THE ANALYSIS OF A NONLINEAR DIFFERENCE EQUATION OCCURRING IN DYNAMICAL SYSTEMS

Thesis by

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ABSTRACT

A difference equation with a cubic nonlinearity is examined. Using a phase plane analysis, both quasi-periodic and chaotically behaving solutions are found. The chaotic behavior is investigated in relation to heteroclinic and homoclinic oscillations of stable and unstable solution manifolds emanating from unstable periodic points. Certain criteria are developed which govern the existence of the stochastic behavior. An approximate solution technique is developed giving expressions for the quasi-periodic solutions close to a stable periodic point and the accuracy of these expressions are investigated. The stability of the solutions is examined and approximate local stability criteria are obtained. Stochastic excitation of a nonlinear difference equation is also considered and an approximate value of the second moment of the solution is obtained.

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1. INTRODUCTION

1.1 Difference Equations

In the construction of mathematical models of physical systems it is usually assumed that all of the independent variables, such as time and space, are continuous. This assumption normally leads to a realistic and justified approximation of the real variables of the system. However, we regularly encounter systems for which this continuous variable assumption cannot be made.

Systems in which one or more variables are inherently discrete are in areas such as population growth [1,2], digital control [3], digital communication networks [4], quantum mechanics, strong focusing of orbiting particles in accelerators [5], and delayed feedback oscillation as in laser emission pulsation [6]. Due to their discrete character, these systems must be modelled by the use of difference equations or mappings. The step sizes of the discrete variables are large enough to prevent the use of differential equations.

Another important use of difference equations arises from the reduction of differential equations to mappings. This reduction is accomplished by considering the theory of flows [7] of differential equations and maps of the type $M : \Gamma_c \rightarrow \Gamma_c$, where Γ_c is a cross-section of the flow of the corresponding differential equations. For systems with two degrees of freedom the map M is called a Poincare map [8,9]. M can always be looked upon as a difference equation.

Through the consideration of Poincare maps, problems such as the motion of satellites [10,11], and the motion of coupled nonlinear oscillators [12] can be analyzed using nonlinear difference equations. Of particular interest, in the use of difference equations, is the seemingly successful modelling of turbulence by the Lorenz equation [13], achieved by truncating the Navier Stokes equations. Through the tuning of the parameters of the Lorenz equation it is possible to obtain chaotic behavior of the solution [14,15,16,17], i.e., turbulence. This behavior can readily be studied by reducing the differential equation to a difference equation by considering a Poincare map of the flow.

In numerical analysis, differential equations are converted to difference equations which in turn can be solved by the use of a digital computer. This conversion can be accomplished through a wide range of discretization methods such as forward, central and backward difference, and the trapezoidal rule. In order that the solution of the difference equation approximate the solution of the differential equation, with acceptable accuracy, the step sizes of the discrete independent variables are usually taken to be small. By letting the step sizes approach zero the behavior of the solution of the difference equations can be made to approach the behavior of the corresponding differential equation. No connections between difference and differential equations are made in this dissertation and hence the step size of the difference equation is left to assume values which are far removed from zero.

To summarize, the reason for studying nonlinear difference equations lies in the fact that they are fascinating mathematical problems in their own right and in addition form important mathematical models for real dynamical systems.

1.2 Statement of the Problem

In this study we will consider second order scalar nonlinear difference equations of the type

$$x_{n+1} = f(x_n, x_{n-1}, n) \tag{1.1}$$

with initial conditions x_0, x_1, where x_n*memberR oppA* integers n.

In the main part of this dissertation $f(x_n, x_{n-1}, n)$ is taken to be

$$f(x_n, x_{n-1}, n) = a x_n - x_{n-1} - b x_n^3 \tag{1.2}$$

The cubic nonlinearity was chosen as it can be characterized as the simplest, odd, analytic nonlinearity. Whenever $f(x_n, x_{n-1}, n)$ is chosen to be given by (1.2) we will call equation (1.1) the discrete Duffing's equation, due to the obvious similarities with the important nonlinear differential Duffing's equation.

In Chapter 2 we conduct a phase plane analysis of the discrete Duffing's equation. The existence and stability of the equilibrium points of equation (1.1) is analyzed in detail. Solution trajectories about a stable fixed point, a center, are examined. It is shown that close to a center, the motions are stable quasi-periodic orbits; however as the initial point is moved farther and farther away from the center, chaotic behavior frequently occurs. This apparent stochastic behavior of the solution is examined in relation to the homoclinic and heteroclinic oscillations of the stable and unstable manifolds emanating from unstable equilibrium or periodic points. Certain criteria are developed, governing the existence of this stochastic behavior.

In Chapter 3 we develop an approximate method, similar to the method of slowly varying parameters used in the analysis of nonlinear differential equations. We also develop approximate stability boundaries for the solutions.

Chapter 4 treats the discrete Mathieu equation obtained by perturbing equation (1.1). Approximate stability boundaries corresponding to the solutions of the discrete Mathieu equation are obtained. Higher order approximations are achieved by the consideration of the third harmonic, occurring due to the nonlinearity. The approximate stability boundaries obtained in Chapter 4 are compared to the boundaries constructed by the use of the method developed in Chapter 3. Finally we conduct a numerical determination of the stability boundaries of the solutions of the discrete Mathieu equation using Floquet theory.

In Chapter 5 we consider linear and nonlinear difference equations driven by stochastic excitation. A general expression for the second moments of the steady state response of a linear difference equation is derived. This expression is used to consider the exact evaluation of the second moment of the response of a nonlinear difference equation. It is also attempted to extend the method of equivalent linearization to cover nonlinear difference equations. An approximate value of the second moment of the response of a nonlinear difference equation is obtained as a root of a third order polynomial.

2 PHASE PLANE ANALYSIS

2.1 Introduction

Chapter 2 will serve as an introductory survey of the many fascinating, but bothering, problematic topics in the theory of nonlinear difference equations. We will, for simplicity, direct our efforts to a difference equation with a cubic nonlinearity. In spite of the simple structure of the equation we will find a suprising amount of intriguing behavior.

By relating the response x_n to x_{n+1} we obtain a two dimension space similar to the phase plane of continuous systems. The phase plane plot can be used as a helpful tool in discussing such properties as equilibrium, stability and periodicity of the solution.

In Section 2.2 we determine the location and the stability of the equilibrium points. The stability of the solution at an equilibrium point is determined by looking at the corresponding linear perturbation equation. Having determined the equilibrium points and their stability, we can, without searching for the detailed behavior, obtain a rough estimate of the behaviour of the system.

In Section 2.3 we construct several phase plane plots in order to analyse the properties of the exact solution of the nonlinear difference equation.

In order to obtain the phase plane plots we will plot x_{n+1} v.s. x_n. By having a sufficiently large set of points, x_n, x_{n+1}, it is possible to achieve, for small enough initial conditions, a distinct pattern in the x_n, x_{n+1} plane. For sufficiently large initial conditions and for certain values of the parameters of the difference equation we obtain unbounded solutions which seemingly behave in a chaotic manner. We will no longer obtain simple and smooth solution point trajectories. In Section 2.4 we will relate this stochastic behavior of the solution to the intersection of separatrices or homoclinic and heteroclinic oscillation.

2.2 Existence and the Stability of Equilibrium Points

Consider the smooth map

$$f : R^n \to R^n \tag{2.1}$$

where f is a diffeomorphism and where R^n is the n-dimensional Euclidean space. If $x_n \in R^n$ we introduce the notation

$$x_{n+1} = f(x_n) \tag{2.2}$$

for all integers n. The point x_n can then be expressed as

$$x_{n+1} = f^{n+1}(x_o) \tag{2.3}$$

where

$$f^{n+1}(x) = f(f^n(x)) \quad . \tag{2.4}$$

A point is said to be periodic if we have

$$x_n = x_{n+P} \tag{2.5}$$

for some integer $P > 0$.

P is called the period of the point if P is the smallest integer satisfying (2.5). The periodic point, x_n, is also called a fixed point of the P^{th} order.

If $P = 1$, the point is called a fixed point of the first order or an equilibrium point. This point is mapped back onto itself in every iteration. We will also call an equilibrium point simply a fixed point, without the order specification. We will here consider a smooth diffeomorphism on R^2. The difference equation

$$x_{n+1} - a\, x_n + x_{n-1} + b\, x_n^3 = 0 \tag{2.6}$$

is such a map which will take the point (x_{n-1}, x_n) in R^2 to the point (x_n, x_{n+1}) in R^2.

In the following we will determine the location, in the R^2 space, and the stability of the equilibrium points of the nonlinear difference Equation (2.6). In order for an equilibrium point to exist we must have

$$x_{n+1} = x_n = x_{n-1} \quad . \tag{2.7}$$

Hence, at the equilibrium points the Equation (2.6) yields

$$x_n - a\,x_n + x_n + x_n^3 = 0$$

which using (2.7) yields either

$$x_n = x_{n+1} = \pm \left[\frac{a-2}{b}\right]^{1/2} \tag{2.8}$$

or

$$x_n = x_{n+1} = 0 \tag{2.9}$$

(2.8) and (2.9) give the location of the equilibrium points in the phase plane. It is clear that if $(a-2)/b < 0$ there will exist no equilibrium points at the locations given by (2.8). In order to determine the local stability of the equilibrium points we superpose a small perturbation, ξ_n, on the existing steady state solution at the equilibrium points.

Pertubation around $x_{n+1} = x_n = 0$; Assume a solution of the form

$$x_n = \xi_n \tag{2.10}$$

where ξ_n is a small perturbation (2.10) into (2.6) yield

$$\xi_{n+1} - a\xi_n + \xi_{n-1} = 0 \tag{2.11}$$

where we have neglected terms of $O(\xi_n^2)$.

Pertubation around $x_c = x_n = x_{n+1} = [(a-2)/b]^{1/2}$; Let

$$x_n = x_c + \xi_n \tag{2.12}$$

Our linear perturbation equation becomes

$$\xi_{n+1}+2(a-3)\xi_n+\xi_{n-1}=0 \quad . \tag{2.13}$$

In order to conduct a systematic survey of the stability of the equilibrium points we start up by considering two cases, I and II.

I). Let a and b satisfy

$$\frac{a-2}{b}\geq 0. \tag{2.14}$$

We can write this as follows

a). $a\geq 2, b>0$

b). $a\leq 2, b<0.$

II). Let a and b satisfy

$$\frac{a-2}{b}< 0. \tag{2.15}$$

Which can be written

a). $a>2, b<0$

b). $a<2, b>0.$

The following two sections will treat these two cases separately.

2.2.1 Case I. Phase Plane for $\frac{a-2}{b}\geq 0$

It is clear from (2.11) that when

$$-2< a< 2 \tag{2.16}$$

the origin is a center and if

$$a > 2 \;\; or \;\; a < 2 \tag{2.17}$$

the origin is a saddle point. From (2.13) we can conclude that if

$$2< a< 4 \tag{2.18}$$

we have a center at

$$x_n = x_{n+1} = \pm x_c \tag{2.19}$$

and if

$$a < 2 \ or \ a > 4 \tag{2.20}$$

we have a saddle point at

$$x_n = x_{n+1} = x_c \ . \tag{2.21}$$

Using these conditions for which centers and saddle points exist, more subclasses of case I can be distinguished.

Case Ia). can be divided up into

i). Saddle point at the origin, centers at

$x_n = x_{n+1} = \pm \ x_c$

for $b > 0$, $2 < a < 4$.

ii). Saddle point at the origin, saddle points at

$x_n = x_{n+1} = \pm \ x_c$

for $b > 0, a > 4$.

Case Ib). can be divided up into

i). Center at the origin, saddle points at

$x_n = x_{n+1} = \pm \ x_c$

for $b < 0$, $-2 < a < 2$.

ii). Saddle point at the origin, saddle point at

$x_n = x_{n+1} = \pm \ x_c$

for $b < 0$, $a < -2$.

2.2.2 Case II. Phase Plane for $\frac{a-2}{b} < 0$

Clearly no equilibrium points can exist at $x_n = x_{n+1} = [\frac{a-2}{b}]^{1/2}$ if $\frac{a-2}{b} < 0$. Hence, the only existing equilibrium point in case II is located at the origin. Cases IIa and IIb can further be divided up into more subcases depending on the stability of the solution at the origin. Case IIb can be divided up into:

ii) Center at the origin.

$$\text{for } b > 0 \,,\ -2 < a < 2.$$

i) Saddle point at the origin.

$$\text{for } b > 0 \,,\ a < -2.$$

For case IIa the only existing equilibrium point is a saddle point at the origin. The different stability regions obtained above are plotted in Figure (2.1).

2.3 Discussion of Periodicity and Stability of Points Close to a Stable Fixed Point.

Phase plane plots of explicit nonlinear difference equations can easily be produced numerically on a digital computer. This can be accomplished by simply stepping through the difference equation given the initial conditions, x_0 and x_1. Since we only obtain discrete points for each successive iteration we must produce a large set of points in the x_n, x_{n+1} plane in order to obtain distinct clear phase plane trajectories. The fact that closed loop trajectories, Figures (2.2) - (2.4), are obtainable close to a center is due to the lack of periodicity or the existance of extremely long periodic solutions. So we have a situation where the solution of the difference equation, x_n, x_{n+1}, cannot exactly be mapped back onto a previous solution point , x_m, x_{m+1}, unless the solution is periodic and then only when $n >> m$. The traced out curve will hence become more and more dense as the solution refuses to be mapped back onto one of the previous points.

It is possible to get an indication of why the solutions should have long periods by considering a linear difference equation. Consider the equation

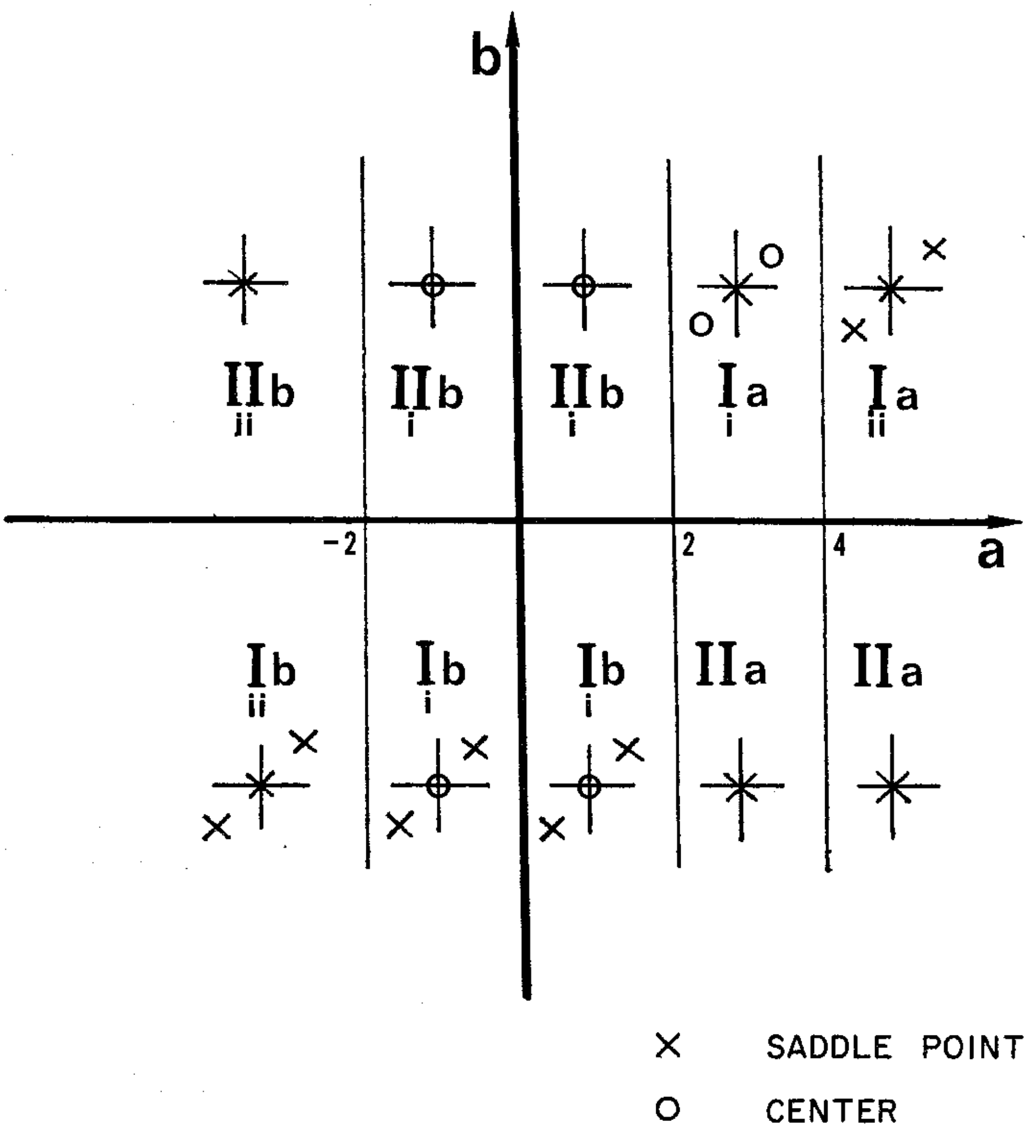


Fig. 2.1 Stability Regions

$$x_{n+1}+x_{n-1}-ax_n=0 \quad . \tag{2.22}$$

Set

$$a=2\cos\vartheta \quad . \tag{2.23}$$

The exact solution is then

$$x_n=A\cos(n\vartheta)+B\sin(n\vartheta) \quad .$$

We assume ϑ to be chosen in such a way that x_n will be periodic with a period of

$$N=\frac{2\pi}{\vartheta} \tag{2.24}$$

where N must be an integer. Hence the solution points will here be mapped perfectly onto the points of the previous period after one revolution in the phase plane.

We will now investigate the consequences of slightly perturbing the value of ϑ. The new value, ϑ', can be written

$$\vartheta'=\vartheta\frac{P}{Q} \tag{2.25}$$

where P and Q are integers and the fraction $\frac{P}{Q}$ is close to 1. The solution to Equation (2.22) can now be written

$$x=A\cos\left(n\frac{P}{Q}\vartheta\right)+B\sin\left(n\frac{P}{Q}\vartheta\right) \quad . \tag{2.26}$$

In order for the solution to be periodic we must have

$$M\frac{P}{Q}\vartheta=R\,2\pi \tag{2.27}$$

where M is the period such that

$$x_n=x_{n+M} \tag{2.28}$$

and R is the smallest integer satisfying Equation (2.27). Equation (2.24) into (2.27) yields

$$R = \frac{P}{Q}\frac{1}{N}M \quad . \tag{2.29}$$

It is clear that if $\frac{P}{QN}$ is a rational fraction, with a highest common factor of one, R must take on the value of P and M must take on the value of QN in order for Equation (2.29) to be satisfied. So by perturbing the value ϑ slightly the period changes from N to QN. This can be a considerable change which can be realized by considering the following example.

Assume the initial period of the solution to be $N = 25$. Changing ϑ slightly by multiplying it by the fraction $P/Q = 1001/1000$ produces, according to (2.29), a new period equal to $M = 25000$. Hence it will now take 25000 iterations, compared to 25 iterations earlier, for the solution to repeat itself.

Figures (2.2) - (2.4) illustrate how the "nonperiodic" quality of the solution makes it feasible to construct smooth closed curves or trajectories around a stable fixed point of the first order. Each figure corresponds to a different stability region, as defined in Section 2.2 and shown in Figure (2.1). Each trajectory is obtained by iterating from one initial point.

One feature in Figures (2.2) - (2.4) with particular interest to us, in the development of an approximate theory, is the distinctness and smoothness of the trajectories close to a stable fixed point. We will return to this fact later in Chapter 3.

A subset C of R^n is invariant under f if $f(C) \subset C$. So judging from the distinct and smooth character of the seemingly closed trajectories in Figures (2.2) - (2.4) we suspect that they are one dimensional invariant subsets under the map corresponding to the difference equation (2.6). If this actually is the case these trajectories must also be one-dimensional invariant manifolds in R^2 since the Map (2.6) is a diffeomorphism in R^2. All our numerical work indicates that this is true. Points

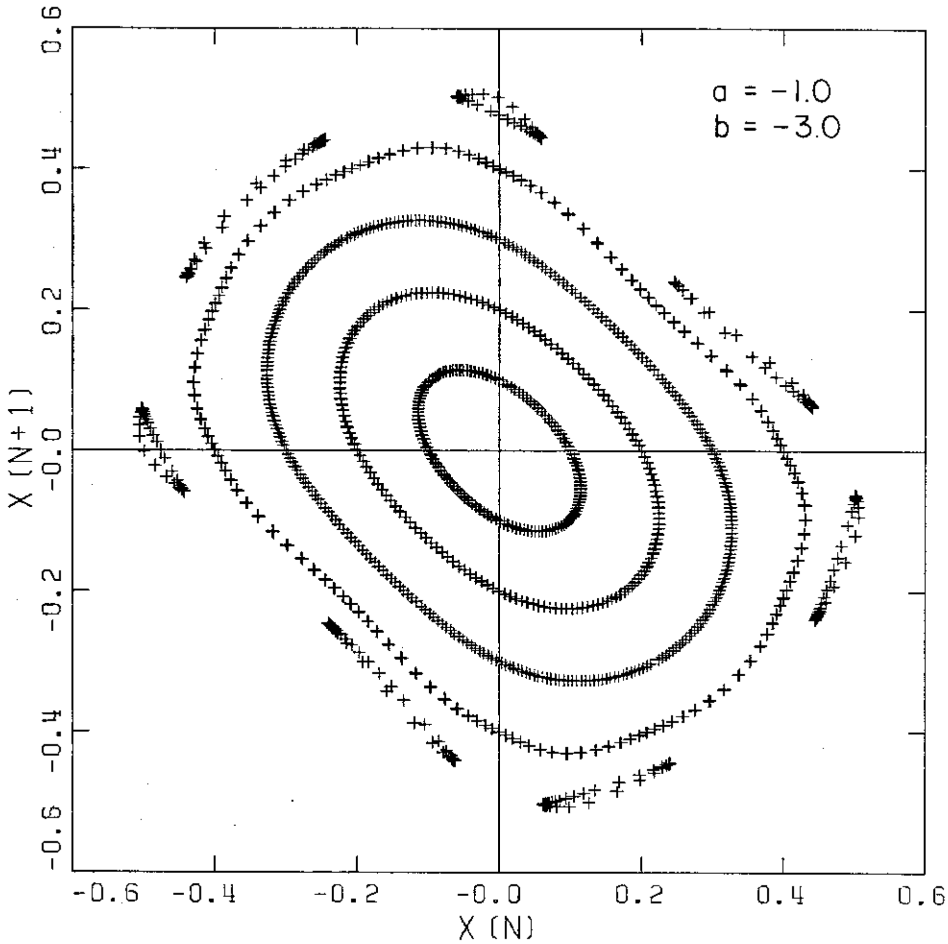


Fig. 2.2 Phase plane of the equation $x_{n+1} - ax_n + x_{n-1} + bx_n^3 = 0$ close to a center at (0.0)

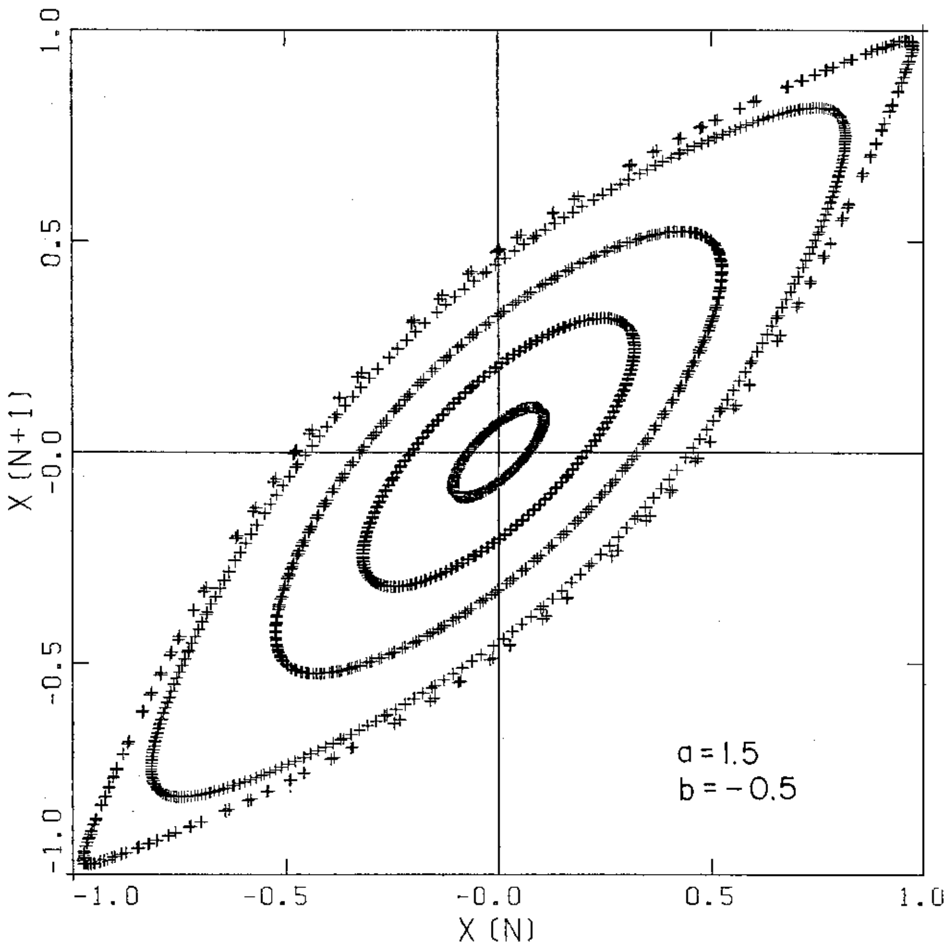


Fig. 2.3 Phase plane of the equation $x_{n+1} - ax_n + x_{n-1} + bx_n^3 = 0$ close to a center at (0.0)

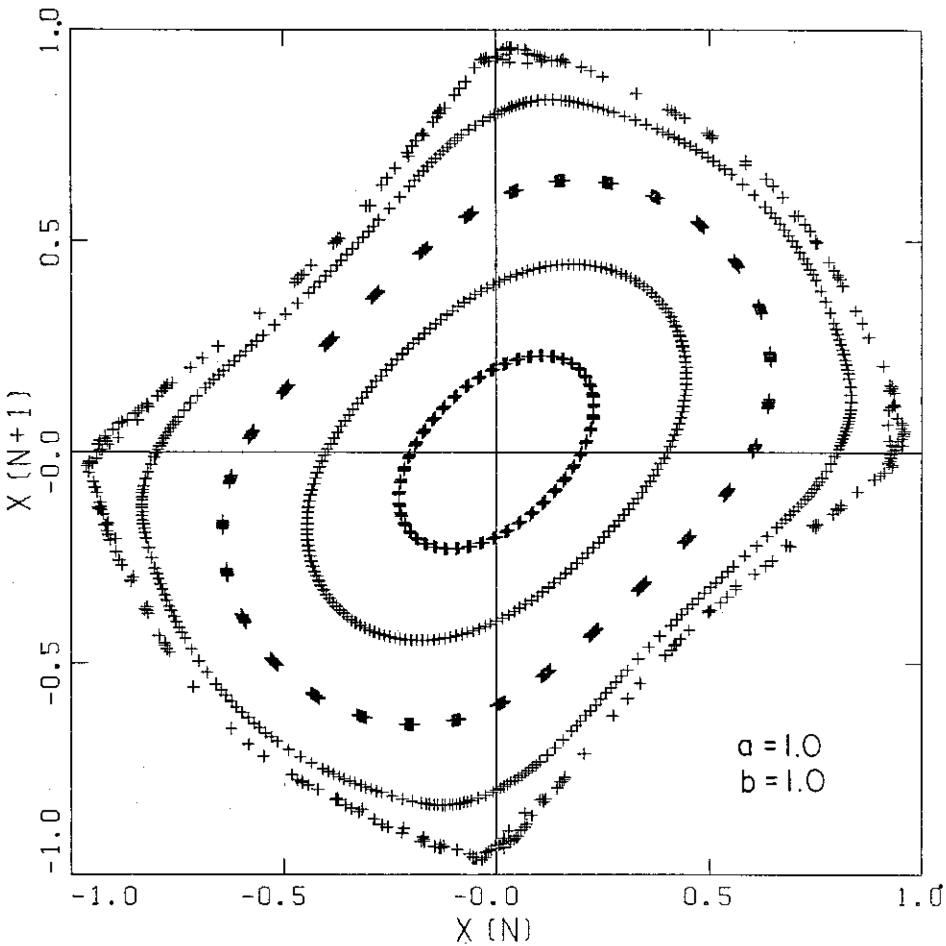


Fig. 2.4 Phase plane of the equation $x_{n+1} - ax_n + x_{n-1} + bx_n^3 = 0$ close to a center at (0,0)

initially situated on one of these smooth curves seem to remain there as n grows to infinity. However, in spite of some effort we have not been able to obtain an expression for these curves nor have we been able to formally prove that closed invariant curves exist for our Map (2.6).

The eigenvalues of the linear map obtained by linearizing the Map (2.6) around an equilibrium point of center type lie on the unit circle in the complex plane. Hence the system belongs to a critical class where stability cannot be determined by the corresponding linear system without the consideration of nonlinear terms. If all the eigenvalues would have been located within the unit circle we could easily have proven that all solutions of the original nonlinear system would have been Liapunov asymptotically stable. The proof builds on the assumption of sufficiently small initial conditions and that

$$\lim_{x_n \to 0} \frac{g(x_n)}{x_n} = 0 \tag{2.30}$$

where $g(x_n)$ is the nonlinear part of equation (2.6). However since our system is critical it is not possible to conclude stability or boundness using the same techniques and we are therefore forced to consider the complete nonlinear equation. So not only are we unable to determine if closed invariant curves exist but we are also unable to determine if the apparently bounded solutions actually are bounded.

The outermost trajectory, in each of the Figures (2.2) - (2.4), can be defined to be the stability boundary which separates bounded and unbounded solutions. The numerical determination of the stability boundary is complicated by the fact that the solution of the difference equation can remain bounded for many thousands of iterations and then suddenly become unstable and blow up. Due to this phenomenon erroneous conclusions concerning the stability of the equation can easily occur. Hence, in order to minimize the error in our stability analysis we are forced to consider a large number of iterations of the difference equation. Since we cannot conclude stability in an analytical sense we will define the solutions which remain within a certain predetermined distance, L , from a center after a given numbers of iterations, S , to be stable.

Defining stability in this manner will lead to stability boundaries that are functions of both L and S. The L dependence is not critical since the solution grows very fast as soon as it tends to become unstable. The dependence on S however, is that much more critical. For example, concluding that the solution is stable due to the fact that after 1000 iterations the solution is still well below the value of L can yield a drastically different stability boundary compared to concluding stability using 100000 iterations. In Figures (2.2) - (2.4) the stability boundaries are found by using S=10000.

Seventh order fixed points very close to the stability boundary can be detected in Figure (2.2). The complex configuration of the stability boundary in Figure (2.3) can be explained by the discrepancy between this boundary and the true stability boundary. The points on our stability boundary will eventually be mapped outside a circle of radius L centered at origin for a certain $n > S$.

In Figures (2.5.a) to (2.5.e) we take a closer look at the behavior of the solution close to the stability boundary. We have chosen to concentrate on case IIb and there to focus on the top center "corner" of the stability boundary. In Figures (2.5.a) and (2.5.b) we have chosen initial conditions that are located inside the stability region. It is seen that the shape of the trajectory changes dramatically as the initial condition approaches the stability boundary, going from Figure (2.5.a) to (2.5.b). A further small perturbation of the initial conditions toward the stability boundary changes the trajectory to some "semiperiodic" appearance, Figure (2.5.c), where the points seem to be influenced by higher order fixed points. Figure (2.5.d) shows the trajectory obtained when the initial conditions are chosen to lie on the stability boundary determined by

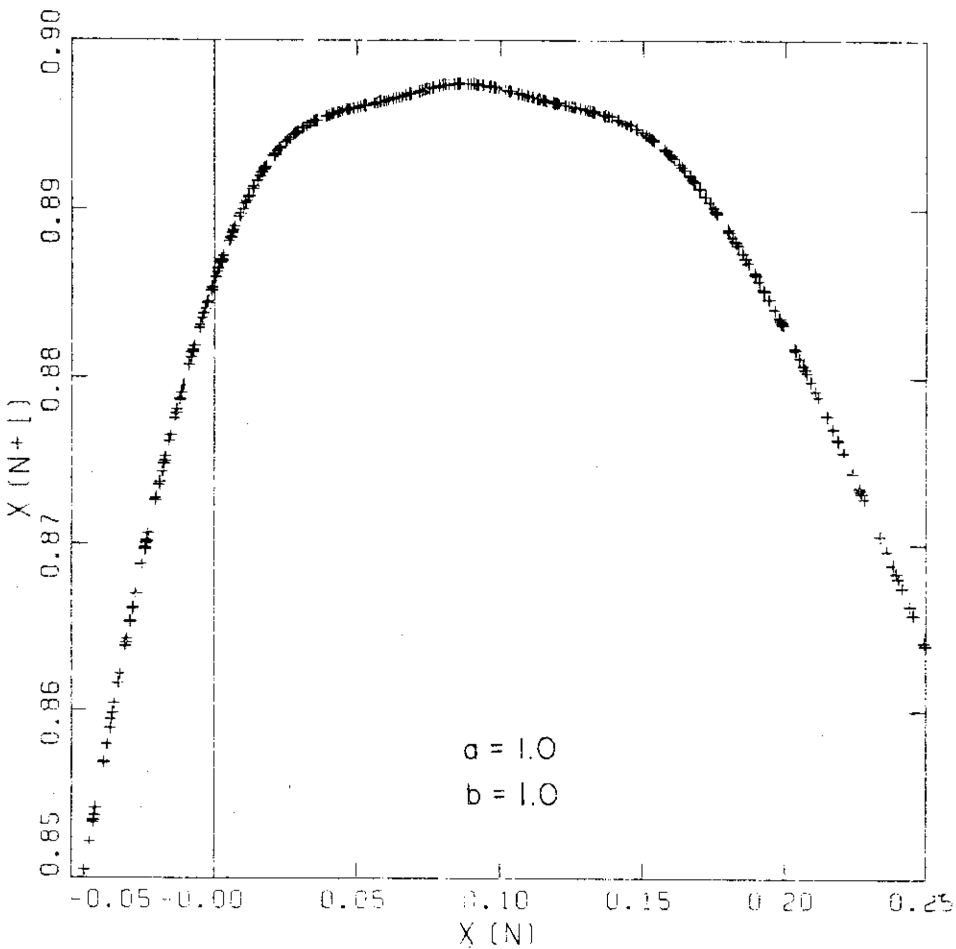


Fig. 2.5.a Detail of phase plane of equation $x_{n+1} - ax_n + x_{n-1} + bx_n^3 = 0$ close to the stability boundary

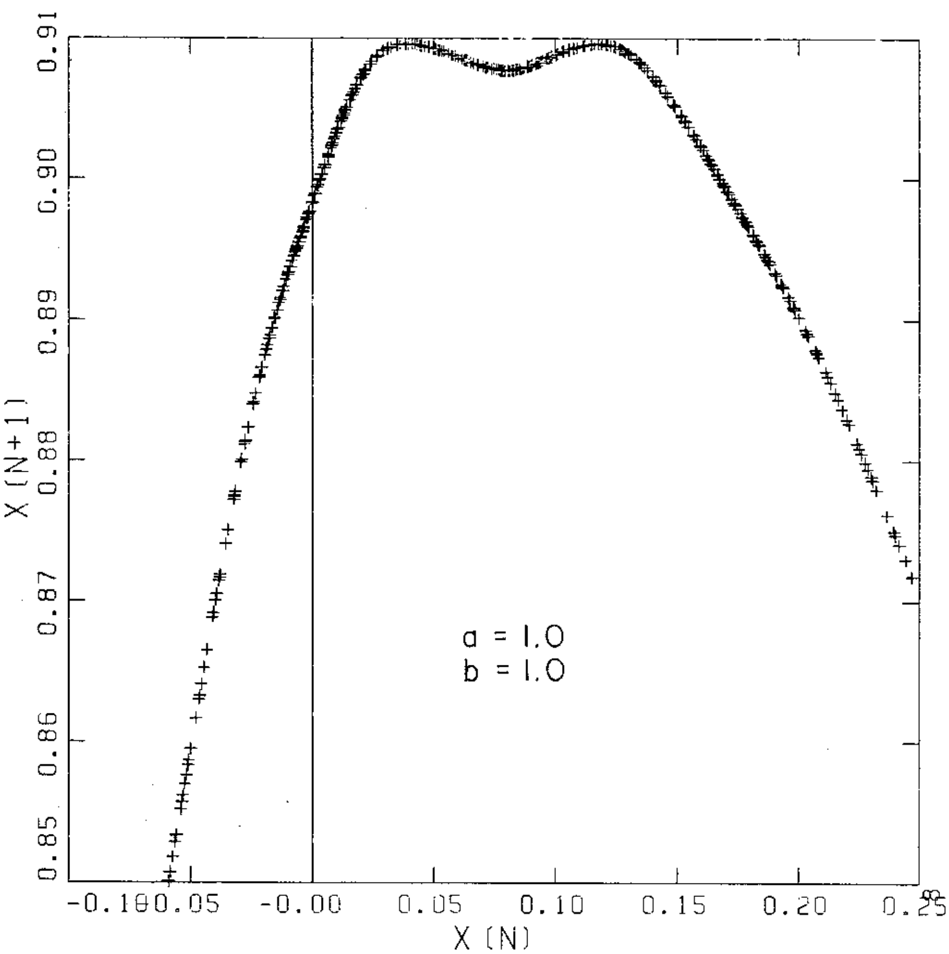


Fig. 2.5b Detail of phase plane of equation $x_{n+1} - ax_n + x_{n-1} + bx_n^3 = 0$ close to the stability boundary.

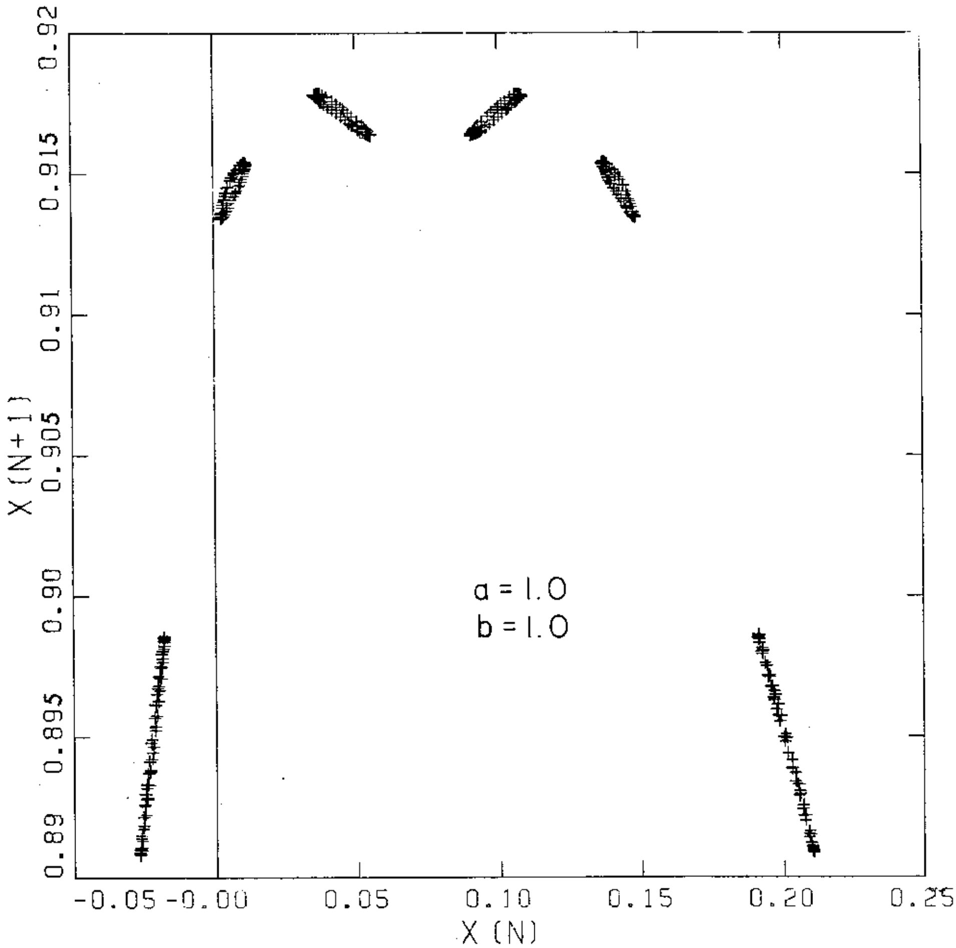


Fig. 2.5c Detail of phase plane of equation $x_{n+1} - ax_n + x_{n-1} + bx_n^3 = 0$ close to the stability boundary

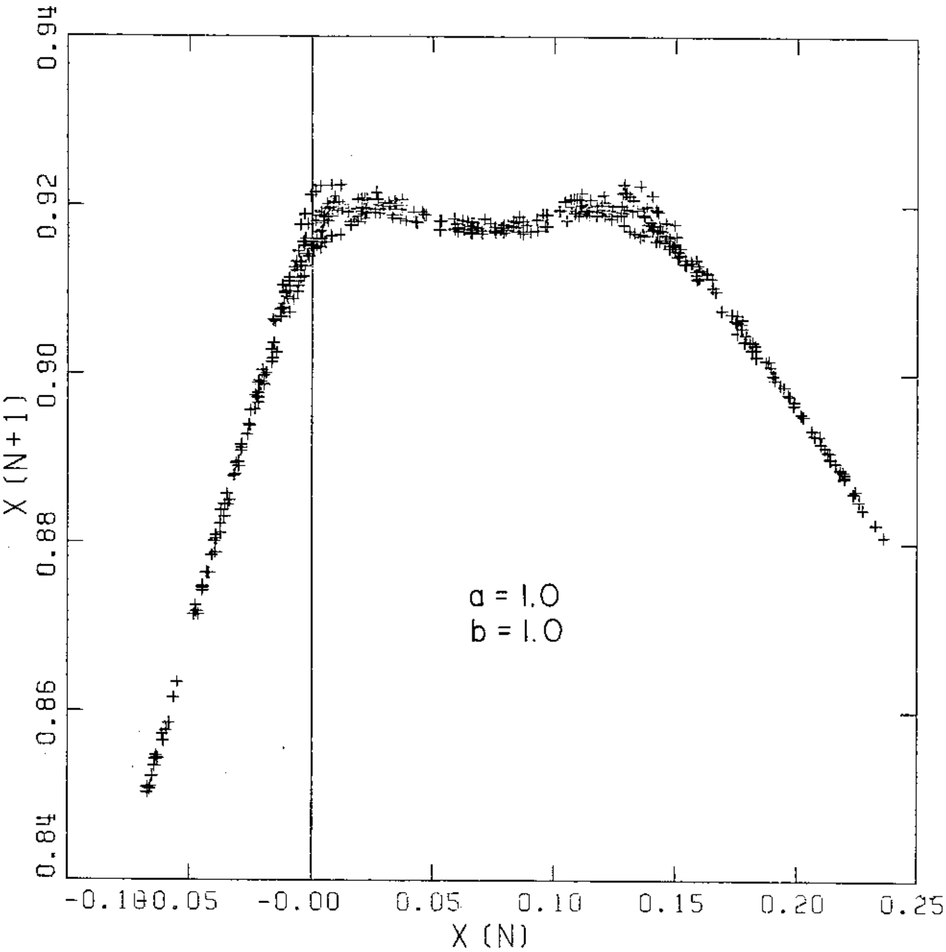


Fig. 2.5.d Detail of phase plane of equation $x_{n+1} - ax_n + x_{n-1} + bx_n^3 = 0$ close to the stability boundary

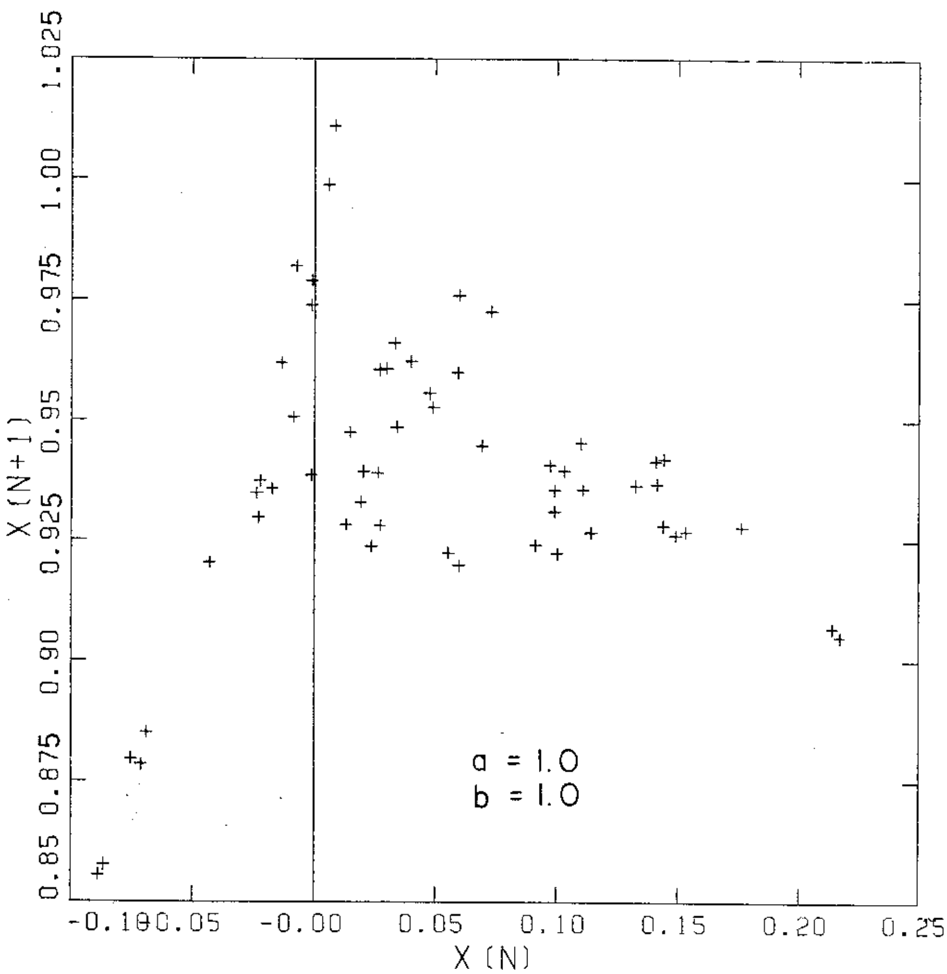


Fig. 2.5.e Detail of phase plane of equation $x_{n+1} - ax_n + x_{n-1} + bx_n^3 = 0$ close to the stability boundary

setting $S = 10000$. Judging from the scatter of the points we can suspect that they are actually located slightly outside the stability boundary and that the solution eventually will grow unstable. In Figure (2.5.e) the initial conditions are clearly located outside the stability region which forces the solution to grow unstable in few hundred steps.

2.4 Stochastic Behavior and Intersecting Separatrices

In this section we will consider the solution in a more global sense. We will study solutions which are located outside of and away from the stability boundaries obtained in Section 2.3. These unbounded solutions behave in a much more complex manner than the apparently bounded solutions discussed in the previous section.

We will first consider Equation (2.6) with $a = -1.0$, $b = -3.0$. This particular choice of the parameters a and b falls into the stability region (Ibi) (see Figure (2.1)). The corresponding bounded solutions were discussed in Section 2.3 . The complex behavior of the unbounded solutions is for this case apparent through the quasi-stochastic scatter of points in the phase plane as displayed in Figure (2.6.a).

As can be seen from Figure (2.6.b) and as determined in Section 2.2 we have here, in addition to the fixed point at the origin, two unstable fixed points of saddle type at (1,1) and at (-1,-1). With each unstable fixed point we have one unstable and stable manifold. We call these manifolds separatrices. Solution points located on the stable separatrix will approach the fixed point after repeated applications of the map or difference Equation (2.6). Solution points located on the unstable separatrix will be mapped away from the fixed point.

By choosing several initial points along the eigenvectors of the linearized system about the fixed point, we can trace out the unstable separatix by iterating forward and the stable separatix by iterating backward. The stable and unstable manifolds can be defined as follows :

$$W_i^s = \{y \in R^2 : \lim_{n\to\infty} f^n(y) \to x_i\} \tag{2.31}$$

$$W_i^u = \{y \in R^2 : \lim_{n\to\infty} f^{-n}(y) \to x_i\} \ . \tag{2.32}$$

where $i=1\ ,2$ and x_i is the location of the i^{th} saddle type fixed point in the phase plane. These separatrices are sketched out and labeled in Figure (2.6.c). From Figures (2.6.b) and (2.6.c) it is seen that the unstable separatrix W_1^u does not simply coincide with the stable separatrix W_2^s. The solution behaves in a more complex and intriguing manner. W_1^u crosses the stable manifold W_2^s and starts to oscillate around W_2^s in an increasingly violent manner. As seen in Figure (2.6.c) W_2^s behaves in a similar manner in relationship to W_1^u. The crossing points of the unstable and stable separatrices are called heteroclinic points and the oscillations mentioned above are called heteroclinic oscillations [2,20,21,22]. (A point of intersection of a stable and unstable manifold emanating from the **same** fixed point is called a homoclinic point [2,18,19,20]. The corresponding oscillations of the manifolds are called homoclinic oscillations.)

Since the map is a diffeomorphism the heteroclinic points must approach the fixed point x_2, along W_2^s, as $n \to \infty$. Similarly the heteroclinic points must approach x_1 along W_1^u , as $n \to -\infty$. So, for example, the heteroclinic point a_1 (Figure (2.6.c)) is mapped to a point farther down the manifold W_2^s closer to the fixed point x_2. Since our map is smooth and onto, a neighborhood of a_1 in W_1^u must be mapped into a neighborhood of the point $f(a_1)$. Hence we must again have a heteroclinic crossing of the manifolds W_1^u and W_2^s at the point $f(a_1)$. We call this point "a_2".

It can easily be shown that the sign of the angle between two intersecting curves in the plane is preserved under the Map (2.6.). Hence points located to the right (left) of an observer traveling down any manifold must always remain to the right (left) for any number of iterations of the map. In order for this to hold we realize that the manifold W_1^u must cross W_2^s between the points a_1 and a_2. In Figure (2.6.c) we call this point "b_1".

Since our map is smooth and onto, it will map a simply connected region into another simply connected region. Hence all the points in the region, A_1, cut out in the plane by the manifolds W_1^u and W_2^s between a_1 and b_1, are mapped into the shaded region A_2. No other points can be mapped into this region. Since the Jacobian determinant of the map is equal to one, the map is said to be area preserving. We therefore have the situation where all the inside regions, labeled A_i $(i = 2,3..)$, must have the same area. Similarly all the outside regions, labeled B_i $(1 = 1,2..)$, must have the same area. (Due to the symmetry between the manifolds, the areas of the outside and the inside regions are also equal.)

As the heteroclinic points approach a fixed point the distance between two consecutive heteroclinic points tends to zero. Hence the amplitude of the oscillation of W_1^u around W_2^s must increase with each mapping in order to preserve the area of each region cut out between the manifolds W_1^u and W_2^s. The "outside",B_i, loops are mapped into thinner and longer regions and all points contained therein must asymtotically approach the unstable manifold W_2^u and tend to $-\infty$ as $n \to \infty$. W_1^u (W_1^s) cannot cross W_2^u (W_1^s) since any intersection point of the two manifolds would be mapped to two different locations, the fixed points x_1 and x_2, as $n \to -\infty(\infty)$. This cannot occur since our Map (2.6) is one to one and onto.

An "inside" loop of a stable manifold, i.e. W_1^u or W_2^u, will eventually, as n grows, be mapped into an outside loop. This transition takes place crossing the diagonal $x_n = -x_{n+1}$, (see Figure (2.6.c)). The same holds true for the inside loops of a stable manifold as $(-n)$ grows.

Unlike an exterior loop of W_1^u, an interior loop of W_1^u cannot increase in height in a simple manner since W_1^u is constrained by W_2^u. Instead the inside loop starts to wrap around in the interior, as n grows, and by so doing it intersects both of the oscillating manifolds, W_1^s, and W_2^s. Hence as $n \to \infty$ more and more of the inside loop of W_1^u will be carried, by the outside loops of W_1^u and W_2^u, to $-\infty$ and ∞ as it also gets thinner and thinner. All this occurs without W_1^u intersecting itself or W_2^u. All points contained in an inside loop will eventually be carried to ∞ or $-\infty$ as $n \to \infty$.

The images of the inside loops start to form an inside envelope, E, as the number of mappings, n, grows. This closed envelope corresponds to our stability boundary for the closed and bounded trajectories discussed in Section 2.3. The inside loops can be mapped arbitrarily close to this envelope but can never cross it. In the exterior of the envelope, the inside loops start to densely fill the remainder of the region cut out by the manifolds. In the interior of the envelope points are mapped along closed and smooth paths (Figures (2.2) and (2.6.c))

By crossing the stability boundary, E (the envelope of the inner loops), leaving the bounded region behind us, we must end up on or in an inside loop of one the four manifolds, W_1^u, $W_1^s (i = 1,2)$. This must be the case since this region becomes dense with inside loops, as $n \to \infty$. In other words, by choosing any location in this region we can always find a $|m|$, sufficiently large, and a closed set L consisting of all points on and within an inside manifold loop such that the set $f^m(L)$ will contain a point at the chosen location. Hence almost all points in the densely filled region will eventually be mapped towards ∞ or $-\infty$ since all points contained in an inside loop must be carried to $\pm\infty$ as $n \to \infty$. (The word "almost" is used here since if a point is located exactly on the stable manifold it will be mapped to its corresponding fixed point. This occurrence, however, is one of zero measure.) When the point is located in an unstable (a stable) manifold loop, the point will first be carried along as the loop wraps (unwraps) itself around the origin.

This explains why we obtain, as mentioned in Section 2.3, points close, but outside, the stability boundary which can remain bounded for several thousand mappings and then suddenly be mapped towards ∞ or $-\infty$. In order for this to occur, the point must be located in an inside loop which must wrap (unwrap) itself around the stable fixed point a large number of times before the loop can carry the point towards $\pm\infty$.

The connection between the stochastic appearance of the mapped points and the occurrence of heteroclinic oscillations can now be treated. Due to the heteroclinic oscillations we obtain, as seen above, a region filled up by inside manifold loops. Points on or within these loops will be mapped around the origin several times in a manner dependent on the loop and the location of the point. Hence, if we choose to plot the images of a few discrete points for repeated maps, and not to plot the manifolds, the points appear to be become scattered in a stochastic manner. However, we classify this as stochastic scatter only because we fail to realize the amazing structure of the inside manifold loops given our limited discrete information. (See Figures (2.6.a) and (2.6.c))

In Figure (2.7) we have $a = 1.5$ and $b = -0.5$. The critical points are of the same type and in the same location as for $a = -1.0$, $b = -3.0$. In spite of this fact we obtain a totally different global behaviour. We have here none of the indications of the existence of heteroclinic oscillations as we had in the previous case. The unstable manifold $W_1^u(W_1^s)$ smoothly coincides with the stable manifold $W_2^s(W_2^u)$. We obtain no chaotic behaviour.

In Figure (2.8.a), $a = 3.0$, $b = 1.0$, we again see clear evidence of stochastic type behaviour. From local analysis, Section 2.3, we have determined the existence of three first order fixed points, a saddle point at the origin and centers at (1,1) and (-1,-1). Figure (2.4.a) was obtained by choosing several points, close to the origin, on the stable, W^s, and unstable, W^u, manifold. Two initial points, one for each center,

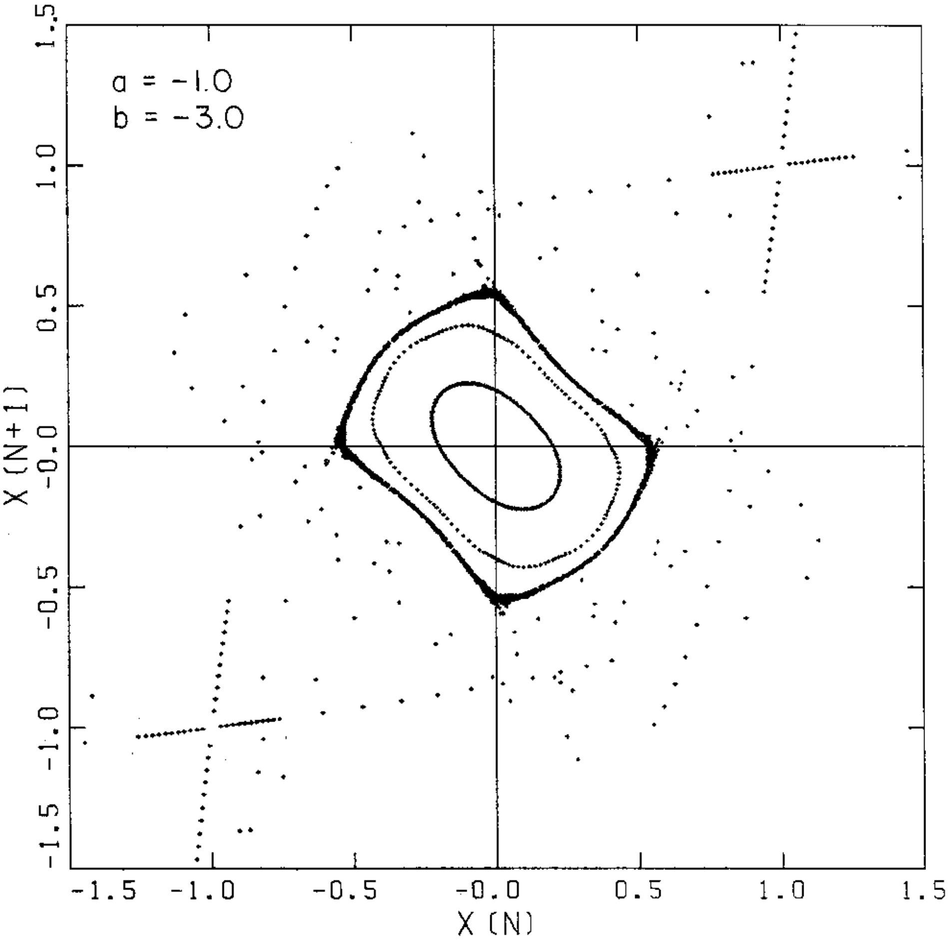


Fig. 2.6.a Phase plane of equation $x_{n+1} - ax_n + x_{n-1} + bx_n^3 = 0$ showing the appearent chaotic scatter of solution points

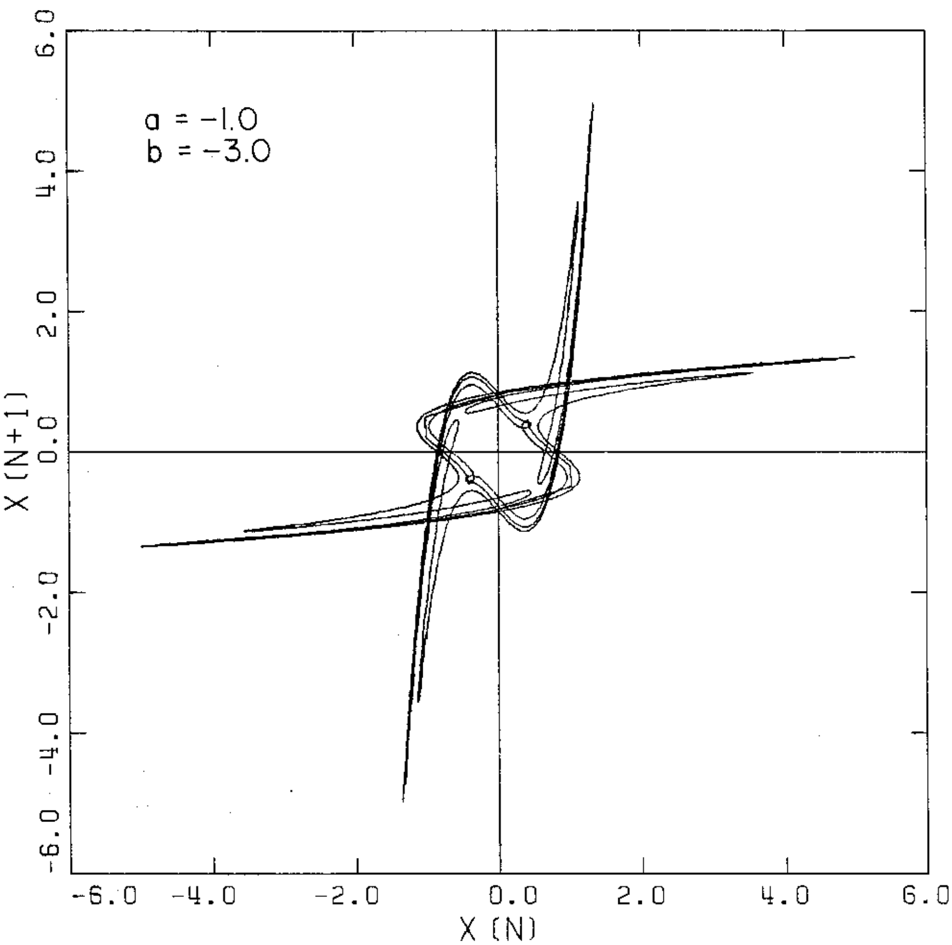


Fig. 2.6.b Phase plane of equation
$x_{n+1} - ax_n + x_{n-1} + bx_n^3 = 0$

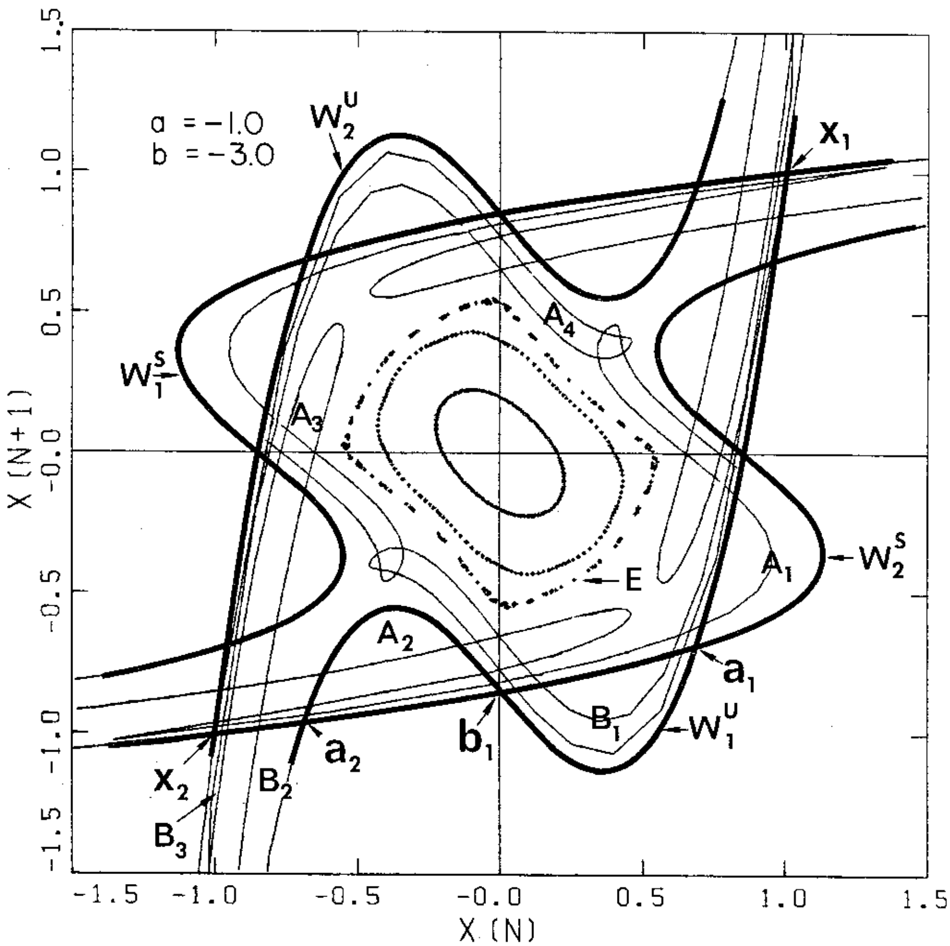


Fig. 2.6 c Phase plane of equation
$x_{n+1} - ax_n + x_{n-1} + bx_n^3 = 0$

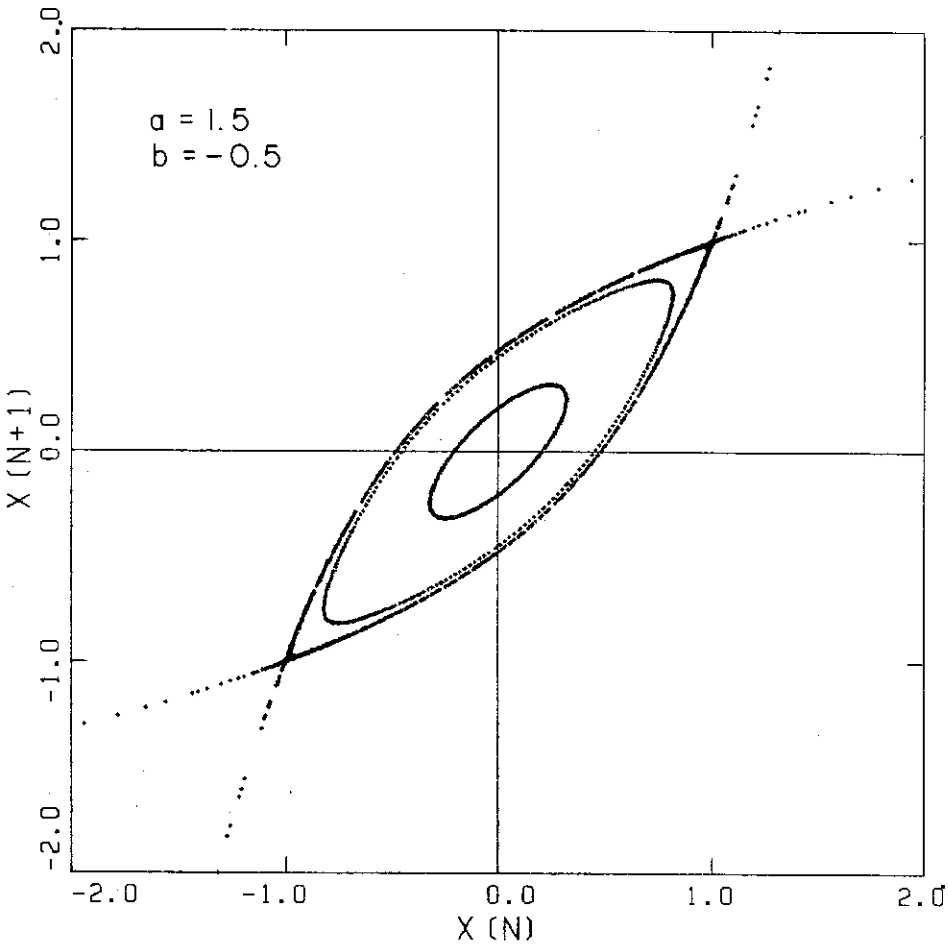


Fig. 2.7 Phase plane of equation

$x_{n+1} - ax_n + x_{n-1} + bx_n^3 = 0$

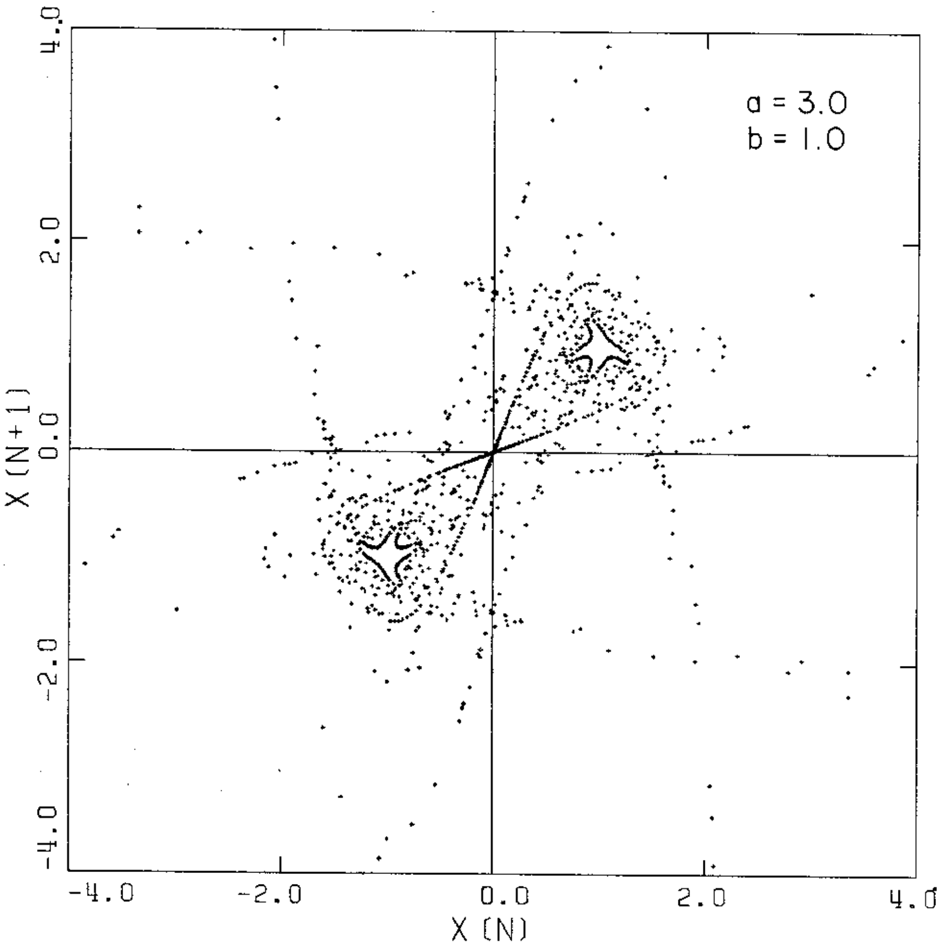


Fig. 2.8a Phase plane of equation
$x_{n+1} - ax_n + x_{n-1} + bx_n^3 = 0$

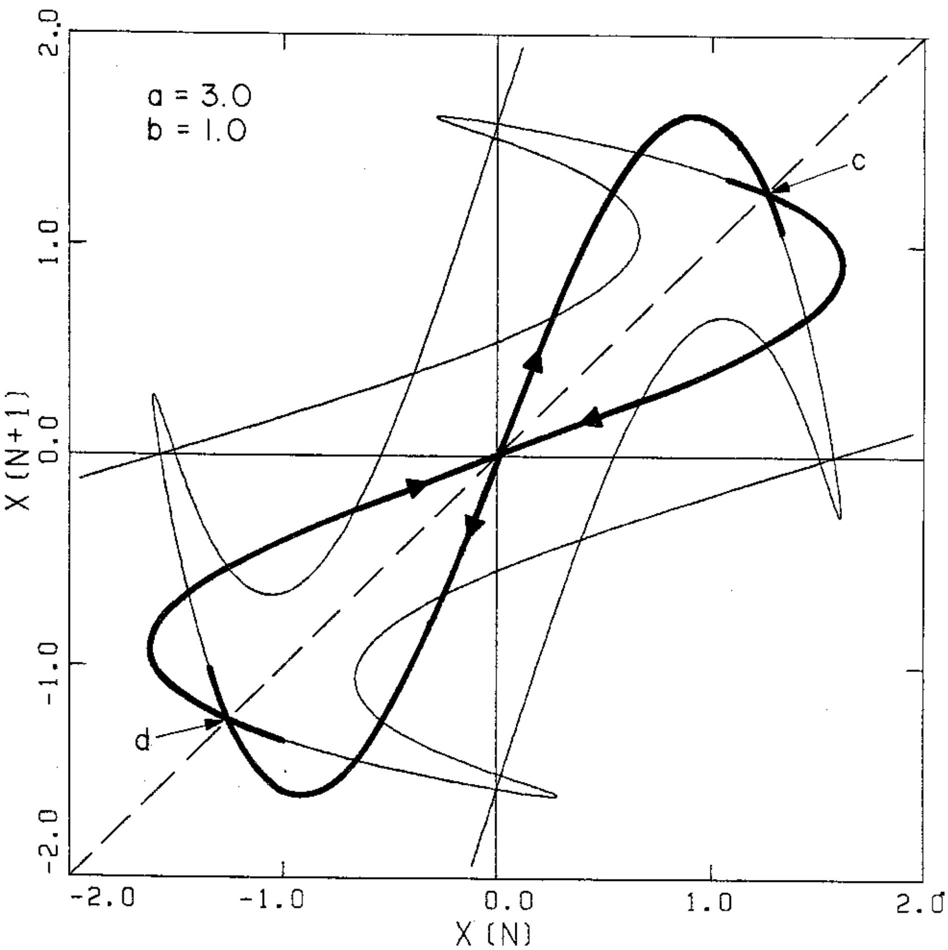


Fig. 2.8.b Phase plane of equation $x_{n+1} - ax_n + x_{n-1} + bx_n^3 = 0$ showing the first few crossings of the manifolds emanating from a saddle point at (0.0)

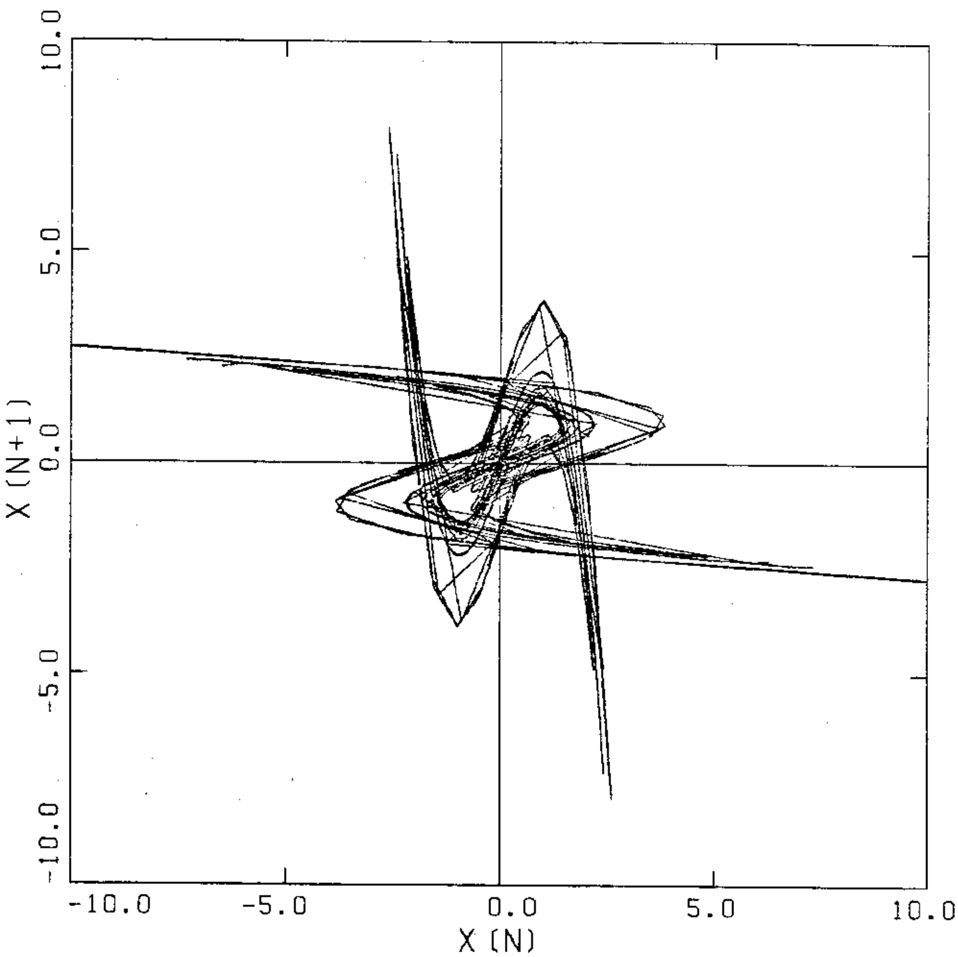


Fig. 2.8.c Phase plane of equation
$x_{n+1} - ax_n + x_{n-1} + bx_n^3 = 0$

were also chosen to lie in the regions of bounded trajectories. Hence the scattered points are all on the stable and unstable manifolds even though the scatter appears to be very chaotic. In Figure (2.8.b) we can see that the manifolds do not have the same slope at the line of symmetry and hence cross at points c and d. This implies, since the manifolds are issued from the same fixed point, that we must have homoclinic crossings. Figure (2.8.c) clearly indicates how the inside homoclinic loops strive to fill the inside space in a dense manner. Again, the loops form envelopes which coincide with the stability boundaries around the centers.

The results above indicate the following. Stochastic behaviour exhibited by a smooth diffeomorphism is caused by homoclinic or heteroclinic oscillations of unstable and stable manifolds.

When do such oscillations occur? We have seen that the occurrence of heteroclinic and homoclinic oscillations for map (2.6) is dependent upon the parameters of the map. It would therefore be desirable to develop a condition that would guarantee the (non)existence of these oscillations. The concept of topological entropy [19] of a map, has been used by Katok [23], Manning [24] and Bowen [25] in order to guarantee homoclinic points. To date we have not been able to adapt the known result, relating the topological entropy of a map to homoclinic oscillations, in order to obtain explicit conditions on the parameters of our map (2.6) guaranteeing the (non)existence of homoclinic oscillations. Further research in this area is continuing. Preliminary results, obtained by considering the slope reversing of an interval of a manifold, indicate that the following must hold: The map (2.6) cannot have any transverse heteroclinic crossings if

$$a - 3b\,x_n^2 > 1 \qquad (2.33)$$

holds true for $x_n \in W_i^u$ and W_i^s, $i = 1, 2$.

As an aside we will give a brief note on a dissipative map $g: R^2 \rightarrow R^2$. A map is dissipative if there exists a set Q, which is an attractor, containing the fixed point x_i. Our Map (2.6) could be converted into a dissipative map by adding an extra x_{n-1} term. It has been shown that for certain values of the parameters of the map g the obtained attracting set appears to be chaotic. Such an attracting set is called a strange attractor [6,26,27]. We make mention of this here because the existence of strange attractors in a dissipative map depends directly on the existence of homoclinic and heteroclinic oscillations.

2.5 Marginally Stable Solutions

During the determination of the stability boundaries in Section 2.3 we also encountered stability regions which can be characterized as lines. The solutions of the difference equation located on these lines are only stable for perturbations in a specific direction, the direction of the line itself. Perturbing the solution any other direction will cause the solution to grow unstable.

It will be shown here that the cause for this phenomenon is that the perturbed system does not have a complete set of linearly independent eigenvectors. It will also be shown that the stability line is in the direction of the ordinary eigenvector.

Consider the case IIb with $a=1$ and $b=1$

$$x_{n+1}-x_n+x_{n-1}+x_n^3=0 \quad . \tag{2.34}$$

Add a small perturbation, ξ_n, to the solution x_{o_n}.

$$x_n = x_{o_n}+\xi_n \quad . \tag{2.35}$$

Substituting the perturbated solution into (2.34) yields

$$\vec{\xi}_{n+1} = \mathbf{A}\vec{\xi}_n \tag{2.36}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 1-3x^2_{o_n} \end{bmatrix} \text{ and } \vec{\xi}_n = \begin{bmatrix} \xi \\ \xi_{n+1} \end{bmatrix} . \tag{2.37}$$

The eigenvalues of A are

$$\lambda_{1,2} = \tfrac{1}{2} \left[-3x^2_{o_n} + 1 \pm \sqrt{9x_{o_n}^{\ 4} - 6x_{o_n}^{\ 2} - 3}\ \right] . \tag{2.38}$$

A necessary condition for dependent eigenvectors is

$$\lambda_1 = \lambda_2 \tag{2.39}$$

so we must have

$$x^2_{o_n} = 1 \tag{2.40}$$

or

$$x^2_{o_n} = -\frac{1}{3} \text{ for } \forall n . \tag{2.41}$$

Hence the only values of x_{o_n} that yield identical eigenvalues are either $x_{o_n} = 1$ or $x_{o_n} = -1$. For these values of x_{o_n} the matrix A does not have a full complement of linearly independent eigenvectors. Both eigenvectors are colinear with the vector

$$\vec{z} = [1,1]^T . \tag{2.42}$$

The generalized eigenvectors can be found in order to write A in Jordan form. The vectors are

$$\vec{\varphi}_1 = [1,-1] \text{ and } \vec{\varphi}_2 = [\alpha, 1-\alpha]^T . \tag{2.43}$$

So we can write

$$\mathbf{A} = \begin{bmatrix} 1 & \alpha \\ -1 & 1-\alpha \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1-\alpha & \alpha \\ 1 & 1 \end{bmatrix}$$

and

$$\mathbf{A}^n = \begin{bmatrix} 1 & \alpha \\ -1 & 1-\alpha \end{bmatrix} \begin{bmatrix} (-1)^n & n(-1)^{n-1} \\ 0 & (-1)^n \end{bmatrix} \begin{bmatrix} 1-\alpha & -\alpha \\ 1 & 1 \end{bmatrix} . \tag{2.45}$$

But from (2.36) we can write

$$\vec{\xi}_{n+m} = \mathbf{A}^m \vec{\xi}_n \tag{2.46}$$

so

$$\vec{\xi}_{n+m} = \begin{bmatrix} 1 & \alpha \\ -1 & 1-\alpha \end{bmatrix} \begin{bmatrix} (-1)^m & m(-1)^{m-1} \\ 0 & (-1)^m \end{bmatrix} \begin{bmatrix} 1-\alpha & \alpha \\ 1 & 1 \end{bmatrix} \vec{\xi}_n \; . \tag{2.47}$$

If we now choose the initial perturbation, $\vec{\xi}_n$, to be parallel to an eigenvector of $\mathbf{A}$,

$$\vec{\xi}_n = [1, -1]^T \; , \tag{2.48}$$

we obtain from (2.45) the following

$$\vec{\xi}_{n+m} = [(-1)^m, -1]^T \; . \tag{2.49}$$

Hence the solution remains bounded for a perturbation along the eigenvector of $\mathbf{A}$. For any other direction of the perturbation the term $m(-1)$ will remain in the expression for $\vec{\xi}_{n+m}$ and therefore will force the solution to grow unstable as $m \to \infty$.

From Equation (2.34) it is clear that we only have one solution with $x_n^2 = 1$ for all n. This solution is periodic, having a period of four iterations in the sequence: (1,1), (1,-1), (-1,-1), (-1,1), (1,1) ,.... . We can conclude from above that the solution around these fourth order fixed points is only stable for small perturbations along a line in the phase plane of slope -1. This agrees very well with our numerical investigation as shown in Figure (2.9).

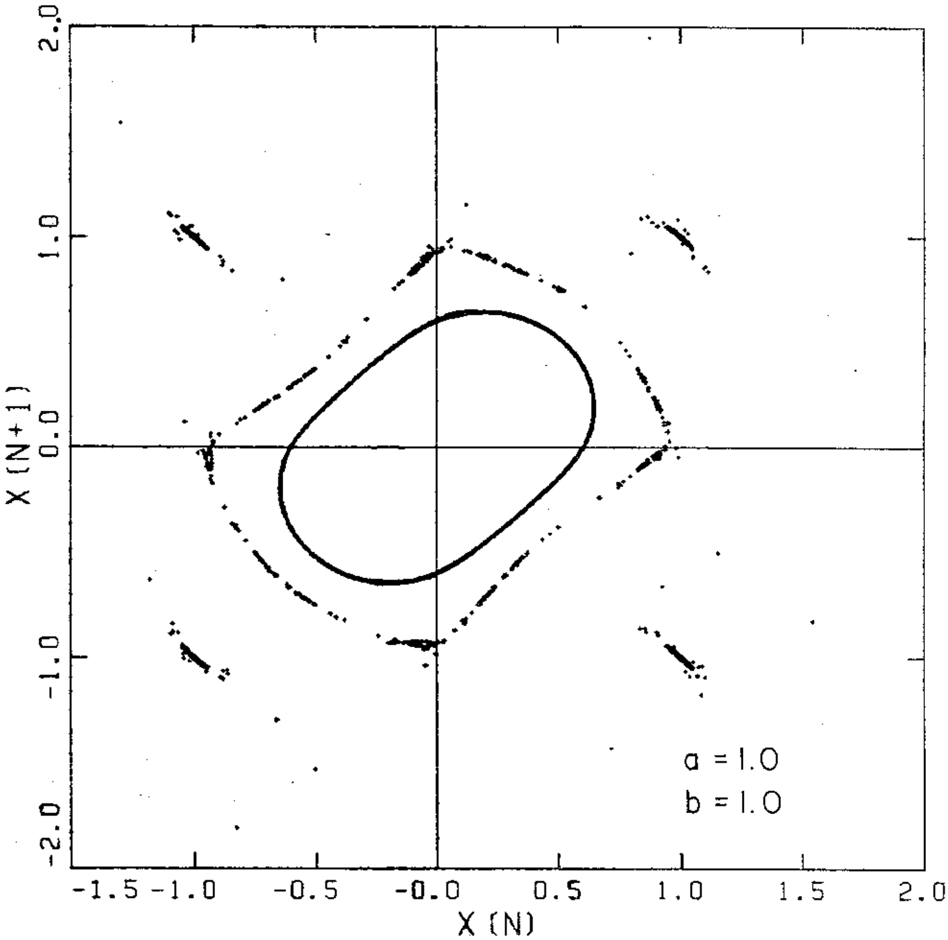


Fig. 2.9 Marginally stable solutions on the phase plane of equation $x_{n+1} - ax_n + x_{n-1} + bx_n^3 = 0$

3 APPROXIMATE SOLUTIONS ; SLOWLY VARYING PARAMETERS

3.1 Introduction

The feasibility of exact analytic determination of solutions of nonlinear difference equations and the corresponding stability criteria is often lacking. As seen previously, even a difference equation of relative simple structure yield solutions of surprising complexity. However, as seen in the previous chapter, points sufficiently close to a stable fixed point are mapped along closed and smooth trajectories encircling the fixed point. Hence by considering solutions only of small magnitude about the fixed point we obtain well behaved solutions. This gives rise to a desire to develop techniques that will yield approximate steady state solutions of small amplitude and corresponding local stability bounds.

We are here going to develop an approximate solution method similar to the method of slowly varying parameters as used in the analysis of nonlinear differential equations. We will, as before, consider the specific case of a difference equation with a cubic nonlinearity. In the first section we obtain an expression for an approximate steady state solution of a homogeneous equation of such a type. In the next two following sections we consider both the main and the ultraharmonic forced response. The stability analysis of the steady state solutions is given in Section 3.5. In Section 3.6 we develop some approximate solutions of higher order.

3.2 The Homogeneous Equation

Consider the equation

$$x_{n+1} - ax_n + x_{n-1} + bx_n^3 = 0 \tag{3.1}$$

with $-2 < a < 2$. Equation (3.1) can be written as

$$\Delta^2 x_n + 2(1-\cos\vartheta)x_n + b\,x_n^3 = 0 \tag{3.2}$$

where

$$a = 2\cos\vartheta \tag{3.3}$$

Δx_n denotes the central difference

$$\Delta x_n = x_{n+½} - x_{n-½} \tag{3.4}$$

and therefore

$$\Delta^2 x_n = x_{n+1} - 2x_n + x_{n-1} \quad . \tag{3.5}$$

First consider the linear case, $b = 0$.

$$x_{n+1} + x_{n-1} - 2\cos\vartheta x_a n = 0. \tag{3.6}$$

This linear difference equation has the exact solution

$$x_n = A\cos(n\vartheta) + B\sin(n\vartheta) \tag{3.7}$$

where the constants A and B are dependent upon the initial conditions of the system governed by (3.6).

By introducing a nonlinearity ($b \neq 0$) the solution of Equation (3.2) can no longer be represented by a simple harmonic function with frequency ϑ. However, for small values of b we can assume the solution to show fairly close similarity to the solution of the linear Equation (3.6). The desired approximate solution of Equation (3.2) is then constructed by letting A and B in (3.7) be slowly varying functions of n, the step number. The solution can be written as

$$x_n = A(n)\cos(n\varphi) + B(n)\sin(n\varphi) \quad . \tag{3.8}$$

In order for x_n (3.8) to be a solution of Equation (3.2) we must force $A(n)$ and $B(n)$ to satisfy certain conditions obtained by substituting (3.8) into Equation (3.2).

The central difference of x_n (3.8) is :

$$\Delta x_n = A(n)\Delta\cos(n\varphi) + B(n)\Delta\sin(n\varphi)$$

$$+\Delta A(n)\cos(n\varphi) + \Delta B(n)\sin(n\varphi) \qquad (3.9)$$

where we have neglected all differences of order two and higher.

The solution x_n (3.8) is a functional of two independent functions of n. Since we have only one equation in two variables we can introduce an arbitrary relationship between $A(n)$ and $B(n)$. It is convenient to choose this auxiliary condition as

$$\Delta A(n)\cos(n\varphi) + \Delta B(n)\sin(n\varphi) = 0 \ . \qquad (3.10)$$

Using (3.10) in (3.9) we have

$$\Delta x_n = A(n)\Delta\cos(\cos\varphi) + B(n)\Delta\sin(n\varphi) \qquad (3.11)$$

so

$$\Delta^2 x_n = A(n)\Delta^2\cos(n\varphi) + B(n)\Delta^2\sin(n\varphi)$$

$$+\Delta A(n)\Delta\cos(n\varphi) + \Delta B(n)\Delta\sin(n\varphi) \qquad (3.12)$$

where we have neglected all differences of order three and higher. Substituting the expression (3.12) for $\Delta^2 x_n$ into Equation (3.2) and noting that

$$\Delta\cos(n\varphi) = -2\sin(n\varphi)\sin(½\varphi)$$

$$\Delta\sin(n\varphi) = \cos(n\varphi)\sin(½\varphi)$$

$$\Delta^2\cos(n\varphi) = -2[1-\cos\varphi]\cos(n\varphi)$$

$$\Delta^2\cos(n\varphi) = -2[1-\cos\varphi]\sin(n\varphi) \qquad (3.13)$$

yields

$$2[A(n)\cos(n\varphi)+B(n)\sin(n\varphi)][\cos\varphi-\cos\vartheta]-$$

$$-2\Delta A(n)\sin(n\varphi)\sin(\tfrac{1}{2}\varphi)+2\Delta B(n)\cos(n\varphi)\sin(\tfrac{1}{2}\varphi)+$$

$$+b\,[A^3(n)\cos^3(n\varphi)+3A^2(n)B(n)\cos^2(n\varphi)\sin(n\varphi)$$

$$+3A(n)B^2(n)\cos(n\varphi)\sin^2(n\varphi)+B^3(n)\sin^3(n\varphi)]=0\;. \tag{3.14}$$

By multiplying Equation (3.14) by $\sin(n\varphi)$ and using the auxiliary condition (3.10) we get

$$2[A(n)\cos(n\varphi)\sin(n\varphi)+B(n)\sin^2(n\varphi)](\cos\varphi-\cos\vartheta)-2\sin(\frac{\varphi}{2})\Delta A(n)$$

$$+b\,[A^3(n)\cos^3(n\varphi)\sin(n\varphi)+3A^2(n)B(n)\cos^2(n\varphi)\sin^2(n\varphi)$$

$$+3A(n)B^2(n)\cos(n\varphi)\sin^3(n\varphi)+B^3(n)\sin^4(n\varphi)]=0\;. \tag{3.15}$$

Similarly multiplication of equation by $\cos(n\varphi)$ yields

$$2\{A(n)\cos^2(n\varphi)+B(n)\sin(n\varphi)\cos(n\varphi)\}(\cos\varphi-\cos\vartheta)$$

$$-2\sin(\tfrac{1}{2}\varphi)\Delta B(n)+b\{A^3(n)\cos^4(n\varphi)+3A^2(n)B(n)\cos^2(n\varphi)\sin(n\varphi)$$

$$+3A(n)B^2(n)\cos^2(n\varphi)\sin^2(n\varphi)+B^3(n)\sin^3(n\varphi)\cos(n\varphi)\}=0\;. \tag{3.16}$$

Since we have assumed $A(n)$ and $B(n)$ to be slowly varying functions with periods of much greater magnitude than $\frac{2\pi}{\varphi}$ we can simplify Equations (3.15) and (3.16) to the following forms

$$A(n)[\cos\varphi-\cos\vartheta]+2\sin(\frac{\varphi}{2})\Delta B(n)+\frac{3}{8}b[A^2(n)+B^2(n)]A(n)=0 \tag{3.17}$$

and

$$B(n)[\cos\varphi - \cos\vartheta] - 2\sin(\frac{\varphi}{2})\Delta A(n) + \frac{3}{8}b[A^2(n) + B^2(n)]B(n) = 0 \quad . \tag{3.18}$$

These two equations determine $A(n)$ and $B(n)$ for all n, given the initial conditions. Proceeding with this determination would require numerical simulation. Since an exact solution of Equation (3.2) could be obtained using the same numerical simulation technique directly on (3.2), we have no interest in determining $A(n)$ and $B(n)$ from (3.17) and (3.18). However, our primary interest is in the steady state solution which can easily be obtained through Equations (3.17) and (3.18) without recourse to numerical techniques.

For steady state solution

$$\Delta A(n) = \Delta B(n) = 0 \quad .$$

Thus Equations (3.17) and (3.18) become

$$A(\cos\varphi - \cos\vartheta) + \frac{3}{8}B(A^2 + b^2)A = 0 \quad . \tag{3.19}$$

$$B(\cos\varphi - \cos\vartheta) + \frac{3}{8}b(A^2 + B^2)B = 0 \tag{3.20}$$

Equations (3.19) and (3.20) are satisfied if

$$A = 0 \tag{3.21}$$

and

$$(\cos\varphi - \cos\vartheta) + \frac{3}{8}b\,B^2 = 0 \quad . \tag{3.22}$$

Equation (3.22) yields the relationship between the amplitude, B, and the frequency, φ, of the steady state response

$$x_n = B\sin(n\varphi) \tag{3.23}$$

The approximate solution is compared to the exact solution, obtained through numerical simulation, in Figure (3.1).

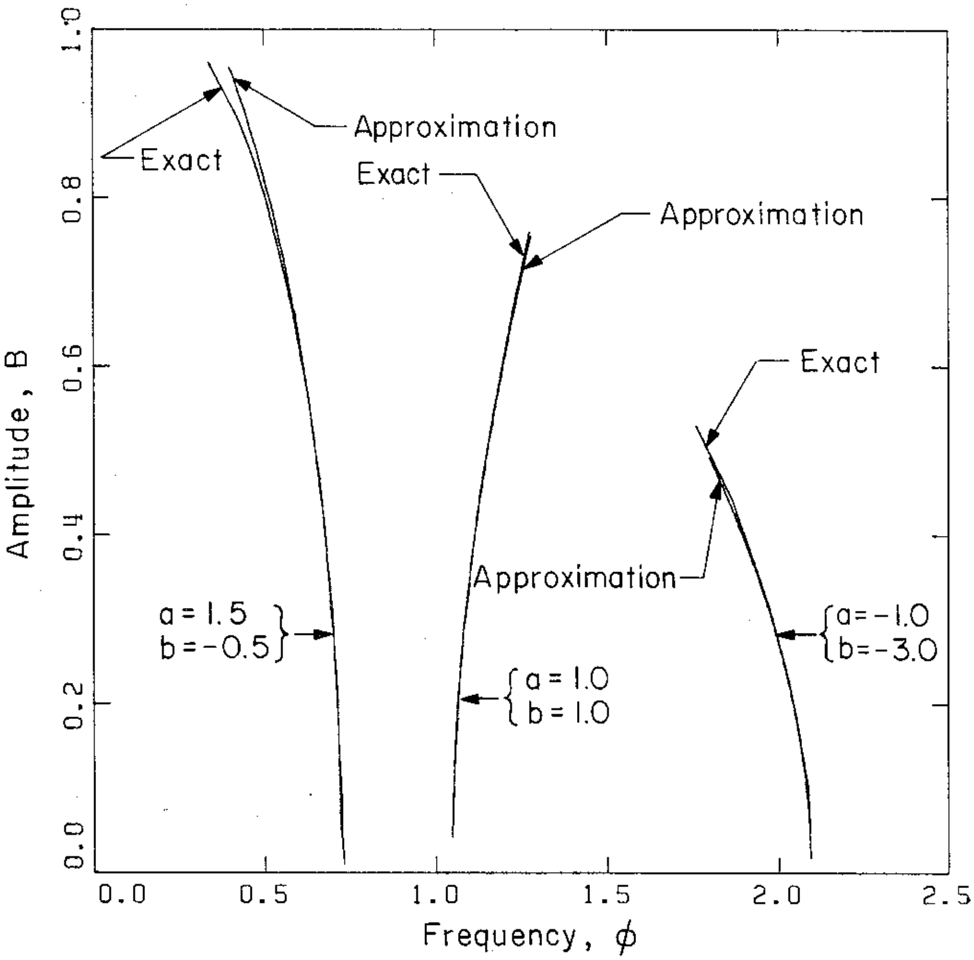


Fig. 3.1 Amplitude as a function of frequency of the steady state solution of the equation $x_{n+1} - ax_n + x_{n-1} + bx_n^3 = 0$

3.3 Forced Oscillations

Consider the forced nonlinear difference equation

$$\Delta^2 x_n + 2(1-\cos\vartheta)x_n + bx_n^3 = bP\sin(n\varphi) \tag{3.24}$$

Where $\varphi \approx \vartheta$. Assume as before a solution of the form

$$x = A(n)\cos(n\varphi) + B(n)\sin(n\varphi) \tag{3.25}$$

Where $A(n)$ and $B(n)$ are slowly varying functions. We will again use the auxiliary condition

$$\Delta A(n)\cos(n\varphi) + \Delta B(n)\sin(n\varphi) = 0 \tag{3.26}$$

So we have

$$\Delta^2 x = A(n)\Delta^2\cos(n\varphi) + B(n)\Delta^2\sin(n\varphi) \tag{3.27}$$

$$+ \Delta A(n)\Delta\cos(n\varphi) + \Delta B(n)\Delta\sin(n\varphi) \quad .$$

Equations (3.25), (3.26), and (3.27) into (3.24) yield

$$2[A(n)\cos(n\varphi)\sin(n\varphi) + B(n)\sin^2(n\varphi)](\cos\varphi - \cos\vartheta)$$

$$-2\sin(\tfrac{1}{2}\varphi)\Delta A(n) + b\,[A^3(n)\cos^3(n\varphi)\sin(n\varphi) + 3A^2(n)B(n)\cos^2(n\varphi)\sin^2(n\varphi)$$

$$+ 3A(n)B^2(n)\cos(n\varphi)\sin^3(n\varphi) + B^3(n)\sin^4(n\varphi)$$

$$= bP\sin^2(n\varphi) \tag{3.28}$$

and

$$2[\,A(n)\cos^2(n\varphi)+B(n)\sin(n\varphi)\cos(n\varphi)\,](\cos\varphi-\cos\vartheta)$$

$$+\,2\sin(\tfrac{1}{2}\,\varphi)\Delta B(n)+b\,\,[A^3(n)\cos^4(n\varphi)+3A^2(n)B(n)\cos^3(n\varphi)\sin(n\varphi)$$

$$+\,3A(n)B^2(n)\cos(n\varphi)\sin^2(n\varphi)+B^3(n)\sin^3(n\varphi)\cos(n\varphi)]$$

$$=P\cos(n\varphi)\sin(n\varphi) \tag{3.29}$$

but since $A(n)$ and $B(n)$ are slowly varying functions of n we have

$$B(n)[\cos\varphi-\cos\vartheta]-2\sin(\frac{\varphi}{2})\Delta A(n)$$

$$+\,\frac{3}{8}b[A^2(n)+B^2(n)]B(n)=\tfrac{1}{2}bP \tag{3.30}$$

and

$$A(n)[\cos\varphi-\cos\vartheta]+2\sin(\frac{\varphi}{2})\Delta B(n)$$

$$+\,\frac{3}{8}b[A^2(n)+B^2(n)]A(n)=0\ \ . \tag{3.31}$$

The steady state solution has constant amplitude and phase with respect to n. Hence for steady state we have

$$\Delta A(n)=\Delta B(n)=0\ \ . \tag{3.32}$$

Set

$$|A(n)|=A_s \ \text{ and } \ |B(n)|=B_s \tag{3.33}$$

The Equations (3.30) and (3.31) can now be written

$$\pm\,B_s[\cos\varphi-\cos\vartheta]+\frac{3}{8}b[A_s^2+B_s^2](\pm\,B_s)=\tfrac{1}{2}\,b\,P \tag{3.34}$$

$$\pm\,A_s[\cos\varphi-\cos\vartheta]+\frac{3}{8}b[A_s^2+B_s^2](\pm\,A_s)=0 \tag{3.35}$$

or

$$A_s = 0 \tag{3.36}$$

$$[\cos\varphi - \cos\vartheta] + \frac{3}{8} b\, B_s^2 = \pm \tfrac{1}{2}\, b\, P/B_s \tag{3.37}$$

Equation (3.37) determines A_s and B_s of the steady state solution

$$x_n = \pm\; B_s \sin(n\varphi)\quad . \tag{3.38}$$

Figures (3.2a) and (3.2b) show the relation between the amplitude B_s and the frequency, φ.

3.4 Ultraharmonic Response

Nonlinear systems often exhibit response frequencies which can significantly deviate from that of the forcing function. In the case of the discrete Duffing Equation, (3.24), we would expect an ultraharmonic response to prevail when the frequency of the forcing term is approximately one third of the linear natural frequency of the system. The ultraharmonic part of the response will then have a frequency which is three times the forcing frequency.

In this section we will find an approximate expression for the response of the discrete system governed by Equation (3.24) for the case when $\varphi \approx \frac{1}{3}\vartheta$.

Consider the equation

$$\Delta^2 x_n + 2(1-\cos\vartheta)x_n + b x_n^3 = b\, P\sin(n\varphi) \tag{3.39}$$

where

$$\varphi \approx \frac{1}{3}\vartheta\ . \tag{3.40}$$

For values of φ close to $\frac{1}{3}\vartheta$ the effect on the main response due to the nonlinearity is minimal. We can therefore write the solution as follows

$$x_n = Q\sin(n\varphi) + A(n)\cos(3n\varphi) + B(n)\sin(3n\varphi) \tag{3.41}$$

where $A(n)$ and $B(n)$ are slowly varying functions of n and where

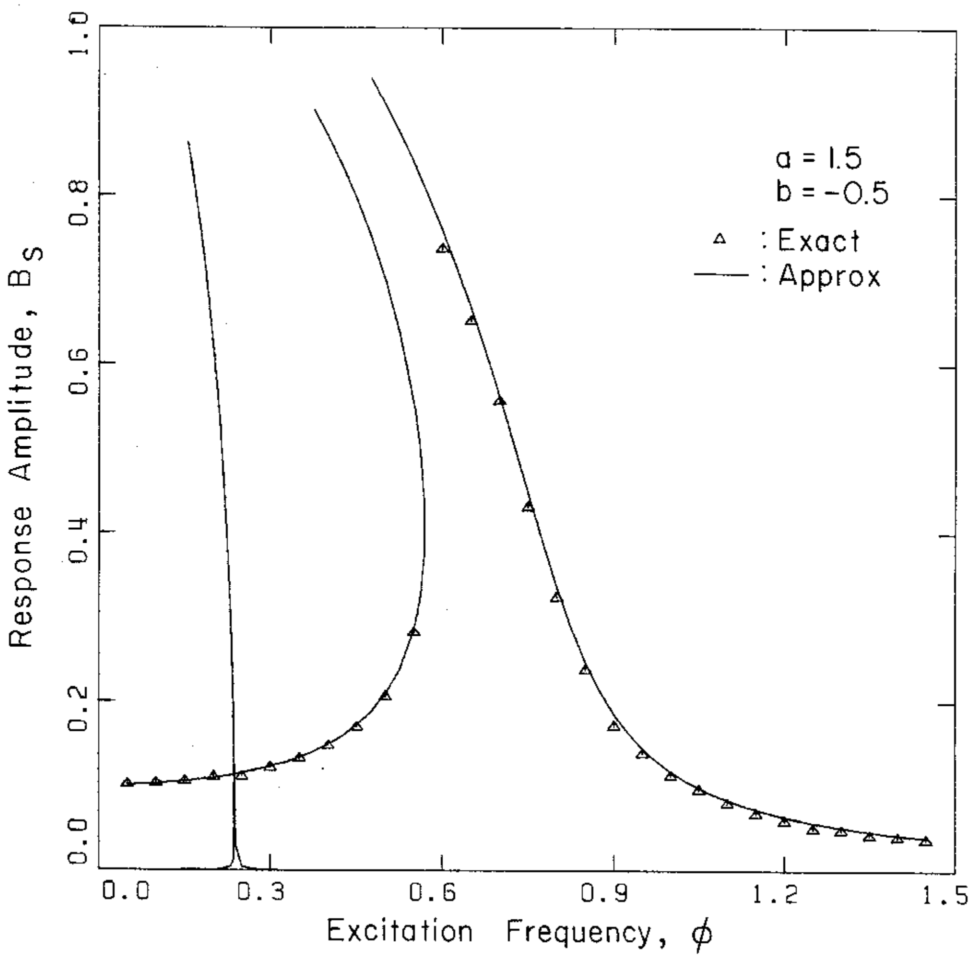


Fig 3.2.a Steady state response amplitude as a function of excitation frequency

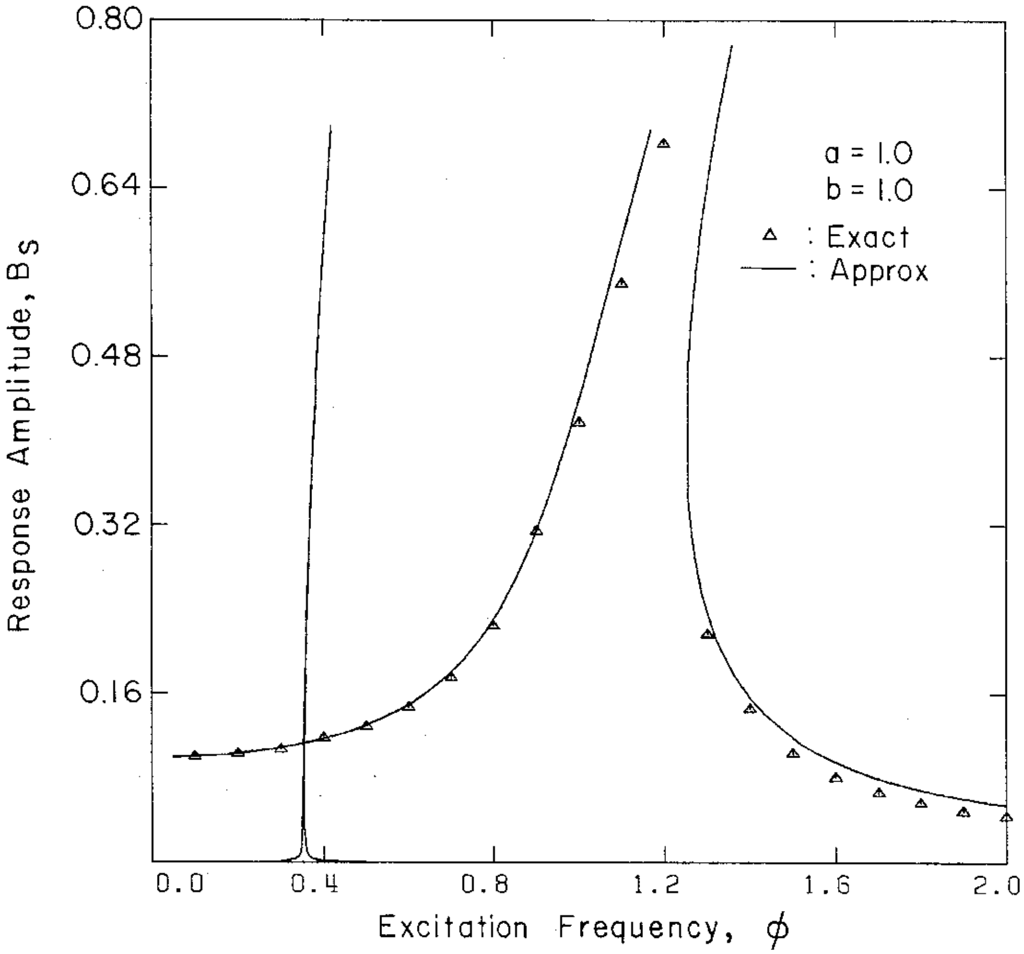


Fig. 3.2.b Steady state response amplitude as a function of excitation frequency

$$x_n = Q\sin(n\varphi) \tag{3.42}$$

satisfies the linear equation

$$\Delta^2 x_n + 2(1-\cos\vartheta)x_n = b\,\sin(n\varphi)\ . \tag{3.43}$$

By substituting (3.42) into Equation (3.43) we obtain

$$Q = bP[2(\cos\varphi - \cos\vartheta)]^{-1}\ . \tag{3.44}$$

The first central difference of x_n is

$$\Delta x_n = Q\Delta\sin(n\varphi) + A(n)\Delta\cos(3n\varphi) + B(n)\Delta\sin(3n\varphi)$$

$$+\Delta A(n)\cos(3n\varphi) + \Delta B(n)\sin(3n\varphi) \tag{3.45}$$

where we have neglected differences of order two and higher.

W e choose our auxiliary condition as

$$\Delta A(n)\cos(3n\varphi) + \Delta B(n)\sin(3n\varphi) = 0\ . \tag{3.46}$$

Hence

$$\Delta^2 x_n = Q\Delta^2\sin(n\varphi) + A(n)\Delta^2\cos(3n\varphi) + B(n)\Delta^2\sin(3n\varphi)$$

$$+\Delta A(n)\Delta\cos(3nPhi) + \Delta B(n)\Delta\sin(3n\varphi) \tag{3.47}$$

Substituting (3.47) into Equation (3.39), multiplying by sin $(3n\varphi)$ and noting that

$$\Delta\cos(3n\varphi) = -2\sin(3n\varphi)sin(\frac{3}{2}\varphi)$$

$$\Delta\sin(3n\varphi) = 2\cos(3n\varphi)sin(\frac{3}{2}\varphi)$$

$$\Delta^2\cos(3n\varphi) = 2[\cos(3\varphi)-1]\cos(3n\varphi)$$

$$\Delta^2\cos(3n\varphi) = 2[\cos(3\varphi)-1]\sin(3n\varphi) \tag{3.48}$$

we obtain after the use of the auxiliary condition (3.46)

$$2[A(n)\cos(3n\varphi)\sin(3n\varphi)+B(n)\sin^2(3n\varphi)][\cos(3\varphi)-\cos\vartheta]$$

$$-2\Delta A(n)\sin(\frac{3}{2}\varphi)+2Q\cos(3n\varphi)\sin(3n\varphi)[\cos\varphi-\cos\vartheta]$$

$$+b[Q\sin(n\varphi)+A(n)\cos(3n\varphi)+B(n)\sin(3n\varphi)]^3\sin(3n\varphi)=$$

$$bP\sin(3n\varphi)\sin(n\varphi)\ . \tag{3.49}$$

Using the fact that $A(n)$ and $B(n)$ are slowly varying functions of n we obtain

$$B(n)[\cos(3\varphi)-\cos\vartheta]-2\sin(\frac{3}{2}\varphi)\Delta A(n)$$

$$+b\,\frac{1}{8}[3B^3(n)-Q^3+3A^2(n)B(n)+6Q^2B(n)]=0\ . \tag{3.50}$$

A second equation relating $A(n)$ and $B(n)$ is obtained in a similar fashion multiplying Equation (3.39) by $\cos(3n\varphi)$.

$$A(n)[\cos(3n\varphi)-\cos\vartheta]+2\sin(\frac{3}{2}\varphi)\Delta B(n)$$

$$+b\,\frac{1}{8}[3A^3(n)+3A(n)B^2(n)+12Q^2A(n)]=0\ . \tag{3.51}$$

The steady state solution is obtained when

$$\Delta A=\Delta B=0\ . \tag{3.52}$$

The Equations (3.50) and (3.51) become

$$B(n)[\cos(3\varphi)-\cos\vartheta]+b\,\frac{1}{8}[3B^3(n)-Q^3+3A^2(n)B(n)+6Q^2B(n)]=0 \tag{3.53}$$

and

$$A(n)[\cos(3\varphi)-\cos\vartheta]+b\frac{1}{8}[3B^3(n)+3A(n)B^2(n)+12Q^2A(n)]=0\ . \tag{3.54}$$

For a nontrivial solution we can set

$$A(n)=0 \tag{3.55}$$

and

$$B(n)(\cos(3\varphi)-\cos\vartheta)+b\frac{1}{8}[3B^3(n)-Q^3+6Q^2B(n)]=0\ . \tag{3.56}$$

Since we have steady state we will call

$$|A(n)|=A_s \tag{3.57}$$

and

$$|B(n)|=B_s \tag{3.58}$$

in order to obtain the following from (3.55) and (3.56)

$$A_s=0 \tag{3.59}$$

and

$$[\cos(3\varphi)-\cos\vartheta]+b\frac{3}{8}[B_s^2+2Q^2]=\pm\ B_s^{-1}\frac{6}{8}Q \tag{3.60}$$

where Q is given by expression (3.44). Equation (3.60) relates the amplitude, B_s, of the solution

$$x_n=Q\sin(n\varphi)\pm\ B_s\sin(3n\varphi) \tag{3.61}$$

to the frequency, φ, of the forcing function. See Figures (3.2a) amd (3.2b).

3.5 Stability of the Steady State Solution

3.5.1 The Main Response

We will here investigate the local stability of the steady state response of the system governed by Equation (3.24) in Section 3.3. In order to perform the investigation, we perturb the solution of Equations (3.30) and (3.31). Hence assume

$$A(n) = \xi_n \tag{3.62}$$

$$B(n) = B + \eta_n$$

where $A(n) = 0$ and $B(n) = B$ is the steady state solution of (3.30) and (3.31). The linearized perturbation equations resulting from substituting (3.62) into (3.30) and (3.31) are:

$$-2\sin\frac{\varphi}{2}\Delta\xi_n + [(\cos\varphi - \cos\vartheta) + \frac{9}{8}b\,B^2]\eta_n = 0 \ . \tag{3.63}$$

$$2\sin\frac{\varphi}{2}\Delta\eta_n + [\cos\varphi - \cos\vartheta) + \frac{3}{8}b\,B^2]\xi_n = 0 \ . \tag{3.64}$$

For simplicity call

$$\nabla_1 = [\cos\varphi - \cos\vartheta] + \frac{3}{8}b\,B^2 \tag{3.65}$$

and

$$\nabla_2 = [\cos\varphi - \cos\vartheta] + \frac{9}{8}b\,B^2 \ . \tag{3.66}$$

Equation (3.64) gives

$$\Delta\xi_n = -2\sin(\frac{\varphi}{2})\,\Delta^2\eta_n\nabla_1^{-1} \ . \tag{3.67}$$

So Equation (3.63) yields

$$\Delta^2\eta_n + \frac{\nabla_1\nabla_2}{4\sin^2(\varphi/2)}\eta_n = 0 \ . \tag{3.68}$$

Now set

$$\eta_{n+1} = \lambda\,\eta_n \ . \tag{3.69}$$

For stability we must require

$$|\lambda| \leq 1 \tag{3.70}$$

Equation (3.69) substituted into Equation (3.68) gives

$$\lambda = 1 - \frac{\nabla_1 \nabla_2}{8\sin^2(\frac{\varphi}{2})} \pm \left[\left[1 - \frac{\nabla_1 \nabla_2}{8\sin^2(\frac{\varphi}{2})}\right]^2 - 1\right]^{1/2} . \tag{3.71}$$

So if

$$\left| 1 - \frac{1}{8} \frac{\nabla_1 \nabla_2}{\sin^2(\frac{\varphi}{2})} \right| \leq 1 \tag{3.72}$$

we have $|\lambda| \leq 1$ and hence a stable solution. The inequality (3.72) is satisfied if

$$0 \leq \nabla_1 \nabla_2 \leq 16\sin^2(\frac{\varphi}{2}) \tag{3.73}$$

Since $\varphi \approx \vartheta$ and $|b|$ is assumed to be small, the right inequality is always satisfied. So we are left with

$$[\cos\varphi - \cos\vartheta + \frac{3}{8} b\, B^2][\cos\varphi - \cos\vartheta + \frac{9}{8} b\, B^2] \geq 0 \ . \tag{3.74}$$

The inequality (3.74) gives a condition on B and φ for the steady state solution (3.38) of the homogeneous difference Equation (3.2) to be locally stable. Local and global stability of the solutions of the homogeneous Equation (3.2) was discussed in the previous chapter.

3.5.2 Stability of the Ultraharmonic Response

Section 3.4 we obtained an approximate solution (3.41) of Equation (3.39) which included the ultraharmonic response. W e arrived at the Equations (3.50) and (3.51) relating the slowly varying functions $A(n)$ and $B(n)$.

$$B(n)[\cos(3\varphi) - \cos\vartheta] - 2\sin(\frac{3}{2}\varphi)\Delta A(n) +$$

$$+ \frac{1}{8} b\,[3B^3(n) - Q^3 + 3A^2(n) + 6Q^2 B(n)] = 0 \tag{3.50}$$

and

$$A(n)[\cos(3\varphi)-\cos\vartheta]+2\sin(\frac{3}{2}\varphi)\Delta B(n)$$

$$+\frac{1}{8}b[3B^3(n)+3A(n)B^2(n)+12Q^2A(n)]=0 \ . \qquad (3.51)$$

In order to determine the local stability condition of the steady state solution of (3.50) and (3.51), hence also an approximate stability condition of the steady state solution of Equation (3.39), we proceed as follows. Let

$$A(n)=\xi_n \qquad (3.75)$$

$$B(n)=B+\eta_n \qquad (3.76)$$

where ξ_n and η_n are small perturbations and $B(n)=B$ and $A(n)=0$ is the steady state solution to (3.50) and (3.51).

Substituting (3.75) and (3.76) into (3.50) and (3.51) yields

$$-2\sin(\frac{3}{2}\varphi)\Delta\xi_n+\Omega_2\eta_n=0 \qquad (3.77)$$

$$2\sin(\frac{3}{2}\varphi)\Delta\eta_n+\Omega_1\xi_n=0 \qquad (3.78)$$

where

$$\Omega_1=\cos(3\varphi)-\cos\vartheta+b\frac{3}{8}\left[B^2+\left(\frac{bP}{\cos\varphi-\cos\vartheta}\right)^2\right] \qquad (3.79)$$

$$\Omega_2=\cos(3\varphi)-\cos\vartheta+b\frac{3}{16}\left[6B^2+\left(\frac{bP}{\cos\varphi-\cos\vartheta}\right)^2\right] \qquad (3.80)$$

Taking the central difference of the Equation (3.78) gives

$$2\sin(\frac{3}{2}\varphi)\Delta^2\eta_n + \Omega_1\Delta\xi_n = 0 \ . \tag{3.81}$$

By solving for $\Delta\xi_n$ in (3.81) and substituting into (3.98) we obtain

$$4\sin^2(\frac{3}{2}\varphi)\Delta^2\eta_n + \Omega_1\Omega_2\eta_n = 0 \ . \tag{3.82}$$

Hence the stability condition becomes

$$\Omega_1\Omega_2 \geq 0 \tag{3.83}$$

or finally

$$\{[\cos(3\varphi) - \cos\vartheta] - \frac{1}{8}b\,[9B^2 + 6Q^2]\}\{[\cos(3\varphi) - \cos\vartheta] + \frac{1}{8}b\,[3B^2 + 12Q^2]\} \geq 0 \ , \tag{3.84}$$

which is our stability criterion for the steady state solution

$$x = Q\sin(n\varphi) + B\sin(3\varphi) \tag{3.85}$$

3.6 Approximate Solutions of Higher Order

3.6.1 Use of the Exact Central Difference Formulation

In Section 3.2 we applied a slowly varying parameter technique to a homogeneous equation. In doing so we used approximate expressions for both the first, Δx_n, and the second, $\Delta^2 x_n$, central difference of x_n. In order to achieve a higher order approximation these approximations will be replaced by exact expressions.

As in Section 3.2 consider the homogeneous Equation (3.15)

$$\Delta^2 x_n + 2(1-\cos\vartheta)x_n + bx_n^3 = 0 \ . \tag{3.2}$$

As before we assume a solution of the form

$$x_n = A(n)\cos(n\varphi) + B(n)\sin(n\varphi) \tag{3.86}$$

Retaining all terms in the expression for the first central difference of x_n gives.

$$\Delta x_n = \Delta\, A(n)\cos(n\varphi) + A(n)\Delta\cos(n\varphi) + \tfrac{1}{2}\Delta A(n)\{\cos(n+\tfrac{1}{2})\varphi - 2\cos(n\varphi)$$

$$+\cos(n-\tfrac{1}{2})\varphi\} + \tfrac{1}{2}\Delta\cos(n\varphi)\{A(n+\tfrac{1}{2}) - 2A(n) + A(n-\tfrac{1}{2})\} + \Delta B(n)\sin(n\varphi)$$

$$+B(n)\Delta\sin(n\varphi) + \tfrac{1}{2}\Delta B(n)\{\sin(n+\tfrac{1}{2})\varphi - 2\sin(n\varphi) + \sin(n-\tfrac{1}{2})\varphi\}$$

$$+\tfrac{1}{2}\Delta\sin(n\varphi)\{B(n+\tfrac{1}{2}) - 2B(n) + B(n-\tfrac{1}{2})\} \ . \tag{3.87}$$

As an auxiliary condition we require

$$\Delta A(n)\{\cos(n+\tfrac{1}{2})\varphi + \cos(n-\tfrac{1}{2})\varphi\}$$

$$+\tfrac{1}{2}\Delta\cos(n\varphi)\{A(n+\tfrac{1}{2}) - 2A(n) + A(n-\tfrac{1}{2})\}$$

$$+\tfrac{1}{2}\Delta B(n)\{\sin(n+\tfrac{1}{2})\varphi + \sin(n-\tfrac{1}{2})\varphi\}$$

$$+\tfrac{1}{2}\Delta\sin(n\varphi)\{AB(n+\tfrac{1}{2}) - 2B(n) + B(n-\tfrac{1}{2})\} = 0 \ . \tag{3.88}$$

Hence

$$\Delta x_n = A(n)\Delta\cos n\varphi + B(n)\Delta\sin n\varphi \ . \tag{3.89}$$

The exact expression for the second difference of x_n becomes

$$\Delta^2 x_n = \tfrac{1}{2}\Delta A(n)\{\Delta\cos(n+\tfrac{1}{2})\varphi + \Delta\cos(n-\tfrac{1}{2})\varphi\}$$

$$+\tfrac{1}{2}\Delta^2\cos(n\varphi)\{A(n+\tfrac{1}{2}) + A(n-\tfrac{1}{2})\}$$

$$+\tfrac{1}{2}\Delta B(n)\{\Delta\sin(n+\tfrac{1}{2})\varphi + \Delta\sin(n-\tfrac{1}{2})\varphi\} \tag{3.90}$$

$$+\tfrac{1}{2}\Delta^2\sin(n\varphi)\{B(n+\tfrac{1}{2}) + B(n-\tfrac{1}{2})\} \ .$$

Using Equations (3.86), (3.13), and (3.90) and noting that

$$\Delta \cos(n+\tfrac{1}{2})\varphi = 2\sin(n+\tfrac{1}{2})\varphi \sin(\tfrac{1}{2}\varphi)$$

$$\Delta \cos(n-\tfrac{1}{2})\varphi = 2\sin(n-\tfrac{1}{2})\varphi \sin(\tfrac{1}{2}\varphi)$$

$$\Delta \sin(n+\tfrac{1}{2})\varphi = 2\cos(n+\tfrac{1}{2})\varphi \sin(\tfrac{1}{2}\varphi)$$

$$\Delta \sin(n-\tfrac{1}{2})\varphi = 2\cos(n-\tfrac{1}{2})\varphi \sin(\tfrac{1}{2}\varphi) \qquad (3.91)$$

the Equation (3.2) yields

$$2[A(n)\cos(n\varphi) + B(n)\sin(n\varphi)](1-\cos\vartheta)$$

$$+ [\{A(n+\tfrac{1}{2}) + A(n-\tfrac{1}{2})\}\cos(n\varphi) + \{B(n+\tfrac{1}{2}) + B(n-\tfrac{1}{2})\}\sin(n\varphi)](\cos\varphi - 1)$$

$$-\Delta A(n)\{\sin(n+\tfrac{1}{2})\varphi + \sin(n-\tfrac{1}{2})\varphi\}\sin(\tfrac{1}{2}\varphi) + \Delta B(n)\{\cos(n+\tfrac{1}{2})\varphi + \cos(n-\tfrac{1}{2})\varphi\}\sin(\tfrac{1}{2}\varphi)$$

$$+ b[A(n)\cos(n\varphi) + B(n)\sin(n\varphi)]^3 = 0 \ . \qquad (3.92)$$

By multiplying Equation (3.92) by $\sin(n\varphi)$ and using the auxilliary condition (3.88) and the fact that $A(n)$ and $B(n)$ are slowly varying functions it is possible to obtain

$$B(n)[1-\cos\vartheta] - (B(n+\tfrac{1}{2}) - B(n) + B(n-\tfrac{1}{2}))(1-\cos\varphi)$$

$$-\Delta A(n)\sin\varphi + \frac{3}{8} b\,[A^2(n) + B^2(n)]B(n) = 0 \ . \qquad (3.93)$$

Repeating the same procedure but multiplying Equation (3.92) by $\cos(n\varphi)$ we achieve

$$A(n)[1-\cos\vartheta] - (A(n+\tfrac{1}{2}) - A(n) + A(n-\tfrac{1}{2})][1-\cos\varphi]$$

$$+\Delta B(n)\sin\varphi + \frac{3}{8} b\,[A^2(n) + B^2(n)]A(n) = 0 \ . \qquad (3.94)$$

The slowly varying parameters, $A(n)$ and $B(n)$, in the solution (3.86) can be determined by the use of the Equations (3.93) and (3.94).

For steady state solutions we have $\Delta A(n) = \Delta B(n) = 0$. Hence

$$B(n)(\cos\varphi - \cos\vartheta) + \frac{3}{8}b\,[A^2(n) + B^2(n)]B(n) = 0 \tag{3.95}$$

$$A(n)(\cos\varphi - \cos\vartheta) + \frac{3}{8}b\,[A^2(n) + B^2(n)]A(n) = 0 \ . \tag{3.96}$$

If we set

$$A(n) = 0 \tag{3.97}$$

we have

$$(\cos\varphi - \cos\vartheta) + \frac{3}{8}b\,B_s^2 = 0 \tag{3.98}$$

where

$$|B(n)| = B_s \tag{3.99}$$

Equation (3.98) determines the amplitude, B_s, of the steady state response. It is realized that Equation (3.98) is identical to Equation (3.22) which was derived with approximate expressions for Δx_n and $\Delta^2 x_n$.

3.6.2 Stability Analysis

In order to determine the local stability of the steady state solution obtained above we perturb the solution as follows

$$A(n) = \xi_n$$

$$B(n) = B + \eta_n \tag{3.100}$$

where ξ_n and η_n are small perturbations. Substituting the perturbed steady state solutions (3.100) into Equation (3.93) and (3.105)

$$\xi_n \nabla_1 + \Delta\eta_n \sin\varphi + (\cos\varphi - 1)(\xi_{n+1/2} + \xi_{n-1/2}) = 0 \tag{3.101}$$

and

$$\eta_n \nabla_2 + \Delta\xi_n \sin\varphi + (\cos\varphi - 1)(\eta_{n+\frac{1}{2}} + \eta_{n-\frac{1}{2}}) = 0 \tag{3.102}$$

where

$$\nabla_1 = 1 - \cos\vartheta + 2\sin^2\left(\frac{\varphi}{2}\right) + \frac{3}{8} b\, B^2 \tag{3.103}$$

$$\nabla_2 = 1 - \cos\vartheta + 2\sin^2\left(\frac{\varphi}{2}\right) + \frac{9}{8} b\, B^2 \ . \tag{3.104}$$

Assume a solution of the form

$$\xi_n = C\, p^{2n} \tag{3.105}$$

$$\eta_n = D\, p^{2n} \ .$$

Hence we have

$$\xi_{n+\frac{1}{2}} = \xi_n p \tag{3.106}$$

$$\eta_{n+\frac{1}{2}} = \eta_n p$$

and the Equations (3.101) and (3.102) yield

$$\xi_{n-\frac{1}{2}}[p\nabla_1 + (\cos\varphi - 1)(p^2+1)] + \eta_{n-\frac{1}{2}}(p^2-1)\sin\varphi = 0 \tag{3.107}$$

$$-\xi_{n-\frac{1}{2}}(p^2-1)\sin\varphi + \eta_{n-\frac{1}{2}}[p\nabla_2 + (\cos\varphi - 1)(p^2+1)] = 0 \ . \tag{3.108}$$

For a nontrivial solution of these equations we must require

$$[(\cos\varphi - 1)(1+p^2) + p\nabla_1][(\cos\varphi - 1)(1+p^2) + p\nabla_2] + (p^2\sin\varphi - \sin\varphi)^2 = 0 \ . \tag{3.109}$$

This can be simplified to yield

$$p^4 + \mathbf{A}\, \mathrm{p}^3 + \mathbf{B}\, \mathrm{p}^2 + \mathbf{A}\ \mathrm{p} + 1 = 0 \tag{3.110}$$

where

$$\mathbf{A} = \frac{-2(\nabla_1 + \nabla_2)\sin^2\left(\frac{\varphi}{2}\right)}{4\sin^4\left(\frac{\varphi}{2}\right) + \sin^2\varphi} \tag{3.111}$$

and

$$\mathbf{B} = -\frac{8\cos\varphi\sin^2(\frac{\varphi}{2}) + \nabla_1\nabla_2}{4\sin^4(\frac{\varphi}{2}) + \sin^2\varphi} \tag{3.112}$$

We are searching for the region in the **A** and **B** plane where the solution of Equation (3.110) satisfies the condition $|p| \le 1$. This region would then, as seen from (3.105), correspond to the region of stability for the steady state solution of our difference equation.

The region for which $|p| \le 1$ is easily determined to be

$$(\mathbf{B} + 2 - 2\mathbf{A})(\mathbf{B} + 2 + 2\mathbf{A}) \ge 0 \ . \tag{3.113}$$

If the expression (3.111) and (3.112) for **A** and **B** are substituted into the inequality (3.113) we get

$$[-8\cos\varphi\sin^2(\frac{\varphi}{2}) + \nabla_1\nabla_2 + 8\sin^4(\frac{\varphi}{2}) + 2\sin^2\varphi + 4(\nabla_1 + \nabla_2)\sin^2(\frac{\varphi}{2})] \times \tag{3.114}$$

$$[-8\cos\varphi\sin^2(\frac{\varphi}{2}) + \nabla_1\nabla_2 + 8\sin^4(\frac{\varphi}{2}) + 2\sin^2\varphi - 4(\nabla_1 + \nabla_2)\sin^2(\frac{\varphi}{2})] \ge 0 \ .$$

Inequality (3.114) can be written as follows

$$\left\{22 + (-24 + 6\cos\vartheta)\cos\varphi + 5\cos(2\vartheta) - 8\cos\vartheta + bB^2[6 - \frac{9}{2}\cos\varphi - \frac{3}{2}\cos\vartheta] + \frac{27}{64}b^2B^4\right\} \times$$

$$\left\{1 - 2\cos\vartheta\cos\varphi + \frac{1}{2}(\cos(2\varphi) + \cos(2\vartheta)) + \frac{3}{2}bB^2(\cos\varphi - \cos\vartheta) + \frac{27}{64}b^2B^4\right\} \ge 0 \tag{3.115}$$

but

$$22 + (-24 + 6\cos\vartheta)\cos\varphi + 5\cos(2\vartheta) - 8\cos\vartheta$$

$$+ bB^2[6 - \frac{9}{2}\cos\varphi - \frac{3}{2}\cos\vartheta] + \frac{27}{64}b^2B^4 \ge 0 \tag{3.116}$$

for $\vartheta \approx \varphi$, which is the region of interest, and $|b|$ small. So for stability we must satisfy

$$1-2\cos\vartheta\cos\varphi+\tfrac{1}{2}(\cos(2\varphi)+\cos(2\vartheta))+\frac{3}{2}bB^2(\cos\varphi-\cos\vartheta)+\frac{27}{64}b^2B^4\geq 0 \qquad (3.117)$$

which can be written

$$[\cos\varphi-\cos\vartheta+\frac{3}{8}b\,B^2][\cos\varphi-\cos\vartheta+\frac{9}{8}b\,B^2]\geq 0 \qquad (3.118)$$

This is exactly the stability criterion for the steady state solution, obtained previously, when approximations for the first and second central difference of x_n were used.

3.6.3 Higher Order Approximation by the Use of the Third Harmonic

Assume a solution to the difference equation

$$\Delta^2 x_n + 2(1-\cos\vartheta)x_n + bx_n^3 = 0 \qquad (3.119)$$

to be of the form

$$x_n = A(n)\cos(n\varphi) + B(n)\sin(n\varphi) + C(n)\cos(3n\varphi) + D(n)\sin(3n\varphi) \quad . \qquad (3.120)$$

Then

$$\Delta x_n = A(n)\Delta\cos(n\varphi) + B(n)\Delta\sin(n\varphi) + C(n)\Delta\cos(3n\varphi) + D(n)\Delta\sin(3n\varphi)$$

$$+\Delta A(n)\cos(n\varphi) + \Delta B(n)\sin(n\varphi) + \Delta C(n)\cos(3n\varphi) + \Delta D(n)\sin(3n\varphi) \quad . \qquad (3.121)$$

Since we have four functions of n describing x_n, we can arbitrarily choose the following as our auxiliary conditions

$$\Delta A(n)\cos(n\varphi) + \Delta B(n)\sin(n\varphi) = 0 \quad , \qquad (3.122)$$

$$\Delta C(n)\cos(3n\varphi) + \Delta D(n)\sin(3n\varphi) = 0 \quad , \qquad (3.123)$$

$$\Delta A(n)\cos(3n\varphi) + \Delta D(n)\sin(n\varphi) = 0 \quad . \qquad (3.124)$$

Using these conditions in the expression for Δx_n (3.121) yields

$$\Delta^2 x_n = A(n)\Delta\cos(n\varphi) + B(n)\Delta\sin(n\varphi) + C(n)\Delta\cos(3n\varphi) + D(n)\Delta\sin(3n\varphi) \quad . \tag{3.125}$$

We also have

$$\Delta^2 x_n = A(n)\Delta^2\cos(n\varphi) + B(n)\Delta^2\sin(n\varphi) + C(n)\Delta^2\cos(3n\varphi) + D(n)\Delta^2\sin(3n\varphi)$$

$$+\Delta A(n)\Delta\cos(n\varphi) + \Delta B(n)\Delta\sin(n\varphi) \tag{3.126}$$

$$+\Delta C(n)\Delta\cos(3n\varphi) + \Delta D(n)\Delta\sin(3n\varphi) \quad .$$

Using the relationships (3.24) and (3.123) and Equations (3.13) and (3.126), the Equation (3.119) becomes

$$2(A(n)\cos(n\varphi) + B(n)\sin(n\varphi))(\cos\varphi - \cos\vartheta) + 2(C(n)\cos(3n\varphi) + D(n)\sin(3n\varphi))\times$$

$$(\cos(3\varphi) - \cos\vartheta) - 2\Delta A(n)\sin(n\varphi)\sin(\frac{\varphi}{2}) + 2\Delta B(n)\cos(n\varphi)\sin(\frac{\varphi}{2})$$

$$-2\Delta C(n)\sin(3n\varphi)\sin(\frac{3}{2}\varphi) + 2\Delta D(n)\cos(3n\varphi)\sin(\frac{3}{2}\varphi)$$

$$+ b[A(n)\cos(n\varphi) + B(n)\sin(n\varphi) + C(n)\cos(3n\varphi) + D(n)\sin(3n\varphi)]^3 = 0 \quad . \tag{3.127}$$

From the auxiliary condition (3.122), (3.123), and (3.124) we have

$$\Delta A(n) = \sin(n\varphi)\cos^{-1}(n\varphi)\Delta B(n) \quad , \tag{3.128}$$

$$\Delta C(n) = \sin(3n\varphi)\cos^{-1}(n\varphi)\Delta B(n) \quad , \tag{3.129}$$

$$\Delta D(n) = \cos(3n\varphi)\cos^{-1}(n\varphi)\Delta B(n) \quad , \tag{3.130}$$

Multiplying Equation (3.127) by $\cos(n\varphi)$ and using Equations (3.128), (3.129), and (3.130) and the fact that $A(n)$, $B(n)$, $C(n)$, and $D(n)$ are all slowly varying functions yield

$$A(n)[\cos\varphi - \cos\vartheta] + \Delta B(n)[\sin(\frac{\varphi}{2}) + \sin(\frac{3}{2}\varphi)]$$

$$+ b[\frac{3}{8}(BA^2 + B^2)A + \frac{1}{4}(C^2 + D^2)A] = 0 \quad . \tag{3.131}$$

Next multiply (3.127) by $sin(n\varphi)$ and use

$$\Delta B(n) = cos(n\varphi)\sin^{-1}(n\varphi)\Delta A$$

$$\Delta C(n) = sin(3n\varphi)\sin^{-1}(n\varphi)\Delta A$$

$$\Delta D(n) = -cos(3n\varphi)\sin^{-1}(n\varphi)\Delta A \qquad (3.132)$$

to get the following

$$B(n)[\cos\varphi - \cos\vartheta] - 2\Delta A(n)[\sin(\frac{\varphi}{2}) + \sin(\frac{3}{2}\varphi)]$$

$$+ b\,[\frac{3}{8}(B^2 + A^2)B + \frac{1}{4}(C^2 + D^2)B] = 0 \ . \qquad (3.133)$$

Where we have again used the fact that $A(n)$, $B(n)$, $C(n)$, $D(n)$ are slowly varying functions.

By a similar procedure we obtain the equations

$$C(n)[\cos(3\varphi) - \cos\vartheta] - 2\Delta C(n)[\sin(\frac{\varphi}{2}) + \sin(\frac{3}{2}\varphi)]$$

$$+ b\,[\frac{3}{8}(C^2 + D^2)C^2 + D^2)C + \frac{1}{4}(A^2 + B^2)C + \frac{1}{8}A^3] = 0 \qquad (3.134)$$

$$D(n)cos(3\varphi) - \cos\vartheta] - 2\Delta C(n)[\sin(\frac{\varphi}{2}) + \sin(\frac{3}{2}\varphi)$$

$$+ b\,[\frac{3}{8}(C^2 + D^2)D + \frac{1}{4}(A^2 + 8^2)D + \frac{1}{8}^3] = 0 \ . \qquad (3.135)$$

The difference Equations (3.131), (3.133), (3.134), and (3.135) determine $A(n)$, $B(n)$, $C(n)$, and $D(n)$.

By setting $\Delta A(n) = \Delta B(n) = \Delta D(n) = 0$ we obtain the following expressions for the steady state solution

$$A\,[\cos\varphi - \cos\vartheta] + b\left[\frac{3}{8}(A^2 + B^2) + \frac{1}{4}(C^2 + D^2)\right]A = 0 \ , \tag{3.136}$$

$$B[\cos\varphi - \cos\vartheta] + b\left[\frac{3}{8}(A^2 + B^2) + \frac{1}{4}(C^2 + D^2)\right]B = 0 \ , \tag{3.137}$$

$$C\,[\cos(3\varphi) - \cos\vartheta] + b\left[\left\{\frac{3}{8}(C^2 + D^2) + \frac{1}{4}(A^2 + B^2)\right\}C + \frac{1}{8}A^3\right] = 0 \ , \tag{3.138}$$

$$D\,[\cos(3\varphi) - \cos\vartheta] + b\left[\left\{\frac{3}{8}(C^2 + D^2) + \frac{1}{4}(A^2 + B^2)\right\}D + \frac{1}{8}B^3\right] = 0 \ . \tag{3.139}$$

Now if we set

$$B = D = 0 \tag{3.140}$$

we get

$$\beta + \frac{1}{8}b\,[3A^2 + 2C^2] = 0 \tag{3.141}$$

and

$$\alpha C + \frac{1}{8}b\,[\{3C^2 + 2A^2C + A^3\}] = 0 \tag{3.142}$$

where

$$\alpha = \cos 3\varphi - \cos\vartheta \ , \tag{3.143}$$

$$\beta = \cos\varphi - \cos\vartheta \ . \tag{3.144}$$

Equation (3.141) yields

$$C^2 = -\frac{4}{b}\beta - \frac{3}{2}A^2 \tag{3.145}$$

Using this expression for C^2 we can eliminate C in Equation (3.142) in order to get

$$\left[-4\alpha^2 + 12\alpha\beta - 9\beta^2\right]\beta + \frac{b}{2}\left[14\alpha\beta - 3\alpha^2 - \frac{57}{4}\beta^2\right]A^2$$

$$+ b^2\left[-\frac{115}{64}\beta + \frac{15}{16}\alpha\right]A^4 - b^3\frac{83}{512}A^6 = 0 \ . \qquad (3.146)$$

This sixth order polynomial determines A given φ and b. C is then determined by Equation (3.145) in order to get the steady state solution

$$x_n = A\cos n\varphi + C\cos 3n\varphi \ . \qquad (3.147)$$

A numerical comparison between the lower order approximate solution (3.23) and the solution obtained here is shown in Figure (3.3).

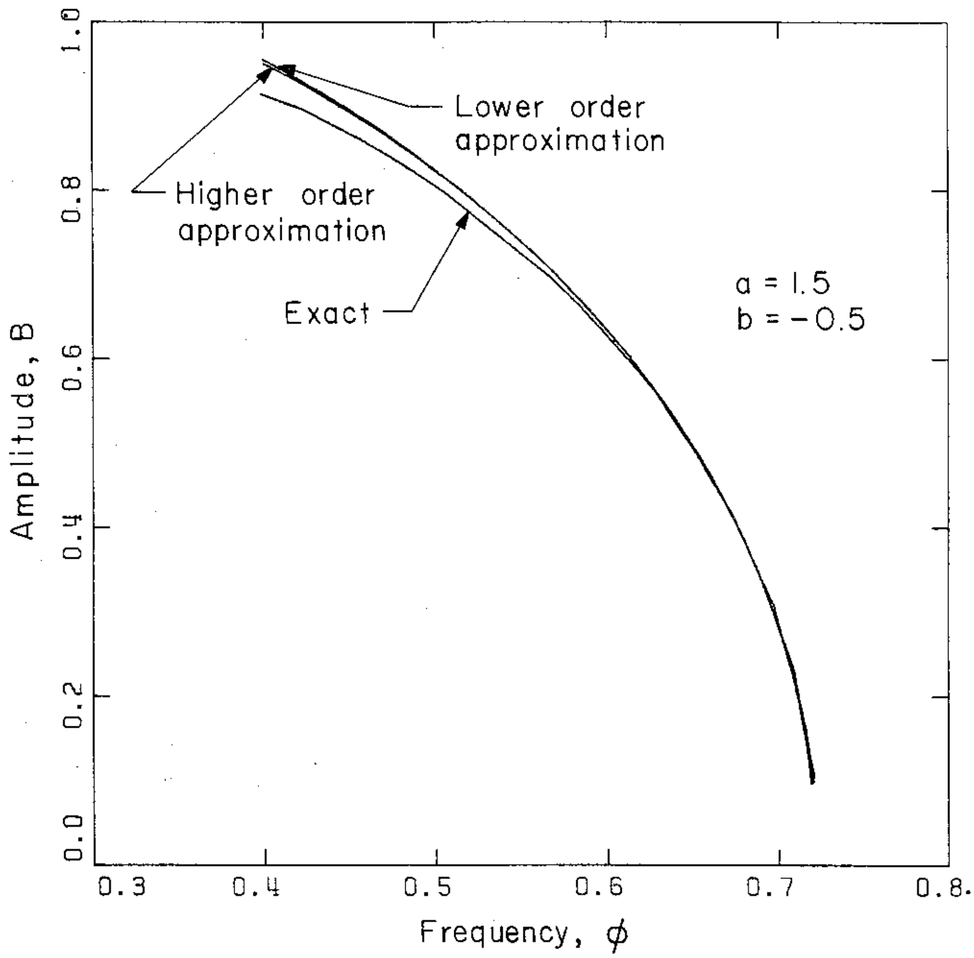


Fig. 3.3 Amplitude as a function of frequency of the steady state solution of the equation $x_{n+1} - ax_n + x_{n-1} + bx_n^3 = 0$

4 THE DISCRETE MATHIEU EQUATION

4.1 Linearization of the Nonlinear Difference Equation

Consider as in the previous chapter the nonlinear difference equation

$$\Delta^2 x_n + (1-\cos\vartheta)x_n + bx_n^3 = bP\sin n\varphi \tag{4.1}$$

Assume x_n^* to be a steady state solution to this equation. Let the pertubation of x_n^* be ξ_n. Hence

$$x_n = \xi_n + x_n^* \ . \tag{4.2}$$

Substitution of (4.2) into Equation (4.1) gives

$$\Delta^2 \xi_n + 2(1-\cos\vartheta)\xi_n + 3b\,(x_n^*)^2 \xi_n = 0 \tag{4.3}$$

where we have neglected terms of $0(\xi^2)$ and higher.

An approximate expression for the steady state solution, x_n^*, was obtained in the previous chapter, Section 3.3. We can therefore write

$$x_n^* = A\cos n\varphi \tag{4.4}$$

Equation (4.3) then becomes

$$\Delta^2 \xi_n + (\alpha + \beta\cos(2n\varphi))\xi_n = 0 \tag{4.5}$$

where

$$\alpha = 2(1-\cos\vartheta) + \frac{3b}{2}A^2 \tag{4.6}$$

and

$$\beta = \frac{3}{2}bA^2 \ . \tag{4.7}$$

We will call Equation (4.5), the discrete Mathieu equation due to the obvious similarity with the well known continuous Mathieu equation.

4.2 Stability Boundaries of the Discrete Mathieu Equation for Small β

4.2.1 Expansion of the Solution

For small values of β it is possible to express the stability boundaries corresponding to the solution of the Equation (4.5), in the α, β plane, as an expansion. Let the boundaries be given by

$$\alpha = \alpha_o + \beta\alpha_1 + \beta^2\alpha_2 + \dots \tag{4.8}$$

and let the solution be given by

$$\xi_n = \xi_{o_n} + \beta\xi_{1_n} + \beta^2\xi_{2_n} + \cdots \tag{4.9}$$

where $\xi_{o_n}, \xi_{1_n} \cdots$ have a period of $\frac{\pi}{\varphi}$ or $\frac{2\pi}{\varphi}$.

Substituting the expansions (4.8) and (4.9) into Equation (4.5) and equating coefficients of like powers of β yield

$$\beta^o: \ \xi_{o_{n+1}} - (2-\alpha_o)\xi_{o_n} + \xi_{o_{n-1}} = 0 \tag{4.10}$$

$$\beta^1: \ \xi_{1_{n+1}} - (2-\alpha_o)\xi_{1_n} + \xi_{1_{n-1}} = -(\alpha_1 + \cos(2n\varphi))\xi_{o_n} \tag{4.11}$$

$$\beta^2: \ \xi_{2_{n+1}} - (2-\alpha_o)\xi_{2_n} + \xi_{2_{n-1}} = -\alpha_2\xi_{o_n} - (\alpha_1 + \cos(2n\varphi))\xi_{1_n} \tag{4.12}$$

$$\beta^3: \ \xi_{3_{n+1}} - (2-\alpha_o)\xi_{3_n} + \xi_{3_{n-1}} = -\alpha_3\xi_{o_n} - \alpha_2\xi_{1_n} - (\alpha_1 + \cos(2n\varphi))\xi_{2_n} \ . \tag{4.13}$$

We will treat each equation separately starting with (4.10).

The solution to the linear difference Equation (4.10) is

$$\xi_{o_n} = A_o \cos(n\psi) + B_o \sin(n\psi) \tag{4.14}$$

where

$$\psi = \cos^{-1}\left[\frac{2-\alpha_o}{2}\right] \ . \tag{4.15}$$

4.2.2 Solutions Having a Period of π/φ

We will first look for solutions having a period of π/φ. Set

$$\alpha_0 = 0 \tag{4.16}$$

The Equation (4.10) has then the periodic solution

$$\xi_{o_n} = A_o \tag{4.17}$$

Using the fact that $\alpha_o = 0$ and $\xi_{o_n} = A_o$ we can write Equation (4.11) as

$$\xi_{1_{n+1}} - 2\xi_{1_n} + \xi_{1_{n-1}} = -A_o(\alpha_1 + \cos(2n\varphi)) \ . \tag{4.18}$$

Since we are looking for periodic solutions of period π/φ we must set

$$\alpha_1 = 0 \ . \tag{4.19}$$

The periodic solutions of Equation (1.5) are then of the form

$$\xi_{1_n} = A_1 \cos(2n\varphi) + C_1 \tag{4.20}$$

where

$$A_1 = \frac{A_o}{2(1-\cos(2\varphi))} \ . \tag{4.21}$$

If we now substitute solutions (4.17) and (4.20) and the values of α_o and α_1 into Equation (4.12) we can write the following

$$\xi_{2_{n+1}} - 2\xi_{2_n} + \xi_{2_{n-1}} = -\alpha_2 A_o - \tfrac{1}{2}A_1 - \tfrac{1}{2}A_1\cos(2n\varphi) - C_1\cos(2n\varphi) \ . \tag{4.22}$$

In order for Equation (4.22) to have periodic solutions we must set

$$-\alpha_2 A_o - \tfrac{1}{2} A_1 = 0 \tag{4.23}$$

which by using (4.21) yields

$$\alpha_2 = -\frac{1}{4}(1-\cos(2\varphi))^{-1} \ . \tag{4.24}$$

By using the two first terms in the expansion for α (4.8) we can now produce an approximate expression for the stability boundary where the solution has a period of $\frac{\pi}{\varphi}$. By substituting the values obtained for α_0, α_1, and α_2 into (4.8) we get as our stability boundary

$$\alpha = -\frac{1}{4}(1-\cos(2\varphi))^{-1}\beta^2 \ . \tag{4.25}$$

We continue our search for additional $\frac{\pi}{\varphi}$ period stability boundaries. By setting

$$\alpha_0 = (2\varphi)^2 \tag{4.26}$$

we obtain from Equation (4.15)

$$\psi = 2\varphi \ . \tag{4.27}$$

Where we have assumed α_0 to be sufficiently small for us to justify the neglect of all terms of $O(\varphi^4)$ and higher. Hence, by using (4.14) we arrive at the following solution to Equation (4.10).

$$\xi_{0_n} = A_0 \cos(2n\varphi) + B_0 \sin(2n\varphi) \tag{4.28}$$

The value of α_0 (4.26) and the solution (4.28) substituted into Equation (4.11) yields the equation

$$\xi_{1_{n+1}} - (2-4\varphi^2)\xi_{1_n} + \xi_{1_{n-1}} = -(\alpha_1 + \cos(2n\varphi))(A_0 \cos(2n\varphi) + B_0 \sin(2n\varphi)) \tag{4.29}$$

In order for the forcing term not to cause resonance we must set

$$\alpha_1 = 0 \ . \tag{4.30}$$

Equation (4.29) can then be written as

$$\xi_{1_{n+1}} - (2-4\varphi^2)\xi_{1_n} + \xi_{1_{n-1}} = -\tfrac{1}{2} A_0 - \tfrac{1}{2} A_0 \cos(4n\varphi) - \tfrac{1}{2} B_0 \sin(4n\varphi) \ . \tag{4.31}$$

The solution to this equation is of the form

$$\xi_{1_n} = A_1 cos(4n\varphi) + B_1 sin(4n\varphi) + C_1 \tag{4.32}$$

where

$$A_1 = -\frac{1}{4} A_o (cos(4\varphi) - cos(2\varphi))^{-1} \tag{4.33}$$

$$B_1 = -\frac{1}{4} A_o (cos(4\varphi) - cos(2\varphi))^{-1} \tag{4.34}$$

$$C_1 = -\frac{Ao}{B\varphi^2} \ . \tag{4.35}$$

Substituting solutions (4.28) and (4.32) and the values of α_o (4.26) and α_1 (4.30) into Equation (4.12) yield

$$\xi_{2_{n+1}} - (2-4\varphi^2)\xi_{2_n} + \xi_{2_{n-1}} = \left[-\alpha_2 A_o + \frac{1}{8} \frac{Ao}{(cos(4\varphi) - cos(2\varphi))} + \frac{Ao}{8\varphi^2} \right] cos(2n\varphi)$$

$$+ \left[-\alpha_2 Bo - \frac{1}{8} \frac{Bo}{cos(4\varphi) - cos(2\varphi)} \right] sin(2n\varphi)$$

$$+ \frac{1}{8} \frac{Ao}{cos(4\varphi) - cos(2\varphi)} cos(6n\varphi) \tag{4.36}$$

$$+ \frac{1}{8} \frac{Bo}{cos(4\varphi) - cos(2\varphi)} sin(6n\varphi) \ .$$

In order for Equation (4.36) to have a periodic solution the following must hold

$$-\alpha_2 Ao + \frac{1}{8} \frac{Ao}{cos(4\varphi) - cos(2\varphi)} + \frac{1}{8} \frac{Ao}{\varphi^2} = 0 \tag{4.37}$$

and

$$-\alpha_2 B_o - \frac{1}{8} \frac{Bo}{cos(4\varphi) - cos(2\varphi)} = 0 \ . \qquad 4.38)$$

These equations are satisfied if

$$A_o = 0 \ \text{and} \ \alpha_2 = \frac{1}{8} \frac{1}{cos(4\varphi) - cos(2\varphi)} + \frac{1}{8\varphi^2} \tag{4.39}$$

or

$$B_o = 0 \text{ and } \alpha_2 = -\frac{1}{8}\frac{1}{cos(4\varphi)-cos(2\varphi)} \quad . \tag{4.40}$$

If we again neglect terms of $O(\varphi^4)$ and higher we get

$$\alpha_2 = -\frac{1}{48}\varphi^{-2} \tag{4.41}$$

or

$$\alpha_2 = \frac{5}{48}\varphi^{-2} \quad . \tag{4.42}$$

The stability boundaries corresponding to a solution of period $\frac{\pi}{\varphi}$ are obtained substituting the determined values of α_o, α_1, and α_2 from (4.26), (4.30), and (4.39) or (4.40) respectively into expansion (4.8)

$$\alpha = 4\varphi^2 + \frac{5}{48}\varphi^{-2}\beta^2 \tag{4.43}$$

and

$$\alpha = 4\varphi^2 - \frac{1}{48}\varphi^{-2}\beta^2 \quad . \tag{4.44}$$

We have so far constructed three stability boundaries corresponding to the solutions of period $\frac{\pi}{\varphi}$ for small β and φ .

4.2.3 Solutions Having a Period of $2\frac{\pi}{\varphi}$.

We here extend our stability analysis by looking for the stability boundaries corresponding to solutions of period $2\frac{\pi}{\varphi}$. Set

$$\alpha_o = \varphi^2 \tag{4.45}$$

then we have from (4.14) and (4.15)

$$\xi_{o_n} = A_o cos(n\varphi) + B_o sin(n\varphi) \tag{4.46}$$

Substituting α_o (4.45) and ξ_{o_n} (4.46) into the difference Equation (4.11) yields

$$\xi_{1_{n+1}}-(2-\varphi^2)\xi_{1_n}+\xi_{1_{n-1}}=-(\alpha_1+\frac{1}{2})A_0\cos(n\varphi)-(\alpha_1-\frac{1}{2})B_0\sin(n\varphi) \tag{4.47}$$

$$-\frac{1}{2}\left[A_0\cos(3n\varphi)+B_0\sin(3n\varphi)\right]$$

In order to have a minimum period of $2\frac{\pi}{\varphi}$ we must set

$$\alpha_1=-\frac{1}{2} \text{ and } B_0=0 \tag{4.48}$$

or

$$\alpha_1=\frac{1}{2} \text{ and } A_0=0 \ . \tag{4.49}$$

So by choosing α_0 and α_1 according to (4.45) and (4.48) or (4.49), respectively, we obtain the following stability boundary

$$\alpha=\varphi^2\pm\frac{1}{2}\beta \ . \tag{4.50}$$

for small β.

4.3 Stability Boundaries of the Discrete Mathieu Equation for Small β up to the Fourth Order in Stepsize.

Consider Equation (4.10)

$$\xi_{0_{n+1}}-(2-\varphi_0)\xi_{0_n}+\xi_{0_{n-1}}=0 \tag{4.10}$$

We are searching for solutions with the minimum period of $2\frac{\pi}{\varphi}$. In the previous sections we chose α_0 such that the exact solution of (4.10) could be approximated, to the second order in φ, by a periodic function with a period of $2\frac{\pi}{\varphi}$. We will here extend this previous second order approximation to a fourth order approximation in φ.

We wish the solution of the difference Equation (4.10) to be of the form

$$\xi_{0_n}=A_0\cos(n\varphi)+B_0\sin(n\varphi) \tag{4.51}$$

Such a solution can be achieved if we choose α_0 according to (4.15). Hence,

$$\psi = \varphi = \cos^{-1}\left[\frac{2-\alpha_o}{2}\right] \tag{4.52}$$

By including the three first terms in the expansion of $\cos(\varphi)$,

$$\cos\varphi = 1-\frac{1}{2}\varphi^2+\frac{1}{24}\varphi^4 \ , \tag{4.53}$$

we can, through a comparison with (4.54), determine α_o as

$$\alpha_o = \varphi^2-\frac{1}{12}\varphi^4 \ . \tag{4.54}$$

We can now substitute ξ_{o_n} (4.51) and α_o (4.54) into Equation (4.11). Hence

$$\xi_{1_{n+1}}-(2-\varphi^2+\frac{1}{12}\varphi^4)\xi_{1_n}+\xi_{1_{n-1}} = \left[\alpha_1+½\right]A_o\cos(n\varphi)-\left[\alpha_1-½\right]B_o\sin(n\varphi)$$

$$-½(A_o\cos(3n\varphi)+B_o\sin(3n\varphi)) \ .$$

In order for Equation (4.54) to have a periodic solution we set

$$\alpha_1 = -\frac{1}{2} \text{ and } B_o = 0 \tag{4.55}$$

or

$$\alpha_1 = \frac{1}{2} \text{ and } A_o = 0 \ . \tag{4.56}$$

Hence the stability boundaries corresponding to the solutions having a period of $2\frac{\pi}{\varphi}$ are

$$\alpha = \varphi^2-\frac{1}{12}\varphi^4+½\,\beta \tag{4.57}$$

or

$$\alpha = \varphi^2-\frac{1}{12}\varphi^4-½\beta \ . \tag{4.58}$$

4.4 The Inclusion of the Third Harmonic

In this section we will derive a stability criterion for the discrete Duffing equation assuming the third harmonic to be included in the steady state solution. By perturbing the nonlinear equation and assuming such a steady state response we will again produce a Mathieu type equation.

In Section 4.1 we saw that by perturbing the Duffing equation we obtained the equation

$$\xi_{n+1} - 2(1-\cos\vartheta)\xi_n + 3b(x_n^*)^2\xi_n = 0 \tag{4.59}$$

By substituting the steady state solution

$$x_n^* = A\cos(n\varphi) + B\cos(3n\varphi) \tag{4.60}$$

where A and B where determined in Section (3.4), we obtain

$$\xi_{n+1} - 2\xi_n + \xi_{n-1} + [\alpha + \beta\cos(2n\varphi) + \gamma\cos(4n\varphi) + \rho\cos(6n\varphi)]\xi_n = 0 \tag{4.61}$$

where

$$\alpha = \frac{3}{2}b(A^2+B^2) + 2(1-\cos\vartheta)$$

$$\beta = \frac{3}{2}b(A^2+2AB)$$

$$\gamma = 3bAB$$

$$\rho = \frac{3}{2}bB^2 \tag{4.62}$$

If we now let

$$\gamma = \Phi\ \beta \tag{4.63}$$

$$\rho = \Theta\ \beta \tag{4.64}$$

we have

$$\xi_{n+1} - 2\xi_n + \xi_{n-1} + [\alpha + \beta(\cos(2n\varphi) + \Phi\cos(4n\varphi) + \Theta\cos(6n\varphi)]\xi_n = 0 \tag{4.65}$$

where

$$\Phi = 2\frac{B}{A+2B} \tag{4.66}$$

and

$$\Theta = \frac{B^2}{A^2+2AB} \tag{4.67}$$

Since β is small the stability boundaries can be written as

$$\alpha = \alpha_o + \beta\alpha_1 + \beta^2\alpha_2 + ... \tag{4.68}$$

and the solution can be written as

$$\xi_n = \xi_{o_n} + \beta\xi_{1_n} + \beta^2\xi_{2_n} + ... \tag{4.69}$$

Substituting α (4.68) and ξ_n (4.69) into Equation (4.65) and equating the coefficients of equal powers of β yield

$$\beta^0: \ \xi_{o_{n+1}} - (2-\alpha_o)\xi_{o_n} + \xi_{o_{n-1}} = 0 \tag{4.70}$$

$$\beta^1: \ \xi_{1_{n+1}} - (2-\alpha_o)\xi_{1_n} + \xi_{1_{n-1}} = [\alpha_1 + \cos(2n\varphi) + \Phi\cos(4n\varphi) + \Theta\cos(6n\varphi)]\xi_{o_n} \tag{4.71}$$

$$\beta^2: \ \xi_{2_{n+1}} - (2-\alpha_o)\xi_{2_n} + \xi_{2_{n-1}} = -\alpha_2\xi_{o_n} - [\alpha_1 + \cos(2n\varphi) + \Phi\cos(4n\varphi) + \Theta\cos(6n\varphi)]\xi_{1_n} \tag{4.72}$$

$$\beta^3: \ \xi_{3_{n+1}} - (2-\alpha_o)\xi_{3_n} + \xi_{3_{n-1}} = -\alpha_3\xi_{o_n} - \alpha_2\xi_{1_n} \tag{4.73}$$

$$-[\alpha_1 + \cos(2n\varphi) + \Phi\cos(4n\varphi) + \Theta\cos(6n\varphi)]\xi_{2_n} \ .$$

We will consider the stability boundaries corresponding to the solutions with a period of $\frac{2\pi}{\varphi}$. For this purpose set

$$\alpha_o = \varphi^2 \ . \tag{4.74}$$

According to Equations (4.14) and (4.15) we then have

$$\xi_{o_n} = A_o\cos(n\varphi) + B_o\sin(n\varphi) \tag{4.75}$$

as the solution to Equation (4.70). Substituting Equations (4.74) and (4.75) into Equation (4.71) yield

$$\xi_{1_{n+1}}-(2-\varphi^2)\xi_{1_n}+\xi_{1_{n-1}}=-\tfrac{1}{2}A_o(2\alpha_1+1)\cos(n\varphi)-\tfrac{1}{2}B_o(2\alpha_1-1)\sin(n\varphi) \tag{4.76}$$

$$-\tfrac{1}{2}A_o(1+\Phi)\cos(3n\varphi)-\tfrac{1}{2}A_o(\Phi+\Theta)\cos(5n\varphi)$$

$$-\tfrac{1}{2}B_o(\Phi-\Theta)\sin(5n\varphi)-\tfrac{1}{2}A_o\Theta\cos(7n\varphi)-\tfrac{1}{2}B_o\sin(7n\varphi) \ .$$

In order to avoid resonance we must set

$$\alpha_1=-\frac{1}{2}, \quad B_o=0 \tag{4.77}$$

or

$$\alpha_1=\frac{1}{2}, \quad A_o=0 \ . \tag{4.78}$$

The solution of Equation (4.76) then takes on the following form

$$\xi_{1_n}=A_{11}\cos(3n\varphi)+A_{12}\cos(5n\varphi)+A_{13}\cos(7n\varphi)+ \tag{4.79}$$

$$B_{11}\sin(3n\varphi)+B_{12}\sin(5n\varphi)+B_{13}\sin(7n\varphi)$$

where

$$A_{11}=\frac{A_o}{4}\frac{1+\Phi}{\cos\varphi-\cos(3\varphi)}, \quad B_{11}=\frac{B_o}{4}\frac{1-\Phi}{\cos\varphi-\cos(3\varphi)} \tag{4.80}$$

$$A_{12}=\frac{A_o}{4}\frac{\Phi+\Theta}{\cos\varphi-\cos(5\varphi)}, \quad B_{12}=\frac{B_o}{4}\frac{\Phi-\Theta}{\cos\vartheta-\cos(5\varphi)}$$

$$A_{13}=\frac{A_o}{4}\frac{\Theta}{\cos\varphi-\cos(7\varphi)}, \quad B_{13}=\frac{B_o}{4}\frac{\Theta}{\cos\varphi-\cos(7\varphi)} \ .$$

By substituting solutions (4.75) and (4.79) and the values of α_o (4.74) and α_1 (4.77), (4.78) into Equation (4.72) it is possible to obtain the following condition for the existence of a periodic solution,

$$\alpha_2 = \frac{1}{8}\left[\frac{(1+\Phi)^2}{\cos\varphi-\cos(3\varphi)}+\frac{(\Phi+\Theta)^2}{\cos\varphi-\cos(5\varphi)}+\frac{\Theta^2}{\cos\varphi-\cos(7\varphi)}\right] \text{ and } B_0=0 \qquad (4.81)$$

or

$$\alpha_2 = -\frac{1}{8}\left[\frac{(1-\Phi)^2}{\cos\varphi-\cos(3\varphi)}+\frac{(\Phi-\Theta)^2}{\cos\varphi-\cos(5\varphi)}+\frac{\Theta^2}{\cos\varphi-\cos(7\varphi)}\right] \text{ and } A_o=0 \ . \qquad (4.82)$$

We now have sufficient information in order to produce an expression for the wanted boundary up to an accuracy of $O(\beta^2)$. The stability boundary for small β is

$$\alpha = \varphi^2 \pm \frac{1}{2}\beta - \frac{1}{8}\left[\frac{(1\pm\Phi)^2}{\cos\varphi-\cos(3\varphi)}+\frac{(\Phi+\Theta)^2}{\cos\varphi-\cos(5\varphi)}+\frac{\Theta^2}{\cos\varphi-\cos(7\varphi)}\right]\beta^2 \ . \qquad (4.83)$$

It is clear that the correction to the stability boundary due to the introduction of the third harmonic is only present in the coefficient of β to the second power.

4.5 Relating the Stability Boundaries of the Mathieu Equation to the Nonlinear Duffing Equation

We recollect that we obtained the discrete Mathieu equation

$$W \ \xi_{n+1}-2\xi_n+\xi_{n-1}+[\alpha+\beta\cos(2n\varphi)]\xi_n = 0 \qquad (4.5)$$

by perturbing the discrete Duffing equation

$$x_{n+1}-2x_n+x_{n-1}+2(1-\cos\vartheta)x_n+bx_n^3 = bP\sin(n\varphi) \qquad (4.1)$$

where

$$\alpha = 2(1-\cos\vartheta)+\frac{3}{2}bA^2 \qquad (4.6)$$

$$\beta = \frac{3}{2}bA^2 \ . \qquad (4.7)$$

From previous Section 4.2.3 we know that two stability boundaries corresponding to solutions of period $2\frac{\pi}{\varphi}$ are given by (4.50)

$$\alpha = \varphi^2 - \frac{1}{2}\beta$$

$$\alpha = \varphi^2 + \frac{1}{2}\beta \ . \qquad (4.50)$$

The stability regions obtained from these boundaries can be stated as:

$$\textbf{Region I}: \qquad \alpha \leq \varphi^2 - \frac{1}{2}\beta \qquad (4.84)$$

$$\textbf{Region II}: \qquad \alpha \geq \varphi^2 + \frac{1}{2}\beta \qquad (4.85)$$

where the regions I and II are shown in Figure (4.1). So one of our stability conditions in region I and II is

$$(\alpha - \varphi^2 - \frac{1}{2}\beta)(\alpha - \varphi^2 + \frac{1}{2}\beta) \geq 0 \qquad (4.86)$$

Substituting the expression for α (4.6) and β (4.7) into the inequality (4.88) yields

$$(\cos\varphi - \cos\vartheta + \frac{3}{8}bA^2)(\cos\varphi - \cos\vartheta + \frac{9}{8}bA^2) \geq 0 \qquad (4.87)$$

The stability criterion (4.89) is identical to the one obtained earlier by the use of the method of slowly varying parameters on the nonlinear Duffing equation.

4.6 Numerical Determination of the Stability Boundaries of the Mathieu Equation Using Floquet Theory

The Mathieu equation

$$\xi_{n+1} - 2\xi_n + \xi_{n-1} + [\alpha + \beta\cos(2n\varphi)]\xi_n = 0 \qquad (4.5)$$

has a fundamental matrix solution $\mathbf{X}_n$ which satisfies

$$\mathbf{X}_{n+1} = \mathbf{A}\mathbf{X}_n \qquad (4.88)$$

$$\mathbf{X}_o = \mathrm{I} \qquad (4.89)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 2-(\alpha+\beta\cos(2n\varphi)) \end{bmatrix} \tag{4.90}$$

Since $\mathbf{A}$ is a periodic matrix we can write the following using Floquet theory

$$\mathbf{X}_n = \mathbf{Q}_n e^{\sigma n} \tag{4.91}$$

where $\mathbf{Q}_n$ is a periodic matrix such that

$$\mathbf{Q}_{n+N} = \mathbf{Q}_n \ . \tag{4.92}$$

Using (4.91) and (4.92) it is possible to show that

$$\mathbf{X}_{n+kN} = \mathbf{X}_n \mathbf{X}_N^k \ . \tag{4.93}$$

Assuming $\mathbf{X}_N$ to be a nondefective matrix we can write

$$\mathbf{X}_{n+kN} = \mathbf{X}_n \mathbf{T} \Lambda^k \mathbf{T}^{-1} \tag{4.94}$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{4.95}$$

and where λ_1 and λ_2 are the eigenvalues of the matrix $\mathbf{X}_N$. It is clear from Equation (4.94) that we must require

$$|\lambda_i| \leq 1 \quad i \in (1,2) \tag{4.96}$$

for the solution to remain bounded. It is also clear that since $\mathbf{A}$ is periodic in N steps, that:

i) if $\lambda_1 = \lambda_2 = 1$ the solution is periodic in N steps

ii) if $\lambda_i = -1$, $i = 1$ or 2 , the solution is periodic in 2N steps.

So a numerical determination of the stability boundaries, in the α, β plane, of the Mathieu equation could follow the following outline:

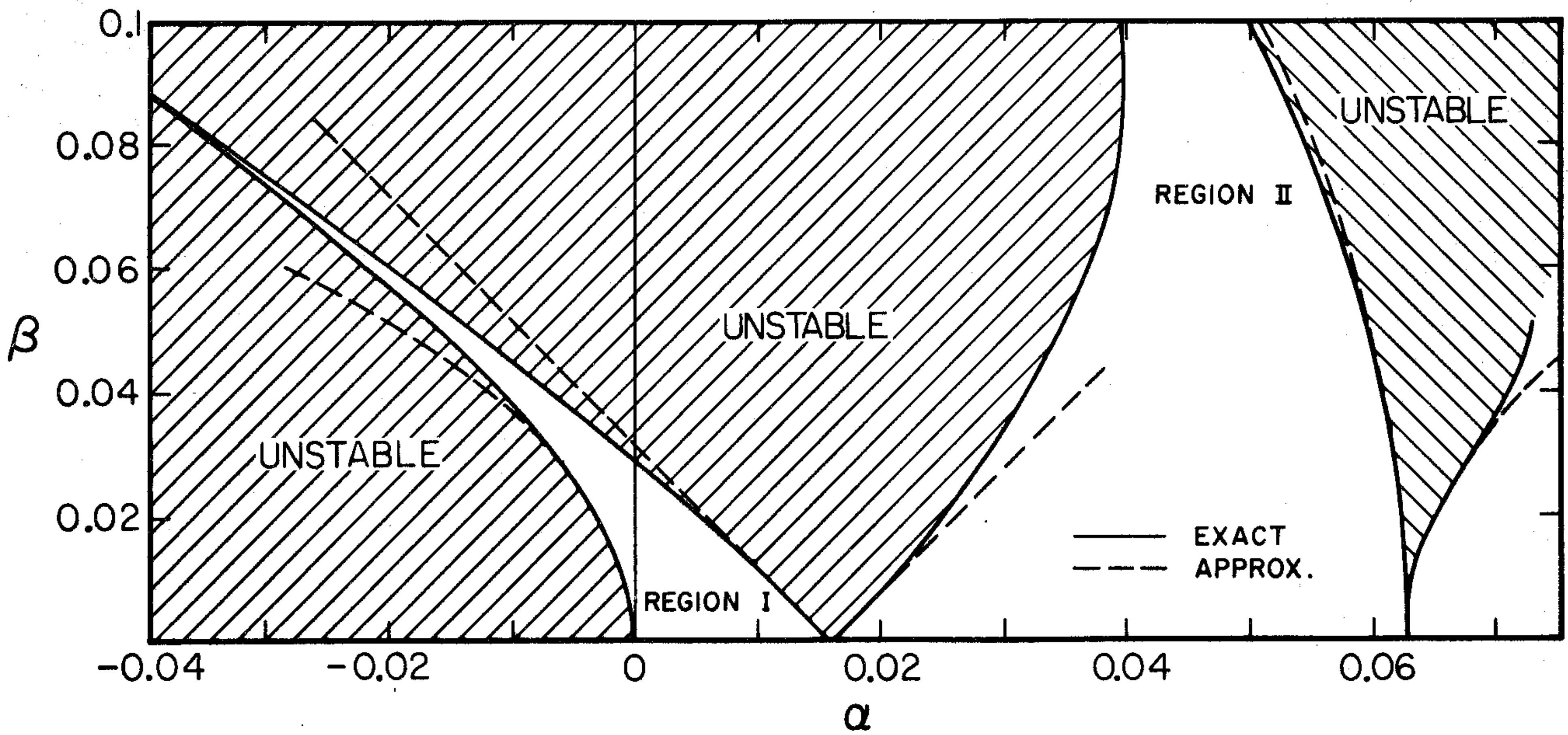
1. Determine $\mathbf{X}_N$ by exact simulation of the difference equation with the initial condition $\mathbf{X}_o = (1,0)^T$ and $\mathbf{X}_o = (0,1)^T$.

2. Obtain the eigenvalues of $\mathbf{X}_N$.

3. Vary α in a systematic manner, holding β

   constant until the desired stability boundary, case i or ii, is reached.

Figure (4.1) shows the numerically obtained stability boundaries corresponding to solutions with a period of both $\frac{\pi}{\varphi}$ and $2\frac{\pi}{\varphi}$. The approximate boundaries produced earlier are also shown in the figure. It is seen that the accuracy of the approximate technique is good for small values of β.



0.1
0.08
0.06
0.04
0.02
β
−0.04
−0.02
0
0.02
0.04
0.06
α
UNSTABLE
UNSTABLE
UNSTABLE
REGION I
REGION II
EXACT
APPROX.

Fig. 4.1

5 STOCHASTIC EXCITATION

5.1 Linear Systems

Before we can approach the task of obtaining exact or approximate solutions to nonlinear difference equations with stochastic forcing functions we must first examine the case for which the equations are linear. Consider the equation

$$x_{n+1}+ax_n+bx_{n-1}=M(n) \tag{5.1}$$

where $M(n)$ is a Gaussian distributed forcing function with

$$E[M(n)]=0 \tag{5.2}$$

and

$$E[M(n)M(p)]=2\delta_{np}B \tag{5.3}$$

where $E[\ ]$ denotes expectation and where

$$\delta_{np}=\begin{cases}1 & n=p\\ 0 & n\neq p\end{cases} \tag{5.4}$$

Due to the statistical description of the forcing function the solution of (5.1) can only be described using the same measures. We will here obtain expessions for the steady state values of the first and second moment of the response, x_n .

Equation (5.1) can be written

$$\begin{bmatrix}x_n\\ x_{n+1}\end{bmatrix}=\begin{bmatrix}0 & 1\\ -b & -a\end{bmatrix}\begin{bmatrix}x_{n-1}\\ x_n\end{bmatrix}+\begin{bmatrix}0\\ M(n)\end{bmatrix} \tag{5.5}$$

If we call

$$\vec{x}_n=\begin{bmatrix}x_{n-1}\\ x_n\end{bmatrix},\ \mathbf{A}=\begin{bmatrix}0 & 1\\ -\mathrm{b} & -\mathrm{a}\end{bmatrix}\ \text{ and }\ \vec{\mathrm{f}}(\mathrm{p})=\begin{bmatrix}0\\ \mathrm{M}(\mathrm{p})\end{bmatrix} \tag{5.6}$$

we have

$$\vec{x}_{n+1}=\mathbf{A}^{n}\vec{x}_1+\sum_{p=1}^{n}\mathbf{A}^{n-p}\vec{f}(p) \tag{5.7}$$

Since we are looking for the steady state solution we can set

$$\vec{x}_1=0 \tag{5.8}$$

hence

$$\vec{x}_{n+1}=\sum_{p=1}^{n}\mathbf{A}^{n-p}\vec{f}(p) \tag{5.9}$$

The mean of $\vec{x}_{n+1}$ is obtained by taking the expectation of Equation (5.9).

$$E[\vec{x}_{n+p}]=\sum_{p=1}^{n}\mathbf{A}^{n-p}E[\vec{f}(p)] \ . \tag{5.10}$$

But from (5.2) we have

$$E[\vec{f}(p)]=0 \tag{5.11}$$

so

$$E[\vec{x}_{n+1}]=0 \tag{5.12}$$

Hence, the first moment of the response of a discrete linear system is equal to zero if the first moment of the excitation is also equal to zero.

We proceed to calculate the variance of the response. The second moment can be obtained by considering

$$E[\vec{x}_{n+1}\vec{x}_{n+1}]^T=\sum_{p=1}^{n}\ \sum_{q=1}^{n}\vec{\mathbf{A}}^{n-p}E[\vec{f}(p)\vec{f}^{T}(q)](\mathbf{A}^T)^{n-q} \tag{5.13}$$

but

$$E[\vec{f}(p)\vec{f}^T(q)]=2B\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\delta_{pq} \ . \tag{5.14}$$

Hence

$$E[\vec{x}_{n+1}\vec{x}_{n+1}^T] = 2B\sum_{p=1}^{n}\mathbf{A}^{n-p}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}(\mathbf{A}^T)^{n-p} \quad . \tag{5.15}$$

Assuming A to be nondefective we can write

$$\mathbf{A} = \mathbf{T}\,\Lambda\,\mathbf{T}^{-1} \tag{5.16}$$

where

$$\Lambda = \begin{bmatrix}\lambda_1 & 0\\0 & \lambda_2\end{bmatrix} \ , \ T = \begin{bmatrix}1 & 1\\ \lambda_1 & \lambda_2\end{bmatrix} \tag{5.17}$$

and where

$$\lambda_1 = (1/2)[-a+\sqrt{a^2-4b}\,]$$

$$\lambda_2 = (1/2)[-a-\sqrt{a^2-4b}\,] \tag{5.18}$$

Substituting (5.16) into (5.15) yields

$$E[x_n^2] = 2B\sum_{p=1}^{n}\frac{1}{(\lambda_2-\lambda_1)^2}\left[\lambda_1^{2(n-p)} - 2(\lambda_1\lambda_2)^{n-p} + \lambda_2^{2(n-p)}\right] \quad . \tag{5.19}$$

If we now let

$$\lambda_1 = \rho\, e^{i\varphi}$$

$$\lambda_2 = \rho\, e^{-i\varphi} \tag{5.20}$$

we have

$$E[x_n^2] = \frac{2B}{\rho^2\sin^2\varphi}\sum_{p=1}^{n}\rho^{2(n-p)}\sin^2[(n-p)\varphi] \quad . \tag{5.21}$$

Since the second moment of the steady state response is desired, let $n \to \infty$ in Equation (5.23). Hence for steady state response we get

$$E[x_n^2] = \frac{2B}{\rho^2\sin^2\varphi}\sum_{p=1}^{n}\rho^{2(n-p)}\sin^2[(n-p)\varphi] \tag{5.22}$$

where

$$q = n - p \tag{5.23}$$

After some manipulation it is possible to write $E[x_n^2]$ as :

$$E[x_n^2] = \frac{B}{\sin^2\varphi}\left[\frac{(1+\rho^2)(1-\cos 2\varphi)}{(1-\rho^2)(1-2\rho^2\cos 2\varphi+\rho^4)}\right] \tag{5.24}$$

but Equations (5.18) and (5.20) yield

$$\rho = \sqrt{b}$$

$$\sin\varphi = \frac{1}{2}\sqrt{4 - \frac{a^2}{b}}$$

$$\cos 2\varphi = \frac{1}{2}\frac{a^2}{b} - 1 \tag{5.25}$$

Hence by substituting Equation (5.25) into (5.24) we finally get

$$E[x_n^2] = 2B\frac{(1+b)}{(1-b)}\,\frac{1}{(1+2b+b^2-a^2)} \tag{5.26}$$

which is the general expression for the second moment of the steady state response of the linear difference Equation (5.1). This result will be used in Section 5.22 in order to obtain an approximate solution to a nonlinear difference equation with stochastic input.

5.2 Nonlinear Systems

5.2.1 Exact Evaluation of the Second Moments

From the theory of stochastic differential equations we know that the probability density function of a Markov process is given by a Fokker-Planck equation [28]. It is therefore tempting to try a similar approach for the case of stochastically forced difference equations. However, the derivation and existence of the Fokker-Planck equation depend upon the capability of being able to let the step size, Δt, approach zero. Clearly, this capability is inherently absent for a specific discrete equation. The stepsize is fixed and cannot be treated as a continuous variable. We can therefore not work with the Fokker-Planck equation as such but are forced to consider a equation of Kolmogorov-Smoluchowski type [28].

Consider the nonlinear difference equation

$$x_{n+1}+g(x_n,x_{n-1})=M(n) \tag{5.27}$$

where $M(n)$ is of Gaussian distribution with zero mean and variance equal to $2B$. We are interested in obtaining the joint probability density, $p(x_{n+1}, x_n)$. We have

$$p(x_{n+1}, x_n)=\int_{-\infty}^{\infty} p(x_{n+1}, x_n, x_{n-1})dx_{n-1} . \tag{5.28}$$

The triple joint probability density, $p(x_{n+1}, x_n, x_{n+1})$ can be written as:

$$p(x_{n+1},x_n,x_{n-1})=p(x_{x+1}|x_n,x_{n-1})p(x_n,x_{n-1}) \tag{5.29}$$

Substituting (5.29) into (5.28) yields

$$p(x_{n+1},x_n)=\int_{-\infty}^{\infty} p(x_{n+1}|x_n, x_{n-1})\,p(x_n, x_{n-1})\,dx_{n-1} \tag{5.30}$$

which can be considered to be a equation of Chapman-Kolmogorov- Smoluchowski type. For steady state we have

$$p(x_{n+1},x_n)=p(x_n,x_{n-1}) \tag{5.31}$$

Hence Equation (5.30) can be considered to be an integral equation for $p(x_{n+1}, x_n)$ during steady state. Our goal is to determine the second moment of the steady state response from this equation.

The conditional probability density $p(x_{n+1}|x_n, x_{n-1})$ is Gaussian since $M(n)$ is of

Gaussian distribution and x_n and x_{n-1} are both considered to be fixed. Hence

$$p(x_{n+1}|x_n,x_{n-1}) = (2\pi)^{-1/2} E[z_{n+1}^2|x_n,x_{n-1}]^{-1/2} \times \qquad (5.32)$$

$$\exp\{-(1/2)z_{n+1}^2 E[z_{n+1}^2|x_n,x_{n-1}]^{-1}\}$$

where

$$z_{n+1} = x_{n+1} - E[x_{n+1}|x_n,x_{n-1}] \qquad (5.33)$$

The first moment of x_{n+1} given x_n and x_{n-1} is

$$E[x_{n+1}|x_n,x_{n-1}] = g(x_n,x_{n-1}) \qquad (5.34)$$

so

$$z_{n+1} = x_{n+1} - g(x_n,x_{n-1}) \qquad (5.35)$$

The conditional second moment $E[z_{n+1}|x_n, x_{n-1}]$ in Equation (5.32) can then be written

$$E[z_{n+1}^2|x_n,x_{n-1}] = E[x_{n+1}^2|x_n,x_{n-1}] - 2E[x_{n+1}g(x_n,x_{n-1})|x_n,x_{n-1}] + E[g^2(x_n,x_{n-1}|x_n,x_{n-1})$$

$$= E[x_{n+1}^2|x_n,x_{n-1}] - E[g^2(x_n,x_{n-1})|x_n,x_{n-1}] + 2E[M(n)g(x_n,x_{n-1})|x_n,x_{n-1}] \qquad (5.36)$$

$$= E[M^2(n)|x_n,x_{n-1}] = 2B \quad ,$$

where Equation (5.27) has been used.

Substituting Equations (5.35) and (5.36) into the expression for $p(x_{n+1}|x_n, x_{n-1})$ (5.32) yields

$$p(x_{n+1}|x_n, x_{n-1}) = (4\pi B)^{-1/2}\exp[-\frac{1}{4B}(x_{n+1} - g(x_n,x_{n-1}))^2] \quad . \qquad (5.37)$$

Hence the integral equation for $p(x_{n+1}, x_n)$ (5.30) becomes

$$p(x_{n+1},x_n) = (4\pi B)^{-1/2} \int_{-\infty}^{\infty} \exp[-\frac{1}{4B}(x_{n+1}-g(x_n,x_{n-1}))^2]p(x_n,x_{n-1})dx_{n-1} \tag{5.38}$$

Since the moment of the process x_n can be obtained from the characteristic function, $M(\vartheta_1, \vartheta_2)$, we proceed to take the Fourier transform of $p(x_{n+1}, x_n)$ in order to obtain $M(\vartheta_1, \vartheta_2)$.

Multiplication of Equation (5.38) by $\exp(i[\vartheta_1 x_{n+1}+\vartheta_2 x_n])$ and integration yield

$$M(\vartheta_1, \vartheta_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[i(\vartheta_1 x_{n+1} + \vartheta_2 x_n)]p(x_{n+1},x_n)dx_{n+1}dx_n \tag{5.39}$$

$$= \int\int_{-\infty}^{\infty}\int \exp[i(\vartheta_1 x_{n+1}+\vartheta_2 x_n)]\exp\left[-\frac{1}{4B}(x_{n+1}-g(x_n,x_{n-1}))^2\right]p(x_n,x_{n-1})dx_{n+1}dx_n dx_{n-1}$$

which simplifies to

$$M(\vartheta_1, \vartheta_2) = \exp[-\vartheta_1^2 B]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} exp[i(\vartheta_1 g(x_n,x_{n-1})+\vartheta_2 x_n)]p(x_n,x_{n-1})dx_n dx_{n-1} \tag{5.40}$$

The evaluation of the second moment of x_n for a general system via Equation (5.38) or Equation (5.40) seems to be a very formidable task which we will here avoid. However, we will use Equation (5.40) to determine the second moment of x_n for a linear difference equation in order to show consistency with the result obtained in Section 5.1.

If we set

$$g(x_n,x_{n-1}) = ax_n+bx_{n-1} \tag{5.41}$$

we have

$$M(\vartheta_1,\vartheta_2) = exp[-\vartheta_1^2 B]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} exp[i((\vartheta_2-a\vartheta_1)x_n-b\vartheta_1 x_{n-1})]p(x_n,x_{n-1})dx_n dx_{n-1} \ . \tag{5.42}$$

Since for steady state Equation (5.31) holds, Equation (5.42) can be written as follows.

$$M(\vartheta_1,\vartheta_2) = exp[-\vartheta_1 B]M(\vartheta_2 - a\vartheta_1, -b\vartheta_1) \tag{5.43}$$

For a Gaussian process with zero mean :

$$M_{XY}(\vartheta_1,\vartheta_2) = \exp\{-\tfrac{1}{2}[E[x^2]\vartheta_1^2+2E[x\,y]\vartheta_1\vartheta_2+E[y^2]\vartheta_2^2]\} \tag{5.44}$$

Hence Equation (5.43) can be written

$$\exp\{-\tfrac{1}{2}\left[E[x_{n+1}^2]\vartheta_1^2+2E[x_{n+1}x_n]\vartheta_1\vartheta_2+E[x_n^2]\vartheta_2^2\right]\} \tag{5.45}$$

$$= \exp[-\vartheta_1 B]\exp\{-\tfrac{1}{2}\left[E[x_{n+1}^2](\vartheta_2-a\vartheta_1^2)+2E[x_{n+1}x_n](\vartheta_2-a\vartheta_1)(-b\vartheta_1)+E[x_n^2](-b\vartheta_1^2)\right]\}$$

By equating coefficient of like powers of $\vartheta_i\vartheta_j$, $i, j \in 1, 2$ we obtain the following three equations

$$E[x_{n+1}^2] = 2B+E[x_{n+1}^2]a^2+2E[x_{n+1}x_n]a\,bE[x_n^2]b^2 \tag{5.46}$$

$$E[x_{n+1}x_n] = E[x_{n+1}^2]a+E[x_{n+1}x_n]b \tag{5.47}$$

$$E[x_n^2] = E[x_{n+1}^2] \tag{5.48}$$

Eliminating $E[x_{n+1}x_n]$ from (5.46) with the use of (5.47) we get

$$E[x_{n+1}^2] = 2B+E[x_{n+1}^2]a^2-\frac{2a^2b}{1+b}E[x_{n+1}^2]+E[x_{n+1}]b^2 \tag{5.49}$$

By using (5.48) we finally obtain

$$E[x_n^2] = 2B\frac{1+b}{1-b}\frac{1}{(1+b)^2-a^2} \tag{5.50}$$

which is consistent with the results produced in Section 5.1.

5.2.2 Equivalent Linearization

The method of equivalent linearization, as applied to stochastic nonlinear differential equations, was developed independently by Booton [29] and Caughey [30]. It is attempted here to extend the applicability of this method to include nonlinear difference equations.

To achieve this task consider the second order nonlinear difference equation driven by a stochastic function.

$$x_{n+1}+ax_n+bx_{n-1}+\varepsilon x_n^3=M(n) \tag{5.51}$$

where

$$E[M(n)]=0 \tag{5.52}$$

and

$$E[M(n)M(p)]=2\delta_{np}B \tag{5.53}$$

δ_{np} is given by (5.4). Assume ε to be a small parameter. We want to write Equation (5.51) as

$$x_{n+1}+a_e x_n+bx_{n-1}+\delta(x_n)=M(n) \tag{5.54}$$

and where a_e is chosen in such a way as to make the equation error, $\delta(x_n)$, as small as possible. One way of achieving this is to minimize the mean square of the equation error, $E[(x)]$. Hence we want to minimize E[δ^2 (x_n)].

$$E[\delta^2(x_n)]=E[(a\,x_n+\varepsilon\,x_n^3-a_e x_n)^2] \tag{5.55}$$

with respect to a_e. So set

$$\frac{\partial E[\delta^2(x_n)]}{\partial a_e}=0\ . \tag{5.56}$$

Substituting the expression for $E[\delta^2(x_n)]$ into Equation (5.56) yields

$$a_e=a+\varepsilon\frac{E[x_n^4]}{E[x_n^2]} \tag{5.57}$$

This specific value of a_e minimizes $E[\delta^2(x_n)]$. In order to get an approximate solution to Equation (5.54) we now simply neglect the minimized term $\delta(x_n)$. Hence we have

$$y_{n+1}+a_e y_n+by_{n-1}=M(n) \tag{5.58}$$

where y is our desired approximate solution. Since $y_n \approx x_n$ for small ε, we will also make the approximation.

$$a_e = a + \varepsilon \frac{E[y_n^4]}{E[y_n^2]} \tag{5.59}$$

Equation (5.58) is a linear difference equation with a Gaussian distributed input, hence the output, y_n, must also be of Gaussian distribution. Therefore

$$E[y_n^4] = 3E[y_n^2]^2 \tag{5.60}$$

Using (5.60) we can rewrite (5.59) as

$$a_e = a + 3E[y_n^2] \tag{5.61}$$

The mean square value of the response of Equation (5.58) is obtained using the theory of linear difference equations derived in the previous sections. Thus

$$E[y_n^2] = 2B\frac{1+b}{1-b}\,\frac{1}{1+2b+b^2-a_e^2} \quad . \tag{5.62}$$

By substituting a_e from (5.61) into Equation (5.62) we get

$$E^3\left[y_n^2\right] + \frac{2a}{3\varepsilon}E^2\left[y_n^2\right] - \frac{1+2b+b^2-a^2}{9\varepsilon^2}E\left[y_n^2\right] + \frac{2B}{9\varepsilon^2}\frac{1+b}{1-b} = 0 \quad . \tag{5.63}$$

So our approximate value for the second moment of the response of the nonlinear difference equation can be obtained from

$$E^3\left[x_n^2\right] + \frac{2a}{3\varepsilon}E^2\left[x_n^2\right] - \frac{1+2b+b^2-a^2}{9\varepsilon^2}E\left[x_n^2\right] + \frac{2B}{9\varepsilon^2}\frac{1+b}{1-b} = 0 \quad . \tag{5.64}$$

6 SUMMARY AND CONCLUSIONS

We have herein conducted an analysis of a difference equation with a cubic non-linearity. The analysed equation is of the form

$$x_{n+1} = f(x_n, x_{n-1}, n) \tag{6.1}$$

where

$$f(x_n, x_{n-1}, n) = a\, x_n - x_{n-1} - b\, x_n^3 + g(n) \quad . \tag{6.2}$$

The solutions of (6.1) were studied using both exact simulation techniques and approximate methods. The approximate methods were developed using slowly varying parameter arguments.

Presented in Chapter 2 was a phase plane analysis of Equation (6.1) with $g(n) = 0$. Phase plane plots were obtained by treating Equation (6.1) as a nonlinear map, which takes the point (x_{n-1}, x_n) in R^2 to the point (x_n, x_{n+1}) in R^2, and by plotting these points for repeated iteration.

The locations of the first order fixed points were obtained and the local stability was discussed. Depending on the value of the parameters of the map, a and b, we classified the fixed points as either centers or saddle points. We found that solution points situated close to a center always exhibit smooth closed trajectories around the center. For initial conditions situated sufficiently far away from a center we obtained unbounded solutions.

For certain values of the parameters of the map (6.1) we obtained points, located in specific regions of the phase plane, which appeared scattered in a stochastic manner. We have concluded that this chaotic behavior is due to homocinic and heteroclinic oscillations of stable and unstable manifolds emanating from the unstable fixed points. Stochastic descriptions of the scattered points have been used in the past due to the failure of relating the scatter of the points with the complicated structure of these oscillating manifolds.

It was discovered that the manifolds mentioned above form closed envelopes around the centers. Inside these envelopes all solution trajectories are smooth and closed.

In Chapter 3 we turned our focus towards approximate solutions of nonlinear difference equations. Still considering, for definiteness, a difference with a cubic non-linearity (6.1), we developed an approximate solution technique using the assumption that the solutions are modulated by slowly varying parameters. Our goal was to obtain approximate solutions of Equation (6.1) in a region in the phase plane where well behaved smooth solution trajectories were known to exist. Hence, the developed technique is only valid for solutions sufficiently close to a stable fixed point.

By assuming a solution of the form

$$x_n = A(n)cos(n\varphi)+B(n)sin(n\varphi) \tag{6.3}$$

where both $A(n)$ and $B(n)$ are slowly varying parameters, we were able to obtain the following relationship between the amplitude of the steady state solution of (6.1) and the frequency, φ, with $g(n)=0$.

$$\cos\varphi - \cos\vartheta + \frac{3}{8}bB^2 = 0$$

$$\tag{6.4}$$

$$A = 0$$

where $a = 2\cos\vartheta$. The agreement with the exact solution was shown to be good.

The approximate solution of the forced equation with $g(n) = b\,P\sin(n\varphi)$ was obtained in the same manner. Again the agreement with the simulated exact solution was shown to be good. The ultraharmonic response due to the nonlinearity of the system was considered in section 3.4.

In order to investigate the local stability of the steady state solutions a pertubation technique was applied. We obtained the following stability criteria corresponding to the steady state solution.

i) Main Response

$$\left[\cos\varphi - \cos\vartheta + \frac{3}{8}bB^2\right]\left[\cos\varphi - \cos\vartheta + \frac{9}{8}bB^2\right] \geq 0 \tag{6.5}$$

ii) Ultraharmonic Response

$$\left[\cos 3\varphi - \cos\vartheta + b\,\frac{1}{8}(9B^2+6Q^2)\right]\times \tag{6.6}$$

$$\left[\cos 3\varphi - \cos\vartheta + b\,\frac{1}{8}(3B^2+12Q^2)\right] \geq 0 \ .$$

In section 3.6 we developed approximate solutions of higher order. In a first attempt of achieving higher order solutions we used the exact central difference formulation. This attempt led to results identical to what had been obtained earlier in Chapter 3. As a second attempt we included the third harmonic, due to the nonlinearity, in the expression of the assumed response of Equation (6.1). By so doing a higher order approximation of the steady state solution was achieved. However, due to the increased complexity of the analysis, the advantage of proceeding with the higher order analysis can be considered to be questionable.

In Chapter 4 we consider the discrete Mathieu equation. By the linearization of Equation (6.1) an equation of the following type was obtained.

$$\Delta^2\xi_n + [\alpha + \beta\cos(2n\varphi)]\xi_n = 0 \ . \tag{6.7}$$

We called this equation "the discrete Mathieu equation".

By letting the stability boundaries be given by

$$\alpha = \alpha_o + \beta\alpha_1 + \beta^2\alpha_2 + \ldots.. \tag{6.8}$$

we obtained, for small β , approximate expressions for the stability boundaries corresponding to solutions with periods of $\frac{\pi}{\varphi}$ and $2\frac{\pi}{\varphi}$. It was concluded that these expressions agreed well with the stability boundaries obtained numerically through the use of Floquet theory and numerical simulation.

In section 4.4 we included the third harmonic of the steady state solution during the linearization of Equation (6.1). This enabled us to obtain the equation

$$\xi_{n+1} - 2\xi_n + \xi_{n-1} + [\,\alpha + \beta\cos(2n\varphi) + \gamma\cos(4n\varphi) + \rho\cos(6n\varphi)] = 0 \ . \tag{6.9}$$

By again expressing the stability boundaries as expansions about α_o we obtained accurate stability boundaries up to $O(\beta^2)$. The inclusion of the third harmonic did introduce a second order correction of the stability boundaries obtained using only the main response.

In section 4.5 it was concluded that the stability criteria obtained through the method of slowly varying parameters are identical to the criteria obtained via the Mathieu equation.

Presented in Chapter 5 was an analysis of stochastic difference equations. The linear equation

$$x_{n+1} + a\,x_n + b\,x_{n-1} = M(n) \tag{6.10}$$

was first considered, where $M(n)$ is a Gaussian distributed forcing function with zero mean and with an autocorrelation, $E[M(n)M(p)]$, equal to $2\delta_{np}B$ .The first and second moment of the steady state response was learned to be

$$E[x_n] = 0 \tag{6.11}$$

and

$$E[x_n^2] = 2B\frac{1+b}{1-b}\frac{1}{1+2b+b^2-a^2} \qquad (6.12)$$

In the case of a nonlinear difference equation we constructed an integral equation of Chapman - Kolmogorov - Smoluchowski type. An implicit expression for the joint characteristic fuction was also developed. No attempt was made of using these results in order to produce an exact evaluation of the second moment of the solution of the nonlinear equation. However, the joint characteristic function expression was used to determine the variance of a solution of a linear difference equation in order to show consistency with the results obtained earlier in Chapter 5.

In section 5.2.2 the method of equivalent linearization was extended for the purpose of obtaining an approximate value of the variance of the response of a nonlinear stochastic difference equation.

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