A COMPUTER FOR ALGEBRAIC FUNCTIONS OF A COMPLEX VARIABLE

Thesis by

Merle Loren Morgan

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ABSTRACT

Any rational algebraic function of a complex variable, and certain irrational functions, can be factored in either of the equivalent forms:

$$F = K_c z^{m_o} (z - z_1)^{m_1} (z - z_2)^{m_2} \cdot \cdot \cdot \cdot$$

or

$$F = K_n z^{m_o} (1 - \frac{z}{z_1})^{m_1} (1 - \frac{z}{z_2})^{m_2} \cdot \cdot \cdot \cdot$$

In these expressions, F is a function of the complex variable z; each m represents a positive or negative constant, and the other letters represent complex constants. A theory is developed for computers for such functions, in which voltages proportional to the logarithmic components of each factor are obtained from the electrical potential distributions on a pair of uniform resistive sheets. The electrical summation of the voltages representing the factors then yields readings of the logarithmic components of the function.

An actual computer, built to test and demonstrate the theory, is described. This computer accepts information in the form of the magnitude (absolute value, modulus) and the phase (angle, argument, amplitude) of each constant and of z in either of the above expressions, and yields answers in the form of the magnitude and phase of the function F. The computer is useful for applications requiring the evaluation of F at a large number of values of z; it is even more valuable for the inverse problem—that of determining by trial (by root locus tracing) the values of z (the roots) for a given value of F.

TABLE OF CONTENTS

INTRODUCTION 1

PART I--MATHEMATICAL CONCEPTS AND COMPUTING AIDS

Complex Quantities--Magnitude and Phase 2

Coordinate Systems for Complex Quantities 3

Rational Algebraic Functions 13

Computing Aids for Algebraic Functions 16

PART II--EVOLUTION OF A NEW COMPUTER

Research Goal 21

Dual Sheets 22

A New Computer Technique 28

Mechanical Requirements in a Practical Computer . 32

Resistive Sheet Materials 34

The Summing Voltmeter 35

PART III--COMPUTERS IN VARIOUS COORDINATE SYSTEMS

Computer Coordinate Systems 40

A Computer in the z Plane 42

A Computer in the 1/z Plane 48

Computers in Log z Coordinates 51

Computers in Which α Is Adjustable 66

Computers in Which α Is Equal to One or Zero . . 73

PART IV--AN ACTUAL COMPUTER

Design and Construction of the Computer 84

Accuracy of the Computer 97

Applications of the Computer 102

Operation of the Computer 106

Calibration of the Computer 108

REFERENCES 113

INTRODUCTION

The differential equations describing a linear electrical or mechanical system[1,2] are commonly transformed into a set of linear algebraic equations in which the real variable, time, has been replaced by a complex variable such as the Laplace transform variable s, the Heaviside operator p, or in the case of sinusoidal single frequency problems, the complex radian frequency $j\omega$. Such a set of linear algebraic equations leads to rational algebraic functions of the complex variable.[3] When quantitative evaluation of such functions is required, as frequently occurs in the development and design of electromechanical equipment[4,5] the numerical calculations are often very involved and tedious. Various machines and techniques have been proposed and used to simplify such calculations. The present computer was developed in an attempt to overcome certain shortcomings of previous computing aids. Although the computer was developed to fill a need in problems involving electric networks and feedback systems (feedback amplifiers and servomechanisms), it is offered as a general purpose computer for rational algebraic functions and certain other functions of a complex variable, with the hope that workers in other fields will also find it useful.

PART I

MATHEMATICAL CONCEPTS AND COMPUTING AIDS

Complex Quantities--Magnitude and Phase

A complex quantity is ordinarily specified by two real quantities, or numbers, called the real and the imaginary components of the complex quantity. Where products of complex quantities are to be computed, it is frequently convenient to use the real and imaginary components of the logarithm of a quantity, instead of the real and imaginary components of the quantity itself. The real component of the logarithm is the logarithm of a real quantity known as the magnitude, absolute value, or modulus. The imaginary component of the logarithm is another real quantity called the argument, amplitude, phase angle, or simply "phase." The magnitude and phase of any complex quantity can thus be considered as an alternate set of components by which the quantity may be specified. In the actual computer which will be described later, these, rather than the real and imaginary components, will be treated as the basic components.

In this thesis, the abbreviation "log" will be used for a logarithm to any base, while "ln" will be used to denote $\log_e$, the natural logarithm. The magnitude and phase of any quantity, for example F, will be written as $|F|$ and $\angle F$

respectively, and the abbreviations Re and Im will be used to stand for "the real component of" and "the imaginary component of" respectively. In terms of these symbols,

$$\underline{Re}(\ln F) = \ln|F| = \text{"log magnitude F"}$$

$$\underline{Im}(\ln F) = \angle F \text{ (radians)} = \text{"phase F"}$$

or written as a single equation, in which $j = \sqrt{-1}$,

$$\ln F = \ln|F| + j\,\angle F \tag{1}$$

The logarithm of a product of complex quantities is equal to the sum of the logarithms of the individual factors. Therefore the magnitude of a product is equal to the product of the magnitudes of the factors, while the phase of a product is the sum of the phase angles of the factors.

Coordinate Systems for Complex Quantities

Since a complex quantity is defined by two real quantities, or components, it can be represented as a point in any two-dimensional coordinate system. The most common coordinate system for a complex variable z is called the "z plane." It is the simple Cartesian coordinate system in which the abscissa $x = \underline{Re}(z)$ and the ordinate $y = \underline{Im}(z)$. In the z plane, the real and imaginary components of z and the magnitude and phase of z are related geometrically as indicated in Figure 1. Two other simple coordinate systems are the "1/z plane" and the "log z plane." These derive their names

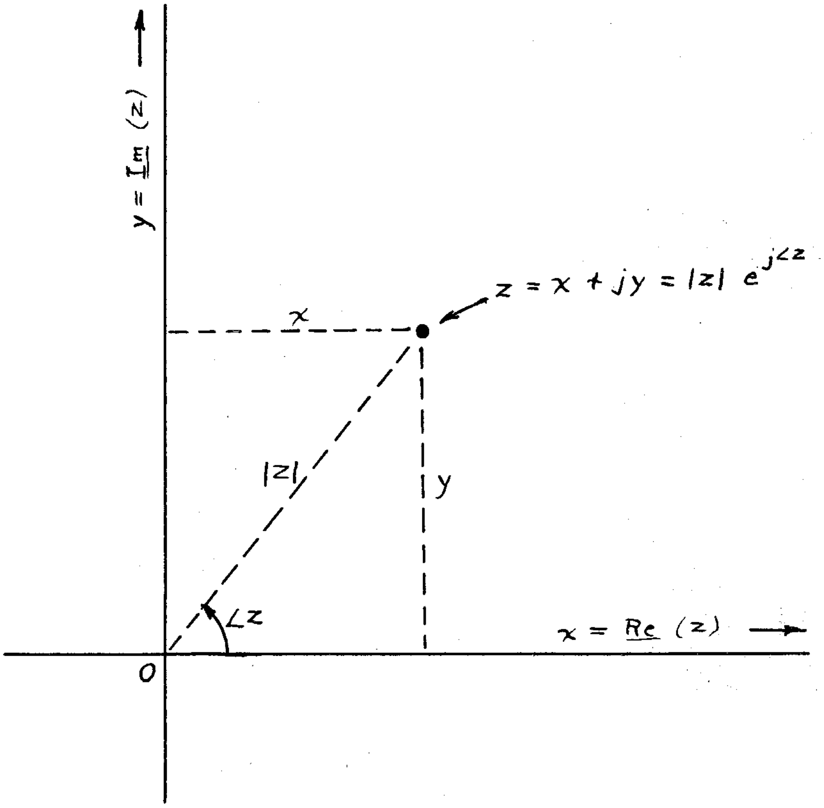


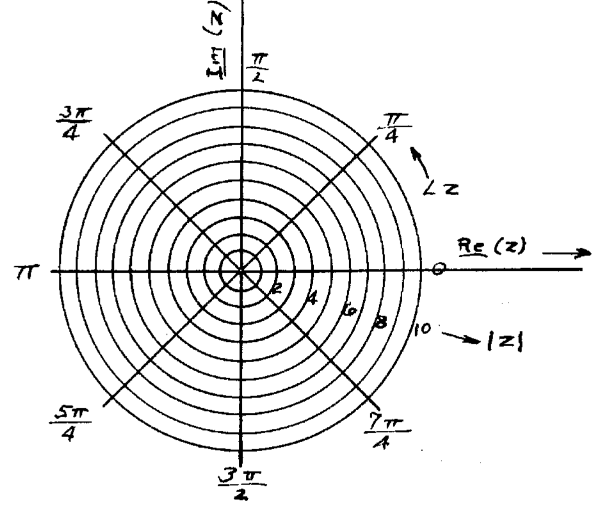
Figure 1

The z Plane

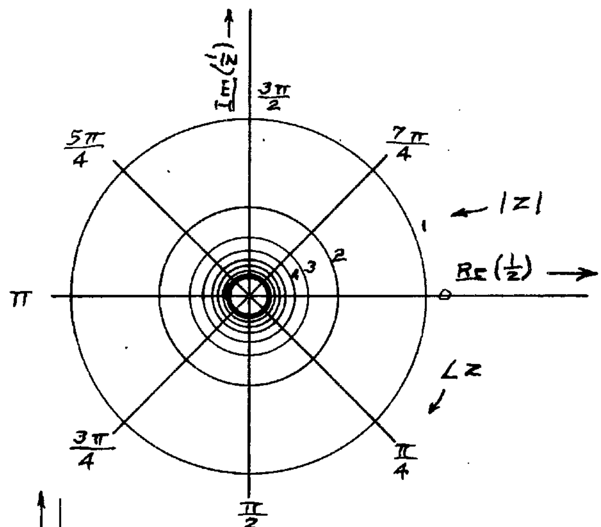
from the fact that the abscissa and the ordinate are the real and imaginary components respectively of $\frac{1}{z}$ and of ln z. If contours of various values of $|z|$ and $\angle z$ are marked off as a new set of coordinates in each of these planes, they appear as shown in Figure 2. In the z plane, $|z|$ and $\angle z$ are seen to be ordinary plane polar coordinates; in the log z plane, ln z and $\angle z$ are the abscissa and ordinate, so that the contours of $|z|$ and $\angle z$ are straight lines corresponding to the calibration of "semi-logarithmic" graph paper.

The above coordinate systems are conformal transformations of one another, since the functions of z relating them are analytic. The coordinate systems in this thesis will be used to describe electric current flow and voltage distribution in uniform resistive sheets; therefore only coordinate systems which are conformally related will be permitted. A change in scale is a conformal transformation if both coordinates are changed in the same proportion; hence any scale can be chosen for a coordinate system, but it is essential that the same scale be used in both directions. In the log z plane it is necessary that the "natural" units—natural logarithms of magnitude and radians of phase angle—have the same scale. But neither natural logarithms nor radians are particularly convenient units for everyday use in plotting z. For practical use, $|z|$ will be indicated on a logarithmic scale marked

The z Plane:



The $\frac{1}{z}$ Plane:



The Log z Plane:

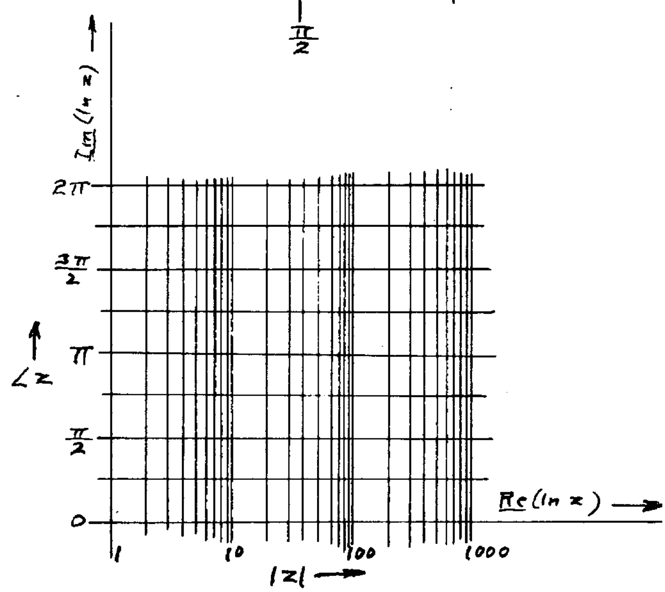


Figure 2

Magnitude and Phase of z in Simple Coordinate Systems

in decimal values of $|z|$; the most useful scale unit is therefore the decade (also called a "cycle" in graph paper)--the distance representing a magnitude ratio of ten. A distance in decades is equal to the $\log_{10}$ of the corresponding magnitude ratio, or $\frac{1}{2.303}$ times the ln of the ratio; thus one decade is equal to 2.303 natural logarithmic distance units. Other logarithmic units which can be used to specify magnitude ratios include the octave, which is a magnitude ratio of two (number of octaves = $\log_2$ of the ratio); the decibel, which for voltage and current ratios (not power ratios) is defined as 1/20 of a decade (number of decibels = 20 x $\log_{10}$ of the voltage or current ratio); and the neper, which is the natural logarithmic unit for voltage and current ratios (number of nepers = ln of the voltage or current ratio). In the $\angle z$ direction, the scale is linear, but the practical scale units are degrees ($^\circ$) or revolutions (multiples of 360° or 2π radians). One radian is equal to 57.30°, so that 1 decade = 2.303 natural units (ln or radians) = 131.93°. One decade is therefore approximately equal to ($\simeq$) 132°, to an accuracy of 0.05%; 3 decades $\simeq 396^\circ$ to the same accuracy; or 3 decades $\simeq 400^\circ$ with an error of about 1%. These and other useful relationships between practical scale units for magnitude and phase are listed in Table I.

Table I

Relation Between Scale Units in Log z Coordinates

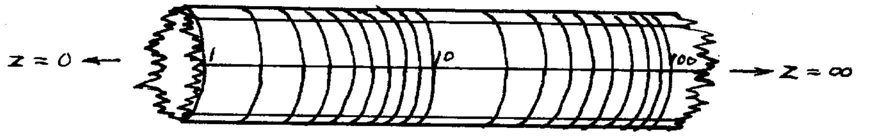
Magnitude Units			Natural Units	Phase Angle Units		
Misc. Units	Decibels	Ratio		Degrees	Useful Approximations	Error of Approximation
1 neper =	8.686db. =	2.718	1.000	57.30°	60° = 1/6 rev.	+5 %
1 octave =	6.02 db. =	2	0.693	39.71°	40° = 1/9 rev.	+0.7%
9 octaves =	54.2 db. =	512	6.239	357.4°	360° = 1 rev.	+0.7%
1 decade =	20 db. =	10	2.303	131.93°	132°	+0.05%
2 decades =	40 db. =	100	4.605	263.9°	270° = 3/4 rev.	+2 %
3 decades =	60 db. =	1000	6.908	395.8°	{ 396° = 1 1/10 rev. { 400° = 1 1/9 rev.	+0.05% +1 %

Most semi-logarithmic graph papers are not at all convenient for plotting degrees versus logarithmic magnitudes in the natural proportion, but Keuffel and Esser No. 358 or 359 - 73 or 73L (3 cycles x 10 to the half-inch, 5th lines accented) is remarkably suited to the purpose. On this graph paper, when each line represents 2° and each accented line 10°, 3 decades and 400° each nominally occupy the same length (10 inches)—an error of only 1% from the correct scale ratio. K. and E. No. 71 or 71L (3 cycles x 10 to the inch, 5th lines accented) is equally accurate when each line represents 4° and each accented line 20°, but this calibration is obviously less convenient to use. Measurements on samples of this graph paper showed a deviation from the nominal scales in the favorable direction, so that the error was actually nearer to one half of one percent. The other K. and E. graph papers with the same logarithmic scale also exhibit useful properties. No. 72 or 72L (3 cycles x 6 to the half-inch) is nominally exactly correct for plotting decibels versus logarithmic magnitudes, when each line represents one-half decibel. And K. and E. No. 112 full-logarithmic graph paper with the same logarithmic scale is available for transferring a decibel plot to a voltage or current plot by tracing the curve onto No. 112 from No. 72 semi-logarithmic paper, or vice versa. Thus all of the Keuffel and Esser full-logarithmic and semi-logarithmic graph papers on which the length of each decade or "cycle" is equal to 3 1/3 inches are well coordinated for various kinds of plots in the natural proportions described in this section. If a semi-logarithmic paper with a smaller scale is desired, K. and E. No. 91 (5 cycles x 10 to the inch, 5th lines accented) is a good choice. It is excellent for decibel plots, since each line represents one decibel in the natural scale ratio. It is not so convenient for log z plots, but it is correct to 1% when each <u>third</u> line represents 20°.

In the z plane and the 1/z plane, each value of z appears only at one point in the infinite plane; but in the log z plane, each value appears an infinite number of times, being repeated at intervals of 2π in the $\angle z$ direction. Therefore, when magnitude and phase are used as the components by which a complex quantity is specified, it must be recognized that any two quantities which are equal in magnitude but differ in phase by a multiple of 2π are in reality not different quantities, but only different representations of the same quantity. It is sufficient for the representation of every value of z to use any strip of the log z plane of width 2π in the $\angle z$ direction but of infinite length in the $|z|$ direction. Such a "log z strip" of width 2π will be called a "single-width" strip; a strip of width 4π, containing every value of z twice, will be called a "double-width" strip; a strip of width π will be called a "half-width" strip.

The surface on which the two coordinates are specified need not be a plane; it can be a surface in two dimensions. Two such curved surface coordinate systems will be mentioned. If a single-width log z strip is wrapped around a tube of unit radius and joined along the sides, the resulting coordinate system will be an infinitely long "log z cylinder." In this coordinate system, ln $|z|$ is measured along the length of the cylinder, and $\angle z$ is the angle about the axis of the cylinder. Another interesting coordinate system is obtained if the infinite ends of the log z cylinder are folded in, i.e., transformed conformally, into plane ends, forming a "cylindrical box" coordinate system on a surface shaped like the familiar tin can. This coordinate system is different from the others which have been described in that it is finite in size, yet all values of z including zero and infinity are contained on it. Furthermore, it can be shown that the ends of the box are portions of the z plane and the 1/z plane respectively; therefore the cylindrical box coordinate system is actually a combination of the three basic coordinate systems in Figure 2. The log z cylinder and the cylindrical box, with $|z|$ and $\angle z$ marked off as coordinates, are illustrated in Figure 3.

The Log z Cylinder:



The Cylindrical Box:

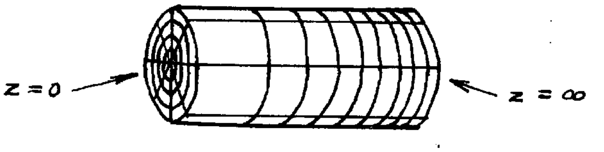


Figure 3

Magnitude and Phase of z in Curved Surface Coordinate Systems

Rational Algebraic Functions

A rational algebraic function of a complex variable is one which can be written in the form

$$\frac{A_o + A_1 z + A_2 z^2 + A_3 z^3 + \cdot \cdot \cdot}{B_o + B_1 z + B_2 z^2 + B_3 z^3 + \cdot \cdot \cdot} \qquad (2)$$

in which z represents the complex variable, and the A's and B's are coefficients which are usually real but may in general be complex. Such a function can be factored into either of two equivalent forms

$$K_c \, z^{m_o} \; \frac{(z - a_1)^{m_1} \, (z - a_2)^{m_2} \cdot \cdot \cdot}{(z - b_1)^{n_1} \, (z - b_2)^{n_2} \cdot \cdot \cdot} \qquad (3c)$$

which will be called the "conventional form" for the factors,

or

$$K_n \, z^{m_o} \; \frac{(1 - \frac{z}{a_1})^{m_1} \, (1 - \frac{z}{a_2})^{m_2} \cdot \cdot \cdot}{(1 - \frac{z}{b_1})^{n_1} \, (1 - \frac{z}{b_2})^{n_2} \cdot \cdot \cdot} \qquad (3n)$$

which will be called the "normalized form" for the factors. When the complex variable z is represented in a coordinate system, the points $z = a_1$, $z = a_2$, etc., are called zeros of the function in either of the above forms, since these are the values of z at which the function is equal to zero. Similarly, the points $z = b_1$, $z = b_2$, etc., are called poles of the function, since these are the values at which the function becomes infinite. Each value of m or n is called the order of the corresponding zero or pole.

The above functions can be written more compactly using the symbol $\prod$ to indicate a product of factors, in a manner analogous to the common use of the symbol $\sum$ to indicate a summation of terms. Using this symbol, and replacing the a's and b's by z_k's,

$$F = K_c \, z^{m_o} \prod_k (z - z_k)^{m_k} \tag{4c}$$

or

$$F = K_n \, z^{m_o} \prod_k \left(1 - \frac{z}{z_k}\right)^{m_k} \tag{4n}$$

The factor z^{m_o} has been written separately in each case so that $z_k \neq 0$. The two expressions are seen to be equivalent, their constants being related by the formula

$$\frac{K_n}{K_c} = \prod_k (-z_k)^{m_k} = (-1)^{\sum m_k} \prod_k z_k^{m_k} \tag{5}$$

For a rational function, each exponent m_k is a positive or negative integer, and K_c or K_n and each z_k is a constant which may in general be complex. Each point $z = z_k$ is then either a zero or a pole of the function, depending on the sign of the corresponding m_k. A pole can therefore be looked upon as a zero of negative order, and vice versa.

The functions encountered in network theory are usually rational functions with real coefficients, in which case K_c or K_n is real, and complex z_k's occur only as complex conjugate pairs. But the computer theory developed in this thesis is not limited to such functions, or even to rational functions. It will handle a function expressed in either of the above forms, in which in general K may be complex, the z_k's need not occur in conjugate pairs, and the m_k's must be real but need not be integers. When any m_k is not an integer, the function is irrational, and the corresponding point z_k is a branch point, rather than a zero or pole. Although the characteristics of branch points in the theory of functions are much different from those of zeros and poles, a branch point represented by a fractional exponent in either of equations (4) may be looked upon for purposes of quantitative evaluation as a hypothetical "zero or pole of fractional order." Therefore it will be understood that whenever zeros and poles are mentioned in this thesis, such branch points are also included.

Computing Aids for Algebraic Functions

One powerful method for the quantitative evaluation of any function which can be expressed in the form of either of equations (4) is the graphical technique. In the z plane, the magnitude and phase of a factor $(z - z_k)$ are simply the length and direction angle respectively of the straight line from z_k to z. Therefore when all the zeros and poles of a function have been plotted, its magnitude for any particular value of z can be determined by measuring the distance from each zero and pole to point z, raising each value to the proper exponent and multiplying on a slide rule. The phase of the function can be determined graphically by measuring with a protractor the direction of the line joining each zero and pole to point z, multiplying each angle by the value of the exponent in the corresponding factor, and adding or subtracting. A number of authors have described applications of the graphical method and the underlying geometrical concepts to various problems in electrical networks and electromechanical feedback systems, or servomechanisms.[6,7,8] Machines operating on these geometrical relationships have also been built in which the multiplication of distances and the summation of angles is performed mechanically or electrically.*[9]

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* A simple mechanical aid for graphical solutions is the "Spirule," available from Walter R. Evans, North American Aviation, Inc., Downey, California. A more complicated electromechanical instrument is the "Complex Plane Analyzer," built by the Technology Instrument Corporation at Acton, Massachusetts.

Another method which has been used to evaluate algebraic functions in the form of equations (4) makes use of the distribution of electrical potential, or voltage, produced on a uniform resistive sheet or electrolytic tray.[9-16] In this method, current sinks and sources in the sheet are located at the zeros and poles of the function, and the current at each of these points is proportional to the exponent on the corresponding factor. When a current flows to a single sink at a point z_k in an infinite uniform sheet, the voltage at any other point z relative to a reference point z_r is proportional to

$$\log |z - z_k| - \log |z_r - z_k|$$

By the principle of superposition, simultaneous currents of magnitude and polarity m_k at each zero or pole z_k will result in a voltage distribution relative to point z_r which is proportional to

$$\sum_k m_k \left(\log|z - z_k| - \log|z_r - z_k|\right)$$

or

$$\log \frac{\prod_k |z - z_k|^{m_k}}{\prod_k |z_r - z_k|^{m_k}}$$

Multiplying the numerator and the denominator by the same arbitrary constant, and writing separately the factor representing any current at $z_k = 0$, this can be written as

$$\log \frac{K_c \, z^{m_o} \prod_k |z - z_k|^{m_k}}{K_c \, z_r^{m_o} \prod_k |z_r - z_k|^{m_k}} \qquad (6c)$$

or

$$\log \frac{K_n \, z^{m_o} \prod_k \left|1 - \frac{z}{z_k}\right|^{m_k}}{K_n \, z_r^{m_o} \prod_k \left|1 - \frac{z_r}{z_k}\right|^{m_k}} \qquad (6n)$$

Comparing equations (6) with equations (4), it is seen that the voltage distribution on the sheet is an analog of log $|F|$. A voltmeter connected between any point z and a reference point z_r will yield a reading proportional to log $\frac{|F(z)|}{|F(z_r)|}$. With suitable logarithmic scale calibration, the voltmeter can be made to read directly the value of $|F|$ relative to its value at $z = z_r$. Thus a large number of relative magnitude values can be obtained as rapidly as the z contact can be moved and the meter read. But when actual values of $|F|$ are required, the relative values must be multiplied by the value at $z = z_r$, which must be obtained by some other means.

Since the voltage distribution on a uniform resistive sheet satisfies Laplace's equation in two dimensions, the voltage can be represented by either the real or the imaginary component of an analytic function of a complex variable, and its value at every point will remain unchanged in any conformal transformation. Since the coordinate systems previously discussed are all conformally related, the electrical potential analog method can theoretically be used in any of them, on an infinite sheet. But the construction of a practical analog on a finite sheet of resistive material requires that boundary conditions be met. The only practical boundary conditions on a sheet are conducting boundaries (electrodes) and insulating boundaries (cuts). The boundaries of the actual sheet must therefore lie along streamlines and equipotential lines in the analog of the function. As a result of the symmetry of rational functions with real coefficients, the lines along which $\underline{/z}$ is a multiple of π (including zero) are always streamlines. Therefore the sides of a log z strip analog of such a function can be cuts along any of these lines.[11] This symmetry requirement could be eliminated by the use of a log z cylinder instead of a log z strip, for functions with complex coefficients. Edge effects in the finite approximation of any of the infinite coordinate systems will be minimized if the actual sheet covers an area much greater than that containing zeros and poles of the function. In log z coordinates, the effect of finite length is negligible when the ends of the strip or cylinder extend only two or three decades beyond the region containing zeros and poles. But for similarly negligible edge effects in the z plane or 1/z plane, the radius of the sheet would have to be very much (a hundred to a thousand times) larger than that of the region containing zeros and poles; on a sheet of reasonable size, the region of interest would be too small for accurate measurements. A technique which has been used to minimize edge effects consists of inverting, or "folding under," the region outside a certain radial distance from the origin, to form a double layer sheet.[12] The two layers are circular regions of the z plane and the 1/z plane joined together at the rim. It can be seen that this is actually a special case of the cylindrical box coordinate system suggested in this thesis, in which the cylindrical surface has been shrunk to zero, bringing the ends of the box together as a double layer sheet. The cylindrical box construction could also be used as a means for eliminating the end effect on a log z cylinder to make the cylinder shorter; zero and pole electrodes on the cylindrical surface could then be located as close to the ends as desired. In the most general case, all three surfaces of a cylindrical box of resistive sheet material could be used in the analog, the locations of electrodes and of voltages being referred to the proper coordinates on each part.

Unfortunately it is not practical in general to set up a potential analog for the phase of a function. It can be shown, however, that the value of the phase can be obtained from the magnitude analog by integrating along a contour the voltage normal to the contour. This integral has been approximated by summing the voltages from a large number of voltage contact pairs distributed along a suitable contour.[12,13]

PART II

EVOLUTION OF A NEW COMPUTER

Research Goal

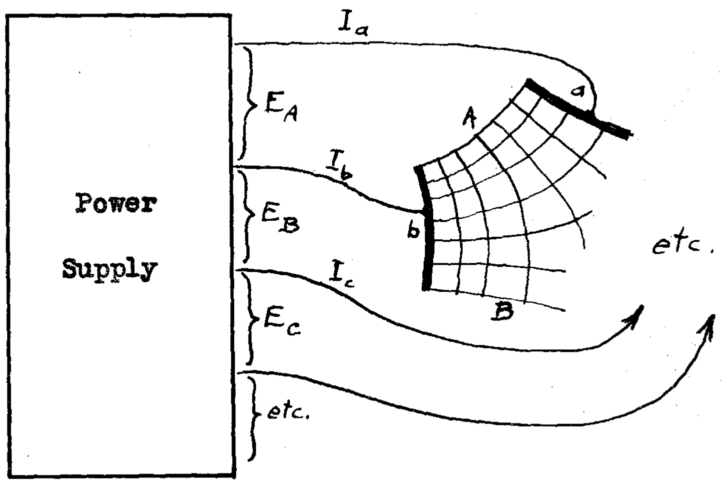
The method of electric potential analogs is seen to have many attractive features, but it has two serious limitations for the practical evaluation of functions. First, it provides no simple method for determining the phase of the function. Second, it yields only relative values of the magnitude of a function; when the absolute value is required, it must be supplemented by some other method. The use of an analog on one of the coordinate surfaces in three dimensions which have been described might provide a slight advantage over the plane analogs previously used, but it would not overcome either of the two principal difficulties.

It is well known that the theory of conjugate analytic functions provides an extremely powerful method for analyzing electric potential distribution and current flow in uniform resistive sheets.[17] Therefore it was felt that further study might lead to some better technique for the evaluation of rational algebraic functions—one in which measurements on electric potential distributions would be used in some manner different from that of the direct analog method. The subsequent investigation was successful, leading to a practical computer for evaluating both the magnitude and the phase of any rational algebraic function expressed in either of the forms of equations (4). This computer is applicable to functions in which complex zeros or poles do not occur in conjugate pairs, even though a log z strip is used rather than a log z cylinder. It is also applicable to irrational functions which can be expressed in the form of equations (4) with fractional exponents.

Dual Sheets

When electric current flows in a uniform layer or sheet of resistive material, the voltage contours or equipotentials and the current streamlines obey the same geometric laws as the contours of the real and imaginary components of an analytic function of a complex variable, since they all satisfy Laplace's equation in two dimensions. If an analytic function F can be found such that either Re(F) or Im(F) is proportional to the voltage at every point on the boundaries of a piece of uniform resistive sheet, the same component of F will describe the voltage distribution over the whole sheet. The boundaries of a sheet ordinarily consist of conducting boundaries (electrodes), which must be segments of equipotential lines, and insulating boundaries (cuts), which must be segments of streamlines. It is possible under certain conditions to interchange the equipotential lines and the streamlines on a sheet, or to construct a congruent pair of sheets on which they are interchanged, by interchanging the conducting and insulating boundaries and supplying the proper voltages and currents. This is illustrated in Figure 4. When this can be done, if the voltage distribution on one sheet is proportional to Re(F), the voltage distribution on the other sheet will be proportional to Im(F). Such a pair of sheets would obviously be useful for the evaluation of both components of F by means of voltage measurements.

For any voltage distribution on a uniform sheet, each curvilinear rectangular element of area formed by equipotentials and streamlines can be looked upon as a resistor, the resistance of which is equal to the resistivity of the sheet times the length of the element divided by the width of the element. If all the elements of a sheet were replaced by their equivalent resistors, an equivalent circuit for the sheet would be obtained. If such equivalent circuits were drawn for a pair of sheets with interchanged equipotentials and streamlines, they would be dual networks;[18] such a pair of sheets will therefore be called "dual sheets." But a dual network exists only for a planar network, and a dual sheet is possible only if the original sheet and the necessary external circuit form a planar network. When the sheet is a simply connected region, i.e., when it is bounded by electrodes and cuts distributed along a single closed contour, the circuit is planar and a dual can be found. Each sheet in Figure 4 is



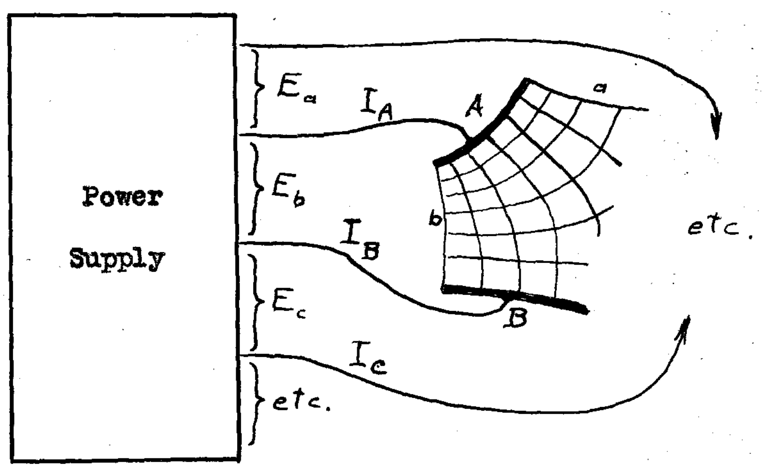


Figure 4

Dual Sheets

such a sheet. But if the sheet is a multiply connected region, i.e., if it is bounded by two or more separate contours, and if a net current flows through an external path between the electrodes on one contour and those on another, the circuit is inherently non-planar. Figure 5 illustrates this kind of sheet. Another way of looking at this limitation is to note that the current flowing through an electrode on one sheet corresponds to the voltage difference between the ends of the corresponding insulating boundary segment on its dual. Hence if the algebraic sum of the currents through all the electrodes on a closed contour on the original sheet is not equal to zero, the sum of the voltage drops around the same contour on the dual sheet would not be equal to zero, which is impossible.

A point current sink (or source) on a sheet would exhibit an infinite voltage; therefore on any actual sheet, a sink must be replaced by an electrode of finite size corresponding to an equipotential line surrounding the theoretical point sink. Any sheet containing current sinks isolated from the other boundaries of the sheet is non-planar, and no dual exists for the sheet as it stands. However, if lines exist which are always streamlines for any distribution of current to be encountered, then the sheet could be cut along any of these lines without disturbing the voltage distribution. If such lines connect all the electrodes on a sheet, the sheet can be made a simply connected region by cutting along such

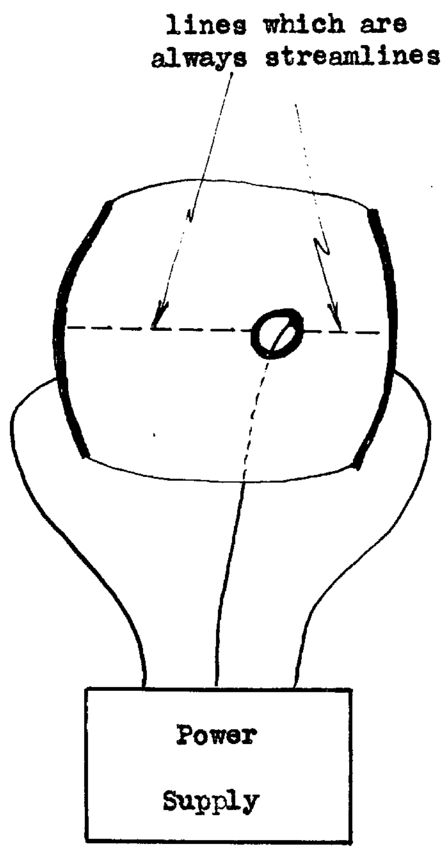
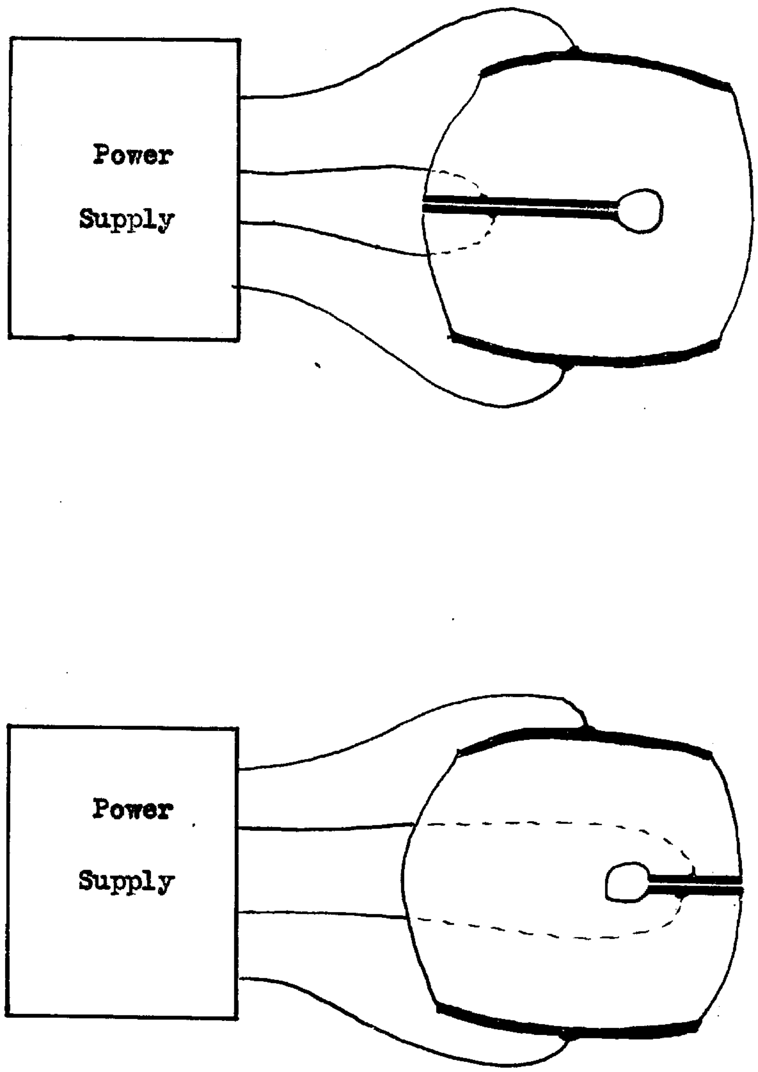


Figure 5

Sheet with Non-Planar Circuit



<u>Figure 6</u>

Alternate Duals of Figure 5

lines, and a dual then exists for the sheet with these cuts. An insulating cut in the original sheet becomes a "conducting cut" in the dual, i.e., a cut faced with a pair of strip electrodes, as though the knife making the cut had caused the edges of the sheet to become conducting as it passed through the sheet. For example, if the dotted lines in Figure 5 are always streamlines, the dual of the sheet will be a sheet which is made a simply connected region by conducting cuts along any of the corresponding lines, as shown in Figure 6. Of course the original sheet of such a dual pair need not actually be cut; it is only necessary that the conducting cuts in the new sheet be located along lines which could have been insulating cuts in the original sheet.

A New Computer Technique

In the direct analog method of evaluating the magnitude of a rational algebraic function F, the voltage is proportional to $\ln |F|$, which is $\underline{Re}(\ln F)$; the voltage on a dual sheet would therefore represent $\underline{Im}(\ln F)$, which is $\angle F$. But since the magnitude sheet contains current sinks and sources at all the zeros and poles of the function, the phase analog sheet would have to have conducting cuts connecting all the zeros and poles along lines which were current streamlines in the magnitude analog. Such a phase analog would not be practical in a computer for analytic functions in general, since each function would have different zero locations and would therefore require the construction of a new sheet with different conducting cuts. If any sort of dual sheet analogs are to be used in a practical computer, it is obvious that each sheet should contain permanent cuts and electrodes. An actual sheet would then be a finite portion of the theoretical infinite sheet bounded by conducting and insulating boundaries along lines which would always have been equipotentials and stream-lines respectively on the infinite sheet.

A method of simplifying the analog so that this is possible is suggested by the principle of reciprocation. In the direct analog method, superimposed currents flowing to sinks at the zeros and from sources at the poles of a function F, each current weighted in proportion to the order of the zero or pole, produce a voltage at every point z which is proportional to log $|F|$ at that value of z. According to the theorem of reciprocation, similar results will be obtained if current flows to a sink at the particular point z of interest, and if the voltages at the zeros and poles of the function are superimposed—i.e., weighted and added in a "summing voltmeter" circuit. The particular value of z at which the function is being evaluated will be called z_o. In this method z_o will be the only point current sink in the sheet on which the magnitude of a function is evaluated, and z_o on the dual sheet will be the only point which must be connected to the external boundaries by a conducting cut. Point z_o will of course be moved about in the coordinate system for the evaluation of the function at different values of z. Since one of the boundaries on either sheet will be a small contour around z_o, and since all the boundaries are to be permanently attached to the sheet, a change in z_o must be accompanied by a movement of the whole sheet relative to the coordinate system.

If, for example, a sheet can be set up with a fixed voltage distribution about a point z_o such that the voltage at any other point z_k, relative to a suitable voltage reference point, is proportional to $\underline{\text{Re}}[\ln(z_o-z_k)]$, then a dual sheet can be set up on which the voltage at z_k is proportional to $\underline{\text{Im}}[\ln(z_o-z_k)]$. Since the real component of the logarithm is the log magnitude and the imaginary component of the logarithm is the phase, the two sheets in any such dual pair will be called the "magnitude sheet" and the "phase sheet" respectively. If in the summing voltmeter each input voltage from a contact at a point z_1, z_2, etc., is multiplied by a corresponding arbitrary positive or negative constant m_1, m_2, etc., before the voltages are added, and if another input to the summing voltmeter is connected to a manually adjustable voltage representing the real or imaginary component of an arbitrary constant ln K, then the voltage reading from the magnitude sheet or the phase sheet in the above example will be proportional to the real or the imaginary component of

$$\ln K + m_1 \ln(z_o - z_1) + m_2 \ln(z_o - z_2) + \cdot \quad \cdot \quad \cdot$$

If the meter is provided with suitable scales, it can be made to read directly the magnitude or phase of a function

$$K\,(z_o - z_1)^{m_1}\,(z_o - z_2)^{m_2}\,\cdot\;\cdot\;\cdot\;\cdot \qquad (7c)$$

Similarly, if a dual pair of sheets can be set up so that the voltage at z_k is proportional to $\underline{Re}\left[\ln(1 - \frac{z_o}{z_k})\right]$ and $\underline{Im}\left[\ln(1 - \frac{z_o}{z_k})\right]$ respectively, then the meter will read the magnitude or phase of the function

$$K\,(1 - \frac{z_o}{z_1})^{m_1}\,(1 - \frac{z_o}{z_2})^{m_2}\;\cdot\;\cdot\;\cdot\;\cdot \qquad (7n)$$

These two examples will be considered as goals in the design of a practical computer, since they correspond to the two forms into which a rational algebraic function commonly is factored, given in equations (4c) and (4n). It will be shown later that such a computer in the z plane coordinate system is applicable to the evaluation of functions factored in the conventional form (4c); a computer in the 1/z plane is applicable to functions in the normalized form (4n); and a computer in log z coordinates is applicable to functions factored in either form.

<u>Mechanical Requirements in a Practical Computer</u>

In a computer using any of the plane coordinate systems already described, it will be shown later than the resistive sheets are to be moved in translation only. The relative motion between the sheet and the coordinate system should therefore for convenience be mechanically constrained against rotation; the requirements for such a mechanism are similar to those for a drafting machine. If a log z cylindrical coordinate system is used, the sheet will be a cylindrical surface; the components of the required relative motion are then a sliding along the axis of the cylinder and a rotation about the axis of the cylinder. In either the plane or the cylindrical case, it will of course be immaterial whether the sheet or the frame representing the coordinate system actually moves. Since the value of the complex variable z at which a function is being evaluated is the position of point z_o in the coordinate system, and since z_o is permanently located on the sheet, the position of the sheet relative to the coordinate system can be calibrated in terms of components of z. If the constraining mechanism is calibrated in this manner, no other coordinate markings are necessary; when a voltmeter contact is to be placed at a particular point z_k, the position can be found by setting $z = z_k$ and placing the contact opposite point z_o on the sheet. It will be helpful, however, if the mechanism is also provided with some means for tracing the relative motion of the sheet (the contour of z) on a piece of paper, since one important application of such a computer is the determination by trial (i.e., by intersection of loci) of the value of z at which a function has a specified value.

Since measurements of the magnitude and the phase of a function are made on separate sheets, each sheet could be provided with a separate set of measuring equipment. This would include a coordinate frame in which voltmeter contacts would be placed at the zeros of the function; a mechanism by which the relative motion would be properly constrained and from which the readings of the complex variable z would be obtained; and a separate summing voltmeter on which the magnitude or the phase of the function would be indicated. This would be a needless duplication of equipment, however, and it would complicate the setting up of a function by requiring the placement of two separate sets of voltmeter contacts at the zeros and poles of the function to be evaluated. The more practical approach is that of providing the summing voltmeter with a double scale for reading either magnitude or phase, and building the equipment so that either sheet can be placed in position in the coordinate system. Again it is immaterial whether the sheets or the coordinate frame are actually moved.

Resistive Sheet Materials

The choice of a material for the resistive sheets requires some consideration. Techniques using voltage measurements in a tray or tank of salt water or other electrolyte have been used for many years for analog solutions to appropriate problems, and they are said to be capable of very accurate results if sufficient care is used in the construction and maintenance of the equipment.[12] However, there are a number of limitations and inconveniences in the use of an electrolytic tray. The bottom of the tray must be extremely flat and very carefully leveled; liquid lost by evaporation must be replaced; the electrodes must be carefully prepared if their contact resistance is to be kept low and uniform; and only alternating current can be used, in order to avoid polarization at the electrodes. In recent years, Teledeltos paper has become increasingly popular as a substitute for an electrolytic tray.[9,13] Although its uniformity of resistance may be questioned, it has been found adequate by various investigators for many analog problems, and it is very easy to use. It can be mounted in any position, and it can be wrapped around curved surfaces; when cemented to a hard insulating surface such as a sheet of Bakelite, it is quite durable. Electrodes with negligible contact resistance are simply painted on it with conducting paint, and either alternating or direct current can be used on it.

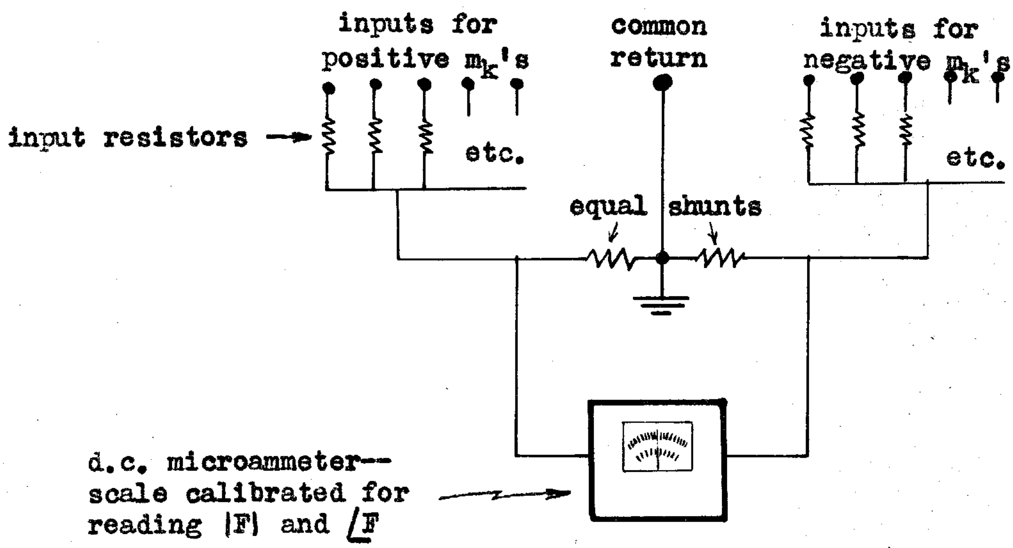
The Summing Voltmeter

In the summing voltmeter circuit required for this type of computer, several input voltages V_k are to be multiplied by arbitrary coefficients m_k and added together. If the meter were actually calibrated in voltage, the meter reading would be

$$V_{meter} = \sum_k m_k V_k \qquad (8)$$

But in a computer the individual input voltages will represent either the log magnitude or the phase of a factor of a function F. Therefore the meter will instead be provided with a logarithmic scale for reading $|F|$ directly, and a linear scale for reading $\angle F$ directly. In order to include factors in both the numerator and the denominator of a function, provision must be made for both positive and negative m's, as illustrated in the basic circuit of Figure 7. A more accurate and sensitive meter circuit using a vacuum-tube feedback summing amplifier can of course be used if desired. For the most general (irrational) function which can be evaluated in this type of computer, each m_k can have any real value; for such functions, the input resistors in the meter circuit would have to be individually adjustable. For rational functions, however, each m_k will be an integer. In this case all the input resistors in Figure 7 can be equal, representing $m_k = \pm 1$, and the voltage for any factor with an exponent with a magnitude greater than one can be connected to $|m_k|$ separate input terminals of the

For D.C.:



For A.C.:

Same as above, except:

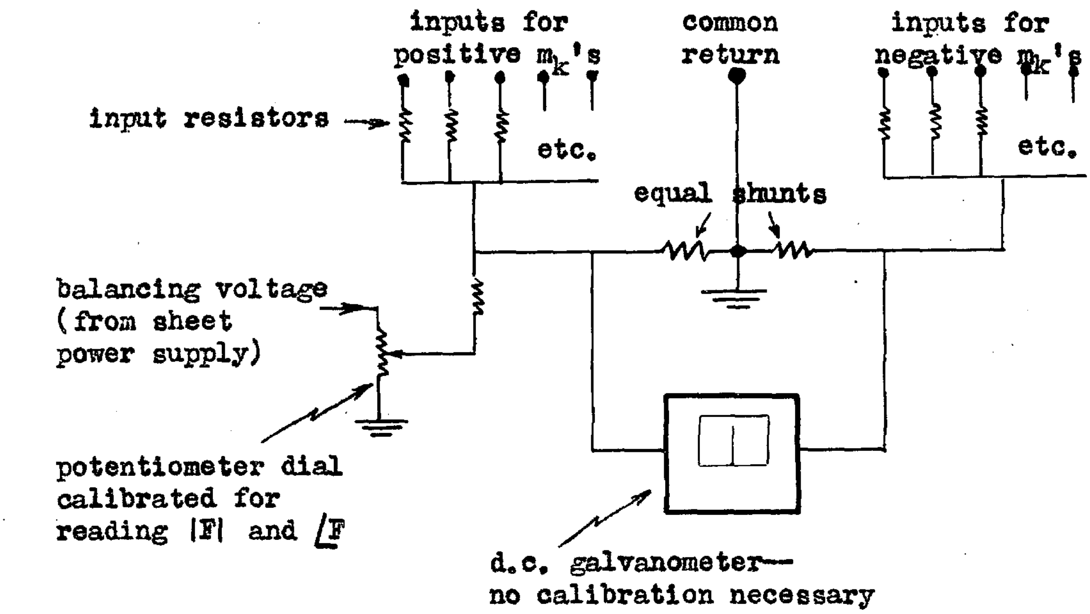
1. d.c. microammeter must be replaced by a.c. meter which will respond to "polarity" or "phase" of a.c. (for example: d.c. meter with synchronous input rectification by a diode switching circuit)

2. shunts may be replaced by transformer with center-tapped primary

Figure 7

Direct-Reading Summing Voltmeter

For D.C.:



For A.C.:

Same as above, except:

1. d.c. galvanometer must be replaced by a.c. null detector which need not recognize polarity of a.c. (for example: a.c. amplifier and tuning eye)

2. shunts may be replaced by transformer with center-tapped primary

Figure 8

Null-Balance Summing Voltmeter

meter circuit. Since rational functions are likely to be encountered much more frequently than irrational functions, it is practical to construct the summing voltmeter circuit with equal internal input resistors to accommodate rational functions, and to add extra resistance externally when a fractional exponent is encountered in a function. The most common fractional exponent is 1/2, for which the external resistance added will be equal to the internal resistance.

The "meter" can be either an actual direct-reading meter as shown in Figure 7 or a null-balance arrangement as shown in Figure 8. Although a direct-reading meter is slightly more convenient in some applications, a computer using a null-balance arrangement is likely to be both simpler and more accurate. It is not the actual voltage reading which represents the function, but rather the ratio of the voltage to the total voltage applied across the electrodes on the sheet. A direct-reading meter will therefore be in error if the power supply voltage changes, but a null-balance circuit will yield readings of the desired ratio independent of the power supply voltage, if the balancing voltage varies in direct proportion to the voltage on the sheet. A further disadvantage of the direct-reading meter is that it must be polarity sensitive, a requirement which complicates the circuit if a.c. voltage is used on the sheets; a null indicator need only indicate the presence or absence of voltage, without regard to polarity.

In practical use in a computer, the manual dial setting required with a null type meter is of little disadvantage, since the time required to make the null adjustment is small compared with the time required to set up the problem. And in locus problems, the null type meter is actually an advantage—in determining the locus of z for a particular value of $|F|$ or $\angle F$, the value is set on the dial and the locus of null indications traced.

Since the range of readings (voltages) which will be encountered for various functions will be very large, it will be necessary to provide meter readings in "ranges" or "bands," so that when the function reaches a value where the meter or the dial runs off the end of the scale, an increment of current can be added which is just sufficient to bring the meter or dial to the other end of the scale. For convenience in reading the different scales, increments representing integral powers of 10 should be used with the magnitude scale, and increments representing multiples of 90°, 180°, or 360° should be used with the phase scale. A difference between two phase readings which is a multiple of 360° is meaningless, the function actually being the same. Therefore increments of 360° added to the phase readings need not be calibrated, but the increments must be provided in the instrument, since voltages over a range representing several times 360° may be encountered.

PART III

COMPUTERS IN VARIOUS COORDINATE SYSTEMS

Computer Coordinate Systems

In a computer of this type, each factor of a function is represented by a voltmeter contact touching the resistive sheet and located at the zero or pole of the factor. The coordinate system in which the zeros and poles are located is defined by the components of some simple function of z--for example, z, 1/z, or ln z. This will be the principal coordinate system of the computer, but since the voltage distribution on the resistive sheet will remain fixed on the sheet as the sheet is moved about in this coordinate system, it will be found convenient in the analysis of the computer to define a second coordinate system fixed on the sheet. For a principal coordinate system described by any function of z, the secondary coordinate system will be defined by the components of the same function of w--namely, w, 1/w, or ln w. The origin in the secondary coordinate system will be taken at point z_0 on the sheet, so that when z_0 is placed at the origin in the principal coordinate system, w will be everywhere equal to z. Since only translation between the two coordinate systems will be found necessary in a computer, rotation will be excluded, and the coordinate transformation equations will be as follows:

Between the z plane and the w plane: $z = z_o + w$

Between the 1/z plane and the 1/w plane: $1/z = 1/z_o + 1/w$

Between the log z and the log w plane or cylinder:

$$\ln z = \ln z_o + \ln w, \text{ or } z = z_o w$$

Since it is desired that z_o be the only point current sink in the theoretical infinite magnitude sheet, the current flowing to z_o must come from infinity on the theoretical sheet. But since the infinite part of the sheet will correspond to different values of z in different coordinate systems, the factors which can be represented on a dual pair of sheets will be different in different coordinate systems.

The factor represented by the voltage at any point z_k will also be dependent on the voltage reference point--the point on the sheet relative to which the voltage at z_k is measured. The voltage reference point can be either a contact fixed in the principal coordinate system or a terminal attached to the sheet. If it is a contact fixed in the principal coordinate system, it will be described by an arbitrary constant z_r in this coordinate system; if it is a terminal attached to the sheet, it will be specified by an arbitrary constant w_r in the secondary coordinate system.

Various possible voltage distributions and voltage reference points will be discussed separately for computers in each of the three basic coordinate systems--z, 1/z, and log z.

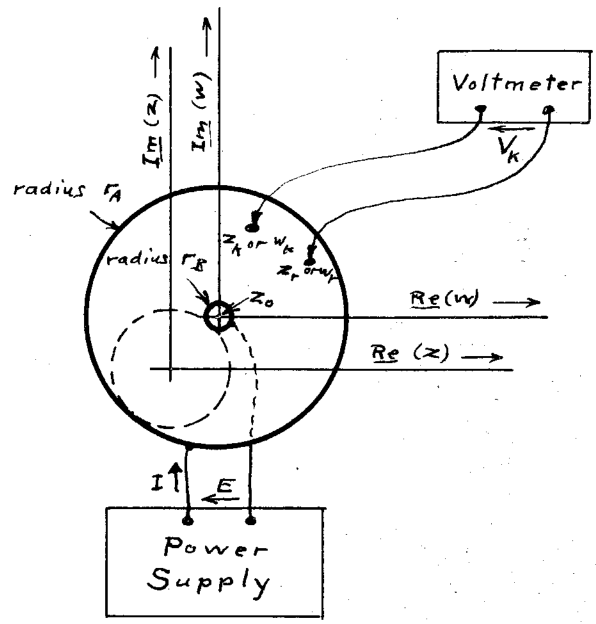
A Computer in the z Plane

In the z plane, each value of z_o appears at only one point; therefore the only possible current distribution on the magnitude sheet is a flow between $z = \infty$ and $z = z_o$ along radial lines through z_o. Any straight line through z_o is therefore suitable for the required conducting cut on the phase sheet. The magnitude and phase sheets for a computer in the z plane will therefore be as illustrated in Figure 9. The sheet must be large enough so that some part of it always covers the region in which the voltmeter contacts are located. It can be seen in Figure 9 that the voltmeter contacts and the point z_o at the center of the sheet can thus be placed anywhere inside the dotted circle half the diameter of the sheet and centered on the origin of the z coordinates.

The voltage and current equations for this computer can be derived by transforming the sheets of Figure 9 into ln w coordinates, as shown in Figure 10. As viewed from the power supply, the equivalent resistance of each rectangular sheet in Figure 10 is equal to the sheet resistivity times the length divided by the width of the sheet. The simplest equivalent circuits for the magnitude and phase sheets are therefore as indicated in Figure 11; their relationship as dual networks inverted about resistance σ is obvious.

On the magnitude sheet in Figure 10, it can be seen that the voltmeter reading V_k at any point w_k, relative to the voltage at a reference point w_r, is

Magnitude Sheet:



Phase Sheet:

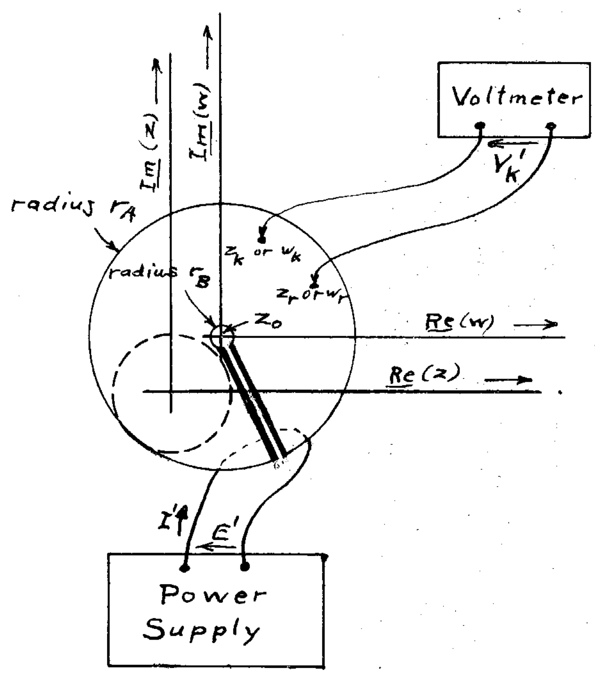
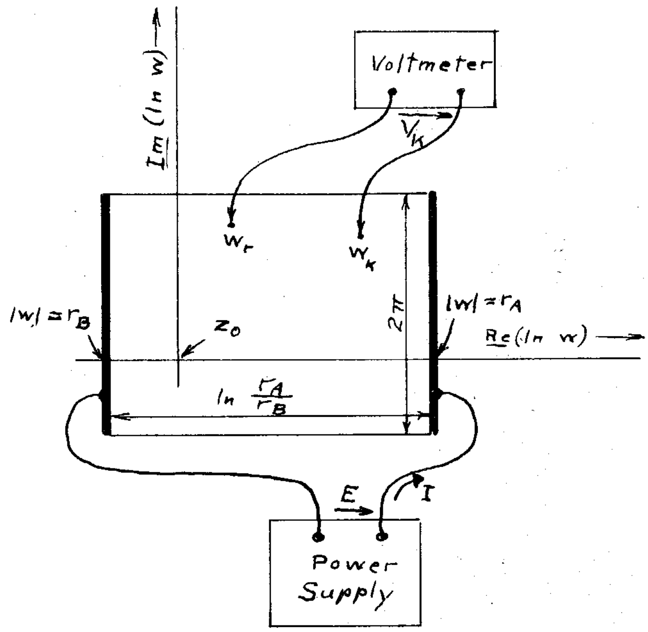


Figure 9

Computer in the z Plane

Transform of Magnitude Sheet:



Transform of Phase Sheet:

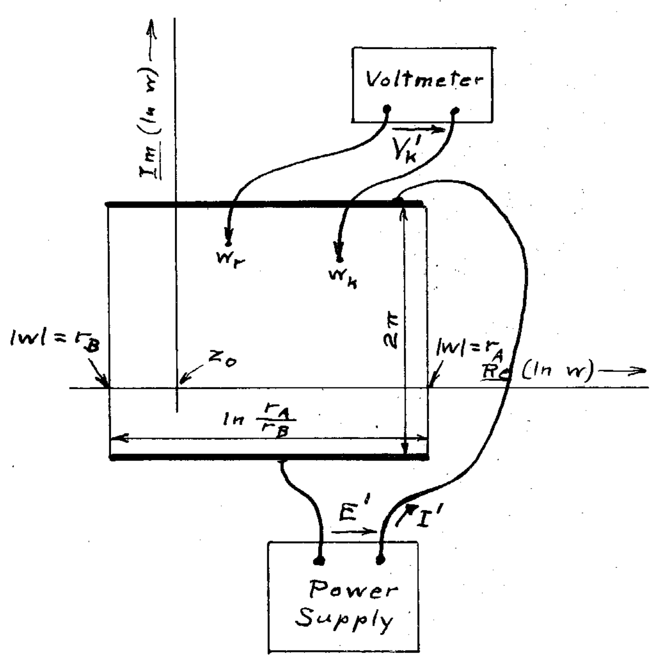
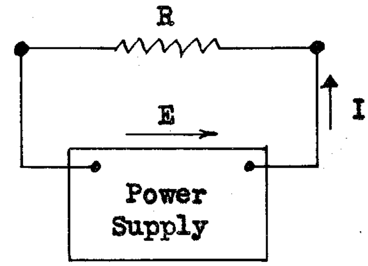


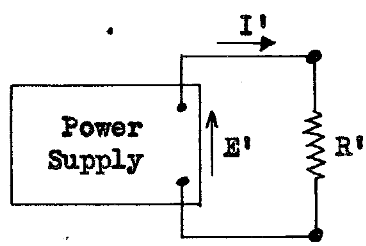
Figure 10

Transformation of Figure 9 into the Log w Plane

Equivalent Circuit of Magnitude Sheet:



Equivalent Circuit of Phase Sheet:



$$\frac{R}{\sigma} = \frac{\sigma}{R'} = \frac{\ln \frac{r_A}{r_B}}{2\pi} = \frac{2.303}{2\pi} \log_{10} \frac{r_A}{r_B}$$

(σ = resistivity of the sheet material, in ohms)

Figure 11

Equivalent Circuits for Figures 9 and 10

$$V_k = E \frac{\underline{Re}(\ln w_k) - \underline{Re}(\ln w_r)}{\ln \frac{r_A}{r_B}} = \frac{\sigma I}{2\pi} \underline{Re}(\ln \frac{w_k}{w_r})$$

On the phase sheet, the voltmeter reading V_k' is

$$V_k' = E' \frac{\underline{Im}(\ln w_k) - \underline{Im}(\ln w_r)}{2\pi} = \frac{E'}{2\pi} \underline{Im}(\ln \frac{w_k}{w_r})$$

Substituting the coordinate transformation $w = z - z_o$, the voltmeter readings are

$$V_k = \frac{\sigma I}{2\pi} \underline{Re}(\ln \frac{z_o - z_k}{z_o - z_r}) \tag{9}$$

and

$$V_k' = \frac{E'}{2\pi} \underline{Im}(\ln \frac{z_o - z_k}{z_o - z_r}) \tag{10}$$

Since z_o represents the particular value of the variable z at which a function is being evaluated in the computer, the above voltages represent the logarithmic components of a basic factor

$$\frac{z - z_k}{z - z_r} \tag{11}$$

If the voltage reference point is a voltmeter contact fixed in the z coordinate system, z_r is an arbitrary constant, and its most useful choice would appear to be point $z_r = 0$, at the origin of the coordinate system. The factor is then

$$\frac{z - z_k}{z}$$

and the function evaluated by the computer would be

$$F = K \prod_k \left(\frac{z - z_k}{z}\right)^{m_k}$$

or

$$F = K \; z^{-\Sigma m_k} \prod_k (z - z_k)^{m_k} \tag{12}$$

Although this function looks very much like that of equation (4c), it is not the same--the exponent on z is not arbitrary.

As z approaches infinity, the function in (12) approaches K as a limit; therefore a computer with this reference point would be limited to functions which approach a constant value as z approaches infinity—i.e., functions which contain no zero or pole at $z = \infty$.

On the other hand, if the reference point is a terminal attached to the sheet, $z_r - z_o = w_r$ is an arbitrary constant, so that the basic factor becomes

$$\frac{z - z_k}{-w_r}$$

In this case the best choice is $w_r = -1$, yielding the simple factor

$$z - z_k$$

One of the voltmeter contacts can be placed at point $z_k = 0$ to represent a factor z; writing this factor separately, the function evaluated by the computer is

$$F = K \; z^{m_o} \prod_k (z - z_k)^{m_k} \qquad (13)$$

Since m_o is an arbitrary weighting in the summing voltmeter just like all the other m_k's, the function in this case is identical to that of equation (4c) and the computer is suitable for the evaluation of any algebraic function factored in the conventional form (4c). It should be noted that the choice $w_r = -1$ does not mean that the terminals on the magnitude and phase sheets must each be located at the complex point $w_r = -1$; the only requirement is that on the magnitude sheet $|w_r| = 1$ and on the phase sheet $\angle w_r = 180^\circ$.

A Computer in the 1/z Plane

In the 1/z plane, each value of z_o appears at only one point, just as in the z plane; therefore the magnitude and phase sheets will look exactly like those for the z plane, but the coordinate system will be different. Figures 9 and 10 can be applied to a computer in the 1/z plane simply by replacing z and w by 1/z and 1/w respectively; Figure 11 is applicable without change. The equations for the voltmeter readings in the 1/z plane will be obtained from (9) and (10) by changing z_o, z_k, and z_r to $1/z_o$, $1/z_k$, and $1/z_r$ respectively, so that

$$V_k = \frac{\sigma I}{2\pi} \underline{\text{Re}}\left(\ln \frac{\frac{1}{z_o} - \frac{1}{z_k}}{\frac{1}{z_o} - \frac{1}{z_r}}\right) = \frac{\sigma I}{2\pi} \underline{\text{Re}}\left(\ln \frac{1 - \frac{z_o}{z_k}}{1 - \frac{z_o}{z_r}}\right) \qquad (14)$$

and $$V_k' = \frac{E'}{2\pi} \underline{\text{Im}}\left(\ln \frac{\frac{1}{z_o} - \frac{1}{z_k}}{\frac{1}{z_o} - \frac{1}{z_r}}\right) = \frac{E'}{2\pi} \underline{\text{Im}}\left(\ln \frac{1 - \frac{z_o}{z_k}}{1 - \frac{z_o}{z_r}}\right) \qquad (15)$$

The voltages therefore represent the logarithmic components of the basic factor

$$\frac{1 - \frac{z}{z_k}}{1 - \frac{z}{z_r}} \qquad (16)$$

If the voltage reference point is fixed in the 1/z coordinate system, the logical choice is $z_r = \infty$, or $1/z_r = 0$, which is the origin of the 1/z coordinate system. The factor then reduces to

$$1 - \frac{z}{z_k}$$

and the function evaluated by the computer would be

$$F = K \prod_k (1 - \frac{z}{z_k})^{m_k} \qquad (17)$$

This function looks almost like equation (4n), but again as in equation (12) there is no arbitrary power of z. In this case, function (17) approaches K as z approaches zero, so that the computer would be limited to functions which approach a constant value as z approaches zero—i.e., functions which contain no zero or pole at $z = 0$.

If the reference point is a terminal attached to the sheet, $1/z_r - 1/z_o = 1/w_r$ is an arbitrary constant, and the basic factor becomes

$$\frac{1 - \frac{z}{z_k}}{-\frac{z}{w_r}}$$

The choice $w_r = -1$ reduces this factor to

$$\frac{1 - \frac{z}{z_k}}{z}$$

In this case, one of the voltmeter contacts can be placed at point $z_k = \infty$ (the origin of the 1/z coordinate system) to represent a factor 1/z. Writing this factor separately, the function evaluated by the computer is

$$F = K \left(\frac{1}{z}\right)^{m_\infty} \prod_k \left(\frac{1 - \frac{z}{z_k}}{z}\right)^{m_k}$$

or

$$F = K \; z^{-(m_\infty + \Sigma m_k)} \prod_k (1 - \frac{z}{z_k})^{m_k} \qquad (18)$$

Since m_∞ is arbitrary, it can be selected to yield any desired value of $m_0 = -(m_\infty + \Sigma m_k)$, so that the function is in effect identical to that of equation (4n) and the computer is suitable for the evaluation of any algebraic function factored in the normalized form (4n).

Computers in Log z Coordinates

It has just been indicated how a practical computer for algebraic functions could be built in either the z plane or the 1/z plane. The development of computers in log z coordinate systems is of much greater interest, however, both because of the useful characteristics of the coordinate system itself, and because of the variety of possibilities for the form of the function computed and the variety of means by which these forms can be attained in a practical computer. A computer for multiplying complex factors in log z coordinates might be likened to a "two-dimensional slide rule." Like the slide rule, such a computer has the useful characteristic that a given error in setting a position corresponds to a fixed relative error (e.g., percent or per unit error) in the quantity represented, regardless of the size of the quantity. The $|z|$ scale in log z coordinates is logarithmic, so that the analogy with a slide rule is obvious; for the extension of the concept to the $\angle z$ scale, it should be noted that a small percent error in $|z|$ corresponds in distance to the same percent of a radian. An error of 1% in $|z|$, for example, carries the same weight as an error in $\angle z$ of .01 radian. The remainder of this thesis will therefore be devoted to the consideration of computers in the log z coordinate system, the goal of the study being the realization of practical computer techniques in this coordinate system for functions in both of the forms (4c) and (4n).

Since both $z = \infty$ and $z = 0$ are infinitely distant in log z coordinates, the range of values of $|z|$ which can be reached must be specified by a maximum value $|z|_{max}$ and a minimum value $|z|_{min}$. If the magnitudes of z_o and all the z_k's are restricted to these limits, they will all be contained in a strip of the log z plane of length $\ln\frac{|z|_{max}}{|z|_{min}}$ and of width 2π. In order that the voltmeter contacts at the z_k's shall not run off the end of the sheet as z_o is moved to different points, the actual sheets must be twice this length. In log z plane coordinates, the sheets should for maximum convenience be of double width also, so that voltmeter contacts will not run off the sides of the sheet as z_o is moved about. It is possible to use single-width strips in a computer, however, by moving each voltmeter contact which runs off the side of the sheet a distance 2π in the $\angle z$ direction to a new position which represents the same value of z_k but lies on the sheet. In log z cylindrical coordinates the sides of the strip are eliminated and this problem is avoided.

The shape of the magnitude and phase sheets in log z coordinates will obviously be quite different from that in z and 1/z coordinates. In the log z plane, each value of z is repeated an infinite number of times; therefore each of the points z_o on the magnitude sheet must be an identical current sink. The current to these sinks can come from $z = \infty$ or $z = 0$, or simultaneously from both in any ratio, as sketched in Figure 12. It can be seen from symmetry that the horizontal

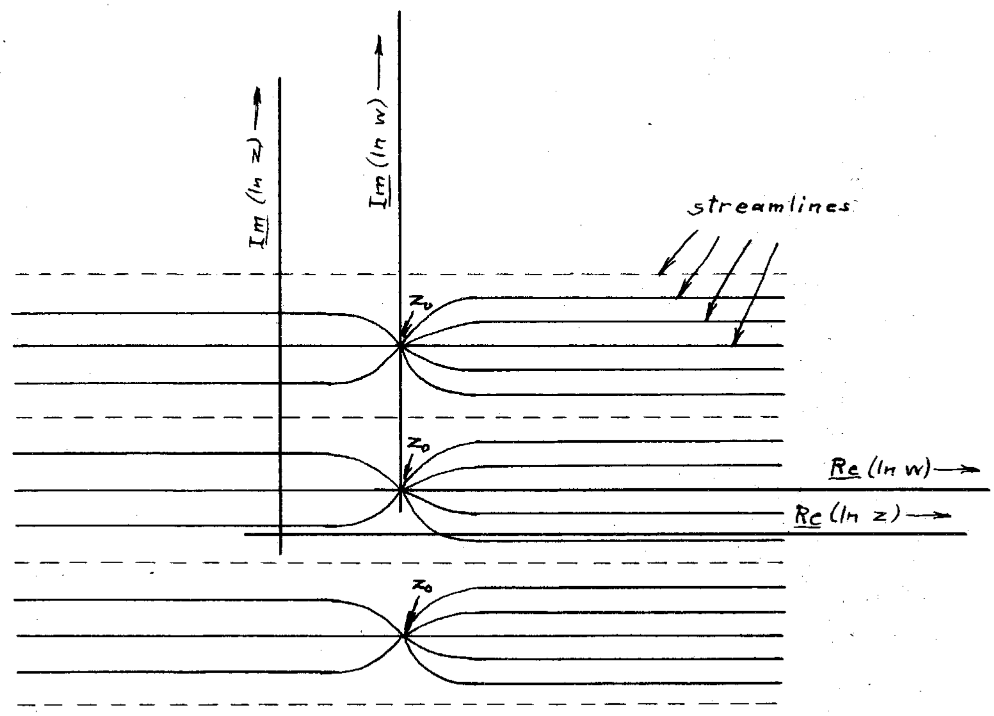


Figure 12

Magnitude Sheet in the Infinite Log z Plane

lines passing either through z_o points or midway between pairs of them will always be streamlines for any current ratio; any of these lines can be insulating cuts or boundaries on an actual magnitude sheet. It is also apparent that any vertical line far enough to the right or the left of the z_o points will be very nearly an equipotential line, and that any very small circle concentric with z_o will be approximately an equipotential line. Consequently, any of these contours can be conducting boundaries on the actual magnitude sheet. Pairs of dual sheets for a computer in the log z plane can therefore be constructed in any of the forms shown in Figure 13 for single-width strips and in Figure 14 for double-width strips. Although the essential conducting cuts in the phase sheets would extend only to the left or to the right from each z_o point, the phase sheets are shown cut completely apart along the lines through these points. When the actual sheet is constructed in this manner, with the adjacent strip electrodes separated by a thin insulating barrier, a pair of adjacent electrodes which are electrically joined in the external circuit corresponds to a continuation of the region across the boundary, while a pair across which voltage is applied represents the essential cut making the region single valued. The dotted lines midway between z_o points on the phase sheets are also always equipotentials; they too can be constructed as conducting cuts if desired.

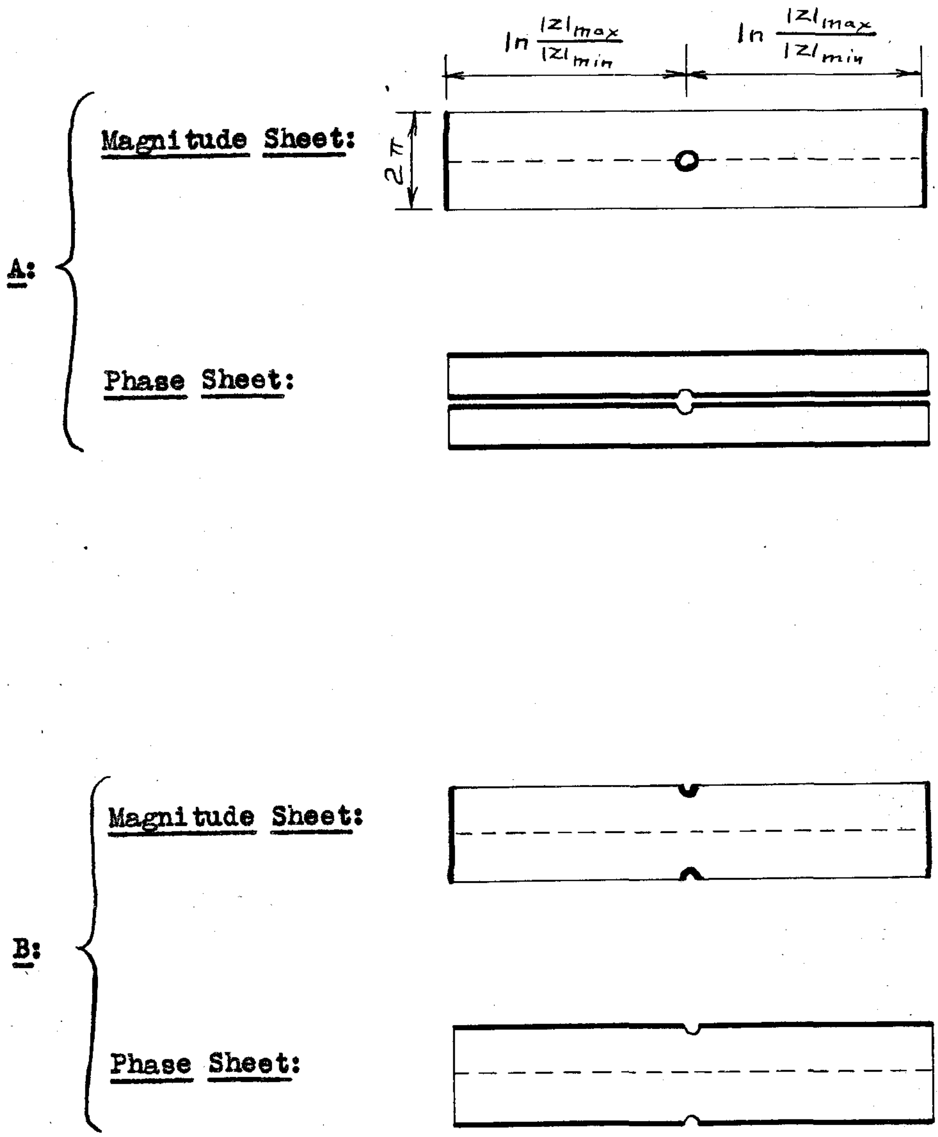


Figure 13

Alternate Single-Width Log z Strips

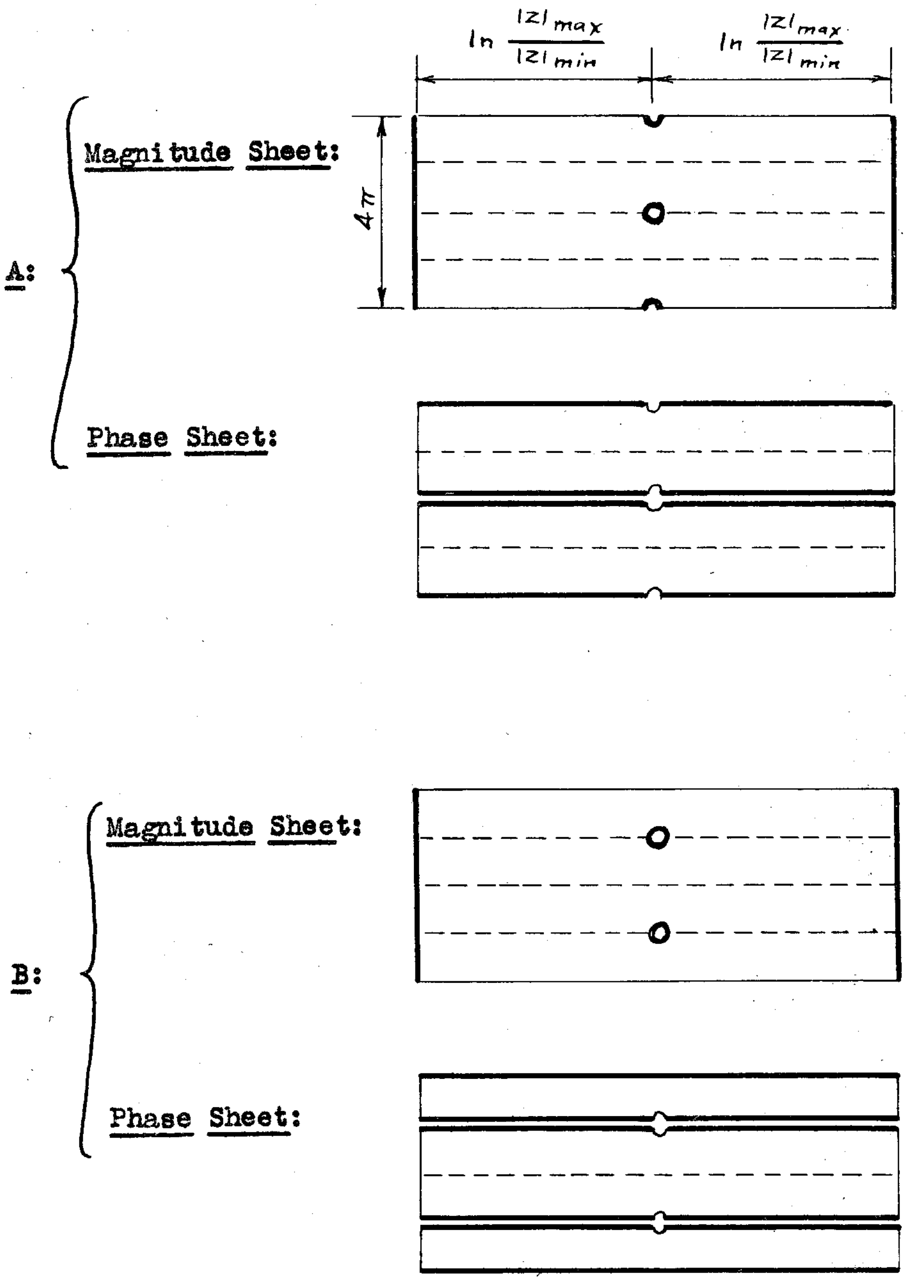


Figure 14

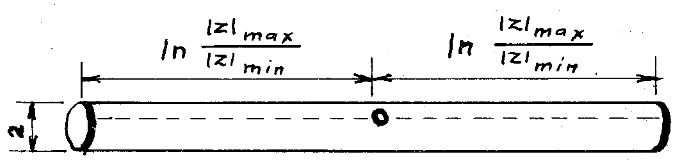
Alternate Double-Width Log z Strips

It is worth while to compare each of the pairs of single-width sheets in Figure 13 with the double-width pair of Figure 14-A. If the circles around the z_0 points in Figure 14-A are made very small, the same sheets can also be used with a suitable change in power supply connections as a single-width pair in the form of either A or B in Figure 13. The coordinate scale when connected as single-width sheets will be magnified to twice the original scale as double-width sheets, so that a "dual scale" computer can be built using sheets in the shape of Figure 14-A.

For a computer in log z cylindrical coordinates, a pair of single-width sheets are wrapped around cylinders or tubes, as shown in Figure 15. The permissible cuts make it possible to assemble a cylinder out of two separate half-cylinders if desired. The electrical characteristics of a computer using a pair of cylindrical sheets are therefore exactly the same as those of one using a pair of strips; the differences are only in the mechanical arrangement of the computer.

On any resistive sheet, a conducting cut, i.e., an actual electrode strip or pair of strips along a line which should always be an equipotential, can be used to improve the accuracy of the voltage distribution. A conducting strip will ensure that the line is actually an equipotential in spite of any non-uniformity in the resistivity of the sheet; and if it is connected to the proper voltage in the power supply, it will ensure that the equipotential is actually at the correct

Magnitude Sheet:



Phase Sheet:

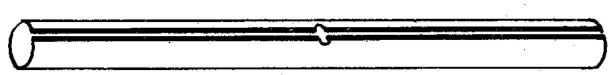


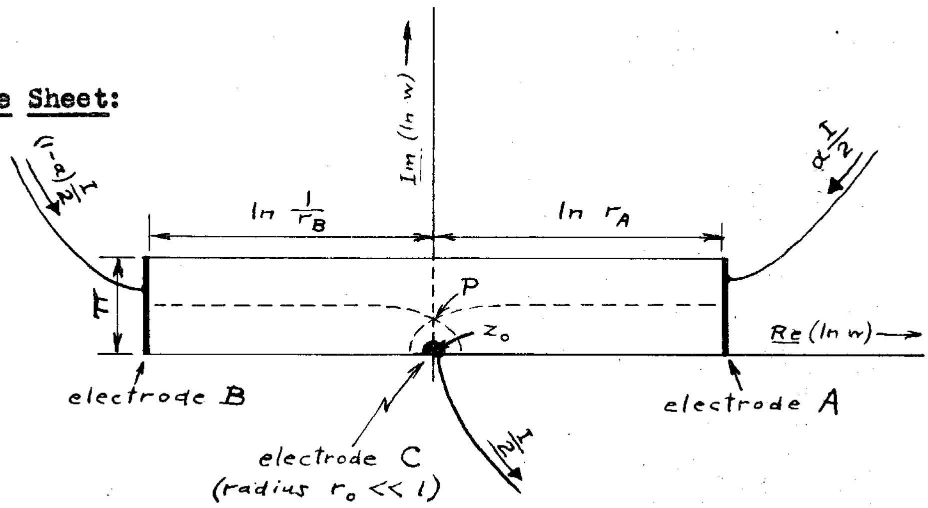
Figure 15

Log z Cylinders

potential in spite of non-uniformity in resistivity. An insulating cut along a line which is always a streamline, however, is a disadvantage. Non-uniformity can cause a discontinuity in voltage to appear between adjacent points on each side of an insulating cut. It is better for the sheet to be continuous, so that any such voltage error will be spread out and reduced in magnitude. Therefore when actual sheets for a computer are built in the form of any of the pairs in Figures 13, 14, or 15, cutting the phase sheet into narrower strips may improve the accuracy of the computer, but cutting the magnitude sheet along the dotted lines is likely to impair the accuracy, and should be avoided whenever possible.

Regardless of whether an actual magnitude or phase sheet is constructed in one piece or in several, the concept of the half-width strip as the basic building block in log z coordinates will be found useful. In representing a sheet by an equivalent circuit as viewed from the external circuit, the equivalent circuit for the actual sheet will be obtained by connecting together the equivalent circuits for the half-width strips of which it is composed. Figure 16 shows the basic half-width strips with currents and voltages labeled for the derivation of the voltage distribution equations. Each of these strips is a three terminal network. If an equivalent T network is chosen to represent the magnitude sheet, its dual Δ will represent the phase sheet. The strips of Figure 16

Magnitude Sheet:



Phase Sheet:

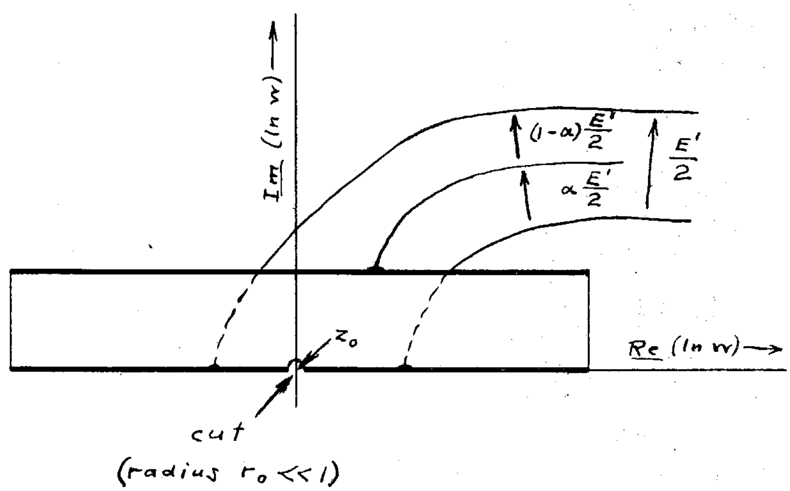
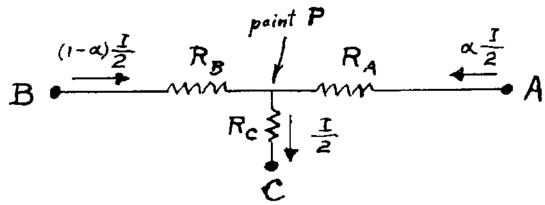


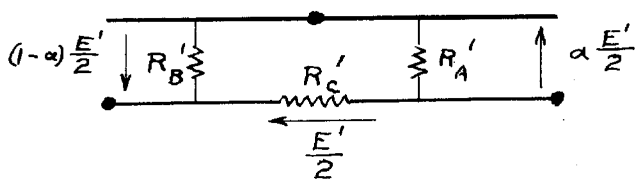
Figure 16

Basic Half-Width Strips

Equivalent Circuit of Half-Width Magnitude Strip:



Equivalent Circuit of Half-Width Phase Strip:



$$\frac{R_A}{\sigma} = \frac{\sigma}{R_A'} = \frac{1}{\pi}\ln r_A = \frac{2.303}{\pi}\log_{10} r_A$$

$$= .733 \times \text{number of decades to right of } z_o$$

$$\frac{R_B}{\sigma} = \frac{\sigma}{R_B'} = \frac{1}{\pi}\ln\frac{1}{r_B} = \frac{2.303}{\pi}\log_{10}\frac{1}{r_B}$$

$$= .733 \times \text{number of decades to left of } z_o$$

$$\frac{R_C}{\sigma} = \frac{\sigma}{R_C'} = \frac{1}{\pi}\ln\frac{1}{r_o} = \frac{2.303}{\pi}\log_{10}\frac{1}{r_o}$$

$$= .733 \times \log_{10}\left(\frac{\text{width of half-width strip}}{\pi \times \text{radius of circle at } z_o}\right)$$

(σ = resistivity of the sheet material, in ohms)

Figure 17

Equivalent Circuits for Figure 16

will therefore be represented by the equivalent circuits of Figure 17. If a current flows between any two electrodes of the magnitude sheet, the equipotential line which is at the same potential as the remaining electrode, in each of the three possible cases, is indicated by a dotted line in Figure 16. The intersection of these lines, marked P, corresponds to the junction of the three resistors in the equivalent T circuit; its voltage will be the same as that of point P in Figure 17 for any possible division of currents between the electrodes. The equations given in Figure 17 for R_A, R_A', R_B, and R_B' are obtained by inspection. The equation for R_C will be determined by transforming the magnitude sheet from the ln w coordinates of Figure 16 into the w coordinate system in Figure 18. In the conformal transformation from the ln w plane to the w plane, the magnification at any point is $\frac{d(w)}{d(\ln w)}$, which is equal to w. In the vicinity of $w = 1$, the magnification is therefore equal to one, so that the small circle of radius r_o around point z_o in the ln w plane transforms (approximately) into a circle of the same radius in the w plane. Since it has already been determined in Figure 16 that $R_B = \frac{\sigma}{\pi} \ln \frac{1}{r_B}$, it is apparent by analogy in Figure 18 that $R_C = \frac{\sigma}{\pi} \ln \frac{1}{r_o}$. R_C' is then the dual of R_C inverted about resistance σ. It can be seen in Figure 18 that point P is located at $w = e^{j\pi/3}$, i.e., at $|w| = 1$ and $\angle w = 60^\circ$; the corresponding point on the adjacent half-width strip is $w = e^{-j\pi/3}$.

Transform of
Magnitude Sheet:

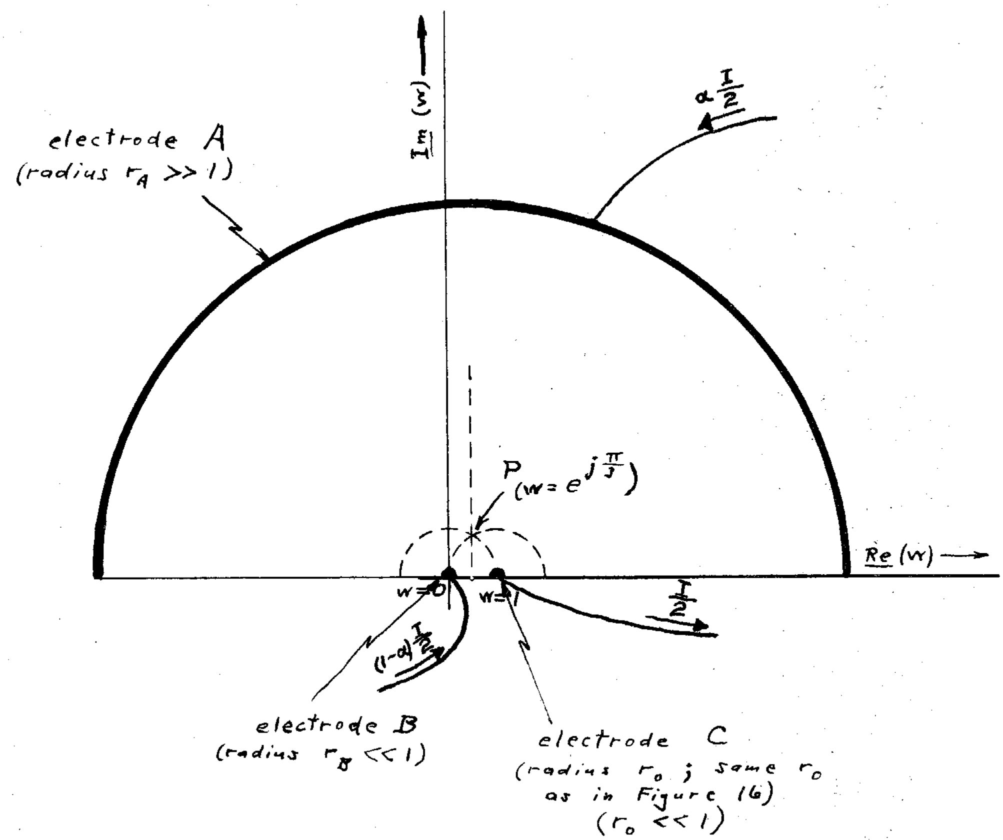


Figure 18

Transformation of the Magnitude Sheet in Figure 16 into the w Plane

The voltage distribution equations for the sheets will be obtained by comparing Figure 18 with Figure 9 and equations (9) and (10). If a current I/2 flows in the half-width magnitude sheet in Figure 18 from electrode A to electrode C (no current to B), the voltage distribution in the w coordinates of *Figure 18 will be exactly the same as the voltage distribution* in the z coordinates in Figure 9 when a current I flows to $z_o = 1$. With this current, the voltage at w_k relative to that at w_r will therefore be obtained by substituting in equation (9) the values $z_o = 1$, $z_k = w_k$, and $z_r = w_r$, resulting in the expression for the voltage

$$V_{k(1)} = \frac{\sigma I}{2\pi} \underline{Re}(\ln \frac{1 - w_k}{1 - w_r})$$

Now, instead, if a current $(1-\alpha)I/2$ flows from electrode B to electrode A (no current to C), the expression for the voltage will be obtained from (9) by replacing I by $(\alpha-1)I$ and substituting the values $z_o = 0$, $z_k = w_k$, and $z_r = w_r$, yielding

$$V_{k(2)} = \frac{\sigma(\alpha-1)I}{2\pi} \underline{Re}(\ln \frac{w_k}{w_r})$$

Since the currents specified in Figure 18 are the superposition of the above two currents, the actual voltage will be the sum of the above expressions:

$$V_k = \frac{\sigma I}{2\pi} \underline{Re} \ln (\frac{w_k}{w_r})^{\alpha-1} \frac{1 - w_k}{1 - w_r}$$

The transformation equation relating log z and log w coordinates is $z = z_o w$; therefore substituting $w_k = z_k/z_o$ and $w_r = z_r/z_o$, the voltage equation in terms of z is

$$V_k = \frac{\sigma I}{2\pi} \, \underline{\mathrm{Re}} \left[\ln \left(\frac{z_k}{z_r}\right)^{\alpha-1} \frac{z_o - z_k}{z_o - z_r} \right] \tag{19}$$

By analogy with equations (9) and (10), the corresponding equation for the phase sheet is

$$V_k' = \frac{E'}{2\pi} \, \underline{\mathrm{Im}} \left[\ln \left(\frac{z_k}{z_r}\right)^{\alpha-1} \frac{z_o - z_k}{z_o - z_r} \right] \tag{20}$$

Replacing z_o by z, the basic factor in the function evaluated by a computer with the currents and voltages in Figure 16 is therefore

$$\left(\frac{z_k}{z_r}\right)^{\alpha-1} \frac{z - z_k}{z - z_r} \tag{21}$$

If the voltage reference point is a contact fixed in the log z coordinate system, z_r in (21) is an arbitrary constant; but if the reference point is a terminal attached to the sheet, w_r is an arbitrary constant. Substituting $z_r = z_o w_r$, the basic factor then becomes

$$\frac{1}{1 - w_r} \left(\frac{z_k}{w_r}\right)^{\alpha-1} \frac{z - z_k}{z^{\alpha}} \tag{22}$$

If all the current in the magnitude sheet and all the voltage on the phase sheet are applied at the right end of the strip, $\alpha = 1$; if they are applied at the left end of the strip, $\alpha = 0$. Computers in log z coordinates will be grouped for discussion into two separate classes--those in which the value of α in (21) or (22) is adjustable, and those in which the value of α is always either 1 or 0.

Computers in Which α Is Adjustable

In this class of computers, α in (21) or (22) is an arbitrary constant. In log z coordinates, z_k in a factor cannot be set equal to either 0 or ∞; consequently (21) does not provide any direct method for introducing the factor z^{m_o} in (4c) or (4n). Function (22), on the other hand, includes in each factor a power of z. Since w_r and α are both arbitrary constants, it will be convenient to define a new arbitrary constant

$$C = \frac{1}{(1 - w_r)\, w_r^{\alpha-1}} \tag{23}$$

In terms of this new constant, (22) reduces to

$$C\, z_k^{\alpha-1} \frac{z - z_k}{z^{\alpha}} \tag{24}$$

If factors in this form are combined in a computer, and if an arbitrary constant factor K_o is inserted by an adjustable voltage control to one input of the summing voltmeter, the function evaluated by the computer is

$$F = K_o\, C^{\Sigma m_k} \left(\prod_k z_k^{m_k}\right)^{\alpha-1} z^{-\alpha \Sigma m_k} \prod_k (z - z_k)^{m_k} \tag{25}$$

This function is the same as either function (4c) or (4n), provided that

$$\alpha = -\frac{m_o}{\Sigma m_k} \tag{26}$$

and in (4c) $$K_c = K_o\, C^{\Sigma m_k} \left(\prod_k z_k^{m_k}\right)^{\alpha-1} \tag{27c}$$

or in (4n) $$K_n = K_o\, (-C)^{\Sigma m_k} \left(\prod_k z_k^{m_k}\right)^{\alpha} \tag{27n}$$

If w_r in (23) could be chosen so that $C = 1$ for all values of α, C would be eliminated from (27c); if it could be chosen so that $C = -1$, C would be eliminated from (27n). It is not possible to find a complex value of w_r which will accomplish either of these results, but this is not necessary. Since each sheet has its own voltage reference terminal, it is only necessary to select a point w_r on the magnitude sheet for which $|C| = 1$ at all values of α, and to select a point on the phase sheet yielding $\angle C = 0°$ for use in (27c), or a point yielding $\angle C = 180°$ for use in (27n). The choices for w_r which will eliminate C are:

On the magnitude sheet: $|w_r| = 1$ and $\angle w_r = \pm 60°$

On the phase sheet
- for (27c): $|w_r| < 1$ and $\angle w_r = 0°$
- for (27n): $|w_r| > 1$ and $\angle w_r = 0°$

The reference point w_r on the magnitude sheet is therefore point P in Figures 16, 17, and 18. With the choices indicated above, equations (27) reduce to

$$K_c = K_o \left(\prod_k z_k^{m_k} \right)^{\alpha - 1} \qquad (28c)$$

$$K_n = K_o \left(\prod_k z_k^{m_k} \right)^{\alpha} \qquad (28n)$$

The remaining step to make the computer practical for evaluating functions in the form of F_c or F_n is the development of a technique for adjusting the K_o control so that K_c or K_n will be equal to the proper value for the function being evaluated. Since the relationship between K_o and K_c or K_n in equations (28) involves all the zeros z_k of the function except those at zero and infinity, the technique for adjusting K_o will obviously involve the positions of all the voltage contacts at the points z_k. If a special reference point and value of z can be chosen such that the voltage at each contact z_k represents a factor $z_k^{\alpha-1}$, the summing voltmeter reading will then be exactly the quantity K_c in (28c). Or if a special reference point and value of z are chosen such that the voltage at each contact represents a factor z_k^{α}, the voltmeter reading will be the quantity K_n in (28n). Function (21) is suitable for this purpose, since it reduces to $z_k^{\alpha-1}$ when $z_r = 1$ and $z \gg z_k$, and it reduces to z_k^{α} when $z_r = 1$ and $z \ll z_k$. If these values of α and z_r are used in the computer, the value of K_c or K_n corresponding to any setting of K_o will be indicated on the meter. The technique for evaluating the magnitude or phase of a function will therefore include the following steps:

1. Place a voltmeter contact at each zero or pole z_k of the function, locating the position by setting $z = z_k$ and placing the contact opposite point z_o on the sheet;
2. Connect each voltmeter contact to a summing voltmeter input adjusted for the exponent m_k of the factor;
3. Adjust the current ratio in the magnitude sheet or the voltage ratio in the phase sheet so that $\alpha = -m_o/\sum m_k$;
4. Switch the common return lead of the summing voltmeter to a voltage reference contact located at $z_r = 1$;
5. For reading K_c, set $z \gg$ every z_k; or for reading K_n, set $z \ll$ every z_k;
6. Set the K_o control so that the meter reading is equal to the magnitude or phase of the given constant K_c or K_n;

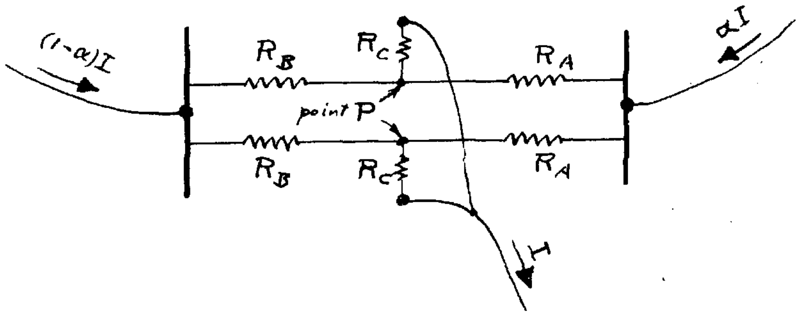
7. Switch the common return lead of the meter to the normal reference terminal (specified on page 67);

8. Set the relative position of the coordinate frame and the sheet to any desired value of z and read the magnitude or phase of F on the meter.

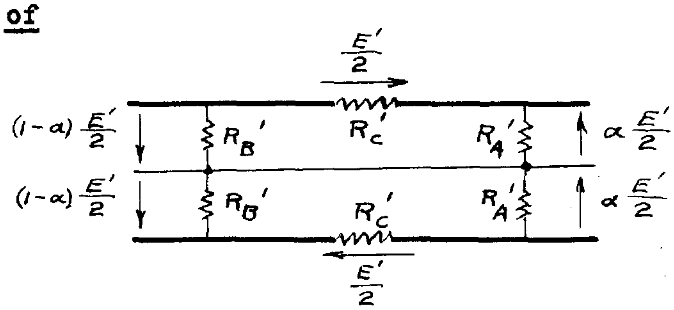
This procedure breaks down if $\sum m_k = 0$ and $m_o \neq 0$—i.e., if the function approaches as a limit the same power of z as z becomes very large that it approaches as z becomes very small. Step 3 would then require that α be set equal to ∞, which is a physical impossibility. This problem can be overcome in practice by modifying the original function at values of $|z|$ well outside the range in which the function is to be evaluated, without changing the function significantly in the desired range of z. For a function in the conventional form (4c), this modification is accomplished by multiplying the function by an extra factor $(\frac{z - z_k}{z})^{m_k}$, in which $z_k \ll z$ for all readings, and m_k is arbitrary. For a function in the normalized form (4n), the modification is accomplished by multiplying by an extra factor $(1 - \frac{z}{z_k})^{m_k}$, in which $z_k \gg z$, and m_k is arbitrary. When the function is thus modified, $\sum m_k$ is no longer equal to zero and the regular procedure is applicable.

As an illustration of a computer with α adjustable, a computer using the sheets of Figure 13-B will be considered. The equivalent circuits of the sheets and the required voltages and currents will then be as given in Figure 19. If only rational algebraic functions are to be evaluated in this computer, m_o and all the m_k's will be small positive or negative

Equivalent Circuit of
Magnitude Sheet:



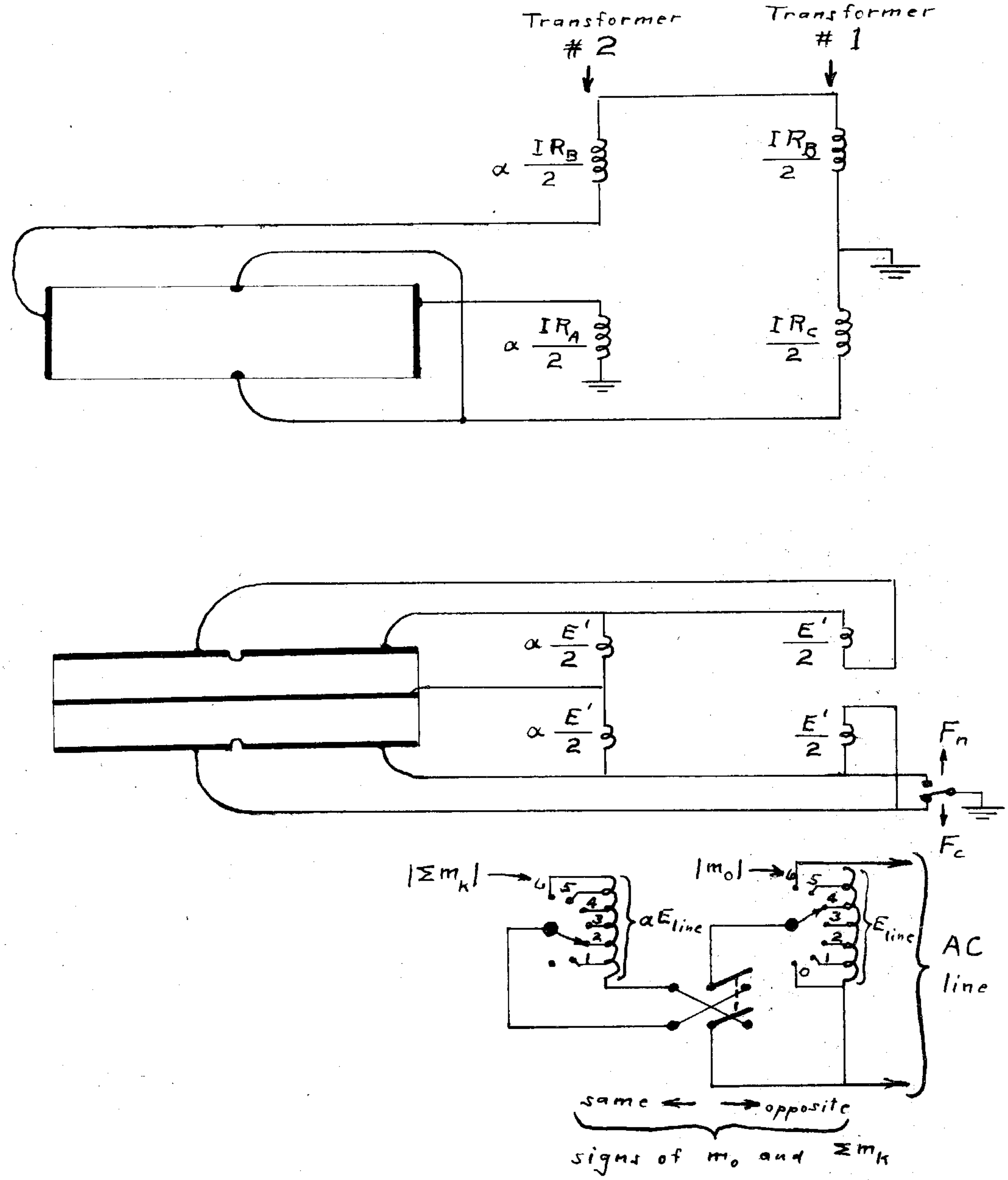
Equivalent Circuit of
Phase Sheet:



(The equations for the R's are given in Figure 15)

Figure 19

Equivalent Circuits for Figure 13-B



Note: $\alpha = -\dfrac{|m_o|}{|\Sigma m_k|}$

Figure 20

Computer with α Adjustable

integers, and an a.c. power supply with controls for α as shown in Figure 20 could be used; for d.c. power, rectifiers could be added to the secondary windings of the transformers. For use with irrational functions (fractional values of m_o and the m_k's), the voltage ratio α would have to be made continuously variable. In Figure 20, ground potential is the required voltmeter reference potential for readings on either sheet; this is true for the magnitude sheet because the reference point P on the sheet (Figure 19) is at the same potential as the grounded point in its power supply (Figure 20). For reading and setting K_c or K_n, the voltmeter reference will of course be transferred to a contact in the coordinate frame at $z_r = 1$ (not shown in the figure).

In the class of computers described in this section, the factor z^{m_o} in either of equations (4) is included in the function by including in each of the other factors a fractional power of z such that the contribution from all the factors results in the desired exponent m_o. This is an elegant solution to the problem, but it involves the use of a rather complicated power supply circuit, and it requires several steps to set a given value K_c or K_n into the computer. It will be found that computers of the second class, in which α is limited to the values one or zero and the factor z^{m_o} is introduced in some other manner, are equally easy to build and adjust, and are simpler to use in practical computation.

Computers in Which α Is Equal to One or Zero

In this class of computers, the current in the magnitude sheet flows to z_0 from only one end of the sheet, and the voltage on the phase sheet is applied only across the corresponding end of the sheet. When the current and voltage are applied at the right-hand ends of the sheets (at $z \simeq \infty$), $\alpha = 1$, and factors (21) and (22) reduce to

$$\frac{z - z_k}{z - z_r} \tag{29}$$

and

$$\frac{z - z_k}{(1-w_r)\, z} \tag{30}$$

When the current and voltage are applied at the left ends of the sheets (at $z \simeq 0$), $\alpha = 0$, and factors (21) and (22) reduce to

$$\frac{1 - \frac{z}{z_k}}{1 - \frac{z}{z_r}} \tag{31}$$

and

$$\frac{1 - \frac{z}{z_k}}{(1 - \frac{1}{w})} \tag{32}$$

These factors would be further simplified by choosing z_r or w_r equal to 0 or ∞; but in log z coordinates this is not physically possible, since these points are infinitely distant off the ends of the strip or cylinder. Such values can be approximated, however, by choosing a very large or very small finite value of z_r or w_r. Then in any sum or difference, if the magnitudes of two quantities differ by a large ratio, the quantity having the smaller magnitude can be considered negligible. The symbol ϵ will be used to denote a very small constant--one which is so small that it can always be neglected in a sum or difference. The symbol $\underline{E}$ will be used to represent a very large constant--one which is so large that all other quantities in a sum or difference can be neglected. The factors resulting when $\alpha = 1$ or ∞ and z_r or $w_r = \epsilon$ or $\underline{E}$, in all possible combinations, are listed in Table II, in the second column. Some of these factors approach a constant value as z approaches zero; the others approach a constant value as z approaches infinity. Therefore, in order to compute a general function in the form of (4c) or (4n), an arbitrary power of z must be introduced in some manner. If this is done, function (4c) can be computed using any of the factors resulting when $\alpha = 1$, and function (4n), when $\alpha = 0$.

One method of introducing an arbitrary power of z is to approximate it by a factor containing a zero or pole z_k having a magnitude either much smaller or much larger than any value of z at which the function is to be evaluated. Limiting values for z_k and the resulting factors are listed in the last two columns of Table II; such a "z factor" can be raised to an arbitrary positive or negative exponent in the summing voltmeter in the same manner as any other factor. For any combination of constants in Table II, however, the desired part of either the general factor or the z factor (or of both) is seen to be associated with a constant part having either a very large or a very small magnitude. Remembering that the

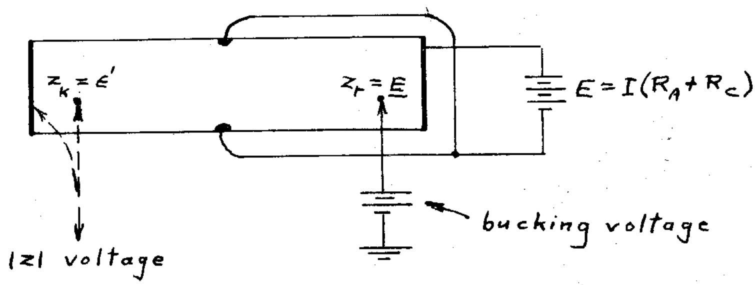
Table II

Simple Factors in Log z Coordinates

	Constants	General Factor	Limiting Value of z_k (for z factor)	z Factor
$\alpha = 1$	$z_r = \epsilon \ll z$ $w_r = \epsilon \ll 1$	$\frac{(z - z_k)}{z}$	$z_k = \underline{E}' \gg z$	$-\underline{E}' \cdot \frac{1}{z}$
	$z_r = \underline{E} \gg z$	$-\frac{1}{\underline{E}} \cdot (z - z_k)$	$z_k = \epsilon' \ll z$	$-\frac{1}{\underline{E}} \cdot z$
	$w_r = \underline{E} \gg 1$	$-\frac{1}{\underline{E}} \cdot \frac{(z - z_k)}{z}$	$z_k = \underline{E}' \gg z$	$\frac{\underline{E}'}{\underline{E}} \cdot \frac{1}{z}$
$\alpha = 0$	$z_r = \underline{E} \gg z$ $w_r = \underline{E} \gg 1$	$(1 - \frac{z}{z_k})$	$z_k = \epsilon' \ll z$	$-\frac{1}{\epsilon'} \cdot z$
	$z_r = \epsilon \ll z$	$-\epsilon \cdot \frac{(1 - \frac{z}{z_k})}{z}$	$z_k = \underline{E}' \gg z$	$-\epsilon \cdot \frac{1}{z}$
	$w_r = \epsilon \ll 1$	$-\epsilon \cdot (1 - \frac{z}{z_k})$	$z_k = \epsilon' \ll z$	$\frac{\epsilon}{\epsilon'} \cdot \frac{1}{z}$

voltages in the computer represent the logarithms of factors, the constant part of a factor can be eliminated by subtracting from the voltage for the whole factor a fixed "bucking voltage" representing the constant part, so that the net series voltage represents the basic part of the factor which is desired. Such a bucking voltage is required only for use with the magnitude sheet. For the phase sheet, the selection of a constant ϵ or $\underline{E}$ having a phase angle of 180° will in each case result in an angle of 0° for the constant part of the factor.

As an illustration of a computer using this method for inserting the z factor, when $\alpha = 1$ and $z_r = \underline{E}$, the general factor is $-\frac{1}{\underline{E}}(z - z_k)$ and the z factor is $-\frac{1}{\underline{E}}z$. A bucking voltage representing $\left|\frac{1}{\underline{E}}\right|$ must therefore be connected in the opposite polarity in series with each voltage contact on the magnitude sheet, including the contact for the z factor at $z_k = \epsilon'$. Since the bucking voltage is the same for all factors, including the z factor, the most practical place for it is in series with the common return lead from the summing voltmeter to the reference point $|z_r| = |\underline{E}|$ on the magnitude sheet. On the phase sheet, the reference contact will be placed at $\angle z_r = \angle \underline{E} = 180°$, so that $\angle -\frac{1}{\underline{E}} = 0°$ and no bucking voltage will be required. Figure 21 shows these connections for the sheets of Figure 13-B. If an adjustable voltage, calibrated as the magnitude or phase of an arbitrary constant K, is connected to one of the inputs of the summing voltmeter, the computer



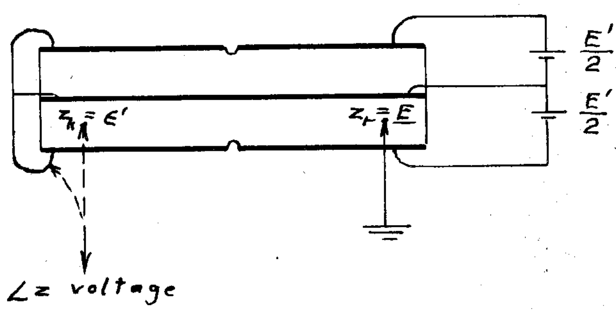


Figure 21

Computer with Bucking Voltage—Conventional Functions

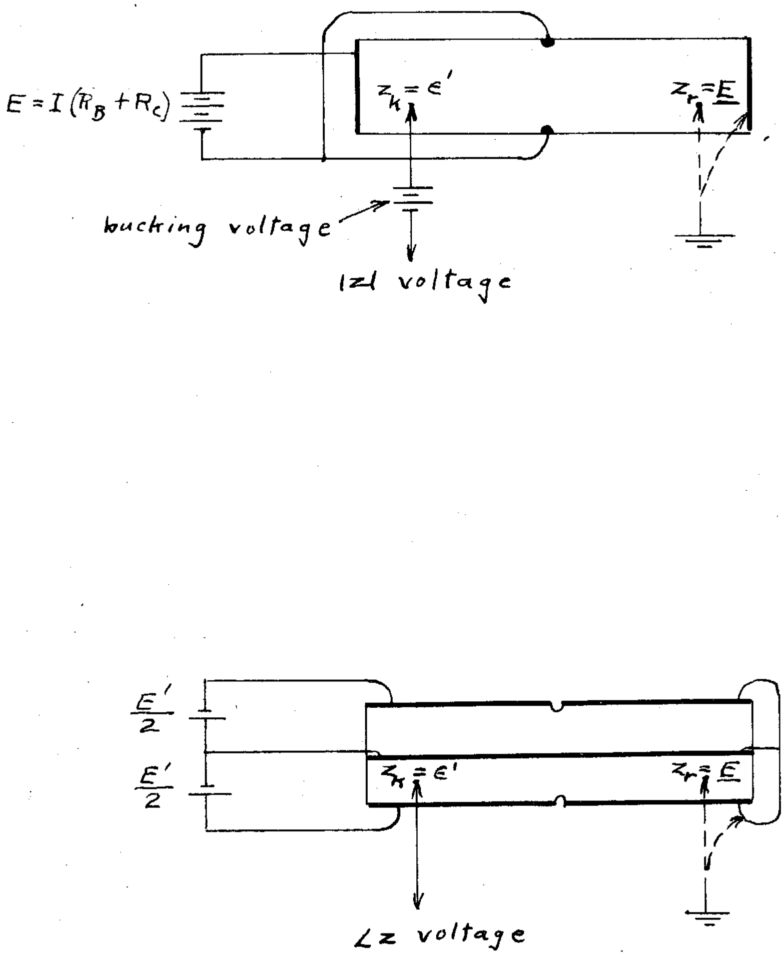


Figure 22

Computer with Bucking Voltage—Normalized Functions

will yield readings of magnitude and phase for functions factored in the conventional form (4c). If the constants are changed to $\alpha = 0$ and z_r or $w_r = \underline{E}$, the general factor is $(1 - \frac{z}{z_k})$ and the z factor is $-\frac{1}{\epsilon'}z$. In this case the bucking voltage representing $\left|\frac{1}{\epsilon'}\right|$ will be connected in series with the voltage contact for the z factor only; the circuit is then as shown in Figure 22. In this form, the computer will yield readings for functions factored in the normalized form (4n). Since the change from the circuit of Figure 21 to that of Figure 22 is a simple switching operation, the same computer can be designed to handle functions factored in either the conventional or the normalized form.

A slightly different approach to the problem of the z factor is the production by some supplementary means of a pair of voltages to represent the factor z—i.e., the production of a pair of voltages proportional to $\ln|z|$ and $\underline{/z}$ respectively. In log z coordinates, these are the two components of the position of the resistive sheet in the coordinate system. Corresponding voltages can therefore be obtained from a pair of linear potentiometers connected across fixed power supply voltages, each ganged with the relative motion in one direction between the coordinate frame and the sheet. To illustrate a computer using potentiometers for the z factor, the selection of constants $\alpha = 1$ and z_r or $w_r = \epsilon$ will be considered first. In this case, the voltage at a z_k contact is seen in Table II to represent a factor $\frac{z - z_k}{z}$. If the potentiometer circuit for each sheet is connected to the reference point z_r or w_r so that the voltage at its tap represents $1/z$, the voltage difference between the contact at z_k and the tap on the potentiometer will represent $(z - z_k)$. Factors in this form will then be evaluated in the computer if the common lead of the summing voltmeter is connected to the potentiometer tap instead of to the reference point on the sheet. And if one of the inputs to the summing voltmeter is connected to the reference point on the sheet, its voltage will be the negative of the potentiometer tap voltage, so that it will represent the factor z. The computer in this form, shown in Figure 23, will therefore evaluate any function in the conventional form (4c). Now if the constants are changed to $\alpha = 0$ and z_r or $w_r = \underline{E}$, and if the common lead from the summing voltmeter is connected to the reference point on the sheet, the voltage at each z_k contact will represent a factor $(1 - \frac{z}{z_k})$. If the potentiometer polarity is reversed from that of the previous case, the voltage at its tap will represent the factor z. With these changes, given in Figure 24, the computer will evaluate a function in the normalized form (4n). Again the change between the two forms is a simple switching operation, so that the same computer will handle functions factored in either form.

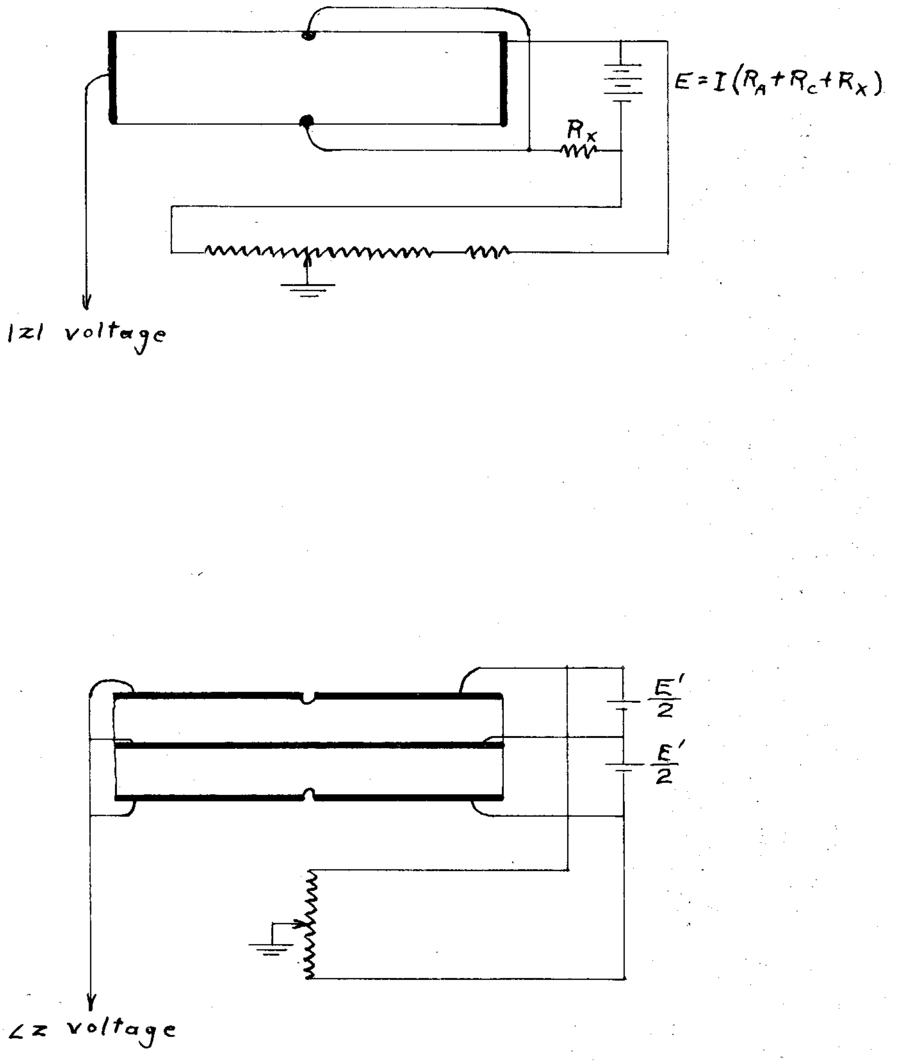


Figure 23

Computer with Potentiometers—Conventional Functions

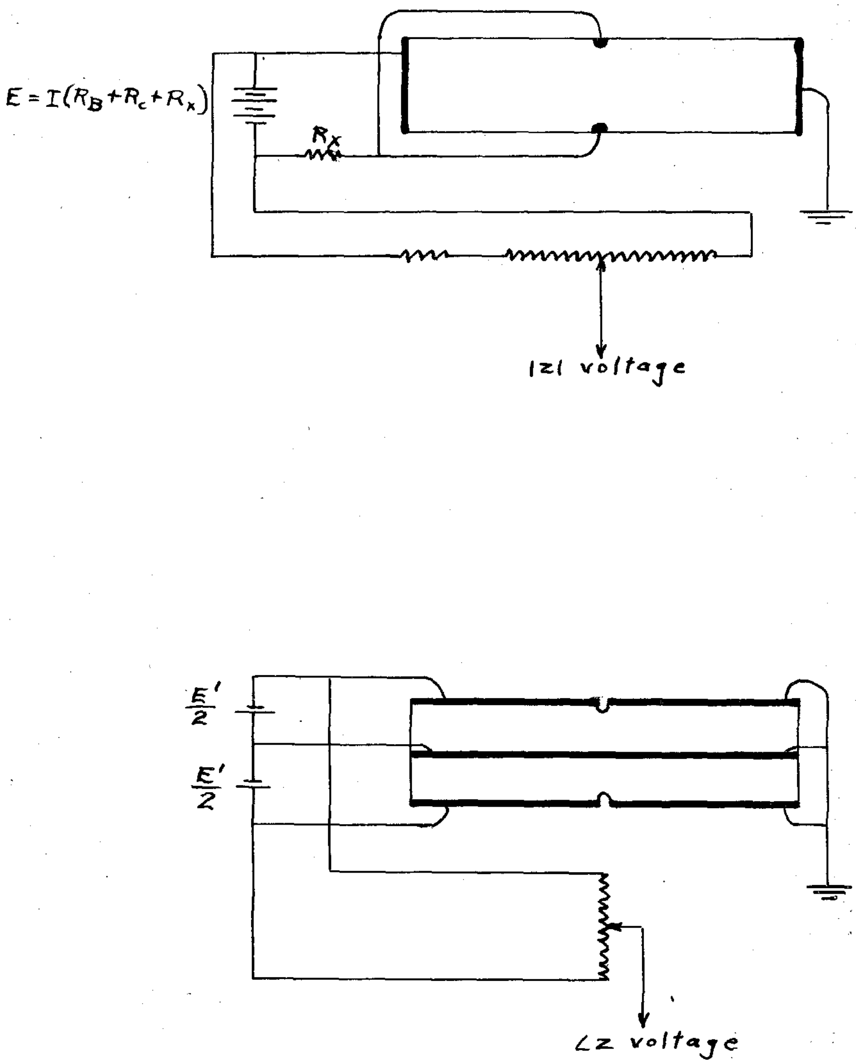


Figure 24

Computer with Potentiometers—Normalized Functions

In either of these computers in which $\alpha = 1$ or 0, the exponent m_o on the z factor is introduced at the summing voltmeter in the same manner as all the other exponents m_k, and the constant K_c or K_n is introduced by a voltage control which is calibrated directly in terms of $|K|$ and $\underline{/K}$. Such a computer will obviously be simpler to use than one requiring a special procedure for setting K_c or K_n into the computer.

PART IV

AN ACTUAL COMPUTER

Design and Construction of the Computer

In order to test the foregoing computer theory, the actual computer in Figure 25 was designed and built. The principal purposes of this experimental model were considered to be the verification of the theory and the demonstration of the usefulness of such a computer. Flexibility of application and expediency in design and construction were given priority over accuracy of computation. Teledeltos paper was therefore chosen for the resistive sheet material, so that direct current could be used on the sheets; this allows the use of a simple direct-reading summing voltmeter without vacuum tubes. The log z coordinate system was selected for the reasons discussed on page 51; the plane form, rather than the cylindrical form, was chosen for simplicity of construction. The sheets were constructed in the pattern of Figure 14-A so that they could be connected as either single-width or double-width strips. On the large scale for z (corresponding to the connection as single-width strips), three decades of $|z|$ and 400° of $\angle z$ each occupy a space of ten inches. This agrees with the scale of K. and E. 3 cycle semi-logarithmic graph paper, as discussed on page 9. The power supply circuit was designed for α limited to the values one and zero, and the potentiometer method of inserting the z factor was used (Figures 23 and 24).

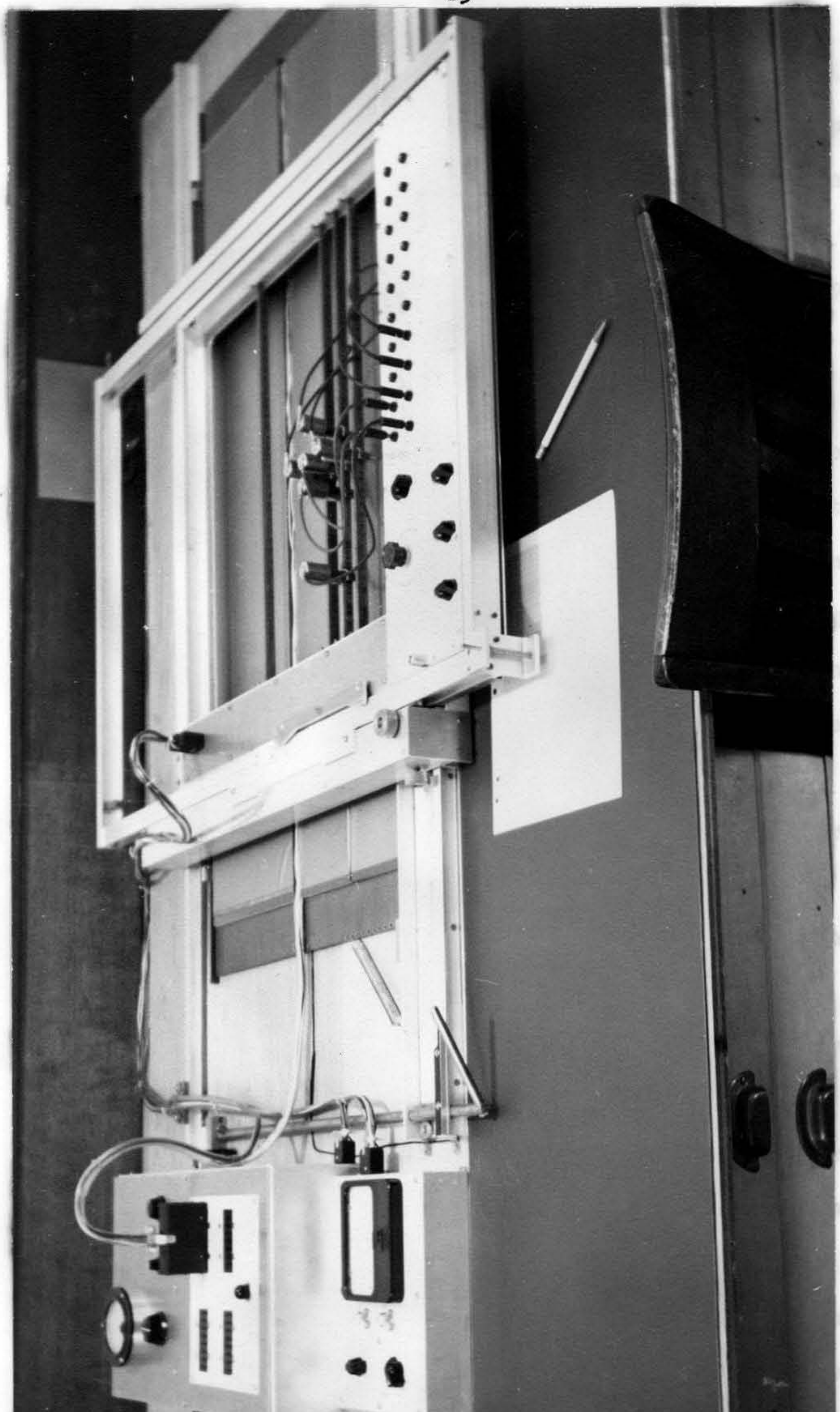


Figure 25

The Computer

The power supply for the sheets is contained in the large box in the upper part of Figure 26. The scale of the computer (single-width or double-width sheet connection) and the form of factoring of the function (corresponding to $\alpha = 1$ or $\alpha = 0$) are selected by plugging the main power cable into the appropriate socket on the top of the power supply. The sheets of Teledeltos paper are cemented to Bakelite sheets, and the magnitude and phase sheets are mounted side by side as shown in Figure 27. The two halves of the phase sheet are constructed as separate pieces. Along the side of each half which is to be placed adjacent to the other half (at the conducting cut down the center of the finished phase sheet), the painted electrodes and a very narrow strip of the active portion of the resistive sheet are folded down over the edge of the Bakelite in order to allow room for the insulating barrier between the two halves.

The sheets are mounted under a movable coordinate frame supporting the voltmeter contacts and carrying a panel on which are located the input jacks for the summing voltmeter and the controls by which the values of the constant K and the exponent on the z factor are set—see Figure 28. The coordinate frame rides on rails in a second frame in which it can be shifted into position over either sheet for evaluating $|F|$ or $\angle F$, as pictured in Figure 29. When the coordinate frame is shifted from one position to the other, the summing voltmeter

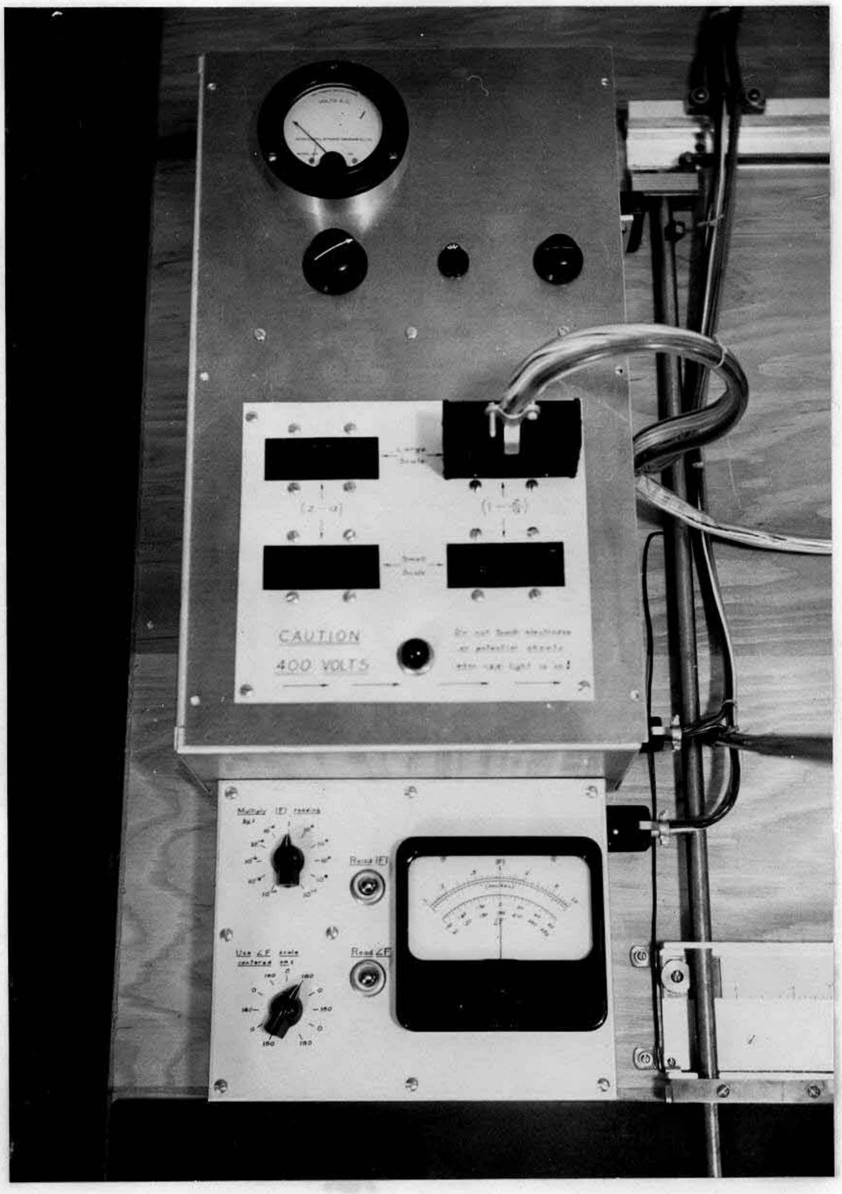


Figure 26

Power Supply and Meter Panels

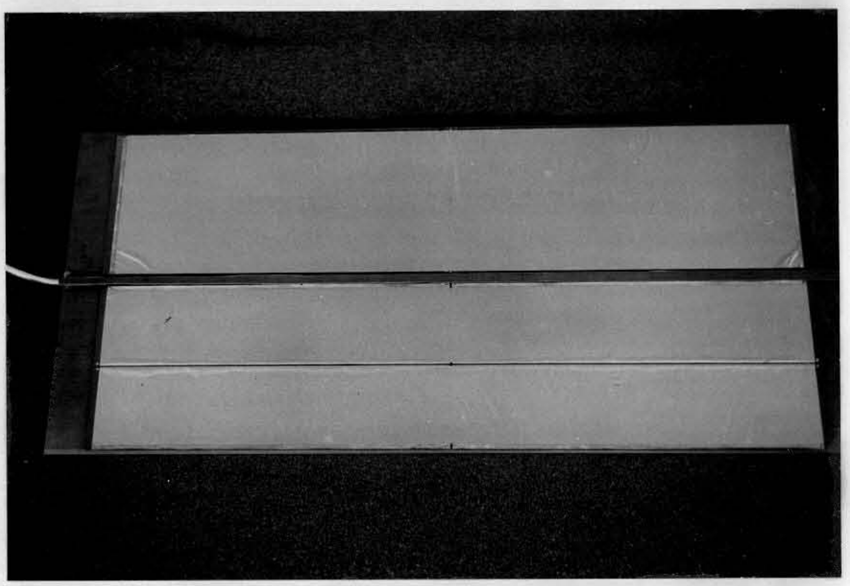


Figure 27

Magnitude and Phase Sheets

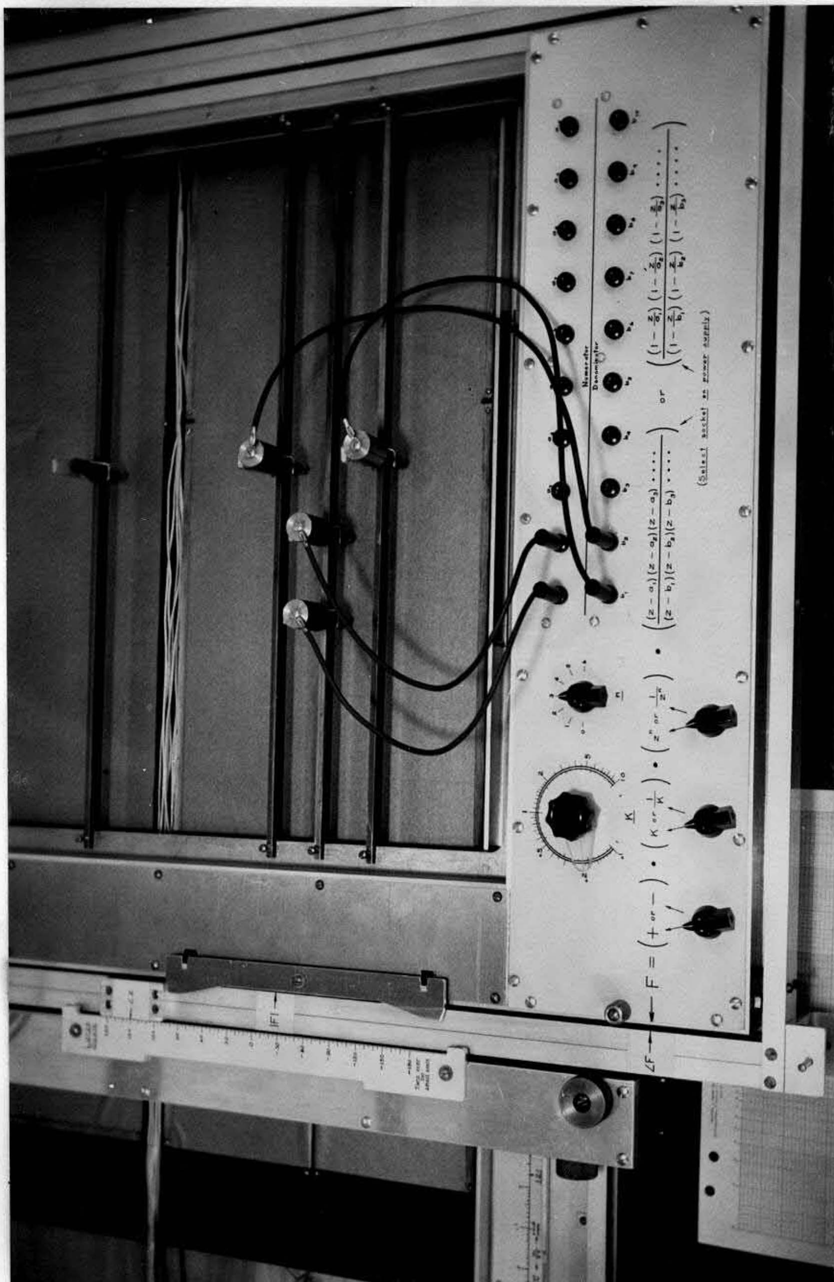
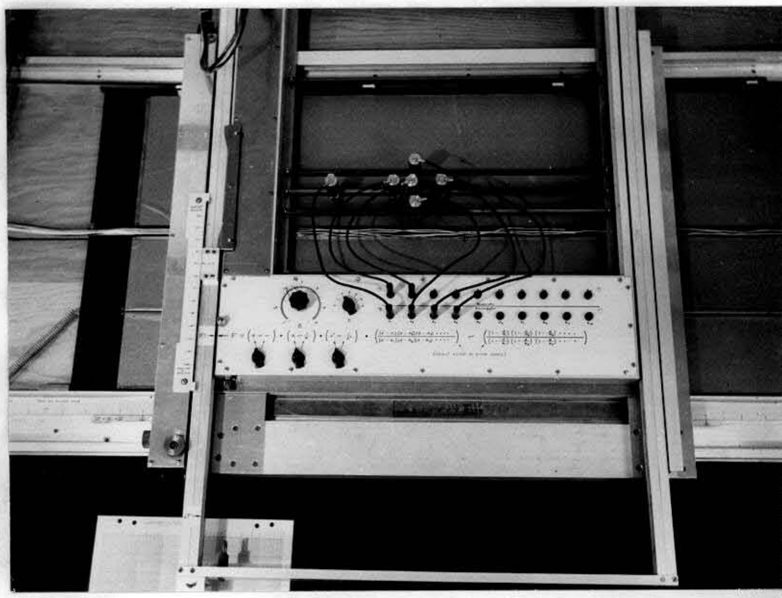


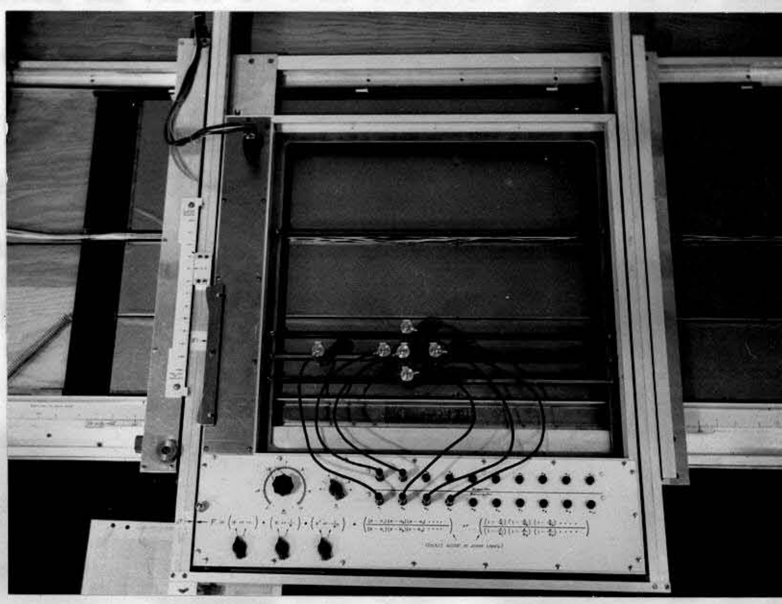
Figure 28

Coordinate Frame and Panel

$|F|$:



$\angle F$:



<u>Figure 29</u>

$|F|$ and $\angle F$ Positions

inputs for the constant K and the factor z are automatically switched between the magnitude and the phase voltages. The second frame rides on rails in a third frame to positions calibrated on a linear scale for $\angle z$, and the third frame rides on rails on the table to positions calibrated on a logarithmic scale for $|z|$. The $|z|$ scale (visible at the left in Figure 29) is provided with a second pointer for reading $\frac{|z|}{2\pi}$; a reverse logarithmic scale (at the right in Figure 29) is provided for reading $\frac{1}{|z|}$. The $|z|$ and $\angle z$ scales are reversible, the large scale being on one side and the small scale on the other side; the scale used must of course agree with the power supply socket which is in use. The position of the second frame represents the value of z at which the function is being evaluated; a retractable stylus is attached to the front left corner of this frame (shown clearly in Figure 25) for tracing out contours or loci of z on a piece of paper on the table.

Each voltmeter contact consists of a small stainless steel rod, which is rounded at the lower end where it touches the sheet, and which carries a weight at its upper end to keep it pressed firmly against the sheet. Such a contact rod is slipped into an insulating tube which is clamped in the proper position on a beam spanning the coordinate frame. A number of insulating tubes can be placed on several beams in the frame as required for a contact at each zero and pole of a function. The position for each contact is located by moving the frame to the position in which $z = z_k$ (on the actual computer panel, z_k is called a or b) and then placing the contact directly over the point z_0 at the center of the sheet. For a first order zero or pole of the function (i.e., for a factor with an exponent +1 or -1), the wire from the voltmeter contact is simply plugged into an appropriate jack on the panel. If the magnitude of the exponent is greater than one, the voltmeter contact must be connected to the corresponding number of input jacks by means of jumpers. For a fractional exponent, the input resistance to the summing voltmeter must be increased in proportion to the reciprocal of the exponent; for the square root, an adapter containing a resistor which is equal to the internal input resistor is inserted in series with the input lead. Figure 30 shows these adapters at two of the input jacks.

When a function has been set up on the computer, the large handle at the operator's left (see Figure 25) is turned; this lifts the sheets up slightly, to touch the voltmeter contacts, and turns on the power to the computer. The a.c. voltmeter at the top of the power supply (Figure 26) indicates the input voltage to the computer; it is adjusted to the red line by means of the knob below the meter. The value of $|F|$ or $\angle F$ is then read on the main meter, which can be seen in the lower part of Figure 26. The small neon bulbs beside the meter show which scale is to be read, and the range switches at the left are used to bring the meter reading on scale as required.

The complete schematic wiring diagram of the computer is given in Figure 31. Since the plug and socket connections between the power supply and the sheets are quite complicated, the basic circuit for each of the four cases is redrawn in Figure 32. The initial calibration of the computer requires the adjustment of all the various resistors in Figures 31 or 32. These resistors theoretically require no further attention after their initial adjustment, unless some other part of the equipment is changed. Unfortunately, the resistance of Teledeltos paper changes with temperature, and possibly with humidity or other environmental conditions. In addition to changes in ambient temperature over all the sheet, local heating is noticeable in the vicinity of z_0 as a result of the concentration of power dissipated in that area. Small changes in resistance will cause little error on the phase sheet, but their effect can be quite serious in the circuit of the magnitude sheet and potentiometer. Consequently, all the resistor adjustments should be checked from time to time, and the resistor in series with the z_0 electrode on the magnitude sheet should be checked and set each time the computer is used. A rheostat was installed for this adjustment--its knob can be seen in Figure 26 on the power supply panel at the right near the top.

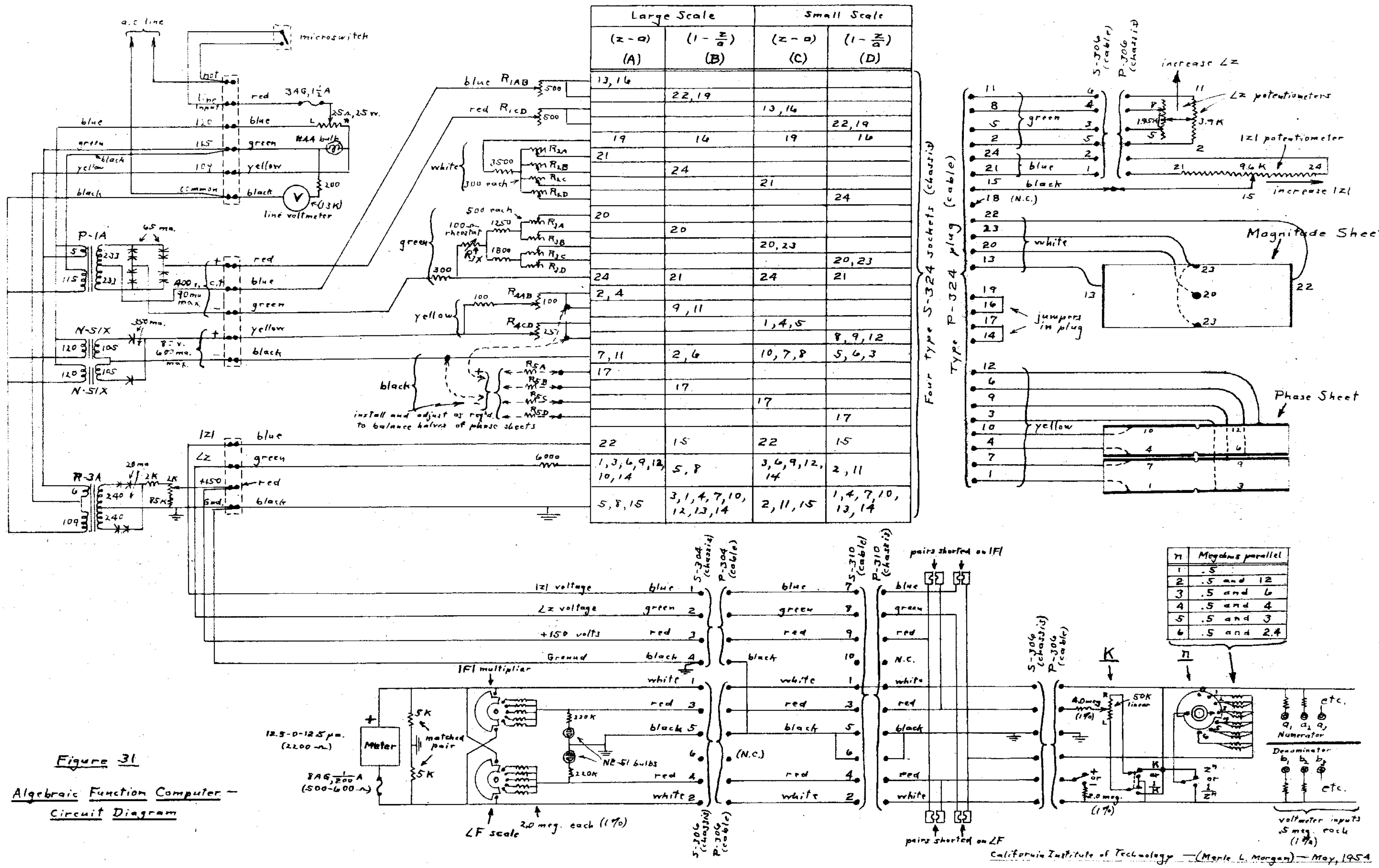


Figure 31

Algebraic Function Computer — Circuit Diagram

California Institute of Technology — (Merle L. Morgan) — May, 1954

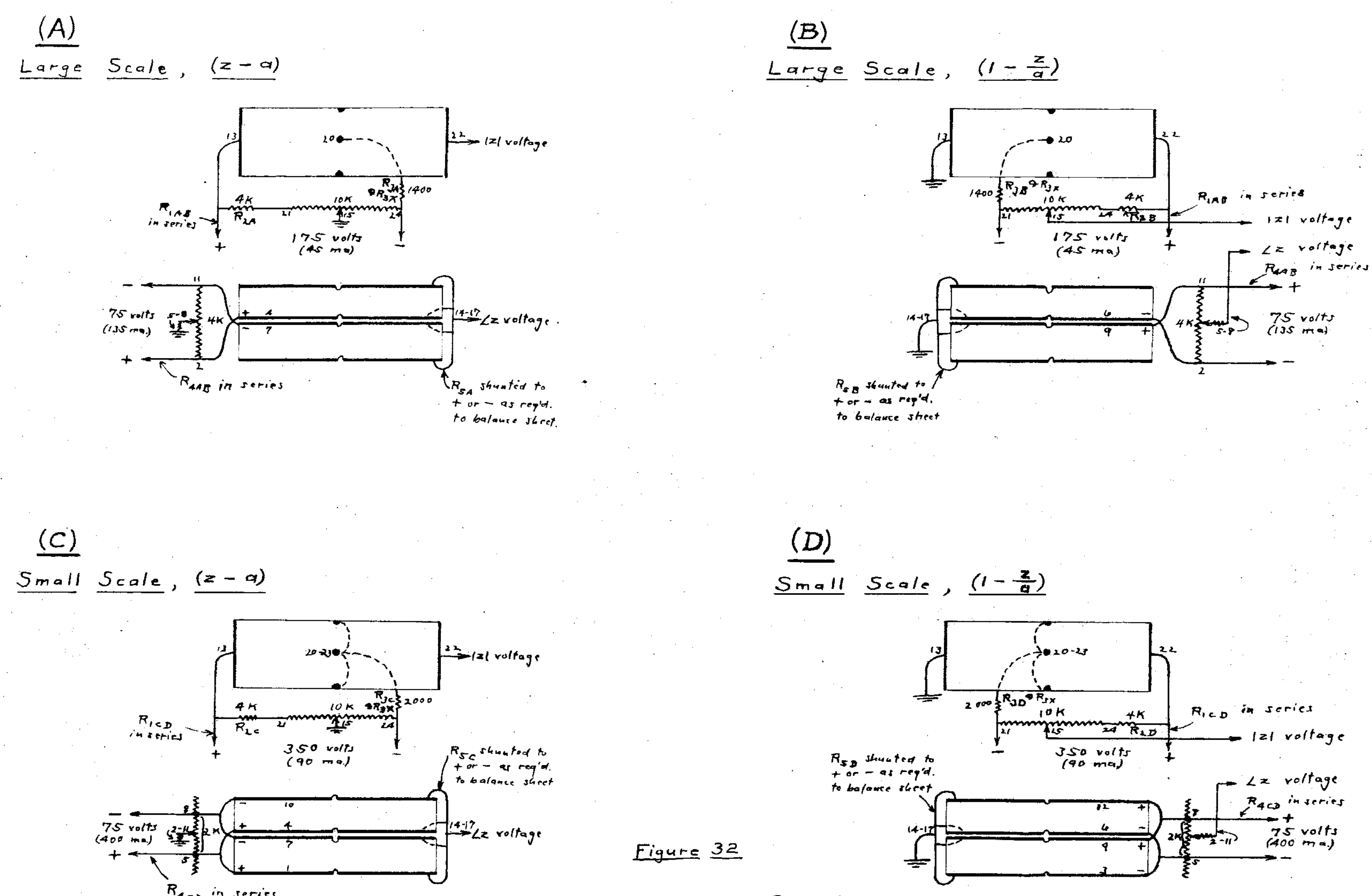


Figure 32

Algebraic Function Computer — Basic Sheet and Power Supply Circuits

Note: The resistance values shown are approximate; see complete circuit diagram for actual specifications.

Nominal voltage per decade on magnitude sheet or per quadrant on phase sheet = 18.75 volts

<u>Accuracy of the Computer</u>

In the actual computer described in the preceding section, the largest errors result from the loading of the circuit (the sheet, the z factor potentiometer, etc.) by the voltmeter. In the interest of simplicity, a sensitive meter (12-0-12 microamperes) was used without amplification, and the voltmeter input resistance was then made as high as possible by operating the sheets at the maximum power level allowable without undue heating in the vicinity of z_0. The resistance of each input is one-half megohm. This value is too low for good accuracy; it should be increased by a factor of at least ten (in this computer) if the effect of circuit loading is to be made negligible. The effective internal resistance of a voltage measured on the magnitude sheet is, at best, in the vicinity of 10 kilohms over most of the sheet. On the phase sheet, it is about 5 kilohms. When the contact is very near a conducting boundary on either sheet, the resistance is lower than these values. As a contact moves about over the sheet, its resistance varies over a range of about 10 kilohms, the values quoted above being the lowest values. The effective internal resistance of the $|z|$ voltage (the $|z|$ potentiometer and its associated circuit) varies between 7 and 11 kilohms at different values of $|z|$; the resistance of the $\angle z$ voltage changes in a similar manner, but the actual resistance is not so large. Roughly speaking, then, the effective internal resistances of the voltages measured by the summing voltmeter range between 0 and 20 kilohms—an average value of 10 kilohms with a variation of ± 10 kilohms. Since the voltmeter input resistors are each .500 megohm (nominally 1%, actually selected much closer), the effective input resistance for which the computer is calibrated is .510 megohm, with a variation of about ± 2% from this assumed value. This means an error of 2% in the natural logarithm of the basic factor before raising to an exponent. Even when the exponent is one, the error can be quite large for a factor of large magnitude. For example, if the magnitude of a factor is 1000, the natural logarithm is about 7; a 2% error in the logarithm corresponds then to a 14% error in the magnitude of the factor.

When a factor is raised to an exponent, the error is multiplied by the value of the exponent. Fortunately, the factors represented by voltage contacts on the sheet are not often raised to higher powers; but higher powers of z are very common. If the z factor voltage were simply connected to a voltmeter input in which the number of .500 megohm resistors is equal to the power of z, the error at higher powers would be very serious. For example, at $|z| = 1000$, a 2% error in $\ln |z|$ corresponds to a 14% error in $|z|$ and an 84% error in $|z|^6$ (the highest power of z in the computer). This difficulty is greatly reduced by shunting each .500 megohm resistor with another resistor of a value such that the total resistance, including the assumed 10 kilohms in the $|z|$ potentiometer circuit, is equal to .510 megohm divided by the exponent. The required values for these compensating resistors are listed on the circuit diagram (Figure 31). The remaining error for $|z|$ results from the fact that the resistance of the $|z|$ potentiometer circuit is not actually constant at 10 kilohms as assumed, but varies between 7 and 11 kilohms. The maximum resistance error is then 3 kilohms, resulting in a 0.6% error in $\ln |z|$, which is approximately a 4% error in $|z|$(a 24% error in $|z|^6$). Over most of the range of $|z|$, the resistance actually lies between 9 and 11 kilohms, resulting in only a 1.4% error in $|z|$ (an 8.4% error in $|z|^6$). The effective internal resistance of the $\angle z$ voltage varies over a smaller range, and its average value is about 4 kilohms. Since the same switch and set of resistors are used for raising both $|z|$ and $\angle z$ to various powers, a fixed resistance of 6 kilohms is added in series with the $\angle z$ circuit, in order to make the average total resistance equal to the 10 kilohm value for which the compensation was calculated.

The second important source of error in the experimental model computer lies in the changing resistance of the sheets. As has already been pointed out, the principal change in resistance occurs in the vicinity of point z_0, as a result of the heating from the power dissipation. There is also apparent, however, a general over-all drift in resistivity over a period of days or weeks. No attempt has been made to explain the cause of this effect; it is not too large, but it does require the recalibration occasionally of the adjustable resistors in the power supply.

The above errors could be practically eliminated in the design of a new computer. For a computer of higher accuracy, it would be advantageous to use a null-balance type voltmeter with very high input resistances and an amplifier type null detector. The sheets could then be operated on a.c. directly from a transformer. Operation of the sheets at a lower power level would eliminate the local heating near z_0, and the use of the null-balance meter circuit would make the actual voltage applied to the sheets unimportant. The elimination of the calibration adjustments relating the potentiometer voltage to the sheet voltage should be helpful in improving the accuracy and simplicity of the computer. This would be possible through the use of the bucking voltage circuits of Figures 21 and 22.

If these changes were made, and if sufficiently accurate resistors were used in the voltmeter circuit, the only important source of error remaining would be the effect of non-uniformity in the sheet resistance. When Teledeltos paper or some other solid sheet material is used in the computer, the uniformity of resistance is beyond the control of the user. In the actual computer, however, the uniformity of the Teledeltos paper appears to be better than was anticipated. In deducing the effect of such non-uniformity, it is useful to conceive of a voltage distribution in which—if the sheet were perfectly uniform—the equipotentials and streamlines would form a perfect grid of squares corresponding to a Cartesian coordinate system for the sheet. The effect of non-uniformity in an actual sheet would be to distort such a grid. The lines of the distorted grid represent a coordinate system in which Laplace's equation is satisfied by any voltage and current distribution. Therefore the voltage on the actual sheet at a point in the distorted coordinate system is the same as the voltage on the theoretical perfect sheet at the corresponding point in the actual perfect coordinate system. A voltage error caused by non-uniformity of the sheet can thus be looked upon as equivalent to the error caused by a corresponding inaccuracy in determining the position at which the voltage is measured. Since the voltage can be set at the desired value at the electrode boundaries on the sheet, it is intuitively obvious that the equivalent displacement error caused by sheet non-uniformity will tend to be the least near the electrodes and the greatest far from the electrodes. For a given equivalent displacement, the actual voltage error will of course be proportional to the component of the voltage gradient in the direction of the equivalent displacement.

Since the effect of sheet non-uniformity tends to be masked by the other errors in the experimental model computer, its magnitude has not been determined to any degree of accuracy. From the measurements which were made, however, it appears that the effect in this computer is represented by a maximum equivalent displacement of approximately ± 0.07" in the relative position of z and z_k. On the large scale of the computer, this is equal to 0.05 natural units, corresponding to an error of 5% or 1/2 db. in either $|z|$ or $|z_k|$, or an error of 3° in either $\angle z$ or $\angle z_k$. On the small scale, the same displacement represents twice as great an error. The mechanical tolerance in the placing of the voltmeter contacts can easily be held to a much smaller value than this, and the electrical tolerances could be improved. Therefore, it would appear that an improved model computer still using Teledeltos paper could be built in which the over-all error in each factor, expressed in terms of equivalent displacement of either z or z_k, would be held to about $\pm$ .05 natural units. This is equal to an error of 5% or 3° in z or z_k.

Applications of the Computer

The most obvious application of the computer is the direct problem in which the function F is to be evaluated at one or more specified values of z. For example, if F is the complex gain or transfer function of an amplifier or a network, the curves of gain and phase shift are simply the magnitude and phase of F evaluated at $z = j\omega$ for a large number of values of the angular frequency ω. When the function is set up on the computer, either of these curves can be obtained by placing the coordinate frame over the appropriate sheet, clamping $\angle z$ at 90°, and plotting the meter readings as a function of $|z|$, which is equal to ω. Alternatively, the curves can be plotted as a function of the actual frequency

$f = \frac{\omega}{2\pi}$, which is marked $\frac{|z|}{2\pi}$ on the computer scale. When the curve is to be plotted on graph paper having the same logarithmic frequency scale as the computer $|z|$ scale, the graph paper can be attached to the computer table under the stylus in the proper position for ω or f to be located mechanically, without the necessity for actually reading the values of $|z|$ or of $\frac{|z|}{2\pi}$.

A somewhat special direct evaluation problem, to which the computer is particularly well suited, is the measurement of the residue[19] at each first order pole of a function. The residue at a first order pole is the value of the function which remains after the corresponding factor has been removed from the denominator of the original function, evaluated at the position of the removed pole. It is obvious, then, that the residue will be obtained on the computer by:

1. Moving the coordinate frame to the position in which the voltmeter contact representing that pole is directly over point z_0 (at the center of the sheet);
2. Removing the voltmeter contact;
3. Reading the remaining function.

The more important application of the computer, however, is probably the inverse problem--that of determining the roots of the equation when F is a given parameter. For this type of problem, a piece of paper is placed under the stylus, and the loci of points at which either the magnitude or the phase of F has the desired value are located experimentally and marked on the paper. After these loci have been located on the paper for one component of F, the coordinate frame is shifted to measure the other component. The point on each locus at which this component also shows the correct value for F is a root of the equation.

The ability of such a computer to find roots quickly is particularly useful in factoring polynomials of high degree.[20] The polynomial is set equal to zero, forming an equation whose roots are the zeros of the function; for example:

$$A_o + A_1 z + A_2 z^2 + A_3 z^3 + A_4 z^4 + A_5 z^5 = 0$$

Half of the terms are then transferred to the other side of the equation and the common power of z factored out:

$$A_o + A_1 z + A_2 z^2 = - z^3 (A_3 + A_4 z + A_5 z^2)$$

Both sides of the equation are then divided by either one of the sides; for example:

$$1 = - z^3 \frac{A_3 + A_4 z + A_5 z^2}{A_o + A_1 z + A_2 z^2}$$

When the original polynomial is of the fifth degree or lower, it is seen that the resulting numerator and denominator can be factored by means of the quadratic formula. The equation can then be set up on the computer and the roots found by the root locus technique described in the preceding paragraph. For a polynomial of higher than fifth degree, the numerator, or the denominator--or both--will be of higher degree than quadratic; each of them can be factored individually with the aid of the computer to put the principal equation in the form which the computer can handle. Any polynomial can therefore be factored with the aid of the computer, in a sufficient number of steps or stages. When such root finding is merely a preliminary stage of a problem to be handled on the computer, the root locations marked on the paper under the stylus can be used directly to locate the voltmeter contacts for the next stage, without the necessity of reading their locations on the z scales.

The computer is directly applicable to the root locus method of servomechanism analysis and design.[8] The figure obtained with the computer is the log transform of the usual graphical root locus plot. The graphical root locus method is limited to the z plane; in it, the frequency range which can be handled at one time is limited, and the relative accuracy of measurements in the vicinity of the origin is comparatively poor. In the log z coordinates of the computer, on the other hand, zeros and poles of the function over an extremely wide frequency range can be included, and the relative accuracy of measurements is the same at all frequencies.

Operation of the Computer

The alternate forms of the function evaluated by the computer are expressed on the computer input panel as follows:

$$(+\text{ or }-)(K\text{ or }\frac{1}{K})\cdot(z^n\text{ or }\frac{1}{z^n})\cdot\left(\frac{(z-a_1)(z-a_2)\text{etc.}}{(z-b_1)(z-b_2)\text{etc.}}\text{ or }\frac{(1-\frac{z}{a_1})(1-\frac{z}{a_2})\text{etc.}}{(1-\frac{z}{b_1})(1-\frac{z}{b_2})\text{etc.}}\right)$$

The steps in setting up a function for evaluation are summarized as follows:

1. Choose the computer scale to be used--to change from large scale to small scale, or vice versa, turn over the $|z|$ and $\angle z$ scales.

2. Plug into the proper plug on the top of the power supply to agree with the scale chosen and the form of the factors in the function to be evaluated.

3. Check and set the computer calibration adjustment as follows:

   a. Set K = 1 (accurately).

   b. Set n equal to any value other than one.

   c. Set the frame in position to read $|F|$.

   d. Set $|z|$ = 1 (accurately).

   e. Press microswitch on right-hand side of power supply, to turn on power without lifting sheets; set calibration adjustment (small knob on top of power supply at right) so that main meter reads $|F|$ = 1, and set a.c. line voltmeter on power supply to red line by means of knob directly below it.

4. Now set the dials on the input panel for the constants in the function to be evaluated--i.e., for the sign, the choice of K or 1/K, the value of K, the choice of z^n or $1/z^n$, and the value of n.

5. For each of the general factors in the numerator and the denominator of the function, connect a contact to a corresponding input jack* and mount the contact in the coordinate frame at the position of the constant a or b of that factor; the proper position is found by setting the frame at (for example) $z = a_1$ (both $|z|$ and $\angle z$)** and placing the contact directly over the center of the sheet.***

When the function has been set up on the computer as described above, $|F|$ or $\angle F$ can be obtained at any desired value of $|z|$ and $\angle z$ by shifting the frame to the proper position, turning the large lever at the left, and reading the answer on the meter--bringing the meter reading on scale, if necessary, by means of the range switch on the meter panel.

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* When a factor is raised to a power, it is a repeated factor, and its contact is connected to the corresponding number of input jacks. Fractional exponents are not provided for in the internal circuit of the computer, but they may be accommodated by adding external series resistance. The effective internal resistance at each input jack, representing an exponent 1, is .51 megohm; when external resistance is added, the exponent is inversely proportional to the total internal and external resistance. Plug-in adapters containing a resistance of .51 megohm are provided for taking the square root of factors; two such adapters can be cascaded for a cube root, etc.

** *If a function is given with factors in the form* $(1+T_1 z)$, *etc.*, the computer form $(1 - \frac{z}{a_1})$ is equivalent when $T_1 = -\frac{1}{a_1}$. The contact position for $(1 - T_1 z)$ is then found by setting the frame at $\angle z = \angle T_1 \pm 180^\circ$, and at $\frac{1}{|z|}$ (on the scale at the right) $= \frac{1}{|a_1|} = |T_1|$.

*** When using the large scale on the computer, it will frequently be necessary to move some of the contacts a distance 360° in the $\angle z$ direction from their original positions, in order to keep them over the sheet as the value of $\angle z$ is changed. Since on this scale the width of each sheet is 360°, this alternate position for a contact can be located by setting the frame in the position in which the contact is at the center of one side of the strip, and then moving the contact to another insulating tube clamped directly over the center of the other side of the strip.

Calibration of the Computer

The only adjustments in the computer which must be checked regularly are the line voltage adjustment and the calibration adjustment on the power supply panel; these adjustments have already been described in the instructions for the operation of the computer. The rest of the computer calibration adjustments should be checked occasionally, however, and if a sheet is replaced as a result of wear or damage, a complete recalibration is of course required. The resistors which require adjustment are mounted on the under side of the power supply panel, and are specified on the circuit diagrams as R_1 through R_5, with additional subscripts A through D indicating which power supply socket is affected by the adjustment of each resistor. A resistor specified with two letter subscripts affects two socket connections; the same setting should be correct for both. The calibration procedures outlined on the following pages are recommended for the adjustment of these resistors. Since K does not enter into the calibration problems, the K dial should be accurately set at 1 during calibration.

Socket A--*Large Scale, (z - a):*

Magnitude Calibration:

1. Set up to evaluate magnitude of $F = \frac{1}{z^6}$ at $|z| = 1$; adjust R_{3A} and/or panel control R_{3X} so that meter reads $|F| = 1$.
2. Set up to evaluate magnitude of $F = \frac{1}{z}$ at $|z| = 100$; read $|F|$ on meter.
3. Set up to evaluate magnitude of $F = \frac{1}{z - 100}$ at $|z|$ very small (frame near left-hand limit); read $|F|$ at various values of $\angle z$--if readings vary, stop at a point where the reading is typical; at this point, adjust R_{2A} so that $|F|$ reading is the same as in step 2 above.
4. Return to step 1 and repeat all three steps in turn, several times if necessary, until no adjustment is required at step 3.
5. Now set up to read magnitude of $F = z$ and read $|F|$ at $|z| = .01$, 1, and 100; adjust R_{1AB} so that the error (if any) is the same at all three points. If this error is appreciable, return to step 1 and repeat.

Phase Calibration:

1. Set up to evaluate phase of $F = z^6$ at $\angle z = 0^\circ$; if meter does not read $\angle F = 0^\circ$, install and/or adjust R_{5A} as required.
2. Set up to evaluate phase *of $F = z$ and read* $\angle F$ at $\angle z = -180^\circ$ and $+180^\circ$; adjust R_{4AB} so that the error (if any) is the same at both points. If this error is appreciable, return to step 1 and repeat both steps.

Socket B--Large Scale, $(1 - \frac{z}{a})$:

Magnitude Calibration:

1. Set up to evaluate magnitude of $F = z^6$ at $|z| = 1$; adjust R_{3B} and/or panel control R_{3X} so that meter reads $|F| = 1$.
2. Set up to evaluate magnitude of $F = z$ at $|z| = .01$; read $|F|$ on meter.
3. Set up to evaluate magnitude of $F = \frac{z}{1 - \frac{z}{.01}}$ at $|z|$ very large (frame near right-hand limit); read $|F|$ at various values of $\angle z$--if readings vary, stop at a point where the reading is typical; at this point, adjust R_{2B} so that $|F|$ reading is the same as in step 2 above.
4. Return to step 1 and repeat all three steps in turn, several times if necessary, until no adjustment is required at step 3.
5. Now set up to read magnitude of $F = z$ and read $|F|$ at $|z| = .01$, 1, and 100; adjust R_{1AB} so that the error (if any) is the same at all three points. If this error is appreciable, return to step 1 and repeat.

Phase Calibration:

1. Set up to evaluate phase of $F = z^6$ at $\angle z = 0^\circ$; if meter does not read $\angle F = 0^\circ$, install and/or adjust R_{5B} as required.
2. Set up to evaluate phase of $F = z$ and read $\angle F$ at $\angle z = -180^\circ$ and $+180^\circ$; adjust R_{4AB} so that the error (if any) is the same at both points. If this error is appreciable, return to step 1 and repeat both steps.

Socket C--Small Scale, $(z - a)$:

Magnitude Calibration:

1. Set up to evaluate magnitude of $F = \frac{1}{z^6}$ at $|z| = 1$; adjust R_{3C} and/or panel control R_{3X} so that meter reads $|F| = 1$.
2. Set up to evaluate magnitude of $F = \frac{1}{z}$ at $|z| = 10^4$; read $|F|$ on meter.
3. Set up to evaluate magnitude of $F = \frac{1}{z - 10^4}$ at $|z|$ very small (frame near left-hand limit); read $|F|$ at various values of $\angle z$--if readings vary, stop at a point where the reading is typical; at this point, adjust R_{2C} so that $|F|$ reading is the same as in step 2 above.
4. Return to step 1 and repeat all three steps in turn, several times if necessary, until no adjustment is required at step 3.
5. Now set up to read magnitude of $F = z$ and read $|F|$ at $|z| = 10^{-4}$, 1, and 10^{+4}; adjust R_{1CD} so that the error (if any) is the same at all three points. If this error is appreciable, return to step 1 and repeat.

Phase Calibration:

1. Set up to evaluate phase of $F = z^6$ at $\angle z = 0^\circ$; if meter does not read $\angle F = 0^\circ$, install and/or adjust R_{5C} as required.
2. Set up to evaluate phase of $F = z$ and read $\angle F$ at $\angle z = -180^\circ$ and $+180^\circ$; adjust R_{4CD} so that the error (if any) is the same at both points. If this error is appreciable, return to step 1 and repeat both steps.

Socket D--Small Scale, $(1 - \frac{z}{a})$:

Magnitude Calibration:

1. Set up to evaluate magnitude of $F = z^6$ at $|z| = 1$; adjust R_{3D} and/or panel control R_{3X} so that meter reads $|F| = 1$.
2. Set up to evaluate magnitude of $F = z$ at $|z| = 10^{-4}$; read $|F|$ on meter.
3. Set up to evaluate magnitude of $F = \frac{z}{1 - \frac{z}{10^{-4}}}$ at $|z|$ very large (frame near right-hand limit); read $|F|$ at various values of $\angle z$--if readings vary, stop at a point where the reading is typical; at this point, adjust R_{2D} so that $|F|$ reading is the same as in step 2 above.
4. Return to step 1 and repeat all three steps in turn, several times if necessary, until no adjustment is required at step 3.
5. Now set up to read magnitude of $F = z$ and read $|F|$ at $|z| = 10^{-4}$, 1 and 10^{+4}; adjust R_{1CD} so that the error (if any) is the same at all three points. If this error is appreciable, return to step 1 and repeat.

Phase Calibration:

1. Set up to evaluate phase of $F = z^6$ at $\angle z = 0°$; if meter does not read $\angle F = 0°$, install and/or adjust R_{5D} as required.
2. Set up to evaluate phase of $F = z$ and read $\angle F$ at $\angle z = -180°$ and $+180°$; adjust R_{4CD} so that the error (if any) is the same at both points. If this error is appreciable, return to step 1 and repeat both steps.

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