I. ON THE REFLECTION OF PLANE WAVES BY STRATIFIED SYSTEMS.
(NORMAL INCIDENCE)

II. THE DETERMINATION OF SEISMIC VELOCITIES IN LAYERS WITH NON-PARALLEL INTERFACES.

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ABSTRACT

Part 1.

The purpose of this study is to establish that the reflection of a plane wave (at normal incidence) by a non-homogeneous layer with properties varying only in the direction of wave propagation may be deduced by a limiting process from the formulae which are valid in the case of discrete homogeneous layers. In this limiting process the number of layers is made to increase without bounds while the thickness of each layer tends to zero in such a way that the total thickness remain constant.

This is done by using the matrix relation which binds the up and down-going wave amplitudes in two layers of a pile of homogeneous layers on top of a semi-infinite homogeneous medium. Then the number of layers is increased as described above and the limit of the matrix relation obtained. It is then verified that the result thus gotten is identical to the quantity obtained by solving the differential equation for the disturbance in a non-homogeneous layer whose properties are the ones which the discrete case is made to tend to.

Part II.

A method for calculating velocities in homogeneous isotropic layers and the position of their interfaces from surface reflection seismic measurements is described. The problem is discussed only in the case of parallel strikes.

Conditions of applicability are plane interfaces and good lateral correlations.

Also the accuracy of the determination of the velocities and the position of the interfaces depends very much on the velocity, the dip and a distance which gives the position of the interface. One should be careful to estimate in each particular case whether the method may safely be used.

TABLE OF CONTENTS

I. On the reflection of plane waves by stratified systems (normal incidence)

page

1. Homogeneous layers with parallel interfaces.

1.1 Introduction.................................... 1

1.2 Relation between the wave amplitudes in different layers.................................... 4

1.2.1 Notations.................................... 4

1.2.2 The fundamental relations.................................... 4

1.3 Other matrix relations.................................... 7

1.4 Reflection coefficients.................................... 9

1.4.1 Definition.................................... 9

1.4.2 Examples.................................... 11

2. Non-homogeneous media.

2.1 Introduction.................................... 14

2.2 Validity of the limiting process.................................... 14

2.2.1 Use of the differential equation.................................... 14

2.2.2 Limit of a product of matrices.................................... 18

2.2.3 Synthesis.................................... 28

Appendixes.

A.1 Computation of the reflection coefficient R_2 by the summation method.................................... 33

A.2 Computation of the reflection method R_3 by the summation method.................................... 34

A.3 The differential equation for a non-homogeneous medium.................................... 36

A.4 On the convergence of infinite products of matrices.................................... 38

A.5 On Lucas polynomials and their computation.... 46

A.6 On the series $\left\langle \sum_{n=0}^{\infty} \begin{pmatrix} 0 & iG \\ iG & \tilde{F} \end{pmatrix}^n \right\rangle$.................................... 50

TABLE OF CONTENTS
(continued)

page

II. The determination of seismic velocities in layers with non-parallel interfaces........................ 59

I. ON THE REFLECTION OF PLANE WAVES BY STRATIFIED SYSTEMS
(NORMAL INCIDENCE)

MAIN SYMBOLS OF PART I.

	page	
a	4	amplitude of down-going wave.
b	4	amplitude of up-going wave.
c	7	$c = a + b$.
d	7	$d = a - b$.
f	22	$f(z) = \frac{1}{2} \frac{d}{dz} (\log \rho V)$.
g	22	$g(z) = \frac{\omega}{V}$.
k	5	ratio of 2 values of ρV : $k_{m+1,m} = \frac{(\rho V)_{m+1}}{(\rho V)_m}$.
s	9 38	source amplitude when source is sinusoidal. in Appendix 4, integer.
t	4	time.
u	4	total displacement of particles.
z	4	depth (positive downward).
α	15	constant.
ρ	4	density.
τ	5	time taken by waves to travel through a layer (designed by subscript) $\tau_m = \frac{z_m - z_{m-1}}{V_m}$.
ω	4	angular frequency of sinusoidal waves.
F	23	integral operator $Fx = \int_{z_1}^{z_2} f\, x\, dz$.
$\tilde{F}$	28	$\tilde{F} = 2F$.
G	23	integral operator $Gx = \int_{z_1}^{z_2} g\, x\, dz$.
I		unit matrix.
I	4	(with a subscript) interface.

MAIN SYMBOLS OF PART I.
(continued)

	page	
M	6	matrix $M_{m+1,m} = \begin{pmatrix} (1+k_{m+1,m})e^{i\omega\tau_m} & (1-k_{m+1,m})e^{i\omega\tau_m} \\ (1-k_{m+1,m})e^{-i\omega\tau_m} & (1+k_{m+1,m})e^{-i\omega\tau_m} \end{pmatrix}$.
P	6	the elements of matrix $\left(\prod_{j=1}^{n-1} M_{j+1,j}\right)\begin{pmatrix}1\\0\end{pmatrix}$.
Q	6	
R	9	reflection coefficient of a system of layers (in mumber indicated by subscript).
U	46	Lucas polynomial of 1st kind.
V	4	velocity of longitudinal waves: $V = \sqrt{\frac{\lambda + 2\mu}{\rho}}$.
Z	15	solution of the differential equation $$\frac{d^2 Z}{dz^2} + \frac{d}{dz}(\log \rho V^2)\frac{dZ}{dz} + \left(\frac{\omega}{V}\right)^2 Z = 0 .$$
$\langle\ \rangle$	23	$\langle$a product of operators F and G$\rangle$ = this product operating on 1.

1. Homogeneous layers with parallel interfaces.

1.1 Introduction.

a) The model.

A section of layers of homogeneous and isotropic materials will be considered. The interfaces are supposed to be parallel.

No absorption is assumed; the materials are supposed to obey Hooke's law.

The last medium down is semi-infinite (see fig. 1).

b) The problem.

The reflection of plane waves at normal incidence will be studied. Their time dependance will be sinusoidal in all that follows.

A source of planes waves being situated at the surface of the pile of layers, find the relation between the amplitude and phase of the disturbance (as it can be measured at the surface) on the one hand and the amplitude and phase of the source on the other hand.

c) The method.

As Sommerfeld points out (Ref.1, p.41) about the problem of light propagation in a system of transparent plates, which has many analogies to the seismic problem, there is a choice between two methods.

Either a summation method can be used, adding up all the multiple reflections, or a boundary-value-type method, fitting waves into each layer so as to satisfy the conditions at the interfaces.

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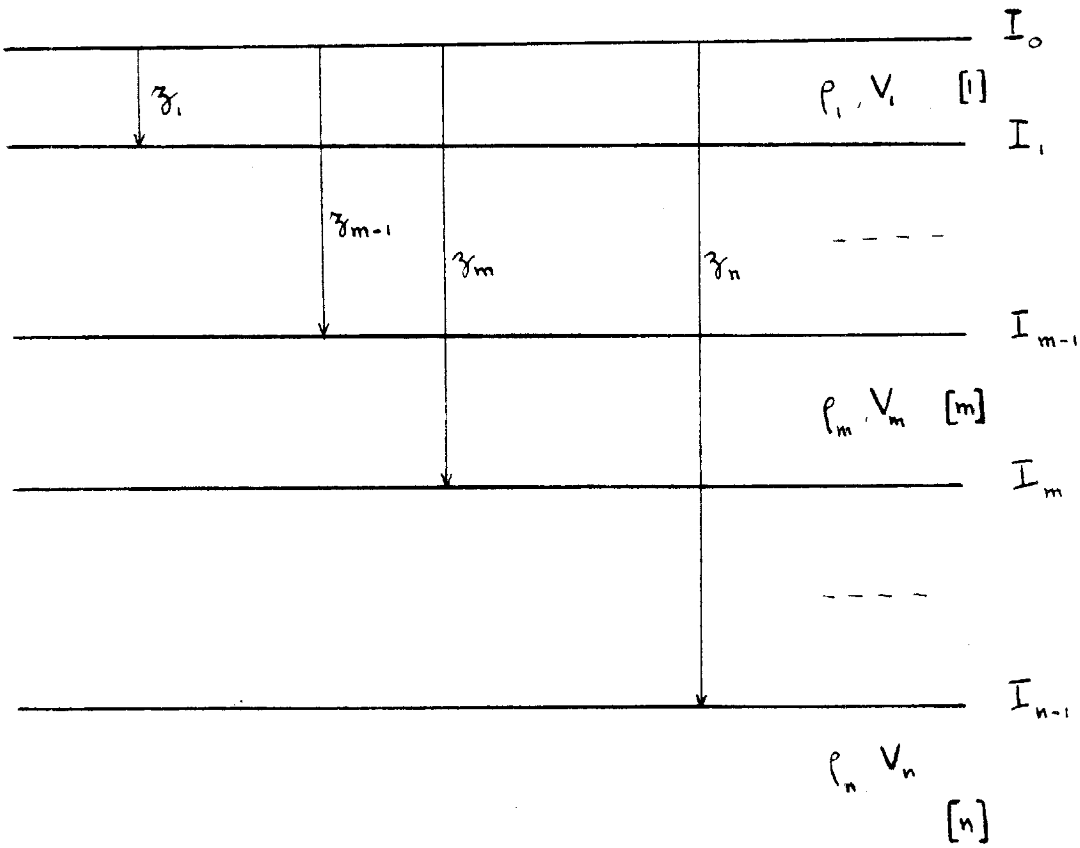

FIG. 1

The first method gives a better insight into what happens to the multiply-reflected waves but is awkward to use when there are many layers.

The other one permits the discovery of general results and will be used here after Lord Rayleigh (Ref. 2).

1.2 Relation between the wave amplitudes in different layers.

1.2.1 Notations.

The interfaces are numbered from 0 to n, if n is the number of media, starting from the upper surface (see fig. 1).

The m^{th} layer is thus bounded by interfaces I_{m-1} and I_m. The depths of the interfaces are z_{m-1} and z_m counted positively downward from the surface I_o.

The material in layer [m] has the density ρ_m and the longitudinal wave velocity V_m.

Let it be recalled that the waves considered are sinusoidal in their time-dependance and that medium [n] extends to infinity downward.

Above the surface I_o there is a vacuum for all our purposes.

1.2.2 The fundamental relation.

Let the amplitude of a particle's motion at depth z and time t in layer m be written as

$$u_m e^{i\omega t} = a_m e^{i\omega\left(t - \frac{z - z_{m-1}}{V_m}\right)} + b_m e^{i\omega\left(t + \frac{z - z_{m-1}}{V_m}\right)} \qquad (1.2.2.1)$$

where a_m and b_m are the amplitudes of waves traveling in [m] respectively down and up, and ω the angular frequency of the waves.

It is important to notice immediately that a and b are in general complex quantities in order to take into account the delays due to the transmission through the various layers.

In all layers expressions similar to that of equation 1.2.2.1 can be written, but since n is semi-infinite

$$b_n = 0 \qquad (1.2.2.2)$$

At interface I_m there must be continuity of displacement of the particles and of normal stresses for all times.

The first condition gives

$$a_m e^{-i\omega\tau_m} + b_m e^{i\omega\tau_m} = a_{m+1} + b_{m+1} \qquad (1.2.2.3)$$

with

$$\tau_m = \frac{z_m - z_{m-1}}{V_m} .$$

The second one, or(*)

$$\rho_m V_m^2 \left(\frac{du_m}{dz}\right)_{z_m} = \rho_{m+1} V_{m+1}^2 \left(\frac{du_{m+1}}{dz}\right)_{z_m} \qquad (1.2.2.4)$$

yields

$$a_m e^{-i\omega\tau_m} - b_m e^{i\omega\tau_m} = k_{m+1,m}\left(a_{m+1} - b_{m+1}\right) \qquad (1.2.2.5)$$

where

$$k_{m+1,m} = \frac{(\rho V)_{m+1}}{(\rho V)_m}$$

The system of equations 1.2.2.3 and 1.2.2.5 can be written in matrix form

$$\begin{pmatrix} a_m \\ b_m \end{pmatrix} = \begin{pmatrix} e^{-i\omega\tau_m} & e^{i\omega\tau_m} \\ e^{-i\omega\tau_m} & -e^{i\omega\tau_m} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ k_{m+1,m} & -k_{m+1,m} \end{pmatrix} \begin{pmatrix} a_{m+1} \\ b_{m+1} \end{pmatrix} .$$

(1.2.2.6)

Hence the following result :

The matrices of the amplitudes in [m] and [m+1] can be obtained one from the other by multiplication by a square matrix,

(*) see Bullen (Ref. 3, p.20)

itself the product of two square matrices.

One of these depends only on ω and τ_m, the other, on the contrary, only on the ratio of ρV on both sides of I_m.

Equation 1.2.2.6 may be put in the form

$$\begin{pmatrix} a_m \\ b_m \end{pmatrix} = \frac{1}{2} M_{m+1,m} \begin{pmatrix} a_{m+1} \\ b_{m+1} \end{pmatrix} \qquad (1.2.2.7)$$

with the following notation

$$M_{m+1,m} = \begin{pmatrix} (1+k_{m+1,m})e^{i\omega\tau_m} & (1-k_{m+1,m})e^{i\omega\tau_m} \\ (1-k_{m+1,m})e^{-i\omega\tau_m} & (1+k_{m+1,m})e^{-i\omega\tau_m} \end{pmatrix}. \qquad (1.2.2.8)$$

Equations 1.2.2.7 and 1.2.2.8 give immediately a relation between matrices

$$\begin{pmatrix} a_m \\ b_m \end{pmatrix}, \quad \begin{pmatrix} a_{m'} \\ b_{m'} \end{pmatrix}$$

whatever m and m' between 0 and n, m<m',

$$\begin{pmatrix} a_m \\ b_m \end{pmatrix} = \left(\frac{1}{2}\right)^{m'-m} \left(\prod_{j=m}^{m'-1} M_{j+1,j}\right) \begin{pmatrix} a_{m'} \\ b_{m'} \end{pmatrix}. \qquad (1.2.2.9)$$

In particular,

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \left(\frac{1}{2}\right)^{n-1} \left(\prod_{j=1}^{n-1} M_{j+1,j}\right) \begin{pmatrix} a_n \\ 0 \end{pmatrix}. \qquad (1.2.2.10)$$

The matrix product that enters the preceding equation is of the form

$$\begin{pmatrix} P \\ Q \end{pmatrix}.$$

This will be used to find the value of the reflection coefficient and to discuss theorems concerning it (see section 1.4).

1.3 Other matrix relations.

Equation 1.2.2.7 is of course not the only possible one that can be written between quantities attached to waves in different layers.

For instance, instead of using quantities a_m and b_m one could use c_m and d_m such that

$$\begin{cases} c_m = a_m + b_m , \\ d_m = a_m - b_m . \end{cases} \tag{1.3.1}$$

Since

$$\begin{pmatrix} c_m \\ d_m \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a_m \\ b_m \end{pmatrix} ,$$

one finds the relation

$$\begin{pmatrix} c_m \\ d_m \end{pmatrix} = \begin{pmatrix} \cos\omega\tau_m & i\,k_{m+1,m} \sin\omega\tau_m \\ i\sin\omega\tau_m & k_{m+1,m}\cos\omega\tau_m \end{pmatrix} \begin{pmatrix} c_{m+1} \\ d_{m+1} \end{pmatrix} . \tag{1.3.2}$$

Another type of matrix relation can be obtained by consideration of

$$\begin{pmatrix} p_m \\ v_m \end{pmatrix}$$

where $p_m e^{i\omega t}$ is the stress across I_m and $v_m e^{i\omega t}$ the particle velocity at I_m.

At $z = z_m$,

$$\begin{pmatrix} p_m \\ v_m \end{pmatrix} = i\omega \begin{pmatrix} -(\rho V)_m e^{-i\omega\tau_m} & (\rho V)_m e^{i\omega\tau_m} \\ e^{-i\omega\tau_m} & e^{i\omega\tau_m} \end{pmatrix} \begin{pmatrix} a_m \\ b_m \end{pmatrix} . \tag{1.3.3}$$

At $z = z_{m+1}$, a similar equation can be written between

$$\begin{pmatrix} p_{m+1} \\ v_{m+1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{m+1} \\ b_{m+1} \end{pmatrix}$$

and by use of relation 1.2.2.7 the equation

$$\begin{pmatrix} p_m \\ v_m \end{pmatrix} = \begin{pmatrix} \cos \omega \tau_{m+1} & -i\,(\rho V)_{m+1} \sin \omega \tau_{m+1} \\ \dfrac{-i}{(\rho V)_{m+1}} \sin \omega \tau_{m+1} & \cos \omega \tau_{m+1} \end{pmatrix} \begin{pmatrix} p_{m+1} \\ v_{m+1} \end{pmatrix} \tag{1.3.4}$$

is obtained.

The square matrix of equation 1.3.4 will be called $\mathcal{M}_{m+1,\,m}$. It has the character of an impedance matrix similar to that of the matrices that relate the matrices of voltage and current at the input and the output of an electrical quadripole.

In particular, the determinant of a matrix $\mathcal{M}$ is equal to 1.

The terms on the main diagonal are numbers whereas on the other diagonal the lower left term has the dimension of an admittance and the upper right the dimension of an impedance (Brillouin, Ref.4, page 203).

This dimension of impedance can be defined as $\frac{p}{v}$, and then we have the well known fact that ρ V is an impedance.

Relation (1.3.4) enables a complete parallelism to be established between the theory of electrical quadripoles and that of a system of layers with parallel interfaces.

1.4 Reflection coefficients.

1.4.1 Definition.

It was seen in section 1.2.2 (equation 10) that the complex amplitudes a_1 and b_1 are given in terms of the amplitude a_n by formulae

$$\begin{cases} a_1 = \dfrac{P}{q^{n-1}} \, a_n , \\ \\ b_1 = \dfrac{Q}{q^{n-1}} \, a_n \end{cases} \tag{1.4.1.1}$$

where P and Q are the two elements of the 2x1 matrix

$$\left(\prod_{\gamma=1}^{n-1} M_{\gamma+1,\gamma} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and n is the number of media.

Suppose we have a sinusoidal source of plane waves acting on the surface I_o with an amplitude s , the reflection coefficient R_n of the whole system of interfaces will be defined as

$$R_n = \frac{b_1}{s} , \tag{1.4.1.2}$$

the ratio of the complex amplitude of the sum of all waves traveling upward in layer [1] , taken at the surface I_o, to the amplitude of the source.

There obviously must exist a relation between a_1 , b_1 and s for the equilibrium - a dynamical one - of the surface I_o.

This relation is best found by considering the problem of an impulsive source.

Let the source produce an amplitude $S(t)$. After multiple reflections, the surface I_o will be subject to two disturbances $A_1(t)$ and $B_1(t)$, analogous to a_1 and b_1 of the case with the sinusoidal source. Suppose $S(t)$ is short enough in time so that the first reflection arrives at I_o after S has died out. Then a relation between $S(t)$, $A_1(t)$, $B_1(t)$ may be found.

These terms are schematically represented on fig. 2 drawn according to principles given by Bewley (Ref. 5) and explained in Appendix 1. $B_1(t)$ is constituted by multiple reflections of $S(t)$ in layer 1 plus multiple reflections in the same layer of impulses coming from below. $A_1(t)$ is the sum of $S(t)$ and a certain term $A'_1(t)$.

$$S(t) = A_1(t) - A'_1(t) . \qquad (1.4.1.3)$$

Consideration of the diagram shows that since a vacuum is supposed to be above I_o, A'_1 is just equal to B_1 (perfect reflection).

Therefore

$$S(t) = A_1(t) - B_1(t) . \qquad (1.4.1.4)$$

Now, writing this equation in Fourier integrals, we have

$$\int_{-\infty}^{\infty} \mathcal{S}(\omega)\, e^{i\omega t}\, d\omega = \int_{-\infty}^{\infty} \mathcal{a}_1(\omega)\, e^{i\omega t}\, d\omega - \int_{-\infty}^{\infty} \mathcal{B}_1(\omega)\, e^{i\omega t}\, d\omega$$

and the equation

$$\mathcal{S}(\omega) = \mathcal{a}_1(\omega) - \mathcal{B}_1(\omega) \qquad (1.4.1.5)$$

must hold. This is obviously true even if the signal $S(t)$ is not short.

In the notations previously used, this can be written

$$s = a_1 - b_1 \qquad (1.4.1.6)$$

which is the relation sought for, valid for all frequencies .

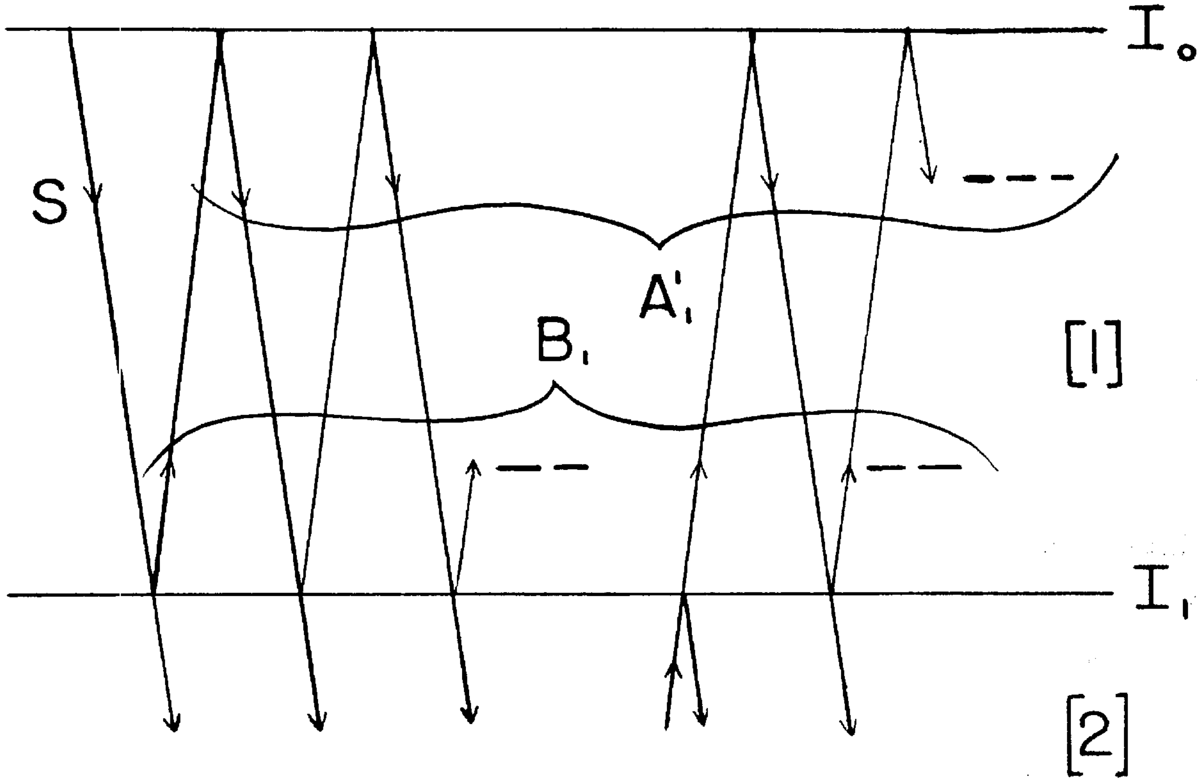

FIG. 2

Putting this in the expression for R_n, we get

$$R_n = \frac{b_1}{a_1 - b_1} . \tag{1.4.1.7}$$

Introducing formulae 1.4.1.1, we obtain

$$R_n = \frac{Q}{P-Q} , \tag{1.4.1.8}$$

or with

$$r_n = \frac{Q}{P} ,$$

$$R_n = \frac{r_n}{1 - r_n} . \tag{1.4.1.9}$$

1.4.2 Examples.

As a verification, the reflection coefficients for systems of one layer and one semi-infinite medium and two layers and one semi-infinite medium will be computed by this method.

In Appendix 1 and 2 the same reflection coefficients will be computed by the summation method and the results may be checked.

In the case of one layer over a semi-infinite medium

$$\begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} (1 + k_{2,1})\, e^{i\omega\tau_1} \\ (1 - k_{2,1})\, e^{-i\omega\tau_1} \end{pmatrix} \tag{1.4.2.1}$$

and

$$R_2 = \frac{r_{1,2}\, e^{-2i\omega\tau_1}}{1 - r_{1,2}\, e^{-2i\omega\tau_1}} \tag{1.4.2.2}$$

with

$$r_{1,2} = \frac{\rho_1 V_1 - \rho_2 V_2}{\rho_1 V_1 + \rho_2 V_2} \quad . \tag{1.4.2.3}$$

This last quantity is naturally the reflection coefficient for a wave traveling in the semi-infinite medium of properties ρ_1, V_1 and being reflected at a boundary with a semi-infinite medium of properties ρ_2, V_2 .

Relation 1.4.2.2 shows the relationship between the two reflection coefficients.

Consider now the case of two layers above a semi-infinite medium.

We have

$$\begin{pmatrix} P \\ Q \end{pmatrix} = (k_{2,1}+1)(k_{3,2}+1) \begin{pmatrix} e^{i\omega(\tau_1+\tau_2)} + r_{1,2} r_{2,3} e^{i\omega(\tau_1-\tau_2)} \\ \\ r_{1,2} e^{i\omega(-\tau_1+\tau_2)} + r_{2,3} e^{-i\omega(\tau_1+\tau_2)} \end{pmatrix} \tag{1.4.2.4}$$

and

$$R_3 = \frac{Q}{P-Q} = \frac{r_{1,2} e^{-2i\omega\tau_1} + r_{2,3} e^{-2i\omega(\tau_1+\tau_2)}}{1 + r_{1,2} r_{2,3} e^{-2i\omega\tau_2} - r_{1,2} e^{-2i\omega\tau_1} - r_{2,3} e^{-2i\omega(\tau_1+\tau_2)}} \tag{1.4.2.5}$$

with

$$r_{2,3} = \frac{\rho_2 V_2 - \rho_3 V_3}{\rho_2 V_2 + \rho_3 V_3} \quad .$$

Various theorems on the reflection coefficients can be deduced, which are close equivalents to those which are obtained in the study of systems of transparent plates or of electrical quadripoles (see the bibliographies in References 6, 7, 8 and 9).

2. Non-homogeneous media.

2.1 Introduction.

The problem of the determination of the waves which are reflected and transmitted by a non-homogeneous layer with parallel interfaces sandwiched between two homogeneous half-spaces when excited by plane waves at normal incidence will be treated using two approaches.

First we will use the differential equation which gives the amplitude of the disturbance in the non-homogeneous medium.

Secondly the non-homogeneous medium will be considered as the limit of a system of homogeneous layers when the thickness of each layer goes to zero, the number of layers going to infinity in such a way that the total thickness remain constant.

Finally the two processes will be found to be equivalent, thus showing that the limiting process which has been often used in the past (Lord Rayleigh, Ref. 2; Abelès, Ref. 6, etc...) without justification is entirely correct in its conclusions.

2.2 Validity of the limiting process.

2.2.1 Use of the differential equation.

Let a slab of inhomogeneous isotropic material II be comprised between two half-spaces of homogeneous media I and III with parallel plane interfaces at abscissae $z=z_1$ and $z=z_2$

(with the z axis perpendicular to the interfaces).

The properties of medium II will be supposed to vary only in the z direction and to be continuous across the interfaces.

Then if a sinusoidal wave arrives in medium I at the boundary $z=z_1$ with normal incidence part of it will go through II and propagate in medium III and part of it will be sent back to infinity in II.

This can be expressed in our notations by

$$u_{I} = a_{I}\, e^{-i\omega\frac{z-z_1}{V_{I}}} + b_{I}\, e^{i\omega\frac{z-z_1}{V_{I}}} , \tag{2.2.1.1}$$

$$u_{II} = \alpha_1 Z_1(z) + \alpha_2 Z_2(z) \tag{2.2.1.2}$$

and

$$u_{III} = a_{III}\, e^{-i\omega\frac{z-z_2}{V_{III}}} \tag{2.2.1.3}$$

with

$$V_{I} = V(z_1) \quad , \quad V_{III} = V(z_2) .$$

In equation 2.2.1.2 , α_1 and α_2 are constants and Z_1 and Z_2 are two independant solutions of the differential equation

$$\frac{d^2Z}{dz^2} + \frac{d}{dz}\left(\log \rho V^2\right)\frac{dZ}{dz} + \left(\frac{\omega}{V}\right)^2 Z = 0 \tag{2.2.1.4}$$

which is derived in Appendix 3.

Notice that in formula A.3.4 we used the quantity $u(z,t)$ which is related to u_{II} by

$$u(z,t) = u_{II}\, e^{i\omega t} .$$

By writing the continuity of particle displacement and normal stress at the two interfaces we obtain the equations.

$$a_I + b_I = \alpha_1 Z_1(z_1) + \alpha_2 Z_2(z_1) , \qquad (2.2.1.5a)$$

$$a_{III} = \alpha_1 Z_1(z_2) + \alpha_2 Z_2(z_2) , \qquad (2.2.1.5b)$$

$$-\frac{i\omega}{V_I} a_I + \frac{i\omega}{V_I} b_I = \left(\alpha_1 \frac{dZ_1(z)}{dz} + \alpha_2 \frac{dZ_2(z)}{dz} \right)_{z_1} , \qquad (2.2.1.5c)$$

$$-\frac{i\omega}{V_{III}} a_{III} = \left(\alpha_1 \frac{dZ_1(z)}{dz} + \alpha_2 \frac{dZ_2(z)}{dz} \right)_{z_2} . \qquad (2.2.1.5d)$$

It may be remarked that on the left-hand side of these equations we have exclusively c_I , d_I and a_{III}. The quantities c and d were defined as $a + b$ and $a - b$ in section 1.3.

By elimination of α_1 and α_2 from the equations 2.2.1.5 we may obtain the quantities $\frac{c_I}{a_{III}}$ and $\frac{d_I}{a_{III}}$ as functions of z_1 and z_2 provided the differential equation 2.2.1.4 is solved.

We get

$$\frac{c_{I}}{a_{III}} = \frac{-Z_1(\gamma_1)\left[\left(\frac{dZ_2}{d\gamma}\right)_{\gamma_2} + \frac{i\omega}{V_{III}} Z_2(\gamma_2)\right] + Z_2(\gamma_1)\left[\left(\frac{dZ_1}{d\gamma}\right)_{\gamma_2} + \frac{i\omega}{V_{III}} Z_1(\gamma_2)\right]}{\left(\frac{dZ_1}{d\gamma}\right)_{\gamma_2} Z_2(\gamma_2) - Z_1(\gamma_2)\left(\frac{dZ_2}{d\gamma}\right)_{\gamma_2}}, \quad (2.2.1.6a)$$

$$\frac{d_{I}}{a_{III}} = \frac{\left(\frac{dZ_1}{d\gamma}\right)_{\gamma_1}\left[\left(\frac{dZ_2}{d\gamma}\right)_{\gamma_2} + \frac{i\omega}{V_{III}} Z_2(\gamma_2)\right] - \left(\frac{dZ_2}{d\gamma}\right)_{\gamma_1}\left[\left(\frac{dZ_1}{d\gamma}\right)_{\gamma_2} + \frac{i\omega}{V_{III}} Z_1(\gamma_2)\right]}{\frac{i\omega}{V_{I}}\left[\left(\frac{dZ_1}{d\gamma}\right)_{\gamma_2} Z_2(\gamma_2) - Z_1(\gamma_2)\left(\frac{dZ_2}{d\gamma}\right)_{\gamma_2}\right]}. \quad (2.2.1.6b)$$

2.2.2 Limit of a product of matrices.

According to equation 1.2.2.9, the matrices $\begin{pmatrix} a \\ b \end{pmatrix}$ in layers [m] and [m'] are related in the following way

$$\begin{pmatrix} a_m \\ b_m \end{pmatrix} = \left(\prod_{j=m}^{m'-1} \left(\frac{1}{2} M_{j+1,j} \right) \right) \begin{pmatrix} a_{m'} \\ b_{m'} \end{pmatrix} . \qquad (2.2.2.1)$$

When we perform the limiting process that leads toward a continuously varying medium, the matrices $\begin{pmatrix} a \\ b \end{pmatrix}$ at the termini will become

$$\begin{pmatrix} a \\ b \end{pmatrix}_{z = z_1} \quad \text{and} \quad \begin{pmatrix} a \\ b \end{pmatrix}_{z = z_2}$$

In order to go to the limit, the thicknesses of the layers are allowed to go to zero while the number of layers goes to infinity, the total thickness remaining constant.

Let us first examine what each matrix $\frac{1}{2} M_{j+1,j}$ becomes when the thickness of the layer it represents becomes arbitrarily small.

It was shown in section 1.2.2 that $\frac{1}{2} M_{j+1,j}$ is equal to

$$\frac{1}{2} \begin{pmatrix} (1 + k_{j+1,j})\, e^{i\omega \frac{\delta z_j}{V_j}} & (1 - k_{j+1,j})\, e^{i\omega \frac{\delta z_j}{V_j}} \\ (1 - k_{j+1,j})\, e^{-i\omega \frac{\delta z_j}{V_j}} & (1 + k_{j+1,j})\, e^{-i\omega \frac{\delta z_j}{V_j}} \end{pmatrix} \qquad (2.2.2.2)$$

with

$$k_{j+1,j} = \frac{(\rho V)_{j+1}}{(\rho V)_j} \quad , \quad \delta z_j = z_j - z_{j-1} .$$

Introducing the notation

$$\delta(\rho V)_\gamma = (\rho V)_{\gamma+1} - (\rho V)_\gamma \quad ,$$

one obtains

$$1 + k_{\gamma+1,\gamma} = 2 + \frac{\delta(\rho V)_\gamma}{(\rho V)_\gamma} \quad ,$$

$$k_{\gamma+1,\gamma} - 1 = \frac{\delta(\rho V)_\gamma}{(\rho V)_\gamma} \; .$$

This makes it possible if log ρV is indefinitely differentiable to write matrix 2.2.2.2 in the form

$$\begin{pmatrix} \left(1 + \frac{1}{2}\frac{d(\log \rho V)}{dz}_\gamma \delta z_\gamma + \cdots\right)\left(1 + i\omega\frac{\delta z_\gamma}{V_\gamma} + \cdots\right) & -\left(\frac{1}{2}\frac{d(\log \rho V)}{dz}_\gamma \delta z_\gamma + \cdots\right)\left(1 + i\omega\frac{\delta z_\gamma}{V_\gamma} + \cdots\right) \\ -\left(\frac{1}{2}\frac{d(\log \rho V)}{dz}_\gamma \delta z_\gamma + \cdots\right)\left(1 - i\omega\frac{\delta z_\gamma}{V_\gamma} + \cdots\right) & \left(1 + \frac{1}{2}\frac{d(\log \rho V)}{dz}_\gamma \delta z_\gamma + \cdots\right)\left(1 - i\omega\frac{\delta z_\gamma}{V_\gamma} + \cdots\right) \end{pmatrix} .$$

(2.2.2.3)

The Taylor's expansions being absolutely convergent for δz_γ small enough, the 4 elements of matrix 2.2.2.3, which are products of such expansions, are themselves absolutely convergent series. They can therefore, each of them, be written in any order without their sums being altered.

In particular, these 4 elements of matrix 2.2.2.3 can be written as power series in δz_γ and matrix 2.2.2.3 is therefore the sum of a power series in δz_γ with matrix coefficients, as follows

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\frac{d(\log \rho V)}{dz}_\gamma + i\frac{\omega}{V_\gamma} & -\frac{1}{2}\frac{d(\log \rho V)}{dz}_\gamma \\ -\frac{1}{2}\frac{d(\log \rho V)}{dz}_\gamma & \frac{1}{2}\frac{d(\log \rho V)}{dz}_\gamma - i\frac{\omega}{V_\gamma} \end{pmatrix} \delta z_\gamma + \cdots \qquad (2.2.2.4)$$

It is then seen that, when $\delta\gamma_{\gamma} \to 0$, matrix 2.2.2.3 goes to the unit matrix I in the way indicated by (2.2.2.4).

In the same time, the product of matrices of equation 2.2.2.1 becomes the product of an infinite number of such power series with matrix coefficients as (2.2.2.4).

It will be shown in Appendix 4 that infinite products of matrices obey the same convergence rules as infinite products of numbers (*).

If this is so and since the series of matrices 2.2.2.4 is absolutely convergent, then the product of an infinite number of such series is absolutely convergent (**), therefore simply convergent (**) and has a certain finite value depending on z_1, z_2, $\rho(z)$, $V(z)$, and ω.

All the $\delta\gamma_{\gamma}$ will be supposed to be equal to a common value δz.

Since series 2.2.2.4 is an absolutely convergent series in δz, the product which was just proved to exist, is an absolutely convergent power series in δz. It can therefore be ordered in any fashion without its value being altered.

Let it then be arranged in the following way : First comes the term that is constant and equal to the unit matrix I , then the sum of the terms in $\delta\gamma$, then the sum of the terms of 2nd order in δz,etc...

Now examine the formation of such terms when the number of layers is m'-m , still finite.

The terms in $\delta\gamma$ are products of m'-m-1 matrices equal to I and the coefficient of $\delta\gamma$ in one of the series 2.2.2.4. Similarly, the terms in $(\delta\gamma)^2$ are the products of m'-m-2 matrices equal to I and the $\delta\gamma$ coefficients of two series 2.2.2.4, plus the products of m'-m-1 I's and the $(\delta\gamma)^2$

(*) see Knopp (Ref. 10, p. 211-221) for these rules.
(**)in the sense of product-convergence (Appendix 4,def.2 and 3)

coefficient of one series 2.2.2.4. And so on ...

Call ${}_{\gamma'}N_{\gamma}$ the matrix coefficients of series 2.2.2.4, the subscript γ' indicating what power of δ_{γ} we are considering the coefficient of (in the series 2.2.2.4 with the index γ)

The constant term of the product is I.

The 1st order term in δ_{γ} is

$$\left(\sum_{\gamma=m}^{m'-1} {}_1N_{\gamma}\right)\delta_{\gamma} . \tag{2.2.2.5}$$

The 2nd order term in δ_{γ} is

$$\left[\sum_{\gamma'=m}^{m'-2}\left[{}_1N_{\gamma'}\left(\sum_{\gamma=\gamma'+1}^{m'-1} {}_1N_{\gamma}\right)\right]+\sum_{\gamma=m}^{m'-1} {}_2N_{\gamma}\right](\delta_{\gamma})^2 \tag{2.2.2.6}$$

and so on .

So far δ_{γ} was kept small but finite and the product did not have an infinite number of terms. The considerations on the convergence of the infinite product have ensured us that the re-ordered form of the finite product will, when $\delta_{\gamma} \to 0$, tend to the same limit no matter how this re-ordering is done.

Therefore, all that remains to do now in order to have the limit of the infinite product is to make δ_{γ} tend to 0 in formulae 2.2.2.5, 2.2.2.6 and the ones that would be obtained by generalizing the law of formation of such terms.

Term 2.2.2.5 has as a limit

$$\int_{z_1}^{z_2} {}_1N\ (z)\ dz \tag{2.2.2.7}$$

and term 2.2.2.6,

$$\int_{z_1}^{z_2} {}_1N\ (z)\ dz \int_{z}^{z_2} {}_1N\ (z')\ dz' . \tag{2.2.2.8}$$

The term of 3rd order in δz would tend to

$$\int_{z_1}^{z_2} {}_1N(z)\,dz \int_{z}^{z_2} {}_1N(z')\,dz' \int_{z'}^{z_2} {}_1N(z'')\,dz'' \tag{2.2.2.9}$$

and so on.

The remarkable feature of this is that the matrix ${}_1N\ (z)$ is only used in the formation of the limit of the infinite product. The other terms remain infinitely small and do not give rise to any integrals.

For the sake of abbreviation write

$$f(z) = \frac{1}{2}\,\frac{d}{dz}(\log \rho v) \quad , \quad g(z) = \frac{\omega}{v}\ . \tag{2.2.2.10}$$

Then

$${}_1N(z) = \begin{pmatrix} f + ig & -f \\ -f & f - ig \end{pmatrix} . \tag{2.2.2.11}$$

Formula 2.2.2.7 becomes

$$\begin{pmatrix} \int_{z_1}^{z_2} f\,dz + i\int_{z_1}^{z_2} g\,dz & -\int_{z_1}^{z_2} f\,dz \\ -\int_{z_1}^{z_2} f\,dz & \int_{z_1}^{z_2} f\,dz - i\int_{z_1}^{z_2} g\,dz \end{pmatrix} \tag{2.2.2.12}$$

and formula 2.2.2.8

$$\int_{z_1}^{z_2} \begin{pmatrix} f + ig & -f \\ -f & f - ig \end{pmatrix} \begin{pmatrix} \int_{z}^{z_2} f\,dz' + i\int_{z}^{z_2} g\,dz' & -\int_{z}^{z_2} f\,dz' \\ -\int_{z}^{z_2} f\,dz' & \int_{z}^{z_2} f\,dz' - i\int_{z}^{z_2} g\,dz' \end{pmatrix} dz\ . \tag{2.2.2.13}$$

And so on for further terms of the infinite product.

To simplify further the writing, the following notations will be introduced.

Let the operators F and G be defined by

$$F\,x(z) = \int_{z_1}^{z_2} f x \, dz \,, \qquad G\,x(z) = \int_{z_1}^{z_2} g x \, dz \,, \tag{2.2.2.14}$$

if f , g and x are functions of z such that the integrals exist.

These operators are linear, which means that

$$(F+G)x = (G+F)x = Fx + Gx \,,$$

$$[(F+G)+F]x = [F+(G+F)]x \,,$$

$$\alpha(Fx) = F(\alpha x) \,.$$

Multiplication will be <u>defined</u> in the following manner

$$(FG)x \qquad \text{written} \qquad FGx = F\int_{z}^{z_2} g x \, dz'$$

and similarly for GFx , and products of a higher number of operators and x .

Finally the notation $\langle F \rangle$ will mean

$$F(1) = \int_{z_1}^{z_2} f \, dz \,,$$

$$\langle FG \rangle = FG1 = \int_{z_1}^{z_2} f \, dz \int_{z}^{z_2} g \, dz' \,,$$

,etc ...

It is easily seen that multiplication of such operators is distributive for

$$(F+G)Fx = F^2x + GFx,$$

$$F(Fx+Gx) = F^2x + FGx$$

and associative for

$$(FG)x = F(Gx)$$

but multiplication is not commutative : in general

$$FGx \neq GFx \, .$$

With these definitions and remarks, matrix 2.2.2.12 may be written in the form

$$\begin{pmatrix} \langle F\rangle + i\langle G\rangle & -\langle F\rangle \\ -\langle F\rangle & \langle F\rangle - i\langle G\rangle \end{pmatrix}$$

or

$$\left\langle \begin{pmatrix} F+iG & -F \\ -F & F-iG \end{pmatrix} \right\rangle \, . \qquad (2.2.2.15)$$

Matrix 2.2.2.13 will be symbolically written

$$\left\langle \begin{pmatrix} F+iG & -F \\ -F & F-iG \end{pmatrix}^2 \right\rangle \, . \qquad (2.2.2.16)$$

This way, the infinite product may be written in the condensed form

$$\left\langle \sum_{n=0}^{\infty} \begin{pmatrix} F+iG & -F \\ -F & F-iG \end{pmatrix}^{n} \right\rangle . \qquad (2.2.2.17)$$

The convention will be adopted that the operator matrix to the power 0 is the unit matrix I.

This matrix 2.2.2.17 is thus the ratio between

$$\begin{pmatrix} a \\ b \end{pmatrix}_{z=z_2} \qquad \text{and} \qquad \begin{pmatrix} a \\ b \end{pmatrix}_{z=z_1}$$

that we proposed to find.

In the following a relation between

$$\begin{pmatrix} c \\ d \end{pmatrix}_{z=z_1} \qquad \text{and} \qquad \begin{pmatrix} c \\ d \end{pmatrix}_{z=z_2}$$

will be needed. ($c=a+b$, $d=a-b$).

It will be obtained by applying our limiting process to relation 1.3.2.

$$\begin{pmatrix} c_j \\ d_j \end{pmatrix} = \begin{pmatrix} \cos \omega \dfrac{\delta z_j}{v_j} & i\, k_{j+1,j} \sin \omega \dfrac{\delta z_j}{v_j} \\ i \sin \omega \dfrac{\delta z_j}{v_j} & k_{j+1,j} \cos \omega \dfrac{\delta z_j}{v_j} \end{pmatrix} \begin{pmatrix} c_{j+1} \\ d_{j+1} \end{pmatrix} . \qquad (2.2.2.18)$$

The square matrix may be written

$$\begin{pmatrix} \left(1-\omega^2\left(\frac{\delta z_j \dot{\gamma}}{V_j}\right)^2+\cdots\right) & i\left(1+\frac{d(\log \rho V)_j}{dz}\delta z_j+\cdots\right)\left(\omega\frac{\delta z_j}{V_j}+\cdots\right) \\ i\left(\omega\frac{\delta z_j}{V_j}+\cdots\right) & \left(1+\frac{d(\log \rho V)_j}{dz}\delta z_j+\cdots\right)\left(1-\left(\omega\frac{\delta z_j}{V_j}\right)^2+\cdots\right) \end{pmatrix} \tag{2.2.2.19}$$

It can be expanded as

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & i\frac{\omega}{V_j} \\ i\frac{\omega}{V_j} & \frac{d}{dz}(\log \rho V)_j \end{pmatrix} \delta z_j + \cdots \tag{2.2.2.20}$$

and the limit of the product of an infinite number of such series is

$$\left\langle \sum_{n=0}^{\infty} \begin{pmatrix} 0 & iG \\ iG & 2F \end{pmatrix}^n \right\rangle \tag{2.2.2.21}$$

with the same definitions and conventions as above.

It may be checked that, in the same manner as

$$\begin{pmatrix} \cos\omega\tau_j & i k_{j+1,j}\sin\omega\tau_j \\ i\sin\omega\tau_j & k_{j+1,j}\cos\omega\tau_j \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\left(\frac{1}{2}M_{j+1,j}\right)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1},$$

in the same way,

$$\begin{pmatrix} 0 & iG \\ iG & 2F \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} F+iG & -F \\ -F & F-iG \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}.$$

Also,

$$\begin{pmatrix} 0 & iG \\ iG & 2F \end{pmatrix}^n = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} F+iG & -F \\ -F & F-iG \end{pmatrix}^n \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}.$$

Notice that the matrices of the sum 2.2.2.21 do not retain the property that matrices $M_{\dot{\gamma}+1,\dot{\gamma}}$ had, namely that the elements on the same diagonal are complex conjugates.

These matrices have the property that the elements of the main diagonal are real and the elements of the other diagonal are pure imaginaries.

2.2.3 Synthesis.

Equation 2.2.1.6a shows that $\frac{c_{I}}{a_{III}}$ is a linear combination of $Z_1(z_1)$ and $Z_2(z_1)$ where $Z_1(z)$ and $Z_2(z)$ are independent solutions of the differential equation 2.2.1.4.

It follows that $\frac{c_{I}}{a_{III}}$, considered as a function of z_1, must satisfy the differential equation 2.2.1.4 where the variable z is replaced by z_1.

By equation 2.2.2.21 we have

$$\begin{pmatrix} c_{I} \\ d_{I} \end{pmatrix} = \left\langle \sum_{n=0}^{\infty} \begin{pmatrix} 0 & iG \\ iG & 2F \end{pmatrix}^n \right\rangle \begin{pmatrix} a_{III} \\ a_{III} \end{pmatrix} \qquad (2.2.3.1)$$

since

$$c_{III} = d_{III} = a_{III} .$$

Therefore the quantity $\frac{c_{I}}{a_{III}}$ as obtained from the limiting process is equal to the sum of the elements of the upper line of the matrix

$$\left\langle \sum_{n=0}^{\infty} \begin{pmatrix} 0 & iG \\ iG & 2F \end{pmatrix}^n \right\rangle$$

or

$$\frac{c_{I}}{a_{III}} = \left\langle \left[\sum_{q=0}^{\infty} (-1)^q \left(G \frac{1}{1-\widetilde{F}} G \right)^q \right] \left(1 + iG \frac{1}{1-\widetilde{F}} \right) \right\rangle , \qquad (2.2.3.2)$$

as computed in Appendix 6 (formula A.6.2.13a).

We will now proceed to show that this function of z_1 satisfies the differential equation 2.2.1.4.

This equation may be written

$$\frac{d^2 Z}{dz^2} + \left(\tilde{f} - \frac{1}{g}\frac{dg}{dz}\right)\frac{dZ}{dz} + g^2 Z = 0 \tag{2.2.3.3}$$

with

$$\tilde{f} = 2f = \frac{d}{dz}(\log \rho V),$$

$$g = \frac{\omega}{V},$$

the same notations as in 2.2.2.10.

Now we have to compute the first two derivatives of the function $\frac{c_I}{a_{III}}$ (2.2.3.2) with respect to z_1.

The first one is

$$\frac{d}{dz_1}\left(\frac{c_I}{a_{III}}\right) = -i\, g(z_1)\, \frac{d_I}{a_{III}} \tag{2.2.3.4}$$

as results from the formula A.6.2.12 , namely

$$\frac{c_I}{a_{III}} = 1 + i\left\langle G\left(\frac{d_I}{a_{III}}\right)\right\rangle .$$

The second derivative is obtained easily if we know

$$\frac{d}{dz_1}\left(\frac{d_I}{a_{III}}\right).$$

From Appendix 6 (formula A.6.2.13b) we know that

$$\frac{d_I}{a_{III}} = \left\langle \frac{1}{1-\tilde{F}} \left[\sum_{q=0}^{\infty} (-1)^q \left(G^2 \frac{1}{1-\tilde{F}} \right)^q \right] (1 + i\,G) \right\rangle \quad . \tag{2.2.3.5}$$

Then

$$\frac{d}{dz_1}\left(\frac{d_I}{a_{III}}\right) = -i\,g(z_1) + g(z_1)\left\langle G\,\frac{d_I}{a_{III}}\right\rangle - \tilde{f}(z_1)\,\frac{d_I}{a_{III}} \tag{2.2.3.6}$$

for

$$\frac{d_I}{a_{III}} = \left\langle \left[\sum_{q=0}^{\infty} (-1)^q \left(G^2 \frac{1}{1-\tilde{F}} \right)^q \right] (1+iG) + \tilde{F}\,\frac{1}{1-\tilde{F}} \left[\sum_{q=0}^{\infty} (-1)^q \left(G^2 \frac{1}{1-\tilde{F}} \right)^q \right] (1+iG) \right\rangle$$

$$= \left\langle \left[1 - G^2 \frac{1}{1-\tilde{F}} \left\{ \sum_{q=0}^{\infty} (-1)^q \left(G^2 \frac{1}{1-\tilde{F}} \right)^q \right\} \right] (1+iG) + \tilde{F}\,\frac{d_I}{a_{III}} \right\rangle$$

$$= 1 + i\langle G \rangle - \left\langle G^2 \frac{d_I}{a_{III}} \right\rangle + \left\langle \tilde{F}\,\frac{d_I}{a_{III}} \right\rangle \quad . \tag{2.2.3.7}$$

Whence

$$\frac{d^2}{dz^2}\left(\frac{c_I}{a_{III}}\right) = i\left[g^2(z_1)\left(i - \left\langle G\,\frac{d_I}{a_{III}} \right\rangle \right) + \tilde{f}(z_1)\,g(z_1)\,\frac{d_I}{a_{III}} - \left(\frac{dg}{dz}\right)_{z_1} \frac{d_I}{a_{III}} \right]$$

(2.2.3.8)

and it may be verified by putting formulae 2.2.3.2 , 2.2.3.4 and 2.2.3.8 into the equation 2.2.3.3, that $\frac{c_I}{a_{III}}$ as computed by the limiting process satisfies the differential equation for the non-homogeneous medium.

Furthermore $\frac{d_I}{a_{III}}$ as evaluated from the limiting process obeys equation 2.2.3.4 which is identical to the equation obtained by differentiating 2.2.1.5a with respect to z_1 and replacing $\left(\alpha_1 \frac{dz_1}{dz} + \alpha_2 \frac{dz_2}{dz}\right)_{z_1}$ by its value taken from 2.2.1.5c.

These conclusions are true for all z_1, for we will assume that the differential equation has no singular point in the range in which we are interested.

If we make z_1 tend toward z_2, keeping the same laws of variation of density and velocity

$$\rho = \rho(z),$$

$$V = V(z),$$

it may be seen by the differential equation approach (see equations 2.2.1.5) that c_I and d_I tend toward a_{III}.

By the limiting process, we also get the same result, as may be checked on the forms 2.2.3.2 and 2.2.3.5.

Therefore the function of z_1, $\frac{c_I}{a_{III}}$, obtained from the limiting process (formula 2.2.3.2), which is a solution of the differential equation and has the same value(1) and the same derivative $(-ig(z_2))$ for $z_1=z_2$ as the corresponding function of z_1 obtained from the differential equation (and given by formula 2.2.1.6a), is identical to this function.

It follows from equation 2.2.3.4 that the value of $\frac{d_I}{a_{III}}$ given by the limiting process (formula 2.2.3.5) is identical to the corresponding function of z_1 obtained by the differential equation (formula 2.2.1.6b).

This is true for all values of z_2.

Hence the final conclusion that the limiting process and the differential equation approach give identical values for $\frac{c_{I}}{a_{III}}$ and $\frac{d_{I}}{a_{III}}$ and therefore also for $\frac{a_{I}}{a_{III}}$ and $\frac{b_{I}}{a_{III}}$.

Appendix 1.

Computation of the reflection coefficient R_2 by the summation method.

Let us make use of a convenient way of keeping track of waves during their multiple reflections which was given by Bewley (Ref. 5)

Time is represented by the abscisse and each interface is drawn perpendicularly to the ordinate axis, not according to its depth, but to the sum of the thickness of the layers between the surface and that interface divided by the corresponding velocities.

In other words, the ordinate of an interface will be proportional to the total time it takes a wave to arrive there from the surface without any reflections.

This way, the travel time curve of each wave is a straight line. All of them have the same slope in absolute value on the diagram.

As a result, one gets simply the sum of reflected waves with their time delays at the surface.

Take now the problem of one layer and one semi-infinite medium (see fig. 3).

The amplitude of the reflected wave at the surface is

$$\Delta r_{1,2}\, e^{-2i\omega\tau_1} + \Delta r_{1,2}^{\,2}\, e^{-4i\omega\tau_1} + \cdots$$

since reflection by I_0 is total.

Therefore

$$R_2 = \frac{r_{1,2}\, e^{-2i\omega\tau_1}}{1 - r_{1,2}\, e^{-2i\omega\tau_1}} \,.$$

Appendix 2.

Computation of the reflection coefficient R_3 by the summation method.

The process of multiple reflection of an amplitude s in a system made up of two layers on top of a semi-infinite medium may be decomposed in the following way :

1) The amplitude s sent by the source is multiply reflected in layer [1] (see fig. 4). A total amplitude As (counted at the surface I_0) is sent back up while an amplitude A's (counted at I_1) penetrates into layer [2].

2) A's is multiply reflected in layer [2] and this process causes a total amplitude Bs (counted at I_0) to be sent back up into layer [1] .

3) then Bs is reflected into Bs by I_0 and steps 1) and 2) can be applied to (A + B)s, yielding an amplitude (A + B)Bs arriving back to I_0.

4) by continuing the same process indefinitely we get finally as a total amplitude reaching I_0:

$$(A+B)s + (A+B)Bs + (A+B)B^2 s + \cdots = \frac{A+B}{1-B}\, s$$

and the reflection coefficient

$$R_3 = \frac{A+B}{1-B} \, .$$

Clearly A equals

$$r_{1,2}\, e^{-2i\omega\tau_1}\left(1 + r_{1,2}\, e^{-2i\omega\tau_1} + \left(r_{1,2}\, e^{-2i\omega\tau_1}\right)^2 + \cdots\right)$$

$$= \frac{r_{1,2}\, e^{-2i\omega\tau_1}}{1 - r_{1,2}\, e^{-2i\omega\tau_1}} = R_2 \, .$$

The computation of B requires more attention.

Was transmitted into [2] , measured at I_1 , the amplitude

$$s(1-r_{1,2})e^{-i\omega\tau_1} + s(1-r_{1,2})(r_{1,2}e^{-3i\omega\tau_1}) + \cdots$$

$$= \frac{s(1-r_{1,2})e^{-i\omega\tau_1}}{1-r_{1,2}e^{-2i\omega\tau_1}} .$$

Was reflected upward, at I_2 , an amplitude equal to

$$\frac{s(1-r_{1,2})e^{-i\omega(\tau_1+\tau_2)}}{1-r_{1,2}e^{-2i\omega\tau_1}}\left(r_{2,3} + r_{2,3}r_{2,1}r_{2,3}e^{-2i\omega\tau_2} + \cdots\right)$$

$$= \frac{s(1-r_{1,2})r_{2,3}e^{-i\omega(\tau_1+\tau_2)}}{(1-r_{1,2}e^{-2i\omega\tau_1})(1-r_{2,1}r_{2,3}e^{-2i\omega\tau_2})} .$$

Of this quantity,

$$\frac{s(1-r_{1,2}^2)r_{2,3}e^{-i\omega(\tau_1+2\tau_2)}}{(1-r_{1,2}e^{-2i\omega\tau_1})(1-r_{2,1}r_{2,3}e^{-2i\omega\tau_2})}$$

enters [1] (measured at I_1).

And finally, arriving at I_0 , we have

$$Bs = \frac{(1-r_{1,2}^2)r_{2,3}e^{-2i\omega(\tau_1+\tau_2)}}{(1-r_{1,2}e^{-2i\omega\tau_1})(1-r_{2,1}r_{2,3}e^{-2i\omega\tau_2})}s .$$

Hence

$$R_3 = \frac{A+B}{1-B} = \frac{r_{1,2}e^{-2i\omega\tau_1} + r_{2,3}e^{-2i\omega(\tau_1+\tau_2)}}{1-r_{1,2}e^{-2i\omega\tau_1} + r_{1,2}r_{2,3}e^{-2i\omega\tau_2} - r_{2,3}e^{-2i\omega(\tau_1+\tau_2)}} .$$

Appendix 3.

The differential equation for a non-homogeneous medium.

In the most general case this equation is

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial}{\partial x_i}(\lambda \theta) + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\right] \tag{A.3.1}$$

where the indices i and j - and a 3rd one k - are used to label the three directions of space (Bullen, Ref. 3, p.21) In this equation u and x denote the displacements and the coordinates in the direction i, j, k;

θ is the dilatation and ρ , λ and μ are the density and Lamé's constants of the material.

Considering plane compressional waves propagating in the $x_3 = z$ direction in a medium whose properties ρ , λ and μ vary only in this same z direction, we have

$$\theta = \frac{\partial u_z}{\partial z},$$

$$\frac{\partial \lambda}{\partial x_1} = \frac{\partial \lambda}{\partial x_2} = \frac{\partial \mu}{\partial x_1} = \frac{\partial \mu}{\partial x_2} = 0,$$

$$u_{x_1} = u_{x_2} = 0 .$$

Then, calling u_z , u, equation A.3.1 becomes

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\rho}\frac{\partial}{\partial z}\left(\rho V^2 \frac{\partial u}{\partial z}\right) . \tag{A.3.2}$$

Were the density constant, this equation would reduce to

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial z}\left(V^2 \frac{\partial u}{\partial z}\right) . \qquad \text{(A.3.3)}$$

Separating the time and space dependances in u , one transforms equation A.3.2 into

$$Z'' + \frac{1}{\rho V^2} \frac{d}{dz}(\rho V^2) Z' + \left(\frac{\omega}{V}\right)^2 Z = 0 \qquad \text{(A.3.4)}$$

with

$$u(z,t) = Z(z)\, e^{i\omega t} .$$

It is interesting to note that equation (A.3.4) depends only on

$$\frac{d}{dz}(\log \rho V^2) \quad \text{and} \quad \frac{\omega}{V} .$$

Another form of it would be the Liouville form

$$\frac{d}{dz}\left(\rho V^2 \frac{dZ}{dz}\right) + \omega^2 \rho Z = 0 \qquad \text{(A.3.5)}$$

depending on ρV^2 and $\rho \omega^2$.

This form makes it easy to use Sturmian theorems on the number of nodal planes in a given non-homogeneous layer.

Appendix 4.

On the convergence of infinite product of matrices.

Definition 1.

The modulus of a matrix A is defined as the matrix whose element r,s (row r, line s) is the modulus of the element r,s of A.

It will be written $|A|$.

Then

$$(|A|)_{r,s} = |(A)_{r,s}| .$$

Definition 2.

An infinite product of matrices $\prod_{\gamma=1}^{\infty} (I + A_{\gamma})$ is said to be convergent if, a number $\varepsilon > 0$ being given, no matter how small, there exists a number $\ell > 0$ such that

$$\left| \prod_{\gamma=\ell}^{m} (I + A_{\gamma}) \right|_{r,s} < \varepsilon$$

for all r,s and for all m's such that $m > \ell$.

Definition 3.

An infinite product of matrices $\prod_{\gamma=1}^{\infty} (I + A_{\gamma})$ is said to be absolutely convergent if the infinite product $\prod_{\gamma=1}^{\infty} (I + |A_{\gamma}|)$ is convergent.

Lemma 1.

When three matrices with non-negative real elements A_1, A_2, A_3 are such that

$$A_1 = A_2 + A_3 ,$$

then

$$(A_2)_{r,s} \leq (A_1)_{r,s} \quad \text{for all } r,s$$

This is obvious since by the law of matrix addition,

$$(A_1)_{r,s} = (A_2)_{r,s} + (A_3)_{r,s}.$$

Lemma 2.

If four matrices A_1, A_2, A_3, A_4 with non-negative real elements are such that, for all r's and s's.

$$(A_1)_{r,s} \leqslant (A_3)_{r,s},$$

$$(A_2)_{r,s} \leqslant (A_4)_{r,s},$$

then

$$(A_1 A_2)_{r,s} \leqslant (A_3 A_4)_{r,s} \quad \text{for all r and s.}$$

In effect, by the law of multiplication of matrices,

$$(A_1 A_2)_{r,s} = (A_1)_{r,p} \cdot (A_2)_{p,s}$$

with Einstein's summation convention.

Similarly

$$(A_3 A_4)_{r,s} = (A_3)_{r,p} \cdot (A_4)_{p,s} \cdot$$

But by hypothesis, the non-negative quantities $(A_1)_{r,p}$, $(A_2)_{p,s}$, $(A_3)_{r,p}$, $(A_4)_{p,s}$ are such that

$$(A_1)_{r,p} \leqslant (A_3)_{r,p},$$

$$(A_2)_{p,s} \leqslant (A_4)_{p,s};$$

therefore $(A_1 A_2)_{r,s}$ is not greater than $(A_3 A_4)_{r,s}$ whatever r and s.

Lemma 3.

Lemma 2 may easily be extended to run as follows.

If n matrices A_1, A_2 .... A_n, $A_{n\ 1}$, ... A_{2n} with non-negative real elements are such that

$$(A_1)_{r,s} \leq (A_{n+1})_{r,s} \,,$$

$$(A_2)_{r,s} \leq (A_{n+2})_{r,s} \,,$$

$$\vdots$$

$$(A_n)_{r,s} \leq (A_{2n})_{r,s} \,,$$

then

$$(A_1 A_2 \ldots A_n)_{r,s} \leq (A_{n+1} A_{n+2} \ldots A_{2n})_{r,s}$$

whatever r and s.

Lemma 4.

If two matrices A_1 and A_2 with non-negative elements are such that

$$(A_1)_{r,s} \leq (A_2)_{r,s} \,,$$

then

$$\left(e^{A_1}\right)_{r,s} \leq \left(e^{A_2}\right)_{r,s} \,.$$

By definition of the exponential of a matrix

$$\left(e^{A_1}\right)_{r,s} = (I)_{r,s} + (A_1)_{r,s} + \frac{(A_1^2)_{r,s}}{2!} + \ldots$$

Since by hypothesis $(A_1)_{r,s} \leq (A_2)_{r,s}$, by lemma 2,

$$(A_1^2)_{r,s} \leq (A_2^2)_{r,s}$$

and by lemma 3

$$(A_1^n)_{r,s} \leq (A_2^n)_{r,s} \,.$$

All the terms of the expansion of $(e^{A_1})_{r,s}$ are smaller or equal to the corresponding term of the expansion of $(e^{A2})_{r,s}$

It follows that

$$(e^{A_1})_{r,s} \leqslant (e^{A_2})_{r,s} .$$

Lemma 5.

If the series of matrices with non-negative real elements $\sum_{\gamma=1}^{\infty} A_\gamma$ converges to a sum L, the matrix $e^{\sum_1^l A_\gamma}$ is bounded by e^L (in other words $\left(e^{\sum_{\gamma=1}^{\infty} A_\gamma} \right)_{r,s} \leq \left(e^{L} \right)_{r,s}$).

If by hypothesis the series ΣA_γ converges to a sum L, the partial sums $\sum_{\gamma=1}^{l} A_\gamma$ are such that their elements are bounded by the elements of same row and line in L:

$$\left(\sum_{\gamma=1}^{l} A_\gamma \right)_{r,s} \leq (L)_{r,s}$$

Then by use of lemma 4

$$\left(e^{\sum_{\gamma=1}^{l} A_\gamma} \right)_{r,s} \leq \left(e^{L} \right)_{r,s}$$

Theorem 1.

A sufficient condition for an infinite product $\prod_{\gamma=1}^{\infty} (I + A_\gamma)$ of matrices with non-negative elements to be convergent is that the series $\sum_{\gamma=1}^{\infty} A_\gamma$ be convergent.

In effect, noticing that

$$I + A_\gamma = e^{A_\gamma} - \sum_{n=2}^{\infty} \frac{A_\gamma^n}{n!} ,$$

we deduce by lemma 1 that

$$(I + A_{\gamma})_{r,s} \leq (e^{A_{\gamma}})_{r,s} .$$

By application of lemma 3,

$$\left(\prod_{\gamma=1}^{\ell} (I + A_{\gamma}) \right)_{r,s} \leq \left(\prod_{\gamma=1}^{\ell} e^{A_{\gamma}} \right)_{r,s}$$

or

$$\left(\prod_{\gamma=1}^{\ell} (I + A_{\gamma}) \right)_{r,s} \leq \left(e^{\sum_{\gamma=1}^{\ell} A_{\gamma}} \right)_{r,s} .$$

Now, if ℓ is allowed to go to infinity and the series $\sum_{\gamma=1}^{\infty} A_{\gamma}$ converges to a sum L, then certainly the partial sum $\sum_{\gamma=1}^{\ell} A_{\gamma}$ is bounded by L.

Because of lemma 5

$$\left(e^{\sum_{\gamma=1}^{\ell} A_{\gamma}} \right)_{r,s} \leq \left(e^{L} \right)_{r,s}$$

and

$$\left(\prod_{\gamma=1}^{\ell} (I + A_{\gamma}) \right)_{r,s} \leq \left(e^{L} \right)_{r,s} .$$

The partial product $\prod_{\gamma=1}^{\ell} (I + A_{\gamma})$ is bounded. As all the elements of A_{γ} are real non-negative, when ℓ increases the partial products increase monotonically.

Therefore the infinite product is convergent. We might add that it converges to a limit such that its element r,s is not superior to $(e^{L})_{r,s}$

Corollary to theorem 1.

A sufficient condition for an infinite product of matrices $\prod_{\gamma=1}^{\infty} (I + A_\gamma)$ to be absolutely convergent is that the series $\sum_{\gamma=1}^{\infty} A_\gamma$ be absolutely convergent.

Lemma 6.

If A and B are two matrices with real or non-real elements, then for all r's and s's

$$|A\,B|_{r,s} \leq (\,|A|\,|B|\,)_{r,s}$$

where $|A|$ and $|B|$ are the moduli of matrices A and B as defined by definition 1.

By the rules of the multiplication of matrices and with the summation convention, we have

$$(\,A\,B\,)_{r,s} = A_{r,p}\,B_{p,s}\,.$$

Then

$$|A\,B|_{r,s} = |A_{r,p}\,B_{p,s}|$$

and

$$(|A|\,|B|)_{r,s} = |A|_{r,p}\,|B|_{p,s}\,.$$

Since

$$|A_{r,p}\,B_{p,s}| \leq |A|_{r,p}\,|B|_{p,s}$$

we deduce that

$$|A\,B|_{r,s} \leq (\,|A|\,.\,|B|\,)_{r,s}\,.$$

Lemma 7.

If n matrices $A_1, A_2, \ldots, A_n$ have real or complex elements,

$$|A_1 \; A_2 \cdots A_n|_{r,s} \leq (|A_1| \; |A_2| \cdots |A_n|)_{r,s}$$

for all r's and s's.

The sign = corresponds to the case of real elements. This can be shown in exactly the same way as lemma 6.

Theorem 2.

If an infinite product of matrices is absolutely convergent, it is also simply convergent.

By hypothesis $\prod_{j=1}^{\infty} (I + |A_j|)$ converges (see definition 3).

Since

$$|I + A_j|_{r,s} \leq (I + |A_j|)_{r,s} \quad ,$$

for all r and s, by lemma 3

$$\left(\prod_{j=l}^{m} |I + A_j|\right)_{r,s} \leq \left(\prod_{j=l}^{m} (I + |A_j|)\right)_{r,s}$$

with $m > l$.

By lemma 7, we can write

$$\left|\prod_{j=l}^{m} (I + A_j)\right|_{r,s} \leq \left(\prod_{j=l}^{m} |I + A_j|\right)_{r,s}$$

and finally we obtain from this inequality and the preceding one the relation

$$\left|\prod_{j=l}^{m} (I + A_j)\right|_{r,s} \leq \left(\prod_{j=l}^{m} (I + |A_j|)\right)_{r,s} \quad ,$$

otherwise written

$$\left|\prod_{j=l}^{m} (I + A_j)\right|_{r,s} \leq \left|\prod_{j=l}^{m} (I + |A_j|)\right|_{r,s}$$

since

$$\left(\prod_{j=l}^{m} (I + |A_j|) \right)_{r,s} = \left| \prod_{j=l}^{m} (I + |A_j|) \right|_{r,s} .$$

Since the product $\prod_{j=1}^{\infty} (I + |A_j|)$ converges, by definition 2, given a number $\varepsilon > 0$, there exists a number $l > 0$ such that

$$\left| \prod_{j=l}^{m} (I + |A_j|) \right|_{r,s} < \varepsilon$$

whatever r and s and m such that $m > l$.

It follows that

$$\left| \prod_{j=l}^{m} (I + A_j) \right|_{r,s} < \varepsilon$$

for the same value of l and also whatever m such that $m > l$.

Therefore the product $\prod_{j=l}^{\infty} (I + A_j)$ is convergent.

Appendix 5.

On Lucas polynomials and their computation.

They can be defined (see Lucas, ref. 11, p. 308) by the recurrence relation

$$U_n = \gamma\, U_{n-1} - \delta\, U_{n-2}$$

where γ and δ are integers positive or negative whose product is not zero.

This definition will be here extended to any values of γ and δ, real or complex.

The initial values have to be given and will be taken by definition

$$U_1 = 1 \quad , \quad U_2 = \gamma \;.$$

If λ_1 and λ_2 are roots of the equation

$$\lambda^2 - \gamma\lambda + \delta = 0 ,$$

$\dfrac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$ is the polynomial $U_n(\gamma, \delta)$.

This is shown easily by writing the first terms n=1, n=2 and showing that the quantity $\dfrac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$ obeys the recurrence formula .

$$n = 1 \quad , \quad \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2} = 1 \;.$$

$$n = 2 \quad , \quad \frac{\lambda_1^2 - \lambda_2^2}{\lambda_1 - \lambda_2} = \lambda_1 + \lambda_2 = \gamma \;.$$

$$\vdots$$

$$n = \dot{\gamma} \quad , \quad \frac{\lambda_1^{\dot{\gamma}} - \lambda_2^{\dot{\gamma}}}{\lambda_1 - \lambda_2} = \gamma\, \frac{\lambda_1^{\dot{\gamma}-1} - \lambda_2^{\dot{\gamma}-1}}{\lambda_1 - \lambda_2} - \delta\, \frac{\lambda_1^{\dot{\gamma}-2} - \lambda_2^{\dot{\gamma}-2}}{\lambda_1 - \lambda_2} \;.$$

The general formula for $U_n(\gamma,\delta)$ is

$$U_n(\gamma,\delta) = \gamma^{n-1} - \binom{n-2}{1}\gamma^{n-3}\delta + \dots + (-1)^j\binom{n-j-1}{j}\gamma^{n-2j-1}\delta^j + \dots$$

It terminates at the term of rank $\frac{n-2}{2}$ if n is even, $\frac{n-1}{2}$ if n is odd.

This development will be verified by recurrence too.

It can be seen to be true for n = 1 and 2.

Let it be true for U_{n-1} and U_{n-2}.

$$U_n = \gamma\left[\gamma^{n-2} - \binom{n-3}{1}\gamma^{n-4}\delta + \dots + (-1)^j\binom{n-j-2}{j}\gamma^{n-2j-2}\delta^j + \dots\right]$$

$$-\delta\left[\gamma^{n-3} - \binom{n-4}{1}\gamma^{n-5}\delta + \dots + (-1)^{j-1}\binom{n-j-2}{j-1}\gamma^{n-2j-1}\delta^{j-1} + \dots\right]$$

and since

$$\binom{n-j-2}{j-1} + \binom{n-j-2}{j} = \binom{n-j-1}{j},$$

the expansion for U_n is verified.

Lucas defines another polynomial (V_n in his notations) which will be denoted W_n here such that it has the same recurrence law

$$W_n = \gamma W_{n-1} - \delta W_{n-2}$$

but different initial values

$$W_1 = \gamma \quad , \quad W_2 = \gamma^2 - 2\delta .$$

It is easily shown by the same methods as above that

$$W_n = \lambda_1^n + \lambda_2^n$$

and that

$$U_{2n} = U_n W_n \,,$$

$$W_{2n} = (W_n)^2 - 2\delta^n .$$

Also

$$U_{n+n'} = \frac{1}{2}\left[U_n W_{n'} + U_{n'} W_n \right] .$$

These last two formulae together with

$$W_n(\gamma,\delta) = \gamma^n - \frac{n}{1}\gamma^{n-2}\delta + \frac{n(n-3)}{1.2}\gamma^{n-4}\delta^2 + \dots + (-1)^j \frac{n}{j}\binom{n-j-1}{j-1}\gamma^{n-2j}\delta^j + \dots$$

and

$$W_{n+n'} = \frac{1}{2}\left[W_n W_{n'} - (\gamma^2 - 4\delta) U_n U_{n'} \right]$$

can be used to compute U_n when n is large. This process is faster than using the general expansion for U_n.

Suppose $U_{23}(\gamma,\delta)$ is needed. We can use the following equations

$$U_{23} = \frac{1}{2}\left(U_{20} W_3 + U_3 W_{20} \right) ,$$

$$U_{20} = U_{10} W_{10}$$

$$= U_5 W_5 \left[(W_5)^2 - 2\delta^5 \right] ,$$

$$W_{20} = (W_{10})^2 - 2\delta^{10}$$

$$= \left[(W_5)^2 - 2\delta^5 \right]^2 - 2\delta^{10} ,$$

$$U_5 = \gamma^4 - 3\gamma^2\delta + \delta^2,$$

$$W_5 = \gamma^5 - 5\gamma^3\delta + 5\gamma\delta^2,$$

$$U_3 = \gamma^2 - \delta,$$

$$W_3 = \gamma^3 - 3\gamma\delta;$$

and U_{23} may be computed quite rapidly.

Appendix 6.

On the series $$\left\langle \sum_{n=0}^{\infty} \begin{pmatrix} 0 & iG \\ iG & \widetilde{F} \end{pmatrix}^n \right\rangle .$$

A. 6. 1. Summation of operator series.

The operators F and G have been defined by the equations 2.2.2.14 and $\widetilde{F}$ is simply $2F$.

The notation $\langle \ \rangle$ and properties of the operators F and G have been defined or deduced also in section 2.2.2.

Equation 2.2.3.10 shows that the quantity $\frac{c_I}{a_{III}}$ is equal to the sum of the elements of the upper line of the matrix

$$\left\langle \sum_{n=0}^{\infty} \begin{pmatrix} 0 & iG \\ iG & \widetilde{F} \end{pmatrix}^n \right\rangle \tag{A.6.1.1}$$

and the quantity $\frac{d_I}{a_{III}}$ is equal to the sum of the elements of the lower line of the same matrix.

So in order to compute $\frac{c_I}{a_{III}}$ and $\frac{d_I}{a_{III}}$,we need to explicit the matrix sum A.6.1.1 in terms of $\widetilde{F}$'s and G's.

This series cannot be summed as a geometric series of ordinary matrices would since this would necessitate the use of inverses of óperator matrices. This in turn would be possible if inverse operators to $\widetilde{F}$ and G could be defined and this in general cannot be done if these inverses are to be of the same type of operators as $\widetilde{F}$ and G .

In order to see that, it can be shown that there is not in general any unit operator of the same type as $\widetilde{F}$ and G . Suppose K be such a unit operator; by definition it satisfies

$$Kx = x$$

or

$$\int_{z_1}^{z_2} kx\,dz = x \tag{A.6.1.2}$$

where k and x are two functions of z such that the integral exist.

This integral equation cannot be solved unless x is a constant. Therefore K cannot be the unit operator we wanted it to be.

And if there is no unit operator, there is no definition possible for inverse operators. Also an operator matrix does not have any eigenvalues or eigenvectors.

Therefore we must resort to other means of computing the sum of series A.6.1.1.

The first step will be to compute the form of the general term of the series

$$\begin{pmatrix} 0 & iG \\ iG & \tilde{F} \end{pmatrix}^n$$

in terms of products of $\tilde{F}$'s and G's . This will give the general form of $\frac{c_I}{a_{III}}$ and $\frac{d_I}{a_{III}}$.

Finally various formulae will be found that tie c_I and d_I.

A. 6. 2. Computation of the general term of the series.

We will here make use of the formal similarity between matrix

$$A = \begin{pmatrix} 0 & iG \\ iG & \tilde{F} \end{pmatrix} \tag{A.6.2.1}$$

and a matrix of numbers.

A matrix of numbers can be diagonalized by a " similarity transformation " and then any power of that matrix is easily computed.

This cannot be done with matrix A but we can nevertheless symbolically write it as

$$\frac{1}{\lambda_2-\lambda_1}\begin{pmatrix} iG & iG \\ \lambda_1 & \lambda_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} -i\lambda_2 G^{-1} & -1 \\ i\lambda_1 G^{-1} & 1 \end{pmatrix} \qquad \text{(A.6.2.2)}$$

which will behave as true so long as we observe the following empirical rules :

a) λ_2 and λ_1 which would be the eigenvalues if A were a matrix of numbers, will by definition be such that $\lambda_1+\lambda_2$ will be put equal to $\widetilde{F}$ and $\lambda_1\lambda_2$ and $\lambda_2\lambda_1$ equal to G^2.

This is exactly parallel to the result we would have, were $\widetilde{F}$ and G numbers. Then the characteristic equation of A would be

$$\lambda^2 - \widetilde{F}\lambda + G^2 = 0$$

and we would have

$$\lambda_1+\lambda_2 = \widetilde{F},$$

$$\lambda_1\lambda_2 = G^2.$$

b) the multiplications in the matrix product A.6.2.3 must be done in order and $\lambda_1\lambda_2 G^{-1}$ will be put equal to G.

c) the division by $\lambda_2-\lambda_1$ may be done any how, from the right or left.

These conventions will, when observed, make the product A.6.2.3 , otherwise absolutely meaningless, represent the operator matrix A.

If they are applied to the product

$$\frac{1}{\lambda_2-\lambda_1}\begin{pmatrix} iG & iG \\ \lambda_1 & \lambda_2 \end{pmatrix}\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}\begin{pmatrix} -i\lambda_2 G^{-1} & -1 \\ i\lambda_1 G^{-1} & 1 \end{pmatrix}, \quad (A.6.2.3)$$

we will obtain the matrix

$$\begin{pmatrix} -G\,\dfrac{\lambda_2^{n-1}-\lambda_1^{n-1}}{\lambda_2-\lambda_1}\,G & iG\,\dfrac{\lambda_2^n-\lambda_1^n}{\lambda_2-\lambda_1} \\ i\,\dfrac{\lambda_2^n-\lambda_1^n}{\lambda_2-\lambda_1}\,G & \dfrac{\lambda_2^{n+1}-\lambda_1^{n+1}}{\lambda_2-\lambda_1} \end{pmatrix}. \quad (A.6.2.4)$$

This can be written

$$\begin{pmatrix} -G\,U_{n-1}(\tilde{F},G^2)\,G & iG\,U_n(\tilde{F},G^2) \\ i\,U_n(\tilde{F},G^2)\,G & U_{n+1}(\tilde{F},G^2) \end{pmatrix} \quad (A.6.2.5)$$

where $U_n(\tilde{F},G^2)$ is the Lucas polynomial of order n of $\tilde{F}$ and G^2. Lucas polynomials of numbers are discussed in Appendix 5 where the original definition of Lucas's was extended to complex numbers. Here a further extension is done by considering polynomials of operators $\tilde{F}$ and G^2. The recurrence relation is

$$U_n = \tilde{F}\,U_{n-1} - G^2 U_{n-2}$$

where the multiplications must be done in that order. This is a convention that has to be observed in order to assign a meaning to the matrix A.6.2.5 , as well as the following further rule.

The terms of the form

$$\binom{n-j-1}{j} \tilde{F}^{\,n-2j-1}\, G^{2j}$$

obtained in the expansion of the Lucas polynomials (and they will all be of this type) must be interpreted as the sum of all the permutations of $n - 2j - 1$ $\tilde{F}$'s with $j(G^2)$'s.

As examples, $2\tilde{F}G^2$ must be put equal to $\tilde{F}G^2 + G^2\tilde{F}$, $3\tilde{F}G^4$ to $G^2\tilde{F}G^2 + \tilde{F}G^4 + G^4\tilde{F}$, etc...

This interpretation is consistently parallel to the expansion we would obtain, were $\tilde{F}$ and G^2 numbers, for the coefficient $\binom{n-j-1}{j}$ is precisely equal to the number of permutations of $n - 2j - 1$ $\tilde{F}$'s with $j(G^2)$'s.

This final convention has the merit that by its use the matrix A.6.2.5 becomes " equal " to the matrix A^n.

It is easy to verify it for small values of n and by recurrence for any value of n (*) , thereby providing a general check and justification for the step taken in writing A^n in the form A.6.2.5.

From this we can immediately write

$$\left\langle \sum_{n=0}^{\infty} \begin{pmatrix} 0 & iG \\ iG & \tilde{F} \end{pmatrix}^{n} \right\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{n=1}^{\infty} \left\langle \begin{pmatrix} -GU_{n-1}G & iGU_n \\ iU_nG & U_{n+1} \end{pmatrix} \right\rangle \qquad (A.6.2.6)$$

(*) Note that by this recurrence demonstration, A^n may be shown to be equal to $A\,A^{n-1}$ and not to $A^{n-1}.A$. For under the form A.6.2.5 A^{n-1} and A do not commute. This is due to the convention on the recurrence relation for Lucas polynomials of $\tilde{F}$ and G for otherwise A can be shown to commute with any of its powers.

with the previously described conventions, which yield the following useful formulae

$$\frac{c_{\mathrm{I}}}{a_{\mathrm{III}}} = 1 + \sum_{n=1}^{\infty} \left\langle -G\, U_n(\tilde{F}, G^2)\, G + i\, G\, U_n(\tilde{F}, G^2) \right\rangle , \quad \text{(A.6.2.7a)}$$

$$\frac{d_{\mathrm{I}}}{a_{\mathrm{III}}} = 1 + \sum_{n=1}^{\infty} \left\langle i\, U_n(\tilde{F}, G^2)\, G + U_{n+1}(\tilde{F}, G^2) \right\rangle , \quad \text{(A.6.2.7b)}$$

with

$$U_j = 0 \ , \qquad \text{when} \quad j \leqslant 0 \ ,$$

$$U_1 = 1 \ ,$$

$$\vdots$$

$$U_n = \tilde{F} U_{n-1} - G^2 U_{n-2} \ , \qquad n \geqslant 1 \ .$$

From this, and noticing that $U_n(\tilde{F}, G^2)$ is an even function of G, if C_{2q}, C_{2q+1}, D_{2q}, D_{2q+1} (q being an integer) are respectively the sums of the terms of $\frac{c_{\mathrm{I}}}{a_{\mathrm{III}}}$ and $\frac{d_{\mathrm{I}}}{a_{\mathrm{III}}}$ which contain $2q$ and $2q+1$ G's, we have

$$C_{2q} = (-1)^q \left\langle G \left[\sum_n \binom{n-q-1}{q-1} \tilde{F}^{\,n-2q}\, G^{2(q-1)} \right] G \right\rangle ,$$

$$n \geqslant 2q > 0 \ , \quad \text{(A.6.2.8a)}$$

$$C_{2q+1} = (-1)^q\, i \left\langle G \left[\sum_n \binom{n-q-1}{q} \tilde{F}^{\,n-2q-1}\, G^{2q} \right] \right\rangle ,$$

$$n \geqslant 2q+1 \geqslant 1 , \quad \text{(A.6.2.8b)}$$

$$D_{2q} = (-1)^q \left\langle \sum_n \binom{n-q}{q} \tilde{F}^{n-2q} G^{2q} \right\rangle ,$$

$$n \geqslant 2q > 0 \ , \qquad \text{(A.6.2.8c)}$$

$$D_{2q+1} = (-1)^q i \left\langle \left[\sum_n \binom{n-q-1}{q} \tilde{F}^{n-2q-1} G^{2q} \right] G \right\rangle ,$$

$$n \geqslant 2q+1 \geqslant 1 \ , \qquad \text{(A.6.2.8d)}$$

These quantities are related by the following equations

$$G \, D_{2q+1} = C_{2q+1} \, G \ , \qquad q \geqslant 0 \ , \qquad \text{(A.6.2.9)}$$

$$C_{2q+2} = - G \, D_{2q} \, G \ , \qquad q > 0 \ , \qquad \text{(A.6.2.10)}$$

$$D_{2q+1} = i \, D_{2q} \, G \ , \qquad q > 0 \ . \qquad \text{(A.6.2.11)}$$

Also

$$\frac{c_I}{a_{III}} = i \left\langle G \frac{d_I}{a_{III}} \right\rangle + 1 \qquad \text{(A.6.2.12)}$$

may be derived directly from the equations A.6.2.7 .

A general form for $\frac{c_I}{a_{III}}$ and $\frac{d_I}{a_{III}}$ may be written which is a function of G and $\frac{1}{1-\tilde{F}}$ only, $\frac{1}{1-\tilde{F}}$ meaning $\sum_{j=0}^{\infty} \tilde{F}^j$.

$$\frac{c_I}{a_{III}} = \left\langle \left[\sum_{q=0}^{\infty} (-1)^q \left(G \frac{1}{1-\tilde{F}} G \right)^q \right] \left(1 + i \, G \frac{1}{1-\tilde{F}} \right) \right\rangle \ , \qquad \text{(A.6.2.13a)}$$

$$\frac{d_I}{a_{III}} = \left\langle \frac{1}{1-\tilde{F}} \left[\sum_{q=0}^{\infty} (-1)^q \left(G^2 \frac{1}{1-\tilde{F}} \right)^q \right] \left(1 + i \, G \right) \right\rangle \qquad \text{(A.6.2.13b)}$$

where, for example, $\left(G^2 \frac{1}{1-\tilde{F}}\right)^q$ means

$$\underbrace{G^2 \frac{1}{1-\tilde{F}} \quad G^2 \frac{1}{1-\tilde{F}} \quad \cdots\cdots \quad G^2 \frac{1}{1-\tilde{F}}}_{q \text{ times}}$$

These two expressions may be checked to be true by verifying that they give for each quantity C_{2q} , C_{2q+1} , D_{2q} , D_{2q+1} exactly the same number of terms as formulæ A.6.2.8 predict without omission nor repetition.

It will suffice to give one example.

D_{2q} is given by A.6.2.13b as

$$(-1)^q \left\langle \frac{1}{1-\tilde{F}} \left(G^2 \frac{1}{1-\tilde{F}}\right)^q \right\rangle$$

or

$$(-1)^q \left\langle \left(\sum_0^\infty \tilde{F}^\gamma\right) \underbrace{G^2 \sum_0^\infty \tilde{F}^\gamma \quad G^2 \sum_0^\infty \tilde{F}^\gamma \cdots\cdots G^2 \sum_0^\infty \tilde{F}^\gamma}_{q \text{ times}} \right\rangle \tag{A.6.2.14}$$

The terms which contain $n - 2q$ $\tilde{F}$'s are $\binom{n-q}{q}$ in number as can be verified for the first values of q and n and by recurrence for higher values. Therefore the expression A.6.2.14 has all the terms given by A.6.2.8c and none other.

Formulæ A.6.2.9 to 12 may be checked on the forms A.6.2.13.

Further relations may be obtained immediately by inspection from the equations A.6.2.13 :

$$\frac{i}{1-\tilde{F}} G^2 D_{2q-1} = D_{2q} G \quad , \tag{A.6.2.15}$$

$$C_{2q+1} = i\, C_{2q}\, G\, \frac{1}{1-\tilde{F}} \quad , \tag{A.6.2.16}$$

$$C_{2q+1} = i\, C_{2q}\, G\, \frac{1}{1-\tilde{F}} \quad . \tag{A.6.2.17}$$

LITERATURE

1. Sommerfeld, A. (1950) Vorlesungen über theoretische Physik, Bd IV, Optik, 1. Aufl., Dieterisch, Wiesbaden.
2. Lord Rayleigh (1912) Propagation of waves through a stratified medium, with special reference to the question of reflection, Proc. Roy. Soc. London, vol. 86A, p. 207-226.
3. Bullen, K. E. (1953) An introduction to the theory of seismology, 2nd. ed., Cambridge University Press.
4. Brillouin, L. (1953) Wave propagation in periodic structures, 2nd. ed., Dover, New York.
5. Bewley, L. V. (1930) (discussion on a paper), Trans. A.I.E.E., vol. 49, p. 1442.
6. Abelès, F. (1950) Recherches sur la propagation des ondes électromagnétiques sinusoidales dans les milieux stratifiés. Application aux couches minces. Ann. de Physique, t. 5, p. 596-640, 706-782.
7. Altenburg, K., Kästner, S. (1953) Wellenausbreitung in geschichteten Medien bei senkrechtem Einfall und die Anwendung auf Leitungstheorie, elektrische Wellen, Optik, Akustik, Wellenmechanik sowie mechanische und elektrische Vierpolketten, Ann. d. Physik, 6. Folge, Bd 13, Heft 1/5, p. 1-43. (Bibliography p. 41-43).
8. Les propriétés optiques des lames minces, Colloque international sur les lames minces solides, 19-23 April 1949 in Marseille, J. de Physique, t. 11, no.7 (1950) p. 305-481 (extensive bibliography p. 461-479).
9. Wolter, H. (1956) Optik dünner Schichten, Handbuch der Physik, Bd. XXIV, 1. Aufl. Springer, Berlin, p. 461-554.
10. Knopp, K. (1922) Theorie und Anwendung der unendlichen Reihen, 1. Aufl., Springer, Berlin.
11. Lucas, E. (1891) Théorie des nombres, 1ère éd., Gauthier-Villars, Paris.

II. THE DETERMINATION OF SEISMIC VELOCITIES IN LAYERS WITH NON-PARALLEL INTERFACES.

INTRODUCTION

The problem of the determination of the velocities in homogeneous isotropic layers separated by plane interfaces has been solved exactly when these are parallel (Dürbaum, Ref. 1; Dix, Ref. 2).

When they are not parallel, approximate methods had to be used.

However such iteration methods as were described by Dürbaum and Dix are not necessary since velocities and dips can be computed by exact formulas to be given in the present paper.

GENERAL FORMULAS

- Hypotheses.

The only case where all strikes are parallel will be considered.

The interfaces are assumed to be plane surfaces and all layers isotropic and homogeneous.

- First layer determination.

By the use of Pflueger's (Ref. 3) formula the velocity can be determined through the knowledge of times and distances at the surface.

Favre's formula (Dix, Ref. 2) makes the calculation of the dip possible from time measurements only.

Finally the position of the interface can easily be determined.

- Determination of any layer knowing the layers above.

The principle of the present method is to determine the layers one by one in the order in which they send reflections to a given set-up of a shot-point and a surface receiver.

Let the velocities in the $n-1$ first layers and the dips and positions of the $n-1$ first interfaces be known.

By shooting reversals the angles at which any ray reflected on the n-th interface starts from and arrives back at the surface may be known (Dix, Ref. 2).

Therefore the branches of any one ray going down from the surface toward the n-th interface and up from this interface toward the surface can be either constructed on a section or computed by methods analogous to those of Dix and Lawlor (Ref. 4). (See also Dürbaum, Ref. 5).

Also the travel-time of any ray inside the n-th layer can be computed.

We are then confronted with the problem : how to find the relative dip θ of the interface I_n with respect to I_{n-1}, the position of point O (see fig. 1) and the velocity V of the n-th layer knowing the travel-times T_1, T_2, etc... of rays between A_1 and A'_1 , A_2 and A'_2, etc... of known position on I_{n-1}.

If A_i and A'_i are corresponding points of the two families of points A_1 , A_2 ... and A'_1 , A'_2 ... and if $x_i = OA_i$ and $x'_i = OA'_i$, we have with $m = \tan\theta$

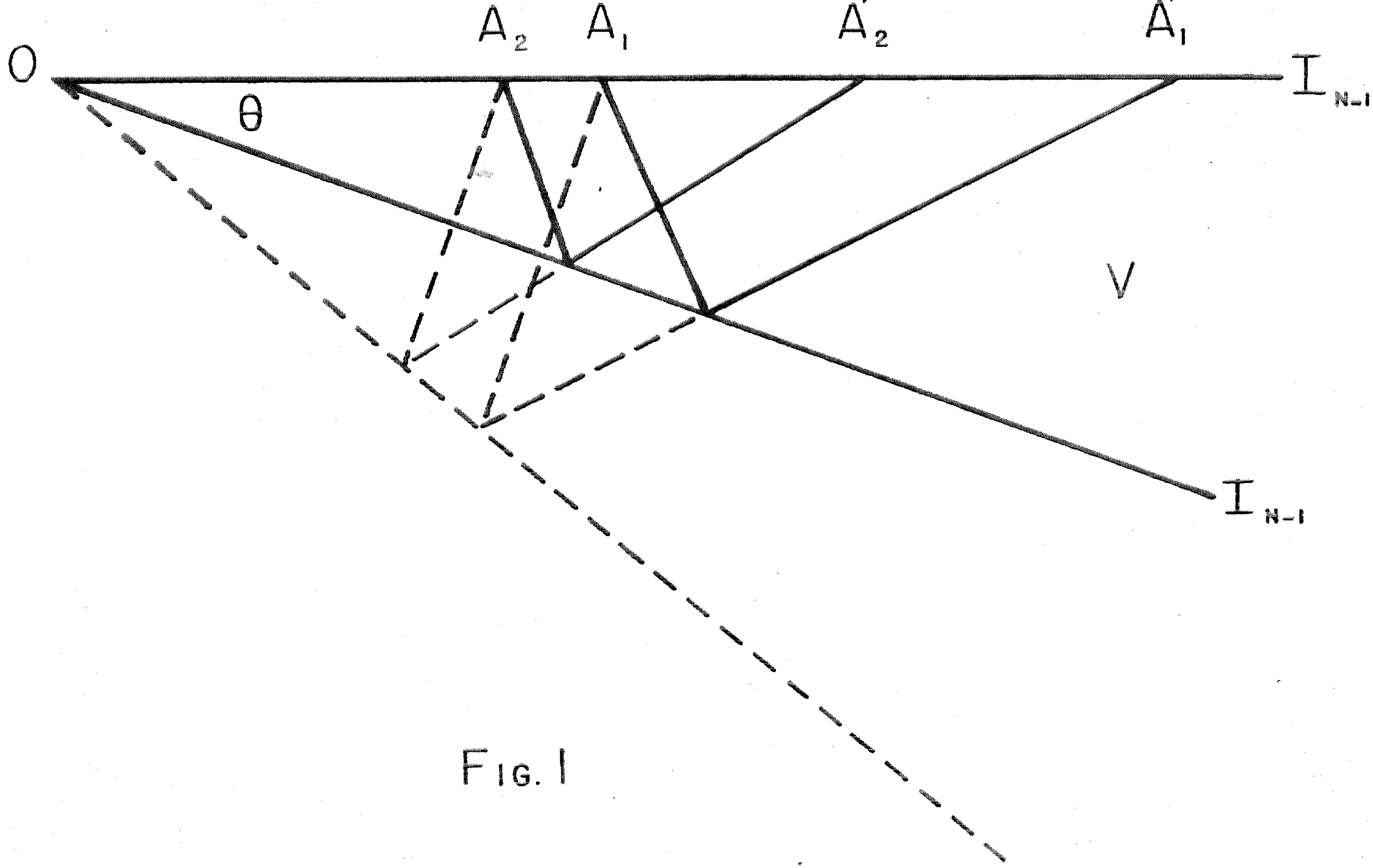

FIG. 1

$$V^2 T_i^2 = 4 m^2 x_i^2 / (1+m^2)^2 + \left[x_i (1-m^2)/(1+m^2) - x'_i \right]^2 . \quad (1)$$

Taking one of the points A as a reference point and calling X its abscissa OA , equation 1 can be rewritten as

$$V^2 T_i^2 = (X + \xi_i)^2 \alpha^2 + \delta_i \left[\delta_i + (X + \xi_i) \alpha^2 \right] \quad (2)$$

where

$$\xi_i = x_i - X ,$$

$$\alpha^2 = 4 m^2 / (1 + m^2)$$

and

$$\delta_i = x'_i - x_i .$$

Three such equations as (2), where the unknowns are V , α and X , will suffice to determine these 3 quantities knowing T_i, ξ_i , δ_i .

Such a system of equations will be written with point A, as a reference $(\xi_1 = 0)$. It will be chosen such that rays 1 and 3 are normally reflected on I_n (fig. 2). This simplifies the equations but, what is more important, just one surface set-up will furnish all the necessary information.

As a matter of fact, although one set-up of two surface points is enough, it will have to be shot once at each end. Furthermore additional receivers will have to be laid on the line near the two shot-points in order to give a time-distance curve from which to get the angles of arrival of the rays at the surface.

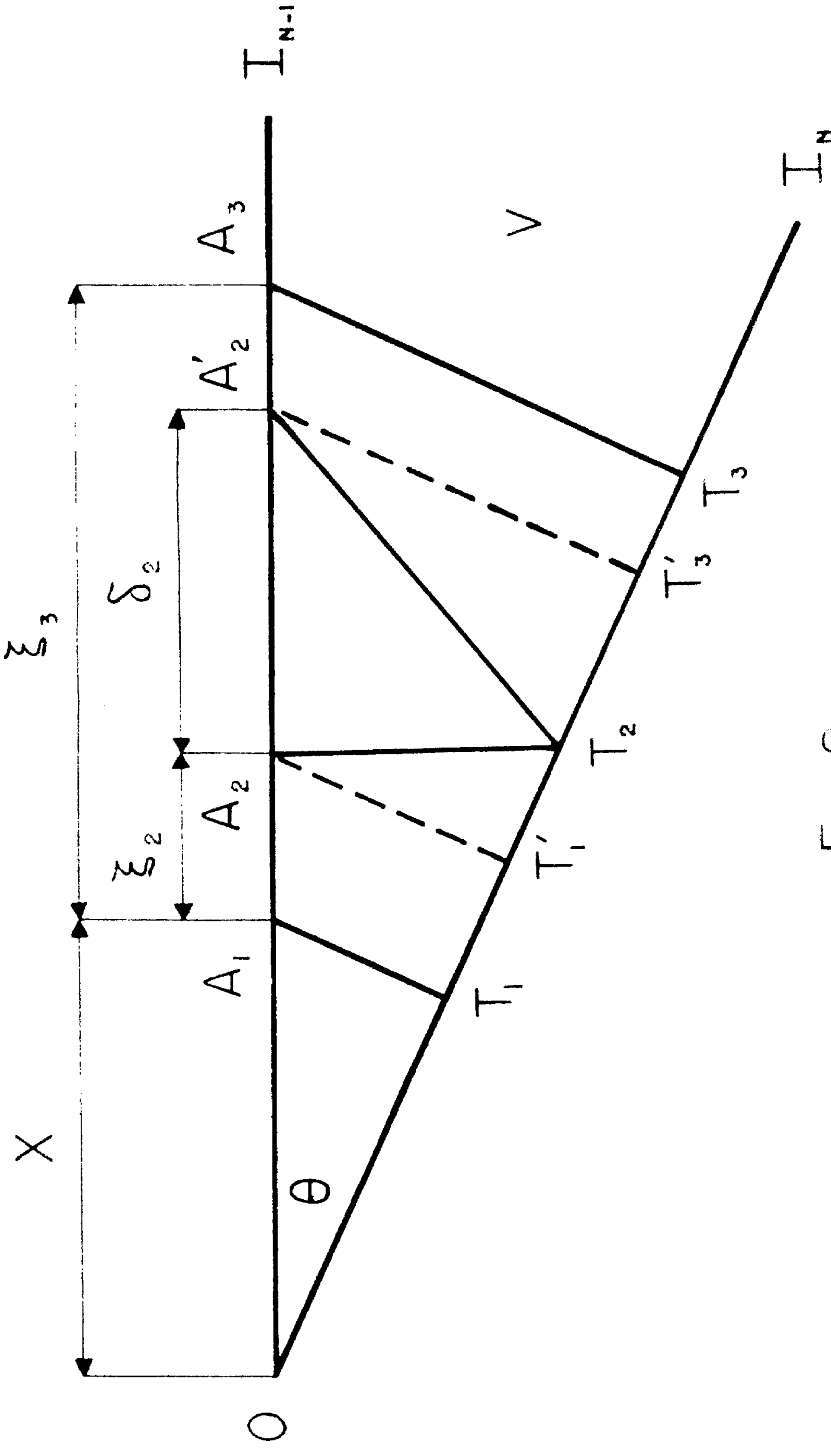

FIG. 2

The system of equations is

$$V^2 T_1^{\,2} = \alpha^2 X^2 , \tag{3}$$

$$V^2 T_2^{\,2} = \alpha^2 \left[X^2 + (2\xi_2 + \delta_2) X + \xi_2 (\xi_2 + \delta_2) \right] + \delta_2^2 , \tag{4}$$

$$V^2 T_3^{\,2} = \alpha^2 \, (X + \xi_3)^2 \tag{5}$$

since $\delta_1 = \delta_3 = 0$ (rays reflected along themselves).

From the system of equations 3, 4, 5 the following can be derived :

$$X = \xi_3 T_1 / (T_3 - T_1) , \tag{6}$$

$$V^2 = X^2 \delta_2^2 / \left[(T_2^2 - T_1^2) X^2 - (2\xi_2 + \delta_2) T_1^2 X - \xi_2 (\xi_2 + \delta_2) T_1^2 \right] , \tag{7}$$

$$\sin\theta = V (T_3 - T_1) / 2\xi_3 = V T_1 / 2X \tag{8}$$

or expressing all quantities in terms of T_1, T_2, T_3, X, ξ_2, ξ_3 and δ_2 ,

$$V^2 = \xi_3^2 \delta_2^2 / \left[\xi_3^2 (T_2^2 - T_1^2) - \xi_3 (2\xi_3 + \delta_2) T_1 (T_3 - T_1) - \xi_2 (\xi_2 + \delta_2) (T_3 - T_1)^2 \right] \tag{9}$$

and

$$\sin^2\theta = \delta_2^2 (T_3 - T_1)^2 / 4 \left[\xi_3^2 (T_2^2 - T_1^2) - \xi_3 (2\xi_3 + \delta_2) T_1 (T_3 - T_1) - \xi_2 (\xi_2 + \delta_2) (T_3 - T_1)^2 \right] . \tag{10}$$

In order to get X, V and θ , use can be made of equations 6, 7 and 8, as follows.

Equations 6 and 8 give X and θ easily and then we can extend Pflueger's formula to obtain V.

Let T'_1 and T'_3 be the times in layer n for a normal reflection on I_n of fictitious rays going through A_2 and A'_2 (fig. 2)

The velocity is given by

$$V^2 = \delta_2^{\,2} / (T_2^{\,2} - T'_1 \, T'_3) \tag{11}$$

and the fictitious times T'_1 and T'_3 are

$$T'_1 = (1 + \xi_2 / X)T_1 \tag{12}$$

and

$$T'_3 = (1 + (\xi_2 + \delta_2)/ X)T_1 = T'_1 + T_1 \, \delta_2/X \tag{13}$$

As can easily be verified, equation 11 is of course but another way of writing (7). This form is the basis of the slide rules construction (see fig. 3).

The rule with two sliders gives the quantity

$$A = T'_1 T'_3$$

by following the procedure here described : Put the arrow of the lower slider (the one that bears the scale of $1 + \frac{\xi + \delta}{X}$) opposite the desired value of $1 + \frac{\xi}{X}$ on the very lowest scale. Then put the arrow of the upper slider (with the scale of T_1) opposite the desired value of $1 + \frac{\xi + \delta}{X}$ and read the value of A on the higher scale opposite the value of T_1 on the upper slider.

When this is done, use the other slide rule. Put the arrow of the slider opposite the value of $T_2^{\,2} - A$, correspon-

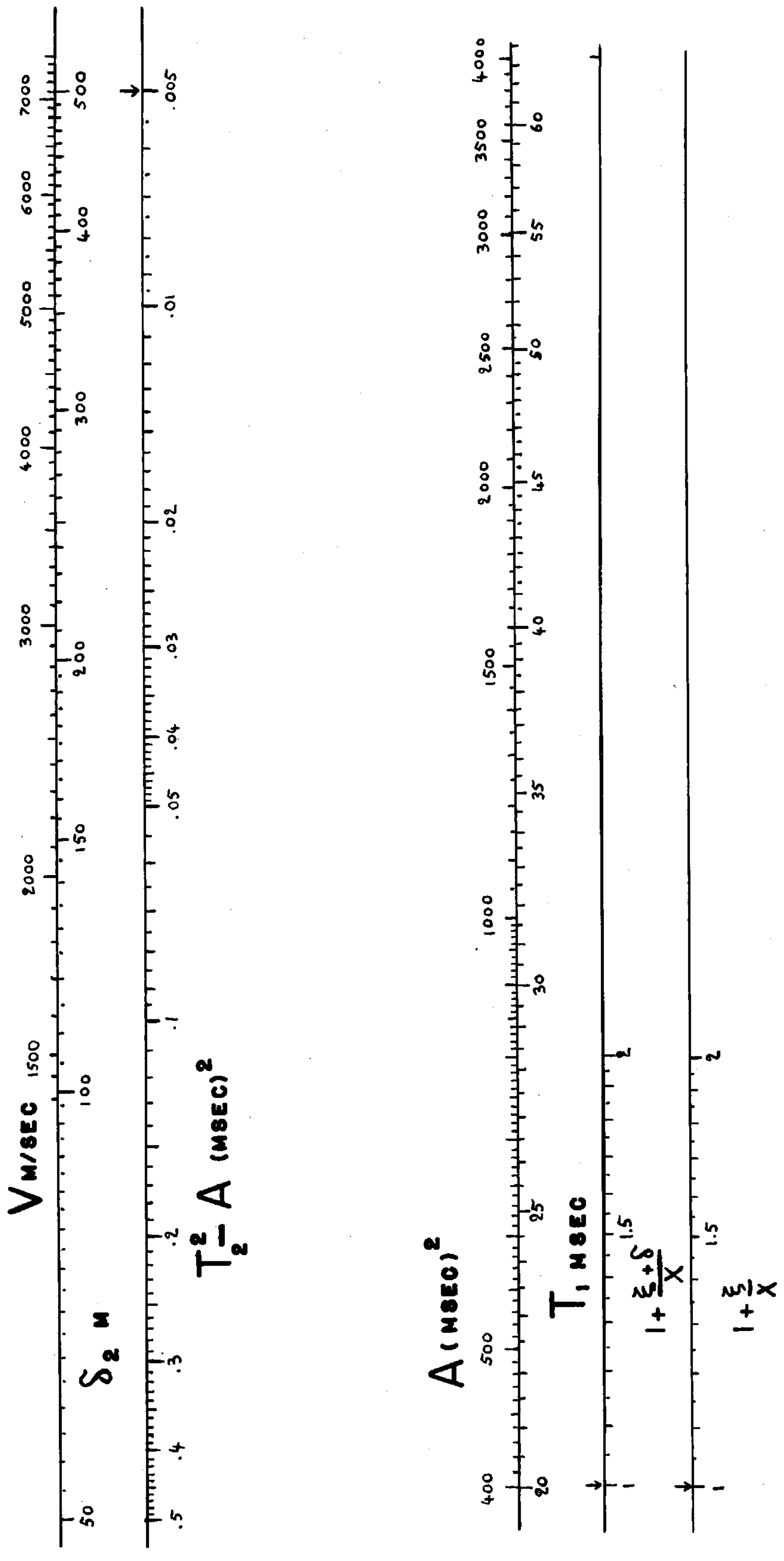

FIG. 3

ding to the given T_1 and the A just calculated and opposite the value of δ_2 on the slider read V on the upper scale.

If a value of a given variable is not on the corresponding scale, one must use the homogeneity properties of formula 11. For instance, suppose δ_1 equals 600m., use $\frac{600m}{3} = 200m$ which is on the scale, read V opposite and multiply this value by 3 .

LIMITATIONS OF THE METHOD

- Accuracy of the method itself.

The errors made in the determination of X , θ and V of a layer depend on the errors made on all the previously calculated layers. This is a very serious disadvantage.

However, when the geological section contains a series of consecutive parallel interfaces the Dix-Dürbaum method can be used for this series with benefit.

Namely, the error on the velocity in any of the layers in the conformable series will be independent of the errors on the velocities in the layers of the same series above. However it will still depend on the errors on the velocities in the layers above that series.

We saw that, at the beginning of the procedure for the determination of each layer, we have to trace 3 rays reflected on the lower interface which limits that layer. Two of these rays are parallel since both of them fall perpendicularly on I_n.

Use can be made of this fact by employing the formula of J. de Caleya (Ref. 6) :

$$2\xi_{3,i} \sin s_i = V_i \ (t_{n,1} - T_{n,3}) \tag{14}$$

where $\xi_{3,i}$ is the distance between the two parallel rays on the interface I_{i-1} , s_i is the angle of the rays in medium i with the normal to I_{i-1} , V_i is the velocity in layer i , $t_{n,1}$ and $t_{n,3}$ the total times along the two rays (numbered 1 and 3) that are going to be reflected normally on I_n.

Applying the formula 14 to the first layer, we get

$$2 \ D \sin s_1 = V_1 \ (t_{n,1} - t_{n,3}) \tag{15}$$

where D is the distance between the shot-points.

This formula can be used for the determination of s , instead of or as a check to the method using the $x^2 - T_x^2$ graph (Dix, Ref. 7, p. 125; first proposed by Green in 1938, Ref. 8)

But in order to determine the angles of departure and arrival of the ray number 2 not reflected normally,no other method than the $x^2 - T_x^2$ graph is available yet.

This method requires for a good accuracy that, by the use of Hansen's field technique, as large a distance D be used as possible (Dix, Ref. 2). But a limit has to be set to it because of the limitations imposed by the hypotheses we set ourselves first. The larger D , the more influence the curvature of the interfaces will have. Furthermore, the more danger there will be not to find good correlations from record to record.

As all errors on previously calculated layers influence the calculation of any layer, an error estimation becomes rapidly involved.

However, an order of magnitude of the errors to be expected can be computed without difficulty for the first layer.

Maximum errors have been computed for the determination of the first layer in various cases. They are given in the following table where D was taken equal to 500m and the <u>maximum</u> errors on D and the times were supposed to be 1m. and .010 sec. respectively (but the times T_1 , T_2 and T_3 do not differ by more than 2 milliseconds).

The figures in each block indicate from top to bottom respectively the maximum error on the velocity, the distance X and the angle θ .

V = 1500m/sec

X \ θ	5°	10°	30°
500m	70m/sec 122m 16'	100m/sec 62m 18'	165m/sec 23m 20'
1000m	180m/sec 270m 13'	340m/sec 140m 13'	
5000m	330m/sec 455m 12'		

$V = 4500 m/sec$

X \ θ	5°	10°	30°
500m	625m/sec 365m 49'	885m/sec 185m 55'	2100m/sec 65m 61'
1000m	1595m/sec 795m 39'	3010m/sec 405m 39'	
5000m	2950m/sec 1330m 35'	5790m/sec 680m 35'	

The blank blocks were left so because they corresponded to reflection times too large for the corresponding events to be recorded (times greater than 2 seconds).

It may be seen that other things being equal, the accuracy on V is better when the dip is small, <u>or</u> X small, <u>or</u> V itself small (that is to say if X and V are kept constant, for small θ , or if V and θ are constant, for small X , or if X and θ are kept constant, for small V). Naturally when V <u>and</u> X <u>and</u> θ are small, the accuracy on V is the best.

The accuracy on X is higher when V is smaller, θ larger, X smaller.

The accuracy on θ is higher for smaller V , larger X , smaller θ .

We may presume that these qualitative results can be extended safely enough to the determination of a second layer.

If this method were used, an error estimation should be done because in inadequate cases, the error may be quite large. For example, see in the case of one layer, for $V = 4500m/sec$ and $X = 5000m$, the error on the velocity is larger than 100%.

It would be difficult to give a general idea of the values of the maximum errors to be expected even in the rather simple case of two layers because of the number of parameters involved.

- Limitations due to hypotheses.

The present method is applicable only when the hypotheses for which it is valid are verified in the geological problem to be explored.

First it was assumed that all strikes are parallel and that the profiles are shot in the plane that contains the dips. If this is not true the method can be extended but to the cost of increased complication and time-consumption.

Secondly the interfaces should be plane and the layers isotropic. This is a weak point of the method since curvature as well as smooth lateral velocity changes may be expected to be quite general. Moreover if Hansen's field set-up is to be used good correlations should be available.

However it is suggested that the procedure above described might be of some help in the cases where layer interfaces are rendered non-parallel either by some sedimentary pinch-out effect or by a thinning of beds due to compaction (Athy, Ref. 9) or to tectonic causes (such as for example structures related to salt dome; see an example in Deecke, Ref. 10).

Possibly also, it might be applied when salt layers have introduced dysharmonic folding of the type of, say, the S. W. Iran oil fields (O' Brien, Ref. 11) although in that particular case it seems rather dubious that the salt layers have retained enough homogeneity for this method to apply.

But in most cases the interfaces are very unlikely to be plane on large distances. It seems then that the present method cannot be used without much caution if meaningful results are to be obtained.

LITERATURE

1. Dürbaum, H, 1954, Zur Bestimmung von Wellengeschwindigkeiten aus reflexionsseismischen Messungen : Geophysical Prospecting, vol. 2, p. 151-167.
2. Dix, C. H., 1955, Seismic Velocities from surface measurements : Geophysics, vol. 20, p. 68-86.
3. Pflueger, J. C., 1954, Delta-T formula for obtaining average seismic velocity to a dipping reflecting bed : Geophysics, vol. 19, p. 339-341.
4. Dix, C. H., and Lawlor, R. C., 1943, Computation of seismic dips below an unconformity : Geophysics, vol. 8, p. 105-118.
5. Dürbaum, H., 1953, Possibilities of constructing true ray paths in reflection seismic interpretation : Geophysical Prospecting, vol. 1, p. 125-139.
6. de Caleya, J., Closs, H., Dürbaum, H., 1952, Zur Berücksichtigung der Brechung in der Reflexionsseismik : Erdöl und Kohle, vol. 12, p. 765-768.
7. Dix, C. H., 1952, Seismic prospecting for oil : Harper and Bros., New York.
8. Green, C. H., 1938, Velocity determinations by means of reflection profiles : Geophysics, vol. 3, p. 295-305.
9. Athy, L. F., 1934, Compaction and its effect on local structure : Problems of petroleum geology, Amer. Assoc. Petrol. Geol., p. 811-823.
10. Deecke, H., 1949, Schichtfolge und Tektonik des Rotliegend-Zechstein-Salzaufbruches von Heide : Erdöl und Tektonik in Nordwestdeutschland, Hannover, p. 173-190.
11. O'Brien, C.A.E., 1948, Tectonic problems of the oilfield belt of S. W. Iran : Eighteenth international geological congress, Part 6, The geology of petroleum, p. 45-58.