BOUNDARY VALUE PROBLEMS OVER SEMI-INFINITE INTERVALS

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ABSTRACT

A theory of existence and uniqueness of bounded solutions of linear and nonlinear boundary value problems over a semi-infinite interval is developed. A numerical method for solving such problems is proposed. The method uses only finite intervals and convergence is proven as the length of the interval goes to infinity. This work is extended to problems over $0 \leq t < \infty$ with a regular singular point at $t = 0$.

The techniques developed are applied to solve three problems.

i) The beam equation representing a semi-infinite pile imbedded in soil. Such problems are of interest in structural and foundation engineering.

ii) An eigenvalue problem representing the solution of the Schrödinger equation for an ion of the hydrogen-molecule with fixed nuclei.

iii) The Navier-Stokes equations for the von Kármán swirling flow. For this problem the existence of multiple solutions has recently been discovered. We discover an additional branch of solutions and reproduce the previous results in a much simpler and more efficient manner. Our results clearly suggest that an infinite family of branches of solutions exist for this problem.

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BOUNDARY VALUE PROBLEMS ON A SEMI-INFINITE INTERVAL

CHAPTER 1

BOUNDARY VALUE PROBLEMS ON A SEMI-INFINITE INTERVAL

1.1 Introduction

We are interested in the numerical solution of the boundary value problem on a semi-infinite interval

$$u' = t^{r-1}\left[B(t)\, u + h(t,u)\right] \quad , \; t_o \leq t < \infty \qquad , \tag{1.1a}$$

$$C_o\, u(t_o) + \lim_{t\to\infty} C_\infty\, u(t) = \alpha \; . \tag{1.1b}$$

Here u and h are vector valued functions of dimension n, $B(t)$ is a matrix valued function of dimension $n \times n$ and

$$B_\infty = \lim_{t\to\infty} B(t) \neq 0$$

i.e., infinity is a singularity of the second kind of rank r, $r \geq 1$, see Coddington and Levinson [5] p. 138 ; C_o and C_∞ are constant matrices of dimension $m \times n$ and α is a constant vector of dimension m.

We will restrict our attention to the case of solutions which satisfy

$$u(t) \text{ is bounded as } t \to \infty \tag{1.1c}$$

We first determine the most general boundary conditions of the form (1.1) b) which can be prescribed in this case.

1.2 Notation For $x \in \mathbb{R}^n$ we use $||\cdot||$ to denote the maximum norm

$$||x|| = \max_{1 \le i \le n} |x_i| \quad ,$$

and for $A \in \mathbb{R}^{m\times n}$ we use $||.||$ to denote the induced maximum norm

$$||A|| = \sup_{||x||=1} ||Ax|| = \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}|$$

1.3 Boundary value problems on a semi-infinite interval. The linear constant coefficient case.

Let us consider problem (1.1) where

$$B(t) \equiv B_\infty \text{ for } t \in [t_o, \infty) \; .$$

Let P be a similarity transformation that takes B_∞ into its Jordan canonical form:

$$A_\infty = P^{-1} B_\infty P \quad . \tag{3.1}$$

We shall call A^+, A^o and A^- the submatrices of A_∞ corresponding to those eigenvalues with positive, zero and negative real parts respectively. Their dimensions will be respectively q_∞, r_∞ and p_∞. We assume finally that the eigenvalue $\lambda = 0$ has geometric multiplicity k_∞. Thus,

$$A_\infty = \begin{bmatrix} \overbrace{A^+}^{q_\infty} & \overbrace{0}^{r_\infty} & \overbrace{0}^{p_\infty} \\ 0 & A^o & 0 \\ 0 & 0 & A^- \end{bmatrix} \begin{matrix} \} \, q_\infty \\ \} \, r_\infty \\ \} \, p_\infty \end{matrix} \tag{3.2}$$

and A^+, A^o, A^- are block diagonal with each block having the familiar Jordan form. For convenience we permute the rows and columns of A^o so that it takes the form

$$\left[\begin{array}{c|ccc} A^o_{11} & & 0 & \\ \hline & \lambda^o_1 & & \\ A^o_{21} & & \ddots & \\ & & & \lambda^o_{s_\infty} \end{array}\right]$$

where s_∞ is the number of different Jordan blocks in A^o and the last k_∞ rows have $\lambda^o_i \equiv 0$. This is done by pushing to the right all the columns whose only elements different from zero are on the diagonal and then doing the proper interchanges of rows such that those elements remain in the diagonal.

We assume that P includes this permutation of columns.

Making the change of variables $u = Py$ we obtain a problem equivalent to (1.1)

$$y(t') = t^{r-1}\left[A_\infty\, y(t) + f(t)\right] \quad , \tag{3.3a}$$

$$D_o\, y(t_o) + \lim_{t\to\infty} D_\infty\, y(t) = \alpha \quad , \tag{3.3b}$$

$$y(t) \text{ bounded as } t \to \infty \tag{3.3c}$$

Here $f(t) = P^{-1}\, h(t)$, $D_o = C_o\, P$ and $D_\infty = C_\infty\, P$.

The homogeneous system associated with (3.3) a) has a fundamental matrix of solutions of the form:

$$Y(t) = \begin{bmatrix} e^{A^+(t^r-t_o^r)/r} & 0 & 0 \\ 0 & e^{A^o(t^r-t_o^r)/r} & 0 \\ 0 & 0 & e^{A^-(t^r-t_o^r)/r} \end{bmatrix} \tag{3.4}$$

Hence any solution of the homogeneous system can be written as

$$y_H = Y(t)\,\xi \quad . \tag{3.5}$$

In order that $y_H(t)$ be bounded as $t \to \infty$ it is necessary and sufficient that

$$\xi = \begin{pmatrix} 0 \\ \xi^o \\ \xi^- \end{pmatrix}$$

where ξ^o has dimension s_∞ and ξ^- has dimension p_∞.

Thus we conclude that the homogeneous problem related to (3.3) (i.e. with $f(t) \equiv 0$, $\alpha \equiv 0$) has $p_\infty + s_\infty$ linearly independent solutions. Any solution of (3.3) can be expressed as

$$y(t) = y_H(t) + w(t) \tag{3.6}$$

where $y_H(t)$ is a bounded solution of the homogeneous problem and $w(t)$ is a particular solution of (3.3).

In order to construct a particular solution of (3.3) we rewrite it in the block form

$$\begin{aligned} w_1'(t) &= t^{r-1}(A^+ w_1(t) + f_1) \\ w_2'(t) &= t^{r-1}(A^o w_2(t) + f_2) \\ w_3'(t) &= t^{r-1}(A^- w_3(t) + f_3) \end{aligned} \tag{3.7}$$

where

$$f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \quad w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

are the natural partitions induced by A_∞.

The variation of parameters formula can be applied to (3.7). In doing this, we set the limits of the integration so that $w(t)$ is bounded as $t \to \infty$ for appropriate $f(t)$. Thus we take as the particular solution:

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \int_\infty^t e^{-A^+(s^r-t^r)/r} \; s^{r-1} \; f_1(s)ds \\ \int_\infty^t e^{-A^o(s^r-t^r)/r} \; s^{r-1} \; f_2(s)ds \\ \int_{t_o}^t e^{-A^-(s^r-t^r)/r} \; s^{r-1} \; f_3(s)ds \end{pmatrix} \tag{3.8}$$

$w(t)$ will be bounded as $t \to \infty$ iff $f(t)$ satisfies certain conditions that we specify in the following lemma.

<u>Lemma 3.9</u>: <u>The nonhomogeneous problem</u> (3.3), <u>where</u> A_∞ <u>is of the form</u> (3.2), <u>has</u> $p_\infty + s_\infty$ <u>linearly independent solutions of the form</u>:

$$y(t) = Y(t) \begin{pmatrix} 0 \\ \binom{o}{\xi_2} \\ \xi^- \end{pmatrix} + \begin{pmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{pmatrix} . \tag{3.10}$$

<u>Here</u> $Y(t)$ <u>is given in</u> (3.4), ξ_2 <u>and</u> ξ^- <u>are constant vectors of dimension</u> s_∞ <u>and</u> p_∞ <u>respectively</u>. <u>In particular</u> $y(t)$ <u>is bounded as</u> $t \to \infty$ <u>if</u> $f(t)$ <u>is differentiable in</u> $[t_o, \infty)$, $f'(t)$ <u>goes to zero as</u> $t \to \infty$; $f_1(t) = O(1)$; $f_2(t) = o\left(\frac{1}{t^{m_o r}}\right)$; <u>and</u> $f_3(t) = O(1)$ <u>as</u> $t \to \infty$. <u>Here</u> m_o <u>is the dimension of the largest Jordan block of</u> A^o.

<u>Proof</u>: We only have to prove that $w(t)$ is bounded as $t \to \infty$. Moreover, in order to get an algebraic representation of the condition "$y(t)$ is bounded as $t \to \infty$", it is necessary to know the behavior of $w(t)$ given in (3.8).

We first notice, using integration by parts, that

$$\int_{\infty}^{t} e^{-A^{+}(s^{r}-t^{r})/r} s^{r-1} f_1(s)\, ds =$$

$$- (A^{+})^{-1} f_1(t) + \int_{\infty}^{t} (A^{+})^{-1} e^{-A^{+}(s^{r}-t^{r})/r} f_1'(s) ds$$

implies

$$\lim_{t\to\infty} w_1(t) = - (A^{+})^{-1} \lim_{t\to\infty} f_1(t) \tag{3.11}$$

since

$$\lim_{t\to\infty} \int_{\infty}^{t} (A^{+})^{-1} e^{-A^{+}(s^{r}-t^{r})/r} f_1'(s) ds$$

$$= \lim_{t\to\infty} (A^{+})^{-1} e^{A^{+}t^{r}/r} \int_{\infty}^{t} e^{-A^{+} s^{r}/r} f_1'(s) ds$$

$$= \lim_{t\to\infty} \frac{e^{A^{+}t^{r}/r}\, e^{-A^{+}t^{r}/r}\, f_1'(t)}{t^{r-1}} = 0 \quad .$$

Similarly,

$$\lim_{t\to\infty} w_3(t) = - (A^{-})^{-1} \lim_{t\to\infty} f_3(t) \quad . \tag{3.12}$$

For the vector $w_2(t) = (w_{2,1}, \ldots, w_{2,r_\infty})$ we have

$$||w_2(t)|| \leq \int_{t^r/r}^{\infty} ||e^{A^o(t^r/r-u)}|| \; ||f_2(u)||du \quad .$$

Then for t sufficiently large

$$||w_2(t)|| \leq k \int_t^{\infty} \left(\frac{t^r}{r} - u\right)^{m_o-1} ||f_2(u)||du$$

for some constant k.

Since $||f_2(t)|| = o\left(\frac{1}{t^{m_o r}}\right)$ as $t \to \infty$ we have

$$\lim_{t\to\infty} ||w_2(t)|| = 0$$

and hence

$$\lim_{t\to\infty} w_2(t) = 0 \quad . \tag{3.13}$$

Thus, all the bounded solutions of system (3.3) satisfy

$$\lim_{t\to\infty} \left[I \mid 0\right] \; y(t) = \hat{f} \; , \tag{3.14}$$

where I is the identity matrix of order $(q_\infty + r_\infty - s_\infty)$ and

$$\hat{f} = \begin{pmatrix} -(A^+)^{-1} \lim_{t\to\infty} f_1(t) \\ 0 \end{pmatrix} \quad . \tag{3.15}$$

Condition (3.14) is exactly a projection into the subspace of bounded solutions of system (3.3). From now on we will refer to condition (3.14) as the "projection condition."

As it should be expected, we only can specify $(p_\infty + s_\infty)$ additional conditions to pick one solution from the set of $(p_\infty + q_\infty)$ linearly independent bounded solutions.

Thus the most general form for specifying those conditions will be

$$D_o\, y(t_o) + \lim_{t\to\infty} D_\infty\, y(t) = \alpha \tag{3.16}$$

where D_o and D_∞ are $(p_\infty + s_\infty)\times n$ constant matrices and α is a $(p_\infty + s_\infty)$ dimensional constant vector.

From the representation (3.6), with $w(t)$ given by (3.8) and $y_H(t)$ given by (3.5), we can deduce the conditions which D_o and D_∞ must satisfy in order that (3.16) and (3.14) guarantee the existence and uniqueness of a solution.

Lemma 3.17: Consider the nonhomogeneous boundary value problem

$$y'(t) = A_\infty\, y(t) + f(t) \tag{3.18a}$$

$$\lim_{t\to\infty} \left[I \mid 0 \right] y(t) = \hat{f} \tag{3.18b}$$

$$D_o\, y(t_o) + \lim_{t\to\infty} D_\infty\, y(t) = \alpha \quad , \tag{3.18c}$$

where: I is the identity matrix of dimension $(q_\infty + r_\infty - s_\infty)$;

$$f = \begin{pmatrix} -(A^+)^{-1} \lim_{t\to\infty} f_1(t) \\ 0 \end{pmatrix};$$ D_o and D_∞ are constant matrices of dimension $(p_\infty + s_\infty)\times n$.

Let us assume that A_∞ and $f(t)$ satisfy the hypotheses of Lemma 3.9. We also require

$$D_\infty R = D_\infty \quad . \tag{3.19}$$

Here R is the projection onto the nullspace of A_∞, i.e.,

$$R = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{n-(k_\infty+p_\infty) \quad k_\infty \quad p_\infty} \begin{matrix} n-(k_\infty+p_\infty) \\ k_\infty \\ p_\infty \end{matrix} \tag{3.20}$$

Then (3.18) has a unique solution provided

$$D_o + D_\infty Y(t) \equiv \left[D_1 \mid D_2\right]$$

has D_2 nonsingular.

Proof: Using (3.5), (3.6) and (3.8) in (3.18) c) gives

$$D_o(\xi + w(t_o)) + \lim_{t\to\infty} D_\infty Y(t)\xi + D_\infty \hat{w} = \alpha \quad . \tag{3.21}$$

Here

$$\hat{w} = \lim_{t\to\infty} w(t) \quad .$$

Since $y(t)$ satisfies condition (3.18b) we must have

$$\xi = \begin{pmatrix} 0 \\ \\ \xi^* \end{pmatrix}$$

where ξ^* is of dimension $(s_\infty + p_\infty)$.

We point out that the components of $Y(t)\xi$ with indices in $(q_\infty + r_\infty - s_\infty) < i < (q_\infty + r_\infty - k_\infty)$ are oscillatory and undamped. Hence their limit as $t \to \infty$ does not exist. On the other hand, the last p_∞ components of $Y(t)\xi$ vanish as $t \to \infty$.

The previous discussion implies that we can only impose conditions on

$$\lim_{t\to\infty} R\, Y(t)\xi \quad .$$

Then (3.18) has unique solution iff

$$D_2\, \xi^* = \alpha - D_\infty \hat{w} - D_o\, w(t_o)$$

is univocally solvable. ∎

1.4 The variable coefficients case

We will consider first the homogeneous problem

$$y' = t^{r-1} A(t) y \quad , \tag{4.1a}$$

$$y(t) \text{ bounded as } t \to \infty \quad . \tag{4.1b}$$

Here $\lim_{t\to\infty} A(t) = A_\infty$, with A_∞ as in (3.2).

For the rest of this paper we will consider the partition of $A(t)$ induced by that of A_∞

$$A(t) = \begin{bmatrix} \overbrace{A_{11}}^{q_\infty} & \overbrace{A_{12}}^{r_\infty} & \overbrace{A_{13}}^{p_\infty} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{matrix} \} \; q_\infty \\ \} \; r_\infty \\ \} \; p_\infty \end{matrix} \tag{4.2}$$

Let us write the equivalent problem

$$y' = t^{r-1} A_\infty y(t) + t^{r-1} (A(t) - A_\infty) y(t) \ , \ t \in [t_\infty, \infty) \tag{4.3a}$$

$$y(t) \text{ bounded as } t \to \infty \ . \tag{4.3b}$$

System (4.3a) can be expressed in block form as

$$\begin{aligned} y_1'(t) &= t^{r-1} A^+ y_1(t) + t^{r-1} \\ &\qquad \{(A_{11}(t) - A^+) y_1(t) + A_{12}(t) y_2(t) + A_{13}(t) y_3(t)\} \\ y_2'(t) &= t^{r-1} A^o y_2(t) + t^{r-1} \\ &\qquad \{A_{21}(t) y_1(t) + (A_{22}(t) - A^o) y_2(t) + A_{23}(t) y_3(t)\} \\ y_3'(t) &= t^{r-1} A^- y_3(t) + t^{r-1} \\ &\qquad \{A_{31}(t) y_1(t) + A_{32}(t) y_2(t) + (A_{33}(t) - A^-) y_3(t)\} \ . \end{aligned} \tag{4.4}$$

Variation of parameters can be applied to (4.4) in order to replace it by integral equations. Again the limits of integration must be set correctly. Thus

$$y_1(t) = -\int_t^\infty e^{-A^+(s^r-t^r)/r}\, s^{r-1} \{(A_{11}(s)-A^+)y_1(s) + A_{12}(s)y_2(s) + A_{13}(s)y_3(s)\}ds \tag{4.5}$$

$$y_2(t) = e^{A^o(t^r-t_\infty^r)/r}\begin{pmatrix}0\\ \xi^o\end{pmatrix} - \int_t^\infty e^{-A^o(s^r-t^r)/r}\, s^{r-1} \{A_{21}(s)y_1(s) + (A_{22}(s)-A^o)y_2(s) + A_{23}(s)y_3(s)\}ds$$

$$y_3(t) = e^{A^-(t^r-t_\infty^r)/r}\,\xi^- + \int_{t_\infty}^t e^{-A^-(s^r-t^r)/r}\, s^{r-1} \{A_{31}(s)y_1(s) + A_{32}(s)y_2(s) + (A_{33}(s)-A^-)y_3(s)\}ds$$

where, again, ξ^o is of dimension s_∞ and ξ^- is of dimension p_∞.

Then any solution of (4.3) would be a solution of the integral equations (4.5) provided the integrals converge. Conversely, any solution of (4.5), where the integrals converge, will be a solution of (4.3).

In the next theorem we will prove that (4.5) has $(p_\infty + s_\infty)$ linearly independent solutions provided $A(t)$ satisfies certain conditions.

Theorem 4.6: *Let* $A(t)$ *be continuous for* $t \varepsilon [t_o, \infty)$. *We also assume that* $A(t)$ *is of the form* (4.2), *and that for sufficiently large* t_∞ *there exist* $\sigma(t)$, $\rho(t)$ *and* $\tau < 1$ *such that for each* $t \varepsilon [t_\infty, \infty)$:

$$||A_{11}(t) - A^{+}|| \le \sigma(t_\infty) \;,\; ||A_{12}(t)|| \le \sigma(t_\infty) \;,\; ||A_{13}(t)|| \le \sigma(t_\infty) \;,$$

$$||A_{21}(t)|| \le \rho(t) \;,\; ||A_{22}(t) - A^{o}|| \le \rho(t) \;,\; ||A_{23}(t)|| \le \rho(t) \;,$$

$$||A_{31}(t)|| \le \sigma(t_\infty) \;,\; ||A_{32}(t)|| \le \sigma(t_\infty) \;,\; ||A_{13}(t)|| \le \sigma(t_\infty) \;,$$

$$\frac{3\sigma(t_\infty)}{\varepsilon} \le \tau < 1 \;,\; 3k \int_t^\infty \left[\frac{s^r - t^r}{r}\right]^{m_o - 1} s^{r-1} \rho(s) ds \le \tau < 1 \;.$$

Here m_o is the dimension of the largest Jordan block of A^o, k is the smallest constant so that $||e^{-A^o(s^r - t^r)/r}|| \le k \left[\frac{s^r - t^r}{r}\right]^{m_o - 1}$, $s \ge t$, $t \in [t_\infty, \infty)$ and $R(\lambda^+) \ge \varepsilon > 0$, $R(\lambda^-) \le -\varepsilon < 0$.

Then the integral equations (4.5) have $p_\infty + s_\infty$ linearly independent solutions. Each of these solutions satisfies $\lim_{t\to\infty} R\, y(t)$ exists and is finite, where R is the projection onto the null space of A_∞.

Proof: Define $y_1^o = 0$, $y_2^o = 0$, $y_3^o = 0$ and

$$y_1^{\nu+1}(t) = -\int_t^\infty e^{-A^+(s^r - t^r)/r} s^{r-1}$$
$$\{(A_{11}(s) - A^-)y_1^\nu(s) + A_{12}(s)y_2^\nu(s) + A_{13}(s)y_3^\nu(s)\}ds$$

$$y^{\nu+1}(t) = e^{A^o(t^r - t_\infty^r)/r} \begin{pmatrix} 0 \\ \xi^o \end{pmatrix} - \int_t^\infty e^{-A^o(s^r - t^r)/r} s^{r-1}$$
$$\{A_{21}(s)y_1^\nu(s) + (A_{22}(s) - A^o)y_2^\nu(s) + A_{23}(s)y_3^\nu(s)\}ds$$

$$y_3^{\nu+1}(t) = e^{A^-(t^r - t_\infty^r)/r} \xi^- - \int_{t_\infty}^t e^{-A^-(s^r - t^r)/r} s^{r-1}$$
$$\{A_{31}(s)y_1^\nu(s) + A_{32}(s)y_2^\nu(s) + (A_{33}(s) - A^-)y_3^\nu(s)\}ds$$

$\nu = 0, 1, 2, \ldots$.

We first notice that

$$y_1^1(t) \equiv 0 \quad ,$$

$$y_2^1(t) = e^{A^o(t^r - t_\infty^r)/r} \begin{pmatrix} 0 \\ \xi^o \end{pmatrix} \quad ,$$

$$y_3^1(t) = e^{A^-(t^r - t_\infty^r)/r} \xi^- \quad .$$

Hence there exists a constant $L \geq 0$ such that $||y_2^1(t)|| \leq L$ and $||y_3^1(t)|| \leq L$ for $t \in [t_\infty, \infty)$.

Then we conclude that $y^1(t)$ is bounded as $t \to \infty$. Moreover,

$$\lim R\, y^1(t) = \begin{pmatrix} 0 \\ \xi^* \\ 0 \end{pmatrix}$$

where ξ^* is the vector formed with the last k_∞ components of ξ^o.

Assume $y^1(t), \ldots, y^\nu(t)$ are bounded as $t \to \infty$. Then $y^{\nu+1}$ is well defined since the integrals converge and

$$||y^{\nu+1}(t)|| \leq \frac{3\sigma(t_\infty)}{\varepsilon} \max_{t_\infty \leq s < \infty} ||y^\nu(s)|| \leq \tau \max_{t_\infty \leq s < \infty} ||y^\nu(s)||$$

$$||y_2^{\nu+1}(t)|| \leq ||\xi^o|| + \{k \int_t^\infty \left(\frac{s^r - t^r}{r}\right)^{m_o - 1} \rho(s) ds\} \max_{t_\infty \leq s < \infty} ||y^\nu(s)||$$

$$\leq L + \tau \max_{t_\infty \leq s < \infty} ||y^\nu(s)||$$

$$||y_3^{\nu+1}(t)|| \leq L + \tau \max_{t_\infty \leq s < \infty} ||y^\nu(s)|| \Rightarrow y^{\nu+1}(t) \text{ is bounded for } t \in [t_\infty, \infty).$$

Since

$$\lim_{t\to\infty} \int_t^\infty e^{-A^o(s^r-t^r)/r} s^{r-1} \{A_{21}(s)y_1^\nu(s) + (A_{22}(s)-A^o)y_2^\nu(s) + A_{23}(s)y_3^\nu(s)\}ds = 0$$

we have

$$\lim_{t\to\infty} R\, y^{\nu+1}(t) = \begin{pmatrix} 0 \\ \xi^* \\ 0 \end{pmatrix} ,$$

where R is given in (3.20).

With $e^{(\nu)}(t) = y^{(\nu)}(t) - y^{(\nu-1)}(t)$ it follows that

$$y^\nu(t) = \sum_{i=1}^{\nu} e^i(t) .$$

We shall show that $\sum_{i=1}^{\infty} e^i(t)$ is uniformly and absolutely convergent for $t \in [t_\infty, \infty)$.

From the integral equation for the $y_k^\nu(t)$ we get:

$$\begin{aligned} ||e_1^i(t)|| &= \left|\left|\int_t^\infty e^{-A^+(s^r-t^r)/r} s^{r-1} \{(A_{11}(s)-A^+)e_1^{i-1}(s) + A_{12}(s)e_2^{i-1}(s) + A_{13}(s)e_3^{i-1}(s)\}ds\right|\right| \\ &\leq \int_t^\infty e^{-\varepsilon(s^r-t^r)/r} s^{r-1}\, 3\sigma(t_\infty)||e^{i-1}(s)||ds \\ &\leq \frac{3\sigma(t_\infty)}{\varepsilon} \max_{t\leq s<\infty} ||e^{i-1}(s)|| , \end{aligned}$$

$$||e_1^i(t)|| \leq \tau \max_{t\leq s<\infty} ||e^{i-1}(s)|| .$$

Similarly:

$$||e_2^i(t)|| = ||\int_t^\infty e^{-A^o(s^r-t^r)/r}$$

$$\{A_{21}(s)e_1^{i-1}(s) + (A_{22}(s)-A^o)e_2^{i-1} + A_{23}(s)e_3^{i-1}(s)\}ds||$$

$$\leq 3k\int_t^\infty \left[\frac{s^r-t^r}{r}\right]^{m_o-1} s^{r-1}\rho(s)||e^{i-1}(s)||ds$$

$$\leq \tau \max_{t\leq s<\infty} ||e^{i-1}(s)||$$

and

$$||e_3^i(t)|| \leq ||\int_{t_\infty}^t e^{-A^-(s^r-t^r)/r} s^{r-1}$$

$$\{A_{31}e_1^{i-1} + A_{32}e_2^{i-1} + (A_{33}-A^-)e_3^{i-1}\}ds||$$

$$\leq \frac{3\sigma(t_\infty)}{\varepsilon} \max_{t_\infty\leq s<\infty} ||e^{i-1}(s)|| \leq \tau \max_{t_\infty\leq s<\infty} ||e^{i-1}(s)|| \quad .$$

Hence

$$||e^i(t)|| \leq L\,\tau^{i-1} \quad , \; t\,\varepsilon\left[t_\infty, \infty\right)$$

and we recall $0 \leq \tau < 1$. Then we conclude that $y(t) = \lim_{\nu\to\infty} y^\nu(t) = \sum_{i=1}^\infty e^i(t)$ exists and it is bounded as $t \to \infty$. Also $y(t)$ can be continued to $t = t_o$ and

$$\lim_{t\to\infty} R\, y(t) = \begin{pmatrix} 0 \\ \xi^* \\ 0 \end{pmatrix} . \blacksquare$$

Let

$$\hat{Y}(t) = (\hat{y}_1(t), \ldots, \hat{y}_{p_\infty + s_\infty}(t)) \quad . \tag{4.7}$$

Here $\hat{y}_i(t), i = 1, \ldots, p_\infty + s_\infty$, are any $p_\infty + q_\infty$ linearly independent solutions of the integral equation (4.5).

Then any bounded solution on $[t_o, \infty)$ of the differential system $y' = A(t)y$ can be expressed as

$$y_H(t) = \hat{Y}(t)\beta \tag{4.8}$$

where we recall $\hat{Y}(t)$ is $n\times(p_\infty + s_\infty)$ and β is a $(p_\infty + s_\infty)$ constant vector.

We will consider now the inhomogeneous problem

$$y' = t^{r-1}(A(t)y + f(t)) \tag{4.9a}$$

$$y(t) \text{ bounded as } t \to \infty \quad . \tag{4.9b}$$

Any solution of (4.9) can be written as

$$y(t) = y_H(t) + w(t)$$

where $w(t)$ is a particular solution of (4.9) and $y_H(t)$ has the form (4.8).

We have, for the existence of $w(t)$,

Lemma 4.10: *Let $A(t)$ and t_∞ be as in Theorem 4.6 and $f(t)$ be as in Lemma 3.9. Then the integral equations*

$$w_1(t) = -\int_t^\infty e^{-A^+(s^r-t^r)/r}\, s^{r-1}\{(A_{11}-A^+)w_1 + A_{12}w_2 + A_{13}w_3 + f_1\}ds \tag{4.11}$$

$$w_2(t) = -\int_t^\infty e^{-A^o(s^r-t^r)/r}\, s^{r-1}\{A_{21}w_1 + (A_{22}-A^o)w_2 + A_{23}w_3 + f_2\}ds$$

$$w_3(t) = \int_{t_\infty}^t e^{-A^-(s^r-t^r)/r}\, s^{r-1}\{A_{31}w_1 + A_{32}w_2 + (A_{33}-A^-)w_3 + f_3\}ds$$

have a unique solution.

Proof: The proof is very similar to that of Theorem 4.6. We only need to prove that the Picard iterates remain bounded as $t \to \infty$. In fact we shall show that:

$$\lim_{t\to\infty} w_1^\nu(t) = (-A^+)^{-1}\,\hat{f}_1 \tag{4.12a}$$

$$\lim_{t\to\infty} w_2^\nu(t) = 0 \tag{4.12b}$$

$$\lim_{t\to\infty} w_3^\nu(t) = (-A^-)^{-1}\,\hat{f}_3 \tag{4.12c}$$

where $\lim_{t\to\infty} f(t) = \hat{f}$.

We take $w^o = 0$ and then

$$w_1^1(t) = -\int_t^\infty e^{-A^+(s^r-t^r)/r}\, s^{r-1}\, f_1(s)ds$$

$$w_2^1(t) = -\int_t^\infty e^{-A^o(s^r-t^r)/r}\, s^{r-1}\, f_2(s)ds$$

$$w_3^1(t) = -\int_{t_\infty}^\infty e^{-A^-(s^r-t^r)/r}\, s^{r-1}\, f_3(s)ds \quad .$$

As in section 3,

$$\lim_{t\to\infty} w_1^1(t) = -\ (A^+)^{-1}\ \hat{f}_1$$

$$\lim_{t\to\infty} w_2^1(t) = 0$$

$$\lim_{t\to\infty} w_3^1(t) = -\ (A^-)^{-1}\ \hat{f}_3 \quad .$$

Now using induction on ν and the fact that $\lim_{t\to\infty} (A(t) - A_\infty) = 0$ we get (4.12). ∎

Letting $\nu \to \infty$ in (4.12) it follows that $w(t)$, the particular solution of (4.9), has the same limiting forms as in the constant coefficient case (3.3). Thus the condition that <u>the solution of</u> (4.9) <u>has to be bounded at infinity</u> can be expressed in algebraic form as the "projection condition":

$$\lim_{t\to\infty} \left[I \mid 0 \right] y(t) = \begin{pmatrix} -(A^+)^{-1} & \hat{f}_1 \\ \\ & 0 \end{pmatrix} . \qquad (4.13)$$

Here I is the identity matrix of dimension $q_\infty + r_\infty - s_\infty$.

Finally we have:

<u>Theorem 4.14</u>: <u>Consider the nonhomogeneous</u> B.V.P.

$$y' = A(t)y + f(t) \tag{4.15a}$$

$$D_o\, y(t_o) + \lim_{t\to\infty} D_\infty\, y(t) = \alpha \tag{4.15b}$$

$$y(t) \text{ bounded as } t \to \infty \quad , \tag{4.15c}$$

<u>where</u>: D_o <u>and</u> D_∞ <u>are</u> $(p_\infty + s_\infty)\times n$ <u>constant matrices</u>, $A(t)$ <u>satisfies the hypotheses of Theorem</u> 4.6, <u>and</u> $f(t)$ <u>satisfies the hypotheses of Lemma</u> 3.9.

<u>Then</u> (4.15) <u>has a unique solution provided that the matrix of order</u> $p_\infty + s_\infty$

$$\left[D_o\, \hat{Y}(t_o) + \lim_{t\to\infty} D_\infty\, R\, \hat{Y}(t)\right]$$

<u>is nonsingular</u>. <u>Here</u> R <u>is as in</u> (3.20) <u>and</u> $\hat{Y}(t)$ <u>as in</u> (4.7).

<u>Proof</u>: The proof follows the same steps of the proof of Lemma 3.17. But in this case $y_H(t)$ and $w(t)$ in (3.6) are given by (4.8) and (4.11) respectively. ∎

1.5 <u>More general linear systems</u>:

The requirements on $A(t)$ and $f(t)$ in Theorem 4.6 and Lemma 3.9 seem very restrictive. In particular this is so when A_∞ has multiple eigenvalues with real part equal to zero. However, in this case it is easy to construct examples which violate the hypotheses of

Theorem 4.6 and/or Lemma 3.9 and have solutions with unpredictable asymptotic behavior, see Cesari [3]. On the other hand, many problems of practical interest, when transformed into the form

$$y' = t^{r-1}\left[A(t)y + f(t)\right] , \tag{5.1}$$

fail to satisfy the requirements mentioned above.

F. de Hoog and R. Weiss, [8], have found a way to overcome some of these difficulties in the case of an irregular singularity at a finite point. We will extend their results to the semi-infinite interval. Let us consider the slightly more general linear system:

$$y'(t) = T(t)\left[B(t)\ y(t) + h(t)\right] \tag{5.2}$$

where

$T = \operatorname{diag}\,(t^{r_1-1} I_1,\ t^{r_2-1} I_2, \ldots,\ t^{r_k-1} I_k)$, the I_j are identity matrices (5.3a)

$r_i \geq 1 \quad i = 1, \ldots, k$ (5.3b)

$M = \lim_{t\to\infty} B(t)$ is in block upper triangular form (5.3c)

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} & \cdots & M_{1k} \\ 0 & M_{22} & M_{23} & \cdots & M_{2k} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & M_{kk} \end{bmatrix} = D + \hat{M}$$

$D = \text{diag}\ (M_{11}, \ldots, M_{kk})$ (5.3d)

Each M_{jj} is a square matrix of the same dimension as I_j (5.3e) and does not have eigenvalues with real part equal to zero for $j = 1, \ldots, k-1$.

Either $M_{kk} \equiv 0$ and $r_k = 1$ or M_{kk} is as in e) and $r_k \geq 1$. (5.3f)

In the rest of this section we will employ the partitions

$$B(t) = \begin{bmatrix} B_{11} & B_{12} & B_{13} \cdots & B_{1k} \\ B_{21} & B_{22} & B_{23} \cdots & B_{2k} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ B_{k1} & & & B_{kk} \end{bmatrix}, \quad y(t) = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_k \end{bmatrix} \text{ and } h(t) = \begin{bmatrix} h_1 \\ h_2 \\ \cdot \\ \cdot \\ \cdot \\ h_k \end{bmatrix}$$

induced by the partition of M given in (5.3c).

It is possible to prove results similar to those in section 4 for the system (5.2) under very mild conditions on the convergence of $B(t)$ and $h(t)$ as $t \to \infty$. In fact,

<u>Theorem 5.4</u>: <u>Let</u> $B(t)$ <u>be continuous for</u> $t \in [t_o, \infty)$. <u>Furthermore let</u> $M = \lim B(t)$ <u>be as in</u> (5.3c). <u>If</u> $M_{kk} \equiv 0$ <u>we assume</u> $B_{ki} = o(\frac{1}{t})$ <u>as</u> $t \to \infty$. <u>Then the homogeneous system associated with</u> (5.2) <u>has</u> $p_\infty + r_\infty$ <u>linearly independent solutions which are bounded as</u> $t \to \infty$, <u>where</u> p_∞ <u>is the number of eigenvalues of</u> M <u>with real part</u> < 0 <u>and</u> r_∞ <u>is the algebraic multiplicity of the eigenvalue zero</u>. <u>Moreover each of those solutions satisfies</u>

$$\lim_{t\to\infty} M\, y(t) = 0 \quad . \tag{5.5}$$

Proof: Using variation of parameters we transform for $t \in [t_\infty, \infty)$ $t_\infty \geq t_o$, the differential system (5.2) into an integral system. Since we are only interested in bounded solutions, the integral system takes the form

$$\begin{aligned} y(t) &= e^{(S(t)-S(t_\infty))D}\, \xi \\ &+ \int_{t_\infty}^{t} e^{(S(t)-S(s))D}\, PT(s)\left[\hat{M}y(s) + (B(s)-M)y(s)\right] ds \\ &- \int_{t}^{\infty} e^{(S(t)-S(s))D} (I-P)T(s)\left[\hat{M}y(s) + (B(s)-M)y(s)\right] ds \quad . \end{aligned} \tag{5.6}$$

Here $S(t) = \int_o^t T(s)ds$, $P = \text{diag}\left[P_1,\ldots, P_k\right]$ and P_i is the projection into the invariant subspace of M_{ii} which is associated with the eigenvalues with real part ≤ 0. The vector ξ satisfies

$$P\xi = \xi \quad . \tag{5.7}$$

We observe that any solution of the homogeneous form of (5.2), which is bounded as $t \to \infty$, will also be a solution of (5.6). Reciprocally, any solution of (5.6), where the integrals converge, is a solution of (5.2) with $h(t) \equiv 0$.

In order to prove that (5.6) has a nontrivial solution we use a

Gauss-Seidel type iteration, thus

$$y^{(0)} = 0$$

and for $\nu = 1, 2, \ldots$

$$y^{(\nu)}(t) = e^{(S(t)-S(t_\infty))D}\,\xi \tag{5.8}$$

$$+ \int_{t_\infty}^{t} e^{(S(t)-S(s))D}\, PT(s)\left[\hat{M}y^{(\nu)}(s)+(B(s)-M)y^{(\nu-1)}(s)\right]ds$$

$$- \int_{t}^{\infty} e^{(S(t)-S(s))D}(I-P)T(s)\left[\hat{M}y^{(\nu)}(s)+(B(s)-M)y^{(\nu-1)}(s)\right]ds.$$

We observe that the integrals are convergent for $\nu = 1$. Moreover, $y^{(1)}(t) = e^{(S(t)-S(t_\infty))D}\,\xi$ and hence there exist a constant $L \geq 0$ so that $||y^{(1)}(t)|| \leq L$.

If $y^{(\nu)}(t)$ is bounded as $t \to \infty$ then the integrals in (5.8) are convergent and hence $y^{(\nu+1)}(t)$ is well defined.

To see what is the behaviour of $y^{(\nu)}(t)$ as $t \to \infty$ we integrate by parts and take limits:

If $M_{kk} \neq 0$ D is nonsingular then

$$\lim_{t\to\infty} y^{(\nu)}(t) = \lim_{t\to\infty}\left[-D^{-1}T^{-1}(t)PT(t)\hat{M}\; y(t)-D^{-1}T^{-1}(t)(I-P)T(t)\hat{M}\; y^{(\nu)}(t)\right].$$

Since $T(t)$ is diagonal and $D + \hat{M} = M$ we get:

$$\lim_{t\to\infty} M\, y^{(\nu)}(t) = 0 \quad .$$

If $M_{kk} = 0$ we do it by components and get

$$\lim_{t\to\infty} y_k^{(\nu)}(t) = \xi_k \quad .$$

Here ξ_k are the last components of the vector ξ.

For $i = 1,\ldots,k-1$

$$\lim_{t\to\infty} y_i^{\nu}(t) = \lim_{t\to\infty}\left[-M_{ii}^{-1}P_i \sum_{j=i+1}^{k} M_{ij}y_j^{(\nu)}(t) - M_{ii}^{-1}(I-P_i)\sum_{j=i+1}^{k} M_{ij}y_j^{(\nu)}(t)\right] ,$$

i.e.,

$$\lim_{t\to\infty} \sum_{j=i}^{k} M_{ij}\, y_j^{(\nu)}(t) = 0 \quad .$$

Then we also have in this case

$$\lim_{t\to\infty} M\, y^{\nu}(t) = 0 \quad . \tag{5.9}$$

In order to prove that the sequence (5.8) is convergent as $\nu \to \infty$ we define:

$$e^{(\nu)}(t) = y^{(\nu)}(t) - y^{(\nu-1)}(t) \ , \ \nu = 1,2,\ldots \quad .$$

Then

$$e^{(\nu)}(t) = \int_{t_\infty}^{t} e^{(T(t)-T(s))D} PT(s)\left[\hat{M}\, e^{(\nu)}(s) + (B(s)-M)e^{(\nu-1)}(s)\right] ds \quad (5.10)$$

$$- \int_{t}^{\infty} e^{(T(t)-T(s))D} (I-P)T(s)\left[\hat{M}\, e^{(\nu)}(s) + (B(s)-M)e^{(\nu-1)}(s)\right] ds \quad .$$

Since $e^{(1)}(t) = y^{(1)}(t)$ we have

$$||e^{(1)}(t)|| \leq L \quad .$$

On the other hand, t_∞ can be chosen large enough so that:

a) $$||B(t) - M|| \leq \sigma(t_\infty) \text{ for } t \in [t_\infty, \infty) \quad .$$

b) $$\text{If } M_{kk} \equiv 0 \text{ then } \int_{t_\infty}^{t} \sum_{i=1}^{k} ||B_{ki}(s)|| \leq \frac{2K\sigma(t_\infty)}{\varepsilon}$$

$$\text{for } t \in [t_\infty, \infty) \quad .$$

c) There exists a constant τ , $0 \leq \tau < 1$, such that

$$\frac{2K\sigma(t_\infty)}{\varepsilon} \sum_{i=0}^{k-1} \left(\frac{2\mu}{\varepsilon}\right)^{i} = \tau < 1 \quad .$$

Here ε and μ are positive constants such that $|\lambda_i(M)| > \varepsilon$ for all eigenvalues of M different from zero and $||M|| \leq \mu$. Hence

If $M_{kk} \neq 0$

$$||e_k^{(\nu)}(t)|| \leq \int_{t_\infty}^{t} e^{-\varepsilon(t^{r_k}-s^{r_k})/r_k} s^{r_k-1} K\sigma(t_\infty) \; ||e^{(\nu-1)}(s)||ds$$

$$+ \int_{t}^{\infty} e^{\varepsilon(t^{r_k}-s^{r_k})/r_k} s^{r_k-1} K\,\sigma(t_\infty) \; ||e^{(\nu+1)}(s)||ds$$

$$\leq (1-e^{-\varepsilon(t^{r_k}-t_\infty^{r_k})/r_k} \frac{K\,\sigma(t_\infty)}{\varepsilon} \max_{t_\infty \leq s<t} ||e^{(\nu-1)}(s)||$$

$$+ \frac{K\sigma(t_\infty)}{\varepsilon} \max_{t \leq s<\infty} ||e^{(\nu-1)}(s)||$$

$$\leq \frac{2K\sigma(t_\infty)}{\varepsilon} \max_{t_\infty \leq s<\infty} ||e^{(\nu-1)}(s)||.$$

If $M_{kk} = 0$

$$||e_k^{(\nu)}(t)|| \leq \int_{t_\infty}^{t} \sum_{i=1}^{k} ||B_{ki}(s) \; e_i^{(\nu-1)}(s)||ds$$

$$\leq \max_{t_\infty \leq s<t} ||e^{(\nu-1)}(s)|| \int_{t_\infty}^{t} \sum_{i=1}^{k} ||B_{ki}(s)||ds$$

$$\leq \frac{2K\sigma(t_\infty)}{\varepsilon} \max_{t_\infty \leq s<\infty} ||e^{(\nu-1)}(s)|| \quad .$$

This implies:

$$||e_k^{(\nu)}(t)|| \leq \tau \max_{t_\infty \leq s<\infty} ||e^{(\nu-1)}(s)||$$

Similarly,

$$||e_{k-1}^{(\nu)}(t)|| \leq \frac{1}{\varepsilon}\left[2K\sigma(t_\infty)\frac{\mu}{\varepsilon} + K\sigma(t_\infty)\right]\max_{t_\infty \leq s < t}||e^{(\nu)}(s)||\left[1-e^{-\varepsilon(t^{r_i}-t_\infty^{r_i})/r_i}\right]$$

$$+ \frac{1}{\varepsilon}\left[2K\sigma(t_\infty)\frac{\mu}{\varepsilon} + K\sigma(t_\infty)\right]\max_{t \leq s < \infty}||e^{(\nu)}(s)||$$

$$\leq \frac{2K\sigma(t_\infty)}{\varepsilon}\left[1 + \frac{2\mu}{\varepsilon}\right]\max_{t_\infty \leq s < \infty}||e^{(\nu)}(s)||$$

$$\leq \tau \max_{t_\infty \leq s < \infty}||e^{(\nu-1)}(s)|| \quad .$$

Finally we get

$$||e_1^{(\nu)}(t)|| \leq \frac{2K\sigma(t_\infty)}{\varepsilon}\sum_{i=0}^{k-1}\left(\frac{2\mu}{\varepsilon}\right)^i \max_{t_\infty \leq s < \infty}||e^{(\nu-1)}(s)||$$

$$\leq \tau \max_{t_\infty \leq s < \infty}||e^{(\nu-1)}(s)|| \quad .$$

Hence

$$||e^{(\nu)}(t)|| \leq \tau \max_{t_\infty \leq s < \infty}||e^{(\nu-1)}(s)|| \leq L\,\tau^\nu \tag{5.11}$$

So $\sum_{i=0}^{\infty} e^{(\nu)}(t)$ is uniformly and absolutely convergent. Since

$$y^{(\nu)}(t) = \sum_{i=0}^{\nu} e^{(i)}(t)$$

we have, $y(t) = \lim_{\nu\to\infty} y^{(\nu)}(t) = \sum_{i=0}^{\infty} e^{(i)}(t)$ is continuous for $t \in [t_\infty, \infty)$, bounded as $t \to \infty$ and it can be continued to $t = t_o < t_\infty$.

From (5.9) we conclude that $y(t)$ satisfies (5.5). ∎

It can also be proven that if $h(t)$ satisfies certain conditions the nonhomogeneous system (5.2) also has $(p_\infty + r_\infty)$ linearly independent solutions.

In fact, any bounded solution of (5.2) can be represented as

$$y(t) = \hat{Y}(t)\ \beta + w(t) \tag{5.12}$$

where $\hat{Y}(t)$ is the $n\times(p_\infty + r_\infty)$ matrix formed with $(p_\infty + r_\infty)$ linearly independent solutions of the integral equation (5.6), and $w(t)$ is the solution of the integral equations

$$\begin{aligned} w(t) &= \int_{t_\infty}^{\infty} e^{(T(t)-T(s))D}\, PT(s)\left[\hat{M}\ w(s) + (B(s)-M)w(s) + h(s)\right] ds \\ &\quad - \int_{t}^{\infty} e^{(T(t)-T(s))D}(I-P)T(s)\left[\hat{M}\ w(s) + (B(s) - M)w(s) + h(s)\right] ds\ . \end{aligned} \tag{5.13}$$

In the next theorem we prove that equations (5.13) actually have a unique solution.

<u>Theorem 5.14</u>: Let $B(t)$, M and t_∞ satisfy the hypotheses of Theorem 5.4. In addition, let $h(t)$ be continuous for $t \ \varepsilon \left[t_o, \infty\right)$, $t_o \le t_\infty$ and bounded as $t \to \infty$. If $M_{kk} = 0$ we also require $h_k(t) = o(\frac{1}{t})$ as $t \to \infty$. Then the integral equations (5.13) have a solution which is a bounded solution of the system (5.2). Moreover,

$$\lim_{t\to\infty} \sum_{j=1}^{k} M_{ij}\, w_i(t) = - \lim_{t\to\infty} h_i(t) \ , \ i = 1,\ldots,k-1 \ . \tag{5.15a}$$

$$\begin{cases} \underline{\text{If}} \ \ M_{kk} \equiv 0 \ \ \underline{\text{then}} \ \ \lim\limits_{t\to\infty} w_k(t) = 0 \ , \\ \underline{\text{If}} \ \ M_{kk} \neq 0 \ \ \underline{\text{then}} \ \ \lim\limits_{t\to\infty} M_{kk}\, w_k(t) = - \lim\limits_{t\to\infty} h_k(t) \ , \end{cases} \tag{5.15b}$$

in other words,

$$\lim_{t\to\infty} w(t) = - (M+R)^{-1} \lim_{t\to\infty} h(t) \ . \tag{5.16}$$

If $M_{kk} \neq 0$ then $R = 0$, if $M_{kk} = 0$ $R = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & I_k \end{array}\right]$.

Proof: The proof is very similar to the one of Theorem 5.4. In fact, we define again a Gauss-Seidel type iteration and start with $w^{(0)} \equiv 0$. It easily follows that every iterate is well defined and bounded as $t \to \infty$. Moreover, $E^{(\nu+1)}(t) = w^{(\nu+1)}(t) - w^{(\nu)}(t)$ satisfies, as the $e^{\nu+1}(t)$, the relation (5.10).

Hence $\sum_{i=0}^{\infty} E^{\nu+1}(t)$ is uniformly and absolutely convergent. Let $w(t) = \lim_{\nu\to\infty} w^{\nu}(t)$.

The last part of the theorem is obtained by taking the limit of $w^{\nu}(t)$ as $t \to \infty$ $\nu = 1,2,\ldots$. ∎

From (5.16) and (5.5) we conclude that any bounded solution of system (5.2) satisfies

$$\lim_{t\to\infty} y(t) = - (M+R)^{-1} \left(\lim_{t\to\infty} h(t) - \lim_{t\to\infty} R\, y(t)\right) \tag{5.17}$$

Hence the projection in this case can be written as

$$\lim_{t\to\infty} Q\, y(t) = -\, Q(M+R)^{-1} \left(\lim_{t\to\infty} h(t) - \lim_{t\to\infty} R\, y(t)\right) \tag{5.18}$$

where $Q = \mathrm{diag}\left[(I-P_1),\ldots,(I-P_k)\right]$ is the projection into the invariant subspace of M associated to the eigenvalues with positive real parts.

The projection condition imposes $n - (p_\infty + r_\infty)$ restrictions on the solution. Then we only can specify $(p_\infty + r_\infty)$ more conditions in order to expect existence and uniqueness of solution for (5.2).

Theorem 5.19: Let $B(t)$ and $h(t)$ satisfy the hypotheses of Theorems 5.4 and 5.14. Consider the boundary conditions

$$D_o\, y(t_o) + \lim_{t\to\infty} D_\infty\, y(t) = \alpha \quad , \tag{5.20}$$

where D_o and D_∞ are $(p_\infty + r_\infty)\times n$ constant matrices. Then (5.2), subject to (5.20) has a unique bounded solution if the matrix

$$C_o\, \hat{Y}(t_o) + \lim_{t\to\infty} D_\infty\, R\, \hat{Y}(t)$$

is nonsingular. ∎

1.6 Nonlinear problems.

K. W. Chang [4] has proved the existence of decaying solutions for a special nonlinear systems of the form

$$u' = f(t, u) \quad . \tag{6.1}$$

He studied the case in which the variational equation of (6.1) has an exponential dichotomy. Our aim in this section is to treat more general systems.

We will be interested in the existence of solutions of (6.1) which satisfy

$$\lim_{t\to\infty} u(t) \text{ exists and it is finite.} \tag{6.2}$$

In all that follows we will call

$$\gamma = \lim_{t\to\infty} u(t) \quad . \tag{6.3}$$

We suppose that the following assumptions hold throughout:

$f(t, u)$ and $f_u(t, u)$ are bounded and continuous for (6.4a)

$(t, u) \in [t_\infty, \infty)\times D$, $D \subset \mathbb{R}^n$ and

$$S_\rho = \{u \mid ||u-\gamma|| < \rho\} \subset D \ .$$

$f_u(t, u) \to T(t)\, A(t)$ as $y \to \gamma$ uniformly on t. Here (6.4b)

$T(t)$ is as in (5.3a) and there exist a nonsingular matrix Q so that

$$B(t) = Q^{-1}\, A(t)Q \tag{6.4c}$$

and

$$M = \lim_{t\to\infty} B(t) \tag{6.4d}$$

is as in (5.3c).

$$f(t, u) = o(||u-\gamma||) \text{ as } u(t) \to \gamma \quad . \tag{6.4e}$$

Let P be the projection into the invariant subspace associated with the eigenvalues of M with negative real parts. We will show that, for every ξ such that $P\xi = \xi$ and t_∞ large enough, the system 6.1 has a solution $u(t)$, $t \in [t_\infty, \infty)$, which satisfies $P\ u(t_\infty) = \xi$ and (6.2).

To simplify the notation in what follows we make some changes of variables:

1) We will assume that $\gamma \equiv 0$. Since the change $v(t) = u(t) - \gamma$ gives a system for v equivalent to (6.1) and $\lim_{t\to\infty} v(t) = 0$.

2) System (6.1) can be rewritten in the equivalent form

$$u'(t) = T(t)\left[A(t)\ u(t) + T^{-1}(t)\ f(t, u) - A(t)\ u(t)\right] \quad .$$

Introducing the change $u(t) = Q\ y(t)$, we get for $y(t)$:

$$y'(t) = T(t)\left[B(t)\ y(t) + g(t, y(t))\right] \quad , \ t \in [t_\infty, \infty) \tag{6.5a}$$

where

$$g(t, y(t)) = Q^{-1}\ T^{-1}(t)\ f(t, Qy) - B(t)\ y(t) \tag{6.5b}$$

$$\lim_{t\to\infty} y(t) = 0 \quad . \tag{6.5c}$$

Theorem 6.6: Let $f(t, y)$ and $f_y(t, y)$ satisfy (6.4) a), b), c), d) and e) where $\gamma \equiv 0$. Moreover, if $M_{kk} \equiv 0$ we also assume that $B_{ki}(t) = o(\frac{1}{t})$ as $t \to \infty$, $i = 1,\dots,k-1$ and that $f_k(t, 0) = o(\frac{1}{t})$ as $t \to \infty$. Then for sufficiently large t_∞ and every ξ such that $P\xi = \xi$, (6.5) has a solution which satisfies $P\,y(t_\infty) = \xi$.

Proof: The proof is very similar to that of Theorem 5.4. In fact, we will follow the steps of that proof.

Using variation of parameters we get for $t \in [t_\infty, \infty)$ the integral equation:

$$y(t) = e^{(T(t)-T(t_\infty))D}\,\xi \tag{6.7}$$

$$+ \int_{t_\infty}^{t} e^{(T(t)-T(s))D}\, PT(s)\left[\hat{M}y(s) + (B(s)-M)y(s) + g(s,y(s))\right] ds$$

$$- \int_{t}^{\infty} e^{(T(t)-T(s))D}(I-P)T(s)\left[\hat{M}y(s) + (B(s)-M)y(s)+g(s,y(s))\right] ds .$$

As in Theorem 5.4 we construct the sequence $y^{(\nu)}(t)$ using a Gauss-Seidel type iteration. We start with

$$y^{(0)}(t) = e^{(T(t) - T(t_\infty))D}\,\xi \quad . \tag{6.8}$$

Hence $\lim_{t\to\infty} y^{(0)}(t) = 0.$

For $\nu = 1,2,\dots$

$$y^{(\nu)}(t) = e^{(T(t)-T(t_\infty))D}\,\xi \tag{6.9}$$
$$+ \int_{t_\infty}^{t} e^{(T(t)-T(s))D}\, PT(s)\Big[\hat{M}y^{(\nu)}(s) + (B(s)-M)y^{(\nu-1)}(s) + g(s,y^{(\nu-1)}(s))\Big]ds$$
$$- \int_{t}^{\infty} e^{(T(t)-T(s))D}(I-P)T(s)\Big[\hat{M}y^{(\nu)}(s) + (B(s)-M)y^{(\nu-1)}(s) + g(s,y^{(\nu-1)}(s))\Big]ds \ .$$

Suppose that $\lim_{t\to\infty} y^{(\nu-1)}(t) = 0$, then $y^{(\nu)}(t)$ is well defined. Moreover taking limits in (6.9) we get

$$\lim_{t\to\infty} (M + R)\, y^{(\nu)}(t) = - \lim_{t\to\infty} g(t, y^{(\nu-1)}(t)) \ .$$

Since $g(t, y(t)) = o(||y(t)||)$ as $||y(t)|| \to 0$ and $y^{(0)}(t)$ decays exponentially we conclude that $(M + R)\, y^{(\nu)}(t) \to 0$ exponentially as $t \to \infty$. Here R is the projection into the nullspace of M. Since $(M + R)$ is nonsingular we can write

$$\lim_{t\to\infty} y^{(\nu)}(t) = 0 \ . \tag{6.10}$$

To prove the convergence of the sequence $\{y^{(\nu)}(t)\}$ we define

$$e^{(\nu)}(t) = y^{(\nu)}(t) - y^{(\nu-1)}(t) \ , \ \nu = 1,\ldots \tag{6.11}$$
$$e^{(0)}(t) = y^{(0)}(t) \ .$$

Then there exists $L > 0$ so that

$$||e^{(0)}|| \leq L \quad .$$

By the mean value theorem we have:

$$g(t,y^{\nu}) - g(t,y^{\nu-1}) = \int_0^1 \frac{\partial g}{\partial y}(t,Q(\theta y^{\nu} + (1-\theta)y^{\nu-1})e^{\nu}(t)d\theta \qquad . \ (6.12)$$

Using the definition of $g(t, y)$ in (6.12):

$$g(t,y^{\nu}) - g(t,y^{\nu-1}) = \int_0^1 \left[Q^{-1}T^{-1}(t)f_y(t,Q(\theta y^{\nu}+(1-\theta)y^{\nu-1})-B(t)\right] e^{(\nu)}(t)d\theta.$$

$$= J^{\nu}(t)\ e^{(\nu)}(t) \quad .$$

Since $f_y(t, y)$ is uniformly continuous as $y \to 0$: given $\delta > 0$, there exist $\rho > 0$ so that if $||\theta\ y^{\nu}(t) + (1-\theta)\ y^{\nu-1}(t)|| \leq \rho$ then

$$||J^{\nu}(t)|| \leq \delta \ , \ t \ \varepsilon\left[t_{\infty}, \infty\right) \quad .$$

At the end of this proof we will show that if L is small enough then $||y^{\nu}|| < \rho$, $\nu = 1,2,\ldots,$ for any given ρ.

Let us suppose that t_{∞} is so large that:

i) $||B(t) - M|| + ||J^{\nu}(t)|| \leq \sigma(t_{\infty})$ for $t \ \varepsilon\left[t_{\infty}, \infty\right)$

ii) If $M_{kk} \equiv 0$ then $\displaystyle\int_t^{\infty} \sum_{i=1}^{k} (||B_{ki}(s)|| + ||J_{ki}(s)||)ds \leq \frac{2K\,\sigma(t_{\infty})}{\varepsilon}$

iii) There exists a constant τ, $0 \leq \tau < 1$, so that

$$\frac{2K\sigma(t_\infty)}{\varepsilon} \sum_{i=0}^{k-1} \left(\frac{2\mu}{\varepsilon}\right)^i = \tau < 1 \quad .$$

Here ε and μ are positive constants such that $|\lambda_i(M)| > \varepsilon$ for all eigenvalues of M different from zero and $||M|| \leq \mu$.

Then, as in Theorem 5.4, we conclude

$$||e_i^{(\nu)}(t)|| \leq \frac{2K\sigma(t_\infty)}{\varepsilon} \sum_{j=0}^{k-i} \left(\frac{2\mu}{\varepsilon}\right)^j \max_{t_\infty \leq s < \infty} ||e^{(\nu-1)}(s)||$$

$$\leq \tau \max_{t_\infty \leq s < \infty} ||e^{(\nu-1)}(s)|| \quad .$$

This implies

$$||e^{(\nu)}(t)|| \leq \tau \max_{t_\infty \leq s < \infty} ||e^{(\nu-1)}(s)|| \leq L\, \tau^\nu \quad .$$

Then

$$y(t) = \lim_{N\to\infty} \sum_{\nu=1}^{N} e^{(\nu)}(t)$$

exists and satisfies (6.5) c).

From (6.11) we have

$$y^{(\nu)}(t) = \sum_{i=1}^{\nu} e^{(i)}(t) \quad .$$

This implies

$$||y^{(\nu)}(t)|| \leq L \frac{1 - \tau^{\nu+1}}{1 - \tau} \quad .$$

Thus, if $\frac{L}{1-\tau} \leq \rho$ we get $||y^{(\nu)}(t)|| \leq \rho$, $\nu = 1,2,\ldots$.

In this case the "projection condition" for $y(t)$ is

$$\lim_{t\to\infty} (I-P)\ y(t) = 0 \text{ , i.e.,}$$
$$\lim_{t\to\infty} (I-P)\ Q^{-1}\ u(t) = 0 \quad . \tag{6.13}$$

We point out that if in the original problem $\gamma \neq 0$, P depends upon $\gamma = \lim_{t\to\infty} u(t)$. If the actual value of γ is not known then (6.13) is nonlinear in $u(t)$.

It is possible to prove for (6.1) results similar to those in Theorem 6.6 with slightly different $T(t)$ and $A(t)$ (recall (6.4b)) in fact:

Theorem 6.14: Let $f(t, y)$ and $f_y(t, y)$ satisfy (6.4a,b,e) where $\gamma \equiv 0$. We also assume that $T(t) = t^r I$, $r \geq 0$ and I identity matrix, $A_\infty = \lim_{t\to\infty} A(t)$ is as in (3.2). Moreover $A_{21} = o(\frac{1}{t^{r(m_o-1)}})$, $(A_{22} - A^o) = o(\frac{1}{t^{r(m_o-1)}})$, $A_{23} = o(\frac{1}{t^{r(m_o-1)}})$, $f_2(t,0) = o(\frac{1}{t^{r(m_o-1)}})$ as $t \to \infty$. Here m_o is the dimension of the largest Jordan block of A^o. Then for sufficiently large t_∞ and every ξ such that $P\xi = \xi$ (6.5) has a solution which satisfies $P\ y(t_\infty) = \xi$. ∎

Note: We have seen that the results of Lemma 3.9 are fundamental for all the conclusions of this chapter. We point out that the same results can be obtained without the requirements on $f'(t)$ (see page 7). We chose the present proof because of its simplicity.

CHAPTER 2

THE NUMERICAL SOLUTION OF BOUNDARY VALUE PROBLEMS ON SEMI-INFINITE INTERVALS

2.1 The linear case.

In this chapter we will consider the boundary value problem with separated boundary conditions

$$y' = t^{r-1}\left[A(t)y + f(t)\right] \ , \ t_o \le t < \infty \ , \qquad (1.1a)$$

$$D_o \, y(t_o) = \alpha \ , \ D_o \text{ is } k_o \times n \qquad (1.1b)$$

$$y(t) \text{ bounded at infinity} \ . \qquad (1.1c)$$

We assume that $\lim_{t\to\infty} A(t) = A_\infty$, with A_∞ as in sections 1.3 and 1.4 (Chapter 1) and $k_o = p_\infty + s_\infty$.

In the previous chapter we saw that under certain conditions (1.1) c) can be written in the algebraic form

$$\lim_{t\to\infty} \left[I \mid 0\right] y(t) = \lim_{t\to\infty} D_\infty \, y(t) = \beta \qquad (1.1d)$$

where I is the identity matrix of dimension $q_\infty + r_\infty - s_\infty$ and

$$\beta = \begin{pmatrix} -(A^+)^{-1} \hat{f}_1 \\ \\ 0 \end{pmatrix} .$$

It is easy to see that the change of dependent variable

$$\hat{y}(t) = y(t) - \begin{pmatrix} -(A^+)^{-1} & \hat{f}_1 \\ 0 & \\ -(A^-)^{-1} & \hat{f}_3 \end{pmatrix}$$

gives an equivalent system with inhomogeneous term which goes to zero as $t \to \infty$, and hence the boundary condition at infinity reduces to $\lim_{t\to\infty} \left[I \mid 0 \right] \hat{y}(t) = 0$. We will assume in the rest of this chapter that this is the case.

Most numerical work on problems of the form (1.1) a), b), c) proceed by replacing the interval $\left[t_o, \infty\right)$ by a finite interval, say $\left[t_o, t_\infty\right]$. However the boundary conditions to be imposed at $t = t_\infty$ are not always chosen correctly.

We propose to solve

$$v' = t^{r-1}\left[A(t)v + f(t)\right] \tag{1.2a}$$

$$D_o\, v(t_o) = \alpha \tag{1.2b}$$

$$D_\infty\, v(t_\infty) = 0 \quad . \tag{1.2c}$$

We will study the error that is made by solving (1.2) instead of (1.1). To do this we assume that (1.1) a), b), d), and (1.2) have unique solutions $y^*(t)$ and $v^*(t)$ respectively.

In order to estimate the error

$$e(t) = y^*(t) - v^*(t) \ , \ t \in \left[t_o, t_\infty\right] \ , \tag{1.3}$$

we first study, as Keller [13], the special case when $A(t)$ and $f(t)$

have "constant tail", i.e.,

$$A(t) \equiv A_\infty ,$$

$$f(t) \equiv 0 \quad , \ t > t_\infty$$

Theorem 1.4: Let $A(t)$ and $f(t)$ be continuous for $t \in [t_o, t_\infty]$ and $A(t) \equiv A_\infty$, $f(t) \equiv 0$ for $t > t_\infty$. Let (1.1) a), b), d) and (1.2) a), b), c) have unique solutions $y^*(t)$ and $v^*(t)$. Then $y^*(t) = v^*(t)$ for $t \in [t_o, t_\infty)$.

Proof: $v^*(t)$ is such that

$$v^*(t_\infty) = \begin{pmatrix} 0 \\ \\ \gamma^* \end{pmatrix} \begin{matrix} \} (n - k_o) \\ \\ \} \quad k_o \end{matrix} .$$

Consider the initial value problem

$$Z'(t) = A_\infty Z(t) \ , \ t_\infty \leq t < \infty \tag{1.5a}$$

$$Z(t_\infty) = \begin{pmatrix} 0 \\ \\ \gamma^* \end{pmatrix} . \tag{1.5b}$$

The solution of (1.5) can be written as

$$Z(t) = e^{A_\infty (t^r - t_\infty^r)/r} \begin{pmatrix} 0 \\ \\ \gamma^* \end{pmatrix} = \begin{pmatrix} 0 \\ \\ \gamma(t) \end{pmatrix} ,$$

with $\gamma(t)$ of dimension k_o. Then we have

$$\lim_{t\to\infty} D_\infty \; Z(t) = \lim_{t\to\infty} \left[I \mid 0 \right] Z(t) = 0 \quad .$$

Thus we conclude that the function

$$w(t) = \begin{cases} v^*(t) & t \; \varepsilon \left[t_o, \; t_\infty\right] \\ \\ Z(t) & t \; \varepsilon \left[t_\infty, \; \infty\right) \end{cases}$$

is a solution of the problem (1.1) a), b), d) and hence $e(t) = y^*(t) - v^*(t) = 0$, $t \; \varepsilon \left[t_o, \; t_\infty\right]$. ∎

For general $A(t)$ and $f(t)$ we have

Theorem 1.6: Let (1.1) a), b), d) satisfy all the hypotheses of Theorem 4.14 (Chapter 1). We also assume $\lim_{t\to\infty} f(t) = 0$ and that (1.2) has a unique solution $v^*(t)$. Then for $t \; \varepsilon \left[t_o, \; t_\infty\right]$

$$e(t) = y^*(t) - v^*(t) = Y(t)Q_\infty^{-1}\begin{pmatrix} 0 \\ -\int_{t_\infty}^{\infty} e^{-A^+(s^r - t_\infty^r)/r}\, s^{r-1} m_1(s)ds \\ -[I|0]\int_{t_\infty}^{\infty} e^{-A^o(s^r - t_\infty^r)/r}\, s^{r-1} m_2(s)ds \end{pmatrix}$$

Here $y^*(t)$ is the unique solution of (1.1), $Q_\infty = \begin{pmatrix} D_o \\ D_\infty \; Y(t_\infty) \end{pmatrix}$,

$Y(t)$ is the fundamental solution of the homogeneous system associated with (1.1) subject to $Y(t_o) = I$ and

$$m(t) = \begin{cases} 0 & t_o \le t \le t_\infty \\ f(t) + (A(t) - A_\infty)y^*(t) \;, & t > t_\infty \end{cases} .$$

Proof: Let us call w^* the solution of the "constant tail" problem

$$w' = \begin{cases} t^{r-1}(A(t)w + f(t)) & t_o \le t \le t_\infty \\ t^{r-1}A_\infty \; w & t > t_\infty \end{cases}$$

$$D_o \; w(t) = \alpha$$

$$\lim_{t\to\infty} D_\infty \; w(t) = 0 \; .$$

Then $E(t) = y^*(t) - w^*(t)$ satisfies

$$E' = \begin{cases} t^{r-1} A(t)E & t_o \leq t \leq t_\infty \\ t^{r-1}(A_\infty E + m(t)) & t > t_\infty \end{cases}$$

$$D_o\ E(t_o) = 0$$

$$\lim_{t\to\infty} D_\infty E(t) = 0 \quad .$$

After a little algebra one gets

$$E(t_\infty) = \begin{pmatrix} -\displaystyle\int_\infty^{t_\infty} e^{-A^+(s^r - t_\infty^r)/r}\, s^{r-1} m_1(s)ds \\ \gamma^o - \displaystyle\int_\infty^{t_\infty} e^{-A^o(s^r - t_\infty^r)/r}\, s^{r-1} m_2(s)ds \\ \gamma^- \end{pmatrix} ,$$

where γ^o and γ^- are constant vectors and $\gamma^o = \begin{pmatrix} 0 \\ \gamma_2^o \end{pmatrix} \begin{matrix} \}\, r_\infty - s_\infty \\ \}\, s_\infty \end{matrix}$

Hence $e(t) = y^*(t) - v^*(t) = E(t)$, $t_o \leq t \leq t_\infty$, is the unique solution of the boundary value problem

$$e' = t^{r-1} A(t)e$$

$$D_o\ e(t_o) = 0$$

$$D_\infty\ e(t_\infty) = \begin{pmatrix} -\displaystyle\int_{t_\infty}^{\infty} e^{-A^+(s^r - t_\infty^r)/r}\, s^{r-1} m_1(s)ds \\ -[I|0] \displaystyle\int_{t_\infty}^{\infty} e^{-A^o(s^r - t_\infty^r)/r}\, s^{r-1} m_2(s)ds \end{pmatrix}$$

where I is of dimension $(r_\infty - s_\infty)$. ∎

From the previous Theorem it is clear that:

Corollary 1.7: If the conditions of Theorem 1.6 are satisfied then for fixed t, $e(t) \to 0$ as $t_\infty \to \infty$, provided the matrix

$$Q(t) = Y(t)Q_\infty^{-1} \begin{bmatrix} 0 \\ --- \\ I \end{bmatrix} \begin{matrix} \} \; p_\infty + s_\infty \\ \\ \} \; q_\infty + (r_\infty - s_\infty) \end{matrix}$$

remains bounded for $t_o \le t \le t_\infty$ as $t_\infty \to \infty$.

Now we will give an example where this is the case. Suppose that

$$A(t) \approx A_\infty + \sum_{k=1} A_k t^{-k}$$

and that all the eigenvalues of A_∞ have geometric multiplicity equal to 1. Then we know from Coddington and Levinson [5] that the homogeneous system has an asymptotic solution of the form

$$\phi = P\, t^R e^Q \quad ,$$

where

$$P = \sum_{k=0} t^{-k} P_k \; , \; P_o = I$$

$$Q = \frac{t^r}{r} Q_o + \frac{t^{r-1}}{r-1} Q_1 + \ldots + t\, Q_{r-1} \quad ,$$

R and Q_i $i = 0, \ldots, r-1$ are diagonal. Moreover $Q_o = A_\infty$.

In this case Q_∞^{-1} has the following form

$$Q_\infty^{-1} = \begin{pmatrix} 0 & \mathcal{Q}^{-1}(t_\infty) \\ & \\ D_2^{-1} & -D_2^{-1} D_1 \mathcal{Q}^{-1}(t_\infty) \end{pmatrix} + 0(\frac{1}{t_\infty})$$

where $D_o = \left[D_1 \mid D_2\right]$, $\mathcal{Q}(t_\infty) = \left[I \mid 0\right] t_\infty^R e^Q$, and I is of dimension q_∞. The dominant term in $\mathcal{Q}(t_\infty)$ has the form $e^{A^+ t_\infty^r / r}$ and therefore we conclude that the matrix $Q(t)$ remains bounded for $t \in \left[t_o, t_\infty\right]$ as $t_\infty \to \infty$.

In conclusion, to solve numerically (1.1) a), b), c) we propose:

1) Pick a suitable large t_∞ (Studies on the problem of choosing the "appropriate" t_∞ for a given problem are being carried out at present.)

2) Obtain the "projection" condition.

3) Solve the resulting two point boundary value problem using any "global" method.

In particular, all our numerical results have been obtained using the TPBVP solver PASVA3, which is described in the Appendix.

We should mention that recently other proofs of convergence of numerical methods for this kind of problems have appeared. Bayliss in [1] proposes a double shooting technique and Franklin and Scott [11] consider a "shooting from infinity" method.

CHAPTER 3

BOUNDARY VALUE PROBLEMS ON A SEMI-INFINITE INTERVAL WITH A REGULAR SINGULAR POINT

3.1 Existence theory for the linear problem.

In this chapter we are interested in the existence of solutions, which are continuous as $t \to 0$ and bounded as $t \to \infty$, of the equation

$$u'(t) = (\frac{M}{t} + C(t))\, u(t) + f(t) \ , \ 0 < t < \infty \ . \tag{1.1}$$

Here M and $C(t)$ are $n \times n$ matrices, $u(t)$ and $f(t)$ are vectors of dimension n. $C(t)$ and $f(t)$ are continuous for $t \in [0, \infty)$ and bounded as $t \to \infty$.

At the end of the chapter we will show the most general boundary conditions which are compatible with the requirements of continuity and boundedness and that also guarantee uniqueness.

To study problem (1.1) we divide the interval $[0, \infty)$ into two parts: $[0, t_o]$, $[t_o, \infty)$ for finite t_o. We first seek solutions which are continuous in $[0, t_o]$, then we continue them through the interval $[t_o, \infty)$, and study their behaviour at infinity.

Thus, let us consider

$$u'(t) = (\frac{M}{t} + C(t))\, u(t) + f(t) \quad 0 < t \leq t_o \tag{1.2}$$

here $C(t)$ and $f(t)$ are continuous for $t \in [0, t_o]$.

$$M = \left(\begin{array}{c|c|c} M^{+} & & \\ \hline & M^{o} & \\ \hline & & M^{-} \end{array}\right) \begin{array}{l} \} \; p_o \\ \} \; r_o \\ \} \; q_o \end{array} , \qquad \begin{array}{ccc} \underbrace{}_{p_o} & \underbrace{}_{r_o} & \underbrace{}_{q_o} \end{array} \tag{1.3}$$

M^{+} and M^{-} are Jordan blocks corresponding to eigenvalues of M with positive and negative real parts. M_o is also a Jordan block but we have interchanged some columns and rows, such that it takes the form

$$M^{o} = \left(\begin{array}{c|c} 0 & M^{o}_{12} \\ \hline 0 & M^{o}_{22} \end{array}\right) \begin{array}{l} \} \; s_o \\ \} \; r_o - s_o \end{array}$$

Here s_o is the geometric multiplicity of the eigenvalue zero.

The existence of continuous solutions for problem (1.2) has been studied extensively by de Hoog and Weiss [7]. Brabston and Keller [2] also consider numerical methods for solving (1.2) subject to boundary conditions of the form

$$B\,u(t) \equiv \lim_{t \to o} \left[B_o(t)\;u(t) + B_1 u(1) - b(t)\right] = 0 \quad .$$

We will give a new proof of existence and uniqueness of a continuous solution for (1.2) subject to boundary conditions of the form

$$C_o\;u(0) + C_1\;y(t_o) = \alpha \quad . \tag{1.4}$$

Here C_o and C_1 are $(p_o + s_o)\times n$ constant matrices and $\alpha \in \mathbb{R}^{(p_o+s_o)}$.

Our proof will use the same techniques employed in the earlier chapters and is of a more elementary structure than the one in [7].

To start the analysis we rewrite (1.2) in block form

$$
\begin{aligned}
u_1'(t) &= \frac{M^+}{t}\, u_1(t) + (C_{11}(t)u_1(t) + C_{12}(t)u_2(t) + C_{13}(t)u_3(t) + f_3(t)) \\
u_2'(t) &= \frac{M^o}{t}\, u_2(t) + (C_{21}(t)u_1(t) + C_{22}(t)u_2(t) + C_{23}(t)u_3(t) + f_2(t)) \\
u_3'(t) &= \frac{M^-}{t}\, u_3(t) + (C_{31}(t)u_1(t) + C_{32}(t)u_2(t) + C_{33}(t)u_3(t) + f_3(t)) \quad .
\end{aligned}
\tag{1.5}
$$

The fundamental solution of $y'(t) = \frac{M}{t}\, y(t)$ is given by $Y(t) = t^M$ for $0 < t \leq t_o$ and $Y(1) = I$. Variation of parameters suggests rewriting (1.5) in the integral form

$$
\begin{aligned}
u_1 &= \left(\frac{t}{t_o}\right)^{M^+} \gamma^+ + t^{M^+} \int_{t_o}^{t} s^{-M^+} \{C_{11}\, u_1 + C_{12}\, u_2 + C_{13}\, u_3 + f_1\} ds \\
u_2 &= \left(\frac{t}{t_o}\right)^{M^o} \gamma^o + t^{M^o} \int_{0}^{t} s^{-M^o} \{C_{21}\, u_1 + C_{22}\, u_2 + C_{23}\, u_3 + f_2\} ds \\
u_3 &= \left(\frac{t}{t_o}\right)^{M^-} \gamma^- + t^{M^-} \int_{0}^{t} s^{-M^-} \{C_{31}\, u_1 + C_{32}\, u_2 + C_{33}\, u_3 + f_3\} ds \quad .
\end{aligned}
$$

Since we are only interested in continuous solutions at zero then:

$$
\gamma^- = 0 \quad \text{and} \quad \gamma^o = \begin{pmatrix} \gamma_1^o \\ 0 \end{pmatrix} \begin{matrix} \} \; s_o \\ \} \; r_o - s_o \end{matrix} \quad . \tag{1.6}
$$

Thus,

$$u_1(t) = (\frac{t}{t_o})^{M^+} \gamma^+ + t^{M^+} \int_{t_o}^{t} s^{-M^+} \{C_{11} u_1 + C_{12} u_2 + C_{13} u_3 + f_1\} ds \tag{1.7}$$

$$u_2(t) = (\frac{t}{t_o})^{M^o} \begin{pmatrix} \gamma_1^o \\ 0 \end{pmatrix} + t^{M_o} \int_0^t s^{-M^o} \{C_{21} u_1 + C_{22} u_2 + C_{23} u_3 + f_2\} ds$$

$$u_3(t) = t^{M^-} \int_0^t s^{-M^-} \{C_{31} u_1 + C_{32} u_2 + C_{33} u_3 + f_3\} ds \quad .$$

Clearly any continuous solution of (1.2) will be a solution of (1.7). Conversely any solution of (1.7) is a solution of (1.2). It will be proven in Theorem 1.8 that every solution of (1.7) is continuous at $t = 0$.

Theorem 1.8: *Let* $C(t)$ *and* $f(t)$ *be continuous for* $t \varepsilon [0, t_o]$ *and* M *be as in* (1.3). *Then the integral equations* (1.7) *have a solution. Hence* (1.2) *has* $(p_o + s_o)$ *linearly independent solutions which are continuous at* $t = 0$.

Proof: We use Picard iterations in (1.7) and start with $u^{(0)}(t) \equiv 0$. We get for $u^{(1)}(t)$

$$\begin{aligned} u_1^{(1)}(t) &= (\frac{t}{t_o})^{M^+} \gamma^+ + t^{M^+} \int_{t_o}^{t} s^{-M^+} f_1(s) ds \\ u_2^{(1)}(t) &= (\frac{t}{t_o})^{M^o} \begin{pmatrix} \gamma_1^o \\ 0 \end{pmatrix} + t \int_0^1 s^{-M^o} f_2(ts) ds \\ u_3^{(1)}(t) &= t \int_0^1 s^{-M^-} f_3(ts) \quad . \end{aligned} \tag{1.9}$$

Since $f(t)$ is continuous at $t = 0$, all the integrals converge. Hence $u^{(1)}(t)$ is well defined. Moreover $u^{(1)}(t)$ is continuous for $t \in [0, t_o]$, and taking limits as $t \to 0$ in (1.9) we get

$$u^{(1)}(0) = \begin{pmatrix} 0 \\ \gamma_1^o \\ 0 \end{pmatrix} .$$

Then $u^{(2)}(t)$ is well defined. Similarly, it can be shown by induction that $u^{(\nu)}(t)$ is well defined and continuous for $t \in [0, t_o]$ and that

$$u_1^{(\nu+1)}(0) = \begin{pmatrix} 0 \\ \gamma_1^o \\ 0 \end{pmatrix} . \tag{1.10}$$

We define

$$e^{(\nu)}(t) = u^{(\nu)}(t) - u^{(\nu-1)}(t) \ , \ \nu = 1, 2, \ldots, \quad .$$

Then $e^{(\nu+1)}(t)$ is given by

$$e_1^{(\nu+1)}(t) = t^{M^+} \int_{t_o}^{t} s^{-M^+} \{C_{11}(s)e_1^{(\nu)}(s) + C_{12}(s)e_2^{(\nu)}(s) + C_{13}(s)e_3^{(\nu)}(s)\}ds,$$

$$e_2^{(\nu+1)}(t) = t^{M^o} \int_{0}^{t} s^{-M^o} \{C_{21}(s)e_1^{(\nu)}(s) + C_{22}(s)e_2^{(\nu)}(s) + C_{23}(s)e_3^{(\nu)}(s)\}ds,$$

$$e_3^{(\nu+1)}(t) = t^{M^-} \int_{0}^{t} s^{-M^-} \{C_{31}(s)e_1^{(\nu)}(s) + C_{32}(s)e_2^{(\nu)}(s) + C_{33}(s)e_3^{(\nu)}(s)\}ds.$$

We have for $t \in [0, 1]$ and some constant $D > 0$,

$$\left|\left|\int_1^t \left(\frac{t}{s}\right)^{M^+} ds\right|\right| \leq D\, t^{\alpha}(1 + |\ln t|^{m-1}) \quad .$$

Here $\alpha = \min_{\lambda} R(\lambda(M^+))$ and m is the dimension of the largest Jordan block in M^+.

Let

$$a = \max\left(\int_0^1 ||s^{-M^o}||ds \;,\; \int_0^1 ||s^{-M^-}||ds\right).$$

Then

$$||e_1^{(\nu+1)}(t)|| \leq 3Dt^{\alpha}(1 + |\ln t|^{m-1}) \max_{0\leq s\leq t_o} (||e^{\nu}(s)||\cdot||C(s)||),$$

$$||e_2^{\nu+1}(t)|| \leq 3at \max_{0\leq s\leq t_o} (||e^{\nu}(s)||\;||C(s)||) \;,$$

$$||e_3^{\nu+1}(t)|| \leq 3at \max_{0\leq s\leq t_o} (||e^{\nu}(s)||\;||C(s)||) \quad .$$

Let t_o be so small that

$$t_o < \max\left(1 \;,\; 1/(3a \max_{0\leq s\leq t_o} ||C(s)||\right) \;,$$

and for $t \in [0, t_o]$

$$3D(\max_{0\leq s\leq t_o} ||C(s)||)t^{\alpha}(1 + |\ln t|^{m-1}) < 1$$

Then we get, in the usual manner, that

$\sum_{\nu=1}^{\infty} ||e^{(\nu)}(t)||$ is uniformly convergent. Hence $\sum_{\nu=1}^{\infty} e^{(\nu)}(t)$ is uniformly and absolutely convergent on $[0, t_o]$. Then we conclude that $\sum_{\nu=1}^{N} e(t) = u^{(N)}(t)$ tends uniformly to a continuous function $u(t)$ which is a solution of the differential equation (1.2) on $[0, t_o]$. Moreover, from (1.9) we get

$$u(0) = \begin{pmatrix} 0 \\ \gamma_1^o \\ 0 \end{pmatrix} \begin{matrix} \} \; p_o \\ \} \; s_o \\ \} \; (n - p_o - s_o) \end{matrix} \tag{1.11}$$

This solution can be continued for $t > t_o$ if $f(t)$ and $C(t)$ are continuous for $t > t_o$. $\blacksquare$

It should be noticed that all the solutions of (1.2) which are continuous at $t = 0$ satisfy (1.11). In this case the projection into the subspace of solutions which are continuous at $t = 0$ is given by

$$Q_o \; u(0) = 0 \quad . \tag{1.12}$$

Here $Q_o = (I - P_o - R_o)$, P_o is the projection into the invariant subspace associated to the eigenvalues of M with real part greater than zero, and R_o is the projection into the nullspace of M.

From (1.12) it is also clear that we can only give s_o additional conditions at zero. So we get:

Theorem 1.13: Consider the two point boundary value problem

$$u' = (\frac{M}{t} + C(t))u + f(t) \;, \; 0 < t \leq t_o \;, \tag{1.14a}$$

$$u(t) \text{ continuous at } t = 0 \;, \tag{1.14b}$$

$$\left[B_o \, u(0) + B_1 \, u(t_o)\right] = \alpha \;. \tag{1.14c}$$

Here M, $C(t)$ and $f(t)$ satisfy the hypotheses of Theorem 1.8; B_o and B_1 are $(p_o + s_o)\times n$ constant matrices; α is a constant vector of dimension $(p_o + s_o)$. This problem has a unique solution for each α iff the $(p_o + s_o)$ order matrix

$$\hat{B} = \left[B_o \, R_o \, U(0) + B_1 \, U(t_o)\right] \tag{1.15}$$

is nonsingular.

Here $U(t)$ is any $n\times(p_o + s_o)$ matrix formed from $(p_o + s_o)$ linearly independent solutions of the integral equation (1.7) with $f(t) \equiv 0$ (homogeneous system).

Proof: From Theorem 1.8 we know that all the solutions of (1.14) a) which are continuous at $t = 0$ satisfy (1.11). Then condition (1.14) c) can be rewritten as

$$B_o \, R_o \, u(0) + B_1 \, u(t_o) = \alpha \;. \tag{1.16}$$

On the other hand, any continuous solution of (1.1) can be represented as

$$u(t) = U(t)\beta + v(t) \quad . \tag{1.17}$$

Here $v(t)$ is the unique solution of the integral equation (1.7) with $\gamma \equiv 0$. Then substitution of (1.17) into (1.16) gives

$$B_o\, R_o\, U(0)\beta + B_1\left[U(t_o)\beta + v(t_o)\right] = \alpha \quad .$$

Hence (1.14) has unique solution iff $\hat{B}$ is nonsingular. ∎

From the previous results and those in Chapter 1 we can deduce when it is possible for the system (1.1) to have solutions which are continuous at zero and bounded as $t \to \infty$.

Before continuing further we state some notation that we will use later on:

$Y(t)$ is the $n\times(p_\infty + s_\infty)$ "fundamental" matrix of solutions of (1.1) which are bounded as $t \to \infty$. (1.18a)

$U(t)$ is the $n\times(p_o + s_o)$ "fundamental" matrix of solutions of (1.1) which are continuous at $t = 0$. (1.18b)

$\hat{U}(t)$ is the matrix formed with those columns of $U(t)$ which are in the range of $Y(t)$. $\hat{U}(t)$ is $n \times k$. (1.18c)

From (1.18) c) we deduce that there exist C_o and C_∞ constant matrices, of dimension $(p_o + s_o)\times k$ and $(p_\infty + s_\infty)\times k$ respectively, such that

$$U(t)C_o = \hat{U}(t) = Y(t)C_\infty \quad . \tag{1.19}$$

Finally we have

Theorem 1.20: Suppose that M, $C(t)$ and $f(t)$ satisfy the hypotheses of Theorem 1.8. We also assume that there exists a similarity transformation P such that

$$A(t) = t^{-r+1} P^{-1}\left[\frac{M}{t} + C(t)\right] P$$

and

$$h(t) = t^{-r+1} P^{-1} f(t)$$

satisfy the hypotheses of Theorem 4.14 (Chapter 1) for some $r \geq 1$. Then if system (1.1) has a continuous bounded solution $u_p(t)$, that solution is unique iff the k defined in (1.18) c) is equal to zero. If $k > 0$ it is necessary to specify k additional conditions to guarantee uniqueness:

$$B_o\, u(0) + \lim_{t\to\infty} B_\infty\, u(t) = \alpha \quad . \tag{1.21}$$

Here B_o and B_∞ are $k \times n$ constant matrices and α is a constant vector of dimension k.

The solution is unique iff

$$\hat{B} = B_o\, R_o\, \hat{U}(0) + \lim_{t\to\infty} B_\infty\, R_\infty\, \hat{U}(t) \tag{1.22}$$

is nonsingular. R_o is the projection into the nullspace of M and

R_∞ <u>is the projection into the nullspace of</u> $A_\infty = \lim_{t\to\infty} A(t)$.

<u>Proof</u>: Any bounded continuous solution of (1.1) can be represented as

$$u(t) = \hat{U}(t)\beta + u_p(t) \quad . \tag{1.23}$$

The matrix $\hat{U}(t)$ is given in (1.18) c), β is a constant vector in $\mathbb{R}^k$ and $u_p(t)$ is the particular solution of (1.1). Substitute (1.23) into (1.21), recall that $u(0) = R_o\, u(0)$ and that the only components of $u(t)$ which can be specified at infinity are $R_\infty\, u(t)$. Then we see that the boundary conditions (1.21) are equivalent to

$$\left[B_o\, R_o\, \hat{U}(0) + \lim_{t\to\infty} B_\infty\, R_\infty\, \hat{U}(t)\right]\beta = \alpha - B_\infty R_\infty \lim_{t\to\infty} u_p(t) - B_o\, u_p(0) . \tag{1.24}$$

The system of algebraic equations (1.24) has a unique solution iff $\hat{B}$ is nonsingular. ■

3.2 <u>Numerical considerations</u>: De Hoog and Weiss [7] have shown that the box scheme and the trapezoidal rule are stable discretizations for system (1.2) when computing solutions which are continuous at $t = 0$. In the case of the trapezoidal rule it is necessary to change equation (1.2) at $t = 0$ to

$$y'(0) = M\, y'(0) + C(0)\, u(0) + f(0) \quad .$$

Thus this scheme only is applicable when M does not have the eigenvalue $\lambda = 1$.

De Hoog and Weiss also mention that, although the trapezoidal rule is still a second order method for (1.2) the discretization error does not have a uniform expansion.

We should mention that our numerical experience, using PASVA3 (see the Appendix), does not support this statement.

CHAPTER 4

NUMERICAL EXAMPLES

4.1 The beam equation.

We have chosen a very interesting problem to start the computational test of the methods we described in the previous chapters.

J. N. Franklin and R. F. Scott [11], have proposed the use of a shooting technique to obtain bounded solutions of the equation

$$w^{(4)}(x) + x^p w(x) = 0 \ , \ x \ \varepsilon [0, \infty) \quad . \tag{1.1}$$

"This equation is satisfied by the deflection $w(x)$ of loaded beams resting on, or imbedded (piles) in soils; x is the distance from the surface.

In many structural and foundation engineering problems consideration must be given to elastically supported beams. When the support is provided by a continuum, a solution in the form of the deflection of the beam as a function of its length is often difficult to obtain. A good representation of the system is achieved by considering the support of the beam to consist of springs continuously distributed along its length.

In the simplest case, the beam has properties uniform along its length. In this case analytical solutions in terms of elementary functions have long been available. However, when the beam or spring properties vary in some fashion along the beam's length, closed-form analytical solutions have not been obtained.

A problem of considerable practical interest is the pile, which consists of a beam imbedded vertically, or near vertically, in the ground, and loaded by a horizontal force or a moment at ground surface. Most soils become stiffer with depth, in some cases linearly, in others with some other power of distance.

For the diameter and length of piles most frequently employed in practice, displacement, moment, and other quantities in this problem die out relatively rapidly with distance from the surface, so that the range of numerical values occurs near the loaded end. Conditions at the pile base are unimportant, and the problem becomes a semi-infinite one."

The well known simplified equation for a beam bending under the action of transverse forces $q(x)$ is

$$EI\, w^{(4)}(x) - q(x) = 0 \ , \ 0 < x < \infty \tag{1.2}$$

where EI is the elastic modulus, w is the deflection, x is the length coordinate taken along the beam's axis. Where the beam is not loaded by external forces $q(x) = k(x)\, w(x)$ is supplied by the foundation material pressing on the beam. If $k(x)$ can be represented as a power of the distance, equation (1.1) is obtained from (1.2) through a change of variables.

For the pile problem there are two possible sets of conditions that can be given at the top of the pile - either the values of w' and w''' or w'' and w''', depending on what is known: the transversal force or the moment at the top of the pile. The complementary conditions are:

$$\lim_{x\to\infty} w(x) = \lim_{x\to\infty} w'(x) = \lim_{x\to\infty} w''(x) = \lim_{x\to\infty} w'''(x) = 0 \quad . \tag{1.3}$$

In order to find the appropriate projection condition for the equation (1.1) when $p > -4$ we rewrite it as a first order system which has the same asymptotic behaviour as $x \to \infty$ as equation (1.1). Making the change of dependent variable (Coddington and Levinson [5] p. 169)

$$y_k = x^{-(k-1)r}\, w^{(k-1)} \ , \ k = 1,2,3,4 \ , \ r = p/4 \quad , \tag{1.4}$$

we get that the vector $y = (y_1, y_2, y_3, y_4)^T$ satisfies

$$y' = x^r A(x) y \quad . \tag{1.5a}$$

Here

$$A(x) \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -rx^{-(r+1)} & 1 & 0 \\ 0 & 0 & -2rx^{-(1+r)} & 1 \\ -1 & 0 & 0 & -3rx^{-(1+r)} \end{pmatrix}$$

and

$$A_\infty = \lim_{x\to\infty} A(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad . \tag{1.5b}$$

A_∞ has the following eigenvalues $\lambda_1 = e^{\pi i/4}$, $\lambda_2 = e^{7\pi i/4}$,

$\lambda_3 = e^{3\pi i/4}$, $\lambda_4 = e^{5\pi i/4}$. We notice that $Re(\lambda_1) > 0$,$Re(\lambda_2) > 0$, $Re(\lambda_3) < 0$, $Re(\lambda_4) < 0$. We conclude that the system (1.5) has two linearly independent solutions which vanish as $x \to \infty$.

The eigenvectors of A_∞ are

$$\begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2}(1+i) \\ i \\ -\frac{\sqrt{2}}{2}(1-i) \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2}(1-i) \\ -i \\ -\frac{\sqrt{2}}{2}(1+i) \end{pmatrix}, \begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2}(1-i) \\ -i \\ \frac{\sqrt{2}}{2}(1+i) \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2}(1+i) \\ i \\ \frac{\sqrt{2}}{2}(1-i) \end{pmatrix} .$$

After some algebraic manipulation, we find that the projection condition for a real solution can be expressed as

$$\lim_{x\to\infty}\left[y_1(x) + \frac{1}{\sqrt{2}}y_2(x) - \frac{1}{\sqrt{2}}y_4(x)\right] = 0 ,$$
$$\lim_{x\to\infty}\left[\frac{1}{\sqrt{2}}y_2(x) + y_3(x) + \frac{1}{\sqrt{2}}y_4(x)\right] = 0 . \qquad (1.6)$$

We observe that by making the change of variables (1.4) we have introduced a singularity at $x = 0$ which did not exist in the original problem. To avoid this diffucilty, we use close to zero a different formulation of (1.1) which has no singularity at $x = 0$. Specifically we use near $x = 0$ the system:

$$\hat{y}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -x^p & 0 & 0 & 0 \end{bmatrix} \hat{y} . \qquad (1.7)$$

The components of $\hat{y}(t)$ are defined by

$$\hat{y}_i(t) = w^{(i-1)}(t) \ , \ i = 1,2,3,4 \, . \tag{1.8}$$

In practice, we choose a suitable x_∞ (in this problem $x_\infty = 7$ was satisfactory) and apply the projection condition at $x = x_\infty$. On the other hand, it is convenient to compute two linearly independent bounded solutions once and for all, such that one can obtain any solution of physical interest as a linear combination of those two. That can be achieved by imposing the initial conditions either $w' = 1$, $w''' = 0$ and $w' = 0$, $w''' = 1$ or $w'' = 1$, $w''' = 0$ and $w'' = 0$, $w''' = 1$. Then for $x \, \varepsilon \left[0, \ 1\right]$ we solve system (1.7) with the two appropriate initial conditions and finally for $x \, \varepsilon \left[1 \ , \ x_\infty\right]$ we solve system (1.5) with the projection condition. This is done simultaneously by making the additional changes of independent variables

$$t = x, \ x \varepsilon \left[0, \ 1\right] \quad \text{and}$$
$$t = \frac{x-1}{x_\infty - 1}, \ x \varepsilon \left[1 \ , \ x_\infty\right] \quad . \tag{1.9}$$

Now $t \, \varepsilon \left[0, \ 1\right]$ is the independent variable for both systems and since we want a continuous solution the additional conditions are

$$\hat{y}_1(1) - y_1(0) = 0$$
$$\hat{y}_2(1) - y_2(0) = 0$$
$$\hat{y}_3(1) - y_3(0) = 0$$
$$\hat{y}_4(1) - y_3(0) = 0 \quad .$$

Figures 1 and 2, correspond to the numerical solution obtained using the two-point boundary solver PASVA3 [a full description of this program is in the Appendix] to solve the systems we just described for $p = \frac{1}{2}$. Since we were interested in making some comparisons with the results of Franklin and Scott [11], we chose the initial conditions properly in order to get their solutions.

In conclusion we solve for $v_1 = y_1$, $v_2 = y_2$, $v_3 = y_3$, $v_4 = y_4$, $v_5 = \hat{y}_1$, $v_6 = \hat{y}_2$, $v_7 = \hat{y}_3$, $v_8 = \hat{y}_4$, $t \in [0, 1]$:

$$v_1'(t) = v_2(t)$$
$$v_2'(t) = v_3(t)$$
$$v_3'(t) = v_4(t)$$
$$v_4'(t) = -t^{4r} \; v_1(t)$$
$$v_5'(t) = (x_\infty - 1)^{r+1} \; (t+1/(x_\infty - 1))^r \; v_6(t)$$
$$v_6'(t) = - \frac{r}{(t+1/(x_\infty - 1))} v_6 + (x_\infty - 1)^{r+1} (t+1/(x_\infty - 1))^r \; v_7(t)$$
$$v_7'(t) = - \frac{2r}{(t+1/(x_\infty - 1))} v_7 + (x_\infty - 1)^{r+1} (t+1/(x_\infty - 1))^r \; v_8(t)$$
$$v_8'(t) = - \frac{3r}{(t+1/(x_\infty - 1))} v_8 - (x_\infty - 1)^{r+1} (t+1/(x_\infty - 1))^r \; v_5(t)$$

with the boundary conditions:

$$v_2(0) = -.886498 \qquad v_4(0) = .290394$$

or

$$v_3(0) = -.584606 \qquad v_4(0) = .892356$$

and

$$v_1(1) - v_5(0) = 0$$

$$v_2(1) - v_6(0) = 0$$

$$v_3(1) - v_7(0) = 0$$

$$v_4(1) - v_8(0) = 0$$

$$\sqrt{2}\, v_5(1) + v_6(1) - v_8(1) = 0$$

$$v_6(1) + \sqrt{2}\, v_7(1) + v_8(1) = 0$$

The numerical results were obtained on a grid of 49 points in the interval $[0, 7]$*. The results agree with those of Franklin and Scott up to 4 significant digits.

It is necessary to emphasize some of the advantages of our approach as compared to that of Franklin and Scott:

* The work done is equivalent of that of solving one system of 4 equations on 98 grid points.

1. It is not necessary "to play" with boundary conditions since the projection condition and the physical requirements form a set of 4 boundary conditions.
2. The results were obtained by running the program only twice.
3. We use a boundary value solver with estimation of global errors and deferred corrections which permit us to obtain accurate solutions efficiently.*

4.2 An eigenvalue problem.

The second example we consider was chosen for two reasons. First it is one of those historic problems that has appeared in the literature for many years because it was difficult to solve with standard techniques. Second, it has a regular singular point at the left endpoint and is on $[0, \infty)$.

Fox in [10] and Conte in [6] consider the following eigenvalue problem related to the solution of the Schrödinger equation for the hydrogen-molecule ion with fixed nuclei:

* I would like to thank Professor Joel Franklin for mentioning this problem to me and Professor Ronald Scott for explaining the physical significance of the problem and offering his numerical results for comparisons.

$$u'' = 2\lambda u - 4\,E_o\,u - \frac{4}{5}\,E_2\,v - \frac{4}{9}\,E_4\,w \tag{2.1}$$

$$v'' = 2\lambda v + \frac{6}{x^2}\,v - 4\,E_2\,u - (4E_o + \frac{8}{7}\,E_2 + \frac{8}{7}\,E_4)v$$

$$- (\frac{8}{7}\,E_2 + \frac{400}{693}\,E_4 + \frac{100}{143}\,E_6)w$$

$$w'' = 2\lambda w + \frac{20}{x^2}\,w - 4\,E_4\,u - (\frac{72}{35}\,E_2 + \frac{80}{77}\,E_4$$

$$+ \frac{180}{143}\,E_6)v - (4\,E_o + \frac{80}{77}\,E_2 + \frac{648}{1001}\,E_4$$

$$+ \frac{80}{143}\,E_6 + \frac{1960}{2431}\,E_8)w$$

$$E_s = \begin{cases} x^s & \text{if } 0 \le x \le 1 \\ x^{-(s+1)} & \text{if } x \ge 1 \end{cases} .$$

Boundary conditions: the solution has to be continuous at $x = 0$ and vanish at infinity.

We should mention at this point that we have not done any analytical study of eigenvalue problems in semi infinite intervals. It seems unlikely that one can compute the eigensolution, using the techniques we have discussed in this thesis, when the problem has a continuous spectrum. But the situation for discrete spectrums looks more promising.

In any event, the problem that we present here has a discrete spectrum. Moreover, it has an isolated eigenvalue close to $\lambda = .3$. We intend to compute that eigenvalue and its corresponding eigenfunction more accurately than has been done so far. In particular, computing the eigenfunction numerically with any reasonable accuracy has been a very difficult task in the past.

To get a first order system equivalent to (2.1), which preserves the regular singularity at zero we make the change of variables

$$\begin{array}{lll} y_1(x) = u(x) & y_2(x) = u'(x) & y_3(x) = v(x) \\ y_4(x) = x\,v'(x) & y_5(x) = w(x) & y_6(x) = x\,w'(x) \end{array} \quad . \tag{2.2}$$

Then we get for y

$$y' = \frac{1}{x} A(x) + f(x,\ y) \tag{2.3a}$$

where

$$M = \lim_{x \to 0} A(x) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 20 & 1 \end{pmatrix} \tag{2.3b}$$

and $f(x,\ y)$ is continuous as $x \to 0$ for continuous $y(x)$.

The eigenpairs of M are

$$\lambda = 0 : \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; \quad \lambda = -2 : \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 0 \\ 0 \end{pmatrix} ;$$

$$\lambda = -4: \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -4 \end{pmatrix} \quad ; \quad \lambda = 3: \begin{pmatrix} 0 \\ 0 \\ 1/3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad ; \quad \lambda = 5: \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/5 \\ 1 \end{pmatrix}$$

Then, after some algebra, we find that a continuous solution at zero must satisfy

$$\begin{aligned} -3\, y_3(0) + y_4(0) &= 0 \quad , \\ -5\, y_5(0) + y_6(0) &= 0 \quad . \end{aligned} \tag{2.4}$$

To obtain the projection condition at infinity we have to make a different change of variables since system (2.3) does not have the same asymptotic behavior as system (2.1).

The proper change of variables is

$$\begin{aligned} \hat{y}_1(x) &= u(x) \qquad \hat{y}_2(x) = u'(x) \qquad \hat{y}_3(x) = v(x) \\ \hat{y}_4(x) &= v'(x) \qquad \hat{y}_5(x) = w(x) \qquad \hat{y}_6(x) = w'(x) \quad . \end{aligned} \tag{2.5}$$

Then, $\hat{y}$ satisfies

$$\hat{y}' = \hat{A}(x)\hat{y} + \hat{f}(x, \hat{y}) \quad , \tag{2.6a}$$

where

$$A_\infty = \lim_{x\to\infty} \hat{A}(x) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 2\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2\lambda & 0 \end{pmatrix} . \qquad (2.6b)$$

The eigenvalues of A_∞ are

$$\mu_1 = -\sqrt{2\lambda} \qquad \mu_2 = \sqrt{2\lambda} \qquad ,$$

both of algebraic multiplicity 3 but geometric multiplicity 3 (we only consider the case $\lambda > 0$).

The corresponding eigenvectors are

$$\begin{pmatrix} 1 \\ -\sqrt{2\lambda} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\sqrt{2\lambda} \\ 0 \\ 0 \end{pmatrix} , \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -\sqrt{2\lambda} \end{pmatrix} \qquad (2.7)$$

and

$$\begin{pmatrix} 1 \\ \sqrt{2\lambda} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ \sqrt{2\lambda} \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ \sqrt{2\lambda} \end{pmatrix} \tag{2.7}$$

Hence the projection condition at infinity becomes

$$\lim_{x\to\infty} (\sqrt{2\lambda}\ \hat{y}_1(x) + \hat{y}_2(x)) = 0 \quad , \tag{2.8a}$$

$$\lim_{x\to\infty} (\sqrt{2\lambda}\ \hat{y}_3(x) + \hat{y}_4(x)) = 0 \quad , \tag{2.8b}$$

$$\lim_{x\to\infty} (\sqrt{2\lambda}\ \hat{y}_5(x) + \hat{y}_6(x)) = 0 \quad . \tag{2.8c}$$

Conditions (2.8) written in the y variables become

$$\lim_{x\to\infty} (\sqrt{2\lambda}\ y_1(x) + y_2(x)) = 0 \tag{2.9a}$$

$$\lim_{x\to\infty} (\sqrt{2\lambda}\ y_3(x) + y_4(x)/x) = 0 \tag{2.9b}$$

$$\lim_{x\to\infty} (\sqrt{2\lambda}\ y_5(x) + y_6(x)/x) = 0 \quad . \tag{2.9c}$$

We actually use the projection conditions (2.9) at a suitable finite value $x = x_M$. For this problem we found $x_M = 13$ quite satisfactory.

We note that continuity at zero and boundedness at infinity only impose 5 boundary conditions for a system of 6 first order equations. For this problem it is possible to specify one additional condition at zero. We take, for convenience,

$$y_1(0) = 0 \quad . \tag{2.10}$$

After all this analysis we have transformed the original problem into an eigenvalue problem in a finite interval. In order to be able to use PASVA3 we introduce as two new variables the eigenvalue λ and a normalisation,

$$\begin{aligned} y_7(x) &= \lambda \quad , \\ y_8(x) &= \int_0^x \sum_{i=1}^{6} y_i^2(x)dx \quad , \end{aligned} \tag{2.11}$$

together with the boundary conditions

$$y_8(0) = 0 \qquad y_8(\infty) = 1 \quad . \tag{2.12}$$

The resultant system is

$$y_1' = y_2$$

$$y_2' = 2\lambda\, y_1 - 4\, E_o\, y_1 - \frac{4}{5}\, E_2\, y_3 - \frac{4}{9}\, E_4\, y_5$$

$$y_3' = \frac{y_4}{x}$$

$$y_4' = \frac{y_4}{x} + 2\lambda x\, y_3 + \frac{6}{x}\, y_3 - 4x\, E_2\, y_1$$

$$- x(4\, E_o + \frac{8}{7}\,(E_2 + E_4))y_3$$

$$- x(\frac{8}{7}\, E_2 + \frac{400}{693}\, E_4 + \frac{100}{143}\, E_6)y_5$$

$$y_5' = \frac{y_6}{x}$$

$$y_6' = \frac{y_6}{x} + 2\lambda x\, y_5 + \frac{20}{x}\, y_5 - 4x\, E_4\, y_1$$

$$- x(\frac{72}{35}\, E_2 + \frac{80}{77}\, E_4 + \frac{180}{143}\, E_6)y_3$$

$$- x(4E_o + \frac{80}{77}\, E_2 + \frac{648}{1001}\, E_4 + \frac{80}{143}\, E_6$$

$$+ \frac{1960}{2431}\, E_8)y_5$$

$$y_7' = 0$$

$$y_8' = \sum_{i=1}^{6} y_i^2(x) \quad .$$

The boundary conditions are:

$$-\ 3\ y_3(0) + y_4(0) = 0$$

$$-\ 5\ y_5(0) + y_6(0) = 0$$

$$y_1(0) = 0$$

$$y_8(0) = 0$$

$$\sqrt{2y_7}\ y_1(x_M) + y_2(x_M) = 0$$

$$\sqrt{2y_7}\ y_3(x_M) + \frac{y_4(x_M)}{x_M} = 0$$

$$\sqrt{2y_7}\ y_5(x_M) + \frac{y_6(x_M)}{x_M} = 0$$

$$y_8(x_M) = 1$$

Conte in [6] computed the eigenfunction associated with the eigenvalue $\lambda = .36004$, which he calculated with 6 significant figures. He used an initial value method to compute the eigenfunction and a root finder procedure to get the value of λ which satisfies the additional boundary conditions.

Figure 3 shows the eigenfunction we obtained for $\lambda = .36004$. Conte used an step size equal to .05 to compute the eigenfunction to at least 4 significant figures in the interval $[0, 15]$. We used PASVA3 to compute the same eigenfunction with 165 mesh points in the interval $[0, 13]$; 3 corrections were required to get Conte's results. Besides the advantages of efficiency that this approach gives it is the first time, to our knowledge, that a theoretical justification has been given for solving (2.1) in a finite interval.

CHAPTER 5

MULTIPLE SOLUTIONS FOR THE VON KÁRMÁN SWIRLING FLOW

5.1 Introduction

The problem of a rotationally symmetric, viscous, incompressible fluid above an infinite rotating disk has been of great interest from both the theoretical and the numerical point of view.

Von Kármán has shown [12] that in the case of axially symmetric flow, the Navier-Stokes equations can be reduced to a system of ordinary differential equations.

For various values of the ratio γ of the angular velocity of the fluid at ∞ and the disk, solutions have been obtained. More recently, D. Dijkstra and P. J. Zandbergen [9] and A. B. White [20] have shown that at least two branches of solutions exist for $\gamma \in (-.16054, .07)$. In fact, these two branches coincide for $\gamma = -.16054$.

From the theoretical point of view Mc Leod has proven several important results:

i) Solutions exist for $\gamma \geq 0$ [15], [17].

ii) Solutions do not exist for $\gamma = -1$ when there is no suction through the plate [16].

iii) A unique solution exists for $\gamma = 1$ [16].

We only treat the case of no suction through the plate. We find another limit point at $\gamma = .07452$. This new branch crosses the axis and continues for negative values of γ. That is, at least three solutions exist for $\gamma \in [0, .07452]$. In fact, we have computed this third branch up to $\gamma = 0$. As we will see later, we have enough evidence to believe that there are infinitely many solutions.

5.2 Governing Equations

In a cylindrical coordinate system (r, ϕ, Z) the disk is the plane $Z = 0$ and the corresponding velocities are

$$u = r\,\Omega\, f'(t)\ ,\ v = r\,\Omega\, g(t)\ ,\ w = -\,2(\nu\Omega)^{\frac{1}{2}}\, f(t)\quad . \tag{2.1}$$

The angular velocity of the disk is Ω and

$$t = Z(\Omega/\nu)^{\frac{1}{2}}\quad .$$

Then the Navier-Stokes equations reduce in this case to

$$f'''(t) + 2\, f(t)\, f''(t) = f'^2(t) + \gamma^2 - g^2(t) \tag{2.2a}$$

$$g''(t) + 2\, f(t)\, g'(t) = 2\, f'(t)\, g(t)\quad . \tag{2.2b}$$

For no suction at the disk the appropriate boundary conditions are:

$$f(0) = 0\ ,\ f'(0) = 0\ ,\ g(0) = 1\quad . \tag{2.3}$$

In (2.3) we have assumed that the angular velocity of the fluid at infinity is $\gamma\Omega$.

The asymptotic behaviour of the solution at infinity has also been studied by Mc Leod. In fact, for $\gamma = 0$ the solution decays exponentially. For $\gamma \neq 0$ the solution is damped oscillatory. Ultimately

$$\begin{pmatrix} f(t) \\ f'(t) \\ f''(t) \\ g(t) \\ g'(t) \end{pmatrix} \rightarrow \begin{pmatrix} f_\infty \\ 0 \\ 0 \\ \gamma \\ 0 \end{pmatrix} \quad \text{as} \quad t \rightarrow \infty \; .$$

Here f_∞ is a constant and depends on the value of the parameter γ.

In order to get the appropriate projection condition for system (2.2) we make the change of dependent variables:

$$\begin{aligned} w_1(t) &= f(t) - f_\infty \\ w_2(t) &= f'(t) \\ w_3(t) &= f''(t) \\ w_4(t) &= g(t) - \gamma \\ w_5(t) &= g'(t) \quad . \end{aligned} \tag{2.4}$$

Then, the vector $w(t)$ is the solution of the first order system

$$w'(t) = A_\infty w(t) + \begin{pmatrix} 0 \\ 0 \\ -2w_1(t)w_3(t) + w_2^2(t) - w_4^2(t) \\ 0 \\ -2w_1(t)w_5(t) + 2w_2(t)w_1(t) \end{pmatrix} \tag{2.5a}$$

$$A_\infty = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2f_\infty & -2\gamma & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2\gamma & 0 & 0 & -2f_\infty \end{bmatrix} \tag{2.5b}$$

The eigenvalues of A_∞ are

$$\lambda = 0 \quad , \tag{2.6}$$

and the roots of the equation

$$\lambda^2(\lambda + 2f_\infty)^2 + 4\gamma^2 = 0 \quad ,$$

i.e.,

$$\lambda = -f_\infty + \frac{1}{\sqrt{2}}\left[\pm\{(f_\infty^4 + 4\gamma^2)^{\frac{1}{2}} + f_\infty^2\}^{\frac{1}{2}} \pm i\{(f_\infty^4 + 4\gamma^2)^{\frac{1}{2}} - f_\infty^2\}^{\frac{1}{2}}\right] .$$

To simplify the notation we write these four roots as:

$$\begin{aligned} \lambda_1 &= -f_\infty + x + i\,y \;,\; \lambda_2 = -f_\infty + x - i\,y \\ \lambda_3 &= -f_\infty - x + i\,y \;,\; \lambda_4 = -f_\infty - x - i\,y \end{aligned} \tag{2.7a}$$

where

$$x = \frac{1}{\sqrt{2}} \{(f_\infty^4 + 4\gamma^2)^{\frac{1}{2}} + f_\infty^2\}^{\frac{1}{2}}$$

$$y = \frac{1}{\sqrt{2}} \{(f_\infty^4 + 4\gamma^2)^{\frac{1}{2}} - f_\infty^2\}^{\frac{1}{2}} \quad . \tag{2.7b}$$

We notice that, for $\gamma \neq 0$, A_∞ has two complex eigenvalues with positive real part, and two complex eigenvalues with negative real part.

From the theoretical considerations of Mc Leod it can be concluded that for $\gamma = 0, f_\infty$ has to be positive. Hence, in this case we have two negative eigenvalues, and the eigenvalue zero has algebraic multiplicity 3 with one Jordan block of dimension 2.

To go further in our study we separate the cases $\gamma = 0$ and $\gamma \neq 0$.

The eigenvectors of A_∞ for $\gamma \neq 0$ are

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & 0 & \lambda_3 & \lambda_4 \\ \lambda_1^2 & \lambda_2^2 & 0 & \lambda_3^2 & \lambda_4^2 \\ -\lambda_1^2 C_1/2\gamma & -\lambda_2^2 C_2/2\gamma & 0 & -\lambda_3^2 C_3/2\gamma & -\lambda_4^2 C_4/2\gamma \\ -\lambda_1^3 C_1/2\gamma & -\lambda_2^3 C_2/2\gamma & 0 & -\lambda_3^3 C_3/2\gamma & -\lambda_4^3 C_4/2\gamma \end{bmatrix} \tag{2.8}$$

Here $C_i = 2f_\infty + \lambda_i$; this implies

$$C_1 = f_\infty + x + iy = -\lambda_4 \qquad C_2 = f_\infty + x - iy = -\lambda_3$$

$$C_3 = f_\infty - x + iy = -\lambda_2 \qquad C_4 = f_\infty - x - iy = -\lambda_1 \tag{2.9}$$

In P we have ordered the eigenvectors so that

$$A = P^{-1} A_\infty P = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & 0 & & \\ & & & \lambda_3 & \\ & & & & \lambda_4 \end{bmatrix}$$

Then the projection condition is

$$\lim_{t\to\infty} q_1^T w(t) = 0 \tag{2.9a}$$

$$\lim_{t\to\infty} q_2^T w(t) = 0 \tag{2.9b}$$

where q_1^T and q_2^T are the first two rows of the matrix P^{-1}.

We notice that the matrix P can be written as

$$P = (p_1, \bar{p}_1, p_2, p_3, \bar{p}_3) \quad ,$$

with $p_2 = \bar{p}_2$. Therefore

$$q_1^T = \bar{q}_2^T \quad .$$

This means that it is only necessary to calculate q_1^T, and since the desired solution is real, the boundary condition at infinity is given by

$$\lim_{t\to\infty} R(q_1^T)\, w(t) = 0$$

$$\lim_{t\to\infty} I(q_1^T)\, w(t) = 0 \quad . \tag{2.10}$$

Let: $q_1^T = (x_1, x_2, x_3, x_4, x_5)$. Then, we have to solve the linear system of equations

$$q_1^T P = (1, 0, 0, 0, 0) \quad .$$

Since $\quad p_2 = (1, 0, 0, 0, 0) \quad$ then

$$x_1 = 0 \quad .$$

The equations for the other components are:

$$x_2 + \lambda_1 x_3 - \frac{\lambda_1 C_1}{2\gamma} x_4 - \frac{\lambda_1^2 C_1}{2\gamma} x_5 = 1/\lambda_1$$

$$x_2 + \lambda_i x_3 - \frac{\lambda_i C_i}{2\gamma} x_4 - \frac{\lambda_i^2 C_i}{2\gamma} x_5 = 0 \qquad i = 2,3,4 \quad . \tag{2.11}$$

Let $\delta_i = - \lambda_i C_i \qquad i = 1, 2, 3, 4 \quad ;$ i.e.,

$\delta_1 = \lambda_1 \lambda_4/2\gamma \qquad \delta_2 = \lambda_2 \lambda_3/2\gamma$

$\delta_3 = \lambda_3 \lambda_2/2\gamma \qquad \delta_4 = \lambda_4 \lambda_1/2\gamma \quad .$

Hence $\quad \delta_1 = \delta_4 \qquad \delta_2 = \delta_3 \quad .$

The equations (2.11) can be written in the form:

$$Q\underset{\sim}{x} = \begin{pmatrix} 1 & \lambda_1 & \delta_1 & \lambda_i\delta_1 \\ 1 & \lambda_2 & \delta_2 & \lambda_2\delta_2 \\ 1 & \lambda_3 & \delta_3 & \lambda_3\delta_3 \\ 1 & \lambda_4 & \delta_4 & \lambda_4\delta_4 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1/\lambda_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad (2.12)$$

To solve (2.12) we reduce Q to lower triangular form. This gives:

$$Q_1 = Q\ T_1\ T_2 \quad .$$

Where

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & (\lambda_2-\lambda_1) & 0 & 0 \\ 1 & (\lambda_3-\lambda_1) & -\dfrac{(\lambda_3-\lambda_1)}{(\lambda_2-\lambda_1)}(\delta_2-\delta_4)+(\delta_2-\delta_4) & 0 \\ 1 & (\lambda_4-\lambda_1) & -\dfrac{(\lambda_4-\lambda_1)}{(\lambda_2-\lambda_1)}(\delta_2-\delta_4) & (\delta_4-\delta_2)(\lambda_4-\lambda_1) \end{bmatrix}$$

$$T_1 = \begin{bmatrix} 1 & -\lambda_1 & -\delta_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\lambda_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} ,$$

$$T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \dfrac{(\delta_4-\delta_2)}{(\lambda_2-\lambda_1)} & -\delta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

Then (2.12) is equivalent to

$$Q_1^{-1} T_2^{-1} T_1^{-1} x = b \quad , \tag{2.13}$$

where

$$b = \begin{pmatrix} 1/\lambda_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} .$$

Defining $v = T_2^{-1} T_1^{-1} x$, we solve (2.13) in two steps:

$$Q_1 v = b \tag{2.14a}$$

$$x = T_1 T_2 v \quad . \tag{2.14b}$$

Then,

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 1/\lambda_1 \\ -1/\lambda_1(\lambda_2-\lambda_1) \\ 1/\lambda_1(\delta_4-\delta_2) \\ -1/\lambda_1(\delta_4-\delta_2)(\lambda_4-\lambda_1) \end{pmatrix} .$$

Hence

$$x_5 = v_4$$

$$x_4 = v_3 - \lambda_1 \; v_4 = \frac{(\lambda_4-\lambda_1) + \lambda_1}{\lambda_1(\delta_4-\delta_2)(\lambda_4-\lambda_1)} = - \lambda_4 \; v_4$$

$$x_3 = v_2 + \frac{\delta_4-\delta_2}{\lambda_2-\lambda_1} v_3 - \delta_2 \; v_4 = - \delta_2 \; v_4$$

$$x_2 = \frac{(\delta_4-\delta_2)(\lambda_4-\lambda_1) - \delta_1(\lambda_4-\lambda_1) - \lambda_1 \; \delta_2}{\lambda_1(\delta_4-\delta_2)(\lambda_4-\lambda_1)}$$

$$= - \frac{\lambda_4\delta_2}{\lambda_1(\delta_4-\delta_2)(\lambda_4-\lambda_1)}$$

$$= \lambda_4 \; \delta_2 \; v_4 \quad .$$

Therefore, conditions (2.9a) are given by

$$\lim_{t\to\infty}\{\lambda_4 \; \delta_2 \; w_2(t) - \delta_2 \; w_3(t) - \lambda_4 \; w_4(t) + w_5(t)\} = 0 \quad ,$$

since $v_4 \neq 0$.

Separation of real and imaginary parts gives

$$h_1(w, \gamma) = \lim_{t\to\infty}\{\frac{xy^2}{\gamma} w_2(t) + (f_\infty + x) \; w_4(t) + w_5(t)\} = 0 \quad ; \qquad (2.15)$$

$$h_2(w, \gamma) = \lim_{t\to\infty}\{-\frac{(f_\infty xy + x^2y)}{\gamma} w_2(t) - \frac{xy}{\gamma} w_3(t) + y\ w_4(t)\} = 0.$$

But $\frac{y}{\gamma} \neq 0$ so:

$$h_2(w, \gamma) = \lim_{t\to\infty}\{-(f_\infty x + x^2)\ w_2(t) - x\ w_3(t) + \gamma\ w_4(t)\} = 0 \quad . \tag{2.16}$$

This simplification is even valid when $\gamma = 0$, since for $|\gamma| << 1$ and $f_\infty \neq 0$

$$\begin{aligned} y &= \frac{1}{\sqrt{2}} (f_\infty^2(1 + 4\gamma^2/f_\infty^4)^{\frac{1}{2}} - f_\infty^2)^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{2}} (f_\infty^2 + \frac{2\gamma^2}{f_\infty^2} - f_\infty^2 + 0(\gamma^4))^{\frac{1}{2}} \\ &= \frac{\gamma}{f_\infty} (1 + 0(\gamma^2)) \ . \end{aligned}$$

Similarly,

$$x = f_\infty(1 + 0(\gamma^2)) \quad .$$

As we will see later, it is important to know the limiting expressions for $h_1(w, \gamma)$ and $h_2(w, \gamma)$ as $\gamma \to 0$. Namely

$$\begin{aligned} h_1(w, 0) &= \lim_{t\to\infty}\{2\ f_\infty\ w_4(t) + w_5(t)\} \\ h_2(w, 0) &= \lim_{t\to\infty}\{2\ f_\infty\ w_2(t) + w_3(t)\} \ . \end{aligned} \tag{2.17}$$

The eigenvectors of A_∞ for $\gamma = 0$ and $f_\infty > 0$ are

$$P = \begin{bmatrix} 1/f_\infty & 0 & 0 & 0 & 1 \\ 0 & 1/f_\infty & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 4f_\infty \\ 0 & 0 & 1/f_\infty & 1/f_\infty & 0 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

We have ordered the eigenvectors so that

$$P^{-1} A_\infty P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 & \lambda_4 \end{bmatrix} \tag{2.18}$$

Then the first three rows of P^{-1} are

$$q_1 = (f_\infty, 0, -1/4f_\infty, 0, 0)$$
$$q_2 = (0, f_\infty, \tfrac{1}{2}, 0, 0)$$
$$q_3 = (0, 0, 0, f_\infty, \tfrac{1}{2}) \quad .$$

We notice that q_2 and q_3 are the limiting forms, as $\gamma \to 0$, of the projection conditions $h_1(w, \gamma)$ and $h_2(w, \gamma)$ given in (2.15) and (2.16).

Now we can state the problem which we will solve:

$$f''' + 2\,f\,f'' = f'^2 + \gamma^2 - g^2 \qquad 0 \leq t < \infty \qquad (2.19a)$$

$$g'' + 2\,f\,g' = 2\,f'\,g$$

with the boundary conditions:

$$f(0) = 0 \qquad f'(0) = 0 \qquad g(0) = 1 \qquad (2.19b)$$

$$\lim_{t\to\infty} \{\frac{xy^2}{\gamma}\, f'(t) + (f(t) + x)(g(t)-\gamma) + g'(t)\} = 0$$

$$\lim_{t\to\infty} \{-\,x(f(t) + x)\, f'(t) - x\, f''(t) + \gamma(g(t)-\gamma)\} = 0\,. \qquad (2.19c)$$

Here x and y are given in (2.7b). γ is a parameter in the problem, and for $\gamma = 0$ instead of (2.19c) we use their limiting forms

$$\lim_{t\to\infty} \{2\,f(t)\,g(t) + g'(t)\} = 0$$

$$\lim_{t\to\infty} \{2\,f(t)\,f'(t) + f''(t)\} = 0\,. \qquad (2.20)$$

We remark that this is the first time, to our knowledge, that the boundary conditions (2.19c) have been used. But more recently, and completely independently, A. B. White [20] has also used some sort of projection condition when solving this swirling flow problem.

Since PASVA3 is a first order solver we have to rewrite (2.19). Thus we actually solve for the variables

$$y_1(t) = f(t) \ , \ y_2(t) = f'(t) \ , \ y_3(t) = f''(t) \ ,$$
$$y_4(t) = g(t) - \gamma \ , \ y_5(t) = g'(t) \ .$$

Then we want to compute the solutions of the first order system

$$\begin{aligned}
y_1'(t) &= y_2(t) \\
y_2'(t) &= y_3(t) \\
y_3'(t) &= -2\,y_1(t)\,y_3(t) + y_2^2(t) - 2\,\gamma\,y_4(t) - y_4^2(t) \\
y_4'(t) &= y_5(t) \\
y_5'(t) &= -2\,y_1(t)\,y_5(t) + 2\,y_2(t)\,(y_4(t) + \gamma)
\end{aligned} \tag{2.21a}$$

subject to the boundary conditions

$$y_1(0) = 0 \qquad y_2(0) = 0 \qquad y_4(0) = 1 - \gamma$$
$$\lim_{t\to\infty} \{\frac{xy^2}{\gamma}\, y_2(t) + (y_1(t) + x)\, y_4(t) + y_5(t)\} = 0 \tag{2.21b}$$
$$\lim_{t\to\infty} \{-\,x(y_1(t) + x)\, y_2(t) - x\, y_3(t) + \gamma\, y_4(t)\} = 0 \ .$$

5.3 Numerical Method.

From the experience of Dijkstra and Zandbergen [9] and A. B. White [20] we know that problem (2.21) has multiple solutions. These authors have computed two different branches of solutions for $\gamma \in [-.16057, .07]$. Moreover there is a "limit point", see section 2 of the Appendix, at $\gamma = -.16057$. All of these computations stop at approximately $\gamma = .07$. Dijkstra and Zandbergen indicate their belief that the branch continues to larger values of γ and also that they have strong evidence that around $\gamma = .0752$ there is another "limit point." On the other hand, White argues that a small "kink" develops near $\gamma = 0.07$.

To solve (2.19) numerically we have introduced in PASVA3 the necessary modifications to carry out the arc length continuation procedure of Keller [14]. In the Appendix there is a detailed description of those modifications.

We still have to say how to choose t_∞, the point where we apply the boundary conditions (2.19c). We know that the behaviour of the solutions change drastically when passing through a limit point. From [9] we have taken the idea of using the integral identities

$$f''^2(0) - g'^2(0) = \int_0^\infty 4\, f(f''^2 - g'^2)dt \quad ,$$

$$f''(0)\; g'(0) + \frac{2}{3}\gamma^3 - \gamma^2 + \frac{1}{3} = \int_0^\infty 4\, f\, f''\, g'\, dt$$

as an additional check on the goodness of t_∞. But to guarantee p digits in our results, we compute the solutions for different values of t_∞ until we have stabilized at least p figures in the solution. All the solutions were obtained on a net with a maximum of 120 points. We observe that, although it was necessary to enlarge the interval every time the branch of solutions crossed a limit point, the "boundary layers" were thicker and hence less points were necessary to reach the desired accuracy.

All our results are accurate to at least 4 significant figures.

5.4 Numerical Results.

As it is described in the Appendix in order to use the arc length continuation procedure we have to start with an isolated solution of (2.21). This is easy because for $\gamma = 1$ we know an exact solution of the problem, namely solid body rotation of the fluid: $f \equiv 0$, $g \equiv 1$.

Using arc length continuation we follow the branches that have been already computed by White [20] and Dijkstra and Zandbergen [9]. On the first branch, from $\gamma = 1$ to $\gamma = 0$ an interval of length 15 was used, from $\gamma = 0$ to the first limit point the length of the interval was enlarged to 25. All the solutions of the second branch were computed on an interval of length 45. For the third branch we needed an interval of length 95.

The numerical results of our computations are given in figures 4 - 15 and table 4.1.

Figures 4,5 and 6 show the bifurcation diagrams corresponding to $-f_\infty, g'(0)$ and $f''(0)$ as functions of γ, $\gamma \in [-.16, .08]$. For plotting purposes, we translated the values of $g'(0)$ and $f''(0)$. Figures 5 and 6 show $(g'(0) + 0.6)$ and $(f''(0) - 0.5)$ respectively. Every point in any of these diagrams correspond to a solution of (2.21).

Table 4.1 contains the numerical values used to plot the third branch of solutions. (For the numerical values of the other two branches see [9] and [20]).

γ	f_∞	$f''(0)$	$g'(0)$
.07452	-.2258	.4940	-.5622
.07451	-.2259	.4940	-.5622
.07450	-.2260	.4940	-.5622
.07447	-.2259	.4940	-.5622
.07437	-.2251	.4941	-.5622
.07435	-.2249	.4941	-.5622
.07433	-.2246	.4941	-.5622
.07414	-.2221	.4941	-.5622
.07	-.1448	.4946	-.5623
.0675	-.1112	.4949	-.5624
.0650	-.08152	.4952	-.5625
.0625	-.05472	.4955	-.5626
.06	-.03037	.4958	-.5627
.0575	-.00816	.4961	-.5627
.0550	.01222	.4963	-.5628
.0525	.03097	.4966	-.5629
.05	.04826	.4968	-.5630
.04556	.07577	.4972	-.5631
.03791	.1150	.4978	-.5633
.02463	.1627	.4985	-.5635
.01089	.1890	.4989	-.5637
.005	.1934	.4990	-.5637
0	.1938	.4990	-.5637

TABLE 4.1. Third Branch of the Swirling Flow Problem

Figures 7, 8 and 9 show the three velocity components for each of the three branches for $\gamma = 0$. The flow corresponding to each of them is shown in figures 10, 11, 12.

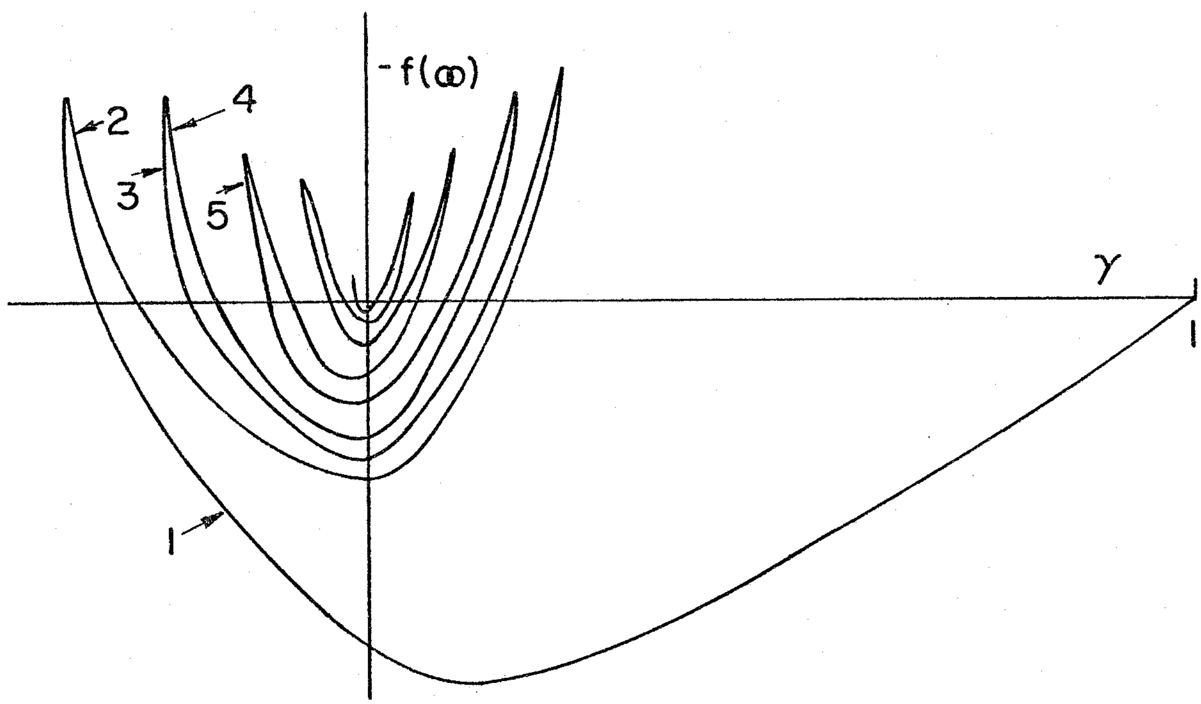
To describe the flow field we will use the concept of a cell. A <u>cell</u> (Szeto [19]) is a region bounded by planes of constant Z that includes only its own recirculating fluid. In other words, it is a region bounded by planes of constant Z at which the Z-velocity is zero.

Thus, we observe that the flow for the first branch has only one cell. The second branch has three cells and the third branch has five cells.

In figures 10, 11, 12 the size of the arrows is proportional to the logarithm of 1 + magnitude of the velocity.

Finally, figures 13, 14 and 15 represent the two solutions for $\gamma = 0.0745$ corresponding to the second and the third branch.

It seems reasonable to conjecture that there are infinitely many solutions for $\gamma = 0$, with their inflow velocities, $-f_\infty$, monotonically converging to zero from below. That is, schematically, the extension of figure 4 is conjectured to be:



The regularity of the added "periods", and cells also suggests this. Indeed based on the solution branches 1, 2, 3 which use boundary conditions at $t_\infty = 15, 45, 95$ we assume that the fourth branch will require $t_\infty \approx 195$.

CHAPTER 6

APPENDIX

6.1 Brief description of PASVA3.

PASVA3 is a modified version of the variable order, variable step code for solving two-point boundary value problems described in [18].

To start the description we consider a general nonlinear two-point boundary value problem:

$$\frac{dw}{dt} - f(t, w) = 0 \ , \ t \in [a, b] \quad , \tag{1.1a}$$

subject to the nonlinear boundary conditions

$$g(w(a) \ , \ w(b)) = 0 \quad , \tag{1.1b}$$

where $w(t)$, $f(t, w)$ and g are d-dimensional vector functions.

It is assumed that the conditions in g are given in the following order

$$g(w(a),w(b)) = \begin{pmatrix} g^{(1)}(w(a)) \\ g^{(2)}(w(a),w(b)) \\ g^{(3)}(w(b)) \end{pmatrix} \begin{matrix} \} \ p \\ \} \ r \\ \} \ q \end{matrix} \quad .$$

In other words, the problem has p initial conditions $g^{(1)}(w(a)) = 0$; r coupled boundary conditions, $g^{(2)}(w(a),w(b)) = 0$; and q end conditions, $g^{(3)}(w(b)) = 0$.

Moreover, it is also assumed that f has at least two continuous derivatives and that there exists an isolated solution $w^*(t)$ to the problem (1.1).

The numerical method used to solve (1.1) is based on approximating $\frac{dw}{dt}$ by the trapezoidal rule on a mesh with $(J+1)$ points in the interval $[a, b]$. Thus, we consider a mesh π of points $\{t_j\}$ $j = 1, \ldots, J+1$ satisfying:

$$a = t_1 < t_2 < \ldots < t_{J+1} = b \quad . \tag{1.2}$$

The trapezoidal rule approximation to (1.1a) is:

$$\frac{W_{j+1} - W_j}{h_j} = \tfrac{1}{2}\left[f(t_j, W_j) + f(t_{j+1}, W_{j+1})\right] , \; j = 1, \ldots, J \;, \tag{1.3a}$$

with the boundary conditions

$$g(W_1, W_{J+1}) = 0 \quad . \tag{1.3b}$$

Here the d-vectors W_j approximate $w(t_j)$, and $h_j = t_{j+1} - t_j$ is the mesh spacing, which is not assumed to be uniform.

Equations (1.3) form a system of $(J+1)\times d$ nonlinear algebraic equations in the same number of unknowns $\{W_{ij}\}\begin{matrix} i = 1, \ldots, d \\ j = 1, \ldots, J+1 \end{matrix}$.

Using further vector notation we refer to (1.3) as the discrete system and write it as:

$$G_\pi(W) = 0 \quad , \tag{1.4}$$

where

$$W = \begin{bmatrix} W_{11} \\ W_{21} \\ \cdot \\ \cdot \\ \cdot \\ W_{d1} \\ W_{12} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ W_{dJ+1} \end{bmatrix} , \quad G_\pi(W) = \begin{bmatrix} g^{(1)}(W_1) \\ W_2 - W_1 - \frac{h_1}{2}(f_1 + f_2) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ W_{J+1} - W_J - \frac{h_J}{2}(f_{J+1} + f_J) \\ g^{(2)}(W_1, W_{J+1}) \\ g^{(3)}(W_{J+1}) \end{bmatrix}$$

(d-vectors →)

with $f_j \equiv f(t_j, w_j)$.

Under mild assumptions system (1.4) will have an isolated solution W^* near $\{w^*(t_j)\}$, provided $h = \max_{j=1,..,J} h_j$ is sufficiently small. Moreover, this discrete approximation will be accurate to order h^2, i.e., there exists a constant c so that:

$$||W^* - w^*|| = \max_{\substack{i=1,..,d \\ j=1,..,J+1}} |W^*_{ij} - w^*_i(t_j)| \leq c\, h^2 \quad . \tag{1.5}$$

The solution of (1.4) is computed using a Newton iteration. If we call $G_w(W)$ the Jacobian matrix of G_π; we have that the iteration is defined by:

$$W^{(0)} \quad \text{given} \tag{1.6a}$$

$$G_w(W^\nu)\Delta W^{\nu+1} = - G_\pi(W^\nu) \tag{1.6b}$$

$$W^{(\nu+1)} = W^{(\nu)} + \Delta W^{(\nu+1)} \quad \nu = 0,1,\dots \quad . \tag{1.6c}$$

We see that for every Newton iteration it is necessary to solve a linear system of algebraic equations, namely system (1.6b). To solve these systems the program constructs a stable LU factorization of G_w, using an alternating partial pivoting strategy that does not destroy the sparse structure of G_w. Then it solves two block triangular systems to obtain $\Delta W^{(\nu+1)}$.

As we pointed out before the discretization (1.3) has order of accuracy h^2, even if a nonuniform mesh is used. Whenever there is a priori information on regions in which the solution $w^*(t)$ might have rapid variations it should be used by considering an appropriate mesh π. The program has an automatic mesh selection procedure which, in the course of the computation, will try to find a good mesh for the problem.

The order of accuracy of the basic method (1.3) will usually be too low. To improve this situation a variable order method, based on deferred corrections, has been implemented in the program. This deferred correction procedure gives, as a by product, a very good estimation of the global error.

All these various techniques are arranged in a somewhat complex structure with a master control program that makes automatic decisions, based on currently available information on when to refine the mesh, when to increase the order, and finally when to stop with a sufficiently accurate result.

6.2 Brief description of the arc length continuation procedure.

The arc length continuation procedure is completely described in Keller [14]. Here we will give a summary of it.

We are interested in two point boundary value problems which depend upon a parameter:

$$G(w, t, \gamma) = \begin{pmatrix} \frac{dw}{dt} - f(t, w, \gamma) \\ \\ g(w(a), w(b), \gamma) \end{pmatrix} = 0 \quad . \tag{2.1}$$

It is assumed that w is in some Banach space B and $\gamma \in \mathbb{R}$.

We start with some definitions:

a) Arc of solutions: is a one parameter family of solutions of (2.1), $(w(s), \gamma(s))$, $s \in [s_1, s_2]$, twice continuously differentiable in the parameter s.

b) Problem (2.1) has an isolated solution, w_o, for $\gamma = \gamma_o$ if $G_w^o \equiv G_w(w_o, \gamma_o)$ is nonsingular and if $G(w, \gamma)$ is continuously differentiable in some ρ_o-sphere about $[w_o, \gamma_o]$.

c) $[w_o, \gamma_o]$ is a regular solution (point) if w_o is an isolated solution of $G(w, t, \gamma_o)$.

d) $\left[w_o, \gamma_o\right]$ is a <u>normal limit solution (point)</u> if:

i) w_o is a solution of $G(w, t, \gamma_o)$

ii) $\dim N(G_w^o) = \operatorname{codim} R(G_w^o) = 1$

iii) $G_\gamma^o \notin R(G_w^o)$

e) $\left[w_o, \gamma_o\right]$ is a simple <u>bifurcation solution (point)</u> if:

i) w_o is a solution of $G(w, t, \gamma_o)$

ii) $\dim N(G_w^o) = \operatorname{codim} R(G_w^o) = 1$

iii) $G_\gamma^o \in R(G_w^o)$.

iv) bifurcation equation is satisfied (too messy to include).

The whole idea of the arc length continuation procedure is to introduce an artificial parameter in the problem and to continue in this parameter. To do this we seek solutions of (2.1) which satisfy the normalisation

$$N_\varphi^1(w, \gamma, s) = \varphi \, ||\dot{w}(s)||^2 + (1-\varphi) \, |\dot{\gamma}(s)|^2 - 1 = 0 \qquad (2.2)$$

where $\varphi \in (0, 1)$ is arbitrary and $\cdot$ denotes differentiation respect to s.

Normalisation (2.2) is a form of arc length. In practice it is more convenient to use

$$N_\varphi(w, \gamma, s) = \varphi \, \dot{w}^*(s_o) \left[w(s) - w(s_o)\right] + (1-\varphi) \, \dot{\gamma}(s_o) \left[\gamma(s) - \gamma(s_o)\right] - (s-s_o) , \qquad (2.3)$$

$s \in \left[s_o, s_1\right)$.

Here we have assume that we know a solution $[w_o, \gamma_o] = [w(s_o), \gamma(s_o)]$ and a "tangent" $[\dot{w}_o, \dot{\gamma}_o]$ satisfying (2.2) and

$$G_w^o \dot{w}_o + G_\lambda^o \dot{\lambda}_o = 0 \quad .$$

Then $\dot{w}^*(s_o) \varepsilon B^*$ is the dual element of $\dot{w}_o = \dot{w}(s_o)$ such that $\dot{w}^*(s_o)\, \dot{w}(s_o) = ||\dot{w}(s_o)||^2$.

We recall that, if G is smooth enough, differentiation of

$$G(w, t, \gamma) = 0\,, \ (w, \gamma) \ \varepsilon \ S_{\rho_o}(w_o, \gamma_o)$$

gives

$$G_w \frac{dw_o}{d\gamma} + G_\gamma = 0 \quad , \tag{2.4a}$$

$$G_w \dot{w}_o + G_\gamma \dot{\gamma}_o = 0 \quad . \tag{2.4b}$$

Hence away from a limit point

$$\dot{w}_o = \dot{\gamma}_o \frac{dw_o}{d\gamma} \quad . \tag{2.5}$$

At a limit point $\dot{w}$ is a vector in the nullspace of G_w since $\dot{\gamma} \equiv 0$ at a limit point.

On the other hand, differentiation of N_ϱ gives

$$\varrho \dot{w}^*(s_o)\, \dot{w}(s) + (1-\varrho)\, \dot{\gamma}(s_o)\dot{\gamma}(s) - 1 = 0 \ , \ s \ \varepsilon \left[s_o, s_1\right) . \tag{2.6}$$

Substituting (2.5) in (2.6) we obtain, for $s = s_o$,

$$\varrho \dot{\gamma}^2(s_o) \frac{dw_o^*}{d\gamma} \frac{dw_o}{d\gamma} + (1-\varrho)\dot{\gamma}^2(s_o) - 1 = 0 \quad ,$$

,i.e.,

$$\dot{\gamma}(s_o) = \pm \frac{1}{\sqrt{1 + \varrho(\frac{dw_o^*}{d\gamma}\frac{dw_o}{d\gamma} - 1)}} \quad . \tag{2.7}$$

To get $(w(s), \gamma(s))$, $s = s_o + \Delta s$, we use Newton's method to solve

$$G(w, \gamma, s) = 0$$

$$N_\varrho(w, \gamma, s) = 0 \quad .$$

Keller proved in [14] that for Δs small enough the Newton iteration

$$\begin{pmatrix} G_w^\nu & G_\gamma^\nu \\ \varrho \dot{w}_o^* & (1-\varrho)\dot{\gamma}_o \end{pmatrix} \begin{pmatrix} \Delta w^{\nu+1} \\ \Delta \gamma^{\nu+1} \end{pmatrix} = - \begin{pmatrix} G^\nu \\ N_\varrho^\nu \end{pmatrix} \tag{2.8}$$

converges quadratically close to regular and normal limit points, provided one uses as an initial iterate

$$\begin{pmatrix} w^o(s) \\ \gamma^o(s) \end{pmatrix} = \begin{pmatrix} w_o \\ \gamma_o \end{pmatrix} + \begin{pmatrix} \dot{w}_o \\ \dot{\gamma}_o \end{pmatrix} \Delta s \quad . \tag{2.9}$$

6.3 Description of PASSIN (a modified version of PASVA3 to solve TPBVP with normal limit points).

Keller in [14] pointed out which were the modifications that have to be introduced in a program like PASVA3 in order to carry out the arc length continuation procedure.

The algorithm is as follows:

i) Given a solution $[w_k, \gamma_k]$ compute $\frac{dw_k}{d\gamma}$, by solving (2.4a). This is done with the same subroutines of PASVA3 that solve the linear system (1.6b).

ii) Use the discretized version of (2.5) and (2.7) to compute $\dot{w}_k$ and $\dot{\gamma}_k$ (changing s_o to s_k).

iii) Call PASVA3 with initial values:

$$\begin{pmatrix} w^o_{k+1} \\ \\ \gamma^o_{k+1} \end{pmatrix} = \begin{pmatrix} w_k \\ \\ \gamma_k \end{pmatrix} + \begin{pmatrix} \dot{w}_k \\ \\ \dot{\gamma}_k \end{pmatrix} \Delta s$$

iv) To obtain $\begin{pmatrix} \Delta w^{\nu+1}_{k+1} \\ \\ \Delta \gamma^{\nu+1}_{k+1} \end{pmatrix}$ PASSIN solves

$$G^\nu_w z^\nu = - G^\nu \quad ,$$

$$G^\nu_w y^\nu = - G^\nu_\gamma \quad .$$

v)

$$\Delta\gamma^{\nu+1} = - (N^\nu_\theta + \theta \dot{w}^*_k z^\nu)/((1-\theta)\dot{\gamma}_k + \theta \dot{w}^*_k y^\nu)$$

$$\Delta w^{\nu+1} = z^\nu + \Delta\gamma^\nu y^\nu$$

$$w^{\nu+1}_{k+1} = w^\nu_k + \Delta w^{\nu+1}$$

$$\gamma^{\nu+1}_{k+1} = \gamma^\nu_k + \Delta\gamma^{\nu+1} \quad .$$

Remarks:

1) y^ν is an approximation to $\frac{dw_{k+1}}{d\gamma_{k+1}}$, so we can use the last y^ν computed in the Newton iteration as $\frac{dw_{k+1}}{d\gamma_{k+1}}$.

2) To solve the indetermination of sign of $\dot{\gamma}_k$ we use the fact (Keller) that the sign $(\det G_w \cdot \dot{\gamma})$ is constant on branches of regular and normal limit points. Thus, we pick a sign for $\dot{\gamma}$ when we start the computations on a branch and change the sign of $\dot{\gamma}$ whenever $\det G_w$ changes sign.

To locate accurately normal limit points we have developed a "modified" chord method to find the zeroes of $\dot{\lambda}(s)$. The standard chord method cannot be used because every time we compute a new solution we change the parameterization of the problem (since we insist in having unit tangent vectors at every s_k respect to different normalisations). There are three ideas in our "modified" chord method:

1) At a limit point γ is an extreme, i.e., either γ is the maximum or minimum value for which there is a solution on the current branch.

2) $\dot{\lambda}(s) = \alpha(s - s_k) + 0(s - s_k)^2$ for s close to a limit point s_k, Szeto [19].

3) The parameterization depends upon the point from where we start the next computation, i.e., if we want to reach $\tilde{\gamma}$, say, and we have values for $[\gamma_1, \dot{\gamma}_1]$, $[\gamma_2, \dot{\gamma}_2]$ then we have two possibilities for the initial guess, namely

$$\tilde{\gamma} = \gamma_1 + \dot{\gamma}_1 \Delta s_1 ,$$

or

$$\tilde{\gamma} = \gamma_2 + \dot{\gamma}_2 \, \Delta s_2$$

In general, $\Delta s_1 \neq \Delta s_2$.

Hence given γ_1 and γ_2 with γ_2 closer to the limit point ($|\dot{\gamma}_2| < |\dot{\gamma}_1|$), say, then the initial guess for the limit point will be

$$\tilde{\gamma}^o = \gamma_2 + \dot{\gamma}_2 \, \Delta s$$

$$\Delta s = \pm \frac{\gamma_1 - \gamma_2}{\dot{\gamma}_2 - \dot{\gamma}_1} \quad .$$

The sign of Δs is picked so that $\tilde{\gamma}^o$ is always closer to the limit point. In other words, choose the sign of Δs to force

$\tilde{\gamma}^o > \gamma_2$ if the limit point is a maximum

$\tilde{\gamma}^o < \gamma_2$ if the limit point is a minimum.

We recall that the limit point is a maximum iff $|\dot{\gamma}_2| < |\dot{\gamma}_1|$ implies $\gamma_2 > \gamma_1$.

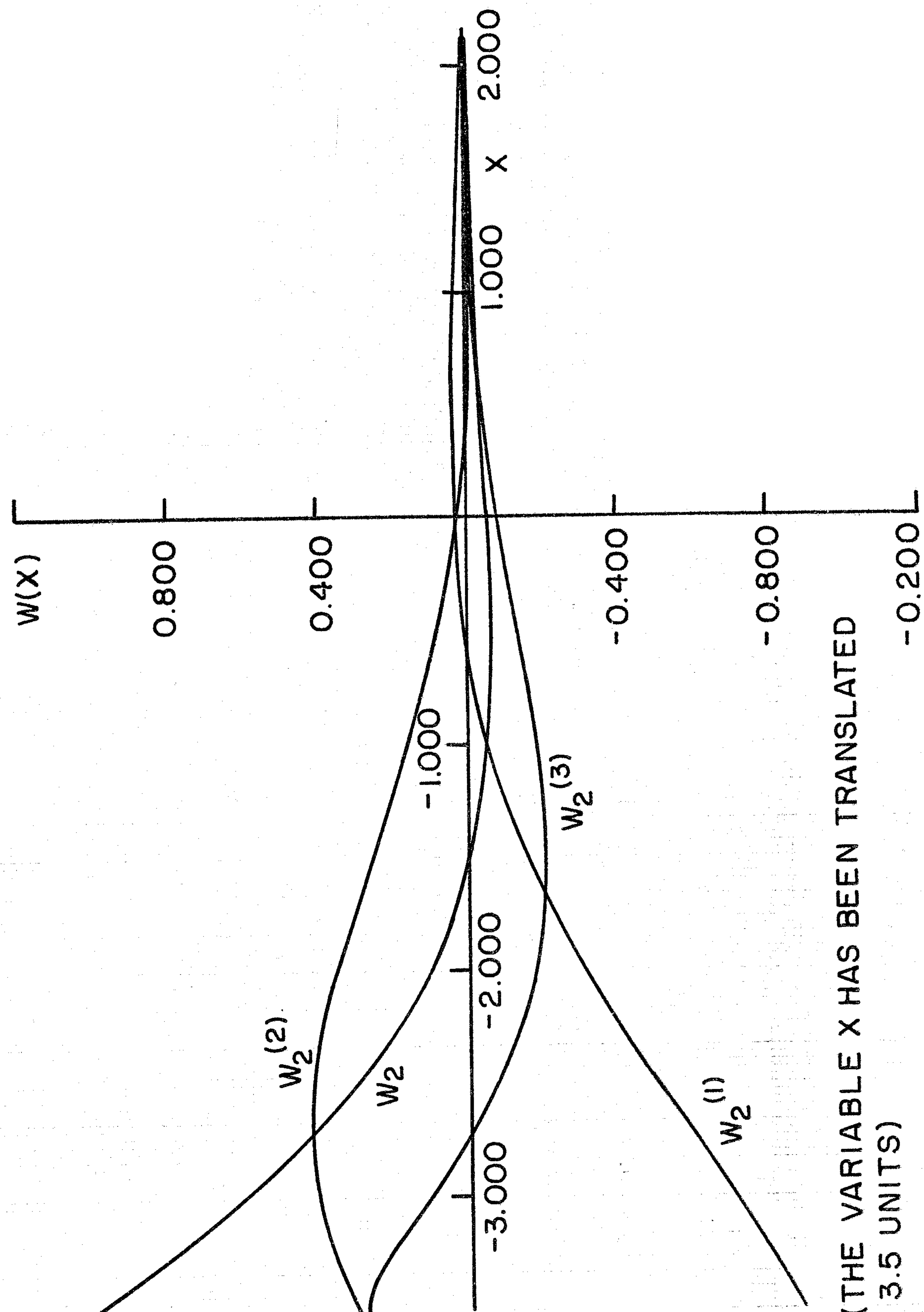

W(X)
0.800
0.400
-0.400
-0.800
-0.200
X
2.000
1.000
-1.000
-2.000
-3.000
$W_2^{(3)}$
$W_2^{(2)}$
W_2
$W_2^{(1)}$
(THE VARIABLE X HAS BEEN TRANSLATED 3.5 UNITS)

FIG. 1

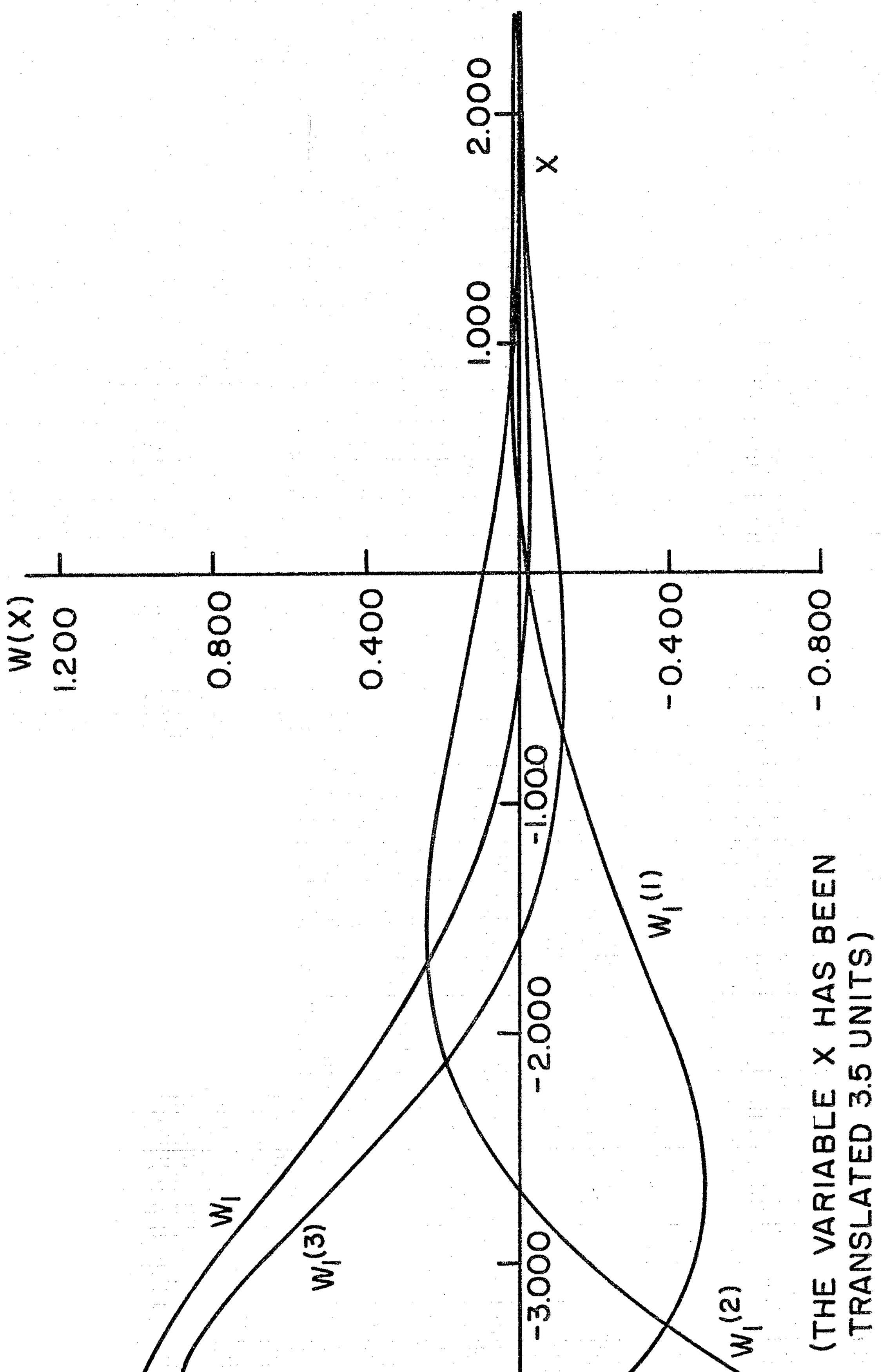

W(X)
1.200
0.800
0.400
-0.400
-0.800
2.000
1.000
-1.000
-2.000
-3.000
X
W_I
$W_I^{(1)}$
$W_I^{(2)}$
$W_I^{(3)}$
(THE VARIABLE X HAS BEEN TRANSLATED 3.5 UNITS)

FIG. 2

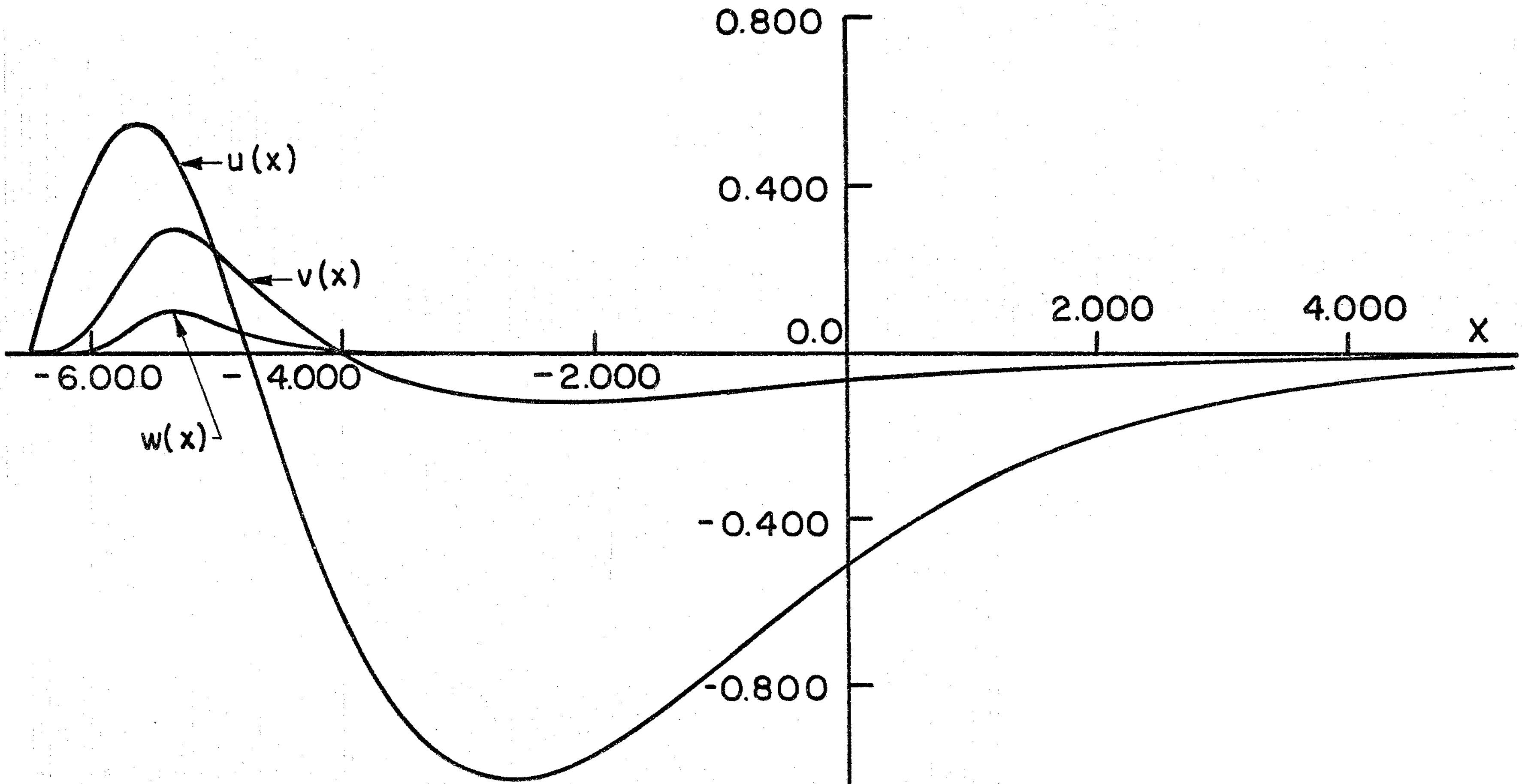
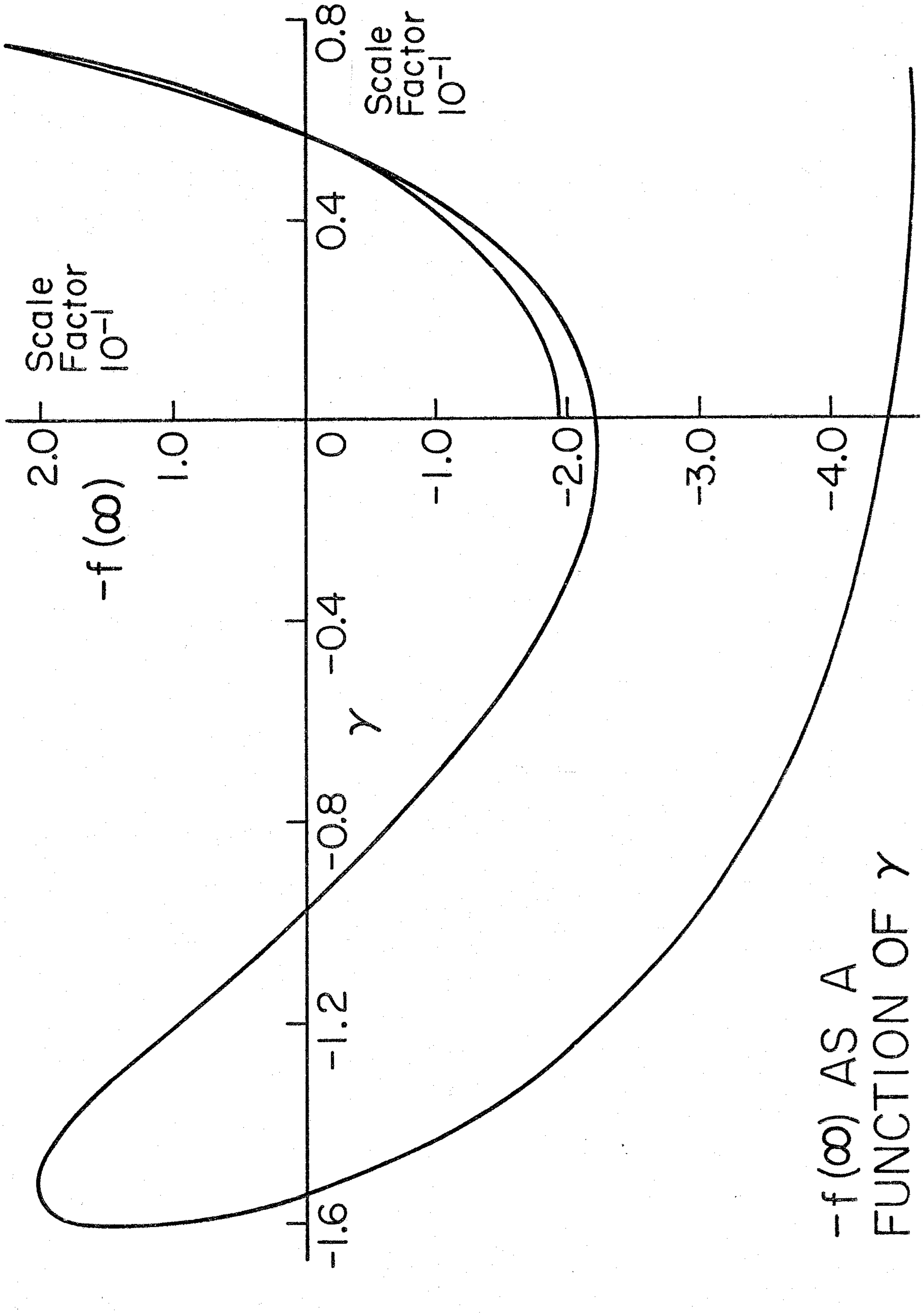


FIG. 3

(THE VARIABLE X HAS BEEN TRANSLATED 6.5 UNITS)



−f (∞) AS A FUNCTION OF γ

FIG. 4

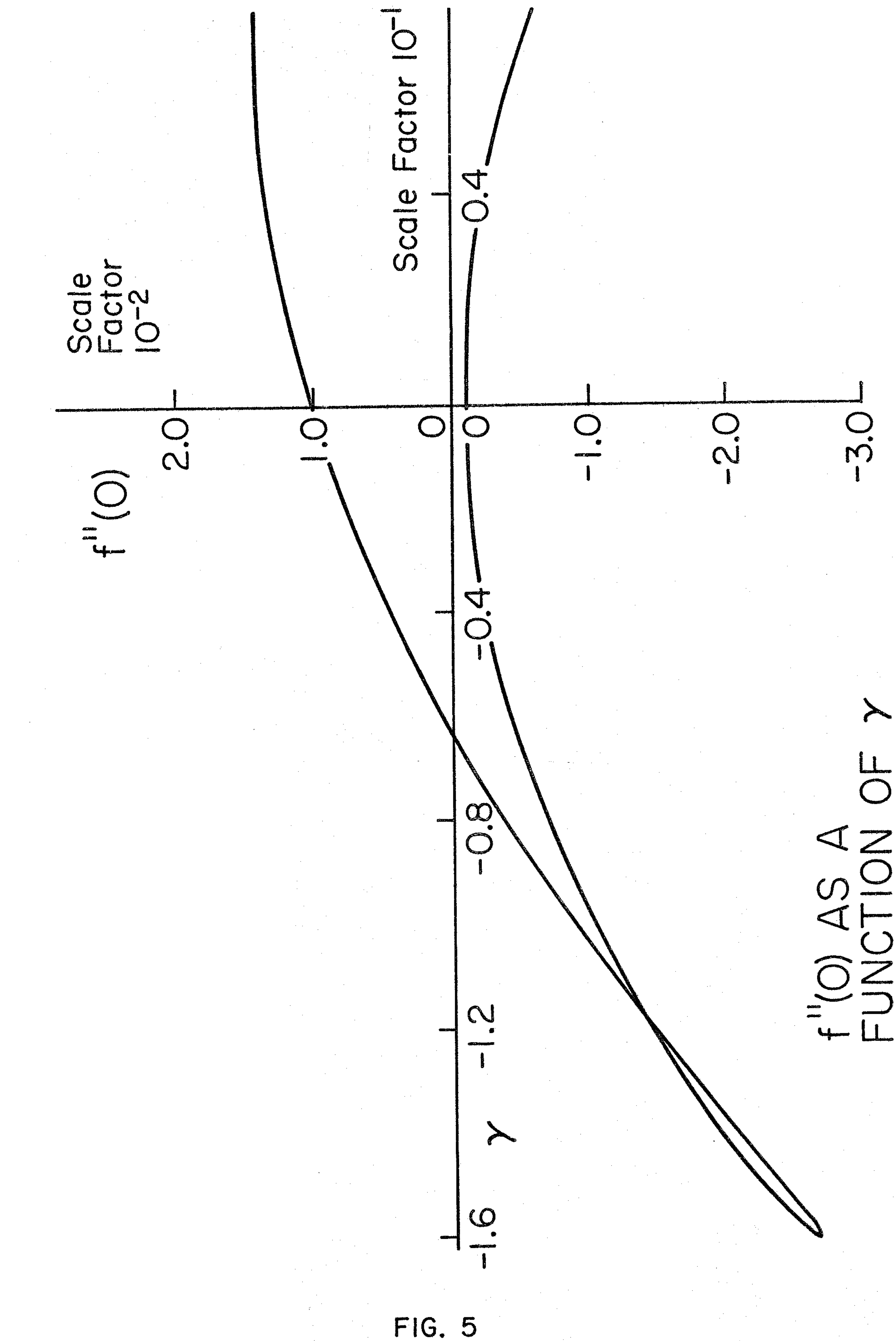

f''(O)
Scale Factor 10^{-2}
Scale Factor 10^{-1}
2.0
1.0
0
-1.0
-2.0
-3.0
0.4
-0.4
-0.8
-1.2
-1.6
γ
f''(O) AS A FUNCTION OF γ

FIG. 5

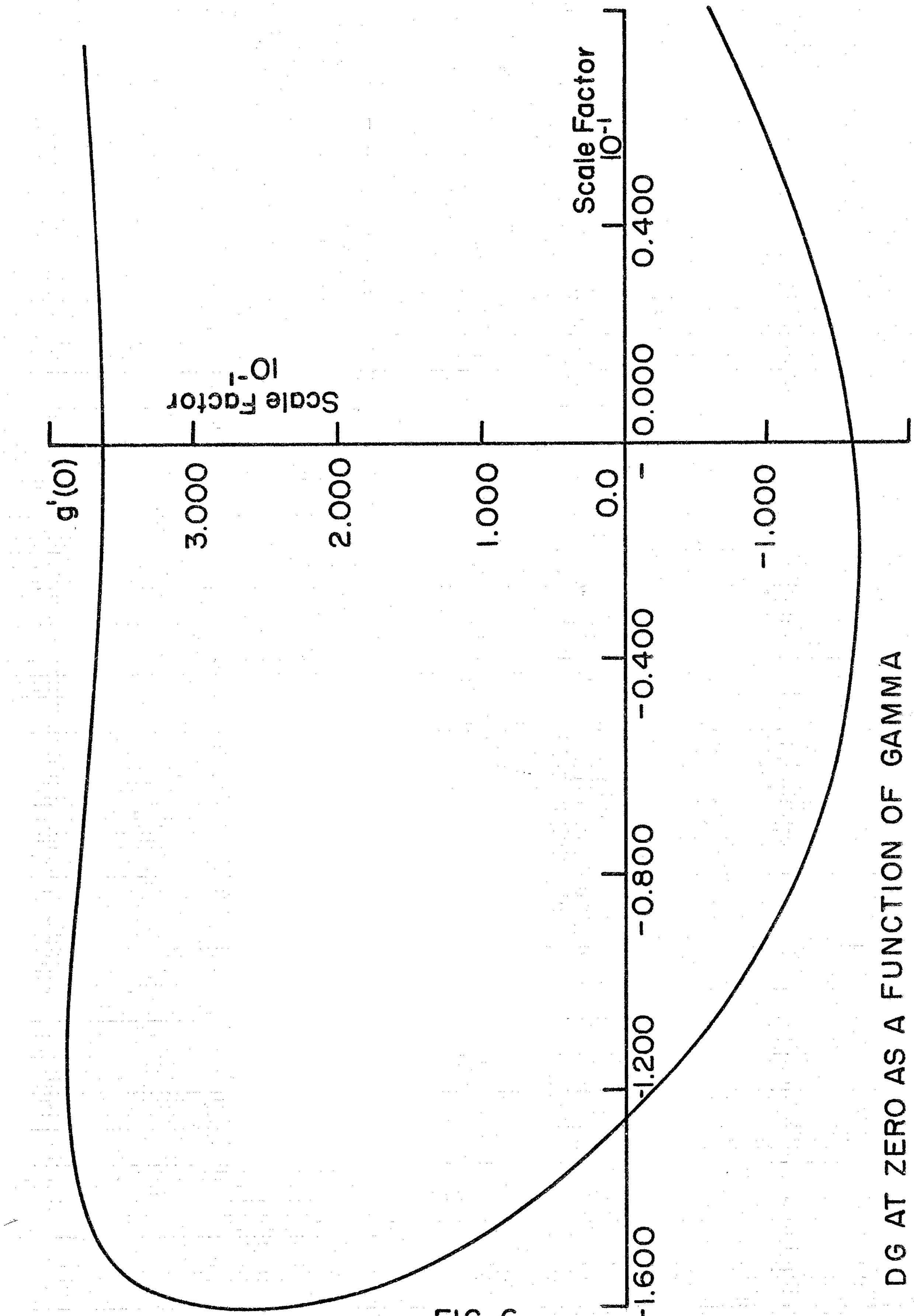
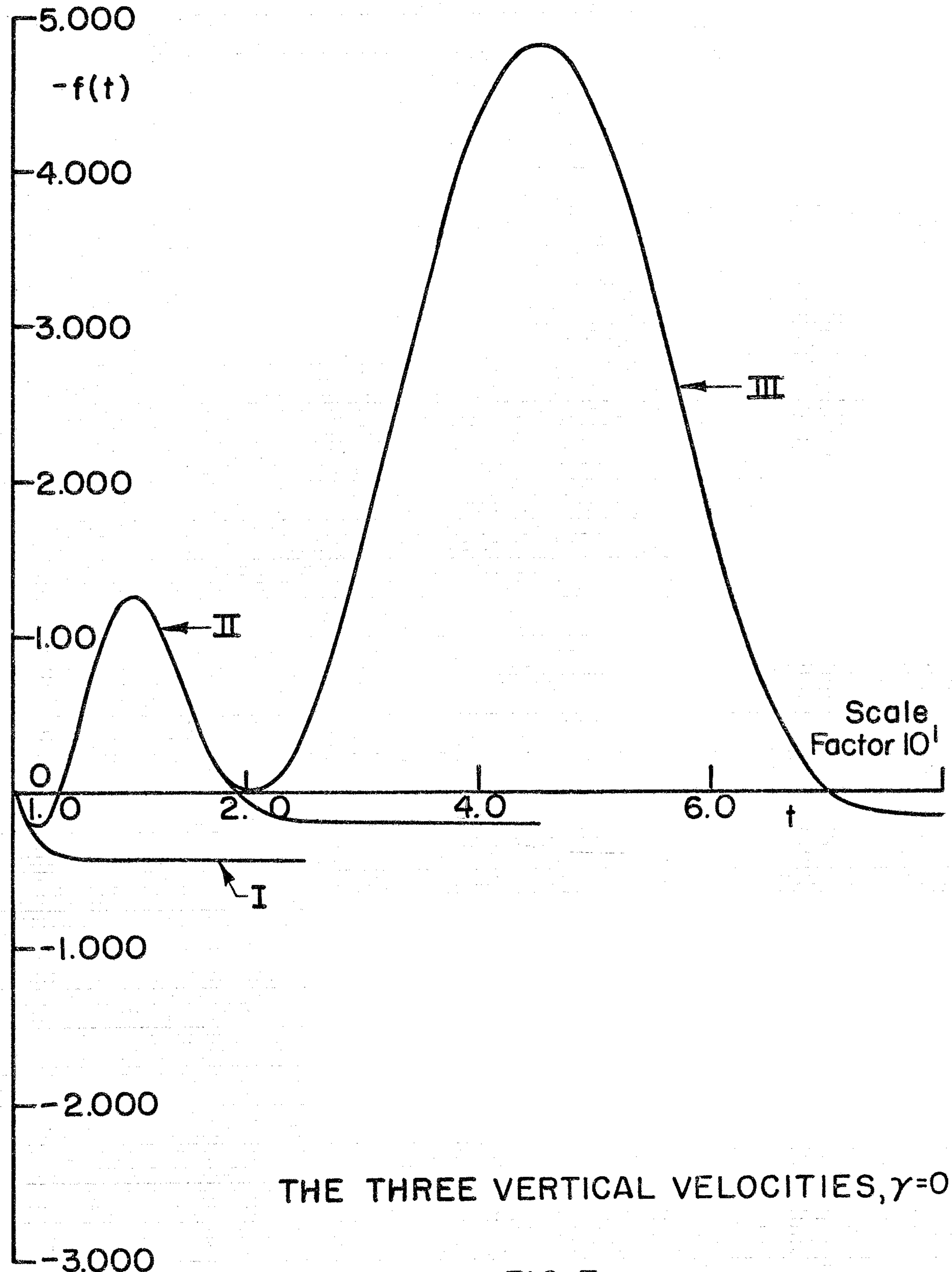


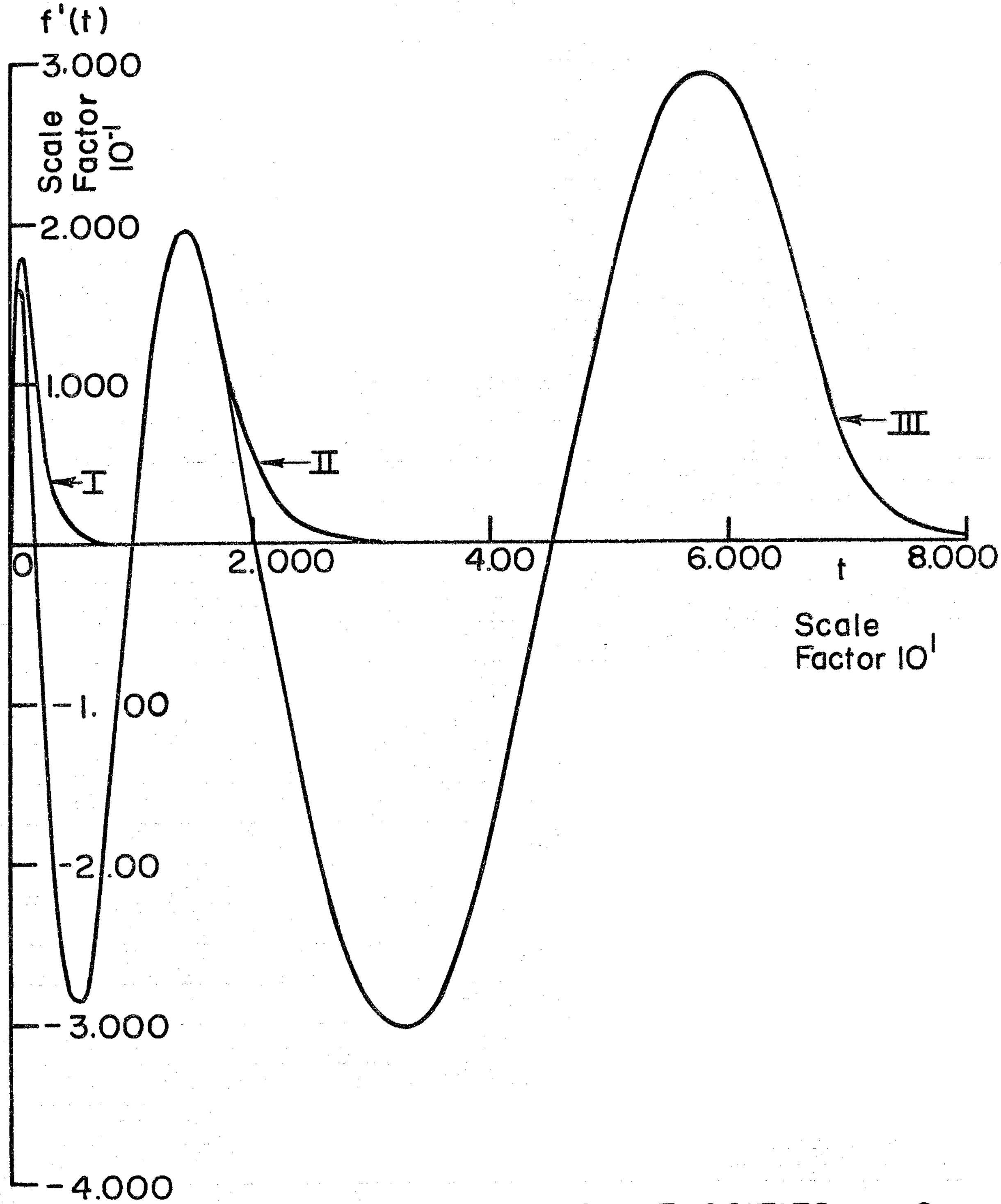
FIG. 6

DG AT ZERO AS A FUNCTION OF GAMMA



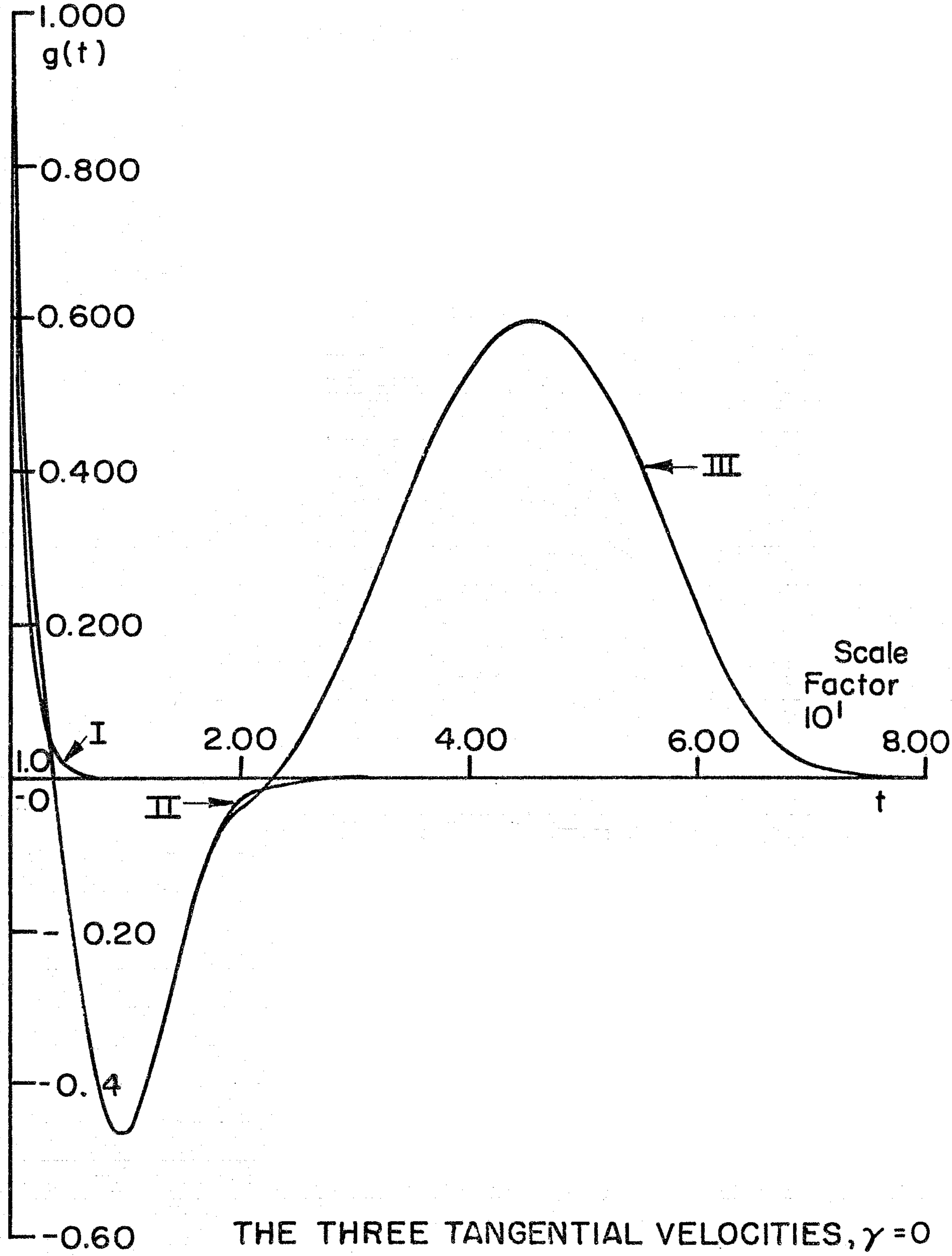
THE THREE VERTICAL VELOCITIES, $\gamma = 0$

FIG. 7



THE THREE RADIAL VELOCITIES, $\gamma = 0$

FIG. 8



THE THREE TANGENTIAL VELOCITIES, $\gamma = 0$

FIG. 9

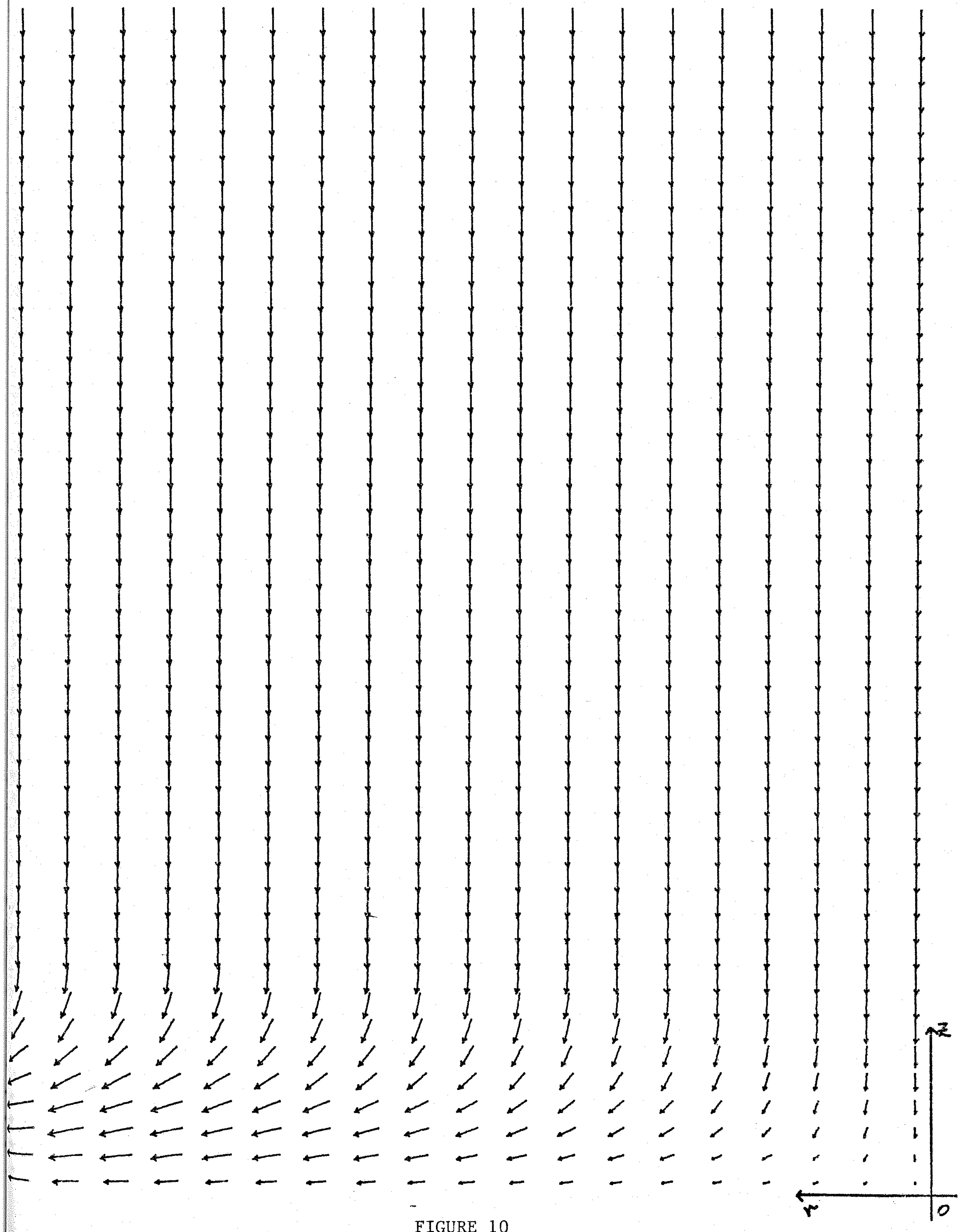


FIGURE 10

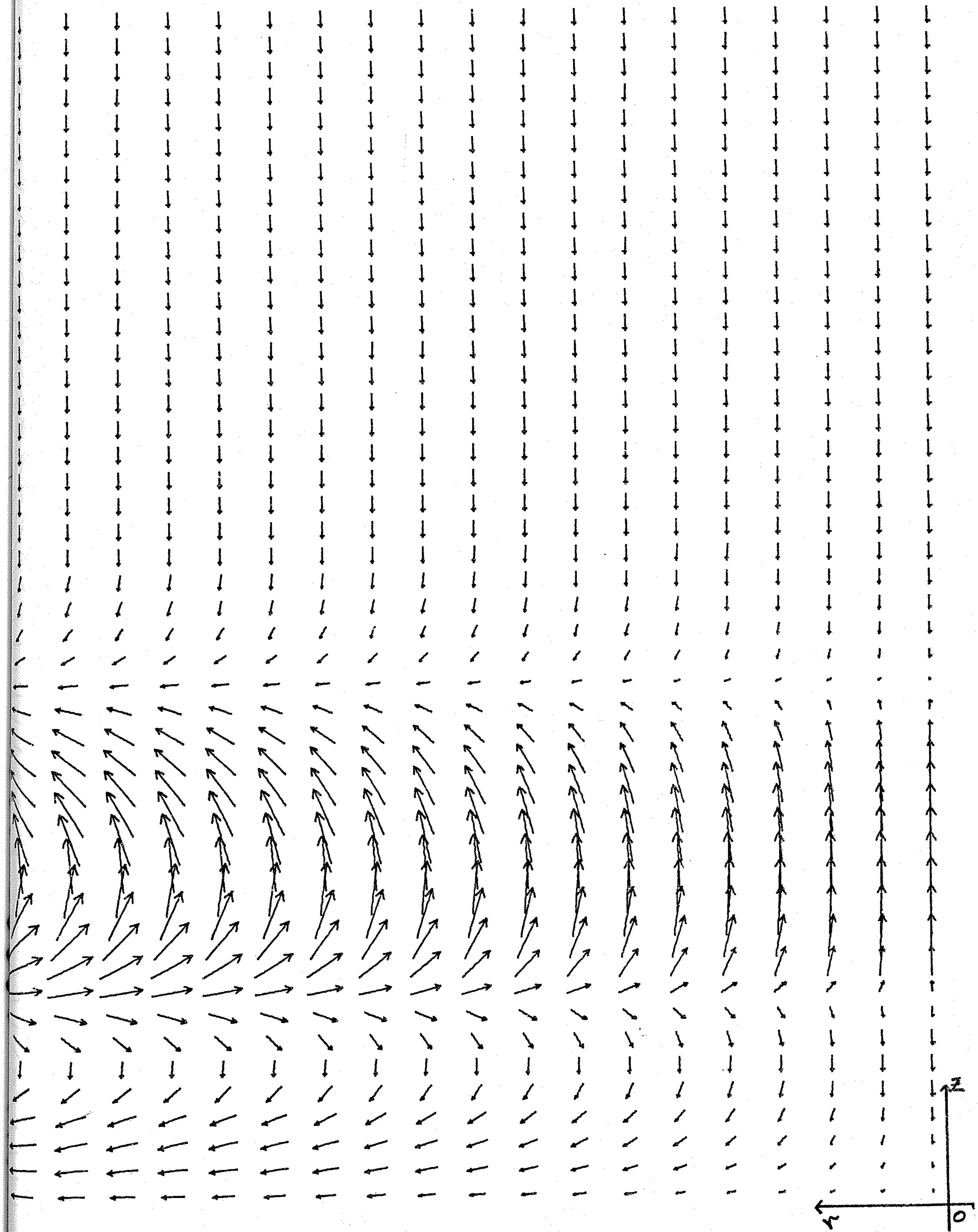


FIGURE 11

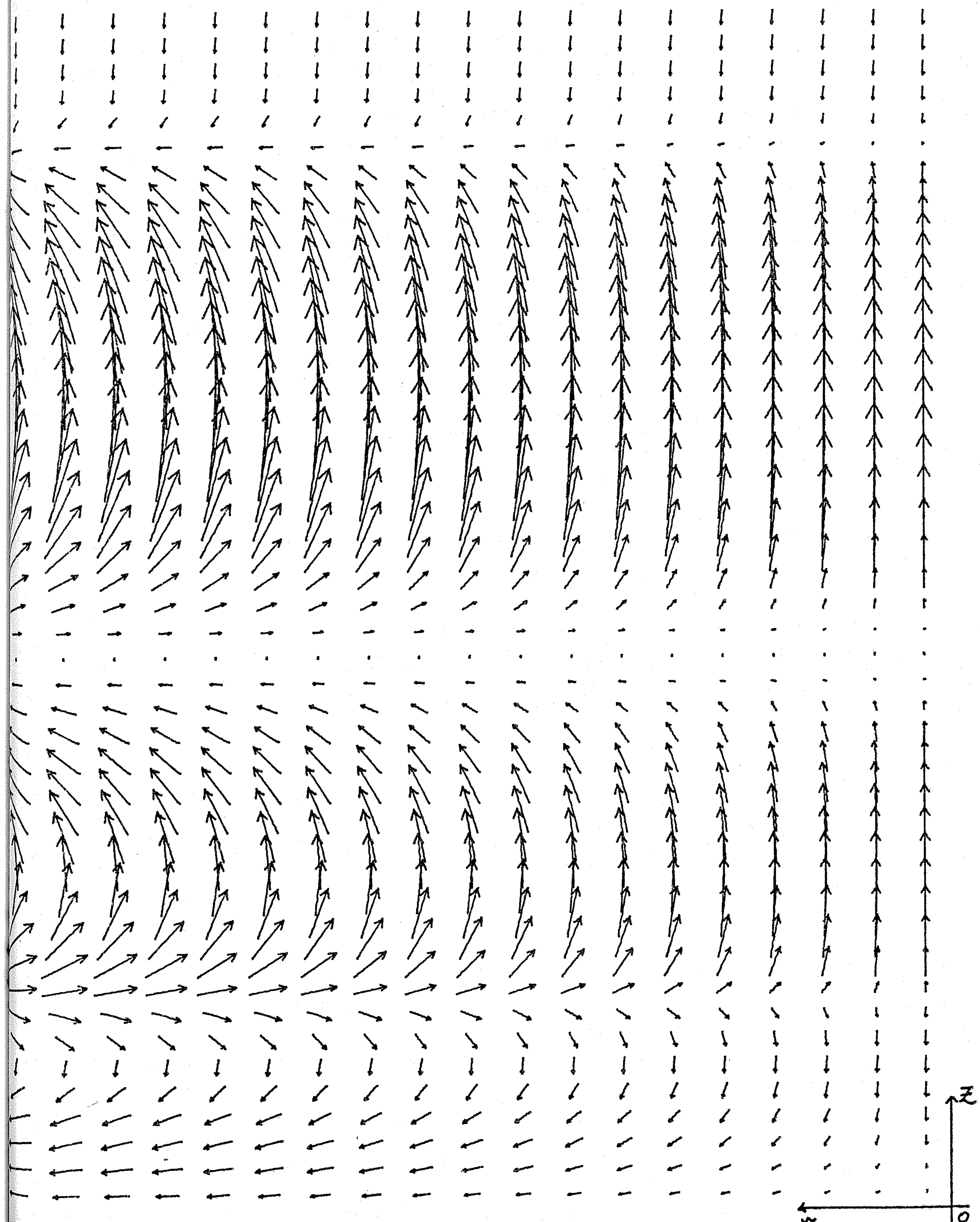


FIGURE 12

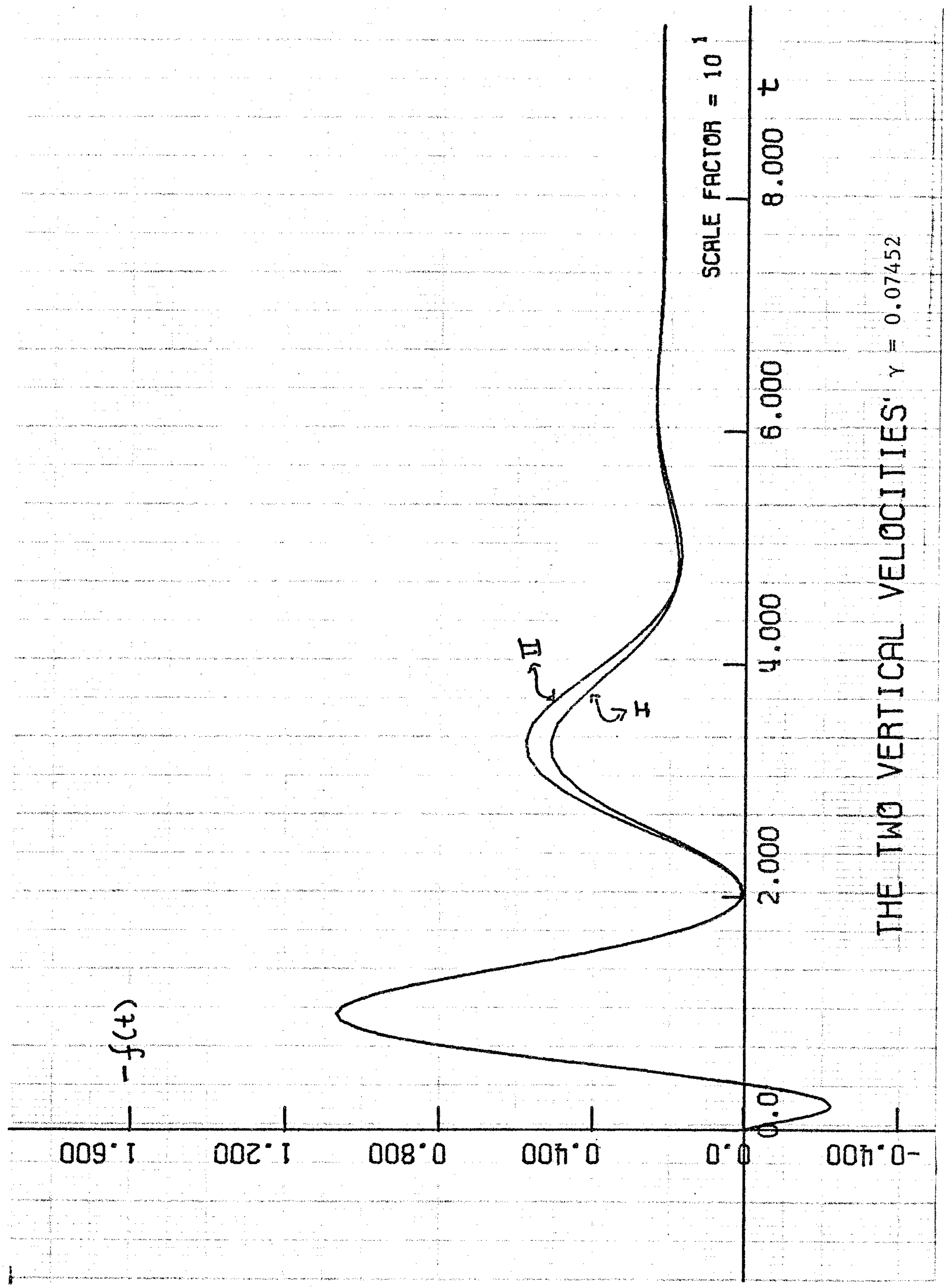

THE TWO VERTICAL VELOCITIES' γ = 0.07452
SCALE FACTOR = 10 1
t
0.0
2.000
4.000
6.000
8.000
-f(t)
-0.400
0.0
0.400
0.800
1.200
1.600
I
II

FIGURE 13

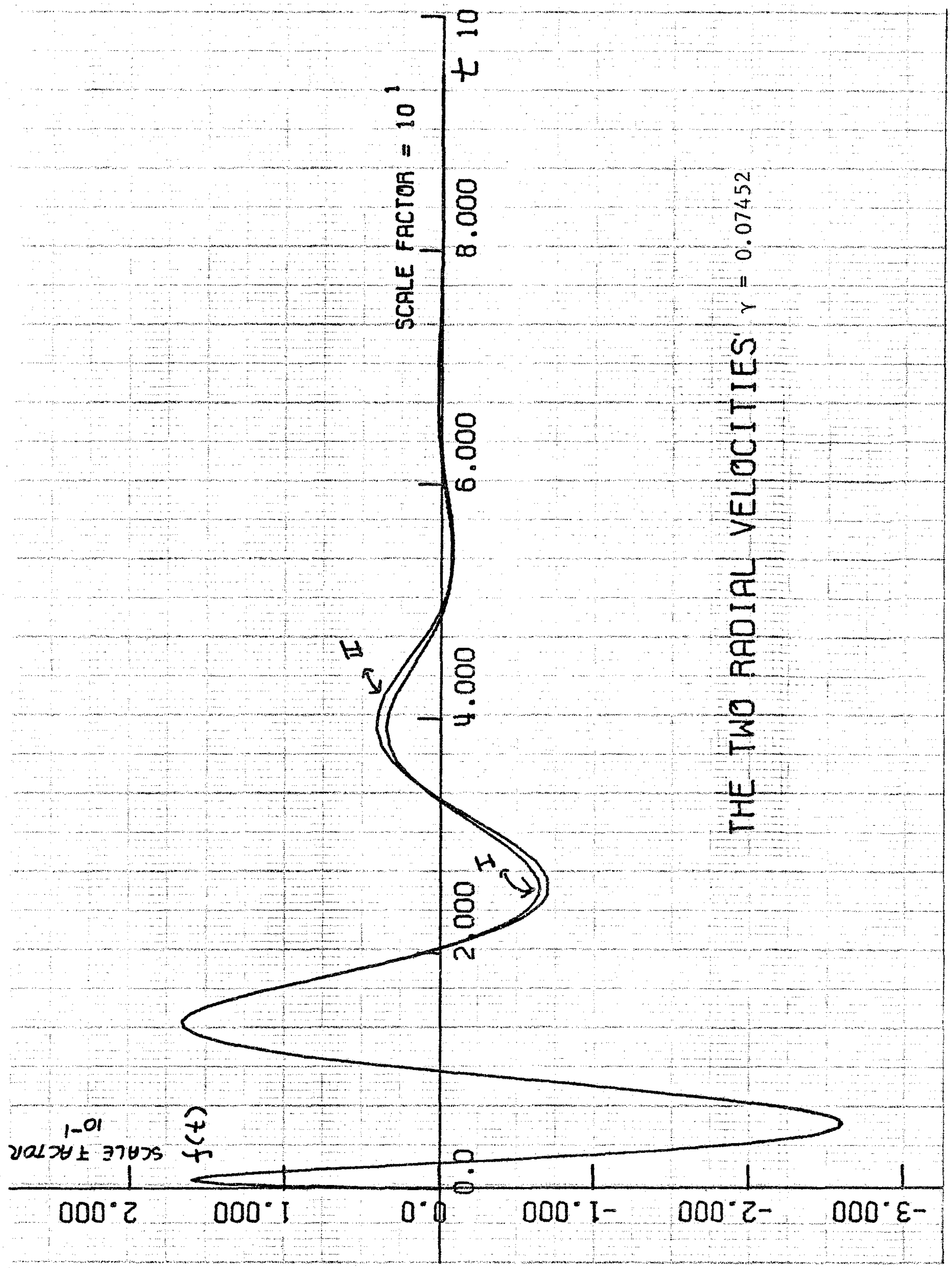

THE TWO RADIAL VELOCITIES γ = 0.07452
SCALE FACTOR = 10 1
t 10
8.000
6.000
4.000
2.000
0.0
f(t)
SCALE FACTOR 10^-1
2.000
1.000
0.0
-1.000
-2.000
-3.000
I
II

FIGURE 14

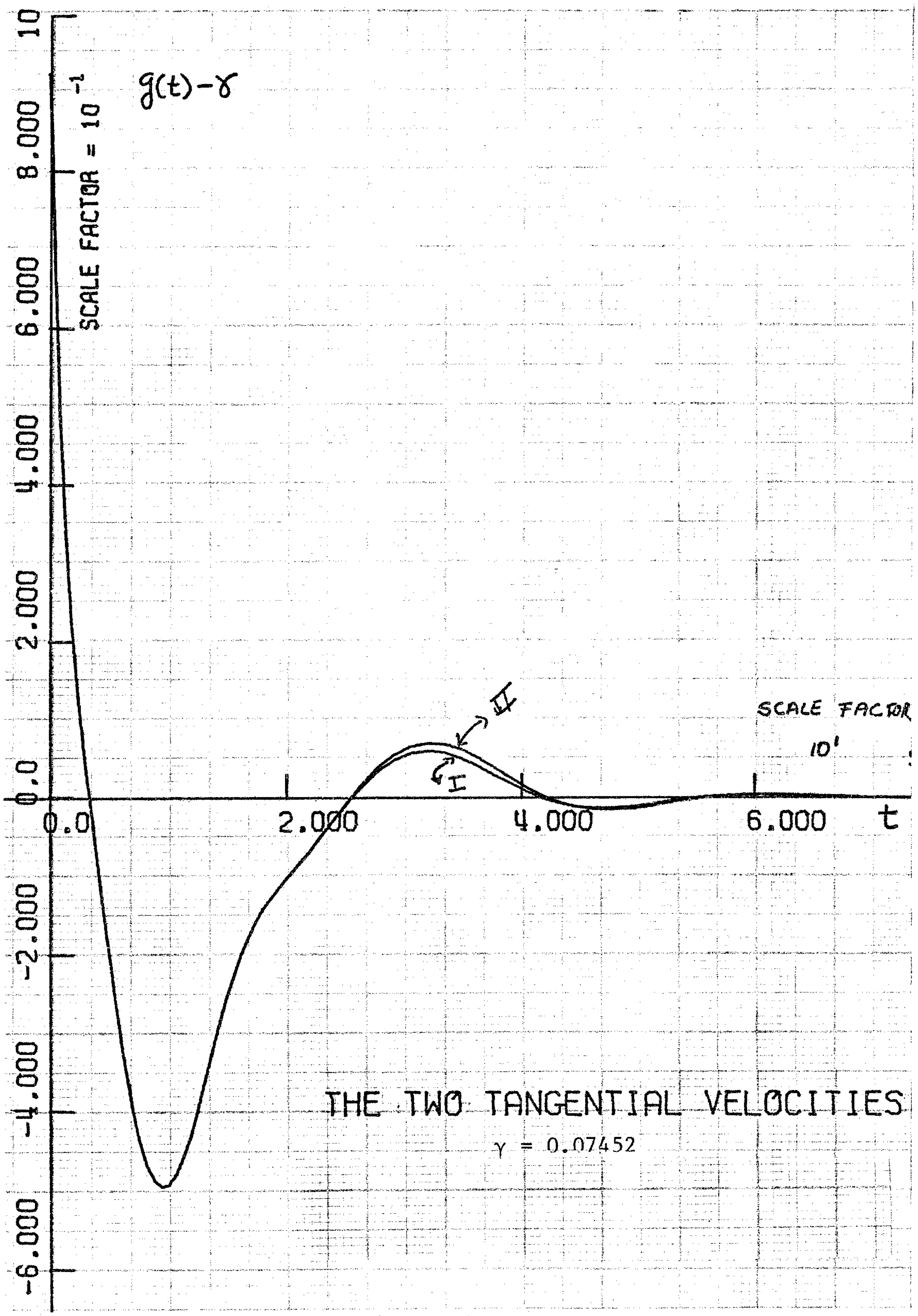

10
8.000
6.000
4.000
2.000
0.0
-2.000
-4.000
-6.000
SCALE FACTOR = 10 -1
g(t)-γ
II
I
SCALE FACTOR
10'
0.0
2.000
4.000
6.000
t
THE TWO TANGENTIAL VELOCITIES
γ = 0.07452

FIGURE 15

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