APPLICATION OF DISPERSION RELATIONS TO PROTON COMPTON SCATTERING

Thesis by

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In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

1957

ACKNOWLEDGEMENTS

The author is greatly indebted to Professor Murray Gell-Mann for suggesting this problem and for giving considerable advice during the course of the ensuing research. The author also wishes to thank Dr. Ricardo Gomez for many discussions of the M.I.T. experimental results and of Compton scattering in general, Dr. G. Bernardini for forwarding to the author the results of the Illinois group, and Drs. V. Z. Peterson and W. S. McDonald for communicating their analysis of neutral pion photoproduction in advance of publication. Finally, the author is very grateful to the National Science Foundation for the predoctoral fellowship which was awarded him for his graduate study.

ABSTRACT

Dispersion theory has been applied to proton Compton scattering at all angles. Unitarity of the S matrix determines the imaginary part of the scattering amplitude; the only absorptive process considered is pion photoproduction. The problem of analytic continuation of amplitudes has been handled in a natural way, which should be valid for scattering energies up to several hundred Mev. Numerical results calculated in a static approximation are in essential agreement with present experimental data.

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Introduction

Even if one accepts present-day field theory - including pions - as valid, it is very difficult to calculate any quantity involving pions, since the size of the pion-nucleon coupling constant renders perturbation methods of dubious value. Hence it is very desirable to seek out properties of field theory not requiring a perturbation expansion.

An example is the derivation of so-called dispersion relations for scattering amplitudes. This originated with the work of Kramers and Heisenberg,[(1)] which relied on the correspondence principle and closely paralleled the classical theory of dispersion. In recent years the concept of "microscopic causality," the commuting of field operators at points separated by a space-like interval, has been applied by Gell-Mann, Goldberger and others[(2)-(4)] to derive dispersion relations within the framework of quantum field theory.

Another important property of field theory is the basic idea of conservation of probability, or, more formally, the unitarity of the S matrix.[(5)] The most familiar theorem related to unitarity is the well-known relation between the imaginary part of a forward scattering amplitude and the total cross-section.

The program of this thesis is as follows: Dispersion theory is to be used to predict the elastic gamma-ray scattering from free protons. Since dispersion relations are conveniently written as relations between the real and imaginary parts of scattering amplitudes, we use the unitarity of the S matrix to deduce the imaginary Compton amplitude, and then get the real part from dispersion theory. The unitarity principle gives the imaginary part of the scattering amplitude as a bilinear form in amplitudes for all processes resulting from gamma-rays incident on protons. The only such process we shall take into consideration is pion photoproduction.

There are at least three significant processes which occur when gamma-rays with energies of several hundred Mev strike photons. The predominant one is electron pair production. The dispersion scattering resulting from this process is known as Delbruck scattering.* Not much is known quantitatively about non-forward Delbruck scattering, but the cross-section is certainly negligible except for extreme forward angles ($\theta_{max} \sim k/m$, where k is the photon energy and m is the electron mass). At such angles experimental difficulties make it impossible to measure Compton scattering anyway.

The second absorptive process is pion photoproduction, which has been studied quite thoroughly for photon energies up to 500 Mev. Since the charged and neutral cross-sections both show a resonance at about 300 Mev, we should be able to obtain reliable low energy results without knowing the photoproduction cross-sections at higher energies; the relevant integrals converge rapidly and contributions from beyond 500 Mev are apparently only small corrections which can be estimated by extrapolating the experimental curves.

Incidentally, one can estimate the effect of this resonance by the Breit-Wigner single-level formula for total cross-section. If the pion and gamma-ray widths are determined from scattering and photoproduction

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* A brief discussion and some references are given in reference (6).

data,[7] one finds that the total gamma-ray cross-section contributed by this resonance at 300 Mev should be about 3.5 times the zero-energy cross-section. The more precise calculations of this thesis, and the few experimental points in this region, suggest that this considerably underestimates the true effect, which is some five or more times the zero-energy cross-section.

Finally, the third process is elastic gamma-ray scattering, which is the process we are trying to predict. Fortunately, the cross-section is several orders of magnitude smaller than photomeson cross-sections, so that it is a reasonable approximation to neglect the absorptive effects of Compton scattering. This procedure is clearly equivalent to neglecting all terms of order e^4 and higher in the Compton scattering amplitude. The situation is quite different for pion scattering; here the dispersion relations become integral equations for the scattering amplitudes, rather than integrals which give the amplitudes directly.

In Section I of this thesis we derive the unitarity relations which give us the imaginary parts of the Compton amplitudes in terms of photoproduction amplitudes. In Section II we discuss the dispersion formalism in general, and its application to Compton scattering amplitudes. The unitarity relations are expressed most naturally in the center-of-momentum (hereafter CM) system, whereas a different coordinate system is used for the dispersion relations; the transformation of amplitudes between these two systems is discussed in Section III. Finally, in Section IV, approximations are made and some numerical computations are given. Several appendices contain formulas and calculations which would be inconvenient interruptions in the main body of the thesis.

I - Unitarity Relations

We begin by mentioning some conventions to be used in this thesis. Units are chosen so that $\hbar = c = 1$. In numerical work M, the proton mass, is also set equal to unity. The unit of cross-section is then

$$(\hbar/Mc)^2 = 4.42 \times 10^{-28} \text{ cm}^2 = 442 \text{ microbarns}$$

$-e$ denotes the rationalized charge of the electron:

$$e^2/4\pi\hbar c \approx 1/137$$

Three-vectors are written $\vec{a}$; if $\vec{a}\cdot\vec{a} = 1$ we often write $\hat{a}$ instead. In general by $\hat{a}$ we mean $\vec{a}/|\vec{a}|$. A four-vector is written a or a_μ; by $a\cdot b$ we mean

$$a_\mu b_\mu = a_o b_o - a_1 b_1 - a_2 b_2 - a_3 b_3 = a_o b_o - \vec{a}\cdot\vec{b}$$

We shall begin with the formal relation

$$S_{fi} = \delta_{fi} - (2\pi)^4 i\, \delta(p_f - p_i)\, R_{fi}$$

relating the R and S matrices for scattering from a state i, with total four-momentum p_i, to a state f with total four-momentum p_f. When we calculate a scattering amplitude, even in the forward direction, we get R, not S, since we are calculating $\lim_{f\to i} S$, so that we do not pick up the δ_{fi} term. Calculation with the well-known Feynman rules gives $-iR$. In this thesis we shall refer to R as the "scattering amplitude."

The cross-section for a process $i\to f$ is

$$\sigma_{fi} = (1/v_i) \sum_f (2\pi)^4 \delta(p_f - p_i) |R_{fi}|^2$$

where the summation extends over all final states observed. For example, if we have a two-particle final state,

$$= \frac{1}{v_i} \int \frac{d^3 p_a}{(2\pi)^3} \int \frac{d^3 p_b}{(2\pi)^3} \quad (2\pi)^4 \, \delta(\vec{p}_a + \vec{p}_b - \vec{p}_i) \, \delta(E_a + E_b - E_i) |R_{fi}|^2$$

$$= \frac{1}{v_i} \int \frac{d^3 p_a}{(2\pi)^3} \quad (2\pi) \, \delta(E_a + E_b - E_i) |R_{fi}|^2$$

$$= \frac{2\pi}{v_i} \rho_f |R_{fi}|^2 \tag{1}$$

which is the usual rule.

We now assume that the S matrix is unitary, which is equivalent to conservation of probability.[5] From the relation

$$S^+ S = 1$$

we deduce immediately

$$R_{fi} - R^*_{if} = -(2\pi)^4 \, i \sum_n \delta(p_n - p) \, R^*_{nf} \, R_{ni} \tag{2}$$

$$(p_i = p_f = p)$$

We shall now specialize: State i is a photon $\vec{k}$, $\hat{e}$ and a proton $-\vec{k}$, E_k. State f is a photon $\vec{k}'$, $\hat{e}'$ and a proton $-\vec{k}'$, E_k. State n is a meson $\vec{q}$, ω_q plus a nucleon $-\vec{q}$, E_q. The notation is fairly conventional, k denoting photon momenta; e, polarizations; q, meson momenta; ω, meson energies, and E, proton energies.

$$\omega_q^2 = q^2 + m^2 \qquad\qquad E_k^2 = k^2 + M^2$$

It is clear we must work in the CM system. If, for example, the initial proton were at rest, the final one would be recoiling at some angle, so that R_{nf} would refer to photoproduction in some odd system, neither lab nor CM nor anything else in particular. The states in (2) are related by their total four-momenta p; for the CM system $\vec{p} = 0$, whereas no such characterization in terms of total momentum defines the lab system.

With these substitutions, (2) becomes

$$R_{fi} - R^*_{if} = -(2\pi)^4 \, i \sum_{ch} \int \frac{d^3q}{(2\pi)^3} \frac{1}{(2\pi)^3} \delta(\omega_q + E_q - E_k - k) \, R^*_{nf} R_{ni}$$

$$= \frac{-i}{4\pi^2} \frac{q\omega_q E_q}{E} \sum_{ch} \int d\Omega_q \, R^*_{nf} R_{ni} \tag{3}$$

$$(E = \omega_q + E_q = k + E_k)$$

Note that we must sum over the possible charges (+ and 0) of the pion. Also we have omitted reference to the nucleon spin states; the σ matrices will take care of themselves if we interpret $*$ as Hermitian conjugation.

We write our photoproduction amplitude as

$$R_{ni} = f_1(k,x) \, \hat{e}\cdot\hat{k}\times\hat{q} + f_2(k,x) \, i\vec{\sigma}\cdot\hat{e} + f_3(k,x) \, \hat{e}\cdot\hat{q} \, i\vec{\sigma}\cdot\hat{k} + f_4(k,x) \, \hat{e}\cdot\hat{q} \, i\vec{\sigma}\cdot\hat{q}$$

$$(x \equiv \hat{k}\cdot\hat{q}) \tag{4}$$

Since R must be of the form $\hat{e}\cdot\vec{A}$, with $\vec{A}$ a pseudovector, the four amplitudes in (4) are the only ones possible. We shall always assume transverse gauge in this thesis; $\hat{k}\cdot\hat{e} = 0$.

In order to substitute (4) into (3) and integrate over q, we must do some integrals, such as

$$I_1 = \int d\Omega_q \, f_1(x') \, f_2(x) \qquad (x \equiv \hat{k}\cdot\hat{q} \, , \; x' \equiv \hat{k}'\cdot\hat{q})$$

Choosing $\hat{k}$ as axis of spherical polar coordinates, and changing the integration variables to x and x', we obtain

$$I_1 = 2\iint \frac{dx\,dx'}{K(x,x',y)} \quad f_1(x')\, f_2(x) \qquad\qquad (y \equiv \hat{k}\cdot\hat{k}')$$

where

$$K^2(x,x',y) = 1 - x^2 - (x')^2 - y^2 + 2xx'y$$

and integration is to be performed only over regions for which K is real.

Proceeding in similar fashion we do the other required integrals; the results are in Appendix A.

We now write the Compton scattering amplitude in the form

$$\begin{aligned} R_{fi} = {} & g_1(k,y)\ \hat{e}\cdot\hat{e}' + g_2(k,y)\ \hat{e}\cdot\hat{k}'\ \hat{e}'\cdot\hat{k} + g_3(k,y)\ i\vec{\sigma}\cdot\hat{e}\times\hat{e}' \\ & + g_4(k,y)\ \hat{e}\cdot\hat{e}'\ i\vec{\sigma}\cdot\hat{k}\times\hat{k}' + g_5(k,y)\ i\vec{\sigma}\,(\hat{e}\cdot\hat{k}'\ \hat{e}'\times k - \hat{e}'\cdot\hat{k}\ \hat{e}\times\hat{k}') \\ & + g_6(k,y)\ i\vec{\sigma}\cdot(\hat{e}\cdot\hat{k}'\ \hat{e}'\times\hat{k}' - \hat{e}'\cdot\hat{k}\ \hat{e}\times\hat{k}) \end{aligned} \tag{5}$$

We have omitted two amplitudes, obtainable by replacing the minus signs by plus signs in the fifth and sixth terms. In general, these amplitudes will be expected to occur, but only in order e^4; we are neglecting such terms. The vanishing of these amplitudes to order e^2 results from two effects: There are no such terms in the Born approximation (i.e., the Klein-Nishina amplitude), and the appearance of such terms in the dispersion contributions is forbidden by the theorem(7) equating the phase of a meson photoproduction matrix element to the corresponding meson scattering phase shift. This theorem breaks down for terms of order e^3 in the photoproduction amplitude, and so terms of order e^4 will appear in the Compton amplitude.

If the amplitude (5) is squared and averaged over spins and polarizations, and if the appropriate density of states is used, we find for the Compton cross-section in the CM system:

$$(d\sigma/d\Omega)_{cm} = (kE_k/2\pi E)^2 \Big[\tfrac{1}{2}|g_1|^2(1+\cos^2\theta) + \tfrac{1}{2}|g_2|^2 \sin^4\theta + (-\mathrm{Re}\ g_1^* g_2$$
$$+\mathrm{Re}\ g_3^* g_4 + 6\mathrm{Re}\ g_5^* g_6 - 2\mathrm{Re}\ g_3^* g_5 - 2\mathrm{Re}\ g_4^* g_6) \sin^2\theta\cos\theta$$
$$- 2\mathrm{Re}\ g_4^* g_5 \sin^2\theta \cos^2\theta + |g_5|^2 \sin^2\theta\ (1+2\cos^2\theta) \tag{6}$$
$$+ \tfrac{1}{2}|g_3|^2\ (3-\cos^2\theta) + \tfrac{1}{2}|g_4|^2 \sin^2\theta\ (1+\cos^2\theta)$$
$$+(3|g_6|^2 - 4\mathrm{Re}\ g_3^* g_6) \sin^2\theta \Big]$$

We shall write this in the form

$$(d\sigma/d\Omega)_{cm} = (e^2/4\pi M)^2\ (A + B\cos\theta + C\cos^2\theta + D\cos^3\theta + E\cos^4\theta) \tag{7}$$

If we now do the integration in (3) by means of the formulas in Appendix A, we obtain the imaginary parts of the amplitudes g_i as bilinear integral forms in the quantities f_i^* and f_i. These integrals are given in Appendix B. The formulas appear to be too cumbersome for direct application.

In the forward direction (y=1) the kernels of Appendix B become singular. For example

$$\frac{1}{K(x,x',1)} = \pi\,\delta(x - x')$$

Working out the limits of the remaining kernels as $y \to 1$, we obtain the formulas of Appendix C for Im $g_i(k,1)$. We have a check available for the first and third of these, since the former must be $(-E/2E_k)$ times the total photomeson cross-section for unpolarized light, and the latter must be $(-E/4E_k)$ times the total cross-section for circular polarization parallel to proton spin minus the total cross-section for polarization opposite to proton spin. Both of these checks, which are derivable from (2), are easily verified. It should be noted that the correct velocity v_i to use in formula (1) is the sum of photon and proton velocities, $1 + (k/E_k)$, if we are in the CM system. (See Appendix F)

Since we need the imaginary parts of g_i at all angles, and the formulas in Appendix B are very complicated, we shall take advantage of the fact that photomeson production at energies up to 500 Mev appears to be capable of description in terms of a very small number of multipole amplitudes. We shall adopt the notation of Gell-Mann and Watson[7] and write

$$R_{ni} = f_{E1}\, i\vec{\sigma}\cdot\hat{e} - f_{M1}\left[\hat{k} \times \hat{e}\cdot\hat{q} - i\,\vec{\sigma}\cdot(\hat{k} \times \hat{e}) \times \hat{q}\right]$$
$$- f_{M3}\left[2\hat{k} \times \hat{e}\cdot\hat{q} + i\vec{\sigma}\cdot(\hat{k} \times \hat{e}) \times \hat{q}\right] + \frac{1}{2} f_{E2}\, i\vec{\sigma}\cdot(\hat{k}\ \hat{e}\cdot\hat{q} + \hat{e}\ \hat{k}\cdot\hat{q}) \qquad (8)$$

Note that we use M3 to mean magnetic dipole in the J=3/2 state, not magnetic octupole.

In the notation of (4),

$$f_1 = f_{M1} + 2 f_{M3} \qquad f_3 = - f_{M1} + f_{M3} + \frac{1}{2} f_{E2}$$

$$f_2 = f_{E1} + x f_{M1} - x f_{M3} + \frac{1}{2} x f_{E2} \qquad f_4 = 0$$

We may now insert (8) directly into the unitarity relation (3), and obtain formulas for Im g_i which are much more tractable than those of Appendix B:

$$\text{Im } g_1 = -(q\omega_q E_q/2\pi E) \sum_{ch} \left(|f_{E1}|^2 + y|f_{M1}|^2 + 2y|f_{M3}|^2 + \frac{1}{6}y|f_{E2}|^2 \right)$$

$$\text{Im } g_2 = -(q\omega_q E_q/2\pi E) \sum_{ch} \left(-|f_{M1}|^2 - 2|f_{M3}|^2 + \frac{1}{6}|f_{E2}|^2 \right)$$

$$\text{Im } g_3 = -(q\omega_q E_q/2\pi E) \sum_{ch} \left(-|f_{E1}|^2 - y|f_{M1}|^2 + y|f_{M3}|^2 - \frac{1}{12} y|f_{E2}|^2 + y f^*_{M3} f_{E2} \right) \qquad (9)$$

$$\text{Im } g_4 = -(q\omega_q E_q/2\pi E) \sum_{ch} \left(-|f_{M1}|^2 + |f_{M3}|^2 - \frac{1}{12}|f_{E2}|^2 - \text{Re } f^*_{M3} f_{E2} \right)$$

$$\text{Im } g_5 = -(q\omega_q E_q/2\pi E) \sum_{ch} \left(-|f_{M1}|^2 + |f_{M3}|^2 + \frac{1}{12}|f_{E2}|^2 \right)$$

$$\text{Im } g_6 = 0$$

II - Dispersion Formulas

Before discussing our specific applications, we shall briefly review the dispersion formalism. It should be emphasized that we are not concerned with the rigorous foundations of dispersion theory, a subject which has received considerable attention recently.[8] The derivations of this section will be heuristic in nature, with little or no mention of necessary and sufficient conditions, etc.

We begin with Dyson's formula for the S matrix (in interaction representation)

$$S = \Sigma_{n=0}^{\infty} \frac{(-i)^n}{n!} \int dt_1 \ldots dt_n \, P\left[H_I(t_1) \ldots H_I(t_n)\right]$$

H_I being the interaction Hamiltonian and P the time-ordering operator. The S matrix element for scattering a photon with four-momentum q and polarization i to a photon q', f , while a proton p goes into a proton p', is

$$S_{fi} = \Sigma_{n=0}^{\infty} \frac{(-i)^n}{n!} \int dt_1 \ldots dt_n \, (\Phi_{p'} \, , \, a_f(\vec{q}\,')P\left[H_I(t_1) \ldots H_I(t_n)\right] a_i^+(\vec{q})\Phi_p)$$

Φ denotes a bare proton; a and a^+ are photon destruction and creation operators. By commuting the a and a^+ through the P-bracket,[9] we obtain (assuming initial and final photon are not identical)

$$S_{fi} = -(4q_o q_o')^{-1/2} \int d^4x\, d^4y\, e^{-iq\cdot x}\, e^{iq'\cdot y}\, (\Psi_{p'}\,,\, P[j_i(x) j_f(y)] \Psi_p)$$

Ψ denotes a physical nucleon, and we are in Heisenberg representation.

Making a change of variable and using translational invariance, this becomes

$$S_{fi} = -(4q_o q_o')^{-1/2} (2\pi)^4 \delta(p+q-p'-q') \int d^4x\, e^{-ix\cdot(q+q')/2}\, (\Psi_{p'}\,, P[j_i(\tfrac{x}{2}) j_f(-\tfrac{x}{2})] \Psi_p)$$

We now follow Goldberger[2] and consider instead of R a new quantity M, defined by

$$M_{fi} = -i(4q_o q_o')^{-1/2} \int d^4x\, e^{-ix\cdot(q+q')/2}\, (\Psi_{p'}\,,\, [j_f(\tfrac{x}{2}),\, j_i(-\tfrac{x}{2})] \Psi_p)\, \eta(x) \quad (10)$$

where

$$\eta(x) = \begin{cases} 1 & \text{if } x_o > 0 \\ 0 & \text{if } x_o < 0 \end{cases}$$

R = M for physical scattering energies, i.e., positive energies such that all momenta are real.

We shall adopt the coordinate system of Figure 1:

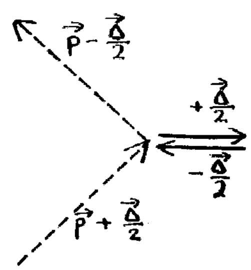


Figure 1

The initial and final protons have three-momenta $-\vec{\Delta}/2$ and $+\vec{\Delta}/2$, respectively; the initial and final photon momenta are $\vec{p}+\frac{\vec{\Delta}}{2}$ and $\vec{p}-\frac{\vec{\Delta}}{2}$.

The photon energy is ℓ. With these substitutions, (10) becomes

$$2\ell M_{fi} = -i\int d^4x\, e^{i\ell t - i\vec{p}\cdot\vec{x}} \left(\Psi_{\frac{\vec{\Delta}}{2}}, \left[j_f(x/2),\ j_i(-x/2)\right]\Psi_{-\frac{\Delta}{2}}\right)\eta(x) \tag{11}$$

We now consider M to be defined by (11) for complex values of ℓ, with Δ^2 being held fixed. Since

$$\vec{p} = p(\ell)\hat{j}, \text{ with } p^2(\ell) = \ell^2 - \frac{\Delta^2}{4}$$

there are branch points at $\ell = \pm\Delta/2$; the precise way we cut the ℓ-plane is not important provided the cuts are confined to the lower half plane. A convenient arrangement of cuts is given in reference (4).

It then follows from the assumption of microscopic causality

$$\left[j_f(x/2),\ j_i(-x/2)\right] = 0 \qquad \text{for} \quad x^2 < 0$$

that $2\ell M$ has no singularities above the real axis. This is the key step in establishing dispersion relations. The conclusion may be rendered plausible by the following reasoning: $2\ell M$ is of the form

$$\int_0^\infty dt \int_{-t}^{t} dx\, e^{i\ell t - ix\sqrt{\ell^2 - (\Delta^2/4)}}\ F(x,t)$$

This is a function of ℓ. If we take its Fourier transform by multiplying by $e^{is\ell}$ and integrating over ℓ from $-\infty$ to $+\infty$, we obtain

$$\int_{-\infty}^{\infty} d\ell \int_0^\infty dt \int_{-t}^{t} dx\, e^{i\ell(s+t) - ix\sqrt{\ell^2 - (\Delta^2/4)}}\ F(x,t)$$

If the real part of s is positive, and if we can interchange the order of integration so as to integrate over ℓ first, we can complete the ℓ contour by a large semicircle in the upper half of the ℓ-plane, and it is easily seen that the integral vanishes. Therefore $2\ell M$ is a function whose Fourier transform vanishes in the upper half plane. With certain mathematical assumptions about boundedness at infinity, it then follows that $2\ell M$ is analytic in the upper half of the ℓ-plane. The necessary and sufficient conditions are discussed in great detail by Toll.[8]

From this fact we can deduce alternative dispersion relations, depending on the behavior of M for large $|\text{Re}\ \ell|$. For example, if $\ell^2 M(\ell)$ is bounded for $|\text{Re}\ \ell| \to \infty$, an elementary contour integration gives

$$2\ell\ M(\ell,\Delta^2) = (i/\pi)\ P\int_{-\infty}^{\infty} \frac{2\ell'\ M(\ell',\Delta^2)}{\ell - \ell'}\ d\ell' \tag{12}$$

where P indicates that the principal value is to be taken at $\ell' = \ell$. If M does not decrease rapidly enough at high energies for (12) to be valid, we may obtain an alternative relation by formally subtracting from (12) the corresponding relation for $\ell=0$ (or any other constant value). Thus, for example,

$$2\ell\ M(\ell,\Delta^2) - \left[2\ell\ M(\ell,\Delta^2)\right]_{\ell=0} = (i\ell/\pi)\ P\int_{-\infty}^{\infty} \frac{2\ell'\ M(\ell',\Delta^2)}{\ell'(\ell - \ell')}\ d\ell' \tag{13}$$

This "subtraction" technique is very important in obtaining dispersion relations. Note that (13) could alternatively have been derived by applying the relation (12) to the function

$$\frac{M(\ell) - M(0)}{\ell}$$

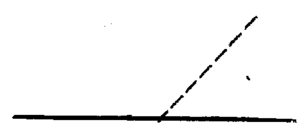
The actual relations which we shall use are obtained by taking the real part of both sides of (12) or (13); thus we obtain the real part of M as an integral involving the imaginary part.

We now write M in a form analogous to (5):

$$M_{fi}(\ell,\Delta^2) = h_1(\ell,\Delta^2)\,\hat{e}\cdot\hat{e}' + h_2(\ell,\Delta^2)\,\hat{e}\cdot\frac{\vec{\Delta}}{2}\,\hat{e}'\cdot\frac{\vec{\Delta}}{2} + h_3(\ell,\Delta^2)\,i\vec{\sigma}\cdot\hat{e}\times\hat{e}'$$
$$+ h_4(\ell,\Delta^2)\,\hat{e}\cdot\hat{e}'\,i\vec{\sigma}\cdot\vec{p}\times\frac{\vec{\Delta}}{2}$$
$$+ h_5(\ell,\Delta^2)\,i\vec{\sigma}\cdot\left(\hat{e}\cdot\frac{\vec{\Delta}}{2}\,\hat{e}'\times\frac{\vec{\Delta}}{2} - \hat{e}'\cdot\frac{\vec{\Delta}}{2}\,\hat{e}\times\frac{\vec{\Delta}}{2}\right) \qquad (14)$$
$$+ h_6(\ell,\Delta^2)\,i\vec{\sigma}\cdot\left(\hat{e}\cdot\frac{\vec{\Delta}}{2}\,\hat{e}'\times\vec{p} + \hat{e}'\cdot\frac{\vec{\Delta}}{2}\,\hat{e}\times\vec{p}\right)$$

Again two amplitudes have been omitted, for reasons discussed after (5).

Since the only important requirement for the derivation of (12) and (13) was analyticity for $\mathrm{Im}\,\ell > 0$, we conjecture without proof that the quantities $h_i(\ell,\Delta^2)$ obey independent dispersion relations. A fact which makes this assumption very plausible is the following: If we attempt to derive the Klein-Nishina formula from dispersion relations, we can formally obtain an imaginary, or absorptive, part of M from the process represented by this diagram:



The process is forbidden by momentum conservation, but it has a non-zero matrix element. We obtain the Klein-Nishina formula exactly if we "factor out" the amplitudes h_i and apply dispersion relations separately to each. See Appendix D for details.

Thus we have the basic relation

$$2\ell \ \mathrm{Re}\ h_i(\ell,\Delta^2) = (1/\pi)\ P\int_{-\infty}^{\infty} \frac{2\ell' \ \mathrm{Im}\ h_i(\ell',\Delta^2)}{\ell' - \ell}\ d\ell' \tag{15}$$

In order to obtain Im h_i for $\ell' < 0$, we consider the symmetries of M_{fi}. If we write the $\eta(x)$ of (11) as

$$\eta(x) = \frac{1}{2}\varepsilon(x) + \frac{1}{2} \quad , \qquad \text{where} \quad \varepsilon(x) \equiv \begin{cases} -1 & \text{if } x_0 < 0 \\ +1 & \text{if } x_0 > 0 \end{cases}$$

M_{fi} becomes the sum of two terms, called its dispersive and absorptive parts, respectively:

$$M_{fi} = D_{fi} + iA_{fi}$$

Correspondingly, each h_i can be written

$$h_i = h_i^{(d)} + ih_i^{(a)}$$

It then follows from (11), by reasoning practically identical to that of Capps and Takeda,[4] that the functions $h_i^{(d)}$ and $h_i^{(a)}$ are real, that for i=1,2 , $h_i^{(d)}$ is an even function of ℓ and $h_i^{(a)}$ is an odd function, and that for i=3-6, $h_i^{(d)}$ is odd while $h_i^{(a)}$ is even.

Therefore (15) becomes

$$2\ell \ \mathrm{Re}\ h_i(\ell) = (2/\pi)\ P\int_0^{\infty} \frac{2\ell'^2 \ \mathrm{Im}\ h_i(\ell')}{\ell'^2 - \ell^2}\ d\ell' \qquad (i=1,2) \tag{16}$$

$$2\ell \ \mathrm{Re}\ h_i(\ell) = (2\ell/\pi)\ P\int_0^{\infty} \frac{2\ell' \ \mathrm{Im}\ h_i(\ell')}{\ell'^2 - \ell^2}\ d\ell' \qquad (i=3\text{-}6) \tag{17}$$

Our formalism has all been derived for fixed Δ^2. We shall expand relations (16) and (17) in powers of Δ^2, applying dispersion relations separately to each power, without worrying about convergence of the series for large Δ^2. We are not particularly interested in large Δ^2 for two reasons: In our numerical applications we will be forced to make approximations which limit us to low energies, and the dispersion formalism used here appears to require serious modification for large Δ^2. At the end of this Section we shall discuss this matter in somewhat more detail.

After expanding into a power series in Δ^2, we shall neglect the terms in Δ^4, Δ^6, . . . This is quite reasonable in view of our emphasis on low energies. If the difficulty mentioned in the preceding paragraph can be resolved, and if it appeared that significantly greater precision could be obtained, terms in Δ^4, . . . could be included in the framework of this thesis. With the assumptions we shall make about photoproduction (Section IV), it is most reasonable to consider only the first two terms.

We noted before that a relation such as (15) is valid only if $h_i(\ell)$ approaches zero rapidly enough for large ℓ. Otherwise we must make a subtraction, as in equation (13). For example, in the forward direction (Δ^2=0) it is well-known that $\lim_{\ell\to 0} h_1(\ell) = e^2/2\ell M$. But the right side of (16) for i=1, Δ^2=0, ℓ=0 is certainly negative, since Im $h_1(\Delta^2=0)$ is just one-half the negative of the total cross-section for unpolarized gamma-rays on protons. Therefore (16) cannot be correct, and for Δ^2=0 we write instead

$$2\ell \ \mathrm{Re}\ h_1(\ell) = (e^2/M) + (2\ell^2/\pi)\ P\int_0^\infty \frac{2\ell' \ \mathrm{Im}\ h_1(\ell')}{\ell'(\ell'^2 - \ell^2)}\, d\ell' \qquad (18)$$

This relation, with a similar one for h_3, has been used by

Gell-Mann and Goldberger[10] in a discussion of Compton scattering for near-forward angles, and by Gell-Mann, Goldberger and Thirring[2] in an extension to all angles by considering the multipoles expected to dominate low-energy Compton scattering. Capps[11] has recently made a similar calculation with the more refined photoproduction data now available. In principle the method of this thesis is much more general, since we apply dispersion relations at all angles, and are not forced to make assumptions about the relative strengths of various multipole amplitudes. In practice, however, we make assumptions about photoproduction which are essentially equivalent to these previous approximations, with the result that our low-energy calculations should closely resemble those of Capps.

It is important to observe that (18) must be valid; we cannot be required to make another subtraction. The reason is that as $\ell \to \infty$, the total cross-section should not increase indefinitely. It follows that $2\ell\left|h_1(\ell)\right|$ must remain finite as $\ell \to \infty$. Therefore $2\ell\left|h_1(\ell)\right|/\ell^2$ is square-integrable at infinity, and satisfies a dispersion relation. We shall use similar reasoning to limit the number of subtractions we must make in other dispersion formulas.

The CM differential cross-section may be written

$$d\sigma = (1/4\pi^2)\ (\ell E_{\frac{\Delta}{2}}/E)^2\ \left|M\right|^2\ d\Omega$$

This formula is derived in Appendix F, formula (F-2). Now consider the limit $\ell \to \infty$, with Δ remaining finite ($\lesssim M$). Then we are considering near-forward scattering, into a solid angle $d\Omega \sim k^{-2}$. The kinematical relations we shall need are

$$E \sim k \sim \ell^{1/2} \qquad p \sim \ell \qquad E_{\Delta/2} \sim M$$

and in order that the total cross-section in this forward cone shall not diverge, we must therefore require $|M| \lesssim 1$ as $\ell \to \infty$. To derive restrictions on the high-energy behavior of the h_i we apply this condition to formula (F-3). Considering each term in (F-3) separately, we deduce the following restrictions as $\ell \to \infty$:

$$\begin{aligned} |h_i| &\lesssim 1 \qquad i=1,2,3,5 \\ |h_i| &\lesssim \ell^{-1} \qquad i=4,6 \end{aligned} \tag{19}$$

There is one possible source of error, however; the h_i may be larger at high energies than (19) suggests, with the dominant terms in the cross-section cancelling each other. It can be seen from (F-3) that this is in fact the case for h_3 and h_5. If $h_5 \sim \ell$ and $h_3 = \Delta^2 h_5 + O(1)$, then $|M|^2$ is still of order unity. If one examines M for different spins and polarizations, we find that this cancellation occurs in every case. The possibility $h_3 \sim h_5 \sim \ell$ can therefore not be excluded by these arguments. We believe it quite unlikely that this relation between h_3 and h_5 at high energies holds (it certainly is not true in Born approximation), but even if it were true our dispersion relations and all succeeding arguments would be unaltered.

We must also consider restrictions in the other direction, requiring us to make subtractions. We pointed out that equation (16) would be incorrect for h_1 in the forward direction, since the low-energy amplitude has the wrong sign. A stronger argument results if we "switch off" all meson effects. Then the dispersion integral vanishes to order e^2, since there are now no absorptive effects of this order. But we should still get the Thomson term

e^2/M. Therefore our unsubtracted formula is wrong. The same reasoning applies to all e^2 terms in the Born approximation.

We may go farther and represent the proton anomalous moment μ_a by a Pauli term in the Hamiltonian. Terms in $e\mu_a$ and μ_a^2 now occur in the scattering amplitude. The following theorem has been proved by Low,[12] and by Gell-Mann and Goldberger[13]: If we set $\ell \sim \Delta \sim$ some low energy $\omega \ll M$, then it follows from field theory, to all orders in a perturbation expansion, that the Born approximation calculation is correct for terms in M of order ω^{-1} and 1. We shall hereafter refer to this theorem as "the low-energy theorem." We shall certainly subtract out such terms, since we are thereby assured of getting the low-energy terms exactly right, and the new integrals will converge faster at high energies. The status of the remaining terms in $e\mu_a$ and μ_a^2 is somewhat dubious; we have no assurance that the validity of a Pauli-type moment extends to such terms.

If we write $\operatorname{Im} h_i(\ell,\Delta^2) = \operatorname{Im} h_i(\ell,0) + \Delta^2 \operatorname{Im} h_i'(\ell,0) + \ldots$, and make use of (19), the most general form for our dispersion relations is

$$2\ell \operatorname{Re} h_1(\ell,\Delta^2) = A_1 + (2\ell^2/\pi)\, P\int_0^\infty \frac{d\ell'}{\ell'(\ell'^2-\ell^2)} \left[2\ell' \operatorname{Im} h_1(\ell',0)\right] + \Delta^2\left(\frac{A_2}{\ell^2} + A_3\right) + (2\ell^2\Delta^2/\pi)\, P\int_0^\infty \frac{d\ell'}{\ell'(\ell'^2-\ell^2)} \times \left[2\ell' \operatorname{Im} h_1'(\ell',0)\right]$$

$$2\ell \ \mathrm{Re}\ h_2(\ell,\Delta^2) = \frac{B_1}{\ell^2} + B_2 + (2\ell^2/\pi)\ P\int_0^\infty \frac{d\ell'}{\ell'(\ell'^2-\ell^2)} \left[2\ell'\ \mathrm{Im}\ h_2(\ell',0)\right]$$

$$+ \Delta^2\left(\frac{B_3}{\ell^4} + \frac{B_4}{\ell^2} + B_5\right) + (2\ell^2\Delta^2/\pi)\ P\int_0^\infty \frac{d\ell'}{\ell'(\ell'^2-\ell^2)} \left[2\ell'\ \mathrm{Im}\ h_2'(\ell',0)\right]$$

$$2\ell \ \mathrm{Re}\ h_3(\ell,\Delta^2) = C_1\ell + (2\ell^3/\pi)\ P\int_0^\infty \frac{d\ell'}{\ell'^2(\ell'^2-\ell^2)} \left[2\ell'\ \mathrm{Im}\ h_3(\ell',0)\right]$$

$$+ \Delta^2\left(\frac{C_2}{\ell} + C_3\ell\right) + (2\ell^3\Delta^2/\pi)\ P\int_0^\infty \frac{d\ell'}{\ell'^2(\ell'^2-\ell^2)} \left[2\ell'\ \mathrm{Im}\ h_3'(\ell',0)\right]$$

$$2\ell \ \mathrm{Re}\ h_4(\ell,\Delta^2) = \frac{D_1}{\ell} + (2\ell/\pi)\ P\int_0^\infty \frac{d\ell'}{\ell'^2-\ell^2} \left[2\ell'\ \mathrm{Im}\ h_4(\ell',0)\right]$$

$$+ \Delta^2\left(\frac{D_2}{\ell^3} + \frac{D_3}{\ell}\right) + (2\ell\Delta^2/\pi)\ P\int_0^\infty \frac{d\ell'}{\ell'^2-\ell^2} \left[2\ell'\ \mathrm{Im}\ h_4'(\ell',0)\right] \tag{20}$$

$$2\ell \ \mathrm{Re}\ h_5(\ell,\Delta^2) = \frac{E_1}{\ell} + E_2\ell + (2\ell^3/\pi)\ P\int_0^\infty \frac{d\ell'}{\ell'^2(\ell'^2-\ell^2)} \left[2\ell'\ \mathrm{Im}\ h_5(\ell',0)\right]$$

$$+ \Delta^2\left(\frac{E_3}{\ell^3} + \frac{E_4}{\ell} + E_5\ell\right) + (2\ell^3\Delta^2/\pi)\ P\int_0^\infty \frac{d\ell'}{\ell'^2(\ell'^2-\ell^2)} \left[2\ell'\ \mathrm{Im}\ h_5'(\ell',0)\right]$$

$$2\ell \ \mathrm{Re}\ h_6(\ell,\Delta^2) = \frac{F_1}{\ell} + (2\ell/\pi)\ P\int_0^\infty \frac{d\ell'}{\ell'^2-\ell^2} \left[2\ell'\ \mathrm{Im}\ h_6(\ell',0)\right]$$

$$+ \Delta^2\left(\frac{F_2}{\ell^3} + \frac{F_3}{\ell}\right) + (2\ell\Delta^2/\pi)\ P\int_0^\infty \frac{d\ell'}{\ell'^2-\ell^2} \left[2\ell'\ \mathrm{Im}\ h_6'(\ell',0)\right]$$

We must now attempt to determine the constants appearing in these dispersion relations. Our first step will be to use the low-energy theorem. This fixes the following values:

$A_1 = e^2/M \qquad A_2 = 0 \qquad B_1 = 0 \qquad B_3 = 0 \qquad C_1 = -2\mu_a^2$

$C_2 = \mu^2 \qquad D_1 = 4\mu^2 \qquad D_2 = 0 \qquad E_1 = 4\mu \left(\frac{e}{M} + \mu_a\right) \qquad E_3 = 0$

$F_1 = 4\mu\mu_a \qquad F_2 = 0$

$$\left(\mu = \frac{e}{2M} + \mu_a \right)$$

This fixes 12 of our 22 constants.

The remaining 10 constants can not be fixed precisely at present. Three alternatives suggest themselves for each:

(a) The constant may be equated to its value in Born approximation.

(b) The constant may be evaluated by eliminating a subtraction. For example, we may suppose that

$$A_3 = (2/\pi)\, P\int_0^\infty \frac{d\ell'}{\ell'}\; 2\ell' \;\mathrm{Im}\; h_1'(\ell',0)$$

in which case the first dispersion equation becomes

$$2\ell \;\mathrm{Re}\; h_1(\ell,\Delta^2) = A_1 + (2\ell^2/\pi)\, P\int_0^\infty \frac{d\ell'}{\ell'(\ell'^2-\ell^2)} \left[2\ell' \;\mathrm{Im}\; h_1(\ell',0)\right]$$
$$+ A_2\Delta^2/\ell^2 + (2\Delta^2/\pi)\, P\int_0^\infty \frac{\ell' d\ell'}{\ell'^2-\ell^2} \left[2\ell' \;\mathrm{Im}\; h_1'(\ell',0)\right]$$

(c) Finally, we may suppose that neither (a) nor (b) is correct, and leave the constant unknown, ultimately to be fitted to experimental data.

In this thesis we have decided to adopt alternative (a), but we shall briefly discuss to what extent (b) may be applicable. If the Born approximation for a constant contains an e^2 term, then (b) cannot be correct, as we mentioned earlier. This applies to A_3, D_3 and E_4, whose Born approximations we give here:

$$A_3 = -\mu^2/M \qquad D_3 = -\mu^2/M^2 \qquad E_4 = -(1/4M^2)\left(\frac{5e^2}{2M^2} + \frac{9e\mu_a}{M} + 6\mu_a^2\right)$$

The remaining seven constants admit all three alternatives. We can not decide unambiguously among these, but we can compare the values assigned to each constant by (a) and by (b). We list these values below; the dispersion integrals have been evaluated numerically with the static approximations of Section IV.

Constant	(a)	(b)
B_2	$\frac{4\mu_a}{M}\left(\frac{e}{M} + \mu_a\right) = .620$	$(2/\pi)\int_0^\infty \frac{d\ell'}{\ell'}\left[2\ell' \text{ Im } h_2(\ell',0)\right] = -3.20$
B_4	$\frac{-2\mu_a}{M}\left(\frac{e}{M} + \mu_a\right) = -.310$	$(2/\pi)\int_0^\infty \ell' d\ell'\left[2\ell' \text{ Im } h_2'(\ell',0)\right] = 0$
B_5	$\frac{-\mu_a}{M^3}\left(\frac{e}{M} + \mu_a\right) = -.155$	$(2/\pi)\int_0^\infty \frac{d\ell'}{\ell'}\left[2\ell' \text{ Im } h_2'(\ell',0)\right] = 0$
C_3	$\frac{\mu_a}{4M^2}\left(\frac{e}{M} + \mu_a\right) = .039$	$(2/\pi)\int_0^\infty \frac{d\ell'}{\ell'^2}\left[2\ell' \text{ Im } h_3'(\ell',0)\right] = .610$
E_2	$\frac{\mu_a}{M^2}\left(\frac{e}{M} + \mu_a\right) = .155$	$(2/\pi)\int_0^\infty \frac{d\ell'}{\ell'^2}\left[2\ell' \text{ Im } h_5(\ell',0)\right] = 2.44$

<u>Constant</u>	<u>(a)</u>	<u>(b)</u>
E_5	$-\frac{5\mu_a}{16M^4}\left(\frac{e}{M}+\mu_a\right) = -.048$	$(2/\pi)\int_0^\infty \frac{d\ell'}{\ell'^2}\left[2\ell' \text{ Im } h_5'(\ell',0)\right] = 0$
F_3	$-\frac{\mu_a}{2M^2}\left(\frac{e}{M}+\mu_a\right) = -.078$	$(2/\pi)\int_0^\infty d\ell'\left[2\ell' \text{ Im } h_6'(\ell',0)\right] = 0$

The integrals are over experimental photomeson cross-sections up to 500 Mev and are unlikely to be in error by more than 20%. No small differences of large quantities are involved. The zeros under (b) result from neglecting meson angular momenta greater than 1 in the photoproduction process.

It appears that (a) and (b) are quite distinct alternatives. In connection with our choosing (a), it is interesting to note that in every case the Born approximation supplies exactly the desired number of constants; all terms involving higher powers of ℓ vanish.

Finally, there is the possibility that C_1 and F_1, two of the constants guaranteed by the low-energy theorem, may in fact be given by dispersion relations, as in alternative (b) above. The two possible sum rules are

$$C_1 : \qquad -2\mu_a^2 = (2/\pi)\int_0^\infty \frac{d\ell'}{\ell'^2}\left[2\ell' \text{ Im } h_3(\ell',0)\right]$$

or $\qquad -.14 = -.240 + .222$

$$F_1 : \qquad 4\mu\mu_a = (2/\pi)\int_0^\infty d\ell'\left[2\ell' \text{ Im } h_6(\ell',0)\right]$$

or $\qquad .456 = .288$

The integral in C_1 reduces to the difference of two nearly equal quantities, which are given separately. The agreement is not satisfactory in either case.

Before leaving this Section, we shall briefly discuss the manner in which the simple dispersion formalism discussed here becomes incorrect above a certain critical photon energy.* We remarked after equation (10) that the Goldberger amplitude M equals the conventional scattering amplitude R for physical scattering energies. However, the integrals in (20) extend from $\ell=0$ to $\ell=\infty$ for any Δ^2; clearly the range $0<\ell<\Delta/2$ is an unphysical energy range. Thus we must examine the behavior of the absorptive part of M for such energies.

It follows from (11) by inserting a complete set of intermediate states Ψ_n, with energies E_n and momenta $\vec{p}_n$, that

$$2\ell A_{fi} = -\frac{1}{2}(2\pi)^4 \sum_n (\Psi_{\frac{\vec{\Delta}}{2}}, j_f(0)\Psi_n)(\Psi_n, j_i(0)\Psi_{-\frac{\vec{\Delta}}{2}})\,\delta(\vec{p}_n-\vec{p})\,\delta(E_n-E_{\frac{\Delta}{2}}-\ell)$$
$$-(\Psi_{\frac{\vec{\Delta}}{2}}, j_i(0)\Psi_n)(\Psi_n, j_f(0)\Psi_{-\frac{\vec{\Delta}}{2}})\,\delta(\vec{p}_n+\vec{p})\,\delta(E_n-E_{\frac{\Delta}{2}}+\ell) \qquad (21)$$

This expression differs from the absorptive part of the true scattering amplitude R by the sign between the two terms within the square brackets. Therefore as long as only the first term is non-zero we are all right, but when the second term begins to contribute, our results cease to be rigorously correct. The first term exists at the following energies:

$$\ell = \ell_b = -\Delta^2/4E_{\Delta/2} \qquad \text{for } \Psi_n = 1 \text{ nucleon}$$

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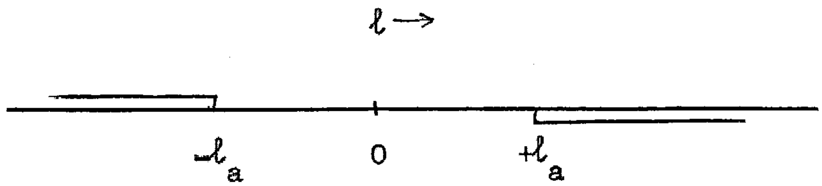
* A good discussion for the case of meson-nucleon scattering is given in reference 4.

$$\ell \geqslant \ell_a = \frac{Mm + (m^2/2) - (\Delta^2/4)}{E_{\Delta/2}} \qquad \text{for } \Psi_n = 1 \text{ nucleon plus 1 meson, etc.}$$

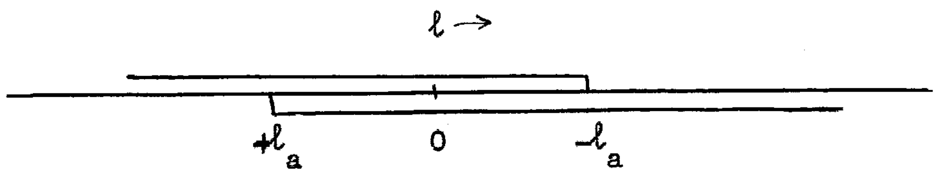
where m is the pion rest mass. Similarly the spectrum of the second term consists of the single point $\ell = -\ell_b$ and a continuous spectrum $\ell \leqslant -\ell_a$.

We do not need to consider the contributions at $\ell = \pm \ell_b$; these just give us the Born approximation, as we have seen earlier in this section. Therefore the criterion for validity of our dispersion relations is $\ell_a > 0$; the spectra of the two terms in (21) are then isolated as shown here:



and in integrating from $\ell=0$ to $\ell=\infty$ we only get contributions from the first term, which has the correct sign.

If $\ell_a < 0$, however, the situation is different. The two spectra now overlap, as shown below:



When we integrate from $\ell=0$ to $\ell=\infty$ we now make an error, since in the interval $0 < \ell < -\ell_a$ we should integrate the absorptive part of M, not of R. In order to remedy this defect, a method needs to be found to relate the absorptive part of M directly to an observed process, as we have done for R in Section I.

Solving the equation $$Mm + (m^2/2) - (\Delta^2/4) = 0$$

we find for $\Delta/2$ a value of about 370 Mev. Thus for momentum transfers in the CM system of less than 370 Mev the two absorptive spectra do not overlap. Since the static approximations of Section IV would be quite unreliable at such a high energy, we shall confine ourselves to lower energies and the limitation discussed above is not of practical importance.

Finally, one more point should be mentioned. For a given value of Δ^2, we must integrate over all ℓ from 0 to ∞. But $\text{Im}\, h_i(\ell,\Delta^2)$ vanishes except in the regions $\ell > \ell_a$, $\ell < -\ell_a$, so that the lower limit of the integral is ℓ_a if $\ell_a > 0$. However, the lowest ℓ which is physically meaningful is $\ell = \Delta/2$. When $\Delta/2 > \ell_a$, which occurs for $\Delta/2 > 130$ Mev, we have the problem of analytically continuing our scattering amplitude to the unphysical range $\ell_a < \ell < \Delta/2$. This can be done by expanding in powers of $\Delta^2/4\ell^2$ and then analytically continuing the amplitude. This amounts to the same thing as the expansion into Legendre polynomials discussed by Capps and Takeda,[4] and is apparently valid until the limit $\ell_a = 0$ is reached; beyond this point such a continuation clearly fails.

III - Transformation of Amplitudes

We have derived in Section I the imaginary parts of our amplitudes in the CM system, and in Section II we have seen that the dispersion formalism is conveniently done in a different system, shown in Figure 1. We must now consider how to transform our amplitudes from one coordinate system to the other.

There are two effects which complicate the transformation.

The first we shall call "spin precession"; this effect, closely related to the well-known Thomas precession of atomic electrons, simply reflects the fact that an electron's spin direction depends on the Lorentz frame in which it is viewed. In other words, "spin up" in one system is not necessarily "spin up" in another.

The second effect has to do with our choice of transverse gauge for the electromagnetic field. The condition $\hat{e}\cdot\hat{k} = 0$ is not relativistically invariant, and so if it is satisfied in the CM system it may not be satisfied in some other system. Therefore the gauge used in the two systems will not be identical; there will be a gauge transformation from one to the other.

We consider the spin precession first. We start with the well-known transformation properties of the Dirac equation. If a Lorentz transformation $(x,t)\rightarrow(x',t')$ is described by

$$\vec{x}' = \vec{x} + \vec{v}\left[\frac{\vec{x}\cdot\vec{v}}{v^2}(\gamma - 1) - \gamma t\right]$$

$$t' = \gamma(t - \vec{v}\cdot\vec{x}) \tag{22}$$

where $\gamma \equiv (1-v^2)^{-1/2}$, then the Dirac spinor u becomes $u'=Su$, where

$$S = \frac{\gamma + 1 - \gamma\vec{\alpha}\cdot\vec{v}}{2(\gamma+1)^{1/2}} \tag{23}$$

This use of the symbol γ should not be confused with the Dirac matrices $\vec{\gamma}$ or γ_μ.

Equation (23) is obtained from the familiar condition $S^{-1}\gamma_\mu S = a_{\mu\nu}\gamma_\nu$, where $a_{\mu\nu}$ is defined by $x'_\mu = a_{\mu\nu}x_\nu$. It then follows that the spinor for a moving proton is

$$u_{\vec{p}} = \frac{\vec{\alpha}\cdot\vec{p} + E_p + M}{[2M(E_p + M)]^{1/2}} u_o = D_{\vec{p}}\, u_o \tag{24}$$

where u_o is the spinor for a proton at rest. The normalization is $\tilde{u}u$ = constant. Note that (24) defines what we mean by "spin up," or any other direction of spin; we transform the proton to rest and then consider its spin direction.

Now denote quantities in two Lorentz systems by the subscripts 1 and 2. Consider a matrix element

$$\tilde{u}_f\, M\, u_i \tag{25}$$

We evaluate this matrix element in practice by setting $u_f = D_f u_{fo}$, $u_i = D_i u_{io}$, and omitting the u_o's. That is, in system 1 we calculate

$$D_{f1}\beta M_1 D_{i1} = \mathcal{M}_1$$

However, if $u_2 = S u_1$, then $S^{-1} M_2 S = M_1$. From this and the relation

$$S^{-1}\beta = \beta S$$

it follows that

$$\mathcal{M}_1 = D_{f1} S\beta M_2 S D_{i1}$$

In system 2 on the other hand, we calculate

$$\mathcal{M}_2 = D_{f2}\beta M_2 D_{i2}$$

Therefore the matrices $\mathcal{M}_1$ and $\mathcal{M}_2$ are connected by the relation

$$\mathcal{M}_2 = T_f^{-1}\mathcal{M}_1 T_i \tag{26}$$

where $T = D_1^{-1} S^{-1} D_2$. Straightforward calculation gives

$$T = \frac{(E_1+M)(\gamma+1) - \gamma\vec{p}_1\cdot\vec{v} + i\vec{\sigma}\cdot\vec{v}\times\vec{p}_1}{\left[2(\gamma+1)(E_1+M)(E_2+M)\right]^{1/2}} \tag{27}$$

We now consider system 1 to be the CM system, and system 2 to be the one shown in Figure 1. The kinematical relations between the two coordinate systems are summarized here:

$$\vec{k} = \frac{\vec{\Delta}}{2} + (E_{\Delta/2}/\rho)\,\vec{p} \qquad \vec{\Delta} = \vec{k} - \vec{k}'$$

$$\vec{k}' = -\frac{\vec{\Delta}}{2} + (E_{\Delta/2}/\rho)\,\vec{p} \qquad E_{\Delta/2} = \left[M^2 + \frac{1}{2}k^2(1-y)\right]^{1/2}$$

$$E_k = (E_{\Delta/2}/\rho)(\ell+E_{\Delta/2}) \qquad \vec{p} = (E/2E_{\Delta/2})(\vec{k}+\vec{k}')$$

$$y = \frac{p^2E^2_{\Delta/2} - \rho^2\Delta^2/4}{\left(\ell E_{\Delta/2} + \frac{\Delta^2}{4}\right)^2} \qquad \rho = E$$

$$E = \rho \qquad \ell = (k/2E_{\Delta/2})\left[2E_k + k(1+y)\right]$$

ρ is defined by $\rho^2 = E^2_{\Delta/2} + 2\ell E_{\Delta/2} + \frac{\Delta^2}{4}$

The velocity of system 2 relative to system 1 is

$$\vec{v} = -\frac{\vec{p}}{\ell+E_{\Delta/2}} = -\frac{\vec{k}+\vec{k}'}{2E_k}$$

With these substitutions, (27) becomes

$$T_i = \frac{(M+E_{\Delta/2})(\ell+E_{\Delta/2}+\rho) + i\vec{\sigma}\cdot\vec{p}\times\vec{\Delta}/2}{\left[(M+\rho)\;(M+E_{\Delta/2})(\ell+E_{\Delta/2}+\rho)\right]^{1/2}} \tag{28}$$

The replacement $\vec{\Delta}\rightarrow-\vec{\Delta}$ converts T_i into T_f.

Direct use of (28) in (26) is tedious; the results can be expressed more simply. If $\mathcal{M}_1 = A + i\vec{\sigma}\cdot\vec{B}$, then $\mathcal{M}_2 = C + i\vec{\sigma}\cdot\vec{D}$, where

$$C = A \cos 2\lambda - B_z \sin 2\lambda$$
$$\vec{D} = A\,\hat{e}_z \sin 2\lambda + \vec{B} - 2\hat{e}_z B_z \sin^2\lambda$$

the z-axis being chosen along the direction of $\vec{p} \times \vec{\Delta}$. λ is defined by

$$\cos 2\lambda = \frac{E^2_{\Delta/2} + ME_k}{E_{\Delta/2}(E_k+M)}$$

We now turn to the problem of the gauge transformation between the two coordinate systems. It is well-known that the polarization four-vector e is always undefined in that the replacement $e \rightarrow e+\lambda k$, λ being an arbitrary number, is without physical meaning.

Consider a photon with momentum $k=(\vec{k},k_o)$ and transverse polarization $e=(\hat{e},0)$; $\hat{e}\cdot\hat{k}=0$. Performing a Lorentz transformation (22) we obtain new four-vectors $k'=(\vec{k}',k'_o)$, $e'=(\vec{e}',e'_o)$; e'_o is no longer zero. However, an equivalent polarization is $e'' = e' - (e'_o/k'_o)\,k'$, in which the time component now vanishes. Carrying this operation through, we find that the polarizations in our two systems are related as follows:

$$\hat{e}_1 = \hat{e}_2 + \frac{\vec{p}\cdot\hat{e}_2}{\ell E_{\Delta/2}+\frac{\Delta^2}{4}}\left(\frac{\vec{\Delta}}{2} + \frac{E_{\Delta/2}+\rho}{\ell+E_{\Delta/2}+\rho}\,\vec{p}\right)$$

The formula for final polarizations is obtained by making the substitutions $\hat{e}_1 \rightarrow \hat{e}'_1$, $\hat{e}_2 \rightarrow \hat{e}'_2$, $\vec{\Delta} \rightarrow -\vec{\Delta}$.

We are now in a position to relate the amplitudes g_i of (5) to the h_i of (14). Since we are considering invariant matrix elements of the form (25), the quantities to which we must apply the above transformations are the scalars kE_kR and $\ell E_{\Delta/2}M$. The resulting formulas, which are rather lengthy, are given in Appendix E. A reasonable computational procedure might be to expand these formulas in powers of Δ^2, since this is the way we have handled the dispersion relations. This, however, leads to very cumbersome formulas, and the numerical work would be of an order of magnitude to be handled by an electronic computer. Furthermore, these transformation relations are kinematical in nature and vary slowly and smoothly with ℓ and Δ^2. This is to be contrasted with the dispersion integrals in (20), which fluctuate quite considerably over the energy range we are considering. In order to get a reasonable estimate of the effects of the absorptive photomeson process, in particular the well-known resonance, we shall therefore make a static approximation in these transformation formulas; ℓ/M and Δ^2/M^2 are neglected.

In this static limit, we have

$$2\ell h_1 = 2kg_1 \qquad 2\ell h_2 = -(4/\ell^2)\ 2kg_2 \qquad 2\ell h_3 = 2kg_3$$

$$2\ell h_4 = -(2/\ell^2)\ 2kg_4 \qquad 2\ell h_5 = -(2/\ell^2)\ 2kg_5 + (2/\ell^2)\ 2kg_6 \tag{29}$$

$$2\ell h_6 = -(2/\ell^2)\ 2kg_5 - (2/\ell^2)\ 2kg_6$$

IV - Numerical Applications and Discussion

Since the unitarity relations, which we shall take in the form (9), involve photoproduction amplitudes, we must deduce these amplitudes from available cross-section data. In this we shall be guided by the successful phenomenological approach of Brueckner, Watson and others, who show that the data strongly suggest, and can be satisfactorily explained in terms of, a resonance in the magnetic dipole amplitude to the state T=J=3/2, plus smaller amplitudes in the other S and P states.

The usual approach is to analyze differential cross-sections in the form

$$(d\sigma/d\Omega)_{cm} = A + B \cos\theta + C \cos^2\theta$$

If we know the three quantities A, B and C as functions of energy, we have enough information to determine any three multipole amplitudes, since the phases of these amplitudes are very closely equal to the corresponding meson scattering phase shifts.[7]

As a preliminary, we arbitrarily set $f_{M1}=0$ and solved for f_{E1}, f_{E2} and f_{M3}; the notation is that of (4). We used the experimental π^o data of Corson, Peterson and McDonald.[14] The results are that f_{E2} is negligible, f_{M3} has the expected resonance behavior, and f_{E1} is quite small, of the order of its Born approximation with a reasonable coupling constant ($g^2/4\pi \sim 10$).

It therefore seems valid to consider the photoproduction amplitudes as Born approximation plus an enhanced M3(T=3/2) matrix element. This was checked by computing the charged amplitudes in this way - the Born approximation matrix elements are now quite large - and comparing with experiment. The agreement is quite satisfactory except that at extreme forward and backward angles the computed cross-section is low; in other words the computed C is too low. This discrepancy is still present if the exact Born approximation is used, rather than its S- and P-wave parts alone.

In view of all this, it seemed very reasonable to neglect all amplitudes except the charged E1 and M3(T=3/2) amplitudes. This is a very good approximation for neutral production, and for charged production these two amplitudes are certainly the dominant ones. The work of Chew and Low[15] also suggests that these two amplitudes should be predominant in low-energy photoproduction. To be sure, we will be doing integrals of photoproduction amplitudes over all energies, but the integrals converge rapidly beyond the resonance and contributions from above 350 Mev are quite small.

Thus as a first approximation we simplify (9) to the following:

$$\begin{aligned}
\mathrm{Im}\, g_1 &= -(q\omega_q E_q/2\pi E)\ \left(\left|f^+_{E1}\right|^2 + 3y\left|f^o_{M3}\right|^2 \right) \\
\mathrm{Im}\, g_2 &= -(q\omega_q E_q/2\pi E)\ \left(-3\left|f^o_{M3}\right|^2 \right) \\
\mathrm{Im}\, g_3 &= -(q\omega_q E_q/2\pi E)\ \left(-\left|f^+_{E1}\right|^2 + \tfrac{3}{2}\, y\left|f^o_{M3}\right|^2 \right) \\
\mathrm{Im}\, g_4 &= -(q\omega_q E_q/2\pi E)\ \left(\tfrac{3}{2}\left|f^o_{M3}\right|^2 \right) \\
\mathrm{Im}\, g_5 &= -(q\omega_q E_q/2\pi E)\ \left(\tfrac{3}{2}\left|f^o_{M3}\right|^2 \right) \\
\mathrm{Im}\, g_6 &= 0
\end{aligned} \qquad (30)$$

With only these photoproduction amplitudes present, the total photoproduction cross-sections are

$$\sigma_o = (q\omega_q E_q E_k/\pi E^2)\ \left(2\left|f^o_{M3}\right|^2 \right)$$

$$\sigma_+ = (q\omega_q E_q E_k/\pi E^2) \left(\left|f^+_{E1}\right|^2 + \left|f^o_{M3}\right|^2 \right)$$

so that (30) may be rewritten in terms of total cross-sections:

$$\begin{aligned}
\text{Im } g_1 &= -(E/2E_k) \left(\sigma_+ - \tfrac{1}{2}\sigma_o + \tfrac{3}{2}\sigma_o y \right) \\
\text{Im } g_2 &= -(E/2E_k) \left(-\tfrac{3}{2}\sigma_o \right) \\
\text{Im } g_3 &= -(E/2E_k) \left(-\sigma_+ + \tfrac{1}{2}\sigma_o + \tfrac{3}{4}\sigma_o y \right) \\
\text{Im } g_4 &= -(E/2E_k) \left(\tfrac{3}{4}\sigma_o \right) \\
\text{Im } g_5 &= -(E/2E_k) \left(\tfrac{3}{4}\sigma_o \right) \\
\text{Im } g_6 &= 0
\end{aligned} \tag{31}$$

The fact that we have been able to express our amplitudes in terms of the two experimental quantities σ_o and σ_+ is a consequence of our limiting ourselves to the two independent amplitudes f^o_{M3} and f^+_{E1}.

We should now convert these Im g_i into Im h_i, as discussed in Section III. For purposes of numerical calculation, however, we shall make the static approximation discussed at the end of that Section. The result is that we are essentially applying dispersion relations to the g_i. We will be guided very significantly, of course, by our discussion of the correct dispersion relations, as regards subtractions and Born approximation terms.

The most direct procedure would be to substitute (25) into the dispersion relations (20), replacing Δ^2 by $2k^2(1-y)$ and ℓ by k, and then integrate over the experimental photomeson cross-sections.

This is very nearly what we shall do, but two points need some discussion. In the first place, we should try to get the low-energy amplitudes correct, and in the second place we should try to understand which energy, lab or CM, to use in various places in the dispersion relations in order to make the least error in our static approximation.

To get the low-energy amplitudes correct, we shall use the Born approximations for the g_i, rather than trying to transform the Born approximations for the h_i. We shall neglect terms of order $(k/M)^2$ and smaller. In order to decide the second question, we refer to the exact forward-direction dispersion relations in the laboratory system.[10,11] The procedure which seems most reasonable is then to set the factor E/E_k of (31) equal to unity, and interpret all energies appearing in or multiplying dispersion integrals as laboratory energies. We equate the free variable in the dispersion integrals to the lab energy corresponding to the CM energy k which is the argument of the rest of the equation.

We thus take for approximate dispersion relations:

$$2k \operatorname{Re} g_1 = (e^2/M) - (2k_L^2/\pi)\int_0^\infty \frac{dk_L'}{k_L'^2-k_L^2}\left[\sigma_+(k_L') + \sigma_o(k_L')\right]$$

$$+ (3k_L^4/\pi)(1-y)\int_0^\infty \frac{dk_L'}{k_L'^2(k_L'^2-k_L^2)}\,\sigma_o(k_L')$$

$$2k \operatorname{Re} g_2 = -(e^2k/M^2) + (3k_L^4/\pi)\int_0^\infty \frac{dk_L'}{k_L'^2(k_L'^2-k_L^2)}\,\sigma_o(k_L')$$

$$2k\ \mathrm{Re}\ g_3 = 2k(\mu^2-\mu_a^2) - 2k\mu^2 y - (2k_L^3/\pi)\int_0^\infty \frac{dk_L'}{k_L'(k_L'^2-k_L^2)}\left[\frac{5}{4}\sigma_o(k_L') - \sigma_+(k_L')\right]$$

$$+ (3k_L^5/2\pi)(1-y)\int_0^\infty \frac{dk_L'}{k_L'^3(k_L'^2-k_L^2)}\ \sigma_o(k_L')$$

$$2k\ \mathrm{Re}\ g_4 = -2k\mu^2 - (3k_L^3/2\pi)\int_0^\infty \frac{dk_L'}{k_L'(k_L'^2-k_L^2)}\ \sigma_o(k_L') \qquad (32)$$

$$2k\ \mathrm{Re}\ g_5 = -2k\mu^2 - (3k_L^3/2\pi)\int_0^\infty \frac{dk_L'}{k_L'(k_L'^2-k_L^2)}\ \sigma_o(k_L')$$

$$2k\ \mathrm{Re}\ g_6 = e\mu k/M$$

We define the following integrals:

$$I_{o,+}(k_L) = (k_L^2/\pi)\int_0^\infty \frac{dk_L'}{k_L'^2-k_L^2}\ \sigma_{o,+}(k_L')$$

$$J_{o,+}(k_L) = (k_L^3/\pi)\int_0^\infty \frac{dk_L'}{k_L'(k_L'^2-k_L^2)}\ \sigma_{o,+}(k_L')$$

$$K_o(k_L) = (k_L^4/\pi)\int_0^\infty \frac{dk_L'}{k_L'^2(k_L'^2-k_L^2)}\ \sigma_o(k_L')$$

$$L_o(k_L) = (k_L^5/\pi)\int_0^\infty \frac{dk_L'}{k_L'^3(k_L'^2-k_L^2)}\ \sigma_o(k_L')$$

Then (32) becomes (with M=1):

$$2k\ \mathrm{Re}\ g_1 = e^2 - 2(I_o + I_+) + 3(1-y)K_o$$

$$2k\ \mathrm{Re}\ g_2 = -e^2 k + 3K_o$$

$$2k \text{ Re } g_3 = 2k(\mu^2 - \mu_a^2) - 2k\mu^2 y - (5/2)J_o + 2J_+ + \frac{3}{2}L_o(1-y)$$
$$2k \text{ Re } g_4 = -2k\mu^2 - \frac{3}{2}J_o \qquad (33)$$
$$2k \text{ Re } g_5 = -2k\mu^2 - \frac{3}{2}J_o$$
$$2k \text{ Re } g_6 = e\mu k$$

The arguments of the dispersion integrals in (33) are understood to be k_L.

The integrals I, J, K, L were computed numerically, using experimental photomeson data up to 500 Mev. The results are graphed in Figures 2 and 3. In our calculations we have made small corrections for extrapolated tails beyond 500 Mev; it was assumed that $\sigma \sim k_L^{-3}$ in this region. The real parts of the amplitudes g_i were computed from (33) and are given in Table 1.

Table 1

k_L(Mev)	k/M	$\frac{2k}{e^2}$ Reg_1	$\frac{2k}{e^2}$ Reg_2	$\frac{2k}{e^2}$ Reg_3	$\frac{2k}{e^2}$ Reg_4	$\frac{2k}{e^2}$ Reg_5	$\frac{2k}{e^2}$ Reg_6
0	0	1	0	0	0	0	0
50	.0507	.959−.001y	−.050	.117−.197y	−.199	−.199	.071
100	.0968	.825−.013y	−.084	.234−.379y	−.396	−.396	.136
140	.1309	.604−.061y	−.070	.372−.525y	−.571	−.571	.183
190	.1709	.211−.298y	.127	.570−.775y	−.885	−.885	.239
230	.2008	.235−.674y	.473	.299−1.029y	−1.234	−1.234	.281
270	.2293	.419−.911y	.682	−.208−1.199y	−1.297	−1.297	.321

In Table 2 we list the imaginary parts of the g_i, obtained from (31) without further approximation:

Table 2

k_L(Mev)	$\frac{2k}{e^2}\,Img_1$	$\frac{2k}{e^2}\,Img_2$	$\frac{2k}{e^2}\,Img_3$	$\frac{2k}{e^2}\,Img_4$	$\frac{2k}{e^2}\,Img_5$	$\frac{2k}{e^2}\,Img_6$
190	−.366−.078y	.078	.366−.039y	−.039	−.039	0
230	−.716−.441y	.441	.716−.221y	−.221	−.221	0
270	−.748−1.719y	1.719	.748−.860y	−.860	−.860	0

We can now use (6) to calculate the coefficients A − E of (7). In Table 3 we list the results, and in Table 4 we give the differential cross-section at representative angles, in units of the Thomson cross-section: $(e^2/4\pi M)^2 = 2.36 \times 10^{-32}\ cm^2$.

Table 3

k_L(Mev)	A	B	C	D	E
0	.50	0	.50	0	0
50	.48	.00	.41	−.05	.00
100	.50	−.10	.24	−.07	.00
140	.56	−.31	.05	−.05	.01
190	1.17	−.73	−.20	.01	.01
230	2.55	−.52	−.51	.08	.00
270	4.81	.65	.06	.03	.01

Table 4 - $(d\sigma/d\Omega)_{cm} \div (e^2/4\pi M)^2$

k_L(Mev)	0^o	45^o	90^o	135^o	180^o
0	1.00	.75	.50	.75	1.00
50	.84	.67	.48	.70	.94
100	.56	.51	.49	.71	.90
140	.26	.35	.56	.83	.98
190	.26	.56	1.17	1.59	1.70
230	1.60	1.96	2.55	2.63	2.48
270	5.56	5.31	4.81	4.37	4.20

The most significant comparison of these predictions with experimental data would appear to be that of Figure 4, where the 90^o excitation curve from Table 4 is plotted against the existing experimental data from MIT[16] and Illinois.[17] In Figure 5 we show the predicted angular distributions at various energies; the experimental data at angles other than 90^o is so fragmentary that no real comparison is possible. In Figure 6 we give the predicted 135^o excitation curve, with the existing experimental points.

We have performed an alternative calculation along the same lines as the one described in this thesis, except for the subtraction procedure (and some minor kinematical differences). In Section II we discussed the alternatives available to us, and chose to make the maximum number of subtractions allowed if our total forward cross-section is to remain finite at high energies. In our previous calculation we made fewer subtractions than would be allowed by our criteria of Section II; subtractions were made only when the subtracted term was guaranteed correct by the low-energy theorem. In several instances, for example the Δ^2 part of the first equation of (20), this meant that we were omitting e^2 terms which must be present even in the absence of mesons. In Figure 7 we give the predictions of this model. It is interesting to note that the excitation curves at 90° for the two models are practically identical up to the highest energies considered. The angular distributions at high energies are markedly different, however. In the calculation discussed in this thesis, forward scattering exceeds backward scattering by 30% at 270 Mev; in our previous calculation the 180° cross-section is nearly twice the 0° cross-section at this energy. Further experimental data at high energies would help distinguish among the alternative subtraction possibilities.

Capps[11] has recently considered this same problem, by a somewhat different approach. He has only used the two forward-direction dispersion relations in the lab system, and has extended his work to other angles by considering the multipole amplitudes expected to dominate the scattering. We have included two angular distributions predicted by Capps in Figure 5. He states that he expects his curves to overestimate the backward scattering by a factor of about 1.5.

Watson, Zachariasen and Karzas are attacking this problem in the framework which Chew and Low[15],[18] have applied to meson scattering and photoproduction.[19] Numerical results are not available at the present time, but are expected shortly.

Finally, we should mention that Compton scattering may be analyzed in terms of multipole amplitudes, although we have not chosen to do so in our calculations. Details of the general analysis are given in Appendix G.

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Appendix A - Useful Integrals

We use the notation $x \equiv \hat{k}\cdot\hat{q}$, $x' \equiv \hat{k}'\cdot\hat{q}$, $y \equiv \hat{k}\cdot\hat{k}'$

$\hat{a}$, $\hat{b}$, $\hat{c}$ are constant vectors.

$K(x,x',y) = 1 - x^2 - x'^2 - y^2 + 2xx'y$

All double integrals range over those values of x and x' for which K is real.

$$\int d\Omega_q \, f_1(x') f_2(x) = 2\iint \frac{dx\, dx'}{K} \, f_1(x') f_2(x)$$

$$\int d\Omega_q \, f_1(x') f_2(x) \, \hat{q}\cdot\hat{a} = 2\iint \frac{dx\, dx'}{K} f_1(x') f_2(x) \left[\left(\frac{x - x'y}{1-y^2}\right) \hat{a}\cdot\hat{k} + \left(\frac{x' - xy}{1-y^2}\right) \hat{a}\cdot\hat{k}' \right]$$

$$\int d\Omega_q \, f_1(x') f_2(x) \hat{q}\cdot\hat{a}\, \hat{q}\cdot\hat{b} = 2\iint \frac{dx\, dx'}{K} f_1(x') f_2(x) \left\{ \left(\frac{K^2}{1-y^2}\right) \hat{a}\cdot\hat{b} \right.$$

$$+ \left[\frac{xx' - y}{1-y^2} + \frac{2yK^2}{(1-y^2)^2} \right] \left(\hat{a}\cdot\hat{k}\, \hat{b}\cdot\hat{k}' + \hat{a}\cdot\hat{k}'\, \hat{b}\cdot\hat{k} \right)$$

$$+ \left[\frac{1 - x'^2}{1-y^2} - \frac{2K^2}{(1-y^2)^2} \right] \hat{a}\cdot\hat{k}\, \hat{b}\cdot\hat{k} \quad + \left[\frac{1 - x^2}{1-y^2} - \frac{2K^2}{(1-y^2)^2} \right] \hat{a}\cdot\hat{k}'\, \hat{b}\cdot\hat{k}'$$

$$\int d\Omega_q\, f_1(x')\, f_2(x)\, \hat{q}\cdot\hat{a}\;\hat{q}\cdot\hat{b}\;\hat{q}\cdot\hat{c} = 2\iint \frac{dx\,dx'}{K}\, f_1(x')\, f_2(x) \Big\{ \frac{x-x'y}{(1-y^2)^2}\, K^2 (\hat{a}\cdot\hat{b}\,\hat{c}\cdot\hat{k}$$

$$+ \hat{a}\cdot\hat{c}\,\hat{b}\cdot\hat{k} + \hat{b}\cdot\hat{c}\,\hat{a}\cdot\hat{k}) + \frac{x'-xy}{(1-y^2)^2}\, K^2 (\hat{a}\cdot\hat{b}\,\hat{c}\cdot\hat{k}' + \hat{a}\cdot\hat{c}\,\hat{b}\cdot\hat{k}' + \hat{b}\cdot\hat{c}\,\hat{a}\cdot\hat{k}')$$

$$+ \frac{x-x'y}{(1-y^2)^3}\left[(x-x'y)^2 - 3K^2\right] \hat{a}\cdot\hat{k}\,\hat{b}\cdot\hat{k}\,\hat{c}\cdot\hat{k} + \frac{x'-xy}{(1-y^2)^3}\left[(x'-xy)^2 - 3K^2\right] \hat{a}\cdot\hat{k}'\,\hat{b}\cdot\hat{k}'\,\hat{c}\cdot\hat{k}'$$

$$+ \frac{1}{(1-y^2)^3}\left[(x-x'y)^2(x'-xy) + (3xy - 2x'y^2 - x')\,K^2\right] (\hat{a}\cdot\hat{k}'\,\hat{b}\cdot\hat{k}\,\hat{c}\cdot\hat{k} + \hat{b}\cdot\hat{k}'\,\hat{a}\cdot\hat{k}\,\hat{c}\cdot\hat{k}$$

$$+ \hat{c}\cdot\hat{k}'\,\hat{a}\cdot\hat{k}\,\hat{b}\cdot\hat{k}) + \frac{1}{(1-y^2)^3}\left[(x'-xy)^2(x-x'y) + (3x'y - 2xy^2 - x)\,K^2\right] \times$$

$$\times (\hat{a}\cdot\hat{k}'\,\hat{b}\cdot\hat{k}'\,\hat{c}\cdot\hat{k} + \hat{b}\cdot\hat{k}'\,\hat{c}\cdot\hat{k}'\,\hat{a}\cdot\hat{k} + \hat{c}\cdot\hat{k}'\,\hat{a}\cdot\hat{k}'\,\hat{b}\cdot\hat{k})$$

Appendix B – General Unitarity Relations

Again we set $K^2 = 1 - x^2 - x'^2 - y^2 + 2xx'y$ and integrate only over regions where K is real. The common factor $-\frac{q\,\omega_q E_q}{8\pi^2 E}$ has been omitted.

$$\mathrm{Im}\; g_1(y) = 2\iint \frac{dx\,dx'}{K}\Big\{\Big[y - xx' - \frac{yK^2}{1-y^2}\Big] f_1^*(x')\,f_1(x) + f_2^*(x')f_2(x)$$

$$+ \frac{yK^2}{1-y^2} f_3^*(x')f_3(x) + \frac{K^2}{1-y^2}\cdot 2\,\mathrm{Re}\, f_2^*(x')f_4(x)$$

$$+ \frac{x'K^2}{1-y^2}\cdot 2\,\mathrm{Re}\, f_3^*(x')f_4(x) + \frac{K^2}{1-y^2} f_4^*(x')f_4(x)\Big\}$$

$$\mathrm{Im}\; g_2(y) = 2\iint \frac{dx\,dx'}{K}\Big\{\Big[\frac{y(y-xx')}{1-y^2} - \frac{(1+y^2)K^2}{(1-y^2)^2}\Big] f_1^*(x')\,f_1(x)$$

$$+ \frac{x'-xy}{1-y^2}\cdot 2\,\mathrm{Re}\, f_2^*(x')f_3(x) + \Big[\frac{y(xx'-y)}{1-y^2} + \frac{2y^2K^2}{(1-y^2)^2}\Big] f_3^*(x')f_3(x)$$

$$+ \Big[\frac{xx'-y}{1-y^2} + \frac{2yK^2}{(1-y^2)^2}\Big]\cdot 2\,\mathrm{Re}\, f_2^*(x')\,f_4(x)$$

$$+ \frac{x'}{(1-y^2)^2}\Big[(x-x'y)(x'-xy) + yK^2\Big]\cdot 2\,\mathrm{Re}\, f_3^*(x')\,f_4(x)$$

$$+ \Big[\frac{xx'-y}{1-y^2} + \frac{2yK^2}{(1-y^2)^2}\Big] f_4^*(x')\,f_4(x)\Big\}$$

$$\mathrm{Im}\ g_3(y) = 2\iint \frac{dx\,dx'}{\overline{K}} \Big\{ -f_2^*(x')\,f_2(x) + \Big[y - xx' - \frac{y\overline{K}^2}{1-y^2}\Big] \times$$

$$\times\ 2\,\mathrm{Re}\ f_1^*(x')\,f_3(x) + \Big[y - xx' - \frac{2y\overline{K}^2}{1-y^2}\Big] f_3^*(x')\,f_3(x)$$

$$-\frac{1}{(1-y^2)^2}\Big[(x-x'y)^2(x'-xy) + \overline{K}^2(3xy - x'y^2 - 2x')\times 2\,\mathrm{Re}\ f_1^*(x')\,f_4(x)$$

$$-\frac{\overline{K}^2}{1-y^2}\cdot 2\,\mathrm{Re}\ f_2^*(x')\,f_4(x) - \frac{1}{(1-y^2)^2}\Big[(x-x'y)^2(x'-xy) + (3xy - 2x'y^2 - x')\overline{K}^2\Big]\times$$

$$\times\ 2\,\mathrm{Re}\ f_3^*(x')\,f_4(x)\Big\}$$

$$\mathrm{Im}\ g_4(y) = 2\iint \frac{dx\,dx'}{\overline{K}} \Big\{ \frac{x'y - x}{1-y^2} \times 2\,\mathrm{Re}\ f_1^*(x')\,f_2(x)$$

$$-\frac{\overline{K}^2}{1-y^2} f_3^*(x')\,f_3(x) + \frac{(x'y-x)\overline{K}^2}{(1-y^2)^2}\ 2\,\mathrm{Re}\ f_1^*(x')\,f_4(x)$$

$$+\frac{(x'y-x)\overline{K}^2}{(1-y^2)^2}\times 2\,\mathrm{Re}\ f_3^*(x')\,f_4(x)\Big\}$$

$$\mathrm{Im}\, g_5(y) = 2\iint \frac{dx\,dx'}{K}\Big\{ \frac{x'y-x}{2(1-y^2)} \cdot 2\,\mathrm{Re}\, f_1^*(x')\, f_2(x)$$

$$+\left[\frac{y(xx'-y)}{2(1-y^2)} + \frac{(1+y^2)K^2}{2(1-y^2)^2}\right]\cdot 2\,\mathrm{Re}\, f_1^*(x')\, f_3(x) + \frac{x'-xy}{2(1-y^2)} \times$$

$$\times\, 2\,\mathrm{Re}\, f_2^*(x')\, f_3(x) - y\left[\frac{y-xx'}{1-y^2} - \frac{2yK^2}{(1-y^2)^2}\right] f_3^*(x')\, f_3(x)$$

$$+\frac{1}{2(1-y^2)^3}\left[(x-x'y)(x'-xy)(-x'+2xy-x'y^2) + K^2(5xy^2+x-2x'y^3-4x'y)\right]\times$$

$$\times\, 2\,\mathrm{Re}\, f_1^*(x')\, f_4(x) + \left[\frac{xx'-y}{2(1-y^2)} + \frac{yK^2}{(1-y^2)^2}\right]\cdot 2\,\mathrm{Re}\, f_2^*(x')\, f_4(x)$$

$$+\frac{1}{2(1-y^2)^3}\left[2y\,(x-x'y)^2(x'-xy) + K^2(5xy^2+x-3x'y^3-3x'y)\right]\cdot 2\,\mathrm{Re}\, f_3^*(x')\, f_4(x)\Big\}$$

$$\mathrm{Im}\, g_6(y) = 2\iint \frac{dx\,dx'}{K}\Big\{\left[\frac{y-xx'}{2(1-y^2)} - \frac{yK^2}{(1-y^2)^2}\right]\cdot 2\,\mathrm{Re}\, f_1^*(x')\, f_3(x)$$

$$+\left[\frac{y-xx'}{1-y^2} - \frac{2yK^2}{(1-y^2)^2}\right] f_3^*(x')\, f_3(x) - \frac{1}{2(1-y^2)^3}\Big[(x-x'y)(x'-xy)(x-2x'y^2+xy^2)$$

$$-K^2(3x'+3x'y^2-6xy)\Big]\cdot 2\,\mathrm{Re}\, f_1^*(x')\, f_4(x) + \left[\frac{1-x^2}{2(1-y^2)} - \frac{K^2}{(1-y^2)^2}\right]\times$$

$$\times\, 2\,\mathrm{Re}\, f_2^*(x')\, f_4(x) - \frac{1}{2(1-y^2)^3}\left[2(x-x'y)^2(x'-xy) + K^2(5xy-5x'y^2+xy^3-x')\right]\times$$

$$\times\, 2\,\mathrm{Re}\, f_3^*(x')\, f_4(x)\Big\}$$

Appendix C - Unitarity Relations in Forward Direction

$$\mathrm{Im}\, g_1(1) = -\frac{q\,\omega_q E_q}{4\pi E}\int_{-1}^{+1} dx \Big\{ \tfrac{1}{2}(1-x^2)\, f_1^*(x) f_1(x) + f_2^*(x) f_2(x)$$

$$+ \tfrac{1}{2}(1-x^2) \times 2\,\mathrm{Re}\, f_2^*(x) f_4(x) + \tfrac{1}{2}(1-x^2)\, f_3^*(x) f_3(x)$$

$$+ \tfrac{1}{2} x (1-x^2) \times 2\,\mathrm{Re}\, f_3^*(x) f_4(x) + \tfrac{1}{2}(1-x^2)\, f_4^*(x) f_4(x) \Big\}$$

$$\mathrm{Im}\, g_2(1) = -\frac{q\,\omega_q E_q}{4\pi E}\int_{-1}^{+1} dx \Big\{ -\tfrac{1}{2}(1-x^2)\, f_1^*(x) f_1(x)$$

$$-\tfrac{1}{8}(1-x^2)^2 f_1^{*\prime}(x) f_1'(x) + \tfrac{1}{2}(1-x^2) \times 2\,\mathrm{Re}\, f_2^{*\prime}(x) f_3(x)$$

$$+\tfrac{1}{8}(1-x^2)^2 \times 2\,\mathrm{Re}\, f_2^{*\prime}(x) f_4'(x) + \tfrac{1}{8}(1-x^2)^2 f_3^{*\prime}(x) f_3'(x)$$

$$+\tfrac{1}{8}(1-x^2)^2 \times 2\,\mathrm{Re}\, f_3^*(x) f_4'(x) + \tfrac{1}{8} x (1-x^2)^2 \times 2\,\mathrm{Re}\, f_3^{*\prime}(x) f_4'(x)$$

$$+\tfrac{1}{8}(1-x^2)^2 f_4^{*\prime}(x) f_4'(x) \Big\}$$

$$\mathrm{Im}\, g_3(1) = -\frac{q\,\omega_q E_q}{4\pi E}\int_{-1}^{+1} dx \Big\{ \tfrac{1}{2}(1-x^2) \times 2\,\mathrm{Re}\, f_1^*(x) f_3(x)$$

$$+\tfrac{1}{2} x (1-x^2) \times 2\,\mathrm{Re}\, f_1^*(x) f_4(x) - f_2^*(x) f_2(x)$$

$$-\tfrac{1}{2}(1-x^2) \times 2\,\mathrm{Re}\, f_2^*(x) f_4(x) \Big\}$$

$$\mathrm{Im}\; g_4(1) = -\frac{g\,\omega_g E_g}{4\pi E}\int_{-1}^{+1} dx \Big\{ -\tfrac{1}{2}(1-x^2) \times 2\,\mathrm{Re}\, f_1^*(x)\, f_2'(x)$$

$$-\tfrac{1}{8}(1-x^2)^2 \times 2\,\mathrm{Re}\, f_1^*(x)\, f_4'(x) - \tfrac{1}{2}(1-x^2)\, f_3^*(x)\, f_3(x)$$

$$-\tfrac{1}{8}(1-x^2)^2 \times 2\,\mathrm{Re}\, f_3^*(x)\, f_4'(x) \Big\}$$

$$\mathrm{Im}\; g_5(1) = -\frac{g\,\omega_g E_g}{4\pi E}\int_{-1}^{+1} dx \Big\{ -\tfrac{1}{4}(1-x^2) \times 2\,\mathrm{Re}\, f_1^*(x)\, f_2'(x)$$

$$+\tfrac{1}{4}(1-x^2) \times 2\,\mathrm{Re}\, f_1^*(x)\, f_3(x) + \tfrac{1}{16}(1-x^2)^2 \times 2\,\mathrm{Re}\, f_1^{*\prime}(x) f_3'(x)$$

$$+\tfrac{1}{96}(1-x^2)^3 \times 2\,\mathrm{Re}\left[f_1^{*\prime}(x)\, f_4''(x) - f_1^{*\prime\prime}(x)\, f_4'(x)\right]$$

$$+\tfrac{1}{4}(1-x^2) \times 2\,\mathrm{Re}\, f_2^{*\prime}(x)\, f_3(x) + \tfrac{1}{16}(1-x^2)^2 \times 2\,\mathrm{Re}\, f_2^{*\prime}(x) f_4'(x)$$

$$+\tfrac{1}{8}(1-x^2)^2\, f_3^{*\prime}(x)\, f_3'(x) + \tfrac{1}{16}(1-x^2)^2 \times 2\,\mathrm{Re}\, f_3^*(x)\, f_4'(x)$$

$$+\tfrac{1}{48}(1-x^2)^3 \times 2\,\mathrm{Re}\, f_3^{*\prime}(x)\, f_4''(x) \Big\}$$

$$\mathrm{Im}\, g_6(1) = -\frac{g\,\omega_q E_q}{4\pi E} \int_{-1}^{+1} dx \left\{ -\frac{1}{16}(1-x^2)^2 \times 2\,\mathrm{Re}\, f_1^{*\prime}(x) f_3^{\prime}(x) \right.$$

$$+ \frac{1}{8}(1-x^2)^2 \times 2\,\mathrm{Re}\, f_1^{*}(x) f_4^{\prime}(x)$$

$$+ \frac{1}{96}(1-x^2)^3 \times 2\,\mathrm{Re}\left[f_1^{*\prime\prime}(x) f_4^{\prime}(x) - f_1^{*\prime}(x) f_4^{\prime\prime}(x) \right]$$

$$+ \frac{1}{16}(1-x^2)^2 \times 2\,\mathrm{Re}\, f_2^{*\prime\prime}(x) f_4(x) - \frac{1}{8}(1-x^2)^2 f_3^{*\prime}(x) f_3^{\prime}(x)$$

$$\left. - \frac{1}{16}(1-x^2)^2 \times 2\,\mathrm{Re}\, f_3^{*\prime}(x) f_4(x) - \frac{1}{48}(1-x^2)^3 \times 2\,\mathrm{Re}\, f_3^{*\prime}(x) f_4^{\prime\prime}(x) \right\}$$

Appendix D - "Derivation" of Klein-Nishina Amplitude by Dispersion Theory

We start with equation (21):

$$2\ell A_{fi} = -(1/2)(2\pi^4)\sum_n (\Psi_{\frac{\vec{\Delta}}{2}}, j_f(0)\Psi_n)(\Psi_n, j_i(0)\Psi_{-\frac{\vec{\Delta}}{2}})\,\delta(\vec{p}_n-\vec{p})\,\delta(E_n-E_{\frac{\Delta}{2}}-\ell)$$

$$-(\Psi_{\frac{\vec{\Delta}}{2}}, j_i(0)\Psi_n)(\Psi_n, j_f(0)\Psi_{-\frac{\vec{\Delta}}{2}})\,\delta(\vec{p}_n+\vec{p})\,\delta(E_n-E_{\frac{\Delta}{2}}+\ell)$$

The states Ψ_n must either contain one fermion or two fermions and one antifermion.

If the first term is to contribute, we must have

$$\vec{p}_n = \vec{p} \quad ; \quad E_n = \ell + E_{\frac{\Delta}{2}}$$

The only possibility is a one-fermion state, with

$$\ell = -(\Delta^2/4E_{\frac{\Delta}{2}}) \quad , \quad E_p = M^2/E_{\frac{\Delta}{2}}$$

Similarly the only contribution from the second term occurs for a one-fermion state, with

$$\ell = +(\Delta^2/4E_{\frac{\Delta}{2}}) \quad , \quad E_p = M^2/E_{\frac{\Delta}{2}}$$

The matrix elements of j are easily worked out:

$$(\Psi_{\vec{k}}, j_i\Psi_{\vec{p}}) = e(\Psi_{\vec{k}}, \tilde{\psi}\vec{\gamma}\cdot\hat{e}_i\,\psi\Psi_{\vec{p}}) = e\,u^*_{\vec{k}}\,\beta\vec{\gamma}\cdot\hat{e}_i\,u_{\vec{p}}$$

$$= \frac{e\left[(E_p+M)\,\vec{\sigma}\cdot\vec{k}\,\vec{\sigma}\cdot\hat{e}_i + (E_k+M)\,\vec{\sigma}\cdot\hat{e}_i\,\vec{\sigma}\cdot\vec{p}\right]}{\left[4E_k(E_k+M)E_p(E_p+M)\right]^{1/2}}$$

The last expression is a matrix in the initial and final spins; we have omitted the spinors.

We therefore find for the contribution of the first term in (21):

$$\frac{-\pi e^2\,\delta\left(\sqrt{M^2+\ell^2-\frac{\Delta^2}{4}}-\ell-E_{\frac{\Delta}{2}}\right)}{4M^3E_{\frac{\Delta}{2}}}\left[M\vec{\sigma}\cdot\frac{\vec{\Delta}}{2}\vec{\sigma}\cdot\hat{e}' + E_{\frac{\Delta}{2}}\vec{\sigma}\cdot\hat{e}'\vec{\sigma}\cdot\vec{p}\right] \times$$

$$\times\left[E_{\frac{\Delta}{2}}\vec{\sigma}\cdot\vec{p}\,\vec{\sigma}\cdot\hat{e} - M\vec{\sigma}\cdot\hat{e}\,\vec{\sigma}\cdot\frac{\vec{\Delta}}{2}\right]$$

Using the identity

$$\vec{\sigma}\cdot\hat{a}\,\vec{\sigma}\cdot\hat{b}\,\vec{\sigma}\cdot\hat{c}\,\vec{\sigma}\cdot\hat{d} = \hat{a}\cdot\hat{b}\hat{c}\cdot\hat{d} - \hat{a}\cdot\hat{c}\hat{b}\cdot\hat{d} + \hat{a}\cdot\hat{d}\hat{b}\cdot\hat{c} + i\vec{\sigma}\cdot(\hat{a}\cdot\hat{b}\hat{c}\times\hat{d} + \hat{c}\cdot\hat{d}\hat{a}\times\hat{b}$$
$$- \hat{a}\cdot\hat{c}\hat{b}\times\hat{d} - \hat{b}\cdot\hat{d}\hat{a}\times\hat{c} + \hat{a}\cdot\hat{d}\hat{b}\times\hat{c} + \hat{b}\cdot\hat{c}\hat{a}\times\hat{d})$$

we find the following contributions to the amplitudes defined in (14):

$$\text{Im } h_1 : -M\Delta^2/4 \qquad\qquad \text{Im } h_4 : E_{\frac{\Delta}{2}}$$

$$\text{Im } h_3 : +M\Delta^2/4 \qquad\qquad \text{Im } h_5 : M + E_{\frac{\Delta}{2}}$$

We have omitted the common factor

$$-(\pi e^2/2M^2E_{\frac{\Delta}{2}})\,\delta\left(\sqrt{M^2+\ell^2-\frac{\Delta^2}{4}}-\ell-E_{\frac{\Delta}{2}}\right)$$

To get the contribution from the second term we

(1) change the overall sign

(2) change the signs of ℓ and p

(3) exchange $\hat{e}$ and $\hat{e}'$.

If we integrate over positive energies ℓ, the second term is the only one whose delta-function is picked up.

The Klein-Nishina amplitudes in this coordinate system are

$$2\ell \;\mathrm{Re}\; h_1 = \frac{e^2}{\ell^2 E^2_{\frac{\Delta}{2}} - \frac{\Delta^4}{16}} \qquad\qquad 2\ell \;\mathrm{Re}\; h_4 = \frac{e^2}{\ell^2 E^2_{\frac{\Delta}{2}} - \frac{\Delta^4}{16}}\;\ell$$

$$2\ell \;\mathrm{Re}\; h_3 = \frac{e^2}{\ell^2 E^2_{\frac{\Delta}{2}} - \frac{\Delta^4}{16}}\;\frac{M\Delta^2\ell}{4E_{\frac{\Delta}{2}}} \qquad\qquad 2\ell \;\mathrm{Re}\; h_5 = \frac{e^2}{\ell^2 E^2_{\frac{\Delta}{2}} - \frac{\Delta^4}{16}}\left(\frac{E_{\Delta/2} + M}{M}\right)\ell$$

It is now easy to verify that relations (16) and (17) are satisfied; (16), with one subtraction, for h_1, and (17) for the other three.

The important fact is that the amplitudes h_i themselves obey the dispersion relations. It would be very difficult to apply the relations to a quantity such as $h_4(\ell)\; i\vec{\sigma}\cdot\vec{p} \times \vec{\Delta}/2$, since $\vec{p}$ would vary with ℓ, and perhaps the polarizations should also vary in order to remain transverse.

Appendix E - Transformation of Amplitudes

$$\ell E_{\Delta/2}\, h_1 = \frac{2(\rho E_{\Delta/2} + M\ell + M E_{\Delta/2})}{(M+\rho)^2}\,(E_k K g_1) + \frac{4\rho E_{\Delta/2}\, p^2 \frac{\Delta^2}{4}}{(M+\rho)^2 \left(\ell E_{\Delta/2} + \frac{\Delta^2}{4}\right)^2}\,(K E_k g_4)$$

$$\left(\rho^2 \equiv E^2_{\Delta/2} + 2\ell E_{\Delta/2} + \frac{\Delta^2}{4}\right)$$

$$\ell E_{\Delta/2} h_2 = -\frac{4\left(\frac{\Delta^2}{4}+2\ell E_{\Delta/2}\right)\left(\rho E_{\Delta/2}+M\ell+ME_{\Delta/2}\right)}{(M+\rho)^2\left(\ell E_{\Delta/2}+\frac{\Delta^2}{4}\right)^2}\left(\kappa E_\kappa g_1\right)$$

$$-\frac{8\ell^2 E_{\Delta/2}^2 \rho^2\left(\rho E_{\Delta/2}+M\ell+ME_{\Delta/2}\right)}{(M+\rho)^2\left(\ell E_{\Delta/2}+\frac{\Delta^2}{4}\right)^4}\left(\kappa E_\kappa g_2\right) + \frac{4\ell^2\rho E_{\Delta/2}}{(M+\rho)^2\left(\ell E_{\Delta/2}+\frac{\Delta^2}{4}\right)^2}\left(\kappa E_\kappa g_3\right)$$

$$-\frac{8\rho E_{\Delta/2}\, p^2 \frac{\Delta^2}{4}\left(\frac{\Delta^2}{4}+2\ell E_{\Delta/2}\right)}{(M+\rho)^2\left(\ell E_{\Delta/2}+\frac{\Delta^2}{4}\right)^4}\left(\kappa E_\kappa g_4\right)$$

$$+\frac{8\rho\ell E_{\Delta/2}}{(M+\rho)^2\left(\ell E_{\Delta/2}+\frac{\Delta^2}{4}\right)^3}\left[\frac{\Delta^2}{4}\left(\ell+E_{\Delta/2}\right) - p^2 E_{\Delta/2} + \frac{2p^2\frac{\Delta^2}{4}E_{\Delta/2}}{\ell E_{\Delta/2}+\frac{\Delta^2}{4}}\right]\left(\kappa E_\kappa g_5\right)$$

$$-\frac{8\rho\ell^2 E_{\Delta/2}}{(M+\rho)^2\left(\ell E_{\Delta/2}+\frac{\Delta^2}{4}\right)^2}\left(\kappa E_\kappa g_6\right)$$

$$\ell E_{\Delta/2} h_3 = -\frac{2p^2\frac{\Delta^2}{4}\left(\frac{\Delta^2}{4}+2\ell E_{\Delta/2}\right)}{(M+\rho)^2\left(\ell E_{\Delta/2}+\frac{\Delta^2}{4}\right)^2}\left(\kappa E_\kappa g_1\right) - \frac{4\ell^2E_{\Delta/2}^2\rho^2 p^2\frac{\Delta^2}{4}}{(M+\rho)^2\left(\ell E_{\Delta/2}+\frac{\Delta^2}{4}\right)^4}\left(\kappa E_\kappa g_2\right)$$

$$+\frac{\ell}{\left(\ell E_{\Delta/2}+\frac{\Delta^2}{4}\right)^2}\left\{E_{\Delta/2}\frac{\Delta^2}{4}+\ell E_{\Delta/2}^2+\rho\frac{\Delta^2}{4}+\frac{2\rho\left(\ell E_{\Delta/2}+\frac{\Delta^2}{4}\right)\left(\ell+E_{\Delta/2}-\rho\right)\left(E_{\Delta/2}-M\right)}{(M+\rho)^2}\right\}\times$$

$$\left(\kappa E_\kappa g_3\right) + \frac{4\rho E_{\Delta/2}\,p^2\frac{\Delta^2}{4}\left(2\ell E_{\Delta/2}+\frac{\Delta^2}{4}\right)\left(\rho E_{\Delta/2}+M\ell+ME_{\Delta/2}\right)}{(M+\rho)^2\left(\ell E_{\Delta/2}+\frac{\Delta^2}{4}\right)^4}\left(\kappa E_\kappa g_4\right)$$

$$-\frac{4\rho\ell E_{\Delta/2}\,p^2}{(M+\rho)^2\left(\ell E_{\Delta/2}+\frac{\Delta^2}{4}\right)^3}\left[E_{\Delta/2}\left(E_{\Delta/2}-M\right)\left(\ell+E_{\Delta/2}-\rho\right)+M\frac{\Delta^2}{4}+\frac{2\frac{\Delta^2}{4}E_{\Delta/2}\left(\rho E_{\Delta/2}+M\ell+ME_{\Delta/2}\right)}{\ell E_{\Delta/2}+\frac{\Delta^2}{4}}\right]\times$$

$$\times\left(\kappa E_\kappa g_5\right) + \frac{2\rho\ell E_{\Delta/2}\left(\ell+E_{\Delta/2}-\rho\right)}{(M+\rho)^2\left(\ell E_{\Delta/2}+\frac{\Delta^2}{4}\right)^3}\left[\frac{\Delta^2}{4}(M+\rho)^2-2\ell\left(E_{\Delta/2}-M\right)\left(\frac{\Delta^2}{4}+\ell E_{\Delta/2}\right)\right]\left(\kappa E_\kappa g_6\right)$$

$$\ell E_{\Delta/2} h_4 = \frac{2}{(M+\rho)^2}(\kappa E_\kappa g_1) - \frac{4\rho E_{\Delta/2}(\rho E_{\Delta/2} + M\ell + ME_{\Delta/2})}{(M+\rho)^2(\ell E_{\Delta/2} + \frac{\Delta^2}{4})^2}(\kappa E_\kappa g_4)$$

$$\ell E_{\Delta/2} h_5 = -\frac{2p^2(\frac{\Delta^2}{4} + 2\ell E_{\Delta/2})}{(M+\rho)^2(\frac{\Delta^2}{4} + \ell E_{\Delta/2})^2}(\kappa E_\kappa g_1) - \frac{4\ell^2 E_{\Delta/2}^2 \rho^2 p^2}{(M+\rho)^2(\ell E_{\Delta/2} + \frac{\Delta^2}{4})^4}(\kappa E_\kappa g_2)$$

$$+ \frac{\rho\ell}{(\ell E_{\Delta/2} + \frac{\Delta^2}{4})^2}\left[1 + \frac{2\ell E_{\Delta/2}(\ell + E_{\Delta/2} - \rho)}{(M+\rho)^2(M + E_{\Delta/2})}\right](\kappa E_\kappa g_3)$$

$$+ \frac{4\rho E_{\Delta/2} p^2 (2\ell E_{\Delta/2} + \frac{\Delta^2}{4})(\rho E_{\Delta/2} + M\ell + ME_{\Delta/2})}{(M+\rho)^2(\ell E_{\Delta/2} + \frac{\Delta^2}{4})^4}(\kappa E_\kappa g_4) - \frac{4\rho\ell E_{\Delta/2}}{(M+\rho)^2(\ell E_{\Delta/2} + \frac{\Delta^2}{4})^3} \times$$

$$\times \left[(\ell + E_{\Delta/2})(\rho E_{\Delta/2} + M\ell + ME_{\Delta/2}) + \frac{E_{\Delta/2} p^2(\ell + E_{\Delta/2} - \rho)}{E_{\Delta/2} + M} + \frac{2p^2 E_{\Delta/2}(\rho E_{\Delta/2} + M\ell + ME_{\Delta/2})}{\ell E_{\Delta/2} + \frac{\Delta^2}{4}}\right](\kappa E_\kappa g_5)$$

$$+ \frac{4\rho\ell E_{\Delta/2}}{(M+\rho)^2(\ell E_{\Delta/2} + \frac{\Delta^2}{4})^3}\left[(\ell + E_{\Delta/2})(\rho E_{\Delta/2} + M\ell + ME_{\Delta/2}) - \frac{E_{\Delta/2} p^2(\ell + E_{\Delta/2} - \rho)}{E_{\Delta/2} + M}\right](\kappa E_\kappa g_6)$$

$$\ell E_{\Delta/2} h_6 = 2\frac{\frac{\Delta^2}{4}(\frac{\Delta^2}{4} + 2\ell E_{\Delta/2})}{(M+\rho)^2(\ell E_{\Delta/2} + \frac{\Delta^2}{4})^2}(\kappa E_\kappa g_1) + \frac{4\ell^2 E_{\Delta/2}^2 \rho^2 \frac{\Delta^2}{4}}{(M+\rho)^2(\ell E_{\Delta/2} + \frac{\Delta^2}{4})^4}(\kappa E_\kappa g_2)$$

$$+ \frac{2\ell E_{\Delta/2}(E_{\Delta/2}^2 + M\rho)}{(M+\rho)^2(\ell E_{\Delta/2} + \frac{\Delta^2}{4})^2}(\kappa E_\kappa g_3) - \frac{4\rho E_{\Delta/2}\frac{\Delta^2}{4}(2\ell E_{\Delta/2} + \frac{\Delta^2}{4})(\rho E_{\Delta/2} + M\ell + ME_{\Delta/2})}{(M+\rho)^2(\ell E_{\Delta/2} + \frac{\Delta^2}{4})^4} \times$$

$$\times (\kappa E_\kappa g_4) - \frac{4\rho\ell E_{\Delta/2}}{(M+\rho)^2(\ell E_{\Delta/2} + \frac{\Delta^2}{4})^3}\left[M^3 + \rho E_{\Delta/2}^2 + M\ell E_{\Delta/2} - \frac{2\frac{\Delta^2}{4}E_{\Delta/2}(\rho E_{\Delta/2} + M\ell + ME_{\Delta/2})}{\ell E_{\Delta/2} + \frac{\Delta^2}{4}}\right] \times$$

$$\times (\kappa E_\kappa g_5) - \frac{4\rho\ell E_{\Delta/2}}{(M+\rho)^2(\ell E_{\Delta/2} + \frac{\Delta^2}{4})^3}\left[E_{\Delta/2}(\rho E_{\Delta/2} + M\ell + ME_{\Delta/2}) - M\frac{\Delta^2}{4}\right](\kappa E_\kappa g_6)$$

Appendix F - Cross-Sections in Arbitrary Coordinate Systems

This discussion is based on part of an article by Moller[20] on the scattering matrix. The problem is to derive a cross-section from a matrix element in an arbitrary coordinate system.

We shall consider the problem of a particle 1 colliding with a particle 2 to give a particle 3 and a particle 4. The generalization to arbitrary numbers of particles is immediate. In the lab system, with particle 2 at rest,

$$d\sigma = (1/4\pi^2)(E_{1L}/p_{1L})|R_L|^2 \int d^3p_{3L} \int d^3p_{4L}\, \delta(\vec{p}_{3L}+\vec{p}_{4L}-\vec{p}_{1L}-\vec{p}_{2L}) \times$$
$$\times\delta(E_{3L}+E_{4L}-E_{1L}-E_{2L}) \qquad \text{(F-1)}$$

We must write this in a covariant way, so that the subscripts L can be removed. In the first place, integration over 3 variables d^3p is not invariant, so we multiply and divide by E_{3L} and E_{4L}, giving

$$d\sigma = (1/4\pi^2)(E_{1L}/p_{1L})|R_L|^2\, E_{3L}E_{4L} \int \frac{d^3p_{3L}}{E_{3L}} \int \frac{d^3p_{4L}}{E_{4L}}\, \delta(\vec{p}_{3L}+\vec{p}_{4L}-\vec{p}_{1L}-\vec{p}_{2L}) \times$$
$$\times\, \delta(E_{3L}+E_{4L}-E_{1L}-E_{2L})$$

Secondly, we recall that $|R|^2$ is not an invariant, since it contains the reciprocal energy of each particle. Therefore we write

$$d \quad = (1/4\pi^2)(1/p_{1L}E_{2L})(|R_L|^2E_{1L}E_{2L}E_{3L}E_{4L}) \int \frac{d^3p_{3L}}{E_{3L}} \int \frac{d^3p_{4L}}{E_{4L}} \times$$
$$\times\, \delta(\vec{p}_{3L}+\vec{p}_{4L}-\vec{p}_{1L}-\vec{p}_{2L})\, \delta(E_{3L}+E_{4L}-E_{1L}-E_{2L})$$

Finally, we must find an invariant which equals $p_{1L}E_{2L}$ in the lab system. Moller suggests the scalar

$$\sqrt{-(1/2)(p_{1\mu}p_{2\nu} - p_{1\nu}p_{2\mu})^2} = \sqrt{(\vec{p}_1E_2 - \vec{p}_2E_1)^2 - (\vec{p}_1 \times \vec{p}_2)^2}$$

$$= E_1E_2\sqrt{(\vec{v}_1 - \vec{v}_2)^2 - (\vec{v}_1 \times \vec{v}_2)^2}$$

Therefore an invariant expression for cross-section is

$$d\sigma = (1/4\pi^2)\left[(\vec{v}_1-\vec{v}_2)^2 - (\vec{v}_1 x \vec{v}_2)^2\right]^{-1/2}\int d^3p_3\int d^3p_4\ |R|^2\ \delta(\vec{p}_3+\vec{p}_4-\vec{p}_1-\vec{p}_2)\ x$$
$$x\ \delta(E_3+E_4-E_1-E_2) \qquad \text{(F-2)}$$

This is the correct generalization of (F-1) to an arbitrary coordinate system.

The "effective velocity" $\sqrt{(\vec{v}_1-\vec{v}_2)^2 - (\vec{v}_1 x \vec{v}_2)^2}$ has the following values in the three coordinate systems with which we are concerned:

Lab: $v_1 = 1$

CM: $|\vec{v}_1 - \vec{v}_2| = v_1 + v_2 = E/E_k$

System of Figure 1 : $(\ell E_{\frac{\Delta}{2}} + \frac{\Delta^2}{4})/\ell E_{\frac{\Delta}{2}}$

The density of states is obtained when we integrate over the energy delta-function of (F-1) or (F-2). Various expressions for the density of states in the CM and lab systems are well known. For Compton scattering in our special system the integration over the delta-functions gives

$$\int d^3p_3\int d^3p_4\ \delta(\vec{p}_3+\vec{p}_4-\vec{p}_1-\vec{p}_2)\ \delta(E_3+E_4-E_1-E_2) = 2\pi\ell\ d(\Delta'/2)$$

It is more convenient to have an element of CM solid angle in our expression. Therefore we obtain a cross-section from M by writing

$$R = (\ell E_{\frac{\Delta}{2}}/kE_k)\ M$$

and substituting into the cross-section formula in the CM system. The result is

$$d\sigma = (\ell^2 E_{\frac{\Delta}{2}}^2/4\pi^2 E^2)\ |M|^2\ d\Omega$$

Finally, for reference we give the summed and averaged $|M|^2$ of (14):

$$\begin{aligned}
\Sigma\ \mathrm{Av}\ |M|^2 = {} & |h_1|^2 \cdot \frac{1}{\ell^4}\left(p^4 + \frac{\Delta^4}{16}\right) + \left(\mathrm{Re}\, h_1^* h_2 - 2\mathrm{Re}\, h_3^* h_4\right)\frac{p^2}{\ell^4}\frac{\Delta^2}{4}\left(\ell^2 - 2\frac{\Delta^2}{4}\right) \\
& + \left(|h_2|^2 - 8\,\mathrm{Re}\, h_5^* h_6\right)\frac{p^4}{\ell^4}\frac{\Delta^4}{16} + |h_3|^2 \frac{1}{\ell^4}\left(\ell^4 + 2p^2\frac{\Delta^2}{4}\right) \\
& + |h_4|^2 \frac{p^2}{\ell^4}\frac{\Delta^2}{4}\left(p^4 + \frac{\Delta^4}{16}\right) + |h_5|^2 \frac{p^2}{\ell^4}\frac{\Delta^4}{16}\left(\ell^2 + 2\frac{\Delta^2}{4}\right) \\
& + |h_6|^2 \frac{p^4}{\ell^4}\frac{\Delta^2}{4}\left(3\ell^2 - 2\frac{\Delta^2}{4}\right) - 2\,\mathrm{Re}\, h_3^* h_5 \frac{p^2}{\ell^4}\frac{\Delta^2}{4}\left(\ell^2 + 2\frac{\Delta^2}{4}\right) \\
& + 2\,\mathrm{Re}\, h_3^* h_6 \frac{p^2}{\ell^4}\frac{\Delta^2}{4}\left(3\ell^2 - 2\frac{\Delta^2}{4}\right) + 2\,\mathrm{Re}\, h_4^* h_5 \frac{p^2}{\ell^4}\frac{\Delta^4}{16}\left(\ell^2 - 2\frac{\Delta^2}{4}\right) \\
& - 2\,\mathrm{Re}\, h_4^* h_6 \frac{p^4}{\ell^4}\frac{\Delta^2}{4}\left(\ell^2 - 2\frac{\Delta^2}{4}\right)
\end{aligned}$$

Appendix G - Multipole Analysis of Compton Scattering

In reference (2) a phase-shift analysis of the scattering of electromagnetic radiation by a particle of spin 1/2 is given. However, the possibility of mixed electric-magnetic scattering (for example, M1-E2 with J=3/2) is omitted. To include this, we merely note that unitarity and time-reversal invariance require the S matrix for a channel of given total angular momentum and parity to have the form

$$\begin{pmatrix} \cos\theta\, e^{2i\delta_1} & i\sin\theta\, e^{i(\delta_1+\delta_2)} \\ i\sin\theta\, e^{i(\delta_1+\delta_2)} & \cos\theta\, e^{2i\delta_2} \end{pmatrix} \qquad \text{(G-1)}$$

where the two rows (or columns) refer to electric and magnetic radiation. The quantities θ, δ_1, δ_2 are real. It then follows in a quite straightforward fashion that the scattering amplitude for circularly polarized light on spin-up particles is

$$\begin{aligned}
\vec{S}(\Omega) = \frac{i}{2K}\sum_J \Big\{ & -\left(\frac{J}{4\pi}\right)^{1/2} C^{J-\frac{1}{2},\frac{1}{2}}_{J_j\pm1,\frac{1}{2}} \left(\cos\theta_{J,J-1/2}\, e^{2i\delta_J^{E,J-1/2}} - 1\right) \vec{X}_{J,J-1/2,\frac{1}{2}\pm1} \\
& \pm\left(\frac{J+1}{4\pi}\right)^{1/2} C^{J+1/2,1/2}_{J_j\pm1,\frac{1}{2}} \left(\cos\theta_{J,J-1/2}\, e^{2i\delta_J^{M,J+1/2}} - 1\right) \vec{Y}_{J,J+1/2,1/2\pm1} \\
& \pm i\sin\theta_{J,J+\frac{1}{2}}\, e^{i\left(\delta_J^{E,J-\frac{1}{2}}+\delta_J^{M,J+1/2}\right)} \left[\left(\frac{J+1}{4\pi}\right)^{1/2} C^{J+\frac{1}{2},\frac{1}{2}}_{J_j\pm1,\frac{1}{2}} \vec{X}_{J,J-\frac{1}{2},\frac{1}{2}\pm1} \mp \left(\frac{J}{4\pi}\right)^{1/2} C^{J-1/2,1/2}_{J_j\pm1,1/2} \vec{Y}_{J,J+1/2,\frac{1}{2}\pm1}\right] \\
& -\left(\frac{J+1}{4\pi}\right)^{\frac{1}{2}} C^{J+\frac{1}{2},\frac{1}{2}}_{J_j\pm1,\frac{1}{2}} \left(\cos\theta_{J,J+\frac{1}{2}}\, e^{2i\delta_J^{E,J+\frac{1}{2}}} - 1\right) \vec{X}_{J,J+\frac{1}{2},\frac{1}{2}\pm1} \qquad \text{(G-2)} \\
& \pm\left(\frac{J}{4\pi}\right)^{1/2} C^{J-\frac{1}{2},\frac{1}{2}}_{J_j\pm1,\frac{1}{2}} \left(\cos\theta_{J,J+\frac{1}{2}}\, e^{2i\delta_J^{M,J-\frac{1}{2}}} - 1\right) \vec{Y}_{J,J-1/2,\frac{1}{2}\pm1} \\
& \pm i\sin\theta_{J,J+\frac{1}{2}}\, e^{i\left(\delta_J^{E,J+\frac{1}{2}}+\delta_J^{M,J-\frac{1}{2}}\right)} \left[\left(\frac{J}{4\pi}\right)^{\frac{1}{2}} C^{J-\frac{1}{2},\frac{1}{2}}_{J_j\pm1,\frac{1}{2}} \vec{X}_{J,J+\frac{1}{2},\frac{1}{2}\pm1} \mp \left(\frac{J+1}{4\pi}\right)^{1/2} C^{J+\frac{1}{2},\frac{1}{2}}_{J_j\pm1,\frac{1}{2}} \vec{Y}_{J,J-\frac{1}{2},\frac{1}{2}\pm1}\right]\Big\}
\end{aligned}$$

For $J=1/2$, $\theta_J=0$ since the only multipoles are E1 and M1 and no mixing occurs. We have defined:

$$\vec{X}_{JLM} = \sum_s C^{L,\frac{1}{2}}_{J;M-s,s}\, \vec{X}_{L,M-s}\, \alpha_s$$

$$\vec{Y}_{JLM} = \sum_s C^{L,\frac{1}{2}}_{J;M-s,s}\, \vec{Y}_{L,M-s}\, \alpha_s$$

$\vec{X}_{LM}$ is the normalized angular distribution of $\vec{A}$ for $(EL)_M$ radiation; $\vec{Y}_{LM}$ similarly characterizes $(ML)_M$ radiation. In terms of vector spherical harmonics,(21)

$$\vec{X}_{LM} = \sqrt{\frac{L+1}{2L+1}}\, \vec{Y}^M_{L,L-1,1} + \sqrt{\frac{L}{2L+1}}\, \vec{Y}^M_{L,L+1,1}$$

$$\vec{Y}_{LM} = \vec{Y}^M_{L,L,1}$$

$\alpha_{\pm 1/2}$ defines the spin of the scatterer in the final state.

If $J=3/2$, the six terms in (G-2) give E1, M2, E1-M2, E2, M1, and E2-M1 scattering, respectively.

We do not feel that a phase-shift analysis of Compton scattering is very meaningful, however, because of the absorptive effects of photomeson production. When this channel is taken into account, the form of (G-1) becomes

$$\begin{pmatrix} A + i\alpha & \sqrt{AB} + i\beta \\ \sqrt{AB} + i\beta & B + i\gamma \end{pmatrix} \qquad \text{(G-3)}$$

where A and B are squares of photoproduction amplitudes, and $A, B, \alpha, \beta, \gamma$ are real. Thus the scattering is now characterized by five real parameters, not the three of (G-1), and we see no natural phase shift description of this scattering. One might try making δ_1 and δ_2 in (G-1) complex, but this form is not equivalent to (G-3).

We shall therefore analyze our amplitude (5) into E1(1/2), M1-E2(3/2), etc. without using phase shifts. We expand R in normalized multipole amplitudes:

$$R = g_{E1}^{1/2} \cdot \frac{1}{\sqrt{8\pi}} \left(\hat{e}\cdot\hat{e}' - i\vec{\sigma}\cdot\hat{e}\times\hat{e}' \right) + g_{M1}^{1/2} \cdot \frac{1}{\sqrt{8\pi}} \Big[\hat{k}\cdot\hat{k}'\hat{e}\cdot\hat{e}' - \hat{e}\cdot\hat{k}'\hat{e}'\cdot\hat{k}$$

$$+ i\vec{\sigma}\cdot\left(-\hat{k}\cdot\hat{k}'\hat{e}\times\hat{e}' - e\cdot\hat{e}'\hat{k}\times\hat{k}' - \hat{e}\cdot\hat{k}'\hat{e}'\times\hat{k} + \hat{e}'\cdot\hat{k}\,\hat{e}\times\hat{k}' \right) \Big]$$

$$+ g_{M1}^{3/2} \cdot \frac{1}{\sqrt{16\pi}} \Big[2\hat{k}\cdot\hat{k}'\hat{e}\cdot\hat{e}' - 2\hat{e}\cdot\hat{k}'\hat{e}'\cdot\hat{k} + i\vec{\sigma}\cdot\left(\hat{k}\cdot\hat{k}'\hat{e}\times\hat{e}' + \hat{e}\cdot\hat{e}'\hat{k}\times\hat{k}' \right.$$

$$\left. + \hat{e}\cdot\hat{k}'\hat{e}'\times\hat{k} - \hat{e}'\cdot\hat{k}\,\hat{e}\times\hat{k}' \right) \Big] + g_{E2}^{3/2} \cdot \frac{1}{\sqrt{16\pi}} \Big[2\hat{k}\cdot\hat{k}'\hat{e}\cdot\hat{e}' + 2\hat{e}\cdot\hat{k}'\hat{e}'\cdot\hat{k}$$

$$+ i\vec{\sigma}\cdot\left(-\hat{k}\cdot\hat{k}'\hat{e}\times\hat{e}' - \hat{e}\cdot\hat{e}'\hat{k}\times\hat{k}' + \hat{e}\cdot\hat{k}'\hat{e}'\times\hat{k} - \hat{e}'\cdot\hat{k}\,\hat{e}\times\hat{k}' \right) \Big]$$

$$+ g_{M1-E2}^{3/2} \sqrt{\frac{3}{8\pi}}\; i\vec{\sigma}\cdot\left(\hat{k}\cdot\hat{k}'\hat{e}\times\hat{e}' - \hat{e}\cdot\hat{e}'\hat{k}\times\hat{k}' \right) + \cdots$$

Then if we define α_i, β_i by

$$g_1=\alpha_1+\beta_1 y \quad , \quad g_3=\alpha_3+\beta_3 y \quad , \quad g_i=\alpha_i \quad (i=2,4,5,6)$$

multipole analysis gives

$$g_{E1}^{1/2} = \frac{1}{3}\sqrt{8\pi}\,(\alpha_1 - 2\alpha_3)$$

$$g_{M1}^{1/2} = \frac{1}{6}\sqrt{8\pi}\,(\beta_1 - \alpha_2 - \beta_3 - \alpha_4 - 2\alpha_5)$$

$$g_{M1}^{3/2} = \frac{1}{12}\sqrt{16\pi}\,(2\beta_1 - 2\alpha_2 + \beta_3 + \alpha_4 + 2\alpha_5)$$

$$g_{E2}^{3/2} = \frac{1}{20}\sqrt{16\pi}\,(2\beta_1 + 2\alpha_2 - 3\beta_3 - 3\alpha_4 + 6\alpha_5)$$

$$g_{M1-E2}^{3/2} = \frac{1}{2}\sqrt{\frac{8\pi}{3}}\,(\beta_3 - \alpha_4)$$

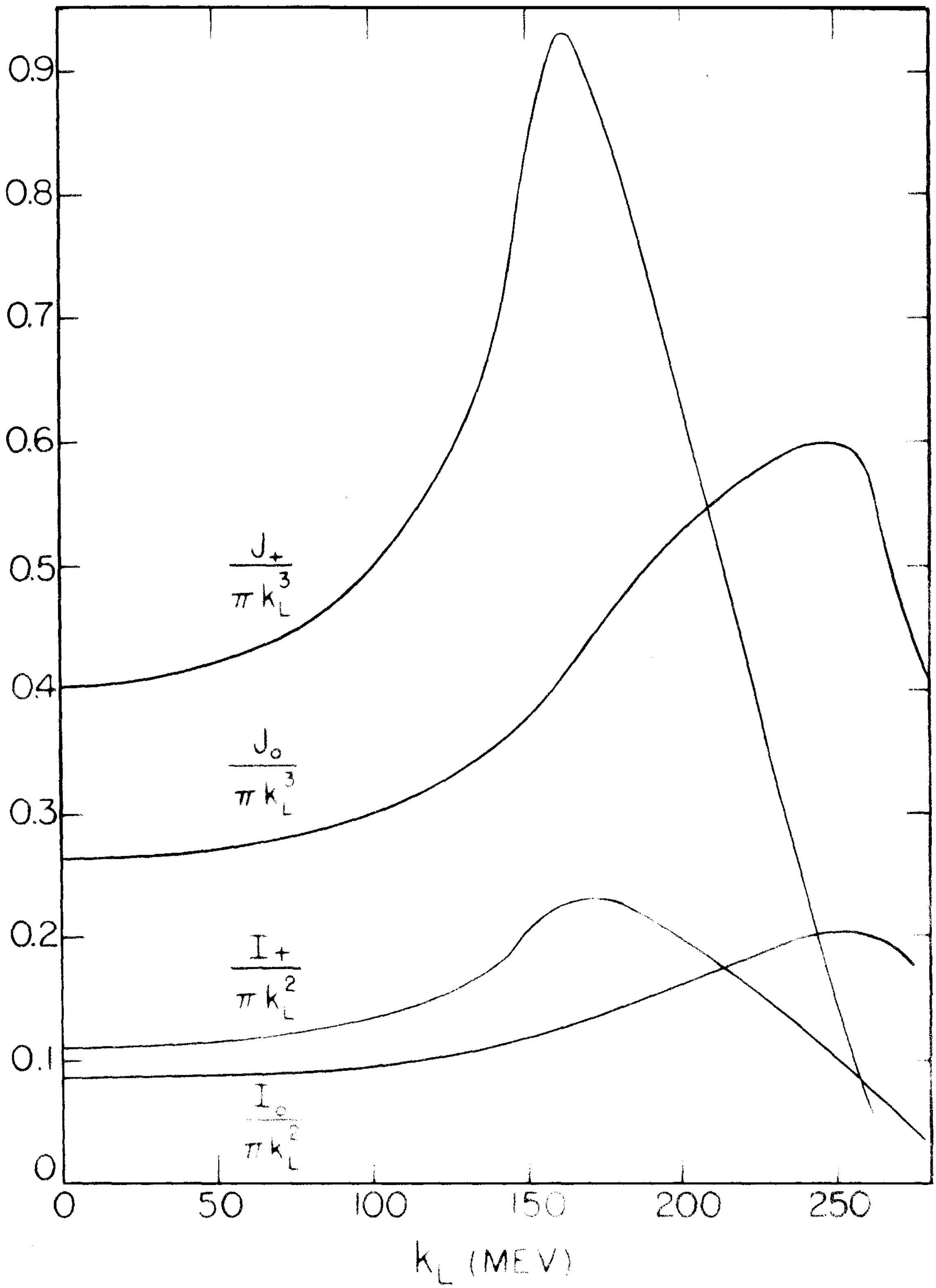

FIGURE 2 – DISPERSION INTEGRALS

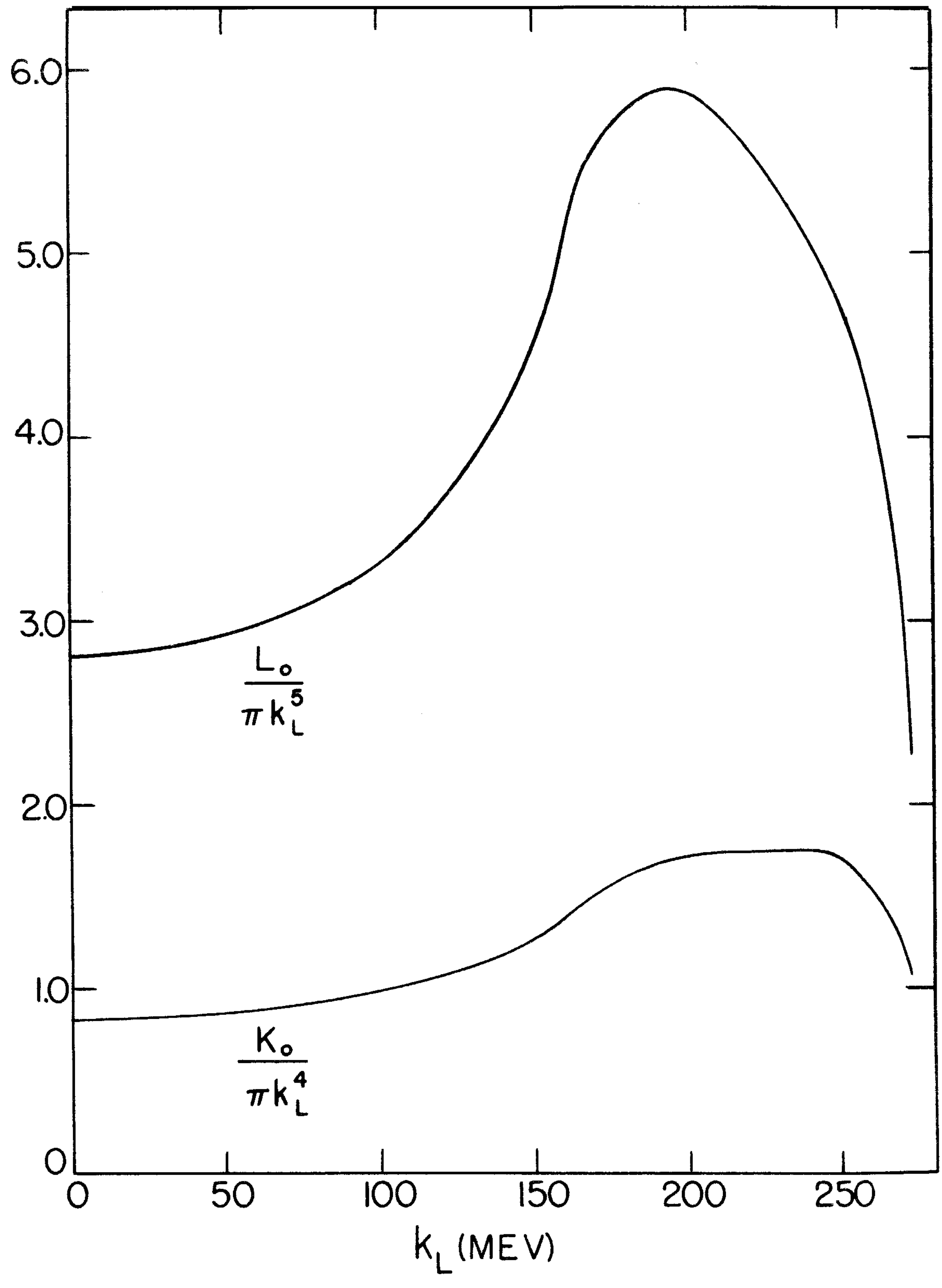

FIGURE 3 — DISPERSION INTEGRALS

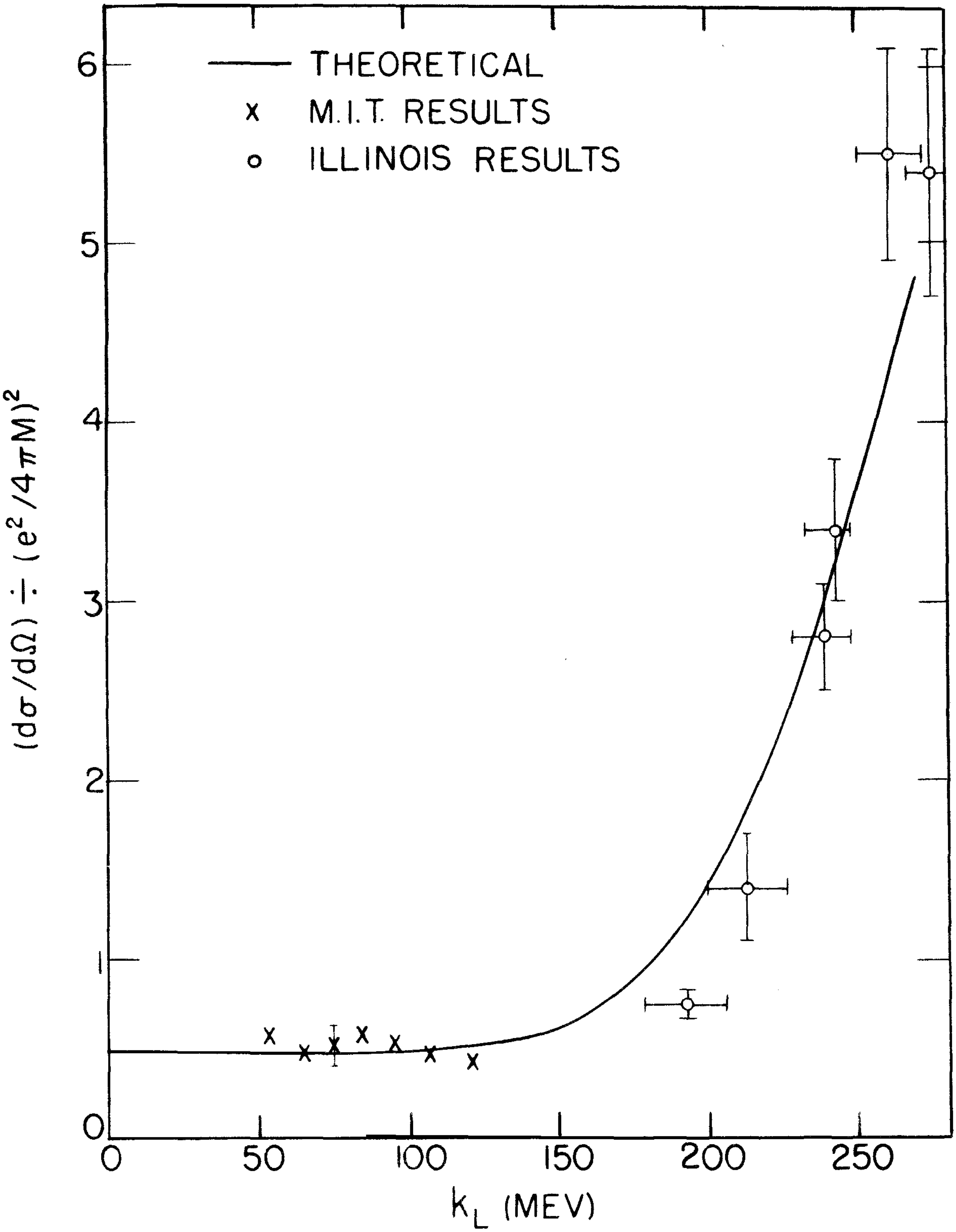

FIGURE 4 — EXCITATION CURVE FOR $\theta_{CM}=90°$

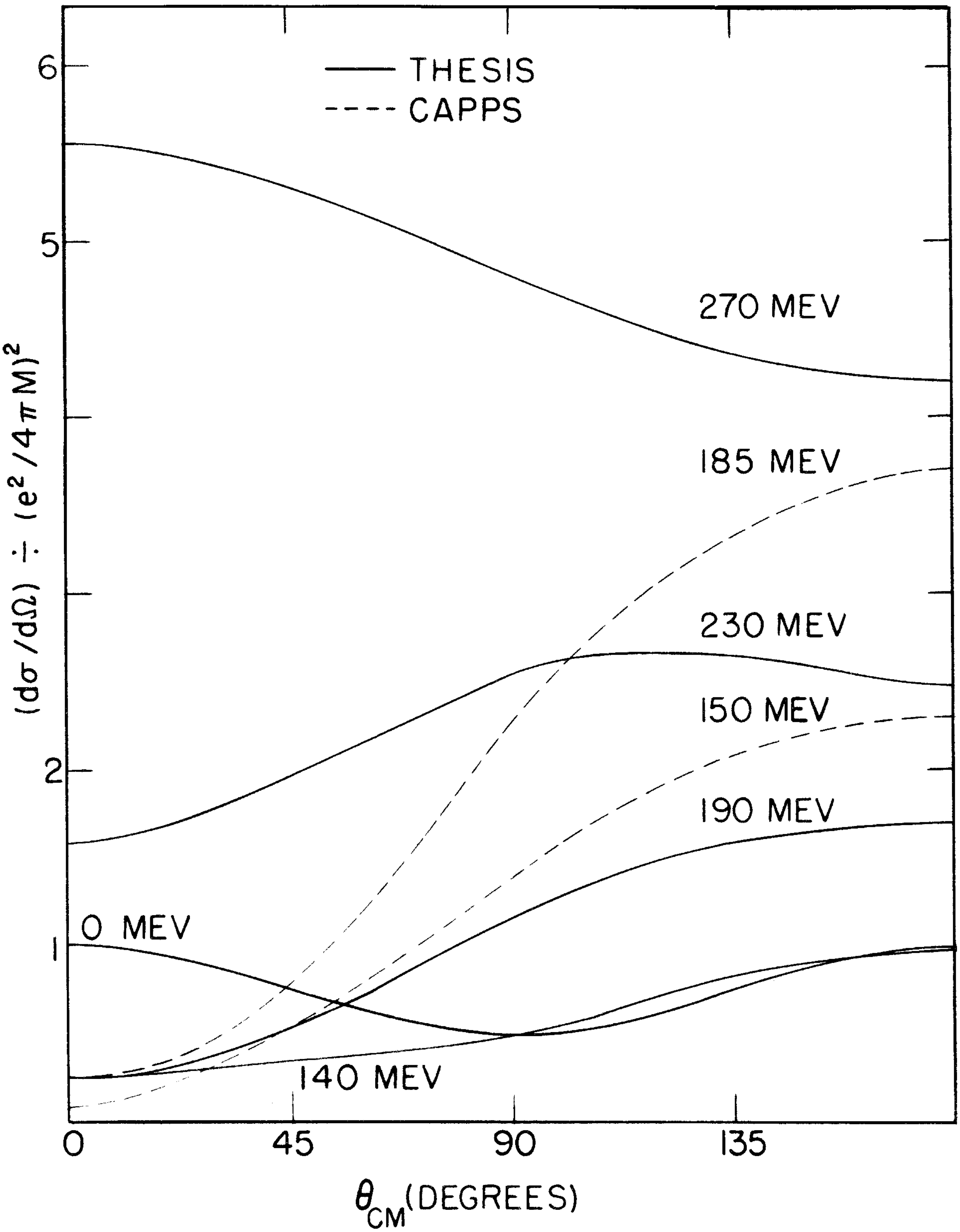

FIGURE 5 - ANGULAR DISTRIBUTION

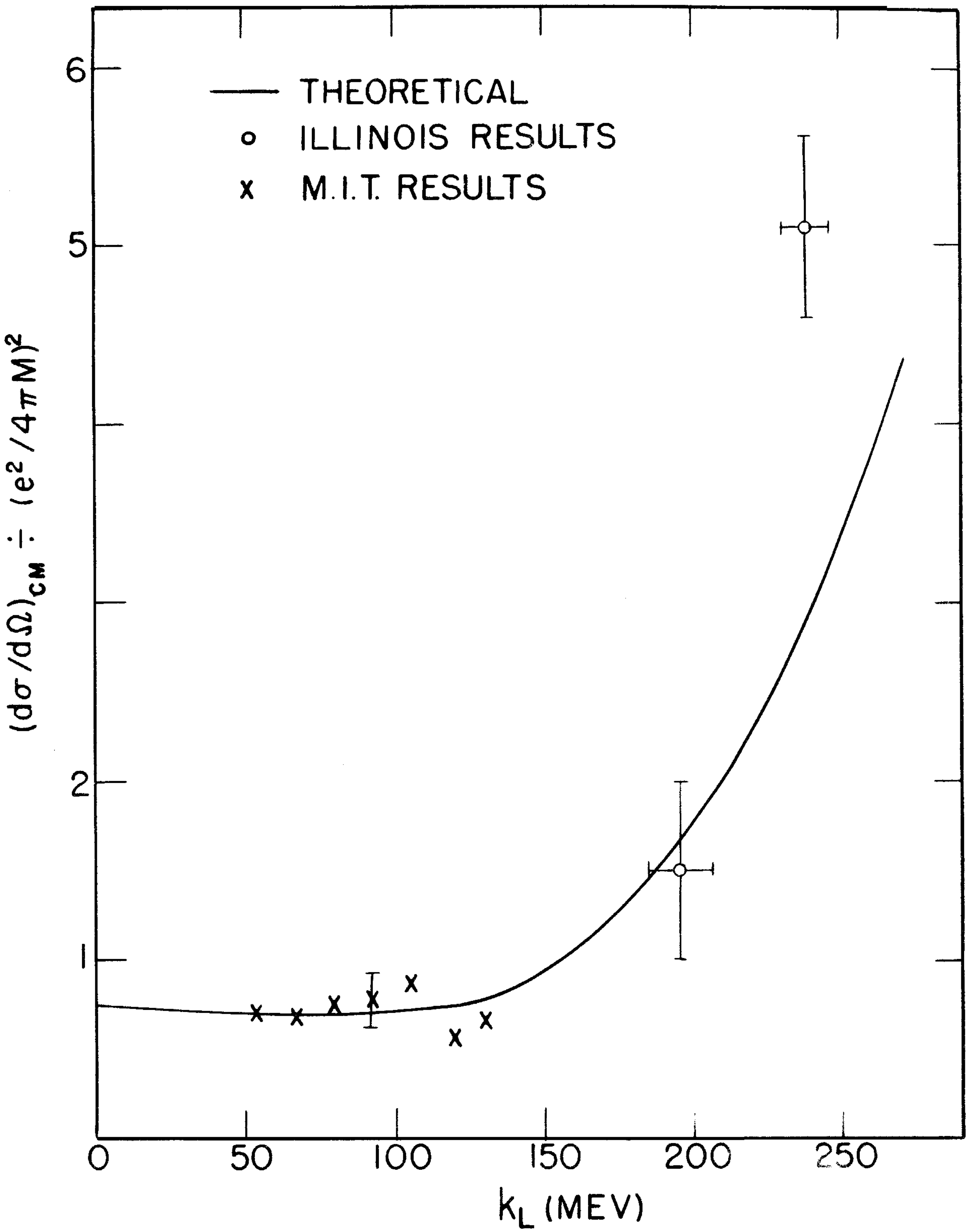

FIGURE 6 – EXCITATION CURVE FOR θ_{CM}=135°

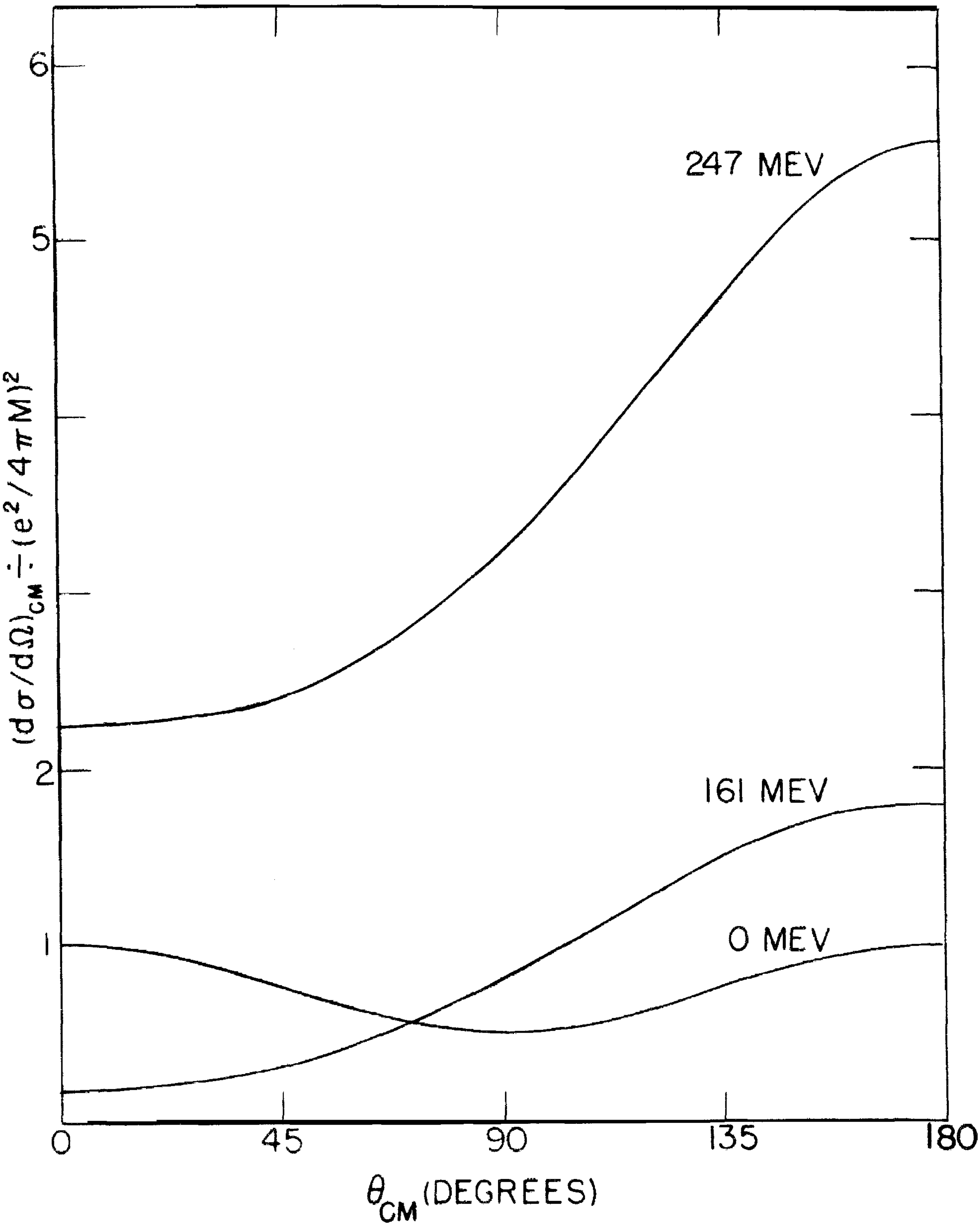

FIGURE 7 – ANGULAR DISTRIBUTIONS OF OLD MODEL