ARITHMETICAL PROPERTIES OF COMBINATORIAL MATRICES

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Abstract

Results are derived on rational solutions to $AA^T = B$, where B is integral and A need not be square. It is shown that in general, provided a rational solution exists, one can be found in which all denominators are a power of two. More general restrictions follow from the corresponding restrictions possible on rational lattices representing integral positive definite quadratic forms of determinant one. Results due to Kneser and others are applied to show that A may be taken as integral if it has no more than seven columns, half-integral if it has no more than sixteen columns.

These results are then applied to three types of matrix completion problems, integral matrices satisfying $AA^T = mI$, partial Hadamard matrices and partial incidence matrices of symmetric block designs. It is found that rational normal completing matrices in which all denominators are powers of two are always possible in the first two cases and almost always possible in the final case.

Using a computer approach, the specific problem of showing that the last seven rows of a partial Hadamard matrix or a partial incidence matrix (with suitable parameters) can always be completed is tackled and it is shown that this is in fact the case, extending results by Marshall Hall for no more than four rows. An appendix lists the computer tabulation which is the basis of this conclusion.

Introduction

A basic--and difficult--problem in combinatorial theory is the completion problem: given an initial set of blocks in a block design, when can additional blocks be added to form a complete design?

The tack to be taken here is a matrix theoretic one. We will form a partial incidence matrix X from the blocks originally specified and complete it, add additional rows of rational elements, to form a normal square matrix A satisfying the requisite incidence equation. We will then attempt to apply rational orthogonal transformations to the added rows to force the added elements to be integral while still preserving the original incidence equations. A similar procedure is to be followed to complete a partial Hadamard matrix or, more generally, any (n-r) by n integral matrix X satisfying $XX^T = mI$ for suitable m.

We begin, in Section 1, by citing results due to Hall, Ryser and others showing when rational solutions of the equations to be considered are possible at all. It will be noted that provided such rational solutions exist, completions are possible under very general circumstances for arbitrary X.

In Section 2, the major results of this thesis are derived. It is found that orthogonal transformations may be applied so as to force the added elements to have denominators which are powers of two. It is further shown that the size of these denominators is dependent on the number of added rows and in fact, if the number of added rows is no more than seven, the added rows can be made integral, if no more than sixteen, half-integral.

In Section 3, we apply these results to general integral matrices X satisfying $XX^T = mI$. Some examples and consequences of these results are described, along with possible lines of further investigation.

In Section 4, we specialize to the case of Hadamard matrices, square n by n matrices with all elements either +1 or -1 and satisfying the equation $HH^T = nI$. The results of Section 2 apply so far as rational completions are concerned; for (1,-1) completions when no more than seven rows are to be added, we turn to the computer and find that indeed, such completions are always possible. A counterexample is cited to show that it is not always possible to add eight (1,-1) rows.

In Section 5, we examine the case of incidence matrices. X is a (0,1) matrix satisfying $XX^T = (k-\lambda)I_{v-r} + \lambda J_{v-r}$. We wish to add r rows of zeros and ones to form a full incidence matrix A satisfying $AA^T = (k-\lambda)I_v + \lambda J_v$. Again, this is not always possible. Nor are rational completions in which all denominators are a power of two, since an additional restriction must be placed on the orthogonal transformation involved to preserve the normality of A. Cases in which this can occur are shown to be highly restricted, however. In the case where no more than seven rows (blocks) are to be added, we are able to show by recourse to the computer results of Section 4 that completions to full incidence matrices are always possible.

An appendix contains a computer generated list of forms and representing matrices used to show that six row completions of Hadamard and incidence matrices are always possible.

We note that the seven row completion results are extensions of similar results by Marshall Hall, who earlier proved that two row completions of integral matrices satisfying $XX^T = mI$, four row completions of partial incidence matrices and four row completions of partial incidence matrices were possible in all cases.

Sections 1, 2 and 5 of this thesis are substantially taken from [1b] and Section 4 is substantially taken from [1a]. Section 3 is an amalgam of results from both papers, with proofs based on the newer results of [1b].

1. Existence Conditions for Rational Completions

The basic theorem for rational completions is the following theorem from the Hall-Ryser paper [2]:

Theorem 1.1. (Hall, Ryser). Suppose that $AA^T = D_1 \oplus D_2$. Here the matrix A is of order n and nonsingular. The matrix D_1 is of order r and D_2 is of order s, where r + s = n. Let X be an arbitrary r by n matrix such that $XX^T = D_1$. Then there exists an n by n matrix Z having X as its first r rows such that $ZZ^T = D_1 \oplus D_2$. This result holds for all fields of characteristic not 2.

For the purposes of this paper, the field in question will be the rationals and A will satisfy one of the two equations $AA^T = mI$ or $AA^T = (k-\lambda)I + \lambda J$ (J is the matrix of all ones).

Existence conditions for rational solutions to $AA^T = mI$ are described by Hall in [3]:

Theorem 1.2. There exists an integral square matrix A of order n such that $AA^T = mI$ if and only if:

i) For n odd, m is a perfect square;

ii) For $n \equiv 2 \pmod 4$, m is the sum of two squares;

iii) For $n \equiv 0 \pmod 4$, m is any positive integer.

The necessary condition for a rational solution is in fact sufficient for an integral solution.

Existence conditions for rational solutions to the incidence equation of a symmetric block design are given in [4]:

Theorem 1.3 (Bruck, Ryser, Chowla). Suppose v, k, λ satisfy $k(k-1) = \lambda(v-1)$. Necessary and sufficient conditions for the existence of a rational v by v matrix A satisfying $AA^T = (k-\lambda)I_v + \lambda J_v$ are:

i) For v even, $(k-\lambda)$ is a perfect square;

ii) For v odd, $z^2 = (k-\lambda)x^2 + (-1)^{(v-1)/2}\lambda y^2$ has a solution in integers x, y, z not all zero.

Conditions for rational completions of appropriate X now follow:

Theorem 1.4. Suppose m, n, satisfy Theorem 1.2 and X is an (n-r) by n matrix satisfying $XX^T = mI_{n-r}$. Then there exists an n by n rational matrix A, with X as its first (n-r) rows and satisfying $AA^T = mI_n$.

Proof: Follows directly from Theorem 1.1.

For the case of rational normal completions of incidence matrices, we again quote the Hall-Ryser paper [2]:

Theorem 1.5 (Hall, Ryser). Suppose that the matrix $B = (k-\lambda)I_v + \lambda J_v$ is rationally congruent to the identity (as established by Theorem 1.3). Let X be a (v-r) by v (0,1) matrix satisfying $XX^T = (k-\lambda)I_{v-r} + \lambda J_{v-r}$ and $XJ_v = kJ_{v-r,v}$. Then there is a rational matrix A having X as its first (v-r) rows and satisfying $AA^T = A^TA = B$, $AJ_v = J_vA = kJ_v$.

2. General Results

In this section, we will be concerned with the matrix equation $YY^T = B$, where Y is rational of size n by r and B is integral of size n by n. In our applications, Y^T will consist of the r rows added to an integral (n-r) by n matrix X to form a normal matrix A satisfying $AA^T = (k-\lambda)I + \lambda J$ or $AA^T = mI$, for suitable values of the parameters. Since A is normal and AA^T and X are both integral, so is B.

We will apply rational transformations to Y of the form $W = YU$, where U is rational and has determinant one. If U is further stipulated to be orthogonal, then we note that $WW^T = YUU^TY^T = YY^T = B$, so the matrix equation is preserved. If we now replace Y by W in A, forming the matrix $\bar{A}$, then $\bar{A}^T\bar{A} = A^TA$. If originally $AA^T = mI$, normality of $\bar{A}$ is now assured; if $AA^T = (k-\lambda)I + \lambda J$, an additional restriction on U is required.

We would like to be able to choose U so that not only does $\bar{A}$ satisfy the same normal matrix equation as A, but so that $\bar{A}$ is entirely integral as well. If this can be done, then in the case of incidence matrices, it is possible to show that the new matrix is in fact (0,1) as required. In the case of Hadamard matrices, it is possible for the added integral elements to be other than +1 or -1, in fact it is possible for there to be an integral completion, but no (1,-1) completion at all.

In any event, we will see that, aside from small values of r, it is not always possible to make W integral. We will show that it is possible to choose an orthogonal transformation to force all denominators of W to be powers of two, the size of which is dependent on r (but note Theorem 5.1, which indicates that it is not always possible to force the denominators to be powers of two when the additional restriction necessary for a normal incidence matrix completion is imposed).

The maximum size of the exponent of two required for a fixed r arises from the denominators of rational lattices representing integral positive definite quadratic forms of determinant one and order r. In particular, we may take a zero exponent if $r \leq 7$ (i.e., W is integral) and one if $r \leq 16$ (i.e., W is half-integral). A list of quadratic forms and matrices representing them is included at the end of the section.

Theorem 2.1. Let Y be a rational n by r matrix where YY^T is integral. Then there is a rational orthogonal matrix U of order r so that 2^eYU is integral for sufficiently large e, i.e., all denominators of YU are a power of two. If then $W = YU$, we have $WW^T = YY^T$.

Proof: Choose $s_o = 2^a s$ (s odd), the smallest integer so that s_oY is integral. Proof by induction on s.

Let p be an odd prime factor of s and let $Y_1 = s_oY$. Then $Y_1Y_1^T \equiv 0 \pmod{p^2}$. Choose $v_1,\ldots,v_k$, row vectors from Y_1 which form a basis of the row space of Y_1 over Z_p. The dual of $\langle v_1,\ldots,v_k\rangle$ is of rank $(r-k)$, but $v_1,\ldots,v_k$ are self-dual over Z_p. Hence, there are $(r-2k)$ further vectors $v_{k+1},\ldots,v_{r-k}$ which together with $v_1,\ldots,v_k$ form a basis of the dual of $\langle v_1,\ldots,v_k\rangle$. Finally, there are k further vectors v_{r-k+1}, $\ldots,v_r$ which complete a basis of Z_p^r.

Since $v_1,\ldots,v_r$ are independent over Z_p, rational representatives will be independent over Q. Form the matrix V by taking the rows of V as the vectors $v_1,\ldots,v_r$. The independence of the rows of V over Z_p insures that p does not divide det V. We have the following equations:

2.1) i) $(v_i,v_j) \equiv 0 \pmod{p^2}$ for $1 \leq i \leq j \leq k$;

ii) $(v_i,v_j) \equiv 0 \pmod{p}$ for $1 \leq i \leq k$, $k < j \leq r-k$.

Let $P = I_k \oplus pI_{r-2k} \oplus p^2I_k$ and consider $B = PV$. It follows immediately that p^r divides det B, but p^{r+1} does not. Furthermore, i) and ii) imply that $BB^T \equiv 0 \pmod{p^2}$, as is easily verified.

Let y be a row of Y_1. Since $v_1,\ldots v_k$ is a basis of the row space of Y_1 over Z_p, we may write $y = \sum_{i=1}^{k} a_iv_i + py_1$, where the a_i are integers and y_1 is another integral row vector. Since each v_i, $i = 1,\ldots,k$, is a row vector from Y_1 and $Y_1Y_1^T \equiv 0 \pmod{p^2}$, we have $(y,v_j) = \sum_{i=1}^{k} a_i (v_i,v_j) + p\,(y_1,v_j) \equiv 0 \pmod{p^2}$, $j = 1,\ldots,k$. But $(v_i,v_j) \equiv 0 \pmod{p^2}$, $i,j = 1,\ldots,k$. Hence $(y_1,v_j) \equiv 0 \pmod{p}$, $j = 1,\ldots,k$ or y_1 is in the dual of $\langle v_1,\ldots,v_k\rangle$. From this we obtain $BY_1^T \equiv 0 \pmod{p^2}$.

Let $q \neq p$ be a prime dividing det B. There are integers $d_1, \ldots, d_r$, with g.c.d. $(d_1, \ldots, d_r) = 1$ so that q divides $\sum_{i=1}^{r} d_i b_i$, where the b_i are the rows of B. It is well known that given these conditions, there is an integral matrix D of determinant one with $(d_1, \ldots, d_r)$ as its first row. Let $B_1 = DB$. Then $B_1 Y_1^T = DBY_1^T \equiv 0 \pmod{p^2}$ and $B_1 B_1^T = DBB^T D^T \equiv 0 \pmod{p^2}$. The first row of B_1 has all entries divisible by q. Divide out the factor q from the first row and consider the resulting matrix B_2. The determinant has been divided by a corresponding factor, however $B_2 Y_1^T \equiv 0 \pmod{p^2}$ and $B_2 B_2^T \equiv 0 \pmod{p^2}$ as before. Continue inductively until the determinant is a power of p. Renaming the new matrix C, we see that in fact $|\det C| = p^r$, by what was established earlier.

Consider the integral matrix $E = p^{-2} CC^T$. E is positive definite and has determinant one. Such a form can be represented by a lattice K, satisfying the condition that $I = J_1, \ldots, J_t = K$ is a chain of lattices on Q^r with J_{i+1} adjacent to J_i, I denoting the identity lattice [5]. This means that there is a rational matrix F so that $FF^T = E$ and all denominators of the entries of F are powers of two. Then $F^T E^{-1} E E^{-1} F = I_r$ and if $C_1 = p^{-1} F^T E^{-1} C$, C_1 is orthogonal and $2^b p C_1$ is integral for b sufficiently large. Now we know that $2^b C_1 Y_1^T$ is integral and in fact $2^b C_1 Y_1^T \equiv 0 \pmod{p}$, given the properties we have established for C.

So consider $Y_2 = YC_1 = 2^{-a} s^{-1} Y_1 C_1^T$. Then $2^{a+b} s_1 Y_2$ is integral where $s_1 = s/p < s$. Inductively, we may apply further orthogonal transformations $C_2, \ldots, C_u$ until $s = 1$. Taking $U = C_1^T \cdots C_u^T$, we see that U is orthogonal and $2^e YU$ is integral for e sufficiently large.

<u>Theorem</u> 2.2. Let Y be a rational n by r matrix where YY^T is integral. Then there is a rational matrix Z with det Z = 1, ZZ^T integral and ZY^T integral.

Proof: By Theorem 2.1, we may choose a rational orthogonal U so that $2^e U^T Y^T$ is integral for e sufficiently large. In fact, choose a minimal e satisfying this condition and let $Y_1 = 2^e YU$. Evidently, $Y_1 Y_1^T \equiv 0 \pmod{4^e}$. We now choose a basis of the integral lattice generated by the rows of Y_1 as follows:

2.2) $B = \{ b_1=v_1,\ldots,b_{k_1}=v_{k_1}, b_{k_1+1}=2v_{k_1+1},\ldots,b_{k_2}=2v_{k_2},\ldots,$
$b_{k_{e-1}+1}=2^{e-1}v_{k_{e-1}+1},\ldots,b_{k_e}=2^{e-1}v_{k_e}, b_{k_e+1}=2^e v_{k_e+1},\ldots\}$

where the v_i are all nonzero (mod 2) and independent over Z_2. Since $Y_1Y_1^T \equiv 0 \pmod{4^e}$, the same holds for the inner product of any two b_i and it follows from this that $(v_i,v_j) \equiv 0 \pmod 4$, $i,j = 1,\ldots,k_e$. The dual of $\langle v_1,\ldots,v_{k_e} \rangle$ over Z_2 is of rank $(r-k_e)$ and contains $v_1,\ldots,v_{k_e}$. We may choose additional vectors $y_{k_e+1},\ldots,y_{r-k_e}$ to complete the dual of $\langle v_1,\ldots,v_{k_e} \rangle$ over Z_2. By adding suitable even multiples of vectors annihilating each of the v_i, $i = 1,\ldots,k_e$, except for a specified v_j, it can be seen that we may in fact stipulate that representative y_i be chosen so that $(v_i,y_j) \equiv 0 \pmod{2^e}$, $i = 1,\ldots, k_e$, $j = k_e+1,\ldots,r-k_e$. Then if we set $z_i = 2^e y_i$, $i = k_e+1,\ldots,r-k_e$, it follows that $(b_i,z_j) \equiv 0 \pmod{4^e}$, $i = 1,\ldots,k_e$, $j = k_e+1,\ldots r-k_e$.

Continue now in a similar fashion to obtain $y_{r-k_e+1},\ldots,y_{r-k_{e-1}}$ completing the dual of $\langle v_1,\ldots,v_{k_{e-1}} \rangle$, which already contains $v_1,\ldots,v_{k_e}$ and $y_{k_e+1},\ldots,y_{r-k_e}$. Representatives should be chosen in such a fashion so that in fact $(v_i,y_j) \equiv 0 \pmod{2^{e-1}}$, for $i = 1,\ldots, k_{e-1}$, $j = r-k_e+1,\ldots,r-k_{e-1}$. Then if we again set $z_i = 2^{e+1}y_i$, $i = r-k_e+1,\ldots,r-k_{e-1}$, we have $(b_i,z_j) \equiv 0 \pmod{4^e}$ for $i = 1,\ldots,k_{e-1}$, $j = r-k_e+1,\ldots,r-k_{e-1}$ immediately. In the remaining cases, for $i = k_{e-1}+1,\ldots,k_e$, we need only write $(b_i,z_j) = (2^e v_i, 2^{e+1}y_j) \equiv 0 \pmod{4^e}$.

Eventually, we determine $y_{k_e+1},\ldots,y_r$ and the corresponding $z_{k_e+1},\ldots,z_r$. Form a matrix Z_1, with rows $b_1,\ldots,b_{k_e},z_{k_e+1},\ldots,z_r$. The inner products (z_i,z_j), $i,j = k_e+1,\ldots,r$ are all zero (mod 4^e) as each z_i has been multiplied by a factor of at least 2^e. This plus what was previously shown gives that $Z_1Z_1^T \equiv 0 \pmod{4^e}$.

We note that the way $b_1,\ldots,b_{k_e}$ were chosen shows that they form a basis of the row space of Y_1, modulo 2^e. Each z_j annihilates each b_i,

$i = 1,\ldots,k_e$ modulo 4^e. Since each z_j has been multiplied by a factor of at least 2^e and the same holds true for the remaining $b_{k_e+1},\ldots,b_m$, where m is the rank of Y_1 over the rationals, it follows that in fact $(b_i,z_j) \equiv 0 \pmod{4^e}$, $i = 1,\ldots,m$, $j = k_e+1,\ldots,r$. Thus $Z_1Y_1^T \equiv 0$ (modulo 4^e).

Consider finally det Z_1. Since $v_1,\ldots,v_{k_e},y_{k_e+1},\ldots,y_r$ are independent over Z_2, we need only count the powers of two factored into the rows of Z_1 to determine the largest power of two dividing det Z_1. We have that $v_1,\ldots,v_{k_1}$ have been multiplied by $2^0 = 1$, $y_{r-k_1+1},\ldots,y_r$ have been multiplied by 2^{2e}. The total contribution from these $2k_1$ rows is $(4^e)^{k_1}$. Likewise, $v_{k_1+1},\ldots,v_{k_2}$ and $y_{r-k_2+1},\ldots,y_{r-k_1}$ contribute 2^1 and 2^{2e-1} respectively, for a total from these $2(k_2-k_1)$ rows of $(4^e)^{k_2-k_1}$. Continuing onward, it can be seen that the contribution of $v_1,\ldots,v_{k_e}$ and $y_{r-k_e+1},\ldots,y_r$ is $(4^e)^{k_e}$. The remaining vectors y_{k_e+1}, $\ldots,y_{r-k_e}$ are all multiplied by a factor of 2^e. Hence the largest power of two dividing det Z_1 is $(4^e)^{k_e}\,(2^e)^{r-2k_e} = 2^{er}$.

Now we proceed as in the proof of Theorem 2.1, multiplying by unimodular matrices on the left and dividing by odd prime factors until we obtain a matrix Z_2 with $|\det Z_2| = 2^{er}$ and the other properties preserved; $Z_2Z_2^T \equiv 0 \pmod{4^e}$, $Z_2Y_1^T \equiv 0 \pmod{4^e}$. If we set $Z_3 = 2^{-e}Z_2$, $|\det Z_3| = 1$, $Z_3Z_3^T$ is integral and $Z_3Y_1^T$ is integral with $Z_3Y_1^T \equiv 0$ (modulo 2^e). Finally, set $Z = \pm Z_3U^T$, sign chosen so that det $Z = 1$. Then $ZZ^T = Z_3U^TUZ_3^T = Z_3Z_3^T$, which is integral, $ZY^T = \pm Z_3U^T(2^{-e}UY_1^T)$ $= \pm 2^{-e}Z_3Y_1^T$, which is also integral.

It follows easily from this that a representative rational W may be chosen, satisfying $WW^T = YY^T$ and with the denominators of W no larger than those of a lattice representing the form ZZ^T, as derived above.

<u>Theorem</u> 2.3. Suppose there is a rational n by r matrix Y satisfying $YY^T = B$, where B is integral. Then there is such a Y with denominators dividing s, where s is the least common denominator of the entries of a rational matrix L representing the form ZZ^T, as determined in

Theorem 2.2, up to integral equivalence.

Proof: By Theorem 2.2, we may choose a rational Z so that det Z = 1, $ZZ^T = E$ is integral and ZY^T is integral. Suppose L represents the form $M^TEM = F$, where M is unimodular. Then $LL^T = F$ and $L^TF^{-1}L = I_r$. Since F and F^{-1} are symmetric, integral and unimodular, we may write $L^TF^{-1}FF^{-1}L = (F^{-1}L)^TF(F^{-1}L) = I_r$. Substituting for F, we find that $(MF^{-1}L)^TE(MF^{-1}L) = I_r$ and finally, substituting for E, we have $(Z^TMF^{-1}L)^T(Z^TMF^{-1}L) = I_r$. Evidently, we have that $Z^TMF^{-1}L$ is orthogonal. Since F^{-1}, M^T and ZY^T are all integral, so is $F^{-1}M^TZY^T$. Thus $W^T = L^TF^{-1}M^TZY^T$ has denominators dividing s, the l.c.m. of the denominators of L. But $WW^T = YUU^TY^T = YY^T$, where $U = Z^TMF^{-1}L$ is orthogonal.

Since the form ZZ^T of Theorem 2.2 is always integral, positive definite and of determinant one, it follows that properties of representations W above carry over from the the corresponding properties of these particular forms.

Using Kneser's results of [6], we may for example state the following as corollaries:

Corollary 2.4. Suppose there is a rational n by r matrix Y satisfying $YY^T = B$, where B is integral and $r \leq 7$. Then there is an integral n by r matrix W so that $WW^T = B$.

Proof: Theorem 2.3, noting that any integral positive definite form of determinant one and order less than or equal to seven is integrally equivalent to the identity, thus allowing us to take s = 1.

Corollary 2.5. Suppose there is a rational n by r matrix Y satisfying $YY^T = B$, where B is integral and $r \leq 16$. Then there is a half-integral matrix W (i.e., 2W is integral) so that $WW^T = B$.

Proof: Kneser lists the indecomposable forms of order 16 or less as I_1, K_8, K_{12}, M_{14}, M_{15}, K_{16} and $L_{16,8}$. It suffices by Theorem 2.3 to find a half-integral lattice representing each of these forms, as any other integral positive definite form of determinant one is equivalent to a direct sum of copies of these.

Kneser in fact gives half-integral representations for K_8, K_{12}, K_{16} and $L_{16,8}$ explicitly and it is not difficult to find such representations for M_{14} and M_{15} as well.

As a curious byproduct of this decomposition, it also follows that in the cases r = 9, 10, 11, 13, we may in fact stipulate that 1, 2, 3 or 1 of the columns of W may be taken as integral, respectively. This is because a form of one of these orders must have I_1, I_2, I_3 or I_1 as a direct summand, respectively.

We list the indecomposable forms and their associated lattice bases (in matrix form) below (zero entries are left blank for clarity:

$$K_{4n}:\ \begin{bmatrix} n & 1 & & & & & & \\ 1 & 4 & 2 & & & & & \\ & 2 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & \cdot & 1 \\ & & & & & & 1 & 2 \end{bmatrix} \qquad \text{lattice:}\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ & 2 & & & & & \\ & 1 & -1 & & & & \\ & & -1 & 1 & & & \\ & & & 1 & -1 & & \\ & & & & \cdot & \cdots & \\ & & & & & \cdot\ -1 & \\ & & & & & -1 & 1 \end{bmatrix}$$

$$M_{14}:\ \begin{bmatrix} 2 & 1 & & & & & & & & & & & & \\ 1 & 2 & 1 & & & & & & & & & & & \\ & 1 & 2 & 1 & & & & & & & & & & \\ & & 1 & 2 & 1 & & & & & & & & & \\ & & & 1 & 2 & 1 & & & & & & & & \\ & & & & 1 & 2 & 2 & & & & & & & \\ & & & & & 2 & 7 & 1 & & & & & & \\ & & & & & & 1 & 14 & 4 & & & & & \\ & & & & & & & 4 & 2 & 1 & & & & \\ & & & & & & & & 1 & 2 & 1 & & & \\ & & & & & & & & & 1 & 2 & 1 & & \\ & & & & & & & & & & 1 & 2 & 1 & \\ & & & & & & & & & & & 1 & 2 & 1 \\ & & & & & & & & & & & & 1 & 2 \end{bmatrix}$$

lattice:

$$\frac{1}{2}\begin{bmatrix}
 & -1 & 1 & & -1 & 1 & & & 1 & -1 & & & 1 & -1 \\
1 & -1 & & 1 & -1 & & & -1 & 1 & & & -1 & 1 & \\
 & -1 & -1 & & -1 & -1 & & & 1 & 1 & & & 1 & 1 \\
-1 & -1 & & -1 & -1 & & 1 & & & 1 & 1 & & & 1 \\
 & -1 & 1 & & -1 & 1 & & & -1 & 1 & & & -1 & 1 \\
1 & -1 & & 1 & -1 & & & 1 & -1 & & & 1 & -1 & \\
 & -2 & -2 & 2 & & & & & -2 & -2 & 2 & 2 & & \\
2 & -2 & -2 & -2 & 2 & 2 & 2 & 2 & -2 & -2 & -2 & -2 & 2 & 2 \\
1 & -1 & & -1 & 1 & & & 1 & -1 & & & -1 & 1 & \\
 & -1 & 1 & & 1 & -1 & & & -1 & 1 & & & 1 & -1 \\
-1 & -1 & & 1 & 1 & & 1 & & & 1 & -1 & & & -1 \\
 & -1 & -1 & & 1 & 1 & & & 1 & 1 & & & -1 & -1 \\
1 & -1 & & -1 & 1 & & & -1 & 1 & & & 1 & -1 & \\
 & -1 & 1 & & 1 & -1 & & & 1 & -1 & & & -1 & 1
\end{bmatrix}$$

M_{15}:

$$\begin{bmatrix}
15 & 4 & & & & & & & & & & & & & \\
4 & 2 & 1 & & & & & & & & & & & & \\
 & 1 & 2 & 1 & & & & & & & & & & & \\
 & & 1 & 2 & 1 & & & & & & & & & & \\
 & & & 1 & 2 & 1 & & & & & & & & & \\
 & & & & 1 & 2 & 1 & & & & & & & & \\
 & & & & & 1 & 2 & 1 & & & & & & & \\
 & & & & & & 1 & 2 & 1 & & & & & & \\
 & & & & & & & 1 & 2 & 1 & & & & & \\
 & & & & & & & & 1 & 2 & 1 & & & & \\
 & & & & & & & & & 1 & 2 & 1 & & & \\
 & & & & & & & & & & 1 & 2 & 1 & & \\
 & & & & & & & & & & & 1 & 2 & 1 & \\
 & & & & & & & & & & & & 1 & 2 & 1 \\
 & & & & & & & & & & & & & 1 & 2
\end{bmatrix}$$

lattice:

$$\frac{1}{2}\begin{bmatrix}
2 & -2 & -2 & 2 & 2 & -2 & -2 & 2 & 2 & -2 & -2 & 2 & 2 & -2 & -2 \\
1 & -1 & & & 1 & -1 & & & 1 & -1 & & & 1 & -1 & \\
 & -1 & 1 & & & -1 & 1 & & & -1 & 1 & & & -1 & 1 \\
-1 & -1 & & & -1 & -1 & & 1 & & & 1 & 1 & & & 1 \\
 & -1 & -1 & & & -1 & -1 & & & 1 & 1 & & & 1 & 1 \\
1 & -1 & & & 1 & -1 & & & -1 & 1 & & & -1 & 1 & \\
 & -1 & 1 & & & -1 & 1 & & & 1 & -1 & & & 1 & -1 \\
-1 & -1 & & 1 & & & 1 & -1 & & & -1 & & 1 & 1 & \\
 & -1 & -1 & & & 1 & 1 & & & -1 & -1 & & & 1 & 1 \\
1 & -1 & & & -1 & 1 & & & 1 & -1 & & & -1 & 1 & \\
 & -1 & 1 & & & 1 & -1 & & & -1 & 1 & & & 1 & -1 \\
-1 & -1 & & & 1 & 1 & & 1 & & & 1 & -1 & & & -1 \\
 & -1 & -1 & & & 1 & 1 & & & 1 & 1 & & & -1 & -1 \\
1 & -1 & & & -1 & 1 & & & -1 & 1 & & & 1 & -1 & \\
 & -1 & 1 & & & 1 & -1 & & & 1 & -1 & & & -1 & 1
\end{bmatrix}$$

$$L_{16,8}: \begin{bmatrix}
3 & 1 & & & & & & & 2 & -1 & -1 & & & & & \\
1 & 4 & 2 & & & & & & 1 & & & & & & & \\
 & 2 & 2 & 1 & & & & & & & & & & & & \\
 & & 1 & 2 & 1 & & & & & & & & & & & \\
 & & & 1 & 2 & 1 & & & & & & & & & & \\
 & & & & 1 & 2 & 1 & & & & & & & & & \\
 & & & & & 1 & 2 & 1 & & & & & & & & \\
 & & & & & & 1 & 2 & 1 & & & & & & & \\
2 & 1 & & & & & & 1 & 6 & 4 & 1 & & & & & \\
-1 & & & & & & & & 4 & 8 & 2 & & & & & \\
-1 & & & & & & & & 1 & 2 & 2 & 1 & & & & \\
 & & & & & & & & & & 1 & 2 & 1 & & & \\
 & & & & & & & & & & & 1 & 2 & 1 & & \\
 & & & & & & & & & & & & 1 & 2 & 1 & \\
 & & & & & & & & & & & & & 1 & 2 & 1 \\
 & & & & & & & & & & & & & & 1 & 2
\end{bmatrix}$$

$$\text{lattice: } \tfrac{1}{2}\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & & & & & & & \\
 & 4 & & & & & & & & & & & & & & \\
 & 2 & -2 & & & & & & & & & & & & & \\
 & & -2 & 2 & & & & & & & & & & & & \\
 & & & 2 & -2 & & & & & & & & & & & \\
 & & & & -2 & 2 & & & & & & & & & & \\
 & & & & & 2 & -2 & & & & & & & & & \\
 & & & & & & -2 & 2 & & & & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & & & & 4 & -4 & & & & & & & \\
 & & & & & & & & -2 & 2 & & & & & & \\
 & & & & & & & & & 2 & -2 & & & & & \\
 & & & & & & & & & & -2 & 2 & & & & \\
 & & & & & & & & & & & 2 & -2 & & & \\
 & & & & & & & & & & & & -2 & 2 & & \\
 & & & & & & & & & & & & & 2 & -2 &
\end{bmatrix}$$

3. Orthogonal Completions

As the first application of the results of Section 2, we will concern ourselves with orthogonal completions, i.e., suppose we have an integral (n-r) by n matrix X satisfying $XX^T = mI_{n-r}$. Theorem 1.4 shows that a rational matrix A of order n with X as its first (n-r) rows may be chosen so that $AA^T = mI_n$. What restrictions can be placed on the added rows?

We will write

$$A = \begin{bmatrix} X \\ Y^T \end{bmatrix}$$

where Y is rational and of size n by r. We note that since $YY^T = A^TA - X^TX$ and the latter quantities are integral, so is the matrix YY^T. All results of Section 2 now apply and any replacement of Y by a matrix W satisfying $WW^T = YY^T$ will preserve the matrix equation $AA^T = A^TA = mI_n$. We obtain immediately that W may be taken so that all denominators are a power of two always, with suitable restrctions on the size of the exponent depending on r.

In particular, it is interesting to note that for $n \leq 8$, if we take X as a single row, with the sum of the squares of the elements of X a suitable m satisfying Theorem 1.2, then (n-1) additional integral rows may always be added so that the resultant matrix A satisfies $AA^T = mI_n$. To be even more specific in the case n = 3, this means that if integers x, y, z are given so that $x^2 + y^2 + z^2 = m^2$, then there are two independent solutions x_i, y_i, z_i, i = 1, 2 to the Diophantine equations $xx_i + yy_i + zz_i = 0$, $x_i^2 + y_i^2 + z_i^2 = m^2$.

For n = 10, if we take X as a row of ones (m = n = 10), it is easy to verify that any start of rows can always be completed, following from the fact that all possible second rows allow a third to be added, leaving seven. These matrices X are used in the construction of so-called Type II solutions to the incidence equation for a projective plane of order 10. That conjecture by Marshall Hall, later verified on the computer by Arthur Rubin, led to Hall's original investigations of integral completions and later to the work described here.

<u>Theorem</u> 3.1. Suppose X is an integral (n-r) by n matrix satisfying $XX^T = mI_{n-r}$, where m, n satisfy the hypotheses of Theorem 1.2. Then there is an n by n matrix A, with X as its first (n-r) rows, satisfying $AA^T = mI$ and with 2^eA integral for sufficiently large e.

Proof: By Theorem 1.1, X has a rational completion

$$3.1)\quad A_o = \begin{bmatrix} X \\ Y^T \end{bmatrix}$$

where $A_oA_o^T = mI$. A_o is necessarily normal, hence since X^TX is integral, so is $YY^T = A_o^TA_o - X^TX$. Theorem 2.1 applies and we may choose W so that 2^eW is integral for e sufficiently large and $WW^T = YY^T$.

But then

$$3.2)\quad A = \begin{bmatrix} X \\ W^T \end{bmatrix}$$

satisfies $A^TA = mI = AA^T$ and we are done.

As a corollary, we have immediately that integral completions of X as in Theorem 3.1 are possible whn $r \leq 7$.

<u>Corollary</u> 3.2. Suppose X is as in Theorem 3.1 and $r \leq 7$. Then X has an integral completion to a matrix A satisfying $AA^T = mI$.

Proof: Corollary 2.4 and the same argument used in the proof of Theorem 3.1.

Marshall Hall proved in [3] the same result for $r \leq 2$.

That $r \leq 7$ is best possible may be seen by taking n = 9, m = 9, r = 8 and letting

$$3.3)\quad X = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]$$

We have $(3I_9)(3I_9)^T = 9I_9$ and yet there are no integral row vectors of length 9 which are both orthogonal to X and have sum squares of the entries equal to 9. This follows from the fact that there are no such solutions mod 2.

There is however a half-integral solution:

3.4) $$A = \frac{1}{2}\begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & -3 & -3 & 0 & 0 & 0 & 0 & 0 \\ 3 & -3 & 3 & -3 & 0 & 0 & 0 & 0 & 0 \\ 3 & -3 & -3 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 3 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 & 3 & -3 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 & 3 & -3 & -3 & 3 & 0 \\ 2 & 2 & 2 & 2 & -1 & -1 & -1 & -1 & -4 \\ 1 & 1 & 1 & 1 & -2 & -2 & -2 & -2 & 4 \end{bmatrix}$$

It seems quite likely that somewhat stronger results may be possible. We consider for instance completions of

3.5) $X = [1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 2 \ 2 \ 2 \ 2]$

A half integral completion is the following:

3.6) $$A = \frac{1}{2}\begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 5 & -5 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -5 & 5 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -5 & -5 & 5 \\ 8 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & 1 & 1 & 1 & 1 \\ -2 & 8 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & 1 & 1 & 1 & 1 \\ -2 & -2 & 8 & -2 & -2 & -2 & -2 & -2 & -2 & 1 & 1 & 1 & 1 \\ -2 & -2 & -2 & 8 & -2 & -2 & -2 & -2 & -2 & 1 & 1 & 1 & 1 \\ -2 & -2 & -2 & -2 & 8 & -2 & -2 & -2 & -2 & 1 & 1 & 1 & 1 \\ -2 & -2 & -2 & -2 & -2 & 8 & -2 & -2 & -2 & 1 & 1 & 1 & 1 \\ -2 & -2 & -2 & -2 & -2 & -2 & 8 & -2 & -2 & 1 & 1 & 1 & 1 \\ -2 & -2 & -2 & -2 & -2 & -2 & -2 & 8 & -2 & 1 & 1 & 1 & 1 \\ -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & 8 & 1 & 1 & 1 & 1 \end{bmatrix}$$

It can be verified that no orthogonal tranformation with all denominators a power of two can be applies to the lower twelve rows from the left and produce any further integral rows. Yet X does have an integral completion:

3.7) $$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ -2 & -2 & -1 & -1 & -1 & -1 & -1 & 0 & 3 & 0 & 1 & 1 & 1 \\ 2 & 2 & -1 & -1 & -1 & -1 & -1 & 1 & 2 & 2 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 & 4 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 4 & -1 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 4 & -1 & -1 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & -1 & -1 & -1 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & -1 & -1 & -1 & -1 & -2 & 0 & 1 & 0 & 0 & 0 \\ -2 & 3 & -1 & -1 & -1 & -1 & -1 & 0 & -2 & 0 & 1 & 1 & 1 \\ 3 & -2 & -1 & -1 & -1 & -1 & -1 & 0 & -2 & 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 2 & -2 & -2 & 3 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 2 & -2 & 3 & -2 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 2 & 3 & -2 & -2 \end{bmatrix}$$

The transformation involved is in fact of denominator 10.

4. Hadamard Completions

The problem of completing appropriate (1,-1) matrices to full Hadamard matrices is probably the most interesting subcase of the preceding combinatorially. Consider the following example from [3]:

$$4.1) \qquad A = \left[\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\
\hline
2 & -2 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
2 & 0 & -2 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & -2 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & 2 & -2 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & -1 & 2 & 0 & -2 \\
0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & 2 & -2 \\
0 & 0 & 0 & -1 & 2 & -1 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 2 & -1 & -1 & 2 & 0 & 0 & 0
\end{array}\right]$$

A satisfies $AA^T = A^TA = 12I$. Apparently then, the first four rows do have an integral completion to a matrix A satisfying $AA^T = 12I$. But they do not have a Hadamard (1,-1) completion. This may be verified by noting that the sum of the first four rows is:

$$4.2) \quad S = 4[1 \;\; 1 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]$$

No vector of 1's and -1's can be orthogonal to such a vector, as is necessary if any (1,-1) rows are to be added.

This example illustrates that an integral completion is not sufficient to guarantee a Hadamard completion and further that the completion of the last seven rows of a (1,-1) Hadamard start is the best that might be expected in general. Hall in [3] was able to show that if no more than four rows remain, a full Hadamard completion is always possible. Here, it will be shown that in fact the last seven rows can be so completed. We will need three lemmas and a largely computational theorem.

<u>Lemma</u> 4.1. Let X be an (n-r) by n matrix, $n \equiv 0 \pmod 4$ with every entry +1 or -1 and satisfying $XX^T = nI_{n-r}$. Suppose Y is a rational completion of X (always possible by Theorems 1.1 and 1.2), i.e.,

$$4.3) \quad A_o = \begin{bmatrix} X \\ Y \end{bmatrix}$$

satisfies $A_oA_o^T = nI_n$. Then the entries of Y^TY are integers of the same parity as r and absolute value no larger than r.

Proof: Since $A_o^T A_o$ is integral and $X^T X$ is integral, then also $Y^T Y = A_o^T A_o - X^T X$ is integral. Furthermore, n is even by assumption, so $A_o^T A_o \equiv 0 \pmod 2$ and since all the entries of X are +1 or -1, $X^T X \equiv (n-r)J \pmod 2$. So $Y^T Y \equiv rJ \pmod 2$. Since the main diagonal entries of $A_o^T A_o$ are n and the main diagonal entries of $X^T X$ are (n-r), the main diagonal entries of $Y^T Y$ are r. Applying the Schwarz inequality to the rows of Y^T shows that the absolute value of any inner product is no larger than r.

Lemma 4.2. Call $B = (b_{ij}) = Y^T Y$, where Y is as in Lemma 4.1. Then for arbitrary i, j, k, $b_{ij} + b_{ik} + b_{jk} \equiv -r \pmod 4$.

Proof: Let x^T, y^T and z^T be three columns from X. Then $(x,x) = (y,y) = (z,z) = (n-r)$. Let $w = x+y+z$. Then since x, y, z have all odd entries, so does w. Then $(w,w) \equiv (n-r) \pmod 8$ as any odd square is congruent to 1 (mod 8). Then $(x,y) + (x,z) + (y,z) = \frac{1}{2}[(w,w) - (x,x) - (y,y) - (z,z)] \equiv -(n-r) \equiv r \pmod 4$, as $n \equiv 0 \pmod 4$. Thus if $C = (c_{ij}) = X^T X$, we have $c_{ij} + c_{ik} + c_{jk} \equiv r \pmod 4$ for any i, j, k. Hence $B \equiv -C \pmod 4$ satisfies the condition stated.

Lemma 4.3. Suppose that one of the off-diagonal entries of $Y^T Y$ is (r-2) or -(r-2). Then there is an orthogonal transformation U so that UY has a row composed entirely of entries +1 or -1.

Proof: We may suppose that the first and second columns of Y have the specified inner product. If the inner product is negative, negate one of the columns, so that we may suppose that the inner product is in fact (r-2). The same orthogonal transformation will suffice. Call the two columns x^T, y^T. Then we have

$$4.4) \quad W = \begin{bmatrix} x \\ y \end{bmatrix}, \quad WW^T = \begin{bmatrix} r & r-2 \\ r-2 & r \end{bmatrix}$$

so that WW^T is nonsingular. Let $D_1 = WW^T$. Adding further rows to W, orthogonal to x and y, it is easy to see that we may obtain a nonsingular rational r by r matrix W' with $W'(W')^T = D_1 \oplus D_2$, some D_2. Now let

$$4.5)\quad Z = \begin{bmatrix} 1 & \cdots & 1 & 1 \\ 1 & \cdots & 1 & -1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

where Z is 2 by r. Then $ZZ^T = D_1 = WW^T$. All the hypotheses of Theorem 1.1 are now satisfied and there is a rational orthogonal U so that $WU = Z$ and hece $U^TW^T = Z^T$. Let $Y_1 = U^TY$. Then the first two columns of Y_1 are the rows of Z and $Y_1Y_1^T = U^TYY^TU = nI_r$ and also $U^TYX^T = 0_r$. Hence if we take

$$4.6)\quad A_1 = \begin{bmatrix} X \\ Y_1 \end{bmatrix}.$$

then $A_1A_1^T = nI_n$ as before. Now let z^T be any other column of Y_1 and let s be its last entry. Evidently, $(a,z) - (b,z) = 2s$. Lemma 4.2 evidently still holds if we replace b by its negative and we obtain

$$4.7)\quad (a,z) - (b,z) - (a,b) = 2s - (r-2) \equiv -r \pmod 4$$

So s is integral and $2s \equiv 2 \pmod 4$ or s is odd. Evidently the last row of Y_1 consists of n odd integers, the sum of whose squares is n. This forces all entries of the last row to be +1 or -1 as desired.

We will now deal with the needed computational result:

Theorem 4.4. Suppose Y is a rational r by n matrix with $r \leq 7$, rank Y = r and $Y^TY = C$ satisfying the following properties:

i) C is integral and $c_{ii} = r$, all i;

ii) $C \equiv rJ \pmod 2$;

iii) $c_{ij} + c_{ik} + c_{jk} \equiv -r \pmod 4$, all i, j, k.

Then there is a (1,-1) matrix $\overline{Y}$ of size r by n so that $\overline{Y}^T\overline{Y} = C$.

Proof: By induction on r. We note that Lemma 4.3 demonstrates that whenever C, as above, has an off-diagonal element of magnitude (r-2), we may apply an orthogonal tranformation to produce a row of 1's and -1's in Y. Deleting this row to produce a new (r-1) by n matrix Y_1, we see that if $C_1 = Y_1^TY_1$, properties (i) and (ii) are satisfied (for $r_1 = (r-1)$) immediately. If x, y, z are all (1,-1), then as in Lemma 4.2, $xy + xz + yz \equiv 3 \pmod 4$; subtracting this from (iii) above shows that (iii) holds for C_1 and r_1 as well. We have thus reduced the problem to the previous case and we may assume, henceforth, that C has no off-diagonal elements of magnitude $(r-2)$.

Case 1. $r = 1, 2, 3, 4$.

Let Y, C be as above. We may assume that C has no elements of magnitude $(r-2)$. (i), (ii) and the Schwarz inequality imply that in fact all elements of C are in fact r, $-r$ or (r even only) zero.

If $r = 1$, Y is already $(1,-1)$ and we are done.

If $r = 2$, choose two independent columns. Their inner product must be zero ($= (r-2)$) and we are done.

If $r = 3$, the inner product of two independent culumns must be 1 or -1 and again we are done.

If $r = 4$, permute columns so that the first four are independent. We may assume that inner products between distinct columns are zero. Apply an orthogonal transformation so that the first four columns are

$$4.8) \quad H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Further columns must have a non-zero inner product with one of these, we may in fact assume that it is 4 or -4. But then the further column must be a duplicate (or the negative of one) of one of the first four columns. So Y is now $(1,-1)$ and we are done.

Case 2. $r = 5$.

Let Y, C be as above. The entries of C must be 1, -1, 3, -3, 5 or -5. If any entry is 3 or -3, we may apply an orthogonal transformation to obtain a row of $(1,-1)$ entries and this then reduces to the case $r = 4$. So suppose there are no entries 3, -3 in C. Let W^T be the 5 by 5 matrix consisting of the first five columns of Y (assumed to be independent). Then $F = WW^T$ has all off-diagonal entries +1 or -1. Negate appropriate rows of W so that the first row and column of F (except for $f_{11} = 5$) consists of all +1 entries. Since $-r \equiv 3 \pmod 4$, (iii) now implies that the remaining off-diagonal elements of F are all +1, i.e., $F = 4I_5 + J_5$. Let

4.9) $$Z = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & -1 \end{bmatrix}$$

Then $ZZ^T = F$. Apply an orthogonal transformation to Y to obtain Y' with first five columns Z^T. Let y be another column of Y'. If Zy has an entry 5 or -5, it must be identical to or a negative of a column of Z^T. Otherwise all entries are +1 or -1. (iii) now implies that all entries are identical, so negating if necessary, we may take them all to be +1. We may solve directly for y and find that

4.10) $y^T = (1/3)\,[1 \;\; 1 \;\; 1 \;\; 1 \;\; -1]$

Evidently, $y^Ty = 5/9 \neq 5$, so this case cannot arise. Thus $\overline{Y}$ is entirely (1,-1) and we are done.

<u>Case</u> 3. $r = 6$.

Let Y, C be as above. The entries of $C = Y^TY$ must come from 0, 2, -2, 4, -4, 6, -6. If any entry is 4 or -4, then, as above, we may reduce to the previous case. Let W^T be the 6 by 6 matrix consisting of the first six (independent) columns of Y. Then if $F = WW^T$, all off-diagonal entries are 0, 2 or -2. At this point a certain amount of hand and computer calculation is involved. Up to permutation and negation of columns of W^T, we find 28 possible F satisfying (iii). Of these, 19 fail to have determinant a perfect integral square (necessary if there is to be a rational solution W as above). For the remaining nine, (1,-1) Z satisfying $ZZ^T = F$ are found in all cases. Let y be another column of Y and let $v = Zy$. v must be consistent with (iii). In addition, $y^Ty = 6$ implies $(Z^{-1}v)^T(Z^{-1}v) = v^TF^{-1}v = 6$. Taking these two conditions into account, it is found that for all admissable v, $y = Z^{-1}v$ is (1,-1). So Y is now (1,-1). Tables of F, Z, v, y are found in the appendix.

<u>Case</u> 4. $r = 7$.

Let Y, C be as above. The entries in $C = Y^TY$ must come from 1, -1, 3, -3, 5, -5, 7, -7. If any entry is 5 or -5, then as above, we may reduce to the previous case. Let W^T be the 7 by 7 matrix consisting of the first seven (independent) columns of Y. Then if $F = WW^T$, all off-diagonal entries are 1, -1, 3 or -3. A computer is used to find all possibilities for F, up to permutation and negation of the rows of W, satisfying (iii) and with determinant a perfect integral square. A total of 167 possibilities are found. In all cases, one or more (1,-1) matrices Z are found so that $ZZ^T = F$.

Let y be a further column of Y and suppose that $v = Zy$. Then as in the previous case, we must have $v^T F^{-1} v = r = 7$. If any entry in v is 7 or -7, then y is identical to or a negative of a column of W^T. Let then $y_1, \ldots, y_k$ be the remaining columns of Y, with no entry of $v_i = Zy_i$ equal to 7 or -7, all i. Negating if necessary, we may assume that the first entry of v_i is 1 or -3 in all cases. (iii) shows that if z is the first column of Z^T, then $(z,y_i) + (z,y_j) + (y_i,y_j) \equiv 1 \pmod 4$. But now we have $(z,y_i) \equiv (z,y_j) \equiv 1 \pmod 4$, all i, j. Hence we now have $(y_i,y_j) \equiv -1 \pmod 4$, all i,j.

Let F be fixed. For each (1,-1) Z satisfying $ZZ^T = F$, determine the subset $V_Z = \{ v : v$ compatible with (iii) and $Z^{-1}v$ is (1,-1) $\}$. Let $V_1, \ldots, V_a$ be all such distinct subsets. Suppose then an incompatible subset $V = \{ v_1', \ldots, v_k' \}$ of v_i' individually compatible with (iii) is found, further satisfying $(v_i')^T F^{-1} v_j'$ ($= (y_i', y_j')$) $\equiv -1 \pmod 4$, all i, j. Then clearly such a subset exists with $k \leq a$. Since a is typically rather small, it was possible to use the computer to check through all such subsets, for all F. It was found that no such incompatible subsets exist for any F.

Hence a (1,-1) completion is always possible in the $r = 7$ case as well.

Tables of F, Z, v, y exist in unpublished form.

It is now immediate that Hadamard completions are possible for $r \leq 7$.

<u>Theorem</u> 4.5. Let X be an (n-r) by n matrix, $n \equiv 0 \pmod 4$ with every entry +1 or -1 and satisfying $XX^T = nI_{n-r}$. Then if $r \leq 7$, there is a Hadamard matrix of order n with X as its first (n-r) rows.

Proof: By Theorems 1.1 and 1.2, there is a rational completion Y of X, as in (4.3). By Lemmas 4.1 and 4.2, Y satisfies the hypotheses of Theorem 4.4. Hence, there is an r by n (1,-1) matrix $\overline{Y}$ satisfying $\overline{Y}^T\overline{Y} = Y^TY$. But now

$$4.11)\quad A_1 = \begin{bmatrix} X \\ \overline{Y} \end{bmatrix}$$

satisfies $A_1^TA_1 = A_o^TA_o = nI = A_1A_1^T$. So A_1 is a full Hadamard matrix with X as its first (n-r) rows.

As noted earlier, Hall proved the same result in [3] for $r \leq 4$.

5. Symmetric Block Designs

The problem of interest here is the rational and, if possible, (0,1) completion of a partial incidence matrix of a symmetric block design.

This is similar to the general representation problem described in Section 2, but there are additional restrictions. Suppose X is a (v-r) by v (0,1) matrix satisfying $XX^T = (k-\lambda)I_{v-r} + \lambda J_{v-r}$, where v, k, λ satisfy the conditions of Theorem 1.3. Then by Theorem 1.5, we may determine a rational matrix

$$A = \begin{bmatrix} X \\ Y^T \end{bmatrix}$$

satisfying $AA^T = A^TA = B = (k-\lambda)I_v + \lambda J_v$. The theorems of Section 2 apply to Y and we may determine a rational orthogonal U so that W = YU satisfies the various properties indicated. To preserve the normality of A, we must further stipulate that $U^TJ_r = J_r$. If A is to be the incidence matrix of a design, we want additionally that Y be (0,1). In this regard, a theorem of Ryser [11] applies, which states that a normal integral matrix satisfying $AA^T = B$, as above, is either the incidence matrix of a design or the negative of one. Hence it suffices to force the entries of W to be integral by a transformation which preserves the normality of A. Here, we will be able to use previous work on Hadamard matrix completions to show that X may be completed to an incidence matrix whenever $r \leq 7$, extending a result by Hall for $r \leq 4$.

Theorem 5.1. Suppose that X is a (v-r) by v (0,1) matrix satisfying $XX^T = (k-\lambda)I_{v-r} + \lambda J_{v-r}$, where v, k, λ satisfy Theorem 1.3. Then there exists a rational normal matrix A, with X as its first (v-r) rows and further satisfying 2^eA is integral for e sufficiently large, except possibly when the following conditions hold:

Let $x_1,\dots,x_v$ denote the column sums of X. For some odd prime p, we must have:

i) p^2 divides r;

ii) $x_i \equiv k \pmod p$, i = 1,...,v.

Proof: By Theorem 1.5, we may choose a rational Y of size v by r so that

5.1) $A = \begin{bmatrix} X \\ Y^T \end{bmatrix}$

satisfies $AA^T = A^TA = B = (k-\lambda)I_v + \lambda J_v$. Append a column of ones, forming the matrix

5.2) $A_o = \begin{bmatrix} X_o \\ Y_o^T \end{bmatrix}$, $X_o = \begin{bmatrix} 1 & \\ \vdots & X \\ 1 & \end{bmatrix}$, $Y_o^T = \begin{bmatrix} 1 & \\ \vdots & Y^T \\ 1 & \end{bmatrix}$

As shown in [2], we must have $AJ_v = J_vA = kA$ and so all inner products between columns of A_o must be integral. The same is clearly true for the columns of X_o, as the entries are all integers, so subtracting, $Y_oY_o^T = A_o^TA_o - X_o^TX_o$, we see that the same must be true for columns of Y_o^T as well. Y_o satisfies the hypotheses of Theorem 2.1 and we may determine a rational orthogonal U so that all denominators of $U^TY_o^T$ are a power of 2. Call the first column of $Y_1^T = U^TY_o^T$ y_1. We note that $(y_1,y_1) = r$, as U is orthogonal. If we can apply a series of orthogonal transformations to y_1, say $U_1^T,\ldots,U_t^T$, so that $U_t^T\ldots U_1^TU^Ty_1 = j$, a column of r ones, and so that no odd factors are introduced into the denominators of Y_1 when the same transformations are applied to this matrix, then if $U_o = UU_1\ldots U_t$, we see that U_o is orthogonal and $U_o^TJ_r = J_r$, as is necessary and sufficient if the completion

5.3) $A_2 = \begin{bmatrix} X \\ Y_2^T = U_o^TY^T \end{bmatrix}$

is to be normal, $A_2A_2^T = A_2^TA_2 = B$.

Choose a minimal f so that $y_2 = 2^fy_1$ is integral. Then $(y_2,y_2) = 4^fr \equiv 0 \pmod 4$, if $f > 0$. It follows that y_2 has 4m odd entries, some m. After permutations and negations (all orthogonal transformations), we put y_2 in the form

5.4) $y_2^T \equiv [1 \quad 1 \quad 1 \quad 1 \quad \ldots] \pmod 4$.

We can now apply the transformation $\overline{U} = H \oplus I_{r-4}$, where

$$5.5)\quad H = \tfrac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Then we have

$$5.6)\quad (\bar{U}y_2)^T \equiv [0 \;\; 0 \;\; 0 \;\; 0 \;\; \ldots] \pmod 2.$$

In this way, we have reduced the weight (number of odd entries) of y_2 by applying an orthogonal transformation of denominator 2. Continue until all entries of y_2 are even. Then a smaller f will suffice to make $2^f y_1$ integral. Inductively, we may continue until $f = 0$, or y_1 itself is integral. Suppose y_1 has an even nonzero entry. Then since y_1 is integral and $(y_1,y_1) = r$, y_1 has at least three zero entries. After permutation, write

$$5.7)\quad y_1^T = [2u \;\; 0 \;\; 0 \;\; 0 \;\; \ldots]$$

Apply the transformation $\bar{U}$ described above to obtain

$$5.8)\quad y_1^T = [\, u \;\; u \;\; u \;\; u \;\; \ldots]$$

where the entries are still integral and there are fewer zero entries than before. Inductively, we may assume that all entries are odd or zero. Suppose we have two odd entries of differing magnitudes. After possible negation, take them both positive and write

$$5.9)\quad y_1^T = [\, v \;\; w \;\; 0 \;\; 0 \;\; \ldots]$$

Again, apply $\bar{U}$ to obtain

$$5.10)\quad y_1^T = [\, v^* \;\; v^* \;\; w^* \;\; w^* \;\; \ldots]$$

where $v^* = \tfrac{1}{2}(v+w)$, $w^* = \tfrac{1}{2}(v-w)$ and both quantities are integral and nonzero. Again, we have fewer zero entries than before.

Inductively, continue until all entries are either zero or of a single odd magnitude, call it x. After negation, we may in fact take

$$5.11)\quad y_1^T = [\, x \;\; x \;\; x \;\; \ldots \;\; x \;\; 0 \;\; \ldots \;\; 0]$$

If $x = 1$, there are no zeros and in fact we have $y_1 = j$ and we have achieved our desired objective. If $x > 1$, then x^2 divides r (as $(y_1,y_1) = r$) and if p is an odd prime dividing x, p^2 divides r. Apply our composite orthogonal transformation U_o^T to the full matrix Y_o^T, so that the first column of $U_o^T Y_o^T$ is our newly obtained y_1^T of (5.11) and the rest of the elements have denominators a power of two. Inner products of columns have not changed and in fact we have $(y_1, y_1^*) = (k-x_i)$, where y_i^* is the i-th column of $U_o^T Y^T$ and x_i is the i-th row sum of X. For large enough g, $2^g y_i^*$ is an integral vector, as is $x^{-1} y_1$. Hence $(x^{-1} y_1) \cdot (2^g y_i^*) = x^{-1} 2^g (k-x_i)$ is integral which, since x is odd and $(k-x_i)$ is integral, means that x divides $(k-x_i)$ or $x_i \equiv k \pmod p$ for any odd prime p dividing x.

If no such odd prime p exists, $x > 1$ is impossible and this completes the proof.

Counterexamples when conditions (i) and (ii) hold do in fact exist. E. T. Parker has shown in [8] that for an arbitrary prime p and suitable $t > 1$, a construction of $(p-1)$ mtually orthogonal Latin squares of order pt is possible, so that the associated partial incidence matrix (for a projective plane of order pt) has no rational normal completion in which p does not divide the denominators of some elements. This is true even though t may be chosen so that a rational normal completion does exist by Theorem 1.5. The partial incidence matrix involved has $p^2 t + pt + 1$ rows and all column sums are 1, $(pt+1)$ or $(p+1)$. Hence p^2 divides $r = p^2(t^2-t)$ and all column sums are congruent to $k = (pt+1) \equiv 1 \pmod p$. So both (i) and (ii) are satisfied, as would be expected.

While it is not always possible to find a rational normal completion of a partial incidence matrix with denominators a power of two, it is possible to find some normal solution to the incidence equation with denominators a power of two whenever a rational solution exists.

Corollary 5.2. Suppose v, k, λ satisfy Theorem 1.3. Then there is a rational normal matrix A such that $AA^T = A^T A = B = (k-\lambda)I_v + \lambda J_v$ and $2^e A$ is integral for sufficiently large e.

Proof: Take X as a single row of k ones and $(v-k)$ zeros in Theorem 5.1. By Theorem 1.5, there is a rational normal completion. As $0 < k < v$, some column sums are one, others zero and thus are incongruent modulo any prime p. Condition (ii) of Theorem 5.1 being violated, we are therefore assured of a rational normal completion in which all denominators are a power of two.

Finally, we indicate a connection to completions without essential denominator, as described in Jones [9], Goldhaber [10] and Parker [8].

Theorem 5.3. Suppose v, k, λ, r, X are as in Theorem 5.1 and there is a rational normal completing matrix A, with X as its first (v-r) rows, $AA^T = A^TA = B = (k-\lambda)I_v + \lambda J_v$ and the further stipulation that no entry of A has as a divisor an odd prime p satisfying conditions (i) and (ii) of Theorem 5.1. Then there is a rational normal completing matrix $\overline{A}$ so that $2^e\overline{A}$ is integral for e sufficiently large.

Proof: Write A as in (5.1) and determine an orthogonal matrix U_o by the process described in Theorem 5.1 so that $U_o^TY^T$ has all denominators a power of two and

5.12) $U_o^Tj = [x \;\; x \;\; \dots \;\; x \;\; 0 \;\; \dots \;\; 0]$

where j is a column of r ones, x is integral, positive and odd. Let p be an odd prime dividing x. We note that in the construction of U_o, as outlined in Theorems 2.1 and 5.1, involved are a series of rational orthogonal transformations in which the denominators have as prime factors only those primes dividing 2s, where s is the l.c.d. of the entries of Y. Hence, if t is the l.c.d. of the entries of U_o, our assumption and Theorem 3.1 imply that p does not divide t.

Set $U_1 = tU_o$, so that U_1 is integral and det $U_1 = t^r$. Evidently, $U_1^Tj \equiv 0 \pmod{p}$, where $j \not\equiv 0 \pmod{p}$. But this is impossible unless p divides det U_1, a contradiction. Hence x = 1 and we are done.

We will now deal with the problem of finding design completions for $r \leq 7$. Hall proves the following theorem in [7].

Theorem 5.4 (Hall). Suppose v, k, λ, r, X are as in Theorem 5.1 and $r \leq 4$. Then X can be extended to a (0,1) matrix A, with X as its first (v-r) rows and satisfying $AA^T = A^TA = (k-\lambda)I_v + \lambda J_v$.

The proof involved is a detailed examination of the cases. However, with a transformation of the partial incidence matrix X, it is possible to use previous work done on Hadamard matrices in Section 4 to obtain the same result for $r \leq 7$.

We recall Theorem 4.4:

Theorem 4.4. Suppose Y is a rational n by r matrix with $r \leq 7$, rank Y $= r$ and $YY^T = C$ satisfying the following properties:

i) C is integral and $c_{ii} = r$, all i;

ii) $C \equiv rJ \pmod 2$;

iii) $c_{ij} + c_{ik} + c_{jk} \equiv -r \pmod 4$, all i, j, k.

Then there is a (1,-1) matrix $\overline{Y}$ of size n by r so that $\overline{Y}\overline{Y}^T = C$.

We will now tranform the (0,1) matrix X into a (1,-1) matrix X_o satisfying (i), (ii) and (iii), thereby allowing us to obtain our result.

Theorem 5.5. Suppose v, k, λ, r, X are as in Theorem 5.1 and $r \leq 7$. Then X can be extended to a (0,1) matrix A with X as its first (v-r) rows and satisfying $AA^T = A^TA = (k-\lambda)I_v + \lambda J_v = B$.

Proof: By Theorem 1.5, there is a rational normal completion of X, i.e., a matrix $\overline{A}$ of size v by v with X as its first (v-r) rows and satisfying $\overline{A}\overline{A}^T = \overline{A}^T\overline{A} = B$, $\overline{A}J_v = J_v A = kJ_v$.

Write $A_o = 2\overline{A} - J_v$. Then $A_oA_o^T = 4\overline{A}\overline{A}^T - 4kJ_v + vJ_v = vJ_v + 4(k-\lambda)(I_v - J_v) = A_o^TA_o$, $A_oJ_v = 2\overline{A}J_v - vJ_v = (2k-v)J_v = J_vA_o$. Write

$$5.13) \quad A_o = \begin{bmatrix} X_o \\ Y_o^T \end{bmatrix}$$

where X_o is (v-r) by v and has all entries 1 or -1 and Y_o is v by r and rational. Since X_o is (1,-1), $X_o^TX_o \equiv (v-r)J_v \pmod 2$. Hence $Y_oY_o^T = A_o^TA_o - X_o^TX_o \equiv vJ_v - (v-r)J_v \equiv rJ_v \pmod 2$. All main diagonal entries of $A_o^TA_o$ are v, those of $X_o^TX_o$ (v-r). Hence the main diagonal entries of $Y_oY_o^T$ are $v-(v-r) = r$. Let x_i, x_j, x_k be three columns from X_o. Let $w = x_i+x_j+x_k$. All entries of w are integral and odd. Hence $(w,w) \equiv (v-r) \pmod 8$. Thus $(x_i,x_j) + (x_i,x_k) + (x_j,x_k) = \frac{1}{2}(\,(w,w) - (x_i,x_i) - (x_j,x_j) - (x_k,x_k)\,) \equiv \frac{1}{2}[(v-r) - 3(v-r)] \equiv -(v-r) \pmod 4$.

Consequently, if $X_o^T X_o = (x_{st})$, we have $x_{ij}+x_{ik}+x_{jk} \equiv -(v-r) \pmod 4$. The corresponding entries a_{ij}, a_{ik}, a_{jk} from $A_o^T A_o$ are all congruent to $v \pmod 4$. Hence, for y_{ij}, y_{ik}, y_{jk} from $Y_o Y_o^T$, we have

5.14) $y_{ij} + y_{ik} + y_{jk} \equiv 3v-[-(v-r)] \equiv 4v-r \equiv -r \pmod 4$

for all i, j, k.

Now we append a column of ones to A_o as follows:

5.15) $$A_1 = \begin{bmatrix} X_1 \\ Y_1^T \end{bmatrix}, \quad X_1 = \begin{bmatrix} 1 & \\ \vdots & X_o \\ 1 & \end{bmatrix}, \quad Y_1^T = \begin{bmatrix} 1 & \\ \vdots & Y_o^T \\ 1 & \end{bmatrix}$$

Consider $C = Y_1 Y_1^T$. All entries have been shown to satisfy (i), (ii) (iii) of Theorem 4.4, except for those entries associated with inner products with the appended column, denoted with the index 0. Y_o has rank r since A_o is nonsingular and so Y_1 has rank r.

Clearly, $c_{oo} = r$ as desired. For any i, $c_{oi} \equiv (2k-v) - (v-r) \equiv r \pmod 2$, as $J_v A_o = (2k-v)J_v$ and X_1 is $(1,-1)$. We know that for any i, j, $x_{oi} + x_{oj} + x_{ij} \equiv -(v-r)$ by the reasoning outlined above. Further, if $i > j > 0$, $a_{oi} + a_{oj} + a_{ij} = (2k-v) + (2k-v) + (v-4(k-\lambda)) \equiv -v \pmod 4$. Hence if $i > j > 0$, $c_{0i} + c_{0j} + c_{ij} \equiv -v - [-(v-r)] \equiv -r \pmod 4$. Hence all hypothese of Theorem 4.4 are satisfied.

Choose $\overline{Y}$ according to Theorem 4.4, stipulating (after negation of columns) that the first row of $\overline{Y}$ is all ones. Then still $\overline{Y}\overline{Y}^T = C$. Call $\overline{Y}_o$ the matrix $\overline{Y}$ with the first row deleted and form

5.16) $$\overline{A}_o = \begin{bmatrix} X_o \\ \overline{Y}_o^T \end{bmatrix}$$

Then $\overline{A}_o^T \overline{A}_o = A_o^T A_o = vJ_v + 4(k-\lambda)(I_v - J_v)$ and $J_v \overline{A}_o^T = (2k-v)J_v$. Set $\overline{A} = \frac{1}{2}(\overline{A}_o + J_v)$. Then $\overline{A}\overline{A}^T = (1/4)(\overline{A}_o \overline{A}_o^T + \overline{A}_o J_v + J_v \overline{A}_o^T + vJ_v) = (k-\lambda)I_v + \lambda J_v$ and $\overline{A}J_v = \frac{1}{2}(\overline{A}_o J_v + vJ_v) = kJ_v$. Further, $\overline{A}$ is $(0,1)$. So $\overline{A}$ is in fact the incidence matrix of a design containing the original X.

Hall gave in [7] an example of three lines of an (11, 5, 2) design with no completion, indicating that $r \leq 7$ is best possible.

5.17)
$$\frac{1}{2}\left[\begin{array}{ccccccccccc}
2 & 2 & 2 & 2 & 2 & & & & & & \\
2 & 2 & & & & 2 & 2 & 2 & & & \\
2 & 2 & & & & & & & 2 & 2 & 2 \\
\hline
2 & -1 & 2 & & 1 & 2 & & 1 & 2 & & 1 \\
2 & -1 & & 2 & 1 & & 2 & 1 & & 2 & 1 \\
 & 1 & 2 & 2 & -1 & & 1 & 2 & 1 & & 2 \\
 & 1 & 2 & & 1 & 2 & 1 & & -1 & 2 & 2 \\
 & 1 & 2 & & 1 & & 2 & 1 & 2 & 2 & -1 \\
 & 1 & & 2 & 1 & 2 & -1 & 2 & 1 & 2 & \\
 & 1 & & 2 & 1 & 2 & 2 & -1 & 2 & & 1 \\
 & 1 & & & 3 & & 1 & 2 & 1 & & 2
\end{array}\right]$$

The design cannot be completed since then the inner product of the first two columns would be two, which is impossible.

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Appendix

Forms for Hadamard six row completions

Listed are the 28 forms mentioned in the case $r = 6$ of Theorem 4.4. Each form is numbered, followed by its determinant and square root. Only those cases in which the determinant is a perfect nonzero integral square need be considered for the purposes of the theorem. In these cases, admissable v are listed to the side of the form F.

Following the form are one or more (1,-1) solutions Z^T to the equation $ZZ^T = F$. To the right of each Z are the vectors $y = Z^{-1}v$, corresponding to the v listed by the form. It may be seen that y is (1,-1) in all cases.

Note that v for which the inner product with a column of Z^T is 4, -4, 6 or -6 need not be considered, as in the first two cases, the problem reduces to the case $r = 5$ and in the other two cases y must be a duplicate or a negative of a column of Z^T.

```
1
6 2 0 0 0 2
2 6 0 0 0 2
0 0 6 2 2 0
0 0 2 6 2 0
0 0 2 2 6 0
2 2 0 0 0 6

DETERMINANT   25600 SQRT 160.000

1 1 1 1 1-1
1 1 1-1-1 1
1 1-1-1 1 1
1 1-1 1-1 1
1-1 1 1 1 1
1-1-1-1-1-1


1 1 1 1-1 1
1 1 1-1 1 1
1 1-1-1-1-1
1 1-1 1 1 1
1-1 1 1 1-1
1-1-1-1-1 1


2
6 2 0 0 0 2
2 6 0 0 0-2
0 0 6 2 2 0
0 0 2 6 2 0
0 0 2 2 6 0
2-2 0 0 0 6

DETERMINANT   20480 SQRT 143.108

IRRATIONAL COMPLETION
```

```
 3
 6 2 0 0 0 0
 2 6 0 0 0 0
 0 0 6 2 2 2
 0 0 2 6 2 2
 0 0 2 2 6 2
 0 0 2 2 2 6
```

DETERMINANT   24576 SQRT 156.767

IRRATIONAL COMPLETION

```
 4
 6 2 0 0 0 0       0    0
 2 6 0 0 0 0       0    0
 0 0 6 2 2 2       2    2
 0 0 2 6 2 2      -2   -2
 0 0 2 2 6-2      -2    2
 0 0 2 2-2 6      -2    2
```

DETERMINANT   16384 SQRT 128.000

```
 1 1 1 1 1-1     1.0 -1.0
 1 1 1-1-1 1     1.0  1.0
 1 1-1-1 1-1    -1.0  1.0
 1 1-1 1-1 1    -1.0 -1.0
 1-1 1 1 1 1    -1.0  1.0
 1-1-1-1-1-1     1.0 -1.0

 1 1 1 1-1 1     1.0 -1.0
 1 1 1-1 1-1     1.0  1.0
 1 1-1-1-1 1    -1.0  1.0
 1 1-1 1 1-1    -1.0 -1.0
 1-1 1 1 1 1    -1.0  1.0
 1-1-1-1-1-1     1.0 -1.0

 5
 6 2 0 0 0 2       0    2
 2 6 0 0 0-2       0   -2
 0 0 6 2 2 0       2    0
 0 0 2 6-2 0      -2    0
 0 0 2-2 6 0      -2    0
 2-2 0 0 0 6       0   -2
```

DETERMINANT   16384 SQRT 128.000

```
1 1 1 1 1-1   -1.0  1.0
1 1 1 1-1 1    1.0 -1.0
1 1-1-1 1 1   -1.0 -1.0
1 1-1-1-1-1    1.0  1.0
1-1 1-1 1 1    1.0  1.0
1-1-1 1-1 1   -1.0  1.0

1 1 1 1 1 1   -1.0 -1.0
1 1 1 1-1-1    1.0  1.0
1 1-1-1 1-1   -1.0  1.0
1 1-1-1-1 1    1.0 -1.0
1-1 1-1 1 1    1.0  1.0
1-1-1 1-1 1   -1.0  1.0

1 1 1 1 1-1   -1.0  1.0
1 1 1-1 1 1    1.0 -1.0
1 1-1-1-1-1    1.0  1.0
1 1-1 1-1 1   -1.0 -1.0
1-1 1 1-1 1    1.0  1.0
1-1-1-1 1 1   -1.0  1.0

1 1 1 1 1 1   -1.0 -1.0
1 1 1-1 1-1    1.0  1.0
1 1-1-1-1 1    1.0 -1.0
1 1-1 1-1-1   -1.0  1.0
1-1 1 1-1 1    1.0  1.0
1-1-1-1 1 1   -1.0  1.0

1 1 1 1-1-1    1.0  1.0
1 1 1-1 1 1    1.0 -1.0
1 1-1-1 1-1   -1.0  1.0
1 1-1 1-1 1   -1.0 -1.0
1-1 1 1 1 1   -1.0  1.0
1-1-1-1-1 1    1.0  1.0

1 1 1 1-1 1    1.0 -1.0
1 1 1-1 1-1    1.0  1.0
1 1-1-1 1 1   -1.0 -1.0
1 1-1 1-1-1   -1.0  1.0
1-1 1 1 1 1   -1.0  1.0
1-1-1-1-1 1    1.0  1.0
```

```
 6
 6 2 0 0 0 0
 2 6 0 0 0 0
 0 0 6 2 2 2
 0 0 2 6-2-2
 0 0 2-2 6-2
 0 0 2-2-2 6

DETERMINANT        0

SINGULAR SYSTEM


 7
 6 2 0 2 2 2
 2 6 0 2 2 2
 0 0 6 0 0 0
 2 2 0 6 2 2
 2 2 0 2 6 2
 2 2 0 2 2 6

DETERMINANT   21504 SQRT 146.642

IRRATIONAL COMPLETION


 8
 6 2 0 2 2 2
 2 6 0 2 2 2
 0 0 6 0 0 0
 2 2 0 6 2 2
 2 2 0 2 6-2
 2 2 0 2-2 6

DETERMINANT   12288 SQRT 110.851

IRRATIONAL COMPLETION
```

```
9
6 2 0 2 2 2       2     2     2     2     2
2 6 0 2 2 2      -2    -2    -2    -2     2
0 0 6 0 0 0       0     0     0     0     0
2 2 0 6 2-2      -2    -2     2     2    -2
2 2 0 2 6-2       2     2    -2    -2    -2
2 2 0-2-2 6      -2     2    -2     2     2

DETERMINANT     9216 SQRT  96.000


1 1 1 1 1-1     1.0 -1.0  1.0 -1.0  1.0
1 1 1 1 1 1    -1.0  1.0 -1.0  1.0 -1.0
1 1-1 1-1 1    -1.0 -1.0  1.0  1.0  1.0
1 1-1-1 1 1     1.0  1.0 -1.0 -1.0  1.0
1-1 1-1-1 1     1.0  1.0  1.0  1.0  1.0
1-1-1 1 1-1     1.0  1.0  1.0  1.0 -1.0

1 1 1 1 1 1    -1.0  1.0 -1.0  1.0 -1.0
1 1 1 1 1-1     1.0 -1.0  1.0 -1.0  1.0
1 1-1 1-1 1    -1.0 -1.0  1.0  1.0  1.0
1 1-1-1 1 1     1.0  1.0 -1.0 -1.0  1.0
1-1 1-1-1 1     1.0  1.0  1.0  1.0  1.0
1-1-1 1 1-1     1.0  1.0  1.0  1.0 -1.0

1 1 1 1-1 1    -1.0 -1.0  1.0  1.0  1.0
1 1 1-1 1 1     1.0  1.0 -1.0 -1.0  1.0
1 1-1 1 1-1     1.0 -1.0  1.0 -1.0  1.0
1 1-1 1 1 1    -1.0  1.0 -1.0  1.0 -1.0
1-1 1 1 1-1     1.0  1.0  1.0  1.0 -1.0
1-1-1-1-1 1     1.0  1.0  1.0  1.0  1.0

1 1 1 1-1 1    -1.0 -1.0  1.0  1.0  1.0
1 1 1-1 1 1     1.0  1.0 -1.0 -1.0  1.0
1 1-1 1 1 1    -1.0  1.0 -1.0  1.0 -1.0
1 1-1 1 1-1     1.0 -1.0  1.0 -1.0  1.0
1-1 1 1 1-1     1.0  1.0  1.0  1.0 -1.0
1-1-1-1-1 1     1.0  1.0  1.0  1.0  1.0
```

```
10
 6 2 0 2 2 2
 2 6 0 2-2 2
 0 0 6 0 0 0
 2 2 0 6 2-2
 2-2 0 2 6 2
 2 2 0-2 2 6


DETERMINANT        0

SINGULAR SYSTEM


11
 6 2 0 2 2 2       2    2    2    2    2
 2 6 0 2-2 2      -2   -2   -2    2    2
 0 0 6 0 0 0       0    0    0    0    0
 2 2 0 6 2-2      -2   -2    2   -2    2
 2-2 0 2 6-2      -2    2    2    2   -2
 2 2 0-2-2 6       2   -2    2    2   -2

DETERMINANT     9216 SQRT  96.000


 1 1 1 1 1-1    -1.0  1.0 -1.0  1.0  1.0
 1 1 1 1-1 1     1.0 -1.0  1.0 -1.0  1.0
 1 1-1 1 1 1    -1.0 -1.0  1.0  1.0 -1.0
 1 1-1-1-1 1     1.0  1.0 -1.0  1.0  1.0
 1-1 1-1 1 1     1.0  1.0  1.0  1.0 -1.0
 1-1-1 1 1-1     1.0  1.0  1.0 -1.0  1.0

 1 1 1 1 1 1    -1.0 -1.0  1.0  1.0 -1.0
 1 1 1-1-1 1     1.0  1.0 -1.0  1.0  1.0
 1 1-1 1 1-1    -1.0  1.0 -1.0  1.0  1.0
 1 1-1 1-1 1     1.0 -1.0  1.0 -1.0  1.0
 1-1 1 1 1-1     1.0  1.0  1.0 -1.0  1.0
 1-1-1-1 1 1     1.0  1.0  1.0  1.0 -1.0

12
 6 2 0 2 2 2
 2 6 0 2-2-2
 0 0 6 0 0 0
 2 2 0 6 2 2
 2-2 0 2 6-2
 2-2 0 2-2 6
```

DETERMINANT -12287

COMPLEX COMPLETION

```
13
 6 2 0 2 2 2
 2 6 0-2-2-2
 0 0 6 0 0 0
 2-2 0 6-2-2
 2-2 0-2 6-2
 2-2 0-2-2 6
```

DETERMINANT -49151

COMPLEX COMPLETION

```
14
 6 2 2 2 2 2
 2 6-2 2-2 2
 2-2 6 2-2 2
 2 2 2 6 2 2
 2-2-2 2 6 2
 2 2 2 2 2 6
```

DETERMINANT -16383

COMPLEX COMPLETION

```
15
 6 2 2 2 2 2
 2 6-2 2-2 2
 2-2 6 2-2 2
 2 2 2 6 2 2
 2-2-2 2 6-2
 2 2 2 2-2 6
```

DETERMINANT -8191

COMPLEX COMPLETION

```
16
 6 2 2 2 2 2
 2 6-2 2-2 2
 2-2 6 2-2 2
 2 2 2 6 2-2
 2-2-2 2 6 2
 2 2 2-2 2 6
```

DETERMINANT -32767

COMPLEX COMPLETION

```
17
 6 2 2 2 2 2
 2 6-2 2-2 2
 2-2 6 2-2 2
 2 2 2 6 2-2
 2-2-2 2 6-2
 2 2 2-2-2 6
```

DETERMINANT -8191

COMPLEX COMPLETION

```
18
 6 2 2 2 2 2
 2 6-2 2-2 2
 2-2 6 2-2-2
 2 2 2 6 2 2
 2-2-2 2 6-2
 2 2-2 2-2 6
```

DETERMINANT -24575

COMPLEX COMPLETION

```
19
 6 2 2 2 2 2
 2 6-2 2-2-2
 2-2 6 2-2-2
 2 2 2 6 2 2
 2-2-2 2 6-2
 2-2-2 2-2 6
```

DETERMINANT -65535

COMPLEX COMPLETION

```
20
 6 2 2 2 2 2
 2 6-2 2 2 2
 2-2 6 2 2 2
 2 2 2 6 2 2
 2 2 2 2 6 2
 2 2 2 2 2 6
```

DETERMINANT   8192 SQRT 90.510

IRRATIONAL COMPLETION

```
21
 6 2 2 2 2 2
 2 6-2 2 2 2
 2-2 6 2 2 2
 2 2 2 6 2 2
 2 2 2 2 6-2
 2 2 2 2-2 6
```

DETERMINANT   0

SINGULAR SYSTEM

```
22
 6 2 2 2 2 2       2    2    2    2    2    2    2    2    2    2
 2 6-2 2 2 2      -2   -2   -2   -2   -2    2    2    2    2    2
 2-2 6 2 2-2      -2    2    2    2    2   -2   -2   -2    2    2
 2 2 2 6 2 2      -2   -2   -2    2    2   -2   -2    2   -2    2
 2 2 2 2 6 2      -2    2    2   -2   -2   -2   -2    2    2   -2
 2 2-2 2 2 6       2   -2    2   -2    2   -2    2    2   -2   -2

DETERMINANT     4096 SURT  64.000


 1 1 1 1 1-1    -1.0  1.0 -1.0  1.0 -1.0  1.0 -1.0  1.0  1.0  1.0
 1 1 1 1 1 1    -1.0 -1.0  1.0 -1.0  1.0 -1.0  1.0 -1.0  1.0  1.0
 1 1-1 1-1 1     1.0 -1.0 -1.0  1.0  1.0  1.0  1.0  1.0 -1.0  1.0
 1 1-1-1 1 1     1.0  1.0  1.0 -1.0 -1.0  1.0  1.0  1.0  1.0 -1.0
 1-1 1 1 1 1     1.0  1.0  1.0  1.0  1.0 -1.0 -1.0  1.0 -1.0 -1.0
 1-1 1-1-1-1     1.0  1.0  1.0  1.0  1.0  1.0  1.0 -1.0  1.0  1.0

 1 1 1 1 1 1    -1.0 -1.0  1.0 -1.0  1.0 -1.0  1.0 -1.0  1.0  1.0
 1 1 1 1 1-1    -1.0  1.0 -1.0  1.0 -1.0  1.0 -1.0  1.0  1.0  1.0
 1 1-1 1-1 1     1.0 -1.0 -1.0  1.0  1.0  1.0  1.0  1.0 -1.0  1.0
 1 1-1-1 1 1     1.0  1.0  1.0 -1.0 -1.0  1.0  1.0  1.0  1.0 -1.0
 1-1 1 1 1 1     1.0  1.0  1.0  1.0  1.0 -1.0 -1.0  1.0 -1.0 -1.0
 1-1 1-1-1-1     1.0  1.0  1.0  1.0  1.0  1.0  1.0 -1.0  1.0  1.0

23
 6 2 2 2 2 2       2    2    2    2    2    2    2    2    2    2
 2 6-2 2 2 2      -2   -2   -2   -2   -2    2    2    2    2    2
 2-2 6 2 2-2      -2    2    2    2    2   -2   -2   -2    2    2
 2 2 2 6 2 2      -2   -2   -2    2    2   -2   -2    2   -2    2
 2 2 2 2 6-2      -2   -2   -2    2    2    2    2   -2    2   -2
 2 2-2 2-2 6       2   -2    2   -2    2   -2    2    2   -2    2

DETERMINANT     4096 SQRT  64.000


 1 1 1 1 1-1    -1.0  1.0 -1.0  1.0 -1.0  1.0 -1.0  1.0  1.0  1.0
 1 1 1 1 1 1    -1.0 -1.0  1.0 -1.0  1.0 -1.0  1.0 -1.0  1.0  1.0
 1 1-1 1 1 1     1.0 -1.0 -1.0  1.0  1.0  1.0  1.0  1.0 -1.0 -1.0
 1 1-1-1-1 1     1.0  1.0  1.0 -1.0 -1.0  1.0  1.0  1.0  1.0  1.0
 1-1 1 1-1 1     1.0  1.0  1.0  1.0  1.0 -1.0 -1.0  1.0 -1.0  1.0
 1-1 1-1 1-1     1.0  1.0  1.0  1.0  1.0  1.0  1.0 -1.0  1.0 -1.0

 1 1 1 1 1 1    -1.0 -1.0  1.0 -1.0  1.0 -1.0  1.0 -1.0  1.0  1.0
 1 1 1 1 1-1    -1.0  1.0 -1.0  1.0 -1.0  1.0 -1.0  1.0  1.0  1.0
 1 1-1 1 1 1     1.0 -1.0 -1.0  1.0  1.0  1.0  1.0  1.0 -1.0 -1.0
 1 1-1-1-1 1     1.0  1.0  1.0 -1.0 -1.0  1.0  1.0  1.0  1.0  1.0
 1-1 1 1-1 1     1.0  1.0  1.0  1.0  1.0 -1.0 -1.0  1.0 -1.0  1.0
 1-1 1-1 1-1     1.0  1.0  1.0  1.0  1.0  1.0  1.0 -1.0  1.0 -1.0
```

```
24
 6 2 2 2 2 2
 2 6-2 2 2 2
 2-2 6 2 2-2
 2 2 2 6-2 2
 2 2 2-2 6-2
 2 2-2 2-2 6
```

DETERMINANT 0

SINGULAR SYSTEM

```
25
 6 2 2 2 2 2
 2 6-2 2 2-2
 2-2 6 2 2-2
 2 2 2 6-2-2
 2 2 2-2 6-2
 2-2-2-2-2 6
```

DETERMINANT 0

SINGULAR SYSTEM

```
26
 6 2 2 2 2 2
 2 6-2 2 2-2
 2-2 6 2-2 2
 2 2 2 6 2 2
 2 2-2 2 6-2
 2-2 2 2-2 6
```

DETERMINANT 0

SINGULAR SYSTEM

```
27
 6 2 2 2 2 2      2    2    2    2    2    2    2    2    2    2
 2 6-2 2 2-2     -2   -2   -2   -2    2    2    2    2    2    2
 2-2 6 2-2 2     -2    2    2    2   -2   -2   -2    2    2    2
 2 2 2 6-2-2     -2   -2    2    2   -2    2    2   -2    2    2
 2 2-2-2 6 2      2   -2   -2    2    2   -2    2    2   -2    2
 2-2 2-2 2 6      2    2   -2    2   -2   -2    2    2    2   -2

DETERMINANT     4096 SQRT  64.000


 1 1 1 1 1 1   -1.0 -1.0 -1.0  1.0 -1.0 -1.0  1.0  1.0  1.0  1.0
 1 1 1 1-1-1   -1.0  1.0  1.0 -1.0  1.0  1.0 -1.0  1.0  1.0  1.0
 1 1-1 1 1-1    1.0 -1.0  1.0  1.0  1.0  1.0  1.0 -1.0 -1.0  1.0
 1 1-1-1 1 1    1.0  1.0 -1.0 -1.0  1.0  1.0  1.0  1.0  1.0 -1.0
 1-1 1 1-1 1    1.0  1.0  1.0  1.0 -1.0  1.0  1.0 -1.0  1.0 -1.0
 1-1 1-1 1 1    1.0  1.0  1.0  1.0  1.0 -1.0 -1.0  1.0 -1.0  1.0

28
 6 2 2 2 2 2      2    2    2    2    2    2    2    2    2    2
 2 6 2 2 2 2     -2   -2   -2   -2   -2   -2    2    2    2    2
 2 2 6 2 2 2     -2   -2   -2    2    2    2   -2   -2   -2    2
 2 2 2 6 2 2     -2    2    2   -2   -2    2   -2   -2    2   -2
 2 2 2 2 6 2      2   -2    2   -2    2   -2   -2    2   -2   -2
 2 2 2 2 2 6      2    2   -2    2   -2   -2    2   -2   -2   -2

DETERMINANT    16384 SQRT 128.000


 1 1 1 1 1-1   -1.0 -1.0  1.0 -1.0  1.0  1.0 -1.0  1.0  1.0  1.0
 1 1 1 1-1 1   -1.0  1.0 -1.0  1.0 -1.0  1.0  1.0 -1.0  1.0  1.0
 1 1 1-1 1 1    1.0 -1.0 -1.0  1.0  1.0 -1.0  1.0  1.0 -1.0  1.0
 1 1-1 1 1 1    1.0  1.0  1.0 -1.0 -1.0 -1.0  1.0  1.0  1.0 -1.0
 1-1 1 1 1 1    1.0  1.0  1.0  1.0  1.0  1.0 -1.0 -1.0 -1.0 -1.0
 1-1-1-1-1-1    1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0
```