ACCELERATION OF COSMIC RAYS
BY HYDROMAGNETIC WAVES

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ABSTRACT

The problem of the electromagnetic acceleration of cosmic rays to high energies by turbulent magnetic fields within the confines of our galaxy is considered. The model of the magnetic field used is essentially that proposed by Fermi, in which the field is assumed to be fairly regular and to run along the spiral arm of the galaxy. The magnetic field plays a dual role, storing or trapping the cosmic rays, and accelerating them when the field is not static. The fluctuating part of the magnetic field is described statistically in terms of a spectral decomposition of the field into hydromagnetic waves of different wavelengths moving in the direction of the primary field.

Two main problems are of concern: (a) the energy distribution of the high energy particles, and (b) the angular distribution of these particles. A partial differential equation of the diffusion type is derived which describes statistically the behavior of an ensemble of particles undergoing accelerating and decelerating interactions (betatron interactions) with the varying magnetic field. In addition to accelerating the particles, the betatron interactions change the component of momentum parallel to the field in a way which depends on the energy change. In addition to these processes, the differential equation accounts for interactions with inhomogeneities in the field whose scale is small compared to the helix radius, as well as removal of particles by nuclear collisions and by diffusion of particles out of the region of the magnetic field.

Solutions to the steady-state diffusion equation show that a power-law energy spectrum results for the high energy particles. The exponent in the power-law spectrum is related to the parameters describing the magnetic field, the mean-square velocity of the turbulent medium, and the mean time for loss of particles by nuclear collisions and diffusion out of the spiral arm. An approximate form of the space-dependent steady-state diffusion equation is solved to estimate the mean time for escape of the particles by diffusion, and to relate this parameter to the length of the spiral arm and to the parameters describing the magnetic field. The results also show that the angular distribution of the particles is inextricably tied up with the energy spectrum, with the degree of anisotropy being determined by the relative effectiveness of the scattering by small scale inhomogeneities which tend to make the distribution isotropic, and the betatron processes which tend to make the distribution highly anisotropic with most of the particles lying in very steep spirals.

It appears from the results that a set of parameters describing the magnetic field can be found which are astronomically plausible, and which give results for the power-law exponent and the anisotropy within the range of values observed experimentally.

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1. INTRODUCTION

It is now widely recognized that high energy cosmic rays may well result from electromagnetic acceleration of particles by turbulent magnetic fields within the confines of our galaxy. The galactic magnetic field plays a dual role: (a) it provides a mechanism for storing or trapping the particles within the galactic boundaries, and (b) accelerates the particles when the field is not static. Many specific mechanisms have been proposed that might accomplish the acceleration, based on widely different models of the magnetic field.

In an early paper by Fermi (1949), the magnetic field was assumed to be very tangled and imbedded in a turbulent gas whose motion dominated the field. In this model there is no preferred direction of the magnetic field. Fermi considered interactions between particles and moving gas clouds in which magnetic fields are imbedded, and showed that the average energy change in a large number of interactions is not zero if one accounts for the fact that collisions between particles and clouds moving in opposite directions are more probable than collisions between particles and clouds moving in the same direction. On the basis of this average effect, Fermi showed that a power law spectrum for the energies is to be expected, and that the exponent would be that observed experimentally provided certain parameters had appropriate values.

In a second paper, Fermi (1954) assumed the magnetic field to be fairly regular and to run along a spiral arm of the galaxy. In this model, the field is assumed to be strong enough to dominate the turbulent motion of the gas. Fermi considered processes in which a particle is trapped between two constrictions in a tube of force, and showed that large amounts of energy will be gained or lost depending on whether the particle is trapped between constrictions that are moving together or apart. The argument is continued to show that a positive increase in energy results when a particle runs through a series of traps even though it is equally likely for a particle to encounter a positive or negative trap. The essence of the argument is that particles trapped between constrictions moving apart are not to be regarded as losing energy because their helices become flatter and the particles more susceptible to future trapping. Thus once trapped, a particle encounters other traps (positive and negative) and escapes only when the energy has been increased. Again, on the basis of this positive average increase in energy, Fermi predicted a power law spectrum for the energy.

Other investigators have found that neither of Fermi's versions seem satisfactory when considered quantitatively using the best available astronomical data. With regard to the first of Fermi's models, Unsöld (1951) found that the values of the parameters necessary to produce the observed energy spectrum of high energy cosmic rays were astronomically implausible. Davis (1956) has pointed out that in the second of Fermi's models, the processes considered lead to a very anisotropic angular distribution of the cosmic ray particles. The observed near-isotropy of cosmic radiation thus seems to require that the anisotropic distribution be smoothed out by other processes. If this is the case, Davis observes, then it will not be true that a sequence of traps produces a large energy gain.

Extension of Fermi's arguments and attempts to remove some of the inherent difficulties have been made by several investigators, (Morrison et al. 1954, Fan 1951, Cocconi 1951) but without notable success. Davis (1954, 1956) has pointed out that other processes, not considered by Fermi, are important in acceleration of cosmic ray particles. In particular he argues that induced electric fields, which are present when the magnetic field strength changes with time, will increase or decrease the energy of a particle. These processes Davis calls "betatron collisions." In considering these processes it is necessary to consider the statistical fluctuations in the number of positive and negative energy changes that individual particles experience. These fluctuations should be considered even though there is a known average effect such as those considered in Fermi's two papers. These fluctuations are most easily described by a differential equation of the diffusion type which describes the rate at which particles enter or leave a given differential energy range. An analysis of this type has been made by Davis based on a model of the field consisting of a uniform field on which are superposed standing hydromagnetic waves. Davis obtains an equation of the form

$$\frac{\partial n}{\partial t} = -\frac{n}{\tau} + \frac{\partial (pn)}{\partial \epsilon} - \frac{\partial}{\partial \epsilon}(D_{\epsilon} n) + \frac{1}{2}\frac{\partial^2}{\partial \epsilon^2}(D_{\epsilon\epsilon} n) + I(\epsilon, t)$$

where $n d\epsilon$ denotes the number of particles in the range $\epsilon \rightarrow \epsilon + d\epsilon$, and where ϵ is the logarithm of the ratio of the total energy of a particle to its rest energy. In this equation, the first term on the right represents the rate at which particles disappear due to nuclear collisions or by diffusion out of the region where cosmic rays are stored. The second term represents the net rate at which particles enter the range $d\epsilon$ due to ionization losses in the interstellar medium. The third and fourth terms represent the net rate at which particles enter the range $d\epsilon$ due to random accelerations and decelerations by betatron collisions. The last term represents the rate at which particles are added by some injection process. The quantities D_ϵ and $D_{\epsilon\epsilon}$ depend on the rate at which the betatron interactions occur and other details of the interaction. Their evaluation is complicated by the fact that the rate at which the energy (or ϵ) changes during a betatron interation depends strongly on the pitch angle of the helix in which the particle is traveling at the time of the interaction.

Davis shows that if D_ϵ and $D_{\epsilon\epsilon}$ can be adequately approximated by constants, then the steady state solution of the above equation is

$$n = n_o e^{-\alpha\epsilon}$$

where

$$\alpha = -\frac{D_\epsilon - p}{D_{\epsilon\epsilon}} + \left[\left(\frac{D_\epsilon - p}{D_{\epsilon\epsilon}}\right)^2 + \frac{2}{D_{\epsilon\epsilon}T}\right]^{1/2}$$

This implies a power law spectrum in the energy w, with the number of particles in the range dw given by

$$n(w)dw = n_o w_o^{\alpha} w^{-(\alpha+1)} dw$$

The value of α required to fit the observed spectrum is between 1.5 and 1.7 (Neher and Stern 1955).

Davis makes an order of magnitude estimate of the value of D_ϵ and $D_{\epsilon\epsilon}$ on the basis of a magnetic field composed of a uniform field on which are superposed two dimensional standing hydromagnetic waves of a fixed frequency and wave length, and shows that the observed energy spectrum is obtained if rms gas velocities of 10 km/sec are present in oscillations whose extensions normal and parallel to the field are of the order of 1 and 7 light years, respectively.

On the basis of his model of the magnetic field, Davis obtains, for extremely relativistic particles, $D_\epsilon = 0$ and

$$D_{\epsilon\epsilon} = \frac{2c\beta_g^2 b}{a^2}\left[\ln\left(\frac{16\pi\rho c^2 a^2}{B_o^2(a^2+b^2)}\right) - \frac{3}{2}\right]$$

where $c^2\beta_g^2$ is the mean square transverse velocity of the turbulent gas, B_o is the unperturbed field strength, a and b are the cell sizes for the oscillations perpendicular and parallel to the undisturbed field, respectively, and ρ is the mass density of the medium.

In computing $D_{\epsilon\epsilon}$, certain angular factors appear which must be averaged over the actual angular distribution of the particles. On the basis of experimental evidence, Davis assumes the angular distribution of the spiraling particles is maintained near isotropy by scattering processes due to inhomogeneities small compared to the helix radius. These scatterings must be numerous enough to produce near isotropy without being so numerous that the particle is less likely to diffuse to the end of the spiral arm than make a nuclear collision.

Although Davis' treatment of the betatron mechanism indicates that this mechanism is an efficient means of accelerating cosmic ray particles, certain questions are left unresolved. The most important of these has to do with the angular distribution of the cosmic ray particles produced by the mechanism. If such a mechanism is plausible it must explain the observed distribution which is known to be isotropic to within a few percent (Davis 1954a). This imposes additional conditions on the parameters describing the magnetic field. It is known that the betatron mechanism by itself tend to make a very anisotropic distribution with most of the particles lying in steep spirals. The scattering processes which tend to make the distribution isotropic must therefore be relatively efficient.

The promising results obtained by Davis make it worthwhile to attempt a much more extensive analysis, taking into account the angular distribution of cosmic ray particles produced by betatron collisions and the "smoothing" of the distribution due to the interaction with inhomogeneities small compared to the helix radius, as well as the boundary effects at the ends of the spiral arm. This thesis is addressed to the above problem.

Any such analysis must start with a particular model of the magnetic field. The model assumed here is essentially that proposed by Fermi in his second paper, a model in which the lines of force run the length of the spiral arm of the galaxy, with the field strength being roughly independent of the distance along the spiral arm. Since the field strength is known to be of the order of 10^{-5} gauss, the radius of the helix of a typical high energy particle (say 10^{17} ev.) is small compared to the radius of the spiral arm (of the order of 1000 light years), and hence the boundary effects at the surface of the arm may be disregarded.

The model adopted consists of a uniform field on which are superposed hydromagnetic oscillations of a wide variety of wavelengths and amplitudes. These oscillations consist of running waves moving in both directions along the uniform field. A statistical description of the hydromagnetic waves is adopted which requires a minimum of special assumptions regarding the field. This model is more general than that assumed by Davis in which a particular form is assumed for the space and time dependence. The present treatment requires only that the space dependence be a stationary random function of the space coordinate measured along the direction of the unperturbed field (at a particular instant of time). The effect of such a field in accelerating cosmic ray particles is most conveniently dealt with by considering the behavior of an ensemble of particles moving in such a field. For a given particle, let θ denote the helix angle, the angle between the momentum vector and the direction of the uniform field and let $\mu = \cos\theta$. Let $\epsilon = \ln\left(\frac{mc^2}{m_o c^2}\right)$, be the logarithm of the ratio of the total energy of a particle to its rest energy, and let z be the distance along the spiral arm. The behavior of the ensemble of particles is then described by the distribution function $W(\mu, \epsilon, z;\ t)$ which, when multiplied by $d\mu d\epsilon dz$, represents the probability that a particle will be found in the range $\mu \rightarrow \mu + d\mu$, $\epsilon \rightarrow \epsilon + d\epsilon$, $z \rightarrow z + dz$ at the time t.

It will be shown that $W(\mu, \epsilon, z; t)$ *satisfies a linear partial* differential equation which is first order in t and z and second order in μ and ϵ. This function contains all the information of physical interest regarding the spatial distribution of particles along the spiral arm, the energy distribution, and the angular distribution. In formulating the differential equation for W we take into account the following physical processes:

(a) Interaction with large scale inhomogeneities (betatron processes), i.e., interactions of the particles with hydromagnetic waves.

(b) Scattering by small scale inhomogeneities.

(c) Absorption of particles by nuclear collisions, and loss of particles by diffusion of particles out of the spiral arm.

(d) Injection of new particles.

These processes are represented by separate terms or groups of terms in the diffusion equation. Of the four processes listed, the first is the most difficult to treat, and requires extended analysis of the dynamics of charged particles moving in inhomogeneous magnetic fields.

Section 2 of this thesis is devoted to a study of relativistic particles moving in slowly varying magnetic fields. The major results of this section are equations of motion for μ and ϵ. As a by-product of the analysis some theorems are established which are generalizations of Alfven's (1950) familiar nonrelativistic magnetic moment theorems to the relativistic case. These theorems are of interest in themselves.

In Section 3 a mathematical model of the hydromagnetic wave field is presented. The equations of motion for μ and ϵ derived in Section 2 and the model of the magnetic field are then used to evaluate certain " diffusion coefficients " which appear in the differential equation defining $W(\mu, \epsilon, z; t)$. The last part of this section is concerned with the treatment of the scattering by small scale inhomogeneities and the absorption and injection processes.

In Section 4 the results of Section 3 are combined to complete the derivation of the diffusion equation, and the appropriate boundary conditions are formulated.

Section 5 is devoted to solution of the steady state diffusion equation in certain special cases of interest. The anisotropy produced by diffusion of particles out of the spiral arm is considered, and the mean time for escape by diffusion out of the spiral arm is estimated and related to parameters describing the magnetic field.

In Section 6 the space independent diffusion equation is considered and asymptotic solutions for large ϵ are obtained by numerical integration.

Section 7 gives a summary of the results obtained from Sections 5 and 6, and the conclusions that can be drawn from them.

2. DYNAMICS OF COSMIC RAY PARTICLES IN SLOWLY VARYING MAGNETIC FIELDS

2.1 Representation of the Field by a Vector Potential

In the following sections we shall consider the motion of a particle of charge q in a magnetic field $\vec{B}$ having the general character described above. We use Gaussian units.

The field will be assumed to consist of a uniform static field $\vec{B} = B_o \vec{e}_z$ and a fluctuating part $\overrightarrow{\Delta B}$ composed of hydromagnetic waves moving in the $\pm z$ directions. We choose the direction of the uniform part of the field as the z-axis. It is a well known result that in such a uniform field a charged particle moves along the magnetic field in a helix whose axis is along the direction of the field and with a radius given by

$$r = \frac{c\, p_t}{q\, B_o} \tag{2.1-1}$$

where c is the velocity of light and p_t is the projection of the momentum on a plane perpendicular to the field. We further split $\overrightarrow{\Delta B}$ into variations of two kinds: (1) those whose scale is small compared to the radius of the spiral motion, and (2) those whose scale is of the order of the radius of the spiral motion or larger. Variations of the two types we will call small and large scale inhomogeneities, respectively.

The major effect of the small scale inhomogeneities is to change the direction of motion of a spiraling particle. The small scale inhomogeneities also affect the energy but to a lesser extent than the large ones*. We neglect the energy changes produced by small scale inhomogeneities.

The large scale inhomogeneities change the energy of a particle through the action of the induced electric field, and at the same time change its direction of motion. The interactions of cosmic ray particles with the large scale variations in the field will be considered in Sections 2 and 3.2. The effects of small scale inhomogeneities will be considered in Section 3.3.

Let us concentrate our attention on the region in the vicinity of one of the large scale inhomogeneities. We assume that the magnetic field is axially symmetric in the region of interest, and that there is no net charge density present.

The magnetic field and the induced electric field must satisfy Maxwell's equations. Those of importance** here are

$$\nabla \cdot \vec{B} = 0 \tag{2.1-2}$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{2.1-3}$$

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* There will be small random electric fields associated with the small scale inhomogeneities if they are not static. This leads to a random walk in the energy of a particle.

** The remaining two of Maxwell's field equations are

$$\nabla \cdot \vec{E} = 4\pi\rho = 0 \qquad \nabla \times \vec{B} = \frac{4\pi}{c}\vec{i}$$

where the displacement current has been neglected in the last equation. For the assumed form for the vector potential, there is a current density present given by

$$\vec{i} = -\frac{c}{4\pi}\nabla^2\vec{A} = -\frac{c}{4\pi}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_\phi}{\partial r}\right) - \frac{A_\phi}{r^2}\right]\vec{e}_\phi$$

By virtue of the above assumptions, we may describe both fields $\vec{E}$ and $\vec{B}$ by a vector potential which has only one component $\vec{A} = A_\phi \vec{e}_\phi$ in a system of cylindrical coordinates (r, ϕ, z). Furthermore, we will take A_ϕ to be independent of ϕ. $\vec{A}$ thus has zero divergence:

$$\nabla \cdot \vec{A} = \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} = 0 \qquad (2.1\text{-}4)$$

The fields $\vec{B}$ and $\vec{E}$ are calculated from the vector potential by*

$$\vec{B} = \nabla \times \vec{A} = \nabla \times \vec{A}_\phi(r, z, t)\, \vec{e}_\phi = \frac{1}{r} \frac{\partial}{\partial r} (r\, A_\phi)\, \vec{e}_z - \frac{\partial A_\phi}{\partial z}\, \vec{e}_r \qquad (2.1\text{-}5)$$

$$\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} = -\frac{1}{c} \frac{\partial A_\phi(r, z, t)}{\partial t}\, \vec{e}_\phi \qquad (2.1\text{-}6)$$

It is easily verified from (2.1-5) that $\nabla \cdot \vec{B} = 0$.

In what follows we shall use, wherever possible, the general form $\vec{A} = A_\phi(r, z, t)\, \vec{e}_\phi$ without explicitly stating the dependence on r, z, and t. In Section 3 we will find it convenient to use the form

$$A_\phi(r, z, t) = \frac{r}{2} B_o + A'_\phi(r, z, t) \qquad (2.1\text{-}7)$$

where $\frac{r}{2} B_o \vec{e}_\phi$ is the vector potential for the uniform static field $\vec{B} = B_o \vec{e}_z$, and A'_ϕ is the vector potential for the fluctuating part of the field.

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* For a given magnetic field $\vec{B}$, the vector potential is defined by (2.1-4) and (2.1-5).

2.2 *The Relativistic Hamiltonian and the Equations of Motion for a Charged Particle in a Cylindrically Symmetric Magnetic Field*

Our first task in studying the dynamics of cosmic ray particles in inhomogeneous magnetic fields will be to find the Hamiltonian function and to formulate the equations of motion in cylindrical coordinates. We start with the Lagrangian for a charged particle in an electro-magnetic field (Landau and Lifschitz 1951) described by the vector potential $\vec{A}$ whose covarient components are A_i ,

$$L = -m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} + \frac{q}{c} A_i \dot{x}^i \tag{2.2-1}$$

where $v^2 = g_{ij} \dot{x}^i \dot{x}^j$ is the velocity of the particle written in terms of the coordinate derivatives and the metric tensor g_{ij} , and summation over repeated indices is implied. (The summation convention will be used throughout this section.)

In cylindrical coordinates

$$(x^1, x^2, x^3) = (r, \phi, z)$$

$$g_{11} = 1 \qquad g_{22} = r^2 \qquad g_{33} = 0$$

$$g_{ij} = 0 \qquad i \neq j \tag{2.2-2}$$

The derivatives of the Lagrangian

$$\frac{\partial L}{\partial \dot{x}_i} = p_i = \frac{m_o g_{ij} \dot{x}^j}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{q}{c} A_i \tag{2.2-3}$$

are the generalized momenta (covarient components of the momentum vector), and we find the Hamiltonian function $H(x^i, p_i, t)$ by eliminating $\dot{x}_i$ between (2.2-3) and

$$H = p_i x^i - L(x^i, \dot{x}^i) \tag{2.2-4}$$

In fields of the type we shall be considering, the only non-zero component of the vector potential is $A_2 = r A_\phi$, hence,

$$p_1 = p_r = \frac{m_o \dot{r}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{m_o \dot{x}^1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.2-5}$$

$$p_2 = p_\phi = \frac{m_o r^2 \dot{\phi}}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{q}{c} r A_\phi = \frac{m_o g_{22} \dot{x}^2}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{q}{c} A_2 \tag{2.2-6}$$

$$p_3 = p_z = \frac{m_o \dot{z}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{m_o \dot{x}^3}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.2-7}$$

Substituting (2.2-5), (2.2-6) and (2.2-7) into the expression for H given by (2.2-4) we get

$$H = \frac{m_o v^2}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{q}{c} A_2 \dot{x}^2 + m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} - \frac{q}{c} A_2 \dot{x}^2 \tag{2.2-8}$$

or

$$H = \frac{m_o c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.2-9}$$

We note that the value of the Hamiltonian function is just the total energy of the particle. To find the equations of motion we must express H in terms of the momenta p_i rather than the velocity. Squaring the expressions (2.2-5), (2.2-6) and (2.2-7) and adding, we find

$$v^2 = \dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2 = \left[p_r^2 + \frac{1}{r^2}(p_\phi - \frac{q}{c}A_2)^2 + p_z^2\right] \frac{(1 - \frac{v^2}{c^2})}{m_o} \qquad (2.2\text{-}10)$$

Squaring (2.2-4) and eliminating v^2 between (2.2-10) and (2.2-9), we get

$$\frac{H^2}{c^2} = p_r^2 + \frac{1}{r^2}(p_\phi - \frac{q}{c}A_2)^2 + p_z^2 + m_o^2c^4 \qquad (2.2\text{-}11)$$

Writing $A_2 = rA_\phi$ we have finally

$$H = \left[c^2p_r^2 + c^2p_z^2 + c^2\left(\frac{p_\phi}{r} - \frac{q}{c}A_\phi\right)^2 + m_o^2c^4\right]^{1/2} \qquad (2.2\text{-}12)$$

We find the equations of motion from the Hamiltonian function in the usual manner using the general formulas

$$\frac{\partial H}{\partial p_i} = \dot{x}^i \qquad \frac{\partial H}{\partial x^i} = -\dot{p}_i \qquad (2.2\text{-}13)$$

Carrying out the required differentiations we get

$$\frac{\partial H}{\partial r} = -\dot{p}_r = \frac{-c^2\left[\frac{p_\phi}{r} - \frac{q}{c}A_\phi\right]\left[\frac{p_\phi}{r} + \frac{q}{c}r\frac{\partial A_\phi}{\partial r}\right]}{rH} \qquad (2.2\text{-}14)$$

$$\frac{\partial H}{\partial \phi} = -\dot{p}_\phi = 0 \qquad (2.2\text{-}15)$$

$$\frac{\partial H}{\partial z} = -\dot{p}_z = -\frac{c^2}{H}\left[\frac{p_\phi}{r} - \frac{q}{c}A_\phi\right]\frac{q}{c}\frac{\partial A_\phi}{\partial z} \tag{2.2-16}$$

$$\frac{\partial H}{\partial p_r} = \dot{r} = \frac{c^2 p_r}{H} \tag{2.2-17}$$

$$\frac{\partial H}{\partial p_z} = \dot{z} = \frac{c^2 p_z}{H} \tag{2.2-18}$$

$$\frac{\partial H}{\partial p_\phi} = \dot{\phi} = \frac{c^2}{rH}\left[\frac{p_\phi}{r} - \frac{q}{c}A_\phi\right] \tag{2.2-19}$$

It follows immediately from equations (2.2-14) through (2.2-19) and (2.1-5) that

$$\dot{p}_z = -\frac{q}{c} r\dot{\phi}\ B_r \tag{2.2-20}$$

$$\dot{p}_r = m r^2 \dot{\phi} + \frac{q}{c} r\dot{\phi} B_z \tag{2.2-21}$$

$$m r^2 \dot{\phi} + \frac{q}{c} r A_\phi = \text{constant} \tag{2.2-22}$$

2.3 Approximate Constants of the Motion for a Charged Particle Moving in a Slowly Varying Inhomogeneous Magnetic Field

2.31 Definition of the Magnetic Moment of a Particle Moving in an Inhomogeneous Field

For later use in discussing the interaction of cosmic ray particles with inhomogeneities in the magnetic field, it will be convenient to have at hand some theorems regarding constants of the motion. For our purposes here we will define a constant of the motion to be any function of the coordinates and the momenta whose total time derivative vanishes. In a static field one such constant of the motion is the Hamiltonian function itself. We shall discuss others presently, first in the non-relativistic limit, and then relativistic case. As a preliminary step we shall develop some useful results concerning magnetic moments.

Consider a particle moving in a magnetic field which varies with both space and time. With little loss of generality we restrict ourselves to cylindrically symmetric magnetic fields* which may be represented by a one-component vector potential $\vec{A} = A_\phi\,(r, z;\ t)\,\vec{e}_\phi$. We further stipulate that the change in the field during one turn of the spiral motion is small. More precisely we assume that

$$\left|(\rho\,\vec{e}_r \cdot \nabla)\,\vec{B}\right| \ll \left|\vec{B}\right| \tag{2.3-1}$$

---

* This restriction is made for computational convenience. It is not necessary that the particle spiral about the axis of symmetry.

and

$$T \left| \frac{\partial \vec{B}}{\partial t} \right| \ll \left| \vec{B} \right| \tag{2.3-2}$$

where ρ is the radius of curvature in a plane perpendicular to the z-axis and T is the period of the motion (or the approximate period if the motion is not exactly periodic). Stated in terms of the components of $\vec{B}$, (2.3-1) and (2.3-2) are

$$\rho \frac{\partial B_z}{\partial r} \ll B_o \qquad \rho \frac{\partial B_r}{\partial r} \ll B_o \tag{2.3-3}$$

$$T \frac{\partial B_z}{\partial t} \ll B_o \qquad T \frac{\partial B_r}{\partial t} \ll B_o \tag{2.3-4}$$

We define a quantity M by

$$M = \frac{c^2}{2HB_z} \left[p_r^{\,2} + \left(\frac{p_\phi}{r} - \frac{q}{c} A_\phi \right)^2 \right] \tag{2.3-5}$$

where H is the Hamiltonian and B_z is the z-component of the magnetic field given by (2.1-5). In a static, uniform field $\vec{B} = B_o \vec{e}_z$, the quantity M reduces to the magnetic moment of the spiraling particle defined by

$$M = \frac{IA}{c} = -\frac{q}{T} \frac{A}{c} \tag{2.3-6}$$

where $I = -\frac{q}{T}$ is the equivalent current, and A is the area of the circle which is the projection of the motion on a plane perpendicular to the z-axis. Equation (2.3-5) may thus be considered to be a generalized definition of the magnetic moment for particles in inhomogeneous fields.

It will be noted that, by definition, the form of M in terms of coordinates and momenta is invarient to translations of the cylindrical coordinate system parallel to itself.* It will now be shown that its numerical value is independent of such transformations by demonstrating the equivalence of (2.3-5) and (2.3-6), an expression that depends only on the motion and not on the coordinate used in describing it or the vector potential used to describe the magnetic field.

Consider for a moment a particle moving in a uniform, static magnetic field. The projection of the motion on a plane perpendicular to the direction of the field (z-axis) is exactly circular with the radius given by (2.1-1). If we choose the axis of the coordinate system along the axis of the helix then in this system r = const. and $p_r = 0$. We do not wish to restrict ourselves to such a coordinate system, however, and hence we must include the radial momentum term in (2.3-5) which in general will not be zero even though the transverse motion is circular. Reference to the following diagram will make this point clear.

---

* To make M invarient to guage transformations in $\vec{A}$, we must replace the definition (2.3-5) by

$$M = \frac{c^2}{2HB_z}\left[(p_r - \frac{q}{c}A_r)^2 + (\frac{p_\phi}{r} - \frac{q}{c}A_\phi)^2\right]$$

and re-define the Hamiltonian by

$$H = \left[c^2(p_z - \frac{q}{c}A_z)^2 + c^2(p_r^2 - \frac{q}{c}A_r^2)^2 + c^2(\frac{p_\phi}{r} - \frac{q}{c}A_\phi)^2 + m_o^2c^4\right]^{1/2}$$

For the particular coordinate system and vector potential we shall use, $A_r = A_z = 0$.

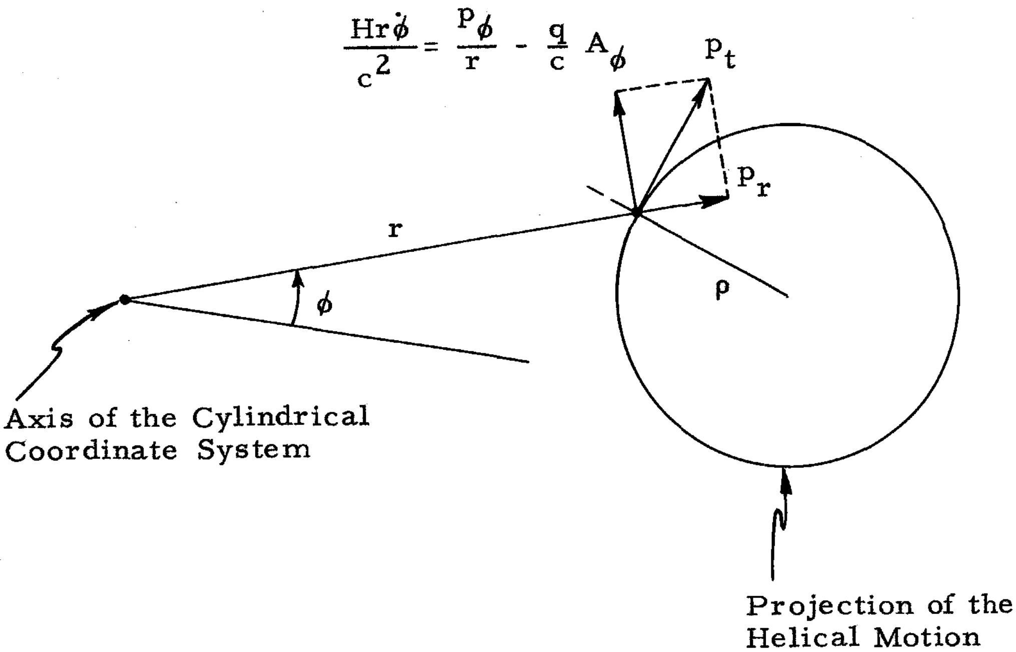

Figure 1. Components of the Momentum Vector and the Coordinates Describing the Motion of a Particle in a Plane Perpendicular to the Magnetic Field.

In coordinate systems with an arbitrary location of the axis as shown in the figure, the quantities r, ϕ, p_r, $\frac{p_\phi}{r} - \frac{q}{c} A_\phi$, etc., are approximately periodic functions of the time if the field is slowly varying.

We now return to the question of the equivalence of (2.3-5) and (2.3-6) for a particle in a uniform, static magnetic field. We note that in such a field H and p_z are constant and hence from (2.3-5) and (2.2-12), M is also constant. We evaluate M at an instant when $p_r = 0$. Then

$$M = \frac{c^2}{2HB_z}\left[\frac{p_\phi}{r} - \frac{q}{c}A_\phi\right]^2 \quad ; \quad p_r = 0 \tag{2.3-7}$$

Using (2.2-19)

$$M = \frac{c^2}{2HB_z}\left[\frac{rH}{c^2}\dot{\phi}\right]^2 = \frac{Hr^2\dot{\phi}^2}{2c^2B_z} \tag{2.3-8}$$

where the quantity $r\dot{\phi}$ is evaluated at the point where $p_r = 0$. Let ρ denote the radius of the helix and ω the angular velocity (cf. Figure 1), then

$$M = \frac{Hr^2\dot{\phi}^2}{2c^2B_z} = \frac{1}{2B_z}\frac{H}{c^2}\rho^2\omega^2 = \frac{m}{2B_z}\rho^2\omega^2 \tag{2.3-9}$$

where m is the relativistic mass. Using the well-known result*

$$\omega = -\frac{qB_z}{mc} \tag{2.3-10}$$

for the angular velocity of a particle in a uniform, static field, (2.3-10) may be written

$$M = -\frac{q}{2c}\rho^2\omega = \frac{-q}{c}\left(\frac{\omega}{2\pi}\right)\pi\rho^2 = -\frac{q}{c}\frac{A}{T} = \frac{IA}{c} \tag{2.3-11}$$

This demonstrates the equivalence of (2.3-5) and (2.3-6).

As a further remark it may be stated that in any slowly varying field the average radial momentum will always be small compared to

---

* This result is easily obtained from (2.2-14) and (2.2-19) by using (2.1-11) and noting that $p_r = 0$ in a coordinate system whose axis is along the axis of the helix.

the total momentum in a coordinate system whose axis is along the axis of the helix. It then follows quite generally from (2.2-14) and (2.2-19) that in such a coordinate system

$$M = -\frac{q}{2c} r^2 \dot{\phi} \qquad p_r^2 \ll p^2 \tag{2.3-12}$$

This result will be important later on.

2.32 The Magnetic Moment in the Non-Relativistic Limit

In the non-relativistic limit, the expression for the magnetic moment given by (2.3-5) becomes

$$M = \frac{p_t^2}{2 m_o B_z} = \frac{W_\perp}{B_z} \tag{2.3-13}$$

where p_t is the component of the momentum transverse to the uniform part of the field and $W_\perp$ is the energy associated with the transverse motion.

By introducing inhomogeneities in the field as perturbations to the motion in a homogeneous, static field, Alfven (1950) has shown that if the slowly varying conditions (2.3-1) and (2.3-2) are satisfied then the non-relativistic magnetic moment defined by (2.3-13) remains constant for perturbations of the following three types:

(1) The field is uniform but varies with time.

(2) The gradient of the field has a component (constant) in the direction of the unperturbed field.

(3) The gradient of the field has a component (constant) perpendicular to the unperturbed field.

Since the radial component of the momentum remains shall (in a coordinate system whose axis is along the axis of the spiral motion) as the particle passes through the inhomogeneity, the transverse motion remains very nearly circular. In the first two cases the radius of the circle changes slowly as the particle moves through the inhomogeneity. In the third case the center of the circle drifts perpendicular to the field following the curvature of the field lines. In the non-relativistic limit the flux through the circle is proportional to M; hence in cases (1) and (2), the particle moves on the surface of a flux tube. We shall refer to this result as the flux theorem.

Alfven's perturbation approach cannot be readily generalized to the relativistic case.* In the following section we shall investigate this case using the Hamiltonian formalism. This approach allows us to find relativistically correct equations of motion for any quantity expressable to terms of the coordinates and momenta, and also allows us to treat the space and time variations of the field simultaneously. Our approach thus treats the case of a static magnetic field as a special case. It will turn out that Alfven's results

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* In Alfven's proofs the kinetic energy of a particle is split into two parts. One part is associated with the component of the velocity in the direction of the unperturbed magnetic field and the other part is associated with the component of the velocity transverse to the magnetic field. This splitting cannot be accomplished in the relativistic case.

It has come to the author's attention that Helwig (1955) has treated the relativistic and non-relativistic cases in some detail, using an entirely different approach from that given here. Helwig shows that the "magnetic moment" is a constant of the motion for relativistic or non-relativistic particles moving in slowly varying fields. However it appears that Helwig's relativistic expression for the magnetic moment does not reduce to $M=IA/c$ for a particle moving in a uniform field, but rather reduces to $M=(m/m_o)(IA/c)$ where m is the relativistic mass and m_o is the rest mass. Helwig's "magnetic moment" is thus proportional to the quantity L_B defined in (2.3-30).

regarding magnetic moments do not hold for relativistic particles moving in time varying fields, although the flux theorem still holds. It will turn out that the appropriate constant of the motion in this case is the magnetic moment multiplied by the total energy.

2.33 The Time-Rate of Change of the Magnetic Moment

In this section we wish to study how the magnetic moment of a particle, moving through an inhomogeneous magnetic field, varies with time. We start with the relativistic Hamiltonian (2.2-12) and the magnetic moment (2.3-5), expressed in terms of the coordinates and momenta. This permits us to calculate the total time rate of change of M (Goldstein 1950) from

$$\frac{dM}{dt} = \sum_i \left[\frac{\partial M}{\partial x^i}\frac{\partial H}{\partial p_i} - \frac{\partial M}{\partial p_i}\frac{\partial H}{\partial x^i}\right] + \frac{\partial M}{\partial t} \tag{2.3-14}$$

or in the usual Poisson bracket notation

$$\frac{dM}{dt} = \left[M, H\right] + \frac{\partial M}{\partial t} \tag{2.3-15}$$

Since M is independent of ϕ, its total time derivative is given by

$$\frac{dM}{dt} = \frac{\partial M}{\partial r}\frac{\partial H}{\partial p_r} - \frac{\partial M}{\partial p_r}\frac{\partial H}{\partial r} + \frac{\partial M}{\partial z}\frac{\partial H}{\partial p_z} - \frac{\partial M}{\partial p_z}\frac{\partial H}{\partial z} + \frac{\partial M}{\partial t} \tag{2.3-16}$$

A somewhat lengthy but straightforward calculation gives

$$\frac{dM}{dt} = \frac{M}{B_z}\,\dot{r}\,\frac{\partial B_z}{\partial r} - \frac{\dot{z}}{B_z}\left[M\frac{\partial B_z}{\partial z} + \left(\frac{q r^2 \dot{\phi}}{2c}\right)\frac{2}{r}\frac{\partial A_\phi}{\partial z}\right] - \frac{1}{B_z}\left[M\frac{\partial B_z}{\partial t} + \left(\frac{q r^2 \dot{\phi}}{2c}\right)\frac{2}{r}\frac{\partial A_\phi}{\partial t}\right] - \frac{M}{H}\frac{\partial H}{\partial t} \tag{2.2-17}$$

where $\dot{r}$, $\dot{z}$, and $\dot{\phi}$ are defined by (2.2-17), (2.2-18) and (2.2-19) respectively.

To simplify this expression we use (2.3-12) and average over one cycle of the motion, obtaining

$$\left\langle \frac{dM}{dt} \right\rangle = \frac{M}{B_z} \left\langle \dot{r} \right\rangle \frac{\partial B_z}{\partial r} - \frac{\dot{z}}{B_z} \left[M \frac{\partial B_z}{\partial z} - M \frac{2}{r} \frac{\partial A_\phi}{\partial z} \right]$$

$$- \frac{1}{B_z} \left[M \frac{\partial B_z}{\partial t} - M \frac{2}{r} \frac{\partial A_\phi}{\partial t} \right] - \frac{M}{H} \frac{\partial H}{\partial t} \qquad (2.3\text{-}18)$$

This expression can be further re-written

$$\frac{dM}{dt} = \frac{M}{B_z} \left\langle \dot{r} \right\rangle \frac{\partial B_z}{\partial r} - \frac{M}{B_z} \frac{d}{dt} \left[B_z - \frac{2}{r} A_\phi \right] - \frac{M}{H} \frac{\partial H}{\partial t} \qquad (2.3\text{-}19)$$

where the total time derivative in the second term on the right is defined by

$$\frac{d}{dt} = \dot{z} \frac{\partial}{\partial z} + \frac{\partial}{\partial t} \qquad (2.3\text{-}20)$$

Several comments may now be made regarding (2.3-19). Reference to the following table shows that the first two terms on the right vanish if either (a) the field is uniform, but changing with time, or (b) if the z-component of $\vec{B}$ is a function only of z and t over the region of interest. In either case the vector potential is related to the z-component of the field by

$$A_\phi = \frac{r}{2} B_z \qquad (2.3\text{-}21)$$

Table I

Relationship Between the Vector Potential and the Components of $\vec{B}$ for Three Types of Fields

Type of Field	Vector Potential	B_z	B_r
Uniform field changing with time	$A_\phi = \frac{r}{2} B_z(t)$	$B_z(t)$	0
Field has a gradient only in z-direction	$A_\phi = \frac{r}{2} B_z(z, t)$	$B_z(z, t)$	$B_r(z, t) = -\frac{r}{2} \frac{\partial B_z}{\partial z}$
General cylindrically symmetric field	$A_\phi (r, z, t)$	$B_z = \frac{1}{r} \frac{\partial}{\partial r}(r A_\phi)$	$B_r = -\frac{\partial A_\phi}{\partial z}$

Turning to the more general case where A_ϕ depends on r as well as z and t , we note that the first term on the right on (2.3-19) may be written approximately

$$\frac{M}{B_z} \left\langle \dot{r} \right\rangle \frac{\partial B_z}{\partial r} = \frac{M}{T} \left(\frac{\Delta\rho}{\rho}\right) \left(\frac{\rho}{B_z} \frac{\partial B_z}{\partial r}\right) \tag{2.3-22}$$

where $\Delta\rho$ is the distance the axis of the spiral moves during one period T of the motion and ρ is the radius of the spiral. In slowly varying fields both $\frac{\Delta\rho}{\rho}$ and $\frac{\rho}{B_z} \frac{\partial B_z}{\partial r}$ are small compared to 1 , and hence this term is negligible.

Passing to the second term on the right of (2.3-19), we note that if

$$B_z(r, z; t) \cong \frac{2}{r} A_\phi (r, z, t) \tag{2.3-23}$$

then this term is also negligible. As mentioned previously, the relationship is exact for fields of the first two types listed in Table I.

The necessary and sufficient condition is that

$$\frac{\partial B_z}{\partial r} = 0 \qquad (2.3\text{-}24)$$

It can easily be shown that the difference between B_z and $\frac{2}{r} A_\phi$ over a region $r_o < r < r_o + \rho$ satisfies the inequality

$$\frac{\left| B_z - \frac{2}{r} A_\phi \right|}{B_o} \leq \frac{1}{3} \frac{\rho}{B_o} \left| \frac{\partial B_z}{\partial r} \right|_{max} \qquad (2.3\text{-}25)$$

where $\left| \frac{\partial B_z}{\partial r} \right|_{max}$ is the maximum value of $\frac{\partial B_z}{\partial r}$ over the region. Applying the slowly varying condition (2.3-3) we see that

$$\frac{\left| B_z - \frac{2}{r} A_\phi \right|}{B_o} \ll \frac{1}{3} \qquad (2.3\text{-}26)$$

We are thus justified in neglecting the second term in (2.3-19).

2.34 An Approximate Constant of the Motion for a Relativistic Particle in a Slowly Varying Field; The Flux Theorem.

After dropping the first two terms of (2.3-19) in accordance with the discussion of the last paragraph, we are left with

$$\frac{dM}{dt} = -\frac{M}{H}\frac{dH}{dt} \tag{2.3-27}$$

This equation may be rewritten

$$\frac{d}{dt}(MH) = 0 \tag{2.3-28}$$

Hence

$$MH = \text{const.} \tag{2.3-29}$$

The quantity MH is therefore a constant of the motion for slow changes in the field.*

This theorem** can be restated in an alternative form which has more physical meaning: Let L_B denote the angular momentum about the axis of the helix. It is easily shown that

$$L_B = \frac{2\,MH}{qc} \tag{2.3-30}$$

and hence $L_B = \text{const.}$

---

* In the non-relativistic limit the magnitude of the Hamiltonian reduces to $m_o c^2$. Hence M = const., in agreement with Alfven's results.

** An alternative proof of this theorem is given in Appendix C.

An alternative form of this result is

$$L_B = \frac{(H^2 - m_o^2 c^4) \sin^2 \theta}{qc B_z} = \text{const.} \qquad (2.3\text{-}31)$$

where θ is the angle between the momentum vector and the z-axis (the spiral pitch angle). This expression follows immediately from (2.3-5) and (2.2-12).

If we compute the flux Φ through the helix using (2.1-1) we find

$$\Phi = \pi \rho^2 B_z = \pi \left[\frac{c p_t^2}{q B_z}\right]^2 B_z = \frac{\pi c^2 p_t^2}{q^2 B_z} \qquad (2.3\text{-}32)$$

This expression may be written in terms of the magnetic moment by using the definition (2.3-5). The result is

$$\Phi = \frac{2\pi}{q^2} MH = \frac{\pi c}{q} L_B \qquad (2.3\text{-}33)$$

Thus, even in the relativistic case, the flux enclosed by the helix remains constant as the particle moves through the field. Stated another way, the particle moves at all times on the surface of a flux tube, provided the particle does not diffuse at right angles to the field due to a lateral gradient in the field.

2.4 The Time Rate of Change of the Logarithmic Energy Parameter and the Spiral Pitch Angle.

In studying the interaction of particles with hydromágnetic waves, two quantities will be of particular interest, the logarithmic energy parameter defined by

$$\epsilon = \ln \left[\frac{H}{m_o c^2}\right] \qquad (2.4\text{-}1)$$

and the spiral pitch angle, defined as the angle between the momentum vector $\vec{p}$ and the z-axis.

To find the time rate of change of ϵ we start with the time rate of change of the Hamiltonian given by

$$\frac{dH}{dt} = \left[H, H\right] + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \qquad (2.4\text{-}2)$$

$$= \frac{c^2}{H}\left(\frac{p_\phi}{r} - \frac{q}{c} A_\phi\right)\left(-\frac{q}{c}\frac{\partial A_\phi}{\partial t}\right)$$

Using (2.2-19) this may be written

$$\frac{dH}{dt} = \left(-\frac{q}{2c} r^2 \dot{\phi}\right) \frac{2}{r} \frac{\partial A_\phi}{\partial t} \qquad (2.4\text{-}3)$$

The first factor on the right will be recognized as the magnetic moment [cf. (2.3-12)]. Therefore, when $\dot{r} = 0$,

$$\frac{dH}{dt} = M \frac{2}{r}\frac{\partial A_\phi}{\partial t} = \frac{c^2}{2H}\frac{p_t^2}{B_z}\frac{\partial}{\partial t}\left(\frac{2A_\phi}{r}\right) \qquad (2.4\text{-}4)$$

where (2.3-5) has been used. Dividing both sides of (2.4-4) by H we may write

$$\frac{1}{H}\frac{dH}{dt} = \frac{1}{(H/m_o c^2)}\frac{d}{dt}\frac{H}{m_o c^2} = \frac{c^2 p_t^2}{2H^2 B_z}\frac{\partial}{\partial t}\frac{2A_\phi}{r} \tag{2.4-5}$$

We restrict ourselves to the extreme relativistic case $(p^2c^2 \gg m_o^2c^4)$ writing

$$H^2 = p^2c^2 \tag{2.4-6}$$

Hence

$$\frac{d\epsilon}{dt} = \frac{1}{2B_z}\frac{p_t^2}{p^2}\frac{\partial}{\partial t}\frac{2A_\phi}{r} = \frac{(1-\mu^2)}{2B_z}\frac{\partial}{\partial t}\frac{2A_\phi}{r} \tag{2.4-7}$$

where $\mu = \cos\theta$ is the cosine of the spiral pitch angle. If the field is sufficiently slowly varying we may replace B_z by its average value B_o and $\frac{2A_\phi}{r}$ by B_z in accordance with the discussion of Section 2.2.

The equation of motion for ϵ then becomes

$$B_o \frac{d\epsilon}{dt} = \frac{1}{2}(1-\mu^2)\frac{\partial B_z}{\partial t} \tag{2.4-8}$$

We are now in a position to find the equation of motion for the spiral pitch angle. From the results of Section 2.3, the quantity

$$L_B = \frac{2MH}{qc} = \frac{(H^2 - m_o^2c^4)\sin^2\theta}{qcB_z} \tag{2.4-9}$$

is a constant of the motion for slow changes in the field. In the extreme relativistic limit we neglect $m_o^2c^4$ in (2.4-9) and write

$$L_B = \frac{H^2(1-\mu^2)}{qcB_z} \tag{2.4-10}$$

Taking the total time derivative* of (2.4-10) we get

$$\frac{1}{L_B}\ \frac{dL_B}{dt} = -\ \frac{1}{B_z}\ \frac{dB_z}{dt} + \frac{2}{H}\ \frac{dH}{dt} - \frac{1}{1-\mu^2}\ \frac{d\mu^2}{dt} = 0 \qquad (2.4\text{-}11)$$

Introducing $\epsilon = \ln\,(H/m_o c^2)$ we get

$$\frac{1}{1-\mu^2}\ \frac{d\mu^2}{dt} = 2\ \frac{d\epsilon}{dt} - \frac{1}{B_z}\ \frac{dB_z}{dt} \qquad (2.4\text{-}12)$$

Eliminating $\frac{d\epsilon}{dt}$ using (2.4-8) and replacing the factor $\frac{1}{B_z}$ by $\frac{1}{B_o}$ we have**

$$\frac{1}{1-\mu^2}\ \frac{d\mu^2}{dt} = \frac{1}{B_o}\ (1-\mu^2)\ \frac{\partial B_z}{\partial t} - \frac{1}{B_o}\ \frac{dB_z}{dt} \qquad (2.4\text{-}13)$$

or

$$B_o\ \frac{d\mu^2}{dt} = (1-\mu^2)^2\ \frac{\partial B_z}{\partial t} - (1-\mu^2)\ \frac{dB_z}{dt} \qquad (2.4\text{-}14)$$

---

* The total time derivative has two parts:

$\frac{d}{dt} = v \cdot \nabla + \frac{\partial}{\partial t}$ The total time derivative of B_z is thus

$$\frac{dB_z}{dt} = \dot{z}\ \frac{\partial B_z}{\partial z} + \frac{\partial B_z}{\partial t}$$ neglecting terms involving $\dot{r}$.

** This result may also be obtained using the Hamiltonian formalism but with considerably more labor. This requires calculating $\frac{d\mu^2}{dt}$ from

$$\frac{d\mu^2}{dt} = \left[\mu^2,\ H\right] + \frac{\partial \mu^2}{\partial t}$$

using

$$\mu^2 = \frac{p_z^{\ 2}}{p^{\ 2}} = \frac{p_z^{\ 2}}{p_r^{\ 2} + p_z^{\ 2} + \left[\frac{p_\phi}{r} - \frac{q}{c}\ A_\phi\right]^2}$$

3. SCATTERING AND ACCELERATION OF COSMIC RAYS BY TURBULENT MAGNETIC FIELDS

3.1 General

A qualitative description of the magnetic field has been given in Section 2.1. As noted there, we consider the magnetic field to be made up of a uniform field $B = B_o \vec{e}_z$ on which is superposed a random part consisting of hydromagnetic waves having a variety of amplitudes and wavelengths. These waves are propagated along the direction of the undisturbed field with velocities that depend on their wavelength, the density of the medium, and the magnetic field strength.

The parameters to be used to describe the state of motion of a particle in this field are the logarithmic energy parameter ϵ, the cosine of the spiral pitch angle μ, and the position coordinate z, measured along the direction of the unperturbed magnetic field. Consider a large number of cosmic ray particles moving through such a magnetic field, and let $W(\mu, \epsilon, z; t)\, d\mu d\epsilon dz$ denote the probability of finding a particle at μ, ϵ, z in the range $d\mu, d\epsilon, dz$, at time t. *We shall show that $W(\mu, \epsilon, z; t)$ satisfies a second order partial* differential equation of the parabolic type which we will refer to as the "diffusion equation." In formulating this differential equation, we regard the following physical processes as fundamentally important:

(a) Betatron Processes. These are interactions with inhomogeneities in the magnetic field (hydromagnetic waves), whose scale is large compared to the helix of the particle. Interactions of this kind change the energy of a particle and its direction of motion through the action of the induced electric field which is present when the magnetic field strength changes with time. These changes take place in a way that depends on the energy of a particle and its direction of motion, and are described by the pair of differential equations (2.4-8) and (2.4-14).

(b) Scattering by small scale inhomogeneities in the field. This type of interaction tends to make the angular distribution isotropic, smoothing out the anisotropy produced by the betatron collisions. It is assumed that this type of scattering does not change the energy appreciably. Calculations are carried out on the assumption that the root-mean-square scattering angle is small and does not depend on the direction of motion.

(c) Absorption of particles by nuclear collisions, and loss of particles by diffusion out of the spiral arm.

(d) Injection of new particles.

Each of these processes will be treated separately in the following sections. As we shall see presently, these processes are represented by separate terms or groups of terms in the diffusion equation. In Section 3.21, and Appendix A, the diffusion equation representing the interaction with large scale inhomogeneities (betatron processes), is derived. Certain coefficients appear in this equation which are related to the mean and mean-square changes in the variables μ, ϵ, and z in a time increment Δt due to the action of the magnetic field. These quantities are appropriately called "diffusion coefficients," and are in general functions of μ, ϵ, and z. Evaluation of these coefficients requires a rather lengthy analysis of the behavior of an ensemble of particles in the fluctuating magnetic field. This analysis is carried out in Section 3.24 using the mathematical model of the magnetic field described in Section 3.22. We shall find it convenient to express the diffusion coefficients in terms of the mean-square turbulent velocity of the medium through which the hydromagnetic waves are propagated. Section 3.23 is devoted to the calculation of this velocity from the model of Section 3.22.

In Section 3.3, scattering by small scale inhomogeneities is considered. Absorption and source terms are considered in Section 3.4.

3.2 Interaction of Cosmic Rays with Large Scale Inhomogeneities in the Magnetic Field

3.21 Description of the Interactions by a Diffusion Equation

It is shown in Appendix A that the behavior of an ensemble of particles whose individual states of motion are described by the variables μ, ϵ, and z, is governed by a partial differential equation of the form*

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial z}\left[D_z W\right] - \frac{\partial}{\partial \mu}\left[D_\mu W\right] - \frac{\partial}{\partial \epsilon}\left[D_\epsilon W\right]$$

$$+ \frac{1}{2}\frac{\partial^2}{\partial z^2}\left[D_{zz} W\right] + \frac{1}{2}\frac{\partial^2}{\partial \epsilon^2}\left[D_{\epsilon\epsilon} W\right] + \frac{1}{2}\frac{\partial^2}{\partial \mu^2}\left[D_{\mu\mu} W\right] \qquad (3.2\text{-}1)$$

$$+ \frac{\partial^2}{\partial \epsilon \partial \mu}\left[D_{\mu\epsilon} W\right] + \frac{\partial^2}{\partial \epsilon \partial z}\left[D_{\epsilon z} W\right] + \frac{\partial^2}{\partial \mu \partial z}\left[D_{\mu z} W\right]$$

where $W(\mu, \epsilon, z; t)$ was defined above as the fraction of the total number of particles in the range $d\mu\, d\epsilon\, dz$. Let x_i, $i = 1, 2, 3$ denote the variables μ, ϵ, z. The diffusion coefficients D_z, D_u, etc., are defined by

$$D_i = \lim_{\Delta t \to o} \frac{\langle \Delta x_i \rangle}{\Delta t} \qquad (3.2\text{-}2)$$

$$D_{ij} = \lim_{\Delta t \to o} \frac{\langle \Delta x_i \Delta x_j \rangle}{\Delta t} \qquad (3.2\text{-}3)$$

---

*The form of this equation is similar to the Fokker-Planck equation in velocity space used in the theory of Brownian motion (Chandrasekhar, 1943). The derivation follows similar lines.

where $\langle \Delta x_i \rangle$ is the average change in x_i in a time Δt and $\langle \Delta x_i \Delta x_j \rangle$ is the average of the product of the changes in x_i and x_j in a time Δt. These diffusion coefficients are, in general, functions of the three independent variables μ, ϵ, and z.

The right side of (3.2-1) represents the net rate at which particles enter the differential range $d\mu d\epsilon dz$ due to interactions with the magnetic field. The first derivative terms on the right describe the average motion or "drift" of an ensemble of particles in the space of the coordinates μ, ϵ, z, and the second derivative terms describe the fluctuations (diffusion) about this average motion.

To apply this equation to the problem of the interaction of particles with large scale inhomogeneities in the magnetic field it is now necessary to evaluate the functions D_μ, D_ϵ, etc. The mathematical description of the magnetic field on which this evaluation is based is considered next.

3.22 A Statistical Model of the Fluctuating Magnetic Field

The purpose of this section is to describe a mathematical model of the fluctuating magnetic field. This model will be applied in Section 3.24 to the problem of evaluating the diffusion coefficients D_{μ}, D_{ϵ}, $D_{\mu\mu}$, etc., appearing in (3.2-1).

It will turn out that the random nature of the fluctuating part of the field is of paramount importance. Since the field is undoubtedly rather chaotic, it is impossible to predict exactly its behavior as a function of time and the space coordinates r and z. A statistical description of the magnetic field is therefore indicated.

We shall focus our attention primarily on the quantity $\frac{\partial B_z}{\partial t}$ which appears in the equations of motion for μ and ϵ, and make use of the Fourier integral to express $\frac{\partial B_z}{\partial t}$ as a superposition of waves of different wave lengths. The spectrum of wave lengths (in the z-direction) assumes a special significance in the model, for if we adopt the assumption that the field is represented adequately by a train of hydromagnetic waves of unchanging shape or a superposition of a finite number of monochromatic waves, then we are quickly lead to the conclusion that the field does not produce any scattering or change the energy of a particle.

To form a basis for the present discussion, we consider first some of the properties of the simplest types of cylindrically symmetric, undamped, hydromagnetic waves. Wave motion of this type is discussed in Appendix B, starting from Maxwell's equations and the hydrodynamic equation of motion for the medium. This treatment is based on the usual linearized theory with the following assumptions:

(1) Displacement current is neglected.

(2) Infinite conductivity of the medium in which the oscillations take place is assumed.

(3) Nonelectromagnetic forces on the medium are neglected.

Some of the results of Appendix B are tabulated here for future reference. The following quantities of interest are associated with a wave propagated in the $+z$ direction. Cylindrical coordinates (r, ϕ, z) are used and all quantities are given in Gaussian units.

Vector Potential:

$$\vec{A} = -B_o\left[AJ_1(k_2r)e^{i(k_1z-\omega t)} + \frac{r}{2}\right]\vec{e}_\phi \tag{3.2-4}$$

Magnetic Field:

$$\begin{aligned}\vec{B} = & -B_oAik_1J_1(k_2r)e^{i(k_1z-\omega t)}\vec{e}_r \\ & +B_o\left[1 + Ak_2J_o(k_2r)e^{i(k_1z-\omega t)}\right]\vec{e}_z\end{aligned} \tag{3.2-5}$$

Current Density:

$$\vec{i} = \frac{Ac\omega^2}{B_o} J_1(k_2r)e^{i(k_1z-\omega t)}\vec{e}_\phi \tag{3.2-6}$$

Velocity of Medium:

$$\vec{v} = i\omega AJ_1(k_2r)e^{i(k_1z-\omega t)}\vec{e}_r \tag{3.2-7}$$

Displacement of Medium:

$$\vec{s} = A J_1(k_2 r) e^{i(k_1 z - \omega t)} \vec{e}_r \qquad (3.2\text{-}8)$$

Electric Field:

$$\vec{E} = \frac{-i\omega A B_o}{c} J_1(k_2 r) e^{i(k_1 z - \omega t)} \vec{e}_\phi \qquad (3.2\text{-}9)$$

Angular Frequency:

$$\omega^2 = \frac{B_o^2}{4\pi \rho_o} (k_1^2 + k_2^2) \qquad (3.2\text{-}10)$$

Phase Velocity of Waves:

$$V^2 = \frac{\omega^2}{k_1^2} = \frac{B_o^2}{4\pi \rho_o} \left[1 + \frac{k_2^2}{k_1^2} \right] \qquad (3.2\text{-}11)$$

In these equations, B_o is the strength of the uniform part of the field, ρ_o is the average mass density of the medium, c is the velocity of light, and A, k_1, k_2 are constants whose physical meaning is obvious.

It is evident from these results that waves of arbitrary wave lengths in the r and z-directions are possible. These waves are propagated with velocities which depend on the ratio of the wave lengths in the two directions*.

It is reasonable to assume that the fluctuating magnetic field in the spiral arm may be described as a superposition of waves of the simple form described above. To simplify the model, we fix the scale

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*This is not the case in rectangular coordinates if the velocity of the medium has only a single component, say $\vec{v} = v_y \vec{e}_y$. Waves of this type are propagated along the primary field with a velocity given by

$$V^2 = \frac{B_o^2}{4\pi \rho_o} \quad \text{(Alfven 1950)} .$$

of the waves in the r-direction* (i.e., we take k_2 to be a fixed constant) and fix our attention on the behavior of $B_z(r, z, t)$ as a function of z. We further assume that $B_z(r, z, t)$ is a stationary random function of z (i.e., the statistical properties of B_z are independent of z). It is well known that such a function does not satisfy the conditions for representation by a Fourier integral (in particular, it cannot be assumed to vanish at $z = \pm\infty$). We therefore introduce a commonly used mathematical artifice which consists of modifying the function so that it is zero outside of the interval $-z_o \leq z \leq z_o$, where z_o is some large but finite value of z, and developing the modified function as a Fourier integral. At a later stage in the analysis, we let $z_o \to \infty$.

We consider waves moving in both $\pm z$ directions and write,

$$B_z(r, z, t) - B_o \tag{3.2-12}$$

$$= B_o k_2 J_o(k_2 r) \int_{-\infty}^{\infty} \left[A_1(k_1) e^{i(k_1 z - \omega t)} + A_1^*(k_1) e^{-i(k_1 z - \omega t)} \right] dk_1$$

where $A_1(k_1)$ and $A_1^*(k_1)$ are the amplitudes for the two waves moving to the right and left along the z-axis. Differentiating with respect to the time we find

$$\frac{\partial B_z(r, z, t)}{\partial t} \tag{3.2-13}$$

$$= B_o k_2 J_o(k_2 r) \int_{-\infty}^{\infty} \left[-i\omega A_1(k_1) e^{i(k_1 z - \omega t)} + i\omega A_1^*(k_1) e^{-i(k_1 z - \omega t)} \right] dk_1$$

---

* We will later average quantities of interest in such a way as to eliminate the radial dependence.

Consider now a particle moving through the field described by (3.2-12). The component of the particle's velocity in the z-direction is $\dot{z} = \mu c$. If the particle is at $z = z_1$ at $t = 0$ then its position at a time t later is given by

$$z = \mu ct + z_1 \tag{3.2-14}$$

as long as μ does not change appreciably. Inserting this approximate expression* into (3.2-13) we have

$$\frac{\partial B_z}{\partial t} \tag{3.2-15}$$

$$= B_o k_2 J_o(k_2 r) \int_{-\infty}^{\infty} \left[-i\omega A_1(k_1) e^{i(k_1 \mu c - \omega)t} + i\omega A_1^*(k_1) e^{-i(k_1 \mu c - \omega)t} \right] dk_1$$

where the factors $e^{\pm i k_1 z_1}$ have been absorbed into $A_1(k_1)$ and $A_1^*(k_1)$. Now as long as

$$|\mu c| \gg \left| \frac{\omega}{k_1} \right| \tag{3.2-16}$$

for all significant values** of k_1 , we may neglect ω in comparison with $k_1 \mu c$ in (3.2-15). This means that we may neglect the explicit time dependence in (3.2-13), writing

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* Small changes in μ , which are correlated with $\frac{\partial B_z}{\partial t}$ lead to second order terms in the diffusion coefficients which cannot be neglected. Equation (3.2-14) will be replaced by a more exact expression in a later section.

** Typical values for the wave velocity $\frac{\omega}{k_1}$ are of the order of 10 km/sec. Except for particles moving in extremely flat spirals ($\mu < 3 \times 10^{-5}$), this assumption is an exceedingly good one.

$$\frac{\partial B_z}{\partial t} = B_o k_2 J_o(k_2 r) \int_{-\infty}^{+\infty} \left[-i\omega A_1(k_1) e^{ik_1 z} + i\omega A_1^*(k_1) e^{-ik_1 z} \right] dk_1 \quad (3.2\text{-}17)$$

A simpler expression may be obtained by rewriting the second term of (3.2-17) as follows: Replace k_1 by $-k_1'$ noting that ω is an even function of k_1,

$$\int_{-\infty}^{+\infty} i\,\omega A_1^*(k_1) e^{-ik_1 z} \, dk_1 = -\int_{+\infty}^{-\infty} i\,\omega A_1^*(-k_1') \, e^{ik_1' z} \, dk_1' \quad (3.2\text{-}18)$$

Now replace the dummy variable k_1' by k_1, and reverse the limits of integration. This gives

$$\int_{-\infty}^{+\infty} i\omega A_1^*(k_1) e^{-ik_1 z} \, dk_1 = \int_{-\infty}^{+\infty} i\omega A_1^*(-k_1) e^{ik_1 z} \, dk_1 \quad (3.2\text{-}19)$$

Equation (3.2-17) now becomes

$$\frac{\partial B_z}{\partial t} = B_o k_2 J_o(k_2 r) \int_{-\infty}^{+\infty} \left[-i\omega \, A_1(k_1) + i\omega \, A_1^*(-k_1) \right] e^{ik_1 z} dk_1 \quad (3.2\text{-}20)$$

If we now define

$$A(k_1) = -A_1(k_1) + A_1^*(-k_1) \quad (3.2\text{-}21)$$

we obtain the simple expression

$$\frac{\partial B_z}{\partial t} = B_o k_2 J_o(k_2 r) \int_{-\infty}^{+\infty} \omega(k_1) \, A(k_1) e^{ik_1 z} \, dk_1 \quad (3.2\text{-}22)$$

We will henceforth denote $\frac{\partial B_z}{\partial t}$ by $\dot{B}(z)$, suppressing the r dependence which is not important.

Since $\dot{B}(z)$ is a stationary random function of z , it has an autocorrelation function $\mathcal{R}(\overline{z})$. This function* has great utility since it will turn out that all of the quantities of interest which we may have occasion to calculate from our model of the field are easily obtained from $\mathcal{R}(\overline{z})$. The next few paragraphs will be concerned with establishing the connection between this function and the Fourier integral representation of the field given in (3.2-22). As we shall see presently, the Fourier transform of $\mathcal{R}(\overline{z})$ is closely related to the amplitude function $A(k_1)$ appearing in this equation.

The autocorrelation function $\mathcal{R}(\overline{z})$ is defined by the ensemble average,

$$\mathcal{R}(\overline{z}) = \widetilde{\dot{B}_1 \dot{B}_2} = \widetilde{\dot{B}(z)\, \dot{B}(z + \overline{z})}$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \dot{B}_1 \dot{B}_2 \, W(\dot{B}_1, \dot{B}_2; \overline{z})\, d\dot{B}_1 \, d\dot{B}_2 \tag{3.2-23}$$

where $W(\dot{B}_1, \dot{B}_2; \overline{z})d\dot{B}_1 d\dot{B}_2$ is the probability of finding a value of $\dot{B}_1$ in the interval $\dot{B}_1 \rightarrow \dot{B}_1 + d\dot{B}_1$ <u>and</u> a value of $\dot{B}_2$ in the interval $\dot{B}_2 \rightarrow \dot{B}_2 + d\dot{B}_2$, when the two intervals are separated by a distance $\overline{z}$. Since $\dot{B}(z)$ is a stationary random variable, the autocorrelation function is not a function of z . An equivalent** definition may be written in terms of an average over z :

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* For a detailed description of the use of autocorrelation functions and the associated spectral densities in the description of random processes, the reader is referred to Laning and Battin (1956).

** The equivalence of the two definitions holds only for stationary random variables. If the random variable under consideration is non-stationary then the definition in terms of an ensemble average must be used.

$$\mathcal{R}(\bar{z}) = \lim_{z_o \to \infty} \frac{1}{2z_o} \int_{-z_o}^{+z_o} \dot{B}(z)\, \dot{B}(z + \bar{z})\, dz \qquad (3.2\text{-}24)$$

The function $\mathcal{R}(z)$ thus gives a measure of the interrelation of $\dot{B}(z)$ measured at two different points separated by a distance $\bar{z}$. We expect that as $\bar{z}$ is increased indefinitely the correlation disappears. Hence, as $\bar{z} \to \infty$, $W(\dot{B}_1, \dot{B}_2; \bar{z})$ becomes independent of $\bar{z}$ and

$$W(\dot{B}_1, \dot{B}_2; \bar{z}) \longrightarrow W_1(\dot{B}_1) W_1(\dot{B}_2) \qquad (3.2\text{-}25)$$

where $W_1(\dot{B})d\dot{B}$ is the probability of finding $\dot{B}$ in the range $\dot{B}$ to $\dot{B} + d\dot{B}$. Thus, from the definition (3.2-23)

$$\mathcal{R}(\infty) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \dot{B}_1 \dot{B}_2 W_1(\dot{B}_1) W_2(\dot{B}_2) d\dot{B}_1 d\dot{B}_2 = (\tilde{\dot{B}})^2 \qquad (3.2\text{-}26)$$

i.e., $\mathcal{R}(\infty)$ is the square of the mean value of $\dot{B}$. We assume that it is equally likely that $\dot{B}$ be positive or negative, and hence

$$\tilde{\dot{B}} = 0 \; ; \quad \mathcal{R}(\infty) = 0 \qquad (3.2\text{-}27)$$

It follows immediately from (3.2-24) that the value of $\mathcal{R}(\bar{z})$ at the origin is the variance of $\dot{B}$:

$$\mathcal{R}(o) = \widetilde{(\dot{B})^2} = \sigma^2_{\dot{B}} \qquad (3.2\text{-}28)$$

To compute the autocorrelation function corresponding to the representation of $\dot{B}$ given by (3.2-22), we recall that the representation

$$\dot{B}(z) = B_o k_2 J_o(k_2 r) \int_{-\infty}^{+\infty} \omega(k_1) A(k_1) e^{ik_1 z} dk_1 \qquad (3.2\text{-}29)$$

is valid only if $\dot{B} = 0$ outside the interval $-z_o \leq z \leq +z_o$. By the Fourier inversion formula

$$B_o k_2 J_o(k_2 r)\omega(k_1)A(k_1) = \frac{1}{2\pi} \int_{-z_o}^{+z_o} \dot{B}(z) e^{-ik_1 z} dz \qquad (3.2\text{-}30)$$

Let $A^*(k_1)$ denote the complex conjugate of $A(k_1)$, then since $\dot{B}(z)$ is real, (3.2-30) indicates that

$$A^*(k_1) = A(-k) \qquad (3.2\text{-}31)$$

From 3.2-24) and (3.2-29)

$$\mathcal{R}(z) = \widetilde{\dot{B}(z)\dot{B}(z + \bar{z})} = \lim_{z_o \to \infty} \frac{1}{2z_o} \int_{-z_o}^{+z_o} \dot{B}(z)\dot{B}(z + z)\, dz$$

$$= B_o^2 k_2^2 J_o^2 (k_2 r) \lim_{z_o \to \infty} \frac{1}{2z_o} \int_{-z_o}^{+z_o} dz \int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dk_1'' \quad X \qquad (3.2\text{-}32)$$

$$\left[\omega(k_1)\omega(k_1'')A(k_1)A(k_1'')\, e^{i(k_1 + k_1'')z + ik_1''\bar{z}} \right]$$

Letting $k_1'' = -k_1'$

$$\mathcal{R}(\overline{z}) = B_o^2 k_2^2 J_o^2 (k_2 r) \lim_{z_o \to \infty} \frac{1}{2z_o} \int_{-z_o}^{+z_o} dz \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty}$$

$$\omega(k_1)\, \omega k_1')\, A^*(k_1')\, A(k_1)\, e^{i(k_1 - k_1')z}\, e^{-ik_1'\overline{z}}\, dk_1\, dk_1'$$

(3.2-33)

$$= B_o^2 k_2^2 J_o^2 (k_2 r) \lim_{z_o \to \infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \omega(k_1)\omega(k_1')A(k_1)A^*(k_1')$$

$$\frac{\sin (k_1 - k_1')z_o}{(k_1 - k_1')\, z_o}\, e^{-ik_1'\overline{z}}\, dk_1\, dk_1'$$

Now introduce the new variable $\xi = (k_1 - k_1')\, z_o$ and rewrite (3.2-33) as

$$\mathcal{R}(\overline{z}) = B_o^2 k_2^2 J_o^2 (k_2 r) \int_{-\infty}^{+\infty} \omega(k_1)\, A(k_1) e^{-ik_1\overline{z}}\, dk_1 \int_{-\infty}^{+\infty} \lim_{z_o \to \infty}$$

(3.2-34)

$$\frac{\omega(k_1 - \frac{\xi}{z_o})\, A^*(k_1 - \frac{\xi}{z_o})}{z_o}\, e^{i\xi \frac{\overline{z}}{z_o}}\, \frac{\sin \xi}{\xi}\, d\xi$$

Performing the limiting operation and then the integration over ξ we obtain

$$\mathcal{R}(\overline{z}) = B_o^2 k_2^2 J_o^2 (k_2 r) \int_{-\infty}^{+\infty} \omega^2(k_1)\, \Phi(k_1) e^{-ik_1\overline{z}}\, dk_1 \qquad (3.2\text{-}35)$$

where

$$\Phi(k_1) = \lim_{z_o \to \infty} \frac{\pi}{z_o} \left| A(k_1) \right|^2 \tag{3.2-36}$$

Except for the factor $B_o^2 k_2^2 J_o^2(k_2 r)$, the function $\Phi(k_1)$ is the spectral density of $B_z(z) - B_o$, and $\omega^2(k_1)\,\Phi(k_1)$ is the spectral density of $\dot{B}(z)$. By virtue of the definition (3.2-36), $\Phi(k_1)$ and $\omega^2(k_1)\,\Phi(k_1)$ are even functions of k_1.

From (3.2-28) and (3.2-35), we note that the mean-square value of $\dot{B}$ is given by the following integral

$$\sigma_{\dot{B}}^2 = B_o^2 k_2^2 J_o^2(k_2 r) \int_{-\infty}^{+\infty} \omega^2(k_1)\,\Phi(k_1)\,dk_1 \tag{3.2-37}$$

Equations (3.2-35) and (3.2-37) form the basis for the statistical description of the fluctuating magnetic field.

For later use it will be convenient to introduce a normalized power spectral density $\phi(k_1)$ defined by

$$\Phi(k_1) = \lim_{z_o \to \infty} \frac{\pi}{z_o} \left| A(k_1) \right|^2 = A^2 \lim_{z_o \to \infty} \frac{\pi}{z_o} \left| a(k_1) \right|^2 = A^2 \phi(k_1) \tag{3.2-38}$$

where the constant A^2 is chosen so that

$$\int_{-\infty}^{+\infty} \omega^2(k_1)\,\phi(k_1) = 1 \tag{3.2-39}$$

holds, and $A(k_1) = Aa(k_1)$.

The corresponding normalized autocorrelation function will be denoted by $R(\overline{z})$ and is defined by

$$R(\overline{z}) = \int_{-\infty}^{+\infty} \omega^2(k_1)\,\phi(k_1)\,e^{-ik_1\overline{z}}\,dk_1 \tag{3.2-40}$$

with $R(0) = 1$. We further define

$$F(z) = \int_{-\infty}^{+\infty} \omega(k_1)\,a(k_1)\,e^{ik_1 z}\,dk_1 \tag{3.2-41}$$

It follows immediately from (3.2-35) and (3.2-38) that the autocorrelation function for $F(z)$ is $R(z)$, i.e.,

$$R(\overline{z}) = \widetilde{F(z)F(z+\overline{z})} \tag{3.2-42}$$

and that

$$\begin{aligned} \mathcal{R}(\overline{z}) &= B_o^2 k_2^2 J_o^2(k_2 r) A^2 \int_{-\infty}^{+\infty} \omega^2(k_1)\,\phi(k_1)\,e^{-ik_1\overline{z}}\,dk_1 \\ &= \mathcal{R}(o) \int_{-\infty}^{+\infty} \omega^2(k_1)\,\phi(k_1)\,e^{-ik_1\overline{z}}\,dk_1 = \sigma_{\dot{B}}^2\,R(\overline{z}) \end{aligned} \tag{3.2-43}$$

where

$$\mathcal{R}(o) = \sigma_{\dot{B}}^2 = B_o^2 k_2^2 J_o^2(k_2 r) A^2 \tag{3.2-44}$$

So far nothing has been said about the form of the autocorrelation function $R(\overline{z})$ or its Fourier transform $\omega^2(k_1)\,\phi(k_1)$.

These quantities depend on a much more detailed knowledge of the galactic magnetic field than is currently available. Fortunately, the results which we shall derive in later sections do not depend on knowing the shape of the spectrum, but only its value at $k_1 = 0^*$.

Although we shall have no specific use for them, we shall give two examples of autocorrelation functions which might reasonably be expected for a magnetic field of the type we have considered. One such a function is

$$R(\bar{z}) = e^{-\gamma |\bar{z}|} \cos k_o \bar{z} \tag{3.2-45}$$

where γ and k_o are positive constants. The corresponding spectral density is

$$\begin{aligned} \omega^2(k_1)\phi(k_1) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\gamma |\bar{z}|} \cos k_o \bar{z}\, e^{-ik_1 \bar{z}}\, d\bar{z} \\ &= \frac{1}{2\pi} \left[\frac{\gamma}{\gamma^2 + (k_o + k_1)^2} + \frac{\gamma}{\gamma^2 + (k_o - k_1)^2} \right] \end{aligned} \tag{3.2-46}$$

This is the spectral density of a random function $F(z)$ having a spectrum of wave numbers distributed over a band about k_o . The constant γ is interpreted as the width of the peak at half maximum.

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* It will be shown later that the diffusion coefficients associated with the betatron mechanism contain the factor $\omega^2(o)\phi(o)$. At first sight it might seem strange that the detailed shape of the spectrum is not needed. Similar results appear in the analysis of noise in electrical control systems and in the theory of Brownian motion. In the latter example the mean-square velocity of a particle undergoing Brownian motion is proportional to $\phi(o)$ where $\phi(k)$ is the spectral density of the random forcing function (Uhlenbeck and Ornstein 1930). It is customary to evaluate $\phi(o)$ with the help of the theorem of equipartition of energy.

Another spectrum which has a predominance of small wave numbers (long wavelengths) is obtained by taking $k_o = 0$ in (3.2-45):

$$R(\bar{z}) = e^{-\gamma |\bar{z}|}$$

The corresponding spectral density is

$$\omega^2(k_1)\,\phi(k_1) = \frac{1}{\pi}\,\frac{\gamma}{\gamma^2 + k_1^2} \quad .$$

3.23 The Mean Square Gas Velocity of the Turbulent Medium

In Section 3.24 we shall find it convenient to have an expression for the mean square gas velocity of the oscillating medium. This involves a simple application of some of the results derived in the preceding section.

It will be noted from (3.2-5) and (3.2-7) that the expressions for the radial velocity v_r and the rate of change of the z-component of the magnetic field $\frac{\partial B_z}{\partial t}$ for a simple wave are of the same formexcept for the radial dependence and a constant multiplying factor. This similarity allows us to write down immediately an expression for v_r analogous to the representation of $\frac{\partial B_z}{\partial t}$ given by (3.2-22). Assuming that v_r vanishes outside the interval $-z_o \leq z \leq +z_o$, we may write

$$v_r = J_1(k_2 r) \int_{-\infty}^{+\infty} \omega(k_1) A(k_1) e^{ik_1 z} \, dk_1 \qquad (3.2\text{-}47)$$

We calculate the mean square velocity $\widetilde{v_r^2}$ from

$$\widetilde{v_r^2} = \lim_{z_o \to \infty} \frac{1}{2z_o} \int_{-\infty}^{+\infty} v_r^2 (z) dz \qquad (3.2\text{-}48)$$

Noting the similarity between (3.2-47) and (3.2-22) we find

$$\widetilde{v_r^2} = J_1^2(k_2 r) \int_{-\infty}^{+\infty} \omega^2(k_1) \, \Phi(k_1) dk_1 \qquad (3.2\text{-}49)$$

Introducing the normalized spectral density defined by (3.2-38) gives

$$v_r^2 = J_1^2(k_2r)A^2 \int_{-\infty}^{+\infty} \omega^2(k_1)\,\phi(k_1)dk_1 = J_1^2(k_2r)A^2 \qquad (3.2\text{-}50)$$

To remove the radial dependence, we must carry out a further averaging process. The average over r is carried out over the region $o \leq r \leq r_1$ where r_1 is chosen so that

$$J_1(k_2r_1) = J_1(\alpha_o) = 0 \qquad (3.2\text{-}51)$$

In what follows we will denote quantities averaged over both z and r by angular brackets.

The mean square gas velocity, denoted by $\left\langle v_g^2 \right\rangle$, is given by the expression

$$\left\langle v_g^2 \right\rangle = A^2 \frac{\int_o^{r_1} J_1^2(k_2r)rdr}{\int_o^{r_1} r\,dr} = \frac{2A^2}{r_1^2} \int_o^{r_1} J_1^2(k_2r)rdr \qquad (3.2\text{-}52)$$

Making the substitution $\alpha = k_2r$ reduces the integral to the form

$$\left\langle v_g^2 \right\rangle = \frac{2A^2}{\alpha_o^2} \int_o^{\alpha_o} J_1^2(\alpha)\alpha\, d\alpha \qquad (3.2\text{-}53)$$

where $\alpha_o = k_2r_1$. The integral is easily evaluated (Smythe 1950) yielding

$$\left\langle v_g^2 \right\rangle = \frac{2A^2}{\alpha_o^2}\left[\frac{\alpha_o^2}{2}\left[J_1^2(\alpha_o) + J_o^2(\alpha_o)\right] - \alpha_o J_1(\alpha_o) J_o(\alpha_o)\right] \quad (3.2\text{-}54)$$

or

$$\left\langle v_g^2 \right\rangle = A^2 J_o^2(\alpha_o) \quad (3.2\text{-}55)$$

We finally wish to note the connection between $\left\langle v_g^2 \right\rangle$ and the mean square value of $\dot{B}$. Carrying out a similar average over the radial dependence of $\sigma.^2$ given by (3.2-37) we find

$$\left\langle \dot{B}^2 \right\rangle = B_o^2 k_2^2 A^2 \frac{\int_o^{r_1^B} J_o^2(k_2 r)\, r dr}{\int_o^{r_1} r dr} \quad (3.2\text{-}56)$$

$$= \frac{2B_o^2 k_2^2 A^2}{\alpha_o^2} \int_o^{\alpha_o} J_o^2(\alpha)\alpha d\alpha = B_o^2 k_2^2 A^2 \left[J_o^2(\alpha_o) + J_1^2(\alpha_o)\right]$$

or

$$\left\langle \dot{B}^2 \right\rangle = B_o^2 k_2^2 A^2 J_o^2(\alpha_o) \quad (3.2\text{-}57)$$

Eliminating the factor $A^2 J_o^2(\alpha_o)$ between (3.2-55) and (3.2-57), we find*

$$\left\langle \dot{B}^2 \right\rangle = B_o^2 k_2^2 \left\langle v_g^2 \right\rangle \quad (3.2\text{-}58)$$

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* Note that it is immaterial which root of $J_1(\alpha_o) = 0$ is chosen in carrying out the averages over r.

3.24 Evaluation of the Diffusion Coefficients D_μ, $D_{\mu\mu}$, D_ϵ, $D_{\epsilon\epsilon}$, $D_{\mu\epsilon}$, D_z, D_{zz}

To evaluate the diffusion coefficients D_μ, $D_{\mu\mu}$, D_ϵ, etc., we begin with the differential equations

$$B_o \frac{d\epsilon}{dt} = \frac{1}{2}(1-\mu^2)\frac{\partial B_z}{\partial t} \qquad (3.2\text{-}59)$$

$$B_o \frac{d\mu}{dt} = \frac{1}{2}\frac{(1-\mu^2)^2}{\mu}\frac{\partial B_z}{\partial t} - \frac{1}{2}\frac{(1-\mu^2)}{\mu}\frac{dB_z}{dt} \qquad (3.2\text{-}60)$$

where $\frac{\partial B_z}{\partial t}$ is to be regarded as a random function of z. In using these equations we must follow a particle through the field. Let z denote the coordinate of the particle. The z-component of the particles velocity is then given by

$$\frac{dz}{dt} = \mu c \qquad (3.2\text{-}61)$$

Evaluated at the position of the particle, $\frac{\partial B_z}{\partial t}$ becomes an implicit function of $\mu(t)$.

The function $\frac{dB_z}{dt}$ in (3.2-60) has a somewhat different character. We note that the integral

$$\int_{t_1}^{t_2} \frac{dB_z}{dt}\, dt = B_z(t_2) - B_z(t_1) \qquad (3.2\text{-}62)$$

represents the net change in the z-component of the field as seen by the particle as it moves through the field. If we assume that the particle starts at t_1 from a point where the field has the value B_o and moves through the field to any other point where the field is also B_o, then the net value of the integral (3.2-62) is zero. We expect therefore that this term will contribute nothing to the diffusion process, and we henceforth disregard it.

In accordance with the discussion of Section 3.22, we may disregard the explicit time dependence of $\frac{\partial B_z}{\partial t}$ as long as the z-component of the particle velocity is large compared with the velocity of propagation of the magnetohydrodynamic waves. This condition is satisfied for all particles except those moving in extremely flat spirals.

From (3.2-22) we have the following representation of $\dot{B}(z)$,

$$\dot{B}(z) = B_o k_2 J_o(k_2 r) \int_{-\infty}^{+\infty} \omega(k_1) A(k_1) e^{ik_1 z} dk_1 \qquad (3.2\text{-}63)$$

Associated with this function are the spectral density defined in terms of $A(k_1)$ by Equation (3.2-36) and the corresponding autocorrelation function $\mathcal{R}(\overline{z})$ defined by Equation (3.2-35). We shall find it convenient in simplifying the computations to introduce the normalized amplitude function $a(k_1)$ defined by

$$A(k_1) = Aa(k_1) \qquad (3.2\text{-}64)$$

and require the constant A to be such that (3.2-39) is satisfied. Introducing this notation into (3.2-63) we have

$$\dot{B}(z) = B_o A k_2 J_o(k_2 r) \int_{-\infty}^{+\infty} \omega(k_1) a(k_1) e^{ik_1 z} dk_1 \qquad (3.2\text{-}65)$$

$$= B_o A k_2 J_o(k_2 r) F(z)$$

This is the same F(z) as defined by (3.2-41). The normalization requirements are now such that

$$R(o) = \widetilde{F^2(z)} = \int_{-\infty}^{+\infty} \omega(k_1)\,\phi(k_1)\,dk_1 = 1 \qquad (3.2\text{-}66)$$

In this notation the differential equations (3.2-59) and (3.2-60) become,

$$\frac{d\epsilon}{dt} = \frac{1}{2}(1 - \mu^2)\, k_2 J_o(k_2 r)\, AF(z) \qquad (3.2\text{-}67)$$

$$\frac{d\mu}{dt} = \frac{(1 - \mu^2)^2}{2\mu}\, k_2 J_o(k_2 r)\, AF(z) \qquad (3.2\text{-}68)$$

To further simplify the notation we define

$$S_o(\mu, r) = \frac{1}{2}(1 - \mu^2)\, k_2 J_o(k_2 r) \qquad (3.2\text{-}69)$$

$$S_1(\mu, r) = \frac{(1 - \mu^2)^2}{2\mu}\, k_2 J_o(k_2 r) \qquad (3.2\text{-}70)$$

and write

$$\frac{d\epsilon}{dt} = S_o AF(z) \qquad (3.2\text{-}71)$$

$$\frac{d\mu}{dt} = S_1 AF(z) \qquad (3.2\text{-}72)$$

To relate z to μ we use

$$\frac{dz}{dt} = \mu\, c \qquad (3.2\text{-}73)$$

An exact integration of equation (3.2-71), (3,2-72), and (3.2-73) is clearly impossible. An approximate solution is obtained by expanding ϵ and μ as a power series in the amplitude A (A is small for slowly varying fields), dropping terms of order A^3 and higher. The approximate solution is adequate for our purposes, since we want a solution which is good only for short times.

We choose the time origin so that $z = 0$ at $t = 0$, let ϵ_o and μ_o denote the values of μ and ϵ at $t = 0$. Let Δt denote an interval of time large compared to the significant correlation time of $F(z(t))$, and let

$$\Delta\epsilon(t) = \epsilon(t) - \epsilon_o \tag{3.2-74}$$

$$\Delta\mu(t) = \mu(t) - \mu_o \tag{3.2-75}$$

We expand $\epsilon(t)$ and $\mu(t)$ in the following manner

$$\epsilon(t) = \epsilon_o + Ay_1(t) + A^2 y_2(t) + \ldots\ldots \tag{3.2-76}$$

$$\mu(t) = \mu_o + Ax_1(t) + A^2 x_2(t) + \ldots\ldots \tag{3.2-77}$$

An expansion for $z(t)$ is obtained by substituting the expansion (3.2-77) into (3.2-73) and integrating. This yields,

$$\begin{aligned} z(t) &= \mu_o ct + Ac \int_o^t x_1(\tau)\, d\tau + A^2 c \int_o^t x_2(\tau)\, d\tau + \ldots \\ &= \mu_o ct + Ac\, z_1(t) + A^2 c\, z_2(t) + \ldots \end{aligned} \tag{3.2-78}$$

Substituting these expansions into the differential equations (3.2-71) and (3.2-72), and expanding, we obtain

$$A\frac{dy_1}{dt} + A^2\frac{dy_2}{dt} + \dots = AS_o\left[\mu_o + Ax_1(t) + \dots\right] F\left[\mu_o ct + Acz_1(t) + \dots\right.$$

$$= AS_o(\mu_o) F(\mu_o ct)$$

$$+ A^2\left[S_o'(\mu_o) F(\mu_o ct) + cS_o(\mu_o) F'(\mu_o ct) z_1(t)\right]$$

$$+ O(A^3) \qquad (3.2\text{-}79)$$

and

$$A\frac{dx_1}{dt} + A^2\frac{dx_2}{dt} + \dots = AS_1\left[\mu_o + Ax_1(t) + \dots\right]$$

$$\times F\left[\mu_o ct + Acz_1(t) + \dots\right]$$

$$= AS_1(\mu_o) F(\mu_o ct)$$

$$+ A^2\left[S'_1(\mu_o) F(\mu_o ct) + cS_1(\mu_o)\right.$$

$$\left. F'(\mu_o ct) z_1(t)\right] + O(A^3) \qquad (3.2\text{-}80)$$

Equating terms of the same order on both sides of these equations, we obtain the following differential equations for $x_1(t)$, $x_2(t)$, $y_1(t)$, $y_2(t)$:

$$\frac{dy_1}{dt} = S_o(\mu_o) F(\mu_o ct) \qquad (3.2\text{-}81)$$

$$\frac{dy_2}{dt} = S'_o(\mu_o) F(\mu_o ct) x_1(t) + cS_o(\mu_o) F'_1(\mu_o ct) z_1(t) \qquad (3.2\text{-}82)$$

$$\frac{dx_1}{dt} = S_1(\mu_o) F(\mu_o ct) \qquad (3.2\text{-}83)$$

$$\frac{dx_2}{dt} = S'_1(\mu_o) F(\mu_o ct) x_1(t) + cS_1(\mu_o) F'(\mu_o ct) z_1(t) \qquad (3.2\text{-}84)$$

From these equations we wish to determine the following ensemble average values,

$$\widetilde{(\Delta\epsilon)} = A\,\widetilde{y_1(\Delta t)} + A^2\,\widetilde{y_2(\Delta t)} + O(A^3) \qquad (3.2\text{-}85)$$

$$\widetilde{(\Delta\epsilon)^2} = A^2\,\widetilde{\left[y_1(\Delta t)\right]^2} + O(A^3) \qquad (3.2\text{-}86)$$

$$\widetilde{(\Delta\mu)} = A\,\widetilde{x_1(\Delta t)} + A^2\,\widetilde{x_2(\Delta t)} + O(A^3) \qquad (3.2\text{-}87)$$

$$\widetilde{(\Delta\mu)^2} = A^2\,\widetilde{\left[x_1(\Delta t)\right]^2} + O(A^3) \qquad (3.2\text{-}88)$$

$$\widetilde{(\Delta\mu\Delta\epsilon)^2} = A^2\,\widetilde{\left[x_1(t)\,y_1(t)\right]} + O(A^3) \qquad (3.2\text{-}89)$$

The averages are to be carried out over an ensemble of particles all having the same value of ϵ and μ at $t = 0$. Integrating (3.2-83) from o to Δt,

$$x_1(\Delta t) = S_1(\mu_o) \int_o^{\Delta t} F(\mu_o ct)\, dt \qquad (3.2\text{-}90)$$

Averaging,*

$$\widetilde{x_1(\Delta t)} = S_1(\mu_o) \int_o^{\Delta t} \widetilde{F(\mu_o ct)}\, dt = 0 \qquad (3.2\text{-}91)$$

since $F(\mu_o ct)$ is a random function having zero mean value.

---

* It is essential here that an ensemble average be used since $x_1(t)$ is in general a nonstationary random function even though $F(\mu_o ct)$ is stationary.

Similarly,

$$\widetilde{y_1(\Delta t)} = S_o(\mu_o) \int_o^{\Delta t} \widetilde{F(\mu_o ct)}\, dt = 0 \qquad (3.2\text{-}92)$$

Thus, all the diffusion coefficients are at least second-order in $\widetilde{A}$.

We shall first deal with the second-order averages $\widetilde{(\Delta\epsilon)^2}$, $\widetilde{(\Delta\mu\Delta\epsilon)}$, and $\widetilde{(\Delta\mu)^2}$ since they are the easiest to obtain. Squaring (3.2-90) and averaging

$$\widetilde{\left[x_1(\Delta t)\right]^2} = S_1^{\,2}(\mu_o) \int_o^{\Delta t} \int_o^{\Delta t} \widetilde{F(\mu_o ct)\, F(\mu_o c\tau)}\, dt\, d\tau \qquad (3.2\text{-}93)$$

The average $\widetilde{F(\mu_o ct)\, F(\mu_o c\tau)}$ is just the autocorrelation function $R(\overline{z})$ with the argument $\overline{z}$ replaced by $\mu_o c(t - \tau)$.

Thus

$$\widetilde{\left[x_1(\Delta t)\right]^2} = S_1^{\,2}(\mu_o) \int_o^{\Delta t} \int_o^{\Delta t} R\left[\mu_o c(t - \tau)\right] dt\, d\tau \qquad (3.2\text{-}94)$$

Introducing the change of variables

$$\eta = t - \tau \qquad t = \frac{\xi}{2} + \eta \qquad (3.2\text{-}95)$$

$$\xi = \frac{t + \tau}{2} \qquad \tau = \frac{\xi}{2} - \eta$$

and the corresponding "area transformation"

$$dt\, d\tau = \left| J \right| \, d\xi \; d\eta \qquad (3.2\text{-}96)$$

where J is the Jacobian,

$$J = \begin{vmatrix} \frac{\partial t}{\partial \xi} & \frac{\partial t}{\partial \eta} \\ \frac{\partial \tau}{\partial \xi} & \frac{\partial \tau}{\partial \eta} \end{vmatrix} = -1 \qquad (3.2\text{-}97)$$

we obtain, using the fact that $R\left[\mu_o c(t-\tau)\right]$ is an even function of its argument,

$$\widetilde{\left[x_1(\Delta t)\right]^2} = 2S_1^2(\mu_o) \int_{\eta=0}^{\eta=\Delta t} \int_{\xi=\frac{\eta}{2}}^{\xi=\Delta t-\frac{\eta}{2}} R(\mu_o c \eta)\, d\xi\, d\eta$$

$$= 2S_1^2(\mu_o)\Delta t \int_0^{\Delta t} \left(1-\frac{\eta}{\Delta t}\right) R(\mu_o c \eta)\, d\eta \qquad (3.2\text{-}98)$$

Now, for Δt somewhat larger than the value of η for which $R(\mu_o c \eta)$ becomes negligible,

$$\widetilde{\left[x_1(\Delta t)\right]^2} = 2S_1^2(\mu_o)\Delta t \int_0^{\infty} R(\mu_o c \eta)\, d\eta + O(\Delta t^2)$$

$$= \frac{2S_1^2(\mu_o)\Delta t}{\left|\mu_o c\right|} \int_0^{\infty} R(\bar{z})\, d\bar{z} + O(\Delta t^2) \qquad (3.2\text{-}99)$$

This result may be written in a simpler form by noting from (3.2-40) that

$$\omega^2(k_1)\,\phi(k_1) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R(\overline{z})\,e^{ik_1\overline{z}}\,d\overline{z} \qquad (3.2\text{-}100)$$

Letting $k_1 = 0$,

$$\pi\omega^2(o)\,\phi(o) = \int_{o}^{\infty} R(\overline{z})\,d\overline{z} \qquad (3.2\text{-}101)$$

Hence,

$$\widetilde{\left[x_1(\Delta t)\right]^2} = \frac{2\pi\omega^2(o)\,\phi(o)\,S_1^2(\mu_o)\,\Delta t}{|\mu_o c|} + O(\Delta t^2) \qquad (3.2\text{-}102)$$

Recalling the definition of $S_1(\mu_o)$ and using (3.2-88),

$$\widetilde{(\Delta\mu)^2} = \frac{\pi A^2 k_2^2\,\omega^2(o)\,\phi(o)\,J_o^2(k_2 r)(1-\mu_o^2)^4\,\Delta t}{2c\,\mu_o^2\,|\mu_o|} + O(\Delta t^2) \quad (3.2\text{-}103)$$

We next average over r in the manner described in Section 3.23, and introduce the mean square gas velocity defined by (3.2-55), obtaining,

$$D_{\mu\mu} = \lim_{\Delta t \to o} \frac{\langle \Delta\mu^2\rangle}{\Delta t} = \frac{\pi\omega^2(o)\,\phi(o)k_2^2\,\langle v_g^2\rangle\,(1-\mu_o^2)^4}{2c\,\mu_o^2\,|\mu_o|} \qquad (3.2\text{-}104)$$

or

$$D_{\mu\mu} = K\,\frac{(1-\mu_o^2)^2}{\mu_o^2\,|\mu_o|} \qquad (3.2\text{-}105)$$

where the constant K is given by

$$K = \frac{\pi\omega^2(o)\,\phi(o)\left\langle v_g^2\right\rangle k_2^2}{2c} = \frac{\pi\omega^2(o)\,\phi(o)}{2c}\;\frac{\left\langle \dot{B}_z^2\right\rangle}{B_o^2} \qquad (3.2\text{-}106)$$

or

$$K = \frac{\pi}{2c}\,\phi(o)\,\frac{\left\langle (B_z - B_o)^2\right\rangle}{B_o^2} \qquad (3.2\text{-}107)$$

In a similar manner we obtain from (3.2-86) and (3.2-89),

$$D_{\epsilon\epsilon} = \lim_{\Delta t \to o} \frac{\left\langle(\Delta\epsilon^2)\right\rangle}{\Delta t} = K\,\frac{(1-\mu_o^2)^2}{|\mu_o|} \qquad (3.2\text{-}108)$$

$$D_{\mu\epsilon} = \lim_{\Delta t \to o} \frac{\left\langle \Delta\mu\Delta\epsilon\right\rangle}{\Delta t} = K\,\frac{(1-\mu_o^2)^3}{\mu_o\,|\mu_o|} \qquad (3.2\text{-}109)$$

To find D_μ and D_ϵ, we return to equations (3.2-81) through (3.2-84). Integrating (3.2-84) from o to Δt, we get

$$x_2(\Delta t) = S_1'(\mu_o)\int_o^{\Delta t} F(\mu_o ct)x_1(t)dt + c\,S_1(\mu_o)\int_o^{\Delta t} F'(\mu_o ct)z_1(t)dt \qquad (3.2\text{-}110)$$

Integrating the last term on the right by parts, we obtain

$$\int_o^{\Delta t} F'(\mu_o ct)z_1(t)dt = \frac{F(\mu_o c\Delta t)z_1(\Delta t)}{\mu_o c} - \frac{1}{\mu_o c}\int_o^{\Delta t} F(\mu_o ct)x_1(t)dt \qquad (3.2\text{-}111)$$

Thus, (3.2-110) becomes

$$x_2(\Delta t) = \left[S_1'(\mu_o) - \frac{S_1(\mu_o)}{\mu_o}\right]\int_o^{\Delta t} F(\mu_o ct)x_1(t)dt + \frac{S_1(\mu_o)}{\mu_o}\,F(\mu_o c\Delta t)z_1(\Delta t) \qquad (3.2\text{-}112)$$

From (3.2-83) we obtain the following expressions for $x_1(t)$, and $z_1(t)$:

$$x_1(t) = S_1(\mu_o) \int_o^t F(\mu_o ct)\, dt \qquad (3.2\text{-}113)$$

$$z_1(t) = \int_o^t x_1(\tau) d\tau = S_1(\mu_o) \int_o^t d\tau \int_o^{\tau} F(\mu_o c\tau')d\tau' \qquad (3.2\text{-}114)$$

Substituting these expressions into (3.2-112) and averaging, we find

$$\widetilde{x_2(\Delta t)} = S_1(\mu_o)\left[S'_1(\mu_o) - \frac{S_1(\mu_o)}{\mu_o}\right] \int_o^{\Delta t} dt \int_o^t \widetilde{F(\mu_o ct)F(\mu_o c\tau)}d\tau$$

$$+ \frac{S_1^2(\mu_o)}{\mu_o} \int_o^{\Delta t} dt \int_o^t \widetilde{F(\mu_o c\Delta t)\, F(\mu_o c\tau)}\, d\tau \qquad (3.2\text{-}115)$$

The last term contributes nothing* to D_μ because the quantities $F(\mu_o c\Delta t)$ and $z_1(\Delta t)$ are not correlated for Δt large compared to the significant correlation time of $F(\mu_o ct)$. Thus, making use of the fact that $F(\mu_o ct)$ has zero mean value

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* Another way of establishing this result is as follows:
It is readily shown that

$$\frac{S_1(\mu_o)}{\mu_o} F(\mu_o c\Delta t) z_1(\Delta t) = \frac{S_1^2(\mu_o)}{\mu_o} \Delta t \int_o^{\Delta t} (1 - \frac{\tau}{\Delta t}) R \;\; \mu_o c(\Delta t - \tau) \;\; d\tau$$

The integrand approaches zero exponentially for Δt large compared to the time over which $R(\mu_o c\tau)$ is appreciable, thus

$$\lim_{\Delta t \to o} \frac{S_1(\mu_o)}{\mu_o} \frac{F(\mu_o c\Delta t)\, z_1(\Delta t)}{\Delta t} = 0$$

$$\widetilde{F(\mu_o c\Delta t)\, z_1(\Delta t)} = \widetilde{F(\mu_o c\Delta t)}\, \widetilde{z_1(\Delta t)} = 0 \qquad (3.2\text{-}116)$$

Therefore

$$x_2(\Delta t) = S_1(\mu_o)\left[S_1'(\mu_o) - \frac{S_1(\mu_o)}{\mu_o}\right] \int_o^{\Delta t} dt \int_o^{t} R\ \ \mu_o c(t-\tau)\ \ d\tau \qquad (3.2\text{-}117)$$

Making the change of variables $\tau' = t-\tau$, $d\tau' = -d\tau$ in the integral over τ reduces this expression to the form

$$x_2(\Delta t) = S_1(\mu_o)\left[S_1'(\mu_o) - \frac{S_1(\mu_o)}{\mu_o}\right] \int_o^{\Delta t} dt \int_o^{t} R(\mu_o c\tau)\, d\tau \qquad (3.2\text{-}118)$$

This double integral may be reduced to a single integral by an integration by parts. The result is

$$\begin{aligned} x_2(\Delta t) &= S_1(\mu_o)\left[S_1'(\mu_o) - \frac{S_1(\mu_o)}{\mu_o}\right] \int_o^{\Delta t} (\Delta t - \tau) R(\mu_o c\tau) d\tau \\ &= S_1(\mu_o)\left[S_1'(\mu_o) - \frac{S_1(\mu_o)}{\mu_o}\right] \Delta t \int_o^{\Delta t} (1 - \frac{\tau}{\Delta t}) R(\mu_o c\tau) d\tau \end{aligned} \qquad (3.2\text{-}119)$$

which is of the same form as (3.2-98) .

For Δt somewhat larger than the value of t for which $R(\mu_o c t)$ becomes negligible

$$\widetilde{x_2(\Delta t)} = S_1(\mu_o)\left[S_1'(\mu_o) - \frac{S_1(\mu_o)}{\mu_o}\right]\Delta t \int_o^\infty R(\mu_o c\tau)d\tau + O(\Delta t^2)$$

$$= S_1(\mu_o)\left[S_1'(\mu_o) - \frac{S_1(\mu_o)}{\mu_o}\right]\frac{\Delta t}{\mu_o c} \int_o^\infty R(\bar{z})d\bar{z} + O(\Delta t)^2 \qquad (3.2\text{-}120)$$

Using the definition of $S_1(\mu_o)$ and (3.2-101) we find

$$\widetilde{x_2(\Delta t)} = -\pi k_2^2 \omega^2(o)\phi(o) J_o^2(k_2 r)\frac{(1-\mu_o^2)^3(1+\mu_o^2)}{2c\mu_o^3 \,|\mu_o|}\Delta t + O(\Delta t^2) \qquad (3.2\text{-}121)$$

Using (3.2-87), (3.2-91)

$$\widetilde{(\Delta\mu)} = -\pi A^2 k_2^2 \omega^2(o)\phi(o) J_o^2(k_2 r)\frac{(1-\mu_o^2)^3(1+\mu_o^2)}{2c\mu_o^3 \,|\mu_o|}\Delta t + O(\Delta t^2) \qquad (3.2\text{-}122)$$

Again averaging over r in the manner described in Section 3.23, and introducing the mean square gas velocity defined by (3.2-55), we obtain

$$D_\mu = \lim_{\Delta t\to o} \frac{\langle\Delta\mu\rangle}{\Delta t} = \frac{-\pi\omega^2(o)k_2^2\phi(o)\left\langle v_g^2\right\rangle(1-\mu_o^2)^3(1+\mu_o^2)}{2c\mu_o^3\,|\mu_o|}$$

$$= -K\frac{(1-\mu_o^2)^3(1+\mu_o^2)}{\mu_o^3\,|\mu_o|} \qquad (3.2\text{-}123)$$

In a similar manner we find

$$\widetilde{y_2(\Delta t)} = S_1(\mu_o)\left[S_o'(\mu_o) - \frac{S_o(\mu_o)}{\mu_o} \right] \Delta t \int_o^\infty R(\mu_o c\tau)d\tau + O(\Delta t^2)$$

(3.2-124)

$$= \frac{-\pi\omega^2(o)k_2^2 \phi(o) \left\langle v_g^2 \right\rangle (1-\mu_o^2)^2(1+\mu_o^2)}{4\, c\, \mu_o^2 \; |\mu_o|} \Delta t + O(\Delta t^2)$$

Using (3.2-85), (3.2-92),

$$D_\epsilon = \frac{-\pi\omega^2(o)k_2^2 \phi(o) \left\langle v_g^2 \right\rangle (1-\mu_o^2)^2(1+\mu_o^2)}{4\, c\, \mu_o^2 \; |\mu_o|} = -\frac{K}{2} \frac{(1-\mu_o^2)^2(1+\mu_o^2)}{\mu_o^2 \; |\mu_o|}$$

(3.2-125)

We next calculate the diffusion coefficients D_z and D_{zz}.

From (3.2-78)

$$\widetilde{\Delta z} = \widetilde{z(\Delta t)} = \mu_o c\Delta t + Ac\, \widetilde{z_1(\Delta t)} + A^2 c\, \widetilde{z_2(\Delta t)} + O(A^3)$$

$$= \mu_o c\Delta t + Ac\, S_1(\mu_o) \int_o^{\Delta t} dt \int_o^{\tau} \widetilde{F(\mu_o c\tau')}\, d\tau' \qquad (3.2\text{-}126)$$

$$+ A^2 c \int_o^{\Delta t} \widetilde{x_2(t)}dt + O(A^3)$$

The second term on the right is zero since $F(\mu_o c\tau)$ has zero mean value and the third term is $c\widetilde{\Delta\mu}$ from (3.2-87). Thus

$$D_z = \lim_{\Delta t \to o} \frac{\langle \Delta z \rangle}{\Delta t} = c(\mu_o + D_\mu) \qquad (3.2\text{-}127)$$

Squaring (3.2-78) and averaging, we find

$$\widetilde{(\Delta z)^2} = \mu_o^2 c^2 (\Delta t)^2 + A^2 c^2 \widetilde{\left[z_1(\Delta t) \right]^2} + \mu_o c \Delta t A c \, \widetilde{z_1(\Delta t)} + O(A^3) \quad (3.2\text{-}128)$$

Computations similar to those given above show that $\widetilde{z_1(\Delta t)} = 0$ and $\left[z_1(\Delta t) \right]^2 = O(\Delta t^3)$, hence

$$D_{zz} = \lim_{\Delta t \to o} \frac{\langle \Delta z^2 \rangle}{\Delta t} = 0 \quad (3.2\text{-}129)$$

In a similar manner we find

$$D_{z\epsilon} = D_{z\mu} = 0 \quad (3.2\text{-}130)$$

This completes the computation of the diffusion coefficients, which we now summarize below, dropping the subscript on μ,

$$D_{\mu} = -K \frac{(1-\mu^2)^3 (1+\mu^2)}{\mu^3 \, |\mu|} \quad (3.2\text{-}131)$$

$$D_{\epsilon} = -\frac{K}{2} \frac{(1-\mu^2)^2 (1+\mu^2)}{\mu^2 \, |\mu|} \quad (3.2\text{-}132)$$

$$D_{\mu\mu} = K \frac{(1-\mu^2)^4}{\mu^2 \, |\mu|} \quad (3.2\text{-}133)$$

$$D_{\epsilon\epsilon} = K \frac{(1-\mu^2)^2}{|\mu|} \quad (3.2\text{-}134)$$

$$D_{\mu\epsilon} = K \frac{(1-\mu^2)^3}{\mu \, |\mu|} \quad (3.2\text{-}135)$$

$$D_z = c(\mu + D_\mu) \tag{3.2-136}$$

$$D_{zz} = D_{z\epsilon} = D_{z\mu} = 0 \tag{3.2-137}$$

Some simple checks on the diffusion coefficients may be found as follows: From (3.2-59) and (3.2-60) we find, dropping the term involving $\frac{dB_z}{dt}$ in accordance with the discussion at the beginning of this section,

$$\frac{d\epsilon}{d\mu} = \frac{\mu}{1 - \mu^2} \tag{3.2-138}$$

Integrating,

$$\epsilon = -\frac{1}{2} \ln(1 - \mu^2) + \text{const.} \tag{3.2-139}$$

This relationship indicates that the particles move along curves in the $\mu - \epsilon$ plane given by (3.2-139). Expanding (3.2-139) about $\epsilon = \epsilon_o$, $\mu = \mu_o$, we find

$$\Delta\epsilon = \frac{\mu_o}{1 - \mu_o^2} \Delta\mu + \frac{1}{2} \frac{(1 + \mu_o^2)}{(1 - \mu_o^2)^2} (\Delta\mu)^2 + \dots. \tag{3.2-140}$$

This equation indicates that

$$D_\epsilon = \frac{\mu}{1 - \mu^2} D_\mu + \frac{1}{2} \frac{(1 + \mu^2)}{(1 - \mu^2)^2} D_{\mu\mu} \tag{3.2-141}$$

Squaring (3.2-140), we find

$$D_{\epsilon\epsilon} = \frac{\mu^2}{(1 - \mu^2)^2} D_{\mu\mu} \tag{3.2-142}$$

and in a similar manner we find

$$D_{\mu\epsilon} = \frac{\mu}{1 - \mu^2} D_{\mu\mu} \tag{3.2-143}$$

The expressions that have derived for the diffusion coefficients satisfy these equations.

3.3 Scattering by Small Scale Inhomogeneities in the Field

Next consider the effects of scattering by small scale inhomogeneities in the field. If the inhomogeneities are small compared to the radius of the spiral motion then it is reasonable to assume that the mean scattering angle $\Delta\theta$ (measured with respect to the initial direction of motion) is zero, and that the mean-square scattering angle $\widetilde{(\Delta\theta)}^2$ is a constant independent of the original direction of motion of the particle before scattering. We will furthermore assume that the root-mean-square scattering angle is small. Let θ' and θ denote the angles between the momentum vector and the z-axis before and after the scattering, respectively, as shown in the following diagram.

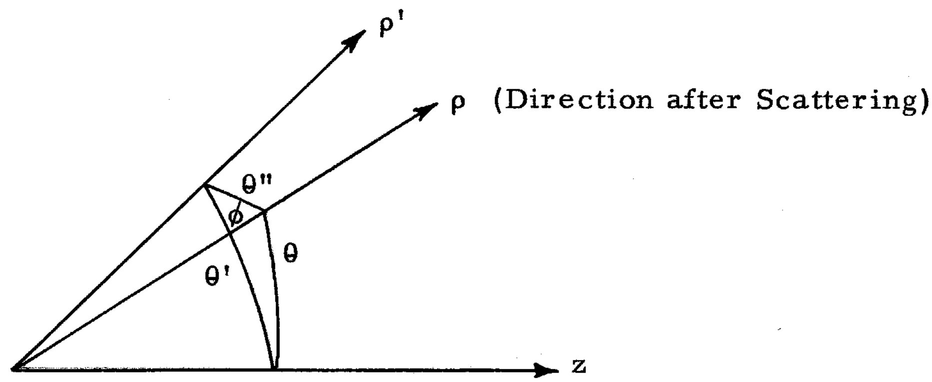

Figure 2. Momentum Vector before and after Scattering by a Small Scale Inhomogeneity.

Let $\alpha(\theta'')\,d\Omega$ be the probability that a particle is scattered from a solid angle at $\theta' = \cos^{-1}\mu'$ into a solid angle $d\Omega$ at $\theta = \cos^{-1}\mu$.

The number of particles scattered into the range $d\mu\; d\epsilon\; dz$ at μ in a time Δt is given by

$$dz\, d\epsilon\, d\mu\, n_2\, \Delta t \int W(\mu', \epsilon)\; \alpha(\theta'')\; d\Omega' \qquad (3.3\text{-}1)$$

The angular distribution factor is normalized so that

$$2\pi \int_0^{\pi} \alpha(\theta) \sin\theta\; d\theta = 1 \qquad (3.3\text{-}2)$$

and n_2 is the number of collisions per second made with small scale inhomogeneities. We will make the assumption that $\alpha(\theta)$ is independent of ϵ and also of z.

The number of particles scattered out of the range $d\epsilon\, d\mu\, dz$ in time Δt is given by

$$W(\mu)\; d\mu\, d\epsilon\, dz\, n_2\; \Delta t \qquad (3.3\text{-}3)$$

Thus, the change in the distribution function in a time Δt due to collisions of this sort is

$$\frac{\partial W}{\partial t}\,\Delta t = -n_2\Delta t\; W(\mu, \epsilon) + n_2\Delta t \int W(\epsilon, \mu')\; \alpha(\theta'')\, d\Omega' \qquad (3.3\text{-}4)$$

Now suppose the scattering is predominantly small angle. From the figure

$$\cos\theta' = \cos\theta\; \cos\theta'' + \sin\theta \sin\theta''\; \cos\phi \qquad (3.3\text{-}5)$$

where $d\Omega' = \sin\theta'' \; d\theta'' \; d\phi$. This equation may be written to second order in θ''

$$\mu' = \mu\left(1 - \frac{\theta''^2}{2}\right) + \theta''\sqrt{1-\mu^2}\;\cos\phi \tag{3.3-6}$$

Inserting this in the expression for $\frac{\partial W}{\partial t}$ we have (suppressing the dependence of W on ϵ and z),

$$\frac{\partial W}{\partial t}\Delta t = n_2\Delta t\left\{ - W(\mu) + \iint W\left[\mu(1 - \frac{\theta''^2}{2}) + \theta''\sqrt{1-\mu^2}\cos\phi\right] \alpha(\theta'')\;\sin\theta''\;d\theta''\;d\phi\right\} \tag{3.3-7}$$

Expanding the function

$$W\left[\mu(1 - \frac{\theta''^2}{2}) + \theta''\sqrt{1-\mu^2}\;\cos\phi\right] \tag{3.3-8}$$

about $\theta'' = 0$ and keeping terms up to the second order in θ'' we have

$$W\left[\mu(1 - \frac{\theta''^2}{2}) + \theta''\sqrt{1-\mu^2}\;\cos\phi\right] = W(\mu) + \frac{\partial W}{\partial\theta''} + \frac{1}{2}\frac{\partial^2 W}{\partial(\theta'')^2} + O(\theta''^3) \tag{3.3-9}$$

where the derivatives are evaluated at $\theta'' = 0$.

We find

$$\left.\frac{\partial W}{\partial\theta''}\right]_{\theta''=0} = \left\{\frac{\partial W}{\partial\mu'}\left[-\mu\theta'' + \sqrt{1-\mu^2}\;\cos\phi\right]\right\}_{\theta''=0} = \sqrt{1-\mu^2}\cos\phi\;\frac{\partial W}{\partial\mu} \tag{3.3-10}$$

and

$$\left.\frac{\partial^2 W}{\partial \theta''^2}\right]_{\theta''=0} = \left\{\frac{\partial^2 W}{\partial \mu'^2}\left[-\mu\theta'' + \sqrt{1-\mu^2}\cos\phi\right]^2 + \frac{\partial W}{\partial \mu'}(-\mu)\right\}_{\theta''=0}$$

$$= (1-\mu^2)\cos^2\phi\,\frac{\partial^2 W}{\partial \mu^2} - \mu\,\frac{\partial W}{\partial \mu} \qquad (3.3\text{-}11)$$

Evaluating the integral over θ'' and ϕ we have

$$\iint W(\mu')\ \alpha(\theta'')\sin\theta''\,d\theta''\ d\phi$$

$$= \iint\left\{W(\mu) + \sqrt{1-\mu^2}\cos\phi\,\frac{\partial W}{\partial \mu}\theta'' + \frac{1}{2}\left[(1-\mu^2)\cos^2\phi\frac{\partial^2 W}{\partial \mu^2} - \mu\frac{\partial W}{\partial \mu}\right]\right.$$

$$\left.\theta''^2\ d(\theta'')\ \sin\theta''\ d\theta''\right\} d\phi$$

$$= W(\mu)\iint \alpha(\theta'')\sin\theta''\,d\theta''\,d\phi + \sqrt{1-\mu^2}\,\frac{\partial W}{\partial \mu}\iint \theta''\sin\theta''\alpha(\theta'')\,d\theta''$$

$$\cos\phi\,d\phi + \frac{1}{2}(1-\mu^2)\frac{\partial^2 W}{\partial \mu^2}\iint \theta''^2\ \alpha(\theta'')\,d\theta''\ \cos^2\phi\,d\phi - \frac{\mu}{2}\,\frac{\partial W}{\partial \mu}$$

$$\iint \theta''\ \alpha(\theta'')\ \sin\theta''\ d\theta''\ d\phi$$

$$= W(\mu) + \frac{\sigma_\theta^2}{2}\left[\frac{1}{2}(1-\mu^2)\,\frac{\partial^2 W}{\partial \mu^2} - \mu\,\frac{\partial W}{\partial \mu}\right] \qquad (3.3\text{-}12)$$

where σ_θ^2 is the mean-square scattering angle given by

$$\sigma_\theta^2 = 2\pi\int_0^\pi \theta''^2\sin\theta''\ \alpha(\theta'')\ d\theta'' = \int (\theta'')^2\ \alpha(\theta'')\ d\Omega'' \qquad (3.3\text{-}13)$$

Thus we find that the rate of change of the distribution function due to collisions of this type is given by

$$\frac{\partial W}{\partial t} = n_2 \frac{\sigma_\theta^2}{2} \frac{\partial}{\partial \mu} \left[\frac{(1-\mu^2)}{2} \frac{\partial W}{\partial \mu} \right] \tag{3.3-14}$$

3.4 Absorption and Source Terms in the Diffusion Equation

Equation (3.2-1) gives the rate of change of the distribution function due to scattering and acceleration of particles by large scale inhomogeneities in the magnetic field. Equation (3.3-14) gives the rate of change of the distribution function due to scattering by small scale inhomogeneities. To account for absorption and injection of new particles we must add additional terms. Assume that the rate of absorption is independent of the energy, the direction of motion, and the position z of the particles. In a time Δt the fraction

$$W \, d\mu \, d\epsilon \, dz \; \frac{\Delta t}{\tau} \qquad (3.4\text{-}1)$$

of the particles disappear due to absorption, where τ is the mean absorption time. Thus the rate of change of the distribution function due to absorption is

$$\left[\frac{\partial W}{\partial t}\right]_{abs.} = - \frac{W}{\tau} \qquad (3.4\text{-}2)$$

Let $Q(\mu, \epsilon, z, t) \, d\mu \, d\epsilon \, dz$ denote the fractional number of particles injected into the distribution in the range $dz \, d\mu \, d\epsilon$ per unit time, at a time t. The rate of change of the distribution function due to injection of new particles is thus

$$\left[\frac{\partial W}{\partial t}\right]_{sources} = Q(\mu, \epsilon, z, t) \qquad (3.4\text{-}3)$$

These terms are incorporated into the diffusion equation in the next section.

4. The Diffusion Equation for the Distribution Function $W(\mu, \epsilon, z; t)$.

4.1 The Time-Dependent Diffusion Equation.

The partial differential equation for the distribution function $W(\mu, \epsilon, z; t)$ may now be obtained by adding together the contributions to $\frac{\partial W}{\partial t}$ from (a) the betatron collision processes, (b) the scattering by small scale inhomogeneities, (c) absorption by nuclear collisions, and (d) injection of new particles. From (3.2-1), (3.3-14), (3.4-2) and (3.4-3)

$$\frac{\partial W}{\partial t} = - \frac{\partial}{\partial z}\left[c(\mu + D_\mu)\, W\right] - \frac{\partial}{\partial \mu}\left[D_\mu W\right] + \frac{\partial}{\partial \epsilon}\left[D_\epsilon W\right] \tag{4.1-1}$$

$$+ \frac{1}{2}\frac{\partial^2}{\partial \mu^2}\left[D_{\mu\mu} W\right] + \frac{1}{2}\frac{\partial^2}{\partial \epsilon^2}\left[D_{\epsilon\epsilon} W\right] + \frac{\partial^2}{\partial \epsilon \partial \mu}\left[D_{\mu\epsilon} W\right]$$

$$+ \frac{n_2 \sigma_\theta^2}{4}\left[(1-\mu^2)\frac{\partial^2 W}{\partial \mu^2} - 2\mu \frac{\partial W}{\partial \mu}\right] - \frac{W}{\tau}$$

$$+ Q(\mu, \epsilon, z; t)$$

where

$$D_\mu = - \frac{K(1-\mu^2)^3 (1+\mu^2)}{\mu^3 |\mu|} \tag{4.1-2}$$

$$D_\epsilon = - \frac{K(1-\mu^2)^2 (1+\mu^2)}{2\mu^2 |\mu|} \tag{4.1-3}$$

$$D_{\epsilon\epsilon} = \frac{K(1-\mu^2)^2}{|\mu|} \tag{4.1-4}$$

$$D_{\mu\mu} = \frac{K(1-\mu^2)^4}{\mu^2 \, |\mu|} \qquad (4.1\text{-}5)$$

$$D_{\mu\epsilon} = \frac{K(1-\mu^2)^3}{\mu \, |\mu|} \qquad (4.1\text{-}6)$$

$$K = \frac{\pi}{c} \, \omega^2(o) \, \phi(o) \, k_2^2 \left\langle v_g^2 \right\rangle = \frac{\pi}{c} \, \omega^2(o) \, \phi(o) \, \frac{\left\langle \dot{B}^2 \right\rangle}{B_o^2} \qquad (4.1\text{-}7)$$

and where D_{zz}, $D_{z\mu}$, $D_{z\epsilon}$ have been set equal to zero in accordance with (3.2-129) and (3.2-130).

In following sections we shall investigate the anisotropy and energy dependence of cosmic rays using the steady-state form of (4.1-1). Boundary conditions for the stead-state equation will be discussed in the next section.

4.2 The Steady-State Equation; Boundary Conditions.

We shall assume that the source strength Q is independent of the time and consider only the steady-state equation obtained by setting $\frac{\partial W}{\partial t} = 0$. We shall furthermore assume Q is independent of z (uniform space distribution of sources). The diffusion equation then becomes

$$- \frac{\partial}{\partial z}\left[c(\mu + D_\mu)W\right] - \frac{\partial}{\partial \mu}\left[D_\mu W\right] - \frac{\partial}{\partial \epsilon}\left[D_\epsilon W\right] + \frac{1}{2}\frac{\partial^2}{\partial \epsilon^2}\left[D_{\epsilon\epsilon} W\right]$$
$$+ \frac{1}{2}\frac{\partial^2}{\partial \mu^2}\left[D_{\mu\mu} W\right] + \frac{\partial^2}{\partial \epsilon \, \partial \mu}\left[D_{\mu\epsilon} W\right] + \frac{n_2 \sigma_\theta^2}{4}\left[(1-\mu^2)\frac{\partial^2 W}{\partial \mu^2} - 2\mu \frac{\partial W}{\partial \mu}\right]$$
$$- \frac{W}{\tau} = -S(\mu, \epsilon) \qquad (4.2\text{-}1)$$

where $S(\mu, \epsilon)$ describes the angular and the energy distribution of the particles injected by the sources. If the sources inject particles isotropically, then $S(\mu, \epsilon)$ is independent of μ. We now consider the problem of formulating boundary conditions for this equation for the case where W is independent of z. Boundary conditions involving z will be considered in Section 5.2.

We begin by deriving expressions for the two components of the flux vector in the μ-ϵ plane. Consider a region $\epsilon_o \rightarrow \epsilon_o + d\epsilon$, $\mu_o \rightarrow \mu_o + d\mu$ in the μ-ϵ plane as shown in the following figure. The steady-state equation (4.2-1) expresses the physical fact that the total number of particles entering this region through the boundaries is equal to the number of new particles injected into the distribution per unit time due to sources, minus the number of particles per unit time. Let J_μ denote the number of particles per second crossing a unit length of the boundary $\mu = \mu_o$ in the positive μ-direction, and let J_ϵ denote the number of particles per second crossing a unit length of the boundary $\epsilon = \epsilon_o$ in the positive ϵ-direction. The net number of particles per unit time entering the rectangle whose sides are $d\mu$ and $d\epsilon$ is given by

$$- \left[J_\mu (\mu + d\mu, \epsilon) - J(\mu, \epsilon) \right] d\epsilon - \left[J_\epsilon (\mu, \epsilon + d\epsilon) - J(\mu, \epsilon) \right] d\mu \quad (4.2\text{-}2)$$

The number of particles entering per second must be balanced by the number absorbed per second

$$- \left[J_\mu (\mu + d\mu, \epsilon) - J_\mu(\mu, \epsilon) \right] d\epsilon - \left[J_\epsilon (\mu, \epsilon + d\epsilon) - J(\mu, \epsilon) \right] d\mu \quad (4.2\text{-}3)$$

$$- \frac{W}{\tau} \, d\mu \, d\epsilon + S(\mu, \epsilon) \, d\mu \, d\epsilon$$

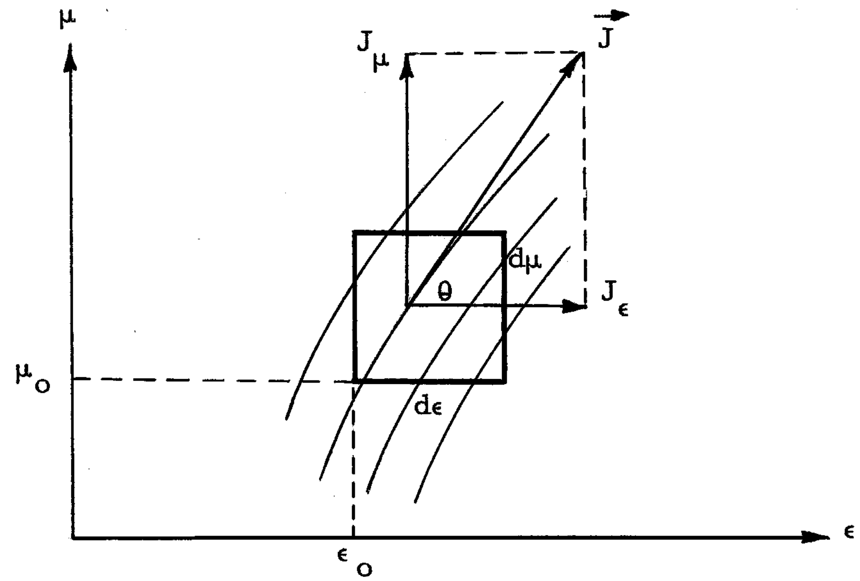

Figure 3. Illustrating the Flux of Particles through a Differential Area in the μ-ε Plane.

or

$$-\frac{\partial J_\mu}{\partial \mu} - \frac{\partial J_\epsilon}{\partial \epsilon} - \frac{W}{\tau} = -S(\mu, \epsilon) \qquad (4.2\text{-}4)$$

Equations (4.2-1) may be written in this form (ignoring the space dependence for the present) by defining

$$-J_\mu = -D_\mu W + \frac{1}{2}\frac{\partial}{\partial\mu}\left[D_{\mu\mu}W\right] + \frac{1}{2}\frac{\partial}{\partial\epsilon}\left[D_{\mu\epsilon}W\right] + k^2(1-\mu^2)\frac{\partial W}{\partial\mu} \qquad (4.2\text{-}5)$$

and

$$-J_\epsilon = -D_\epsilon W + \frac{1}{2}\frac{\partial}{\partial\epsilon}\left[D_{\epsilon\epsilon}W\right] + \frac{1}{2}\frac{\partial}{\partial\mu}\left[D_{\mu\epsilon}W\right] \qquad (4.2\text{-}6)$$

where

$$k^2 = \frac{n_2 \sigma_\theta^2}{4} .$$

Considering only the betatron terms*, it should be possible from these expressions to determine the direction of the flux vector at any point in the μ-ϵ plane. Dividing J_μ by J_ϵ we find

$$\frac{J_\mu}{J_\epsilon} = \frac{-D_\mu W + \frac{1}{2}\frac{\partial}{\partial \mu} D_{\mu\mu} W + \frac{1}{2}\frac{\partial}{\partial \epsilon} D_{\mu\epsilon} W}{-D_\epsilon W + \frac{1}{2}\frac{\partial}{\partial \mu} D_{\epsilon\epsilon} W + \frac{1}{2}\frac{\partial}{\partial \mu} D_{\mu\epsilon} W}$$

$$= \frac{\frac{D_{\mu\mu}}{2}\frac{\partial W}{\partial \mu} + \left[\frac{D'_{\mu\mu}}{2} - D_\mu\right] W + \frac{D_{\mu\epsilon}}{2}\frac{\partial W}{\partial \epsilon}}{\frac{D_{\mu\epsilon}}{2}\frac{\partial W}{\partial \mu} + \left[\frac{D'_{\mu\epsilon}}{2} - D_\epsilon\right] W + \frac{D_{\epsilon\epsilon}}{2}\frac{\partial W}{\partial \epsilon}}$$

$$= \frac{\frac{(1-\mu^2)^4}{2\mu^2 |\mu|}\frac{\partial W}{\partial \mu} - \frac{1}{2}\frac{(1-\mu^2)^3 (1+3\mu^2)}{\mu^3 |\mu|} W + \frac{1}{2}\frac{(1-\mu^2)^3}{\mu |\mu|}\frac{\partial W}{\partial \epsilon}}{\frac{1}{2}\frac{(1-\mu^2)^3}{\mu |\mu|}\frac{\partial W}{\partial \mu} - \frac{1}{2}\frac{(1-\mu^2)^2 (1+3\mu^2)}{\mu^2 |\mu|} W + \frac{1}{2}\frac{(1-\mu^2)^2}{|\mu|}\frac{\partial W}{\partial \epsilon}}$$

$$= \frac{1-\mu^2}{\mu} \qquad (4.2\text{-}7)$$

where (4.1-2) through (4.1-6) have been used. If θ_F is the angle between the flux vector $\vec{J}$ and the ϵ-axis, then

$$\tan \theta_F = \frac{1-\mu^2}{\mu} \qquad (4.2\text{-}8)$$

---

* The remaining terms lead to a flux in the μ-direction.

This expression again verifies the conjecture made in Section 3.2 that the betatron processes produce a flux along the lines

$$\epsilon = \epsilon_o - \frac{1}{2} \ln (1 - \mu^2) \qquad (4.2\text{-}9)$$

in the μ-ϵ plane.

The boundary conditions on (4.2-1) are conveniently expressed in terms of the flux components J_μ and J_ϵ. These conditions are

(a) J_μ and J_ϵ must vanish as $\epsilon \rightarrow \infty$.

(b) J_μ must vanish at $\mu = \pm 1$.

(c) J_μ must vanish at $\mu = 0$.

In addition $W(\mu, \epsilon)$ must satisfy the following conditions

(d) $W(\mu, \epsilon) \geq 0, \quad 0 \leq \epsilon < \infty, \quad -1 \leq \mu \leq 1$.

(e) The integral

$$\int_{\epsilon=0}^{\infty} \int_{\mu=-1}^{+1} W(\mu, \epsilon)\, d\mu\, d\epsilon$$

must be finite.

The physical reasons for conditions (a), (b), (d) and (e) are evident. Condition (c) follows from the fact that J_μ is an odd function of μ and the condition that J_μ have no discontinuity at $\mu = 0$. As we shall see later, condition (b) will be satisfied for any solution of (4.2-1) which is finite at $\mu = \pm 1$, and, in addition, $J_\epsilon(\mu)$ will vanish at $\mu = 0$.

The above conditions do not constitute a complete set of boundary conditions since we have not specified any conditions at $\epsilon = 0$. We may safely ignore this boundary condition*, however, since the behavior of $W(\mu, \epsilon)$ for large values of ϵ (the region of primary interest) is not influenced appreciably by the behavior near $\epsilon = 0$.

---

* The boundary condition at $\epsilon = 0$ is difficult to formulate for several reasons. In the first place the diffusion equation is not expected to be particularly accurate at low energies, especially if the diffusion coefficients given in (4.1-2) through (4.1-6) are used. (It will be recalled that these coefficients were developed from equations which are accurate only in the extreme relativistic limit.) Furthermore, we have neglected processes that may be important at low energies, such as ionization losses, and the detailed nature of the energy distribution of the newly injected particles.

5. ANISOTROPY PRODUCED BY BOUNDARY EFFECTS.

5.1 General

In the following section we shall consider the problem of the anisotropy near the boundaries of the magnetic field. Since most particles move in helices whose radius of curvature is small compared to the radial dimension of the spiral arm, we may ignore the problem of diffusion of particles out of the magnetic field in the radial direction and consider only diffusion in the direction of the magnetic field.

In treating this problem we assume, in accordance with experimental observations, that the angular distribution of the particles in the region far from the ends of the spiral arm is kept very nearly isotropic by the scattering from small scale inhomogeneities. We may thus ignore the scattering by the betatron mechanism without making significant errors.

Ignoring the possibility of non-uniform space distributions of sources, we shall show that the term involving z in (4.1-1) is unimportant except near the ends of the spiral arm where particles cease to be accelerated and scattered. It will turn out that this term may be replaced, under certain condition, by a term $\frac{W(\mu, \epsilon)}{\tau_D}$ which accounts for loss of particles by diffusion out of the ends of the spiral arm. The diffusion time constant τ_D depends on the scattering coefficient $\frac{n_2 \sigma_\theta^2}{4}$ and on the length of the spiral arm. A calculation of τ_D is made in Section 5.3 for the case of a uniform space distribution of sources along the length of the spiral arm.

5.2 Anisotropy Produced by Diffusion of Particles Out of the Spiral Arm

Under the assumptions given in the previous section, (4.1-1) reduces to the following equation for an arbitrary space distribution of sources, which inject new particles isotropically

$$- c\mu \frac{\partial F}{\partial z} + \frac{n_2 \sigma_\theta^2}{4} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial F}{\partial \mu} \right] - \frac{F}{\tau} = - S(z) \qquad (5.2-1)$$

To solve (5.2-1) we use a familiar method from the "one-velocity" theory of neutron diffusion, expanding $F(\mu, z)$ in a series of Legendre Polynomials:

$$F(\mu, z) = \sum_{n=o}^{\infty} A_n(z) \ P_n(\mu) \qquad (5.2-2)$$

In the present analysis, we shall eventually retain only the first three terms in the series. The first two coefficients $A_o(z)$ and $A_1(z)$ have obvious physical interpretations in terms of the density of particles $\rho(z)$ and the flux $J(z)$:

$$A_o(z) = \frac{1}{2} \rho(z) = \frac{1}{2} \int_{-1}^{+1} F(\mu, z) \, d\mu \qquad (5.2-3)$$

$$A_1(z) = \frac{3}{2c} J(z) = \frac{3}{2} \int_{-1}^{+1} \mu \, F(\mu, z) \, d\mu \qquad (5.2-4)$$

Substituting the series (5.2-2) into the diffusion equation and using the well-known formula

$$\frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial P_n(\mu)}{\partial \mu} \right] = - n(n+1) P_n(\mu) \qquad (5.2-5)$$

we find

$$- \sum_{n=o}^{\infty} \left[c \frac{\partial A_n(z)}{\partial z} \mu P_n(\mu) + \frac{n_2 \sigma_\theta^2}{4} n(n+1) A_n(z) P_n(\mu) + \frac{1}{\tau} A_n(z) P_n(\mu) \right]$$

$$= -S(z) \qquad (5.2\text{-}6)$$

Multiplying by $P_m(\mu)$ and integrating over μ from -1 to $+1$

$$\sum_{n=o}^{\infty} c \frac{\partial A_n(z)}{\partial z} \int_{-1}^{+1} \mu P_n(\mu) P_m(\mu) + A_n(z) \left[\frac{n_2 \sigma_\theta^2}{4} n(n+1) + \frac{1}{\tau} \right]$$

$$\int_{-1}^{+1} P_n(\mu) P_m(\mu) \, d\mu$$

$$= -S(z) \int_{-1}^{+1} P_m(\mu) \, d\mu = -S(z) \int_{-1}^{+1} P_m(\mu) P_o(\mu) \, d\mu \qquad (5.2\text{-}7)$$

Making use of the orthogonality properties of $P_n(\mu)$

$$\int_{-1}^{+1} P_n(\mu) P_m(\mu) \, d\mu = \frac{2}{2m+1} \delta_{m,n} \qquad (5.2\text{-}8)$$

(5.2-7) becomes

$$\frac{(2m+1)}{2} c \sum_{n=0}^{\infty} \frac{\partial A_n(z)}{\partial z} \int_{-1}^{+1} \mu P_n(\mu) P_m(\mu) \, d\mu + k^2 m(m+1) + \frac{1}{\tau} A_m(z)$$

$$= S(z) \delta_{m,o} \qquad (5.2\text{-}9)$$

where the Kronecker δ notation has been used, and where

$$k^2 = \frac{n_2 \sigma_\theta^2}{4} .$$

The remaining integration may be performed by making use of the recursion formula (Morse and Feshback 1953)

$$\mu P_m(\mu) = \frac{1}{2m+1} \left[(m+1) P_{m+1}(\mu) + m P_{m-1}(\mu) \right] \tag{5.2-10}$$

Multiplying this expression by $P_n(\mu)$, integrating, and making use of (5.2-8) we find

$$\int_{-1}^{+1} \mu P_m(\mu) P_n(\mu) d\mu = \frac{2}{2m+1} \left[\frac{m+1}{2m+3} \delta_{n,m+1} + \frac{m}{2m-1} \delta_{n,m-1} \right] \tag{5.2-11}$$

Inserting this expression in (5.2-9), and summing over n gives the following infinite set of equations for determining the functions $A_n(z)$.

$$c \frac{m+1}{2m+3} \frac{\partial A_{m+1}(z)}{\partial z} + c \frac{m}{2m-1} \frac{\partial A_{m-1}(z)}{\partial z} + \left[k^2 m(m+1) + \frac{1}{\tau} \right] A_m(z)$$

$$= S(z)\, \delta_{m,o} \tag{5.2-12}$$

This equation holds for all values of m if we adopt the convention that $A_n(z) = 0$ when n is negative. The first three members of the set are

$$\frac{c}{3} \frac{\partial A_1(z)}{\partial z} + \frac{1}{\tau} A_o(z) = S(z) \tag{5.2-13}$$

$$\frac{2c}{5} \frac{\partial A_2(z)}{\partial z} + c \frac{\partial A_o(z)}{\partial z} + \left(2k^2 + \frac{1}{\tau}\right) A_1(z) = 0 \qquad (5.2\text{-}14)$$

$$\frac{3c}{7} \frac{\partial A_3(z)}{\partial z} + \frac{2c}{3} \frac{\partial A_1(z)}{\partial z} + \left(6k^2 + \frac{1}{\tau}\right) A_2(z) = 0 \qquad (5.2\text{-}15)$$

To simplify the notation further we define

$$a_1 = 2k^2 + \frac{1}{\tau} = \frac{n_2 \sigma_\theta^2}{2} + \frac{1}{\tau} \qquad (5.2\text{-}16)$$

$$a_2 = 6k^2 + \frac{1}{\tau} = \frac{3}{2} n_2 \sigma_\theta^2 + \frac{1}{\tau} \qquad (5.2\text{-}17)$$

A reasonably accurate solution is obtained by dropping all terms higher than A_2. The equations for determining A_o, A_1, and A_2 may then be rewritten

$$\frac{c}{3} \frac{\partial A_1(z)}{\partial z} + \frac{A_o(z)}{\tau} = S(z) \qquad (5.2\text{-}18)$$

$$\frac{2c}{5} \frac{\partial A_2(z)}{\partial z} + c \frac{\partial A_o(z)}{\partial z} + a_1 A_1(z) = 0 \qquad (5.2\text{-}19)$$

$$\frac{2c}{3} \frac{\partial A_1(z)}{\partial z} + a_2 A_2(z) = 0 \qquad (5.2\text{-}20)$$

We may conveniently solve these equations by making use of the Fourier transform. Let

$$A_n(p) = \int_{-\infty}^{+\infty} A_n(z)\, e^{-pz}\, dz \qquad (n = 0, 1, 2) \qquad (5.2\text{-}21)$$

$$S(p) = \int_{-\infty}^{+\infty} S(z)\, e^{-pz}\, dz \qquad (5.2\text{-}22)$$

where p denotes $i\omega$. Multiplying the three equations by e^{-pz} and integrating from $-\infty$ to $+\infty$, we obtain*

$$\frac{c}{3} p\, A_1(p) + \frac{1}{\tau} A_o(p) = \dot{S}(p) \tag{5.2-23}$$

$$\frac{2c}{5} p\, A_2(p) + c\, p\, A_o(p) + \alpha_1 A_1(p) = 0 \tag{5.2-24}$$

$$\frac{2c}{3}\, p\, A_1(p) + \alpha_2 A_2(p) = 0 \tag{5.2-25}$$

Solving these equations for A_o, A_1, and A_2, we find

$$A_o(p) = \frac{-\tau \left[1 - \lambda_o^2\, \delta p^2\right] S(p)}{\Lambda^2 p^2 - 1} \tag{5.2-26}$$

$$A_1(p) = \frac{3\lambda_o^2}{c} \frac{p\, S(p)}{\Lambda^2 p^2 - 1} \tag{5.2-27}$$

$$A_2(p) = -5\tau\delta\lambda_o^2 \frac{p^2\, S(p)}{\Lambda^2 p^2 - 1} \tag{5.2-28}$$

where

$$\Lambda^2 = \lambda_o^2 (1 + \delta) \tag{5.2-29}$$

$$\delta = \frac{4}{5\tau\alpha_2} = \frac{4}{5\left[\frac{6}{\tau} \frac{n_2 \sigma_\theta^2}{4} + 1\right]} \tag{5.2-30}$$

---

* It is assumed that $A_n(z)$, $n = 0, 1, 2$ and $S(z) \rightarrow 0$ as $|z| \rightarrow \infty$. This condition is reasonable physically and is a necessary condition for the existence of the Fourier transforms of these quantities.

$$\lambda_o^2 = \frac{c^2 \tau}{3 a_1} = \frac{c^2 \tau}{6\left[\frac{n_2 \sigma_\theta^2}{4} + \frac{1}{2\tau}\right]} \qquad (5.2\text{-}31)$$

It will be noted that the constants Λ and λ_o have the dimensions of length, and δ is dimensionless. We shall adopt the language of neutron diffusion theory and refer to Λ and λ_o as "diffusion lengths". The physical significance of these quantities will become apparent presently.

Now let $2a$ denote the length of the spiral arm, and consider a uniform distribution of sources over $-a \leq z \leq a$. If there are no sources outside of this interval then we may write

$$\begin{cases} S(z) = S_o & -a \leq z \leq a \\ S(z) = 0 & |z| > a \end{cases} \qquad (5.2\text{-}32)$$

$$S(p) = \int_{-\infty}^{+\infty} S(z)\, e^{-pz}\, dz = S_o \int_{-a}^{+a} e^{-pz}\, dz = \frac{2S_o}{p} \sinh(pa) \qquad (5.2\text{-}33)$$

Applying the Fourier inversion formula to (5.2-26) we find for this case

$$A_o(z) = \frac{-\tau}{\Lambda^2} \frac{2S_o}{2\pi j} \int_{-j\infty}^{+j\infty} \frac{\left[1 - \lambda_o^2 \delta p^2\right] \sinh(pa)\, e^{pz}}{p\left[p^2 - \frac{1}{\Lambda^2}\right]}\, dp$$

$$= \frac{2S_o \tau}{\Lambda^2 \pi} \int_o^{\infty} \frac{\left[1 + \lambda_o^2\, \delta \omega^2\right] \sin \omega a \cos \omega z}{\omega\, (\omega^2 + \frac{1}{\Lambda^2})}\, d\omega \qquad (5.2\text{-}34)$$

This integral is easily evaluated using the calculus of residues or evaluated from Tables (Erdelyi, et. al. 1954). The result is

$$A_o(z) = \tau S_o \left[1 - \left(\frac{1}{1+\delta}\right) e^{-\frac{a}{\Lambda}} \cosh\left(\frac{z}{\Lambda}\right) \right] \qquad -a \leq z \leq a \qquad (5.2\text{-}35)$$

$$= \frac{\tau S_o}{1+\delta} \; e^{-\frac{z}{\Lambda}} \sinh\left(\frac{a}{\Lambda}\right) \qquad |z| \geq a \qquad (5.2\text{-}36)$$

where (5.2-29) has been used to eliminate λ_o. We find in a similar manner,*

$$A_1(z) = \frac{3\Lambda S_o}{c(1+\delta)} \; e^{-\frac{a}{\Lambda}} \sinh\left(\frac{z}{\Lambda}\right) \qquad -a \leq z \leq a \qquad (5.2\text{-}37)$$

$$= \frac{3\Lambda S_o}{c(1+\delta)} \; e^{-\frac{z}{\Lambda}} \sinh\left(\frac{a}{\Lambda}\right) \qquad z \geq a \qquad (5.2\text{-}38)$$

$$= \frac{3\Lambda S_o}{c\,(1+\delta)} \; e^{+\frac{z}{\Lambda}} \sinh\left(\frac{a}{\Lambda}\right) \qquad z \leq -a \qquad (5.2\text{-}39)$$

and

$$A_2(z) = \frac{-5\tau\,\delta\, S_o}{1+\delta} \; e^{-\frac{a}{\Lambda}} \cosh\left(\frac{z}{\Lambda}\right) \qquad -a \leq z \leq a \qquad (5.2\text{-}40)$$

$$= \frac{-5\tau\,\delta\, S_o}{1+\delta} \; e^{-\frac{z}{\Lambda}} \sinh\left(\frac{a}{\Lambda}\right) \qquad |z| \geq a \qquad (5.2\text{-}41)$$

---

* It will be noted from the above expressions that $A_o(z)$ and $A_1(z)$ (and hence the density $\rho(x)$ and the flux $J(x)$) are continuous functions, but that $A_2(z)$ is discontinuous at $z = \pm a$. This is, of course, a consequence of discontinuity in the source density at these two points.

The constant λ_o appearing in the above equations in the diffusion length we would have obtained if we had neglected the term involving $A_2(z)$ in (5.2-13) and (5.2-14). Retaining this term does not alter the form of the solutions obtained, but corrects the diffusion length by the factor $(1 + \delta)^{1/2}$.

It is evident from these expressions that if the diffusion length Λ is small compared to the half-length a of the spiral arma then $\rho(z)$ is essentially constant and the flux $J(z)$ is essentially zero except within a few diffusion lengths of points $z = \pm a$. If we take the opposite extreme and let $\Lambda \rightarrow \infty$, then the distribution function $F(\mu, z)$ contains significant first and second harmonic terms. It can be shown that in this case the density $\rho(z)$ approaches the limiting form

$$\rho(z) = A(z^2 - b^2) \tag{5.2-42}$$

where A and b are constants and the flux $J(z)$ becomes a linear function of z .

5.3 The Mean Time for Escape by Diffusion

The above expressions for $A_o(z)$ and $A_1(z)$ will now be used to compute the mean time for a cosmic ray particle to escape by diffusion out of the spiral arm. Considerable care must be exercised in using these expressions, as we will see presently.

Let $\frac{dN_D}{dt}$ denote number of particles escaping per unit time due to diffusion, and let N_o denote the total number of particles present. In the steady state, both of these quantities are constant and their ratio

$$\frac{1}{\tau_D} = \frac{1}{N_o} \frac{dN_D}{dt} \tag{5.3-1}$$

is the reciprocal of the mean time for escape. The rate $\frac{dN_D}{dt}$ at which the particles escape is conveniently evaluated by computing the total flux crossing the boundaries, and N_o is evaluated from the density function $\rho(z)$.

Before proceding with the calculation we make some preliminary remarks regarding the behavior of $\rho(z)$ and $J(z)$ near the boundary. From (5.2-3)

$$\rho(z) = 2\tau S_o\left[1 - \left(\frac{1}{1+\delta}\right) e^{-a/\Lambda} \cosh\left(\frac{z}{\Lambda}\right)\right] \quad -a \leq z \leq a \tag{5.3-2}$$

$$\rho(z) = \frac{2\tau S_o}{1+\delta}\, e^{-|z|/\Lambda} \sinh\frac{a}{\Lambda} \qquad |z| > a \tag{5.3-3}$$

From (5.2-4) the flux $J(z)$ in the region $-a \leq z \leq a$, is given by

$$J(z) = \frac{2 \Lambda S_o}{1 + \delta} e^{-a/\Lambda} \sinh \frac{z}{\Lambda} \tag{5.3-4}$$

The behavior of $\rho(z)$ near the boundary is illustrated in the following diagram, which is drawn for the case where $\Lambda \ll a$.

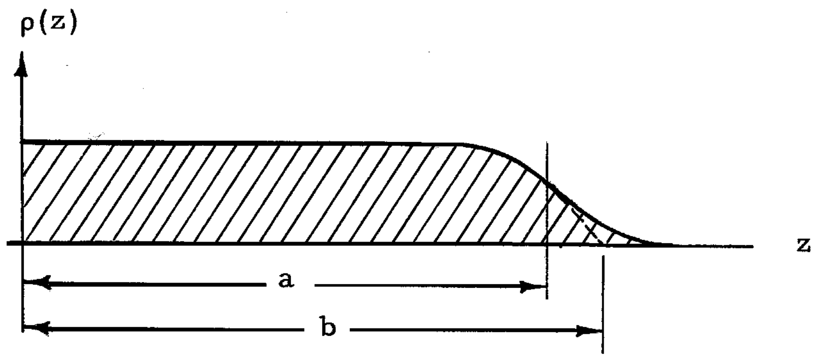

Figure 4. Behavior of the Density $\rho(z)$ Near the Boundary of the Spiral Arm.

We define an "effective boundary" of the spiral arm by extrapolating the function (5.3-2) into the region $|z| > a$ and noting that it becomes zero at $z = \pm b$ where

$$\frac{1}{1 + \delta} e^{-a/\Lambda} \cosh \frac{b}{\Lambda} = 1 \tag{5.3-5}$$

The shape of the extrapolated curve is shown by the dotted line in the above diagram.

In evaluating the number of particles per unit time diffusing out of the spiral arm we must evaluate the flux at $z = \pm b$ rather

$z = \pm a$, to avoid counting those particles which cross $z = \pm a$ but later return.

Eliminating the factor $\frac{1}{1+\delta} e^{-a/\Lambda}$ in (5.3-2) and (5.3-4) using (5.3-5), we obtain

$$\rho(z) = 2\tau S_o \left[1 - \frac{\cosh \frac{z}{\Lambda}}{\cosh \frac{b}{\Lambda}} \right] \tag{5.3-6}$$

$$J(z) = 2 \Lambda S_o \frac{\sinh \frac{z}{\Lambda}}{\cosh \frac{b}{\Lambda}} \tag{5.3-7}$$

From (5.3-7) the total number of particles crossing $z = \pm b$, is

$$\frac{dN_D}{dt} = \left| J(+b) \right| + \left| J(-b) \right| = 4 \Lambda S_o \tanh \frac{b}{\Lambda} \tag{5.3-8}$$

and from (5.3-6),

$$N_o = \int_{-b}^{+b} \rho(z) dz = 2\tau S_o \int_{-b}^{+b} \left[1 - \frac{\cosh \frac{z}{\Lambda}}{\cosh \frac{b}{\Lambda}} \right] dz \tag{5.3-9}$$

$$= 4\tau S_o b \left[1 - \frac{\Lambda}{b} \tanh \frac{b}{\Lambda} \right]$$

Hence,

$$\tau_D = \left[\frac{1}{N_o} \frac{dN_D}{dt} \right]^{-1} = \tau \left[\frac{b}{\Lambda} \operatorname{ctnh} \frac{b}{\Lambda} - 1 \right] \tag{5.3-10}$$

It is of some interest to examine the behavior of τ_D as $\tau \rightarrow \infty$

(no absorption by nuclear collisions). Expanding $\frac{b}{\Lambda}$ ctnh $\frac{b}{\Lambda}$ in a power series about $\frac{b}{\Lambda} = 0$,

$$\tau_D = \frac{\tau b^2}{3\Lambda^2}\left[1 - \frac{1}{15}\frac{b}{\Lambda} + \frac{2}{305}\left(\frac{b}{\Lambda}\right)^2 + \ldots\right] \qquad (5.3\text{-}11)$$

and letting $\tau \rightarrow \infty$, we note, from (5.2-29) and (5.2-30), that $\frac{b}{\Lambda} \rightarrow 0$ but $\frac{\tau b^2}{3\Lambda^2}$ approaches a finite value which is given by*

$$\tau_D = \frac{5b^2 n_2 \sigma_e^2}{18c^2} \qquad (5.3\text{-}12)$$

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* This expression agrees within a factor of 2 with a similar expression derived by Davis (1956) by a quite different procedure.

6. PROPERTIES AND SOLUTIONS OF THE STEADY-STATE DIFFUSION EQUATION

6.1 Behavior of the Solution as a Function of μ and ϵ

In the following sections we shall undertake the problem of finding solutions of the steady-state diffusion equation (4.2-1). There is very little hope of obtaining solutions in closed form, even if the space dependence of $W(\mu, \epsilon, z)$ is neglected and a simple angular distribution function for the injected particles is assumed. Series solutions, while possible, are not too useful due to their slow convergence*. We must, therefore, consider other methods.

Since the major interest in the solutions of (4.2-1) lies in the region $\epsilon \gg 1$, one is naturally led to inquire if asymptotic solutions can be found. The conventional method of finding asymptotic solutions is simply to drop those terms unimportant at large ϵ, thereby obtaining a simpler equation which is more amenable to solution. This procedure is impossible in the present case since the coefficients do not depend on the variable ϵ. To obtain a solution valid for large ϵ, we make use of the fact that the solution for

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* In principle, one could expand the solution in a series of Legendre Polynomials, in which the coefficients depend on ϵ :

$$W(\mu, \epsilon) = \sum_{n=0}^{\infty} A_n(\epsilon) P_n(\mu)$$

The major difficulty with such an approach is the practical one of not being able to carry out the required manipulations necessary to compute the functions $A_n(\epsilon)$. If one could be content with computing the first few coefficients in the series, then this approach might prove feasible. It is almost surely the case, however, that a large number of terms are required to adequately represent the solution, particularly in the region near $\mu = 0$, where the solution departs radically from isotropy.

large ϵ does not depend significantly on the source distribution $S(\mu, \epsilon)$. Stated another way, the diffusion processes smooth out the irregularities due to the injection processes, so that the solution for large ϵ is rather insensitive to particular form chosen for $S(\mu, \epsilon)$. In this region of the $\mu - \epsilon$ plane there are no sources, hence $W(\mu, \epsilon)$ satisfies the homogeneous form of (4.2-1).

It will be shown later that the solution in this region is of the form

$$W(\mu, \epsilon) = A e^{-\alpha \epsilon} f(\mu) \tag{6.1-1}$$

A numerical procedure has been devised for computing $f(\mu)$ and the constant α, as a function of the constant parameters appearing in the diffusion equation. The numerical procedure will be described later.

Before proceding, it will be useful to form a qualitative picture of the solution as a function of μ and ϵ. As an aid to visualizing the solution, we may regard the function $W(\mu, \epsilon)$ as defining a two-dimensional surface projecting above the $\mu - \epsilon$ plane, over the region $0 \leq \epsilon < \infty$, $-1 \leq \mu \leq 1$. Since the sources are located near $\epsilon = 0$, we expect that the surface will be sharply peaked near $\epsilon = 0$, with the behavior of the surface near $\epsilon = 0$ depending strongly on the source strength, $S(\mu, \epsilon)$.

As ϵ increases, we expect W to decrease to zero in an approximately exponential fashion for a fixed value of μ. It will be noted that the coefficients appearing in (4.2-1) do not depend on ϵ. For large ϵ the shape of the contours along lines of constant ϵ are therefore expected to be independent of ϵ. $W(\mu, \epsilon)$ should then reduce to the form of a product of a function of μ only and a function of ϵ only.

The behavior of $W(\mu, \epsilon)$ as a function of μ is somewhat more complicated. To form an intuitive picture of the solution, we must consider the scattering of the cosmic ray particles by the betatron process and the small scale inhomogeneities in the magnetic field.

It was shown in Section 3.2 that the form of the diffusion coefficients D_μ, D_ϵ, $D_{\mu\mu}$, etc., was consistent with the statement that the betatron mechanism causes particles to move along the curves

$$\epsilon = \epsilon_o - \frac{1}{2} \ln(1 - \mu^2) \tag{6.1-2}$$

in the $\mu - \epsilon$ plane.

It can be shown that the particles tend to drift* along these curves to higher energies (and steeper spirals). It is not surpri-

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* We may visualize the interaction of the cosmic ray particles with the large scale inhomogeneities as a random walk along the curves $\epsilon = \epsilon_o - \frac{1}{2} \ln(1 - \mu^2)$ with a step size that depends on μ. The fact that there is a steady drift means that it is more likely for a particle to step in the direction of larger ϵ than for a particle to step in the opposite direction. The effect of the small scale inhomogeneities is to scatter particles from one curve to another. The importance of the steady drift lies in the fact that it provides an efficient mechanism for acceleration. Even if the steady drift were absent or in the opposite direction, some particles could still reach high energies by the random walk process, but the efficiency of this mechanism for producing a considerable number of high energy particles would be considerably reduced.

sing, *in view of this statement, that the betatron mechansim* by itself (no absorption or scattering by small scale inhomogeneities) produces a very anisotropic distribution with most of the particles lying in very steep spirals. However, with a sufficient amount of scattering by small scale inhomogeneities, the distribution may still be nearly isotropic (except near $\mu = 0$). In the region of space where boundary effects are negligible, the degree of anisotropy is determined by the relative effectiveness of the two processes. This is conveniently measured by the ratio

$$\lambda = \frac{n_2 \sigma_\theta^2 / 4}{K} = \frac{c n_2 \sigma_\theta^2}{2\pi \phi(o) \omega^2(o) \left\langle v_g^2 \right\rangle k_2^2} \qquad (6.1\text{-}3)$$

The nature of the diffusion coefficients D_μ, D_ϵ , etc., near $\mu = 0$ have a marked effect on the shape of the surface $W(\mu, \epsilon)$. For any value of ϵ , it can be shown that $W(\mu, \epsilon) \to 0$ as $\mu \to 0$. This means there is a deep cleft in the surface along the line $\mu = 0$. As the parameter λ increases, we expect that the cleft will become very narrow and that the distribution will be almost isotropic (independent of μ) except in the vicinity of $\mu = 0$. As $\lambda \to \infty$, we expect the distribution to become isotropic.

6.2 Approximations for Large Values of ϵ

The arguments of the preceding section lead us to suspect that the solution to (4.2-1) for $\epsilon >> 1$, is of the form

$$W(\mu, \epsilon) = A e^{-\alpha \epsilon} f(\mu) \tag{6.2-1}$$

where A and α are constants. It will be shown presently that a solution to (6.2-1) of this form is possible provided that the parameter α is chosen properly. We assume that ϵ is large enough that there are no sources in the region of interest. We may thus set

$$S(\mu, \epsilon) = 0 \tag{6.2-2}$$

and require that (6.2-1) be a solution of

$$\begin{aligned} &-\frac{\partial}{\partial\mu}\left[D_{\mu} W\right] - \frac{\partial}{\partial\epsilon}\left[D_{\epsilon} W\right] + \frac{1}{2}\frac{\partial^2}{\partial\epsilon^2}\left[D_{\epsilon\epsilon} W\right] + \frac{1}{2}\frac{\partial^2}{\partial\mu^2}\left[D_{\mu\mu} W\right] \\ &+ \frac{\partial^2}{\partial\epsilon\,\partial\mu}\left[D_{\mu\epsilon} W\right] + \frac{n_2\sigma_\theta^2}{4}\left[(1-\mu^2)\frac{\partial^2 W}{\partial\mu^2} - 2\mu\frac{\partial W}{\partial\mu}\right] - \frac{W}{\tau} = 0 \end{aligned} \tag{6.2-3}$$

Substituting $e^{-\alpha\epsilon} f(\mu)$ for $W(\mu, \epsilon)$, carrying out the indicated differentiations, and dividing through by $e^{-\alpha\epsilon}$, this equation becomes

$$\begin{aligned} &\left[\frac{D_{\mu\mu}}{2} + \frac{n_2\sigma_\theta^2}{4}(1-\mu^2)\right]\frac{d^2 f}{d\mu^2} + \left[D_{\mu\mu}' - D_{\mu} - \alpha D_{\mu\epsilon} - 2\left(\frac{n_2\sigma_\theta^2}{4}\right)\mu\right]\frac{df}{d\mu} \\ &+ \left[\frac{D_{\mu\mu}''}{2} - D_{\mu}' - \alpha(D_{\mu\epsilon}' - D_{\epsilon}) + \frac{\alpha^2}{2} D_{\epsilon\epsilon} - \frac{1}{\tau}\right] f = 0 \end{aligned} \tag{6.2-4}$$

where the prime denotes differentiation with respect to μ . Using Equations (4.1-2) through (4.1-6), we find

$$D'_{\mu\mu} - D_{\mu} = \frac{-2K(1-\mu^2)^3 \; (1+2\mu^2)}{\mu^3 \; |\mu|} \tag{6.2-5}$$

$$\frac{D''_{\mu\mu}}{2} - D'_{\mu} = \frac{2K(1-\mu^2)^2 \; (3\mu^4 + 2\mu^2 + 1)}{\mu^4 \; |\mu|} \tag{6.2-6}$$

$$D'_{\mu\epsilon} - D_{\epsilon} = \frac{-K(1-\mu^2)^2 \; (3+7\mu^2)}{2\mu^2 \; |\mu|} \tag{6.2-7}$$

Substituting these expressions in (6.2-5), collecting terms, and dividing through by

$$\frac{D_{\mu\mu}}{2} + \frac{n_2\sigma_\theta^2}{4}(1-\mu^2)$$

results in the equation

$$f'' + A(\mu)f' + B(\mu)f = 0 \tag{6.2-8}$$

where

$$A(\mu) = \frac{D'_{\mu\mu} - D_{\mu} - \alpha D_{\mu\epsilon} - 2\left(\frac{n_2\sigma_\theta^2}{4}\right)\mu}{\frac{D_{\mu\mu}}{2} + \frac{n_2\sigma_\theta^2}{4}(1-\mu^2)}$$

$$= -\frac{2(1-\mu^2)^3\left[2+(4+\alpha)\mu^2\right] + 4\lambda\mu^5}{\mu(1-\mu^2)\left[(1-\mu^2)^3 + 2\lambda\mu^3\right]} \tag{6.2-9}$$

$$B(\mu) = \frac{\frac{\alpha^2}{2} D_{\epsilon\epsilon} - \alpha(D'_{\mu\epsilon} - D_{\epsilon}) + \frac{D''_{\mu\mu}}{2} - D'_{\mu} - \frac{1}{\tau}}{\frac{D_{\mu\mu}}{2} + \frac{n_2\sigma_\theta^2}{4}(1-\mu^2)}$$

$$= \frac{(1-\mu^2)^2\left[(\alpha^2 + 7\alpha + 12)\mu^4 + (3\alpha + 8)\mu^2 + 4\right] - 2\beta\mu^5}{\mu^2(1-\mu^2)\left[(1-\mu^2)^3 + 2\lambda\mu^3\right]} \qquad (6.2\text{-}10)$$

where the expressions for $A(\mu)$ and $B(\mu)$ involving β and λ hold for $0 \leq \mu \leq 1$, and where (See Equations (4.1-7) and Section 3.3 for definitions of the parameters),

$$\lambda = \frac{n_2\sigma_\theta^2}{4K} \qquad (6.2\text{-}11)$$

$$\beta = \frac{1}{K\tau} \qquad (6.2\text{-}12)$$

If we include the loss of particles by diffusion out of the spiral arm, then τ is defined by (5.3-10). If τ_D denotes the mean time for escape by diffusion out of the spiral arm, and τ_N denotes the mean time for absorption by nuclear collisions, then approximately,

$$\beta = \frac{1}{K\tau} = \frac{1}{K}\left[\frac{1}{\tau_D} + \frac{1}{\tau_N}\right] \qquad (6.2\text{-}13)$$

It is easily shown that the Equation (6.2-4) is unchanged if it is replaced everywhere it appears by $-\mu$. The function $f(\mu)$ is therefore an even function of μ.

The boundary conditions on $W(\mu, \epsilon)$ as $\epsilon \rightarrow \infty$ are automatically satisfied by the assumed exponential form for the energy dependence. The function $f(\mu)$ must also satisfy certain conditions. These will be discussed in Section 6.4.

6.3 Power Series Solutions Valid Near $\mu = 0$ and $\mu = \pm 1$

It is of considerable interest to examine the behavior of the solution to (6.2-8) in the vicinity of $\mu = 0$, and $\mu = \pm 1$. Power series methods will be used. We first examine the behavior near $\mu = 0$, writing (6.2-8) as

$$\mu^2 \frac{d^2 f}{d\mu^2} + \mu P(\mu) \frac{df}{d\mu} + Q(\mu) f = 0 \tag{6.3-1}$$

where $P(\mu)$ and $Q(\mu)$ are rational functions of μ which are regular at $\mu = 0$. Writing N_P and N_Q for the numerators of $P(\mu)$ and $Q(\mu)$, and noting from (6.2-9) and (6.2-10) that $P(\mu)$ and $q(\mu)$ have the same denominator $D(\mu)$,

$$\mu^2 D(\mu) \frac{d^2 f}{d\mu^2} + \mu N_P(\mu) \frac{df}{d\mu} + N_Q(\mu) f = 0 \tag{6.3-2}$$

where

$$\begin{aligned} D(\mu) &= (1 - \mu^2)^4 + 2\lambda\mu^3(1 - \mu^2) \\ &= 1 - 4\mu^2 + 2\lambda\mu^3 + 6\mu^4 - 2\lambda\mu^5 - 4\mu^6 + \mu^8 \end{aligned} \tag{6.3-3}$$

$$\begin{aligned} N_P(\mu) &= -2(1 - \mu^2)^3 \left[2 + (4 + \alpha)\mu^2 \right] - 4\lambda\mu^5 \\ &= -4 + \mu^2 \left[12 - 2K_o \right] + \mu^4 \left[-12 + 6K_o \right] - 4\lambda\mu^5 \\ &\quad + \mu^6 \left[4 - 6K_o \right] + \mu^8 \left[2K_o \right] \end{aligned} \tag{6.3-4}$$

$$N_Q(\mu) = (1 - \mu^2)^2 \left[K_1\mu^4 + K_2\mu^2 + 4 \right] - 2\beta\mu^5$$

$$= 4 + \mu^2\left[-8 + K_2 \right] + \mu^4\left[4 - 2K_2 + K_1 \right] - 2\beta\mu^5$$

$$+ \mu^6\left[-2K_1 + K_2 \right] + K_1\mu^8 \qquad (6.3\text{-}5)$$

and where

$$\begin{aligned} K_o &= 4 + \alpha \\ K_1 &= \alpha^2 + 7\alpha + 12 \\ K_2 &= 3\alpha + 8 \end{aligned} \qquad (6.3\text{-}6)$$

If we proceed in the usual manner to solve (6.3-2) by substituting the power series

$$f = \sum_{n=0}^{\infty} a_n \mu^{n+\sigma} \qquad a_o \neq 0 \qquad (6.3\text{-}7)$$

we obtain the indicial equation

$$\sigma^2 + \left[P(o) - 1 \right] \sigma + Q(o) = 0 \qquad (6.3\text{-}8)$$

or

$$\sigma^2 - 5\sigma + 4 = (\sigma - 1)(\sigma - 4) = 0 \qquad (6.3\text{-}9)$$

We thus obtain two formal power series solutions to (6.3-1) of the form

$$f_2 = \mu^4 \left[1 + \sum_{n=1}^{\infty} a_n \mu^n \right] \qquad (6.3\text{-}10)$$

$$f_1 = \mu \left[1 + \sum_{n=1}^{\infty} a'_n \mu^n \right] \qquad (6.3\text{-}11)$$

Since the roots of the indicial equation differ by an integer, the possibility exists that the second solution ($\sigma = 1$), contains a term involving $\ln \mu$. If this is the case, the series (6.3-11) does not converge in the neighborhood of $\mu = 0$.

The simplest way to settle the question of the convergence of (6.3-11) is to attempt to determine, in a straightforward manner, the coefficients in the two power series. If $f_1(\mu)$ contains a term involving $\ln \mu$, then it will be impossible to find the coefficients a'_n.

Rather than use (6.3-7) we prefer to assume the form

$$f = \sum_{n=0}^{\infty} a_n \mu^n \qquad (6.3\text{-}12)$$

with no restrictions on a_n. Substituting this expression in (6.3-2) and collecting powers of μ we find

$$\sum_{o}^{\infty} a_n \mu^n \Big\{ n(n-1) - 4n + 4 + \mu^2 \Big[-4n(n-1) + (12 - 2K_o)\, n - 8 + K_2 \Big]$$

$$+ \mu^3 \Big[2\lambda\, n(n-1) \Big] + \mu^4 \Big[6n(n-1) + (-12 + 6K_o) n + 4 - 2K_2 + K_1 \Big]$$

$$+ \mu^5 \Big[-2\lambda\, n(n-1) - 4\lambda n - 2\beta \Big]$$

$$+ \mu^6 \Big[-4n(n-1) + (4 - 6K_o)\, n - 2K_1 + K_2 \Big]$$

$$+ \mu^8 \Big[n(n-1) + 2K_o\, n + K_1 \Big] \Big\} = 0 \qquad (6.3\text{-}13)$$

which may be rewritten

$$\sum_{n=o}^{\infty} a_n\, \mu^n\, (n-4)(n-1)$$

$$+ \sum_{n=2}^{\infty} a_{n-2}\, \mu^n \Big[-4\,(n-2)(n-3) + (12 - 2K_o)(n-2) - 8 + K_2 \Big]$$

$$+ \sum_{n=3}^{\infty} a_{n-3}\, \mu^n \Big[2\lambda\, (n-3)(n-4) \Big]$$

$$+ \sum_{n=4}^{\infty} a_{n-4}\, \mu^n \Big[6(n-4)(n-5) + (-12 + 6K_o)(n-4) + 4 - 2K_2 + K_1 \Big]$$

$$+ \sum_{n=5}^{\infty} a_{n-5}\,\mu^n \left[-2\lambda\,(n-5)\,(n-6) - 4\lambda\,(n-5) - 2\beta \right]$$

$$+ \sum_{n=6}^{\infty} a_{n-6}\,\mu^n \left[-4\,(n-6)\,(n-7) + (4 - 6K_o)\,(n-6) - 2K_1 + K_2 \right]$$

$$+ \sum_{n=8}^{\infty} a_{n-8}\,\mu^n \left[(n-8)\,(n-9) + 2K_o(n-8) + K_1 \right] \qquad (6.3\text{-}14)$$

Successively letting $n = 0,\ 1,\ \ldots.$ we find

$$n = 0 \qquad a_o = 0 \qquad (6.3\text{-}15)$$

$$n = 1 \qquad a_1 = \text{arbitrary} \qquad (6.3\text{-}16)$$

$$n = 2 \qquad a_2 = 0 \qquad (6.3\text{-}17)$$

$$n = 3 \qquad -2a_3 + K_o a_1 = 0 \qquad (6.3\text{-}18)$$

$$n = 4 \qquad a_4 = \text{arbitrary} \qquad (6.3\text{-}19)$$

$$n = 5 \qquad 4a_5 - 3K_o a_3 + K_1 a_1 = 0 \qquad (6.3\text{-}20)$$

$$n = 6 \qquad 10a_6 + a_4\left[-32 - 5\alpha \right] + 12\lambda\, a_3 + a_1\left[-4\lambda - 2\beta \right] = 0 \qquad (6.3\text{-}21)$$

$$n = 7 \qquad 18\,a_7 + a_5\left[-60 - 7\alpha \right] + 24\lambda\, a_4 + \left[72 + 19\alpha + \alpha^2 \right] a_3 + a_1\left[-36 - 17\alpha - 2\alpha^2 \right] = 0 \qquad (6.3\text{-}22)$$

$$n = 8 \qquad 28a_8 + a_6\left[-96 - 9\alpha\right] + 40\lambda a_5 + a_4\left[120 + 25\alpha + \alpha^2\right]$$

$$+ a_3\left[-24\lambda - 2\beta\right] = 0 \qquad (6.3\text{-}23)$$

$$n \geq 9 \qquad a_n(n-4)(n-1) + a_{n-2}\Big[-4(n-2)(n-3) + (12-2K_o)(n-2)$$

$$-8 + K_2\Big] + a_{n-3}\, 2\lambda\,(n-3)(n-4)$$

$$+ a_{n-4}\left[6(n-4)(n-5) + (n-4)(-12+6K_o) + 4 - 2K_2 + K_1\right]$$

$$+ a_{n-5}\left[-2\lambda(n-5)(n-6) - 4\lambda(n-5) - 2\beta\right]$$

$$+ a_{n-6}\left[-4(n-6)(n-7) + (4-6K_o)(n-6) - 2K_1 + K_2\right]$$

$$+ a_{n-8}\left[(n-8)(n-9) + 2K_o(n-8) + K_1\right] \qquad (6.3\text{-}24)$$

The terms in the solution can be grouped into two sets, the coefficients in one set being proportional to a_1 and those in the second set being proportional to a_4. Since a_1 and a_4 are both arbitrary, the solution obtained is the general solution to (6.3-2) and can be written

$$f = a_1 f_1 + a_4 f_2 \qquad (6.3\text{-}25)$$

where

$$f_1 = \mu + a'_2 \mu^2 + \ldots. \qquad (6.3\text{-}26)$$

$$f_2 = \mu^4 + a''_5 \mu^5 + \ldots. \qquad (6.3\text{-}27)$$

The coefficients $a_2, \dots ; a_5, \dots,$ are easily obtained by step-by-step use of the recursion formulas (6.3-15) through (6.3-24).

We next examine the behavior of the solution of (6.3-2) near $\mu = \pm 1$. We assume a power series solution of the form

$$f = \sum_{n=0}^{\infty} b_n (\mu - 1)^{n+\sigma} \qquad b_o \neq 0 \tag{6.3-28}$$

The indicial equation is immediately found to be

$$\sigma^2 = 0 \tag{6.3-29}$$

Thus, only one solution of (6.2-9) has a power series expansion about $\mu = 1$. Let $f_3(\mu)$ denote this solution. Since $\sigma = 0$ is a double root of the indicial equation, the second linearly independent solution $f_4(\mu)$ must involve logarithmic factors of the form $\ln(\mu - 1)$ (Whittaker and Watson 1946).

The power series representation of $f_3(\mu)$ near $\mu = 1$ may be found by writing (6.2-9) in the form

$$(1-\mu^2) \frac{d^2 f}{d\mu^2} + R(\mu) \frac{df}{d\mu} + S(\mu) f = 0 \tag{6.3-30}$$

where

$$R(\mu) = \frac{-2(1-\mu^2)^3 \left[2 + (4+\alpha)\mu^2\right] - 4\lambda \mu^5}{\mu \left[(1-\mu^2)^3 + 2\lambda \mu^3\right]} \tag{6.3-31}$$

$$S(\mu) = \frac{(1-\mu^2)^2 \left[(\alpha^2+7\alpha+12)\mu^4 + (3\alpha+8)\mu^2 + 4\right] - 2\beta \mu^5}{\mu^2 \left[(1-\mu^2)^3 + 2\lambda \mu^3\right]} \tag{6.3-32}$$

Note that $R(\mu)$ and $S(\mu)$ are regular at $\mu = 1$. Repeatedly differentiating (6.3-30) and setting $\mu = 1$ we find

$$f_3(1) = \text{arbitrary} \tag{6.3-33}$$

$$R(1)\, f'_3(1) + S(1)\, f_3(1) = 0 \tag{6.3-34}$$

$$\left[-2 + R(1)\right] f''_3(1) + \left[R'(1) + S(1)\right] f'_3(1) + S'(1) f_3(1) = 0 \tag{6.3-35}$$

Using (6.3-31) and (6.3-32) we find

$$f'_3(1) = -\frac{\beta}{2\lambda}\, f(1) \tag{6.3-36}$$

$$f''_3(1) = -\frac{\beta}{8\lambda}\left[2 + \frac{\beta}{\lambda}\right] f(1) \tag{6.3-37}$$

The solution near $\mu = 1$ is thus given by

$$f_3(\mu) = f_3(1) + f'_3(1)(\mu - 1) + \frac{f''_3(1)}{2!}(\mu - 1)^2 + \dots$$

$$= f_3(1)\left[1 + \frac{\beta}{2\lambda}(1-\mu) - \frac{\beta}{16\lambda}\left(2 + \frac{\beta}{\lambda}\right)(1-\mu)^2 + \dots\right]$$

(6.3-38)

The results of the preceding analysis may be summarized as follows:

(a) The general solution of (6.2-9) is of the form

$$f = a_1 f_1(\mu) + a_2 f_2(\mu) \tag{6.3-39}$$

where

$$f_1(\mu) = \mu + a'_2 \mu^2 + a'_3 \mu^3 + \ldots . \qquad (6.3\text{-}40)$$

and

$$f_2(\mu) = \mu^4 + a''_5 \mu^5 + a''_6 \mu^6 + \ldots . \qquad (6.3\text{-}41)$$

and where the coefficients $a_2, a_3, \ldots; a_5, a_6, \ldots .$ may be obtained from the recursion formulas (6.3-15) through (6.3-24).

(b) The general solution of (6.2-9) may also be expressed as

$$f(\mu) = A f_3(\mu) + B f_4(\mu) \qquad (6.3\text{-}42)$$

where $f_3(\mu)$ is of the form

$$f_3(\mu) = \sum_{n=0}^{\infty} b_n (\mu - 1)^n \qquad (6.3\text{-}43)$$

with the first three coefficients given in (6.3-38). The solution $f_4(\mu)$ contains terms involving $\ln(\mu - 1)$, and hence is not an admissible solution.

(c) There is, therefore, only one well behaved solution to (6.2-9) which may be written in the form

$$F(\mu) = a_1 f_1(\mu) + a_4 f_2(\mu) = A f_3(\mu) \qquad (6.3\text{-}43)$$

Obviously there must be some relationship of a_1 and a_4 for (6.3-43) to hold. This relationship will be discussed in Section 6.5 after a more detailed examination of boundary conditions.

6.4 Formulation of the Boundary Conditions Involving the Flux in the μ-ϵ Plane.

Next consider the two components of the flux given by (4.2-5) and (4.2-6). Substituting $e^{-\alpha\epsilon} f(\mu)$ for W we find

$$J_\epsilon(\mu) = e^{-\alpha\epsilon}\left[-D_\epsilon + \frac{D'_{\mu\epsilon}}{2} - \alpha\frac{D_{\epsilon\epsilon}}{2} \quad f(\mu) + \frac{D_{\mu\epsilon}}{2}\frac{df}{d\mu}\right] \qquad (6.4\text{-}1)$$

$$= K\, e^{-\alpha\epsilon}\ \frac{(1-\mu^2)^2}{2\mu^3}\left[\mu(1-\mu^2)\frac{df}{d\mu} - \left[1+(3+\alpha)\mu^2\right] f\right]$$

and

$$J_\mu(\mu) = e^{-\alpha\epsilon}\left[\frac{D'_{\mu\mu}}{2} - D_\mu - \alpha\frac{D'_{\mu\epsilon}}{2}\ f(\mu) + \frac{D_{\mu\mu}}{2} + k^2(1-\mu^2)\ \frac{df}{d\mu}\right. \qquad (6.4\text{-}2)$$

$$= K e^{-\alpha\epsilon}\left[\frac{(1-\mu^2)}{2\mu^3}\left[(1-\mu^2)^3 + 2\lambda\mu^3\right]\frac{df}{d\mu} - \frac{(1-\mu^2)^3}{2\mu^4}\left[1+(3+\alpha)\mu^2\right] f\right]$$

Both of these expressions must vanish at $\mu = 0$. We shall first show that $J_\epsilon(\mu)$ vanishes at $\mu = 0$ for any solution of (6.2-9). As noted above, any solution of (6.2-9) may be written in the form

$$f = a_1\left[\mu + a'_2\mu^2 + a'_3\mu^3 + a'_5\mu^5 + \dots\right] \qquad (6.4\text{-}3)$$

$$+ a_4\left[\mu^4 + a''_5\mu^5 + \dots\right]$$

where a_1 and a_4 are arbitrary constants. Near $\mu = 0$

$$f(\mu) = a_1 \left[\mu + a'_2 \mu^2 + a'_3 \mu^3 \right] + a_4 \mu^4 + O(\mu^5) \qquad (6.4\text{-}4)$$

$$f'(\mu) = a_1 \left[1 + 2a'_2 \mu + 3a'_3 \mu^2 + 5a'_5 \mu^4 \right] + a_4 \left[4\mu^3 + a''_5 \mu^4 \right]$$
$$+ O(\mu^5) \qquad (6.4\text{-}5)$$

From (6.2-30) through (6.2-33)

$$a'_2 = 0$$
$$a'_3 = \frac{K_o}{2} = 2 + \frac{\alpha}{2}$$
$$a'_5 = \frac{1}{8} \left[3K_o^2 - 2K_1 \right] = \frac{1}{8}(\alpha^2 + 10\alpha + 24)$$
$$a''_5 = 0 \qquad (6.4\text{-}6)$$

Hence

$$f(\mu) = a_1 \left[\mu + (2 + \frac{\alpha}{2}) \mu^3 + \frac{1}{8}(\alpha^2 + 10\alpha + 24) \mu^5 \right] + a_4 \mu^4 + O(\mu^6) \qquad (6.4\text{-}7)$$

$$f'(\mu) = a_1 \left[1 + 3(2 + \frac{\alpha}{2}) \mu^3 + \frac{5}{8}(\alpha^2 + 10\alpha + 24) \mu^4 \right] + 4a_4 \mu^3 + O(\mu^5) \qquad (6.4\text{-}8)$$

Using these results in (6.4-1) we find

$$J_\epsilon = K e^{-\alpha\epsilon} \left[\frac{3}{2} a_4 \mu + O(\mu^2) \right] \qquad (6.4\text{-}9)$$

which shows that $J_\epsilon(\mu)$ vanishes at $\mu = 0$ for any solution of the form (6.2-9).

We next consider the behaviour of $J_\mu(\mu)$ near $\mu = 0$. Substituting the above expression for $f(\mu)$ and $f'(\mu)$ into (6.4-2) we find

$$J_\mu(\mu) = \frac{K}{2} e^{-\alpha\epsilon} \left[(3a_4 + 2\lambda a_1) + O(\mu^2) \right] \tag{6.4-10}$$

The flux $J_\mu(\mu)$ thus vanishes at $\mu = 0$ if and only if

$$a_4 = -\frac{2}{3} \lambda a_1 \tag{6.4-11}$$

If (6.4-11) is satisfied, then the first non-zero term in $J_\mu(\mu)$ is $O(\mu^2)$. A short computation gives

$$J_\mu(\mu) = \frac{a_1}{2} \left[\beta - \alpha\lambda \right] \mu^2 + O(\mu^3) \tag{6.4-12}$$

which shows that there is no discontinuity in $J_\mu(\mu)$ or its first derivative at $\mu = 0$.

6.5 Analytical Solutions for the Special Case $\alpha = \beta = 0$.

In order to shed further light on the nature and properties of the solutions to (6.2-9), we consider the special case obtained by setting $\alpha = \beta = 0$. It is possible in this case to obtain analytical solutions of the differential equation. The value of treating this special case will become evident presently.

The differential equation obtained by setting $\alpha = \beta = 0$ is [cf. (6.2-10) and (6.2-11)]

$$\left[\frac{D_{\mu\mu}}{2} + k^2(1-\mu^2)\right] f'' + \left[D'_{\mu\mu} - D_{\mu} - 2k^2\right] f' + \left[\frac{D_{\mu\mu}}{2} - D'_{\mu}\right] f = 0 \tag{6.5-1}$$

To solve (6.5-1) we first note that the terms involving k^2 may be written in the following form

$$k^2(1-\mu^2)\frac{d^2f}{d\mu^2} - 2k^2\mu\frac{df}{d\mu} = \frac{d^2}{d\mu^2}\left[k^2(1-\mu^2)f\right] + \frac{d}{d\mu}\left[k^2\mu f\right] \tag{6.5-2}$$

Hence we may rewrite (6.5-1) in the following form

$$\frac{d^2}{d\mu^2}\left[\left(\frac{D_{\mu\mu}}{2} + k^2(1-\mu^2)\right)F\right] + \frac{d}{d\mu}\left[(-D_{\mu} + 2k^2\mu)F\right] = 0 \tag{6.5-3}$$

Integrating once

$$\frac{d}{d\mu}\left[\left(\frac{D_{\mu\mu}}{2} + k^2(1-\mu^2)\right)F\right] + (-D_{\mu} + 2k^2\mu)F = C_1 \tag{6.5-4}$$

or

$$\left[\frac{D_{\mu\mu}}{2} + k^2(1-\mu^2)\right] f' + \left[\frac{D'_{\mu\mu}}{2} - D_\mu\right] f = C_1 \qquad (6.5\text{-}5)$$

where the prime denotes differentiation with respect to μ. We may evaluate C_1 immediately by noting that the left side vanishes at $\mu = \pm 1$, provided F is finite there. Thus $C_1 = 0$, and we are left with

$$\frac{1}{F}\frac{dF}{d\mu} = \frac{d}{d\mu}\ln F = \frac{-D'_{\mu\mu} - 2D_\mu}{D_{\mu\mu} + 2k^2(1-\mu^2)} = g(\mu) \qquad (6.5\text{-}6)$$

From (4.1-2) and (4.1-5) we find for $\mu > 0$.

$$g(\mu) = \frac{(1-\mu^2)^2\,(1+3\mu^2)}{\mu\left[(1-\mu^2)^3 + 2\lambda\mu^3\right]} \qquad (6.5\text{-}7)$$

where λ is defined by (6.1-3).

Integrating (6.5-6) from μ to 1 we obtain

$$F(\mu) = F(1)\,\exp\left[-\int_\mu^1 g(x)\,dx\right]$$

$$= F(1)\,\exp\left[-\int_\mu^1 \frac{(1-x^2)^2\,(1+3x^2)\,dx}{x\left[(1-x^2)^3 + 2\lambda x^3\right]}\right]$$

(6.5-8)

The constant $F(1)$ may be chosen to satisfy the normalization condition $\int_{-1}^{+1} F(\mu)\, d\mu = 1$.

Graphs of $F(\mu)$ (un-normalized) are shown in Figure 5, for various values of λ . As $\lambda \rightarrow \infty$, the distribution becomes isotropic with

$$F(\mu) = F(1) = \frac{1}{2} \tag{6.5-9}$$

For λ very large but finite the distribution is approximately isotropic except near $\mu = 0$.

It will also be noted that for any finite value of λ , the solution must be zero at $\mu = 0$. This statement follows immediately by letting $\mu \rightarrow 0$ in (6.5-8).

In the neighborhood of the origin where $\mu \ll \left(\frac{1}{2\lambda}\right)^{1/3}$ it is easily verified that $F(\mu)$ is approximately a linear function of μ.

Some interesting conclusions may now be drawn from the preceding results. We have shown that a single "well behaved" solution to (6.5-1) exists. The second solution to the differential equation contains a logarithmic term $\ln(1-\mu)$, and hence cannot be an admissible solution. It will be noted that by setting $\alpha = \beta = 0$, we have not changed the nature of the singularities in the coefficients of (6.2-9). The power series solutions obtained in Section 6.3 may thus be specialized to the present case simply by setting $\alpha = \beta = 0$ in the recursion formulas. In particular, the power series (6.2-50) is an expansion of the "well behaved" solution (6.5-8) about $\mu = +1$.

We next note that when $\alpha = 0$, the expression for the

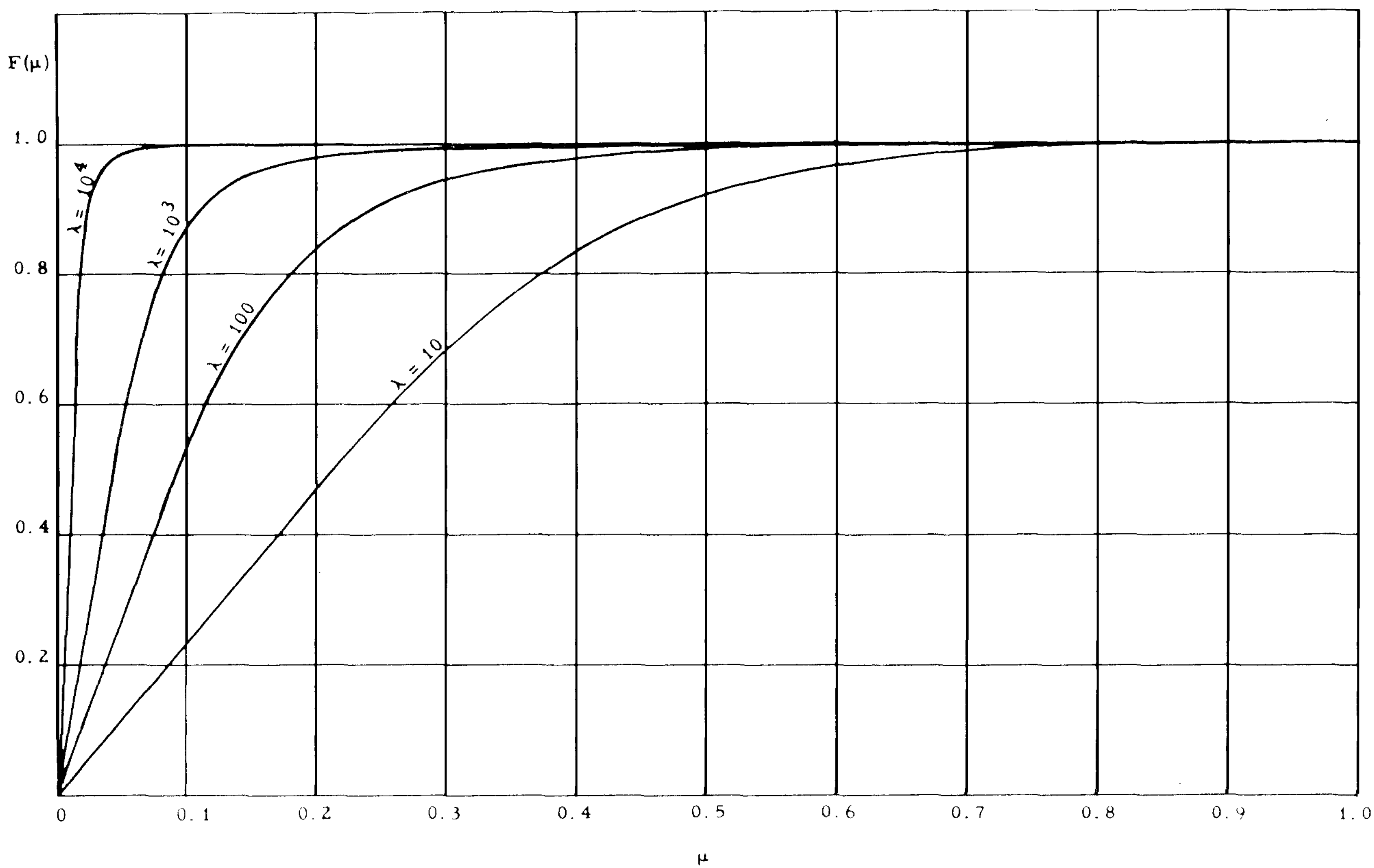

FIGURE 5. The Angular Distribution Function $F(\mu)$ for Several Values of λ.

flux $J_\mu(\mu)$ given by (6.4-2) reduces to

$$J_\mu(\mu) = e^{-\alpha\epsilon}\left[\left(\frac{D_{\mu\mu}}{2} + k^2(1-\mu^2)\right)\frac{df}{d\mu} + \left(\frac{D'_{\mu\mu}}{2} - D_\mu\right) f\right] \qquad (6.5\text{-}10)$$

The quantity in brackets is zero by (6.5-5), and hence the solution given by (6.5-8) gives zero flux $J_\mu(\mu)$ everywhere in the interval $0 \leq \mu \leq 1$. The condition (6.4-11) relating the two power series solutions about $\mu = 0$ is therefore satisfied. (This statement may also be verified directly by expanding the solution (6.5-8) about $\mu = 0$ and examining the coefficient of the μ^4 term.)

Thus in the special case $\alpha = \beta = 0$, the "well behaved" solution satisfies the condition

$$a_4 = -\frac{2}{3}\lambda a_1 \qquad (6.5\text{-}11)$$

necessary to make $J_\mu(0) = 0$. In the general case when α and β are non-zero, the condition (6.5-11) does not insure a solution which is finite at $\mu = 1$. An additional condition must therefore be imposed on the solutions to (6.2-8). This condition takes the form of a constraint on the parameters α, λ, β

$$g(\alpha, \lambda, \beta) = 0 \qquad (6.5\text{-}12)$$

It has not been possible to determine analytically the form of the function $g(\alpha, \lambda, \beta)$. Nevertheless solutions to (6.2-8) can be found which satisfy the required boundary conditions and hence (6.5-12). The problem of finding such solutions is considered in the following section.

6.6 Numerical Solutions for Large ϵ

From the results of the previous section it is evident that one of the two independent solutions to (6.2-9) is well behaved, i.e., it is finite and non-negative everywhere in the interval $0 \leq \mu \leq 1$. In general, such a solution will not satisfy the boundary condition $J_{\mu}(0) = 0$ unless the parameters α, β, λ have been chosen properly. The procedure employed for finding solutions which satisfy this boundary condition is to fix the parameters α and λ, and compute a set of solutions for different values of β by numerical integration*. One then selects from this set of solutions the one which satisfies the condition $J_{\mu}(0) = 0$.

The numerical procedure used is a step-by-step process based on the well known Runge-Kutta method for integrating systems of first order differential equations, with Gill's special treatment which greatly reduces cumulative roundoff errors (Gill 1951). Since the computations required are extremely tedious, they have been carried out with the aid of a digital computer.

To carry out the numerical integration, one must start at a point $\mu = \mu_0$ with the value of the function f and its first derivative. Because of the nature of the singularities in the coefficients of the differential equation at $\mu = 0$ and $\mu = \pm 1$, the

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* The problem of finding the appropriate solutions must be dealt with numerically since solutions to (6.2-9) cannot be found in closed form except in the special case where $\alpha = \beta = 0$. The power series solutions which we have developed as an aid to determining properties of the solutions are not useful for computing complete solutions because of their slow convergence.

solution cannot be started at these points. It has been found convenient, therefore, to start the solution by using the power series (6.3-38) to compute f and f' at $\mu = 0.995$, assigning the value 1 to f at $\mu = 1$. The solution is then continued by numerical integration using a computing interval of 2^{-9}. As μ becomes small ($\mu = 0.005$) the integration becomes inaccurate due to the singularities in the coefficients. The solution is therefore stopped at this point. Since it is known that the solution varies linearly with μ for small μ, the solution obtained is easily extrapolated to $\mu = 0$.

In principal one could start the solution at $\mu = 0$, using the power series (6.3-40) and (6.3-41), and proceeding by step-by-step numerical integration to $\mu = 1$. However, this is difficult in practice. One is faced with the problem of determining the proper linear combination of the two power series solutions which will result in a solution which is finite at $\mu = 1$, and at the same time satisfy the boundary condition $J(\mu) = 0$. The proper linear combination is not known a priori (except in the case $\alpha = \beta = 0$), and hence must be determined by trial and error. The procedure outlined above avoids this difficulty.

In carrying out the numerical integration it is most convenient to fix the values of α and λ, and determine the value of β that results in a solution satisfying the required boundary condition. Values of α have been chosen that agree with experimental observations. The results of Neher and Stern (1955) indicate that α lies in the range 1.5 to 1.7. For each of the values $\alpha = 1.5$ and 1.7 solutions have been computed for

$\lambda = 10, 100, 1000$. Curves showing the behavior of $f(\mu)$ for the higher value of α are given in Figure 6. The values of $\beta = \frac{1}{K\tau}$ corresponding to these solutions are shown in the accompanying table and in Figure 7. The solutions for $\alpha = 1.5$ have not been plotted since they differ very little from the solutions for $\alpha = 1.7$.

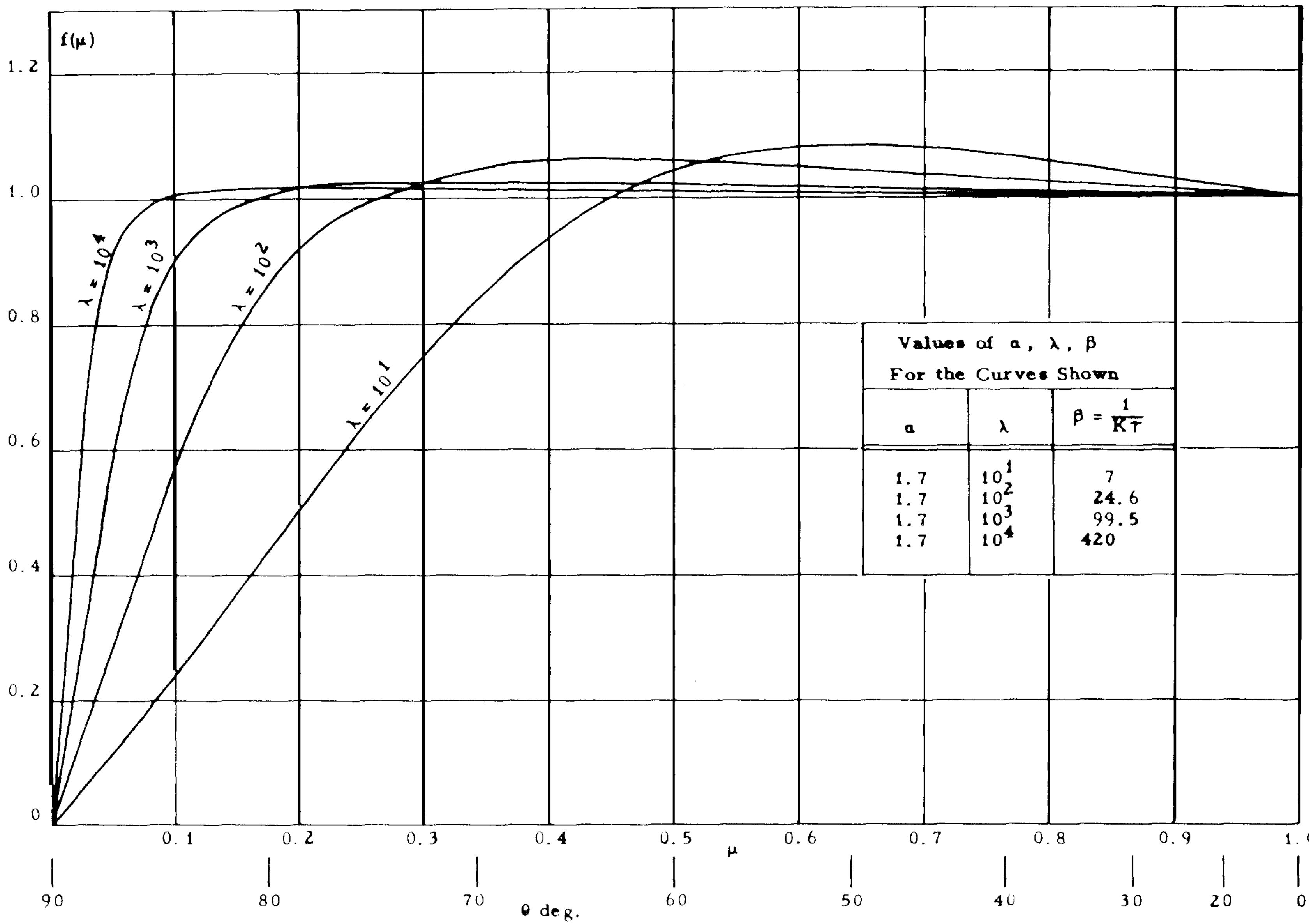

FIGURE 6. The Angular Distribution Function $f(\mu)$ for $\alpha = 1.7$; $\lambda = 10^1, 10^2, 10^3, 10^4$

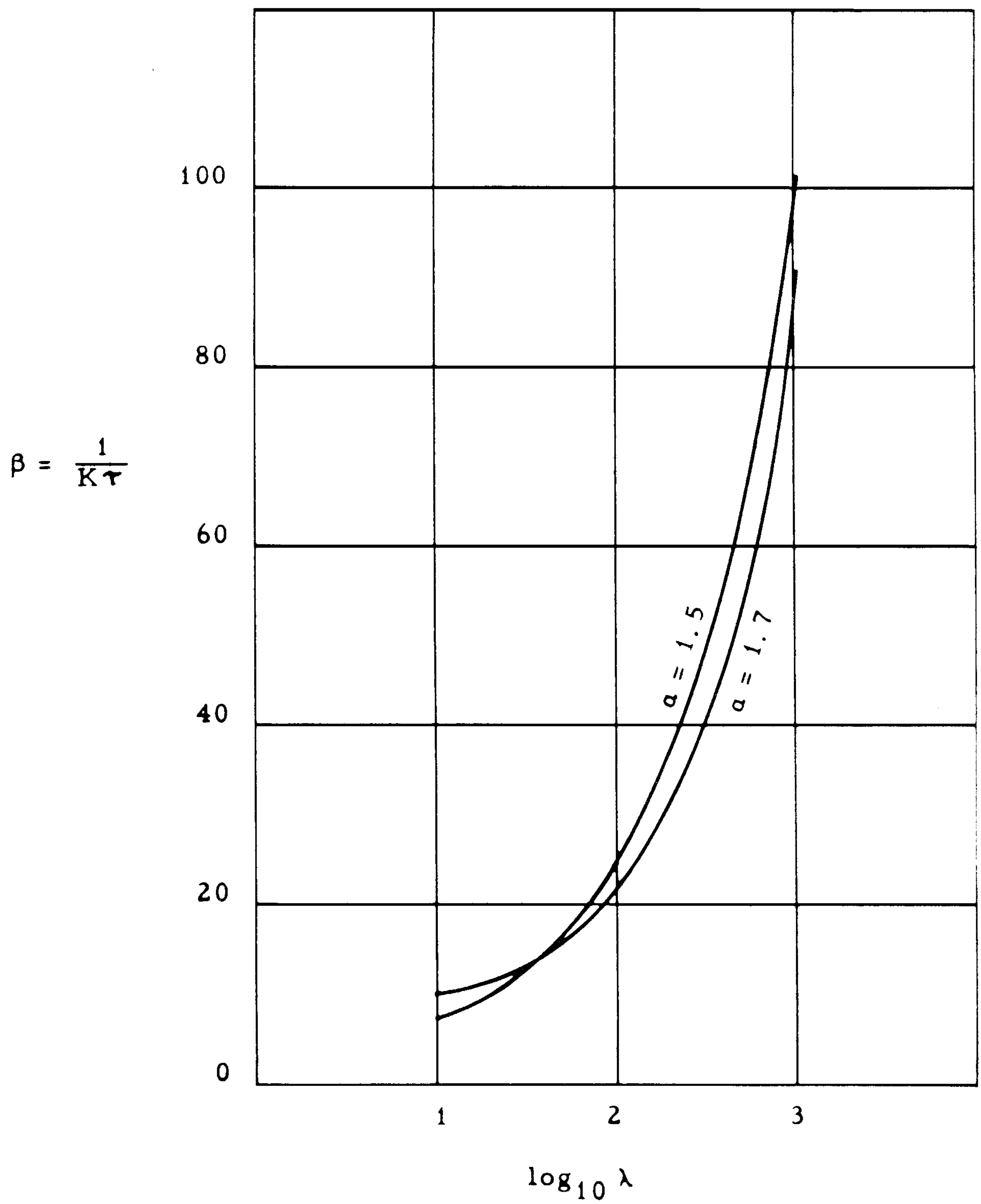

FIGURE 7. The Parameter β as a Function of λ for $a = 1.5$ and 1.7

7. SUMMARY AND CONCLUSIONS

In previous sections, a diffusion equation has been derived which describes statistically the behavior of an ensemble of particles moving in the galactic magnetic field. This equation has been derived on the basis of a model in which particles interact with large scale, time varying, inhomogeneities in the magnetic field (betatron interactions), and with small scale inhomogeneities which change the direction of motion of the particles. Two main problems have concerned us: (a) the energy distribution of the high energy particles, and (b) the angular distribution of these particles.

In Section 6 solutions to the steady-state, space-independent* diffusion equation have been given which are valid in the high energy region. The results have shown that the angular distribution of the cosmic ray particles produced by the betatron mechanism is inextricably tied up with the energy dependence, and hence we cannot consider either separately. The results also show that the degree of anisotropy is determined by the relative effectiveness of the scattering by small scale inhomogeneities which tend to make the distribution isotropic, and the betatron processes which tend to make the distribution highly anisotropic with most of the particles lying in very steep spirals. The relative effectiveness of these two processes is measured by the ratio (See Section 6.1),

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* It has been shown in Section 5 that the anisotropy produced by diffusion of particles out of the spiral arm is small except near the boundaries of the magnetic field. It is thus a valid approximation in the region far from the boundaries of the field to neglect the space dependent terms.

A nonuniform space distribution of sources also leads to anisotropy. This problem has been considered by Fan (1951) and Morrison, Olbert, and Rossi (1954).

$$\lambda = \frac{n_2 \sigma_\theta^2/4}{K} = \frac{c\, n_2 \sigma_\theta^2}{2\pi \phi(o) \omega^2(o) \left\langle v_g^2 \right\rangle k_2^2} \tag{7.1-1}$$

where $\frac{n_2 \sigma_\theta^2}{4}$ is a parameter associated with the scattering by small scale inhomogeneities, and K is a parameter associated with the betatron mechanism. The degree of anisotropy is strongly dependent on the value of λ, with the distribution becoming more and more isotropic as the value of λ increases.

We have shown that for large ϵ, the probability of finding a particle in the range $\epsilon \rightarrow \epsilon + d\epsilon$, $\mu \rightarrow \mu + d\mu$, is given by

$$W(\mu, \epsilon) d\mu\, d\epsilon = A e^{-\alpha \epsilon} f(\mu) \tag{7.1-2}$$

where A and α are constants, and $f(\mu)$ describes the angular distribution of the particles. The probability of finding a particle in the energy range $w \rightarrow w + dw$ is thus

$$n(w) dw = A' e^{-(\alpha + 1)w} dw \tag{7.1-3}$$

The parameter α depends on the values of K and λ which are determined by the properties of the magnetic field, and also on the parameter τ which describes the rate at which particles are removed from the distribution by nuclear collisions and diffusion out of the spiral arm. If τ_N denotes the mean time for absorption of a particle by a nuclear collision and τ_D denotes the mean time for diffusion out of the spiral arm, then approximately

$$\tau^{-1} = \tau_D^{-1} + \tau_N^{-1} \tag{7.1-4}$$

It is known from experimental observations that the value of α lies in the range 1.5 to 1.7 (Neher and Stern 1955). For these extreme values of α, the parameter $\beta = \frac{1}{K\tau}$ has been computed for several values of λ corresponding to different degrees of anisotropy. The results are summarized in the following table.

TABLE II

Values of α, λ, and $\beta = \frac{1}{K\tau}$ Determined From the Steady-State Diffusion Equation

α	λ	$\beta = \frac{1}{K\tau}$	α	λ	$\beta = \frac{1}{K\tau}$
1.5	10^1	10	1.7	10^1	7
1.5	10^2	21.4	1.7	10^2	24.6
1.5	10^3	87.5	1.7	10^3	99.5
			1.7	10^4	420

The angular distribution functions $f(\mu)$ corresponding to the last four of these cases are plotted in Figure 6.

All of these cases are consistent with experimental knowledge of the parameter α, but not with the observed anisotropy of cosmic rays. A lower limit on λ is imposed by experimental results which indicate that the flux of cosmic rays is isotropic to within a few percent (Davis 1954a).

A rough measure of the anisotropy may be obtained by expanding the function $f(\mu)$ in a series of Legendre Polynomials

$$f(\mu) = \sum_{n=o}^{\infty} A_n P_n(\mu) \quad . \tag{7.1-5}$$

Since $f(\mu)$ is an even function of μ, $A_n = 0$ for odd n. Table III gives the values of A_2/A_o and A_4/A_o for the solutions plotted in Figure 6.

TABLE III

Values of A_2/A_o and A_4/A_o for the Functions $f(\mu)$ Plotted in Figure 6

α	λ	A_2/A_o	A_4/A_o
1.7	10^1	0.642	-0.686
1.7	10^2	0.237	-0.361
1.7	10^3	0.109	-0.271

The results given in Table III indicate that λ must be greater than 10^3 to agree with experimental evidence.

Additional conditions on the parameters are imposed by the hypothesis that the cosmic rays diffuse to the end of the spiral arm more rapidly than they suffer nuclear collisions. (Such an assumption is required to explain the physically observed fact that the energy distribution of the high energy particles is independent of their mass.) An upper limit for τ is thus set by the requirement that τ be of the order of the mean time for a nuclear collision of the heavist components of cosmic rays, and is about 4×10^6 years. (Morrison, et al. 1954).

By virtue of the above hypothesis we may neglect τ_N^{-1} in comparison with τ_D^{-1} in (7.1-4). The parameter τ_D may then be estimated from [Cf. (5.3-12)],

$$\tau_D = \frac{5}{18} \left[\frac{b^2}{c^2}\right] n_2 \sigma_\theta^2 \qquad (7.1\text{-}6)$$

where $2b$ is the length of the spiral arm, and c is the velocity of light. Reasonable values for b lie in the range 3×10^5 to 10^6 light years.

Given any three of the five parameters τ_D, $\lambda = n_2\sigma_\theta^2/4K$, α, $\beta = 1/K\tau_D$, b; it is possible to compute the other two using (7.1-6) and Table II. The results for several cases of interest are given in Table IV.

Figure 8 shows the variation of b with λ, computed for $\alpha = 1.7$, $\tau_D = 10^6$ and 4×10^6 years. The values of β are taken from Figure 7.

If we take $\lambda > 10^3$ then it appears from these results that it is possible to find values of the parameters which agree with the observed energy spectrum of the high energy cosmic rays and at

TABLE IV

Values of β, K, $n_2\sigma_\theta^2$, b Computed for $\alpha = 1.7$; $\lambda = 10, 10^2, 10^3$; $\tau_D = 10^6, 4 \times 10^6$ yrs.

α	λ	$\beta = \frac{1}{K\tau_D}$	τ_D yrs	K yrs^{-1}	$n_2\sigma_\theta^2$ yrs^{-1}	b light years
1.7	10^4	420	4×10^6	6.0×10^{-10}	2.4×10^{-5}	7.7×10^5
1.7	10^3	99.5	4×10^6	2.5×10^{-9}	1.0×10^{-5}	1.3×10^6
1.7	10^2	24.6	4×10^6	1.0×10^{-8}	4.0×10^{-6}	1.9×10^6
1.7	10^1	7	4×10^6	3.6×10^{-8}	1.4×10^{-6}	3.4×10^6
1.7	10^4	420	10^6	2.4×10^{-9}	9.6×10^{-5}	1.9×10^5
1.7	10^3	99.5	10^6	1.0×10^{-8}	4.0×10^{-5}	3.0×10^5
1.7	10^2	24.6	10^6	4.0×10^{-8}	1.6×10^{-5}	4.7×10^5
1.7	10^1	7	10^6	1.4×10^{-7}	5.6×10^{-7}	8.0×10^5

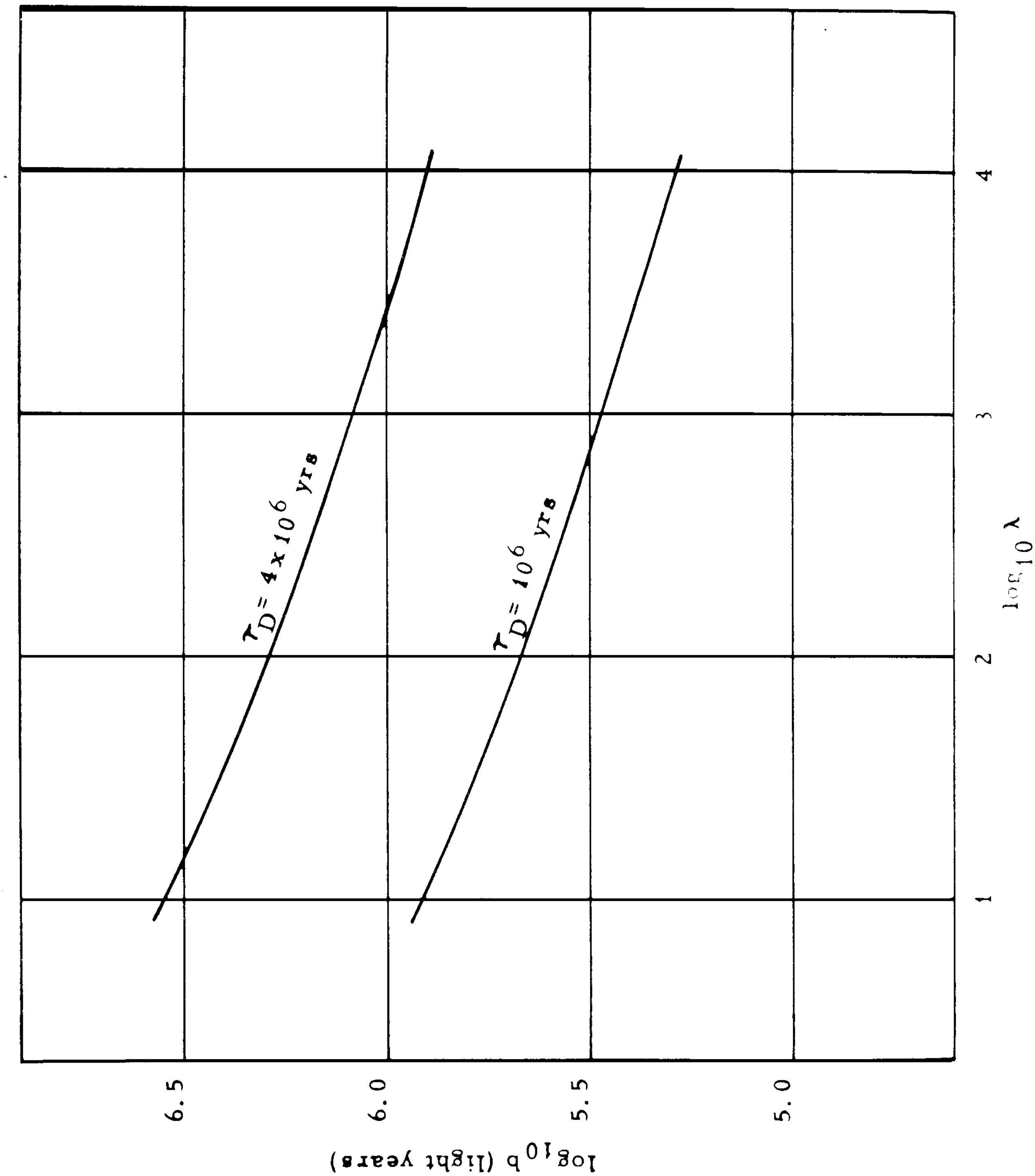

FIGURE 8. The Parameter b as a Function of λ for $\tau_D = 10^6$, 4×10^6 yrs.

the same time allow the cosmic rays to diffuse to the end of the spiral arm in times of the order of the mean time for a nuclear collision by the heaviest atom. The rms deflection angles required to keep the distribution isotropic to within a few percent correspond to a few milliradians at the rate of one deflection per year.

It is not possible to relate the parameter K to other parameters describing the magnetic field unless we make some assumption regarding the spectrum of the oscillations in the field. We shall do this as a matter of illustration only, using the spectrum given by (3.2-46), without implying that this equation correctly describes the magnetic field. A short calculation based on (3.2-46) shows that

$$K = 2\pi \beta_g^2 \left(\frac{b^2}{a^2} \right) \frac{\delta}{b} \qquad (7.1\text{-}7)$$

where b and a are the mean wavelengths of the hydromagnetic waves transverse and parallel to the primary magnetic field, respectively; $c^2 \beta_g^2 = \langle v_g^2 \rangle$ is the mean-square gas velocity of the medium; and $2\delta b$ is the width of the wavelength spectrum at half maximum. The units of distance are light years.

If we take $K = 1/4 \times 10^{-8}$ yrs.$^{-1}$, corresponding to $\lambda = 10^3$ and $\tau_D = 4 \times 10^6$ yrs., in Table IV and, $\beta_g = 3 \times 10^{-5}$ (Davis 1956), then a consistent set of values satisfying (7.1-7) $a = b = 1$ light year, and $\delta = 0.4$. These values appear to be reasonable. However, values of a and b much smaller than about one light year are not reasonable due to the large amount of viscous damping associated with such waves (Davis 1956).

The complete answer to the question of the plausibility of the values of K and $n_2\sigma_\theta^2$ derived from our model of the acceleration mechanism must await further evidence on the nature of the galactic magnetic field.

APPENDIX A

Derivation of the Diffusion Equation from the Exact Integral Equation for W

The purpose of this Appendix is to derive the partial differential equation describing the behavior of an ensemble of particles undergoing scattering and acceleration by inhomogeneities in the field whose scale is large compared to the radius of curvature of the spiral of a typical particle. Additional terms in the diffusion equation are needed to represent the scattering by small scale static inhomogeneities, absorption, and injection of new particles. These modifications are made in Sections 3.3 and 3.4.

Let t denote time, z distance along the direction of the unperturbed magnetic field, μ the cosine of the spiral pitch angle, ϵ the logarithmic energy parameter defined by $\epsilon = \ln \frac{H}{m_0 c^2}$, where H is the total energy of a particle and $m_0 c^2$ is the rest energy. Let $W(\mu, \epsilon, z; t)\, d\mu\, d\epsilon\, dz$ be the probability of finding a particle with μ in the range $\mu \rightarrow \mu + d\mu$, ϵ in the range $\epsilon \rightarrow \epsilon + d\epsilon$, z in the range $z \rightarrow z + dz$, at the time t.

Consider the problem of relating the probability distribution function at a time $t + \Delta t$ to the distribution function at time t. These distribution functions are related through the joint transition probability

$$P(\mu, \epsilon, z; \Delta\mu, \Delta\epsilon, \Delta z)\, d(\Delta\mu)\, d(\Delta\epsilon)\, d(\Delta z) \qquad (1)$$

which represents the probability of μ changing by an amount $\Delta\mu$,

<u>and</u> ϵ changing by an amount $\Delta\epsilon$, <u>and</u> z changing by an amount Δz, in the time Δt. It is reasonable to assume that P is a function of Δt, but not of t. The usual rules of the probability calculus then permit us to write the following exact integral equation for W.

$$W(\mu, \epsilon, z; t+\Delta t) = \iiint W(\mu-\Delta\mu, \epsilon-\Delta\epsilon, z-\Delta z; t) \quad \times \tag{2}$$
$$P(\mu-\Delta\mu, \epsilon-\Delta\epsilon, z-\Delta z; \Delta\mu, \Delta\epsilon, \Delta z) d(\Delta\mu) d(\Delta\epsilon) d(\Delta z)$$

where the integration is carried out over the complete range of $\Delta\mu$, $\Delta\epsilon$, Δz. Now if we expand this equation in terms of the changes $\Delta\mu$, $\Delta\epsilon$, Δz, and pass to the limit in an appropriate manner, we obtain a partial differential equation describing the change of W with time. Terms up to the second order in $\Delta\mu$, etc., will be retained. This gives an adequate representation for times large compared to the time over which the changes $\Delta\mu$, $\Delta\epsilon$, etc., are correlated.

It will be convenient to write

$$\begin{aligned} x_1 &= \mu & \Delta x_1 &= \Delta\mu \\ x_2 &= \epsilon & \Delta x_2 &= \Delta\epsilon \\ x_3 &= z & \Delta x_3 &= \Delta z \end{aligned} \tag{3}$$

We first expand the functions W and P in the integrand of (2),

$$W(x_1-\Delta x_1, x_2-\Delta x_2, x_3-\Delta x_3; t) = W(x_1, x_2, x_3; t)$$
$$- \sum_{i=1}^{3} \frac{\partial W}{\partial x_i} \Delta x_i + \frac{1}{2} \sum_{i,j=1}^{3} \frac{\partial^2 W}{\partial x_i \partial x_j} \Delta x_i \Delta x_j + O(\Delta x_i^3) \tag{4}$$

$$P(x_1-\Delta x_1, x_2-\Delta x_2, x_3-\Delta x_3; \Delta x_1, \Delta x_2, \Delta x_3) = P(x_1, x_2, x_3; \Delta x_1, \Delta x_2, \Delta x_3) - \sum_{i=1}^{3} \frac{\partial P}{\partial x_i} \Delta x_i + \frac{1}{2} \sum_{i,j=1}^{3} \frac{\partial^2 P}{\partial x_i \partial x_j} \Delta x_i \Delta x_j + O(\Delta x_i^3) \qquad (5)$$

Next expand the left hand side of (2), and retain terms to first order in Δt ,

$$W(\mu, \epsilon, z; t+\Delta t) = W(\mu, \epsilon, z; t) + \frac{\partial W}{\partial t} \Delta t + O(\Delta t^2) \qquad (6)$$

Substituting these expansions in (2), we have

$$W(x_1, x_2, x_3; t) + \frac{\partial W}{\partial t} \Delta t + O(\Delta t^2)$$

$$= \iiint \left[W(x_1, x_2, x_3; t) - \sum_{i=1}^{3} \frac{\partial W}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{3} \frac{\partial^2 W}{\partial x_i \partial x_j} \Delta x_i \Delta x_j \right] x$$

$$x \left[P(x_1, x_2, x_3; \Delta x_1, \Delta x_2, \Delta x_3) - \sum_{i=1}^{3} \frac{\partial P}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{3} \frac{\partial^2 P}{\partial x_i \partial x_j} \Delta x_i \Delta x_j \right.$$

$$\left. + O(\Delta x_i^3) \right] d(\Delta x_1)\, d(\Delta x_2)\, d(\Delta x_3) \qquad (7)$$

which can be written,

$$W(x_1, x_2, x_3; t) + \frac{\partial W}{\partial t}\Delta t + O(\Delta t^2) =$$

$$= \iiint W(x_1, x_2, x_3; t)\, P(x_1, x_2, x_3; \Delta x_1, \Delta x_2, \Delta x_3) d(\Delta x_1) d(\Delta x_2) d(\Delta x_3)$$

$$- \iiint \sum_{i=1}^{3} \frac{\partial W}{\partial x_i}\Delta x_i\, P(x_1, x_2, x_3; \Delta x_1, \Delta x_2, \Delta x_3) d(\Delta x_1) d(\Delta x_2) d(\Delta x_3)$$

$$+ \frac{1}{2} \iiint \sum_{i,j=1}^{3} \frac{\partial^2 W}{\partial x_i \partial x_j} \Delta x_i \Delta x_j\, P(x_1, x_2, x_3; \Delta x_1, \Delta x_2, \Delta x_3) \times$$

$$d(\Delta x_1)\, d(\Delta x_2)\, d(\Delta x_3) \tag{8}$$

$$- \iiint W(x_1, x_2, x_3; t) \left[\sum_{i=1}^{3} \frac{\partial P}{\partial x_i} \Delta x_i \right] d(\Delta x_1 d(\Delta x_2) d(\Delta x_3)$$

$$+ \frac{1}{2} \iiint W(x_1, x_2, x_3; t) \left[\sum_{i,j=1}^{3} \frac{\partial^2 P}{\partial x_i \partial x_j} \Delta x_i \Delta x_j \right] d(\Delta x_i) d(\Delta x_2) d(\Delta x_3)$$

$$+ \iiint \left[\sum_{i=1}^{3} \frac{\partial P}{\partial x_i} \Delta x_i \right] \left[\sum_{i=1}^{3} \frac{\partial W}{\partial x_i} \Delta x_i \right] d(\Delta x_1) d(\Delta x_2) d(\Delta x_3)$$

We now consider simplifications of the various terms. The first term on the right can be written (assuming P is normalized)

$$W(x_1, x_2, x_3; t) \iiint P(x_1, x_2, x_3; \Delta x_1, \Delta x_2, \Delta x_3) d(\Delta x_1) d(\Delta x_2) d(\Delta x_3)$$

$$= W(x_1, x_2, x_3; t) \tag{9}$$

The second term on the right can be rewritten as

$$- \sum_{i=1}^{3} \frac{\partial W}{\partial x_i} \iiint \Delta x_i P(x_1, x_2, x_3; \Delta x_1, \Delta x_2, \Delta x_3) d(\Delta x_1) d(\Delta x_2) d(\Delta x_3)$$

$$= - \sum_{i=1}^{3} \frac{\partial W}{\partial x_i} \left\langle \Delta x_i \right\rangle \tag{10}$$

where

$$\left\langle \Delta x_i \right\rangle = \iiint \Delta x_i P(x_1, x_2, x_3; \Delta x_1, \Delta x_2, \Delta x_3) d(\Delta x_1) d(\Delta x_2) d(\Delta x_3) \tag{11}$$

Note that in general $\left\langle \Delta x_i \right\rangle$ is a function of x_1, x_2, x_3. Similarly, the third term may be written

$$\frac{1}{2} \sum_{i,j=1}^{3} \frac{\partial^2 W}{\partial x_i \partial x_j} \iiint \Delta x_i \Delta x_j P(x_1, x_2, x_3; \Delta x_1, \Delta x_2, \Delta x_3) d(\Delta x_1) d(\Delta x_2) d(\Delta x_3)$$

$$= \frac{1}{2} \sum_{i,j=1}^{3} \frac{\partial^2 W}{\partial x_i \partial x_j} \left\langle \Delta x_i \Delta x_j \right\rangle \tag{12}$$

where

$$\left\langle \Delta x_i \Delta x_j \right\rangle = \iiint \Delta x_i \Delta x_j P(x_1, x_2, x_3; \Delta x_1, \Delta x_2, \Delta x_3) d(\Delta x_1) d(\Delta x_2) d(\Delta x_3) \tag{13}$$

Differentiating the expressions for $\left\langle \Delta x_i \right\rangle$ and $\left\langle \Delta x_i \Delta x_j \right\rangle$ we obtain

$$\frac{\partial}{\partial x_i} \left\langle \Delta x_i \right\rangle = \iiint \Delta x_i \frac{\partial P}{\partial x_i} d(\Delta x_1) d(\Delta x_2) d(\Delta x_3) \tag{14}$$

and,

$$\frac{\partial^2}{\partial x_i \partial x_j} \langle \Delta x_i \Delta x_j \rangle = \iiint \Delta x_i \Delta x_j \frac{\partial^2 P}{\partial x_i \partial x_j} d(\Delta x_1) d(\Delta x_2) d(\Delta x_3) \quad (15)$$

These expressions may be used to simplify the last three terms of (8). The fourth term becomes

$$\begin{aligned} &- W(x_1, x_2, x_3; t) \iiint \sum_{i=1}^{3} \frac{\partial P}{\partial x_i} \Delta x_i \, d(\Delta x_1)\, d(\Delta x_2)\, d(\Delta x_3) \\ &= - W(x_1, x_2, x_3; t) \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \langle \Delta x_i \rangle \end{aligned} \quad (16)$$

The fifth term becomes

$$\begin{aligned} &\frac{1}{2} W(x_1, x_2, x_3; t) \iiint \sum_{i,j=1}^{3} \left[\frac{\partial^2 P}{\partial x_i \partial x_j} \Delta x_i \Delta x_j \right] d(\Delta x_1) d(\Delta x_2) d(\Delta x_3) \\ &= + \frac{1}{2} W(x_1, x_2, x_3; t) \sum_{i,j=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j} \langle \Delta x_i \Delta x_j \rangle \end{aligned} \quad (17)$$

Similarly, the last term can be written

$$\iiint \sum_{i,j=1}^{3} \frac{\partial P}{\partial x_i} \frac{\partial W}{\partial x_j} \Delta x_i \Delta x_j \, d(\Delta x_1 d(\Delta x_2) d(\Delta x_3) \quad (18)$$

$$= \sum_{i=1}^{3} \frac{\partial W}{\partial x_i} \left[\iiint \sum_{i=1}^{3} \frac{\partial P}{\partial x_i} \Delta x_i \Delta x_j \right] d(\Delta x_1) d(\Delta x_2) d(\Delta x_3)$$

$$= \sum_{i=1}^{3} \frac{\partial W}{\partial x_i} \left[\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \Delta x_i \Delta x_j \right] = \sum_{i,j=1}^{3} \frac{\partial W}{\partial x_j} \frac{\partial}{\partial x_i} \langle \Delta x_i \Delta x_j \rangle$$

Using Equations (9) through (18) in (8), we have

$$\frac{\partial W}{\partial t}\Delta t + O(\Delta t^2) = -\sum_{i=1}^{3} \frac{\partial W}{\partial x_i}\langle \Delta x_i \rangle + \frac{1}{2}\sum_{i,j=1}^{3} \frac{\partial^2 W}{\partial x_i \partial x_j}\langle \Delta x_i \Delta x_j \rangle$$

$$- W\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\langle \Delta x_i \rangle + \frac{W}{2}\sum_{i,j=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j}\langle \Delta x_i \Delta x_j \rangle$$

$$+ \sum_{i,j=1}^{3} \frac{\partial W}{\partial x_i}\frac{\partial}{\partial x_j}\langle \Delta x_i \Delta x_j \rangle + O(\Delta x_i^{3}) \qquad (19)$$

We can further simplify (19) by noting that

$$-\sum_{i=1}^{3} \frac{\partial W}{\partial x_i}\langle \Delta x_i \rangle - W\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\langle \Delta x_i \rangle = -\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left[W\langle \Delta x_i \rangle \right] \qquad (20)$$

and that

$$\frac{1}{2}\sum_{i,j=1}^{3} \frac{\partial^2 W}{\partial x_i \partial x_j}\langle \Delta x_i \Delta x_j \rangle + \frac{W}{2}\sum_{i,j=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j}\langle \Delta x_i \Delta x_j \rangle \qquad (21)$$

$$+ \sum_{i,j=1}^{3} \frac{\partial W}{\partial x_j}\frac{\partial}{\partial x_i}\langle \Delta x_i \Delta x_j \rangle = \frac{1}{2}\sum_{i,j=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j}\left[W\langle \Delta x_i \Delta x_j \rangle \right]$$

Thus (19) becomes

$$\frac{\partial W}{\partial t}\,\Delta t + O(\Delta t^2) = -\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left[W \langle \Delta x_i \rangle \right] + \frac{1}{2}\sum_{i,j=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j}\left[W \langle \Delta x_i \Delta x_j \rangle \right]$$

$$+ \; O(\langle \Delta x_i \Delta x_j \Delta x_k \rangle) \qquad (22)$$

Dividing by Δt and taking the limit as $\Delta t \rightarrow o$, we obtain finally

$$\frac{\partial W}{\partial t} = -\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left[D_i W \right] + \frac{1}{2}\sum_{i,j=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j}\left[D_{ij} W \right] \qquad (23)$$

where

$$D_i = \lim_{\Delta t \rightarrow o} \frac{\langle \Delta x_i \rangle}{\Delta t} \qquad (24)$$

and

$$D_{ij} = \lim_{\Delta t \rightarrow o} \frac{\langle \Delta x_i \Delta x_j \rangle}{\Delta t} \qquad (25)$$

APPENDIX B

Some Properties of Cylindrically Symmetric Hydromagnetic Waves

In this Appendix we will discuss some of the properties of the simplest types of cylindrically symmetric hydromagnetic waves. We start from Maxwell's equations neglecting the displacement current. In Gaussian units

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{i} \tag{1}$$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \tag{2}$$

$$\nabla \cdot \vec{B} = 0 \tag{3}$$

$$\vec{i} = \sigma \left[\vec{E} + \frac{1}{c} \vec{v} \times \vec{B} \right] \tag{4}$$

In (4) σ represents the conductivity and v the velocity of the medium. To these equations we must add the hydrodynamic equation of motion for the medium. In this simplified treatment we will neglect the non-electromagnetic forces. The hydrodynamic equation of motion is then

$$\rho \frac{d\vec{v}}{dt} = \frac{1}{c} (\vec{i} \times \vec{B}) \tag{5}$$

where ρ is the mass density of the medium. We make the further assumption that the conductivity of the medium is infinite. This requires from (4) that

$$\vec{E} = -\frac{1}{c} \vec{v} \times \vec{B} \tag{6}$$

since the current density $\vec{i}$ must remain finite. We introduce a vector potential $\vec{A}$ from which the fields $\vec{E}$ and $\vec{B}$ are calculated by

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} \tag{7}$$

$$\vec{B} = \nabla \times \vec{A} \tag{8}$$

We seek solutions of the above equations which describe cylindrically symmetric waves moving in the direction of the primary magnetic field $B_o \vec{e}_z$. Such waves may be described by a vector potential having only a ϕ-component. Furthermore all of the parameters describing the field will be independent of the coordinate ϕ. We attempt to find a solution by assuming a vector potential of the form

$$\vec{A} = A_\phi \vec{e}_\phi = \left[\frac{r}{2} B_o - B_o A\, g(r)\, e^{i(k_1 z - \omega t)}\right] \vec{e}_\phi \tag{9}$$

which represents a wave moving to the right along the z-axis. Taking the curl of (9) gives

$$\begin{aligned} \vec{B} &= B_r \vec{e}_r + B_z \vec{e}_z = -\frac{\partial A_\phi}{\partial z} \vec{e}_r + \frac{1}{r}\frac{\partial}{\partial r}(r A_\phi)\, \vec{e}_z \\ &= -B_o A i k_1\, g(r)\, e^{i(k_1 z - \omega t)}\, \vec{e}_r + B_o\left[1 + A f(r)\, e^{i(k_1 z - \omega t)}\right] \vec{e}_z \end{aligned} \tag{10}$$

where

$$f(r) = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\, g(r)\Big) \tag{11}$$

The simplest radial dependence is obtained by taking $g(r) = J_1(k_2 r)$. It then follows that

$$f(r) = k_2 J_o(k_2 r) \tag{12}$$

The vector potential and magnetic field are then given by

$$A_\phi = -B_o A J_1(k_2 r) e^{i(k_1 z - \omega t)} + \frac{r}{2} B_o \tag{13}$$

$$\vec{B} = -B_o A i k_1 J_1(k_2 r) e^{i(k_1 z - \omega t)} \vec{e}_r$$

$$+ B_o \left[1 + A k_2 J_o(k_2 r) e^{i(k_1 z - \omega t)} \right] \vec{e}_z \tag{14}$$

The current density corresponding to (13) is given by (1). Making use of the fact that $\nabla \cdot \vec{A} = 0$ we may write

$$\nabla \times \vec{B} = \nabla \times \nabla \times \vec{A} = -\nabla^2 \vec{A} = \frac{+4\pi}{c} i_\phi \vec{e}_\phi \tag{15}$$

In cylindrical coordinates this equation becomes

$$\nabla^2 \vec{A} = \vec{e}_\phi \left[\nabla^2 A_\phi - \frac{A_\phi}{r^2} \right] = \frac{-4\pi}{c} i_\phi \vec{e}_\phi \tag{16}$$

or

$$\frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial A_\phi}{\partial r} \right] - \frac{A_\phi}{r^2} + \frac{\partial^2 A_\phi}{\partial z^2} = \frac{-4\pi}{c} i_\phi \tag{17}$$

A short computation shows that the current density is

$$\frac{4\pi}{c} i_\phi = B_o A (k_1^2 + k_2^2) J_1(k_2 r) e^{i(k_1 z - \omega t)} \tag{18}$$

Substituting this expression in (5) and making the linearizing approximation that $\vec{B} \cong B_o \vec{e}_z$ yields

$$\frac{d\vec{v}}{dt} = \frac{\vec{i} \times \vec{B}_o}{\rho_o c} = \frac{i_\phi B_o}{\rho_o c} \vec{e}_r$$

$$= \frac{B_o^2 A}{4\pi \rho_o} (k_1^2 + k_2^2) \; J_1(k_2 r) \; e^{i(k_1 z - \omega t)} \vec{e}_r \qquad (19)$$

Integrating this equation gives the velocity of the medium as a function of the space coordinates and the time

$$\vec{v} = v_r \vec{e}_r = \frac{B_o^2 A}{4\pi \rho_o} (k_1^2 + k_2^2) \; J_1(k_2 r) \; \frac{e^{i(k_1 z - \omega t)}}{-i\omega} \vec{e}_r \qquad (20)$$

Substituting this expression in (6) and again making the linearizing approximation that $\vec{B} \cong B_o \vec{e}_z$ gives

$$-c\vec{E} = \frac{\partial \vec{A}}{\partial t} = \frac{B_o^3 A}{4\pi \rho_o} (k_1^2 + k_2^2) \; J_1(k_2 r) \; \frac{e^{i(k_1 z - \omega t)}}{-i\omega} \vec{e}_\phi \qquad (21)$$

Integrating, we find $\vec{A}$ to be

$$\vec{A} = \left[\frac{-B_o^3 A}{4\pi \rho_o \omega^2} (k_1^2 + k_2^2) \; J_1(k_2 r) \; e^{i(k_1 z - \omega t)} + \frac{r}{2} B_o \right] \vec{e}_\phi \qquad (22)$$

with the constant of integration representing the uniform part of the field. Comparing (22) with (13) we find that a solution of the form (13) is possible if ω^2 is related to k_1 and k_2 by

$$\omega^2 = \frac{B_o^2}{4\pi \rho_o} (k_1^2 + k_2^2) \qquad (23)$$

APPENDIX C

A Second Proof that $L_B = \frac{2MH}{qc}$ is a Constant of the Motion for Relativistic Particles Moving in Slowly Varying Magnetic Fields

The theorem that $L_B = \frac{2MH}{qc}$ is a constant of the motion will be proved here only for the case where the magnetic field is uniform, but varies with time. Similar proofs can be given for the case where the magnetic field has a gradient along or parallel to the undisturbed field.

The magnetic moment M is defined by Equation (2.3-5). Using (2.2-17) and (2.2-19), M can be written in the following form:

$$M = \frac{c^2}{2HB_z}\left[p^2 + \left[\frac{p_\phi}{r} - \frac{q}{c}A_\phi\right]^2\right] = \frac{H}{2c^2 B_z}\left[\dot{r}^2 + r^2\dot{\phi}^2\right] \quad (1)$$

Let $v_\perp$ and $v_\parallel$ be the components of the particles velocity perpendicular and parallel to the magnetic field, respectively. Then

$$M = \frac{\frac{1}{2}mv_\perp^2}{B_z} \quad (2)$$

where

$$m = \frac{H}{c^2} = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{m_o}{\sqrt{1 - \frac{(v_{\parallel}^2 + v_{\perp}^2)}{c^2}}} = \gamma m_o \quad (3)$$

is the relativistic mass.

For the case under consideration (uniform magnetic field varying with time),

$$A_{\phi} = \frac{r}{2} B_z(t) \quad (4)$$

$$B_r = \frac{\partial A_{\phi}}{\partial z} = 0 \quad (5)$$

From (5) and (2.2-16),

$$p_z = mv_{\parallel} = m_o \gamma v_{\parallel} = \text{constant} \quad (6)$$

Differentiating (2), (3), and (6)

$$\frac{dM}{M} = \frac{d\gamma}{\gamma} = \frac{2dv_{\perp}}{v_{\perp}} - \frac{dB_z}{B_z} \quad (7)$$

$$\frac{d\gamma}{\gamma} = \frac{\gamma^2}{c^2}\left[v_{\parallel}\ dv_{\parallel} + v_{\perp}\ dv_{\perp}\right] \quad (8)$$

$$\frac{d\gamma}{\gamma} + \frac{dv_{\parallel}}{v_{\parallel}} = 0 \quad (9)$$

From (8) and (9)

$$\frac{c^2}{\gamma^2}\frac{d\gamma}{\gamma} = v_{\parallel}\, dv_{\parallel} + v_{\perp} dv_{\perp} = -v_{\parallel}{}^2 \frac{d\gamma}{\gamma} + v_{\perp}\, dv_{\perp} \tag{10}$$

which may be written

$$\frac{dv}{v} = \frac{1}{v_{\perp}{}^2}\left[\frac{c^2}{\gamma^2} + v_{\parallel}{}^2\right]\frac{d\gamma}{\gamma} \tag{11}$$

Making use of the definition of γ we may write,

$$\frac{1}{\gamma^2} = 1 - \frac{v_{\perp}{}^2 + v_{\parallel}{}^2}{c^2} \tag{12}$$

or

$$\frac{c^2}{\gamma^2} = c^2 - v_{\perp}{}^2 - v_{\parallel}{}^2 \tag{13}$$

Hence,

$$\frac{dv_{\perp}}{v_{\perp}} = \frac{c^2 - v_{\perp}{}^2}{v_{\perp}{}^2}\frac{d\gamma}{\gamma} \tag{14}$$

Now, consider the quantity $L_B = \frac{2MH}{qc}$. Differentiating, and using (3),

$$\frac{dL_B}{L_B} = \frac{d(MH)}{MH} = \frac{dM}{M} + \frac{d\gamma}{\gamma} \tag{15}$$

Using (2) to compute $\frac{dM}{M}$ we find,

$$\frac{dL_B}{L_B} = 2\frac{d\gamma}{\gamma} + 2\frac{dv_{\perp}}{v_{\perp}} - \frac{dB_z}{B_z} \tag{16}$$

The term involving $\frac{dv_\perp}{v_\perp}$ may be eliminated by (14), giving,

$$\frac{dL_B}{L_B} = 2\frac{d\gamma}{\gamma}\left[1 + \frac{c^2 - v_\perp^2}{v_\perp^2}\right] - \frac{dB_z}{B_z}$$

$$= \frac{2c^2}{v_\perp^2}\frac{d\gamma}{\gamma} - \frac{dB_z}{B_z} \tag{17}$$

Next note that

$$\frac{dH}{dt} = \frac{d}{dt}(mc^2) = \frac{d}{dt}(m_o c^2 \gamma) = m_o c^2 \frac{d\gamma}{dt} \tag{18}$$

and also from (2.2-12), (2.3-5), and (2.2-19), that

$$\frac{dH}{dt} = M\frac{dB_z}{dt} \tag{19}$$

Hence, using (2) and (3),

$$\frac{m_o \gamma v_\perp^2}{2B_z m_o c^2 \gamma}\frac{dB_z}{dt} = \frac{1}{\gamma}\frac{d\gamma}{dt} \tag{20}$$

which may be written,

$$\frac{v_\perp^2}{2c^2}\frac{dB_z}{B_z} = \frac{d\gamma}{\gamma} \tag{21}$$

Using this relationship to eliminate $\frac{d\gamma}{\gamma}$ in (17), we find,

$$\frac{dL_B}{L_B} = \frac{2c^2}{v_\perp^2}\frac{v_\perp^2}{2c^2}\frac{dB_z}{B_z} - \frac{dB_z}{B_z} = 0 \tag{22}$$

which gives us finally,

$$L_B = \frac{2MH}{qc} = \text{constant.} \tag{23}$$

REFERENCES

Alfven, 1950, "Cosmical Electrodynamics," Oxford, Clarendon Press, pp. 19 - 23.

Chandrasekhar, S., 1943, Rev. Mod. Phys. 15, 1.

Cocconi, G., 1951, Phys. Rev. 83, 1193.

Davis, L., 1954, Phys. Rev. 93, 947.

Davis, L., 1954a, Phys. Rev. 96, 743.

Davis, L., 1956, Phys. Rev. 101, 351.

Erdelyi, et. al., 1954, "Tables of Integral Transforms," Vol. 1, Bateman Manuscript Project, California Institute of Technology, McGraw-Hill Book Co., New York, page 19.

Fermi, E., 1949, Phys. Rev. 75, 1169.

Fermi, E., 1954, Astrophys. J. 119, 1.

Fan, C. Y., 1951, Phys. Rev. 82, 211.

Gill, S., 1951, Cambridge Phil. Soc. Proc. 47, 96.

Goldstein, 1950, "Classical Mechanics," Addison-Wesley Press, Inc., Cambridge, Mass.

Helwig, 1955, Zeit. for Naturforshung, Vol. 10A, Heft 7, pp. 508-516.

Landau and Lifshitz, 1951, "The Classical Theory of Fields," Addison-Wesley Press, Inc., Cambridge, Mass.

Laning and Battin, 1956, "Random Processes in Automatic Control," McGraw-Hill Book Co., New York.

Morrison, Olbert, and Rossi, 1954, Phys. Rev. 94, 440.

Morse and Feshback, 1953, "Methods of Theoretical Physics," McGraw-Hill Book Co., New York, page 600.

Neher, H. V., and Stern, E. A., 1955, Phys. Rev. 98, 845.

Smythe, 1950, "Static and Dynamic Electricity," McGraw-Hill Book Co., New York, page 176.

Uhlenbeck, G. E., and Ornstein, L. S., 1930, Phys. Rev. 36, 823.

Unsold, A., 1951, Phys. Rev. 82, 857.

Whittaker and Watson, 1946, "Modern Analysis," Cambridge, The University Press, page 201.