THE EFFECTS OF NONSTATIONARY AERODYNAMICS ON THE RIGID-BODY DYNAMIC STABILITY OF AN AIRPLANE

Thesis by

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ABSTRACT

A first order in frequency theory is developed for the aerodynamic loads on a harmonically oscillating thin wing of finite aspect ratio in a subsonic compressible flow. The downwash in the vicinity of a horizontal tail behind such a wing is also evaluated to the same order in frequency. The results are then used to determine the stability derivatives of a conventional-type airplane and to set up the stick-free longitudinal equations of motion including the unsteady flow effects.

An important conclusion of this study is that, within the limitations of a "lifting-strip" theory, the airloads on the oscillating finite span wing are linear in frequency in the neighborhood of zero frequency. This is in contrast with the two-dimensional results which show a logarithmic singularity there.

As an example of a practical application, calculations are made of the frequency, damping and transient responses of the stick-free longitudinal motion of an F-80A airplane and the results compared with those obtained using quasi-steady aerodynamic coefficients. The indications are that, while nonsteady flow considerations show considerable influence upon the control surface motion, they have a negligibly small effect upon the airplane motion.

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SYMBOLS AND NOTATIONS

a - velocity of sound in undisturbed fluid, ft/sec.

Al - translational displacement, ft.

$\mathcal{R}$ - aspect ratio

bl - distance from midchord of wing to airplane c.g. (negative for c.g. forward of midchord), ft.

B - angle of rotation about midchord, rad.

c - lifting surface chord length, ft.

C_j, $j = 1, 2, 3, 4, 5$ - coefficients of frequency quartic defined by eqns. (175) and (176)

$C(\frac{1}{2})$ - number of cycles to reduce amplitude to $\frac{1}{2}$ defined by $C(\frac{1}{2}) = T(\frac{1}{2})/P$

C - flap deflection angle, rad.

$\mathcal{C}(\Omega, M)$ - function defined by eqn. (55)

dl - distance from hinge line to c.g. of elevator, ft.

f - parameter defined by $f = \frac{M}{\beta^2} k$

F - Cicala's function defined by eqn. (E-3)

F_M - function defined by eqn. (D-3)

G - function defined by eqn. (30)

h_e - parameter defined by $h_e = \frac{4 m_e d}{\rho_0 S_w c_w}$

H - defined by equation of lifting surface $z = H(x,y,t)$

$H_0^{(2)}$, $H_1^{(2)}$ - Hankel functions of second kind and of zeroth and first order respectively

$i = \sqrt{-1}$

i_B - parameter defined by $i_B = \frac{4 I_y}{m c_w^2}$

i_e - parameter defined by $i_e = \frac{4 I_e}{m c_w^2}$

I_e - moment of inertia of elevator about hinge line, slugs ft^2

I_y - moment of inertia of airplane about y axis, slugs ft^2

J_0, J_1 - Bessel functions of first kind and of zeroth and first order respectively

k - reduced frequency defined by $k = \omega l/U$

K - function defined by eqn. (29)

l - semichord, ft.

$l_e l$ - distance from airplane c.g. to hinge line, ft.

l_B - distance from airplane c.g. to base of fuselage, ft.

l_N - distance from airplane c.g. to nose of fuselage, ft.

l_T - distance from airplane c.g. to midchord of horizontal tail, ft.

L - lift force, lbs.

L_A', L_A'', L_B', L_B'', L_C', L_C'' - functions defined by eqns. (106), (107), (108) and (109)

m - mass of airplane, slugs

m_e - mass of elevator, slugs

$\mathcal{M}$ - Mach number defined by $\mathcal{M} = U/a$

M - pitching moment, ft. lbs.

M_A', M_A'', M_B', M_B'', M_C', M_C'' - functions defined by eqns. (110), (111), (112) and (113)

$\mathfrak{m}$ - function of Mach number defined by eqn. (B-10)

N - hinge moment, ft. lbs.

N_A', N_A'', N_B', N_B'', N_C', N_C'' - functions defined by eqns. (114), (115), (116) and (117) p - pressure, lbs./ft.^{2}

P - period of the oscillation defined by $P = 2\pi/\omega$, sec.

q - pitch rate defined by $q = d\theta/dt$, rad/sec.

Q - function defined by eqn. (39)

R_A, R_W, R_R - regions in the X, Y plane (airfoil, wake, remainder)

s - ratio of semispan to semichord at midspan

S - lifting surface area, ft.^{2}

S_B - base area of fuselage, ft.^{2}

$S(\Omega, M, s)$ - function defined by eqns. (72) and (75)

t - time, sec.

$T(\frac{1}{2})$ - time to reduce amplitude by $\frac{1}{2}$, sec.

U - velocity of undisturbed fluid in x-direction, ft/sec.

V_F - volume of fuselage, ft.^{3}

w - component of velocity in z-direction, ft/sec.

w_A', w_A'', w_B', w_B'' - functions defined by eqns. (137), (138) and (139)

W - function defined by eqn. (13)

x, y, z - Cartesian coordinates

X, Y, Z - dimensionless coordinates defined by eqn. (2) (flutter axes)

X', Y', Z' - stability axes

X_L - coordinate of leading edge of lifting surface

X_T - coordinate of trailing edge of lifting surface

β = $\sqrt{1 - M^2}$

γ - Euler's constant, γ = 1.781072

Γ - function defined by eqn. (18)

δ - elevator deflection angle, rad.

ε - downwash angle, rad.

η_T - horizontal tail efficiency factor

θ - pitch angle, rad.

λ - operator defined by $\lambda = \frac{\ell}{U}\,\frac{d}{dt}$

$\Lambda_1(X,\xi)$, $\Lambda_2(X,\lambda)$ - functions defined following eqn. (46)

$\mu(\Omega,M)$ - function defined by eqn. (53)

μ_1 - parameter defined by $\mu_1 = \frac{4m}{\rho_0 S_W c_W}$

π - differential pressure distribution, lbs/ft^2

ρ - density, slugs/ft^3

$\sigma(\Omega, M, s)$ - function defined by eqns. (73) and (74)

φ - hinge position, $\varphi = \pi$ times % chord from leading edge

Φ - transformed velocity potential defined by eqn. (4)

Φ_j - functions of hinge position defined by eqn. (118)

ω - circular frequency of oscillation, rad/sec.

Ω - parameter defined by $\Omega = k/\beta^2$

SUBSCRIPTS

$(\ )_0$ - conditions in the undisturbed flow

$(\ )_W$ - wing characteristics

$(\ )_F$ - fuselage characteristics

$(\ )_T$ - horizontal tail characteristics

$(\ )_{T.O.}$ - "tail-off" characteristics

SUPERSCRIPTS

$(\ )^{(2)}$ - two-dimensional (infinite span)

$(\ )^{(3)}$ - three-dimensional (finite span)

Barred quantities $\overline{(\ )}$ are defined by $(\ ) = \overline{(\ )}e^{i\omega t}$.

Several other symbols are defined in the text for use only in the immediate vicinity of their definition.

INTRODUCTION

A knowledge of the dynamic stability of an airplane is a prerequisite to the evaluation of its handling characteristics and maneuverability. The need for the accurate determination of an airplane's dynamic qualities has become increasingly apparent with the advent of modern high-speed configurations and their concomitant reduction of stability margin in comparison with low-speed designs. Furthermore, the recent trends toward automatic flight controls necessitate an understanding of the airplane's dynamic transfer function in order to avoid instabilities due to coupling between the control system and the aircraft.

Currently, calculations on the dynamic stability of aircraft are generally made using "quasi-steady" aerodynamic coefficients. The quasi-steady theory assumes that the aerodynamic loads are linearly dependent upon the angular positions and the velocities of the airfoil surface producing them. The forces and moments are then predicted at any instant of the motion as if the airfoil were in steady motion under the conditions pertaining at that instant. This procedure leads to simultaneous differential equations with constant coefficients from which the frequency and damping of free motion and the dynamic responses to given inputs are easily determined.

Quasi-steady theory, however, neglects the effects of certain aerodynamic lags, i.e., the transition time required for the wing loading to attain its new steady-state level when, for example, the angle of attack of the airplane is suddenly changed. When rapid maneuvers are considered, the aerodynamic derivatives can no longer be considered constant but must be variable functions of the acceleration. Unsteady aerodynamic theory accounts for the time lag between the growth of circulation and the motion generating it by considering the influence of the trailing and shed vortices in the wake of the wing.

When the unsteady air forces are considered, however, the aerodynamic derivatives become functions of the time history of the motion and are no longer expressible in terms of simple, easily computed quantities as in the steady problem. In general, this makes the solution of the resulting nonlinear equations extremely difficult, if not impossible. Besides, at the present time, the solution of the aerodynamic reactions cannot be expressed as a function of any arbitrary motion. The only known solutions are for responses to a step function or to sinusoidal motion of an airfoil.

The theoretical evaluation of air forces on oscillating wings has been the subject for considerable research during the past thirty years. Throughout the greater part of this time, efforts have been primarily concentrated on the incompressible-flow case. Functions that describe the growth of unsteady airloads and their phase relationship to the corresponding motion have been tabulated for different types of motion by Wagner (1), von Kármán and Sears (2), Theodorsen (3), Luke and Dengler (4), and many others.

These results for incompressible flow were significant contributions to the understanding of unsteady aerodynamic theory but present day high-speed aircraft have brought a need for knowing the effects of the compressibility of air on the unsteady air forces. These effects cannot be accounted for by applying simple transformations such as the Prandtl-Glauert factor of steady flow since the finite transmission velocity of disturbances causes not only a change in the magnitudes of the forces but also in the phase lags between the instantaneous forces and the associated instantaneous wing motion.

The methods of calculating these forces in a two-dimensional, compressible, subsonic flow in most general use are based upon the integral equation of Possio (5) and lead to approximative solutions (6, 7, 8, 9, 10). Another approach to the boundary-value problem for this case developed independently by Haskind (11), by Reissner (12) and by Timman and van de Vooren (13) leads to a mathematically exact solution but utilizes series of Mathieu functions.

In view of these remarks, it is obvious that before any extensive investigation of the effects of nonsteady flow can be undertaken some reasonable simplifications must be made. One such simplification was suggested by Luke and Dengler (4). They proposed that the results obtained by considering simple harmonic motion in incompressible flow be applied to damped or divergent oscillations by replacing the reduced frequency in Theodorsen's circulation function C(k) (3) with the appropriate complex argument. Although it can be proven that this is mathematically justifiable for divergent motion, Theodorsen's method of solution is not valid for damped oscillations, and the suggestion has been criticized on this basis by van de Vooren (14), Laitone (15), and others.

The condition that the motion be divergent implies that, downstream, the vorticity in the wake decreases exponentially. Otherwise the derivations not only fail from a mathematical point of view but, for damped motion, become physically meaningless. However, Greidanus and van Heemert (16), by physical reasoning based on the damping effect of viscosity on the vorticity in the wake and by mathematical arguments based on analytic continuation, have shown that a physical interpretation can be given the results even in the excluded cases, provided the damping is small.

Furthermore, the results obtained with Luke and Dengler's method compare favorably, when the damping is small, with those derived by more rigorous methods (17). Their work indicates that for low damping and small reduced frequencies, the functional dependence of the Theodorsen function on the reduced frequency is independent of the amount of damping. There is no reason to suspect that compressibility will affect the essence of these arguments, and, consequently, the philosophy of this simplification will be utilized in this thesis.

Even though simple harmonic motion may be assumed in order to derive the corresponding air forces, however, the resulting aerodynamic derivatives are such complicated functions of frequency that their extensive use is rather impractical. Besides, an investigation made by the author (18) showed that for the frequency range considered (frequency less than 8 rad. per sec. for an F-80A corresponding to a reduced frequency of less than 0.1), the unsteady flow effects on the longitudinal stick-fixed, short period motion were negligible except for the phase lag in the wing downwash acting on the tail. However, even discounting the obvious case of aero-elastic problems, there are instances in which the frequencies of the motions involved are sufficiently high to warrant some consideration of the unsteady aerodynamic reactions particularly with the introduction of the compressibility of the air. This may be true, for example, in stick-free stability analyses or in automatic control through control surface hinge moments such as might be obtained by a servo-operated tab. Nevertheless, the interest lies in the lower frequency range in rigid-body stability work, and further simplification of the problem would seem to be in order. Although it may not be possible to neglect nonsteady flow effects entirely, an approximate solution of this problem for low frequency oscillations leading to simple expressions for the derivatives would be of considerable advantage over tabulated functions necessitating interpolation.

For two-dimensional, incompressible flow Goland (19) has proposed using polynomial approximations for the unsteady aerodynamic forces near zero frequency. This makes a most attractive simplification since the simultaneous differential equations which arise in the quasi-steady case are solved by factoring a characteristic polynomial. Appealing though it may be, such a Taylor-series expansion is not permissible since Theodorsen's function has a logarithmic singularity in slope at the origin. However, Reissner's corresponding function (20) for the three-dimensional, incompressible flow problem does not seem to have this logarithmic singularity when the aspect ratio is small enough. As a matter of fact, plots of Reissner's function vs. frequency appear to be linear near the origin. It will be shown in this thesis that, within the limitations of the theory, this is precisely the result for the finite aspect ratio problem not only for incompressible flow but for compressible flow as well.

The purpose of this study is to evaluate the importance of nonstationary flow effects on the dynamic stability of conventional-type aircraft at subsonic speeds. In view of the preceding remarks a prerequisite to such an investigation is the formulation of a sufficiently accurate first order in frequency theory for the aerodynamic loads on a harmonically oscillating wing of finite span in a compressible flow. It is with this problem that the present study begins.

NONSTEADY AIRFOIL THEORY

A. Formulation of the Boundary-Value Problem of an Oscillating Lifting Surface

It is assumed that a nearly plane surface is placed in a nonviscous fluid which has a uniform velocity U in the direction of the positive x-axis far upstream of the body. The disturbances created by the lifting surface are assumed to be infinitesimal with respect to the asymptotic velocity and the equations of motion are linearized by retaining only the linear terms in the perturbation velocities. The equation for the velocity potential of the oscillating airfoil in a subsonic compressible flow can then be written in the familiar form

$$(1-M^2)\phi_{xx}+\phi_{yy}+\phi_{zz}-\frac{2M}{a}\phi_{xt}-\frac{1}{a^2}\phi_{tt}=0 \tag{1}$$

where a is the velocity of sound in the undisturbed fluid, and M is equal to U/a.

With the transformations

$$X=\frac{x}{\ell}\ ; \quad Y=\beta\frac{y}{\ell}\ ; \quad Z=\beta\frac{z}{\ell} \quad \text{and } T=\frac{Ut}{\ell} \tag{2}$$

where $\beta=\sqrt{1-M^2}$

eqn. (1) becomes

$$\phi_{XX}+\phi_{YY}+\phi_{ZZ}-\frac{2M^2}{\beta^2}\phi_{XT}-\frac{M^2}{\beta^2}\phi_{TT}=0 \tag{3}$$

Now if simple harmonic motion is considered and the following substitution is made in eqn. (3):

$$\phi=e^{iMfX}e^{ikT}\bar{\Phi}(X,Y,Z) \tag{4}$$

where f is related to the circular and reduced frequencies by

$$f = \frac{\omega \ell}{a\beta^2} = M\Omega = \frac{M}{\beta^2} k \tag{5}$$

then

$$\Phi_{XX} + \Phi_{YY} + \Phi_{ZZ} + f^2 \Phi = 0 \tag{6}$$

Under the condition that the changes in density and pressure are small compared with the undisturbed density and pressure ρ_0 and p_0 , the pressure can be written in terms of the velocity potential as follows:

$$\bar{p} = -\rho_0 \left(i\omega \bar{\phi} + U \frac{\partial \bar{\phi}}{\partial x} \right) \tag{7}$$

Using eqns. (2) and (4) in eqn. (7) gives

$$\bar{p} = -\frac{\rho_0 U}{\ell} \left(ik\Phi + iMf\Phi + \frac{\partial \Phi}{\partial X} \right) e^{iMfX} \tag{8}$$

but

$$ik + iMf = i\beta^2\Omega + iM^2\Omega = i\Omega \tag{9}$$

and therefore

$$\bar{p} = -\frac{\rho_0 U}{\ell} \left(i\Omega\Phi + \frac{\partial \Phi}{\partial X} \right) e^{iMfX} \tag{10}$$

The boundary conditions which are appropriate for the wave equation (6) are specified as follows: The prescribed oscillatory motion of the airfoil determines the normal component of velocity of the fluid on the surface. If the surface is defined by z = H(x,y,t) and the boundary condition is satisfied on the x,y-plane rather than on the lifting surface itself which is consistant with the previous linearization, then on the airfoil

$$w(x,y,0) = \left(\frac{\partial \phi}{\partial z}\right)_{z\to 0} = \frac{\partial H}{\partial t} + U\frac{\partial H}{\partial x} \tag{11}$$

Now

$$\left(\frac{\partial \bar{\phi}}{\partial z}\right)_{z\to 0} = \frac{\beta}{l} e^{iMfx}\left(\frac{\partial \bar{\phi}}{\partial Z}\right)_{Z\to 0} = i\omega\bar{H} + \frac{U}{l}\frac{\partial \bar{H}}{\partial x} = \frac{U}{l}\left(ik\bar{H} + \frac{\partial \bar{H}}{\partial X}\right) \tag{12}$$

and, for convenience in later work, introduce a new variable defined by

$$W = \frac{l}{\beta} e^{-iMfX} e^{-i\omega t} w \tag{13}$$

then on the airfoil

$$\bar{W}(X,Y,0) = \left(\frac{\partial \bar{\phi}}{\partial Z}\right)_{Z\to 0} = \frac{U}{\beta} e^{-iMfX}\left(ik\bar{H} + \frac{\partial \bar{H}}{\partial X}\right) \tag{14}$$

The fact that the pressure is an odd function of z and that it must be continuous over all of the x,y-plane except the portion containing the lifting surface results in the following condition in the wake region of the airfoil when applied to eqn. (10).

$$i\Omega\bar{\phi} + \frac{\partial \bar{\phi}}{\partial X} = 0 \tag{15}$$

or in the wake

$$\phi(X, Y, 0) = \frac{\bar{\Gamma}^{(3)}(Y)}{2} e^{-i\Omega X} \tag{16}$$

Along any line Y = constant which does not pass through the wing region $\Gamma^{(3)}(Y) = 0$ from the condition of undisturbed flow at $X = -\infty$. For the wake region, however, $\Gamma^{(3)}(Y)$ is in general not equal to zero. Therefore

$$\phi(X, Y, 0) = 0 \quad \text{outside the wing and wake regions}$$

and

$$\phi(X, Y, 0) = \frac{\bar{\Gamma}^{(3)}(Y)}{2} e^{-i\Omega X} \quad \text{in the wake.}$$

Then by using eqns. (15) and (16), it may be shown that, in the wake

$$\frac{\partial \phi}{\partial X} = -\frac{i\Omega \bar{\Gamma}^{(3)}(Y)}{2} e^{-i\Omega X} \tag{17}$$

or since $\phi = 0$ outside the wing and wake regions

$$\bar{\Gamma}^{(3)}(Y) e^{-i\Omega X_T} = 2\int_{X_L}^{X_T} \frac{\partial \phi(X, Y, 0)}{\partial X} \, dX \tag{18}$$

It is assumed that the Kutta-Joukowsky condition is a reasonable way to take account of the effect of viscosity even in the unsteady problem. Consequently the condition is imposed that along the trailing edge of the airfoil the pressure remains finite, i.e.,

$$i\Omega\phi + \frac{\partial \phi}{\partial X} \quad \text{is finite along } X = X_T(Y) \tag{19}$$

In order to assure that $\phi \sim \mathcal{F}(x,y,z)\, e^{i(\omega t - fr)}$ as $z \to \infty$ and, therefore, that the waves are traveling away from the source of the disturbance, the additional condition is imposed that

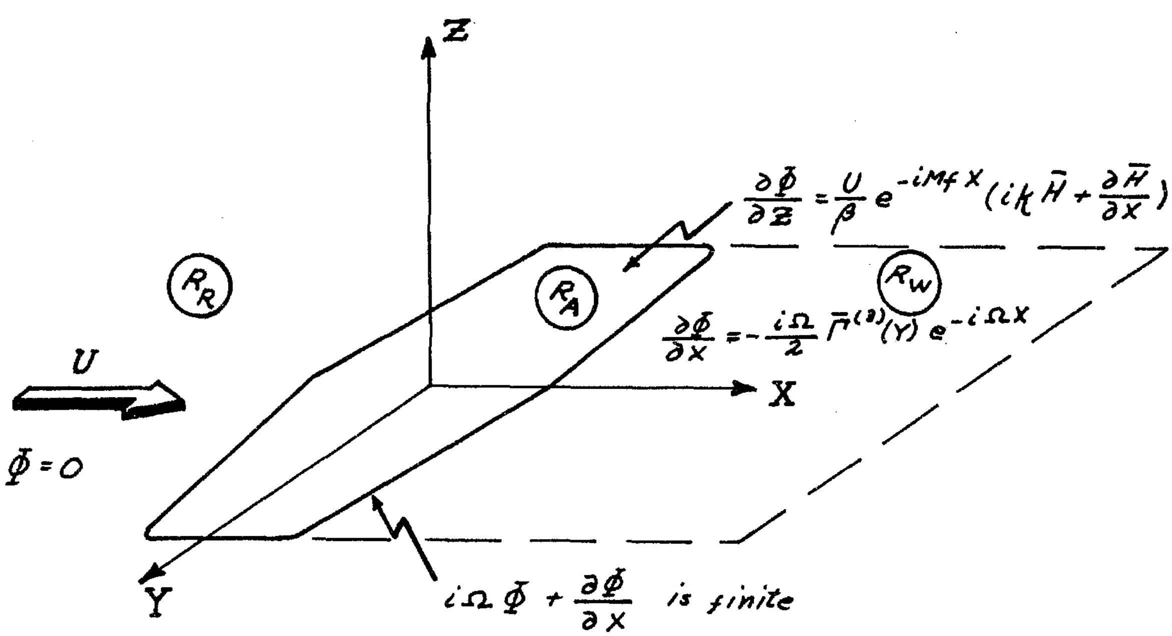
$$\Phi(X,Y,Z) \sim \mathcal{F}(X,Y,Z)\, e^{-ifr} \qquad \text{as } Z \to \infty \tag{20}$$

where $r^2 \equiv X^2 + Y^2 + Z^2$ and $\mathcal{F} \to 0$ as $Z \to \infty$

In summary then, the following problem has been posed: The wave equation

$$\Phi_{XX} + \Phi_{YY} + \Phi_{ZZ} + f^2 \Phi = 0 \tag{21}$$

is to be solved in the half space $Z \geq 0$ with boundary conditions as follows: (see accompanying sketch)



(i) On the airfoil: ($Z = 0$; X, Y in R_A)

$$\left(\frac{\partial \bar{\phi}}{\partial Z}\right)_{Z \to 0} = \overline{W}(X, Y, 0) = \frac{U}{\beta} e^{-iM f X}\left(ik\bar{H} + \frac{\partial \bar{H}}{\partial X}\right) \tag{22}$$

(ii) In the wake: ($Z = 0$; X, Y in R_W)

$$\frac{\partial \bar{\phi}(X, Y, 0)}{\partial X} = -\frac{i\Omega}{2} \bar{\Gamma}^{(s)}(Y)\, e^{-i\Omega X} \tag{23}$$

(iii) At the trailing edge: ($Z = 0$; $X = X_T$)

$$i\Omega\bar{\phi} + \frac{\partial \bar{\phi}}{\partial X} \quad \text{is finite} \tag{24}$$

(iv) In the remainder of the plane $Z = 0$, i.e.,

X, Y in R_R, $\bar{\phi} = 0$

and

(v) As $Z \to \infty$:

$$\bar{\phi} \sim \mathcal{F}(X, Y, Z)\, e^{-ifr} \tag{25}$$

B. Reduction of the Problem to an Integral Equation for the Circulation

A solution of eqn. (21) which satisfies the condition of eqn. (25) and which represents $\partial \bar{\phi}/\partial x$ within the half space $Z > 0$ in terms of the values of $\partial \bar{\phi}/\partial x$ on the boundaries is given by

$$\frac{\partial \bar{\phi}}{\partial x} = \frac{1}{2\pi}\iint \frac{\partial \bar{\phi}(\xi,\eta,0)}{\partial \xi} \frac{\partial}{\partial z}\left(\frac{e^{-ifr}}{r}\right) d\xi\, d\eta \tag{26}$$

where

$$r^2 \equiv (X - \xi)^2 + (Y - \eta)^2 + Z^2$$

From this solution to the wave equation an expression will be derived which relates the prescribed downwash on the airfoil to integrals over the Z = 0 plane of functions of the transformed velocity potential, $\bar{\phi}$. The double integration over the wing region will subsequently be reduced to a single integral through the approximations of a "lifting-strip" theory. Finally, after neglecting terms of the order of frequency squared, an approximate integral equation for the spanwise distribution of circulation will be derived which is comparable to the Prandtl lifting line integral equation in steady flow.

In Appendix A, it is shown that the following expression for $\partial \bar{\phi}/\partial z$ within the half space $Z > 0$ can be determined from eqn. (26):

$$\frac{\partial \Phi}{\partial Z} = \frac{1}{\pi}\int \frac{\partial \Phi(\xi,Y,0)}{\partial \xi}\left\{\frac{\partial}{\partial X}\left[\frac{i\pi}{2} H_0^{(2)}\left(f\sqrt{(X-\xi)^2+Z^2}\right)\right]\right.$$

$$\left. + f^2\int_{-\infty}^{X} \frac{i\pi}{2} H_0^{(2)}\left(f\sqrt{(X'-\xi)^2+Z^2}\right)dX'\right\} d\xi$$

$$+ \frac{1}{2\pi}\iint \frac{\partial^2\Phi(\xi,\eta,0)}{\partial\xi\,\partial\eta}\left\{\frac{|Y-\eta|}{Y-\eta}\frac{\partial}{\partial X}\int_{-\infty}^{-|Y-\eta|}\frac{e^{-if\sqrt{(X-\xi)^2+\varsigma^2+Z^2}}}{\sqrt{(X-\xi)^2+\varsigma^2+Z^2}}\,d\varsigma\right.$$

$$+\frac{|Y-\eta|}{Y-\eta} f^2 \int_{-\infty}^{X}\int_{-\infty}^{-|Y-\eta|}\frac{e^{-if\sqrt{(X'-\xi)^2+\varsigma^2+Z^2}}}{\sqrt{(X'-\xi)^2+\varsigma^2+Z^2}}\,d\varsigma\, dX'$$

$$\left.+\int_{-\infty}^{X}\frac{\partial}{\partial\eta}\left[\frac{e^{-if\sqrt{(X'-\xi)^2+(Y-\eta)^2+Z^2}}}{\sqrt{(X'-\xi)^2+(Y-\eta)^2+Z^2}}\right]dX'\right\} d\xi\, d\eta \qquad (27)$$

In order to utilize the boundary conditions as specified in eqns. (22) and (23), it will be necessary to consider the limiting value of eqn. (27) as $Z \rightarrow 0$. In the cases where the region of integration is over the airfoil surface, however, the integrands become infinite at $Z = 0$, $X = \xi$, $Y = \eta$. Reissner (21) has discussed this problem for incompressible flow and has shown that the Cauchy principal values are the proper interpretations in these cases. The same result may be shown to be true in this case of subsonic compressible flow and consequently the following integral equation may be written for the downwash on the surface of a harmonically oscillating airfoil in a subsonic compressible flow:

$$\overline{W}(X,Y,0) = \frac{U}{\beta} e^{-iMfX}\left(ik\overline{H} + \frac{\partial \overline{H}}{\partial X}\right)$$
$$= \frac{1}{\pi} \oint_{X_L}^{\infty} \frac{\partial \bar{\phi}(\xi,Y,0)}{\partial \xi} K\left[(X-\xi);f\right] d\xi$$
$$+ \frac{1}{2\pi} \oint\!\!\oint_{WING} \frac{\partial^2 \bar{\phi}(\xi,\eta,0)}{\partial \xi \partial \eta} G\left[(X-\xi),(Y-\eta);f\right] d\xi d\eta$$
$$- \frac{i\Omega}{4\pi} \iint_{WAKE} \frac{d}{d\eta}(\bar{\Gamma}^{(3)}) e^{-i\Omega\xi} G\left[(X-\xi),(Y-\eta);f\right] d\xi d\eta \tag{28}$$

where

$$K = \frac{\partial}{\partial X}\left[\frac{i\pi}{2} H_0^{(2)}(f|X-\xi|)\right] + f^2 \int_{-\infty}^{X} \frac{i\pi}{2} H_0^{(2)}(f|X'-\xi|) dX' \tag{29}$$

and

$$G = \frac{|Y-\eta|}{Y-\eta}\left[\frac{\partial}{\partial X}\int_{-\infty}^{-|Y-\eta|} \frac{e^{-if\sqrt{(X-\xi)^2+\zeta^2}}}{\sqrt{(X-\xi)^2+\zeta^2}} d\zeta \right.$$
$$\left. + f^2 \int_{-\infty}^{X}\int_{-\infty}^{-|Y-\eta|} \frac{e^{-if\sqrt{(X'-\xi)^2+\zeta^2}}}{\sqrt{(X'-\xi)^2+\zeta^2}} d\zeta dX'\right]$$
$$+ \int_{-\infty}^{X} \frac{\partial}{\partial \eta}\left[\frac{e^{-if\sqrt{(X'-\xi)^2+(Y-\eta)^2}}}{\sqrt{(X'-\xi)^2+(Y-\eta)^2}}\right] dX' \tag{30}$$

The first integral on the right-hand side of eqn. (28) is evaluated in Appendix B neglecting terms of order frequency squared so that

$$\frac{1}{\pi}\oint_{Y_L}^{\infty} \frac{\partial \bar{\phi}(\xi,Y,0)}{\partial \xi} K\left[(X-\xi);f\right] d\xi \doteq \frac{1}{\pi}\oint_{X_L}^{X_T} \frac{\partial \bar{\phi}(\xi,Y,0)}{\partial \xi} \frac{d\xi}{X-\xi}$$
$$- \frac{i\Omega \bar{\Gamma}^{(3)}(Y)}{2\pi}\left(\int_{X_T}^{\infty} \frac{e^{-i\Omega\xi}}{X-\xi} d\xi - \mathcal{M} e^{-i\Omega X}\right) \tag{31}$$

where

$$\mathcal{M} = \ln \frac{M}{2} - \beta \ln \frac{1-\beta}{M}$$

To evaluate the double integral over the wing region a "lifting-strip" theory as proposed by Reissner (20, 21) in the incompressible problem will be employed. For sufficiently high aspect ratio $|Y-\eta| \gg |X-\xi|$ over most of the airfoil surface. It is assumed that, over the portion of the wing where the inequality $|Y-\eta| \gg |X-\xi|$ is not valid, $\partial^2 \bar{\Phi}/\partial\xi\partial\eta$ varies sufficiently slowly with η so that it may be taken as effectively constant. However, since the kernel G is an odd function of $|Y-\eta|$ the contribution of this part of the wing area to the value of the entire integral can be neglected. This approximation is comparable to the lifting-line theory for the stationary airfoil but because of the importance of the chordwise variation of the induced velocity, it is not possible to speak of a lifting-line in unsteady flow.

The function $G[(X-\xi),(Y-\eta);f]$ is evaluated to first order in frequency terms in Appendix C. Therefore, to first order in frequency and under the additional assumptions of the lifting-strip theory the double integral over the wing region in eqn. (28) may be written:

$$\iint_{WING} \frac{\partial^2 \bar{\Phi}(\xi,\eta,0)}{\partial\xi\partial\eta} G \, d\xi \, d\eta \doteq \iint_{WING} \frac{\partial^2 \bar{\Phi}(\xi,\eta,0)}{\partial\xi\partial\eta}\left(-\frac{i\pi f}{2}\frac{|Y-\eta|}{Y-\eta} + \frac{1}{Y-\eta}\right) d\xi \, d\eta \tag{32}$$

Since the factor of $\partial^2\bar{\phi}/\partial\xi\partial\eta$ does not depend on ξ , the integration with respect to this variable may be carried out explicitly as follows:

$$\int_{x_L}^{x_T} \frac{\partial^2\bar{\phi}(\xi,\eta,0)}{\partial\xi\partial\eta}\,d\xi = \frac{\partial}{\partial\eta}\int_{x_L}^{x_T}\frac{\partial\bar{\phi}}{\partial\xi}d\xi - \left(\frac{\partial\bar{\phi}}{\partial\xi}\right)_{\xi=x_T}\frac{dx_T}{d\eta} + \left(\frac{\partial\bar{\phi}}{\partial\xi}\right)_{\xi=x_L}\frac{dx_L}{d\eta}$$

and so using the condition that $\partial\bar{\phi}/\partial\xi$ is zero immediately ahead of the leading edge and eqns. (18) and (23) results in

$$\int_{x_L}^{x_T} \frac{\partial^2\bar{\phi}(\xi,\eta,0)}{\partial\xi\partial\eta}\,d\xi = \frac{1}{2}e^{-i\Omega x_T}\frac{d}{d\eta}\,\bar{\Gamma}^{(3)}(\eta) \tag{33}$$

Therefore, if eqn. (33) is introduced into eqn. (32)

$$\iint_{WING} \frac{\partial^2\bar{\phi}(\xi,\eta,0)}{\partial\xi\partial\eta}\,G\,d\xi d\eta \doteq \oint \frac{e^{-i\Omega x_T}}{2}\frac{d}{d\eta}\left[\bar{\Gamma}^{(3)}\right]\frac{|Y-\eta|}{Y-\eta}\left(\frac{1}{|Y-\eta|}-\frac{i\pi f}{2}\right)d\eta \tag{34}$$

Although the assumptions of lifting-strip cannot be applied to the wake region, the approximation to G derived in Appendix C is used in Appendix D to reduce this double integration in eqn. (28) to the form

$$\iint_{WAKE} \frac{d}{d\eta}\left[\bar{\Gamma}^{(3)}\right]e^{-i\Omega\xi}\,G\,d\xi d\eta$$
$$\doteq e^{-i\Omega X}\int\frac{d}{d\eta}\left[\bar{\Gamma}^{(3)}\right]F_M\,(Y-\eta)\,d\eta$$
$$+\oint \frac{e^{-i\Omega x_T}-e^{-i\Omega X}}{i\Omega}\,\frac{d}{d\eta}\left[\bar{\Gamma}^{(3)}\right]\frac{|Y-\eta|}{Y-\eta}\left(\frac{1}{|Y-\eta|}-\frac{i\pi f}{2}\right)d\eta \tag{35}$$

where F_M is defined in Appendix D. This function is evaluated approximately by neglecting terms of first-order and higher in frequency in Appendix E where it is found that

$$F_M(z) \doteq \frac{|z|}{z}\left\{1 - \frac{i\pi}{2} - \ln 2\gamma|z| + \frac{\pi M}{2} + m\right\} \tag{36}$$

Therefore, with the results presented in eqns. (31), (34), (35), and (36) and with the introduction of

$$Y = s\beta Y^* \tag{37}$$

eqn. (28) may be written, for the case of a rectangular wing where $X_T = -X_L = 1$ independent of Y, as follows:

$$\begin{aligned} \overline{W}(x,Y^*,o) \doteq\ & \frac{1}{\pi}\oint_{-1}^{1} \frac{\partial\overline{\Phi}(\xi,Y^*,o)}{\partial\xi}\,\frac{d\xi}{x-\xi} \\ & - \frac{i\Omega\overline{\Gamma}^{(3)}(Y^*)}{2\pi}\int_{1}^{\infty}\frac{e^{-i\Omega\xi}}{x-\xi}\,d\xi \\ & + \frac{i\Omega\overline{\Gamma}^{(3)}(Y^*)e^{-i\Omega x}}{2\pi}\,m - e^{-i\Omega x}\oint_{-1}^{1}\frac{d\overline{\Gamma}^{(3)}(\eta^*)}{d\eta^*}\,Q\,d\eta^* \end{aligned} \tag{38}$$

where

$$\begin{aligned} Q(|Y^*-\eta^*|;\Omega,s,M) =\ & \frac{1}{4\pi}\left\{-\frac{1}{\beta s(Y^*-\eta^*)}\right. \\ & \left. + \frac{i\pi M\Omega}{2}\,\frac{|Y^*-\eta^*|}{Y^*-\eta^*} + i\Omega F_M\left[\beta s\Omega(Y^*-\eta^*)\right]\right\} \end{aligned}$$

or by using the approximation to F_M given by eqn. (36) this may be written

$$Q(|Y^*-\eta^*|;\Omega,s,M) \doteq -\frac{1}{4\pi\beta s(Y^*-\eta^*)}$$
$$+\frac{i\Omega}{4\pi}\frac{|Y^*-\eta^*|}{Y^*-\eta^*}\left(1-\frac{i\pi}{2}+\mathcal{M}+\pi M-\ln 2\gamma\beta s\Omega|Y^*-\eta^*|\right) \tag{39}$$

In using the approximation of eqn. (36) to the function F_M, an additional limitation has been imposed upon this theory which is fairly obvious from the derivations of Appendix E. Due to the combined manner in which frequency and span appear in the preceding expressions, the assumption of low frequency implies that the span (or the aspect ratio) shall not be too large. Consequently, it is not possible, in what follows, to permit the span to become infinitely large and expect to obtain the two-dimensional result in the limit.

It now becomes necessary to solve eqn. (38) for $\partial\Phi/\partial x$ in terms of $\overline{W}$ and $\overline{\Gamma}^{(3)}$. The solution of the integral equation (38) may be obtained by the method discussed by Söhngen (22) and Reissner (20, 21). If

$$g(z) = \frac{1}{2\pi}\oint_{-1}^{1}\frac{f(\zeta)}{z-\zeta}\,d\zeta \tag{40}$$

where f(1) is finite then, by integral equation theory,

$$f(z) = -\frac{2}{\pi}\sqrt{\frac{1-z}{1+z}}\oint_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\;\frac{g(\zeta)}{z-\zeta}\,d\zeta \tag{41}$$

So, by rearranging eqn. (38) to conform with eqn. (40), it may be shown that

$$\frac{1}{2}\overline{W}(X,Y^*_,0) + \frac{i\Omega\overline{\Gamma}^{(3)}(Y^*)}{4\pi}\int_{1}^{\infty}\frac{e^{-i\Omega\xi}}{X-\xi}\,d\xi - \frac{i\Omega\overline{\Gamma}^{(3)}(Y^*)e^{-i\Omega X}}{4\pi}\mathcal{M}$$

$$+\frac{e^{-i\Omega X}}{2}\oint_{-1}^{1}\frac{d\overline{\Gamma}^{(3)}}{d\eta^*}Q\,d\eta^* \doteq \frac{1}{2\pi}\oint_{-1}^{1}\frac{\partial\bar{\phi}(\xi,Y^*_,0)}{\partial\xi}\;\frac{d\xi}{X-\xi} \tag{42}$$

and since $\partial\bar{\phi}/\partial\xi$ is finite at $\xi = 1$ by eqn. (24), then according to eqn. (41):

$$\frac{\partial\bar{\phi}(X,Y^*_,0)}{\partial X} \doteq -\frac{2}{\pi}\sqrt{\frac{1-X}{1+X}}\Bigg\{\frac{1}{2}\oint_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\;\frac{\overline{W}(\xi,Y^*_,0)}{X-\xi}\,d\xi$$

$$+\frac{i\Omega\overline{\Gamma}^{(3)}(Y^*)}{4\pi}\oint_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\int_{1}^{\infty}\frac{e^{-i\Omega\lambda}}{(\xi-\lambda)(X-\xi)}\,d\lambda\,d\xi$$

$$+\left[\frac{1}{2}\oint_{-1}^{1}\frac{d\overline{\Gamma}^{(3)}}{d\eta^*}Q\,d\eta^* - \frac{i\Omega\overline{\Gamma}^{(3)}(Y^*)}{4\pi}\mathcal{M}\right]\oint_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\;\frac{e^{-i\Omega\xi}}{X-\xi}\,d\xi\Bigg\} \tag{43}$$

Since it may be shown that

$$\oint_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\int_{1}^{\infty}\frac{e^{-i\Omega\lambda}}{(\xi-\lambda)(X-\xi)}\,d\lambda\,d\xi = -\pi\int_{1}^{\infty}\sqrt{\frac{\lambda+1}{\lambda-1}}\;\frac{e^{-i\Omega\lambda}}{X-\lambda}\,d\lambda$$

eqn. (43) may be written

$$\frac{\partial\bar{\phi}(X,Y^*;0)}{\partial X} \doteq -\frac{2}{\pi}\sqrt{\frac{1-X}{1+X}}\left\{\frac{1}{2}\oint_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\;\frac{\bar{w}(\xi,Y^*;0)}{X-\xi}\,d\xi\right.$$

$$-\frac{i\Omega\bar{\Gamma}^{(3)}(Y)}{4\pi}\int_{1}^{\infty}\sqrt{\frac{\lambda+1}{\lambda-1}}\;\frac{e^{-i\Omega\lambda}}{X-\lambda}\,d\lambda$$

$$\left.+\left[\frac{1}{2}\oint_{-1}^{1}\frac{d\bar{\Gamma}^{(3)}}{d\eta^*}Q\,d\eta^*-\frac{i\Omega\bar{\Gamma}^{(3)}(Y^*)}{4\pi}m\right]\oint_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\;\frac{e^{-i\Omega\xi}}{X-\xi}\,d\xi\right\} \tag{44}$$

Eqn. (44) will now be integrated with respect to X. It will be useful in this regard to have available the following relations:

$$\left.\begin{aligned}\oint_{-1}^{X}\sqrt{\frac{1-X'}{1+X'}}\;\frac{dX'}{\xi-X'} &= \frac{\pi}{2}+\sin^{-1}X+\sqrt{\frac{1-\xi}{1+\xi}}\;\Lambda_1(X,\xi)\\ &= \pi \text{ when } X=1\end{aligned}\right\} \tag{45}$$

$$\left.\begin{aligned}\oint_{-1}^{X}\sqrt{\frac{1-X'}{1+X'}}\;\frac{dX'}{\lambda-X'} &= \frac{\pi}{2}+\sin^{-1}X+\sqrt{\frac{\lambda-1}{\lambda+1}}\;\Lambda_2(X,\lambda)\\ &= \pi\left(1-\sqrt{\frac{\lambda-1}{\lambda+1}}\right) \text{ when } X=1\end{aligned}\right\} \tag{46}$$

where

$$\Lambda_1(X,\xi) = \frac{1}{2}\ln\frac{1-X\xi+\sqrt{1-\xi^2}\sqrt{1-X^2}}{1-X\xi-\sqrt{1-\xi^2}\sqrt{1-X^2}}$$

$$\Lambda_2(X,\lambda) = 2\tan^{-1}\left[\sqrt{\frac{(1-X)(\lambda+1)}{(1+X)(\lambda-1)}}\right]-\pi$$

By using these relations and the condition that Φ is zero just upstream of the leading edge, the integration of eqn. (44) with respect to X may be written:

$$\int_{-1}^{X}\frac{\partial\Phi(X',Y_,^*0)}{\partial X'}dX' \doteq -\frac{2}{\pi}\left\{-\frac{1}{2}\int_{-1}^{1}\left[\sqrt{\frac{1+\xi}{1-\xi}}\left(\frac{\pi}{2}+\sin^{-1}X\right)\right.\right.$$
$$\left.+\Lambda_1\right]\overline{W}(\xi,Y_,^*0)d\xi + \frac{i\Omega\overline{\Gamma}^{(3)}(Y^*)}{4}\int_{1}^{\infty}\left[\sqrt{\frac{\lambda+1}{\lambda-1}}\left(\frac{\pi}{2}+\sin^{-1}X\right)\right.$$
$$\left.+\Lambda_2\right]e^{-i\Omega\lambda}d\lambda - \left[\frac{1}{2}\oint_{-1}^{1}\frac{d\overline{\Gamma}^{(3)}}{d\eta^*}Qd\eta^*\right.$$
$$\left.\left.-\frac{i\Omega\overline{\Gamma}^{(3)}(Y^*)}{4\pi}m\right]\oint_{-1}^{1}\left[\sqrt{\frac{1+\xi}{1-\xi}}\left(\frac{\pi}{2}+\sin^{-1}X\right)+\Lambda_1\right]e^{-i\Omega\xi}d\xi\right\} \tag{47}$$

For the particular case when X = 1, eqn. (47) reduces to:

$$\int_{-1}^{1}\frac{\partial\Phi(X',Y_,^*0)}{\partial X'}dX' \doteq \oint_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\,\overline{W}(\xi,Y_,^*0)d\xi$$
$$-\frac{i\Omega\overline{\Gamma}^{(3)}(Y^*)}{2}\int_{1}^{\infty}\left(\sqrt{\frac{\lambda+1}{\lambda-1}}-1\right)e^{-i\Omega\lambda}d\lambda$$
$$+\left[\oint_{-1}^{1}\frac{d\overline{\Gamma}^{(3)}}{d\eta^*}Qd\eta^* - \frac{i\Omega\overline{\Gamma}^{(3)}(Y^*)}{2\pi}m\right]\oint_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\,e^{-i\Omega\xi}d\xi \tag{48}$$

It may be shown that

$$\int_{1}^{\infty}\left(\sqrt{\frac{\lambda+1}{\lambda-1}}-1\right)e^{-i\Omega\lambda}d\lambda = -\frac{\pi}{2}\left[H_1^{(2)}(\Omega)+iH_0^{(2)}(\Omega)\right]-\frac{e^{-i\Omega}}{i\Omega}$$
$$=\left[-\frac{i}{\Omega}-\ln\frac{\gamma\Omega}{2}-\frac{i\pi}{2}+\frac{i\Omega}{2}\ln\frac{\gamma\Omega}{2}-\frac{\pi\Omega}{4}-\frac{i\Omega}{4}\right.$$
$$\left.+\frac{\Omega^2}{4}\ln\frac{\gamma\Omega}{2}+O(\Omega^2)\right]-\frac{e^{-i\Omega}}{i\Omega} \tag{49}$$

and that

$$\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\,e^{-i\Omega\xi}d\xi = \pi\left[J_0(\Omega)-iJ_1(\Omega)\right]$$
$$=\pi\left[1-\frac{i\Omega}{2}+O(\Omega^2)\right] \tag{50}$$

After eqns. (49) and (50) are substituted into eqn. (48) and only terms of first order in frequency and first order times its logarithm are retained, then eqn. (48) becomes:

$$\int_{-1}^{1} \frac{\partial \phi(X', Y^*, 0)}{\partial X'} dX' \doteq \oint_{-1}^{1} \sqrt{\frac{1+\xi}{1-\xi}}\, \overline{W}(\xi, Y^*)\, d\xi$$
$$- \frac{i\Omega \bar{\Gamma}^{(3)}(Y^*)}{2}\left(-\ln \frac{\gamma\Omega}{2} - \frac{i}{\Omega} - \frac{i\pi}{2}\right) + \frac{\bar{\Gamma}^{(3)}(Y^*) e^{-i\Omega}}{2}$$
$$+ \left[\pi \oint_{-1}^{1} \frac{d\bar{\Gamma}^{(3)}}{d\eta^*} Q\, d\eta^* - \frac{i\Omega \bar{\Gamma}^{(3)}(Y^*)}{2} \mathcal{M}\right]\left(1 - \frac{i\Omega}{2}\right) \tag{51}$$

From eqn. (18), however

$$\int_{-1}^{1} \frac{\partial \phi(X', Y^*, 0)}{\partial X'} dX' = \frac{\bar{\Gamma}^{(3)}(Y^*) e^{-i\Omega}}{2}$$

Consequently, after rearranging terms in eqn. (51), an integral equation for $\bar{\Gamma}^{(3)}(Y^*)$ which is comparable to the integral equation of Prandtl's lifting-line theory (23) may be written as follows:

$$\bar{\Gamma}^{(3)}(Y^*) + \mu(\Omega, M) \oint_{-1}^{1} \frac{d\bar{\Gamma}^{(3)}}{d\eta^*} Q\, d\eta^* = \bar{\Gamma}^{(2)}(Y^*) \tag{52}$$

where

$$\mu = -\frac{2\pi\left(1 - \frac{i\Omega}{2}\right)}{1 + \frac{\pi\Omega}{2} - i\Omega \ln \frac{\gamma\Omega}{2} + i\Omega \mathcal{M}} \doteq -2\pi \left[C(\Omega, M) - \frac{i\Omega}{2}\right] \tag{53}$$

$$\bar{\Gamma}^{(2)}(Y^*) = \frac{2 \oint_{-1}^{1} \sqrt{\frac{1+\xi}{1-\xi}}\, \overline{W}(\xi, Y^*)\, d\xi}{1 + \frac{\pi\Omega}{2} - i\Omega \ln \frac{\gamma\Omega}{2} + i\Omega \mathcal{M}}$$
$$\doteq 2C(\Omega, M) \oint_{-1}^{1} \sqrt{\frac{1+\xi}{1-\xi}}\, \overline{W}(\xi, Y^*)\, d\xi$$

and

$$C(\Omega,M) \doteq 1-\frac{\pi\Omega}{2}+i\Omega\left(\ln\frac{\gamma\Omega}{2}-m\right) \tag{55}$$

$\bar{\Gamma}^{(2)}(Y^*)$ is the distribution of $\bar{\Gamma}$ according to two dimensional theory. $\oint_{-1}^{1}\frac{d\bar{\Gamma}^{(3)}}{d\eta^*}Q\,d\eta^*$ represents the effect of finite span. The function $C(\Omega,M)$ (a plot of which appears in Fig. 1) will be discussed later.

C. Solving for the Circulation Function

It now remains to establish some convenient scheme for solving eqn. (52) for the circulation function $\bar{\Gamma}^{(3)}$. An approximate method of solution for this integral equation has been suggested by Reissner (20) for the corresponding incompressible flow problem which still seems satisfactory in the present case. After introducing the transformation $Y^* = \cos\theta$ and $\eta^* = \cos\theta_1$, it is assumed that the spanwise distribution of the circulation function, $\bar{\Gamma}^{(3)}(Y^*)$, may be approximated by a trigonometric series of the form

$$\bar{\Gamma}^{(3)}(\theta) = \sum_{n=1}^{\infty} A_n \frac{\sin n\theta}{n} \tag{56}$$

Then from eqn. (52)

$$\sum_{n=1}^{\infty} A_n \left[\frac{\sin n\theta}{n} - \mu(\Omega, M) \oint_0^{\pi} Q \cos n\theta_1 \, d\theta_1 \right] = \bar{\Gamma}^{(2)}(\theta) \tag{57}$$

This relation is further simplified by assuming that the circulation function may be satisfactorily approximated by considering only the first term of the series expansion of eqn. (56), i.e., an elliptic distribution of the form:

$$\bar{\Gamma}^{(3)}(\theta) \doteq A_1 \sin\theta \tag{58}$$

Reissner states that this gives sufficiently accurate results for a rigid elliptical airfoil in translating or pitching motion or for a uniform deflection of a full span flap.

Therefore, eqn. (57) becomes

$$A_1\left[\sin\theta - \mu(\Omega, M)\oint_0^{\pi} Q\cos\theta_1 d\theta_1\right] = \bar{\Gamma}^{(2)}(\theta) \tag{59}$$

and satisfying this integral equation at the midspan point, i.e., at $\theta = \frac{\pi}{2}$ gives

$$A_1\left[1 - \mu(\Omega, M)\oint_0^{\pi} Q\cos\theta_1 d\theta_1\right] = \bar{\Gamma}^{(2)}(\theta = \tfrac{\pi}{2})$$

Let $\bar{\Gamma}^{(2),(3)}(0)$ be the respective values at midspan so that

$$\frac{\bar{\Gamma}^{(3)}(0)}{\bar{\Gamma}^{(2)}(0)} \doteq \frac{1}{1-\mu(\Omega, M)\oint_0^{\pi} Q\cos\theta_1 d\theta_1} \tag{60}$$

where from eqn. (39), Q at the midspan point is given by

$$Q(\cos\theta_1;\Omega, s, M) \doteq \frac{1}{4\pi s\beta\cos\theta_1} + \frac{i\Omega}{4\pi}\frac{|\cos\theta_1|}{\cos\theta_1}\Big(-\mathcal{M} - \pi M$$

$$-1 + \frac{i\pi}{2} + \ln 2\gamma s\beta\Omega|\cos\theta_1|\Big) \tag{61}$$

so that

$$\oint_0^{\pi} Q\cos\theta_1 d\theta_1 \doteq \frac{1}{4s\beta} + \frac{i\Omega}{2\pi}\left(-\mathcal{M} - \pi M - 2 + \frac{i\pi}{2} + \ln 4\gamma s\beta\Omega\right) \tag{62}$$

and μ (Ω , M) is given by eqn. (53).

D. Pressure Distribution

Having obtained a solution for the circulation function, it is now possible to derive an expression for the pressure distribution. From eqn. (10) the following relation is valid for determining the pressure:

$$\frac{\bar{p}\ell}{\rho_0 U} = -e^{iM^2\Omega X}\left(i\Omega\Phi + \frac{\partial\Phi}{\partial X}\right) \tag{10}$$

where $\partial\Phi/\partial X$ is given by eqn. (44) and Φ is evaluated in Appendix F from eqn. (47) as

$$\begin{aligned}\int_{-1}^{X}\frac{\partial\Phi(X',Y^*,0)}{\partial X'}dX' &\doteq \frac{1}{\pi}\oint_{-1}^{1}\Lambda_1\bar{W}(\xi,Y^*)d\xi \\ &-\frac{\bar{\Gamma}^{(3)}(Y^*)}{2\pi}\int_{1}^{\infty}e^{-i\Omega\lambda}\frac{\partial\Lambda_2}{\partial\lambda}d\lambda + \frac{1}{\pi i\Omega}\left[\oint_{-1}^{1}\frac{d\bar{\Gamma}^{(3)}}{d\eta^*}Qd\eta^*\right. \\ &\left.-\frac{i\Omega\bar{\Gamma}^{(3)}(Y^*)}{2\pi}m\right]\int_{-1}^{1}e^{-i\Omega\xi}\frac{\partial\Lambda_1}{\partial\xi}d\xi\end{aligned} \tag{63}$$

Combining the relations of eqns. (44) and (63) according to eqn. (10) results in the following:

$$\begin{aligned}\frac{\bar{p}\ell}{\rho_0 U} \doteq \frac{e^{iM^2\Omega X}}{\pi}&\left\{\oint_{-1}^{1}\left(\sqrt{\frac{1-X}{1+X}}\sqrt{\frac{1+\xi}{1-\xi}}\frac{1}{X-\xi} - i\Omega\Lambda_1\right)\bar{W}(\xi,Y^*)d\xi\right. \\ &-\frac{i\Omega\bar{\Gamma}^{(3)}(Y^*)}{2}\int_{1}^{\infty}\left(\sqrt{\frac{1-X}{1+X}}\sqrt{\frac{\lambda+1}{\lambda-1}}\frac{1}{X-\lambda} - \frac{\partial\Lambda_2}{\partial\lambda}\right)e^{-i\Omega\lambda}d\lambda \\ &+\left[\oint_{-1}^{1}\frac{d\bar{\Gamma}^{(3)}}{d\eta^*}Qd\eta^* - \frac{i\Omega\bar{\Gamma}^{(3)}(Y^*)}{2\pi}m\right]\oint_{-1}^{1}\left(\sqrt{\frac{1-X}{1+X}}\sqrt{\frac{1+\xi}{1-\xi}}\frac{1}{X-\xi}\right. \\ &\left.\left.-\frac{\partial\Lambda_1}{\partial\xi}\right)e^{-i\Omega\xi}d\xi\right\}\end{aligned} \tag{64}$$

However, from the definitions of the functions Λ_1 and Λ_2

$$\sqrt{\frac{1-x}{1+x}}\sqrt{\frac{\lambda+1}{\lambda-1}}\;\frac{1}{x-\lambda}-\frac{\partial\Lambda_2}{\partial\lambda}=-\sqrt{\frac{1-x}{1+x}}\;\frac{1}{\sqrt{\lambda^2-1}}$$

and

$$\sqrt{\frac{1-x}{1+x}}\;\sqrt{\frac{1+\xi}{1-\xi}}\;\frac{1}{x-\xi}-\frac{\partial\Lambda_1}{\partial\xi}=-\sqrt{\frac{1-x}{1+x}}\;\frac{1}{\sqrt{1-\xi^2}}$$

so that eqn. (64) may be simplified to

$$\begin{aligned}\frac{\bar{p}\ell}{\rho_o U}\doteq\frac{e^{iM^2\Omega x}}{\pi}\Bigg\{&\oint_{-1}^{1}\left(\sqrt{\frac{1-x}{1+x}}\sqrt{\frac{1+\xi}{1-\xi}}\;\frac{1}{x-\xi}-i\Omega\Lambda_1\right)\overline{W}(\xi,Y^*)d\xi\\&+\frac{i\Omega\bar{\Gamma}^{(3)}(Y^*)}{2}\int_1^\infty\sqrt{\frac{1-x}{1+x}}\;\frac{e^{-i\Omega\lambda}}{\sqrt{\lambda^2-1}}d\lambda-\left[\oint_{-1}^{1}\frac{d\bar{\Gamma}^{(3)}}{d\eta^*}Q\,d\eta^*\right.\\&\left.-\frac{i\Omega\bar{\Gamma}^{(3)}(Y^*)}{2\pi}\,\eta\right]\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\;\frac{e^{-i\Omega\xi}}{\sqrt{1-\xi^2}}d\xi\Bigg\}\end{aligned}\tag{65}$$

Then by using the integral definitions of the Bessel and Hankel functions, $J_o\,(\Omega)$ and $H_o^{(2)}\,(\Omega)$, this may be written

$$\begin{aligned}\frac{\bar{p}\ell}{\rho_o U}\doteq\frac{e^{iM^2\Omega x}}{\pi}\Bigg\{&\oint_{-1}^{1}\left(\sqrt{\frac{1-x}{1+x}}\sqrt{\frac{1+\xi}{1-\xi}}\;\frac{1}{x-\xi}-i\Omega\Lambda_1\right)\overline{W}(\xi,Y^*)d\xi\\&+\frac{i\Omega\bar{\Gamma}^{(3)}(Y^*)}{2}\sqrt{\frac{1-x}{1+x}}\left[-\frac{i\pi}{2}H_o^{(2)}(\Omega)\right]-\left[\oint_{-1}^{1}\frac{d\bar{\Gamma}^{(3)}}{d\eta^*}Q\,d\eta^*\right.\\&\left.-\frac{i\Omega\bar{\Gamma}^{(3)}(Y^*)}{2\pi}\,\eta\right]\sqrt{\frac{1-x}{1+x}}\;\pi J_o(\Omega)\Bigg\}\end{aligned}\tag{66}$$

or

$$\begin{aligned}\frac{\bar{p}\ell}{\rho_o U}\doteq\frac{e^{iM^2\Omega x}}{\pi}\Bigg[&\oint_{-1}^{1}\left(\sqrt{\frac{1-x}{1+x}}\sqrt{\frac{1+\xi}{1-\xi}}\;\frac{1}{x-\xi}\right.\\&\left.-i\Omega\Lambda_1\right)\overline{W}(\xi,Y^*)d\xi-\sqrt{\frac{1-x}{1+x}}\,S\Bigg]\end{aligned}\tag{67}$$

where, after introducing the relation of eqn. (52) for the integral occurring in the last term of eqn. (66), the function S is given by

$$S = -\frac{i\Omega}{2}\bar{\Gamma}^{(3)}(Y^*)\left[-\frac{i\pi}{2}H_0^{(2)}(\Omega)\right] + \left[\frac{\bar{\Gamma}^{(2)}-\bar{\Gamma}^{(3)}}{\mu(\Omega,M)}\right.$$
$$\left. -\frac{i\Omega}{2\pi}\bar{\Gamma}^{(3)}\mathcal{M}\right]\pi J_0(\Omega)$$
$$= \frac{\pi\Omega}{2}\bar{\Gamma}^{(2)}\left\{-\frac{1}{2}H_0^{(2)}(\Omega) - \frac{i}{\pi}\mathcal{M}J_0(\Omega)\right.$$
$$\left. +\left(1-\frac{\bar{\Gamma}^{(3)}}{\bar{\Gamma}^{(2)}}\right)\left[\frac{1}{2}H_0^{(2)}(\Omega) + \frac{2J_0(\Omega)}{\Omega\mu(\Omega,M)} + \frac{i}{\pi}\mathcal{M}J_0(\Omega)\right]\right\} \qquad (68)$$

By using the value of $\bar{\Gamma}^{(2)}(Y^*)$ given by eqn. (54) it may be shown that:

$$S \doteq \left\{C(\Omega,M) - 1 + \left(\frac{\bar{\Gamma}^{(3)}}{\bar{\Gamma}^{(2)}} - 1\right)\left[C(\Omega,M) + \frac{i\Omega}{2}\right]\right\}\oint_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\,\bar{W}(\xi,Y^*)\,d\xi \qquad (69)$$

where $C(\Omega, M)$ has been defined previously by eqn. (55).

By substituting eqn. (69) into eqn. (67), the result is finally obtained that

$$\frac{\bar{p}\ell}{\rho_0 U} \doteq \frac{e^{iM^2\Omega X}}{\pi}\left\langle \oint_{-1}^{1}\bar{W}(\xi,Y^*)\left(\sqrt{\frac{1-X}{1+X}}\sqrt{\frac{1+\xi}{1-\xi}}\,\frac{1}{X-\xi} - i\Omega\Lambda_1\right.\right.$$
$$\left. + \sqrt{\frac{1-X}{1+X}}\sqrt{\frac{1+\xi}{1-\xi}}\right)d\xi - \sqrt{\frac{1-X}{1+X}}\left\{C(\Omega,M) + \left(\frac{\bar{\Gamma}^{(3)}}{\bar{\Gamma}^{(2)}} - 1\right)\left[C(\Omega,M)\right.\right.$$
$$\left.\left.\left. + \frac{i\Omega}{2}\right]\right\}\oint_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\,\bar{W}(\xi,Y^*)\,d\xi\right\rangle \qquad (70)$$

or

$$\frac{\bar{p}\ell}{\rho_0 U} \doteq \frac{e^{iM^2\Omega X}}{\pi}\left\{\oint_{-1}^{1}\bar{W}(\xi,Y^*)\left[\sqrt{\frac{1-X}{1+X}}\sqrt{\frac{1+\xi}{1-\xi}}\,\frac{1}{X-\xi} - i\Omega\Lambda_1\right.\right.$$
$$\left.\left. + \sqrt{\frac{1-X}{1+X}}\sqrt{\frac{1+\xi}{1-\xi}} - \sqrt{\frac{1-X}{1+X}}\sqrt{\frac{1+\xi}{1-\xi}}\,S(\Omega,M,s)\right]d\xi\right\} \qquad (71)$$

where

$$\mathbf{S}(\Omega, M, s) = \mathbf{C}(\Omega, M) + \sigma(\Omega, M, s) \tag{72}$$

and

$$\sigma(\Omega, M, s) \doteq \left(\frac{\bar{F}^{(3)}}{\bar{F}^{(2)}} - 1\right)\left[\mathbf{C}(\Omega, M) + \frac{i\Omega}{2}\right] \tag{73}$$

Eqn. (70) shows that, as in the incompressible case, the aerodynamic span effect manifests itself as an additive correction, σ , to the basic two-dimensional function, $\mathbf{C}$ within the limitations of the lifting-strip theory.

Upon substituting the approximate expressions derived in eqns. (53), (55), (60) and (62) into eqn. (73), it may be established that, to terms of first order in frequency and first order times the logarithm of frequency, σ is given by

$$\sigma \doteq -\frac{2}{\beta\mathcal{R}+2}\left\{1 - \frac{\pi\Omega(\beta\mathcal{R}+2)}{4} + \frac{i\Omega}{2(\beta\mathcal{R}+2)}\left[(\beta\mathcal{R}+2)^2\left(-\mathcal{M} + \ln\frac{\gamma\Omega}{2}\right) + (\beta\mathcal{R})^2(-2 - \pi M + \ln 2\pi\beta\mathcal{R}) + 2\right]\right\} \tag{74}$$

where the aspect ratio for wing of elliptical planform is $\mathcal{R} = 4s/\pi$.

By introducing eqns. (55) and (74) into eqn. (72), the following approximate expression is obtained for $\mathbf{S}$:

$$\mathbf{S}(\Omega, M, \mathcal{R}) \doteq \frac{\beta\mathcal{R}}{\beta\mathcal{R}+2} + \frac{2i\Omega}{(\beta\mathcal{R}+2)^2}\left[(\beta\mathcal{R})^2\left(1 - \frac{1}{2}\ln 2\pi\beta\mathcal{R} + \frac{\pi M}{2}\right) - 1\right] \tag{75}$$

In later work it will be convenient to refer to the real and imaginary components of this expression. Hence the following notation is introduced. Let

$$\left.\begin{aligned} &\mathcal{S} = \mathcal{S}' + i\mathcal{S}'' \\ &\text{where} \\ &\mathcal{S}' = \frac{\beta \mathcal{R}}{\beta \mathcal{R} + 2} \\ &\text{and} \\ &\frac{\mathcal{S}''}{\Omega} = \frac{2}{(\beta \mathcal{R} + 2)^2}\left[(\beta \mathcal{R})^2\left(1 - \frac{1}{2}\ln 2\pi\beta\mathcal{R} + \frac{\pi M}{2}\right) - 1\right] \end{aligned}\right\} \qquad (76)$$

It is worthwhile to pause at this point to give more careful consideration to the implications of this function $\mathcal{S}$, which is plotted in Fig. 2. $\mathcal{S}(\Omega, M, \mathcal{R})$ may be recognized as the three-dimensional analog of the function $\mathcal{C}(\Omega, M)$ which is the compressible version of Theodorsen's function $C(k)$. As a matter of fact for zero Mach number $\mathcal{C}(\Omega, M)$ represents the first few terms of the series expansion of Theodorsen's function in the neighborhood of the origin. It should be noted that, just as in the incompressible case, the imaginary part of $\mathcal{C}(\Omega, M)$ has a logarithmic singularity in its slope at zero frequency. The same is not true, however, for the finite span function $\mathcal{S}$ which is, fortunately, linear in frequency within the limitations of the assumptions in its derivation. It is this conclusion which justifies the work of this thesis. If the logarithmic term had not cancelled out, the assumption of the existence of a first order in frequency theory would have lacked the desired mathematical rigor in the same sense as Goland's polynomial approximation in the two-dimensional incompressible problem which was discussed in the introduction to this thesis.

From the expression which has been derived for the pressure on the upper surface of the airfoil in eqn. (71), the differential pressure distribution may be written:

$$\pi(x,Y,t) \doteq -\frac{2\rho_0 U e^{i\omega t}}{\pi \ell} e^{iM^2\Omega x}\left\{\oint_{-1}^{1} \overline{W}(\xi,Y)\left[\sqrt{\frac{1-X}{1+X}}\sqrt{\frac{1+\xi}{1-\xi}}\,\frac{1}{X-\xi} - i\Omega\Lambda_1 + \sqrt{\frac{1-X}{1+X}}\sqrt{\frac{1+\xi}{1-\xi}} - \sqrt{\frac{1-X}{1+X}}\sqrt{\frac{1+\xi}{1-\xi}}\, S(\Omega,M,R)\right]d\xi\right\} \qquad (77)$$

or by introducing the definition of $\overline{W}$ given by eqn. (13) and by considering airfoil motions which are independent of spanwise position, it may be shown that

$$\pi(X,t) \doteq -\frac{2\rho_0 U e^{i\omega t}}{\pi\beta} e^{iM^2\Omega x}\left\{\oint_{-1}^{1} \overline{w}(\xi) e^{-iM^2\Omega\xi}\left[\sqrt{\frac{1-X}{1+X}}\sqrt{\frac{1+\xi}{1-\xi}}\,\frac{1}{X-\xi} - \frac{i\Omega}{2}\ln\left(\frac{1-X\xi+\sqrt{1-\xi^2}\sqrt{1-X^2}}{1-X\xi-\sqrt{1-\xi^2}\sqrt{1-X^2}}\right) + \sqrt{\frac{1-X}{1+X}}\sqrt{\frac{1+\xi}{1-\xi}} - S\sqrt{\frac{1-X}{1+X}}\sqrt{\frac{1+\xi}{1-\xi}}\right]d\xi\right\} \qquad (78)$$

Upon introducing the transformations $X = -\cos\theta$, $\xi = -\cos\theta_1$, and after some trigonometric manipulations, eqn. (78) may be written

$$\pi(\theta,t) \doteq -\frac{2\rho_0 U e^{i\omega t}}{\pi\beta} e^{-iM^2\Omega\cos\theta} \int_0^{\pi} \bar{w}(\theta_1) e^{iM^2\Omega\cos\theta_1} \Bigg[\frac{\sin^2\theta_1}{\sin\theta(\cos\theta_1 - \cos\theta)}$$

$$+ \frac{i\Omega}{2} \sin\theta_1 \ln\left(\frac{1-\cos(\theta-\theta_1)}{1-\cos(\theta+\theta_1)}\right)$$

$$+ \left(\cot\theta - S\cot\frac{\theta}{2}\right)(1-\cos\theta_1)\Bigg] d\theta_1 \qquad (79)$$

For purposes of practical application, it is convenient to express the downwash function $\bar{w}(\theta)$ in a Fourier series of the form

$$\bar{w}(\theta) = U e^{-iM^2\Omega\cos\theta}\left(P_0 + 2\sum_{n=1}^{\infty} P_n \cos n\theta\right) \qquad (80)$$

where

$$P_0 = \frac{1}{\pi U}\int_0^{\pi} e^{iM^2\Omega\cos\theta_1}\, \bar{w}(\theta_1)\, d\theta_1 \qquad (81)$$

and

$$P_n = \frac{1}{\pi U}\int_0^{\pi} e^{iM^2\Omega\cos\theta_1}\, \bar{w}(\theta_1)\cos n\theta_1\, d\theta_1 \qquad (82)$$

Then by application of the generalized Parseval formula as discussed by Greidanus and van Heemert (16), eqn. (79) for the differential pressure distribution may be expressed in terms of the Fourier coefficients of the downwash. The process is described briefly in Appendix G for the sake of completeness, but for the mathematical details the reader is referred to the above mentioned reference.

Upon the introduction of the results presented in Appendix G into eqn. (79), the following expression may finally be obtained for the differential pressure distribution in terms of the Fourier coefficients of the prescribed downwash on the airfoil.

$$\pi(\theta,t) \doteq \frac{2\rho_0 U^2 e^{i\omega t}}{\beta} e^{-iM^2\Omega\cos\theta} \Big\{ \Big[S(P_0 - P_1) + P_1 \Big] \cot\frac{\theta}{2} + 2\sum_{n=1}^{\infty} \Big[\frac{i\Omega}{2n}(P_{n-1} - P_{n+1}) - P_n \Big] \sin n\theta \Big\} \qquad (83)$$

E. Forces and Moments

Expressions for the lift and moments may now be derived in terms of the Fourier coefficients of the downwash. The total lift force acting on the airfoil is given by

$$L = \frac{S}{2}\int_0^{\pi} \pi(\theta,t)\sin\theta\, d\theta$$

where S is the area of the wing. Substituting eqn. (83) into this integral for the lift results in

$$L = \frac{\rho_0 U^2 S e^{i\omega t}}{\beta}\int_0^{\pi}\Bigg\{\Big[S(P_0 - P_1) + P_1\Big]\cot\frac{\theta}{2} + 2\sum_{n=1}^{\infty}\Big[\frac{i\Omega}{2n}(P_{n-1} - P_{n+1}) - P_n\Big]\sin n\theta\Bigg\} e^{-iM^2\Omega\cos\theta}\sin\theta\, d\theta \qquad (84)$$

In the following work it is convenient to use a notation introduced by Timman and van de Vooren (13) where

$$R_n^*(M^2\Omega, \varphi) = \int_{\varphi}^{\pi} e^{-iM^2\Omega\cos\theta}\cos n\theta\, d\theta \qquad (85)$$

Then

$$\int_0^{\pi} e^{-iM^2\Omega\cos\theta}\cot\frac{\theta}{2}\sin\theta\, d\theta = \int_0^{\pi} e^{-iM^2\Omega\cos\theta}(1+\cos\theta)\, d\theta$$

$$= R_0^*(M^2\Omega, 0) + R_1^*(M^2\Omega, 0) \qquad (86)$$

and

$$\int_0^{\pi} e^{-iM^2\Omega\cos\theta} \sin n\theta \sin\theta d\theta = \int_0^{\pi} e^{-iM^2\Omega\cos\theta}\left[\frac{1}{2}\cos(n-1)\theta - \frac{1}{2}\cos(n+1)\theta\right]d\theta$$

$$= \frac{1}{2}\left[R^*_{n-1}(M^2\Omega,0) - R^*_{n+1}(M^2\Omega,0)\right] \tag{87}$$

so that, by using eqns. (86) and (87), eqn. (84) for the lift may be written

$$L = \frac{\rho_0 U^2 S e^{i\omega t}}{\beta}\left\{\left[S(P_0-P_1)+P_1\right]\left[R^*_0(M^2\Omega,0)+R^*_1(M^2\Omega,0)\right] + \sum_{n=1}^{\infty}\left[\frac{i\Omega}{2n}(P_{n-1}-P_{n+1})-P_n\right]\left[R^*_{n-1}(M^2\Omega,0)-R^*_{n+1}(M^2\Omega,0)\right]\right\} \tag{88}$$

Similarly the moment about the midchord of the airfoil is

$$M = -\frac{S\ell}{2}\int_0^{\pi} \pi(\theta,t)\cos\theta \sin\theta d\theta$$

$$= \frac{\rho_0 U^2 S\ell e^{i\omega t}}{\beta}\int_0^{\pi}\left\{\left[S(P_0-P_1)+P_1\right]\cot\frac{\theta}{2} + 2\sum_{n=1}^{\infty}\left[\frac{i\Omega}{2n}(P_{n-1}-P_{n+1})-P_n\right]\sin n\theta\right\} e^{-iM^2\Omega\cos\theta}\cos\theta\sin\theta d\theta \tag{89}$$

and after evaluating the integrals as before in terms of the function defined in eqn. (85), this may be written finally

$$\mathcal{M} = \frac{\rho_0 U^2 S \ell e^{i\omega t}}{\beta}\Bigg\{\Big[S(\rho_0 - \rho_1) + \rho_1\Big]\Big[R_1^*(M^2\Omega, 0) + \frac{1}{2}R_2^*(M^2\Omega, 0) + \frac{1}{2}R_0^*(M^2\Omega, 0)\Big] + \frac{1}{2}\sum_{n=1}^{\infty}\Big[\frac{i\Omega}{2n}(\rho_{n-1} - \rho_{n+1}) - \rho_n\Big]\Big[R_{n-2}^*(M^2\Omega, 0) - R_{n+2}^*(M^2\Omega, 0)\Big]\Bigg\} \tag{90}$$

The final aerodynamic quantity to be considered is the moment of a control surface about its hinge line at $\theta = \varphi$. This moment may be written

$$\mathcal{N} = \frac{S\ell}{2}\int_{\varphi}^{\pi} \pi(\theta, t)(\cos\varphi - \cos\theta)\sin\theta\, d\theta$$

$$= -\frac{\rho_0 U^2 S\ell e^{i\omega t}}{\beta}\int_{\varphi}^{\pi}\Bigg\{\Big[S(\rho_0 - \rho_1) + \rho_1\Big]\cot\frac{\theta}{2} + 2\sum_{n=1}^{\infty}\Big[\frac{i\Omega}{2n}(\rho_{n-1} - \rho_{n+1}) - \rho_n\Big]\sin n\theta\Bigg\} e^{-iM^2\Omega\cos\theta}(\cos\varphi - \cos\theta)\sin\theta\, d\theta \tag{91}$$

or finally

$$\mathcal{N} = -\frac{\rho_0 U^2 S\ell e^{i\omega t}}{\beta}\Bigg\langle\Big[S(\rho_0 - \rho_1) + \rho_1\Big]\Big\{\Big[R_0^*(M^2\Omega, \varphi) + R_1^*(M^2\Omega, \varphi)\Big]\cos\varphi - R_1^*(M^2\Omega, \varphi) - \frac{1}{2}R_0^*(M^2\Omega, \varphi) - \frac{1}{2}R_2^*(M^2\Omega, \varphi)\Big\} + \sum_{n=1}^{\infty}\Big[\frac{i\Omega}{2n}(\rho_{n-1} - \rho_{n+1}) - \rho_n\Big]\Big\{\Big[R_{n-1}^*(M^2\Omega, \varphi) - R_{n+1}^*(M^2\Omega, \varphi)\Big]\cos\varphi - \frac{1}{2}\Big[R_{n-2}^*(M^2\Omega, \varphi) - R_{n+2}^*(M^2\Omega, \varphi)\Big]\Big\}\Bigg\rangle \tag{92}$$

Consistent with the order of accuracy considered in this thesis, the following approximations to the functions $R_n^*(M^2\Omega,\varphi)$ will be made which are valid to first order in frequency.

$$\left.\begin{aligned}
R_0^*(M^2\Omega,\varphi) &\doteq \pi-\varphi+iM^2\Omega\sin\varphi \\
R_1^*(M^2\Omega,\varphi) &\doteq -\sin\varphi - iM^2\Omega\left[\frac{1}{2}(\pi-\varphi)-\frac{1}{4}\sin 2\varphi\right] \\
R_n^*(M^2\Omega,\varphi) &\doteq -\frac{\sin n\varphi}{n}+iM^2\Omega\left[\frac{\sin(n-1)\varphi}{2(n-1)} + \frac{\sin(n+1)\varphi}{2(n+1)}\right];\ n\geq 2
\end{aligned}\right\} \quad (93)$$

and for $\varphi=0$

$$\left.\begin{aligned}
R_0^*(M^2\Omega,0) &\doteq \pi \\
R_1^*(M^2\Omega,0) &\doteq -iM^2\Omega\frac{\pi}{2} \\
R_n^*(M^2\Omega,0) &\doteq 0\ ;\ n\geq 2
\end{aligned}\right\} \quad (94)$$

F. Fourier Coefficients of the Normal Velocity Distribution for Particular Airfoil Motions

For an airfoil which is undergoing uniform vertical translation and a rotation about its midchord

$$H = e^{i\omega t}(A\ell + Bx) \tag{95}$$

and so from eqn. (11) the normal velocity on the surface is

$$w = \frac{\partial H}{\partial t} + U\frac{\partial H}{\partial x} = Ue^{i\omega t}\left[ik(A - B\cos\theta) + B\right] \tag{96}$$

By defining another function similar to that of eqn. (85) such that

$$R_n(M^2\Omega, \varphi) = \int_{\varphi}^{\pi} e^{iM^2\Omega\cos\theta_1}\cos n\theta_1\, d\theta_1 \tag{97}$$

where to first order in frequency

$$\left.\begin{aligned}
R_0(M^2\Omega,\varphi) &\doteq \pi - \varphi - iM^2\Omega\sin\varphi \\
R_1(M^2\Omega,\varphi) &\doteq -\sin\varphi + iM^2\Omega\left(\frac{\pi-\varphi}{2} - \frac{1}{4}\sin 2\varphi\right) \\
R_n(M^2\Omega,\varphi) &\doteq -\frac{\sin n\varphi}{n} - iM^2\Omega\left[\frac{\sin(n+1)\varphi}{2(n+1)}\right. \\
&\qquad \left. + \frac{\sin(n-1)\varphi}{2(n-1)}\right]; \; n \geq 2
\end{aligned}\right\} \tag{98}$$

and by substituting eqn. (96) into eqns. (81) and (82), it may be established that

$$P_o = \frac{1}{\pi}\left[(ikA+B)R_o(M^2\Omega,0) - ikBR_1(M^2\Omega,0)\right] \tag{99}$$

and

$$P_n = \frac{1}{\pi}\left\{(ikA+B)R_n(M^2\Omega,0) - \frac{ik}{2}B\left[R_{n+1}(M^2\Omega,0) + R_{n-1}(M^2\Omega,0)\right]\right\} \; ; \; n \geq 1 \tag{100}$$

For rotation of a full-span control surface with hinge position φ at its leading edge

$$\left.\begin{aligned} H &= 0 \quad \text{for} \quad 0 \leq \theta \leq \varphi \\ H &= C\ell(\cos\varphi - \cos\theta)e^{i\omega t} \quad \text{for} \quad \varphi \leq \theta \leq \pi \end{aligned}\right\} \tag{101}$$

and therefore

$$\left.\begin{aligned} w &= 0 \quad \text{for} \quad 0 \leq \theta \leq \varphi \\ w &= UC\left[1 + ik(\cos\varphi - \cos\theta)\right]e^{i\omega t} \quad \text{for} \quad \varphi \leq \theta \leq \pi \end{aligned}\right\} \tag{102}$$

The relation between φ and the hinge position in terms of percent of chord is shown in Fig. 3.

The Fourier coefficients corresponding to this motion are

$$P_0 = \frac{C}{\pi}\Big[R_0(M^2\Omega,\varphi) + ik\cos\varphi\, R_0(M^2\Omega,\varphi) - ikR_1(M^2\Omega,\varphi)\Big] \tag{103}$$

and

$$\begin{aligned} P_n = \frac{C}{\pi}\Big\{ & R_n(M^2\Omega,\varphi) + ik\cos\varphi\, R_n(M^2\Omega,\varphi) \\ & - \frac{ik}{2}\Big[R_{n+1}(M^2\Omega,\varphi) + R_{n-1}(M^2\Omega,\varphi)\Big]\Big\} \; ; \; n \geq 1 \end{aligned} \tag{104}$$

In summary, then, the Fourier coefficients for combinations of all three types of motion, after taking into account the approximate expressions of eqn. (98), are to first order in frequency

$$\left.\begin{aligned} P_0 &\doteq ikA + B + \frac{C}{\pi}\Big\{\pi - \varphi + i\Omega\Big[(\pi-\varphi)(1-M^2)\cos\varphi \\ &\qquad - (1-2M^2)\sin\varphi\Big]\Big\} \\ P_1 &\doteq (ikA)\frac{iM^2\Omega}{2} - \frac{i\Omega}{2}B(1-2M^2) + \frac{C}{\pi}\Big\{-\sin\varphi \\ &\qquad + i\Omega\Big[-\frac{(\pi-\varphi)}{2}(1-2M^2) - \frac{1}{4}\sin 2\varphi\Big]\Big\} \\ &\text{and for } n \geq 2: \\ P_n &\doteq \frac{C}{\pi}\Big\{-\frac{\sin n\varphi}{n} + \frac{i\Omega}{2}\Big[\frac{[1-(n+1)M^2]}{n(n-1)}\sin(n-1)\varphi \\ &\qquad - \frac{[1+(n-1)M^2]}{n(n+1)}\sin(n+1)\varphi\Big]\Big\} \end{aligned}\right\} \tag{105}$$

G. Aerodynamic Derivatives

Let the total lift force be written in the form

$$L = -\pi \rho_0 U^2 S e^{i\omega t}\left[ikA(L_A' + iL_A'') + B(L_B' + iL_B'') + C(L_C' + iL_C'')\right] \qquad (106)$$

where the notation L_A', L_A'', etc. is the obvious symbolism first introduced in eqn. (76). Then after substituting eqns. (94) and (105) into eqn. (88), it may be shown that

$$L_A' = \frac{1}{\beta}(-S') \; ; \; \frac{L_A''}{\Omega} = \frac{1}{\beta}\left(-\frac{S''}{\Omega} - \frac{1}{2} + M^2 \frac{\beta AR}{\beta AR + 2}\right) \qquad (107)$$

$$L_B' = \frac{1}{\beta}(-S') \; ; \; \frac{L_B''}{\Omega} = \frac{1}{\beta}\left(-\frac{S''}{\Omega} - \frac{1}{2} + \frac{3M^2 - 1}{2} \; \frac{\beta AR}{\beta AR + 2}\right) \qquad (108)$$

and

$$L_C' = -\frac{1}{\beta}(\Phi_1 S') \; ; \; \frac{L_C''}{\Omega} = -\frac{1}{\beta}\left[\Phi_1 \frac{S''}{\Omega} + \frac{1}{2}\Phi_3 + \frac{1}{2}\Phi_2 \; \frac{\beta AR}{\beta AR + 2} + M^2\left(\frac{1}{2}\Phi_5 - \Phi_{20} \; \frac{\beta AR}{\beta AR + 2}\right)\right] \qquad (109)$$

Plots of these lift force derivatives versus aspect ratio for several Mach numbers are presented in Figs. 4 through 14.

Similarly, writing the total moment about the mid-chord in the form

$$\mathcal{M} = \pi \rho_0 U^2 S \ell e^{i\omega t}\Big[ikA(\mathcal{M}_A' + i\mathcal{M}_A'') + B(\mathcal{M}_B' + i\mathcal{M}_B'')$$

$$+ C(\mathcal{M}_C' + i\mathcal{M}_C'')\Big] \tag{110}$$

and substituting eqns. (94) and (105) into eqn. (90) results in

$$\mathcal{M}_A' = \frac{1}{\beta}\left(\frac{1}{2}S'\right) ; \frac{\mathcal{M}_A''}{\Omega} = \frac{1}{\beta}\left[\frac{1}{2}\frac{S''}{\Omega} + \frac{M^2}{4}\left(1 - 3\,\frac{\beta \mathit{AR}}{\beta \mathit{AR} + 2}\right)\right] \tag{111}$$

$$\mathcal{M}_B' = \frac{1}{\beta}\left(\frac{1}{2}S'\right) ; \frac{\mathcal{M}_B''}{\Omega} = \frac{1}{\beta}\left[\frac{1}{2}\frac{S''}{\Omega} - \frac{1}{4} + \frac{1}{2}M^2 + \left(\frac{1}{4} - M^2\right)\frac{\beta \mathit{AR}}{\beta \mathit{AR} + 2}\right] \tag{112}$$

$$\mathcal{M}_C' = \frac{1}{\beta}\left(\frac{1}{2}\Phi_1 S' - \frac{1}{2}\Phi_5\right) ; \frac{\mathcal{M}_C''}{\Omega} = \frac{1}{\beta}\Big[\frac{1}{2}\Phi_1\frac{S''}{\Omega} + \frac{1}{4}\Phi_{21}$$

$$+ \frac{1}{4}\Phi_2\,\frac{\beta \mathit{AR}}{\beta \mathit{AR} + 2} + M^2\Big(\Phi_{10}$$

$$- \frac{1}{2}\Phi_{12}\,\frac{\beta \mathit{AR}}{\beta \mathit{AR} + 2}\Big)\Big] \tag{113}$$

Plots of these moment derivatives versus aspect ratio for several Mach numbers are presented in Figs. 15 through 25.

Finally, by writing the total hinge moment in the form

$$\mathcal{N} = \pi \rho_0 U^2 S \ell e^{i\omega t}\Big[ikA(\mathcal{N}_A' + i\mathcal{N}_A'') + B(\mathcal{N}_B' + \mathcal{N}_B'')$$

$$+ C(\mathcal{N}_C' + i\mathcal{N}_C'')\Big] \tag{114}$$

and by substituting eqns. (93) and (105) into eqn. (92) then it may be shown that

$$N_A' = -\frac{1}{\beta}\left(\frac{1}{2}\Phi_8 S'\right); \quad \frac{N_A''}{\Omega} = -\frac{1}{\beta}\left[\frac{1}{2}\Phi_8 \frac{S''}{\Omega} + \frac{1}{4}\Phi_4 + \frac{M^2}{2}\left(\Phi_{13} + \Phi_{14}\frac{\beta A\!R}{\beta A\!R+2}\right)\right] \tag{115}$$

$$N_B' = -\frac{1}{\beta}\left(\frac{1}{2}\Phi_8 S'\right); \quad \frac{N_B''}{\Omega} = -\frac{1}{\beta}\left[\frac{1}{2}\Phi_8 \frac{S''}{\Omega} + \frac{1}{4}\Phi_9 + \frac{1}{4}\Phi_8\frac{\beta A\!R}{\beta A\!R+2} + M^2\left(\Phi_{15} + \Phi_{16}\frac{\beta A\!R}{\beta A\!R+2}\right)\right] \tag{116}$$

$$N_C' = -\frac{1}{\beta}\left(\frac{1}{2}\Phi_1\Phi_8 S' + \frac{1}{2}\Phi_{19}\right); \quad \frac{N_C''}{\Omega} = -\frac{1}{\beta}\left[\frac{1}{2}\Phi_1\Phi_8\frac{S''}{\Omega} + \frac{1}{4}\Phi_{11} + \frac{1}{4}\Phi_2\Phi_8\frac{\beta A\!R}{\beta A\!R+2} + M^2\left(\Phi_{17} + \Phi_{18}\frac{\beta A\!R}{\beta A\!R+2}\right)\right] \tag{117}$$

Plots of these hinge moment derivatives versus aspect ratio for several Mach numbers and hinge positions are presented in Figs. 26 through 42.

The hinge position functions Φ_j are defined as follows:

$$
\left.
\begin{aligned}
\Phi_1 &= \frac{1}{\pi}(\pi - \varphi + \sin\varphi) \\
\Phi_2 &= \frac{1}{\pi}\left[\pi - \varphi + 2(\pi-\varphi)\cos\varphi + 2\sin\varphi + \frac{1}{2}\sin 2\varphi\right] \\
\Phi_3 &= \frac{1}{\pi}\left(\pi - \varphi + \frac{1}{2}\sin 2\varphi\right) \\
\Phi_4 &= \frac{1}{\pi}\left[2(\pi-\varphi)\cos\varphi + \frac{3}{2}\sin\varphi + \frac{1}{6}\sin 3\varphi\right] \\
\Phi_5 &= \frac{1}{\pi}\left(\sin\varphi - \frac{1}{2}\sin 2\varphi\right) \\
\Phi_6 &= \frac{1}{\pi}\left[2(\pi-\varphi) + \sin\varphi + \sin 2\varphi - \frac{1}{3}\sin 3\varphi\right] \\
\Phi_7 &= \frac{1}{\pi}\left(\frac{1}{2}\sin\varphi + \frac{1}{2}\sin 2\varphi\right) \\
\Phi_8 &= (\Phi_2 - 2\Phi_3) \\
\Phi_9 &= \frac{1}{\pi}\left[\pi - \varphi + 2(\pi-\varphi)\cos\varphi + \sin\varphi \right.\\
&\qquad \left. + \frac{1}{2}\sin 2\varphi + \frac{1}{3}\sin 3\varphi\right] \\
\Phi_{10} &= \frac{1}{\pi}\left[\frac{1}{2}(\pi-\varphi) + \frac{5}{8}\sin\varphi - \frac{1}{24}\sin 3\varphi\right] \\
\Phi_{11} &= \frac{1}{\pi^2}\left[\frac{1}{8} + (\pi-\varphi)^2 + 2(\pi-\varphi)^2\cos\varphi \right.\\
&\qquad + \frac{5}{2}(\pi-\varphi)\sin\varphi + (\pi-\varphi)\sin 2\varphi \\
&\qquad + \frac{1}{2}(\pi-\varphi)\sin 3\varphi + \frac{1}{2}\cos\varphi \\
&\qquad \left. - \frac{1}{2}\cos 3\varphi - \frac{1}{8}\cos 4\varphi\right] \\
\Phi_{12} &= \frac{1}{\pi}\left[2(\pi-\varphi) + (\pi-\varphi)\cos\varphi + 3\sin\varphi\right] \\
\Phi_{13} &= \frac{1}{\pi}\left[-\frac{1}{2}(\pi-\varphi) + \frac{3}{4}\sin\varphi - \frac{1}{4}\sin 2\varphi \right.\\
&\qquad \left. - \frac{1}{12}\sin 3\varphi\right]
\end{aligned}
\right\} \quad (118)
$$

$$\left.\begin{aligned}
\Phi_{14} &= \frac{1}{\pi}\Big[\frac{3}{2}(\pi-\varphi) - 2(\pi-\varphi)\cos\varphi - \frac{9}{4}\sin\varphi \\
&\qquad + \frac{3}{4}\sin 2\varphi + \frac{1}{12}\sin 3\varphi\Big] \\
\Phi_{15} &= \left(\Phi_{13} - \frac{1}{2\pi}\sin\varphi\right) \\
\Phi_{16} &= \frac{1}{\pi}\Big[\pi-\varphi - \frac{3}{2}(\pi-\varphi)\cos\varphi - \frac{13}{8}\sin\varphi \\
&\qquad + \frac{1}{2}\sin 2\varphi + \frac{1}{24}\sin 3\varphi\Big] \\
\Phi_{17} &= \frac{1}{\pi^2}\Big[\frac{5}{16} - \frac{1}{2}(\pi-\varphi)^2 - \frac{1}{2}(\pi-\varphi)\sin\varphi \\
&\qquad - \frac{1}{2}(\pi-\varphi)\sin 2\varphi - \frac{1}{6}(\pi-\varphi)\sin 3\varphi \\
&\qquad - \frac{1}{4}\cos\varphi - \frac{1}{3}\cos 2\varphi + \frac{1}{4}\cos 3\varphi + \frac{1}{48}\cos 4\varphi\Big] \\
\Phi_{18} &= \frac{1}{\pi^2}\Big[-\frac{21}{16} + \frac{1}{2}(\pi-\varphi)^2 - (\pi-\varphi)^2\cos\varphi \\
&\qquad - \frac{1}{2}(\pi-\varphi)^2\cos 2\varphi - \frac{5}{4}(\pi-\varphi)\sin 2\varphi \\
&\qquad + \frac{1}{6}(\pi-\varphi)\sin 3\varphi + \frac{3}{8}\cos\varphi + \frac{4}{3}\cos 2\varphi \\
&\qquad - \frac{3}{8}\cos 3\varphi - \frac{1}{48}\cos 4\varphi\Big] \\
\Phi_{19} &= \frac{1}{\pi}(\pi-\varphi-\sin\varphi)\Phi_5 \\
\Phi_{20} &= \left(\Phi_1 + \frac{1}{2}\Phi_2 + \Phi_7\right) \\
\Phi_{21} &= \left(\Phi_3 - \Phi_6\right)
\end{aligned}\right\} \begin{array}{l}(118)\\ \text{cont.}\end{array}$$

Similar expressions have been tabulated by Küssner and Schwarz (24) and by Theodorsen and Garrick (25).

It is interesting to note where these results fit in with respect to some of the other theories proposed.

Comparisons can be made in three limiting cases. First is that of zero frequency in which case the results of this thesis reduce to precisely the same as are obtained by using Prandtl's lifting-line theory. Second is a comparison with previous results for the two-dimensional compressible case, to which an accurate solution has been obtained by Dietze (6). Radok (26) has obtained an approximate solution to this problem in which he supposedly consistently retained all terms of order less than frequency squared. His result is incorrect since, by erroneously starting with a Laplace equation rather than the wave equation, he loses a term which is a function of Mach number and first order in frequency. In the plots of Fig. 43, Radok's notation has been retained for ease of comparison. Shown in Fig. 43 are the real and imaginary components of two-dimensional lift force due to translational displacement as a function of reduced frequency. Also plotted are the exact results of Dietze for Mach numbers of 0, 0.5 and 0.7. Radok's and the results of this thesis are exactly the same for the imaginary component in this notation and agree very well with Dietze's theory. A very definite discrepancy exists, however, in the values of the real component and for all Mach numbers the results of this thesis are in better agreement with the exact theory than are those of Radok. It is also interesting to note where quasi-steady theory fits into this picture. The real part in this notation is taken to be zero for all frequencies and Mach numbers. For the imaginary part a straight line is assumed which is tangent to the exact curve at zero frequency for each Mach number.

The third comparison which can be made is between the results of this thesis for zero Mach number and the three-dimensional incompressible results of Reissner (20). Plotted in Fig. 44 are the real and imaginary components of $\mathcal{S}$ for the case of zero Mach number in comparison with Reissner's more exact theory for aspect ratios of 1.5, 2, 3, 6 and infinity. Note that the agreement is very good particularly for the finite aspect ratios.

WING WASH EFFECTS IN THE VICINITY OF THE HORIZONTAL TAIL

The tail behaves in a manner similar to the wing except that, in addition to considering the actual motion of the tail surface, it is necessary to recognize that it is in an oscillating stream due to the wing wake. Cowley and Glauert (27) made the first attempt to account for this unsteady flow condition by approximating the downwash lag by the time required for the flow to travel from the wing to the tail. This effect depends on the translational acceleration dw/dt and therefore appears as acceleration derivatives $M_{\dot{w}}$ and $Z_{\dot{w}}$ in the quasi-steady theory. It is important to note, however, that they arise from a different cause than that usually associated with the acceleration derivatives obtained from unsteady flow theory which are due solely to apparent mass effects.

Unfortunately, as noted by the author (18), the phase lag of the downwash is the most important unsteady flow effect in the dynamic stability analysis of conventional-type aircraft. It would therefore be worthwhile to obtain a better approximation than can be expected from the direct time delay of Cowley and Glauert. Moreover, their method only takes account of the lag due to an angle-of-attack change of the wing, whereas any net change in lift such as might be produced by a pure pitching motion with $\alpha = 0$

must affect the downwash in the wing's wake.

In this section an expression for the downwash in the wake of an oscillating wing will be derived which is comparable to the theory for the airfoil.

With some small changes with respect to the integrals to be considered as principal valued, an expression for the downwash at any point in the plane of the wing has already been derived in eqn. (28). For present purposes this may be written

$$
\begin{aligned}
\overline{W}(X,Y,0) \doteq {} & \frac{1}{\pi}\oint_{X_L}^{\infty} \frac{\partial \phi(\xi,Y,0)}{\partial \xi} K\left[(X-\xi);f\right] d\xi \\
& + \frac{1}{2\pi}\iint_{WING} \frac{\partial^2 \phi(\xi,\eta,0)}{\partial \xi \partial \eta} G\left[(X-\xi),(Y-\eta);f\right] d\xi d\eta \\
& - \frac{i\Omega}{4\pi}\oint\!\!\oint_{WAKE} \frac{d}{d\eta}\left[\overline{\Gamma}^{(3)}\right] e^{-i\Omega\xi} G\left[(X-\xi),(Y-\eta);f\right] d\xi d\eta \qquad (119)
\end{aligned}
$$

where K and G are given by eqns. (29) and (30), respectively. The reasoning which makes the principal-value interpretation of the pertinent integrations valid is the same as is used in the wing theory, and hence, no additional complications may be expected.

In the wing theory it was possible to simplify certain of the double integrations of eqn. (28) by resorting to a "lifting-strip" theory. The assumption that $|Y-\eta| \gg |X-\xi|$ over most of the wing seemed quite justifiable in that problem. In the present case, however, where the point of interest may lie several chord lengths downstream of the wing, this assumption is no longer valid. Rather it will be assumed that the tail length is large with respect to the chord length of the wing and, hence, the chordwise variation of the circulation around the wing has a negligible effect upon the downwash at the horizontal tail. In effect this assumes that the bound vorticity is concentrated along some line, say $X = X_a$, in the wing.

With this assumption, and by using eqns. (18) and (B-3), eqn. (119) may be written

$$\begin{aligned}\bar{w}(x,Y,0) \doteq & \frac{1}{\pi}\oint_{X_L}^{\infty}\frac{1}{X-\xi}\,\frac{\partial\bar{\phi}(\xi,Y,0)}{\partial\xi}\,d\xi - \frac{1}{\pi}\oint_{X_L}^{\infty}\frac{1}{X-\xi}\frac{\partial}{\partial\xi}\left\{\bar{\Phi}(\xi,Y,0)\left[1\right.\right.\\ & \left.\left. + \frac{i\pi f|X-\xi|}{2}H_1^{(2)}(f|X-\xi|)\right]\right\}d\xi\\ & + \frac{1}{4\pi}e^{-i\Omega X_T}\int\frac{d}{d\eta}\left[\bar{\Gamma}^{(3)}\right]G\left[(X-X_a),(Y-\eta);f\right]d\eta\\ & - \frac{i\Omega}{4\pi}\oint\!\!\oint_{WAKE}\frac{d}{d\eta}\left[\bar{\Gamma}^{(3)}\right]G\left[(X-\xi),(Y-\eta);f\right]d\xi\,d\eta \end{aligned} \tag{120}$$

The following approximation, consistent with the assumptions discussed above, will be made to the first term on the right hand side of eqn. (120):

$$\begin{aligned}\frac{1}{\pi}\oint_{X_L}^{\infty}\frac{1}{X-\xi}\,\frac{\partial\bar{\phi}(\xi,Y,0)}{\partial\xi}\,d\xi \doteq & \frac{1}{\pi(X-X_a)}\int_{X_L}^{X_T}\frac{\partial\bar{\phi}}{\partial\xi}\,d\xi\\ & + \frac{1}{\pi}\oint_{X_T}^{\infty}\frac{1}{X-\xi}\,\frac{\partial\bar{\phi}}{\partial\xi}\,d\xi\end{aligned}$$

So upon introducing eqns. (17) and (18), and by neglecting frequency-squared terms, this expression may be written

$$\frac{1}{\pi}\oint_{X_L}^{\infty}\frac{1}{X-\xi}\,\frac{\partial\bar{\Phi}(\xi,Y,0)}{\partial\xi}\,d\xi \doteq \frac{\bar{F}^{(3)}(Y)e^{-i\Omega X_T}}{2\pi(X-X_a)} - \frac{i\Omega\bar{F}^{(3)}(Y)e^{-i\Omega X}}{2\pi}\left[\frac{i\pi}{2}+\ln\gamma\Omega(X-X_T)\right] \qquad (121)$$

It is shown in Appendix H that the second term on the right-hand side of eqn. (120) may be written to first-order terms in frequency as follows:

$$-\frac{1}{\pi}\oint_{X_L}^{\infty}\frac{\partial}{\partial\xi}\left\{\bar{\Phi}(\xi,Y,0)\left[1+\frac{i\pi f|X-\xi|}{2}H_1^{(2)}(f|X-\xi|)\right]\right\}\frac{d\xi}{X-\xi} \doteq -\frac{i\Omega\bar{F}^{(3)}(Y)e^{-i\Omega X}}{2\pi}\,\mathcal{M} \qquad (122)$$

It is necessary now to approximate $G[(X-\xi),(Y-\eta);f]$ to first order in frequency. This has been accomplished previously. By introducing the results obtained in Appendix C, it may be established that

$$G[(X-X_a),(Y-\eta);f] \doteq \frac{\sqrt{(X-X_a)^2+(Y-\eta)^2}}{(X-X_a)(Y-\eta)}+\frac{1}{Y-\eta} - \frac{|Y-\eta|}{(Y-\eta)(X-X_a)}-\frac{i\pi f|Y-\eta|}{2(Y-\eta)} \qquad (123)$$

and that

$$G[(X-\xi),(Y-\eta);0]=\frac{\sqrt{(X-\xi)^2+(Y-\eta)^2}}{(X-\xi)(Y-\eta)}+\frac{1}{Y-\eta}-\frac{|Y-\eta|}{(Y-\eta)(X-\xi)} \qquad (124)$$

Now substitute eqn. (124) into the double integral over the wake region of eqn. (120) and consider the integration with respect to ξ.

$$\oint_{X_T}^{\infty} e^{-i\Omega\xi}\left[\frac{\sqrt{(X-\xi)^2+(Y-\eta)^2}}{(X-\xi)(Y-\eta)}+\frac{1}{Y-\eta}-\frac{|Y-\eta|}{(Y-\eta)(X-\xi)}\right]d\xi = \lim_{\epsilon\to 0}\left(\int_{X_T}^{X-\epsilon}\cdots+\int_{X+\epsilon}^{\infty}\cdots d\xi\right)$$

$$\doteq \lim_{\epsilon\to 0}\left\{\int_{\epsilon}^{X-X_T}\left[\frac{\sqrt{\lambda^2+(Y-\eta)^2}}{\lambda(Y-\eta)}+\frac{1}{Y-\eta}-\frac{|Y-\eta|}{\lambda(Y-\eta)}\right]d\lambda\right.$$
$$\left.+\int_{\epsilon}^{\infty} e^{-i\Omega X}e^{-i\Omega\lambda}\left[-\frac{\sqrt{\lambda^2+(Y-\eta)^2}}{\lambda(Y-\eta)}+\frac{1}{Y-\eta}+\frac{|Y-\eta|}{\lambda(Y-\eta)}\right]d\lambda\right\}$$

$$=\frac{|Y-\eta|}{Y-\eta}\ln\left[\frac{2|Y-\eta|}{|Y-\eta|+\sqrt{(X-X_T)^2+(Y-\eta)^2}}\right]+\frac{\sqrt{(X-X_T)^2+(Y-\eta)^2}}{Y-\eta}$$
$$+\frac{X-X_T}{Y-\eta}-\frac{|Y-\eta|}{Y-\eta}+\frac{|Y-\eta|}{Y-\eta}e^{-i\Omega X}F(\Omega|Y-\eta|) \tag{125}$$

where F $(\Omega|Y-\eta|)$ has been evaluated approximately in Appendix D as

$$F(\Omega|Y-\eta|)\doteq 1-\frac{i\pi}{2}-\ln 2\gamma\Omega|Y-\eta| \tag{126}$$

Upon introducing eqns. (121), (122), (123), (125), and (126) into eqn. (120), it may be shown that, to terms of order Ω ln Ω, eqn. (120) becomes

$$\overline{W}(X,Y,0)\doteq\frac{\overline{F}^{(3)}(Y)e^{-i\Omega X_T}}{2\pi(X-X_a)}-\frac{i\Omega\overline{F}^{(3)}(Y)e^{-i\Omega X}}{2\pi}\left[\frac{i\pi}{2}\right.$$
$$\left.+\ln\gamma\Omega(X-X_T)+M\right]+\frac{1}{4\pi}\oint\frac{d}{d\eta}\left[\overline{F}^{(3)}\right]\frac{|Y-\eta|}{Y-\eta}\left\{\frac{\sqrt{(X-X_a)^2+(Y-\eta)^2}}{(X-X_a)|Y-\eta|}\right.$$
$$+\frac{1}{|Y-\eta|}-\frac{1}{X-X_a}-i\Omega\left[-\frac{\pi M}{2}-\frac{X_T\sqrt{(X-X_a)^2+(Y-\eta)^2}}{(X-X_a)|Y-\eta|}\right.$$
$$-\frac{X}{|Y-\eta|}+\frac{X_T}{X-X_a}+\frac{i\pi}{2}+\ln\gamma\Omega-\frac{\sqrt{(X-X_T)^2+(Y-\eta)^2}}{|Y-\eta|}$$
$$\left.\left.+\ln\left(|Y-\eta|+\sqrt{(X-X_T)^2+(Y-\eta)^2}\right)\right]\right\}d\eta \tag{127}$$

For the evaluation of the downwash along the line $Y = 0$, i.e., in the plane of symmetry of the aircraft, eqn. (127) may be written:

$$\begin{aligned}\overline{W}(X,0,0) \doteq \; & \frac{\bar{\Gamma}^{(3)}(0)e^{-i\Omega X_T}}{2\pi(X-X_a)} - \frac{i\Omega\bar{\Gamma}^{(3)}(0)e^{-i\Omega X}}{2\pi}\Big[\frac{i\pi}{2} \\ & + \ln \gamma\Omega(X-X_T) + \eta\Big] + \frac{1}{4\pi}\oint_{-\beta s}^{\beta s}\frac{d}{d\eta}\Big[\bar{\Gamma}^{(3)}\Big]\frac{|\eta|}{\eta}\Big\{-\frac{1}{|\eta|} \\ & + \frac{1}{X-X_a} - \frac{\sqrt{(X-X_a)^2+\eta^2}}{(X-X_a)|\eta|} + i\Omega\Big[-\frac{i\pi}{2} + \frac{\pi M}{2} \\ & + \frac{X_T\sqrt{(X-X_a)^2+\eta^2}}{(X-X_a)|\eta|} + \frac{X}{|\eta|} + \frac{\sqrt{(X-X_T)^2+\eta^2}}{|\eta|} \\ & - \frac{X_T}{X-X_a} - \ln\gamma\Omega - \ln\Big(|\eta| + \sqrt{(X-X_T)^2+\eta^2}\Big)\Big]\Big\}d\eta \end{aligned} \tag{128}$$

Upon introducing the evaluation of the integral with respect to η as given in Appendix I into eqn. (128), and by retaining terms to first order in frequency, it may be shown that

$$\begin{aligned}\overline{W}(X,0,0) \doteq \; & \frac{\bar{\Gamma}^{(3)}(0)}{2\pi(X-X_a)} - \frac{i\Omega X_T\bar{\Gamma}^{(3)}(0)}{2\pi(X-X_a)} + \frac{\bar{\Gamma}^{(3)}(0)}{2\pi}\Big\langle -\frac{1}{X-X_a} \\ & + \frac{\pi}{2\beta s} + \frac{\sqrt{(X-X_a)^2+(\beta s)^2}}{(X-X_a)\beta s}E(k) + i\Omega\Big\{-\frac{\pi M}{2} \\ & - \frac{\pi X}{2\beta s} - \eta + \frac{\sqrt{(X-X_T)^2+(\beta s)^2}}{\beta s}\Big[K(k_1) - 2E(k_1)\Big] \\ & - \frac{X_T\sqrt{(X-X_a)^2+(\beta s)^2}}{(X-X_a)\beta s}E(k)\Big\}\Big\rangle \end{aligned} \tag{129}$$

where K (k) and E (k) are complete elliptic integrals of the first and second kind respectively and where k and k_1 are defined in Appendix I. Hence, with the substitution of $s = \frac{\pi}{4} \mathit{AR}$ as before:

$$\overline{W}(X,0,0) \doteq \frac{\mathit{A}^{(1)}(0)}{2\pi} \left\langle \frac{2}{\beta \mathit{AR}} + \frac{\sqrt{(X-X_a)^2+\left(\frac{\pi\beta \mathit{AR}}{4}\right)^2}}{(X-X_a)\,\frac{\pi\beta \mathit{AR}}{4}} E(k) \right.$$
$$+ i\Omega \left\{ -\frac{\pi M}{2} - \mathcal{M} - \frac{2X}{\beta \mathit{AR}} \right.$$
$$+ \frac{\sqrt{(X-X_T)^2+\left(\frac{\pi\beta \mathit{AR}}{4}\right)^2}}{\frac{\pi\beta \mathit{AR}}{4}} \left[K(k_1) - 2E(k_1) \right]$$
$$\left.\left. - \frac{X_T \sqrt{(X-X_a)^2+\left(\frac{\pi\beta \mathit{AR}}{4}\right)^2}}{(X-X_a)\,\frac{\pi\beta \mathit{AR}}{4}} E(k) \right\} \right\rangle \tag{130}$$

where

$$k = \sqrt{\frac{1}{\left(\frac{X-X_a}{\frac{\pi}{4}\beta \mathit{AR}}\right)^2+1}}$$

$$k_1 = \sqrt{\frac{1}{\left(\frac{X-X_T}{\frac{\pi}{4}\beta \mathit{AR}}\right)^2+1}}$$

It has previously been assumed that the tail length is large with respect to the chord length hence, also, that $X_T - X_a$ is small with respect to X. It will therefore be assumed that $X_T \doteq X_a$ which implies that as far as the effect upon the downwash in the vicinity of the horizontal tail is concerned, the wake region may be considered as starting at the position of the bound vortex rather than at the trailing edge of the wing. If, in addition, it is assumed that the frequency of the wake is sufficiently low so that the wave length of the stream is large with respect to the chord length of the horizontal tail, then the effect of the chordwise variation of downwash on the tail may be neglected. Let l_T be the distance from the forward quarter chord point of the wing (assumed position of the bound vortex) to the mid-chord point of the horizontal tail in the plane of symmetry of the aircraft. Then with $X - X_a = l_T/l$ and $X_a \doteq X_T \doteq -\frac{1}{2}$

$$\overline{W}(l_T,0,0) \doteq \frac{\overline{\Gamma}^{(3)}(0)}{2\pi}\Bigg\langle\frac{2}{\beta \mathcal{R}} + \frac{l}{l_T}\,\frac{E(k)}{k} + i\Omega\Big\{-\frac{\pi M}{2} - \mathfrak{M} - \frac{2l_T}{\beta \mathcal{R} l} + \frac{1}{\beta \mathcal{R}} + \frac{1}{k}\Big[K(k) - 2E(k) + \frac{1}{2}\frac{l}{l_T}E(k)\Big]\Big\}\Bigg\rangle \tag{131}$$

It has already been determined in eqn. (60) that

$$\overline{\Gamma}^{(3)}(0) \doteq \frac{\overline{\Gamma}^{(2)}(0)}{1-\mu(\Omega,M)\oint_0^{\pi} Q\cos\theta_1\,d\theta_1} \tag{60}$$

where by eqns. (54) and (13)

$$\overline{\Gamma}^{(2)} \doteq 2\frac{l}{\beta}\,\mathbf{C}(\Omega,M)\oint_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\;\overline{w}(\xi)\,e^{-iM^2\Omega\xi}\,d\xi \tag{132}$$

Upon substituting $\xi = -\cos\theta_1$, and by introducing eqns. (81) and (82), eqn. (132) may be written

$$\overline{\Gamma}^{(2)} \doteq 2\pi\frac{lU}{\beta}\,\mathbf{C}(\Omega,M)(P_0 - P_1) \tag{133}$$

By substituting eqns. (53), (55), (62), (133), and

$s = \frac{\pi}{4} A\!R$ into eqn. (60), the following expression may be derived for $\bar{\Gamma}^{(3)}(0)$ to first order in frequency:

$$\bar{\Gamma}^{(3)}(0) \doteq \frac{2\pi U \ell A\!R}{\beta A\!R + 2}\left(P_0 - P_1\right)\left[1 + i\Omega\left(\frac{\pi M \beta A\!R}{\beta A\!R + 2}\right.\right.$$

$$\left.\left. + \frac{2\beta A\!R + 1}{\beta A\!R + 2} - \frac{\beta A\!R}{\beta A\!R + 2} \ln 2\pi\beta A\!R\right)\right] \qquad (134)$$

The Fourier coefficients of the downwash have been evaluated to first-order frequency terms in eqn. (105) so that for uniform translation and rotation of the wing

$$P_0 - P_1 \doteq ikA\left(1 - \frac{iM^2\Omega}{2}\right) + B\left[1 + \frac{i\Omega}{2}(1 - 2M^2)\right] \qquad (135)$$

By introducing eqns. (134) and (135) into eqn. (131), by letting

$$\overline{W}(x,0,0) = \frac{\ell}{\beta} e^{-iM^2\Omega x}\, \bar{w}(x,0,0)$$

and by retaining terms only up to the first order in frequency, an expression for the downwash is obtained as follows:

$$\bar{w}(\ell_T, 0) \doteq \frac{U\beta A\!R}{\beta A\!R + 2}\left\langle (ikA + B)\left[\frac{2}{\beta A\!R} + \frac{\ell}{\ell_T}\frac{E(k)}{k}\right]\right.$$

$$+ i\Omega(ikA + B)\left\{-\frac{\pi M}{2} - m - \frac{2\ell_T}{\beta A\!R \ell} + \frac{1}{\beta A\!R} + \frac{1}{k}\left[K(k)\right.\right.$$

$$\left. - 2E(k) + \frac{1}{2}\frac{\ell}{\ell_T}E(k)\right] + \left[\frac{2}{\beta A\!R} + \frac{\ell}{\ell_T}\frac{E(k)}{k}\right]\left[\frac{2\beta A\!R + 1}{\beta A\!R + 2}\right.$$

$$\left.\left. + \frac{\pi M \beta A\!R}{\beta A\!R + 2} - \frac{\beta A\!R}{\beta A\!R + 2}\ln 2\pi\beta A\!R + \left(\frac{\ell_T}{\ell} - \frac{1}{2}\right)M^2\right]\right\}$$

$$\left. + i\Omega\left[-(ikA)\frac{M^2}{2} + \frac{B}{2}(1 - 2M^2)\right]\left[\frac{2}{\beta A\!R} + \frac{\ell}{\ell_T}\frac{E(k)}{k}\right]\right\rangle \qquad (136)$$

If eqn. (136) is written in the form

$$w(\ell_T,0,0) = U e^{i\omega t}\left[ikA\left(w_A' + i w_A''\right) + B\left(w_B' + i w_B''\right)\right] \tag{137}$$

then

$$\left.\begin{aligned}
w_A' &\doteq \frac{\beta A\!\!R}{\beta A\!\!R+2}\left[\frac{2}{\beta A\!\!R} + \frac{\ell}{\ell_T}\frac{E(k)}{k}\right] \\
\frac{w_A''}{\Omega} &\doteq \frac{\beta A\!\!R}{\beta A\!\!R+2}\left\{-m - \frac{\pi M}{2} - \frac{2\ell_T}{\beta A\!\!R \ell} + \frac{1}{\beta A\!\!R} + \frac{1}{k}\left[K(k)\right.\right. \\
&\quad \left. -2E(k) + \frac{1}{2}\frac{\ell}{\ell_T}E(k)\right] + \left[\frac{2}{\beta A\!\!R} + \frac{\ell}{\ell_T}\frac{E(k)}{k}\right]\left[\frac{\pi M \beta A\!\!R}{\beta A\!\!R+2}\right. \\
&\quad \left.\left. + \frac{2\beta A\!\!R+1}{\beta A\!\!R+2} - \frac{\beta A\!\!R}{\beta A\!\!R+2}\ln 2\pi\beta A\!\!R + M^2\left(\frac{\ell_T}{\ell} - 1\right)\right]\right\}
\end{aligned}\right\} \tag{138}$$

$$\left.\begin{aligned}
w_B' &\doteq w_A' \\
\frac{w_B''}{\Omega} &\doteq \frac{w_A''}{\Omega} + \left(\frac{\beta A\!\!R}{\beta A\!\!R+2}\right)\left(\frac{\beta^2}{2}\right)\left[\frac{2}{\beta A\!\!R} + \frac{\ell}{\ell_T}\frac{E(k)}{k}\right]
\end{aligned}\right\} \tag{139}$$

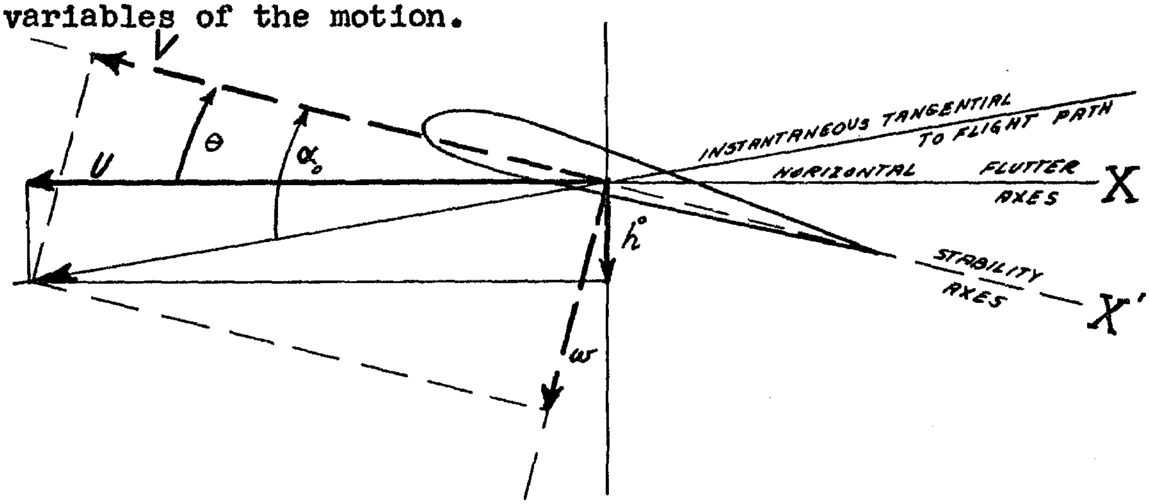
Plots of these downwash derivatives versus aspect ratio for several Mach numbers and tail lengths are presented in Figs. 45 through 56.

It is worthwhile to note that in the steady case these expressions agree with results presented in Durand (28) for the downwash in the vicinity of a horizontal tail.

STABILITY DERIVATIVES

In estimating the aerodynamic forces in the preceding sections, it has been convenient to use an axes system which is parallel and perpendicular to the average motion of the airfoil. Hence, the X axis is always horizontal and the wing rotates about a point on this axis through the angle B or the point moves downward a distance A1 from this horizontal reference axis. This axis system shall be referred to as "flutter axes" and is the system used by Theodorsen where A1 and B of this report correspond to Theodorsen's $\bar{h}$ and $\bar{\alpha}$ respectively.

In dynamic stability analyses, however, it is more convenient to use the stability or fixed wind axes which are fixed with respect to the airfoil with the origin at the center of rotation and the X'-axis always directed along the wing chord. In this system the velocities w and $q = d\theta/dt$ rather than the displacements are the dependent variables of the motion.



By referring to the above sketch the relations among the respective velocities of the two systems are seen to be

$$\left.\begin{aligned} \mathring{h} &= w\cos\theta - V\sin\theta \\ &\text{and} \\ U &= V\cos\theta + w\sin\theta \end{aligned}\right\} \quad (140)$$

U , h , and θ are precisely the U , A1 and B used in the preceding sections and V and w are velocities along the stability axes. α_0 is the ordinary instantaneous angle of attack equal to w/V .

If θ is small, $w \ll V$ and $\dot{h} \ll U$ then eqn. (140) may be approximated by

$$\left.\begin{aligned} \mathring{h} &\doteq w - V\theta \\ &\text{and} \\ U &\doteq V \end{aligned}\right\} \quad (141)$$

To the same order of approximation the difference between the lift force perpendicular to the average wind direction and the lift force referred to the stability axes is negligible. Also the moments are unaffected by the transformation and so, to define the forces and moments in terms of w and θ of the stability axes system, it is only necessary to replace $i\omega \ell A$, B and C by $\bar{w} - U\bar{\theta}$, $\bar{\theta}$ and $\bar{\delta}$, respectively, in eqns. (106), (110), and (114).

For an airfoil which is undergoing uniform translation and rotation about the point x = bl, the negative lift force may be obtained from eqn. (106) as

$$\begin{aligned}\bar{Z} &= \pi\rho_0 U^2 S\left[ik(A-Bb)L_A + BL_B\right] \\ &= \pi\rho_0 U^2 S\left[\frac{i\omega l}{U}AL_A + B\left(L_B - \frac{i\omega l}{U}bL_A\right)\right] \\ &= \pi\rho_0 U^2 S\left[\frac{\bar{w}}{U}L_A + \bar{\theta}\left(L_B - L_A - \frac{i\omega l b}{U}L_A\right)\right]\end{aligned} \tag{142}$$

If the wing characteristics are denoted by the subscript $(\ )_W$ and if it is recalled that only undamped oscillations are being considered so that $\dot{w} = i\omega w$ and $\dot{\theta} = i\omega\theta$, then eqn. (142) may be written

$$Z_W = \frac{1}{2}\rho_0 U^2 S_W\left[\frac{w}{U}Z_{W\left(\frac{w}{U}\right)} + \frac{\dot{w}l}{U^2}Z_{W\left(\frac{\dot{w}l}{U^2}\right)} + \frac{\dot{\theta}l}{U}Z_{W\left(\frac{\dot{\theta}l}{U}\right)}\right] \tag{143}$$

where

$$\left.\begin{aligned} Z_{W\left(\frac{w}{U}\right)} &= 2\pi L'_{A_W} \\ Z_{W\left(\frac{\dot{w}l}{U^2}\right)} &= \frac{2\pi}{\beta^2}\left(\frac{L_A''}{\Omega}\right)_W \\ Z_{W\left(\frac{\dot{\theta}l}{U}\right)} &= \frac{2\pi}{\beta^2}\left[\left(\frac{L_B''}{\Omega}\right)_W - \left(\frac{L_A''}{\Omega}\right)_W - \beta^2 b L'_{A_W}\right]\end{aligned}\right\} \tag{144}$$

Similarly the moment about the point of rotation x = bl for an airfoil undergoing uniform translation and rotation may be obtained from eqn. (110) in the form

$$\bar{M} = \pi\rho_0 U^2 \ell S\left[ik(A - Bb)M_A + BM_B\right] - \bar{Z}b\ell$$

$$= \pi\rho_0 U^2 \ell S\left\{\frac{\bar{w}}{U}(M_A - bL_A) + \bar{\theta}\left[M_B - M_A - b(L_B - L_A) - \frac{i\omega\ell}{U}b(M_A - bL_A)\right]\right\} \tag{145}$$

Hence, for the wing, eqn. (145) may be written

$$M_W = \frac{1}{4}\rho_0 U^2 c_W S_W\left[\frac{w}{U}M_{W_{\left(\frac{w}{U}\right)}} + \frac{\dot{w}\ell}{U^2}M_{W_{\left(\frac{\dot{w}\ell}{U^2}\right)}} + \frac{\dot{\theta}\ell}{U}M_{W_{\left(\frac{\dot{\theta}\ell}{U}\right)}}\right] \tag{146}$$

where

$$\left.\begin{aligned}
M_{W_{\left(\frac{w}{U}\right)}} &= 2\pi(M'_{A_W} - bL'_{A_W}) \\
M_{W_{\left(\frac{\dot{w}\ell}{U^2}\right)}} &= \frac{2\pi}{\beta^2}\left[\left(\frac{M''_A}{\Omega}\right)_W - b\left(\frac{L''_A}{\Omega}\right)_W\right] \\
M_{W_{\left(\frac{\dot{\theta}\ell}{U}\right)}} &= \frac{2\pi}{\beta^2}\left\{\left(\frac{M''_B}{\Omega}\right)_W - \left(\frac{M''_A}{\Omega}\right)_W + b\left[\left(\frac{L''_A}{\Omega}\right)_W \right.\right. \\
&\qquad \left.\left. - \left(\frac{L''_B}{\Omega}\right)_W - \beta^2(M'_{A_W} - bL'_{A_W})\right]\right\}
\end{aligned}\right\} \tag{147}$$

The same procedure is now applied to the horizontal tail forces and moments after introducing the proper area, chord length, efficiency factor, and reduced frequencies.

δ corresponds to C and denotes the deflection of a full-span elevator. The tail characteristics are denoted by the subscript $(\ )_T$.

If the tail length is great enough, it is reasonable to assume that the rotation of the horizontal tail can be neglected and that rotation of the airplane as a whole about its center of gravity causes a pure translational motion of the tail for small disturbances.

Hence

$$Z_T = \eta_T \pi \rho_0 U^2 S_T \left[\left(\frac{w}{U} + \frac{l_T \mathring{\theta}}{U} - \varepsilon \right) L_{A_T} + \delta L_{C_T} \right] \tag{148}$$

where ε is the downwash angle which may be obtained from eqn. (137) in the form

$$\bar{\varepsilon} = \frac{i\omega l A}{U} \, \omega_A + B \omega_B$$

$$= \frac{\bar{w}}{U} \omega_A + \bar{\theta} (\omega_B - \omega_A) \tag{149}$$

Substitution of eqn. (149) into eqn. (148) results in

$$Z_T = \eta_T \pi \rho_0 U^2 S_T \left[\left(\frac{w}{U} + \frac{l_T \mathring{\theta}}{U} - \frac{w}{U} \omega_A + \theta \omega_A - \theta \omega_B \right) L_{A_T} + \delta L_{C_T} \right]$$

which may be written as

$$Z_T = \frac{1}{2} \rho_0 U^2 S_W \left[\frac{w}{U} Z_{T\left(\frac{w}{U}\right)} + \frac{\mathring{w} l}{U^2} Z_{T\left(\frac{\mathring{w} l}{U^2}\right)} + \frac{\mathring{\theta} l}{U} Z_{T\left(\frac{\mathring{\theta} l}{U}\right)} \right.$$
$$\left. + \delta Z_{T(\delta)} + \frac{\mathring{\delta} l}{U} Z_{T\left(\frac{\mathring{\delta} l}{U}\right)} \right] \tag{150}$$

where

$$
\left.
\begin{aligned}
Z_{T_{\left(\frac{w}{U}\right)}} &= 2\pi \frac{S_T}{S_W} L'_{A_T} (1 - w'_A) \eta_T \\
Z_{T_{\left(\frac{\dot{w}\ell}{U^2}\right)}} &= \frac{2\pi}{\beta^2} \frac{S_T}{S_W} \left[\frac{c_T}{c_W} \left(\frac{L''_A}{\Omega} \right)_T (1 - w'_A) \right. \\
&\qquad \left. - \frac{w''_A}{\Omega} L'_{A_T} \right] \eta_T \\
Z_{T_{\left(\frac{\dot{\theta}\ell}{U}\right)}} &= \frac{2\pi}{\beta^2} \frac{S_T}{S_W} L'_{A_T} \left(\beta^2 \frac{\ell_T}{\ell} - \frac{w''_B}{\Omega} + \frac{w''_A}{\Omega} \right) \eta_T \\
Z_{T_{(\delta)}} &= 2\pi \frac{S_T}{S_W} L'_{C_T} \eta_T \\
Z_{T_{\left(\frac{\dot{\delta}\ell}{U}\right)}} &= \frac{2\pi}{\beta^2} \frac{S_T c_T}{S_W c_W} \left(\frac{L''_{C_T}}{\Omega} \right)_T \eta_T
\end{aligned}
\right\} \quad (151)
$$

Similarly the moment about the point of rotation due to the horizontal tail is

$$
\begin{aligned}
M_T = \eta_T \frac{\pi}{2} \rho_0 U^2 c_T S_T \Bigg[& \left(\frac{w}{U} + \frac{\ell_T \dot{\theta}}{U} - \varepsilon \right) \left(M_{A_T} + \frac{2\ell_T}{c_T} L_{A_T} \right) \\
& + \delta M_{C_T} + \delta \frac{2\ell_T}{c_T} L_{C_T} \Bigg]
\end{aligned}
\quad (152)
$$

which, upon substituting eqn. (149), may be written

$$
\begin{aligned}
M_T = \frac{1}{4} \rho_0 U^2 c_W S_W \Bigg[& \frac{w}{U} M_{T_{\left(\frac{w}{U}\right)}} + \frac{\dot{w}\ell}{U^2} M_{T_{\left(\frac{\dot{w}\ell}{U^2}\right)}} \\
& + \frac{\dot{\theta}\ell}{U} M_{T_{\left(\frac{\dot{\theta}\ell}{U}\right)}} + \delta M_{T_{(\delta)}} + \frac{\dot{\delta}\ell}{U} M_{T_{\left(\frac{\dot{\delta}\ell}{U}\right)}} \Bigg]
\end{aligned}
\quad (153)
$$

where

$$\left.\begin{aligned}
\mathcal{M}_{T\left(\frac{w}{U}\right)} &= 2\pi \frac{S_T c_T}{S_W c_W}\left(\mathcal{M}'_{A_T} + \frac{2l_T}{c_T}\mathcal{L}'_{A_T}\right)\left(1-w'_A\right)\eta_T \\
\mathcal{M}_{T\left(\frac{\dot{w}l}{U^2}\right)} &= \frac{2\pi}{\beta^2}\frac{S_T c_T}{S_W c_W}\left\{\frac{c_T}{c_W}\left[\left(\frac{\mathcal{M}''_A}{\Omega}\right)_T + \frac{2l_T}{c_T}\left(\frac{\mathcal{L}''_A}{\Omega}\right)_T\right]\left(1-w'_A\right)\right. \\
&\quad \left. - \frac{w''_A}{\Omega}\mathcal{M}'_{A_T} - \frac{w''_A}{\Omega}\frac{2l_T}{c_T}\mathcal{L}'_{A_T}\right\}\eta_T \\
\mathcal{M}_{T\left(\frac{\dot{\theta}l}{U}\right)} &= \frac{2\pi}{\beta^2}\frac{S_T c_T}{S_W c_W}\left(\beta^2\frac{l_T}{l} + \frac{w''_A}{\Omega} - \frac{w''_B}{\Omega}\right)\left(\mathcal{M}'_{A_T}\right. \\
&\quad \left. + \frac{2l_T}{c_T}\mathcal{L}'_{A_T}\right)\eta_T \\
\mathcal{M}_{T(\delta)} &= 2\pi \frac{S_T c_T}{S_W c_W}\left(\mathcal{M}'_{C_T} + \frac{2l_T}{c_T}\mathcal{L}'_{C_T}\right)\eta_T \\
\mathcal{M}_{T\left(\frac{\dot{\delta}l}{U}\right)} &= \frac{2\pi}{\beta^2}\frac{S_T c_T^2}{S_W c_W^2}\left[\left(\frac{\mathcal{M}''_C}{\Omega}\right)_T + \frac{2l_T}{c_T}\left(\frac{\mathcal{L}''_C}{\Omega}\right)_T\right]\eta_T
\end{aligned}\right\} \tag{154}$$

Finally the total hinge moment for a full-span elevator may be obtained from eqn. (114) in the form

$$\bar{N} = \eta_T \frac{\pi}{2}\rho_0 U^2 c_T S_T\left[ik\left(A + \frac{l_T}{l}B\right)N_A + CN_C\right]$$

or as in the preceding equations

$$N = \eta_T \frac{\pi}{2}\rho_0 U^2 c_T S_T\left[\left(\frac{w}{U} + \frac{l_T\dot{\theta}}{U} - \varepsilon\right)N_A + \delta N_C\right] \tag{155}$$

Eqn. (155) may be written in the more convenient form

$$N = \frac{1}{4}\rho_0 U^2 c_W S_W\left[\frac{w}{U}N_{\left(\frac{w}{U}\right)} + \frac{\dot{w}l}{U^2}N_{\left(\frac{\dot{w}l}{U^2}\right)} + \frac{\dot{\theta}l}{U}N_{\left(\frac{\dot{\theta}l}{U}\right)} + \delta N_{(\delta)} + \frac{\dot{\delta}l}{U}N_{\left(\frac{\dot{\delta}l}{U}\right)}\right] \tag{156}$$

where

$$
\left.
\begin{aligned}
N_{\left(\frac{w}{U}\right)} &= 2\pi \frac{S_T c_T}{S_W c_W} N_A'(1-w_A')\eta_T \\
N_{\left(\frac{\dot{w}l}{U^2}\right)} &= \frac{2\pi}{\beta^2}\frac{S_T c_T}{S_W c_W}\left[\frac{c_T}{c_W}\left(\frac{N_A''}{\Omega}\right)_T (1-w_A') \right. \\
&\qquad \left. - \frac{w_A''}{\Omega} N_A'\right]\eta_T \\
N_{\left(\frac{\dot{\theta}l}{U}\right)} &= \frac{2\pi}{\beta^2}\frac{S_T c_T}{S_W c_W}\left(\beta^2 \frac{l_T}{l} + \frac{w_A''}{\Omega} - \frac{w_B''}{\Omega}\right) N_A' \eta_T \\
N_{(\delta)} &= 2\pi \frac{S_T c_T}{S_W c_W} N_c' \eta_T \\
N_{\left(\frac{\dot{\delta}l}{U}\right)} &= \frac{2\pi}{\beta^2}\frac{S_T c_T^2}{S_W c_W^2}\left(\frac{N_c''}{\Omega}\right)_T \eta_T
\end{aligned}
\right\} \quad (157)
$$

The primary advantage to setting up the forces and moments in this manner is that the resultant stability derivatives as defined in eqns. (144), (147), (151), (154), and (157) are independent of frequency. The problem of solving the longitudinal equations of motion of the airplane including the first order in frequency unsteady flow effects is, therefore, precisely the same as is encountered in the quasi-steady approximation.

For estimating the effects of the fuselage, it is convenient to use the unsteady-slender-body-theory results as discussed by Schalin (29) from which the following expressions may be obtained.

$$Z_F = \frac{1}{2}\rho_0 U^2 S_W \left[\frac{w}{U} Z_{F_{\left(\frac{w}{U}\right)}} + \frac{\dot{w} l}{U^2} Z_{F_{\left(\frac{\dot{w} l}{U^2}\right)}} + \frac{\dot{\theta} l}{U} Z_{F_{\left(\frac{\dot{\theta} l}{U}\right)}} \right] \tag{158}$$

where

$$\left.\begin{aligned} Z_{F_{\left(\frac{w}{U}\right)}} &= -2\frac{S_B}{S_W} \\ Z_{F_{\left(\frac{\dot{w} l}{U^2}\right)}} &= -2\frac{\text{VOLUME}}{S_W l} \\ Z_{F_{\left(\frac{\dot{\theta} l}{U}\right)}} &= -2\frac{S_B l_B}{S_W l} \end{aligned}\right\} \tag{159}$$

and l_B is the distance aft from the center of gravity to the base and S_B is the base area.

Also the moment about the center of gravity is

$$M_F = \frac{1}{4}\rho_0 U^2 S_W c_W \left[\frac{w}{U} M_{F_{\left(\frac{w}{U}\right)}} + \frac{\dot{w} l}{U^2} M_{F_{\left(\frac{\dot{w} l}{U^2}\right)}} + \frac{\dot{\theta} l}{U} M_{F_{\left(\frac{\dot{\theta} l}{U}\right)}} \right] \tag{160}$$

where

$$\left.\begin{aligned} M_{F_{\left(\frac{w}{U}\right)}} &= 4\left(\frac{\text{VOLUME} - S_B l_B}{S_W c_W}\right) \\ M_{F_{\left(\frac{\dot{w} l}{U^2}\right)}} &= \frac{4}{S_W c_W l}\int_{-l_B}^{l_N} S x\,dx \\ M_{F_{\left(\frac{\dot{\theta} l}{U}\right)}} &= -\frac{4}{S_W c_W l}\left(S_B l_B^2 + \int_{-l_B}^{l_N} S x\,dx\right) \end{aligned}\right\} \tag{161}$$

and l_N is the distance from the center of gravity to the nose.

LONGITUDINAL EQUATIONS OF MOTION

In developing the equations of motion in a form amenable to including the unsteady flow effects, the following assumptions will be made:

(i) The effect of variations in the forward speed on the longitudinal motion will be neglected.

(ii) No elastic deformations will be considered, i.e., the aircraft is assumed infinitely rigid.

(iii) The aerodynamic forces derived for harmonic oscillations will be assumed applicable to damped motions.

(iv) The control system is assumed to have no internal damping or stiffness.

(v) The inertia forces and moments due to angular acceleration of the control surface will be neglected in the equations expressing the dynamic balance of normal forces and pitching moments.

(vi) Small deviations from a steady flight path will be assumed.

With the above approximations the summation of forces acting on the airplane in the Z-direction may be written as

$$m(\dot{w}-U\dot{\theta}) = Z = \frac{1}{2}\rho_0 U^2 S_W \left[\frac{w}{U} Z_{\left(\frac{w}{U}\right)} + \frac{\dot{w}\ell}{U^2} Z_{\left(\frac{\dot{w}\ell}{U^2}\right)} + \frac{\dot{\theta}\ell}{U} Z_{\left(\frac{\dot{\theta}\ell}{U}\right)} + \delta Z_{(\delta)} + \frac{\dot{\delta}\ell}{U} Z_{\left(\frac{\dot{\delta}\ell}{U}\right)}\right] \tag{162}$$

where m is the airplane mass.

If the following symbols are introduced:

$$\dot{\theta} = q \tag{163}$$

$$\lambda = \frac{\ell}{U}\frac{d}{dt} \tag{164}$$

$$\mu_1 = \frac{4m}{\rho_0 S_W c_W} \tag{165}$$

then eqn. (162) becomes, after some rearranging of terms:

$$\left\{Z_{\left(\frac{w}{U}\right)} + \left[Z_{\left(\frac{\dot{w}\ell}{U^2}\right)} - \mu_1\right]\lambda\right\}\frac{w}{U} + \left[Z_{\left(\frac{q\ell}{U}\right)} + \mu_1\right]\frac{q\ell}{U} + \left[Z_{(\delta)} + Z_{\left(\frac{\dot{\delta}\ell}{U}\right)}\lambda\right]\delta = 0 \tag{166}$$

Similarly the summation of pitching moments about the center of gravity may be written

$$I_y\ddot{\theta} = M = \frac{1}{4}\rho_0 U^2 c_W S_W \left[\frac{w}{U} M_{\left(\frac{w}{U}\right)} + \frac{\dot{w}\ell}{U^2} M_{\left(\frac{\dot{w}\ell}{U^2}\right)} + \frac{\dot{\theta}\ell}{U} M_{\left(\frac{\dot{\theta}\ell}{U}\right)} + \delta M_{(\delta)} + \frac{\dot{\delta}\ell}{U} M_{\left(\frac{\dot{\delta}\ell}{U}\right)}\right] \tag{167}$$

or

$$\left[M_{\left(\frac{w}{U}\right)} + M_{\left(\frac{\dot{w}l}{U^2}\right)}\lambda\right]\frac{w}{U} + \left[M_{\left(\frac{ql}{U}\right)} - \mu_1 i_B \lambda\right]\frac{ql}{U}$$

$$+\left[M_{(\delta)} + M_{\left(\frac{\dot{\delta}l}{U}\right)}\lambda\right]\delta = 0 \tag{168}$$

where

$$i_B = \frac{4I_y}{m c_w^2} \tag{169}$$

and I_y is the moment of inertia of the airplane about an axis along the wing span through the center of gravity.

Finally the elevator hinge moment balance equation may be written

$$I_e \ddot{\delta} + m_e dl(\dot{w} - U\dot{\theta}) + (I_e + l_e l m_e dl)\ddot{\theta} = N$$

$$= \frac{1}{4}\rho_0 U^2 c_w S_w \left[\frac{w}{U} N_{\left(\frac{w}{U}\right)} + \frac{\dot{w}l}{U^2} N_{\left(\frac{\dot{w}l}{U^2}\right)}\right.$$

$$\left. + \frac{\dot{\theta}l}{U} N_{\left(\frac{\dot{\theta}l}{U}\right)} + \delta N_{(\delta)} + \frac{\dot{\delta}l}{U} N_{\left(\frac{\dot{\delta}l}{U}\right)}\right] \tag{170}$$

or

$$\left\{N_{\left(\frac{w}{U}\right)} + \left[N_{\left(\frac{\dot{w}l}{U^2}\right)} - h_e\right]\lambda\right\}\frac{w}{U} + \left[N_{\left(\frac{ql}{U}\right)} + h_e\right.$$

$$\left. - (\mu_1 i_e + h_e l_e)\lambda\right]\frac{ql}{U}$$

$$+\left[N_{(\delta)} + N_{\left(\frac{\dot{\delta}l}{U}\right)}\lambda - \mu_1 i_e \lambda^2\right]\delta = 0 \tag{171}$$

where I_e is the moment of inertia of the control system about the hinge line, m_e is the mass of the elevator, dl is the distance of the center of gravity of the elevator aft of the hinge line, $l_e l$ is the distance of the hinge line aft of the center of gravity of the airplane, and

$$i_e = \frac{4 I_e}{m c_w^2} \tag{172}$$

$$h_e = \frac{4 m_e d}{\rho_0 S_w c_w} \tag{173}$$

The characteristic determinant is therefore

$$\begin{vmatrix} Z_{\left(\frac{w}{U}\right)} + \left[Z_{\left(\frac{\dot{w}l}{U^2}\right)} - \mu_1\right]\lambda & Z_{\left(\frac{ql}{U}\right)} + \mu_1 & Z_{(\delta)} + Z_{\left(\frac{\dot{\delta}l}{U}\right)}\lambda \\ M_{\left(\frac{w}{U}\right)} + M_{\left(\frac{\dot{w}l}{U^2}\right)}\lambda & M_{\left(\frac{ql}{U}\right)} - \mu_1 i_B \lambda & M_{(\delta)} + M_{\left(\frac{\dot{\delta}l}{U}\right)}\lambda \\ N_{\left(\frac{w}{U}\right)} + \left[N_{\left(\frac{\dot{w}l}{U^2}\right)} - h_e\right]\lambda & \begin{matrix} N_{\left(\frac{ql}{U}\right)} + h_e \\ -(\mu_1 i_e + h_e l_e)\lambda \end{matrix} & N_{(\delta)} + N_{\left(\frac{\dot{\delta}l}{U}\right)}\lambda - \mu_1 i_e \lambda^2 \end{vmatrix} \tag{174}$$

and the corresponding frequency equation is a quartic

$$C_1\lambda^4 + C_2\lambda^3 + C_3\lambda^2 + C_4\lambda + C_5 = 0 \tag{175}$$

where

$$
\left.
\begin{aligned}
C_1 &= \mu_1^2 i_B i_e \left[Z_{\left(\frac{\dot{w}\ell}{V^2}\right)} - \mu_1 \right] \\[2ex]
C_2 &= \mu_1^2 i_B i_e Z_{\left(\frac{w}{V}\right)} + \left[Z_{\left(\frac{\dot{w}\ell}{V^2}\right)} - \mu_1 \right] \left[-\mu_1 i_e M_{\left(\frac{q\ell}{V}\right)} \right. \\
&\quad \left. - \mu_1 i_B N_{\left(\frac{\dot{\delta}\ell}{V}\right)} + (\mu_1 i_e + h_e \ell_e) M_{\left(\frac{\dot{\delta}\ell}{V}\right)} \right] \\
&\quad + M_{\left(\frac{\dot{w}\ell}{V^2}\right)} \left[\mu_1 i_e \left(Z_{\left(\frac{q\ell}{V}\right)} + \mu_1 \right) - (\mu_1 i_e \right. \\
&\quad \left. + h_e \ell_e) Z_{\left(\frac{\dot{\delta}\ell}{V}\right)} \right] + \mu_1 i_B Z_{\left(\frac{\dot{\delta}\ell}{V}\right)} \left[N_{\left(\frac{\dot{w}\ell}{V^2}\right)} - h_e \right] \\[2ex]
C_3 &= Z_{\left(\frac{w}{V}\right)} \left[-\mu_1 i_e M_{\left(\frac{q\ell}{V}\right)} - \mu_1 i_B N_{\left(\frac{\dot{\delta}\ell}{V}\right)} + (\mu_1 i_e \right. \\
&\quad \left. + h_e \ell_e) M_{\left(\frac{\dot{\delta}\ell}{V}\right)} \right] + \left[Z_{\left(\frac{\dot{w}\ell}{V^2}\right)} - \mu_1 \right] \left[M_{\left(\frac{q\ell}{V}\right)} N_{\left(\frac{\dot{\delta}\ell}{V}\right)} - \mu_1 i_B N_{(\delta)} \right. \\
&\quad \left. - \left(N_{\left(\frac{q\ell}{V}\right)} + h_e \right) M_{\left(\frac{\dot{\delta}\ell}{V}\right)} + (\mu_1 i_e + h_e \ell_e) M_{(\delta)} \right] \\
&\quad + M_{\left(\frac{w}{V}\right)} \left[\mu_1 i_e \left(Z_{\left(\frac{q\ell}{V}\right)} + \mu_1 \right) - (\mu_1 i_e + h_e \ell_e) Z_{\left(\frac{\dot{\delta}\ell}{V}\right)} \right] \\
&\quad + M_{\left(\frac{\dot{w}\ell}{V^2}\right)} \left[\left(N_{\left(\frac{q\ell}{V}\right)} + h_e \right) Z_{\left(\frac{\dot{\delta}\ell}{V}\right)} - \left(Z_{\left(\frac{q\ell}{V}\right)} \right. \right. \\
&\quad \left. \left. + \mu_1 \right) N_{\left(\frac{\dot{\delta}\ell}{V}\right)} - (\mu_1 i_e + h_e \ell_e) Z_{(\delta)} \right] \\
&\quad + \mu_1 i_B N_{\left(\frac{w}{V}\right)} Z_{\left(\frac{\dot{\delta}\ell}{V}\right)} + \left[N_{\left(\frac{\dot{w}\ell}{V^2}\right)} - h_e \right] \left[\left(Z_{\left(\frac{q\ell}{V}\right)} \right. \right. \\
&\quad \left. \left. + \mu_1 \right) M_{\left(\frac{\dot{\delta}\ell}{V}\right)} - M_{\left(\frac{q\ell}{V}\right)} Z_{\left(\frac{\dot{\delta}\ell}{V}\right)} + \mu_1 i_B Z_{(\delta)} \right]
\end{aligned}
\right\} \quad (176)
$$

$$
\left.\begin{aligned}
C_4 = {} & Z_{\left(\frac{w}{U}\right)}\left[M_{\left(\frac{q\ell}{U}\right)} N_{\left(\frac{\dot{\delta}\ell}{U}\right)} + (\mu_1 i_e + h_e \ell_e) M_{(\delta)}\right. \\
& \left. - \mu_1 i_B N_{(\delta)} - \left(N_{\left(\frac{q\ell}{U}\right)} + h_e\right) M_{\left(\frac{\dot{\delta}\ell}{U}\right)}\right] + \left[Z_{\left(\frac{\dot{w}\ell}{U^2}\right)}\right. \\
& \left. - \mu_1\right]\left[M_{\left(\frac{q\ell}{U}\right)} N_{(\delta)} - \left(N_{\left(\frac{q\ell}{U}\right)} + h_e\right) M_{(\delta)}\right] \\
& - M_{\left(\frac{w}{U}\right)}\left[\left(Z_{\left(\frac{q\ell}{U}\right)} + \mu_1\right) N_{\left(\frac{\dot{\delta}\ell}{U}\right)} - \left(N_{\left(\frac{q\ell}{U}\right)} + h_e\right) Z_{\left(\frac{\dot{\delta}\ell}{U}\right)}\right. \\
& \left. + (\mu_1 i_e + h_e \ell_e) Z_{(\delta)}\right] + M_{\left(\frac{\dot{w}\ell}{U^2}\right)}\left[\left(N_{\left(\frac{q\ell}{U}\right)}\right.\right. \\
& \left.\left. + h_e\right) Z_{(\delta)} - \left(Z_{\left(\frac{q\ell}{U}\right)} + \mu_1\right) N_{(\delta)}\right] + N_{\left(\frac{w}{U}\right)}\left[\left(Z_{\left(\frac{q\ell}{U}\right)}\right.\right. \\
& \left.\left. + \mu_1\right) M_{\left(\frac{\dot{\delta}\ell}{U}\right)} - M_{\left(\frac{q\ell}{U}\right)} Z_{\left(\frac{\dot{\delta}\ell}{U}\right)} + \mu_1 i_B Z_{(\delta)}\right] \\
& + \left[N_{\left(\frac{\dot{w}\ell}{U^2}\right)} - h_e\right]\left[\left(Z_{\left(\frac{q\ell}{U}\right)} + \mu_1\right) M_{(\delta)} - M_{\left(\frac{q\ell}{U}\right)} Z_{(\delta)}\right] \\
C_5 = {} & Z_{\left(\frac{w}{U}\right)}\left[M_{\left(\frac{q\ell}{U}\right)} N_{(\delta)} - \left(N_{\left(\frac{q\ell}{U}\right)} + h_e\right) M_{(\delta)}\right] \\
& - M_{\left(\frac{w}{U}\right)}\left[\left(Z_{\left(\frac{q\ell}{U}\right)} + \mu_1\right) N_{(\delta)} - \left(N_{\left(\frac{q\ell}{U}\right)}\right.\right. \\
& \left.\left. + h_e\right) Z_{(\delta)}\right] + N_{\left(\frac{w}{U}\right)}\left[\left(Z_{\left(\frac{q\ell}{U}\right)} + \mu_1\right) M_{(\delta)}\right. \\
& \left. - M_{\left(\frac{q\ell}{U}\right)} Z_{(\delta)}\right]
\end{aligned}\right\} \quad \begin{matrix}(176)\\ \text{cont.}\end{matrix}
$$

APPLICATION TO AN F-80A AIRPLANE

In order to be able to focus attention on the effects of unsteady flow considerations as precisely as possible, it is advisable to adjust the theoretically determined stability derivatives so that at zero frequency of oscillation they conform with the corresponding experimentally determined static stability derivatives. This adjustment is particularly important when aerodynamic theory is to be used on control surfaces since leading- and trailing-edge shapes, aerodynamic balance, gaps, and boundary-layer characteristics have such predominant effects upon the hinge-moment derivatives that large differences exist between theoretical and experimental results even in the steady case. In the following application, therefore, certain of the calculated derivatives will be replaced by their measured counterparts in a way such that the theoretical variations of the airloads with frequency are unaffected but the correct static airloads are produced at zero frequency.

Furthermore the relatively small contribution of the fuselage will be taken into account approximately by lumping it with the wing contribution. The resultant derivatives will be referred to as "tail-off" derivatives and will be designated by the subscript $(\ )_{T.O.}$. As before, the contributions of the horizontal tail will be designated by the subscript $(\ )_T$.

Jet-damping will be neglected on the basis of the results of the previous study by the author (18).

Below is a tabulation of the theoretical stability derivatives, the experimental values by which they are to be replaced, and the numbers of the corresponding figures which show their comparisons.

THEORETICAL DERIVATIVE - replaced by -	EXPERIMENTAL DERIVATIVE	FIGURE NO.
$-2\pi L'_{A_W} + 2\frac{S_B}{S_W}$	$(C_{L_\alpha})_{T.O.}$	57
$\pi(M'_{A_W} - bL'_{A_W}) + 2\left(\frac{V_F - S_B l_B}{S_W c_W}\right)$	$(C_{M_\alpha})_{T.O.}$	58
$-2\pi L'_{A_T}$	$(C_{L_\alpha})_T$	59
$-2\pi L'_{C_T} \eta_T \frac{S_T}{S_W}$	C_{L_δ}	60
$\pi\left(M'_{A_T} + \frac{2l_T}{c_T} L'_{A_T}\right)$	$(C_{M_\alpha})_T$	61
$\pi\left(M'_{C_T} + \frac{2l_T}{c_T} L'_{C_T}\right)\eta_T \frac{S_T c_T}{S_W c_W}$	C_{M_δ}	62
$\pi N'_A$	$(C_{H_\alpha})_T$	63
$\pi N'_C$	$(C_{H_\delta})_T$	64
ω'_A	ε_α	65

The F-80A airplane to be used in this example is a low-wing, fighter type, powered by a turbo-jet engine exhausting through the extreme aft end of the fuselage. The gross weight is 10,000 lbs., the center of gravity location is at 27% of the mean aerodynamic wing chord and it is operating at an altitude of 20,000 feet. The additional required data are as follows:

S_W - wing area = 237 ft.^{2}

S_T - horizontal tail area = 43.5 ft.^{2}

c_W - mean aerodynamic chord of the wing = 6.72 ft.

c_T - mean aerodynamic chord of the horizontal tail = 3.12 ft.

AR_W - aspect ratio of the wing = 6.38

AR_T - aspect ratio of the horizontal tail = 5.56

I_y - moment of inertia of airplane about y axis = 13,780 slugs ft.^{2}

bl - distance from midchord of the wing to airplane c.g. (negative for c.g. forward of mid-chord) = -1.55 ft.

l_T - distance from airplane c.g. to midchord of horizontal tail = 15.14 ft.

S_e - elevator area (total) = 8.70 ft.

m_e - mass of elevator = 1.07 slugs

I_e - moment of inertia of elevator about hinge line = 0.11 slugs ft.^{2}

dl - distance from hinge line to c.g. of elevator $\doteq$ 0 ft.

$l_e l$ - distance from airplane c.g. to hinge line = 16.33 ft.

$\frac{\varphi}{\pi}$ - $1 - \frac{\text{elevator chord}}{\text{horizontal tail chord}} = 0.75$

S_B - base area of fuselage = 1.77 ft.^{2}

l_B - distance from airplane c.g. to base of fuselage = 17.25 ft.

V_F - volume of fuselage $\doteq$ 350 ft.^{3}

η_T - efficiency factor for the horizontal tail = 0.90

$$\mu_1 = \frac{4m}{\rho_0 S_w c_w} = 616.48$$

$$i_B = \frac{4I_y}{m c_w^2} = 3.93$$

$$i_e = \frac{4I_e}{m c_w^2} = 3.13 \times 10^{-4}$$

$$h_e = \frac{4 m_e d}{\rho_0 S_w c_w} \doteq 0$$

The required stability derivatives are evaluated according to the following formulas:

$$Z_{T.O.\left(\frac{w}{V}\right)} = -\left(C_{L_\alpha}\right)_{T.O.}$$

$$Z_{T.O.\left(\frac{\dot{w}l}{V^2}\right)} = -\left(C_{L_\alpha}\right)_{T.O.}\left(\frac{1}{\beta^2 L'_{AW}}\right)\left(\frac{L''_A}{\Omega}\right)_W$$

$$Z_{T.O.\left(\frac{ql}{V}\right)} = \left(C_{L_\alpha}\right)_{T.O.}\, b\left\{1 - \frac{1}{\beta^2 b L'_{AW}}\left[\left(\frac{L''_B}{\Omega}\right)_W - \left(\frac{L''_A}{\Omega}\right)_W\right]\right\}$$

$$\mathcal{M}_{T.O.\left(\frac{w}{U}\right)} = 2\left(C_{M_\alpha}\right)_{T.O.}$$

$$\mathcal{M}_{T.O.\left(\frac{\dot{w}\ell}{U^2}\right)} = 2\left(C_{M_\alpha}\right)_{T.O.}\left[\frac{1}{\beta^2\left(\mathcal{M}'_{A_W} - bL'_{A_W}\right)}\right]\left[\left(\frac{\mathcal{M}''_A}{\Omega}\right)_W - b\left(\frac{L''_A}{\Omega}\right)_W\right]$$

$$\mathcal{M}_{T.O.\left(\frac{q\ell}{U}\right)} = -2\left(C_{M_\alpha}\right)_{T.O.}\, b\left\{1 - \frac{1}{b\beta^2\left(\mathcal{M}'_{A_W} - bL'_{A_W}\right)}\left[\left(\frac{\mathcal{M}''_B}{\Omega}\right)_W - \left(\frac{\mathcal{M}''_A}{\Omega}\right)_W + b\left(\frac{L''_A}{\Omega}\right)_W - b\left(\frac{L''_B}{\Omega}\right)_W\right]\right\}$$

$$Z_{T\left(\frac{w}{U}\right)} = -\left(C_{L_\alpha}\right)_T \frac{S_T}{S_W}\left(1-\varepsilon_\alpha\right)\eta_T$$

$$Z_{T\left(\frac{\dot{w}\ell}{U^2}\right)} = -\left(C_{L_\alpha}\right)_T \frac{S_T}{S_W}\;\frac{\eta_T}{\beta^2 L'_{A_T}}\left[\frac{c_T}{c_W}\left(\frac{L''_A}{\Omega}\right)_T\left(1-\varepsilon_\alpha\right) - \varepsilon_\alpha L'_{A_T}\;\frac{1}{\omega'_A}\;\frac{\omega''_A}{\Omega}\right]$$

$$Z_{T\left(\frac{q\ell}{U}\right)} = -\left(C_{L_\alpha}\right)_T \frac{S_T}{S_W}\frac{\ell_T}{\ell}\eta_T\left[1 + \frac{\varepsilon_\alpha}{\beta^2\frac{\ell_T}{\ell}}\;\frac{1}{\omega'_A}\left(\frac{\omega''_A}{\Omega} - \frac{\omega''_B}{\Omega}\right)\right]$$

$$Z_{T(\delta)} = -C_{L_\delta}$$

$$Z_{T\left(\frac{\dot{\delta}\ell}{U}\right)} = -C_{L_\delta}\frac{c_T}{c_W}\;\frac{1}{\beta^2 L'_{C_T}}\left(\frac{L''_C}{\Omega}\right)_T$$

$$\mathcal{M}_{T\left(\frac{w}{U}\right)} = 2\left(C_{M_\alpha}\right)_T \frac{S_T\, c_T}{S_W\, c_W}\left(1-\varepsilon_\alpha\right)\eta_T$$

$$\mathcal{M}_{T\left(\frac{\dot{w}\ell}{U^2}\right)} = 2\left(C_{M_\alpha}\right)_T \frac{S_T\, c_T}{S_W\, c_W}\;\frac{\eta_T}{\beta^2\left(\mathcal{M}'_{A_T} + \frac{2\ell_T}{c_T}L'_{A_T}\right)}\left\{\frac{c_T}{c_W}\left[\left(\frac{\mathcal{M}''_A}{\Omega}\right)_T + \frac{2\ell_T}{c_T}\left(\frac{L''_A}{\Omega}\right)_T\right]\left(1-\varepsilon_\alpha\right) - \frac{\omega''_A}{\Omega}\;\frac{\varepsilon_\alpha}{\omega'_A}\left[\mathcal{M}'_{A_T} + \frac{2\ell_T}{c_T}L'_{A_T}\right]\right\}$$

$$\mathcal{M}_{T\left(\frac{\dot{q}l}{U}\right)} = 2\left(C_{M_\alpha}\right)_T \frac{S_T c_T}{S_W c_W} \frac{l_T}{l} \eta_T \left[1 + \frac{\epsilon_\alpha}{\beta^2 \frac{l_T}{l}} \frac{1}{\omega_A'}\left(\frac{\omega_A''}{\Omega} - \frac{\omega_B''}{\Omega}\right)\right]$$

$$\mathcal{M}_{T(\delta)} = 2 C_{M_\delta}$$

$$\mathcal{M}_{T\left(\frac{\dot{\delta}l}{U}\right)} = 2 C_{M_\delta} \frac{c_T}{c_W} \frac{1}{\beta^2\left(M_{C_T}' + \frac{2l_T}{c_T} L_{C_T}'\right)} \left[\left(\frac{M_C''}{\Omega}\right)_T + \frac{2l_T}{c_T}\left(\frac{L_C''}{\Omega}\right)_T\right]$$

$$N_{\left(\frac{w}{U}\right)} = 2\left(C_{H_\alpha}\right)_T \frac{S_T c_T}{S_W c_W} \left(1 - \epsilon_\alpha\right) \eta_T$$

$$N_{\left(\frac{\dot{w}l}{U^2}\right)} = 2\left(C_{H_\alpha}\right)_T \frac{S_T c_T}{S_W c_W} \frac{\eta_T}{\beta^2 N_{A_T}'} \left[\frac{c_T}{c_W}\left(\frac{N_A''}{\Omega}\right)_T \left(1 - \epsilon_\alpha\right) - \frac{\epsilon_\alpha}{\omega_A'} \frac{\omega_A''}{\Omega} N_{A_T}'\right]$$

$$N_{\left(\frac{ql}{U}\right)} = 2\left(C_{H_\alpha}\right)_T \frac{S_T c_T}{S_W c_W} \frac{l_T}{l} \eta_T \left[1 + \frac{1}{\beta^2 \frac{l_T}{l}} \frac{\epsilon_\alpha}{\omega_A'}\left(\frac{\omega_A''}{\Omega} - \frac{\omega_B''}{\Omega}\right)\right]$$

$$N_{(\delta)} = 2\left(C_{H_\delta}\right)_T \frac{S_T c_T}{S_W c_W} \eta_T$$

$$N_{\left(\frac{\dot{\delta}l}{U}\right)} = 2\left(C_{H_\delta}\right)_T \frac{S_T c_T^2}{S_W c_W^2} \frac{\eta_T}{\beta^2 N_{C_T}'} \left(\frac{N_C''}{\Omega}\right)_T$$

The total airplane stability derivatives are given by

$$Z_{\left(\frac{w}{U}\right)} = Z_{T.O.\left(\frac{w}{U}\right)} + Z_{T\left(\frac{w}{U}\right)}$$

$$Z_{\left(\frac{\dot{w}l}{U}\right)} = Z_{T.O.\left(\frac{\dot{w}l}{U^2}\right)} + Z_{T\left(\frac{\dot{w}l}{U^2}\right)}$$

$$Z_{\left(\frac{q l}{U}\right)} = Z_{T.O.\left(\frac{q l}{U}\right)} + Z_{T\left(\frac{q l}{U}\right)}$$

$$Z_{(\delta)} = Z_{T(\delta)}$$

$$Z_{\left(\frac{\dot{\delta} l}{U}\right)} = Z_{T\left(\frac{\dot{\delta} l}{U}\right)}$$

$$M_{\left(\frac{w}{U}\right)} = M_{T.O.\left(\frac{w}{U}\right)} + M_{T\left(\frac{w}{U}\right)}$$

$$M_{\left(\frac{\dot{w} l}{U^2}\right)} = M_{T.O.\left(\frac{\dot{w} l}{U^2}\right)} + M_{T\left(\frac{\dot{w} l}{U^2}\right)}$$

$$M_{\left(\frac{q l}{U}\right)} = M_{T.O.\left(\frac{q l}{U}\right)} + M_{T\left(\frac{q l}{U}\right)}$$

$$M_{(\delta)} = M_{T(\delta)}$$

$$M_{\left(\frac{\dot{\delta} l}{U}\right)} = M_{T\left(\frac{\dot{\delta} l}{U}\right)}$$

$$N_{\left(\frac{w}{U}\right)}$$

$$N_{\left(\frac{\dot{w} l}{U^2}\right)}$$

$$N_{\left(\frac{q l}{U}\right)}$$

$$N_{(\delta)}$$

$$N_{\left(\frac{\dot{\delta} l}{U}\right)}$$

These are then used to evaluate the coefficients of the characteristic equation according to the following formulas:

$$C_1 = \mu_1^2 i_B i_e \mathrm{I}$$

$$C_2 = \mu_1^2 i_B i_e Z_{\left(\frac{w}{U}\right)} + \mathrm{I} \times \mathrm{II} + M_{\left(\frac{\mathring{w} l}{U^2}\right)} \times \mathrm{III} + \mu_1 i_B Z_{\left(\frac{\mathring{\delta} l}{U}\right)} \times \mathrm{IV}$$

$$C_3 = Z_{\left(\frac{w}{U}\right)} \times \mathrm{II} + \mathrm{I} \times \mathrm{V} + M_{\left(\frac{w}{U}\right)} \times \mathrm{III} + M_{\left(\frac{\mathring{w} l}{U^2}\right)} \times \mathrm{VI}$$

$$+ \mu_1 i_B N_{\left(\frac{w}{U}\right)} Z_{\left(\frac{\mathring{\delta} l}{U}\right)} + \mathrm{IV} \times \mathrm{VII}$$

$$C_4 = Z_{\left(\frac{w}{U}\right)} \times \mathrm{V} + \mathrm{I} \times \mathrm{VIII} + M_{\left(\frac{w}{U}\right)} \times \mathrm{VI} + M_{\left(\frac{\mathring{w} l}{U^2}\right)} \times \mathrm{IX}$$

$$+ N_{\left(\frac{w}{U}\right)} \times \mathrm{VII} + \mathrm{IV} \times \mathrm{X}$$

$$C_5 = Z_{\left(\frac{w}{U}\right)} \times \mathrm{VIII} + M_{\left(\frac{w}{U}\right)} \times \mathrm{IX} + N_{\left(\frac{w}{U}\right)} \times \mathrm{X}$$

where

$$\mathrm{I} = Z_{\left(\frac{\mathring{w} l}{U^2}\right)} - \mu_1$$

$$\mathrm{II} = -\mu_1 i_e M_{\left(\frac{q l}{U}\right)} - \mu_1 i_B N_{\left(\frac{\mathring{\delta} l}{U}\right)} + R M_{\left(\frac{\mathring{\delta} l}{U}\right)}$$

$$\mathrm{III} = \mu_1 i_e S - R Z_{\left(\frac{\mathring{\delta} l}{U}\right)}$$

$$\mathrm{IV} = N_{\left(\frac{\mathring{w} l}{U^2}\right)} - h_e$$

$$V = M_{\left(\frac{ql}{U}\right)} N_{\left(\frac{\dot{\delta} l}{U}\right)} - \mu_1 i_B N_{(\delta)} - T M_{\left(\frac{\dot{\delta} l}{U}\right)} + R M_{(\delta)}$$

$$VI = T Z_{\left(\frac{\dot{\delta} l}{U}\right)} - S N_{\left(\frac{\dot{\delta} l}{U}\right)} - R Z_{(\delta)}$$

$$VII = S M_{\left(\frac{\dot{\delta} l}{U}\right)} - M_{\left(\frac{ql}{U}\right)} Z_{\left(\frac{\dot{\delta} l}{U}\right)} + \mu_1 i_B Z_{(\delta)}$$

$$VIII = M_{\left(\frac{ql}{U}\right)} N_{(\delta)} - T M_{(\delta)}$$

$$IX = T Z_{(\delta)} - S N_{(\delta)}$$

$$X = S M_{(\delta)} - M_{\left(\frac{ql}{U}\right)} Z_{(\delta)}$$

$$R = \mu_1 i_e + h_e l_e$$

$$S = Z_{\left(\frac{ql}{U}\right)} + \mu_1$$

$$T = N_{\left(\frac{ql}{U}\right)} + h_e$$

It now remains to solve the quartic equation

$$C_1 \lambda^4 + C_2 \lambda^3 + C_3 \lambda^2 + C_4 \lambda + C_5 = 0$$

Given the roots in the form

$$\lambda_{1,2} = \alpha_1 \pm i\beta_1$$

$$\lambda_{3,4} = \alpha_2 \pm i\beta_2$$

then the pertinent interesting quantities which may be evaluated for the two modes are the natural frequencies

$$\omega_{n_1} = \frac{2U}{c_w} \beta_1$$

$$\omega_{n_2} = \frac{2U}{c_w} \beta_2$$

and the times to $\frac{1}{2}$ amplitude

$$T(\tfrac{1}{2})_1 = \frac{0.693\, c_w}{U(-2\alpha_1)}$$

$$T(\tfrac{1}{2})_2 = \frac{0.693\, c_w}{U(-2\alpha_2)}$$

The period and number of cycles to $\frac{1}{2}$ amplitude of the motion may be determined from

$$P = \frac{2\pi}{\omega_n}$$

and

$$C(\tfrac{1}{2}) = \frac{T(\frac{1}{2})}{P}$$

After applying Laplace Transform techniques to the equations of motion, the response of the airplane and its control surface due to a suddenly released constant elevator deflection may be obtained from the solution of the following matrix equation:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \frac{w}{U}(s) \\ \frac{ql}{U}(s) \\ \delta(s) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \delta_o$$

where δ_o is the initial elevator deflection and

$$a_{11} = Z_{\left(\frac{w}{U}\right)} + \left[Z_{\left(\frac{\dot{w}l}{U^2}\right)} - \mu_1 \right] \frac{l}{U} s$$

$$a_{12} = Z_{\left(\frac{ql}{U}\right)} + \mu_1$$

$$a_{13} = Z_{(\delta)} + Z_{\left(\frac{\dot{\delta}l}{U}\right)} \frac{l}{U} s$$

$$a_{21} = M_{\left(\frac{w}{U}\right)} + M_{\left(\frac{\dot{w}l}{U^2}\right)} \frac{l}{U} s$$

$$a_{22} = M_{\left(\frac{ql}{U}\right)} - \mu_1 i_B \frac{l}{U} s$$

$$a_{23} = M_{(\delta)} + M_{\left(\frac{\dot{\delta}l}{U}\right)} \frac{l}{U} s$$

$$a_{31} = N_{\left(\frac{w}{U}\right)} + \left[N_{\left(\frac{\dot{w}l}{U^2}\right)} - h_e \right] \frac{l}{U} s$$

$$a_{32} = N_{\left(\frac{ql}{U}\right)} + h_e - (\mu_1 i_e + h_e l_e) \frac{l}{U} s$$

$$a_{33} = N_{(\delta)} + N_{\left(\frac{\dot{\delta}l}{U}\right)} \frac{l}{U} s - \mu_1 i_e \frac{l^2}{U^2} s^2$$

$$b_1 = \frac{l}{U} \left\{ \left[Z_{\left(\frac{\dot{w}l}{U^2}\right)} - \mu_1 \right] \frac{(w/U)_o}{\delta_o} + Z_{\left(\frac{\dot{\delta}l}{U}\right)} \right\}$$

$$b_2 = \frac{l}{U} \left[M_{\left(\frac{\dot{w}l}{U^2}\right)} \frac{(w/U)_o}{\delta_o} - \mu_1 i_B \frac{(ql/U)_o}{\delta_o} + M_{\left(\frac{\dot{\delta}l}{U}\right)} \right]$$

$$b_3 = \frac{l}{U} \left\{ \left[N_{\left(\frac{\dot{w}l}{U^2}\right)} - h_e \right] \frac{(w/U)_o}{\delta_o} - (\mu_1 i_e + h_e l_e) \frac{(ql/U)_o}{\delta_o} \right.$$
$$\left. + N_{\left(\frac{\delta l}{U}\right)} - \mu_1 i_e \frac{l}{U} s \right\}$$

The solutions for both the angle of attack and the control deflection may be written in the form

$$\frac{A_3 s^3 + A_2 s^2 + A_1 s + A_0}{[(s+\alpha)^2 + \beta^2][(s+\sigma)^2 + \gamma^2]}$$

for which the inverse Laplace Transform is

$$\frac{1}{\beta}\sqrt{\frac{[1] + \beta^2 [2]^2}{[(\sigma-\alpha)^2 + \gamma^2 - \beta^2]^2 + 4(\sigma-\alpha)^2}}\; e^{-\alpha t} \sin(\beta t + \psi_1)$$

$$+ \frac{1}{\gamma}\sqrt{\frac{[3]^2 + \gamma^2 [4]^2}{[(\sigma-\alpha)^2 + \beta^2 - \gamma^2]^2 + 4(\sigma-\alpha)^2}}\; e^{-\sigma t} \sin(\gamma t + \psi_2)$$

where

$$\psi_1 = \tan^{-1}\frac{\beta[2]}{[1]} - \tan^{-1}\frac{2(\sigma-\alpha)}{(\sigma-\alpha)^2 + \gamma^2 - \beta^2}$$

$$\psi_2 = \tan^{-1}\frac{\gamma[4]}{[3]} - \tan^{-1}\frac{2(\alpha-\sigma)}{(\sigma-\alpha)^2 + \beta^2 - \gamma^2}$$

$$[1] = A_0 - A_1\alpha + A_2(\alpha^2 - \beta^2) + A_3\alpha(3\beta^2 - \alpha^2)$$

$$[2] = A_1 - 2A_2\alpha + A_3(3\alpha^2 - \beta^2)$$

$$[3] = A_0 - A_1\sigma + A_2(\sigma^2 - \gamma^2) + A_3\sigma(3\gamma^2 - \sigma^2)$$

$$[4] = A_1 - 2A_2\sigma + A_3(3\sigma^2 - \gamma^2)$$

It is desirable to obtain some indication of the importance of considering nonsteady aerodynamics in the calculation of longitudinal, stick-free stability characteristics of a conventional fighter type airplane such as the F-80. For this purpose, the results of the preceding analysis will be compared with those obtained by the currently used procedure employing quasi-steady aerodynamic derivatives.

As was discussed previously in Section III, the effect of the downwash lag is accounted for in the quasi-steady theory through a moment due to rate of change of angle of attack. This term will be denoted by $\mathcal{M}^{*}_{\left(\frac{\dot{\omega} l}{U^2}\right)}$ and is estimated from the theory of Cowley and Glauert (27) as follows:

$$\mathcal{M}^{*}_{\left(\frac{\dot{\omega} l}{U^2}\right)} = -4\left(C_{L_\alpha}\right)_T \frac{S_T l_T^2}{S_W c_W^2} \eta_T \varepsilon_\alpha$$

Hence, if, in the equations of motion derived for the nonsteady aerodynamic theory, $\mathcal{M}_{\left(\frac{\dot{\omega} l}{U^2}\right)}$ is replaced by $\mathcal{M}^{*}_{\left(\frac{\dot{\omega} l}{U^2}\right)}$ and the zero frequency values are used for all the other derivatives, then the equations of motion according to quasi-steady theory are obtained.

$$\left[Z_{\left(\frac{w}{U}\right)} - \mu_1\lambda\right]\frac{w}{U} + \left[Z_{\left(\frac{q\ell}{U}\right)} + \mu_1\right]\frac{q\ell}{U} + Z_{(\delta)}\delta = 0$$

$$\left[M_{\left(\frac{w}{U}\right)} + M^{*}_{\left(\frac{\dot{w}\ell}{U^2}\right)}\lambda\right]\frac{w}{U} + \left[M_{\left(\frac{q\ell}{U}\right)} - \mu_1 i_B\lambda\right]\frac{q\ell}{U} + M_{(\delta)}\delta = 0$$

$$\left[N_{\left(\frac{w}{U}\right)} - h_e\lambda\right]\frac{w}{U} + \left[N_{\left(\frac{q\ell}{U}\right)} + h_e - (\mu_1 i_e + h_e\ell_e)\lambda\right]\frac{q\ell}{U}$$

$$+ \left[N_{(\delta)} - \mu_1 i_e\lambda^2\right]\delta = 0$$

The roots of the resultant characteristic equation are compared to those obtained from nonsteady flow theory in Figs. 66 and 67 in terms of ω_n, $T(\frac{1}{2})$, P and $C(\frac{1}{2})$ versus Mach number. The transient responses due to a suddenly released constant elevator deflection according to the two theories are shown for a Mach number of 0.70 in Figs. 68 and 69.

DISCUSSION AND CONCLUSIONS

Formulas and procedures have been derived which permit the design engineer to estimate the influence of unsteady aerodynamics on the subsonic, dynamic, longitudinal stability characteristics of a rigid conventional airplane. As discussed in Section IID, three-dimensionality actually permits a simplified, approximate analysis of nonsteady aerodynamics which cannot be rigorously justified for the two-dimensional problem.

In addition to the requirements discussed in the text that the aspect ratio should be neither too large nor too small and that the reduced frequency should be small with respect to unity, the theory of this thesis must be limited to Mach numbers below the divergence Mach number at which, in steady flow, the slope of the lift curve falls abruptly. Since this divergence Mach number cannot be predicted theoretically, the upper limit of accuracy on the theory cannot be fixed in general. Furthermore, airfoil thickness and/or transverse displacement effects become very important in this transonic regime.

In the development of the theory of this thesis, airfoil thickness has been completely neglected. This assumption is consistent with the approximation of linearized theory and any attempt to take account of thickness effects would not be compatible with the linearization. Furthermore, for modern airfoil shapes, a theory which allows for thickness without taking account of the effects of viscosity may actually result in poorer agreement with experimental values than a simple flat plate theory.

It is dangerous to attempt to draw any definite conclusions regarding the importance of aerodynamic lags on the basis of a single numerical example. So large a number of airplane parameters become intricately involved in the stability mechanism that their relative influences are practically inseparable. It would not be wise, therefore, to apply the results of this thesis indiscriminately to drastically different configurations such as highly swept wing or tailless aircraft. The F-80A is sufficiently typical of conventional, high-speed, straight-wing, fighter aircraft, however, to warrant consideration of the results of the preceding analysis.

In the example considered, the long-period, <u>stick-free</u> modes have reduced frequencies of the order of 0.01 - 0.02 which are comparable to the reduced frequencies of the short-period, <u>stick-fixed</u> modes. The dynamic similarity of these two cases is due to the fact that the elevator characteristics have very little influence in the former and do not enter at all in the latter. Although nonsteady flow considerations have considerable influence on the damping of the short-period, stick-free modes (reduced frequencies of the order of 0.13) as indicated by Figs. 66 and 68, it is apparent from Figs. 67 and 69 that they have relatively little effect on the long-period, stick-free modes. Furthermore, even though the dynamic characteristics of the control surface are quite different, as shown by the elevator motions in Fig. 68, the "low-pass" filter effect of the airplane itself prevents these differences from exerting any appreciable influence on the airplane motions as demonstrated in the angle-of-attack responses of Fig. 69.

It is important to point out again that, in the numerical example of this thesis, no damping due to structure or friction has been included in the control surface degree-of-freedom. Consequently, damping of this motion is primarily dependent upon the value of $N_{\left(\frac{\dot{\delta l}}{v}\right)}$ which is fundamentally a nonsteady aerodynamic derivative. This fact accounts for the tremendous differences between the results of the two theories as exhibited in Figs. 66 and 68. In terms of the change in damping ratio, the effect would have been much less pronounced if some structural damping had been considered in the analysis. Nevertheless, due to the relatively high frequencies involved, a precise aerodynamic treatment of the problem of estimating the response characteristics of the control surface should include unsteady flow effects. The design engineer must accept these results, however, with the same confidence as he does any theoretical control surface derivative -- whether it be static or dynamic. The results of a nonstationary flow theory would be used most practically as frequency corrections on emperical zero-frequency data whenever possible.

In summary, the indications are, therefore, that at least for the longitudinal motion of an F-80A type airplane;

(a) Nonsteady flow considerations, while showing considerable influence upon the control surface motion, have a negligibly small effect upon the airplane motion just as was previously concluded for the longitudinal stick-fixed problem discussed by the author in reference 18.

(b) In both the long-period and short-period longitudinal stick-free modes, aerodynamic lag effects manifest themselves particularly in the dampings and very little in the natural frequencies of the motions.

(c) For both modes, nonsteady aerodynamic theory predicts a higher value of damping than does quasi-steady theory.

(d) Nonsteady aerodynamics should be considered by the autopilot design engineer in his estimations of control surface dynamics but may be neglected when evaluating handling characteristics and maneuverability of the airplane.

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APPENDIX A

AN EXPRESSION FOR $\partial\Phi/\partial Z$

Given the required solution of the wave equation as in eqn. (26):

$$\frac{\partial\Phi}{\partial X} = \frac{1}{2\pi}\iint \frac{\partial\Phi(\xi,\eta,0)}{\partial\xi}\frac{\partial}{\partial Z}\left(\frac{e^{-ifr}}{r}\right)d\xi\, d\eta \tag{26}$$

where $r^2 \equiv (X-\xi)^2 + (Y-\eta)^2 + Z^2$ then differentiation of this equation with respect to Z results in the following:

$$\frac{\partial^2\Phi}{\partial Z\partial X} = \frac{1}{2\pi}\iint \frac{\partial\Phi(\xi,\eta,0)}{\partial\xi}\frac{\partial^2}{\partial Z^2}\left(\frac{e^{-ifr}}{r}\right)d\xi\, d\eta \tag{A-1}$$

But e^{-ifr}/r is a solution of eqn. (21) and, therefore, eqn. (A-1) may be written:

$$\frac{\partial^2\Phi}{\partial X\partial Z} = -\frac{1}{2\pi}\iint \frac{\partial\Phi(\xi,\eta,0)}{\partial\xi}\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}+f^2\right)\left(\frac{e^{-ifr}}{r}\right)d\xi\, d\eta \tag{A-2}$$

or

$$\frac{\partial^2\Phi}{\partial X\partial Z} = -\frac{1}{2\pi}\iint \frac{\partial\Phi(\xi,Y,0)}{\partial\xi}\left(\frac{\partial^2}{\partial X^2}+f^2\right)\left(\frac{e^{-ifr}}{r}\right)d\xi\, d\eta$$

$$-\frac{1}{2\pi}\iint\left[\frac{\partial\Phi(\xi,\eta,0)}{\partial\xi}-\frac{\partial\Phi(\xi,Y,0)}{\partial\xi}\right]\left(\frac{\partial^2}{\partial X^2}+f^2\right)\left(\frac{e^{-ifr}}{r}\right)d\xi\, d\eta$$

$$-\frac{1}{2\pi}\iint \frac{\partial\Phi(\xi,\eta,0)}{\partial\xi}\frac{\partial^2}{\partial Y^2}\left(\frac{e^{-ifr}}{r}\right)d\xi\, d\eta \tag{A-3}$$

After integrating the last term by parts and noting that $\partial^2/\partial Y^2 = \partial^2/\partial \eta^2$ then it may be shown that

$$\frac{\partial^2 \Phi}{\partial X \partial Z} = -\frac{1}{2\pi}\iint \frac{\partial \Phi(\xi, Y, 0)}{\partial \xi}\left(\frac{\partial^2}{\partial X^2} + f^2\right)\left(\frac{e^{-ifr}}{r}\right) d\xi\, d\eta$$

$$-\frac{1}{2\pi}\iint \left[\frac{\partial \Phi(\xi,\eta,0)}{\partial \xi} - \frac{\partial \Phi(\xi, Y, 0)}{\partial \xi}\right]\left(\frac{\partial^2}{\partial X^2}\right.$$

$$\left. + f^2\right)\left(\frac{e^{-ifr}}{r}\right) d\xi\, d\eta$$

$$-\frac{1}{2\pi}\iint \frac{\partial^2 \Phi(\xi,\eta,0)}{\partial \xi \partial \eta}\,\frac{\partial}{\partial Y}\left(\frac{e^{-ifr}}{r}\right) d\xi\, d\eta \qquad \text{(A-4)}$$

The first double integral on the right-hand side of eqn. (A-4) may be integrated with respect to η using the definition of the Hankel function

$$\int_{-\infty}^{\infty} \frac{e^{-if\sqrt{(X-\xi)^2 + (Y-\eta) + Z^2}}}{\sqrt{(X-\xi)^2 + (Y-\eta)^2 + Z^2}}\, d\eta$$

$$= -i\pi H_0^{(2)}\left(f\sqrt{(X-\xi)^2 + Z^2}\right) \qquad \text{(A-5)}$$

In the second double integral on the right-hand side of eqn. (A-4), it may be shown that

$$\int_{-\infty}^{\infty}\left[\frac{\partial \Phi(\xi,\eta,0)}{\partial \xi} - \frac{\partial \Phi(\xi, Y, 0)}{\partial \xi}\right]\left(\frac{e^{-ifr}}{r}\right) d\eta$$

$$= -\int_{-\infty}^{\infty} \frac{\partial^2 \Phi(\xi,\eta,0)}{\partial \xi \partial \eta}\,\frac{|Y-\eta|}{Y-\eta}\int_{-\infty}^{-|Y-\eta|} \frac{e^{-if\sqrt{(X-\xi)^2 + \zeta^2 + Z^2}}}{\sqrt{(X-\xi)^2 + \zeta^2 + Z^2}}\, d\zeta\, d\eta \qquad \text{(A-6)}$$

The introduction of eqns. (A-5) and (A-6) into eqn. (A-4) results in

$$\frac{\partial^2 \Phi}{\partial X \partial Z} = \frac{i}{2}\int \frac{\partial \Phi(\xi, Y, 0)}{\partial \xi}\left(\frac{\partial^2}{\partial X^2} + f^2\right) H_0^{(2)}\left(f\sqrt{(X-\xi)^2 + Z^2}\right) d\xi$$
$$+ \frac{1}{2\pi}\iint \frac{\partial^2 \Phi(\xi, \eta, 0)}{\partial \xi \, \partial \eta}\left[\frac{|Y-\eta|}{Y-\eta}\left(\frac{\partial^2}{\partial X^2}\right.\right.$$
$$\left. + f^2\right)\int_{-\infty}^{-|Y-\eta|} \frac{e^{-if\sqrt{(X-\xi)^2 + \varsigma^2 + Z^2}}}{\sqrt{(X-\xi)^2 + \varsigma^2 + Z^2}} d\varsigma$$
$$\left. + \frac{\partial}{\partial \eta}\left(\frac{e^{-ifr}}{r}\right)\right] d\xi d\eta \qquad \text{(A-7)}$$

Now by integrating eqn. (A-7) with respect to X from $-\infty$ to X and by using the condition of undisturbed flow far upstream of the airfoil, an expression for $\partial\Phi/\partial Z$ is derived as

$$\frac{\partial \Phi}{\partial Z} = \frac{i}{2}\int \frac{\partial \Phi(\xi, Y, 0)}{\partial \xi}\left\{\frac{\partial}{\partial X}\left[H_0^{(2)}\left(f \sqrt{(X-\xi)^2 + Z^2}\right)\right]\right.$$
$$\left. + f^2 \int_{-\infty}^{X} H_0^{(2)}\left(f\sqrt{(X'-\xi)^2 + Z^2}\right) dX'\right\} d\xi$$
$$+ \frac{1}{2\pi}\iint \frac{\partial^2 \Phi(\xi, \eta, 0)}{\partial \xi \partial \eta}\left\{\frac{|Y-\eta|}{Y-\eta}\frac{\partial}{\partial X}\int_{-\infty}^{-|Y-\eta|} \frac{e^{-if\sqrt{(X-\xi)^2 + \varsigma^2 + Z^2}}}{\sqrt{(X-\xi)^2 + \varsigma^2 + Z^2}} d\varsigma\right.$$
$$+ \frac{|Y-\eta|}{Y-\eta} f^2 \int_{-\infty}^{X}\int_{-\infty}^{-|Y-\eta|} \frac{e^{-if\sqrt{(X'-\xi)^2 + \varsigma^2 + Z^2}}}{\sqrt{(X'-\xi)^2 + \varsigma^2 + Z^2}} d\varsigma dX'$$
$$\left. + \int_{-\infty}^{X} \frac{\partial}{\partial \eta}\left[\frac{e^{-if\sqrt{(X'-\xi)^2 + (Y-\eta)^2 + Z^2}}}{\sqrt{(X'-\xi)^2 + (Y-\eta)^2 + Z^2}}\right] dX'\right\} d\xi d\eta \qquad \text{(A-8)}$$

APPENDIX B

APPROXIMATE EVALUATION OF $\oint_{X_L}^{\infty} \frac{\partial \bar{\Phi}}{\partial \xi} K d\xi$

Consider the first integral on the right-hand side of eqn. (28) and integrate by parts using the condition that $\bar{\Phi}$ is zero ahead of the wing leading edge. Then

$$\oint_{X_L}^{\infty} \frac{\partial \bar{\Phi}(\xi, Y, 0)}{\partial \xi} K\left[(X-\xi); f\right] d\xi$$

$$= \oint_{X_L}^{\infty} \frac{\partial \bar{\Phi}(\xi, Y, 0)}{\partial \xi}\left[-\frac{i\pi f}{2}\frac{|X-\xi|}{X-\xi} H_1^{(2)}(f|X-\xi|) + \frac{i\pi f}{2}\int_{-\infty}^{f|X-\xi|} H_0^{(2)}(|\zeta|)\,d\zeta\right] d\xi$$

$$= \bar{\Phi}(\xi, Y, 0)\left[-\frac{i\pi f}{2}\frac{|X-\xi|}{X-\xi} H_1^{(2)}(f|X-\xi|) + \frac{i\pi f}{2}\int_{-\infty}^{f|X-\xi|} H_0^{(2)}(|\zeta|)\,d\zeta\right]\Bigg|_{X_L}^{\infty} + \frac{i\pi f}{2}\oint_{X_L}^{\infty} \bar{\Phi}(\xi, Y, 0)\left\{\frac{\partial}{\partial \xi}\left[\frac{|X-\xi|}{X-\xi} H_1^{(2)}(f|X-\xi|)\right] + f\frac{|X-\xi|}{X-\xi} H_0^{(2)}(f|X-\xi|)\right\} d\xi$$

$$= \frac{i\pi f}{2}\int_{X_L}^{X} \bar{\Phi}(\xi, Y, 0)\left\{\frac{\partial H_1^{(2)}(f|X-\xi|)}{\partial \xi} + f\left[-\frac{1}{f}\frac{\partial H_1^{(2)}(f|X-\xi|)}{\partial \xi} + \frac{H_1^{(2)}(f|X-\xi|)}{f(X-\xi)}\right]\right\} d\xi$$

$$- \frac{i\pi f}{2}\int_{X}^{\infty} \bar{\Phi}(\xi, Y, 0)\left\{\frac{\partial H_1^{(2)}(f|X-\xi|)}{\partial \xi} + f\left[-\frac{1}{f}\frac{\partial H_1^{(2)}(f|X-\xi|)}{\partial \xi} - \frac{H_1^{(2)}(f|X-\xi|)}{f(X-\xi)}\right]\right\} d\xi$$

$$= \frac{i\pi f}{2}\oint_{X_L}^{\infty} \bar{\Phi}(\xi, Y, 0)\ \frac{H_1^{(2)}(f|X-\xi|)}{X-\xi}\, d\xi \qquad \text{(B-1)}$$

However, consider the following:

$$\frac{\partial}{\partial \xi}\left[\Phi \cdot |x-\xi| \cdot H_1^{(2)}(f|x-\xi|)\right] = \Phi\left[|x-\xi| \frac{\partial H_1^{(2)}(f|x-\xi|)}{\partial \xi} - \frac{|x-\xi|}{x-\xi} H_1^{(2)}(f|x-\xi|)\right] + |x-\xi| H_1^{(2)}(f|x-\xi|) \frac{\partial \Phi}{\partial \xi}$$

Therefore

$$\oint_{x_L}^{\infty} \frac{1}{|x-\xi|} \frac{\partial}{\partial \xi}\left[\Phi \cdot |x-\xi| \cdot H_1^{(2)}(f|x-\xi|)\right] d\xi$$

$$= \oint_{x_L}^{\infty} \frac{\partial}{\partial \xi}\left[\Phi H_1^{(2)}(f|x-\xi|)\right] d\xi - \oint_{x_L}^{\infty} \Phi \frac{H_1^{(2)}(f|x-\xi|)}{x-\xi} d\xi$$

$$= \Phi H_1^{(2)}(f|x-\xi|)\Big|_{x_L}^{\infty} - \oint_{x_L}^{\infty} \Phi \frac{H_1^{(2)}(f|x-\xi|)}{x-\xi} d\xi$$

$$= -\oint_{x_L}^{\infty} \Phi(\xi, Y, 0) \frac{H_1^{(2)}(f|x-\xi|)}{x-\xi} d\xi \qquad \text{(B-2)}$$

which is recognized as the integral occurring in eqn. (B-1), and so

$$\oint_{x_L}^{\infty} \frac{\partial \Phi(\xi, Y, 0)}{\partial \xi} K\left[(x-\xi); f\right] d\xi = -\frac{i\pi f}{2} \oint_{x_L}^{\infty} \frac{1}{|x-\xi|} \frac{\partial}{\partial \xi}\left[\Phi \cdot |x-\xi| \cdot H_1^{(2)}(f|x-\xi|)\right] d\xi$$

$$= \oint_{x_L}^{\infty} \frac{1}{|x-\xi|} \frac{\partial \Phi(\xi, Y, 0)}{\partial \xi} d\xi - \oint_{x_L}^{\infty} \frac{1}{|x-\xi|} \frac{\partial}{\partial \xi}\left\{\Phi(\xi, Y, 0)\left[1 + \frac{i\pi f|x-\xi|}{2} H_1^{(2)}(f|x-\xi|)\right]\right\} d\xi$$

$$= \oint_{x_L}^{\infty} \frac{1}{|x-\xi|} \frac{\partial \Phi(\xi, Y, 0)}{\partial \xi} d\xi - \oint_{x_L}^{x_T} \frac{1}{|x-\xi|} \frac{\partial}{\partial \xi}\left\{\Phi(\xi, Y, 0)\left[1 + \frac{i\pi f|x-\xi|}{2} H_1^{(2)}(f|x-\xi|)\right]\right\} d\xi + \frac{\bar{\Pi}^{(3)}(Y)}{2} \int_{x}^{x_T} \frac{1}{\xi - x} \frac{\partial}{\partial \xi}\left\{e^{-i\Omega\xi}\left[1 + \frac{i\pi f|x-\xi|}{2} H_1^{(2)}(f|x-\xi|)\right]\right\} d\xi - \frac{\bar{\Pi}^{(3)}(Y)}{2} \int_{x}^{\infty} \frac{1}{\xi - x} \frac{\partial}{\partial \xi}\left\{e^{-i\Omega\xi}\left[1 + \frac{i\pi f|x-\xi|}{2} H_1^{(2)}(f|x-\xi|)\right]\right\} d\xi \qquad \text{(B-3)}$$

since Φ (X, Y, 0) $= \frac{1}{2} \bar{\Gamma}^{(3)}(Y) e^{-i\Omega X}$ in the wake region by eqn. (16).

The integrand of the second and third integrals on the right-hand side of eqn. (B-3) may be expanded as follows:

$$1 + \frac{i\pi f|X-\xi|}{2} H_1^{(2)}(f|X-\xi|) = 1 + \frac{i\pi f|X-\xi|}{2} \Big\{ \frac{f|X-\xi|}{2} + \frac{2i}{\pi f|X-\xi|} + \frac{if|X-\xi|}{2\pi} - \frac{if|X-\xi|}{\pi} \ln \frac{\gamma f|X-\xi|}{2} + O(f^3) \Big\}$$

$$\doteq \frac{i\pi f^2(X-\xi)^2}{4} - \frac{f^2(X-\xi)^2}{4} + \frac{f^2(X-\xi)^2}{2} \ln \frac{\gamma f|X-\xi|}{2} \qquad \text{(B-4)}$$

By putting $z = \Omega(\xi - X)$ in the fourth integral on the right-hand side of eqn. (B-3), it may be written

$$-\frac{\Omega \bar{\Gamma}^{(3)}(Y) e^{-i\Omega X}}{2} \int_0^\infty \frac{1}{z} \frac{\partial}{\partial z} \Big\{ \Big[1 + \frac{i\pi M|z|}{2} H_1^{(2)}(Mz) \Big] e^{-iz} \Big\} dz$$

$$= -\lim_{\varepsilon \to 0} \frac{\Omega \bar{\Gamma}^{(3)} e^{-i\Omega X}}{2} \int_\varepsilon^\infty \frac{1}{z} \Big\{ -ie^{-iz} + \frac{i\pi M e^{-iz}}{2} \Big[-iz H_1^{(2)}(Mz) + H_1^{(2)}(Mz) + z \frac{\partial H_1^{(2)}(Mz)}{\partial z} \Big] \Big\} dz$$

$$= -\lim_{\varepsilon \to 0} \frac{\Omega \bar{\Gamma}^{(3)} e^{-i\Omega X}}{2} \int_\varepsilon^\infty \frac{1}{z} \Big\{ -ie^{-iz} + \frac{i\pi M z e^{-iz}}{2} \Big[-i H_1^{(2)} + M H_0^{(2)} \Big] \Big\} dz$$

$$= -\lim_{\varepsilon \to 0} \frac{\Omega \bar{\Gamma}^{(3)} e^{-i\Omega X}}{2} \int_\varepsilon^\infty \Big\{ -\frac{ie^{-iz}}{z} - \frac{\pi}{2} \frac{\partial}{\partial z} \Big[e^{-iz} H_0^{(2)}(Mz) \Big] - \frac{i\pi}{2} \beta^2 H_0^{(2)}(Mz) e^{-iz} \Big\} dz \qquad \text{(B-5)}$$

Consider each of the integrals involved in eqn. (B-5):

$$\int_{\varepsilon}^{\infty} \frac{e^{-iz}}{z}\,dz \doteq -\ln \gamma\varepsilon - \frac{i\pi}{2} + O(\varepsilon) \tag{B-6}$$

$$-\frac{\pi}{2}\Big[e^{-iz} H_0^{(2)}(Mz)\Big]_{\varepsilon}^{\infty} = \frac{\pi}{2} e^{-i\varepsilon} H_0^{(2)}(M\varepsilon) \tag{B-7}$$

$$\doteq \frac{\pi}{2} - i \ln \frac{\gamma M \varepsilon}{2}$$

$$\int_0^{\infty} H_0^{(2)}(Mz) e^{-iz}\,dz = -\frac{2}{\pi\beta} \ln \frac{1-\beta}{M} \tag{B-8}$$

Therefore, by substituting eqns. (B-6), (B-7), and (B-8) into eqn. (B-5) and by taking the limit as $\varepsilon \to 0$, the result is obtained that

$$-\frac{\Omega \bar{\Pi}^{(3)}(Y) e^{-i\Omega X}}{2} \int_0^{\infty} \frac{1}{z} \frac{\partial}{\partial z} \left\{\left[1 + \frac{i\pi M|z|}{2} H_1^{(2)}(Mz)\right] e^{-iz}\right\} dz$$

$$= \frac{i\Omega \bar{\Pi}^{(3)}(Y) e^{-i\Omega X}}{2} \mathcal{M} \tag{B-9}$$

where

$$\mathcal{M} = \ln \frac{M}{2} - \beta \ln \frac{1-\beta}{M} \tag{B-10}$$

Now, if only terms of order less than frequency squared are considered and if the appropriate results of eqns.

(B-4) and (B-9) are substituted into eqn. (B-3), then it may be shown that

$$\oint_{X_L}^{\infty} \frac{\partial \bar{\Phi}(\xi,Y,0)}{\partial \xi} K\left[(X-\xi);f\right] d\xi$$

$$\doteq \oint_{X_L}^{\infty} \frac{\partial \bar{\Phi}(\xi,Y,0)}{\partial \xi} \frac{d\xi}{X-\xi} + \frac{i\Omega \bar{F}^{(3)}(Y) e^{-i\Omega X}}{2} \mathcal{M} \qquad \text{(B-11)}$$

By splitting up the interval of integration in the first term on the right hand side of eqn. (B-11) and by using eqn. (23), this equation becomes:

$$\oint_{X_L}^{\infty} \frac{\partial \bar{\Phi}}{\partial \xi} K\, d\xi \doteq \oint_{X_L}^{X_T} \frac{\partial \bar{\Phi}}{\partial \xi} \frac{d\xi}{X-\xi}$$

$$- \frac{i\Omega \bar{F}^{(3)}}{2} \left(\int_{X_T}^{\infty} \frac{e^{-i\Omega \xi}}{X-\xi} d\xi - \mathcal{M} e^{-i\Omega X} \right) \qquad \text{(B-12)}$$

APPENDIX C

APPROXIMATE EVALUATION OF G

Let the function defined in eqn. (30) be written:

$$G = \frac{|Y-\eta|}{Y-\eta}(I_1 + I_2) + I_3 \tag{C-1}$$

where

$$I_1 = \frac{\partial}{\partial X}\int_{-\infty}^{-|Y-\eta|} \frac{e^{-if\sqrt{(X-\xi)^2+\zeta^2}}}{\sqrt{(X-\xi)^2+\zeta^2}}\,d\zeta \tag{C-2}$$

$$I_2 = f^2\int_{-\infty}^{X}\int_{-\infty}^{-|Y-\eta|} \frac{e^{-if\sqrt{(X'-\xi)^2+\zeta^2}}}{\sqrt{(X'-\xi)^2+\zeta^2}}\,d\zeta\, dX' \tag{C-3}$$

$$I_3 = \int_{-\infty}^{X}\frac{\partial}{\partial \eta}\left[\frac{e^{-if\sqrt{(X'-\xi)^2+(Y-\eta)^2}}}{\sqrt{(X'-\xi)^2+(Y-\eta)^2}}\right]dX' \tag{C-4}$$

Each of these will now be evaluated to terms of first order in frequency.

$$I_1 = \frac{\partial}{\partial X}\left(\int_{-\infty}^{0} + \int_{0}^{-|Y-\eta|}\right)\frac{e^{-if\sqrt{(X-\xi)^2+\zeta^2}}}{\sqrt{(X-\xi)^2+\zeta^2}}\,d\zeta$$

$$\doteq \frac{\partial}{\partial X}\left\{-\frac{i\pi}{2}H_0^{(2)}(f|X-\xi|) + \int_0^{-|Y-\eta|}\left[\frac{1}{\sqrt{(X-\xi)^2+\zeta^2}} - if\right]d\zeta\right\}$$

$$\doteq \frac{i\pi f}{2}\frac{|X-\xi|}{X-\xi}H_1^{(2)}(f|X-\xi|) + \frac{\partial}{\partial X}\left\{\ln\left[\frac{-|Y-\eta| + \sqrt{(X-\xi)^2+(Y-\eta)^2}}{|X-\xi|}\right]\right.$$

$$\left. + if|Y-\eta|\right\}$$

$$\doteq \frac{i\pi f}{2}\frac{|X-\xi|}{X-\xi}\left\{\frac{2i}{\pi f|X-\xi|} + \cdots\right\} + \frac{|Y-\eta|}{(X-\xi)\sqrt{(X-\xi)^2+(Y-\eta)^2}}$$

$$I_1 \doteq -\frac{1}{X-\xi} + \frac{|Y-\eta|}{(X-\xi)\sqrt{(X-\xi)^2+(Y-\eta)^2}} \tag{C-5}$$

$$I_2 = f^2 \int_{-\infty}^{-|Y-\eta|} \left(\int_{-\infty}^{0} + \int_{0}^{X-\xi} \right) \frac{e^{-if\sqrt{(X'')^2+\xi^2}}}{\sqrt{(X'')^2+\xi^2}}\, dX''\, d\xi$$

$$= f^2 \int_{-\infty}^{-|Y-\eta|} \left[-\frac{i\pi}{2} H_0^{(2)}(f|\xi|) \right] d\xi + f^2 \int_0^{X-\xi} \left(\int_{-\infty}^{0} \right.$$

$$\left. + \int_0^{-|Y-\eta|} \right) \frac{e^{-if\sqrt{(X'')^2+\xi^2}}}{\sqrt{(X'')^2+\xi^2}}\, d\xi\, dX''$$

$$= -\frac{i\pi f^2}{2} \int_{-\infty}^{-|Y-\eta|} H_0^{(2)}(f|\xi|)\, d\xi - \frac{i\pi f^2}{2} \int_0 H_0^{(2)}(f|X''|)\, dX''$$

$$+ f^2 \int_0^{X-\xi} \int_0^{-|Y-\eta|} \frac{e^{-if\sqrt{(X'')^2+\xi^2}}}{\sqrt{(X'')^2+\xi^2}}\, d\xi\, dX''$$

$$= -\frac{i\pi f}{2} \left[\int_{-\infty}^{-f|Y-\eta|} H_0^{(2)}(|\lambda|)\, d\lambda + \int_0^{f(X-\xi)} H_0^{(2)}(|\lambda|)\, d\lambda + O(f^2) \right]$$

$$I_2 \doteq -\frac{i\pi f}{2} \tag{C-6}$$

and

$$I_3 = \frac{\partial}{\partial \eta} \int_{-\infty}^{X-\xi} \frac{e^{-if\sqrt{\lambda^2+(Y-\eta)^2}}}{\sqrt{\lambda^2+(Y-\eta)^2}}\, d\lambda$$

$$\doteq \frac{\partial}{\partial \eta} \left\{ \int_{-\infty}^{0} \frac{e^{-if\sqrt{\lambda^2+(Y-\eta)^2}}}{\sqrt{\lambda^2+(Y-\eta)^2}}\, d\lambda + \int_0^{X-\xi} \left[\frac{1}{\sqrt{\lambda^2+(Y-\eta)^2}} - if \right] d\lambda \right\}$$

$$\doteq \frac{\partial}{\partial \eta} \left[-\frac{i\pi}{2} H_0^{(2)}(f|Y-\eta|) \right] + \int_0^{X-\xi} \frac{(Y-\eta)\, d\lambda}{\left[\lambda^2+(Y-\eta)^2\right]^{3/2}}$$

$$\doteq -\frac{i\pi f}{2} \frac{|Y-\eta|}{Y-\eta} H_1^{(2)}(f|Y-\eta|) + (Y-\eta) \left[\frac{\lambda}{(Y-\eta)^2 \sqrt{\lambda^2+(Y-\eta)^2}} \right]_0^{X-\xi}$$

$$\doteq -\frac{i\pi f}{2} \frac{|Y-\eta|}{Y-\eta} \left(\frac{2i}{\pi f |Y-\eta|} + \ldots \right) + \frac{X-\xi}{(Y-\eta)\sqrt{(X-\xi)^2+(Y-\eta)^2}}$$

$$I_3 \doteq \frac{1}{Y-\eta} + \frac{X-\xi}{(Y-\eta)\sqrt{(X-\xi)^2+(Y-\eta)^2}} \tag{C-7}$$

APPENDIX D

REDUCTION OF $\iint\limits_{\text{WAKE}} \frac{d}{d\eta}(\bar{\Gamma}^{(3)})\, e^{-i\Omega\xi}\, G\, d\xi\, d\eta$

TO AN APPROXIMATING SINGLE INTEGRAL

Consider the double integration over the wake region in eqn. (28):

$$\iint\limits_{\text{WAKE}} \frac{d}{d\eta}(\bar{\Gamma}^{(3)})\, e^{-i\Omega\xi}\, G\, d\xi\, d\eta = \int \left\{ \int_X^{\infty} \cdots d\xi - \int_X^{X_T} \cdots d\xi \right\} d\eta \qquad \text{(D-1)}$$

The first integral on the right-hand side of eqn. (D-1) may be written

$$\int \frac{d}{d\eta}(\bar{\Gamma}^{(3)}) \left(\int_X^{\infty} e^{-i\Omega\xi}\, G\, d\xi \right) d\eta$$

$$= \int \frac{d}{d\eta}(\bar{\Gamma}^{(3)}) \left[\int_0^{\infty} e^{-i\Omega(X+\sigma)} G(-\sigma, Y-\eta; f)\, d\sigma \right] d\eta$$

$$= e^{-i\Omega X} \int \frac{d}{d\eta}(\bar{\Gamma}^{(3)})\, F_M\, (Y-\eta)\, d\eta \qquad \text{(D-2)}$$

where

$$F_M = \int_0^{\infty} e^{-i\Omega\sigma} G(-\sigma, Y-\eta; f)\, d\sigma$$

$$= \frac{|Y-\eta|}{Y-\eta} \int_0^{\infty} e^{-i\Omega\sigma} \left[\int_{-\infty}^{-|Y-\eta|} \frac{\sigma e^{-if\sqrt{\sigma^2+\xi^2}}}{\sigma^2+\xi^2} \left(\frac{1}{\sqrt{\sigma^2+\xi^2}} + if \right) d\xi \right.$$

$$+ f^2 \int_{-\infty}^{-\sigma} \int_{-\infty}^{-|Y-\eta|} \frac{e^{-if\sqrt{\tau^2+\xi^2}}}{\sqrt{\tau^2+\xi^2}}\, d\xi\, d\tau$$

$$\left. + \int_{-\infty}^{-\sigma} \frac{|Y-\eta|\, e^{-if\sqrt{\tau^2+(Y-\eta)^2}}}{\tau^2+(Y-\eta)^2} \left(\frac{1}{\sqrt{\tau^2+(Y-\eta)^2}} + if \right) d\tau \right] d\sigma \qquad \text{(D-3)}$$

This function is evaluated approximately in Appendix E.

The second integral on the right-hand side of eqn. (D-1) is comparable to the integration over the wing region discussed in Appendix C, and so

$$\oint \frac{d}{d\eta}(\bar{\Gamma}^{(3)}) \left(\int_X^{X_T} e^{-i\Omega\xi} G \, d\xi \right) d\eta \doteq \oint \frac{d}{d\eta}(\bar{\Gamma}^{(3)}) \frac{|Y-\eta|}{Y-\eta} \left(\frac{1}{|Y-\eta|} - \frac{i\pi f}{2} \right) \int_X^{X_T} e^{-i\Omega\xi} \, d\xi \, d\eta$$

$$\doteq -\oint \frac{d}{d\eta}(\bar{\Gamma}^{(3)}) \frac{|Y-\eta|}{Y-\eta} \left(\frac{1}{|Y-\eta|} - \frac{i\pi f}{2} \right) \frac{e^{-i\Omega X_T} - e^{-i\Omega X}}{i\Omega} \, d\eta \qquad \text{(D-4)}$$

Therefore, upon introducing eqns. (D-2) and (D-4) into eqn. (D-1), it is found that

$$\iint_{\text{WAKE}} \frac{d}{d\eta}(\bar{\Gamma}^{(3)}) e^{-i\Omega\xi} G \, d\xi \, d\eta$$

$$\doteq e^{-i\Omega X} \int \frac{d}{d\eta}(\bar{\Gamma}^{(3)}) F_M (Y-\eta) \, d\eta$$

$$+ \oint \frac{e^{-i\Omega X_T} - e^{-i\Omega X}}{i\Omega} \frac{d}{d\eta}(\bar{\Gamma}^{(3)}) \frac{|Y-\eta|}{Y-\eta} \left[\frac{1}{|Y-\eta|} - \frac{i\pi f}{2} \right] d\eta \qquad \text{(D-5)}$$

APPENDIX E

APPROXIMATE EVALUATION OF THE FUNCTION F_M

The function F_M which arises in Appendix D may be written

$$F_M(z) = \frac{|z|}{z}\int_0^\infty e^{-i\sigma}\left[\int_{|z|}^\infty \frac{\sigma e^{-iM\sqrt{\sigma^2+\zeta^2}}}{\sigma^2+\zeta^2}\left(\frac{1}{\sqrt{\sigma^2+\zeta^2}}+iM\right)d\zeta\right.$$

$$+ M^2\int_\sigma^\infty\int_{|z|}^\infty \frac{e^{-iM\sqrt{\tau^2+\zeta^2}}}{\tau^2+\zeta^2}\,d\zeta\,d\tau$$

$$\left.+\int_\sigma^\infty \frac{|z|\,e^{-iM\sqrt{\tau^2+z^2}}}{\tau^2+z^2}\left(\frac{1}{\sqrt{\tau^2+z^2}}+iM\right)d\tau\right]d\sigma \qquad \text{(E-1)}$$

where $z = \Omega\,(Y - \eta)$. This function will now be evaluated approximately such that terms of the order of z will be neglected.

Let $F_M(z)$ be written as follows:

$$F_M(z) = \frac{|z|}{z}\,F(|z|) + \Delta F_M(z) \qquad \text{(E-2)}$$

where

$$F(|z|) = \int_0^\infty e^{-i\lambda}\left(\frac{1}{|z|}+\frac{1}{\lambda}-\frac{\sqrt{z^2+\lambda^2}}{|z|\lambda}\right)d\lambda \qquad \text{(E-3)}$$

is the corresponding function which occurs in the incompressible problem and is often referred to as the Cicala function (30).

$$F(z) = \int_0^{|z|} e^{-i\lambda}\left(\frac{1}{\lambda} - \frac{\sqrt{z^2+\lambda^2}}{|z|\lambda}\right)d\lambda + \int_{|z|}^{\infty} e^{-i\lambda}\left(\frac{1}{|z|} - \frac{\sqrt{z^2+\lambda^2}}{|z|\lambda}\right)d\lambda$$

$$+ \int_{|z|}^{\infty} \frac{e^{-i\lambda}}{\lambda}d\lambda + \int_0^{|z|} \frac{e^{-i\lambda}}{|z|}d\lambda$$

$$= -\int_0^{|z|} \frac{e^{-i\lambda}}{\lambda} \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2(2n+2)}\left(\frac{\lambda^2}{z^2}\right)^{n+1} d\lambda$$

$$- \int_{|z|}^{\infty} \frac{e^{-i\lambda}}{|z|} \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2(2n+2)}\left(\frac{z^2}{\lambda^2}\right)^{n+1} d\lambda$$

$$+ \int_{|z|}^{\infty} \frac{e^{-i\lambda}}{\lambda}d\lambda + \int_0^{|z|} \frac{e^{-i\lambda}}{|z|}d\lambda$$

$$= -\sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2(2n+2)\,|z|^{2n+2}} \int_0^{|z|} e^{-i\lambda}\lambda^{2n+1} d\lambda$$

$$- \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!\ |z|^{2n+1}}{2^{2n}(n!)^2(2n+2)} \int_{|z|}^{\infty} \frac{e^{-i\lambda}}{\lambda^{2n+2}} d\lambda$$

$$+ \int_{|z|}^{\infty} \frac{e^{-i\lambda}}{\lambda}d\lambda + \int_0^{|z|} \frac{e^{-i\lambda}}{|z|}d\lambda$$

$$F(|z|) = -\sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2 (2n+2)^2}$$

$$-\sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2 (2n+1)(2n+2)} - \ln \gamma |z|$$

$$-\frac{i\pi}{2} + 1 + \text{terms which} \to 0 \text{ as } z \to 0$$

$$\doteq -\sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2 (2n+2)} \left(1 - \frac{2n+1}{2n+2}\right)$$

$$-\sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2 (2n+2)} \left(\frac{2n+2}{2n+1} - 1\right) - \ln \gamma |z| - \frac{i\pi}{2} + 1$$

$$= -\sum_{m=1}^{\infty} (-1)^m \frac{(2m)!}{2^{2m}(m!)^2 (2m)}$$

$$-\sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2 (2n+1)} - \ln \gamma |z| - \frac{i\pi}{2} + 1$$

$$F(|z|) \doteq 1 - \frac{i\pi}{2} - \ln 2\gamma |z| \qquad \text{(E-4)}$$

$$\Delta F_M(z) = \frac{|z|}{z}\left\{\int_0^\infty e^{-i\sigma}\int_{|z|}^\infty\left[\frac{\sigma e^{-iM\sqrt{\sigma^2+\xi^2}}}{(\sigma^2+\xi^2)^{3/2}} - \frac{\sigma}{(\sigma^2+\xi^2)^{3/2}}\right.\right.$$

$$\left.+\frac{iM\sigma e^{-iM\sqrt{\sigma^2+\xi^2}}}{\sigma^2+\xi^2}\right]d\xi\, d\sigma$$

$$+M^2\int_0^\infty e^{-i\sigma}\int_\sigma^\infty\int_{|z|}^\infty \frac{e^{-iM\sqrt{\tau^2+\xi^2}}}{\sqrt{\tau^2+\xi^2}}\, d\xi\, d\tau\, d\sigma$$

$$+\int_0^\infty e^{-i\sigma}\int_\sigma^\infty\left[\frac{|z|e^{-iM\sqrt{\tau^2+z^2}}}{(\tau^2+z^2)^{3/2}} - \frac{|z|}{(\tau^2+z^2)^{3/2}}\right.$$

$$\left.\left.+\frac{iM|z|e^{-iM\sqrt{\tau^2+z^2}}}{\tau^2+z^2}\right]d\tau\, d\sigma\right\} \qquad \text{(E-5)}$$

$$= \frac{|z|}{z}\left(I_1 + I_2 + I_3\right)$$

Let

$I_1 = I_{1a} + I_{1b}$ where

$$I_{1a} = \int_0^\infty e^{-i\sigma}\int_{|z|}^\infty\left[\frac{\sigma e^{-iM\sqrt{\sigma^2+\xi^2}} - \sigma}{(\sigma^2+\xi^2)^{3/2}}\right]d\xi\, d\sigma$$

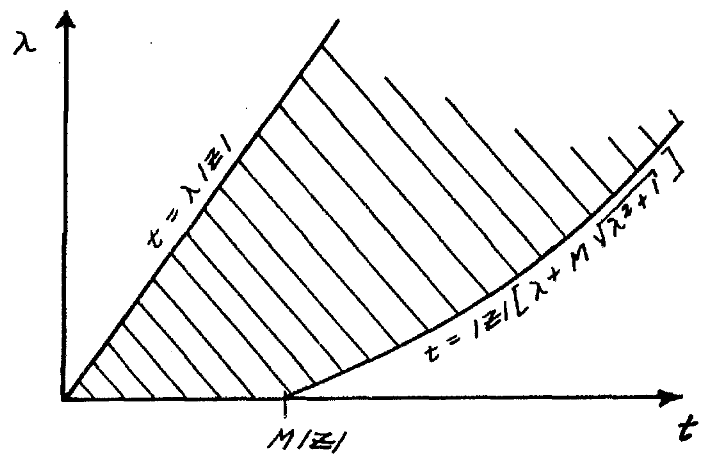
Let $\sigma = |z|\lambda$

$$I_{1a} = \int_0^\infty \frac{\lambda}{(\lambda^2+1)^{3/2}}\int_{|z|}^\infty\left[\frac{e^{-i\xi(\lambda+M\sqrt{\lambda^2+1})} - e^{-i\lambda\xi}}{\xi}\right]d\xi\, d\lambda$$

$$= \int_0^\infty \frac{\lambda}{(\lambda^2+1)^{3/2}}\left[\int_{(\lambda+M\sqrt{\lambda^2+1})|z|}^\infty \frac{e^{-it}}{t}\,dt - \int_{\lambda|z|}^\infty \frac{e^{-it}}{t}\,dt\right]d\lambda$$

$$= \int_0^\infty \frac{\lambda}{(\lambda^2+1)^{3/2}}\int_{\lambda|z|}^{(\lambda+M\sqrt{\lambda^2+1})|z|} \frac{e^{-it}}{t}\,dt\, d\lambda$$

The region of integration for I_{1a} is shown in the sketch below:



By referring to this sketch, the order of integration may be changed so that

$$I_{1a} = \int_0^{M|z|} \frac{e^{-it}}{t}\, dt \int_0^{t/|z|} \frac{\lambda\, d\lambda}{(\lambda^2+1)^{3/2}} + \int_{M|z|}^{\infty} \frac{e^{-it}}{t}\, dt \int_{\lambda_1}^{t/|z|} \frac{\lambda\, d\lambda}{(\lambda^2+1)^{3/2}}$$

where

$$\lambda_1 = \frac{t - M\sqrt{t^2 + z^2(1-M^2)}}{(1-M^2)\,|z|}$$

or

$$I_{1a} = \overbrace{-\int_0^{M|z|} \frac{e^{-it}}{t}\left[\frac{1}{\sqrt{\frac{t^2}{z^2}+1}} - 1\right] dt}^{I_{1a'}} \; \overbrace{-\int_{M|z|}^{\infty} \frac{e^{-it}}{t\sqrt{\frac{t^2}{z^2}+1}}\, dt}^{I_{1a''}}$$

$$\underbrace{+|z| \int_{M|z|}^{\infty} \frac{e^{-it}\left(Mt + \sqrt{t^2 + z^2(1-M^2)}\right)}{t(t^2+z^2)}\, dt}_{I_{1a'''}}$$

$$I_{1a'} = -\int_0^M e^{-i|z|\mu} \sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}(-1)^n \mu^{2n-1}\, d\mu$$

$$= -\sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}(-1)^n \frac{M^{2n}}{2n} + \text{terms which} \to 0 \text{ as } z \to 0$$

$$I_{1a'} \doteq \ln\left(\frac{1+\sqrt{1+M^2}}{2}\right)$$

$$I_{1a''} = -\int_{M|z|}^{\infty} \frac{e^{-it}}{t\sqrt{\left(\frac{t}{z}\right)^2+1}} = -\int_M^1 \frac{e^{-i|z|\mu}}{\mu\sqrt{\mu^2+1}}d\mu - |z|\int_{|z|}^{\infty} \frac{e^{-it}}{t^2\sqrt{1+\left(\frac{z}{t}\right)^2}}dt$$

$$= -\int_M^1 \frac{e^{-i|z|\mu}}{\mu\sqrt{\mu^2+1}}d\mu - \sum_{n=0}^{\infty}(-1)^n \frac{(2n)!\,|z|^{2n+1}}{2^{2n}(n!)^2}\int_{|z|}^{\infty}\frac{e^{-it}}{t^{2n+2}}dt$$

$$= \ln\left(1+\sqrt{2}\right) - \ln\left(\frac{1+\sqrt{1+M^2}}{M}\right) - \sum_{n=0}^{\infty}(-1)^n \frac{(2n)!}{2^{2n}(n!)^2(2n+1)}$$

$$+ \text{terms which} \to 0 \text{ as } z \to 0$$

$$I_{1a''} \doteq -\ln\left(\frac{1+\sqrt{1+M^2}}{M}\right)$$

$$I_{1a'''} = |z|\int_{M|z|}^{\infty} \frac{e^{-it}\left[Mt+\sqrt{t^2+z^2(1-M^2)}\right]}{t\,(t^2+z^2)}dt$$

which will be used in this form later.

Hence

$$I_{1a} \doteq \ln\left(\frac{1+\sqrt{1+M^2}}{2}\right) - \ln\left(\frac{1+\sqrt{1+M^2}}{M}\right) + I_{1a}'''$$

$$= \ln \frac{M}{2} + I_{1a}'''$$

$$I_{1b} = iM \int_0^\infty e^{-i\sigma} \sigma \int_{|z|}^\infty \frac{e^{-iM\sqrt{\sigma^2+\varsigma^2}}}{\sigma^2+\varsigma^2} \, d\varsigma \, d\sigma$$

Let $\sigma = |z| \lambda$

$$I_{1b} = iM \int_{|z|}^\infty \int_0^\infty \frac{\lambda e^{-i\varsigma(\lambda + M\sqrt{\lambda^2+1})}}{\lambda^2+1} \, d\lambda \, d\varsigma$$

$$= iM \int_{|z|}^\infty \int_0^\infty \frac{d}{d\lambda}\left[e^{-i\varsigma(\lambda+M\sqrt{\lambda^2+1})}\right] \frac{i\lambda\sqrt{\lambda^2+1}}{(\lambda^2+1)(\sqrt{\lambda^2+1}+M\lambda)\varsigma} \, d\lambda \, d\varsigma$$

$$I_{1b} = M \int_{|z|}^\infty \int_0^\infty \frac{e^{-i\varsigma(\lambda+M\sqrt{\lambda^2+1})}}{\varsigma} \left[\frac{1}{(\lambda^2+1)(\sqrt{\lambda^2+1}+M\lambda)^2}\right.$$

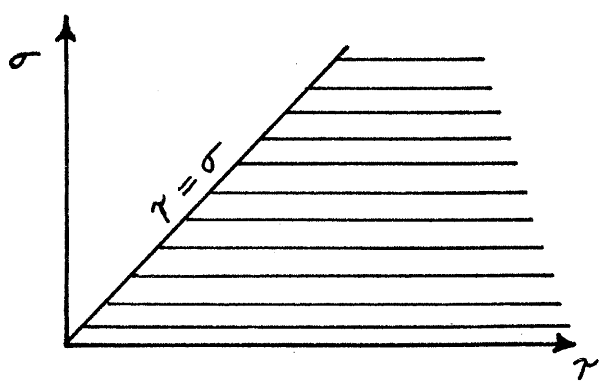
$$\left. - \frac{\lambda^2}{(\lambda^2+1)^{3/2}(\sqrt{\lambda^2+1}+M\lambda)}\right] d\lambda \, d\varsigma$$

which will be used in this form later.

$$I_2 = M^2 \int_0^{\infty} e^{-i\sigma} \int_{\sigma}^{\infty} \int_{|z|}^{\infty} \frac{e^{-iM\sqrt{\tau^2+\zeta^2}}}{\sqrt{\tau^2+\zeta^2}}\, d\zeta\, d\tau\, d\sigma$$

$$\frac{\partial I_2}{\partial |z|} = -M^2 \int_0^{\infty} e^{-i\sigma} \int_{\sigma}^{\infty} \frac{e^{-iM\sqrt{\tau^2+z^2}}}{\sqrt{\tau^2+z^2}}\, d\tau\, d\sigma$$

The region of integration for $\partial I_2/\partial |z|$ is shown in the sketch below.



$$\frac{\partial I_2}{\partial |z|} = -M^2 \int_0^{\infty} \frac{e^{-iM\sqrt{\tau^2+z^2}}}{\sqrt{\tau^2+z^2}} \int_0^{\tau} e^{-i\sigma}\, d\sigma\, d\tau$$

$$= -iM^2 \int_0^{\infty} \frac{e^{-i\tau} e^{-iM\sqrt{\tau^2+z^2}}}{\sqrt{\tau^2+z^2}}\, d\tau$$

$$+ iM^2 \int_0^{\infty} \frac{e^{-iM\sqrt{\tau^2+z^2}}}{\sqrt{\tau^2+z^2}}\, d\tau$$

Let $r = |z|\lambda$ in the first integral.

$$\frac{\partial I_2}{\partial |z|} = -iM^2 \int_0^\infty \frac{e^{-i|z|(\lambda + M\sqrt{\lambda^2+1})}}{\sqrt{\lambda^2+1}} d\lambda + iM^2 \int_0^\infty \frac{e^{-iM\sqrt{r^2+z^2}}}{\sqrt{r^2+z^2}} dr$$

$$= -iM^2 \int_0^\infty \frac{e^{-i|z|(\lambda+M\sqrt{\lambda^2+1})}}{\sqrt{\lambda^2+1}} d\lambda - \frac{\pi}{2} M^2 H_0^{(2)}(M|z|)$$

$$I_2 = iM^2 \int_{|z|}^\infty \int_0^\infty \frac{e^{-i\varsigma(\lambda+M\sqrt{\lambda^2+1})}}{\sqrt{\lambda^2+1}} d\lambda d\varsigma + \frac{\pi}{2} M^2 \int_{|z|}^\infty H_0^{(2)}(M\varsigma) d\varsigma$$

$$= iM^2 \int_{|z|}^\infty \int_0^\infty \frac{e^{-i\varsigma(\lambda+M\sqrt{\lambda^2+1})}}{\sqrt{\lambda^2+1}} d\lambda d\varsigma$$

$$+ \frac{\pi M}{2} \int_0^\infty H_0^{(2)}(\lambda) d\lambda - \frac{\pi}{2} M^2 \int_0^{|z|} H_0^{(2)}(M\varsigma) d\varsigma$$

$$= iM^2 \int_{|z|}^\infty \int_0^\infty \frac{e^{-i\varsigma(\lambda+M\sqrt{\lambda^2+1})}}{\sqrt{\lambda^2+1}} d\lambda d\varsigma + \frac{\pi M}{2}$$

$+$ *terms which* $\to 0$ *as* $z \to 0$

$$\doteq -M^2 \int_{|z|}^\infty \left\{ \frac{1}{\varsigma} \int_0^\infty \frac{d}{d\lambda}\left[e^{-i\varsigma(\lambda+M\sqrt{\lambda^2+1})}\right] \frac{d\lambda}{\sqrt{\lambda^2+1}+M\lambda} \right\} d\varsigma + \frac{\pi M}{2}$$

$$= -M^2 \int_{|z|}^\infty \left[-e^{-iM\varsigma} + \int_0^\infty e^{-i\varsigma(\lambda+M\sqrt{\lambda^2+1})} \frac{\lambda + M\sqrt{\lambda^2+1}}{\sqrt{\lambda^2+1}\,(\sqrt{\lambda^2+1}+M\lambda)^2} d\lambda \right] \frac{d\varsigma}{\varsigma}$$

$$+ \frac{\pi M}{2}$$

$$I_2 \doteq M^2 \int_{|z|}^\infty \frac{e^{-iM\varsigma}}{\varsigma} d\varsigma$$

$$- M^2 \int_0^\infty \left[\frac{\lambda + M\sqrt{\lambda^2+1}}{\sqrt{\lambda^2+1}\,(\sqrt{\lambda^2+1}+M\lambda)^2} \right] \int_{|z|}^\infty \frac{e^{-i\varsigma(\lambda+M\sqrt{\lambda^2+1})}}{\varsigma} d\varsigma d\lambda + \frac{\pi M}{2}$$

Combine this result with I_{1b}.

$$I_2 + I_{1b} \doteq M^2 \int_{|z|}^{\infty} \frac{e^{-iM\varsigma}}{\varsigma}\, d\varsigma + M \int_0^{\infty} \left[-\frac{\lambda^2}{(\lambda^2+1)^{3/2}\left(\sqrt{\lambda^2+1}+M\lambda\right)} \right.$$

$$+ \frac{1}{(\lambda^2+1)\left(\sqrt{\lambda^2+1}+M\lambda\right)^2}$$

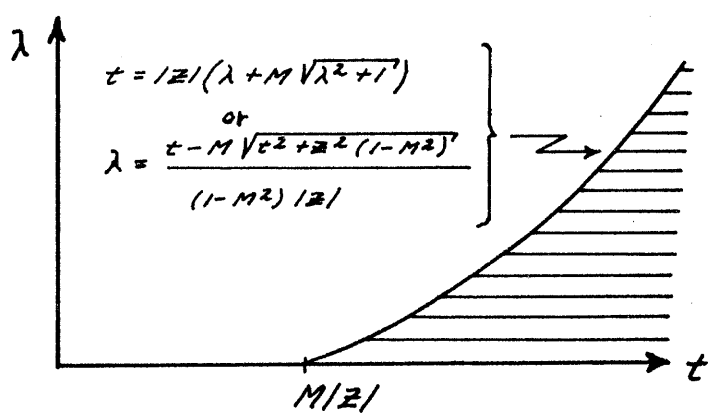
$$\left. - \frac{M\left(\lambda+M\sqrt{\lambda^2+1}\right)}{\sqrt{\lambda^2+1}\left(\sqrt{\lambda^2+1}+M\lambda\right)^2} \right] \int_{|z|}^{\infty} \frac{e^{-i\varsigma\left(\lambda+M\sqrt{\lambda^2+1}\right)}}{\varsigma}\, d\varsigma\, d\lambda + \frac{\pi M}{2}$$

$$= M^2 \int_{|z|}^{\infty} \frac{e^{-iM\varsigma}}{\varsigma}\, d\varsigma + M \int_0^{\infty} \left[-\frac{1}{\lambda^2+1} \right.$$

$$+ \frac{1}{(\lambda^2+1)^{3/2}\left(\sqrt{\lambda^2+1}+M\lambda\right)}$$

$$\left. + \frac{(1-M^2)}{(\lambda^2+1)\left(\sqrt{\lambda^2+1}+M\lambda\right)^2} \right] \int_{|z|\left(\lambda+M\sqrt{\lambda^2+1}\right)}^{\infty} \frac{e^{-it}}{t}\, dt\, d\lambda + \frac{\pi M}{2}$$

The region of integration for the double integral is shown in the sketch below:



By referring to this sketch, the order of integration may be changed so that

$$I_2 + I_{1b} \doteq M^2 \int_{|z|}^{\infty} \frac{e^{-iM\zeta}}{\zeta}\, d\zeta$$

$$+ M \int_{M|z|}^{\infty} \frac{e^{-it}}{t} \int_0^{\lambda_1} \left[-\frac{1}{\lambda^2+1} + \frac{1}{(\lambda^2+1)^{3/2}\left(\sqrt{\lambda^2+1}+M\lambda\right)} \right.$$

$$\left. + \frac{(1-M^2)}{(\lambda^2+1)\left(\sqrt{\lambda^2+1}+M\lambda\right)^2} \right] d\lambda\, dt + \frac{\pi M}{2}$$

$$I_2 + I_{1b} \doteq -M|z| \int_{M|z|}^{\infty} \frac{e^{-it}\left[Mt + \sqrt{t^2+z^2(1-M^2)}\right]}{(t^2+z^2)\sqrt{t^2+z^2(1-M^2)}}\, dt + \frac{\pi M}{2}$$

Integrate by parts using

$$\mu = \frac{Mt + \sqrt{t^2+z^2(1-M^2)}}{(t^2+z^2)\sqrt{t^2+z^2(1-M^2)}} \quad \text{and} \quad dv = e^{-it}dt .$$

Then the following result is obtained:

$$I_2 + I_{1b} \doteq -M|z| \left\{ ie^{-it} \frac{\left[Mt + \sqrt{t^2+z^2(1-M^2)}\right]}{(t^2+z^2)\sqrt{t^2+z^2(1-M^2)}} \Bigg|_{M|z|}^{\infty} \right.$$

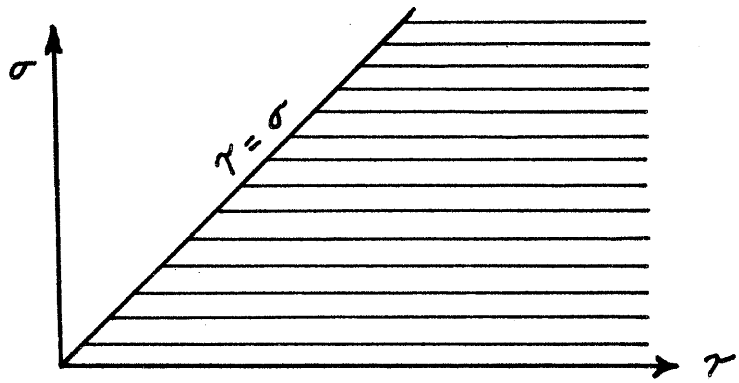
$$- iM \int_{M|z|}^{\infty} e^{-it} \left[\frac{-2t^4 + z^2(z^2-t^2)(1-M^2)}{(t^2+z^2)^2\left[t^2+z^2(1-M^2)\right]^{3/2}} \right] dt$$

$$\left. + 2i \int_{M|z|}^{\infty} \frac{t}{(t^2+z^2)^2}\, dt \right\} + \frac{\pi M}{2}$$

which will be used in this form later.

$$I_3 = |z| \int_0^\infty e^{-i\sigma} \int_\sigma^\infty \left[\frac{e^{-iM\sqrt{\tau^2+z^2}}}{(\tau^2+z^2)^{3/2}} - \frac{1}{(\tau^2+z^2)^{3/2}} + \frac{iMe^{-iM\sqrt{\tau^2+z^2}}}{\tau^2+z^2} \right] d\tau \, d\sigma$$

The region of integration for I_3 is shown in the sketch below:



By referring to this sketch, the order of integration may be changed so that

$$I_3 = |z| \int_0^\infty \left[\frac{e^{-iM\sqrt{\tau^2+z^2}} - 1}{(\tau^2+z^2)^{3/2}} + \frac{iMe^{-iM\sqrt{\tau^2+z^2}}}{\tau^2+z^2} \right] \int_0^\tau e^{-i\sigma} \, d\sigma \, d\tau$$

$$= i|z| \int_0^\infty \left[\frac{e^{-i\tau - iM\sqrt{\tau^2+z^2}}}{(\tau^2+z^2)^{3/2}} + \frac{iMe^{-i\tau - iM\sqrt{\tau^2+z^2}}}{\tau^2+z^2} - \frac{e^{-i\tau}}{(\tau^2+z^2)^{3/2}} - \frac{e^{-iM\sqrt{\tau^2+z^2}}}{(\tau^2+z^2)^{3/2}} - \frac{iMe^{-iM\sqrt{\tau^2+z^2}}}{\tau^2+z^2} + \frac{1}{(\tau^2+z^2)^{3/2}} \right] d\tau$$

Let this be written

$$I_3 = I_{3a} + I_{3b} + I_{3c} + I_{3d} + I_{3e} + I_{3f}$$

$$I_{3a} = i z^2 \int_0^\infty \frac{e^{-i|z|(\lambda + M\sqrt{\lambda^2+1})}}{|z|^3 \, (\lambda^2+1)^{3/2}} \, d\lambda$$

Let $t = |z|(\lambda + M\sqrt{\lambda^2+1})$ then

$$I_{3a} = (1-M^2)^2 |z| \int_{M|z|}^\infty \frac{e^{-it}}{(Mt - \sqrt{t^2+z^2(1-M^2)})^2 \sqrt{t^2+z^2(1-M^2)}} \, dt$$

$$= i|z| \int_{M|z|}^\infty \frac{e^{-it}[Mt + \sqrt{t^2+z^2(1-M^2)}]^2}{(t^2+z^2)^2 \sqrt{t^2+z^2(1-M^2)}} \, dt$$

By combining I_{3a} with $I_2 + I_{1b}$ obtained previously, the following expression results:

$$I_{1b} + I_2 + I_{3a} \doteq \frac{iMe^{-iM|z|}}{|z|}$$

$$+ i|z|(1-M^2) \int_{M|z|}^\infty \frac{e^{-it}}{[t^2+z^2(1-M^2)]^{3/2}} \, dt + \frac{\pi M}{2}$$

The integral occurring in this expression will be evaluated approximately for two cases. Consider first the case when $\sqrt{1-M^2} \geq M$.

$$i|z|(1-M^2)\int_{M|z|}^{\infty}\frac{e^{-it}}{[t^2+z^2(1-M^2)]^{3/2}}\,dt$$

$$=i|z|(1-M^2)\int_{M|z|}^{\sqrt{1-M^2}\,|z|}\frac{e^{-it}}{|z|^3(1-M^2)^{3/2}\left[\frac{t^2}{z^2(1-M^2)}+1\right]^{3/2}}\,dt$$

$$+i|z|(1-M^2)\int_{\sqrt{1-M^2}\,|z|}^{\infty}\frac{e^{-it}}{t^3\left[1+\frac{z^2(1-M^2)}{t^2}\right]^{3/2}}\,dt$$

$$=i\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)!}{2^{2n}(n!)^2 z^{2n+2}(1-M^2)^{n+\frac{1}{2}}}\int_{M|z|}^{\sqrt{1-M^2}\,|z|}e^{-it}t^{2n}dt$$

$$+i\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)!\,|z|^{2n+1}(1-M^2)^{n+1}}{2^{2n}(n!)^2}\int_{\sqrt{1-M^2}\,|z|}^{\infty}\frac{e^{-it}}{t^{2n+3}}\,dt$$

$$=i\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)!}{2^{2n}(n!)^2}\left[\frac{1}{(2n+1)|z|}-\frac{i\sqrt{1-M^2}}{2n+1}\right.$$

$$-\frac{M^{2n+1}}{(1-M^2)^{n+\frac{1}{2}}(2n+1)|z|}+\frac{iM^{2n+2}}{(1-M^2)^{n+\frac{1}{2}}(2n+1)}$$

$$-\frac{iM^{2n+2}}{(1-M^2)^{n+\frac{1}{2}}(2n+1)(2n+2)}+\frac{1}{(2n+2)|z|}$$

$$\left.-\frac{i\sqrt{1-M^2}}{(2n+2)}\right]+\text{terms which}\rightarrow 0\text{ as }z\rightarrow 0$$

$$\doteq\frac{i}{|z|}-\frac{iM}{|z|}-M^2+1$$

For the case when $M \geq \sqrt{1-M^2}$:

$$i|z|(1-M^2)\int_{M|z|}^{\infty} \frac{e^{-it}}{\left[t^2+z^2(1-M^2)\right]^{3/2}}\,dt$$

$$= i|z|(1-M^2)\int_{M|z|}^{\infty} \frac{e^{-it}}{t^3\left[1+\frac{z^2(1-M^2)}{t^2}\right]^{3/2}}\,dt$$

$$= i\sum_{n=0}^{\infty}(-1)^n \frac{(2n+1)!\,|z|^{2n+1}(1-M^2)^{n+1}}{2^{2n}(n!)^2}\int_{M|z|}^{\infty}\frac{e^{-it}}{t^{2n+3}}\,dt$$

$$= i\sum_{n=0}^{\infty}(-1)^n \frac{(2n+1)!}{2^{2n}(n!)^2}\left[\left(\frac{1-M^2}{M^2}\right)^{n+1}\frac{1}{(2n+2)|z|}\right.$$

$$\left. -\left(\frac{1-M^2}{M^2}\right)^{n+1}\frac{iM}{(2n+2)} - \left(\frac{1-M^2}{M^2}\right)^{n+1}\frac{iM}{(2n+2)(2n+1)}\right]$$

$$+ \text{terms which} \rightarrow 0 \text{ as } z \rightarrow 0$$

$$\doteq \frac{i}{|z|} - \frac{iM}{|z|} - M^2 + 1$$

But this is precisely the result which was obtained for the case when $\sqrt{1-M^2} \geq M$, hence for all $M<1$

$$I_{1b} + I_2 + I_{3a} \doteq \frac{\pi M}{2} + 1 + \frac{i}{|z|}$$

$$I_{3b} = -M \int_0^\infty \frac{e^{-i|z|(\lambda + M\sqrt{\lambda^2+1})}}{\lambda^2+1} d\lambda$$

Let $\quad t = |z|(\lambda + M\sqrt{\lambda^2+1})$ then

$$I_{3b} = M \int_{M|z|}^\infty \frac{e^{-it(1-M^2)|z|}}{\left[Mt - \sqrt{t^2+z^2(1-M^2)}\right]\sqrt{t^2+z^2(1-M^2)}} dt$$

$$= -M|z| \int_{M|z|}^\infty \frac{e^{-it}\left[Mt + \sqrt{t^2+z^2(1-M^2)}\right]}{(t^2+z^2)\sqrt{t^2+z^2(1-M^2)}} dt$$

By combining I_{3b} with $I_{1a'''}$ obtained previously, it may be shown that

$$I_{1a'''} + I_{3b} = |z| \int_{M|z|}^\infty \frac{e^{-it(1-M^2)}}{t\sqrt{t^2+z^2(1-M^2)}} dt$$

If this integral is evaluated by the same procedure as was just used for the integral occurring in $I_{1b} + I_2 + I_{3a}$, i.e., by considering the two cases when $\sqrt{1-M^2} \geq M$ and when $M \geq \sqrt{1-M^2}$, it is found that, to the required order of approximation, the results are again the same for both cases. Therefore, it may be shown that, for all $M < 1$

$$I_{1a'''} + I_{3b} \doteq \sqrt{1-M^2} \ln \frac{1+\sqrt{1-M^2}}{M}$$

$$I_{3c} = -i|z|\int_0^\infty \frac{e^{-it}}{(t^2+z^2)^{3/2}}\,dt$$

$$= -i|z|\int_0^{|z|} \frac{e^{-it}}{(t^2+z^2)^{3/2}}\,dt - i|z|\int_{|z|}^\infty \frac{e^{-it}}{(t^2+z^2)^{3/2}}\,dt$$

$$= -\frac{i}{z^2}\int_0^{|z|} \frac{e^{-it}}{\left(\frac{t^2}{z^2}+1\right)^{3/2}}\,dt - i|z|\int_{|z|}^\infty \frac{e^{-it}}{t^3\left(1+\frac{z^2}{t^2}\right)^{3/2}}\,dt$$

$$= -i\sum_{n=0}^\infty (-1)^n \frac{(2n+1)!}{2^{2n}(n!)^2|z|^{2n+2}}\int_0^{|z|} e^{-it}t^{2n}\,dt$$

$$-i\sum_{n=0}^\infty (-1)^n \frac{|z|^{2n+1}(2n+1)!}{2^{2n}(n!)^2}\int_{|z|}^\infty \frac{e^{-it}}{t^{2n+3}}\,dt$$

$$= -i\sum_{n=0}^\infty (-1)^n \frac{(2n+1)!}{2^{2n}(n!)^2}\left[\frac{1}{(2n+1)|z|} - \frac{i}{2n+1}\right.$$

$$\left. + \frac{1}{(2n+2)|z|} - \frac{i}{2n+2}\right] + \text{terms which} \rightarrow 0 \text{ as } z \rightarrow 0$$

$$I_{3c} \doteq -\frac{i}{|z|} - 1$$

$$I_{3d} = -\frac{i}{|z|}\int_0^{\infty} \frac{e^{-iM|z|\sqrt{\lambda^2+1}}}{(\lambda^2+1)^{3/2}}\, d\lambda$$

Let $t = M|z|\sqrt{\lambda^2+1}$ then

$$I_{3d} = -iM^2|z| \int_{M|z|}^{\infty} \frac{e^{-it}}{t^3\sqrt{1-\frac{M^2z^2}{t^2}}}\, dt$$

$$= -i\sum_{n=0}^{\infty} \frac{(2n)!\, M^{2n+2}|z|^{2n+1}}{2^{2n}(n!)^2} \int_{M|z|}^{\infty} \frac{e^{-it}}{t^{2n+3}}\, dt$$

$$= -i\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}\left[\frac{1}{(2n+2)|z|} - \frac{iM}{(2n+2)}\right.$$

$$\left. - \frac{iM}{(2n+2)(2n+1)}\right] + \textit{terms which} \to 0 \textit{ as } z \to 0$$

$$I_{3d} \doteq -\frac{i}{|z|} - \sum_{n=0}^{\infty} \frac{(2n)!\, M}{2^{2n}(n!)^2(2n+1)}$$

$$I_{3e} = M\int_0^{\infty} \frac{e^{-iM|z|\sqrt{\lambda^2+1}}}{\lambda^2+1}\, d\lambda$$

Let $t = M|z|\sqrt{\lambda^2+1}$ then

$$I_{3e} = M^2|z| \int_{M|z|}^{\infty} \frac{e^{-it}}{t^2\sqrt{1-\frac{M^2z^2}{t^2}}}\, dt$$

$$= \sum_{n=0}^{\infty} \frac{M^{2n+2}|z|^{2n+1}(2n)!}{2^{2n}(n!)^2} \int_{M|z|}^{\infty} \frac{e^{-it}}{t^{2n+2}}\, dt$$

$$= \sum_{n=0}^{\infty} \frac{(2n)!\, M}{2^{2n}(n!)^2(2n+1)} + \textit{terms which} \to 0 \textit{ as } z \to 0$$

$$I_{3f} = \frac{i}{|z|}\int_0^\infty \frac{d\lambda}{(\lambda^2+1)^{3/2}}$$

$$I_{3f} = \frac{i}{|z|}$$

From the preceding definitions, $\Delta F_M(z)$ may be written as

$$\Delta F_M(z) = \frac{|z|}{z}\Big(I_{1a} + I_{1b} + I_2 + I_{3a} + I_{3b} + I_{3c} + I_{3d} + I_{3e} + I_{3f}\Big)$$

and by introducing the approximations derived above, the following result is obtained:

$$\Delta F_M(z) = \frac{|z|}{z}\left(\ln\frac{M}{2} + \frac{\pi M}{2} + \sqrt{1-M^2}\,\ln\frac{1+\sqrt{1-M^2}}{M}\right)$$

$$+ \text{ terms which} \to 0 \text{ as } z \to 0$$

or

$$\Delta F_M(z) \doteq \frac{|z|}{z}\left(\frac{\pi M}{2} + \mathcal{M}\right) \qquad \text{(E-6)}$$

where $\mathcal{M}$ has previously been defined as

$$\mathcal{M} = \ln\frac{M}{2} - \beta\ln\frac{1-\beta}{M}$$

APPENDIX F

EVALUATION OF $\int_{-1}^{X} \frac{\partial \Phi}{\partial X'} dX'$

In order to obtain an expression for Φ it is convenient to write eqn. (47) in the following form by using eqn. (48)

$$\int_{-1}^{X} \frac{\partial \Phi(X',Y^*,0)}{\partial X'} dX' - \frac{\bar{\Gamma}^{(3)}(Y^*)e^{-i\Omega}}{2\pi}\left(\frac{\pi}{2} + \sin^{-1}X\right)$$

$$\doteq \frac{1}{\pi}\oint_{-1}^{1}\left[\sqrt{\frac{1+\xi}{1-\xi}}\left(\frac{\pi}{2}+\sin^{-1}X\right)+\Lambda_1\right]\overline{W}(\xi,Y^*)d\xi$$

$$-\frac{i\Omega\bar{\Gamma}^{(3)}(Y^*)}{2\pi}\int_{1}^{\infty}\left[\sqrt{\frac{\lambda+1}{\lambda-1}}\left(\frac{\pi}{2}+\sin^{-1}X\right)+\Lambda_2\right]e^{-i\Omega\lambda}d\lambda$$

$$+\frac{2}{\pi}\left[\frac{1}{2}\oint_{-1}^{1}\frac{d\bar{\Gamma}^{(3)}}{d\eta^*}Q\,d\eta^* - \frac{i\Omega\bar{\Gamma}^{(3)}(Y^*)}{4\pi}m\right]\oint_{-1}^{1}\left[\sqrt{\frac{1+\xi}{1-\xi}}\left(\frac{\pi}{2}\right.\right.$$

$$\left.\left.+\sin^{-1}X\right)+\Lambda_1\right]e^{-i\Omega\xi}d\xi - \frac{1}{\pi}\oint_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\left(\frac{\pi}{2}\right.$$

$$\left.+\sin^{-1}X\right)\overline{W}(\xi,Y^*)d\xi + \frac{i\Omega\bar{\Gamma}^{(3)}(Y^*)}{2\pi}\int_{1}^{\infty}\left[\sqrt{\frac{\lambda+1}{\lambda-1}}\right.$$

$$\left.-1\right]\left[\frac{\pi}{2}+\sin^{-1}X\right]e^{-i\Omega\lambda}d\lambda - \frac{1}{\pi}\left[\oint_{-1}^{1}\frac{d\bar{\Gamma}^{(3)}}{d\eta^*}Q\,d\eta^*\right.$$

$$\left.-\frac{i\Omega\bar{\Gamma}^{(3)}(Y^*)}{2\pi}m\right]\oint_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\left(\frac{\pi}{2}+\sin^{-1}X\right)e^{-i\Omega\xi}d\xi$$

$$=\frac{1}{\pi}\oint_{-1}^{1}\Lambda_1\overline{W}(\xi,Y^*)d\xi - \frac{i\Omega\bar{\Gamma}^{(3)}(Y^*)}{2\pi}\int_{1}^{\infty}\left[\Lambda_2\right.$$

$$\left.+\frac{\pi}{2}+\sin^{-1}X\right]e^{-i\Omega\lambda}d\lambda + \left[\frac{1}{\pi}\oint_{-1}^{1}\frac{d\bar{\Gamma}^{(3)}}{d\eta^*}Q\,d\eta^*\right.$$

$$\left.-\frac{i\Omega\bar{\Gamma}^{(3)}(Y^*)}{2\pi^2}m\right]\oint_{-1}^{1}\Lambda_1 e^{-i\Omega\xi}d\xi \qquad \text{(F-1)}$$

Integration by parts of the integral of the second term on the right-hand side of eqn. (F-1) gives

$$\int_1^\infty \left(\Lambda_2 + \frac{\pi}{2} + \sin^{-1}X\right) e^{-i\Omega\lambda}\, d\lambda$$

$$= \left(\Lambda_2 + \frac{\pi}{2} + \sin^{-1}X\right) \frac{ie^{-i\Omega\lambda}}{\Omega}\Bigg|_1^\infty - \int_1^\infty \frac{ie^{-i\Omega\lambda}}{\Omega} \frac{\partial \Lambda_2}{\partial \lambda}\, d\lambda$$

$$= \frac{e^{-i\Omega}}{i\Omega}\left(\frac{\pi}{2} + \sin^{-1}X\right) + \frac{1}{i\Omega}\int_1^\infty e^{-i\Omega\lambda} \frac{\partial \Lambda_2}{\partial \lambda}\, d\lambda$$

since

$$\Lambda_1(X, \pm 1) = 0\,; \qquad \Lambda_2(X, 1) = 0$$

and

$$\Lambda_2(X, \infty) = -\left(\frac{\pi}{2} + \sin^{-1}X\right)$$

If the last integral of eqn. (F-1) is also integrated by parts then

$$\oint_{-1}^{1} \Lambda_1 e^{-i\Omega\xi} d\xi = \Lambda_1 \frac{ie^{-i\Omega\xi}}{\Omega}\Bigg|_{-1}^{1} + \int_{-1}^{1} \frac{e^{-i\Omega\xi}}{i\Omega} \frac{\partial \Lambda_1}{\partial \xi}\, d\xi$$

$$= \frac{1}{i\Omega}\int_{-1}^{1} e^{-i\Omega\xi} \frac{\partial \Lambda_1}{\partial \xi}\, d\xi$$

Therefore, by introducing these results into eqn. (F-1), it may be written as

$$\int_{-1}^{X} \frac{\partial \Phi(X', Y^*, 0)}{\partial X'} dX' - \frac{\bar{\pi}^{(3)}(Y^*) e^{-i\Omega}}{2\pi}\left(\frac{\pi}{2} + \sin^{-1} X\right)$$

$$\doteq \frac{1}{\pi} \oint_{-1}^{1} \Lambda_1 \overline{W}(\xi, Y^*) d\xi - \frac{\bar{\pi}^{(3)}(Y^*) e^{-i\Omega}}{2\pi}\left(\frac{\pi}{2} + \sin^{-1} X\right)$$

$$- \frac{\bar{\pi}^{(3)}(Y^*)}{2\pi} \int_{1}^{\infty} e^{-i\Omega\lambda} \frac{\partial \Lambda_2}{\partial \lambda} d\lambda + \frac{1}{i\pi\Omega}\left[\oint_{-1}^{1} \frac{d\bar{\pi}^{(3)}}{d\eta^*} Q d\eta^*\right.$$

$$\left. - \frac{i\Omega \bar{\pi}^{(3)}(Y^*)}{2\pi} \eta \right] \int_{-1}^{1} e^{-i\Omega\xi} \frac{\partial \Lambda_1}{\partial \xi} d\xi$$

or

$$\int_{-1}^{X} \frac{\partial \Phi(X', Y^*, 0)}{\partial X'} dX' \doteq \frac{1}{\pi} \oint_{-1}^{1} \Lambda_1 \overline{W}(\xi, Y^*) d\xi$$

$$- \frac{\bar{\pi}^{(3)}(Y^*)}{2\pi} \int_{1}^{\infty} e^{-i\Omega\lambda} \frac{\partial \Lambda_2}{\partial \lambda} d\lambda$$

$$+ \frac{1}{i\pi\Omega}\left[\oint_{-1}^{1} \frac{d\bar{\pi}^{(3)}}{d\eta^*} Q d\eta^*\right.$$

$$\left. - \frac{i\Omega \bar{\pi}^{(3)}(Y^*)}{2\pi} \eta \right] \int_{-1}^{1} e^{-i\Omega\xi} \frac{\partial \Lambda_1}{\partial \xi} d\xi \qquad \text{(F-2)}$$

APPENDIX G

APPLICATION OF PARSEVAL'S FORMULA

The first integral in eqn. (79) is the product of two functions

$$e^{iM^2\Omega\cos\theta_1}\,\bar{\varpi}(\theta_1)\sin\theta_1 \quad \text{and} \quad \ln\left(\frac{1-\cos(\theta-\theta_1)}{1-\cos(\theta+\theta_1)}\right)$$

both of which are quadratically integrable.

The function

$$\ln\frac{1-\cos(\theta-\theta_1)}{1-\cos(\theta+\theta_1)}$$

is odd, and, therefore, its Fourier expansion contains only sine terms.

$$\int_0^{\pi}\ln\left[\frac{1-\cos(\theta-\theta_1)}{1-\cos(\theta+\theta_1)}\right]\sin n\theta_1\,d\theta_1$$

$$= -\frac{2\pi}{n}\sin n\theta$$

and so the Fourier coefficients of $\ln\dfrac{1-\cos(\theta-\theta_1)}{1-\cos(\theta+\theta_1)}$ are

$$A_n = -\frac{4}{n}\sin n\theta \quad ; \qquad n = 1, 2, 3, \cdots$$

Similarly the Fourier coefficients of $e^{iM^2\Omega\cos\theta_1}\bar{\omega}(\theta_1)\sin\theta_1$ are

$$A_n' = \frac{2}{\pi}\int_0^\pi \bar{\omega}(\theta_1)\sin\theta_1\sin n\theta_1\, e^{iM^2\Omega\cos\theta_1}\,d\theta_1$$

$$= \frac{1}{\pi}\int_0^\pi \bar{\omega}(\theta_1)\Big[\cos(n-1)\theta_1 - \cos(n+1)\theta_1\Big]e^{iM^2\Omega\cos\theta_1}\,d\theta_1$$

$$= U\left(P_{n-1} - P_{n+1}\right) \quad ; \qquad n = 1, 2, 3, \cdots$$

So that by Parseval's formula

$$\int_0^\pi e^{iM^2\Omega\cos\theta_1}\,\bar{\omega}(\theta_1)\sin\theta_1 \ln\frac{1-\cos(\theta-\theta_1)}{1-\cos(\theta+\theta_1)}\,d\theta_1$$

$$= \frac{\pi}{2}\sum_{n=1}^{\infty} A_n A_n'$$

$$= -2\pi U\sum_{n=1}^{\infty}\left(P_{n-1} - P_{n+1}\right)\frac{\sin n\theta}{n} \tag{G-1}$$

Now consider the second integral of eqn. (79) in the same manner. The Fourier coefficients of

$$\frac{\sin^2\theta_1}{\sin\theta(\cos\theta - \cos\theta_1)}$$

are given by

$$B_n' = \frac{2}{\pi}\oint_0^\pi \frac{\sin^2\theta_1}{\sin\theta}\,\frac{\cos n\theta_1}{\cos\theta - \cos\theta_1}\,d\theta_1 \qquad ; \; n = 2, 3, \cdots$$

$$= \frac{1}{\pi\sin\theta}\oint_0^\pi \frac{1-\cos 2\theta_1}{\cos\theta - \cos\theta_1}\cos n\theta_1\,d\theta_1$$

$$= \frac{1}{\pi \sin\theta} \oint_0^{\pi} \frac{\cos n\theta_1 \, d\theta_1}{\cos\theta - \cos\theta_1} - \frac{1}{\pi \sin\theta} \oint_0^{\pi} \frac{\cos n\theta_1 \cos 2\theta_1}{\cos\theta - \cos\theta_1} d\theta_1$$

$$= -\frac{\sin n\theta}{\sin^2\theta} - \frac{1}{2\pi \sin\theta}\left[\oint_0^{\pi} \frac{\cos(n+2)\theta_1}{\cos\theta - \cos\theta_1} d\theta_1 \right.$$

$$\left. + \oint_0^{\pi} \frac{\cos(n-2)\theta_1}{\cos\theta - \cos\theta_1} d\theta_1 \right]$$

$$= \frac{1}{\sin\theta}\left[-\frac{\sin n\theta}{\sin\theta} + \frac{1}{2}\frac{\sin(n+2)\theta}{\sin\theta} + \frac{1}{2}\frac{\sin(n-2)\theta}{\sin\theta}\right]$$

$$= -\frac{\sin n\theta}{\sin^2\theta}\left(1 - \cos 2\theta\right)$$

$$B_n' = -2\sin n\theta \quad ; \qquad n = 2, 3, \cdots$$

For $n = 0$:

$$B_0' = \frac{1}{\pi \sin\theta} \oint_0^{\pi} \frac{1 - \cos 2\theta_1}{\cos\theta - \cos\theta_1} d\theta_1$$

$$= 2\frac{\cos\theta}{\sin\theta}$$

For $n = 1$:

$$B_1' = \frac{1}{\pi \sin\theta} \oint_0^{\pi} \frac{\cos\theta_1 \, d\theta_1}{\cos\theta - \cos\theta_1}$$

$$- \frac{1}{2\pi \sin\theta}\left[\oint_0^{\pi} \frac{\cos 3\theta_1}{\cos\theta - \cos\theta_1} d\theta_1 + \oint_0^{\pi} \frac{\cos(-\theta_1)}{\cos\theta - \cos\theta_1} d\theta_1\right]$$

$$= -\frac{1}{\sin\theta}\left(1 - \frac{1}{2}\frac{\sin 3\theta}{\sin\theta} - \frac{1}{2}\right)$$

$$= \frac{\cos 2\theta}{\sin\theta}$$

Similarly the Fourier coefficients of $\bar{\omega}(\theta_1)e^{iM^2\Omega\cos\theta_1}$ are

$$B_n = \frac{2}{\pi}\int_0^\pi e^{iM^2\Omega\cos\theta_1}\,\bar{\omega}(\theta_1)\cos n\theta_1\,d\theta_1 = 2UP_n$$

so that by again applying Parseval's formula, it may be shown that

$$\frac{1}{\sin\theta}\oint_0^\pi e^{iM^2\Omega\cos\theta_1}\,\bar{\omega}(\theta_1)\sin^2\theta_1\,\frac{d\theta_1}{\cos\theta-\cos\theta_1}$$

$$=\frac{\pi}{2}\left(\frac{B_0B_0'}{2}+\sum_{n=1}^{\infty}B_nB_n'\right)$$

$$=\pi U\left(P_0\cot\theta+\frac{P_1}{\sin\theta}-2\sum_{n=1}^{\infty}P_n\sin n\theta\right) \qquad \text{(G-2)}$$

APPENDIX H

APPROXIMATE EVALUATION OF $\oint_{X_L}^{\infty} \frac{\partial}{\partial \xi}\left\{\bar{\Phi}\left[1+\frac{i\pi f|X-\xi|}{2} H_1^{(2)}(f|X-\xi|)\right]\right\}\frac{d\xi}{X-\xi}$

The second term on the right-hand side of eqn. (120) may be written:

$$\frac{1}{\pi}\oint_{X_L}^{\infty} \frac{1}{X-\xi}\frac{\partial}{\partial \xi}\left\{\bar{\Phi}(\xi,Y,0)\left[1+\frac{i\pi f|X-\xi|}{2} H_1^{(2)}(f|X-\xi|)\right]\right\}d\xi$$

$$\doteq \frac{1}{\pi(X-X_a)}\left[1+\frac{i\pi f|X-X_a|}{2} H_1^{(2)}(f|X-X_a|)\right]\int_{X_L}^{X_T}\frac{\partial\bar{\Phi}(\xi,Y,0)}{\partial\xi}d\xi$$

$$+\lim_{\varepsilon\to 0}\left\langle \frac{\bar{\Pi}^{(3)}}{2\pi}\int_{X_T}^{X-\varepsilon}\frac{1}{X-\xi}\frac{\partial}{\partial\xi}\left\{e^{-i\Omega\xi}\left[1\right.\right.\right.$$

$$\left.\left.+\frac{i\pi f|X-\xi|}{2}H_1^{(2)}(f|X-\xi|)\right]\right\}d\xi+\frac{\bar{\Pi}^{(3)}}{2\pi}\int_{X+\varepsilon}^{\infty}\frac{1}{X-\xi}\frac{\partial}{\partial\xi}\left\{e^{-i\Omega\xi}\left[1\right.\right.$$

$$\left.\left.\left.+\frac{i\pi f|X-\xi|}{2}H_1^{(2)}(f|X-\xi|)\right]\right\}d\xi\right\rangle$$

$$=\frac{1}{\pi(X-X_a)}\left[1+\frac{i\pi f|X-X_a|}{2} H_1^{(2)}(f|X-X_a|)\right]\int_{X_L}^{X_T}\frac{\partial\bar{\Phi}(\xi,Y,0)}{\partial\xi}d\xi$$

$$+\lim_{\varepsilon\to 0}\left\langle \frac{\Omega\bar{\Pi}^{(3)}e^{-i\Omega X}}{2\pi}\int_{\Omega(X-X_T)}^{\Omega\varepsilon}\frac{1}{z}\frac{\partial}{\partial z}\left\{e^{iz}\left[1\right.\right.\right.$$

$$\left.\left.+\frac{i\pi Mz}{2}H_1^{(2)}(Mz)\right]\right\}dz-\frac{\Omega\bar{\Pi}^{(3)}e^{-i\Omega X}}{2\pi}\int_{\Omega\varepsilon}^{\infty}\frac{1}{z}\frac{\partial}{\partial z}\left\{e^{-iz}\left[1\right.\right.$$

$$\left.\left.\left.+\frac{i\pi Mz}{2}H_1^{(2)}(Mz)\right]\right\}dz\right\rangle \tag{H-1}$$

Consider first the integral

$$\int_{\Omega(x-x_T)}^{\Omega\epsilon} \frac{1}{z}\frac{\partial}{\partial z}\left\{e^{iz}\left[1+\frac{i\pi M z}{2}H_1^{(2)}(Mz)\right]\right\}dz$$

$$=\int_{\Omega(x-x_T)}^{\Omega\epsilon} \frac{1}{z}\left\{ie^{iz}+\frac{i\pi M e^{iz} z}{2}\left[iH_1^{(2)}+MH_0^{(2)}\right]\right\}dz$$

$$=\int_{\Omega(x-x_T)}^{\Omega\epsilon}\left\{\frac{ie^{iz}}{z}+\frac{\pi}{2}\frac{\partial}{\partial z}\left[e^{iz}H_0^{(2)}(Mz)\right]\right.$$

$$\left.-\frac{i\pi}{2}\beta^2 H_0^{(2)}(Mz)e^{iz}\right\}dz \qquad \text{(H-2)}$$

Evaluate each term of eqn. (H-2) to the order of Ω ln Ω as follows:

$$\int_{\Omega(x-x_T)}^{\Omega\epsilon}\frac{ie^{iz}}{z}dz \doteq i\left[\ln\frac{\epsilon}{x-x_T}+i\Omega\epsilon-i\Omega(x-x_T)+O(\Omega^2)\right] \qquad \text{(H-3)}$$

$$\frac{\pi}{2}\int_{\Omega(x-x_T)}^{\Omega\epsilon}\frac{\partial}{\partial z}\left[e^{iz}H_0^{(2)}(Mz)\right]dz$$

$$=\frac{\pi}{2}\left[e^{i\Omega\epsilon}H_0^{(2)}(M\Omega\epsilon)-e^{i\Omega(x-x_T)}H_0^{(2)}(M\Omega(x-x_T))\right]$$

$$\doteq\frac{\pi}{2}\left[-\frac{2i}{\pi}\ln\frac{\gamma M\Omega\epsilon}{2}+i\Omega\epsilon+\frac{2\Omega\epsilon}{\pi}\ln\frac{\gamma M\Omega\epsilon}{2}\right.$$

$$+\frac{2i}{\pi}\ln\frac{\gamma M\Omega(x-x_T)}{2}-i\Omega(x-x_T)$$

$$\left.-\frac{2\Omega(x-x_T)}{\pi}\ln\frac{\gamma M\Omega(x-x_T)}{2}+O(\Omega^2)\right] \qquad \text{(H-4)}$$

$$\frac{i\pi}{2}\beta^2\int_{\Omega(X-X_T)}^{\Omega\varepsilon} H_0^{(2)}(Mz)\,e^{iz}\,dz$$

$$= \frac{i\pi}{2}\beta^2\Big[\Omega\varepsilon - \frac{2i}{\pi}\Omega\varepsilon \ln\frac{\gamma M\Omega\varepsilon}{2} + \frac{2i}{\pi}\Omega\varepsilon$$

$$- \Omega(X-X_T) + \frac{2i}{\pi}\Omega(X-X_T)\ln\frac{\gamma M\Omega(X-X_T)}{2}$$

$$- \frac{2i}{\pi}\Omega(X-X_T) + O(\Omega^2)\Big] \qquad \text{(H-5)}$$

By combining the results of eqns. (H-3), (H-4), and (H-5) according to eqn. (H-2), then it is found that

$$\int_{\Omega(X-X_T)}^{\Omega\varepsilon} \frac{1}{z}\frac{\partial}{\partial z}\left\{e^{iz}\left[1+\frac{i\pi Mz}{2}H_1^{(2)}(Mz)\right]\right\}dz$$

$$= M^2\Omega\Big[-\varepsilon + (X-X_T) + \frac{i\pi}{2}\varepsilon - \frac{i\pi}{2}(X-X_T)$$

$$+ \varepsilon\ln\frac{\gamma M\Omega\varepsilon}{2} - (X-X_T)\ln\frac{\gamma M\Omega(X-X_T)}{2} + O(\Omega^2)\Big] \qquad \text{(H-6)}$$

The last integral of eqn. (H-1) may be evaluated by considering the results presented in eqns. (B-5), (B-6), (B-7), and (B-8) so that

$$\lim_{\varepsilon\to 0}\int_{\Omega\varepsilon}^{\infty} \frac{1}{z}\frac{\partial}{\partial z}\left\{e^{-iz}\left[1+\frac{i\pi Mz}{2}H_1^{(2)}(Mz)\right]\right\}dz \doteq -i\mathcal{M} \qquad \text{(H-7)}$$

where $\mathcal{M}$ has been defined by eqn. (B-10).

Furthermore, in the first term on the right-hand side of eqn. (H-1)

$$1+\frac{i\pi f|X-X_a|}{2}H_1^{(2)}(f|X-X_a|)=O(\Omega^2 \ln \Omega)$$

Therefore, finally

$$\frac{1}{\pi}\oint_{X_L}^{\infty}\frac{1}{X-\xi}\frac{\partial}{\partial \xi}\left\{\Phi\left[1+\frac{i\pi f|X-X_a|}{2}H_1^{(2)}(f|X-\xi|)\right]\right\}d\xi$$

$$\doteq \frac{i\Omega \bar{\pi}^{(3)}e^{-i\Omega X}}{2\pi}\mathcal{M} \qquad \text{(H-8)}$$

APPENDIX I

EVALUATION OF THE INTEGRATION OVER THE WING SPAN OCCURRING IN THE DOWNWASH PROBLEM

The integrand of the integral over the span in eqn. (128) is an even function of η and, therefore, the integral may be written

$$\lim_{\varepsilon \to 0} 2\int_{\varepsilon}^{\beta s} \frac{d}{d\eta}(\bar{\Gamma}^{(3)}) \left\{ -\frac{1}{\eta} + \frac{1}{X-X_a} - \frac{\sqrt{(X-X_a)^2+\eta^2}}{(X-X_a)\eta} \right.$$
$$+ i\Omega\left[-\frac{i\pi}{2} + \frac{\pi M}{2} - \frac{X_T}{X-X_a} - \ln\gamma\Omega + \frac{X_T\sqrt{(X-X_a)^2+\eta^2}}{(X-X_a)\eta} \right.$$
$$\left.\left. + \frac{X}{\eta} + \frac{\sqrt{(X-X_T)^2+\eta^2}}{\eta} - \ln\left(\eta + \sqrt{(X-X_T)^2+\eta^2}\right)\right]\right\} d\eta \qquad \text{(I-1)}$$

Integrate this expression by parts and approximate $\bar{\Gamma}^{(3)}$ by the elliptic spanwise distribution as given by eqn. (58). Then, since $\bar{\Gamma}^{(3)}(\beta s) = 0$, eqn. (I-1) becomes

$$\lim_{\varepsilon \to 0} 2\left\{ -\bar{\Gamma}^{(3)}(0)\left[-\frac{1}{\varepsilon} + \frac{1}{X-X_a} - \frac{\sqrt{(X-X_a)^2+\varepsilon^2}}{(X-X_a)\varepsilon} + \frac{\pi\Omega}{2} \right.\right.$$
$$+ \frac{i\pi M\Omega}{2} - \frac{i\Omega X_T}{X-X_a} - i\Omega\ln\gamma\Omega + i\Omega X_T\frac{\sqrt{(X-X_a)^2+\varepsilon^2}}{(X-X_a)\varepsilon}$$
$$\left. + \frac{i\Omega X}{\varepsilon} + \frac{i\Omega\sqrt{(X-X_T)^2+\varepsilon^2}}{\varepsilon} - i\Omega\ln\left(\varepsilon + \sqrt{(X-X_T)^2+\varepsilon^2}\right)\right]$$
$$- \frac{\bar{\Gamma}^{(3)}(0)}{\beta s}\int_{\varepsilon}^{\beta s} \sqrt{(\beta s)^2-\eta^2}\left[\frac{1}{\eta^2} + \frac{X-X_a}{\eta^2\sqrt{(X-X_a)^2+\eta^2}}\right.$$
$$\left.\left. - \frac{i\Omega X_T(X-X_a)}{\eta^2\sqrt{(X-X_a)^2+\eta^2}} - \frac{i\Omega X}{\eta^2} - \frac{i\Omega\sqrt{(X-X_T)^2+\eta^2}}{\eta^2}\right] d\eta\right\} \qquad \text{(I-2)}$$

Evaluate the integrals in the following way:

$$\int_{\varepsilon}^{\beta s} \frac{\sqrt{(\beta s)^2-\eta^2}}{\eta^2}\,d\eta = \left[-\frac{\sqrt{(\beta s)^2-\eta^2}}{\eta} - \sin^{-1}\frac{\eta}{\beta s}\right]_{\varepsilon}^{\beta s}$$

$$= -\frac{\pi}{2} + \frac{\sqrt{(\beta s)^2-\varepsilon^2}}{\varepsilon} + \sin^{-1}\frac{\varepsilon}{\beta s}$$

The remaining integrals are of elliptic type. Formula numbers correspond to those used by Byrd and Friedman (31).

$$\int_{\varepsilon}^{\beta s} \frac{\sqrt{(\beta s)^2-\eta^2}}{\eta^2\sqrt{(x-x_a)^2+\eta^2}}\,d\eta = g\int_0^{u_1} tn^2 u\, du \qquad (213.10)$$

where

$$g = \frac{1}{\sqrt{(x-x_a)^2+(\beta s)^2}}$$

$$am\, u_1 = \cos^{-1}\left(\frac{\varepsilon}{\beta s}\right)$$

$$cn\, u_1 = \cos\varphi = \frac{\varepsilon}{\beta s}$$

$$\int_0^{u_1} tn^2 u\, du = \frac{1}{k'^2}\Big[dn\, u\; tn\, u - E(u)\Big]_0^{u_1} \qquad (316.02)$$

$$= \frac{1}{k'^2}\Big[dn\, u_1\; tn\, u_1 - E(u_1)\Big]$$

$$k^2 = \frac{(\beta s)^2}{(x-x_a)^2+(\beta s)^2}$$

$$k'^2 = 1-k^2 = \frac{(x-x_a)^2}{(x-x_a)^2+(\beta s)^2}$$

$$dn\, u_1 = \sqrt{k'^2+k^2 cn^2 u_1} = \sqrt{\frac{(x-x_a)^2}{(x-x_a)^2+(\beta s)^2} + \frac{\varepsilon^2}{(x-x_a)^2+(\beta s)^2}}$$

$$tn\, u_1 = \frac{sn\, u_1}{cn\, u_1} = \frac{\sqrt{1-(\varepsilon/\beta s)^2}}{\varepsilon/\beta s}$$

so

$$\int_{\epsilon}^{\beta s} \frac{\sqrt{(\beta s)^2 - \eta^2}}{\eta^2 \sqrt{(x-x_a)^2 + \eta^2}}\, d\eta$$

$$= \frac{\sqrt{(x-x_a)^2 + (\beta s)^2}}{(x-x_a)^2} \left\{ \frac{\sqrt{(x-x_a)^2 + \epsilon^2}}{\sqrt{(x-x_a)^2 + (\beta s)^2}} \cdot \frac{\sqrt{(\beta s)^2 - \epsilon^2}}{\epsilon} - E(k) \right\}$$

$$= \frac{\sqrt{(x-x_a)^2 + \epsilon^2}\sqrt{(\beta s)^2 - \epsilon^2}}{(x-x_a)^2 \epsilon} - \frac{\sqrt{(x-x_a)^2 + (\beta s)^2}}{(x-x_a)^2} E(k)$$

as $\epsilon \to 0$

$$\int_{\epsilon}^{\beta s} \frac{\sqrt{(\beta s)^2 - \eta^2}}{\sqrt{(x-x_T)^2 + \eta^2}}\, d\eta = \frac{(\beta s)^2}{\sqrt{(x-x_T)^2 + (\beta s)^2}} \int_0^{u_1} sn^2 u\, du \qquad (213.03)$$

$$\int_0^{u_1} sn^2 u\, du = \frac{(x-x_T)^2 + (\beta s)^2}{(\beta s)^2} \left[u_1 - E(u_1) \right] \qquad (310.02)$$

$$= \frac{(x-x_T)^2 + (\beta s)^2}{(\beta s)^2} \left[F(\varphi, k_1) - E(u_1) \right]$$

$$= \frac{(x-x_T)^2 + (\beta s)^2}{(\beta s)^2} \left[K(k_1) - E(k_1) \right] \quad \text{as } \epsilon \to 0$$

so

$$\int_{\epsilon}^{\beta s} \frac{\sqrt{(\beta s)^2 - \eta^2}}{\sqrt{(x-x_T)^2 + \eta^2}}\, d\eta = \sqrt{(x-x_T)^2 + (\beta s)^2} \left[K(k_1) - E(k_1) \right]$$

where

$$k_1 = \frac{(\beta s)^2}{(x-x_T)^2 + (\beta s)^2}$$

$$\int_{\varepsilon}^{\beta s} \frac{\sqrt{(\beta s)^2-\eta^2}\,\sqrt{(x-x_T)^2+\eta^2}}{\eta^2}\,d\eta$$

$$= (x-x_T)^2 \int_{\varepsilon}^{\beta s} \frac{\sqrt{(\beta s)^2-\eta^2}}{\eta^2\sqrt{(x-x_T)^2+\eta^2}}\,d\eta + \int_{\varepsilon}^{\beta s} \frac{\sqrt{(\beta s)^2-\eta^2}}{\sqrt{(x-x_T)^2+\eta^2}}\,d\eta$$

$$= \frac{\sqrt{(x-x_T)^2+\varepsilon^2}\,\sqrt{(\beta s)^2-\varepsilon^2}}{\varepsilon} + \sqrt{(x-x_T)^2+(\beta s)^2}\left[K(k_1)-2E(k_1)\right]$$

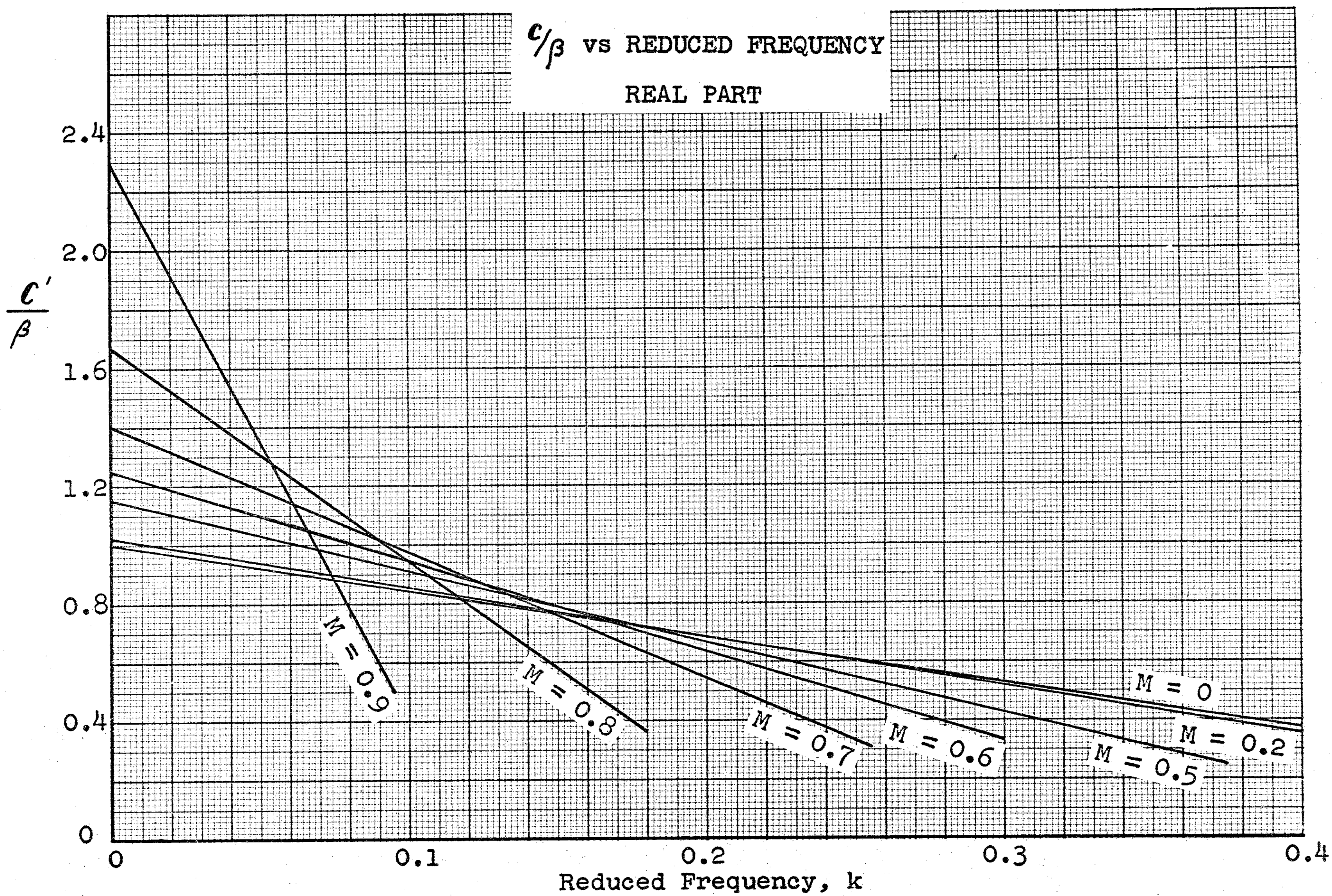
With these results eqn. (I-2) becomes finally

$$-2\bar{\Gamma}^{(3)}(0)\left\langle \frac{1}{x-x_a} - \frac{\pi}{2\beta s} - \frac{\sqrt{(x-x_a)^2+(\beta s)^2}}{(x-x_a)\beta s}E(k) + i\Omega\left\{\frac{\pi M}{2} + \frac{\pi x}{2\beta s} - \frac{i\pi}{2} - \frac{x_T}{x-x_a} - \ln \gamma\Omega(x-x_T) - \frac{\sqrt{(x-x_T)^2+(\beta s)^2}}{\beta s}\left[K(k_1) - 2E(k_1)\right] + \frac{x_T\sqrt{(x-x_a)^2+(\beta s)^2}}{(x-x_a)\beta s}E(k)\right\}\right\rangle \qquad \text{(I-3)}$$

where K(k) and E(k) are complete elliptic integrals of the first and second kind respectively and where

$$k^2 = \frac{(\beta s)^2}{(x-x_a)^2+(\beta s)^2}$$

$$k_1^2 = \frac{(\beta s)^2}{(x-x_T)^2+(\beta s)^2}$$



C/β vs REDUCED FREQUENCY
REAL PART
C′/β
2.4
2.0
1.6
1.2
0.8
0.4
0
0.1
0.2
0.3
0.4
Reduced Frequency, k
M = 0.9
M = 0.8
M = 0.7
M = 0.6
M = 0.5
M = 0.2
M = 0

FIG. 1

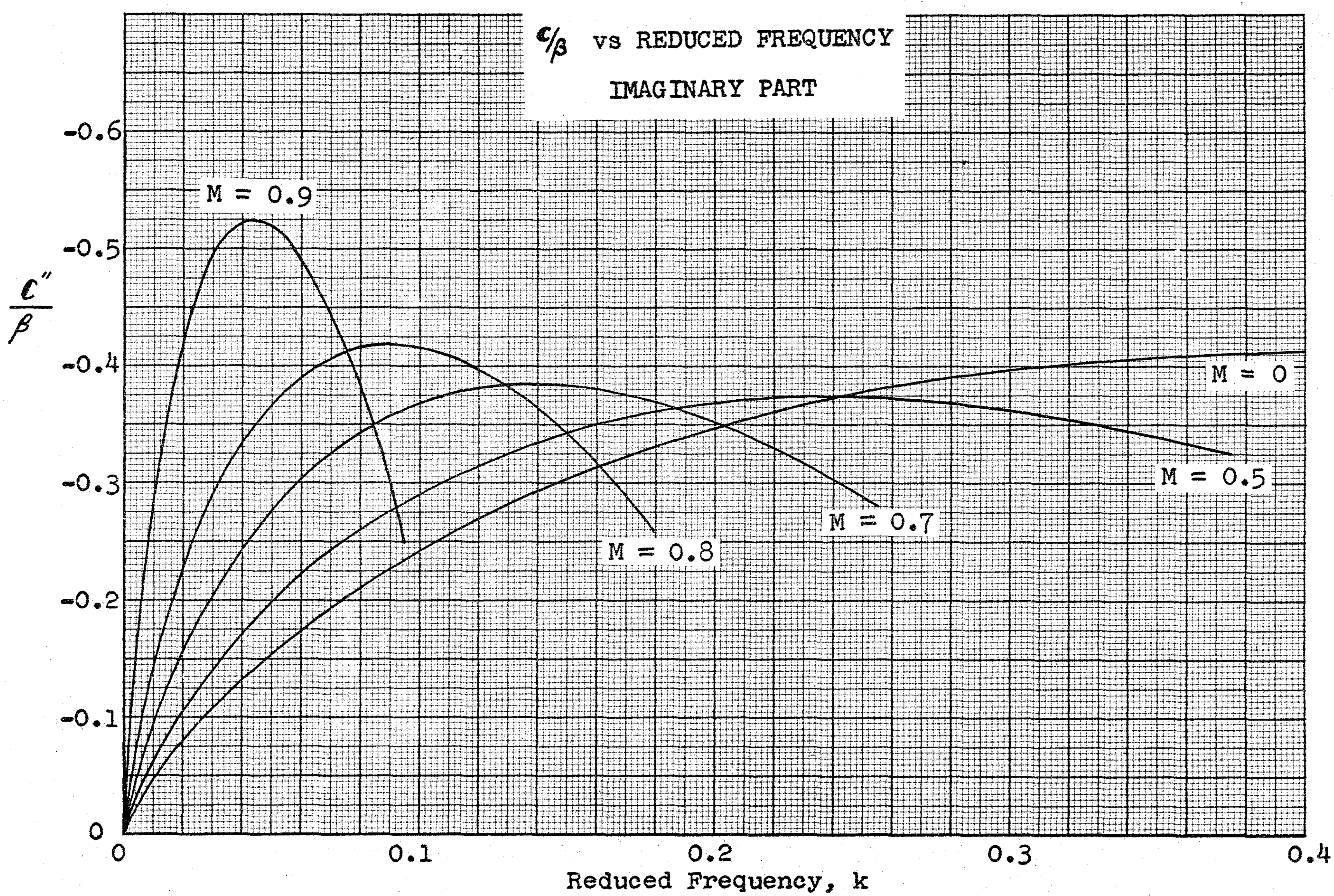

$\mathcal{C}/\beta$ vs REDUCED FREQUENCY
IMAGINARY PART
$\frac{\mathcal{C}''}{\beta}$
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
Reduced Frequency, k
M = 0.9
M = 0.8
M = 0.7
M = 0.5
M = 0

FIG. 1

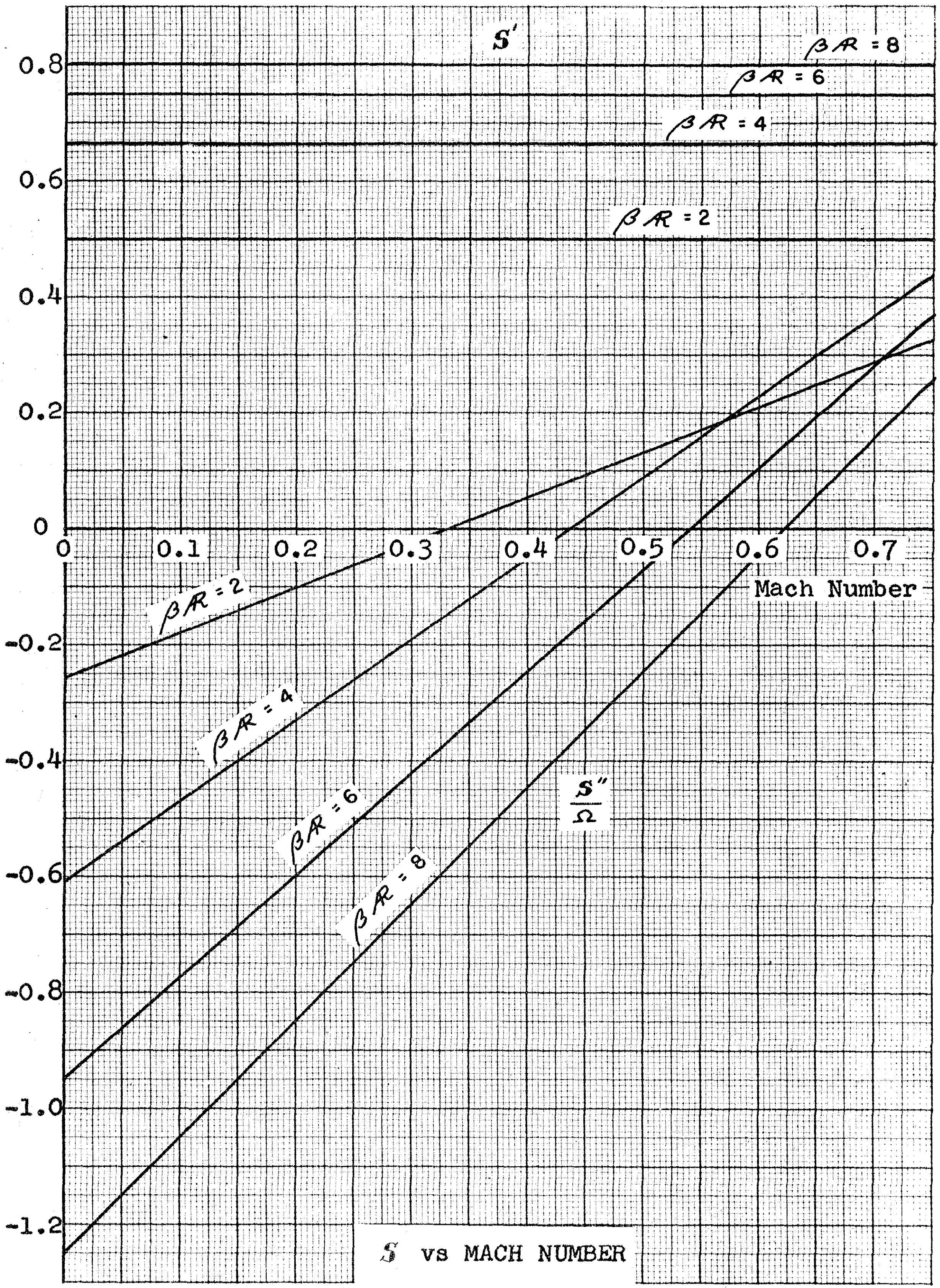

S'
βA = 8
βA = 6
βA = 4
βA = 2
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1.0
-1.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
Mach Number
βA = 2
βA = 4
βA = 6
βA = 8
S''/Ω
S vs MACH NUMBER

FIG. 2

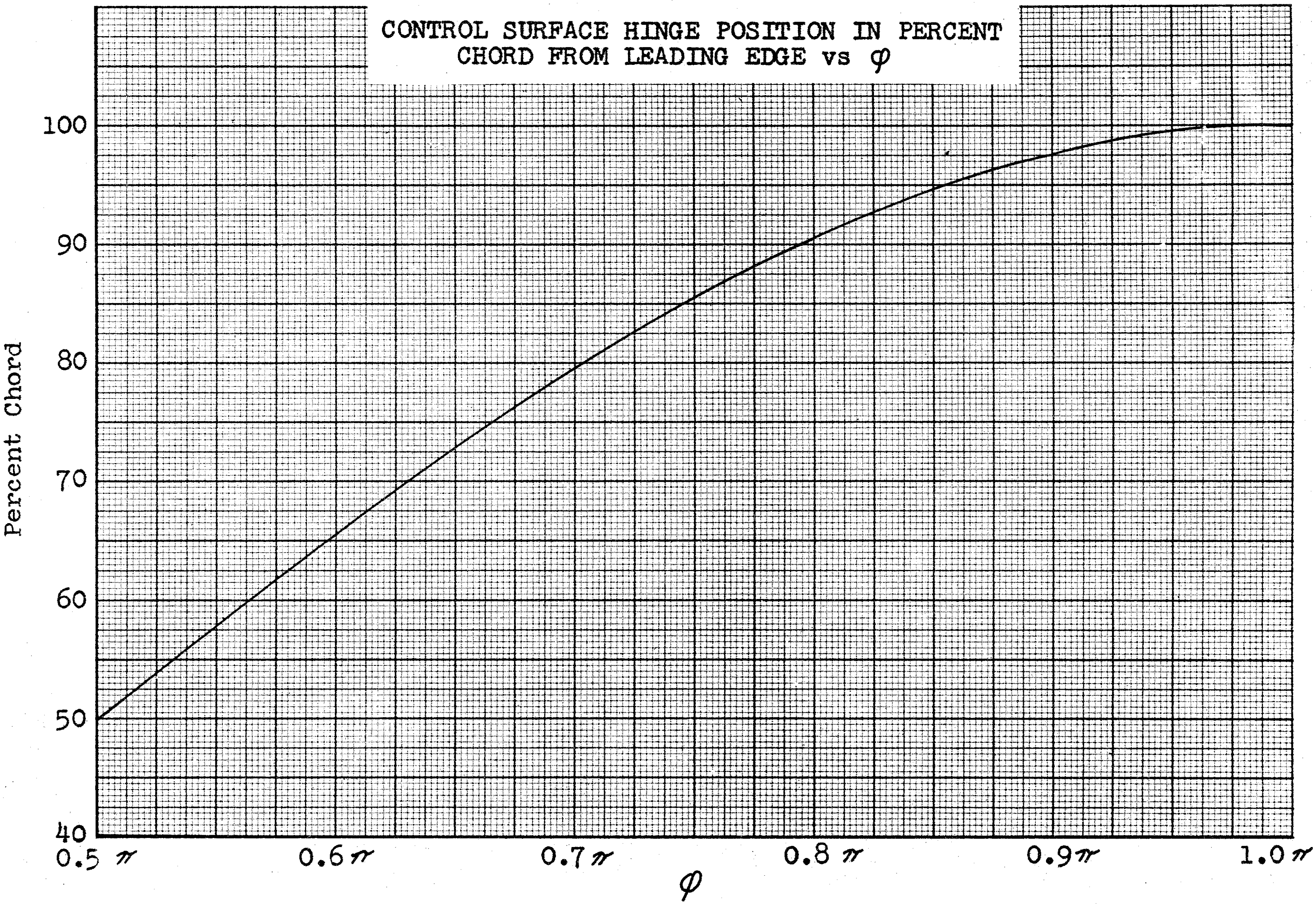
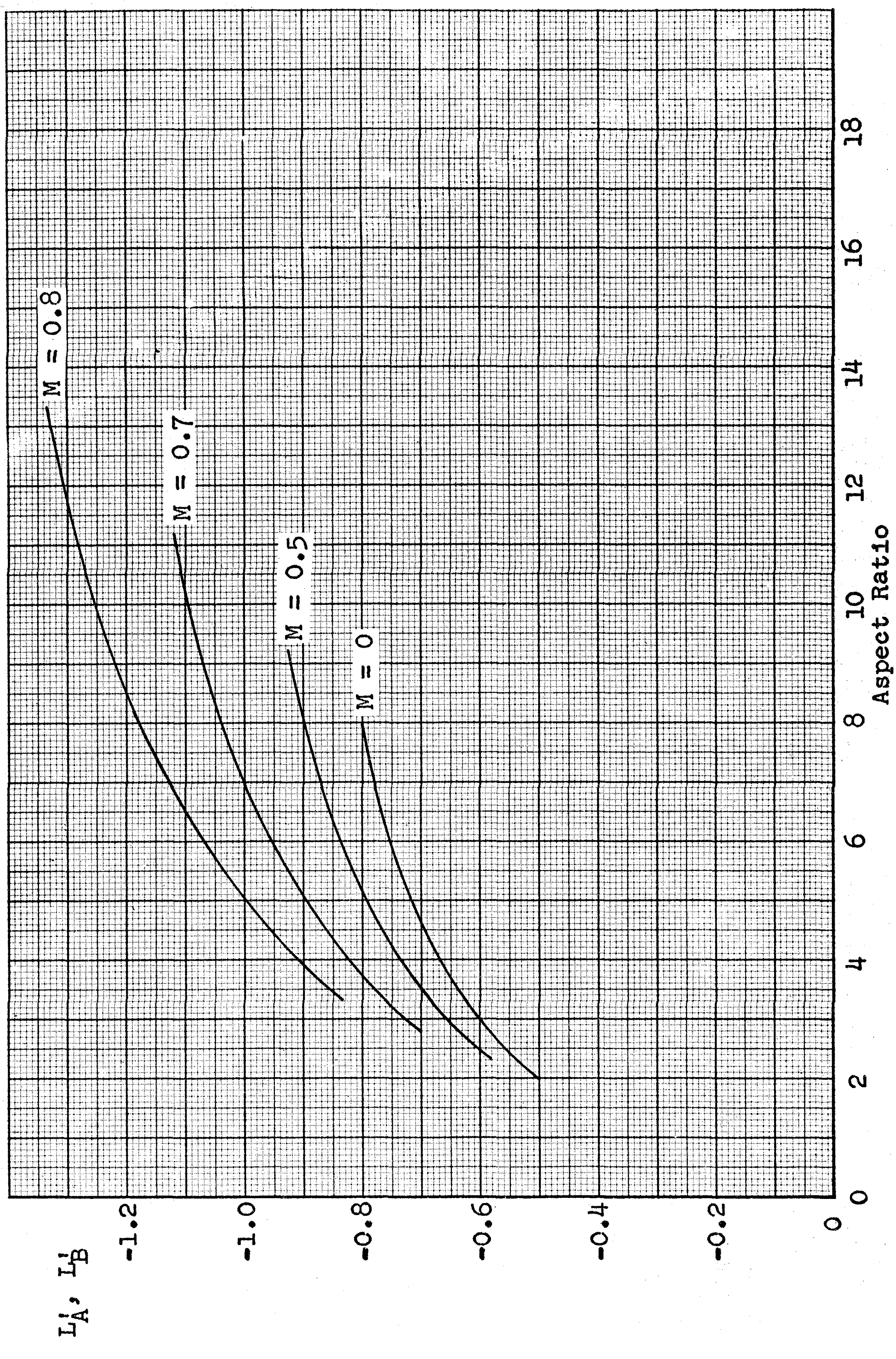

CONTROL SURFACE HINGE POSITION IN PERCENT
CHORD FROM LEADING EDGE vs φ
Percent Chord
100
90
80
70
60
50
40
0.5 π
0.6 π
0.7 π
0.8 π
0.9 π
1.0 π
φ

FIG. 3

FIGURES 4 THROUGH 14

LIFT FORCE DERIVATIVES vs ASPECT RATIO FOR

MACH NUMBERS OF 0, 0.5, 0.7 AND 0.8



M = 0.8
M = 0.7
M = 0.5
M = 0
Aspect Ratio
L'_A, L'_B
0
2
4
6
8
10
12
14
16
18
-1.2
-1.0
-0.8
-0.6
-0.4
-0.2
0

FIG. 4

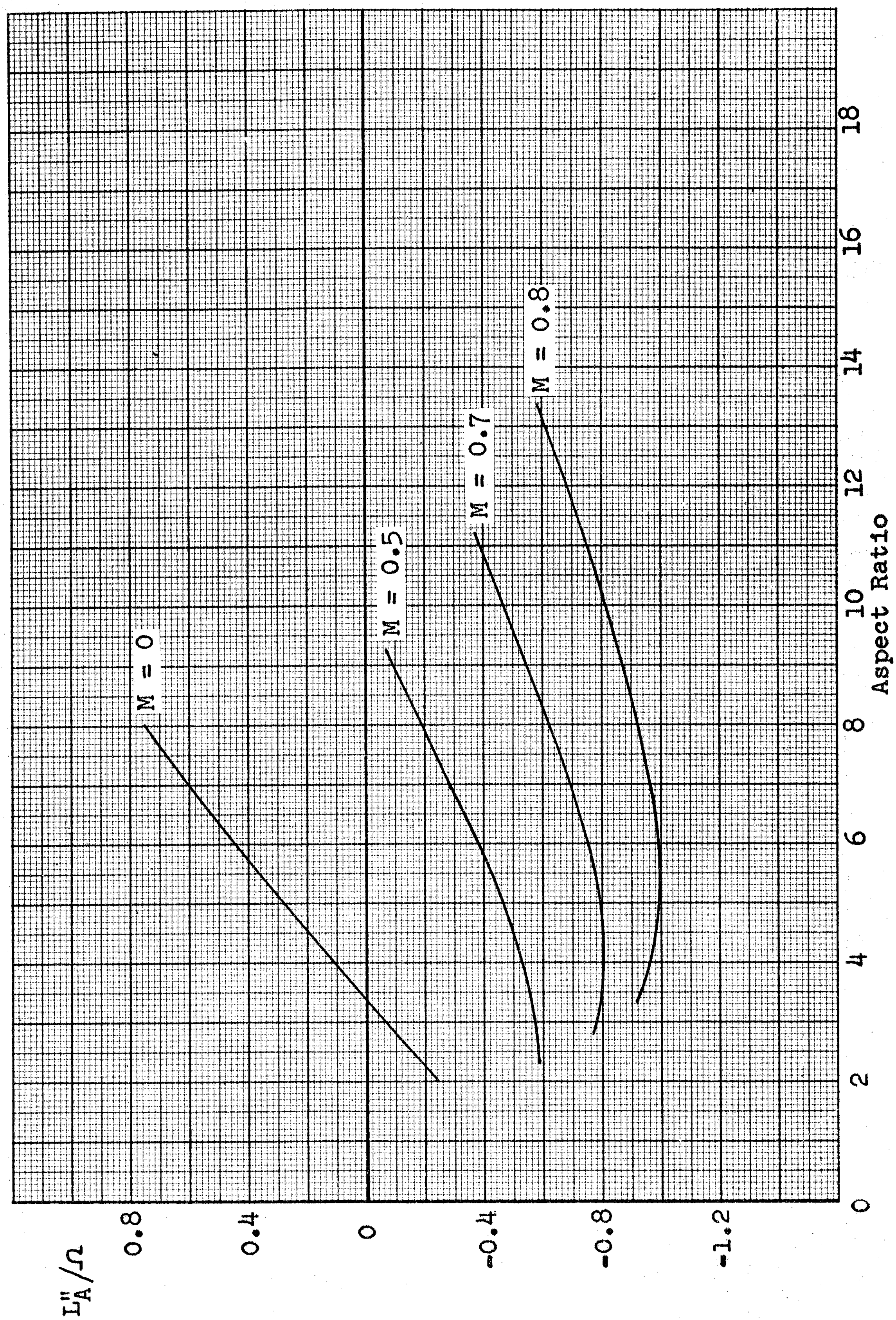

L''_A/Ω
0.8
0.4
0
-0.4
-0.8
-1.2
0
2
4
6
8
10
12
14
16
18
Aspect Ratio
M = 0
M = 0.5
M = 0.7
M = 0.8

FIG. 5

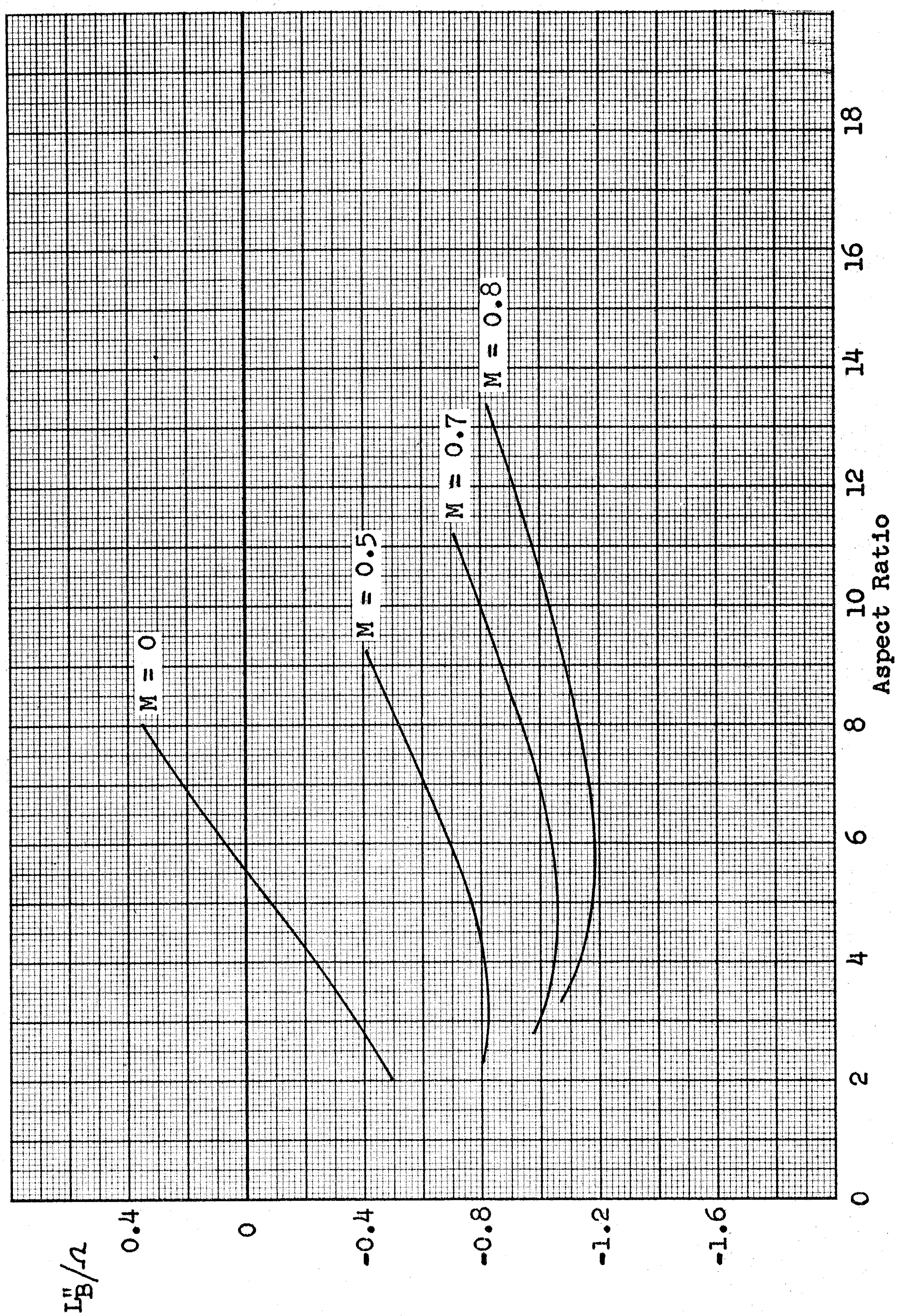
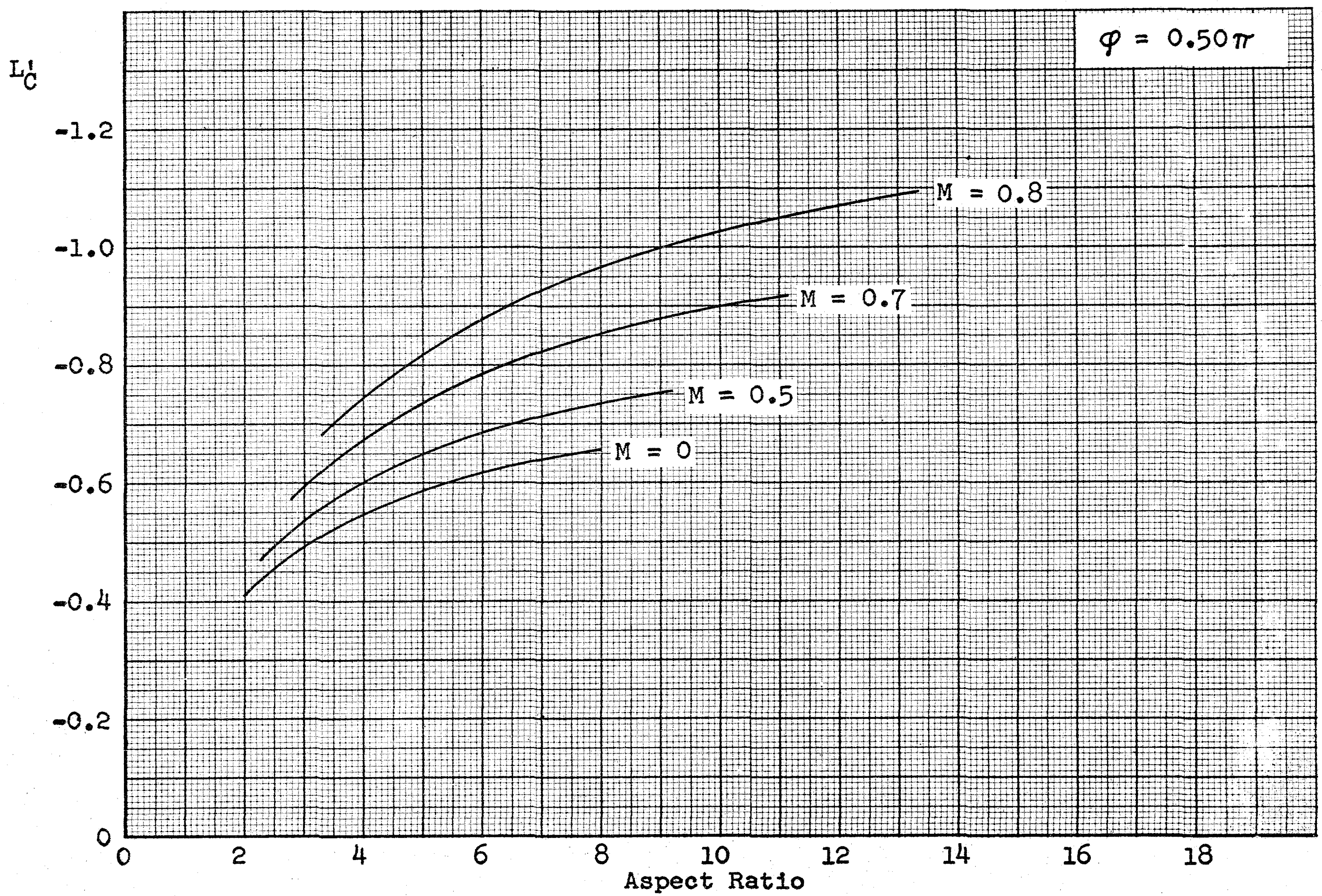

L''_B/Ω
0.4
0
-0.4
-0.8
-1.2
-1.6
0
2
4
6
8
10
12
14
16
18
Aspect Ratio
M = 0
M = 0.5
M = 0.7
M = 0.8

FIG. 6



$\varphi = 0.50\pi$
L'_C
-1.2
-1.0
-0.8
-0.6
-0.4
-0.2
0
M = 0.8
M = 0.7
M = 0.5
M = 0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 7

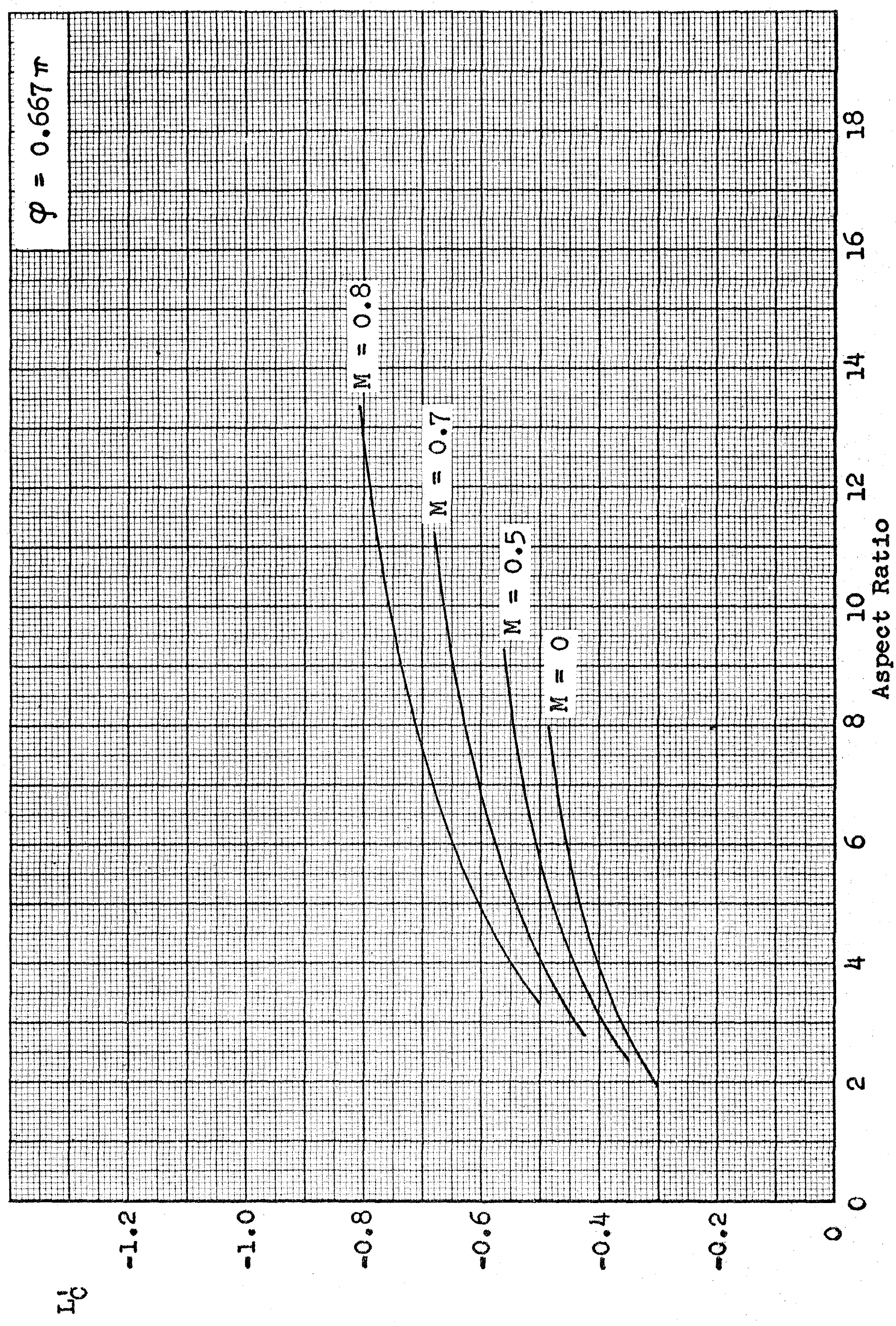
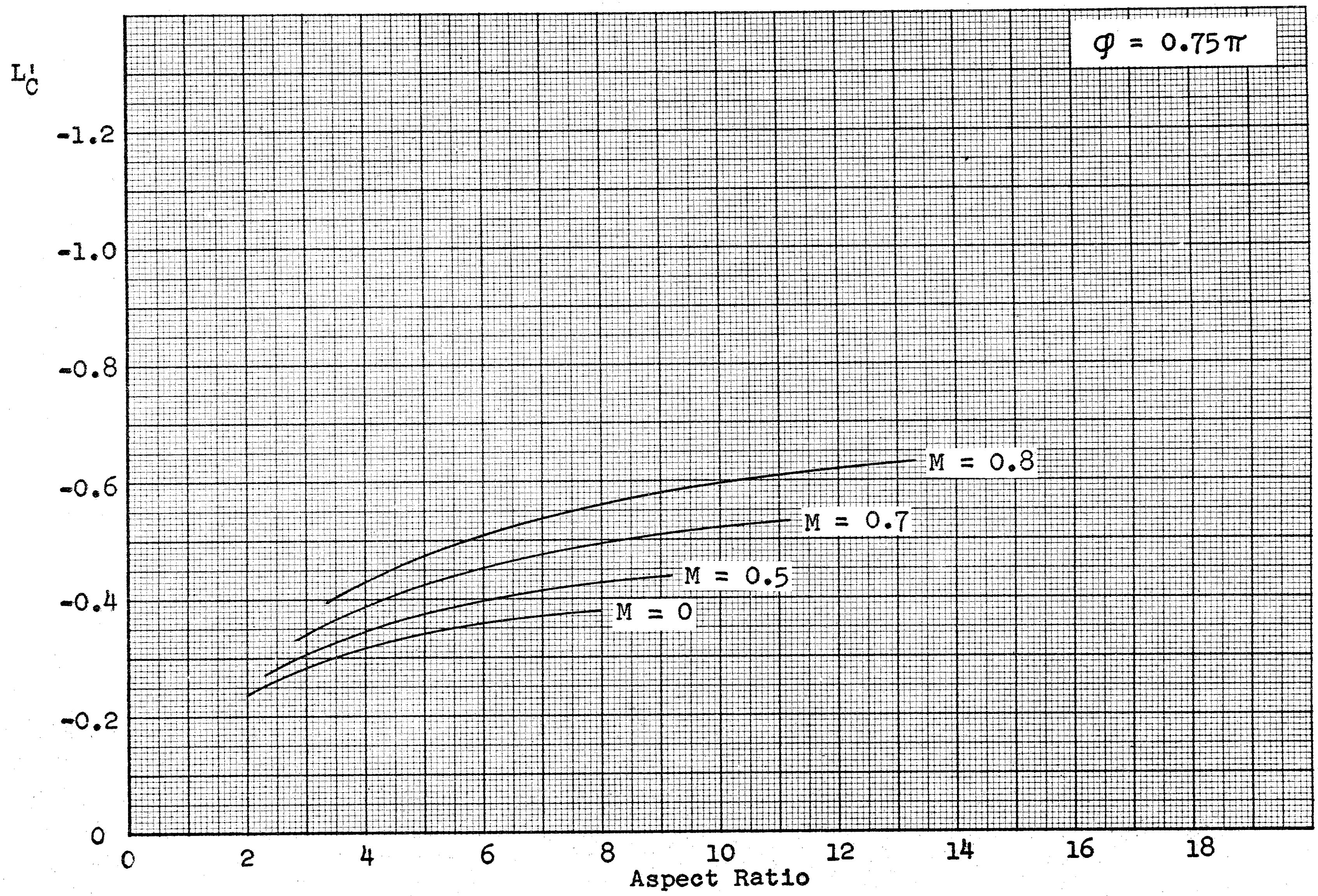

φ = 0.667π
M = 0.8
M = 0.7
M = 0.5
M = 0
Aspect Ratio
0
2
4
6
8
10
12
14
16
18
L'c
0
-0.2
-0.4
-0.6
-0.8
-1.0
-1.2

FIG. 8



$\varphi = 0.75\pi$
L'_C
-1.2
-1.0
-0.8
-0.6
-0.4
-0.2
0
M = 0.8
M = 0.7
M = 0.5
M = 0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 9

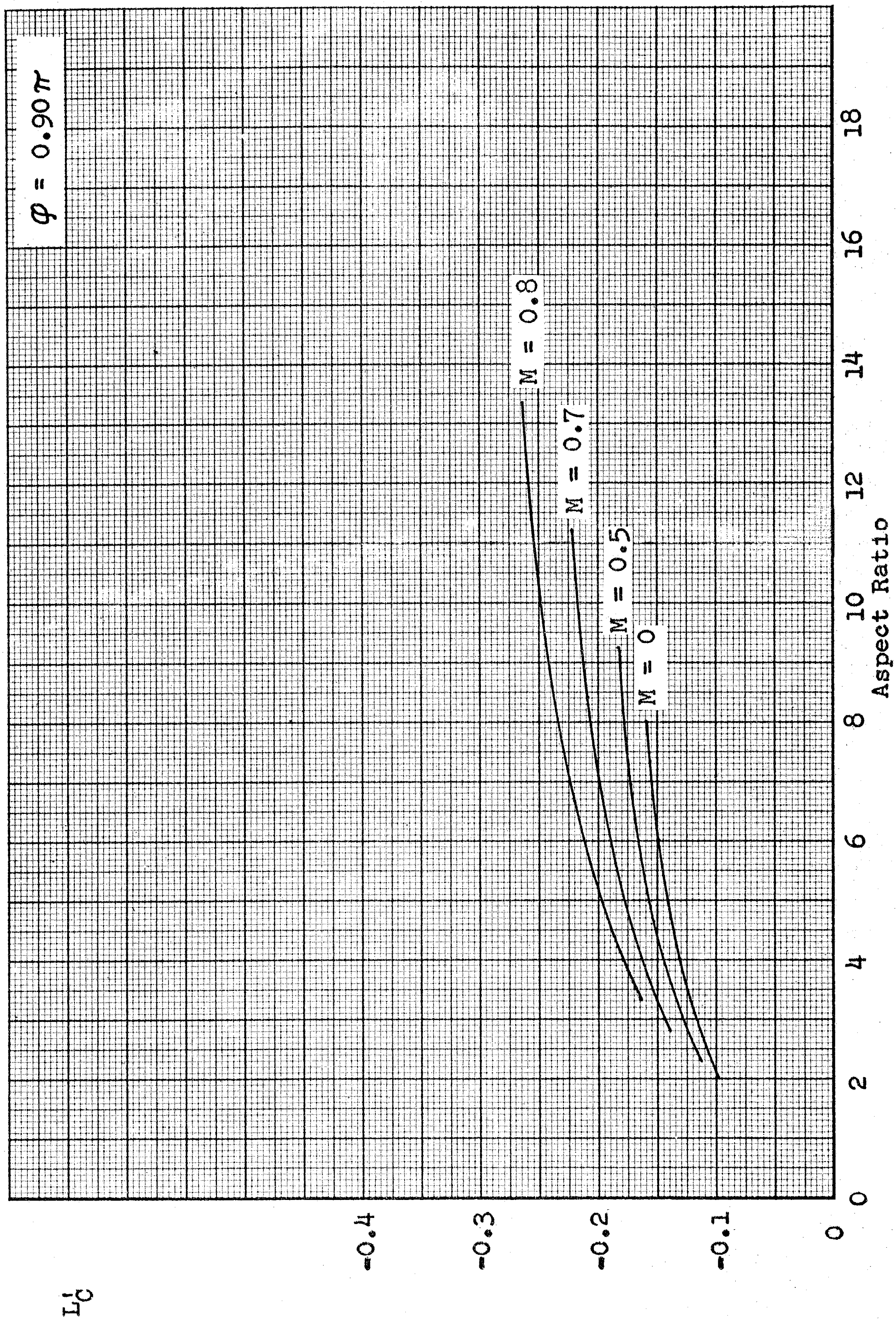
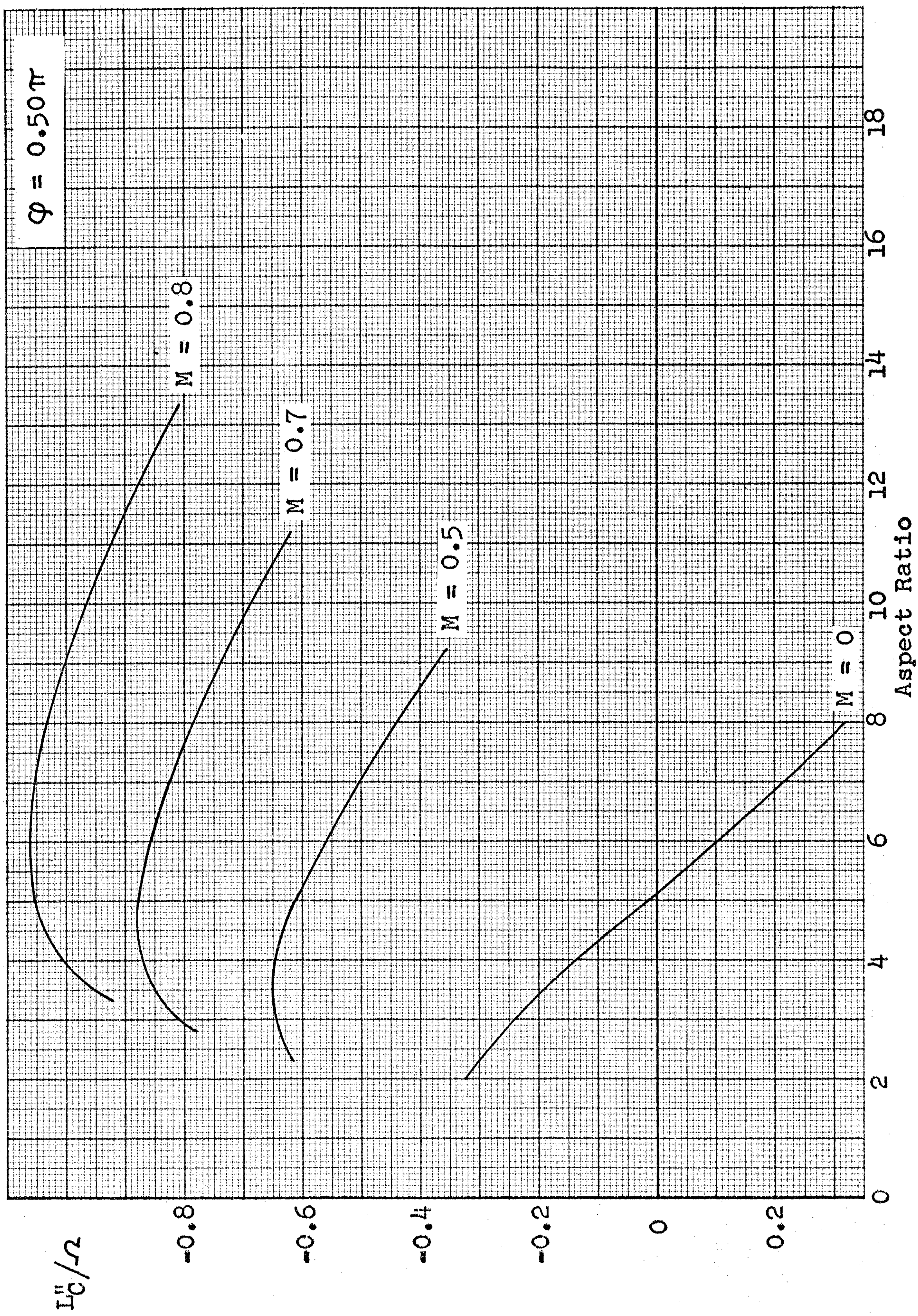

φ = 0.90π
M = 0.8
M = 0.7
M = 0.5
M = 0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio
0
-0.1
-0.2
-0.3
-0.4
L'C

FIG. 10



$\varphi = 0.50\pi$
M = 0.8
M = 0.7
M = 0.5
M = 0
Aspect Ratio
0
2
4
6
8
10
12
14
16
18
L_C''/Ω
-0.8
-0.6
-0.4
-0.2
0
0.2

FIG. 11

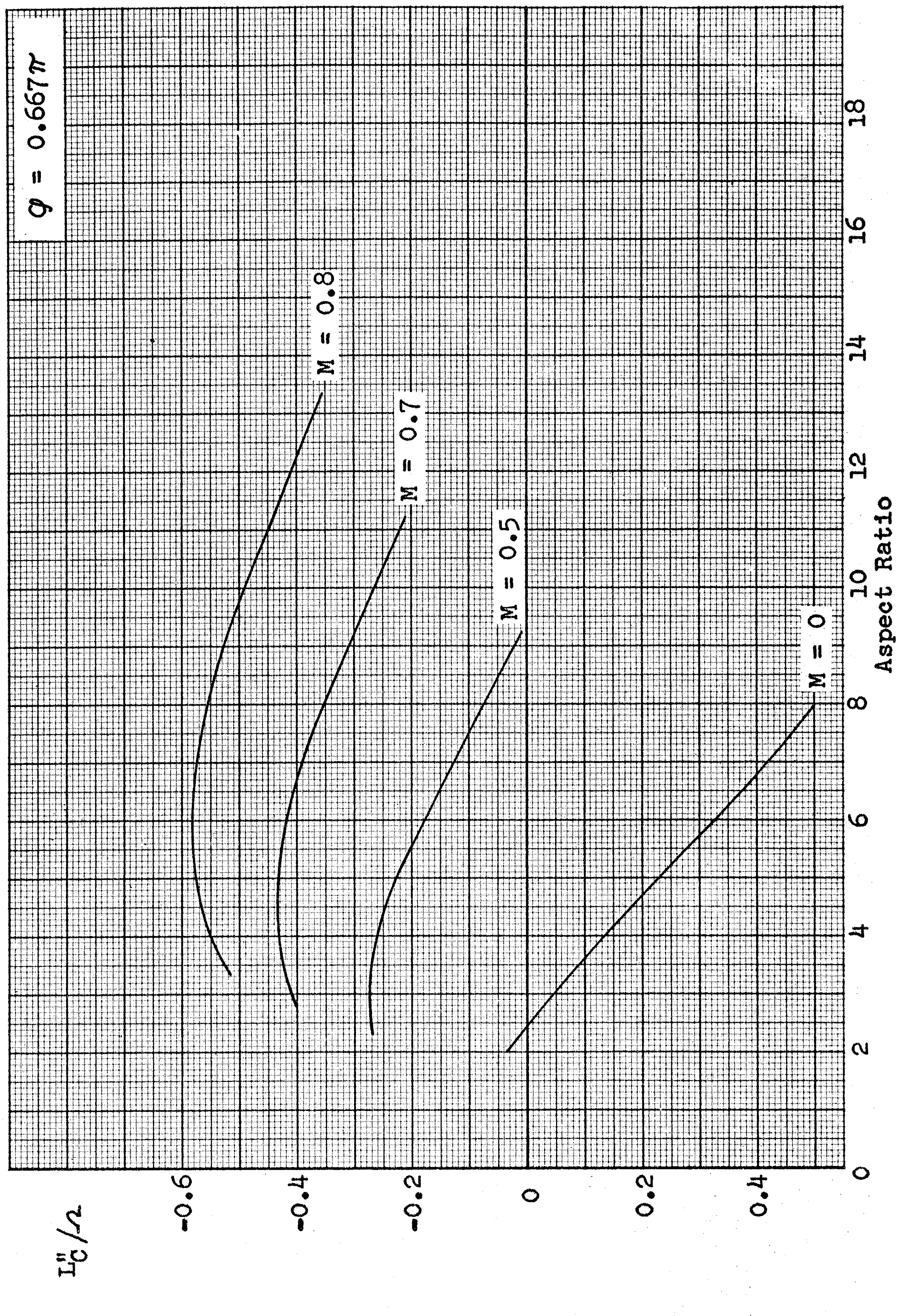
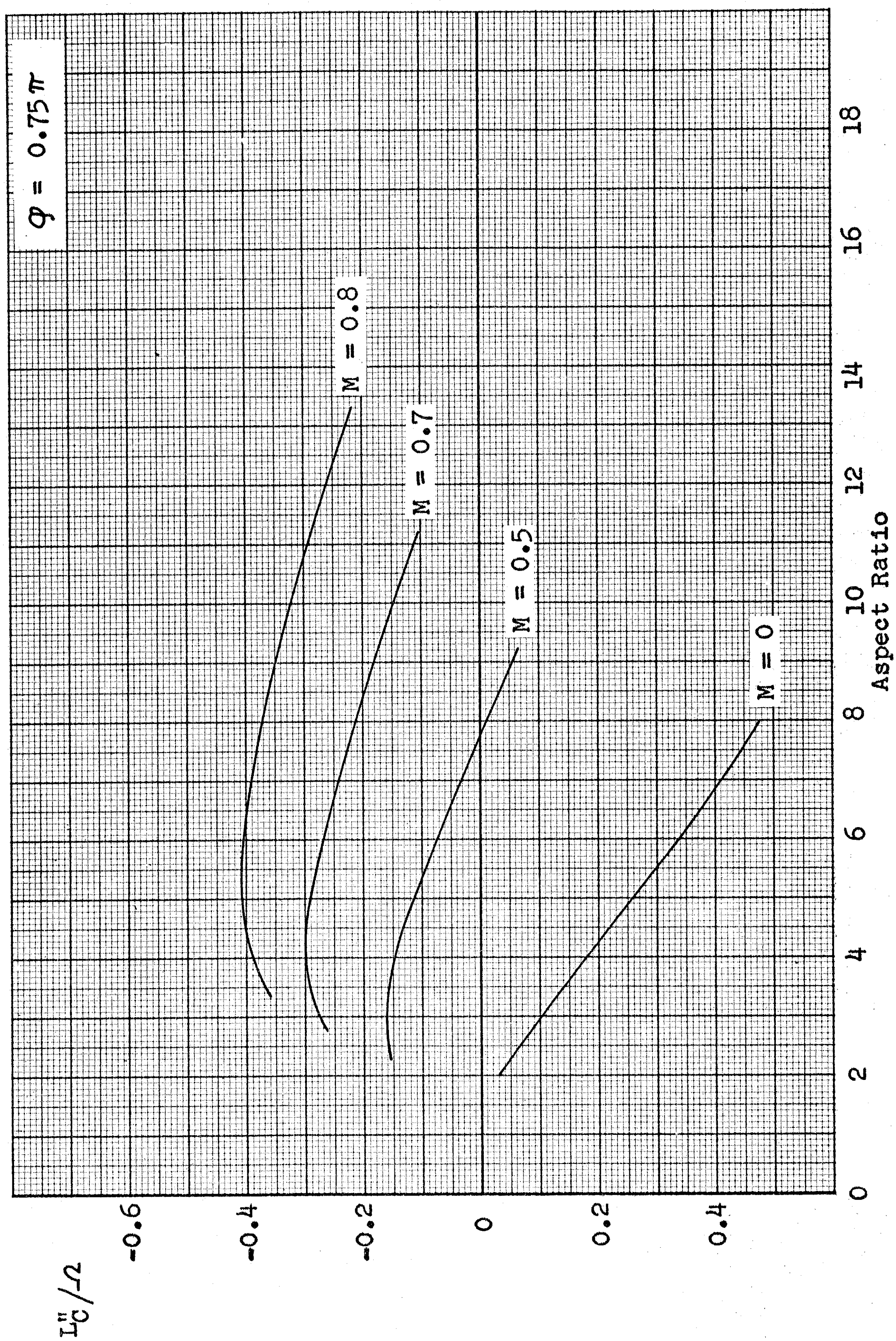
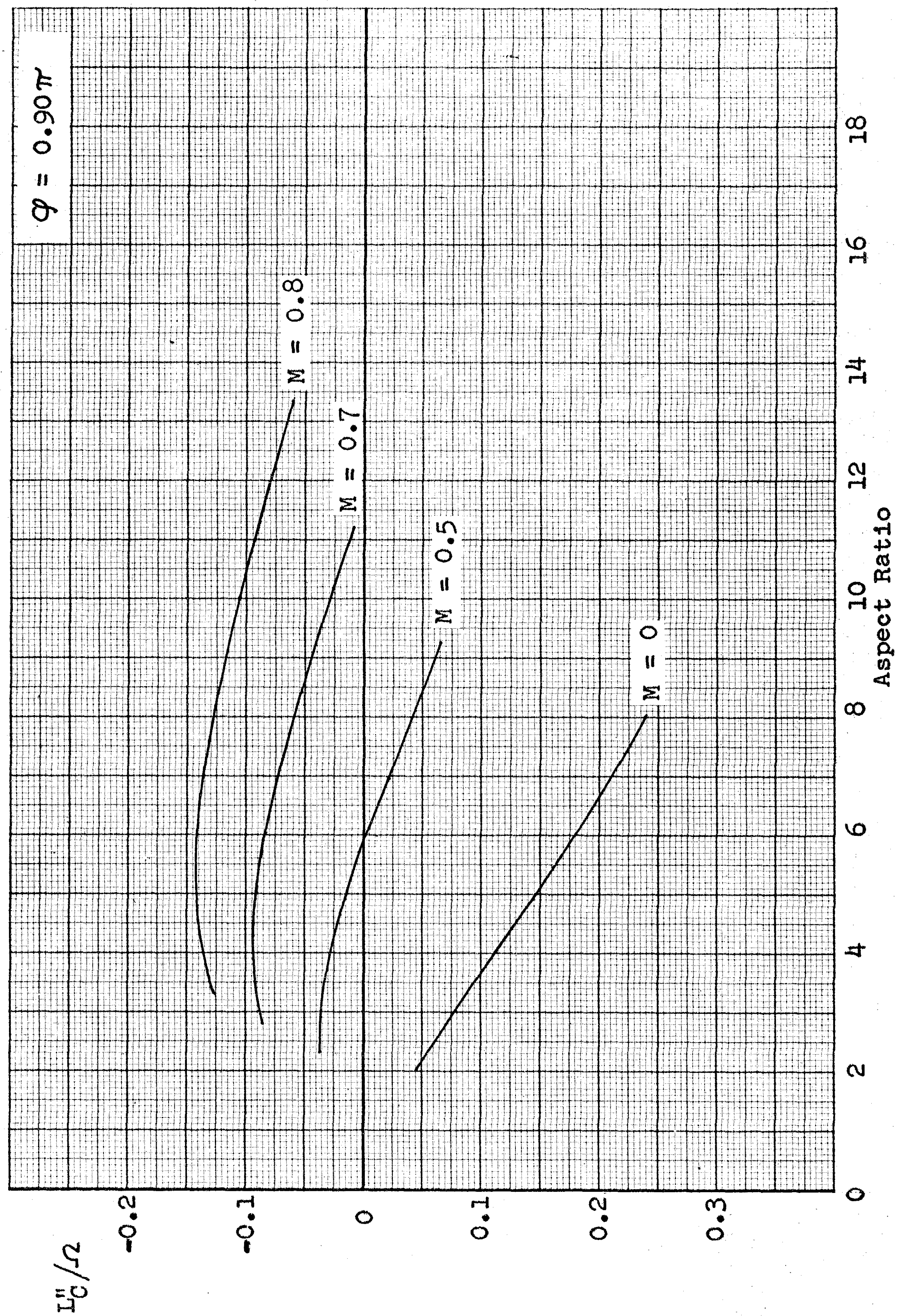

φ = 0.667π
M = 0.8
M = 0.7
M = 0.5
M = 0
L''_C/Ω
-0.6
-0.4
-0.2
0
0.2
0.4
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 12



$\varphi = 0.75\pi$
M = 0.8
M = 0.7
M = 0.5
M = 0
L''_C/Ω
-0.6
-0.4
-0.2
0
0.2
0.4
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 13



$\varphi = 0.90\pi$
M = 0.8
M = 0.7
M = 0.5
M = 0
L_C''/Ω
-0.2
-0.1
0
0.1
0.2
0.3
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 14

FIGURES 15 THROUGH 25

PITCHING MOMENT DERIVATIVES vs ASPECT RATIO

MACH NUMBERS OF 0, 0.5, 0.7 AND 0.8

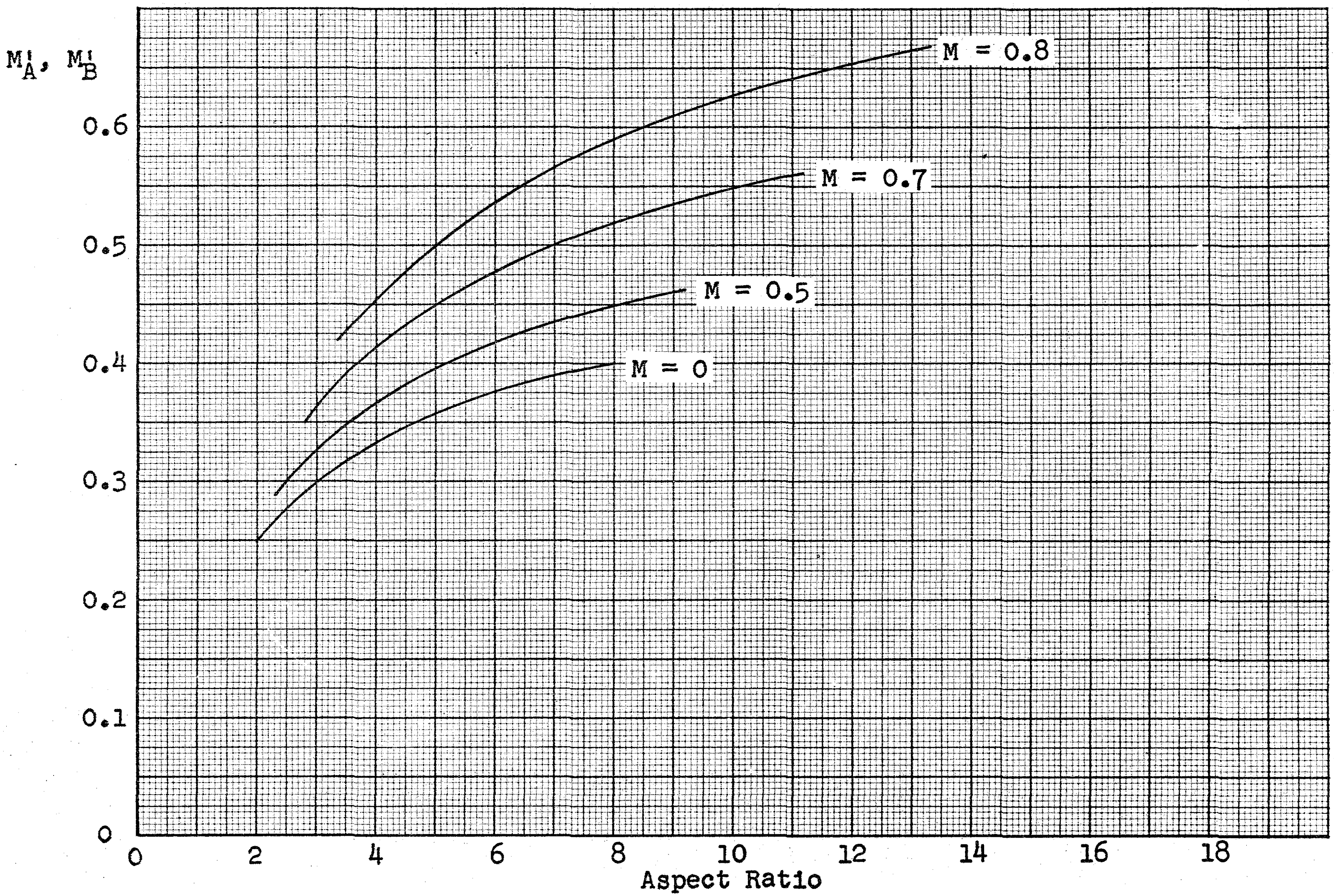

M_A', M_B'
0.6
0.5
0.4
0.3
0.2
0.1
0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio
M = 0.8
M = 0.7
M = 0.5
M = 0

FIG. 15

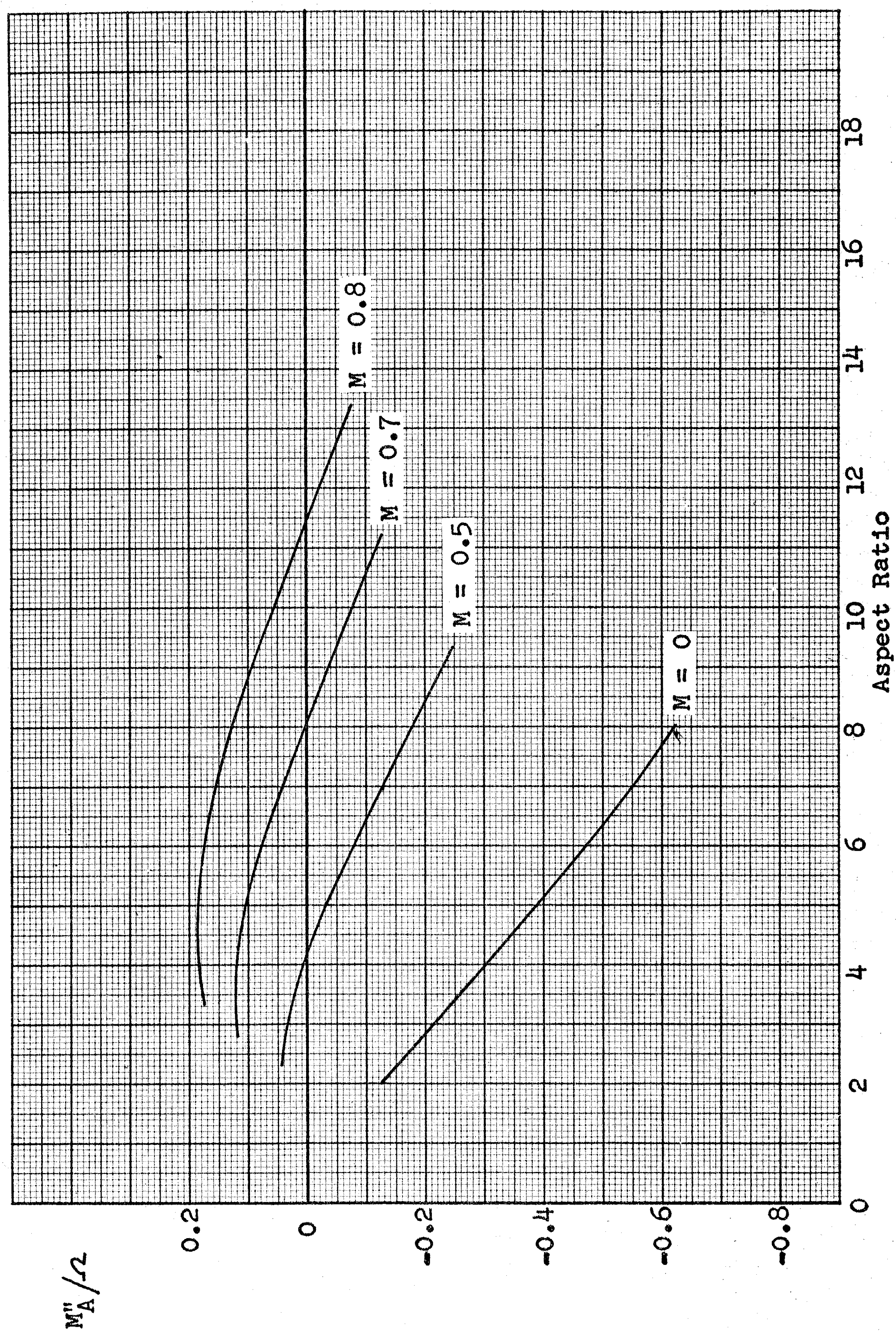

M = 0.8
M = 0.7
M = 0.5
M = 0
Aspect Ratio
0
2
4
6
8
10
12
14
16
18
0.2
0
-0.2
-0.4
-0.6
-0.8
M''_A/Ω

FIG. 16

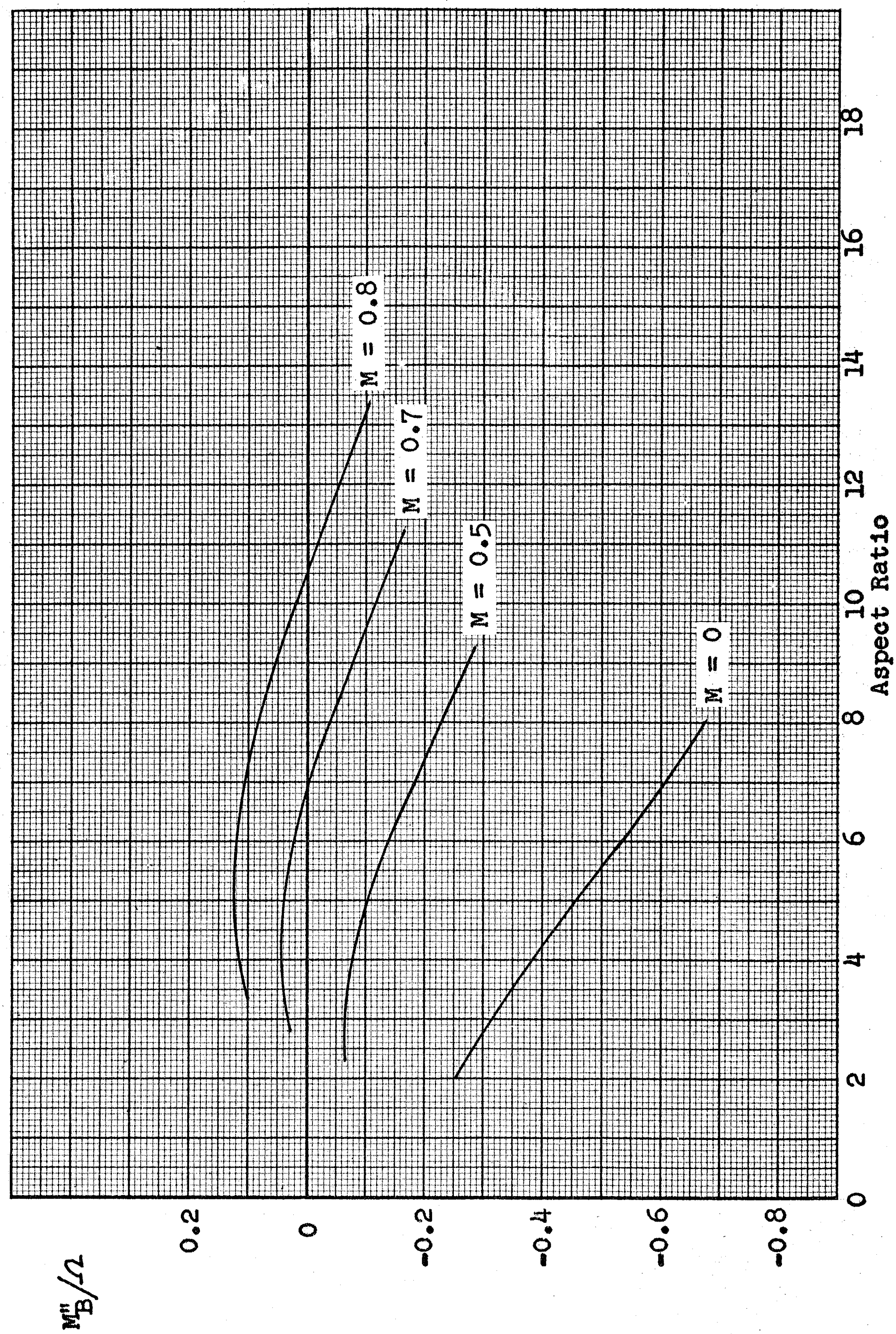

M''_B/Ω
0.2
0
-0.2
-0.4
-0.6
-0.8
0
2
4
6
8
10
12
14
16
18
Aspect Ratio
M = 0.8
M = 0.7
M = 0.5
M = 0

FIG. 17

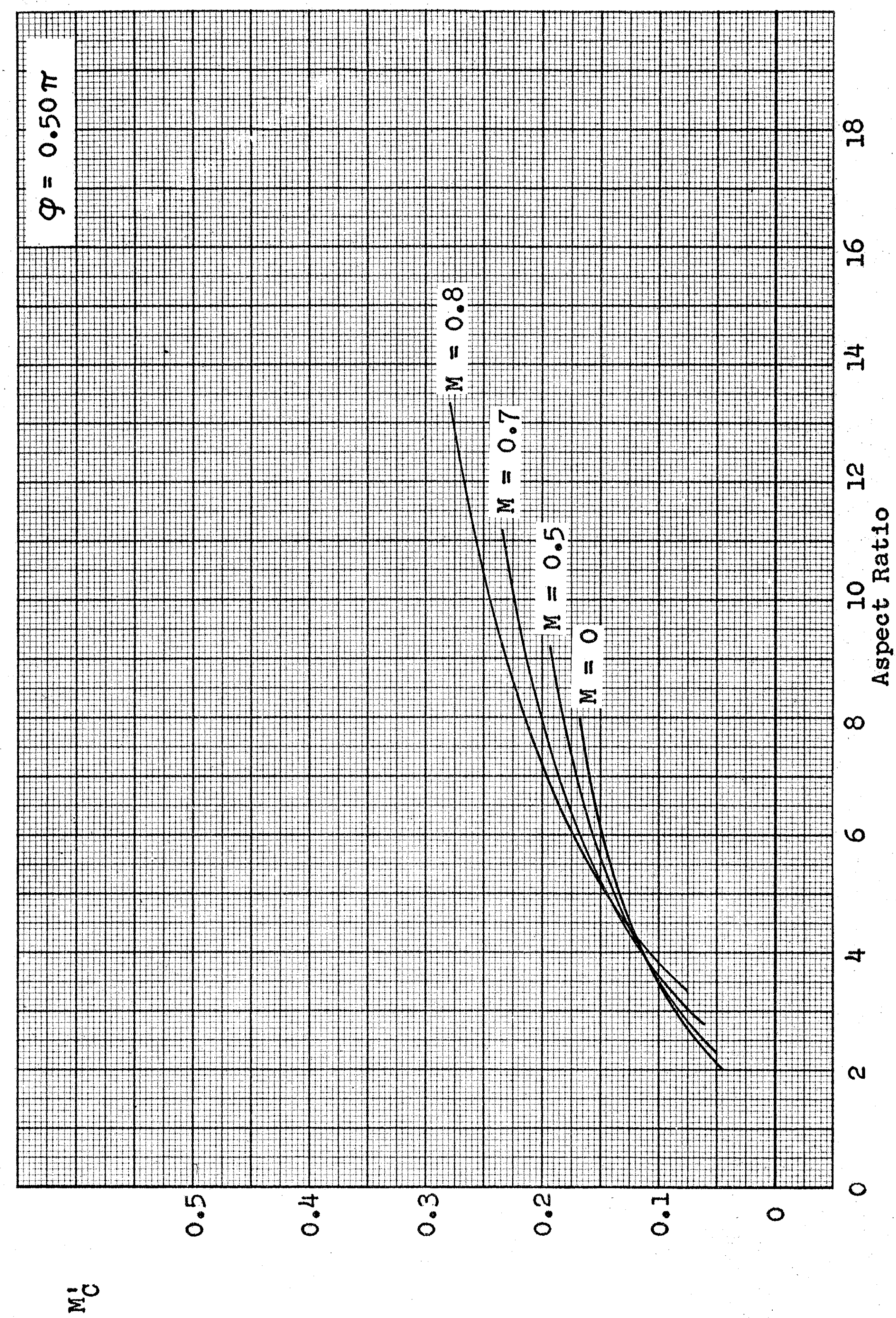

φ = 0.50π
M = 0.8
M = 0.7
M = 0.5
M = 0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio
0
0.1
0.2
0.3
0.4
0.5
M'C

FIG. 18

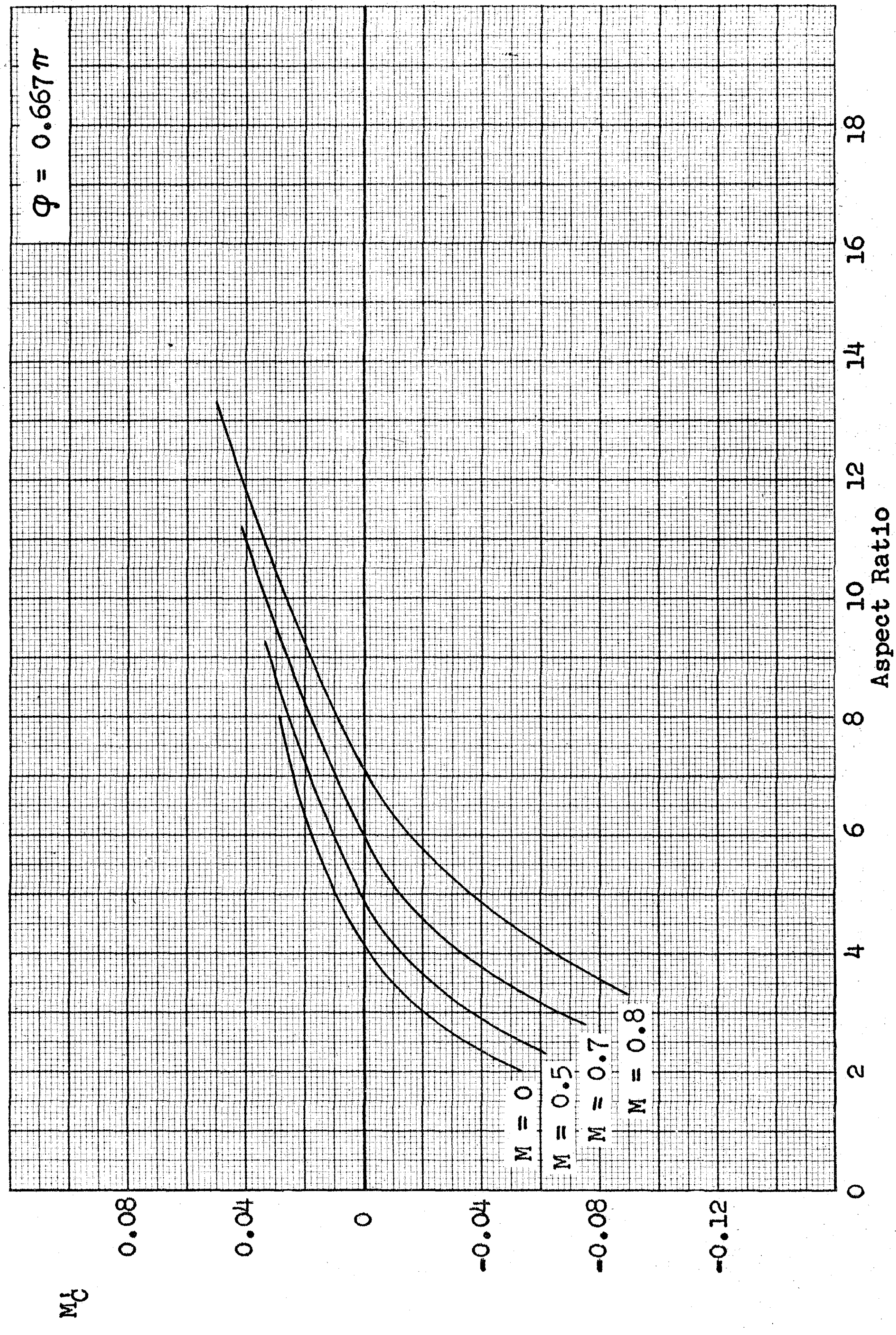
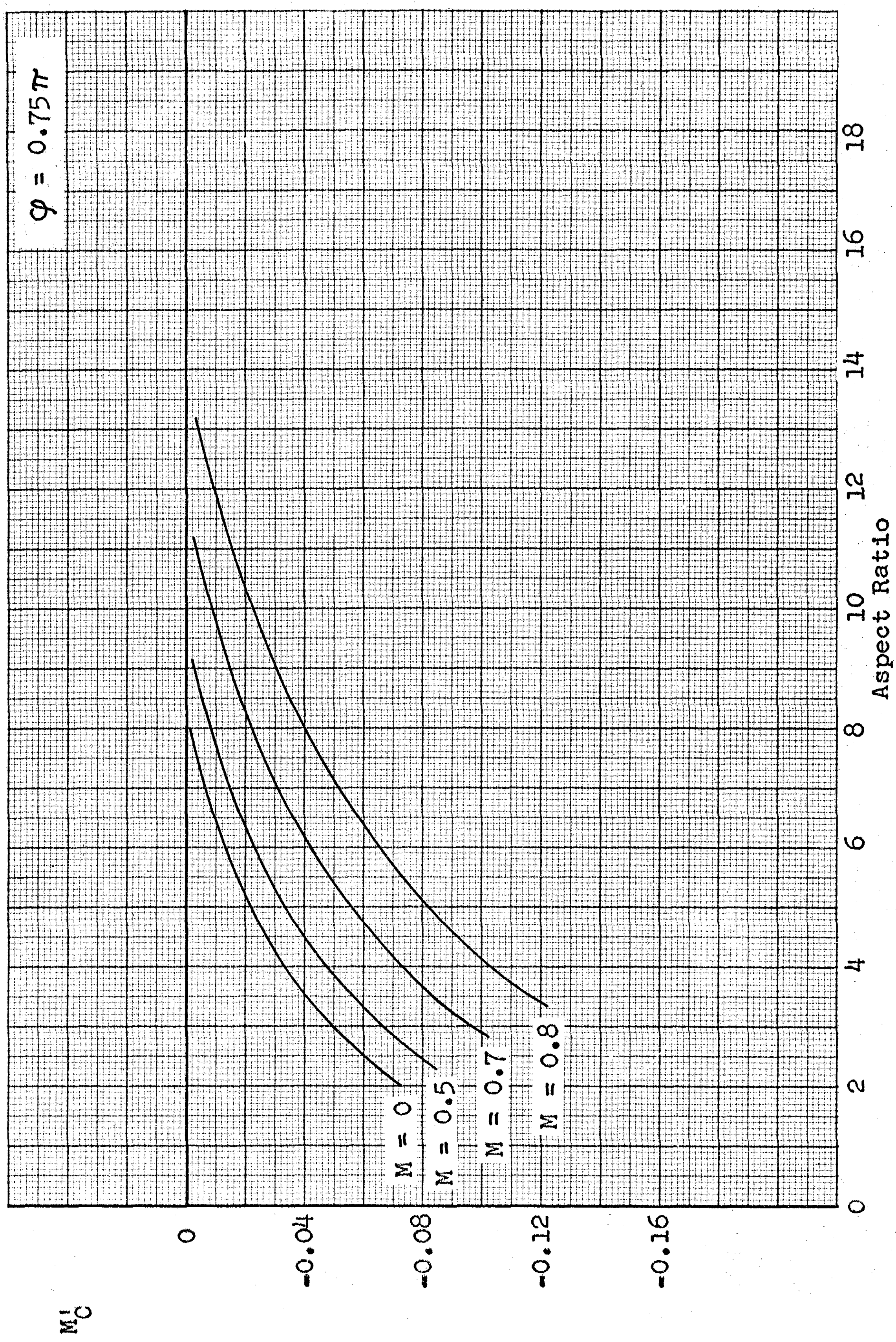

φ = 0.667π
M = 0
M = 0.5
M = 0.7
M = 0.8
0.08
0.04
0
-0.04
-0.08
-0.12
M'C
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 19



$\varphi = 0.75\pi$
M = 0
M = 0.5
M = 0.7
M = 0.8
0
2
4
6
8
10
12
14
16
18
Aspect Ratio
0
-0.04
-0.08
-0.12
-0.16
M'_C

FIG. 20

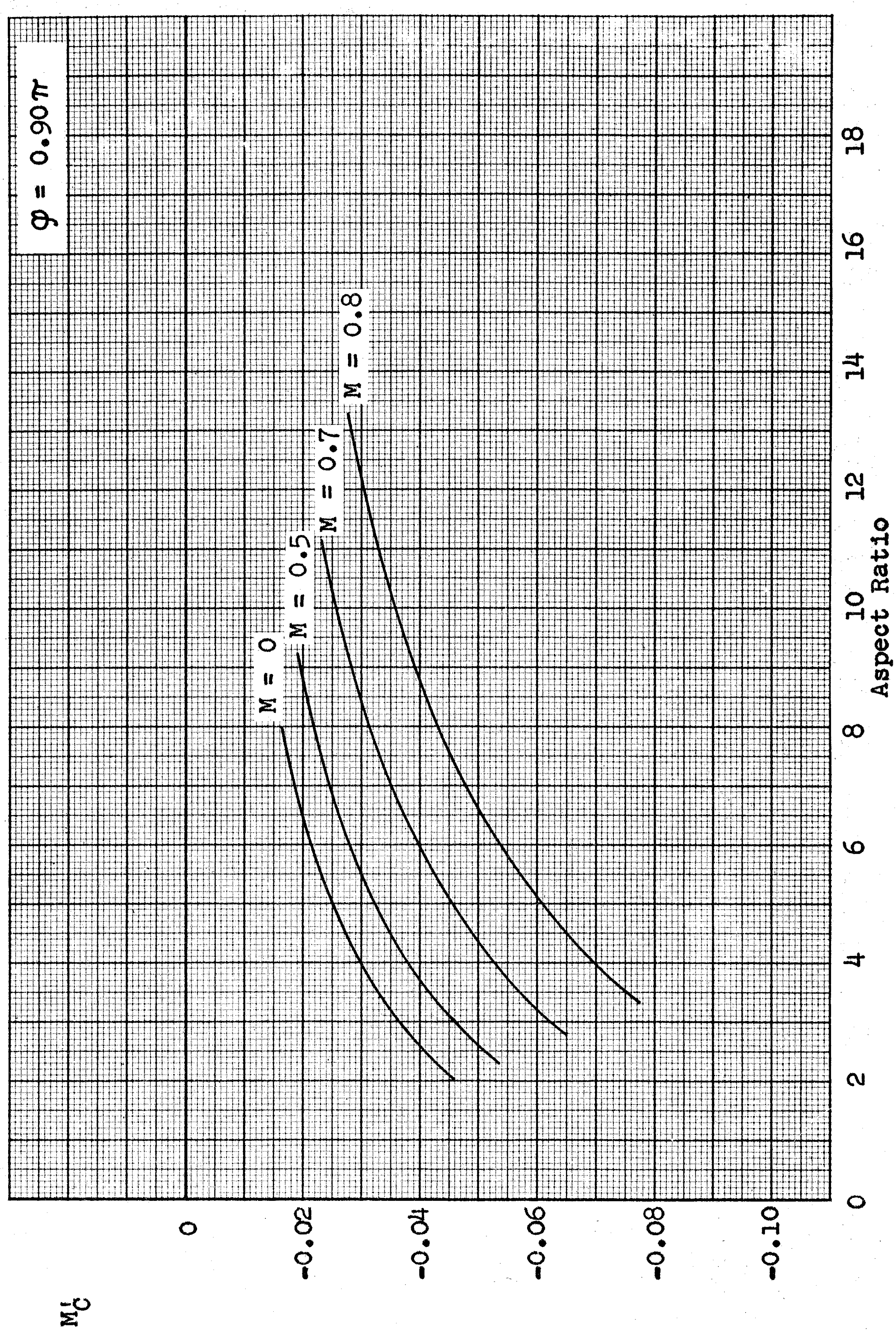

φ = 0.90π
M = 0
M = 0.5
M = 0.7
M = 0.8
Aspect Ratio
0 2 4 6 8 10 12 14 16 18
M'_C
0
-0.02
-0.04
-0.06
-0.08
-0.10

FIG. 21

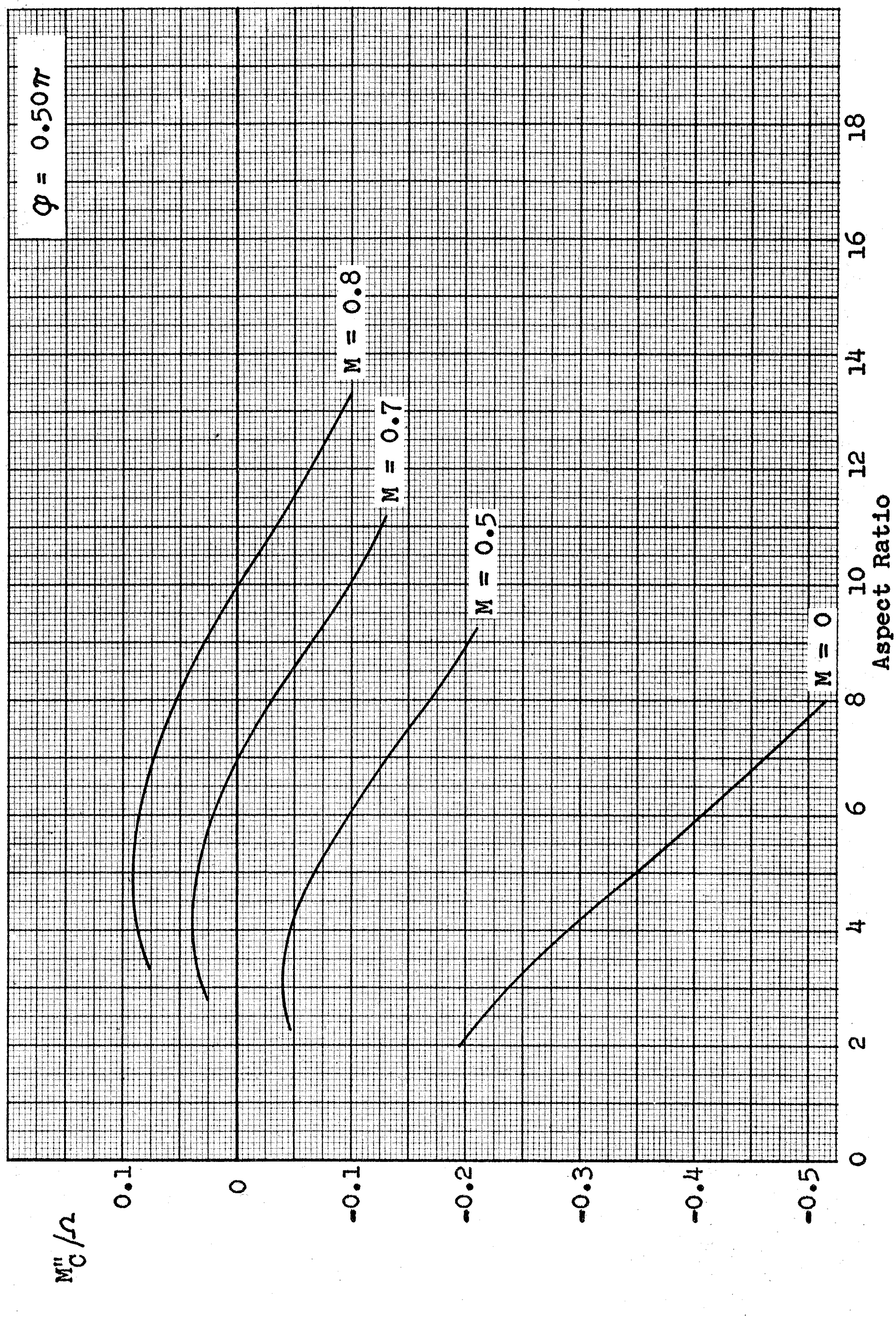
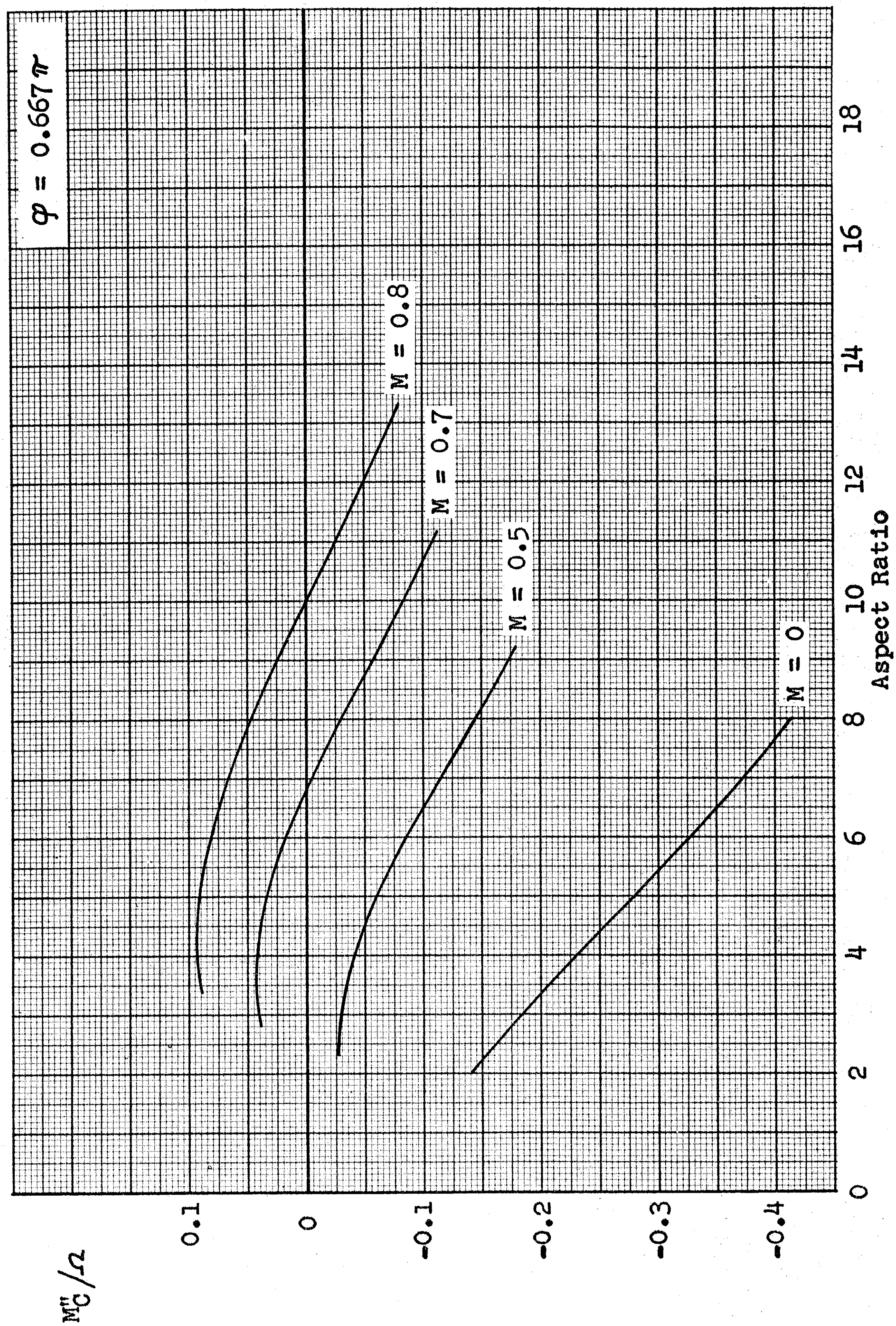

$\varphi = 0.50\pi$
M = 0.8
M = 0.7
M = 0.5
M = 0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
M''_C/Ω

FIG. 22



$\varphi = 0.667\pi$
M = 0.8
M = 0.7
M = 0.5
M = 0
M_C''/Ω
0.1
0
-0.1
-0.2
-0.3
-0.4
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 23

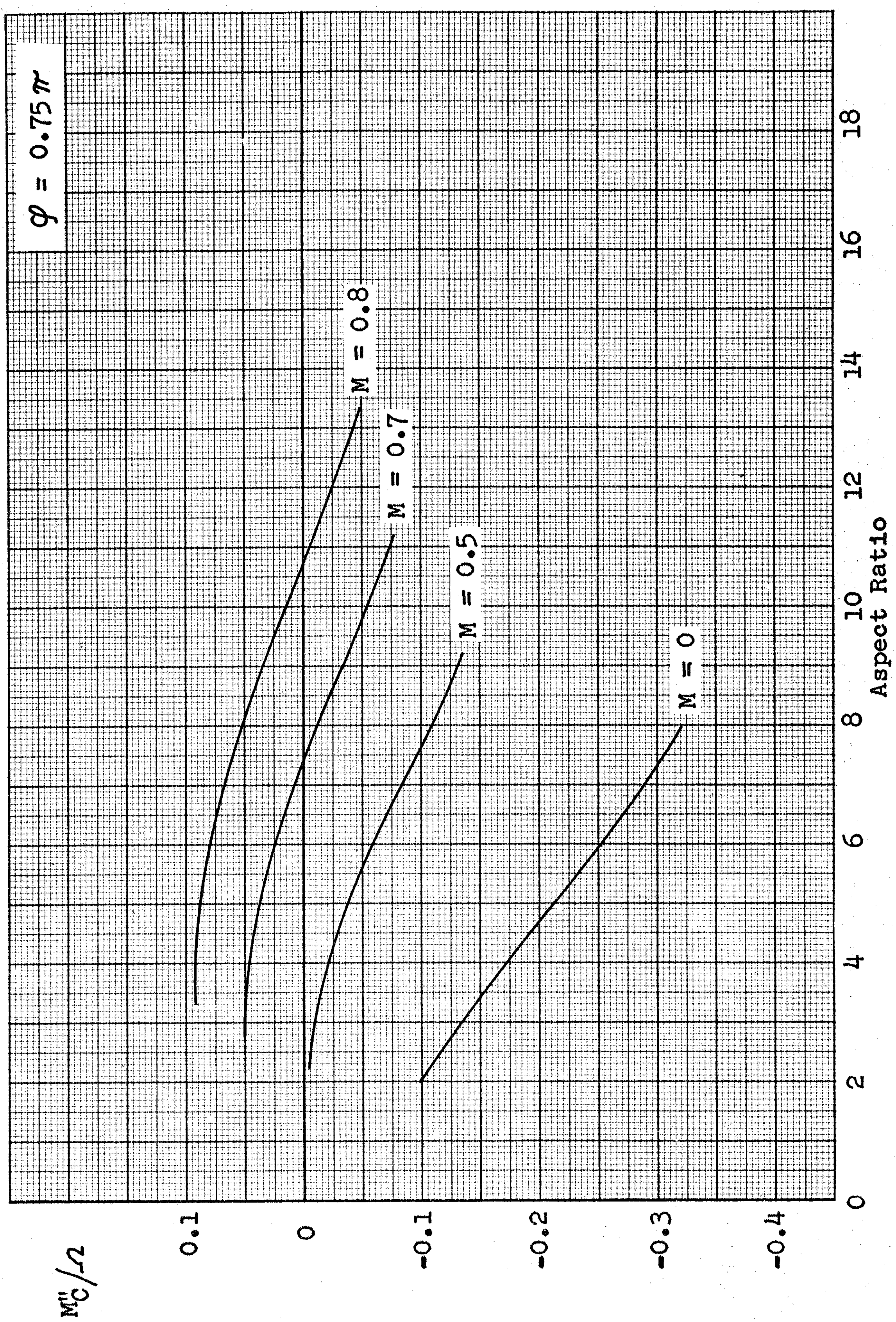
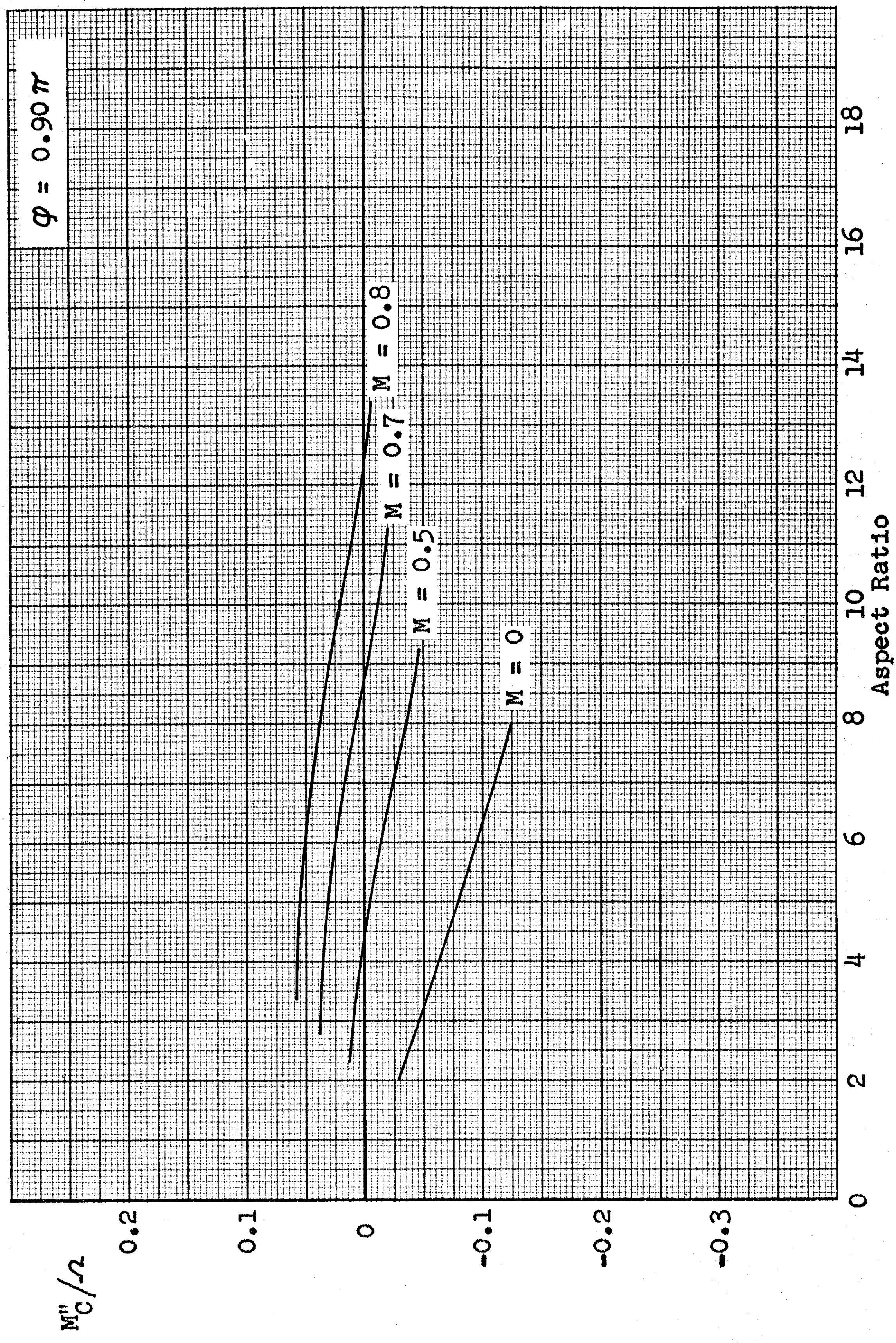

$\varphi = 0.75\pi$
M = 0.8
M = 0.7
M = 0.5
M = 0
M_C''/Ω
0.1
0
-0.1
-0.2
-0.3
-0.4
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 24



φ = 0.90 π
M = 0.8
M = 0.7
M = 0.5
M = 0
0.2
0.1
0
-0.1
-0.2
-0.3
M''_C/Ω
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

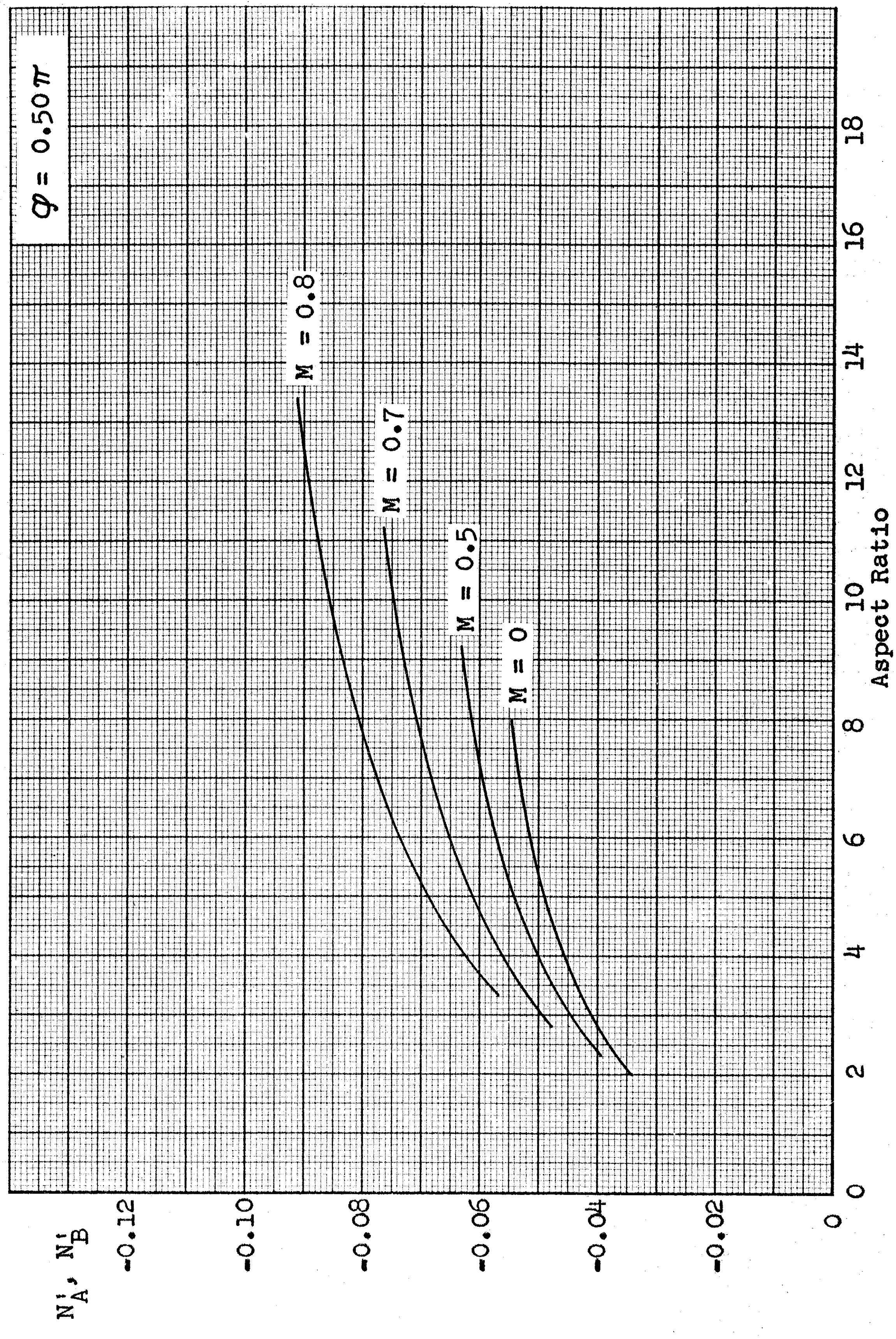
FIG. 25

FIGURES 26 THROUGH 42

HINGE MOMENT DERIVATIVES vs ASPECT RATIO FOR

MACH NUMBERS OF 0, 0.5, 0.7 and 0.8 AND FOR

HINGE POSITIONS OF $\varphi = 0.50\pi, 0.667\pi, 0.75\pi, 0.90\pi$



$\varphi = 0.50\pi$
M = 0.8
M = 0.7
M = 0.5
M = 0
N'_A, N'_B
-0.12
-0.10
-0.08
-0.06
-0.04
-0.02
0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 26

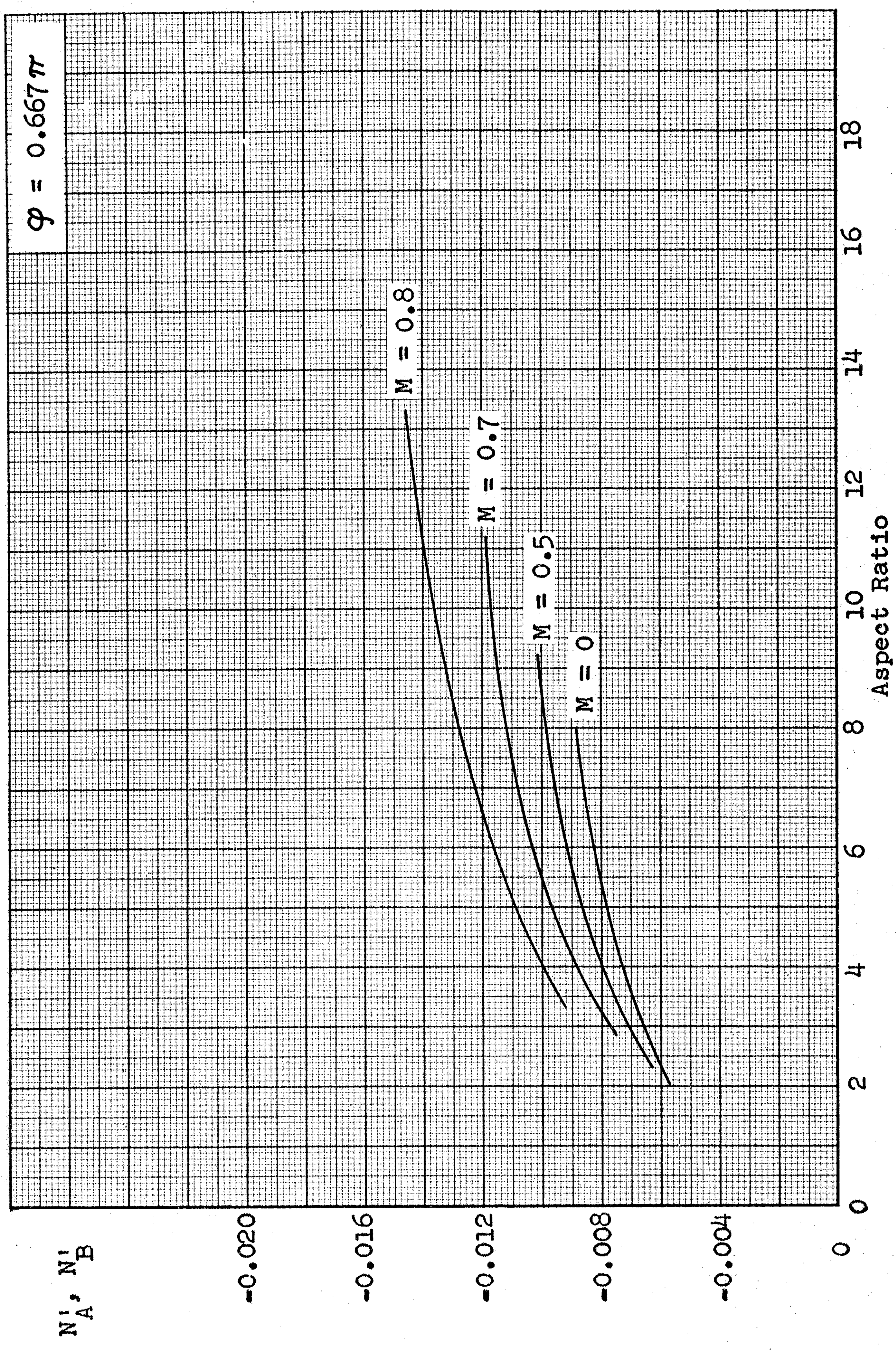
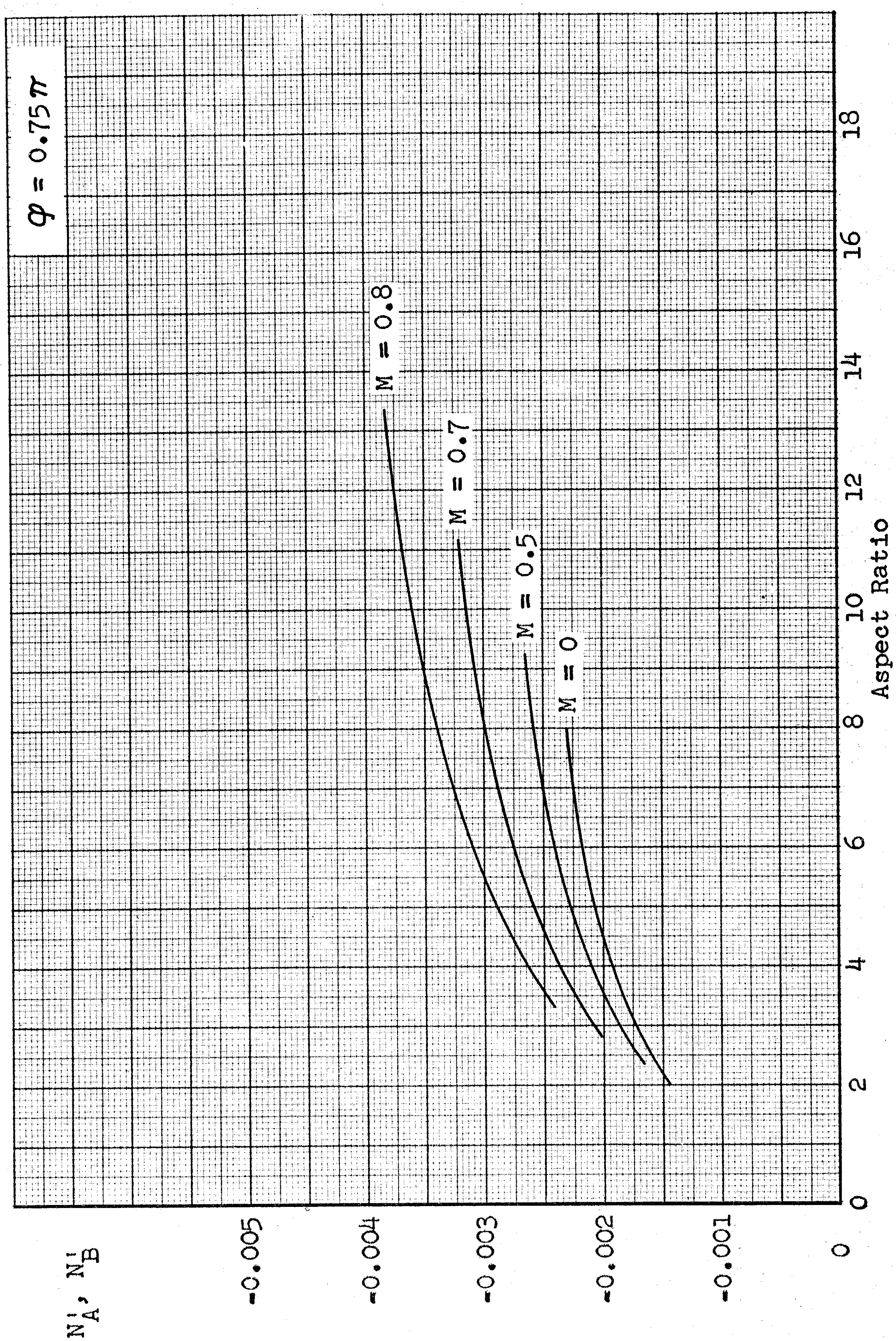

$\varphi = 0.667\pi$
M = 0.8
M = 0.7
M = 0.5
M = 0
N'_A, N'_B
-0.020
-0.016
-0.012
-0.008
-0.004
0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 27



$\varphi = 0.75\pi$
M = 0.8
M = 0.7
M = 0.5
M = 0
N'_A, N'_B
-0.005
-0.004
-0.003
-0.002
-0.001
0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 28

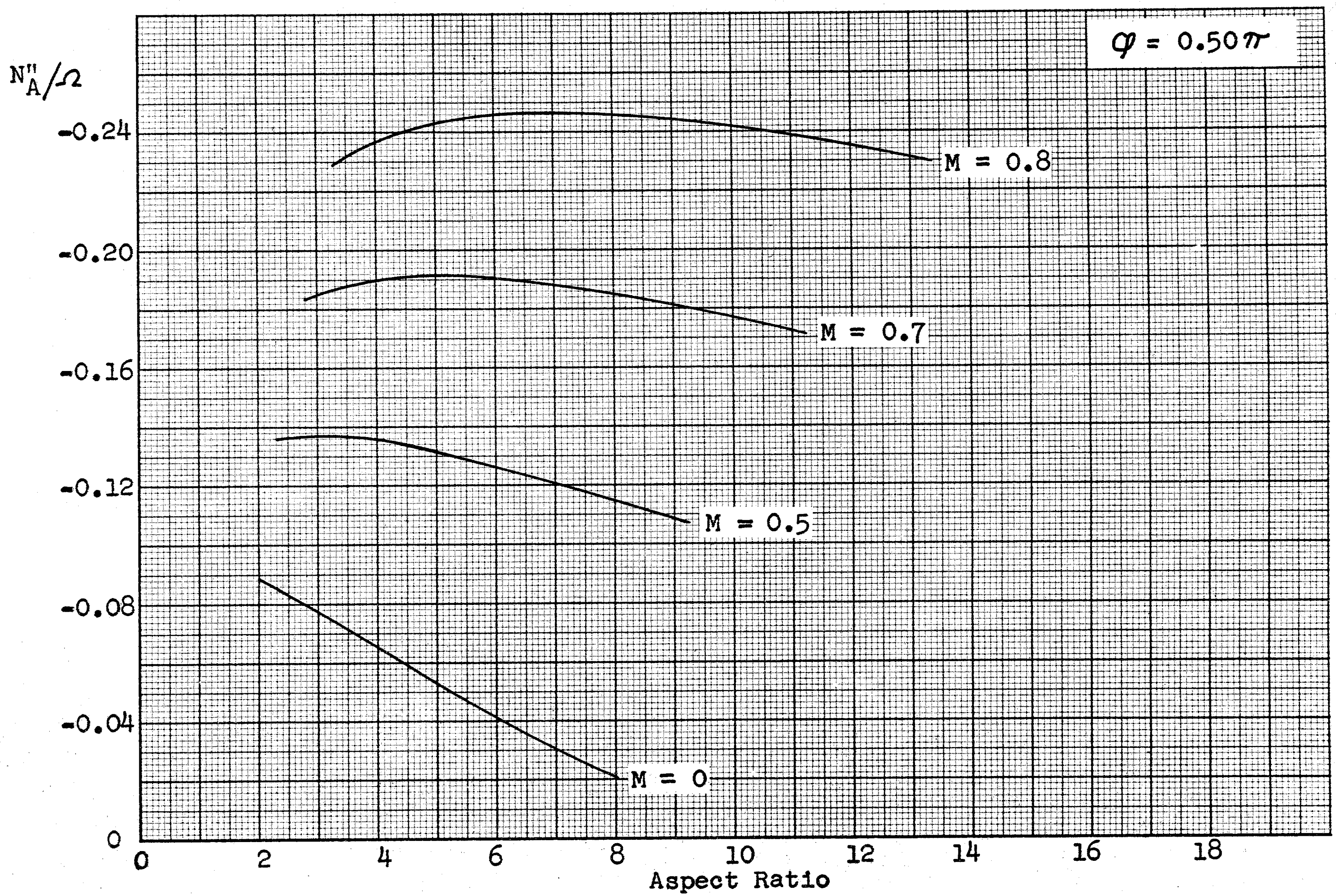
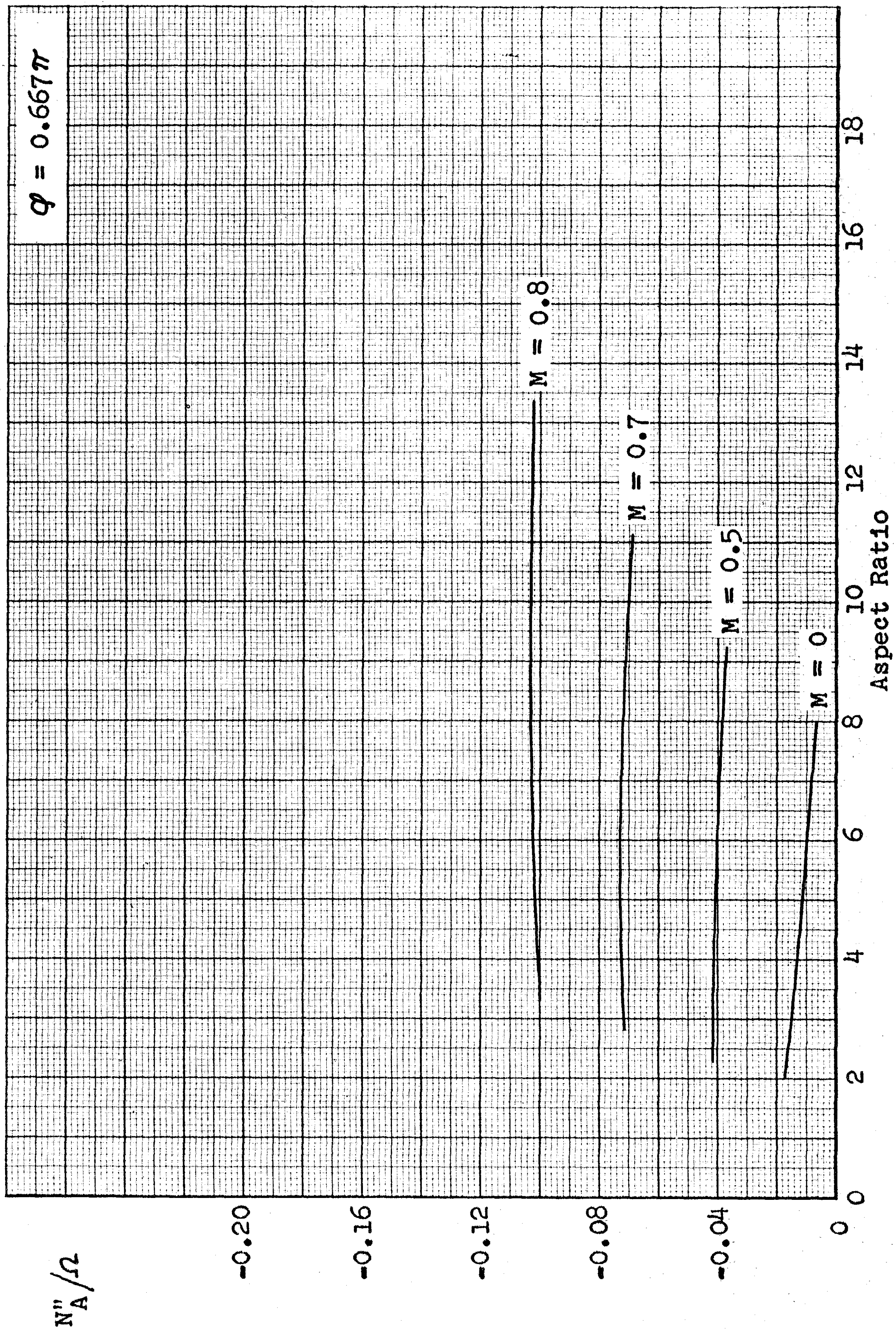

$\varphi = 0.50\pi$
N''_A/Ω
-0.24
-0.20
-0.16
-0.12
-0.08
-0.04
0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio
M = 0.8
M = 0.7
M = 0.5
M = 0

FIG. 29



$\varphi = 0.667\pi$
M = 0.8
M = 0.7
M = 0.5
M = 0
Aspect Ratio
0
2
4
6
8
10
12
14
16
18
N''_A/Ω
0
-0.04
-0.08
-0.12
-0.16
-0.20

FIG. 30

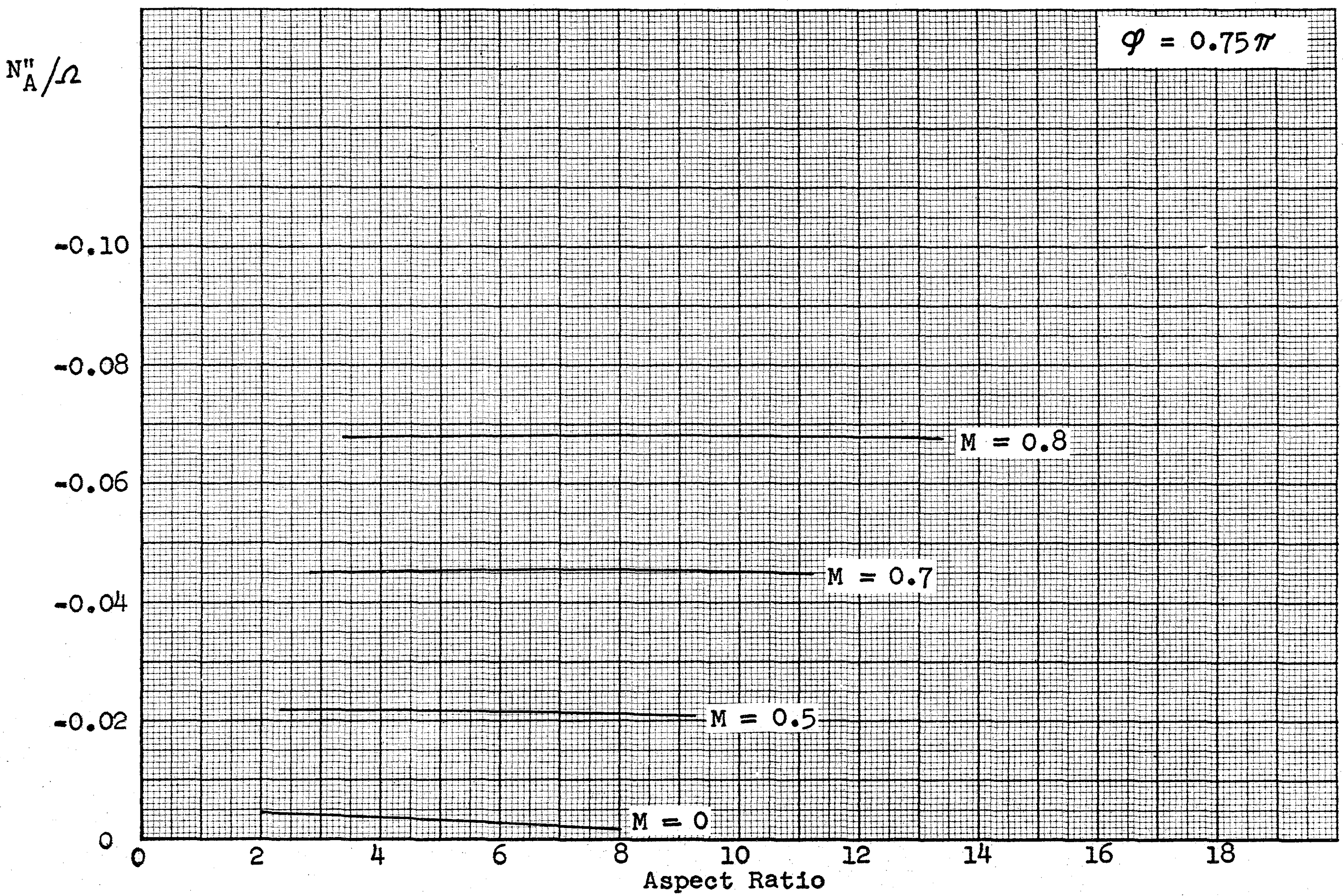

N''_A/Ω
$\varphi = 0.75\pi$
-0.10
-0.08
-0.06
-0.04
-0.02
0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio
M = 0.8
M = 0.7
M = 0.5
M = 0

FIG. 31

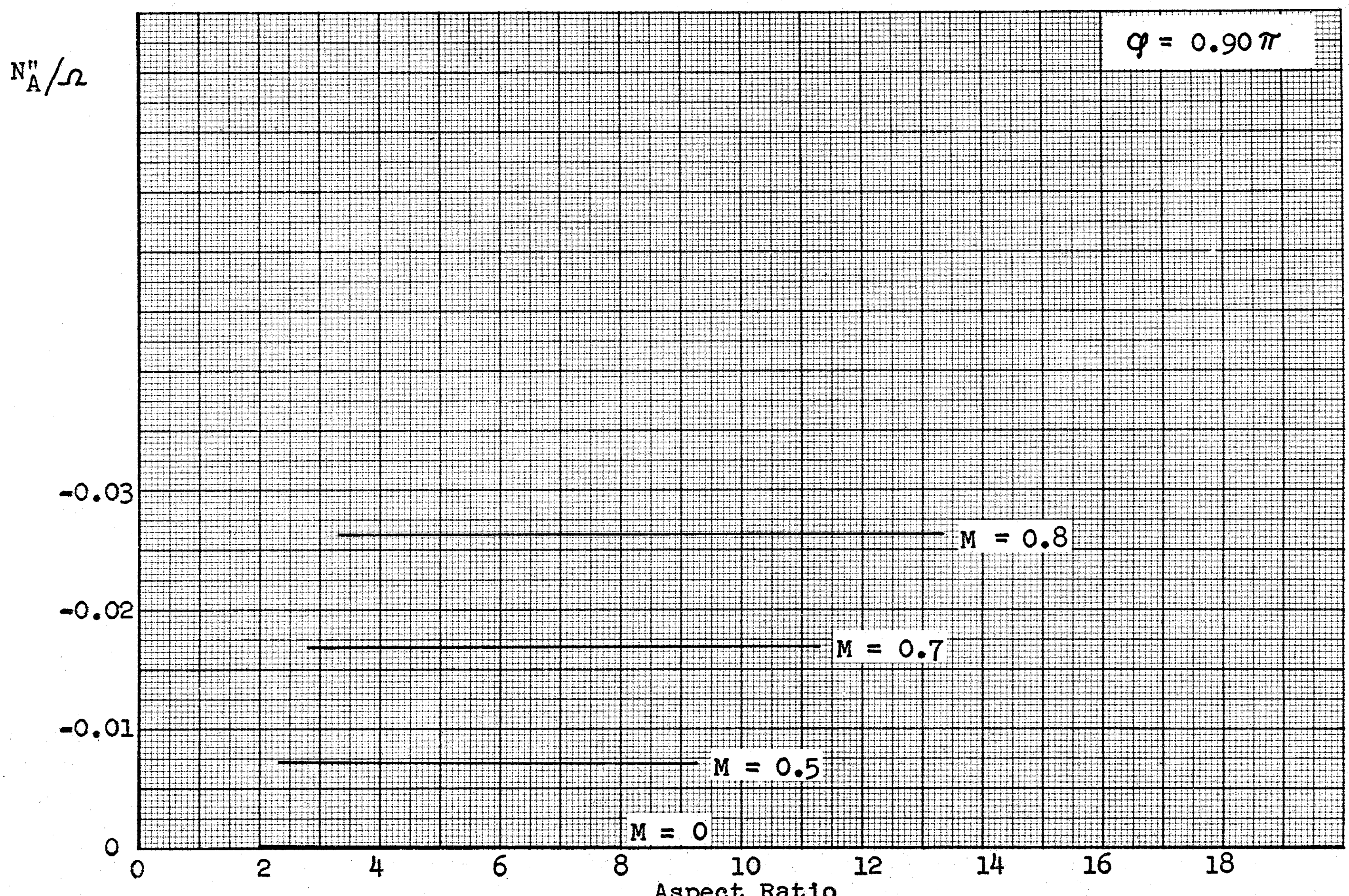
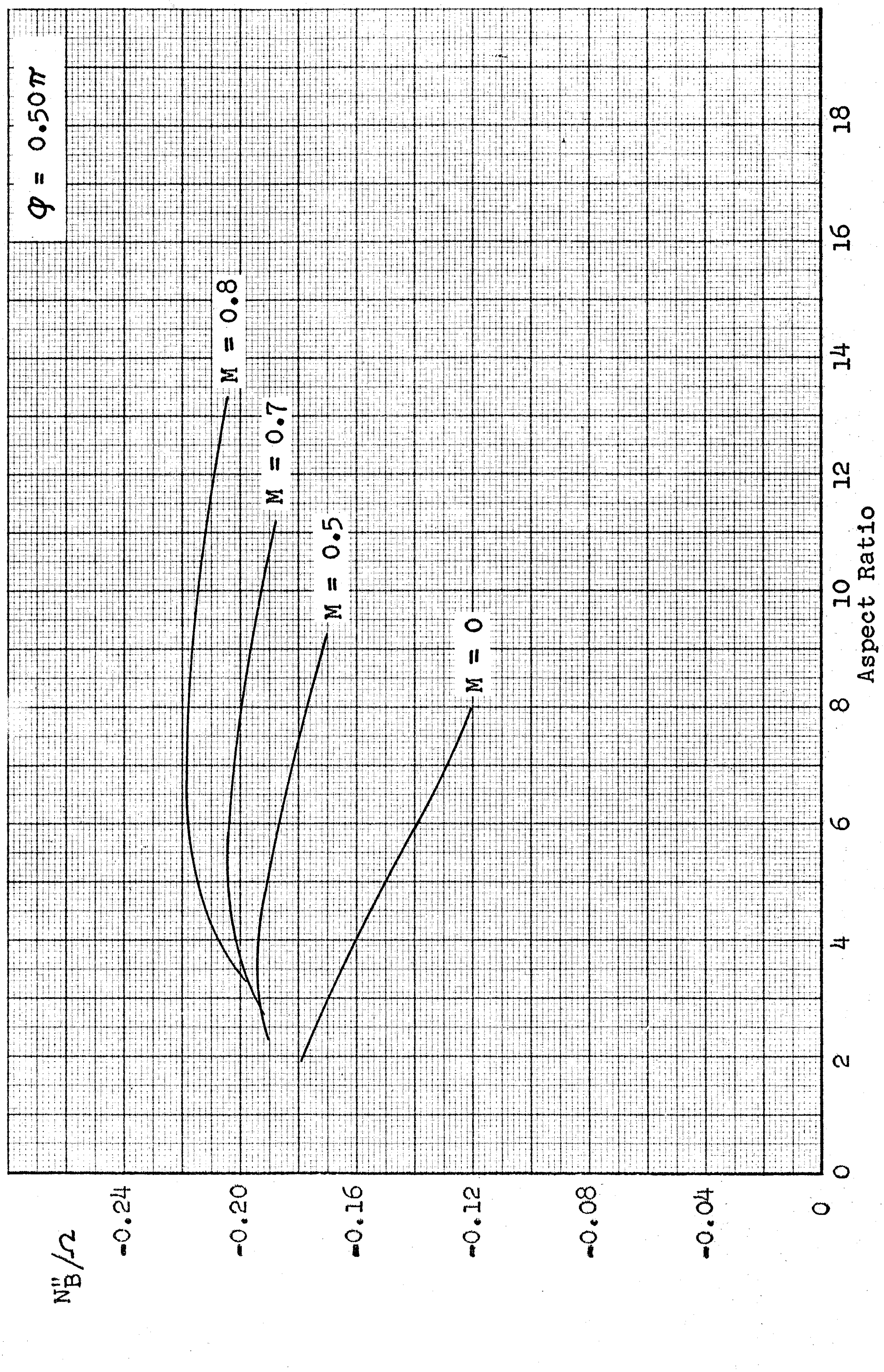
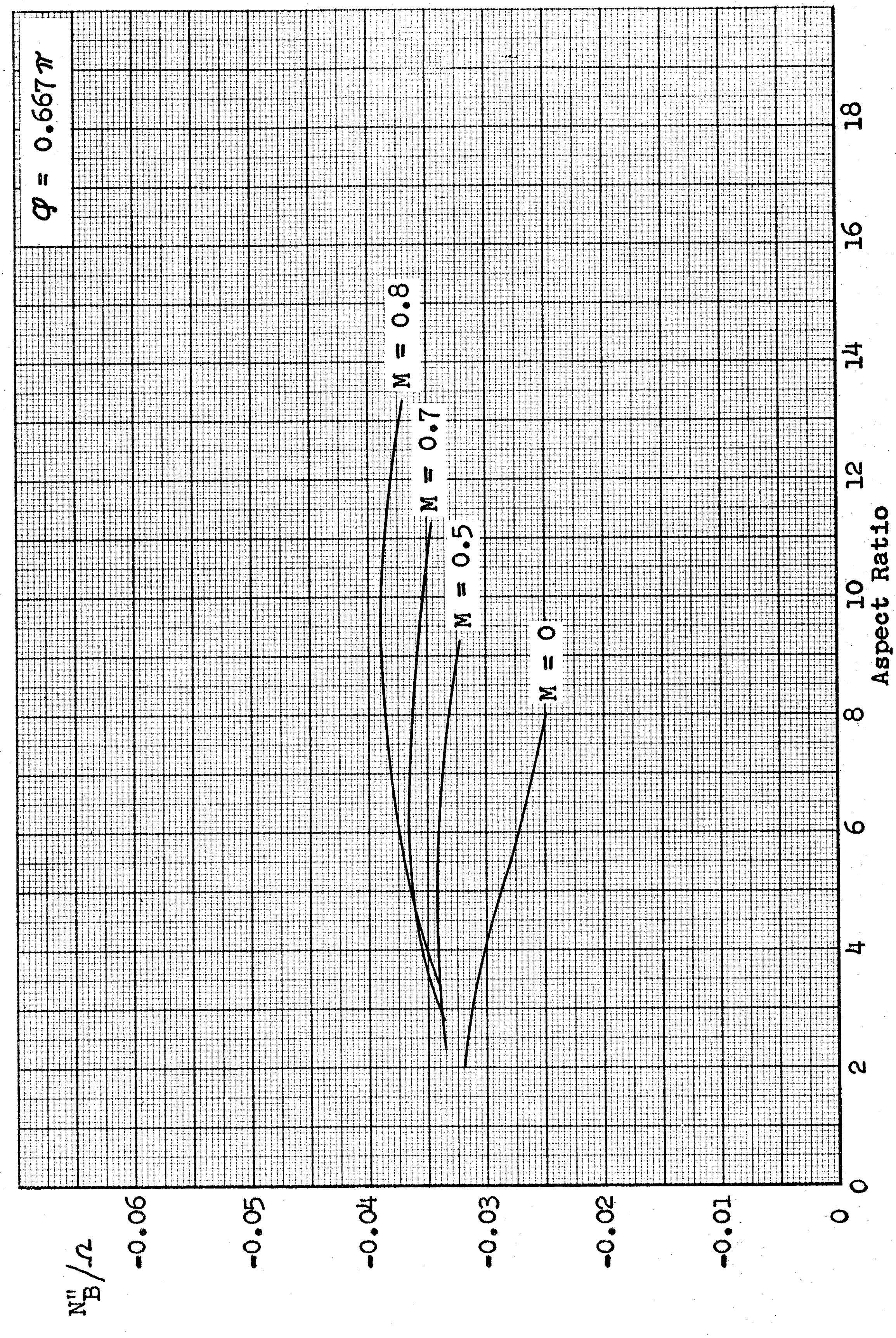

φ = 0.90 π
N''_A/Ω
-0.03
-0.02
-0.01
0
M = 0.8
M = 0.7
M = 0.5
M = 0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 32



φ = 0.50π
M = 0.8
M = 0.7
M = 0.5
M = 0
N''_B/Ω
-0.24
-0.20
-0.16
-0.12
-0.08
-0.04
0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 33



$\varphi = 0.667\pi$
M = 0.8
M = 0.7
M = 0.5
M = 0
N''_B/Ω
-0.06
-0.05
-0.04
-0.03
-0.02
-0.01
0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 34

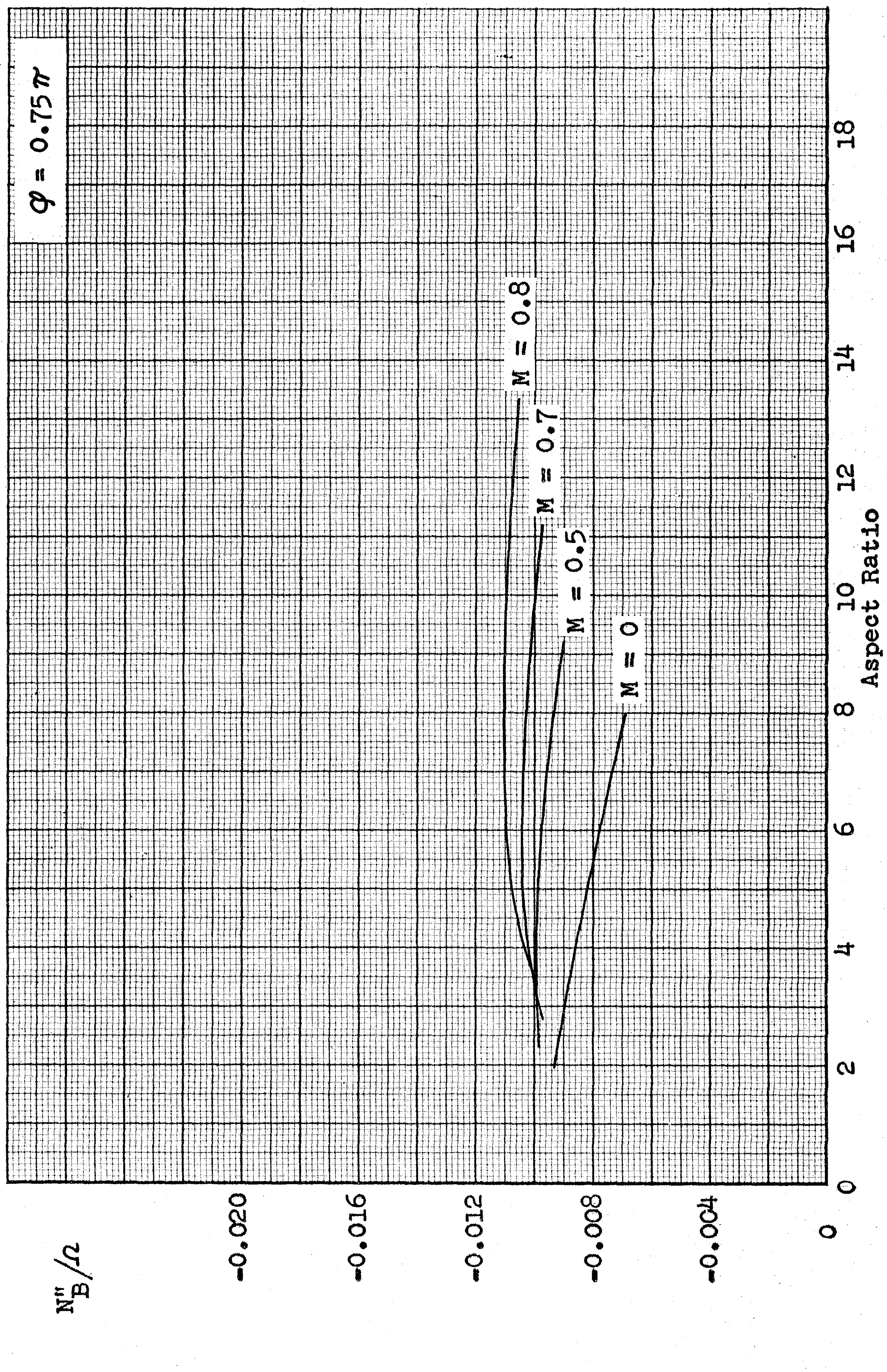

φ = 0.75π
M = 0.8
M = 0.7
M = 0.5
M = 0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio
N"B/Ω
-0.020
-0.016
-0.012
-0.008
-0.004
0

FIG. 35

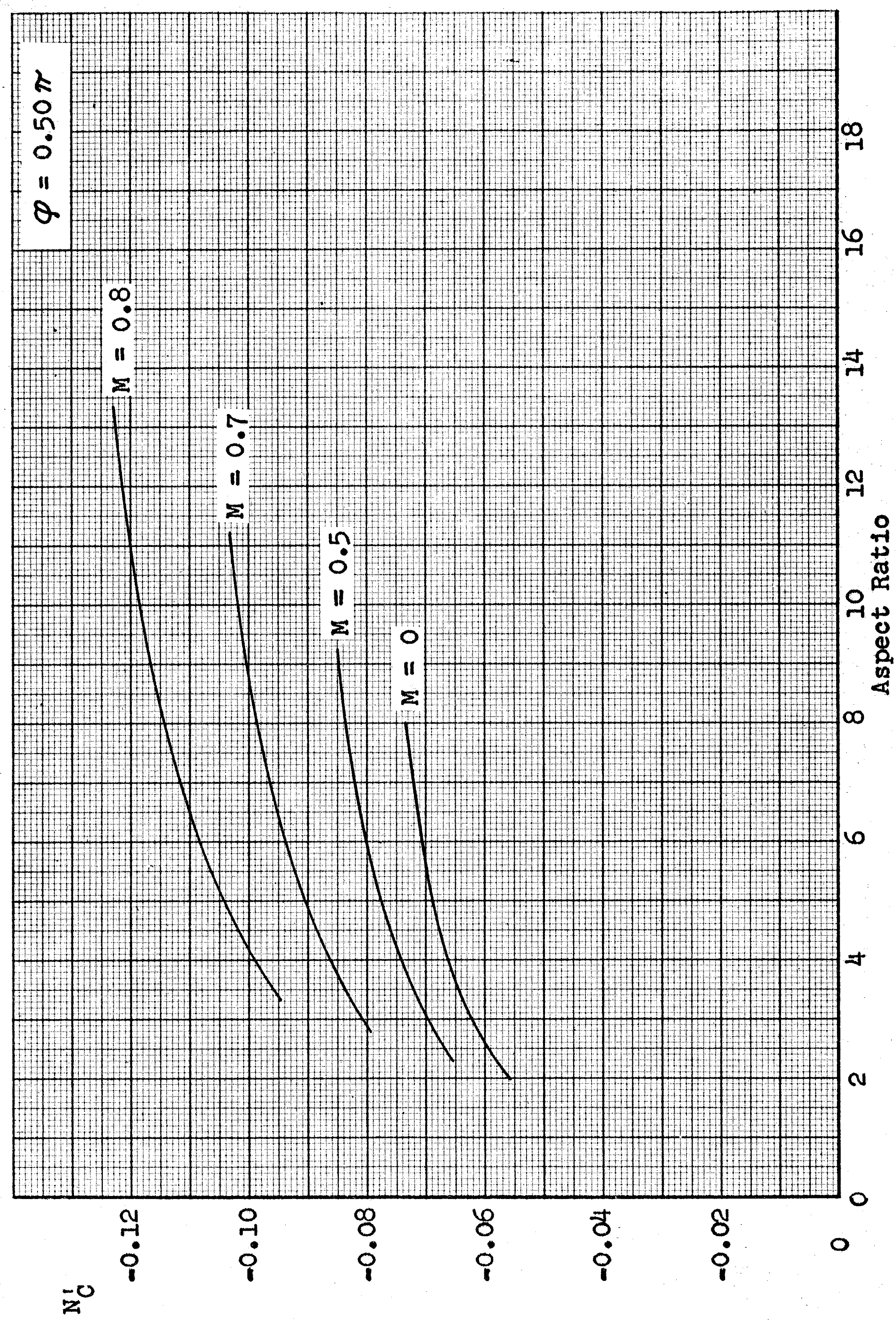

$\varphi = 0.50\pi$
M = 0.8
M = 0.7
M = 0.5
M = 0
N'_C
-0.12
-0.10
-0.08
-0.06
-0.04
-0.02
0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 36

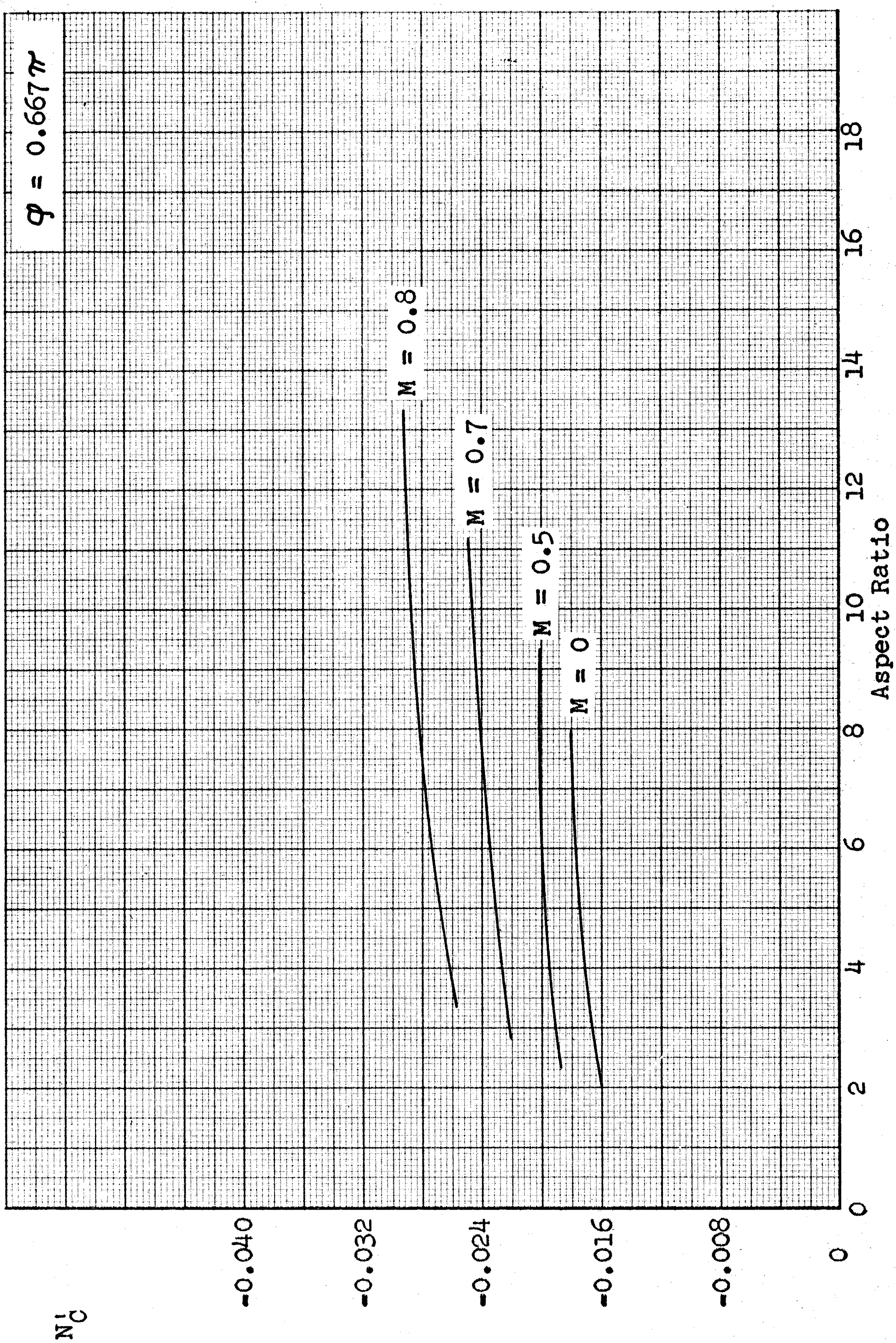
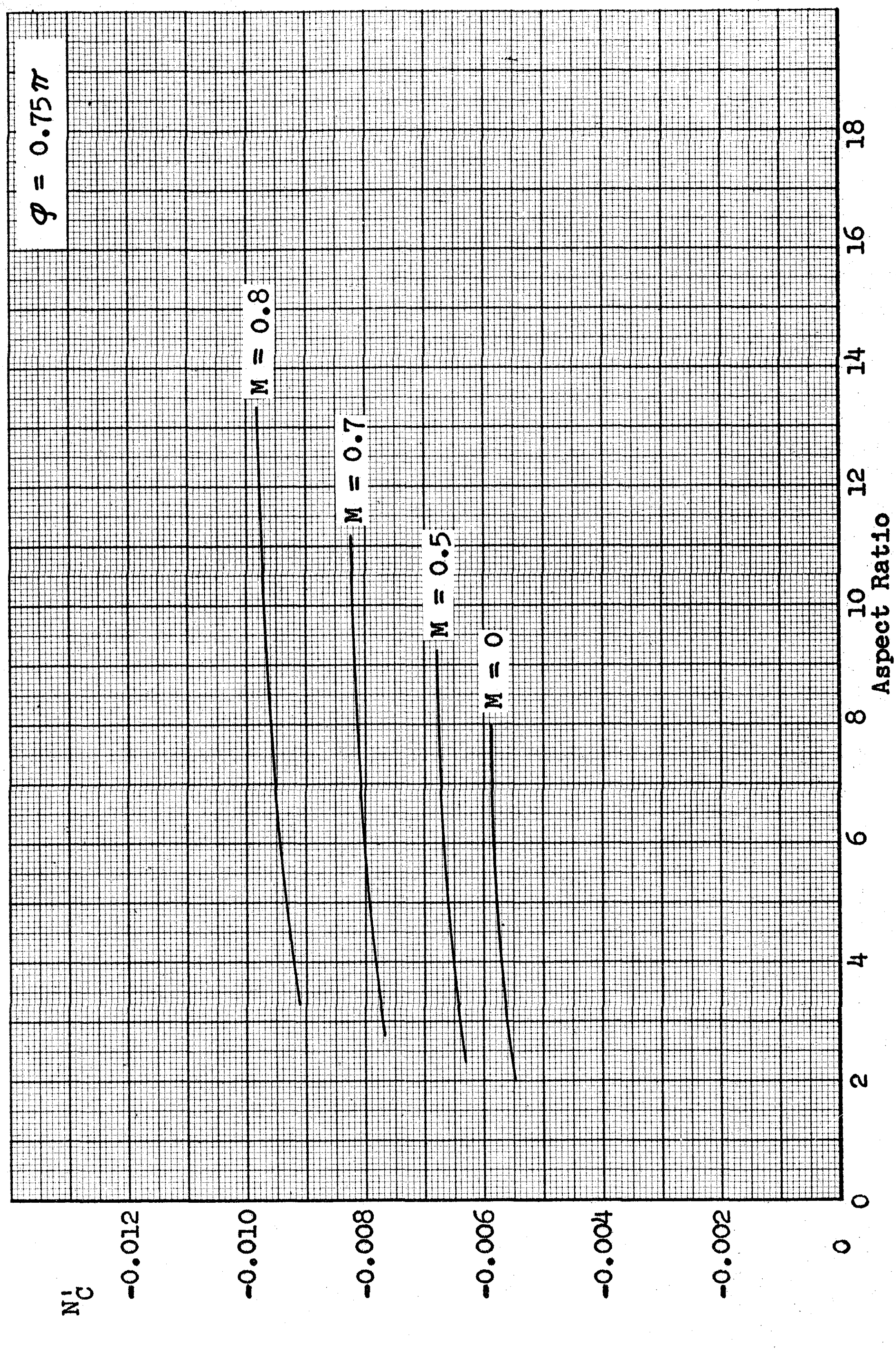
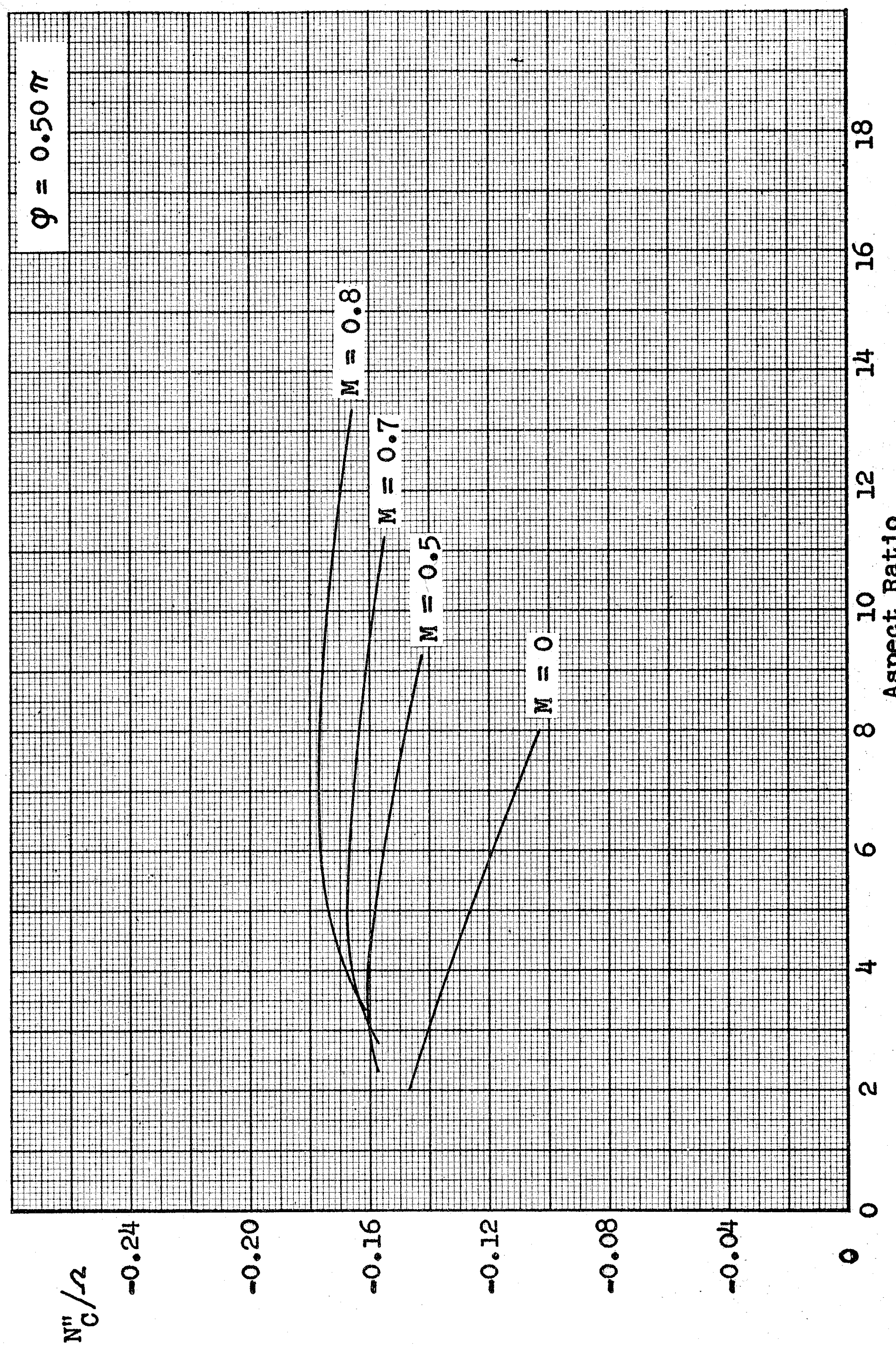

φ = 0.667π
M = 0.8
M = 0.7
M = 0.5
M = 0
N'C
-0.040
-0.032
-0.024
-0.016
-0.008
0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 37



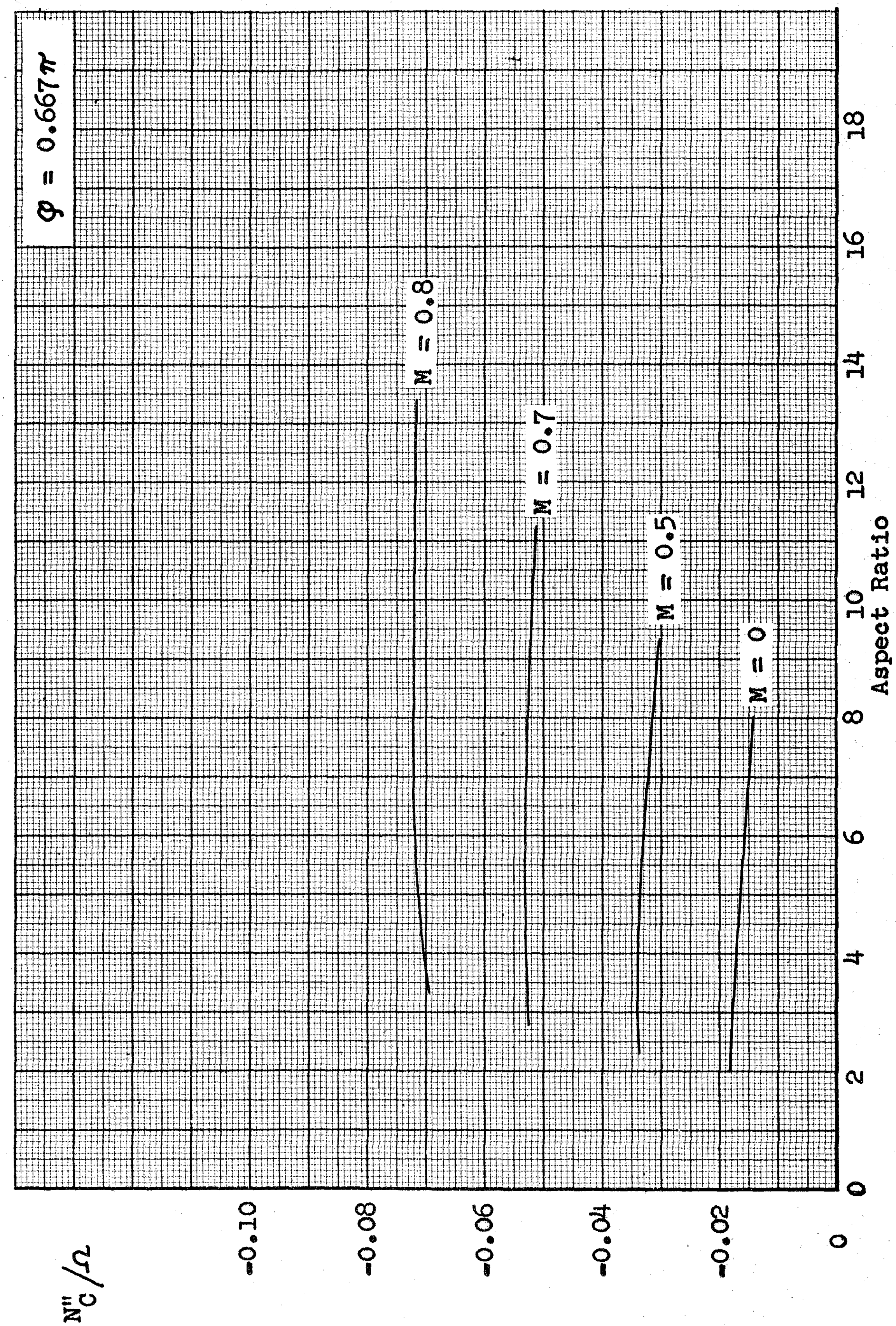
$\varphi = 0.75\pi$
M = 0.8
M = 0.7
M = 0.5
M = 0
N'_C
-0.012
-0.010
-0.008
-0.006
-0.004
-0.002
0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 38



$\varphi = 0.50\pi$
M = 0.8
M = 0.7
M = 0.5
M = 0
Aspect Ratio
0
2
4
6
8
10
12
14
16
18
N''_C/Ω
-0.24
-0.20
-0.16
-0.12
-0.08
-0.04
0

FIG. 39



$\varphi = 0.667\pi$
M = 0.8
M = 0.7
M = 0.5
M = 0
N''_C/Ω
-0.10
-0.08
-0.06
-0.04
-0.02
0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 40

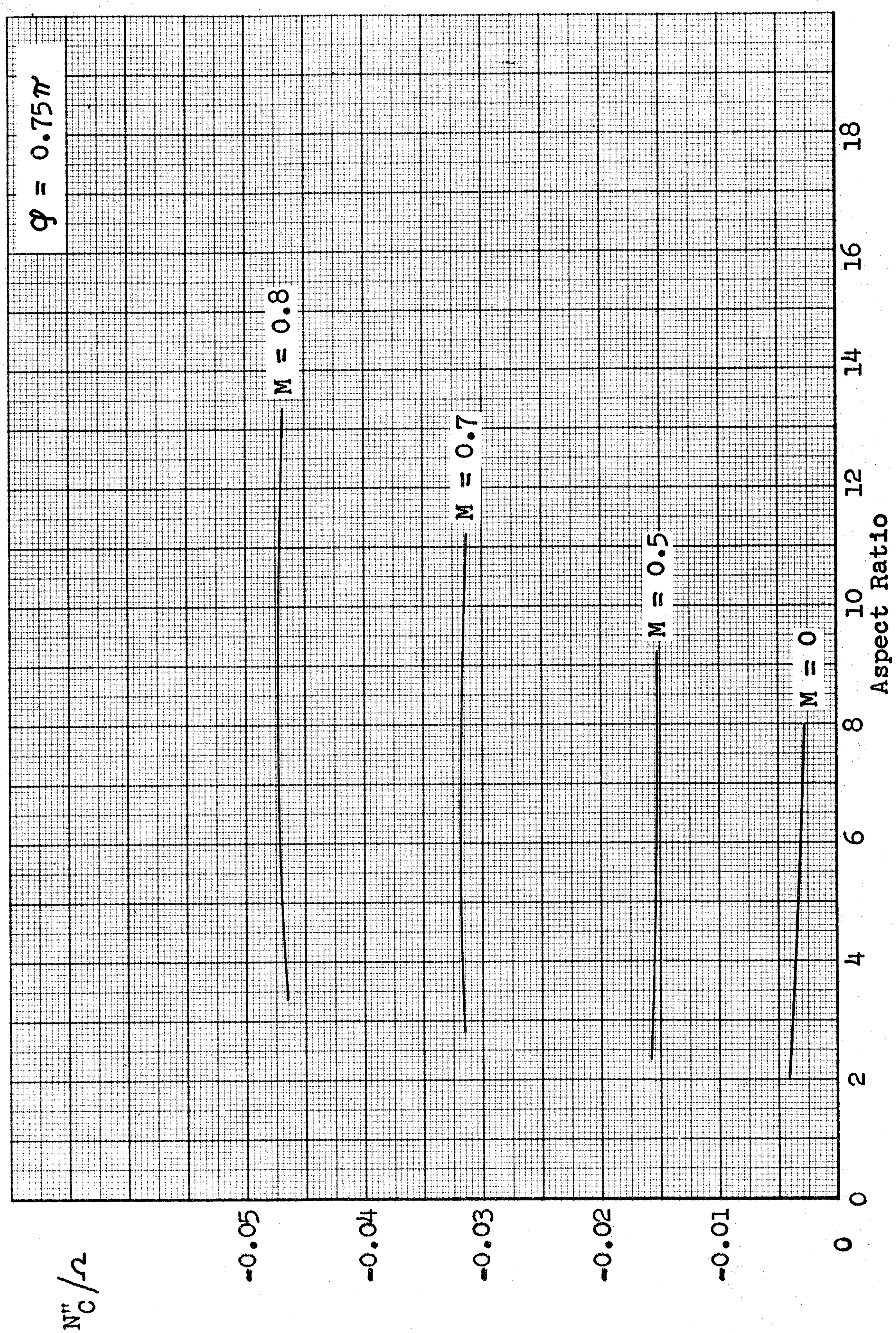

$\varphi = 0.75\pi$
M = 0.8
M = 0.7
M = 0.5
M = 0
N_C''/Ω
-0.05
-0.04
-0.03
-0.02
-0.01
0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 41

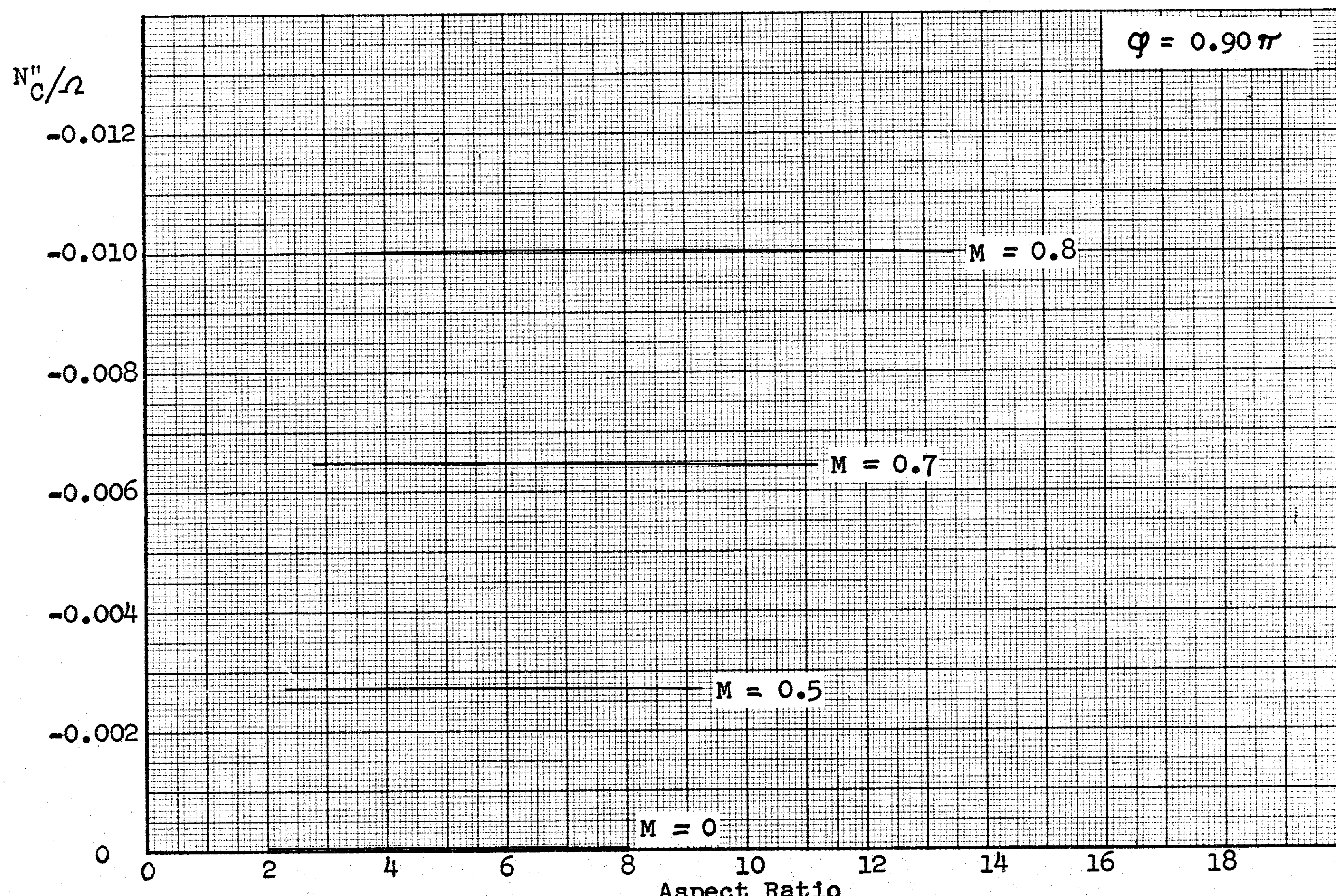

$\varphi = 0.90\pi$
N''_C/Ω
-0.012
-0.010
-0.008
-0.006
-0.004
-0.002
0
M = 0.8
M = 0.7
M = 0.5
M = 0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 42

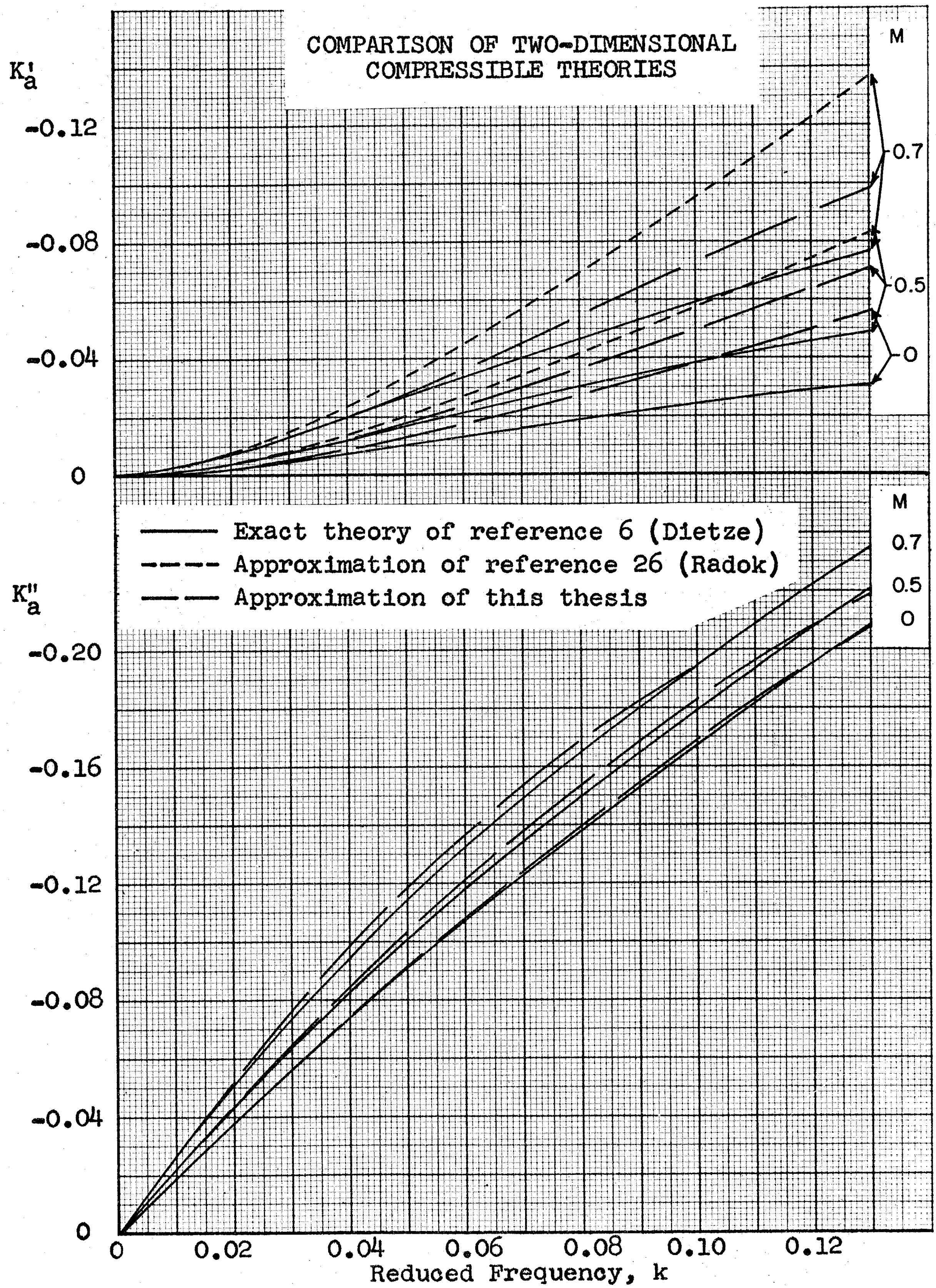

COMPARISON OF TWO-DIMENSIONAL COMPRESSIBLE THEORIES
K_a'
-0.12
-0.08
-0.04
0
M
0.7
0.5
0
Exact theory of reference 6 (Dietze)
Approximation of reference 26 (Radok)
Approximation of this thesis
K_a''
-0.20
-0.16
-0.12
-0.08
-0.04
0
M
0.7
0.5
0
0
0.02
0.04
0.06
0.08
0.10
0.12
Reduced Frequency, k

FIG. 43

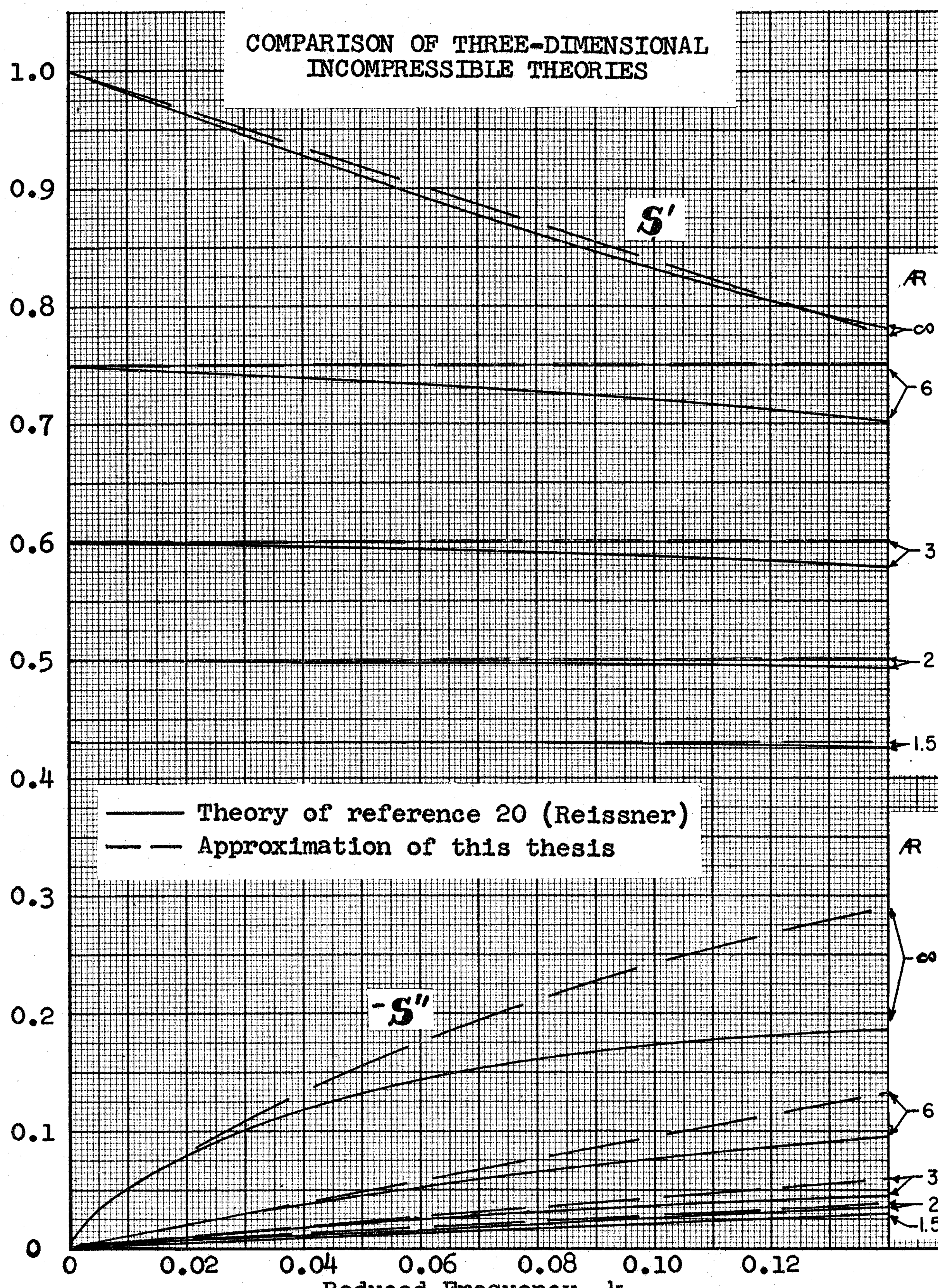

COMPARISON OF THREE-DIMENSIONAL INCOMPRESSIBLE THEORIES
S'
-S''
AR
∞
6
3
2
1.5
Theory of reference 20 (Reissner)
Approximation of this thesis
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.02
0.04
0.06
0.08
0.10
0.12
Reduced Frequency, k

FIG. 44

FIGURES 45 THROUGH 56

DOWNWASH DERIVATIVES vs ASPECT RATIO FOR MACH NUMBERS OF 0, 0.5, 0.7 AND 0.8 AND FOR TAIL LENGTHS OF $l_T/l = 4, 6, 8, 10$

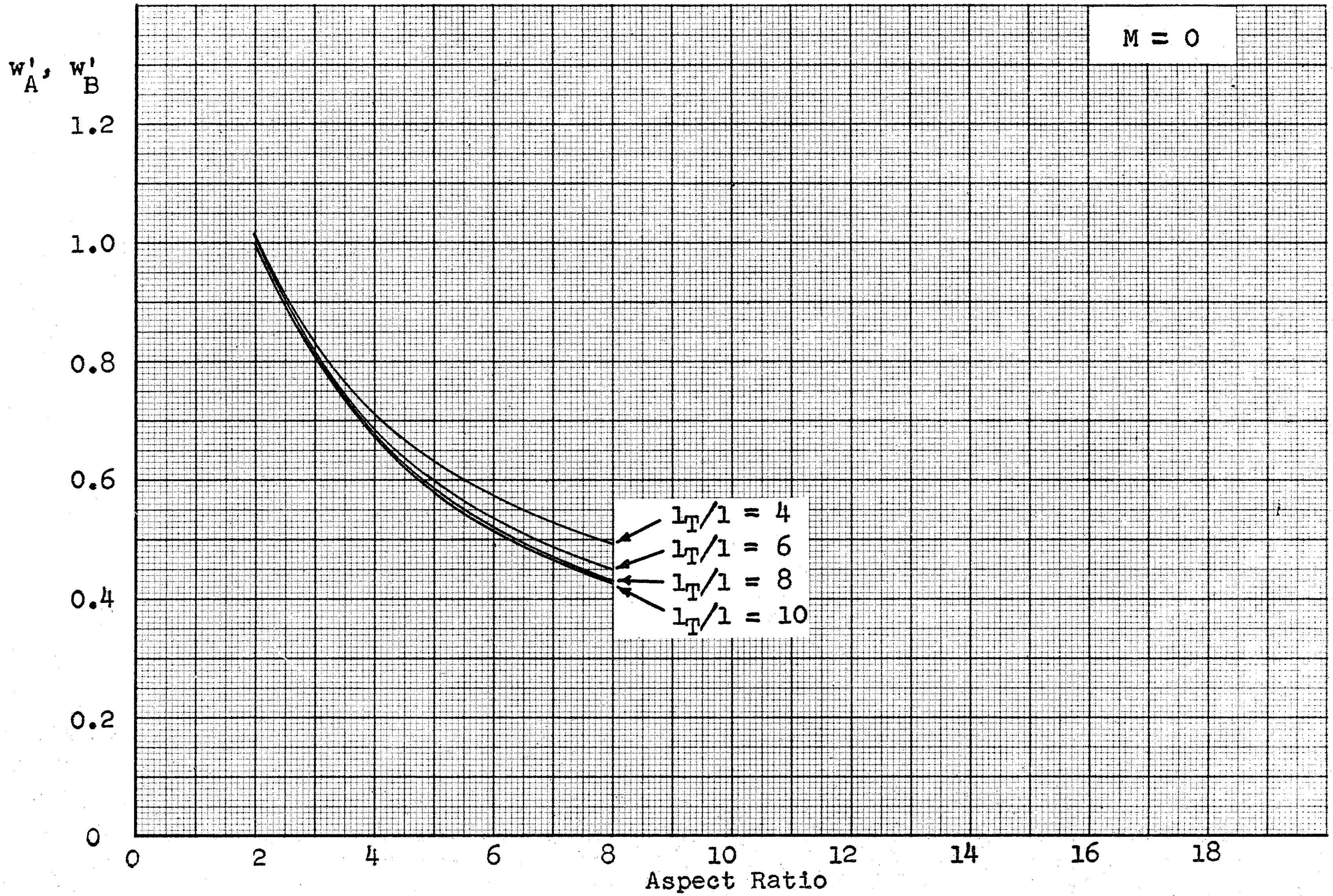

M = 0
w'_A, w'_B
1.2
1.0
0.8
0.6
0.4
0.2
0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio
$l_T/l = 4$
$l_T/l = 6$
$l_T/l = 8$
$l_T/l = 10$

FIG. 45

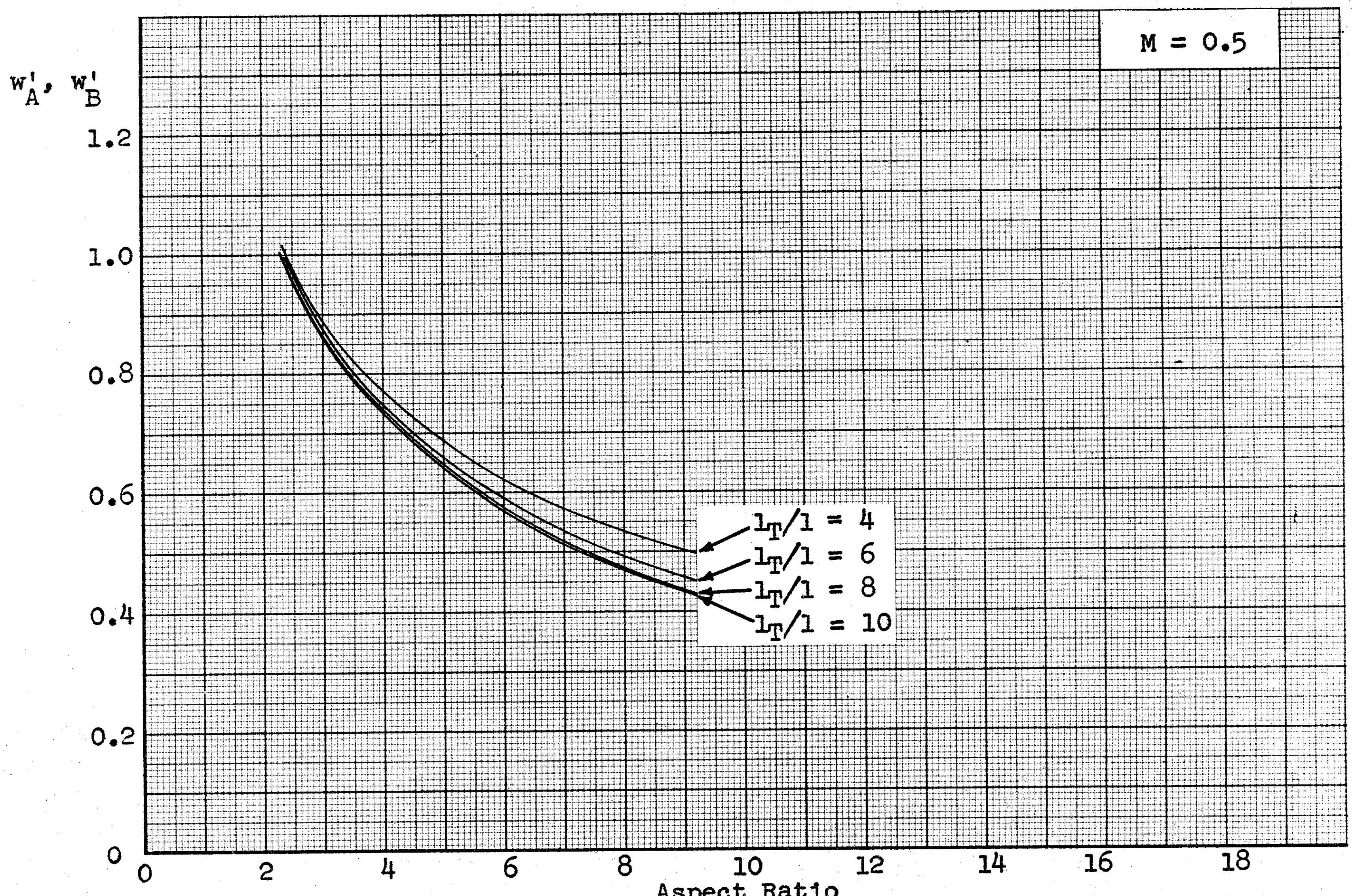
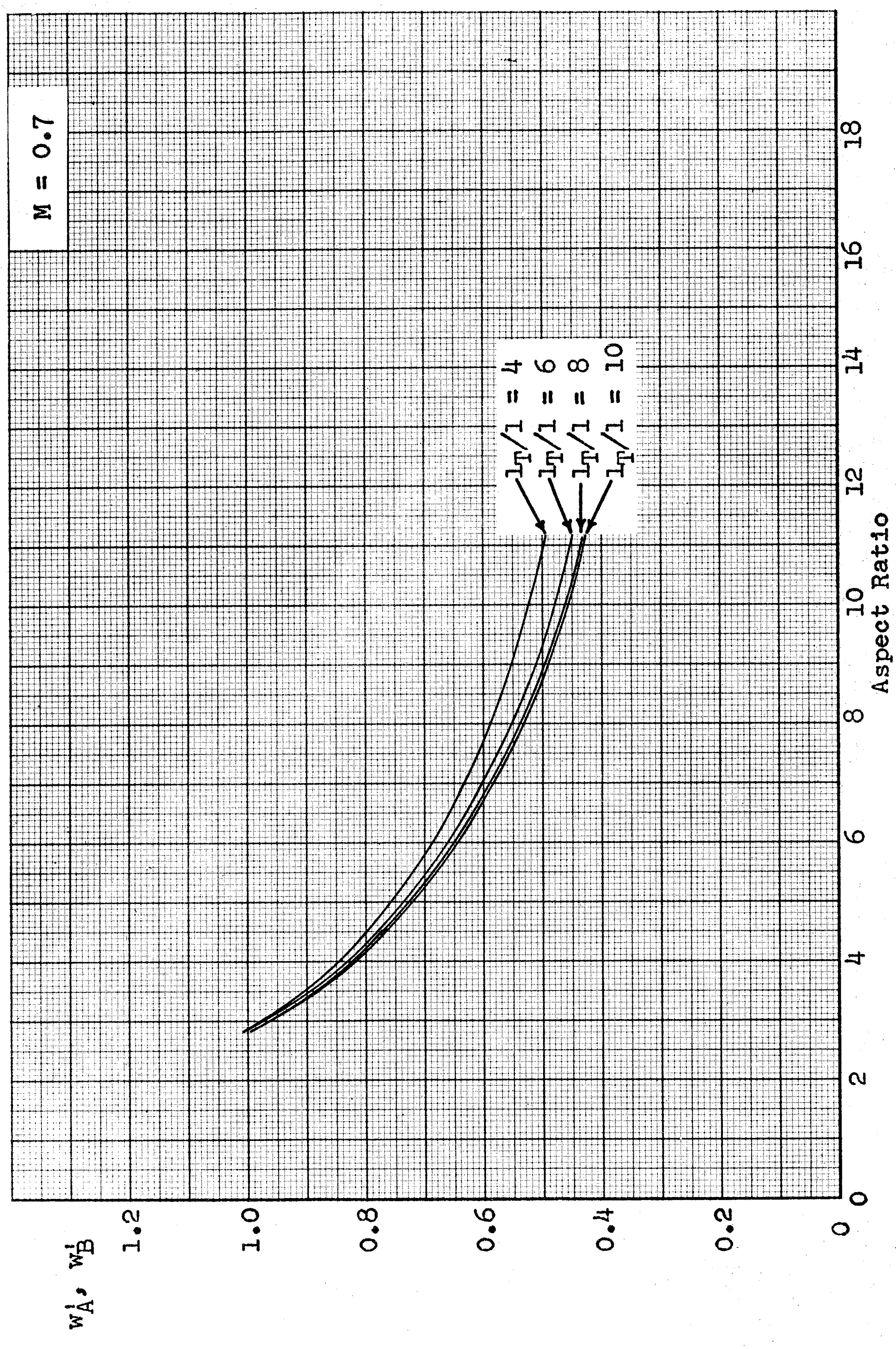

M = 0.5
w_A', w_B'
1.2
1.0
0.8
0.6
0.4
0.2
0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio
$l_T/l = 4$
$l_T/l = 6$
$l_T/l = 8$
$l_T/l = 10$

FIG. 46



M = 0.7
$l_T/l = 4$
$l_T/l = 6$
$l_T/l = 8$
$l_T/l = 10$
w'_A, w'_B
1.2
1.0
0.8
0.6
0.4
0.2
0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 47

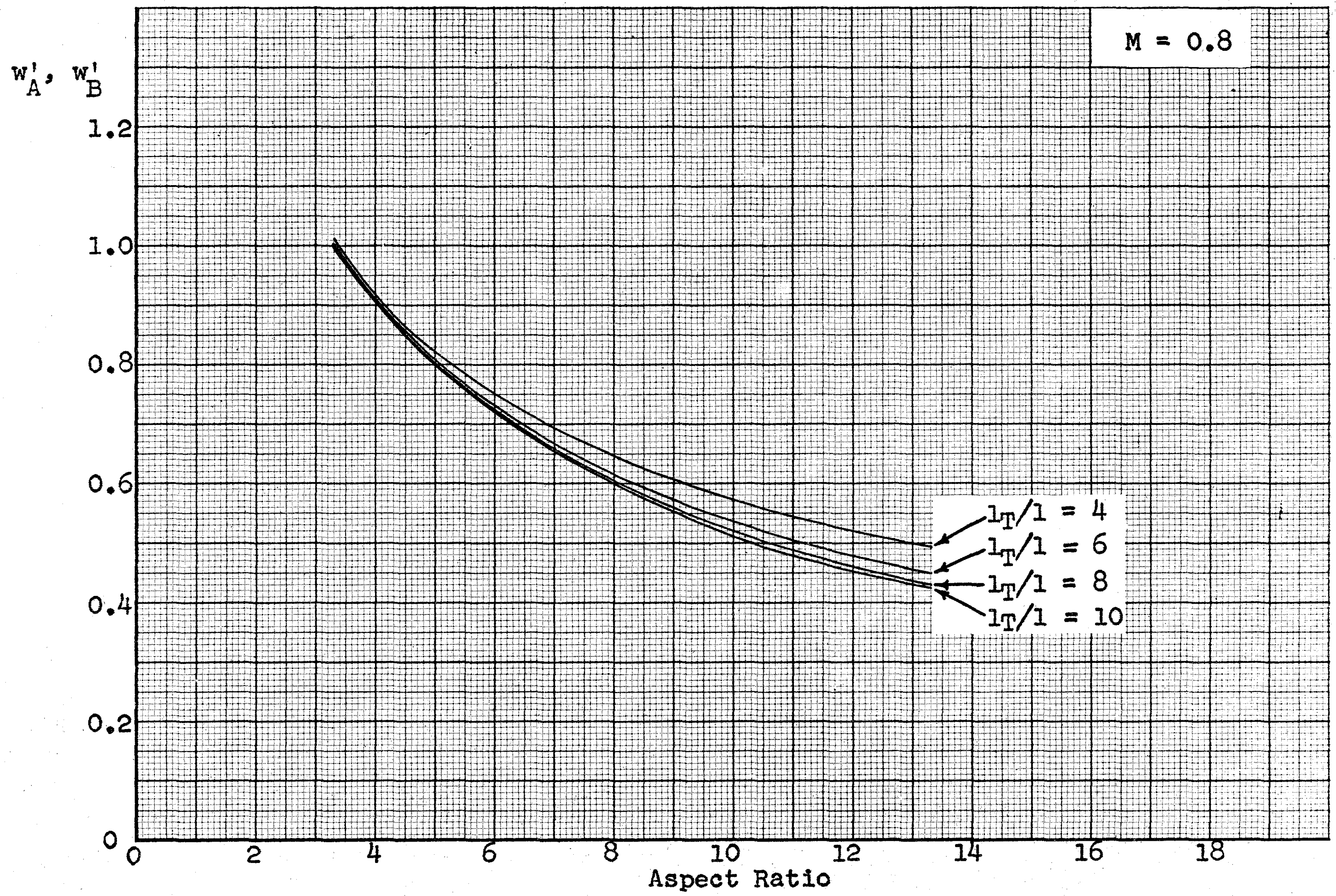

M = 0.8
w'_A, w'_B
1.2
1.0
0.8
0.6
0.4
0.2
0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio
l_T/l = 4
l_T/l = 6
l_T/l = 8
l_T/l = 10

FIG. 48

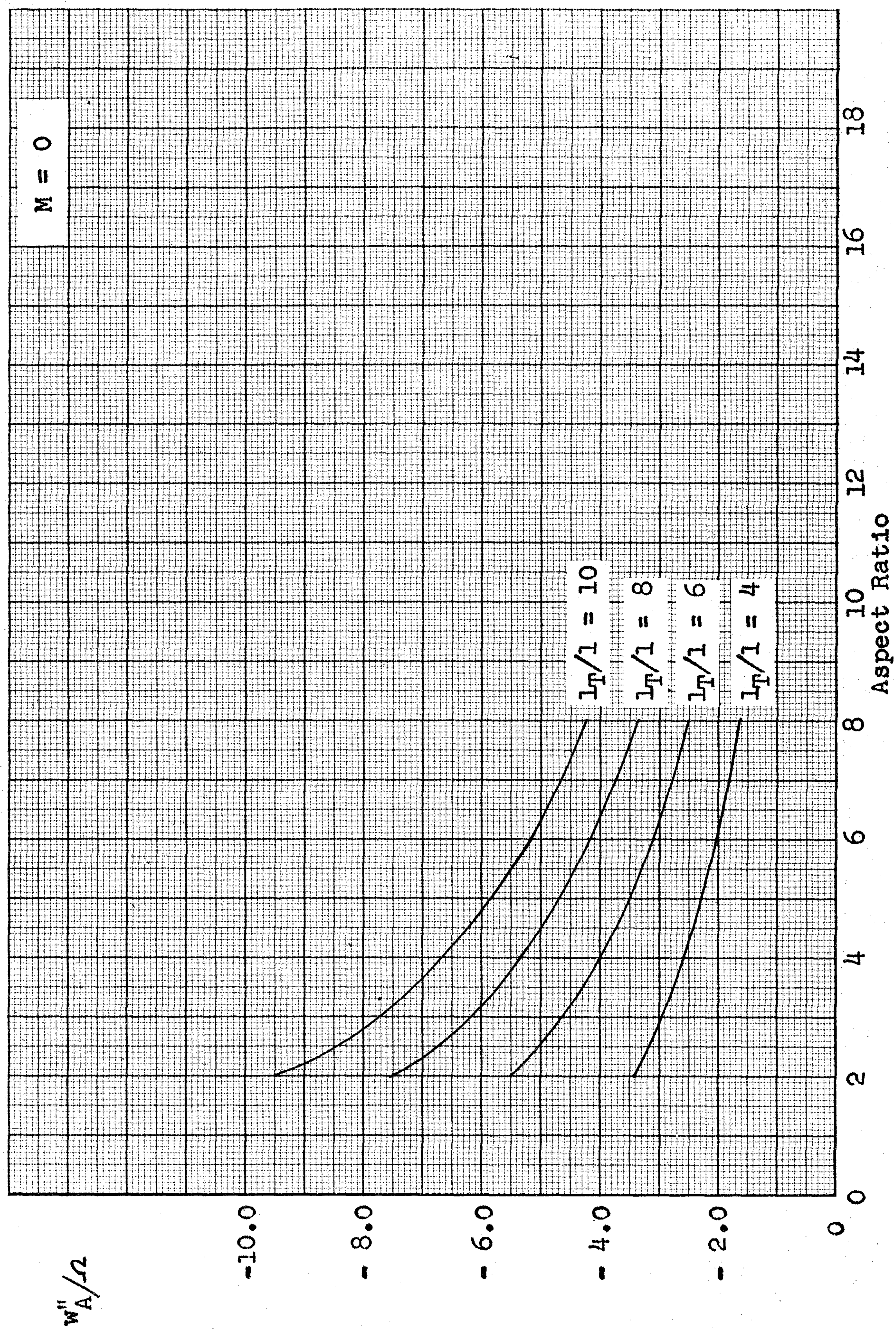
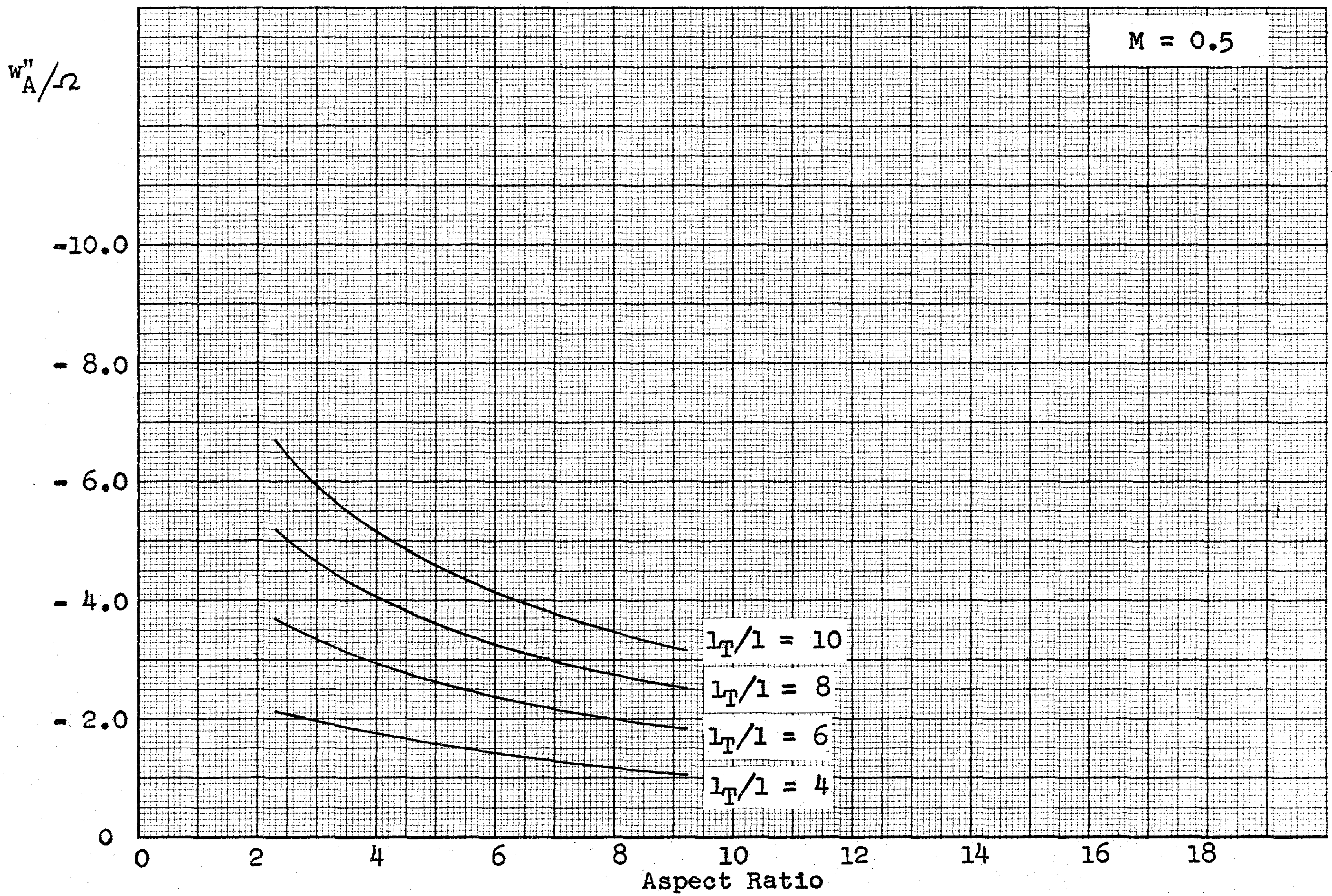

M = 0
l_T/l = 10
l_T/l = 8
l_T/l = 6
l_T/l = 4
w''_A/Ω
-10.0
- 8.0
- 6.0
- 4.0
- 2.0
0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 49



M = 0.5
w''_A/Ω
-10.0
- 8.0
- 6.0
- 4.0
- 2.0
0
l_T/l = 10
l_T/l = 8
l_T/l = 6
l_T/l = 4
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 50

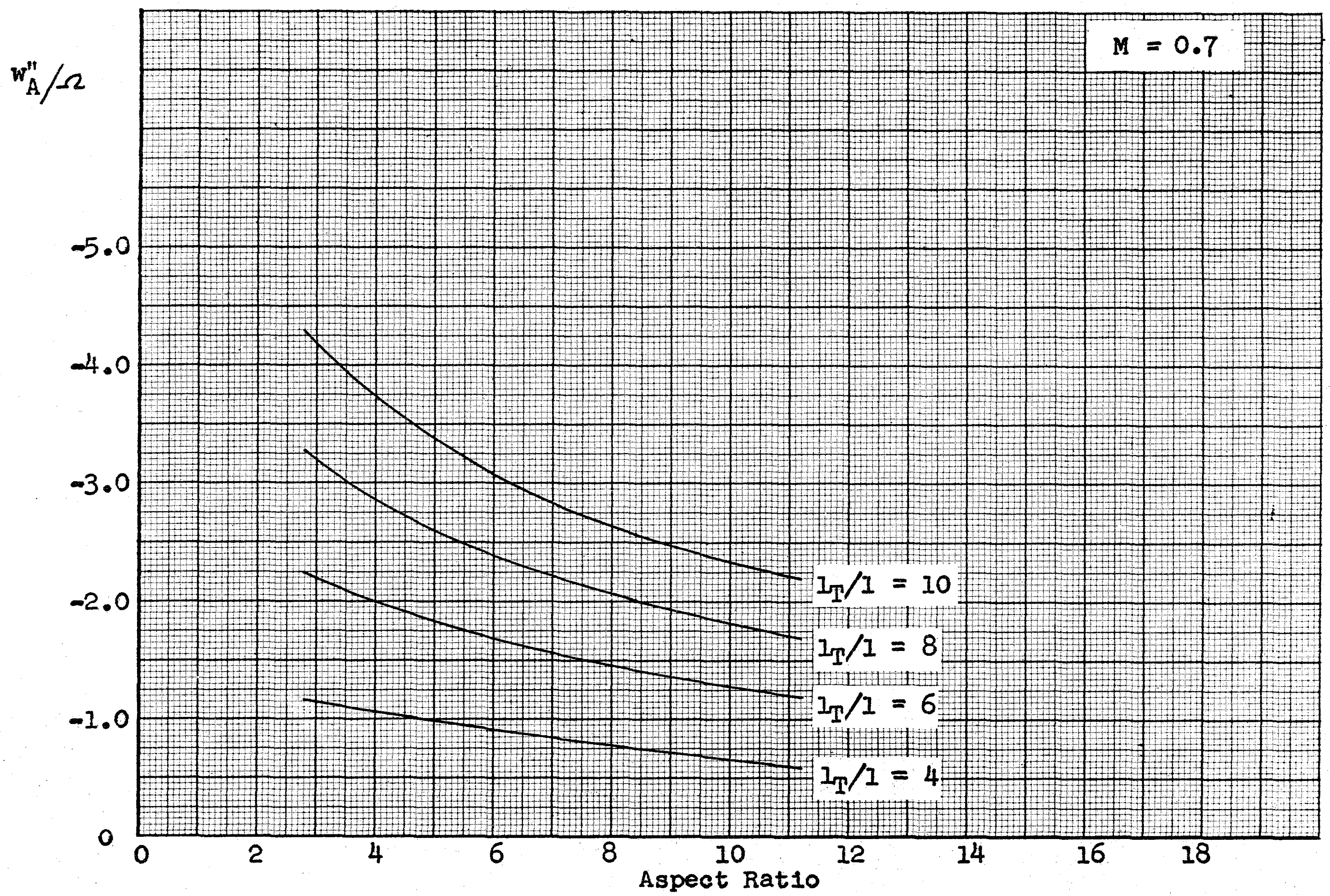

W''_A/Ω
M = 0.7
-5.0
-4.0
-3.0
-2.0
-1.0
0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio
$l_T/l = 10$
$l_T/l = 8$
$l_T/l = 6$
$l_T/l = 4$

FIG. 51

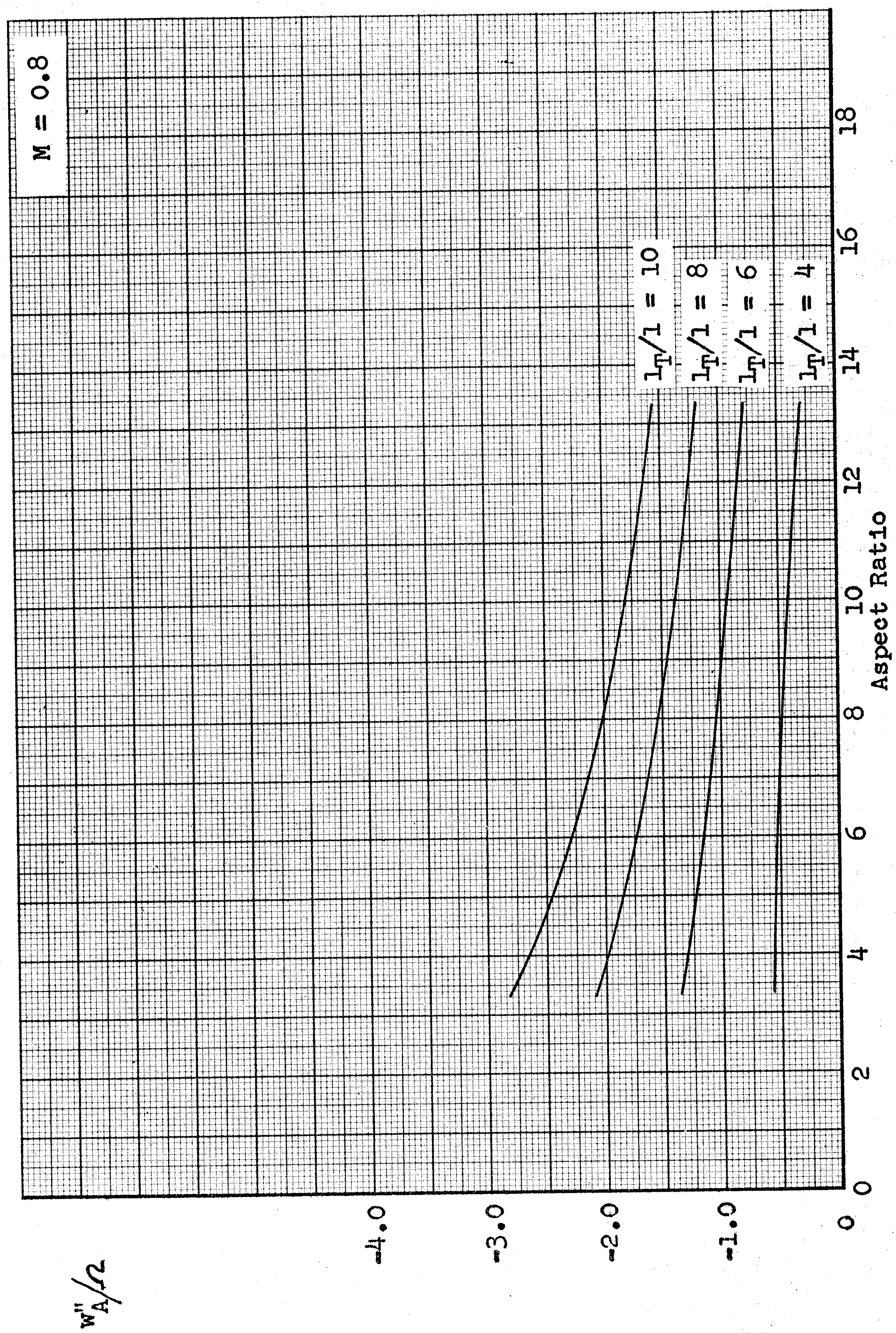

M = 0.8
l_T/l = 10
l_T/l = 8
l_T/l = 6
l_T/l = 4
w''_A/Ω
-4.0
-3.0
-2.0
-1.0
0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 52

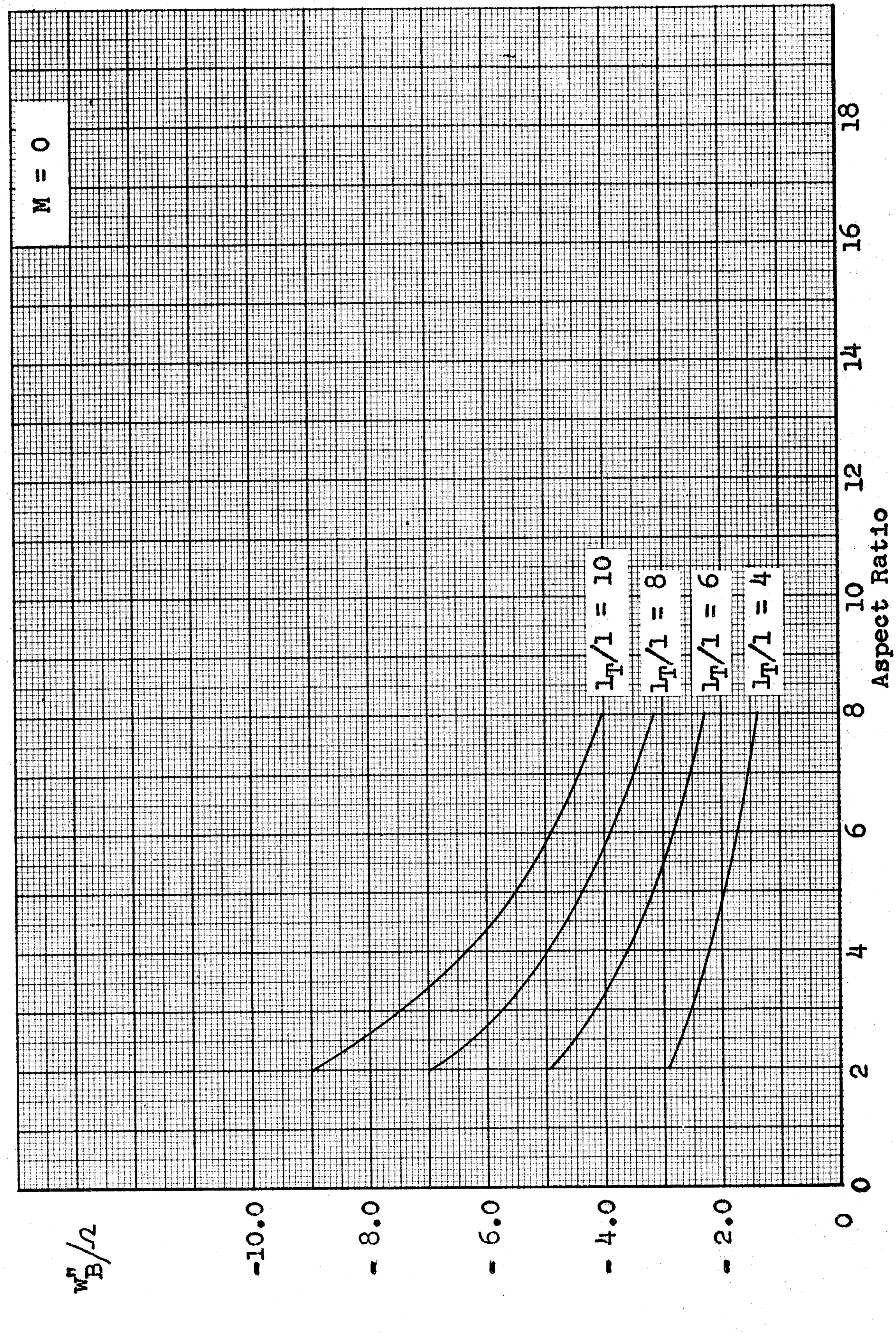
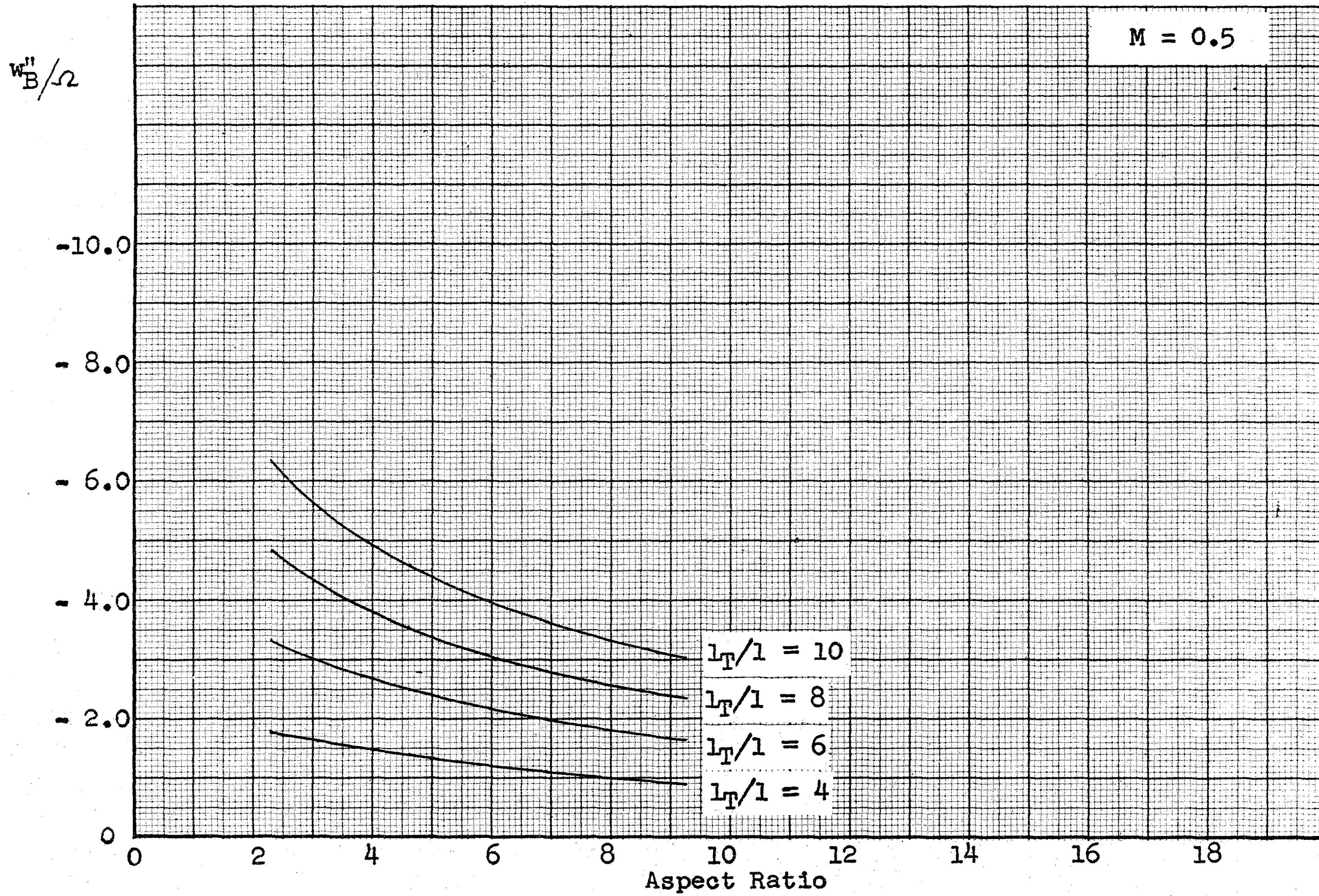
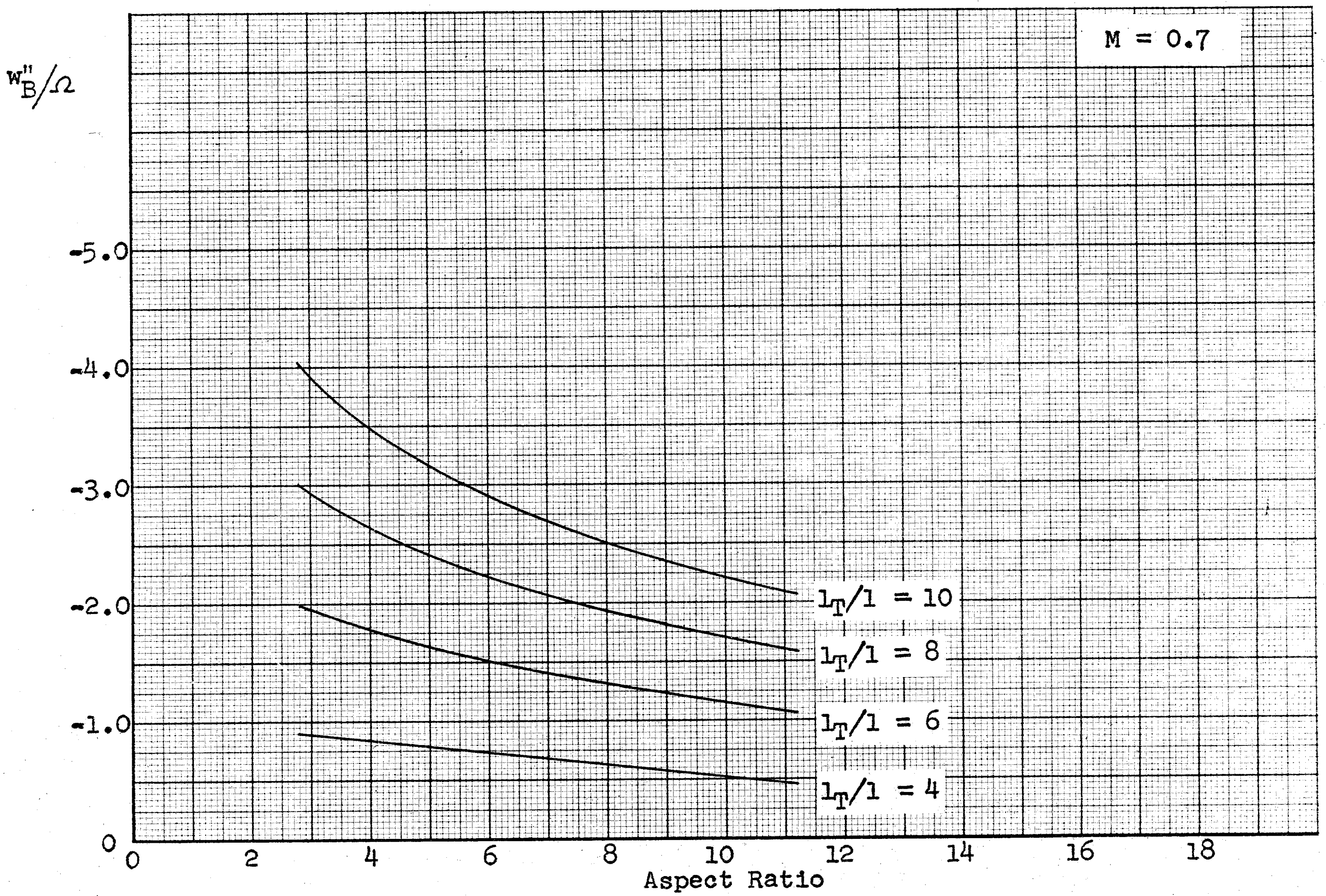

M = 0
l_T/l = 10
l_T/l = 8
l_T/l = 6
l_T/l = 4
w"_B/Ω
-10.0
- 8.0
- 6.0
- 4.0
- 2.0
0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 53



M = 0.5
w''_B/Ω
-10.0
- 8.0
- 6.0
- 4.0
- 2.0
0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio
$l_T/l = 10$
$l_T/l = 8$
$l_T/l = 6$
$l_T/l = 4$

FIG. 54



M = 0.7
w''_B/Ω
-5.0
-4.0
-3.0
-2.0
-1.0
0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio
$l_T/l = 10$
$l_T/l = 8$
$l_T/l = 6$
$l_T/l = 4$

FIG. 55

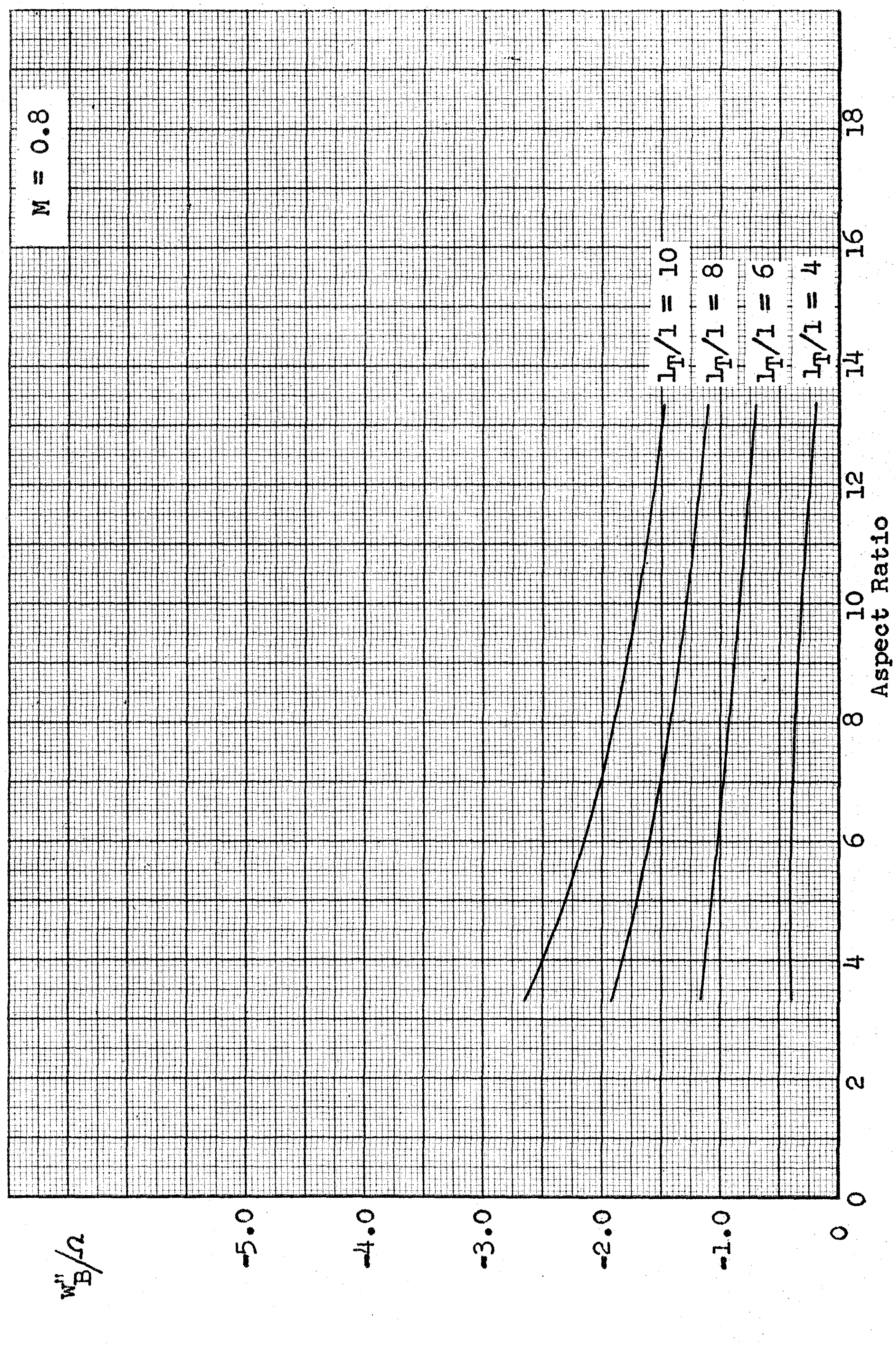
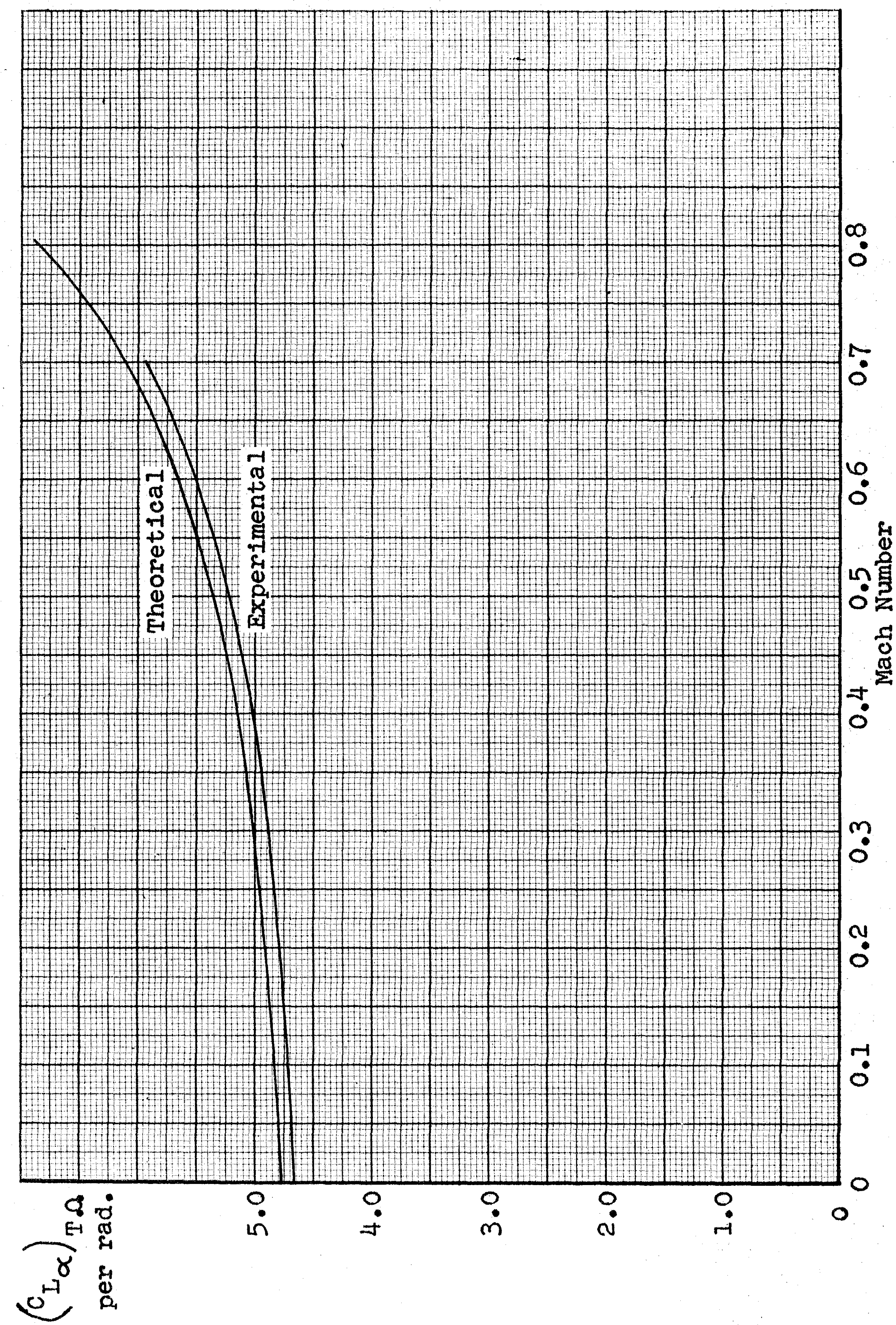

M = 0.8
$l_T/l = 10$
$l_T/l = 8$
$l_T/l = 6$
$l_T/l = 4$
w''_B/Ω
-5.0
-4.0
-3.0
-2.0
-1.0
0
0
2
4
6
8
10
12
14
16
18
Aspect Ratio

FIG. 56

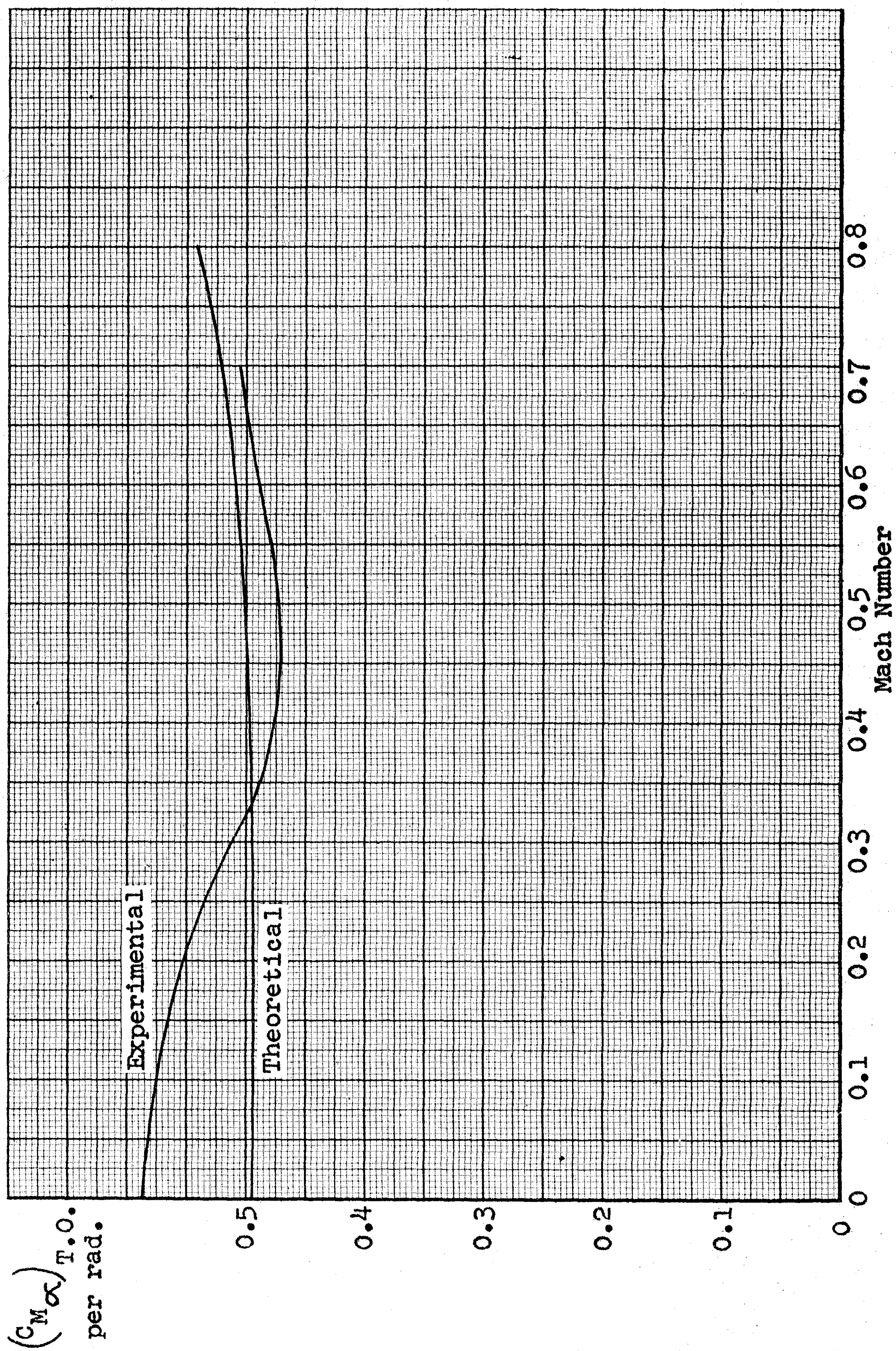
FIGURES 57 THROUGH 65

COMPARISONS OF EXPERIMENTAL AND THEORETICAL AERODYNAMIC DERIVATIVES FOR AN F-80A AIRPLANE



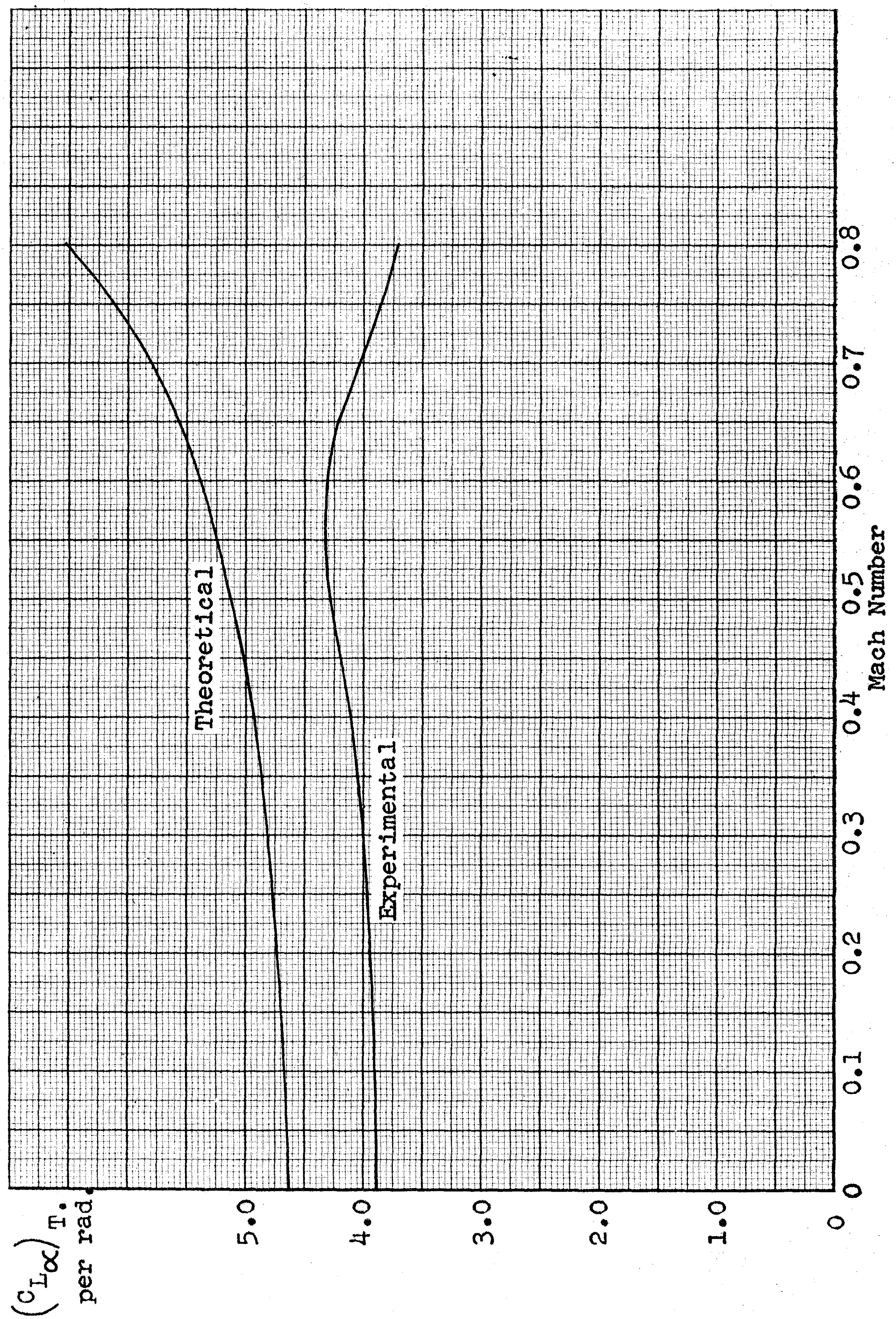
$(C_{L_\alpha})_{T\Omega}$ per rad.
5.0
4.0
3.0
2.0
1.0
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
Mach Number
Theoretical
Experimental

FIG. 57



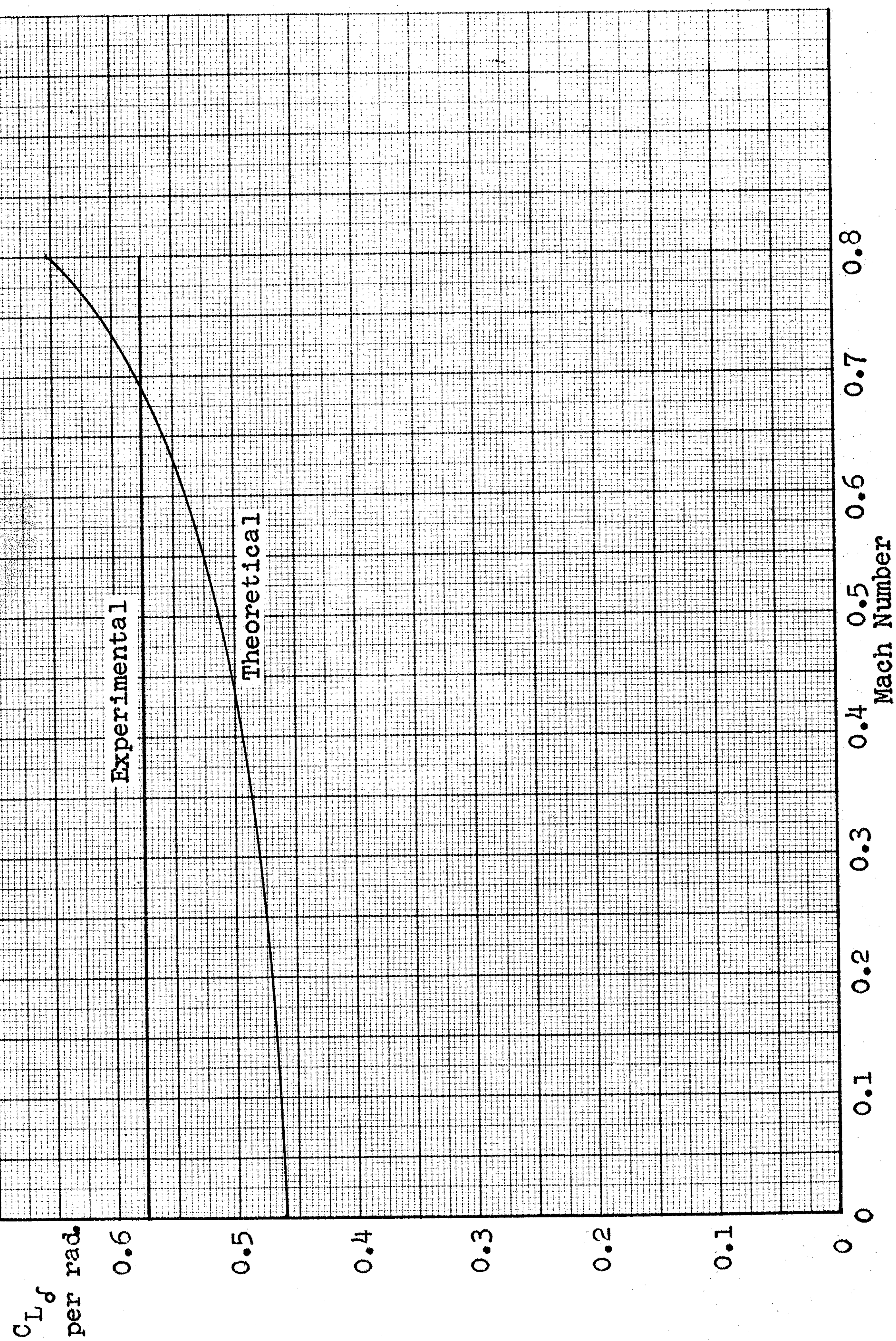
$(C_{M_\alpha})_{T.O.}$
per rad.
Experimental
Theoretical
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
Mach Number

FIG. 58



$(C_{L_\alpha})_{T.}$ per rad
5.0
4.0
3.0
2.0
1.0
0
Theoretical
Experimental
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
Mach Number

FIG. 59



C_{L_δ} per rad.
0.6
0.5
0.4
0.3
0.2
0.1
0
Experimental
Theoretical
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
Mach Number

FIG. 60

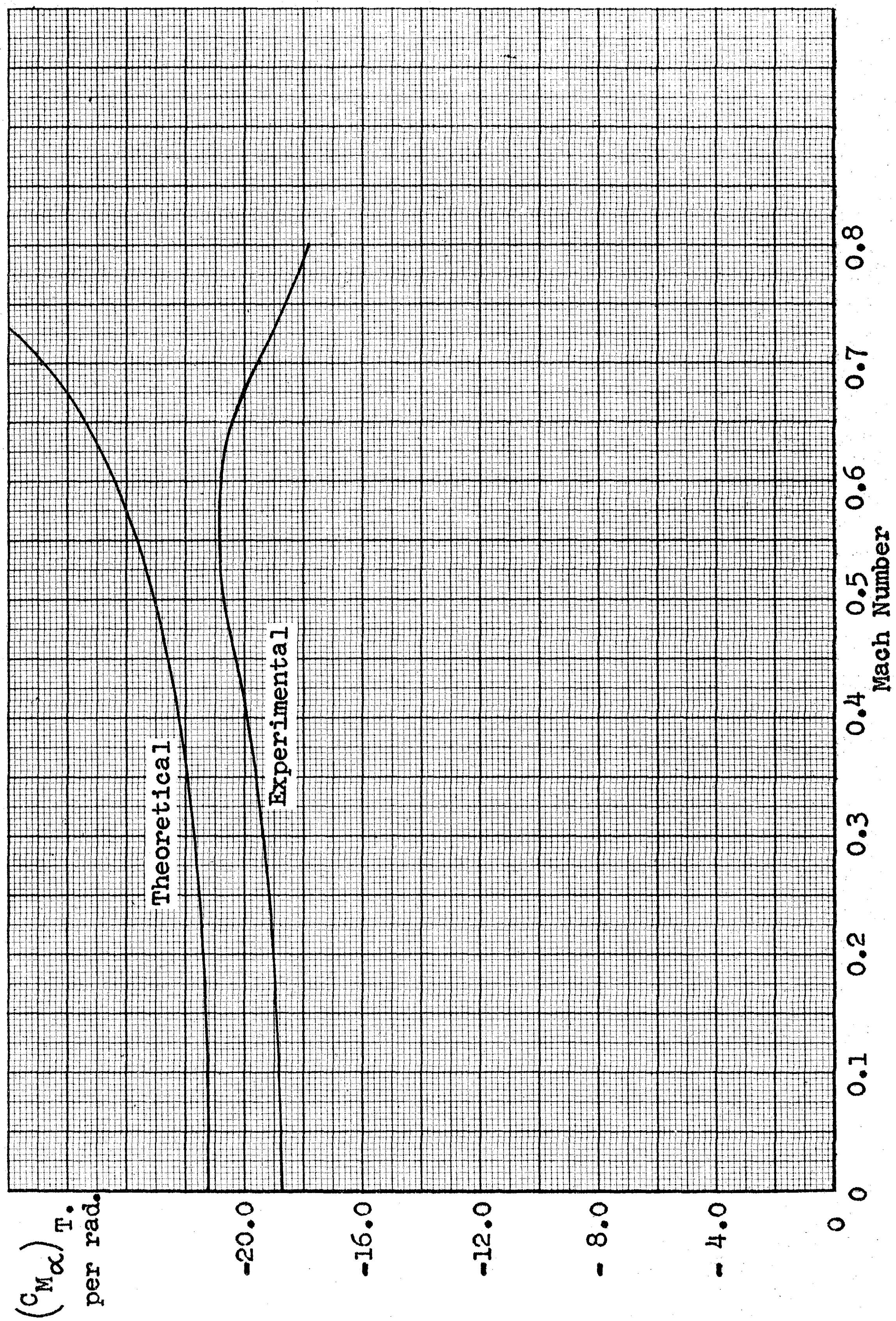

$(C_{M_\alpha})_{T.}$ per rad.
-20.0
-16.0
-12.0
- 8.0
- 4.0
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
Mach Number
Theoretical
Experimental

FIG. 61

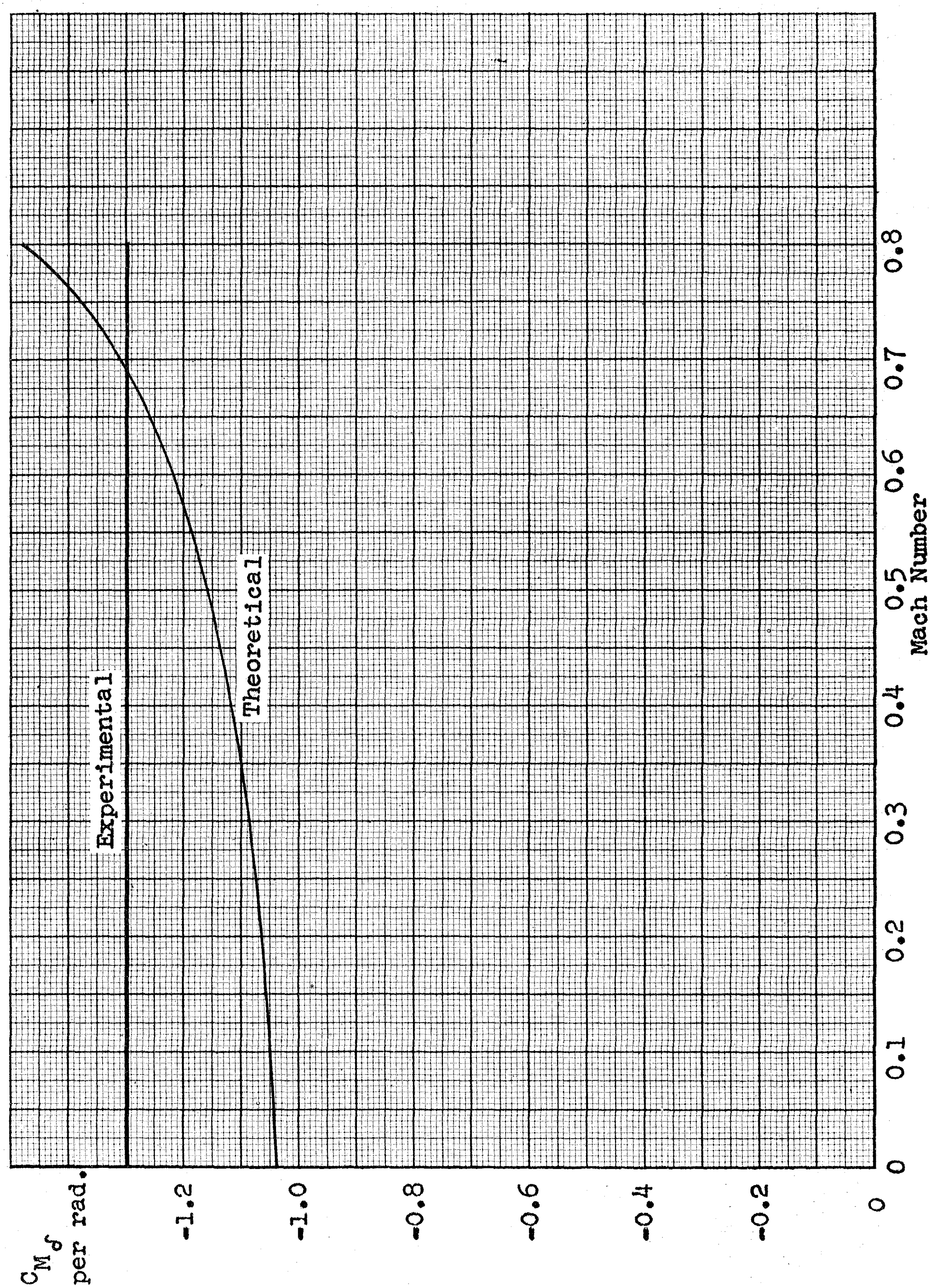

C_{M_δ}
per rad.
-1.2
-1.0
-0.8
-0.6
-0.4
-0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
Mach Number
Experimental
Theoretical

FIG. 62

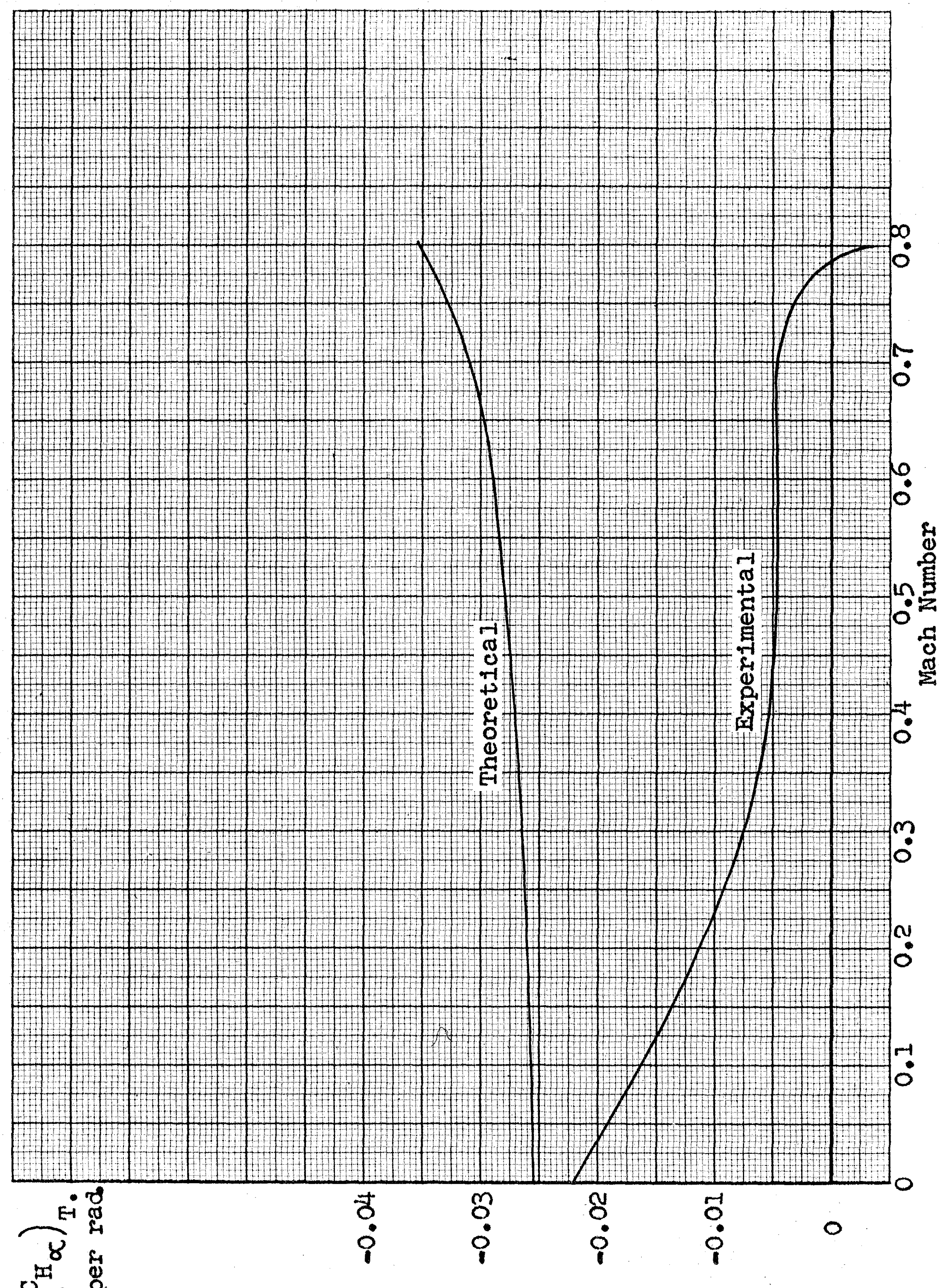
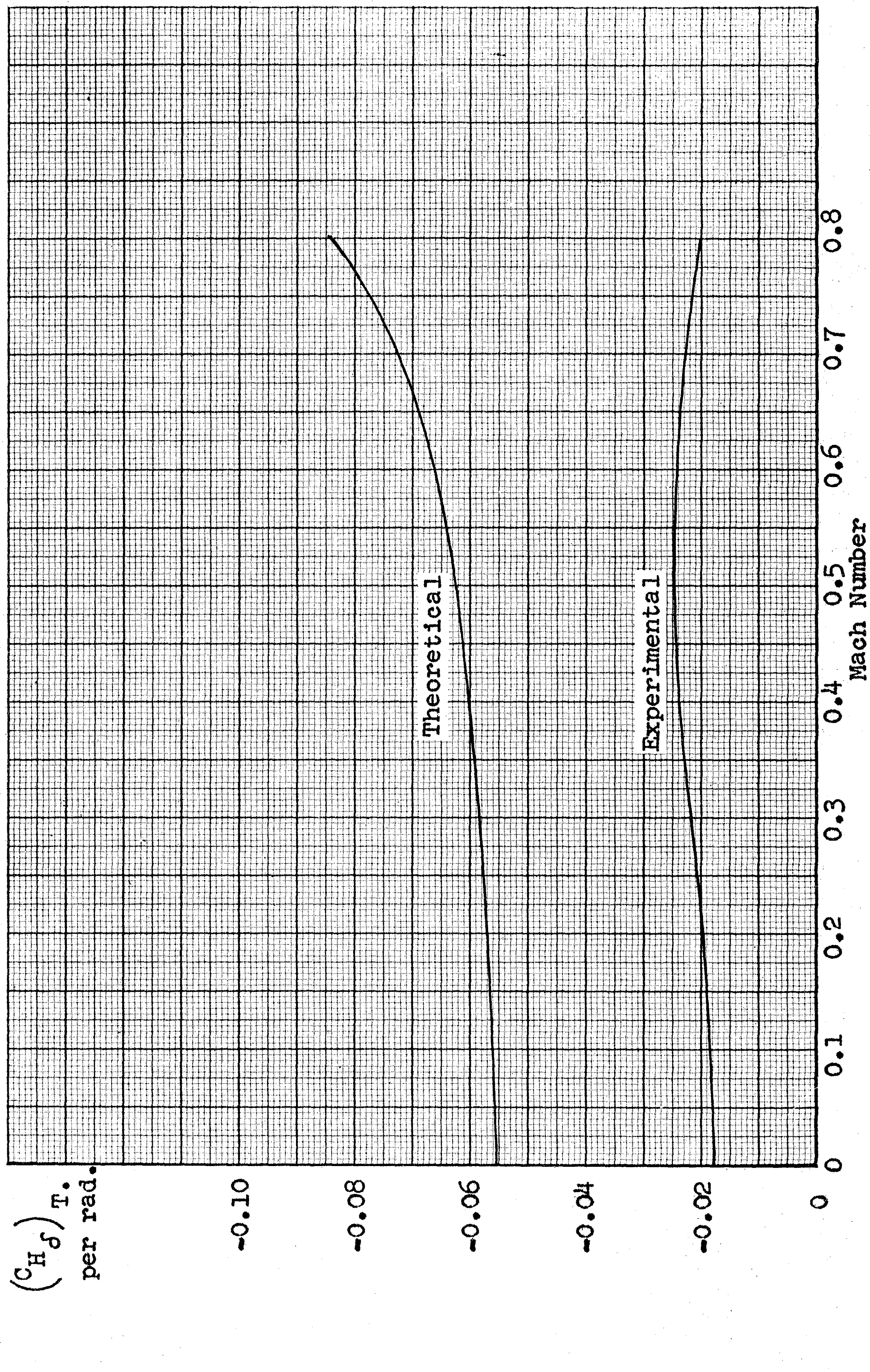

$\left(C_{H_\alpha}\right)_T$.
per rad
-0.04
-0.03
-0.02
-0.01
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
Theoretical
Experimental
Mach Number

FIG. 63



$(C_{H_\delta})_T$ per rad.
-0.10
-0.08
-0.06
-0.04
-0.02
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
Mach Number
Theoretical
Experimental

FIG. 64

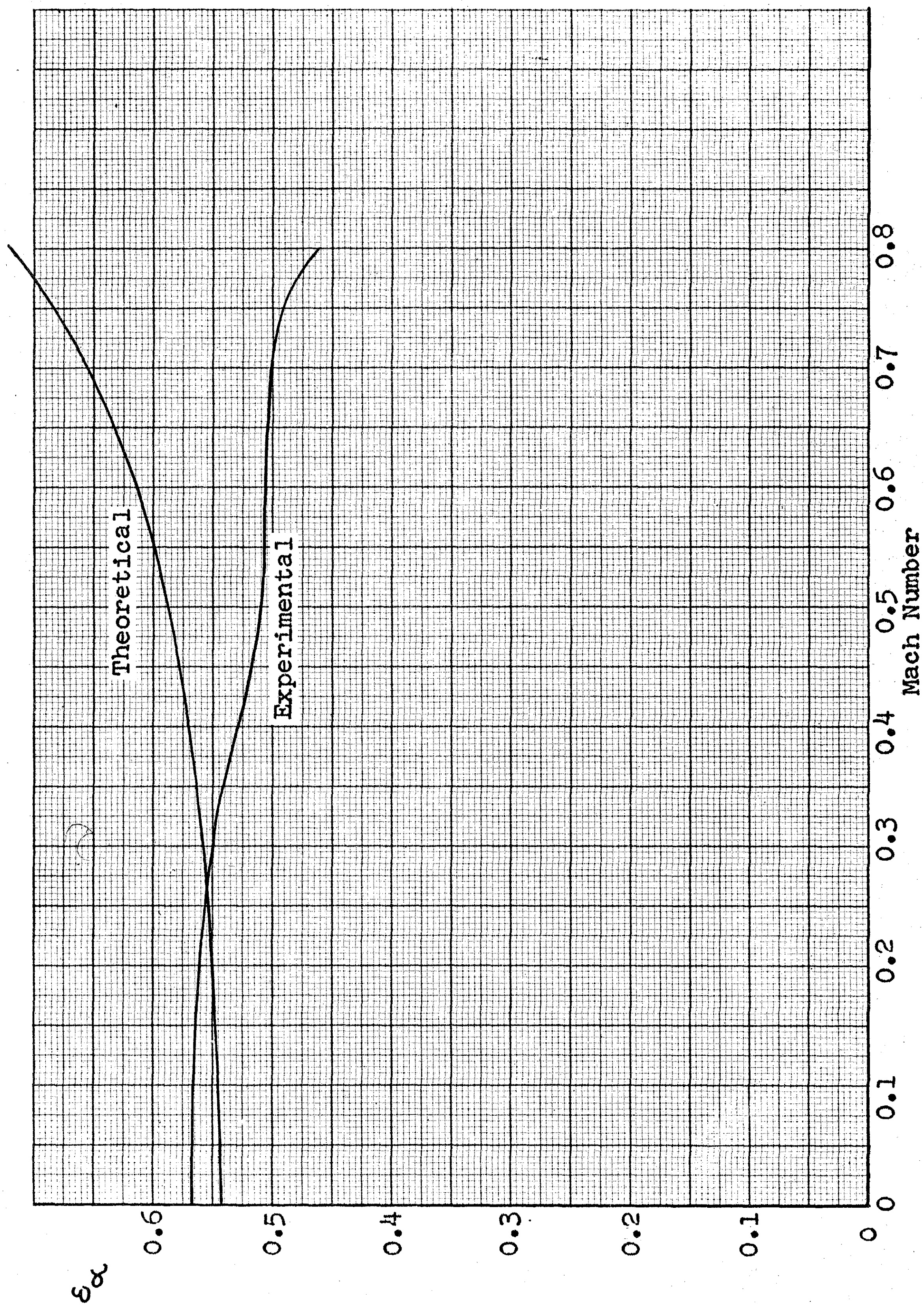

Theoretical
Experimental
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8
Mach Number
0 0.1 0.2 0.3 0.4 0.5 0.6
ε_α

FIG. 65

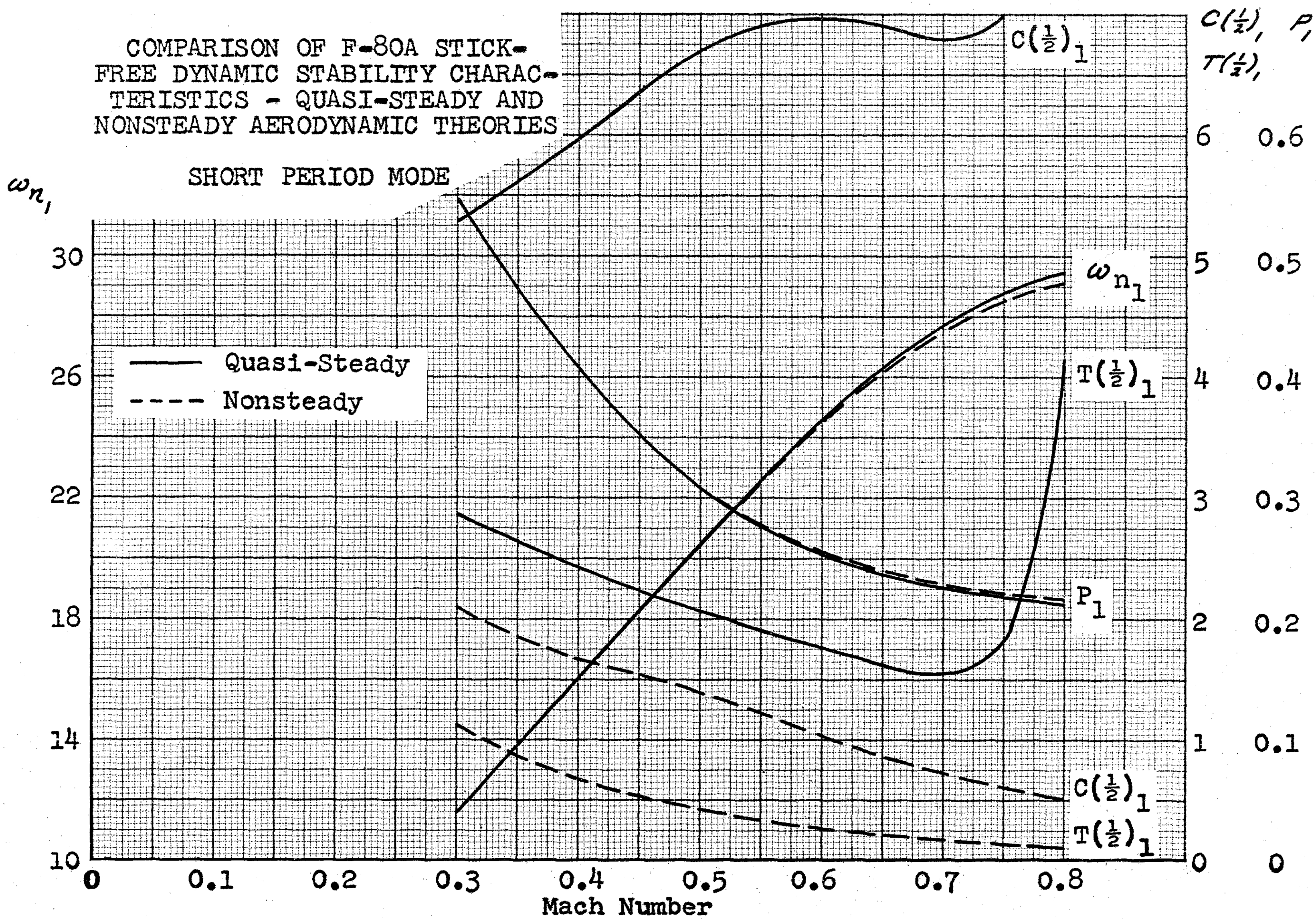

COMPARISON OF F-80A STICK-FREE DYNAMIC STABILITY CHARACTERISTICS - QUASI-STEADY AND NONSTEADY AERODYNAMIC THEORIES
SHORT PERIOD MODE
Quasi-Steady
Nonsteady
ω_{n_1}
$C(\frac{1}{2})_1$
$T(\frac{1}{2})_1$
P_1
$C(\frac{1}{2})_1$, $T(\frac{1}{2})_1$, P_1
Mach Number
30
26
22
18
14
10
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
6
5
4
3
2
1
0
0.6
0.5
0.4
0.3
0.2
0.1
0

FIG. 66

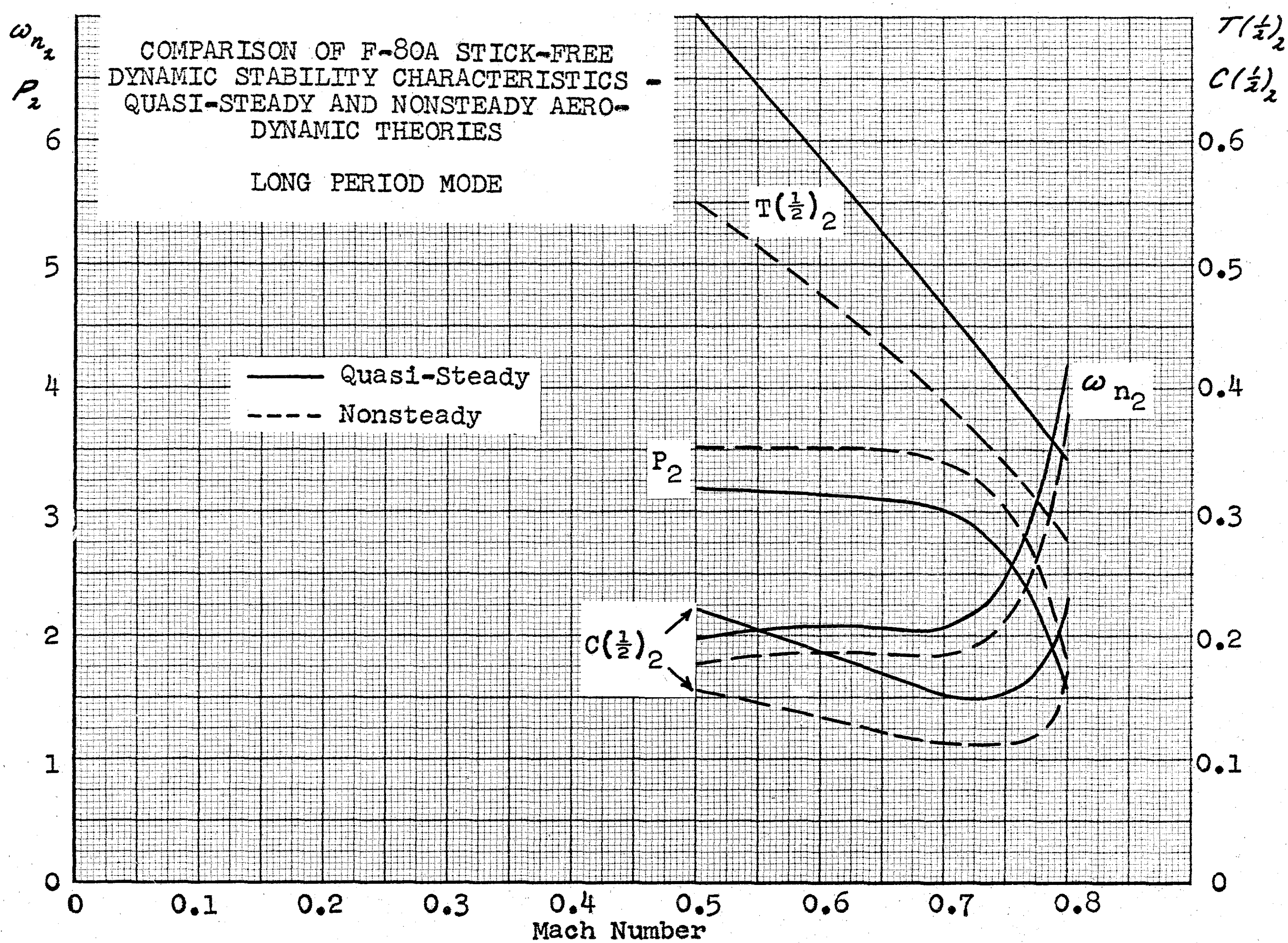

COMPARISON OF F-80A STICK-FREE DYNAMIC STABILITY CHARACTERISTICS - QUASI-STEADY AND NONSTEADY AERO-DYNAMIC THEORIES
LONG PERIOD MODE
Quasi-Steady
Nonsteady
ω_{n_2}
P_2
$T(\frac{1}{2})_2$
$C(\frac{1}{2})_2$
ω_{n_2}
P_2
$T(\frac{1}{2})_2$
$C(\frac{1}{2})_2$
0
1
2
3
4
5
6
0
0.1
0.2
0.3
0.4
0.5
0.6
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
Mach Number

FIG. 67

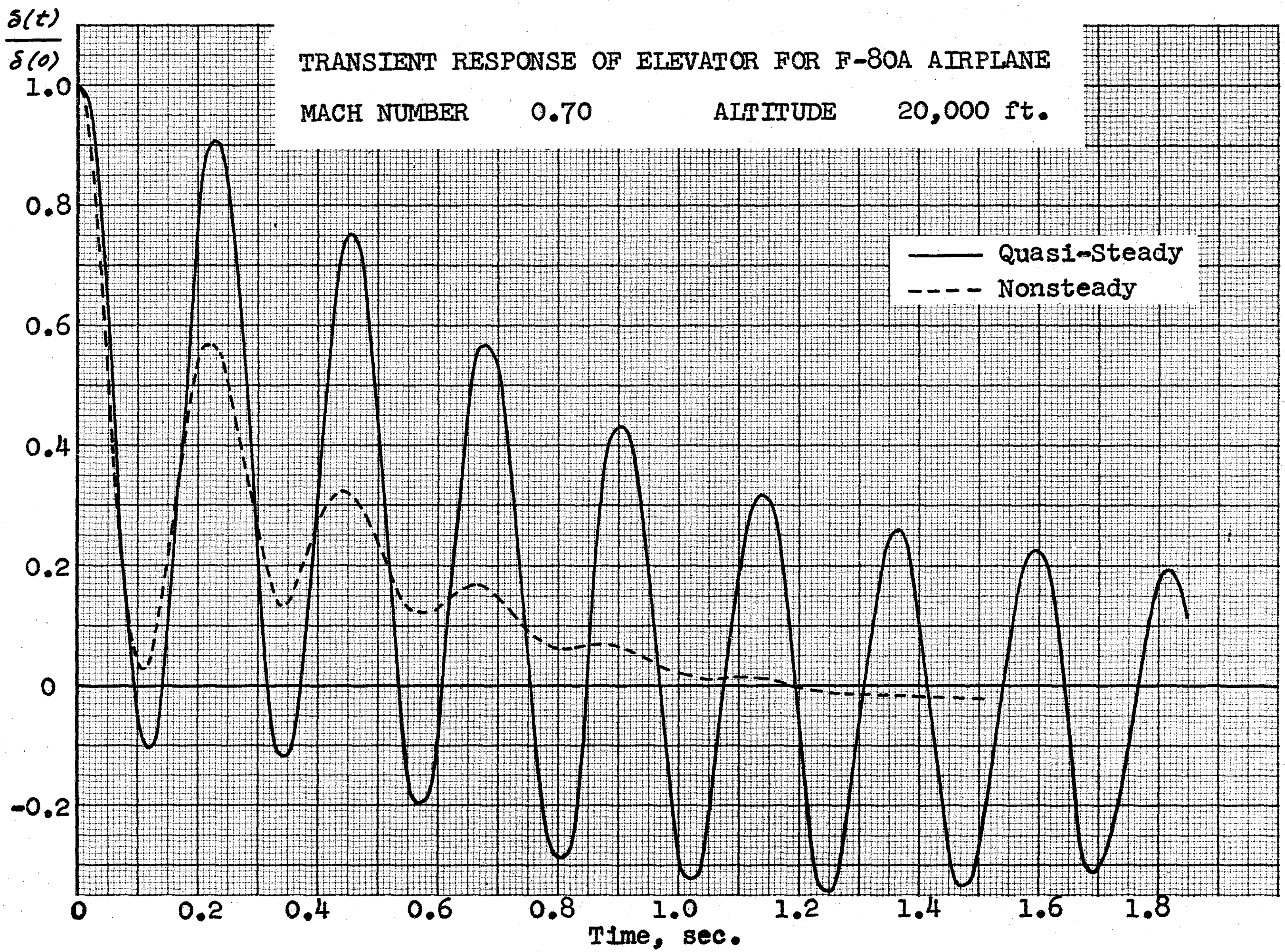

TRANSIENT RESPONSE OF ELEVATOR FOR F-80A AIRPLANE
MACH NUMBER 0.70 ALTITUDE 20,000 ft.
Quasi-Steady
Nonsteady
δ(t)/δ(0)
1.0
0.8
0.6
0.4
0.2
0
-0.2
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
Time, sec.

FIG. 68

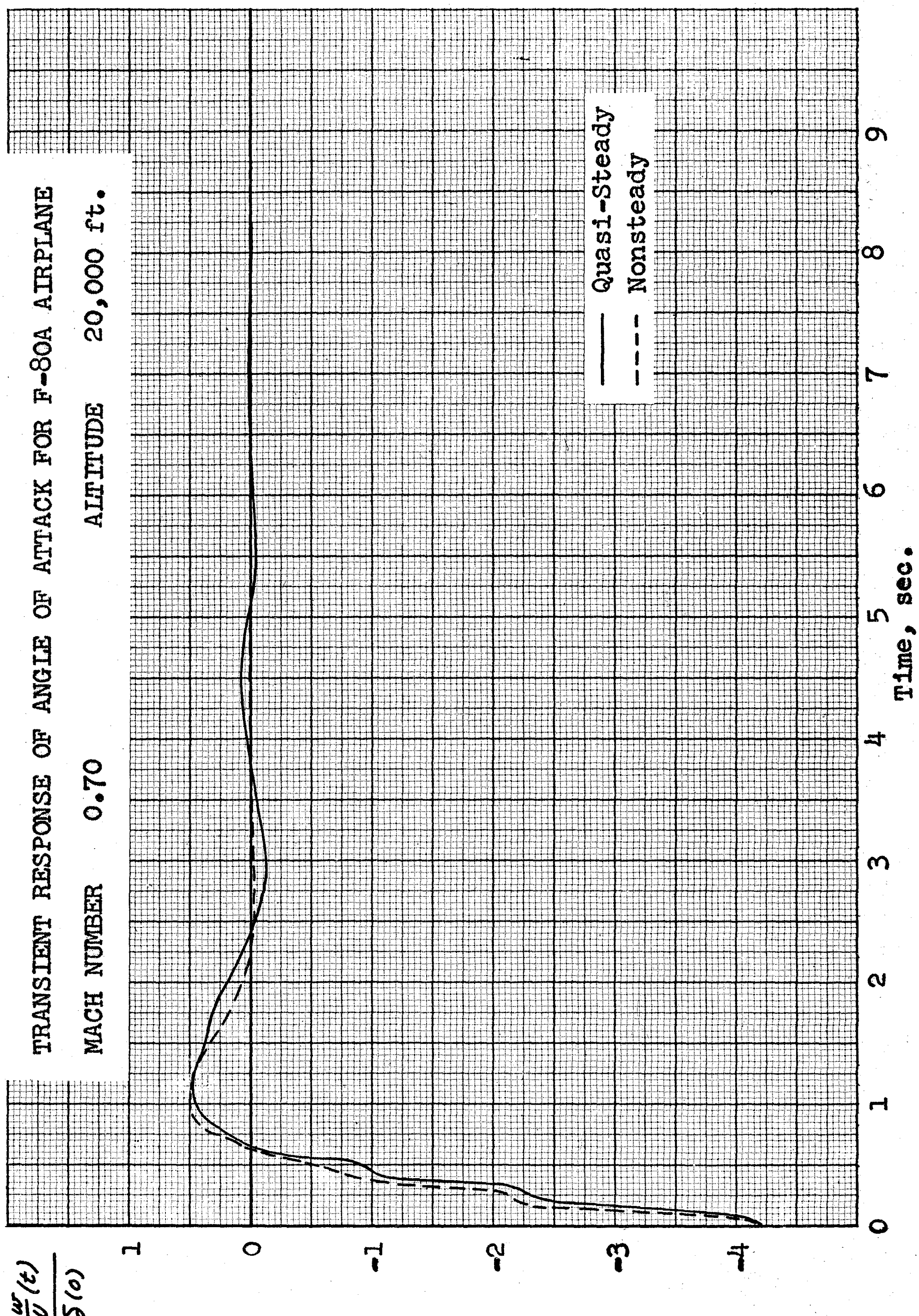

TRANSIENT RESPONSE OF ANGLE OF ATTACK FOR F-80A AIRPLANE
MACH NUMBER 0.70
ALTITUDE 20,000 ft.
Quasi-Steady
Nonsteady
$\frac{w}{U}(t) / \delta(0)$
1
0
-1
-2
-3
-4
0
1
2
3
4
5
6
7
8
9
Time, sec.

FIG. 69

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