THE HYDRODYNAMICS OF SPHERICAL CAVITIES IN THE NEIGHBORHOOD OF A RIGID PLANE

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ABSTRACT

The velocity potential in a perfect fluid is found for a sphere which is translating in a direction normal to a rigid plane and which is simultaneously undergoing a change in its radius. The solution of the problem is conveniently pursued in the bispherical coordinate system. The kinetic energy of the fluid is determined.

The dynamics of the motion of the translating sphere of variable radius is then described in terms of a Lagrangian which is formed from the kinetic energy of the fluid field and from the potential energy of the spherical cavity. The general equations of motion are exhibited and are solved in two cases of physical interest where approximations may be applied: (1) the case of an air bubble undergoing small oscillations because of a time varying external pressure, and (2) the case of a cavitation bubble collapsing so rapidly that the translational velocity may be neglected.

For the cases in which the dynamics of the problem are specifically determined, pressure effects on the rigid plane are expressed in terms of the dynamic variables of the cavity. It is suggested that these results will serve to aid in the further quantitative experimental investigation of cavitation damage.

The most important functions are evaluated numerically and are presented in a series of tables.

TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

I. GENERAL REMARKS

Introduction

A study of the dynamics of spherical cavities in the neighborhood of a rigid wall would seem to arise in a natural way out of the general program of cavitation research being carried on at the Hydrodynamics Laboratory. Early experiments with the high velocity water tunnel were designed so one could follow the history of cavitation bubbles found in the low pressure regions existing on the surface of a submerged ogive. The early work was analyzed by Plesset[1] with the aid of a theory which assumed that the bubbles were isolated and remote from any solid body. The analysis presented here may be regarded as another one of the many refinements of the original theory that have come out of this laboratory. Of particular interest in a practical way is the fact that the present analysis makes available the pressure distribution across the rigid wall. Such knowledge provides a first understanding of the process of cavitation damage. Of course, it must be remarked that this analysis presupposes an incompressible fluid; consequently when the fluid velocities approach that of the speed of sound in water, the analysis will fail.

The details of the dynamics of the cavity will, of course, depend on the boundary condition at infinity and upon the specific nature of the cavity itself. One would normally be interested in holding the pressure at infinity equal to the constant atmospheric pressure prevailing in the laboratory. On the other hand, a pressure at infinity varying sinusoidally with time is of interest in analyzing the interaction of bubbles with an ultrasonic field. Further, the cavity itself, depending upon the experimental conditions, can contain either water vapor or air.

The dynamics of these two bubble types exhibit a qualitative difference: the air bubble is stable; the vapor bubble is not. Fortunately, all such differences serve merely to alter the potential energy of the system.

Whatever the form of the potential energy of the system, the general problem divides itself into two parts. First it is necessary to find the kinetic energy of the velocity field. Once the kinetic energy is known, it is easy to form a Lagrangian. Second we must investigate the dynamic equations of motion.

To solve the potential problem one can separately consider (a) the potential produced by a sphere of fixed radius moving normal to a rigid wall, and (b) the potential produced by an expanding sphere whose center is a fixed distance from the rigid wall. The total velocity potential, the sum of the solutions of (a) and (b), describes the velocity potential produced by a spherical cavity which is undergoing translational and radial motion. These two problems in hydrodynamics are, of course, so well known as to have, by now, accumulated a certain venerable dignity. The classical solution was published by Basset[2] and others[3,4] in the later part of the last century. However, the classical work on these problems all relied on the seemingly natural approach of superimposing an infinite succession of image sources, calculated to fit the boundary conditions on both the plane and the sphere.

The classical technique for the solution of the relevant potential problems has several disturbing features. The most serious difficulties are that: (1) an explicit description of the field is unobtainable; and, (2) since the images yield for the potential a power series which diverges when the sphere and plane are in contact, a discussion of this important limiting situation becomes impossible. Of course, the limitations of the classical analysis were unimportant, until the development of the current interest in cavitation.

Another approach to the field problem is to seek a natural coordinate system which matches the geometry of the sphere and the plane. D. Endo[5] used such a coordinate system for the solution of a related potential problem having somewhat different boundary conditions. He used the bispherical coordinate system to solve the problem of two solid spheres, a fixed distance apart, lying in a uniform velocity stream which made an angle α with the line of centers of the spheres. Apparently unaware of Endo's work, Karal[6] in 1953 used the bispherical coordinate system to get the velocity potential of a solid sphere moving parallel to a plane. Of course, Karal's solution was a particular case of D. Endo's. It seems to be a peculiarity of this coordinate system that attempts at solution lead to sets of difference equations. Both authors left their results in the form of indicated iterations.

The bispherical coordinate is used here to solve the potential problem of (a) *the normally translating sphere*, and of (b) *the dilating sphere*. Thus, a complete description of the velocity field is made available. The solution of neither of those potential problems appears in the literature. Since it was also possible to solve the inevitable difference equations, an explicit expression becomes available for the velocity potential and hence for the Lagrangian. Once the Lagrangian is at hand, the equation of motion of the cavity and all other relevant physical quantities can be derived.

It is necessary to remark at this point that the proposed analysis contains several assumptions. We assume that the fluid in question is a perfect one, incompressible and nonviscous. Viscosity and compressibility enter in general only at small radii and high velocities; more important, we assume that the cavity retains its spherical shape in spite of the forces exerted on the bubble by the interacting wall. Photographic physical evidence exists to support the view that the distortion is not serious,[1] and the theoretical work of Plesset and Mitchell[7] on the surface perturbations of isolated bubbles discusses the surface stability of the cavity. Nevertheless, it should be recognized that the spherical surface is to be regarded as a first approximation of the true surface. The free surface problem could be formulated in the bispherical system by regarding the surface distortion as a perturbation of the spherical bubble. However, one is eventually led to an infinite family of difference equations, each member of which, itself, has an infinite number of terms. An analysis of the free surface problem was not pursued.

The Bispherical Coordinate System

The bispherical coordinate system is formed by three families of mutually orthogonally intersecting surfaces. The first two of these families, characterized by the parameters μ and η, consist of intersecting spherical surfaces. The third family of surfaces are planes which are labeled by the azimuthal angle ϕ.

The virtue of this coordinate system is that there is included among the spherical surfaces one whose radius is infinite. This particular surface is identified by the parameter $\mu = 0$, and is, itself, a plane. Thus, if one allows one of the spheres, $\mu = \mu_o \neq 0$, to coincide with the surface of the spherical cavity, the surface $\mu = 0$ will coincide with the neighboring rigid plane. It then becomes possible to solve the potential problems associated with this geometry by prescribing appropriate boundary conditions over the plane and the sphere.

Surfaces of the bispherical system can be generated by rotating the two dimensional bipolar coordinate system about its z-axis. The bipolar coordinate system arises from a Schwartz transformation and is described, for instance, by Smythe.[8] The transformation equations for the bispherical coordinates are catalogued by Morse and Feshbach.[9] For completeness those technical matters are repeated here.

Consider a two-dimensional y, z Cartesian coordinate system. We locate on the z-axis at the position $(y = 0,\ z = \pm\, a)$, two polar points. Let us pass an arbitrary circle through the polar points. Those points lying on the major arc of this circle will subtend an angle η with the polar points. The major arc is given the coordinate label η. The minor arc is labeled $(\pi - \eta)$. The line joining the polar points is characterized by $\eta = \pi$. A circle of radius a centered at the origin will also pass through the polar points; this circle is characterized by $\eta = \pi/2$. The foregoing construction fills the y, z plane with a one parameter family of circles. If d is the radial distance from the origin, then in region $d < a$, $\pi/2 < \eta \leq \pi$. In the region $d \geq a$, $0 \leq \eta \leq \pi/2$.

It is now possible to construct another set of circles orthogonal to the first set. We characterize the second family of circles by the parameter μ. The center of the circle, $\mu = \mu_o$, lies on the z-axis at a distance $b = a \coth \mu_o$ from the origin. The circle μ_o has a radius $r = a\left|\operatorname{csch} \mu_o\right|$. Thus the region $\mu > 0$ lies above the y-axis and the region $\mu < 0$ lies below the y-axis. The circle $\mu = 0$ coincides with the y-axis. The degenerate circles $\mu = \pm\, \infty$ are the polar points. The three-dimensional system is completed by rotating the y, z plane about the z-axis, and the third coordinate becomes the familiar azimuthal angle ϕ.

The preceding geometrical construction leads to the transformation equations:[9]

$$x = \frac{a \sin\eta \cos\phi}{\cosh\mu - \cos\eta} \quad ; \quad y = \frac{a \sin\eta \sin\phi}{\cosh\mu - \cos\eta} \quad ; \quad z = \frac{a \sinh\mu}{\cosh\mu - \cos\eta} \tag{1.1}$$

The metric is no longer Cartesian, and it develops that

$$h_\mu = h_\eta = a/(\cosh\mu - \cos\eta) \; ; \qquad h_\phi = a \sin\eta/(\cosh\mu - \cos\eta). \tag{1.2}$$

We want to solve Laplace's equation in this system. This becomes

$$\nabla^2\psi = \frac{1}{h^3_\mu}\left[\frac{\partial}{\partial\mu}\left(h_\mu \frac{\partial\psi}{\partial\mu}\right) + \frac{1}{\sin\eta}\frac{\partial}{\partial\eta}\left(h_\mu \sin\eta \frac{\partial\psi}{\partial\eta}\right) + \frac{h_\mu}{\sin^2\eta}\frac{\partial^2\psi}{\partial\phi^2}\right] = 0. \tag{1.3}$$

If we write

$$\psi = \sqrt{\cosh\mu - \cos\eta}\; F \tag{1.4}$$

Eq. (1.3) separates into three ordinary differential equations. After going through the usual procedure it becomes possible to write the following general expression for ψ :

$$\psi = \sqrt{\cosh\mu - \cos\eta} \sum_n \sum_m P_n^m(\cos\eta)\left[A_n^m \sin m\phi + B_n^m \cos m\phi\right] e^{\pm(n+\frac{1}{2})\mu}. \tag{1.5}$$

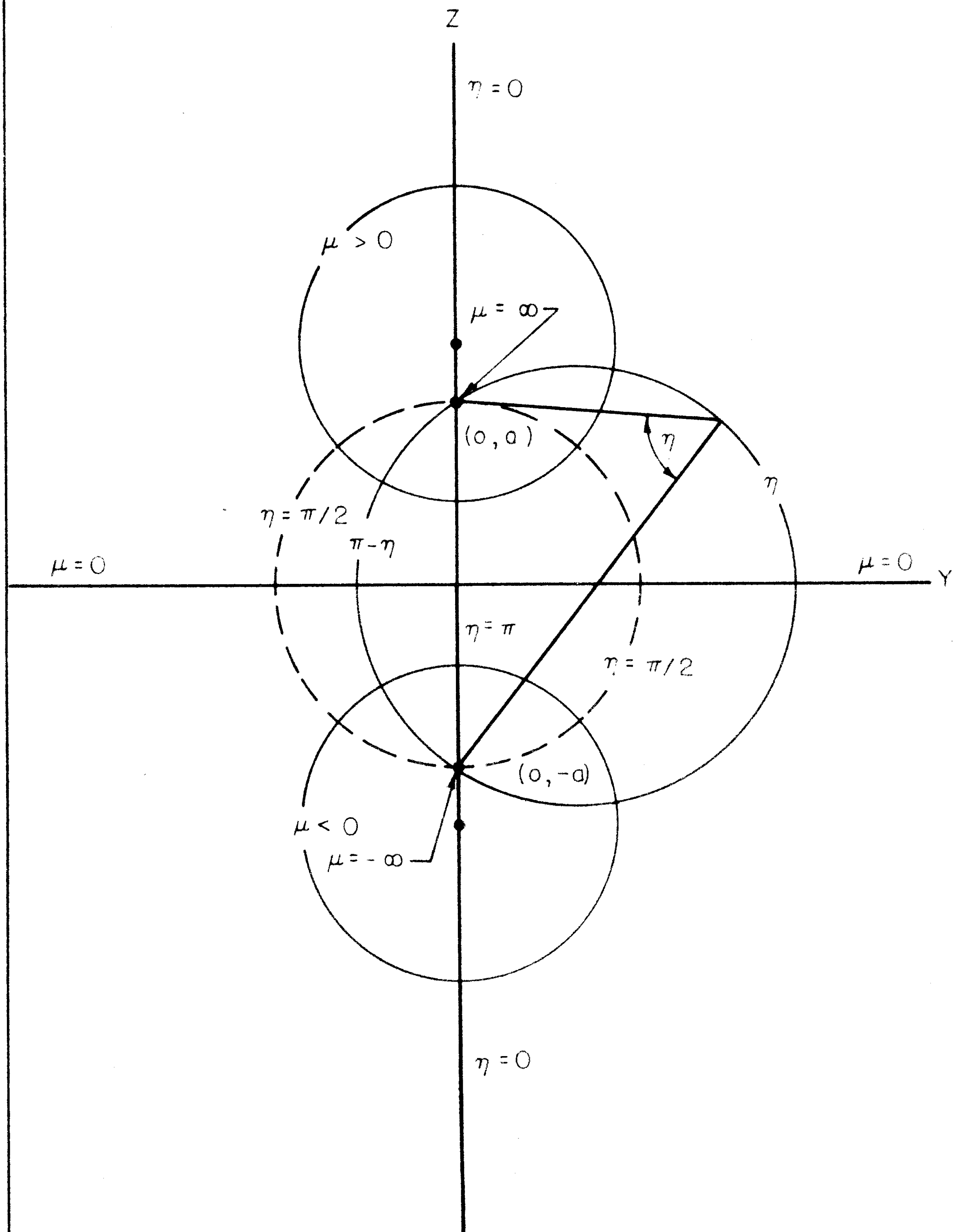
In the subsequent analysis we will need an expression for $(\cosh\mu - \cos\eta)^{-j+1/2}$, j, an integer. It is sufficient to evaluate

$$\frac{1}{\sqrt{\cosh\mu - \cos\eta}} = \frac{\sqrt{2}\, e^{-\mu/2}}{\sqrt{1 - 2e^{-\mu}\cos \; + e^{-2\mu}}} = \sqrt{2}\sum_{n=0}^{\infty} e^{-(n+1/2)\mu} P_n(\cos\eta). \tag{1.6}$$

We are now in a position to proceed with a solution of the boundary value problems.

FIGURE I

A SECTION OF THE BISPHERICAL COORDINATE SYSTEM IN THE YZ PLANE ($\phi = \pi/2$)



II. THE TRANSLATING SPHERE OF FIXED RADIUS

The Velocity Potential of the Translating Sphere

Let ψ_1 be the velocity potential for a sphere of fixed radius r, translating with a velocity $\dot{b}\bar{k}$, in a direction normal to a rigid plane. ψ_1 is specified in a coordinate system fixed to the plane. Choosing a sign convention, we require that the velocity, $\bar{V}$, at any point in the field be given by

$$\bar{V} = -\nabla\psi_1 . \tag{2.1}$$

The symmetry of the problem requires that the potential be independent of the azimuthal angle ϕ. The boundary condition on the rigid plane,

$$\left(\frac{\partial\psi_1}{\partial z}\right)_{\mu=0} = \left(\frac{\partial\psi_1}{\partial\mu}\right)_{\mu=0} = 0 \tag{2.2}$$

is satisfied by further specializing ψ_1 to the form

$$\psi_1 = \sqrt{\cos\mu - \cos\eta}\sum_{n=0}^{\infty} A_n P_n(\cos\eta)\cosh(n+\tfrac{1}{2})\mu . \tag{2.3}$$

If we now transform to a congruent coordinate system which differs from the first only in that it is translating with the velocity of the sphere, then the velocity potential in the new coordinate system becomes,

$$\psi_1' = \psi_1 + \dot{b}z . \tag{2.4}$$

We consider that the solid sphere, of radius r, is instantaneously at a distance b from the rigid plane. It then follows that the translating sphere coincides with the coordinate

$$\mu_o = \cosh^{-1}(b/r) ; \tag{2.5a}$$

also

$$a = r \sinh \mu_o = (b^2 - r^2)^{1/2} \tag{2.5b}$$

defines the scale of the coordinate system.

Now, an observer in the coordinate system which translates with the sphere will find that the fluid flow is always tangential to the sphere at its surface. Thus, the following boundary condition is imposed on ψ_1',

$$\left(\frac{\partial \psi_1'}{\partial \mu}\right)_{\mu=\mu_o} = 0 \, . \tag{2.6}$$

If one makes use of the transformation equation for z, the condition (2.6) becomes equivalent to

$$a\dot{b}\left[\frac{\cosh\mu_o}{\cosh\mu_o - \cos\eta} - \frac{\sinh^2\mu_o}{(\cosh\mu_o - \cos\eta)^2}\right] + \frac{\sqrt{\cosh\mu_o - \cos\eta}}{2} \cdot$$

$$\left\{\sum_o^\infty A_n(2n+1)\, P_n \sinh(n+\tfrac{1}{2})\,\mu_o + \frac{\sinh\mu_o}{(\cosh\mu_o - \cos\eta)} \sum_o^\infty A_n P_n \cosh(n+\tfrac{1}{2})\,\mu_o\right\} = 0. \tag{2.7}$$

Somewhat more conveniently,

$$\frac{2\dot{b}a\,(\cosh\mu_o \cos\eta - 1)}{(\cosh\mu_o - \cos\eta)^{3/2}} = \sum_o^\infty A_n P_n(\cos\eta)\Big[\cosh(n+\tfrac{1}{2})\,\mu_o \sinh\mu_o +$$

$$+ (2n+1)\sinh(n+\tfrac{1}{2})\,\mu_o \cosh\mu_o - (2n+1)\cos\eta \sinh(n+\tfrac{1}{2})\,\mu_o\Big]. \tag{2.8}$$

After some further manipulation, Eq. (2.8) can be recast in the form

$$\frac{2a\dot{b}(\cosh\mu_o \cos\ \ -1)}{(\cosh\mu_o - \cos\eta)^{3/2}} = \sum_o^\infty P_n(\cos\eta)\, R_n \tag{2.9}$$

where

$$R_n = (n+1)(A_n - A_{n+1}) \sinh(n+3/2)\,\mu_o + n(A_n - A_{n-1}) \sinh(n-1/2)\,\mu_o \, . \tag{2.10}$$

By making the natural definition of

$$C_n = A_n - A_{n-1} , \tag{2.11}$$

a further simplicity in form for R_n is achieved:

$$R_n = nC_n \sinh(n-\tfrac{1}{2})\mu_o - (n+1)C_{n+1} \sinh(n+\tfrac{3}{2})\mu_o . \tag{2.12}$$

It is clear that an expansion in Legendre Polynomials is now needed for the expression on the left-hand side of equation (2.9).

By differentiating both sides of equations (1.6), one finds that

$$\frac{1}{(\cosh\mu_o - \cos\eta)^{3/2}} = \frac{\sqrt{2}}{\sinh\mu_o} \sum_o^\infty (2n+1)\, e^{-(n+\frac{1}{2})\mu_o} P_n(\cos\eta) , \tag{2.13}$$

and also

$$\frac{\cosh\mu_o \cos\eta}{(\cosh\mu_o - \cos\eta)^{3/2}} = \sqrt{2}\,\coth\mu_o \sum_o^\infty \left[ne^{-(n-\frac{1}{2})\mu_o} + (n+1)e^{-(n+\frac{3}{2})\mu_o} \right] P_n(\cos\eta). \tag{2.14}$$

Using (2.13) and (2.14), we can find the expansion for the left-hand side of (2.9).

$$\frac{2a\dot{b}(\cosh\mu_o \cos\eta - 1)}{(\cosh\mu_o - \cos\eta)^{3/2}} = 2\sqrt{2}\, a\dot{b} \sum_{n=o}^\infty S_n P_n(\cos\eta) ,$$

where

$$S_n = n e^{-(n-\frac{1}{2})\mu_o} - (n+1)\, e^{-(n+\frac{3}{2})\mu_o} . \tag{2.15}$$

If we compare equations (2.15) and (2.9), the orthogonality of the Legendre Polynomials will require that

$$R_n = 2\sqrt{2}\, a\dot{b} S_n . \tag{2.16}$$

Putting $\alpha = 2\sqrt{2}\, a\dot{b}$, a difference equation emerges,

$$-(n+1)\,C_{n+1}\sinh(n+\tfrac{3}{2})\,\mu_o + nC_n\sinh(n-\tfrac{1}{2})\,\mu_o = \alpha\left(n\,e^{-(n-\frac{1}{2})\mu_o} - (n+1)\,e^{-(n+\frac{3}{2})\mu_o}\right). \tag{2.17}$$

We must now search for a particular solution to the difference equation. If the first and second terms of (2.17) had exhibited exactly the same functional dependence on the index n, a solution would be immediate. Such a fortunate circumstance does not quite exist. However, we observe that the functional behavior of the two terms on the left is very similar to that of the two terms on the right. We can exploit this symmetry by remarking that equation (2.17) is equivalent to the set of equations

$$\begin{aligned} C_n\sinh(n-\tfrac{1}{2})\,\mu_o &= \alpha\left(e^{-(n-\frac{1}{2})\mu_o} + \frac{h_n}{n}\right) \\ C_n\sinh(n+\tfrac{1}{2})\mu_o &= \alpha\left(e^{-(n+\frac{1}{2})\mu_o} + \frac{h_{n-1}}{n}\right) \\ h_o &= 0 \end{aligned} \tag{2.18}$$

h_n is, of course, some new unknown function introduced merely for convenience.

C_n may now be eliminated from (2.18) to yield,

$$\left(\frac{h_{n-1}}{n} + e^{-(n+\frac{1}{2})\mu_o}\right)\sinh(n-\tfrac{1}{2})\mu_o = \left(e^{-(n-\frac{1}{2})\mu_o} + \frac{h_n}{n}\right)\sinh(n+\tfrac{1}{2})\,\mu_o\quad . \tag{2.19}$$

Setting,

$$H_n = \sinh(n+\tfrac{1}{2})\;h_n\;, \tag{2.20}$$

it follows immediately that

$$\begin{aligned} H_n &= H_{n-1} + n\left(e^{-(n+\frac{1}{2})\mu_o}\sinh(n-\tfrac{1}{2})\,\mu_o - e^{-(n-\frac{1}{2})\mu_o}\sinh(n+\tfrac{1}{2})\,\mu_o\right) \\ &= \sum_{k=o}^{n} k\left(e^{-(k+\frac{1}{2})\mu_o}\sinh(k-\tfrac{1}{2})\,\mu_o - e^{-(k-\frac{1}{2})\mu_o}\sinh(k+\tfrac{1}{2}\mu_o)\right). \end{aligned} \tag{2.21}$$

Using equations (2.20) and (2.18) we find, after some manipulation, that

$$C_n = a \left[\frac{e^{-(n+\frac{1}{2})\mu_o}}{\sinh(n+\frac{1}{2})\mu_o} - \frac{(n-1)\sinh\mu_o}{2\sinh(n+\frac{1}{2})\mu_o \sinh(n-\frac{1}{2})\mu_o} \right] \qquad (2.22)$$

The validity of (2.22) can be verified by direct substitution into (2.17).

We shall find it convenient to define the set of functions

$$\gamma_n(\mu_o) = \frac{e^{-(n+\frac{1}{2})\mu_o}}{\sinh(n+\frac{1}{2})\mu_o} \quad . \qquad (2.23)$$

Since

$$\gamma_{n-1} - \gamma_n = \frac{\sinh\mu_o}{\sinh(n+\frac{1}{2})\mu_o \sinh(n-\frac{1}{2})\mu_o} , \qquad (2.24)$$

C_n reduces to the more tractable form

$$C_n = \frac{a}{2}\left[(n+1)\gamma_n - (n-1)\gamma_{n-1}\right] \quad . \qquad (2.25)$$

To evaluate A_n we recognize that equation (2.11) is equivalent to

$$A_n = A_o + \sum_{k=1}^{n} C_k \; . \qquad (2.26)$$

It should be observed that if A_n is a solution to our original difference equation, so also is $A_n + k$, where k is any fixed constant. A_o may, itself, then be regarded as an arbitrary constant which is a solution of the homogeneous difference equation. We shall have to evaluate A_o from other considerations.

First it is necessary to find an explicit expression for A_n. Making use of the result for C_n, it follows that

$$A_n = A_o + \frac{a}{2} \sum_1^n \left[(k+1)\gamma_k - (k-1)\gamma_{k-1}\right] \quad = A_o + \frac{a}{2}(n\gamma_n + \sum_1^n \gamma_k) \tag{2.27}$$

A_o may now be evaluated by recalling that each A_n is a coefficient of a term in an infinite series. Furthermore, A_n is just that factor of the term which causes the infinite series to converge. It follows that a necessary condition for the convergence of (2.3) is that

$$\lim_{n \to \infty} A_n = 0 \, . \tag{2.28}$$

This condition requires that

$$A_o = - \frac{a}{2} \sum_1^{\infty} \gamma_k(\mu_o) \, . \tag{2.29}$$

Thus we finally arrive at an expression for the coefficients in the terms of the potential functions, or

$$A_n = \frac{a}{2}\left[n\gamma_n - \sum_{n+1}^{\infty} \gamma_k\right] = \sqrt{2}\, r\dot{b} \sinh\mu_o \left[n\gamma_n(\mu_o) - \sum_{n+1}^{\infty} \gamma_k(\mu_o)\right] \, . \tag{2.30}$$

The dimensionless coefficient

$$A'_n(\mu_o) = \sqrt{2} \sinh\mu_o \left[n\gamma_n(\mu_o) - \sum_{n+1}^{\infty} \gamma_k(\mu_o)\right] \tag{2.31}$$

is presented in extensive tabular form for various values of μ_o and n in Table I.

As might be expected

$$\lim_{\mu_o \to \infty} A_n \sim n e^{-2n\mu_o} \sqrt{2}\, \dot{b} r \to 0 \, . \tag{2.32}$$

On the other hand, as $\mu_o \to 0$, A_n leads to the divergent harmonic series, which usually is associated with a logarithmic singularity. On physical grounds we have good reason to expect difficulties when $\mu_o = 0$. When the moving sphere is in contact with the rigid plane, the boundary conditions become contradictory at the point of contact. The boundary condition on the plane requires that the point of contact be a stagnation point; the boundary condition on the sphere requires that the velocity of the fluid at the point of contact be equal to the translational velocity of the sphere. Nevertheless the singularity is a weak one, and as we shall see, the kinetic energy of the field remains finite even in this limiting case.

To further illustrate the nature of this solution a contour plot of the isopotentials of the function,

$$\frac{\psi_1}{\sqrt{2}\, a\, \dot{b}} = \sqrt{\frac{\cosh\mu - \cos\eta}{2\sinh\mu_o}} \sum_{n=0}^{\infty} A_n'(\mu_o) \cosh(n+\tfrac{1}{2})\mu \quad P_n(\cos\eta) \tag{2.33}$$

is presented in Figure II.

For completeness we write out in detail the expressions for the velocity potential in the fixed coordinate system

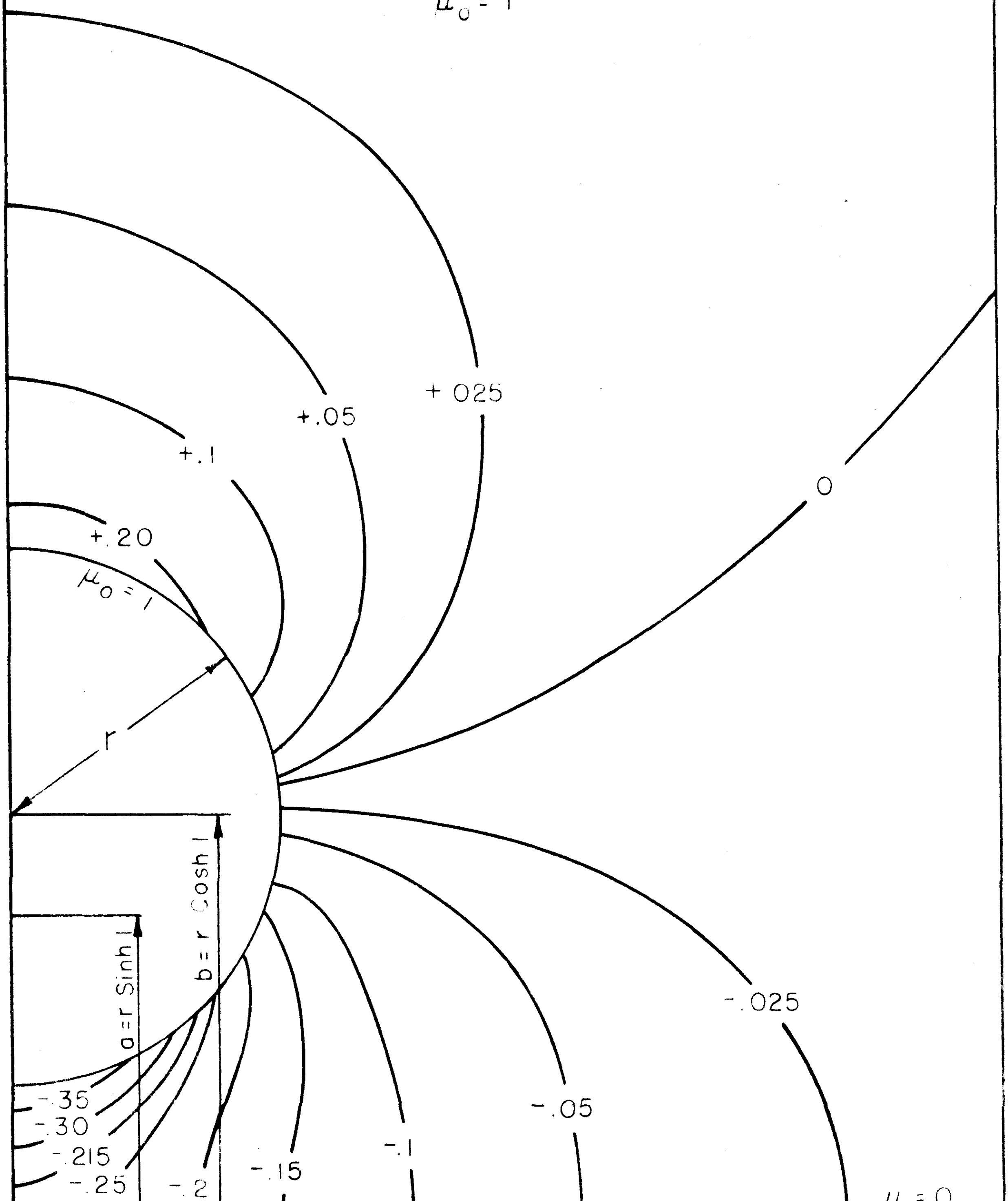
$$\psi_1(\mu,\eta;\mu_o) = \sqrt{2}\, r\, \dot{b} \sinh\mu_o \sqrt{\cosh\mu - \cos\eta} \;\bullet$$
$$\sum_{0}^{\infty}\left[n\gamma_n(\mu_o) - \sum_{n+1}^{\infty}\gamma_k(\mu_o)\right]\cosh(n+\tfrac{1}{2})\mu\, P_n(\cos\eta)\,. \tag{2.34}$$

This completes the solution of the potential problem of the translating sphere. We should perhaps remark that the velocity at any point in space may now be easily obtained by the use of equation (2. 1). The velocity distribution over the rigid plane is exhibited explicitly in a later section.

FIGURE 2

ISOPOTENTIALS OF $\Psi_1/\sqrt{2}\, a\dot{b}$ FOR THE TRANSLATING SPHERE OF FIXED RADIUS

$\mu_o = 1$



Kinetic Energy of the Velocity Field for the Translating Sphere

As indicated in the first section, early attempts at the solution of the potential problem by the image method yielded quite crude results for the kinetic energy of the velocity field. During World War II, this problem became of very real practical interest for the analysis of underwater explosions. In a confidential report which was later declassified, the Applied Mathematics Group of New York University[10] made a good attack on the kinetic energy problem. Though once again image techniques were employed, these authors developed an indicated solution for the kinetic energy which did not make use of a detailed knowledge of the potential function. This solution suffered from the fact that it was not explicit but required the use of a lengthy iteration calculation which converged slowly in the region of small μ_o. Of course, the potential function and other quantities derivable from it were not available. Nevertheless the work was adequate for an analysis of the dynamics of gas-filled explosion cavities. Up to the accuracy with which they were presented in the report, the AMG values for the kinetic energy agree exactly with the results we shall develop here.

The kinetic energy of the fluid may be expressed by

$$T_1 = \frac{1}{2} \rho \int_s \psi_1 \frac{\partial \psi_1}{\partial n} \, ds \, . \tag{2.35}$$

The integration is to be carried out over all surfaces enclosing the fluid.

Because of the boundary conditions, this integral vanishes over the surface at infinity and over the rigid plane. Thus it is merely necessary to perform the integration over the translating sphere. When a proper account is taken of the metric quantities, h_μ, h_ϕ and h_η, the expression for the kinetic energy becomes

$$T_1 = \pi \rho a \int_{-1}^{1} \frac{1}{\cosh \mu_o - w} \psi_1 \left(\frac{\partial \psi_1}{\partial \mu}\right)_{\mu=\mu_o} dw \ . \tag{2.36}$$

We have set w equal to $\cos \eta$.

The boundary condition on the surface of the sphere provides a simple expression for the derivative, $\left(\frac{\partial \psi_1}{\partial \mu}\right)_{\mu=\mu_o}$

$$\left(\frac{\partial \psi_1}{\partial \mu}\right)_{\mu=\mu_o} = -\dot{b}\left(\frac{\partial z}{\partial \mu}\right)_{\mu=\mu_o} = a\dot{b} \ \frac{(\cosh \mu_o \ \cos\eta - 1)}{(\cosh \mu_o - \cos\eta)^2} \ . \tag{2.37}$$

Then, (2.36) becomes

$$T_1 = \pi \rho a^2 \dot{b} \int_{-1}^{1} \left[\frac{(w \cosh \mu_o - 1)}{(\cosh \mu_o - w)^{5/2}} \sum_{n=o}^{\infty} A_n \cosh(n+\tfrac{1}{2})\mu_o \ P_n(w) \right] dw \ . \tag{2.38}$$

It is now necessary to develop an expansion in Legendre Polynomials for the first factor inside the integral. As before, it is possible to make use of equation (1.6) and its derivatives. By also calling upon some standard relationship between the Legendre Polynomials, one deduces that

$$\frac{w \cosh \mu_o - 1}{(\cosh \mu_o - w)^{5/2}} = \frac{\sqrt{2}}{3} \sum_{n=o}^{\infty} P_n(w) \ e^{-(n+\frac{1}{2})\mu_o} (2n+1) \left[(2n+1) - \frac{\cosh \mu_o}{\sinh \mu_o} \right] . \tag{2.39}$$

If one uses equation (2.39), then (2.38) becomes an integration over a double sum of Legendre Polynomials. Because of the orthonormality properties of the Polynomials, the integrations yield

$$T_1 = \frac{2\pi \rho a^2 \dot{b} \sqrt{2}}{3} \sum_{n=o}^{\infty} A_n \ e^{-(n+\frac{1}{2})\mu_o} \ \frac{(2n+1) \sinh \mu_o - \cosh \mu_o}{\sinh \mu_o} \cosh(n+\tfrac{1}{2})\mu_o . \tag{2.40}$$

Equivalently,

$$T_1 = \frac{2\pi\rho a^2 \sqrt{2}\,\dot{b}}{3\sinh\mu_o} \sum_{n=o}^{\infty} A_n \left[n\,e^{-(n-\frac{1}{2})\mu_o} - (n+1)\,e^{-(n+\frac{3}{2})\mu_o} \right] \cosh(n+\tfrac{1}{2})\mu_o \,. \tag{2.41}$$

In equation (2. 30) we have available an expression for A_n. If the coefficient of A_n in equation (2.41) is reduced to exponential form, and if the expression for A_n is used, the summation in equation (2.41) becomes

$$\frac{a}{4}\Bigg\{ e^{\mu_o} \sum_{n=o}^{\infty}\Big[n^2\gamma_n - n \sum_{k=n+1}^{\infty}\gamma_k \Big] - e^{-\mu_o} \sum_{n=o}^{\infty}\Big[n(n+1)\gamma_n - (n+1)\sum_{k=n+1}^{\infty}\gamma_k \Big] +$$
$$+ \sum_{n=o}^{\infty}\Big[n^2\gamma_n - n\sum_{k=n+1}^{\infty}\gamma_k \Big] e^{-2n\mu_o} + \sum_{n=o}^{\infty} e^{-2(n+1)\mu_o}\Big[(n+1)\sum_{k=n+1}^{\infty}\gamma_k - n(n+1)\gamma_n \Big]\Bigg\}. \tag{2.42}$$

After some uninteresting manipulation, double summations can be reduced and the expression for the kinetic energy becomes

$$T_1 = \pi\rho r^3 \dot{b}^2/3 \;\; G(\mu_o)$$

where

$$G(\mu_o) = 2\sinh^3\mu_o \sum_{n=1}^{\infty} \frac{n(n+1)(1+2e^{-(2n+1)\mu_o})\,e^{-(n+\frac{1}{2})\mu_o}}{\sinh(n+\frac{1}{2})\,\mu_o} \,. \tag{2.43}$$

Though equation (2.43) is relatively simple, it presents convergence difficulties at $\mu_o = 0$. Since this is a zero times infinity form at $\mu_o = 0$, there is some hope that the singularity is removable. Certainly, solely on physical grounds, a finite energy is to be expected. We now proceed to reduce (2.43) to a form which converges over the entire range of μ_o. $G(\mu_o)$ separates into

$$G(\mu_o) = \sinh^3\mu_o \left[-8\sum_{n=o}^{\infty} n(n+1)\,e^{-(2n+1)\mu_o} + 6\sum_{n=o}^{\infty} \frac{n(n+1)\,e^{-(n+\frac{1}{2})\mu_o}}{\sinh(n+\frac{1}{2})\,\mu_o} \right] \tag{2.44}$$

It is possible to express the first of these sums in a closed form.

If we let

$$g(\mu_o) = \sum_o^{\infty} e^{-2n\mu_o} = 1/1-e^{-2\mu_o} \qquad (2.45)$$

then,

$$\frac{dg}{d\mu_o} = -2 \sum_{n=o}^{\infty} n\, e^{-2n\mu_o}, \qquad (2.46)$$

and thus

$$e^{\mu_o} \frac{d}{d\mu_o} e^{-2\mu_o} \frac{d}{d\mu_o} g(\mu_o) = 4 \sum_{n=o}^{\infty} n(n+1)\, e^{-(2n+1)\mu_o} . \qquad (2.47)$$

If we perform upon the right-hand side of (2.45) the differentiations indicated in (2.47), we eventually conclude that

$$4 \sum_{n=o}^{\infty} n(n+1)\, e^{-(2n+1)\mu_o} = \frac{1}{(\sinh \mu_o)^3} . \qquad (2.48)$$

The expression for $G(\mu_o)$ now becomes,

$$G(\mu_o) = -2 + 6 \sinh^3 \mu_o \sum_{n=o}^{\infty} \frac{n(n+1)\, e^{-(n+\frac{1}{2})\mu_o}}{\sinh\, (n+\frac{1}{2})\mu_o} . \qquad (2.49)$$

The summation in (2.49) may be rewritten as follows:

$$\sum_o^{\infty} \frac{n(n+1)\, e^{-(n+\frac{1}{2})\mu_o}}{\sinh\, (n+\frac{1}{2})\mu_o} = 2 \sum_o^{\infty} \frac{n(n+1)\, e^{-(2n+1)\mu_o}}{1 - e^{-(2n+1)\mu_o}} =$$

$$= 2 \sum_{n=o}^{\infty} \sum_{r=o}^{\infty} n(n+1)\, e^{-(r+1)(2n+1)\mu_o} \qquad (2.50)$$

If we use (2.50) and (2.48), (2.49) becomes the very simple form

$$G(\mu_o) = -2 + 3 \sum_{r=1}^{\infty} \left(\frac{\sinh \mu_o}{\sinh r\, \mu_o}\right)^3 = 1 + 3 \sum_{r=2}^{\infty} \left(\frac{\sinh \mu_o}{\sinh r\, \mu_o}\right)^3 . \qquad (2.51)$$

It can be shown that, to any desired order in r/b, the resulting value for the kinetic energy is consistent with that derived from the classical approximate solution by images.

In accompanying tables, values of the function $G(\mu_o)$ are tabulated for various values of μ_o. We observe that

$$\lim_{\mu_o \to \infty} G(\mu_o) = 1 \qquad (2.52)$$

When $\mu_o \to \infty$ the kinetic energy reduces to that of a free sphere. The form in (2.51) has the additional virtue that it can be evaluated when the sphere is in contact with the plane $(\mu_o = 0)$. In that case

$$G(0) = 1 + 3 \sum_{r=1}^{\infty} \frac{1}{(r+1)^3} = 1.606213 . \qquad (2.53)$$

For convenience, we now define the quantity

$$f_1(\mu_o) = \sum_{r=2}^{\infty} \left(\frac{\sinh \mu_o}{\sinh r \mu_o} \right)^3 , \qquad (2.54)$$

so

$$G(\mu_o) = 1 + 3 f_1(\mu_o) .$$

$f_1(\mu_o)$ is a measure of the effect of the wall on the kinetic energy. It can be noted that the convergence of $f_1(\mu_o)$ is poorest at $\mu_o = 0$. Even there, the convergence is like $1/k^3$, which is certainly respectable. For larger values of μ_o the convergence is exponential.

The Force on the Translating Sphere of Fixed Radius

Since all the work done by the sphere is used to increase the kinetic energy of the velocity field, the force, F_z, that the fluid exerts on the sphere in the positive direction is immediately given by

$$F_z = -\frac{1}{\dot{b}}\frac{dT_1}{dt} = \frac{\pi r^3 \rho}{3}\left[-2\ddot{b}G(\mu_o) - 3\dot{b}^2\left(\frac{\partial f_1}{\partial b}\right)_r\right]. \tag{2.55}$$

Since

$$\left(\frac{\partial f_1}{\partial b}\right)_r = \frac{df_1}{d\mu_o}\left(\frac{\partial \mu_o}{\partial b}\right)_r = \frac{1}{r \sinh \mu_o}\frac{df_1}{d\mu_o}, \tag{2.56}$$

then,

$$F_z = \dot{b}^2 \pi r^2 \rho \left[-\frac{2}{3}\frac{\ddot{b}r}{\dot{b}^2}G(\mu_o) - \frac{1}{\sinh \mu_o}\frac{df_1}{d\mu_o}\right]. \tag{2.57}$$

Performing the differentiation, we find

$$-\frac{1}{\sinh \mu_o}\frac{df_1}{d\mu_o} = 3\sum_{r=2}^{\infty}\left(\frac{\sinh \mu_o}{\sinh r\mu_o}\right)^2\left(\frac{r \coth r\mu_o - \coth \mu_o}{\sinh r \mu_o}\right) = P_z \tag{2.58}$$

P_z, a dimensionless stress, is just the average stress the fluid exerts over the projected area of the sphere if the sphere is translating in a fluid of unit density with a uniform velocity of unit magnitude. P_z is positive for all values of μ_o. Thus the solid sphere is always repelled by the wall if it moves with a uniform velocity which is normal to the wall. The sign of the velocity is irrelevant. This is, of course, a classical result.

We observe, in passing, that the force becomes infinite when $\mu_o = 0$. This is a pleasant guarantee that the sphere will not penetrate the plane. Values of P_z have been calculated and tables of these results are included.

The Speed of a Freely Translating Sphere of Fixed Radius

If the sphere of fixed radius is permitted to translate freely, then of course the total energy is a constant, and in this case it is equal to the kinetic energy

$$E_1 = T_1 = \frac{\pi \rho \dot{b}^2 r^3}{3} G(\mu_o) . \tag{2.59}$$

$G(\mu_o)$ increases near the wall. Since E_1 is constant, the speed must decrease as the sphere approaches the wall, and must increase as the sphere recedes from the wall. This is precisely the result of the previous subsection. If $\dot{b}_\infty$ is the speed of the sphere when it is infinitely far from the wall, then, since $G(\infty) = 1$, we have

$$\dot{b} = \dot{b}_\infty / \sqrt{G(\mu_o)} . \tag{2.60}$$

Values of $\left[G(\mu_o)\right]^{-1/2}$ have been calculated, and a table of these values is included. Table III shows, in particular, that the ratio of the speed of the sphere when it is touching the plane to its speed at infinity is .789.

III. THE EXPANDING SPHERE OF FIXED CENTER

The Velocity Potential of the Expanding Sphere of Fixed Center

We now seek the velocity potential produced by a sphere whose center lies at a fixed distance, b, from a rigid plane, and whose radius, r, changes at a given rate $\dot{r}$. We use a bispherical coordinate system fixed to the rigid plane $(\mu = 0)$, and so constructed that μ_o instantaneously coincides with the surface of the sphere. Thus equations (2.5) of the previous section still apply. If we let ψ_2 be the velocity potential in the fixed coordinate system, then the boundary condition over the plane

$$\left(\frac{\partial \psi_2}{\partial z}\right)_{\mu=0} = \left(\frac{\partial \psi_2}{\partial \mu}\right)_{\mu=0} = 0 \tag{3.1}$$

specializes the velocity potential to

$$\psi_2 = \sqrt{\cosh \mu - \cos \eta} \sum_{n=o}^{\infty} B_n P_n(\cos \eta) \cosh (n+\tfrac{1}{2}) \mu \,. \tag{3.2}$$

The sign convention has been chosen so that, if $\overline{V}$ is the velocity anywhere in space,

$$\overline{V} = -\nabla \psi_2 \,. \tag{3.3}$$

The boundary condition on the moving surface of the sphere is that the radial component of the relative velocity between the fluid and the surface must vanish over the surface of the sphere. This is equivalent to

$$\dot{r} = \overline{V}_r = \frac{\cosh\mu_o - \cos\eta}{a} \left(\frac{\partial\psi_2}{\partial\mu}\right)_{\mu=\mu_o} =$$

$$= \frac{(\cosh\mu_o - \cos\eta)^{1/2}}{2a} \left[(\cosh\mu_o - \cos\eta) \sum_{n=o}^{\infty} B_n (2n+1) P_n \sinh(n+\tfrac{1}{2})\mu_o + \right.$$

$$\left. + \sum_{n=o}^{\infty} B_n P_n \sinh\mu_o \cosh(n+\tfrac{1}{2})\mu_o \right] . \qquad (3.4)$$

Terms in P_n and $\cos\eta\, P_n$ arise. If we use the recursion formulae for the Legendre Polynomials, equation (3.4) can be transformed into

$$\frac{2\dot{r}a}{(\cosh\mu_o - \cos\ )^{1/2}} = \sum_{n=o}^{\infty} P_n U_n , \qquad (3.5)$$

where

$$U_n = (n+1)(B_n - B_{n+1}) \sinh(n+\tfrac{3}{2})\mu_o + n(B_n - B_{n-1})\sinh(n-\tfrac{1}{2})\mu_o . \qquad (3.6)$$

We make the substitution,

$$\begin{aligned} D_n &= n(B_n - B_{n-1}) \\ D_o &= 0 \\ B_n &= B_o + \sum_{1}^{n} D_k/k \end{aligned} \qquad (3.7)$$

to get

$$U_n = D_n \sinh(n-\tfrac{1}{2})\mu_o - D_{n+1} \sinh(n+\tfrac{3}{2})\mu_o . \qquad (3.8)$$

But from equation (1.6) we can conclude that

$$\frac{2\dot{r}a}{(\cosh\mu_o - \cos\eta)^{1/2}} = a\dot{r}\, 2\sqrt{2} \sum_{n=o}^{\infty} e^{-(n+\frac{1}{2})\mu_o} P_n(\cos\eta) . \qquad (3.9)$$

If we use (3.9), (3.8) and (3.5), the orthogonality of the Legendre Polynomials leads us to another difference equation

$$U_n = D_{n+1} \sinh(n+\tfrac{3}{2})\mu_o - D_n \sinh(n-\tfrac{1}{2})\mu_o = -\beta e^{-(n+\frac{1}{2})\mu_o} \qquad (3.10)$$

where

$$\beta = 2a\sqrt{2}\,\dot{r} = 2\sqrt{2}\,\dot{r}\,r\,\sinh\mu_o \ .$$

The form of the solution of the analogous difference equation in Section 2 suggests the substitution,

$$D_n = \frac{d_n}{\sinh(n-\frac{1}{2})\mu_o \, \sinh(n+\frac{1}{2})\mu_o} \ . \qquad (3.11)$$

It then follows that

$$d_{n+1} - d_n = -\beta e^{-(n+\frac{1}{2})\mu_o} \sinh(n+\tfrac{1}{2})\mu_o = -\frac{\beta}{2}(1-e^{-(2n+1)\mu_o}) \ . \qquad (3.12)$$

The solution of (3.12) is immediate:

$$d_n = -\frac{\beta}{2}(n - \sum_{o}^{n-1} e^{-(2k+1)\mu_o}) = -\frac{\beta}{2}\left(n - \frac{1-e^{-2n\mu_o}}{e^{\mu_o}-e^{-\mu_o}}\right) \ . \qquad (3.13)$$

Since d_n is now determined, it is possible to find B_n by making use of equations (3.7) and (3.11). After some algebra, one finds that

$$B_n = B_o - \frac{\beta}{2\sinh\mu_o}\left[\Upsilon_o(\mu_o) - \Upsilon_n(\mu_o) - \frac{1}{2\cosh\frac{\mu_o}{2}} \cdot \sum_{k=1}^{n} \frac{e^{-\mu_o/2}\Upsilon_{k-1}(\mu_o) + e^{\mu_o/2}\Upsilon_k(\mu_o)}{k}\right] \qquad (3.14)$$

It may be recalled that $\Upsilon_n(\mu_o)$ has been defined in equation (2.23).

As in the previous section, we are left with a solution with one arbitrary constant. The arbitrary constant, B_o, is the solution of the homogeneous difference equation. We proceed as we did in Section 2 and remark that

$$\lim_{n\to\infty} B_n = 0 \tag{3.15}$$

is a necessary condition for the convergence of the series in (3.2).

Equation (3.15) determines B_o

$$B_o = \frac{\beta}{2\sinh\mu_o}\left[\Upsilon_o(\mu_o) - \frac{1}{2\cosh\frac{\mu_o}{2}}\sum_{k=1}^{\infty}\frac{e^{-\frac{\mu_o}{2}}\Upsilon_{k-1}(\mu_o) + e^{\frac{\mu_o}{2}}\Upsilon_k(\mu_o)}{k}\right], \tag{3.16}$$

and hence B_n

$$B_n = \frac{\beta}{2\sinh\mu_o}\left[\Upsilon_n(\mu_o) - \frac{1}{2\cosh\frac{\mu_o}{2}}\sum_{k=n+1}^{\infty}\frac{e^{-\frac{\mu_o}{2}}\Upsilon_{k-1}(\mu_o) + e^{\frac{\mu_o}{2}}\Upsilon_k(\mu_o)}{k}\right]. \tag{3.17}$$

Just as in Section 2, B_n vanishes as $\mu_o \to \infty$, and diverges when $\mu_o = 0$.

Values of the dimensionless coefficient B_n' have been calculated for a large range of μ_o's and n's. These values are presented in tables. B_n' is defined by

$$B_n'(\mu_o) = \frac{B_n}{\dot{r}r\sqrt{2}} = \Upsilon_n(\mu_o) - \frac{1}{2\cosh\frac{\mu_o}{2}}\sum_{k=n+1}^{\infty}\frac{e^{-\frac{\mu_o}{2}}\Upsilon_{k-1}(\mu_o) + e^{\frac{\mu_o}{2}}\Upsilon_k(\mu_o)}{k}. \tag{3.18}$$

For completeness we exhibit the full expression of the velocity potential of the expanding sphere

$$\psi_2(\mu,\eta;\mu_o) = \sqrt{\cosh\mu - \cos\eta}\,(\sqrt{2}\, r\dot{r}) \cdot$$

$$\sum_{n=o}^{\infty}\left\{\left[\Upsilon_n(\mu_o) - \frac{1}{2\cosh\frac{\mu_o}{2}}\sum_{k=n+1}^{\infty}\frac{e^{-\frac{\mu_o}{2}}\Upsilon_{k-1}(\mu_o) + e^{\frac{\mu_o}{2}}\Upsilon_k(\mu_o)}{k}\right]\cdot\right.$$

$$\left. P_n(\cos\eta)\cosh(n+\tfrac{1}{2})\mu\right\} . \tag{3.19}$$

This, in principle, solves the potential problem of the expanding sphere. The velocity of any point in space is given by equation (3.3). Explicit results for the velocity distribution over the rigid plane will be presented in a later section.

The Kinetic Energy of the Field Produced by an Expanding Sphere

The determination of the kinetic energy, T_2, of the field produced by the expanding sphere of fixed center is analogous in technique to the corresponding calculation of T. One engages in a somewhat tedious algebraic exercise, using the recursion formulas for the Legendre Polynomials and their derivatives, until eventually a convenient form emerges. As far as possible, the technical details of this uninteresting exercise will be suppressed in the following section. There are, however, two points of minor interest which should be considered.

First, one's general experience with solutions of Laplace's equation brings to mind the fact that field energies can usually be expressed as quadratic forms of the coefficients which appear in the expansion for the potential function. A quadratic form does not appear in the calculation of the preceding section, because the expansion coefficients were explicitly evaluated. In this section the quadratic form will be exhibited.

Second, it develops that the convergence of the kinetic energy function near $\mu_o = 0$ is not as good as in the previous case. Indeed, some delicacy is required in order to evaluate T_2 when the sphere and plane are in contact.

By following the same reasoning as in Section 2, one obtains an expression for the kinetic energy which is analogous to equation (2.38),

$$T_2 = \pi \rho a^2 \dot{r} \int_{-1}^{1} \frac{1}{(\cosh \mu_o - w)^{3/2}} \sum_{n=o}^{\infty} B_n \cosh(n+\tfrac{1}{2}) \mu_o \, P_n(w) \, dw \,. \qquad (3.20)$$

The factor multiplying the summation must be expanded in Legendre Polynomials. The integration may then be performed, and one obtains

$$T_2 = \frac{\pi \rho a}{\sinh \mu_o} \sum_{n=o}^{\infty} \beta e^{-(n+\frac{1}{2})\mu_o} B_n \cosh (n+\frac{1}{2})\mu_o \quad . \tag{3.21}$$

At this point we depart a bit from the procedure of the preceding section. We do not substitute for B_n, but instead we replace $\beta e^{-(n+1/2)\mu_o}$ by its value obtained from the difference equation. The kinetic energy then becomes,

$$T_2 = \frac{\pi \rho a}{\sinh \mu_o} \sum_{n=1}^{\infty} \left[n(B_n - B_{n-1}) \sinh(n-\frac{1}{2})\mu_o - (n+1)(B_{n+1} - B_n) \sinh(n+\frac{3}{2})\mu_o \right] \bullet B_n \cosh(n+\frac{1}{2})\mu_o \quad . \tag{3.22}$$

After some algebraic manipulation, (2.33) finally reduces to the quadratic form

$$T_2 = \pi \rho r^3 \dot{r}^2 \sum_{n=o}^{\infty} \left[\sinh \mu_o (B_n')^2 + n(B_n' - B_{n-1}')^2 \sinh 2n\mu_o \right]. \tag{3.23}$$

The dimensionless coefficient B_n' has been defined in equation (3.19).

Although equation (3.23) exhibits the kinetic energy as a quadratic form in the B_n's, it is not suitable for calculation. We shall define the quantity $f_2(\mu_o)$ by the equation

$$T_2 = 2\pi \rho \dot{r}^2 r^3 (1 + f_2(\mu_o)) \quad . \tag{3.24}$$

We now return to equation (3.21) and make the appropriate substitution for B_n. After a series of rather tedious computations similar to those in Section 2, one can conclude that,

$$f_2(\mu_o) = \sum_{n=o}^{\infty} \frac{\sinh \mu_o}{\sinh (r+2)\mu_o} + \ln(1-e^{-2\mu_o}) -$$

$$- \sum_{n=o}^{\infty} \frac{e^{-2(n+1)\mu_o} \sinh^2(\mu_o/2)}{(n+1)\sinh(n+\frac{1}{2})\mu_o \sinh(n+\frac{3}{2})\mu_o} , \qquad (3.25)$$

$\mu_o > 0$.

As might be expected, we find that

$$\lim_{\mu_o \to \infty} f_2(\mu_o) = 0 . \qquad (3.26)$$

We recognize that $f_2(\mu_o)$ is a measure of the effect of the wall upon the kinetic energy of the field, and equation (3.26) expresses the fact that when the sphere is far from the wall the kinetic energy reduces to that produced by a free sphere. Values of $f_2(\mu_o)$ are presented in accompanying tables.

One must now consider with a little care, the problem of evaluating f_2 at $\mu_o = 0$. The series of equation (3.25) converges only if $\mu_o > 0$. However, on physical grounds, we know that some value for $f_2(0)$ must exist, and certainly

$$f_2(0) = \lim_{\mu_o \to 0^+} f_2(\mu_o) . \qquad (3.27)$$

We shall use (3.25) to find

$$\lim_{\mu_o \to 0^+} f_2(\mu_o) .$$

The difficulty with the series in (3.25) occurs in the first two terms. Our procedure will be to approximate the first two terms by a closed form which is continuous at $\mu_o = 0$. If we have an adequate estimate of the error introduced by this approximation, it will be possible to evaluate the limit we seek.

Let us rewrite the first summation as,

$$\sum_{r=o}^{\infty} \frac{\sinh \mu_o}{\sinh (r+2)\mu_o} = s_n(\mu_o) + r_n(\mu_o) , \tag{3.28}$$

where

$$s_n = \sum_{o}^{n} \frac{\sinh \mu_o}{\sinh (r+2)\mu_o} , \qquad r_n = \sum_{n+1}^{\infty} \frac{\sinh \mu_o}{\sinh(r+2)\mu_o} . \tag{3.29}$$

The remainder term r_n may be bounded by two integrals,

$$I_n \geq r_n \geq I_{n+1} \tag{3.30}$$

where

$$I_n = \int_n^{\infty} \frac{\sinh \mu_o \, dx}{\sinh(x+2)\mu_o} , \qquad I_{n+1} = \int_n^{\infty} \frac{\sinh \mu_o \, dx}{\sinh(x+3)\mu_o} . \tag{3.31}$$

The error due to replacing r_n by one of its bounding integrals is given by

$$\left| r_n - I_n \right| \leq \sinh \mu_o \int_n^{n+1} \frac{dx}{\sinh(x+2)\mu_o} \leq \frac{\sinh \mu_o}{\sinh(n+2)\mu_o} \leq \frac{1}{n+2} . \tag{3.32}$$

The integral in question may be evaluated exactly.

$$I_n = \frac{\sinh \mu_o}{\mu_o} \int_{(n+2)\mu_o}^{\infty} \frac{d\lambda}{\sinh \lambda} = \frac{\sinh \mu_o}{\mu_o} \ln \frac{e^{(n+2)\mu_o} + 1}{e^{(n+2)\mu_o} - 1} . \tag{3.33}$$

Thus we may say

$$f_2(\mu_o) \approx \sum_{r=o}^{n} \frac{\sinh \mu_o}{\sinh(r+2)\mu_o} + \frac{\sinh \mu_o}{\mu_o} \ln \frac{e^{(n+2)\mu_o}+1}{e^{(n+2)\mu_o}-1} + \ln(1-e^{-2\mu_o}) -$$

$$- \sum_{k=o}^{\infty} e^{-2(n+1)\mu_o} \frac{\sinh^2 \mu_o/2}{(k+1)\sinh(k+\frac{1}{2})\mu_o \sinh(k+\frac{3}{2})\mu_o} \tag{3.34}$$

The error in the estimate of (3.34) is independent of μ_o, and depends only on n. This error is bounded in equation (3.32). We now take the limit of equation (3.34) as μ_o goes to zero. We find

$$\lim_{\mu_o \to 0^+} f_2(\mu_o) \approx \sum_{r=0}^{n} \frac{1}{(r+2)} + \ln 2 + \lim_{\mu_o \to 0} \ln \frac{1-e^{-2\mu_o}}{e^{(n+2)\mu_o}-1} -$$

$$- \sum_{k=o}^{\infty} \frac{1}{(k+1)(2k+1)(2k+3)} = \ln 4 - 1 + \left[\sum_{r=o}^{n+1} \frac{1}{r+1} - \ln(n+2) \right] -$$

$$- \sum_{k=o}^{\infty} \frac{1}{(k+1)(2k+1)(2k+3)} \tag{3.35}$$

The error of the estimate in (3.35) is still given by (3.32). We can now make the estimation error vanish by letting n go to infinity. The limit of the function involving n is an expression of Euler's constant. Thus

$$f_2(0) = \ln 4 - 1 + \ln \gamma - \sum_{k=o}^{\infty} \frac{1}{(k+1)(2k+1)(2k+3)} . \tag{3.36}$$

It also happens that

$$\ln 4 - 1 = \sum_{k=o}^{\infty} \frac{1}{(k+1)(2k+1)(2k+3)} . \tag{3.37}$$

Thus we are finally led to the simple result

$$f_2(0) = \ln \gamma = .57722 \tag{3.38}$$

IV. THE SPHERE WITH A GENERAL MOTION

The Potential of the Sphere with a General Motion

As we have mentioned in the first section, the potential produced by the sphere which is undergoing both a translational and radial motion, is the sum of the potentials of the two previous sections. We may immediately write down the potential ψ_3 of this general motion.

$$\psi_3 = \psi_1 + \psi_2 = r\sqrt{\cosh\mu - \cos\eta}\sum_{n=o}^{\infty}\left[\dot{b}A_n'(\mu_o) + \sqrt{2}\,\dot{r}\,B_n'(\mu_o)\right]\cosh(n+\tfrac{1}{2})\mu P_n(\cos\eta). \tag{4.1}$$

$A_n'(\mu_o)$ and $B_n'(\mu_o)$ have been determined in Sections 2 and 3 respectively.

The Kinetic Energy of the Field Produced by the Sphere with a General Motion

We shall find that the kinetic energy of the field, in the case of the general motion, contains an interaction term in addition to those terms which arise separately from the translational and radial motion. The expression for the kinetic energy is

$$T = \frac{1}{2}\rho\int_s(\psi_1+\psi_2)\frac{\partial}{\partial n}(\psi_1+\psi_2)\,ds = T_1 + T_2 + \frac{\rho}{2}\int_s\left(\psi_1\frac{\partial\psi_2}{\partial n} + \psi_2\frac{\partial\psi_1}{\partial n}\right)ds. \tag{4.2}$$

T_1 and T_2 have been determined in Sections 2 and 3 respectively.

We define the interaction kinetic energy, T_3, to be

$$T_3 = \frac{\rho}{2}\int_s\left(\psi_1\frac{\partial\psi_2}{\partial n} + \psi_2\frac{\partial\psi_1}{\partial n}\right)ds\,. \tag{4.3}$$

By Green's theorem it follows immediately that

$$T_3 = \rho\int_s\psi_1\frac{\partial\psi_2}{\partial n}\,ds\,. \tag{4.4}$$

Because of the boundary conditions on the plane the integral vanishes over the plane. Thus T_3 extends only over the surface of the sphere, defined by $\mu = \mu_o$. The normal derivative, because of the boundary conditions on the sphere, is equal to $\dot{r}$. After a proper account is taken of the noncartesian metric, equation (4.4) becomes

$$T_3 = 2\pi\rho\dot{r}a^2 \int_{-1}^{1} \sum_{n=o}^{\infty} A_n \cosh(n+\tfrac{1}{2})\mu_o P_n(w)/(\cosh\mu_o - w)^{3/2} \quad dw \tag{4.5}$$

where w is set equal to $\cos\eta$.

The irrational factor is expanded in Legendre Polynomials by the same technique used in the previous sections. The integration can then be performed over the double sum of Polynomials, and one obtains

$$T_3 = \frac{4\sqrt{2}\,\pi\rho\dot{r}a^2}{\sinh\mu_o} \sum_{n=o}^{\infty} A_n e^{-(n+\frac{1}{2})\mu_o} \cosh(n+\tfrac{1}{2})\mu_o \,. \tag{4.6}$$

A_n is replaced by the use of equation (2.30). After some lengthy algebraic manipulation, one arrives at a final form

$$T_3 = -4\pi\rho\dot{r}\dot{b}r^3 \left[e^{-\mu_o} - \frac{1}{2} \sum_{n=o}^{\infty} e^{-(n+1)\mu_o} \left(\frac{\sinh\mu_o}{\sinh(r+1)\mu_o} \right)^2 - \frac{e^{\mu_o}-1}{2} \sum_{n=o}^{\infty} e^{-2(n+1)\mu_o} \frac{\sinh\mu_o}{\sinh(r+1)\mu_o} \right] . \tag{4.7}$$

For convenience, we define the dimensionless kinetic energy of interaction by

$$T_3 = -4\pi\rho\dot{r}\dot{b}r^3 f_3(\mu_o) \,;$$

$$f_3 = e^{-\mu_o} - \frac{1}{2}\sum_{n=o}^{\infty} e^{-(n+1)\mu_o}\left(\frac{\sinh\mu_o}{\sinh(n+1)\mu_o}\right)^2 - \frac{e^{\mu_o}-1}{2} \bullet$$

$$\sum_{n=o}^{\infty} e^{-2(n+1)\mu_o} \frac{\sinh\mu_o}{\sinh(n+1)\mu_o} \quad . \tag{4.8}$$

Physically, one expects f_3 to vanish as $\mu_o \to \infty$, since f_3 is produced entirely by the interaction between the sphere with the plane. This presumption is indeed correct.

$$\lim_{\mu_o \to \infty} f_3(\mu_o) \sim \frac{e^{-2\mu_o}}{2} \to 0 \, . \tag{4.9}$$

When $\mu_o = 0$, f_3 exists, and may be evaluated.

$$f_3(0) = 1 - \frac{1}{2}\sum_{n=o}^{\infty}\frac{1}{(n+1)^2} = 1 - \frac{\pi^2}{12} = .177533 \tag{4.10}$$

Values of $f_3(\mu_o)$ have been computed for various values of μ_o and are presented in Table III.

Finally, for completeness, we write out the kinetic energy expression in terms of all the dimensionless kinetic energies.

$$T = 2\pi\rho r^3 \left\{ \left[1 + 3f_1(\mu_o)\right] \frac{\dot{b}^2}{6} - 2\dot{r}\dot{b} f_3(\mu_o) + (1 + f_2(\mu_o))\,\dot{r}^2 \right\} . \tag{4.11}$$

V. GENERAL REMARKS ON THE DYNAMICS OF THE SYSTEM

The Equations of Motion

If the gravitational effects are neglected, the potential energy, U, of the system will be independent of b. The exact form of U will depend upon the physical nature of the cavity and upon the conditions at infinity. We shall not here specialize U(r). The Lagrangian of the system can, then, be written as

$$L = T - U = 2\pi\rho r^3 \left[(1+3f_1(\mu_o)) \frac{\dot{b}^2}{6} - 2\dot{b}\dot{r} f_3(\mu_o) + (1+f_2(\mu_o))\dot{r}^2 \right] - U(r) \tag{5.1}$$

Because of equation (2.5), it follows that

$$\left(\frac{\partial}{\partial b}\right)_r = \frac{1}{r \sinh\mu_o} \frac{d}{d\mu_o} \; ; \quad \left(\frac{\partial}{\partial r}\right)_b = -\frac{\coth\mu_o}{r} \frac{d}{d\mu_o} . \tag{5.2}$$

The equations of motion follow immediately.

$$\frac{d}{dt} 2r^3 \left[\dot{r}(1+f_2(\mu_o)) - \dot{b} f_3(\mu_o) \right] = 3r^2 \left[(1+3f_1)\frac{\dot{b}^2}{6} - 2\dot{b}\dot{r} f_3 + (1+f_2)\dot{r}^2 \right]$$
$$- r^2 \coth\mu_o \left[\frac{\dot{b}^2}{2} \frac{df_1}{d\mu_o} - 2\dot{b}\dot{r} \frac{df_3}{d\mu_o} + \dot{r}^2 \frac{df_2}{d\mu_o} \right] - \frac{1}{2\pi\rho} \frac{dU}{dr} , \tag{5.2a}$$

and

$$\frac{d}{dt} 2r^3 \left[\dot{b}(1+3f_1(\mu_o))/6 - \dot{r} f_3(\mu_o) \right] = \frac{r^2}{\sinh\mu_o} \left[\frac{\dot{b}^2}{2} \frac{df_1}{d\mu_o} - 2\dot{b}\dot{r} \frac{df_3}{d\mu_o} + \dot{r}^2 \frac{df_2}{d\mu_o} \right] . \tag{5.2b}$$

Equations (5.2) and (2.5) give, in principle, a description of the dynamics of the spherical cavity. The function f_1, f_2, and f_3 have been exhibited and tabulated.

Because of the complicated form of the dimensionless kinetic energies, it does not seem possible to solve the equations of motion in their full generality. One must now turn either to a numerical integration or to some approximation method. Approximate solutions for two cases of physical interest are presented in subsequent sections.

The Pressure Distribution Across the Plane

The pressure P anywhere in the velocity field is given by Bernoulli's equation

$$P - P_{\infty} = \rho \frac{\partial \psi_3}{\partial t} - \frac{1}{2} \rho \left(\overline{V} \cdot \overline{V} \right) . \tag{5.3}$$

ρ is the density of the fluid under consideration; P_{∞} is the pressure at infinity. The time derivative of the velocity potential follows from equation (4.1).

$$\frac{\partial \psi_3}{\partial t} = \sqrt{\cosh\mu - \cos\eta} \sum_{n=0}^{\infty} \Big[r \dot{b} \frac{d}{dt} A_n'(\mu_o) + \sqrt{2}\, \dot{r} r \frac{d}{dt} B_n'(\mu_o) + \\ + (r \ddot{b} + \dot{r} \dot{b}) A_n'(\mu_o) + \sqrt{2}\, (\dot{r}^2 + r \ddot{r}) B_n'(\mu_o) \Big] \cosh(n+\tfrac{1}{2})\mu\, P_n(\cos\eta) . \tag{5.4}$$

If we make use of (5.2), we recognize that

$$\frac{d}{dt} A_n' = \frac{1}{r \sinh\mu_o} \frac{d A_n'}{d\mu_o} \left(\dot{b} - \dot{r} \cosh\mu_o \right)$$

$$\frac{d}{dt} B_n' = \frac{1}{r \sinh\mu_o} \frac{d B_n'}{d\mu_o} \left(\dot{b} - \dot{r} \cosh\mu_o \right) . \tag{5.5}$$

To find the derivatives of the A_n' s and B_n' s we will need the result that

$$\frac{d\gamma_n}{d\mu_o} = \frac{d}{d\mu_o} \; \frac{2}{e^{(2n+1)\mu_o} - 1} = \frac{-(2n+1)}{2\sinh^2(n+\frac{1}{2})\mu_o} \tag{5.6}$$

It then follows from equation (2. 31) that

$$\frac{dA'_n}{d\mu_o} = \coth\mu_o A'_n(\mu_o) - \frac{\sinh\mu_o}{\sqrt{2}}\left[\frac{n(2n+1)}{\sinh^2(n+\frac{1}{2})\mu_o} - \sum_{k=n+1}^{\infty} \frac{(2k+1)}{\sinh^2(k+\frac{1}{2})\mu_o}\right] \tag{5.7}$$

Similarly, it follows from equation (3. 18) that

$$\frac{dB'_n}{d\mu_o} = \frac{d\gamma_n}{d\mu_o} + \frac{\sinh\mu_o/2}{4\cosh^2\frac{\mu_o}{2}} \sum_{k=n+1}^{\infty} \frac{e^{-\frac{\mu_o}{2}}\gamma_{k-1} + e^{\frac{\mu_o}{2}}\gamma_k}{k} + \frac{1}{4\cosh\frac{\mu_o}{2}} \bullet$$

$$\sum_{k=n+1}^{\infty} \frac{e^{-\frac{\mu_o}{2}}\gamma_{k-1} - e^{\frac{\mu_o}{2}}\gamma_k}{k} - \frac{1}{2\cosh\frac{\mu_o}{2}} \bullet$$

$$\sum_{k=n+1}^{\infty} \left(e^{-\frac{\mu_o}{2}} \frac{d\gamma_{k-1}}{d\mu_o} + e^{\frac{\mu_o}{2}} \frac{d\gamma_k}{d\mu_o}\right)/k \; . \tag{5.8}$$

It takes some manipulation to reduce (5.8) to its final form

$$\frac{dB'_n}{d\mu_o} = -\frac{n+\tanh\mu_o/2}{\sinh^2(n+\frac{1}{2})\mu_o} + \sum_{k=n+1}^{\infty} \operatorname{csch}^2(k+\tfrac{1}{2})\mu_o - \frac{\sinh\mu_o}{2} \bullet$$

$$\sum_{k=n+1}^{\infty} \frac{e^{-k\mu_o}\sinh k\mu_o}{k\sinh^2(k+\frac{1}{2})\mu_o \sinh^2(k-\frac{1}{2})\mu_o} \tag{5.9}$$

Though (5.3) is perfectly general, we are primarily interested in the pressure on the rigid plane. We thus evaluate (5.4) when $\mu = 0$.

$$\left(\frac{\partial\psi_3}{\partial t}\right)_{\mu=0} = \sqrt{1-\cos\eta}\sum_{n=o}^{\infty}\Big[(\dot{b}-\dot{r}\cosh\mu_o)\,\mathrm{csch}\,\mu_o\left(\dot{b}\frac{dA_n'}{d\mu_o}+\sqrt{2}\,\dot{r}\frac{dB_n'}{d\mu_o}\right)+$$

$$+(r\ddot{b}+\dot{r}\dot{b})A_n'(\mu_o)+\sqrt{2}(\dot{r}^2+r\ddot{r})B_n'(\mu_o)\Big]P_n(\cos\eta) \qquad (5.10)$$

The velocity of the fluid over the rigid plane is also required. The boundary conditions tell us that the velocity vector must lie parallel to the plane. Furthermore, the symmetry of the problem makes it clear that the velocity will be directed radially from the point on the plane which lies directly beneath the center of the sphere. We will define two nondimensional velocities V_1 and V_2: $\dot{r}V_2$ is the radially outward directed velocity over the plane which arises from the expansion (or contraction) of the sphere; and $-\dot{b}V_1$ is the radially outward directed velocity over the plane which arises from the translational motion of the sphere. Since both velocities lie along parallel vectors, the total velocity over the plane is just the algebraic sum of these two, $\dot{r}V_2 - \dot{b}V_1$. Taking the gradient of the appropriate potential function we have

$$V_1 = -\frac{(1-\cos\eta)}{\dot{b}\,r\sinh\mu_o}\left(\frac{\partial\psi_1}{\partial\eta}\right)_{\mu=0} \quad ; \quad V_2 = \frac{(1-\cos\eta)}{\dot{r}\,r\sinh\mu_o}\left(\frac{\partial\psi_2}{\partial\eta}\right)_{\mu=0} \qquad (5.11)$$

We use equation (2.5) to obtain

$$V_1 = \frac{(1-\cos\eta)^{\frac{1}{2}}\sin\eta}{\sinh\mu_o}\left[(1-\cos\eta)\sum_{n=1}^{\infty}A_n'\frac{dP_n(\cos\eta)}{d\cos\eta}-\frac{1}{2}\sum_{n=o}^{\infty}A_n'P_n\right]$$

(5.12)

Equivalently,

$$V_1 = \frac{(1-\cos\eta)^{\frac{1}{2}}\sin\eta}{2\sinh\mu_o}\left[-\sum_{n=1}^{\infty} A_n'\left(P_n - 2\frac{dP_n}{d\cos\eta} + 2\cos\frac{dP_n}{d\cos}\right) - A_o'P_o\right] \tag{5.13}$$

If we make use of the relationships between the Polynomials, we have

$$V_1 = -\frac{(1-\cos\eta)^{\frac{1}{2}}\sin\eta}{2\sinh\mu_o}\left[A_o'P_o + \sum_{n=1}^{\infty} A_n'\left(\frac{dP_{n+1}}{d\cos\eta} + \frac{dP_{n-1}}{d\cos\eta} - 2\frac{dP_n}{d\cos\eta}\right)\right] \tag{5.14}$$

Since

$$P_n^1(\cos\eta) = \sin\eta\,\frac{dP_n}{d\cos\eta}\,, \tag{5.15}$$

we finally obtain, after some reshuffling,

$$V_1 = \frac{(1-\cos\eta)^{\frac{1}{2}}}{2\sinh\mu_o}\sum_{n=1}^{\infty}\left[(A_n'-A_{n-1}') + (A_n'-A_{n+1}')\right]P_n^1(\cos\eta)\,. \tag{5.16}$$

Similarly, the analagous expression for V_2 is

$$V_2 = \frac{(1-\cos\eta)^{\frac{1}{2}}}{\sqrt{2}\sinh\mu_o}\sum_{n=1}^{\infty}\left[(B_{n+1}'-B_n') + (B_{n-1}'-B_n')\right]P_n^1(\cos\eta)\,. \tag{5.17}$$

More explicit forms may be obtained for V_1 and V_2 by substituting the appropriate values for the $A_n's$ and for the $B_n's$. We will once again have to call on the recursion formulas for the Polynomials. One eventually arrives at final forms for V_1 and V_2.

$$V_1 = \frac{(1-\cos\eta)^{\frac{3}{2}}}{\sqrt{2}}\sum_{n=1}^{\infty}(2n+1)\,\Upsilon_n(\mu_o)\,P_n^1(\cos\eta) \tag{5.18}$$

and

$$V_2 = \frac{(1-\cos\eta)^{\frac{1}{2}}}{\sqrt{2}\sinh\mu_o}\left[\frac{\sin\eta}{\sinh\mu_o} - (1-\cos\eta)\bullet\right.$$

$$\left.\sum_{n=1}^{\infty} \frac{2n+1}{n+1}\gamma_n\left(1 + \frac{e^{\mu_o/2}}{2n\cosh\mu_o/2}\right) P_n^1(\cos\eta)\right] \quad . \qquad (5.19)$$

Values of V_1 and V_2 have been calculated and are presented in the accompanying tables.

Since,

$$\overline{V}\cdot\overline{V} = (\dot{r}\,V_2 - \dot{b}\,V_1)^2 \qquad (5.20)$$

we can now use equations (5.3), (5.10), (5.18), and (5.19) to describe the pressure distribution across the rigid plane.

Some particular pressure effects will be calculated in more detail in the subsequent sections.

VI. THE DYNAMICS OF AN AIR BUBBLE SUBJECTED TO A SINUSOIDALLY VARYING PRESSURE

The Solution of the Linearized Equations of Motion of the Oscillating Air Bubble

In attempting to arrive at an approximate solution to the general equations of motion, one first considers the technique of small oscillations. The motion of an air bubble is particularly amenable to this technique since, for any given external pressure, the air bubble has some equilibrium radius. In the following analysis we of course assume that pressure differences are transmitted instantaneously. This is equivalent to saying that the frequency of oscillation must be "low" (i.e. the product of frequency times the bubble radius must be much less than the speed of sound in the fluid). Since the speed of sound of water is about 1.4×10^5 cm/sec, the limitation of compressibility does not become important until frequencies of the order of tens of kilocycles are reached.

We consider that the pressure at infinity, P_∞, makes small sinusoidal oscillations of frequency ω, about some fixed pressure P_o. We let r_o and b_o be the equilibrium values of r and b which correspond to P_o. This leads us to define the quantities ϵ, ξ, and δ by the following relationships

$$P_\infty = P_o(1 + \epsilon e^{i\omega t}) \; ; \; r = r_o(1 + \xi) \; ; \; b = b_o(1 + \delta) \; . \tag{6.1}$$

A suitable dynamic potential for the gas filled bubble is

$$U(r) = \frac{4}{3}\pi r^3 P_\infty + \left(P_o + \frac{2\sigma}{r_o}\right) \frac{4\pi}{3} \frac{r_o^{3\gamma} r^{3(1-\gamma)}}{(\gamma-1)} + 4\pi\sigma r^2 \; . \tag{6.2}$$

The first term is the potential of the pressure at infinity. γ is, as usual, used to denote the ratio of the specific heat at constant pressure to the specific heat at constant volume, for the gas within the bubble. σ is the surface tension of the gas-liquid interface. The second term in (6.2) is a measure of the work done in adiabatically compressing the gas within the bubble. The third term is the surface energy. It can be seen by differentiating (6.2), that when $P_\infty = P_o$, the bubble is in stable equilibrium when $r = r_o$.

We now proceed to write a Lagrangian which is accurate to second order in ϵ, ξ, δ, and their time derivatives. Because of the form of the Lagrangian, it will be sufficient to take

$$\mu_o = \cosh^{-1}(b_o/r_o) = \text{a constant} \tag{6.3}$$

If we make use of (6.1), (6.2) and (4.11), and if we neglect constants terms and terms higher than second order, the Lagrangian becomes

$$L = 2\pi\rho r_o^3\left\{\frac{b_o^2}{6}\dot{\delta}^2\left[1 + 3f_1(\mu_o)\right] - 2\, r_o b_o f_3(\mu_o)\,\dot{\delta}\dot{\xi} + r_o^2\left[1 + f_2(\mu_o)\right]\dot{\xi}^2\right\} - 4\pi r_o^3 P_o \epsilon\xi e^{i\omega t} - \frac{\xi^2}{2}(4\pi r_o^3)\left[3\gamma P_o + \frac{2\sigma}{r_o}(3\gamma - 1)\right]$$

(6.4)

The linearized equations of motion follow immediately from (6.4)

$$\frac{d}{dt}\left\{\frac{1 + 3f_1(\mu_o)}{3}\,\dot{\delta b_o} - 2r_o\dot{\xi}\, f_3(\mu_o)\right\} = 0 \tag{6.5a}$$

$$\frac{d}{dt}\rho\left\{\left[1 + f_2(\mu_o)\right]\dot{\xi} r_o^2 - r_o b_o f_3(\mu_o)\dot{\delta}\right\} = -\epsilon e^{i\omega t} - \xi\left[3\gamma P_o + \frac{2\sigma}{r_o}(3\gamma - 1)\right] \tag{6.5b}$$

We shall now define the following quantities

$$\varkappa^2 = \left[3\gamma P_o + \frac{2\sigma}{r_o}(3\gamma - 1)\right] / \quad r_o^2 \rho \tag{6.6a}$$

$$\lambda^2 = k^2 \varkappa^2 = \left[\frac{1 + 3f_1}{(1 + 3f_1)(1 + f_2) - 6f_3^2}\right] \varkappa^2 \tag{6.6b}$$

$$c = \frac{3\gamma P_o}{3\gamma P_o + \frac{2\sigma}{r_o}(3\gamma - 1)} \tag{6.6c}$$

In terms of the definitions of (6.6), the solutions of equation (6.5) become

$$\xi = \frac{c}{3\gamma} \frac{\lambda^2}{\omega^2 - \lambda^2} \quad \epsilon e^{i\omega t} \tag{6.7a}$$

$$\delta = 6 \frac{r_o}{b_o} \frac{f_3}{1 + 3f_1} \quad \xi \tag{6.7b}$$

c is a dimensionless constant which is nearly equal to one for most cases of interest. λ is the natural frequency of vibration of the gas bubble near the rigid plane. $\varkappa$ is the natural frequency of vibration of a free gas bubble. We see that k has been defined as a measure of the effect of the wall upon the natural frequency of vibration. When $\mu_o = \infty$, and the bubble is remote from the wall, $k = 1$. The value of k at $\mu_o = 0$ is 0.830. The effect of the wall is to decrease the natural frequency of the bubble. This is then consistent with the physical notion that the wall tends to increase the inertia of the fluid field.

It is probably relevant to note that if we apply equation (6.6a) to a one centimeter free water bubble under one atmosphere pressure we get a natural frequency of about 1/3 k.c. This is certainly well within the limits set by the compressibility condition.

Equation (6.7a) shows the standard resonance behavior. For low frequencies the bubble radius is 180° out of phase with the external pressure. This result is in accord with our physical intuition. We should remark that the singularity at $\lambda = \omega$ would disappear were we to consider the energy losses that are always present because of viscosity and acoustic radiation.

Equation (6.7b) shows that the vertical motion of the bubble is in phase with the radial motion. When the bubble expands it also recedes from the plane; when it contracts it approaches the plane. The factor in front of ξ in equation (6.7b) goes to zero rapidly as r_o/b_o approaches zero. This is consistent with the notion that the translational and radial motions of the free bubble are not coupled.

Pressure Effects on the Plane Due to the Oscillating Air Bubble

Now that the equations of motion are solved we can easily discuss the pressure effects across the plane. Since we have made a linearization approximation, equation (5.3) specializes to

$$p = P - P_{\infty} = \rho \left(\frac{\partial \psi_3}{\partial t}\right)_{\mu_o = 0} \tag{6.8}$$

Of course, p is the amount by which the pressure at any point on the plane exceeds the pressure at infinity.

We now use equation (4.1) and (6.7), and once again neglect second order terms. It then follows that (6.8) becomes

$$p = -\xi \rho r_o^2 \omega^2 \sqrt{1 - \cos\eta} \sum_{n=0}^{\infty} \left[\frac{6f_3(\mu_o)}{1 + 3f_1(\mu_o)} A_n'(\mu_o) + \sqrt{2}\, B_n'(\mu_o) \right] P_n(\cos\eta)$$

(6.9)

Of course, the magnitude and time dependence of ξ is given by equation (6.7).

We define $F_\omega(\eta;\mu_o)$ by

$$-\frac{p}{\xi \rho r_o^2 \omega^2} = F_\omega(\eta;\mu_o) = \sqrt{1 - \cos\eta} \sum_{n=0}^{\infty} \left[\frac{6f_3(\mu_o)}{1+3f_1(\mu_o)} A_n'(\mu_o) + \sqrt{2}\, B_n'(\mu_o) \right] P_n(\cos\eta)$$

(6.10)

In particular, we may find F_ω at $\eta = \pi$. This is the point on the plane which lies directly below the center of the sphere.

$$F_\omega(\pi, \mu_o) = \sqrt{2} \sum_{n=0}^{\infty} (-1)^n \left[\frac{6f_3(\mu_o)}{1+3f_1(\mu_o)} A_n'(\mu_o) + \sqrt{2}\, B_n'(\mu_o) \right] \qquad (6.11)$$

$F_\omega(\pi, \mu_o)$ is computed for various values of μ_o and is presented in the accompanying tables.

The distribution over the plane, $F_\omega(\eta;\mu_o)$, is also computed for fixed μ_o, and the different values of η. The results are presented in the accompanying table.

We have used η as a natural measure of position on the rigid plane. It will help to remember that if d is the radial distance from $(\eta = \pi, \mu = 0)$ to any point on the rigid plane, then

$$d = r_o \sinh \mu_o \cot \eta/2 \qquad (6.12)$$

VII. THE DYNAMICS OF THE COLLAPSING CAVITATION BUBBLE

Approximate Solution of the Equations of Motion of the Cavitation Bubble

Cavitation damage is produced by the very high velocity fields that arise during the collapse of vapor bubbles. The bubbles are formed in some region of low pressure and then collapse in some region of high pressure. The details of such a collapse of course depend upon the time behavior of the pressure at infinity. A good understanding of the phenomenon can be obtained by assuming that the pressure at infinity is constant, and that the pressure within the cavity is just the equilibrium vapor pressure corresponding to the temperature of observation. If we let p_a be the excess of the atmospheric pressure over the vapor pressure, the potential energy of the system becomes

$$U = \frac{4}{3}\pi r^3 p_a + 4\pi\sigma r^2 \tag{7.1}$$

When we search for an approximation to the general equations of section 5, we find a helpful clue in the results of section 6. Equation (6.7b) is independent of the forcing function, and can be expected to hold for a sufficiently small period of time if the space derivatives of the kinetic energy functions remain finite. Now the coefficient of ξ in (6.7b) is always less than one, and rapidly goes to zero as r/b decreases. Initially, then, $\dot{b}$ is of less magnitude than $\dot{r}$. Thus for a collapsing bubble r will decrease much faster than b. This compounds the initial effect of (6.7b), and, insofar as equation (6.7b) is correct, $\dot{b}$ becomes negligible with respect to $\dot{r}$. This conjecture is verified by experimental observations[1]: a bubble will collapse to a zero radius; yet during the entire period of observation its position with respect to a neighboring wall remains nearly constant. We therefore make the approximation that b is constant over the period of collapse, and solve the resulting equation. With this approximation the total energy, E, of the system reduces to

$$E = T + U = \frac{4}{3}\pi r^3(p_a + \frac{3\sigma}{r}) + 2\pi\rho r^3\left[1 + f_2(\mu_o)\right]\dot{r}^2 \tag{7.2}$$

It is convenient to define the dimensionless quantities R, s, and τ

$$r = Rr_m \quad ; \quad s = \frac{3\sigma}{3\sigma + r_m p_a} \quad ; \quad t^2 = \left[3\rho r_m^3/2(r_m p_a + 3\sigma)\right]\tau^2 \tag{7.3}$$

If r_m is the radius of the cavity just as the collapse begins, then E, which is a constant, is given by

$$E = \frac{4}{3}\pi r_m^3 (p_a + \frac{3\sigma}{r_m}) \tag{7.4}$$

When (7.2) is divided by (7.4) the following equation results

$$1 = R^3(1 - s) + sR^2 + (1 + f_2)\dot{R}^2 R^3 \tag{7.5}$$

$\dot{R}$ is the derivative of the dimensionless radius, R, with respect to the dimensionless time, τ.

Equation (7.5) may be solved for $\dot{R}$ to give a complete description of the dynamics in terms of the dimensionless radius of the cavity

$$\dot{R} = -\sqrt{\frac{1 - (1-s)R^3 - sR^2}{R^3(1 + f_2)}} \quad , \ R \leq 1 \tag{7.6}$$

We can, if we wish, integrate (7.6) to give time dependence of the radius

$$\tau = \int_R^1 \sqrt{\frac{x^3\left[1 + f_2(\mu_o)\right]}{1 - (1-s)x^3 - sx^2}}\, dx \tag{7.7}$$

In particular the total dimensionless time of collapse is

$$\tau_{total} = \int_0^1 \sqrt{\frac{x^3(1 + f_2)}{1 - (1-s)x^3 - sx^2}}\, dx \tag{7.8}$$

The integration of (7.7) must be done numerically. This computation was not performed. We mention for completeness that

$$f_2(\mu_o) = f_2\left(\cosh^{-1}\left[\frac{1}{R} \cosh \mu_m\right]\right) \tag{7.9}$$

where $\cosh \mu_m = b/r_m$.

When the bubble is far from the wall, $f_2 = 0$ and all the equations reduce to those familiar in the cavitation theory of the free bubble. It can be seen that the presence of the wall serves, through f_2, to increase the inertia of the fluid and to increase the characteristic times of the problem.

This result fits qualitatively with the experimental facts. When the free-bubble theory is used to compare with data taken near an ogival surface, the times of collapse are underestimated.[1]

Pressure Effects of the Collapsing Cavitation Bubble

Since equation (7.6) determines the velocity, and hence the acceleration, of the bubble wall in terms of the radius as a parameter, one can, in principle, use (7.6) and (5.3) to get a complete description of the pressure across the rigid plane, in terms of the dimensionless radius of the bubble. Then, if desired, this can be translated into a time dependence with the use of equation (7.7). All the functions necessary for this computation have been exhibited in section 5.

Also of interest in cavitation damage is the pressure impulse, I, which is defined by

$$I(\eta) = \int_0^t (P - P_\infty)\, dt \tag{7.10}$$

Since the point on the plane directly under the center of the bubble, ($\mu = 0$, $\eta = \pi$) is a stagnation point, (5.3) may be immediately integrated there.

$$I(\eta = \pi) = \int_0^t \rho \frac{\partial \psi_2}{\partial t}\, dt = \rho \psi_2(\mu_o; \mu = 0, \eta = \pi) = 2r\dot{r}\rho \sum_0^\infty (-1)^n B_n'(\mu_o) \tag{7.11}$$

We have used the fact that since $\dot{r} = 0$ when $t = 0$, ψ_2 also vanished at the lower limit of integration.

If we make use of equation (7.3) it follows that

$$I(\eta = \pi) = 2\sqrt{\frac{r_m(r_m p_a + 3\sigma)\rho}{3}}\; R\dot{R}\sqrt{2} \sum_{n=0}^{\infty} (-1)^n B_n'(\mu_o) \tag{7.12}$$

For convenience, we also define the dimensionless pressure impulse, J.

$$J = \frac{I}{2\sqrt{\frac{\rho r_m(r_m p_a + 3\sigma)}{3}}} \tag{7.13}$$

Now, if we use equation (7.6), it follows that

$$J = -\sqrt{\frac{2\left[1-(1-s)R^3 - sR^2\right]}{R\left[1+f_2(\mu_o)\right]}}\ \sum_{n=0}^{\infty} (-1)^n B_n'(\mu_o) \tag{7.14}$$

Clearly J vanishes when R = 1. Furthermore, by considering the asymptotic values of B_n, it follows that when R approaches 0,

$$J \sim -\sqrt{\frac{2R\left[1-(1-s)R^3 - sR^2\right]}{\cosh^2\mu_m - R^2}}\ , \quad \cosh\mu_m = \frac{b}{r_m} \tag{7.15}$$

Thus J also vanishes when R is 0. We conclude that J must have an extremum between 0 and 1. Indeed if we use the asymptotic value for J and take s = 0, $\mu_m \gg 1$, we see that the extremum occurs when $R = (0.25)^{1/3}$. If we recall that J is an integral of the pressure, the existence of an extremum in J indicates that the pressure on the point of the plane we are considering actually changes sign during the collapse of the bubble. J is computed as a function of R for various initial values, μ_m, with s = 0. The results are presented in figure 3. Of course during the collapse μ_o varies, since b is fixed but r changes. μ_o is calculated in the same manner as it is calculated in equation (7.9).

As yet, no quantitative experiments on cavitation damage related to this analysis seem to be available. Some preliminary qualitative work was done by Ellis[11] at the Hydrodynamics Laboratory. It is believed that this analysis will be of use in helping to plan future experiments in the field of cavitation damage.

FIGURE III

THE DIMENSIONLESS IMPULSIVE PRESSURE PRODUCED BENEATH A COLLAPSING CAVITATION BUBBLE

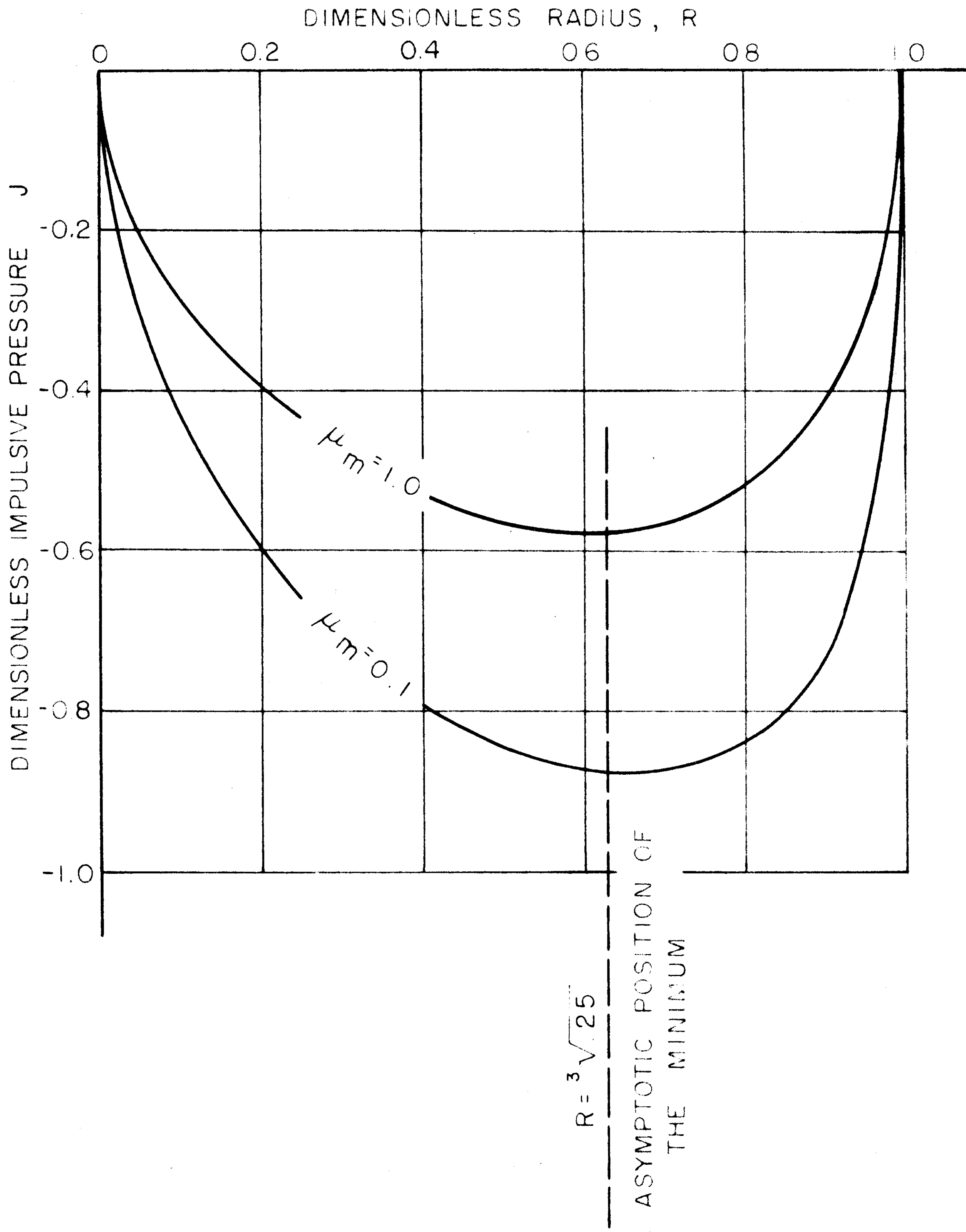


Table I
Values of the Coefficients $A'_n(\mu_o)$

Coefficient	$\mu_o = .01$	$\mu_o = .02$	$\mu_o = .05$	$\mu_o = .1$	$\mu_o = .2$
A'_0	-5.52266	-4.55105	-3.28652	-2.36753	-1.52857
A'_1	-4.17432	-2.7213	-1.43811	- .74793	- .14322
A'_2	-2.93893	-2.0225	- .917949	- .24739	.15834
A'_3	-2.48191	-1.5925	- .56377	- .00819	.24108
A'_4	-2.15047	-1.2876	- .33189	.1290	.24577
A'_5	-1.89234	-1.0556	- .16996	.20074	.22045
A'_6	-1.68244	- .87141	- .05298	.23691	.185471
A'_7	-1.50667	- .72095	.03312	.25104	.150032
A'_8	-1.3563	- .59553	.096977	.25136	.118120
A'_9	-1.2257	- .48940	.14432	.24304	.0911406
A'_{10}	-1.1108	- .39855	.17909	.22947	.0692261
A'_{11}	-1.0087	- .32011	.20414	.21297	.0519196
A'_{12}	- .91726	- .2519	.22159	.19506	.0385341
A'_{13}	- .83474	- .1922	.23303	.17682	.028349
A'_{14}	- .75991	- .1398	.23972	.15890	.020701
A'_{15}	- .69169	- .09357	.24262	.14178	.015019
A'_{16}	- .62924	- .05270	.24255	.1258	.010836
A'_{17}	- .57188	- .0164	.24009	.1110	.0077854
A'_{18}	- .51901	.0155	.23575	.09745	.0055614
A'_{19}	- .47007	.04410	.22998	.08524	.0039606
A'_{20}	- .42480	.06941	.22307	.07427	.002811

Table I
Values of the Coefficients $A'_n(\mu_o)$
(Continued)

Coefficient	$\mu_o = .3$	$\mu_o = .5$	$\mu_o = .7$	$\mu_o = .9$	$\mu_o = 1$
A'_0	-1.09188	-.627132	-.387550	-.248197	-2.00218×10^{-1}
A'_1	.088323	.219512	.211267	.170171	14.8104×10^{-2}
A'_2	.24037	.191597	.150904	.0588352	4.15890×10^{-2}
A'_3	.22637	.111564	.0430544	.0149679	8.62705×10^{-3}
A'_4	.176072	.0566712	.0144988	.00335175	15.7683×10^{-4}
A'_5	.125757	.0267326	.00454209	.000699579	2.68896×10^{-4}
A'_6	.0853985	.0120252	.00135928	13.9712×10^{-5}	4.39039×10^{-5}
A'_7	.0560360	.00523491	3.94249×10^{-4}	2.70750×10^{-5}	6.95852×10^{-6}
A'_8	.0358563	.00222504	1.11788×10^{-4}	5.13344×10^{-6}	10.7934×10^{-7}
A'_9	.0224981	.000928770	3.11588×10^{-5}	9.57313×10^{-7}	16.4697×10^{-8}
A'_{10}	.0139098	.000382245	8.56973×10^{-6}	1.76223×10^{-7}	2.48096×10^{-8}
A'_{11}	.00849110	.000155548	2.33174×10^{-6}		
A'_{12}	.00512962	6.27148×10^{-5}	6.28879×10^{-7}		
A'_{13}	.00315373	2.50921×10^{-5}	1.68365×10^{-7}		
A'_{14}	.00183609	9.97430×10^{-6}			
A'_{15}	.00108703	3.94277×10^{-6}			
A'_{16}	.000640462	1.55103×10^{-6}			
A'_{17}	.000375897	6.0828×10^{-7}			
A'_{18}	.00022001	2.3715×10^{-7}			
A'_{19}	.00012858				
A'_{20}	.000042790				

Table I
Values of the Coefficients $A'_n(\mu_o)$
(Continued)

Coefficient	$\mu_o = 1.2$	$\mu_o = 1.5$	$\mu_o = 2$	$\mu_o = 3$	$\mu_o = 4$
A'_0	-.131597	-.0711625	-.0259654	-.00350591	-4.74277×10^{-4}
A'_1	.108267	.0641478	.0250165	.00348854	4.73960×10^{-4}
A'_2	.0201620	.00649105	.000922804	17.3138×10^{-6}	3.18137×10^{-7}
A'_3	.00278500	.000488840	25.4323×10^{-6}	64.4016×10^{-9}	1.60093×10^{-10}
A'_4	.000339691	3.25938×10^{-5}	6.23175×10^{-7}		
A'_5	3.87164×10^{-5}	20.3382×10^{-7}	14.0161×10^{-9}		
A'_6	4.22903×10^{-6}	12.1724×10^{-7}			
A'_7	4.48676×10^{-7}	7.07924×10^{-8}			
A'_8	4.66011×10^{-8}				
A'_9	4.76266×10^{-9}				

Table II
Values of the Coefficients $B'_n(\mu_o)$

Coefficient	$\mu_o = .01$	$\mu_o = .02$	$\mu_o = .05$	$\mu_o = .1$	$\mu_o = .2$
B'_0	4.6477	4.4513	3.531	2.8640	2.17853
B'_1	3.8145	3.1305	2.242	1.59834	1.00086
B'_2	3.2879	2.6103	1.743	1.13174	.59963
B'_3	2.9517	2.2808	1.434	.85497	.384933
B'_4	2.7044	2.0402	1.213	.66647	.254494
B'_5	2.5091	1.8515	1.044	.529103	.170713
B'_6	2.3480	1.6970	.90896	.425006	.115362
B'_7	2.2110	1.5667	.79820	.344105	$.78237 \times 10^{-1}$
B'_8	2.0922	1.4545	.70537	.280148	$.531401 \times 10^{-1}$
B'_9	1.9875	1.3563	.62639	.228975	$.361042 \times 10^{-1}$
B'_{10}	1.8939	1.2694	.55842	.187676	$.245206 \times 10^{-1}$
B'_{11}	1.8094	1.1915	.49938	.154138	$.166420 \times 10^{-1}$
B'_{12}	1.7326	1.1213	.44773	.126773	$.112849 \times 10^{-1}$
B'_{13}	1.6623	1.0575	.402271	.104371	$.764540 \times 10^{-2}$
B'_{14}	1.5973	.9991	.362062	$.85986 \times 10^{-1}$	$.517505 \times 10^{-2}$
B'_{15}	1.5373	.9456	.326351	$.708711 \times 10^{-1}$	$.350003 \times 10^{-2}$
B'_{16}	1.4814	.8962	.294527	$.584278 \times 10^{-1}$	$.236540 \times 10^{-2}$
B'_{17}	1.4291	.8505	.266086	$.481751 \times 10^{-1}$	$.159758 \times 10^{-2}$
B'_{18}	1.3802	.8079	.240605	$.397224 \times 10^{-1}$	$.107842 \times 10^{-2}$
B'_{19}	1.3342	.7683	.217728	$.327514 \times 10^{-1}$	$.727681 \times 10^{-3}$
B'_{20}	1.2907	.7314	.197152	$.270012 \times 10^{-1}$	$.490492 \times 10^{-3}$

Table 11
Values of the Coefficients $B'_n(\mu_o)$
(Continued)

Coefficient	$\mu_o = .3$	$\mu_o = .5$	$\mu_o = .7$	$\mu_o = .9$	$\mu_o = 1.0$
B'_0	1.809446	1.348638	1.05209	.839445	.75340
B'_1	.689498	.361980	.199431	.111815	$.83891 \times 10^{-1}$
B'_2	.350230	.131015	$.497255 \times 10^{-1}$	$.189997 \times 10^{-1}$	$.117173 \times 10^{-1}$
B'_3	.189811	$.489683 \times 10^{-1}$	$.126851 \times 10^{-1}$	$.324824 \times 10^{-2}$	$.163710 \times 10^{-2}$
B'_4	.105202	$.185347 \times 10^{-1}$	$.321888 \times 10^{-2}$	$.549962 \times 10^{-3}$	$.226371 \times 10^{-3}$
B'_5	$.587260 \times 10^{-1}$	$.699242 \times 10^{-2}$	$.810843 \times 10^{-3}$	$.92427 \times 10^{-4}$	$.310863 \times 10^{-4}$
B'_6	$.328793 \times 10^{-1}$	$.262557 \times 10^{-2}$	$.203130 \times 10^{-3}$	$.154610 \times 10^{-4}$	$.425111 \times 10^{-5}$
B'_7	$.183824 \times 10^{-1}$	$.98178 \times 10^{-3}$	$.506918 \times 10^{-4}$	$.257851 \times 10^{-5}$	$.579816 \times 10^{-6}$
B'_8	$.102952 \times 10^{-1}$	$.365919 \times 10^{-3}$	$.126171 \times 10^{-4}$	$.429179 \times 10^{-6}$	$.789443 \times 10^{-7}$
B'_9	$.571232 \times 10^{-2}$	$.136045 \times 10^{-3}$	$.313445 \times 10^{-5}$	$.713366 \times 10^{-7}$	$.107355 \times 10^{-7}$
B'_{10}	$.317589 \times 10^{-2}$	$.504849 \times 10^{-4}$	$.777652 \times 10^{-6}$	$.118473 \times 10^{-7}$	$.145860 \times 10^{-8}$
B'_{11}	$.176184 \times 10^{-2}$	$.187077 \times 10^{-4}$	$.192736 \times 10^{-6}$		
B'_{12}	$.97555 \times 10^{-3}$	$.692460 \times 10^{-5}$	$.9895 \times 10^{-7}$		
B'_{13}	$.568302 \times 10^{-3}$	$.256095 \times 10^{-5}$	$.2484 \times 10^{-8}$		
B'_{14}	$.299279 \times 10^{-3}$	$.94652 \times 10^{-6}$			
B'_{15}	$.165374 \times 10^{-3}$	$.34969 \times 10^{-6}$			
B'_{16}	$.91330 \times 10^{-4}$	$.12919 \times 10^{-6}$			
B'_{17}	$.504233 \times 10^{-4}$	$.47767 \times 10^{-7}$			
B'_{18}	$.278402 \times 10^{-4}$	$.17714 \times 10^{-7}$			
B'_{19}	$.153818 \times 10^{-4}$				
B'_{20}	$.482713 \times 10^{-5}$				

Table II
Values of the Coefficients $B'_n(\mu_o)$
(Continued)

Coefficient	$\mu_o = 1.2$	$\mu_o = 1.5$	$\mu_o = 2.0$	$\mu_o = 3.0$	$\mu_o = 4.0$
B'_0	.610364	.448687	.271003	$.99580 \times 10^{-1}$	$.366313 \times 10^{-1}$
B'_1	$.472389 \times 10^{-1}$	$.198802 \times 10^{-1}$	$.462947 \times 10^{-2}$	$.240695 \times 10^{-3}$	$.121724 \times 10^{-4}$
B'_2	$.443462 \times 10^{-2}$	$.102127 \times 10^{-2}$	$.866482 \times 10^{-4}$	$.601624 \times 10^{-6}$	$.409715 \times 10^{-8}$
B'_3	$.413307 \times 10^{-3}$	$.518745 \times 10^{-4}$	$.160596 \times 10^{-6}$	$.148954 \times 10^{-8}$	1.37655×10^{-12}
B'_4	$.381406 \times 10^{-4}$	$.261413 \times 10^{-5}$	$.296239 \times 10^{-7}$		
B'_5	$.349972 \times 10^{-5}$	$.131083 \times 10^{-6}$			
B'_6	$.320071 \times 10^{-6}$	$.644447 \times 10^{-8}$			
B'_7	$.292121 \times 10^{-7}$	$.3285 \times 10^{-9}$			
B'_8	$.266258 \times 10^{-8}$				
B'_9	$.24246 \times 10^{-9}$				

Table III
The Dimensionless Kinetic Energies

μ_o	$f_1(\mu_o)$	$G(\mu_o) = 1 + 3f_1(\mu_o)$	$\frac{1}{\sqrt{1 + 3f_1(\mu_o)}}$	$f_2(\mu_o)$	$f_3(\mu_o)$
4.0	.000006	1.000018	.999991	1.83096×10^{-2}	1.61430×10^{-4}
3.0	.000123	1.000369	.999816	4.96693×10^{-2}	1.43278×10^{-3}
2.0	.002353	1.007059	.996489	1.33058×10^{-1}	8.85291×10^{-3}
1.5	.009709	1.029127	.985747	2.14146×10^{-1}	.0228289
1.2	.021636	1.064908	.969045	2.79461×10^{-1}	.0390940
1.0	.035716	1.107148	.950380	3.30727×10^{-1}	.0548407
.9	.045319	1.135957	.938251	3.58287×10^{-1}	.0644499
.7	.070824	1.212472	.908164	4.16071×10^{-1}	.088738
.5	.105628	1.316878	.871419	4.74743×10^{-1}	.114793
.3	.148507	1.445521	.831740	5.29000×10^{-1}	.141500
.2	.170940	1.512820	.813030	5.51769×10^{-1}	.158776
.1	.190966	1.572898	.797351	5.7189×10^{-1}	.171328
.05	.198415	1.595245	.791747	5.7482×10^{-1}	.174387
.02	.201298	1.603894	.789609	5.7634×10^{-1}	.177034
.01	.201879	1.605637	.789180	5.7720×10^{-1}	
0	.202071	1.606213	.789040	5.7722×10^{-1}	.177533

Table IV

Values of $V_1(\eta;\mu_o)$

η	$\mu_o = 4.0$	$\mu_o = 2.0$	$\mu_o = 1.0$	$\mu_o = 0.50$	$\mu_o = 0.20$
0°	.000000	.000000	.000000	.000000	.000000
15°	$.424804 \times 10^{-7}$	$.187480 \times 10^{-4}$	$.696757 \times 10^{-3}$		
30°	$.639907 \times 10^{-6}$	$.279804 \times 10^{-3}$	$.963758 \times 10^{-2}$	.125882	1.023
45°	$.292442 \times 10^{-5}$	$.126009 \times 10^{-2}$	$.386968 \times 10^{-1}$		
60°	$.798598 \times 10^{-5}$	$.337630 \times 10^{-2}$	$.902005 \times 10^{-1}$	.594904	2.08
75°	$.160693 \times 10^{-4}$	$.664639 \times 10^{-2}$	.152931		
90°	$.260602 \times 10^{-4}$	$.105302 \times 10^{-1}$	.209216	.860742	2.45
105°	$.355368 \times 10^{-4}$	$.140502 \times 10^{-1}$	.243659		
120°	$.414268 \times 10^{-4}$	$.160125 \times 10^{-1}$	.247626	.797569	2.14
135°	$.410526 \times 10^{-4}$	$.155823 \times 10^{-1}$	.219591		
150°	$.331662 \times 10^{-4}$	$.124158 \times 10^{-1}$	.163541	.472593	1.24
165°	$.185618 \times 10^{-4}$	$.688878 \times 10^{-2}$	$.871036 \times 10^{-1}$		
180°	.000000	.000000	.000000	.000000	.000000

TABLE V

Values of $V_2(\eta;\mu_o)$

η	$\mu_o = 4.0$	$\mu_o = 2.0$	$\mu_o = 1.0$	$\mu_o = 0.50$	$\mu_o = 0.20$
0^o	.000000	.000000	.000000	.000000	.000000
15^o	$.453600 \times 10^{-4}$	$.256355 \times 10^{-2}$	$.240799 \times 10^{-1}$		
30^o	$.173741 \times 10^{-3}$	$.976818 \times 10^{-2}$	$.882188 \times 10^{-1}$	.367948	1.230
45^o	$.363238 \times 10^{-3}$	$.202546 \times 10^{-1}$	.172537		
60^o	$.581138 \times 10^{-3}$	$.320618 \times 10^{-1}$	.254773	.79444	2.144
75^o	$.788978 \times 10^{-3}$	$.429975 \times 10^{-1}$	.317735		
90^o	$.948522 \times 10^{-3}$	$.510230 \times 10^{-1}$	.351777	.93767	2.475
105^o	$.102769 \times 10^{-2}$	$.545738 \times 10^{-1}$	.353738		
120^o	$.100555 \times 10^{-2}$	$.527682 \times 10^{-1}$	.324838	.81363	2.145
135^o	$.875703 \times 10^{-3}$	$.454920 \times 10^{-1}$	.268988		
150^o	$.647295 \times 10^{-3}$	$.333688 \times 10^{-1}$	.191731	.46984	1.24
165^o	$.343882 \times 10^{-3}$	$.176424 \times 10^{-1}$	$.99653 \times 10^{-1}$		
180^o	.000000	.000000	.000000	.000000	.000000

Table VI

$P_z(\mu_o)$, The Dimensionless Repulsive Stress on the Translating Sphere of Fixed Radius

μ_o	$P_z(\mu_o)$
.01	4.0311
.02	3.3229
.05	2.40325
.1	1.72798
.2	1.08423
.3	.742227
.5	.377945
.7	.198153
.9	.10308
1.0	7.3734×10^{-2}
1.2	3.6992×10^{-2}
1.5	1.2532×10^{-2}
2.0	1.8814×10^{-3}
3.0	3.6513×10^{-5}

Table VII

The Dimensionless Pressure Distribution, $F_\omega(\eta,\mu_o)$ Produced over the Plane by the Oscillating Air Bubble

Values of F_ω when $\eta = \pi$

μ_o	$F_\omega(\eta = \pi,\mu_o)$
.3	1.81159
.5	1.71375
1.0	1.22760
1.5	.835225
2.0	.529184
3.0	.198595
4.0	.073237

Values of F_ω when $\mu_o = 1$

η	$F_\omega(\eta, \mu_o = 1)$
0°	.000000
15°	.220322
30°	.424184
45°	.609196
60°	.773072
75°	.891032
90°	.991789
105°	1.070431
120°	1.130344
135°	1.176039
150°	1.186691
165°	1.221849
180°	1.22760

Table VIII

The Dimensionless Impulsive Pressure, J, Produced Beneath a Collapsing Cavitation Bubble

μ	μ_m=.1		μ_m=.3		μ_m = .5		μ_m=1.0	
	R	−J	R	−J	R	−J	R	−J
.1	1.00000	.00000						
.3	.96141	.52466	1.00000	.00000				
.5	.89125	.75080	.92702	.61427	1.00000	.00000		
1.0	.65129	.87852	.67744	.84054	.73076	.76126	1.00000	.00000
1.5	.42722	.81038	.44437	.79038	.47935	.75160	.65596	.57701
2.0	.26713	.67837	.27785	.66432	.29973	.63786	.41015	.53333
3.0	.099824	.43377	.10383	.42530	.11205	.40937	.15327	.34962
4.0	.036793	.26755	.038270	.26233	.0412825	.25257	.056492	.21592
∞	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000

μ	μ_m = 1.5		μ_m=2.0		μ_m=3.0	
	R	−J	R	−J	R	−J
1.5	1.00000	.00000				
2.0	.62528	.38915	1.00000	.00000		
3.0	.23366	.28186	.37369	.21839	1.00000	.00000
4.0	.086122	.17431	.13773	.13810	.36858	.082611
∞	.00000	.00000	.00000	.00000	.00000	.00000

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