boundary conditions so that the only role of the boundary is to make the volume finite; consequently we take $-\frac{1}{2\pi}\log|\vec{r}-\vec{r}'|$ as our Green's function. We also require that our

vorticity distribution $G(\omega)$ be such that the magnitude of the vorticity is bounded by some $|\omega|_{max}$.

Our Hamiltonian takes the form[10,11] $\mathcal{H} = \frac{1}{2}\int_\Omega d^2\vec{r}\ \vec{u}^2(\vec{r})$. This non-negative quantity is the kinetic energy of the fluid once we rescale $\vec{r}$ to be dimensionless and set the density to 1. We integrate by parts and ignore the contribution of the boundary to obtain

$$\mathcal{H} = -\frac{1}{4\pi}\int_\Omega d^2\vec{r}\int_\Omega d^2\vec{r}'\ \omega(\vec{r})\omega(\vec{r}')\log|\vec{r}-\vec{r}'|. \tag{5}$$

We next write down a canonical partition function: $\int \mathcal{D}^g\omega \exp\{-\mathcal{H}(\omega)/T\}$. The superscript $g$ refers to the fact that we integrate over configurations that have a given vorticity density function $g(\omega) = dG(\omega)/d\omega$. For purposes of counting states, we need to regularize our functional integral. We do so by incorporating a lattice spacing, $a$. Our Hamiltonian becomes:

$$\mathcal{H}^a = -\frac{a^4}{4\pi}\sum_{i\neq j}\omega_i\omega_j\log|\vec{r}_i-\vec{r}_j| \quad + \quad \text{(self-energy)} \tag{6}$$

where the $i,j$ take values on a lattice of side $a$ in the region $\Omega$, and the $\omega_i$ are averages of $\omega(\vec{r})$ over lattice boxes of side $a$. The total self-energy scales as $a^2\log(a^2)$ and so its contribution to the Hamiltonian vanishes as $a \to 0$. Up to a factor of $a^4$, our regularized Hamiltonian looks like that of the point vortex gas (1), but it is distinguished by the underlying lattice, which is required in order to impose the conservation laws.

To understand the effect of the regularization of the partition function on the functional integration, it is easiest to consider an example. Take $g(\omega)$ to have the form $(\pi-\alpha)\delta(\omega)+\alpha\delta(\omega-1)$, where $\pi$ is the area of $\Omega$. That is, $G(\omega)$ describes a vorticity distribution with the property that the area upon which the vorticity takes the value 1 is $\alpha$; the vorticity vanishes elsewhere. Then with lattice spacing $a$ we obtain $N = \pi/a^2$ lattice points, upon which the vorticity takes value 1 on $\alpha/a^2$ points and value 0 on the remaining points. The functional integration varies the vorticity field over all possible

ways of allocating the $\alpha/a^2$ 1's and $N - \alpha/a^2$ 0's among the $N$ lattice sites, with each site occupied by exactly one 1 or 0. It is clear that $G(\omega)$ is approached exactly as $a \to 0$. The limiting process, in which $N \to \infty$ at constant total system volume and the distance of closest approach of two vortices $a \to 0$ at the same rate, distinguishes our system from the point vortex gas. For a continuous $G(\omega)$, we slice the range of the vorticity field into intervals, and choose the relative numbers of lattice sites on which the vorticity falls within a given interval, so as to converge to $G(\omega)$ in the limit of vanishing lattice spacing.

We now outline our argument that the partition function converges to a well-defined and non-trivial limit as the lattice spacing vanishes. In fact, we can derive an explicit condition that the equilibria must satisfy. The reason we can do so is that, for a certain class of vorticity distributions, we can prove that a mean-field theory is exact, as one might anticipate from the long-range nature of the interaction. This class of vorticity distributions consists of those for which $|\omega|_{max}$ is finite.

The validity of mean-field theory is a consequence of four factors: *(i)* the strong constraint imposed by the conservation of a $G(\omega)$ of this type; *(ii)* the independence of the range of the potential on the lattice spacing $a$; *(iii)* the smoothness of the potential away from the source; and *(iv)* the mild divergence of the potential at the source.

Our proof divides into two steps. The first step is to argue that given a $G(\omega)$, we can approximate the energy to within accuracy $\varepsilon$ by considering only structure above a fixed length scale $\ell$. We obtain the Hamiltonian $\mathcal{H}^\ell$ given by (6) but with $\ell$ replacing $a$ and the $\omega_i$ now averages over boxes of side $\ell$. The scale $\ell$ is determined by $|\omega|_{max}$ and $\varepsilon$; we allow $\varepsilon$ and $\ell$ to vanish at the end of the calculation. Then $\mathcal{H}^\ell$ approximates the energy to the desired accuracy *uniformly* over the set of configurations allowed by $G(\omega)$. A consequence of uniform convergence is that we do not care about correlations on scales smaller than $\ell$,

so long as we satisfy the constraints imposed by $G(\omega)$. We take $\ell \to 0$ at the end of the calculation. For notational simplicity we shall not write this limit explicitly.

Our second step is to calculate the entropy $S$ of a system with a given vorticity field and lattice cut-off $a$ by regarding lattice points within a distance $\ell$ of each other as independent. The entropy is dominated by the large number of isoenergetic configurations of the $(\ell/a)^2$ vortices, and may be explicitly calculated.

We may view $g(\sigma)$ as determining the total number of squares of side $a$ on which the vorticity takes a value very close to $\sigma$. We define the quantity $\rho(\sigma, \vec{r})$ as the density of squares of vorticity $\sigma$ within a distance $\ell$ of $\vec{r}$. $\rho$ must satisfy two conditions: *(c1)* $\int_{-\infty}^{\infty} d\sigma\ \rho(\sigma, \vec{r}) = 1$ which enforces incompressibility; and *(c2)* $\int_\Omega d^2\vec{r}\ \rho(\sigma, \vec{r}) = g(\sigma)$, which correctly normalizes the density. Note that $\omega(\vec{r})$, the vorticity density, is given by $\int_{-\infty}^{\infty} d\sigma\ \sigma\rho(\sigma, \vec{r})$.

Now we can write the partition function in terms of $\rho$:

$$\int \mathcal{D}\rho \exp -a^{-2}\left\{\mathcal{H}(\rho)/\bar{T} - \bar{S}(\rho, g)\right\} \tag{7}$$

here $\bar{T} = a^{-2}T$ and $S(\rho, g) = a^{-2}\bar{S}(\rho, g)$ is the logarithm of the number of ways of generating $\rho$, given a vorticity density function $g$ regularized with cut-off $a$. Since we may regard the $(\ell/a)^2$ vortex squares of side $a$ that lie in a box of side $\ell$ around $\vec{r}$, and that yield $\rho(\sigma, \vec{r})$, as uncorrelated, we obtain the entropy of an ideal gas:

$$\bar{S}(\rho, g) = -\int_\Omega d^2\vec{r} \int_{-\infty}^{\infty} d\sigma\ \rho(\sigma, \vec{r}) \log \rho(\sigma, \vec{r}) \tag{8}$$

which does not depend on $a$. The quantity in braces in (7) then does not depend on $a$, and in the limit of vanishing lattice spacing the integral is concentrated where this quantity is

minimized. Stationary points of this quantity occur at:

$$\rho(\sigma,\vec{r}) = \exp\left\{ -\frac{\sigma}{\bar{T}}\psi(\vec{r}) + \mu(\sigma)\right\}[\int_{-\infty}^{\infty} d\sigma' \ \exp\left\{ -\frac{\sigma'}{\bar{T}}\psi(\vec{r}) + \mu(\sigma')\right\}]^{-1} \qquad (9)$$

where $\mu(\sigma)$ are Lagrange multipliers implicitly defined by the constraints (*c2*). Using (2) we see that mimima of (7) correspond to minima of the free energy:

$$\mathcal{F}^g(\psi) \equiv \int d^2\vec{r}\ \{(\vec{\nabla}\psi)^2/2 + \bar{T}\log\int_{-\infty}^{\infty} d\sigma\ \exp\left\{ -\frac{\sigma}{\bar{T}}\psi(\vec{r}) + \mu(\sigma)\right\}\} \qquad (10)$$

where $\psi$ must satisfy the boundary conditions. Equation (10) is the free energy for a generalized Ising model with logarithmic interactions; it can be independently derived from the Hamiltonian for the two-dimensional Euler fluid using a Kac-Hubbard-Stratanovitch transformation[15], where the constraints are imposed upon the Hamiltonian by Lagrange multipliers.

We can draw several conclusions from our argument. (1) It was necessary to require that $T$ vanish along with the lattice spacing. It is $\bar{T}$ that determines the energy. (2) It is easy to see that for a neutral system where $g$ is symmetric about the origin, $\psi \equiv 0$ minimizes $\mathcal{F}^g(\psi)$ for $\bar{T} \geq 0$. There are no non-trivial solutions with non-negative $\bar{T}$ in this case. (3) In general, the vorticity density function $g_d$ derived from the $\omega(\vec{r})$ that yields the above minimum is not the same as $g$. We know that

$$\int_\Omega d^2\vec{r}\,\omega(\vec{r}) = \int_{-\infty}^{\infty} d\sigma\ \sigma g(\sigma) \qquad (11)$$

but no other moment of the vorticity is necessarily the same for both $g$ and $g_d$.

Put another way, suppose our fluid evolves from smooth initial conditions with vorticity distribution $G_b$, a "bare" vorticity distribution. The evolving flow is stretched and folded, a process that effectively disperses the smoothly distributed vorticity into smaller and smaller scales. Asymptotically in time $t$, a measurement on scales large compared to

the arbitrarily small scales into which the vorticity is dispersed will yield a distribution $G_d(t)$ that will converge to $G_d$, the "dressed" vorticity distribution, as $t \to \infty$. Since $G_d$ measures averages, it need not coincide with $G_b$. The energy and one-body integrals are conserved, since they are long-wavelength properties.

Although $G_d$ in general differs from $G_b$, a trivial consequence of our arguments is that, at a given energy, $G_d$ yields the same equilibrium solution as $G_b$. Furthermore, the given energy turns out to be precisely the maximum energy compatible with $G_d$. It follows that the configuration would be dynamically stable[10]. In other words, if we consider the process of solving equation (10) to obtain the dressed distribution from a given bare distribution and energy as a mapping, then $G_d$ is a zero-temperature fixed point of the mapping. A physical implication of this result, which we call the "dressed vorticity corollary," is that for a fluid *in statistical equilibrium*, coarse-grained quantities suffice to determine the equilibrium. This observation suggests that our equilibria might persist in the presence of a viscosity acting to smear the small scales. An equivalent way of stating our result is that the long-time dynamics of an inviscid fluid will evolve to a configuration that is a *global* extremum of the energy, subject to satisfying the long-time (dressed) vorticity distribution.

We work out a simple example to show the relation of our work to previous results. We use same form of the distribution $G$ as in our example of functional integration, and consider for convenience Dirichlet boundary conditions on a disc, with $\bar{T} < 0$. We find from (9)

$$\omega^{\alpha,\bar{T}}(\vec{r}) = -\vec{\nabla}^2\psi = \frac{\alpha \exp\{-\psi/\bar{T} + \mu_1(\alpha,\bar{T})\}}{(\pi-\alpha)\exp\mu_0(\alpha,\bar{T}) + \alpha\exp\{-\psi/\bar{T} + \mu_1(\alpha,\bar{T})\}} \tag{12}$$

where the chemical potentials $\mu_{0,1}(\alpha,\bar{T})$ are used to enforce *(c2)*. This equation describes the statistical equilibrium of an inviscid fluid with our specified vorticity distribution and

temperature $\bar{T}$, and is new in this context. A related, but distinct, equation has been derived by a number of previous authors for point vortices using a mean-field argument[5]. Their equation is a special case of ours, as we can see by taking the limit $\alpha \to 0$, at the same time scaling the vorticity so as to keep the total circulation constant. Fixing $\bar{T}$ less than $-1/8\pi$, the mean-field collapse temperature for point vortices[5], we obtain

$$\hat{\omega}^{\bar{T}}(\vec{r}) = -\vec{\nabla}^2\hat{\psi} = \{\exp -\hat{\psi}/\bar{T}\}\{\int_\Omega d^2\vec{r}\ \exp -\hat{\psi}/\bar{T}\}^{-1} \tag{13}$$

Here $\hat{\omega}^{\bar{T}}(\vec{r})$ denotes the normalized density of points. For $0 \geq \bar{T} > -1/8\pi$, the solutions of (13) collapse to a point[5] in contrast to the solutions of equation (12), which remain continuous and finite[16].

We remark that conservation laws and fields linear in $\omega(\vec{r})$ do not affect our formulation, which we expect to be applicable to a wide variety of Hamiltonian systems possessing infinite families of Casimirs, among them many of those described in Ref. 9. In particular, it may be relevant to the two-dimensional guiding-center plasma[5].

More generally, a $G$ with $|\omega|_{max}$ unbounded may be physically relevant. Our mean-field argument may fail in this case, because $\bar{S}$ diverges as $a$ vanishes, the energy no longer necessarily converges uniformly in $\ell$, and/or the self-energy contribution can no longer be ignored. These considerations lead us to expect that $G$ exist such that $T$ is finite as the lattice spacing $a \to 0$. Such regimes are of interest because we could couple them to thermal (e.g. molecular) degrees of freedom.

We learned after this manuscript was submitted that equation (12) and its generalizations had been derived earlier by Lynden-Bell in the context of stellar dynamics[17]. There the particles interact by a gravitational potential. The equation of motion is the collisionless Boltzmann equation, and the conserved density is a function of both space and velocity degrees of freedom. Interpretation of the Lynden-Bell equilibrium is problematic

in stellar dynamics, since in contrast to two dimensions, in three dimensions equilibria do not exist under physical boundary conditions.

The origin of this work was the suggestion by M.C. Cross that Marcus' dynamical simulations[18] of Jupiter's Red Spot might be explained in the terms of statistical mechanics. We hope in the future to address the application of these methods to the Spot.

The author would like to thank M.C. Cross and P.B. Weichman for advice and discussions. The author received partial support from the Shell Foundation and the NSF under grant number DMR-8715474.

[a] Address after October 1, 1990: AT&T Bell Laboratories - Murray Hill, NJ 07974.

[1]G. Kirchhoff, *Lectures on Mathematical Physics, Mechanics*, (Leipzig, B.G. Teubner, 1877).

[2]D. Forster, D.R. Nelson, and M.J. Stephen, Phys. Rev. A **16**, 732 (1977). See footnote 6 of this reference.

[3]For reviews and references see: P.G. Saffman and G.R. Baker, Ann. Rev. Fluid. Mech. **11**, 95 (1979); A. Leonard, J. Comput. Phys. **37**, 289 (1980).

[4]L. Onsager, Nuovo Cimento Suppl. **6**, 279 (1949).

[5]For reviews and references see: R.H. Kraichnan and D. Montgomery, Rep. Prog. Phys. **43**, 547 (1980).

[6]J. Fröhlich and D. Ruelle, Comm. Math. Phys. **87**, 1 (1982).

[7]R.H. Kraichnan, Phys. Fluids **10**, 1417 (1967).

[8]R. Salmon, Ann. Rev. Fluid Mech. **20**, 225 (1988).

[9]D. Holm, J.E. Marsden, T. Ratiu and A. Weinstein, Phys. Rep. C **123**, 1 (1985).

[10]V.I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, New York, 1978).

[11]P.J. Olver, *Applications of Lie Groups to Differential Equations* (Springer-Verlag, New York, 1986); V.E. Zakharov, Zh. Eksp. Teor. Fiz. **60**, 1714 (1971).

[12]That is, up to symmetry breaking.

[13]Equilibrium statistical mechanics requires a Liouville theorem; the viewpoint of Ref. 10 furnishes a simple way to see that one exists. The fluid flow is given by geodesics

on the group of volume-preserving diffeomorphisms on $\Omega$. The Liouville theorem is then equivalent to the group property: the action of an element of the group preserves volumes in phase space. For an alternative and more direct argument, see T.D. Lee, Q. Appl. Math. **10**, 69 (1952).

[14]Euler flow also preserves connectivity (see Ref. 10). This property is shared by phase space flows, which can nonetheless be mixing. So connectivity is not in itself an obstacle to ergodicity.

[15]J. Miller and P.B. Weichman, manuscript in preparation.

[16] We would like to emphasize the distinction between point vortices and a continuous vorticity field: whereas both the point vortex gas and a continuous vorticity field are sources for an incompressible *velocity* field, *the point vortex gas is itself compressible*, whereas a continuous vorticity field is *incompressible*.

[17]As kindly pointed out to us by E. Ott and D. Montgomery. See D. Lynden-Bell, Mon. Not. R. astr. Soc. **136**, 101 (1967); S. Tremaine, M. Henon, and D. Lynden-Bell, Mon. Not. R. astr. Soc. **219**, 285 (1986); J. Binney and S. Tremaine, *Galactic Dynamics* (Princeton, Princeton University Press, 1987); W. Saslaw, *Gravitational Physics of Stellar and Galactic Systems* (Cambridge, Cambridge University Press, 1985).

[18]P.S. Marcus, Nature **331**, 693 (1988); J. Fluid Mech. **215**, 393 (1990).

## Appendix B

### Statistical equilibrium calculations for Red spot-like flows

The mean-field equations that we derived in chapter 5 depend in a complicated way upon a large number of parameters. We don't know of a practical way to solve them directly, except in very simple cases. For a single species of point vortex in a disc with Dirichlet boundary conditions, an analytic solution is well-known (Ostriker, 1964; Stodolkiewicz, 1963; Montgomery et al., 1979). For up to two non-zero charge species, a relaxational algorithm for axisymmetric equilibria (effectively a one-dimensional problem) has been developed by M.C. Cross. In the single-species point charge limit, Smith (1989) has employed a Newton continuation method that applies to (asymmetric) two-dimensional charge distributions. When required to deal with a large or infinite number of charge species, which is ordinarily the case for a physical problem, we are currently unable to compute equilibria by means of a deterministic, well-controlled algorithm. Rather, we must resort to stochastic methods, in particular Monte Carlo. We have not tried hard to work out an alternative solution method, and see no reason to conclude that, simply because we know of no better way to solve the equations, none exists. On the other hand, Monte Carlo can not be considered completely satisfactory for our purposes, since extracting useful information in the general case seems to be rather time-consuming. We are grateful to M.C. Cross for suggesting the use of Monte Carlo methods in this system.

We begin with a summary of the Monte Carlo computation method, since readers with a fluid mechanics background might not be familiar with the technique. By Monte Carlo, we refer to a class of rejection sampling methods related to the Metropolis algorithm (Metropolis et al., 1953). These methods are based upon a stochastic dynamics that evolves the system toward a statistically stationary equilibrium state. The artificial dynamics is designed to sample the Boltzmann distribution $\exp -\beta\mathcal{H}$, with statistics improving in time (or number of steps for a discrete dynamics). The stochastic dynamics needs to be chosen so that (1) it is ergodic; and (2) it satisfies detailed balance:

$$\mathcal{W}(X,Y)\exp -\beta\mathcal{H}(X) = \mathcal{W}(Y,X)\exp -\beta\mathcal{H}(Y) \tag{4–1}$$

where $\mathcal{W}(X,Y)$ is the probability that the system moves to state $Y$ at the next step, given that it is currently in state $X$ (Kalos, 1986). We may divide $\mathcal{W}(X,Y)$ into two independent parts: the conditional probability $\mathcal{P}(X,Y)$ that a move from $X$ to $Y$ is proposed, given that the system is in state $X$, and the conditional probability that a proposed transition (or move) is accepted, $\mathcal{A}(X,Y)$. If, as is customary, we choose $\mathcal{P}(X,Y)$ constant for a set of $Y$ related to $X$ by an elementary move, and zero otherwise, then $\mathcal{W}(X,Y) = \mathcal{A}(X,Y)$. A variety of choices for $\mathcal{W}(X,Y)$ are possible; the canonical Monte Carlo (Metropolis) method stipulates

$$\mathcal{W}(X,Y) = \max\{1, \exp -\beta\mathcal{H}(Y)/\exp -\beta\mathcal{H}(X)\} \tag{4–2}$$

Once our Monte Carlo dynamics has relaxed the system to equilibrium, we may calculate sought-after quantities by averaging over the configurations given by the dynamics, provided we are careful to average only over configurations separated by sufficiently many time steps that the correlations between them are small.

A closely related "microcanonical" Monte Carlo method has been suggested by Creutz (1983). He partitions the energy between the system of interest and an additional

degree of freedom, the "demon." Moves are restricted to the phase space determined by conservation of the sum of the energies of system and demon.

Our calculations (Miller, unpublished) have made use of both canonical and microcanonical methods. We treat the energy and angular momentum separately. We have found it convenient to calculate with either (1) microcanonical moves separately in both energy and angular momentum; or (2) canonical moves in the energy but microcanonical moves in the angular momentum. Generally speaking, calculations either way agree in the regimes in which we expect them both to work.

Since the long-range interactions and boundary conditions lead to a time-consuming particle calculation, we compute the energy of a fluid configuration by solving the the Poisson equation. We use a square lattice of sites contained within an annulus. Each move consists of independent exchanges of two pairs of lattice sites, each lattice site randomly chosen on the lattice.

Time limitations on our Sparcstation have confined us to a crude $32 \times 32$ square lattice, many of whose sites don't even fall within the annulus. A test case on a disc in which we compare a numerical solution by Cross' method to the Monte Carlo result (figure 1) yields pointwise agreement of the vorticity fields to within about 8%. While Monte Carlo on such a coarse grid yields satisfactory results for the parameter values used in this calculation, for regimes in which the vortices are more tightly packed the error involved in the energy calculation becomes so large that the results only have qualitative value. Numerical work on larger grids is in progress, and until and results described below can only be viewed as indicative of the true equilibrium solutions.

We turn now to the Red Spot, which for our purposes coincides with the persistent spot of cyclonic vorticity in Marcus' dynamical simulations (Marcus, 1988, 1990). Marcus

carries out his calculation on a flat annulus with rigid boundaries. The effect of planetary curvature and rotation is incorporated by the $\beta$-plane approximation (Pedlosky, 1986). On the $\beta$-plane, we may define a *potential vorticity*, $\omega_p$:

$$\omega_p(\vec{r}) = \omega(\vec{r}) + \beta r$$

The Euler dynamics on the $\beta$-plane advectively conserve the potential vorticity $\omega_p$, and not $\omega$. For our purposes, we may account for the $\beta$-plane by adding to the Hamiltonian an external potential $-\beta r^3$, and by replacing $\omega$ with $\omega_p$ in our statistical mechanics.

All parameters for both the dynamical and statistical equilibrium calculations are determined uniquely by the initial conditions. We refer the reader to Marcus (1990) for a full discussion of the numerical methods which produce the long-time vorticity profiles shown in figures 2 and 5. He uses a spectral collocation method, and claims that the energy, circulations and angular momentum are conserved to within a negligible fraction of the numerical error of our own calculations. We are grateful to Changhoon Lee for providing us with the simulation results shown in figures 1, 2, 4, and 5. Figures 1 and 4 show the potential vorticity profiles for two distinct initial conditions. Figures 2 and 5 show respective long-time stationary solutions obtained by dynamical simulation from these initial conditions. Figures 3 and 6 show the results of the Monte Carlo calculations described above. We draw your attention to the fact that, in these statistical equilibrium calculations, *there are no adjustable parameters*. All conserved quantities are determined by the initial conditions.

## FIGURE CAPTIONS

Fig. 1 Comparison of a direct solution (smooth curve) of the differential equation (5-65) to a Monte Carlo calculation (jagged curve) for the same values of the conserved quantities. The domain is the unit disc with rigid boundaries.

$$g(\sigma) = 0.265\delta(\sigma - 1) + 0.735\delta(\sigma)$$

energy: $0.0108\pi$; angular momentum: $0.0988\pi$

Fig. 2 Simulation I: initial conditions. The grey levels denote evenly spaced potential vorticity values ranging from $-1.0$ (lightest) to 1.0 (darkest).

$$\vec{u} = \hat{u} + \vec{u}_p; \qquad \hat{u} = (0, \beta r^2/3); \qquad \beta = 1; \qquad \Gamma_i = 0$$
$$\omega_p = \nabla \times \vec{u}_p = (1/2)\{\tanh 10(r - 11/15) - \tanh 10(r - 14/15)\}$$
$$R_o = 4/3; \qquad R_i = 1/3$$

Fig. 3 Simulation I: long-time vorticity profile from dynamical simulation.

Fig. 4 Simulation I: statistical equilibrium profile from Monte Carlo calculation.

Fig. 5 Simulation II: initial conditions. The grey levels denote evenly spaced potential vorticity values, extending from $-1.0$ (lightest) to 1.0 (darkest).

$$\vec{u} = \hat{u} + \vec{u}_p; \qquad \hat{u} = (0, \beta r^2/3); \qquad \beta = 1; \qquad \Gamma_i = 0$$
$$\omega_p = \nabla \times \vec{u}_p = f(r, \theta, \theta_0) - f(r, \theta, \theta_0 + \pi)$$
$$f(r, \theta, \hat{\theta}) = (1/2)\{\tanh 0.857\left(\eta(r, \theta, \hat{\theta}) + 1\right) - \tanh 0.857\left(\eta(r, \theta, \hat{\theta}) - 1\right)\}$$
$$\eta(r, \theta, \hat{\theta}) = \left[\frac{(5/6)(\theta - \hat{\theta})}{0.35}\right]^2 + \left[\frac{r - 5/6}{0.175}\right]^2$$
$$R_o = 4/3; \qquad R_i = 1/3$$

Fig. 6 Simulation II: long-time vorticity profile from dynamical simulation.

Fig. 7 Simulation II: statistical equilibrium profile from Monte Carlo calculation.

**Figure 1**

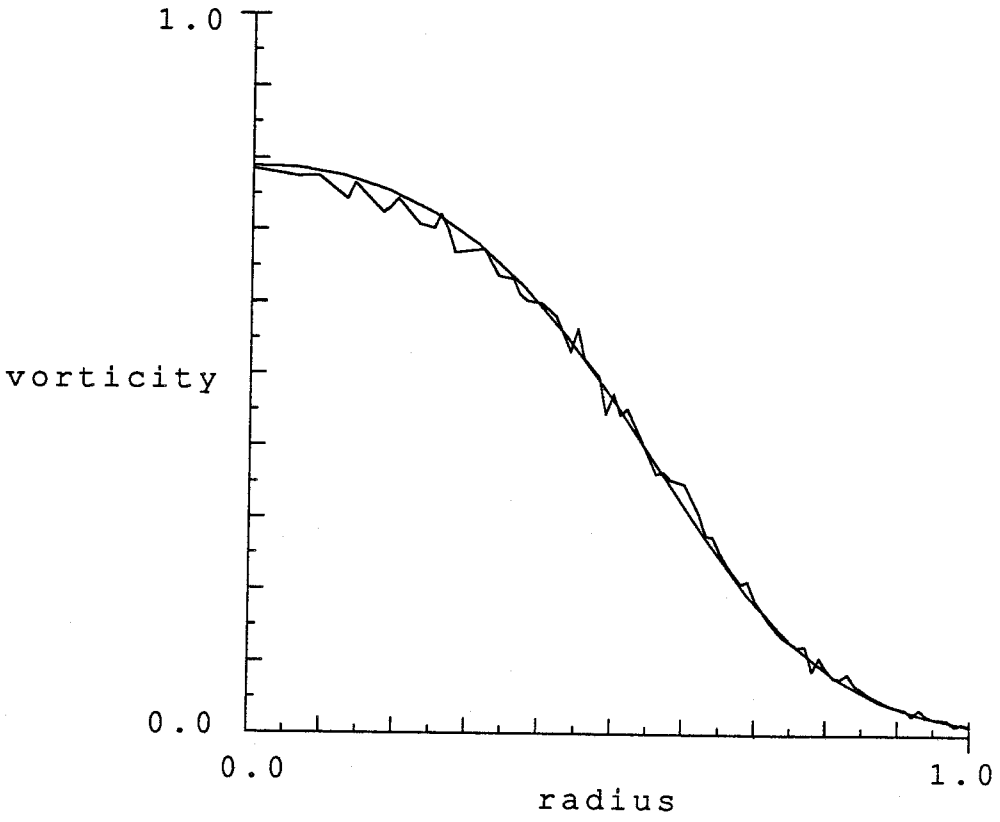

**Figure 2**

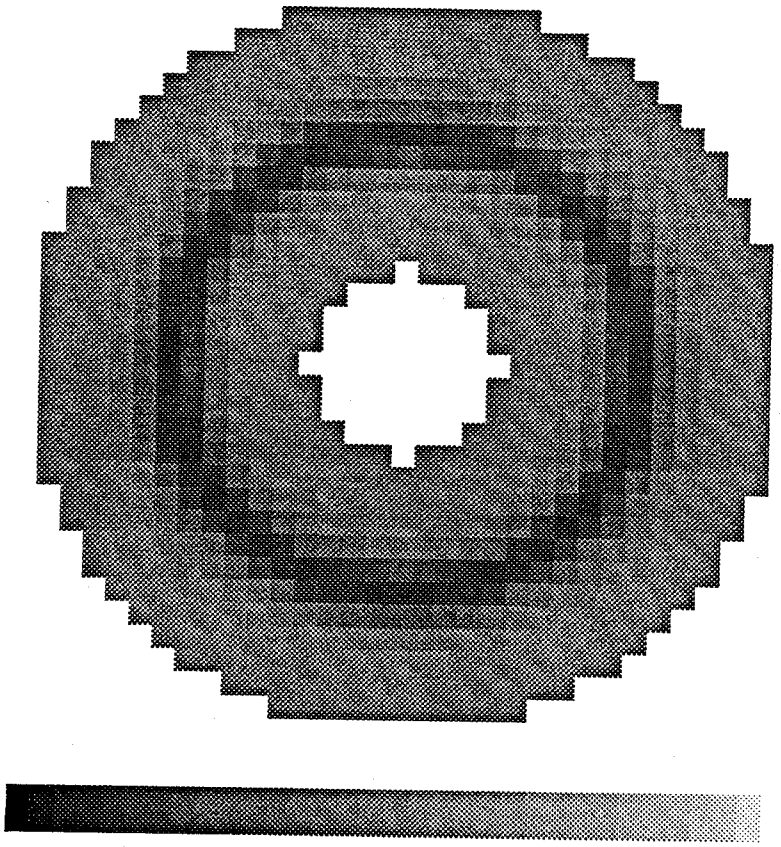

**Figure 3**

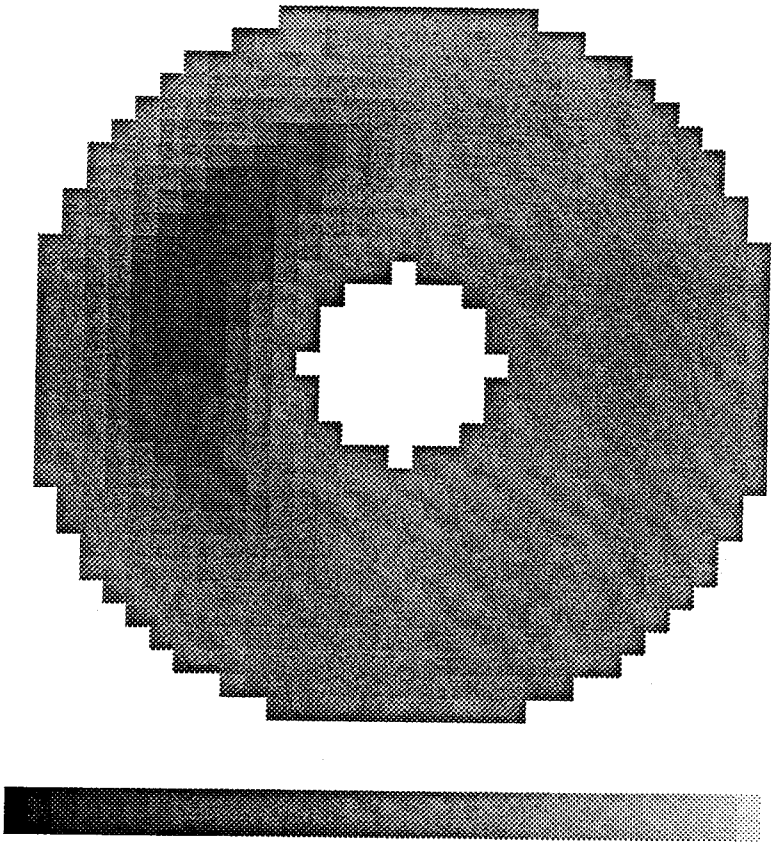

**Figure 4**

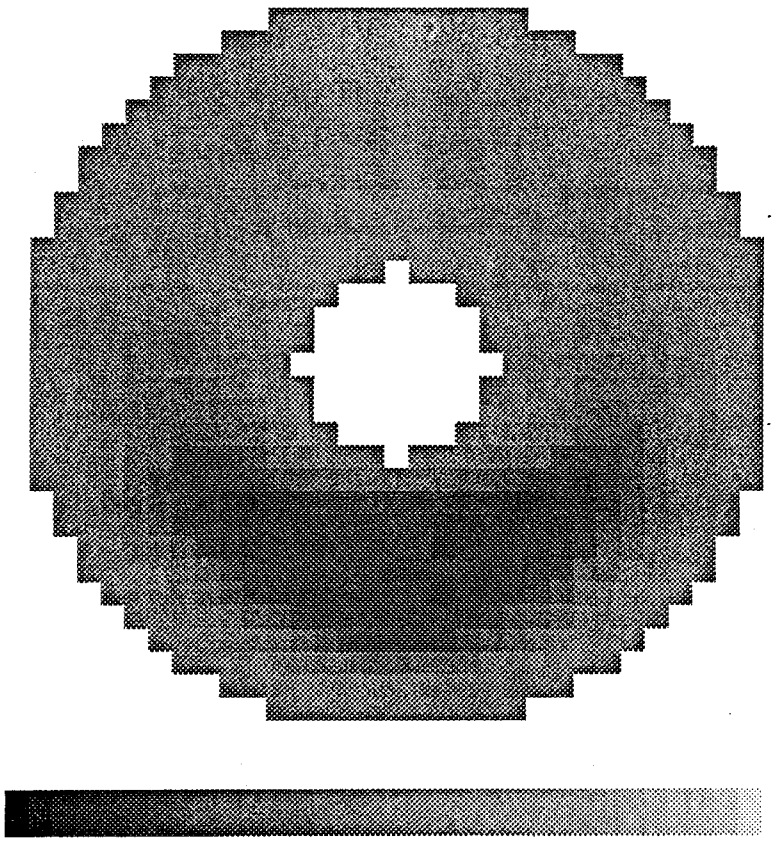

**Figure 5**

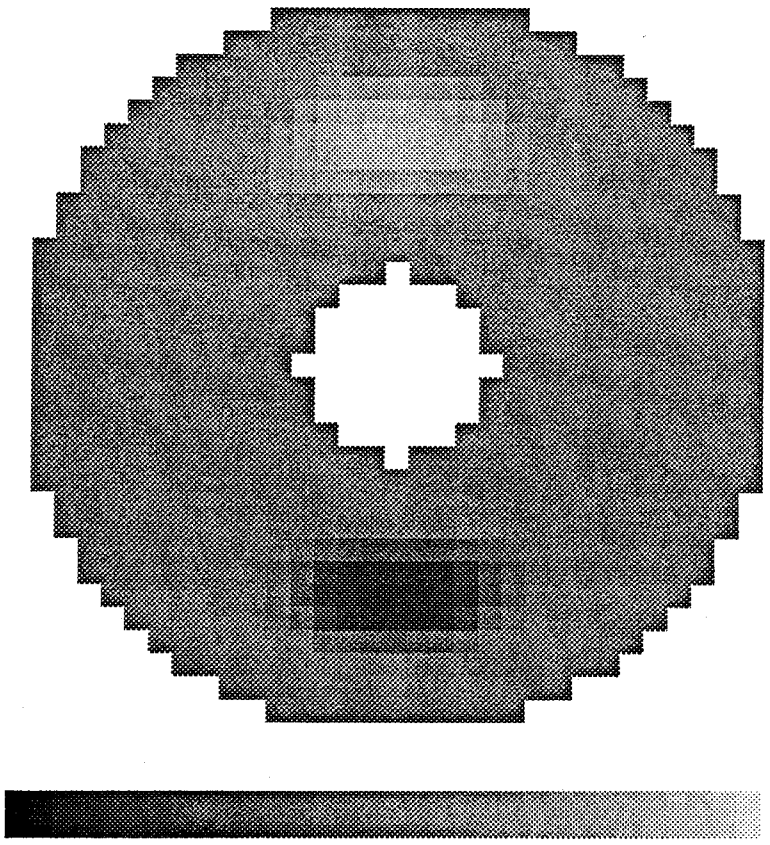

**Figure 6**

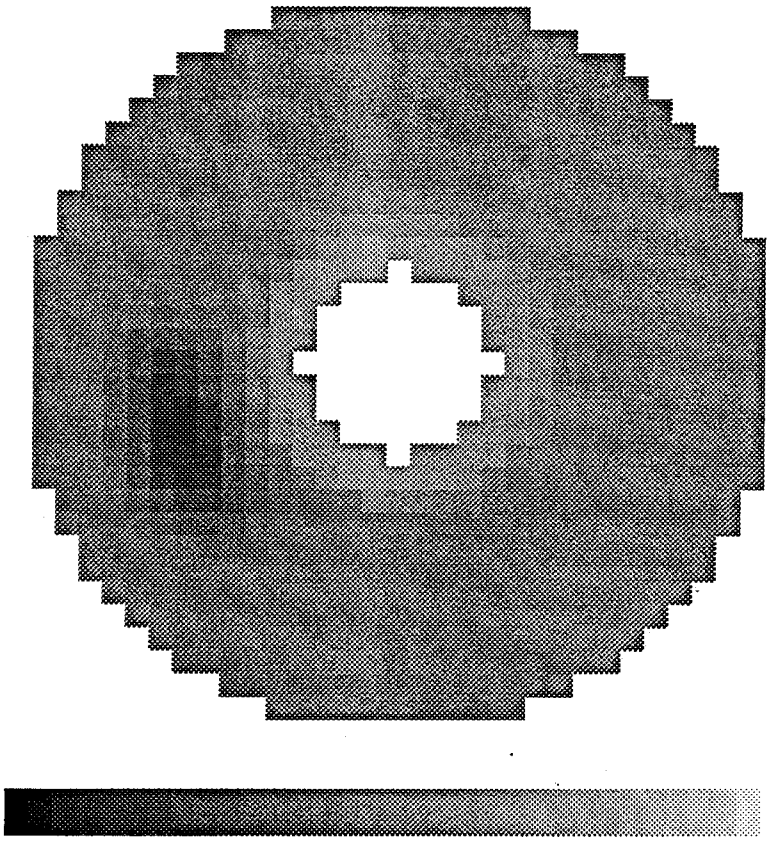

**Figure 7**