A UNIFIED APPROACH

TO

TWO-TERMINAL NETWORK SYNTHESIS

Thesis by

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In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1960

DEDICATION

To my dear children, Peter, aged 6, and Alice, aged 4, without whose cheerful lack of cooperation and understanding this thesis would have been completed months earlier; and to my unselfish wife who, if this enterprise had lasted much longer, would probably have left me.

ABSTRACT

The advantages are considered of developing dual approaches to two terminal lumped constant network synthesis. The first is the usual S-plane approach emphasizing the positive - real concept; the second treats the reflection coefficient, involves unitary modular functions, and is related to the older wave filter theory. The second viewpoint is developed in some detail and used to unify the various existing methods of two-terminal synthesis on a common mathematical basis. The synthesis techniques of Foster, Cauer, Brune, Bott-Duffin, Failkow-Gerst, and Miyata are re-examined in terms of the reflection coefficient and some alternate methods of synthesis are suggested. It is found that all of these synthesis techniques may be derived as applications of the Schwarz' lemma.

ACKNOWLEDGEMENTS

The original suggestion for the investigation presented here was that of Dr. W. H. Pickering who pointed out the need for clarification and codification of the various approaches to network synthesis. The intent at the start of the investigation was to develop techniques which would move network synthesis as a topic more nearly into the area of the electronic designer and to minimize those aspects which made it the domain of the applied mathematician. It became clear, however, that preliminary to such a task was the requirement that a common basis for synthesis be established--a concept or group of concepts which would provide a common basis for such apparently diverse techniques as those of Brune, Darlington, Miyata and others. The material presented here demonstrates that the use of the reflection coefficient serves as such a common basis. Hence it may be presumed that this investigation does significantly advance the understanding of the field of two-terminal synthesis insofar as it does unite previously diverse approaches.

It is evident from the nature of the investigation presented here that much of the guidance and inspiration was provided by material in the literature. The Fialkow-Gerst article[20] and the Caratheodory text[12] were the two works which originally directed the writer's thinking to the importance of the reflection coefficient concept.

It will be found that there are articles in the literature which touch upon topics similar to those developed here and the question arises as to the extent to which they have served as source material. Belevitch[33,34] in two letters in the Correspondence Section of I.R.E.-PGCT calls attention to the simpler derivation of Bott-Duffin that results from use of the reflection coefficient and extends his case to include the Fialkow-Gerst extension of Bott-Duffin synthesis. Although Belevitch's notation is quite different, his treatment is essentially the same as that of the material of Section IV.2. The work in that section was derived from Fialkow and Gerst's paper directly. Because of the difference in notation the similarity to Belevitch's work was not appreciated at the time.

On the other hand a work not directly used in the study but one which served as a definite source of inspiration was the work on the application of geometry to circuit theory by E. Folke Bolinder.[35] The referenced publication covers a great deal of Bolinder's work and includes an extensive bibliography. The volume and calibre of the work served to spur the writer on in his much more restricted task.

Although the writer has earlier taken exception to some of Kuh's comments, he felt that Kuh by his insight into Miyata synthesis and his ability to compare and contrast different synthesis methods was an important aid in increasing the writer's understanding of the field. The exceptions taken to Kuh's viewpoint are minor and do not detract from his contribution.

The writer is indebted to a number of people for helpful comments and criticisms. In connection with the unimodular bounded

concept and non-Euclidean mathematics Dr. Morgan Ward of the Mathematics Department of the California Institute of Technology aided by his advice and interest. The synthesis aspects of the work gained from many stimulating discussions and penetrating comments given by Dr. E. C. Ho of the Hughes Aircraft Company and the University of California in Los Angeles. Dr. George Cooper of the Electrical Engineering Department of Purdue University read critically major portions of the completed text. Dr. John Aseltine of Space Technology Laboratories and the University of California in Los Angeles made a curious but valuable contribution for which the writer is grateful. Dr. Aseltine was associated with the writer only in a very early position of the task and he discouraged the writer from exploring some less significant topics which seemed interesting at the time.

The final typing and organization of this work took place at Aeronutronic, a Division of Ford Motor Company and was completed due to the conscientious care and hard work of Miss Jean Hiatt. Figures and formulas were prepared by Mas Uemura of the Radar Department of that organization.

Lastly, the writer must acknowledge his debt to his thesis advisor, Dr. W. H. Pickering of the Jet Propulsion Laboratory and the Electrical Engineering Department of the California Institute of Technology. That debt is too large and many sided to be easily described. Those who know him will not be surprised to learn that, in spite of the pressure of national affairs, Dr. Pickering always found time to discuss this investigation and, in spite of the many demands on his mental energies, was able to exert a firm and effective control over the development.

TABLE OF CONTENTS

PART I

INTRODUCTION

SECTION I.1

PRESENT STATUS OF TWO-TERMINAL SYNTHESIS

The problem of two-terminal network synthesis may be stated as follows: Given a suitable mathematical expression in the complex frequency variable s , to realize a network consisting of the usual resistance, inductance, or capacitance elements including, if necessary, transformers. The resulting network is required to have the property that the ratio of its input voltage to current will be just the prescribed mathematical expression. It is immaterial whether the problem involves an impedance function as just described or an admittance function in which case the input current to voltage ratio, the reciprocal of that above, is involved. In the general problem covering both cases the function involved is often referred to as an immittance.

Modern driving point immittance synthesis techniques are essentially mathematical programs the use of which starting with a given positive real function $Z(s)$, and following a prescribed step-by-step procedure, results in a two-terminal network configuration of passive, linear, bilateral, lumped elements which will perform as described above.

The basic problem of establishing that, in theory such a synthesis may always be performed for a given rational, positive real function, was carried out by Brune.[1] For particular arrangements of the poles and zeros of the given function, notably alternating in position along the j-axis in the s-plane in the manner required for a reactance function, the straightforward synthesis methods of Foster[2] and Cauer[3] are applicable and well known. The former removes individual complex conjugate pole pairs to obtain a series of additive simpler expressions summing to the given function. Each of the simpler expressions may then be synthesized by inspection. The latter method removes poles at zero and infinity in a continued fraction expansion which leads to the ladder network realization. Both methods are capable of extension, one such extension being that which leads to RL or RC networks when the poles and zeros of the given expression are on the negative real axis in one of the two prescribed arrangements.[4] Further, one may extend the methods of Foster and Cauer in a variety of ways which, while important in themselves to the obtaining of special results, are not fundamental in terms of basic principles. One such special technique is referenced.[5]

If poles placed generally in the left half plane are to be removed in a Foster-like procedure, care must be exercised to establish that such operations may be performed in the case under consideration without creating a remainder function which is non-realizable. Guillemin[4], in Chapter 9 of his synthesis text, develops the conditions that must exist between a pole and its residue in order that the pole may be removed as one of a complex conjugate pair, in a Foster-like procedure without prejudicing the possibility of later synthesizing the remainder function.

For the general problem of two-terminal synthesis, in which the poles and zeros neither fall into a prescribed pattern nor obey the criteria for separate removal, a variety of special techniques have been developed. An organized, detailed enumeration of these procedures developing one from another in an orderly manner has not thus far been available. Textbooks generally describe the techniques as separate and isolated topics, although advanced practitioners in the field undoubtedly see many close relationships between the various methods. As Darlington points out in a review article,[6] there is not available any real comprehensive theory of equivalent networks.

The need, then, at the present time is the development of the logical links between the different synthesis techniques rather than the creation of new alternate methods of synthesis. The goals are not mutually exclusive, for progress in the former task should lead directly to progress in the latter one. The purpose of this writing is the development of these logical links, the establishment of a common mathematical basis into which the important existing synthesis methods may be made to fit in a systematic scheme. To develop the approach and determine the tools needed for the task, the historical development of the field is briefly reviewed.

SECTION I.2

A POSSIBLE ALTERNATIVE DEVELOPMENT

A critical evaluation of the historical development of the field of two-terminal synthesis reveals the existence of two somewhat independent approaches to the subject. Early work in filter design, motivated by the needs of the telephone industry, was the outgrowth of the artificial line theory of Pupin and Campbell. Generally referred to as wave filter theory, it deals with the design of reactive sections to be used in signal shaping in connection with communications equipment. The composite filters introduced by Zobel are made up of chains of reactive sections all having matched image impedances but different transfer constants. Darlington's doctorate thesis[7] contains a more complete discussion of these topics as well as a list of references. A brief review of the essential characteristics of Zobel filters is contained in Smythe.[8] It is of particular interest, for present purposes, to note that the input impedance of a Zobel filter section is always matched to the output impedance of the section immediately preceding it.

The work influenced by Pupin and Campbell and described above emphasized the importance of the reflection coefficient as one of the concepts which developed naturally in a study which commenced with consideration of transmission lines. The work of Foster and Cauer placed less emphasis on the reflection coefficient but rather directed attention to the locations in the s-plane of the zeros and poles of the driving point immittance. Brune, in his fundamental work, followed this tendency of Foster and Cauer. Investigations after Brune in the field of two-terminal network synthesis were almost all carried through in terms of s-plane analysis.

It is difficult to make any absolute statements in regard to the comparative merits of the two approaches to two-terminal synthesis mentioned above. The difference is a matter of degree rather than kind for the proponents of the two viewpoints have lived and worked in close geographic proximity exchanging ideas freely as they mutually developed the field. Although one man's name may be associated with wave filters and another's with the "exact synthesis techniques," there has never been, in fact, any clear cut boundary. The situation might be compared to that in modern physics where one man's name is associated with the particle approach and another's with wave mechanics. In actual practice, however, all physicists use the methods of both approaches freely as suits their immediate needs.

There is one aspect of the matter on which a firm stand may be taken. As in any area of scientific effort, a distinct loss occurs when an alternative viewpoint is overlooked or slighted. If the various approaches to the subject are truly equivalent then any results obtained by one approach may be duplicated using another. Properties and theorems which are obscure in their implications in one framework may, however, sometimes seem clear and intuitively obvious when expressed in terms of an alternate viewpoint. It is apparent that, after Brune, in the study of the two-terminal synthesis problem the s-plane approach was emphasized and the concepts developed out of the transmission line approach to networks received somewhat less attention. The situation suggests that it might be worthwhile to re-examine some of the modern synthesis techniques in terms of the concepts that are associated with the older transmission line approach.

It was in the spirit of the foregoing remarks that it was initially decided to explore the implications of the work of Brune and those who followed him when the work was expressed in terms of the reflection coefficient rather than in the s-plane terms used by Brune. It became evident that such an investigation was capable of providing the common mathematical basis for modern two-terminal synthesis called for in Section I.1.

The proposed investigation may be illustrated diagrammatically. As mentioned, it appears that Brune was strongly influenced by Foster and Cauer which led to the emphasis on the s-plane in his and in subsequent work. Sketched out, the development was:

Foster and Cauer $\longrightarrow$ Brune $\longrightarrow$ s-Plane Emphasis (I.2-1)

The possibility is that an alternative development might have occurred leading Brune to express his results in terms of another viewpoint. This development is:

Pupin and Campbell $\longrightarrow$ Brune $\longrightarrow$ Reflection Coefficient Emphasis (I.2-2)

In accordance with the foregoing reasoning, Part II of this writing develops the basic properties of the reflection coefficient. Part III develops the special properties of the reflection coefficients of reactance functions and presents a reactance function synthesis technique. Part IV interprets the important existing synthesis techniques in terms of the reflection coefficient concept. Finally, in Part V the results are summarized. The interrelations of these various synthesis techniques, called for in Section I.1, are presented diagrammatically; some of the advantages of the present viewpoint are enumerated, and finally, additional associated topics worthy of continued investigation are suggested.

PART II

THE REFLECTION COEFFICIENT VIEWPOINT

II.1 The Reflection Coefficient - A General Description

II.2 The Realizability Conditions

II.3 Useful Characteristics of Realizable Reflection Coefficients

SECTION II.1

THE REFLECTION COEFFICIENT - A GENERAL DESCRIPTION

The present section is expository in nature. It presents the definition of the reflection coefficient and develops, in broad outline the characteristics of this network function. Section II.2 which follows establishes the necessary and sufficient conditions that a function must satisfy if it is to be the reflection coefficient for a realizable, passive, linear two-terminal network. In Section II.3 some of the more useful characteristics of these realizable reflection coefficients are developed. Many of the ideas developed in these three sections are later applied directly to the synthesis problem. It is possible, however, to scan the material returning later to specific items as they are referenced in connection with synthesis techniques.

The reflection coefficient is defined in terms of the notation of Figure II.1-1 as:

$$\rho(s) = \frac{\frac{E_o(s)}{2} - E_Z(s)}{\frac{E_o(s)}{2}} \qquad \text{(II.1-1)}$$

$$\rho(s) = \frac{1 - \frac{Z(s)}{R_1}}{1 + \frac{Z(s)}{R_1}}$$

From the preceding, Z(s) may be written in terms of ρ(s) as:

$$\frac{Z(s)}{R_1} = \frac{1 - \rho(s)}{1 + \rho(s)} \qquad \text{(II.1-2)}$$

and the symmetry between $\frac{Z(s)}{R_1}$ and ρ (s) is to be noted in the two relationships.

In Figure II.1-2 ρ (s) is illustrated for those cases where it is more convenient to deal in terms of admittances and current sources rather than impedances and voltage sources. Examples where this is the case arise in later sections of the study. Note in both Figures II.1-1 and -2 that the various voltages and currents existing in the circuits may be expressed in terms of the reflection coefficient. The expressions given in the two figures lend themselves to partition into incident and reflected wave components and it is instructive to consider the situation as ρ approaches the limiting values of zero and plus and minus unity.

Some straightforward mathematics based on Figure II.1-1 reveals the outstanding characteristic of the reflection coefficient--that it is unimodular bounded in a defined region. The defined region will vary as various mappings are later applied. To repeat for emphasis: the single outstanding

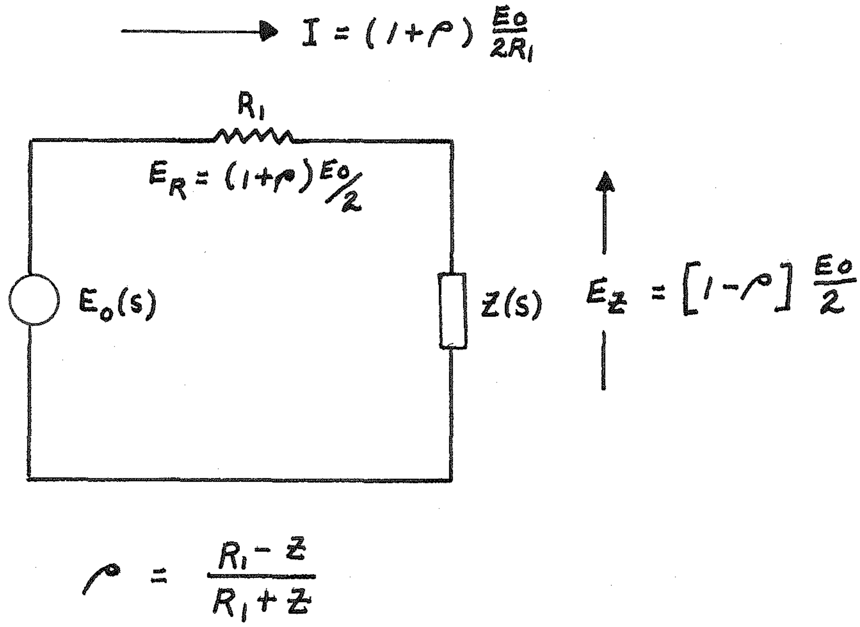

FIGURE II.1-1

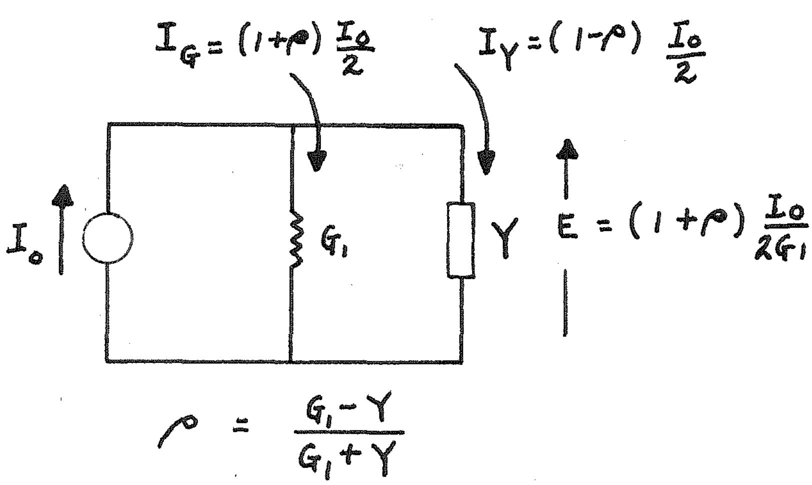

FIGURE II.1-2

characteristic of the reflection coefficient is that it is unimodular bounded, in other words $|\rho(S)| \leq 1$, in a defined region. This characteristic will be found to play the same prominent role that the positive-real concept assumes in the usual approach to synthesis.

The development is as follows: The power into $Z(\omega)$ of Figure II.1-1 is:

$$P_Z(\omega) = U(\omega) \left| \frac{E_0}{R_1 + Z(j\omega)} \right|^2 \qquad \text{(II.1-3)}$$

where

$$Z(j\omega) = U(\omega) + jV(\omega)$$

The maximum possible power into this impedance is:

$$P_{MAX} = \frac{E_0^2}{4R_1} \qquad \text{(II.1-4)}$$

The power "lost" due to mismatch is:

$$P_{MAX} - P_Z = \frac{E_0^2}{4R_1} - U(\omega) \left| \frac{E_0}{R_1 + Z(j\omega)} \right|^2$$

$$= \left| \frac{(R_1 - U)^2 + V^2}{(R_1 + U) + V^2} \right| \frac{E_0^2}{4R_1} \qquad \text{(II.1-5)}$$

$$= \left| \rho(j\omega) \right|^2 P_{MAX}$$

From which it follows that:

$$\left| \rho(j\omega) \right| \leq 1 \tag{II.1-6}$$

Reference to equation II.1-1 reveals that the above may be generalized to:

$$\left| \rho(s) \right| \leq 1 \quad \text{for} \quad \mathrm{Re}\left[Z(s)\right] \geq 0$$

or from the positive real nature of Z(s):

$$\left| \rho(s) \right| \leq 1 \quad \text{for} \quad \mathrm{Re}\left[s\right] \geq 0 \tag{II.1-7}$$

This is the unimodular bounded characteristic of $\rho(s)$ previously mentioned. It will be shown in Section II.3 that equation II.1-7 is both a necessary and sufficient relationship.

The theory of unimodular bounded functions is well developed in the mathematical literature of analytic functions of a complex variable as is brought out in Section III.1. The usual treatment involves functions which are unimodular bounded inside the unit circle in the plane of the independent variable. To take advantage of theorems developed in this notation and because the notation is convenient in the present work, the following change of variable is introduced:

$$P = \frac{1 - S}{1 + S} \tag{II.1-8}$$

$$S = \frac{1 - P}{1 + P} \tag{II.1-9}$$

Equation II.1-7 then becomes:

$$\left| \rho(p) \right| \leq 1 \text{ for } |p| \leq 1 \tag{II.1-10}$$

Note that equations II.1-1, 2, 8, and 9 are all bilinear transformations which transform circles into circles. The mapping relationships between the p- and s-planes are illustrated in Figures II.1-3 and 4 along with a simple example of a concept to be developed in detail in Section IV.1. The location of the axis of zero real part for an impedance consisting of an inductance in series with a resistance is shown in Figure II.1-3. As one learns something of the nature of the immittance by noting the relationship of the two lines $s = j\omega$ and $Z(j\omega)$ so in the p-plane the relationship of these two circles yields the same information. The mapping Z(p) always maps the unit circle onto a portion of itself. As will be developed in Section IV.1, the foregoing is a statement of Pick's interpretation of Schwarz' lemma. The same information is presented in an alternate form in Figures II.1-5 and 6.

The mappings introduced in the present discussion serve to illustrate the relationships involved in the four bilinear

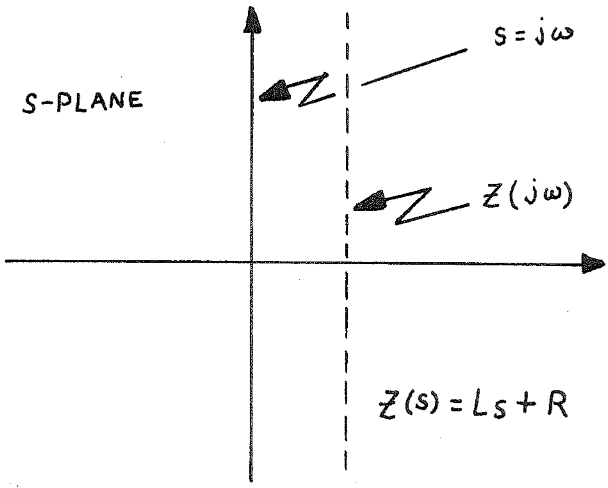

FIGURE II.1-3

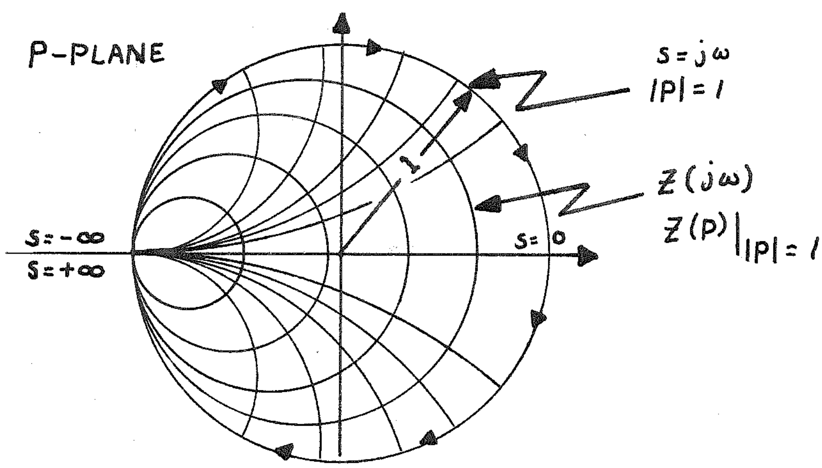

FIGURE II.1-4

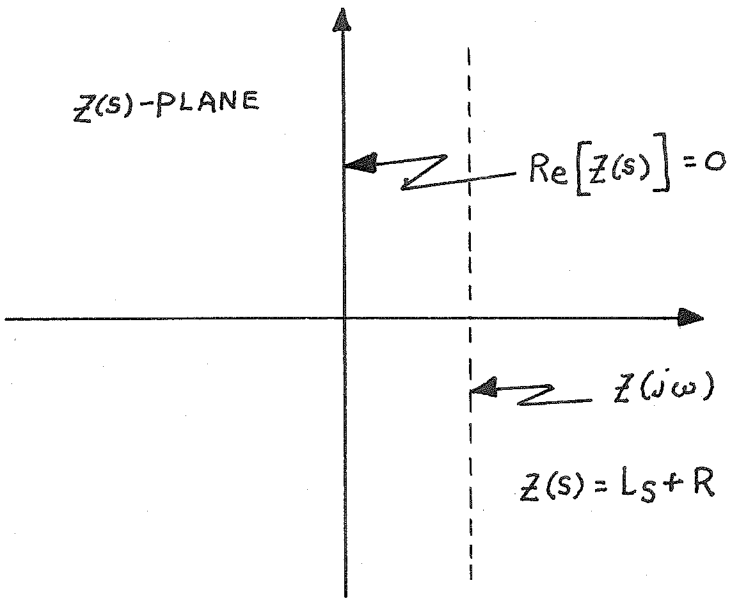

FIGURE II.1-5

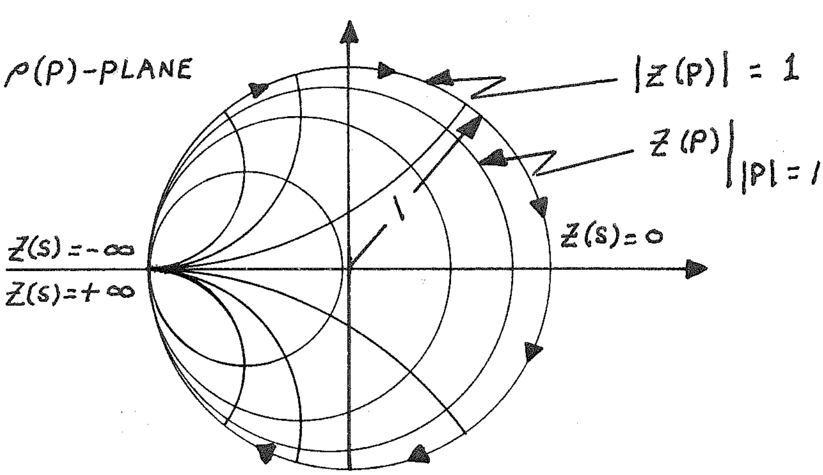

FIGURE II.1-6

equations II.1-1,2,8, and 9. In the work that follows these mappings will be referred to continuously and the concepts presented in a discursive manner will be developed rigorously and in detail, preparatory to being applied to the synthesis problem.

SECTION II.2

THE REALIZABILITY CONDITIONS

The necessary and sufficient conditions that Z(s) be realizable as a driving point immittance of a circuit composed of a finite number of lumped constant linear elements are that it be a rational, positive real function of s. That Z(s) be rational requires that it be capable of being expressed as a ratio of two polynomials in s. That Z(s) be positive real requires: first, that it be real for s real; and second, that the real part of Z(s) be non-negative for the real part of s non-negative.

From the definition of $\rho(s)$, see equation II.1-1, it follows that for realizability it is necessary that $\rho(s)$ be a real, rational function of s and, further, that $\rho(s)$ be unimodular bounded for s not in the left half plane. These results follow directly from the properties of Z(s) described above.

Using equation II.1-2 it can be shown that the properties of being real, rational, and unimodular bounded are sufficient to derive a realizable immittance from the given reflection coefficient. If $\rho(s)$ is real and rational it follows immediately that Z(s) shares the same properties. When $\rho(s)$ is unimodular bounded, then the real part of Z(s) is non-negative as is seen from the following:

$$\rho(s_0) = A + jB \quad ; \quad \left|\sqrt{A^2+B^2}\right| < 1$$

$$\frac{Z(s_0)}{R} = \frac{1 - A - jB}{1 + A + jB}$$

$$Re\left|\frac{Z(s_0)}{R}\right| = \frac{1-(A^2+B^2)}{1+(A^2+B^2)}$$

Hence, for $|\rho(s)| \leq 1$, the real part of Z(s) is non-negative. The preceding results are summarized in Table II.2-1 in which the necessary and sufficient conditions for realizability are given for both $\rho(s)$ and $\rho(P)$.

Item	Z(s)	$\rho(s)$	$\rho(P)$
Rationality			
a)	Z(s) must be rational	$\rho(s)$ must be rational	$\rho(P)$ must be rational
Positive - Reality			
b)	Z(s) real for s real	$\rho(s)$ real for s real	$\rho(P)$ real for P real
c)	$Re[Z(s)] \geq 0$ for $R_e(s) \geq 0$	$\lvert\rho(s)\rvert \leq 1$ for $Re(s) \geq 0$	$\lvert\rho(P)\rvert \leq 1$ for $(P) \leq 1$

Necessary and Sufficient Conditions for Realizability

TABLE II.2-1

Several authors give procedures for testing for positive reality. The tests, which contain redundant items, are designed to simplify the problem of eliminating from consideration those immittance functions which are not realizable. Parallel procedures may be developed for the reflection coefficient. To develop such procedures, however, involves material which is developed in Section II.3. Accordingly the results are presented here without proof and the appropriate material in Section II.3 is referenced.

Tuttle's[9] procedure for testing Z(s) for realizability is presented in Table II.2-2 along with the parallel operations for testing $\rho(s)$. The Guillemin test[4] and its parallel are presented in Table II.2-3.

Item	$Z(s)$	$\rho(s)$
1	Coefficients of Z(s) must be real	Coefficients of $\rho(s)$ must be real
2a	Behavior of Z(s) as $s \to \infty$ must be of one of the forms ks, k, or $\frac{k}{s}$, k being a real positive constant	Behavior of $\rho(s)$ as $s \to \infty$ must be of one of the forms: $\left[-1 + \frac{2R_1}{ks}\right]$, $\left[\frac{R_1 - k}{R_1 + k}\right]$, or $\left[1 - \frac{2k}{R_1 s}\right]$ and these apply respectively to the three forms of Z(s) on the left. (See Item b, Table II.3-3 and associated discussion)
b	Z(s) can have no poles in the right half s-plane	$\lvert\rho(s)\rvert \neq -1$ and $R_e(s) > 0$ and finite R_1
c	Any poles of Z(s) on the s-plane axis of imaginaries must be simple and have real positive residues	Approach of $\rho(s)$ to -1 due to s-plane imaginary axis poles of Z(s) must be of the form: $\lim_{s \to s_o} \rho(s) \to -1 + \frac{2R_1}{k}(s - s_o)$ where k is the residue of the pole at s_o. (See Item 2a of Table II.3-3 and associated discussion)
3	$R_e\left[Z(s)\right] \geq 0$ for $s = j\omega$	$\lvert\rho(s)\rvert \leq 1$ for $s = j\omega$

<u>Realizability Test Based on Tuttle Text</u>

TABLE II.2-2

ITEM	Z(s)	$\rho(s)$
A	Analytic in RHP	$\lvert \rho(s) \rvert < 1$ for $Re[s] > 0$
B	$Re[Z(j\omega)] \geq 0$	$\lvert \rho(j\omega) \rvert \leq 1$
C	j-axis poles simple and have positive real residue	See 2(c) of Table II.2-2 preceding

REALIZABILITY TEST BASED ON GUILLEMIN TEXT

TABLE II.2-3

SECTION II.3

USEFUL CHARACTERISTICS OF REALIZABLE REFLECTION COEFFICIENTS

In the present section some of the characteristics of realizable reflection coefficients which are useful in synthesis will be developed. First, those characteristics which aid in the recognition and manipulation of these network functions are discussed and then summarized in Table II.3-1. Next, the loci of the zeros and poles of the reflection coefficient are examined and that material is presented in tabular form in Table II.3-2. Finally, Table II.3-3 presents the results of the discussion of the behavior of the reflection coefficient in the vicinities of the singularities of Z(s).

Characteristics of $\rho(s)$ Useful in Manipulations:

The most useful properties of the reflection coefficient are those by means of which its realizability may be established. This matter has already been covered in Section II.2 and hence, for completeness, the positive real concept and the unimodular bounded are presented as the first item of Table II.3-1 but without further comment.

The negative of a realizable reflection coefficient is realizable just as is the reciprocal of a realizable immittance function. This follows from the fact that the negative of a realizable reflection coefficient is the reflection coefficient of the dual of the original immittance, if the dual is taken with respect to R_1:

$$-\rho(s) = -\frac{R_1 - Z(s)}{R_1 + Z(s)} = \frac{R_1 - \frac{R_1^2}{Z(s)}}{R_1 + \frac{R_1^2}{Z(s)}} \qquad \text{(II.3-1)}$$

The useful mapping theorem of Brune[1] - that a realizable immittance function of a realizable immittance function is itself a realizable immittance function has an analog in terms of reflection coefficients. Consider the following function:

$$\rho(p) = \rho_1\left[\rho_2(p)\right]$$

where $\rho_1(p)$ and $\rho_2(p)$ are realizable reflection coefficients. To determine, under these conditions, whether $\rho(p)$ is a realizable reflection coefficient apply the tests of Table II.2-1. Since $\rho_1(p)$ and $\rho_2(p)$ are rational functions of p, it follows that $\rho(p)$ is also a rational function of p. Further, since $\rho_1(p)$ and $\rho_2(p)$ are real for p real, the same property holds for $\rho(p)$. Finally, if

$$\left.\begin{aligned} |\rho_1(p)| &\leq 1 \\ |\rho_2(p)| &\leq 1 \end{aligned}\right\} \text{ for } |p| \leq 1$$

Then

$$|\rho(p)| = \left|\rho_1\left[\rho_2(p)\right]\right| \leq 1$$

since a unimodular bounded function of a unimodular bounded function is also a unimodular bounded function. From the foregoing, $\rho(p)$ satisfies the conditions that it be a realizable reflection coefficient.

Loci of Singularities of $\rho(s)$

The locations of the singularities of $\rho(s)$ are now examined in terms of the locations of the singularities of Z(s). Since:

$$Z(\bar{s}) = \overline{Z(s)} \tag{II.3-3}$$

it follows that:

$$\rho(\bar{s}) = \overline{\rho(s)} \tag{II.3-4}$$

and as the singularities of Z(s), when not real, must occur in complex conjugate pairs, the same property must hold for $\rho(s)$.

Zeros of $\rho(s)$ must satisfy the following equation:

$$\frac{Z(s)}{R_1} = \frac{N(s)}{R_1 D(s)} = 1 \tag{II.3-5}$$

The above is typical of the forms that arise in the theory of feedback control systems and to which Evan's root locus techniques[10] may be applied. Equation II.3-5 may be written:

$$\frac{\prod_{i=1}^{\nu} (s - s_{oi})}{R_1 \prod_{i=1}^{\nu} (s - s_{pi})} = 1 \tag{II.3-6}$$

from which it is clear that the loci of the zeros of $\rho(s)$ must originate at the zeros of Z(s) for R_1 vanishingly small and terminate at the poles of Z(s) as R_1 grows without bounds.

Item	Z(s)	$\rho(s)$
a)	A driving point function for linear, lumped constant, passive networks is: rational; real for s real; and of non-negative real part for s of non-negative real part	The reflection coefficient for a linear, lumped constant, passive network is: rational; real for s real; and unimodular bounded for s of non-negative real part
b)	The reciprocal of a realizable immittance function is again a realizable immittance function	The reciprocal of a realizable reflection coefficient is non-realizable since the unimodular bound requirement is violated
c)	The negative of a realizable immittance function is non-realizable since the p.r. property is violated	The negative of a realizable reflection coefficient is again a realizable reflection coefficient
d)	If $Z_1(s)$ and $Z_2(s)$ are realizable immittance functions, then $Z_1[Z_2(s)]$ is again a realizable immittance function	If $\rho_1(p)$ and $\rho_2(p)$ are realizable reflection coefficients, then $\rho_1[\rho_2(p)]$ is also a realizable reflection coefficient

<u>Characteristics of $\rho(s)$ Useful in Manipulation</u>

TABLE II.3-1

Similarly, the loci of the poles of ρ(s) must satisfy the equation:

$$\frac{Z(s)}{R_1} = \frac{N(s)}{R_1 D(s)} = \frac{\prod_{i=1}^{\nu}(s - s_{oi})}{R_1 \prod_{i=1}^{\nu}(s - s_{pi})} = -1 \qquad \text{(II.3-7)}$$

and again, the loci of the poles originate at the zeros and terminate at the poles of Z(s) as R_1 grows from zero to extremely large values.

There is still a third set of loci of interest in network synthesis and that is the set of lines for which the real part of Z(s) vanishes and for which $|\rho(s)| = 1$. These loci satisfy the equations:

$$\frac{Z(s)}{R_1} = \frac{N(s)}{R_1 D(s)} = \frac{\prod_{i=1}^{\nu}(s - s_{oi})}{R_1 \prod_{i=1}^{\nu}(s - s_{pi})} = e^{j\frac{\pi}{2}(2n+1)} \qquad \text{(II.3-8)}$$

where:

$n = 0, 1, 2 \ldots\ldots$

For this system, too, the loci originate at the zeros and terminate at the poles of Z(s).

From equation II.3-6, 7, and 8 it follows that at any zero or pole of Z(s) in the s-plane there are four loci originating or terminating respectively. The situation is illustrated in Figures II.3-1 and 2 for a simple reactance function:

$$Z(s) = \frac{s}{s^2 + \omega^2} \qquad \text{(II.3-9)}$$

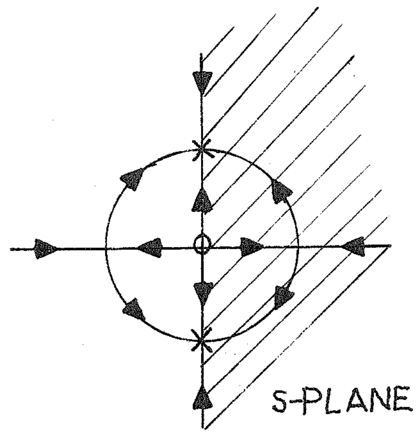

FIGURE II.3-1

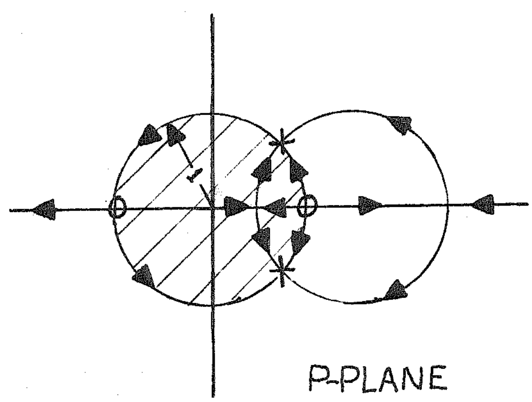

FIGURE II.3-2

The amount of detail presented in Figure II.3-2 makes it somewhat difficult to see the important features involved. These are as follows:

a) Since, for a reactance function, as in II.3-9 above, the loci of zero real part of Z(s) coincide with the s-plane axis of imaginaries, their images coincide with the image of that axis which, in the ρ-plane, is the circumference of the unit circle.

b) The loci of zero real part of Z(s) divide the s-plane into two regions, one in which the real part of Z(s) is positive, which must include the right half plane, and the other in which it is negative. Further, the loci of zeros of $\rho(P)$ or $\rho(s)$ are constrained to be in the former region and the loci of poles are constrained to be in the latter region.

c) By symmetry, the zeros and poles of $\rho(s)$ in the s-plane, for the reactance function of equation II.3-9, are complementary with regard to the imaginary axis. That is, in terms of the usual two dimensional potential analog,[11] the j-axis is an equipotential line in the field of the line charges represented by the zeros and poles of $\rho(s)$. A unit positive line charge is associated with the singularity of one type, and a unit negative line charge with the other. The bilinear transformation, which defines the reflection coefficient, preserves this relationship. Hence the unit circle circumference is an equipotential under the effect of the images of the charges representing the singularities of $\rho(P)$.

Similar relationships obtain for immittances which are not reactance functions. Consider the RL immittance:

$$Z(s) = \frac{s(s+a)}{(s+b)} \qquad a = 2b \qquad \text{(II.3-10)}$$

The two mappings involved are presented in Figures II.3-3 and II.3-4. The example selected might be thought to be too complex for clarity of illustration. A simpler example of a non-reactance function has, however, been partially mapped already in Figures II.1-3 through 6. The features to note in the present example are:

a) The loci of zero real part of Z(s) do not coincide with the circumference of the P -plane unit circle for Z(s) not a reactance function. Indeed, these loci are not necessarily circles at all although such is the case in the present example which was selected for ease in sketching.

b) The region of the P -plane for which the real part of Z(s) is positive necessarily includes the interior of the unit circle which is the image of the right half s-plane. The loci of zeros of $\rho(P)$ or $\rho(s)$ are no longer constrained to fall within the P -plane unit circle, as is the case with reactance functions, but are limited to the larger region of positive real part of Z(s). Poles of $\rho(P)$ are still restricted to the region of negative real part of $\rho(P)$ which is only a portion of the region of the P -plane exterior to the unit circle.

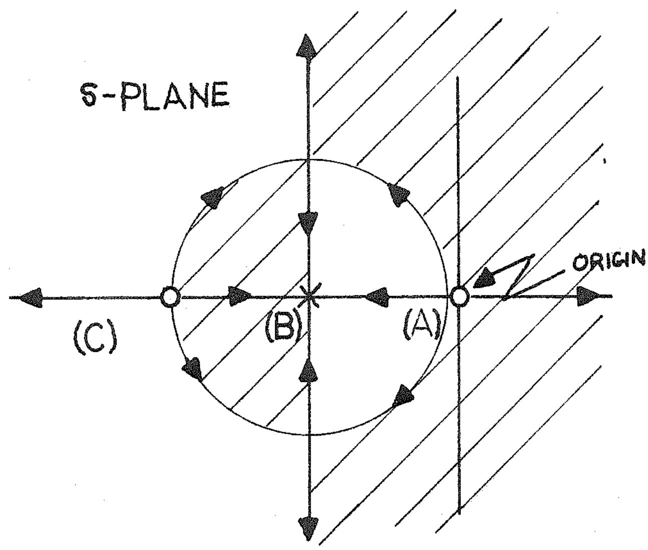

FIGURE II.3-3

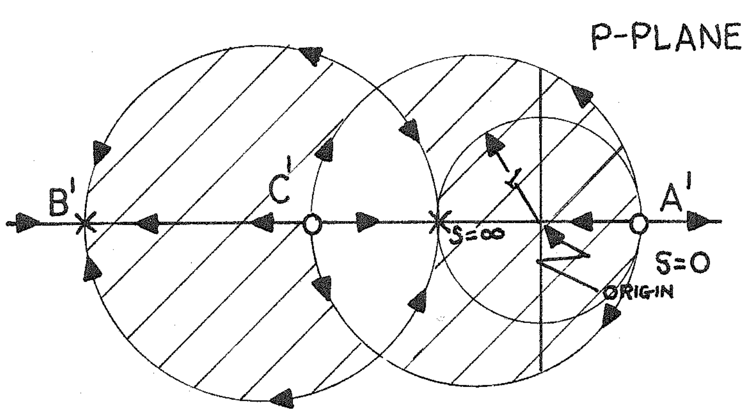

FIGURE II.3-4

c) The loci of zero real part of Z(s) remain equipotentials under the influence of the line charges associated with the singularities of ρ (s) but, as mentioned, these loci are no longer simple shapes.

The material presented in the preceding discussion is summarized in Table II.3-2.

Behavior of ρ (s) Near Singularities of Z(s):

By inspection of the defining equation II.1-1, it can be seen that zeros and poles of Z(s) condense on the points +1 and -1, respectively, in the ρ (s) plane. As is well known, the concept of positive reality restricts the behavior of Z(s) at certain singularities, notably those on the s-plane axis of imaginaries. See items 2a and 2c of Table II.2-2. It follows that these restrictions must imply limitations on the behavior of ρ (s) near the points $\pm$ 1.

Consider the behavior of ρ (s) as s approaches s_p where:

$$\lim_{s \to s_p} Z(s) = \lim_{s \to s_p} \frac{k}{s - s_p} \qquad \text{(II.3-11)}$$

and k is real and positive

Then:

$$\lim_{s \to s_p} \rho(s) = \lim_{s \to s_p} \frac{R_1 - Z(s)}{R_1 + Z(s)} \qquad \text{(II.3-12)}$$

Item	Loci	Characteristics
a	Loci of zeros of real part of Z(s)	Divides s-plane into two regions, one of positive real part of Z(s); the second of negative real part of Z(s). The right half s-plane is part of the former region and, for reactance functions coincides with it. These loci map into the ρ(P) plane unit circle circumference and, in the case of reactance functions, into the P -plane unit circle circumference as well.
b	Loci of zeros of reflection coefficient	Constrained to the region of positive real part of Z(s). Hence may be interior to or exterior to unit circle in P -plane.
c	Loci of poles of reflection coefficient	Constrained to the region of negative real part of Z(s). Hence must be exterior to unit circle in P -plane.
d	Relationship of above three loci types	Loci of zeros of real part of Z(s) are equipotentials in the two dimensional potential analog where line charges are associated with the singularities of ρ(s). All three loci types originate at zeros of Z(s) and terminate at its poles.

Behavior of Singularities of Reflection Coefficient

TABLE II.3-2

$$\lim_{s \to s_p} \rho(s) = \lim_{s \to s_o} \frac{R_1 - \frac{k}{s-s_p}}{R_1 + \frac{k}{s-s_p}} = \lim_{s \to s_p} \frac{\left[\frac{R_1}{k}(s-s_p) - 1\right]}{1 + \frac{R_1}{k}(s-s_p)}$$

(II.3-13)

$$= \lim_{s \to s_p} \left[-1 + \frac{2R_1}{k}(s-s_p) \right]$$

Similarly, in the region of a zero, where:

$$\lim_{s \to s_o} Z(s) = \lim_{s \to s_o} k(s-s_o)$$

There, one writes

$$\lim_{s \to s_o} \rho(s) = \lim_{s \to s_o} \frac{R_1 - Z(s)}{R_1 + Z(s)} = \lim_{s \to s_o} \frac{1 - \frac{k}{R_1}(s-s_o)}{1 + \frac{k}{R_1}(s-s_o)}$$

(II.3-14)

$$= \lim_{s \to s_o} \left[1 - \frac{2k}{R_1}(s-s_o) \right]$$

These conditions are necessary. It is now established that they are sufficient to insure the proper behavior of Z(s). Consider Z(s) as $s \to s_p$ where ρ (s) behaves as described in II.3-13.

$$\lim_{s \to s_p} \frac{Z(s)}{R_1} = \lim_{s \to s_p} \frac{1 - \rho(s)}{1 + \rho(s)}$$

$$= \lim_{s \to s_p} \frac{2 - \frac{2R_1}{k}(s-s_p)}{\frac{2R_1}{k}(s-s_p)}$$

$$= \lim_{s \to s_p} \frac{k}{R_1(s-s_p)}$$ (II.3-15)

$$\therefore \lim_{s \to s_p} Z(s) = \lim_{s \to s_p} \frac{k}{s-s_p}$$

Similarly, near S_0 where $\rho(s)$ behaves as described in II.3-14.

$$\lim_{s \to s_0} \frac{Z(s_0)}{R_1} = \lim_{s \to s_0} \frac{1-\rho(s)}{1+\rho(s)}$$

$$= \lim_{s \to s_0} \frac{\frac{2k}{R_1}(s-s_0)}{2+\frac{2k}{R_1}(s-s_0)} \qquad \text{(II.3-16)}$$

$$= \lim_{s \to s_0} \frac{k}{R_1}(s-s_0)$$

$$\lim_{s \to s_0} Z(s) = \lim_{s \to s_0} k(s-s_0)$$

These results are summarized and applied to the immittance function, reflection coefficient relationship in Table II.3-3.

Item	$Z(s)$	$\rho(s)$
a	Any poles of Z(s) on the s-plane axis of imaginaries must be simple and have real, positive residues	The approach of $\rho(s)$ to the -1 point due to an s-plane imaginary axis poles of Z(s) must be of the form: $\lim_{s \to s_p} \rho(s) = \lim_{s \to s_p} \left[1 + \frac{2R_1}{k}(s - s_p) \right]$ where k is the residue of the poles and must be real and positive.
b	The behavior of Z(s) as $s \to 0$ or ∞ must be of one of the forms ks, k, or k/s where k is a positive real constant	The behavior of $\rho(s)$ as $s \to 0$ must be of one of the forms: $\left[1 - \frac{2k}{R_1} s \right]$ $\left[\frac{R_1 - k}{R_1 + k} \right]$ $\left[-1 + \frac{2R_1}{k} s \right]$ and these apply respectively to the three forms for Z(s) on the left. The behavior of $\rho(s)$ as $s \to \infty$ must be of one of the forms: $\left[-1 + \frac{2R_1}{ks} \right]$ $\left[\frac{R_1 - k}{R_1 k} \right]$ $\left[1 - \frac{2k}{R_1 s} \right]$ and these apply respectively to the three forms for Z(s) on the left, k must be real and positive in all the above six forms.

<u>Behavior of $\rho(s)$ Near Singularities of Z(s)</u>

TABLE II.3-3

PART III

REACTANCE FUNCTIONS

SECTION III.1

REACTANCE FUNCTIONS AND UNIT FUNCTIONS

With equation II.1-8 a special class of unimodular bounded functions was introduced, namely those which have modulus unity everywhere on the boundary of the unit circle. Such functions, which have particular significance to circuit theory, are called unit functions or E-functions (from the German - Einheitsfunktionen); the notation is that of Caratheodory.[12] One may write the general form of a unit function of n^{th} degree as:

$$^{*}\quad E_m(p) = e^{i\phi} \prod_{\nu=1}^{m} \frac{\beta_\nu - p}{1 - \bar{\beta}_\nu p} \qquad \text{(III.1-1)}$$

A unit function of n^{th} degree is a rational function whose numerator and denominator are polynomials of degree n.

The reflection coefficients of reactance functions map the s-plane axis of imaginaries into the boundary of the unit circle in the $\rho(p)$ plane, since the boundary of the unit circle is always the locus of $R_e\left[\,Z(s)\,\right] = 0$. Thus both E-functions and the reflection coefficients of reactance functions, denoted as $\rho_N(p)$, share

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* Dr. E. C. Ho of the University of California in Los Angeles has pointed out to the writer that functions of this form appear in several places in the literature of network synthesis. Investigation reveals that they are used by Norton[13], Darlington[7], and Guillemin[4] in applications where the behavior $F(\omega) = 1/F(1/\omega)$ is required. Guillemin in the latter reference notes the early use of these functions in a monograph by Cauer.[14]

the property:

$$\left.\begin{aligned} |E_m(P)| &= 1 \\ |\rho_n(P)| &= 1 \end{aligned}\right\} \quad \text{for} \quad |P| = 1 \tag{III.1-2}$$

It will be shown, in fact, that the E-functions, with certain restrictions, may be identified with such reflection coefficients. The restrictions involve the arbitrary phase factor for, from item b of Table II.3-1, $\rho_n(P)$ must be real for P real. In Sections II.2 and II.3 it was convenient to speak of ρ(s) since comparison was constantly being made with Z(s). Here attention is focused on the unimodular bounded characteristic:

$$|\rho(P)| \leq 1 \quad \text{for} \quad |P| \leq 1 \tag{II.1-9}$$

and the P -plane is often referenced, hence $\rho(P)$ is used. The change in variables causes no difficulty. Note, further, that to this point it has been proposed that the reflection coefficient viewpoint be considered in synthesis and the properties of the reflection coefficient have been examined. The present discussion begins applying and evaluating that viewpoint, a task to which the remainder of the paper is devoted. In view of the restriction mentioned above, the only forms of the E-function which can be accepted as realizable reflection coefficients are:

$$E_m(P) = \pm \prod_{\nu=1}^{m} \frac{P_\nu - P}{1 - \bar{P}_\nu P} \qquad |P_\nu| < 1 \tag{III.1-3}$$

To demonstrate first that the E-functions do satisfy (III.1-2), consider the single factor:

$$F(P) = \frac{P_K - P}{1 - \bar{P}_K P} \qquad \text{(III.1-4)}$$

$$[F(P)]^2 = \frac{P_K - P}{1 - \bar{P}_K P} \cdot \frac{\bar{P}_K - \bar{P}}{1 - P_K \bar{P}}$$

$$= \frac{A^2 + B^2 - 2AB\cos\gamma}{1 + A^2B^2 - 2AB\cos\gamma} \qquad \text{(III.1-5)}$$

where:

$$P = Ae^{i\alpha}$$
$$P_K = Be^{i\beta}$$
$$\gamma = \alpha - \beta$$

By inspection, $|F(P)| = 1$ when $|P| = 1$. It remains to show that $F(P)$ is unimodular bounded, that is:

$$|F(P)| < 1 \qquad \text{for } |P| < 1 \qquad \text{(III.1-6)}$$

The proof requires verifying that:

$$1 > \frac{[A^2 + B^2 - 2AB\cos\gamma]^{\frac{1}{2}}}{[1 + A^2B^2 - 2AB\cos\gamma]^{\frac{1}{2}}} \qquad \text{(III.1-7)}$$

or:

$$\frac{1}{A\left[1 + \left(\frac{B}{A}\right)^2 - 2\left(\frac{B}{A}\right)\cos\gamma\right]^{\frac{1}{2}}} > \frac{1}{\left[1 + A^2B^2 - 2AB\cos\gamma\right]^{\frac{1}{2}}} \qquad \text{(III.1-8)}$$

The steps taken and those to follow presume that $A \geq B$; if $B > A$ factor B out of the left hand side of equation III.1-7 rather than A and parallel the following development to arrive at the same conclusion. Using the expansion for reciprocal distance in spherical coordinates[8], the above becomes:

$$\frac{1}{A} \sum \left(\frac{B}{A}\right)^n P_n(\cos \gamma) > \sum (AB)^n P_n(\cos \gamma)$$

where $P_n(\cos \gamma)$ is the Legendre coefficient.

It follows that:

$$\frac{1}{A^{n+1}} > A^n$$

which is true for $A < 1$, hence condition, equation III.1-6, is satisfied. Hence, the E-functions of III.1-3 satisfy the conditions of Table II.3-1 which they must in order to be realizable reflection coefficients.

That the E-functions are the reflection coefficients for reactance functions will now be shown. Using equation III.1-3 in II.1-2:

$$\frac{Z(P)}{R_1} = \frac{1 \mp \prod_{\nu=1}^{n} \left(\frac{P_\nu - P}{1 - \bar{P}_\nu P}\right)}{1 \pm \prod_{\nu=1}^{n} \left(\frac{P_\nu - P}{1 - \bar{P}_\nu P}\right)} \qquad \text{(III.1-9)}$$

By straightforward manipulation and substitution of variable, one obtains:

$$\frac{Z(s)}{R_1} = \frac{\prod (s+s_\nu) \mp \prod (s-s_\nu)}{\prod (s+s_\nu) \pm \prod (s-s_\nu)} \qquad \text{(III.1-10)}$$

Equation III.1-10 is recognized as a reactance function when it is noted, for the case of the upper signs, that its numerator and denominator are the odd and even parts, respectively if n is even or odd, of a Hurwitz polynomial. Interchange the words even and odd in the foregoing for the case of the lower signs. The Hurwitz polynomial involved is:

$$H(s) = \prod_{\nu=1}^{n} (s + s_\nu) \qquad \text{(III.1-11)}$$

In arriving at equation III.1-10 it was presumed that the S_ν occur in complex conjugate pairs, which must be the case if the E-function is a realizable reflection coefficient. The foregoing development leads to the following theorem:

<u>Theorem I</u>; The unit functions:

$$E_n(P) = \pm \prod_{\nu=1}^{n} \frac{P_\nu - P}{1 - \bar{P}_\nu P} \qquad \text{(III.1-3)}$$

provided they meet the criteria for realizability, are the reflection coefficients of purely reactive driving point immittances.

From the form of III.1-3 and the realizability restrictions which require singularities to occur in complex conjugate pairs, the zeros and poles of the E-function plot as shown in Figure III.1-1. This behavior was brought out in the discussion of Figure II.2-2 from a different viewpoint. The s-plane behavior of the singularities of the E-function are shown in Figure III.1-2. The results of the discussion are summarized in Table III.1-1.

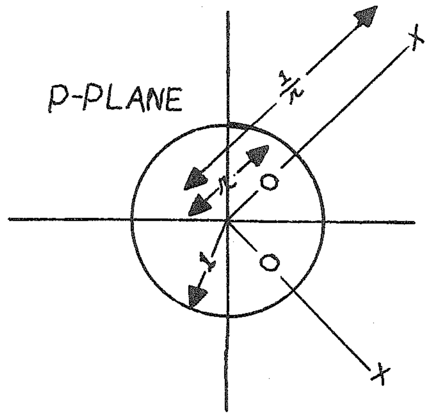


FIGURE III.1-1

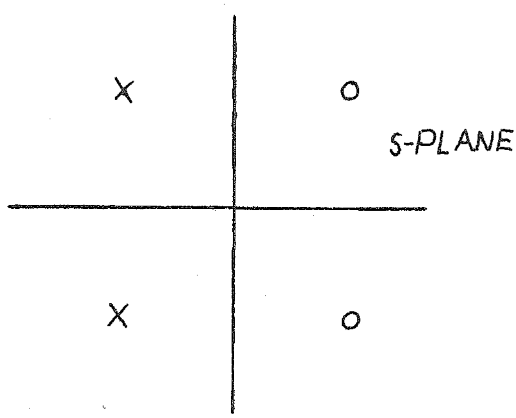


FIGURE III.1-2

	$Z_r(s)$	$\rho_r(p)$
a)	Reactive driving point immittances have a numerator and denominator polynomials which are the odd and even parts, respectively and conversely, of a Hurwitz polynomial.	The reflection coefficients of purely reactive networks are always E-functions.
b)	Zeros and poles of reactive driving point immittances are restricted to the s-plane imaginary axis along which they occur in complex conjugate pairs and alternate.	Zeros and poles of the reflection coefficients of reactive networks occur in complex conjugate pairs and are in complementary positions in respect to the p-plane unit circle; that is the product of the amplitudes of a pole and zero on the same radius vector must equal unity.

BEHAVIOR OF REACTIVE NETWORKS

TABLE III.1-1

SECTION III.2

A REACTANCE FUNCTION SYNTHESIS TECHNIQUE

Caratheodory[12] presents a theorem concerning unit functions which may be used as a basis for a synthesis procedure. While not of practical value in itself, the technique is interesting: first, as it establishes an initial point for synthesis studies; second, as an aid in the development of insight into the synthesis process; third, as it brings out the relationship between lumped constant and distributed parameter systems; and, finally, as it also incidentally casts a light on some of the approximation techniques that have been used. This last characteristic will be discussed very briefly as the subject of approximations does not fall within the scope of the present investigation.

The theorem of interest reads as follows:

> "The totality of unit functions is part of a normal family, the limit functions of which are not necessarily unit functions, though they must be functions of bound one. Thus, for example, the sequence of unit functions $f_\nu(p) = p^\nu$, $\nu = 1, 2, - - - -$ converges to the constant zero."

The foregoing, except for a trivial change in notation, is quoted directly from Caratheodory. He goes on to say that one may assign to every function which is unimodular bounded in the unit circle, a sequence of unit functions: $E_1(p)$, $E_2(p)$, - - - - - that converge to a given function. From this sequence of functions one obtains finally a unit function of infinite degree such that, if one had used a realizable reflection coefficient as the given function, then:

$$\lim_{n \to \infty} E_n(p) = \rho(p) \qquad \text{(III.2-1)}$$

The theorem implies that any realizable reflection coefficient may be written as the limit of a family of E-functions. An equivalent statement is that any realizable network may be developed as a network made up of an infinite number of reactive elements. The theorem makes good physical sense when one recalls that transmission lines may be approximated by lumped circuit elements, an infinite number of infinitely small elements being required to give an exact equivalent. In regard to dissipative elements the effect as far as a driving point source is concerned is the same whether power is lost in localized heating or by being sent down an infinite dissipationless line. The technique under discussion obviously uses the latter method for handling power dissipation.

Caratheodory's procedure for finding the family of E-functions to associate with ρ(p) so that the property expressed in equation III.2-1 is realized, is as follows:

First, expand ρ(p) as a power series.

$$\rho(p) = \sum_{i=0}^{\infty} a_i p^i \qquad \text{(III.2-2)}$$

Next, obtain the function $\phi(P)$ and expand it in a power series. $\phi(P)$ is defined and expanded as follows:

$$\phi(P) = \frac{a_0 - \rho(P)}{P\left[1 - \bar{a}_0 \rho(P)\right]} = \sum_{i=0}^{\infty} b_i P^i \qquad \text{(III.2-3)}$$

It can be shown that $\phi(P)$ is unimodular bounded since $\rho(P)$ has that property by virtue of being a realizable reflection coefficient. Further, following Caratheodory, the E-function, the first $(n-1)$ of whose power series coefficients are identical with those of $\rho(P)$ has the following property:

$$E_{n+1}(P) = \frac{a_0 - P\, E_n^*(P)}{1 - \bar{a}_0\, P\, E_n^*(P)} \qquad \text{(III.2-4)}$$

where:

$E_n^*(P)$ is the E-function of degree n whose first n-1 power series coefficients are identical with those of $\phi(P)$.

$E_{n+1}(P)$ is the E-function of degree n + 1 whose first n power series coefficients are identical with those of $\rho(P)$.

It follows, then, that:

$$\lim_{n \to \infty} E_{n+1}(P) = \rho(P)$$

and just as an ideal transmission line made as long as one pleases always has a purely reactive input impedance, yet in the limit as the length becomes infinite, the input impedance becomes resistive--so, in this case, all members of the family $E_n(p)$ are E-functions except that the limit function is not itself necessarily an E-function.

Caratheodory shows that the foregoing mathematical properties may be made the basis of an orderly iterative program for developing consecutive members of the family $E_n(p)$.

To start the process, $E_1(p)$, the first degree unit function which has the a_0 of III.2-2 as the first term of its power series is:

$$E_1(P) = \frac{a_0 - P}{1 - \bar{a}_0 P} \qquad \text{(III.2-5)}$$

Similarly:

$$E_1^*(P) = \frac{b_0 - P}{1 - \bar{b}_0 P}$$

and

$$E_2(P) = \frac{a_0 - P\,E_1^*(P)}{1 - \bar{a}_0 P E_1^*(P)}$$

$$= \frac{a_0 - (a_0 \bar{b}_0 + b_0)\,P + P^2}{1 - (\bar{a}_0 b_0 + \bar{b}_0)\,P + \bar{a}_0 P^2}$$

To develop an $E_2^*(p)$ so that $E_3(p)$ can be obtained requires the creation of a new function

$$\psi(P) = \frac{b_0 - \phi(P)}{P\left[1 - \bar{b}_0\,\phi(P)\right]}$$

for which an $E_1^{**}(p)$ may be written and from which $E_2^*(p)$ and finally $E_3(p)$ may be derived. The process, then, is repetitive.

If it is true that any unitary modular bounded function may be expressed as the limiting member of a family of unit functions, or equivalently, that any driving point immittance can be realized by a circuit consisting of an infinite number of reactive elements, then such a realization may be obtained for the case of a pure resistance.

Consider Z(s) = R and let R_1 = R so that:

$$\rho(P) = \frac{R_1 - R}{R_1 + R} = 0$$

Then, following the procedure described:

$$\phi(P) = 0$$
$$\psi(P) = 0$$

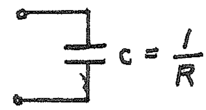
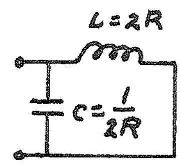
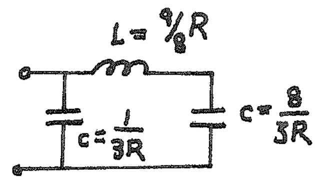
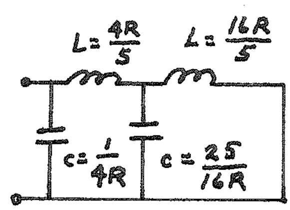
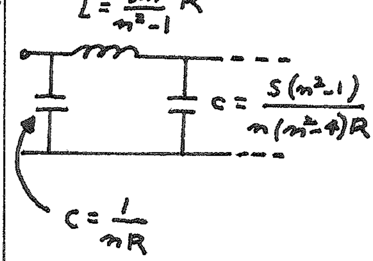
and:

$$E_1(P) = -P$$
$$E_1^*(P) = -P$$
$$\therefore E_2(P) = P^2$$
$$E_1^{**}(P) = -P$$
$$E_2^*(P) = P^2$$
$$E_3(P) = -P^3$$

By inspection, these unit functions are members of the family:

$$\therefore E_n(P) = (-P)^n \qquad \text{(III.2-6)}$$

each of which satisfy the requirement that its first (n-1) power series terms match those of ρ(p). Accordingly, Table III.2-1 has been developed and from it the network consisting of an infinite number of purely reactive elements which has a purely resistive input may be expressed as the ratio of the even and odd parts of the binomial expansion of $(s + 1)^n$.

n	$E_n(P)$	$E_n(S)$	$Z_n(S)$	CIRCUIT
1	$-P$	$-\frac{1-S}{1+S}$	$\frac{R}{S}$	
2	$(-P)^2$	$\left(\frac{S-1}{S+1}\right)^2$	$\frac{2RS}{S^2+1}$	
3	$(-P)^3$	$\left(\frac{S-1}{S+1}\right)^3$	$\frac{(3S^2+1)}{S^3+3S}$ - - - - - - $\cfrac{R}{\frac{1}{3}S+\cfrac{1}{\frac{9}{8}S+\cfrac{1}{\frac{8}{3}S}}}$	
4	$(-P)^4$	$\left(\frac{S-1}{S+1}\right)^4$	$\frac{[4S^3+4S]R}{S^4+6S^2+1}$ - - - - - - $\cfrac{4R}{\frac{1}{S}+\cfrac{1}{\frac{1}{5}S+\cfrac{1}{\frac{25}{4}S+\cfrac{1}{\frac{4}{5}S}}}}$	
n	$(-P)^n$	$\left(\frac{S-1}{S+1}\right)^n$	$\frac{R\left[nS^{n-1}+\frac{(n)(n-1)(n-2)}{(2)(3)}S^{n-3}+\cdots\right.}{\left.S^n+\frac{n(n-1)}{2}S^{n-2}+\cdots\right]}$ - - - - - - $\cfrac{R}{\frac{1}{n}S+\cfrac{1}{\frac{3n}{n^2-1}S+\cfrac{1}{\frac{5(n^2-1)}{n(n^2-4)}S+\cdots}}}$	

Family of Reactive Networks Whose Limit is R
TABLE III.2-1

With the expansion for a pure resistance available as in Table III.2-1, it is in principle established that such a realization may be obtained for any driving point immittance. It is always possible to realize the immittance by one of the more orthodox techniques and then substitute the appropriate infinite network of Table III.2-1 wherever a dissipative element appears.

If the method is applied to reactance functions it automatically terminates in an identity. Consider, for example, $Z(s) = s$:

Let $R_1 = 1$ so that

$$\begin{aligned} \rho(p) &= p \\ \phi(p) &= -1 \\ \psi(p) &= 0 \end{aligned} \qquad \text{(III.2-7)}$$

Then:

$$\begin{aligned} E_1(p) &= -p \\ E_2(p) &= p \end{aligned} \qquad \text{(III.2-8)}$$

and recognizing the identity between III.2-7 and -8 one halts the process.

If a more complicated reactance function is considered:

$$Z(s) = \frac{Ls}{LCs^2+1}$$

$$\text{(III.2-9)}$$

$$\rho(s) = \frac{R_1 LCs^2 - Ls + R_1}{R_1 LCs^2 + Ls + R_1}$$

Let $R_1 = 1/2 \sqrt{L/C}$ so that:

$$\rho(s) = \frac{(\tau s - 1)^2}{(\tau s + 1)^2} \qquad \text{where } \tau = \sqrt{LC}$$

Then:

$$\rho(P) = \left(\frac{\left[\frac{\tau - 1}{\tau + 1} \right] - P}{1 - \left[\frac{\tau - 1}{\tau + 1} \right] P} \right)^2$$

which is immediately recognizable as an E-function.

The method is by no means simple or elegant to apply, for consider the simple impedance Z(s) = Ls + R:

Let R_1 = R, so that:

$$\rho(s) = \frac{-Ls}{2R + Ls}$$

Then:

$$\rho(P) = \frac{-L}{2R+L} \frac{1-P}{1 + \frac{(2R-1)}{(2R+1)}P} = -\beta \frac{1-P}{1+\alpha R} \qquad \text{(III.2-10)}$$

where:

$$\alpha = \frac{2R-L}{2R+L} \quad \text{and} \quad \beta = \frac{L}{2R+L}$$

Expanding equation III.2-10:

$$\rho(P) = -\beta\left[1 - (1+\alpha)P + \alpha(1-\alpha)P^2 + \alpha^2(1-\alpha)P^3 ----\right] \qquad \text{(III.2-11)}$$

Applying equation III.2-3:

$$\phi(P) = \frac{-\beta(1+\alpha)}{1-\beta^2}\left[\frac{1}{1 + \frac{\alpha+\beta^2}{1-\beta^2}P}\right] = -\gamma\left[\frac{1}{1+\delta P}\right] \qquad \text{(III.2-12)}$$

where:

$$\gamma = \frac{L}{R+L} \qquad \delta = \frac{R}{R+L}$$

Applying equation III.2-5 to (p) and (p) in turn:

$$E_1(P) = \frac{-\beta + P}{1 + \beta P} \qquad \text{(III.2-13)}$$

$$E_1^*(P) = \frac{-\gamma + P}{1 + \gamma P} \qquad \text{(III.2-14)}$$

By straightforward substitution:

$$Z_1(s) = \frac{R+L}{S} \tag{III.2-15}$$

And using equation III.2-4:

$$E_2(P) = \frac{-\beta - P\,E_1^*(P)}{1 + \beta P\,E_1^*(P)}$$

$$= \frac{p^2 + \gamma(1-\beta)p - \beta}{-\beta p^2 + \gamma(1-\beta)\,p + 1} \tag{III.2-16}$$

From which:

$$Z_2(s) = \frac{2R\,(R+L)\,s}{(R-L)\,s^2 + (R+L)} \tag{III.2-17}$$

To carry this work further leads to appreciable labor. An answer may be obtained by inspection using other techniques. Translating the s-plane axis of imaginaries to the location of the zero of the function yields a simple reactance function in the new plane and the technique may be said to have been applied. Alternately, an inductance in series with the circuit of Table III.2-1 gives an immediate solution. The above development demonstrates, however, the difficulties involved in finding an appropriate E-function family directly for even relatively simple immittance functions.

It is shown in the discussion in the section which follows that in spite of the difficulties involved in the manipulations in application of the technique described here, that the method is important for conceptual reasons and leads logically into approaches from which the usual exact synthesis techniques may be derived. Further, while the Caratheodory approach guarantees that a solution may be found it must be appreciated that, as in all synthesis techniques, the solution is not a unique one and alternate solutions for the identical problem are always possible.

SECTION III.3

THE SIGNIFICANCE & LIMITATIONS OF REACTANCE FUNCTION SYNTHESIS

Reactance function synthesis serves, for conceptual and pedagogical purposes, as the primary synthesis technique. For these needs its practical shortcomings are not particularly important. Its use depends only upon the most obvious property of the reflection coefficient - that it be unimodular bounded on the p-plane unit circle. Even the shortcomings of the method militate in favor of considering it as a starting point in synthesis for those difficulties focus attention on the need for special handling of the dissipative elements of a network if a finite number of elements realization is to result.

Brune's synthesis method has been thought of, in the past, as the basic technique since, historically, it was used to prove the sufficiency of the p.r. condition. The method, however, is basically an advanced topic and involves the recognition of special properties of network functions and treatment of those functions in a manner which always results in the appearance of unity coupled transformers. As such the method falls into a class also occupied by the techniques of Darlington and Miyata. The latter synthesis technique avoids the transformers by recognition of further special properties of the network functions. The topic is treated in detail in Section IV.3.

It appears that the E-function expansion shows promise as a basis for approximation techniques. Although approximation methods do not constitute a part of the present investigation, a little conjecture is not inappropriate. The usual approximation methods can be related to the potential analog approach[11] and are concerned with obtaining a desired result over a portion of the frequency spectrum. E-function synthesis appears to hold more possibility as a time domain than as a frequency domain approximation. That this is so, is evident when one recalls that a very long and lossless transmission line looks like an infinite line until that time when the reflection from the far end has returned.

None of the reactance functions obtained in the E-function expansion are equal to the desired function with the exception of the limiting member of the family of reactance functions. In the case of the resistance expansion illustrated in Table III.2-1, the series of alternating imaginary axis zeros and poles can, by no stretch of the imagination, be considered equivalent to a resistance. As long as the sequence of elements is finite, there will eventually be returned reflections and the input immittance assumes its proper reactive form. It is clear, however, that until the time that the return reflection does appear the possibility that the network may approximate a constant resistance does exist. Additional work beyond that presented in the preceding section would be required to establish the technique as a useful one.

There is a possibility that E-function synthesis may be applied to approximations in the frequency domain. The effect of the return reflection may be minimized by including a small amount of dissipation with each reactive element. Thus if the series of poles and zeros of Table III.2-1 are displaced slightly into the left half plane and if they are sufficient in number, it becomes reasonable that the effect along the axis of imaginaries will approximate that of a constant resistance.

Lastly, as a matter of interest, it is noted that Leo Storch in his approach to approximating a constant delay[15] essentially makes use of an E-function expansion to approximate the irrational reactive impedance $Z(s) = \tanh(s)$.

PART IV

IMPLICATIONS OF THE SCHWARZ' LEMMA

SECTION IV.1

SCHWARZ' LEMMA - A MEASURE OF DISSIPATION

The single important point to be developed in the present section is one of the implications of which may be readily seen by inspection. It is this, that the Schwarz' lemma is at least a qualitative measure of the amount of dissipation represented in a realizable reflection coefficient. The Schwarz' lemma, in p-plane terms, states that any unimodular bounded function may be written as the product of an E-function and another unimodular bounded function. Stated mathematically:

$$\rho(p) = E(p)\ \rho_1(p) \qquad \text{(IV.1-1)}$$

In Section III.1 it was seen that the E-function represents a pure reactance and maps the p-plane unit circle interior on to itself. Presuming that a combination of E(p) and $\rho_1(p)$ can be found so that both are realizable, then $\rho_1(p)$ must represent all the lossy elements of $\rho(p)$ and, further, must be responsible for the fact that the p-plane unit circle interior maps onto only a portion of itself in the $\rho(p)$ transformation.

Since E-functions are always readily realizable as reactance networks, see Theorem I of Section III.1, equation IV.1.1 evidently shows promise as an approach to synthesis. This is particularly true in view of its property of factoring out the portions of the function involving resistive elements in their realizations, since it was seen in the E-function expansion method of Sections III.2 and III.3 that these are the elements that cause the difficulty. It is shown in Section IV.2 that equation IV.1-1 applied as is, with only a few straightforward steps taken to insure the realizability of $\rho_1(p)$ leads to the Bott-Duffin and Fialkow-Gerst family of synthesis techniques. With a few further refinements, as explained in Section IV.3, the equation leads to the Foster, Cauer, Brune, and Miyata methods. The point is made finally that all known synthesis techniques, with the sole exception of the E-function expansion method, are representations of Schwarz' lemma in the form given in equation IV.1-1.

The fact that the closeness of approach of the image of the Z(s)-plane axis of imaginaries to the circumference of the unit circle in the ρ -plane is a measure of the dissipation in the network, has a direct analogy in s-plane analysis. The planes involved are shown in Figure IV.1. Evidently in the s-plane the location of the image of the Z(s)-plane, axis of imaginaries with respect to the Z(s) plane axis of imaginaries contains the same information. The advantage to the ρ -plane viewpoint is that the relationship may be expressed in a simple mathematical form as given in equation IV.1-1.

It was mentioned in the opening paragraph of this section that the only point to be made here was that the Schwarz' lemma was a measure of the dissipation in a circuit. That point has been made and discussed. The remainder of the section essentially repeats the material already presented and adds nothing new. The concept just

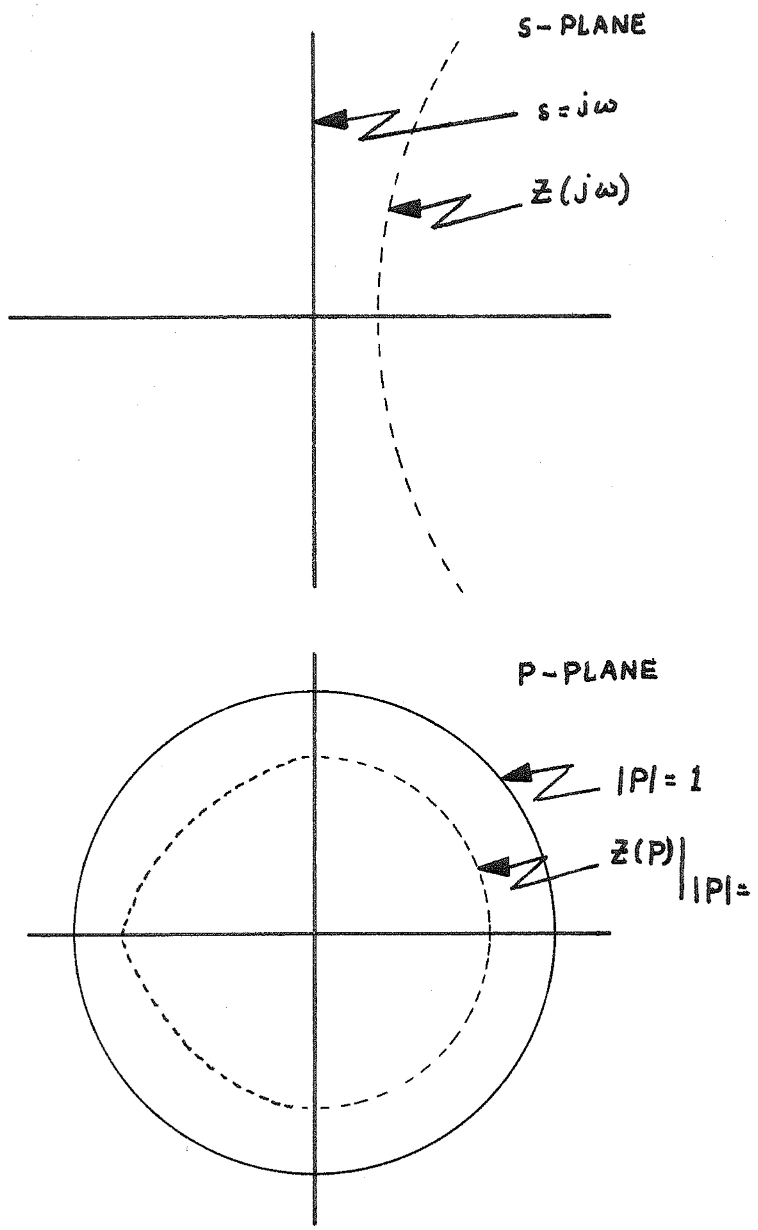


FIGURE IV.1-1

developed is more rigorously developed within the framework of non-Euclidean geometry. Since no new material is introduced and, with non-Euclidean geometry, a branch of mathematics is applied which is not in common use in network studies, some justification for the procedure is required.

As brought out in Figure IV.1-1, the concept described here has an analogy in the s-plane. It is also clear from comments made earlier that the intention is to demonstrate that the Schwarz' lemma is the cornerstone of driving point synthesis techniques. The basic work in the field of driving point synthesis would appear to be that of Brune, although admittedly, the field is characterized by regular and gradual growth resulting from the investigations of many people over an extended period. In his contribution[1], Brune explores the implications of the Schwarz' lemma and concludes from that study that, for positive real functions:

$$|\arg Z(s)| \leq |\arg s| \quad \text{when} \quad |\arg s| \leq \pi/2 \qquad \text{(IV.1-2)}$$

Evidently the foregoing expresses mathematically and in more generality the observations made here in regard to the s-plane significance of the Schwarz' lemma. Brune's results while interesting are not in as simple a form as those of the -plane as a comparison of equations IV.1-1 and IV.1-2 reveals. In addition the s-plane result expressed in IV.1-2 is not as immediately applicable to synthesis as is equation IV.1-1. This latter statement as to the applicability of equation IV.1-1 will be demonstrated in Section IV.2 and IV.3. The situation demonstrates the utility of having several viewpoints or approaches to a subject - in the present case, the s-plane and p-plane approaches. It is seldom evident from the beginning that one approach will bring out all the implications of a result that are evident in the alternate approach. In view of the importance of Brune's work and because of the interest here in contrasting the two viewpoints it seems worthwhile to parallel Brune's non-Euclidean s-plane analysis with a similar p-plane development.

A second reason for extending the analysis is a more tenuous one. In the present investigation the Schwarz' lemma is used as a basis for the development of synthesis techniques and its property of revealing the extent to which dissipation is present in a network is pointed out. No deep investigation of this property is made and no attempt is carried out to develop it on a quantitative basis. It can be conjectured that such a development of a quality factor similar to the "Q" factor commonly used for resonant circuits might yield a parameter useful in determining the nature of the network function before detailed computations are made. For example, it is pointed out in Section IV.3 that Kuh has developed criteria, based on the location in the s-plane of the zeros of the even part of an immittance function, for ascertaining the extent to which Miyata's synthesis technique may be applied. The criteria are not complete in that they cover regions of zero locations which cannot be so realized and other regions which can, but include a region in which it is not evident to which class the function belongs. Whether or not a quality factor developed along the lines indicated here would be superior to Kuh's criterion is a matter for further investigation.

A third reason for the non-Euclidean analysis is that it introduces a non-linear scale of distances which, in many applications, reflects the importance of dissipation as a function of the location of the immittance singularities in the s-plane. Thus if dealing with high-Q circuits whose zeros and poles are close to the s-plane axis of imaginaries, as, for example, with crystal filters, a small amount of dissipation may be extremely important. In other applications where the zeros and poles are well into the left half plane, the matter is usually not as critical. This topic is obviously related to the possibility of the development of a quality factor suggested above.

To re-capitulate:

The basic results of this section are presented in equation IV.1-1 and the discussion associated with it. A non-Euclidean analysis of the significance of that equation is presented for the following reasons:

1. Such an analysis parallels one presented in Brune's work and so serves to compare and contrast the s-plane and $\mathbf{P}$ -plane viewpoints.

2. It seems worthwhile recording all important characteristics of equation IV.1-1 because of its present significance to synthesis, brought out in Sections IV.2 and IV.3, and the possibility of the future development of additional uses for the equation.

3. The non-Euclidean scale of distance developed by the analysis to be presented would appear to reflect the relative importance of small amounts of dissipation in different parts of the $\mathbf{P}$ -plane.

If somewhat elaborate explanations have been made as to the reasons for a non-Euclidean analysis, the analysis is itself presented with little preamble. A discussion of the implications of non-Euclidean geometry would be too lengthy and carry the present discussion too far afield. The study began with the development by Lobatschewsky, Bolyai, and Gauss of systems of geometry for which Euclid's parallel hypothesis did not hold. In the course of this work there were developed alternate concepts of the straight line and of distance, which concepts depend on the nature of the curvature of the surface being considered. A simple well-known example is that the sphere on which great circles are the "straight lines" along which distance is measured in degrees. Several texts which have been helpful in gaining an understanding of the concepts are referenced[16,17,18].

The Caratheodory text[12] has been the most useful single source.

Pick makes a non-Euclidean interpretation of the Schwarz' lemma of equation IV.1-1 which reads as follows:

> "Any function $\rho(P)$ of bound one maps the non-Euclidean plane $|P|<1$ onto itself, or onto a part of itself, in such a way that the non-Euclidean distance of two image points under the mapping never exceeds the non-Euclidean distance between their pre-images. If these two distances are equal for even one pair of image points and the corresponding pair of original points, then the mapping must be a non-Euclidean motion that leaves all distances invariant."

The foregoing, with a trivial change in notation to conform to the present writing is quoted from Caratheodory[12]. The concept of non-Euclidean distance will be developed briefly, the proof of the statement outlined, and it will be seen that the lemma suggests means for handling dissipative elements in synthesizing networks.

The non-Euclidean plane referred to in the theorem above is the unit circle $|P|<1$ in which straight lines are defined to be the arcs of circles orthogonal to the unit circle and the distance between two points, denoted as $D(P_1\ P_2)$ is defined as:

$$D\left(P_1, P_2\right) = 2 \tanh^{-1} \left| \frac{P_1 - P_2}{1 - \bar{P}_1 P_2} \right| \qquad \text{(IV.1-3)}$$

The above concept of distance has the property that distance increases monotonically as the two points separate, equals zero when they coincide, and approaches infinity as one of the points approaches the horizon of the plane. The horizon is simply the boundary, the circumference of the unit circle.

The plane in question is illustrated in Figure IV.1-2. As a matter of interest and to emphasize the non-Euclidean aspects of the matter, in that figure two lines through a common point X and parallel to the same line AB have been constructed. The lines are parallel to AB in the sense that they meet that line only at infinity which is the circumference of the unit circle. To demonstrate this sense of infinity consider the distance of the point P_2 from the origin by letting P_1 be zero in equation IV.1-3.

$$D(0, P_2) = 2 \tanh^{-1} |P_2| \qquad \text{(IV.1-4)}$$

By the relationship[19]:

$$\tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x} \qquad \text{(IV.1-5)}$$

This may be written:

$$D(0, P_2) = \log \frac{1+|P_2|}{1-|P_2|} \qquad \text{(IV.1-6)}$$

and an examination of either form IV.1-4 or 6 reveals the rapid change in the scale of distance as P_2 moves out along a radius from the origin.

To demonstrate Pick's interpretation, consider $\rho(P)$, a unimodular bounded function which at some P_0 assumes the value ρ_0.

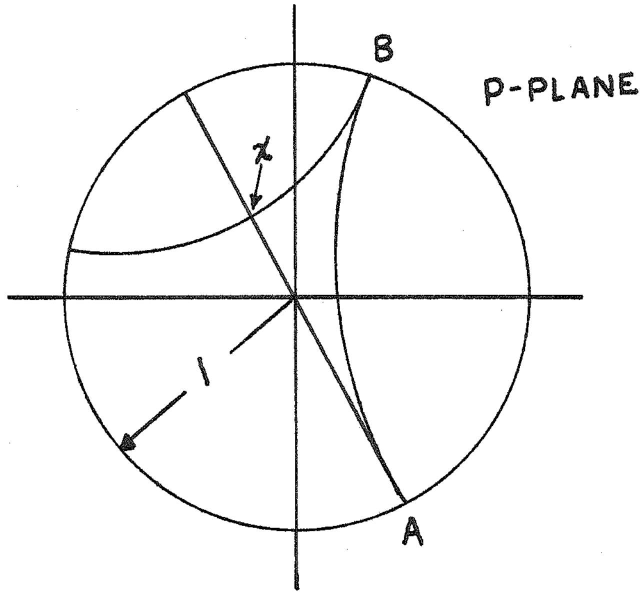


FIGURE IV.1-2

Then, if $\rho(P) \neq 1$, the expression

$$\frac{\rho_0 - \rho(P)}{1 - \bar{\rho}_0 \rho(P)} \qquad \text{(IV.1-7)}$$

is also unimodular bounded. By application of the Schwarz' lemma it may be written:

$$\frac{\rho_0 - \rho(P)}{1 - \bar{\rho}_0 \rho(P)} = \left[\frac{P_0 - P}{1 - \bar{P}_0 P}\right] g(P) \qquad \text{(IV.1-8)}$$

where the E-function on the right side of the above expression has been selected to have a zero and pole corresponding to those of the function on the left and $g(P)$ is necessarily unitary modular bounded. It follows that:

$$\left|\frac{\rho_0 - \rho(P)}{1 - \bar{\rho}_0 \rho(P)}\right| \leq \left|\frac{P_0 - P}{1 - \bar{P}_0 P}\right| \qquad \text{(IV.1-9)}$$

From equation IV.1-3:

$$\tanh\left[\tfrac{1}{2} D(\rho_0, \rho)\right] = \left|\frac{\rho_0 - \rho(P)}{1 - \bar{\rho}_0 \rho(P)}\right| \qquad \text{(IV.1-10)}$$

$$\tanh\left[\tfrac{1}{2} D(P_0, P)\right] = \left|\frac{P_0 - P}{1 - \bar{P}_0 P}\right| \qquad \text{(IV.1-11)}$$

Hence equation IV.1-9 may be written:

$$D(\rho_0, \rho) \leq D(P_0, P) \qquad \text{(IV.1-12)}$$

From the above, Pick's interpretation of Schwarz' lemma follows directly. The case for which distances are left invariant corresponds to $g(P)$ of IV.1-8 being of the form $e^{i\phi}$. Expressing these results in mathematical form, the following theorem is obtained:

Theorem II: Any reflection coefficient $\rho(p)$ maps the interior of the p-plane unit circle onto itself or onto a portion of itself. If the former is true the reflection coefficient is necessarily an E-function and hence by Theorem I represents a purely reactive immittance. If the latter is true, the extent to which the two mappings fail to overlap is a measure of the extent to which the associated immittance fails to be purely reactive.

As previously mentioned, Brune[1] carried through a treatment parallel in many respects to the foregoing to arrive at the important theorem:

> "If $Z(\lambda)$ is a positive real function $|\arg Z(\lambda)| \leq |\arg \lambda|$ for all values of λ satisfying $0 < |\arg \lambda| \leq \frac{\pi}{2}$ The equality signs can only hold simultaneously, unless they hold identically."

In his development of the above, Brune quotes the Pick interpretation of the Schwarz' lemma as:

> "If the function W of γ has no essential singularities for values of γ within the circle K_γ, and takes on values which lie only in the interior of another circle K_W, then all non-Euclidean distances, elements of arcs and arcs are shortened in the conformal mapping by $W(\gamma)$. If one such mapping remains unchanged, all remain unchanged and W is a linear function of γ."

Brune's non-Euclidean space is the right half s-plane, his "straight lines," the arcs of circles orthogonal to the s-plane axis of imaginaries. His distance measure is defined as the logarithm of the cross-ratio of the two points involved taken with the two points which are the intersection with the axis of imaginaries of a circle through the original two points and orthogonal to the axis of imaginaries. His is the Poincare' representation as compared to that used here which is sometimes referred to as the Klein representation. Obviously Brune's non-Euclidean space maps into the non-Euclidean space of Figure IV.1-2 by the definition of p:

$$p = \frac{1 - s}{1 + s} \qquad \text{(II.1-8)}$$

and his straight lines into the straight lines used here since the mapping is conformal. Finally the distance concepts may be related by application of equation IV.1-5.

SECTION IV.2

DIRECT APPLICATION OF SCHWARZ' LEMMA - RICHARDS THEOREM

To consider the synthesis possibilities of Schwarz' lemma, write equation IV.1-1:

$$\rho(P) = E(P)\,\rho_1(P) \tag{IV.1-1}$$

in the s-plane representation. It becomes:

$$Z(s) = \frac{1}{\dfrac{1}{Z_1(s)} + \dfrac{1}{\left[\dfrac{R_1^2}{Z_2(s)}\right]}} + \frac{1}{\dfrac{1}{Z_2(s)} + \dfrac{1}{\left[\dfrac{R_1^2}{Z_1(s)}\right]}} \tag{IV.2-1}$$

where:

$$\frac{Z(s)}{R_1} = \frac{1 - \rho(s)}{1 + \rho(s)}$$

$$\frac{Z_1(s)}{R_1} = \frac{1 - \rho_1(s)}{1 + \rho_1(s)}$$

$$\frac{Z_2(s)}{R_1} = \frac{1 - E(s)}{1 + E(s)}$$

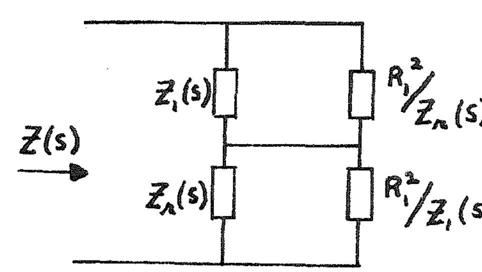


FIGURE IV.2-1

By inspection of equation IV.2-1, the balanced bridge circuit of Figure IV.2-1 is seen to be the network representation for the Schwarz' lemma. Equation IV.1-1 is useful in the general synthesis problem only if the function $Z_1(s)$ is easier to synthesize than was the original Z(s). The network is useful in design for with $Z_1 = R_1$, it becomes one of the Norton constant resistance networks[13]. To return to the synthesis problem, solve IV.1-1 for

$\rho_1(p)$ substituting the general form of E(p) from equation III.1-3:

$$\rho_1(p) = \left[\pm \prod_{\nu=1}^{n} \frac{1 - \bar{P}_\nu P}{P_\nu - P} \right] \rho(P) \qquad \text{(IV.2-2)}$$

Consider E(p) having only one factor in its numerator and denominator or at most two since complex conjugates must occur in pairs. In either case only one factor need be considered in the discussion. $\rho_1(p)$ for realizability can have no poles inside the unit circle from item c of Table II.3-1. However, $|P_\nu| < 1$. Hence the factor $(P_\nu - P)$ must be cancelled by a like factor in the numerator of $\rho(p)$ if $\rho_1(p)$ is to be realizable. Hence:

$$\rho(P_\nu) = 0 \qquad \text{(IV.2-3)}$$

Further, to insure $\rho_1(p)$ be not merely realizable but that it be less complex than $\rho(p)$, it is necessary that the factor $(1 - \bar{P}_\nu P)$ also cancel with a similar factor in the numerator of $\rho(p)$. Hence:

$$\rho(\bar{P}_\nu^{-1}) \longrightarrow \infty \qquad \text{(IV.2-4)}$$

Equations IV.2-3 and 4 require respectively that:

$$\left. \begin{aligned} Z(s_\nu) &= R_1 \\ Z(-\bar{s}_\nu) &= -R_1 \end{aligned} \right\} \qquad \text{(IV.2-5)}$$

Since s_ν is always taken with its complex conjugate, the above is equivalent to:

$$Z(s_\nu) + Z(-s_\nu) = 0 \qquad \text{(IV.2-6)}$$

Equation IV.2-6 then states as a condition that $\rho_1(P)$ be realizable and less complex than $\rho(P)$, that the E-function be constructed so that S_ν occurs at a zero of the even part of Z(s) at which Z(s) is equal to a positive real number.

The foregoing brief treatment contains the essence of Richards' Theorem[21], Bott-Duffin synthesis[22], and the Fialkow-Gerst synthesis[20] which latter work contains the two preceding contributions and generalizes on them. The development presented here depended upon three distinct aspects of the situation:

1. The Schwarz' lemma, equation IV.1-1 assured the possibility of factoring an E-function out of the reflection coefficient expression.

2. The choice of S_ν and R_1 can be made so that the remaining expression, after factoring an E-function, is a realizable reflection coefficient.

3. The product of two realizable reflection coefficients may be realized in circuit form as a balanced bridge.

It is important to recognize the separate existence of the three factors above for they may be used one or two at a time to obtain additional results. The first and third are formalized as theorems as follows:

> <u>Theorem III</u>: Realizable reflection coefficients of degree higher than the first in numerator and denominator may be written as the product of an E-function and a reflection coefficient of lower degree.

> <u>Theorem IV</u>: The product of two reflection coefficients may be realized in circuit form as a balanced bridge. The relationships involved may be obtained by inspection of equations IV.1-1 and IV.2-1 and Figure IV.2-1. It is not necessary that one of the factors in the product be an E-function as shown in the referenced example.

To complete the Fialkow-Gerst synthesis procedure one must consider the realization procedure when, at the zero of the even part, Z(s) is not a positive real number. In the case when it is a pure imaginary on the s-plane axis of imaginaries, that is:

$$Z(s)\Big|_{s=j\omega_0} = j\omega_0 L \qquad \text{(IV.2-7)}$$

which is the Bott-Duffin case, the following variation of the technique is appropriate: Write the Schwarz' lemma of equation IV.1-1 as: $\rho(s) = E(s)\,\rho_2(s)$

$$\frac{Z(k)-Z(s)}{Z(k)+Z(s)} = \frac{k-s}{k+s} \quad \frac{Z(k)-Z_2(s)}{Z(k)+Z_2(s)} \qquad \text{(IV.2-8)}$$

Then:

$$\rho_2(s) = \frac{k+s}{k-s} \quad \frac{Z(k)-Z(s)}{Z(k)+Z(s)} \tag{IV.2-9}$$

It is seen by inspection that $Z_2(s)$ is of the same degree as $Z(s)$. However, if:

$$Z(k) = Lk \tag{IV.2-10}$$

then, also by inspection $\rho_2(s) = 1$ at $s = j\omega_o$ hence $Z_2(s)$ must have a zero on the s-plane imaginary axis and so it and its dual in the bridge circuit may be easily reduced. It is well established in the literature that a k satisfying IV.2-10 may always be found. The reactance function L_s yields the E-function required in equation IV.2-9. It has already been mentioned that the Schwarz' lemma has other applications in circuit theory beyond those in synthesis. Recall that it can be used to realize constant resistance networks. In the present case the lemma was used first in equation IV.2-8 to obtain from a given immittance, another having more desirable properties.

A similar technique applies when, at the zero of the even part, $Z(s_\nu)$ is a complex quantity. Using:

$$Z_2(s) = \frac{k\,Z(k)}{s}$$

and

$$Z_3(s) = Z(k)\,Z_2(s)$$

write Schwarz' lemma in the form:

$$\rho(s) = \frac{s-k}{s+k} \quad \frac{1-Z_2(s)}{1+Z_2(s)}$$

It can be shown that a value of k can always be found for which $Z_2(s,k)$ has a zero of its even part at s_ν and is equal to a positive real constant there. Hence $Z_2(s)$ may be reduced by the method previously described.

Details and proof are presented in the Fialkow-Gerst paper[20]. Note in reading that paper that Fialkow and Gerst use Schwarz' lemma in the form given in Theorem 294 of Polya' and Szegoe[23]. This is essentially a combination of equations IV.1-1 and II.1-3 of the present work. The simpler and more commonly used form of Schwarz' lemma written in the present notation is:

$$\rho(P) = P \cdot \rho_1(P)$$

and involves only the simplest E-function. It is Polya' and Szegoe's Theorem 280.

SECTION IV.3

THE SIMPLIFIED FORM OF THE SCHWARZ' LEMMA CAUER, FOSTER, BRUNE, DARLINGTON, AND MIYATA SYNTHESIS

Basic Principle

As was pointed out in Section IV.2, the direct application of the Schwarz' lemma at a zero of the even part of the immittance function at which the function assumes a real positive value leads directly to a synthesis procedure, since at such points:

$$R_1 - Z(s_\nu) = R_1 + Z(s_\nu) = 0$$

This property, that a pole and zero of the reflection coefficient be complementary with regard to the s-plane imaginary axis, is just the requirement that they be singularities of an E-function, see Figures III.1-1 and 2. It follows, then, from Schwarz' lemma, that these terms may be immediately factored out of the expression for the reflection coefficient leaving the remaining expression simpler to realize. It follows, then, that, thru use of the bridge realization for the product of reflection coefficients, a practical synthesis technique results.

The concepts described above are easily explained in terms of the s-plane root locus plot of the zeros and poles of the reflection coefficient. Refer to Figures II.2-2a and b. The poles and zeros of $\rho(s)$ originate at the zeros of Z(s) for small R_1 and approach the poles of Z(s) as R_1 grows without bound. In the Fialkow-Gerst work[20] these loci are essentially examined for the value of R_1 at which at least one pole-zero pair assumes the required complementary relationship. Since the value of R_1 required is just that at a zero of the even part of the immittance the search of the locus is actually performed implicitly without actually drawing the locus, but merely by examining the expression for the zeros of the even part of the immittance.

There is one situation where no detailed examination is required to determine the value of R_1 and s_ν. As R_1 grows without bound, the pole-zero pairs of the reflection coefficient converge on the poles of Z(s). If some of these poles of Z(s) are located on the axis of imaginaries in the s-plane, then, in the limit as described, the pole and zero of ρ(s) converge on the axis of imaginaries and assume the required complementary relationship with regard to that axis. Conditions of positive reality prevent these root loci from approaching the imaginary axis along paths other than ones orthogonal to it.

In the special case just described, when R_1 grows without bounds and the singularities of the reflection coefficient approach s_ν, an imaginary axis pole of Z(s), the equations of Section IV.2 are modified as follows: Equation IV.2-5 becomes:

$$Z(s_\nu) = Z(-\bar{s}_\nu) \longrightarrow \infty \qquad \text{(IV.3-1)}$$

and equation IV.2-1 simplifies to:

$$Z(s) = Z_r(s) + Z_1(s) \qquad \text{(IV.3-2)}$$

Hence, the balanced bridge relationship of IV.2-1 has become simply the equation for two impedances in series or two admittances in parallel.

It is the foregoing situation which is the basis for Foster and Cauer synthesis as well as for those of Miyata, Brune, and Darlington. The methods for developing Foster and Cauer synthesis in Schwarz' lemma terms are straightforward and the required concepts are brought out in a discussion of Miyata and Brune synthesis. Accordingly, these two former methods will not be discussed explicitly, the material necessary for their understanding being contained in the discussion of the more complex Brune and Miyata techniques.

The discussion diverges for a few paragraphs to cover an essential point of mathematics. In the simplification of equation IV.2-1 to the form of IV.3-2, that is taking the balanced bridge to the simpler series network form there arises a question of defining the value of such terms as:

$$\lim_{R_1 \to \infty} \frac{R_1^2}{Z_{\mathcal{N}}(s)}$$

at points in the complex plane where $Z_{\mathcal{N}}(s)$ itself has a pole. Implicit in the foregoing treatment has been the assumption that the R_1^2 term dominates so that the above expression grows without bound in the limit. Hence the reciprocal of such terms vanishes leading to the simpler form. On the other hand, R_1 may be so large that $\rho(s)$:

$$\rho(s) = \frac{R_1 - Z(s)}{R_1 + Z(s)} \qquad \text{(II.1-1)}$$

reduces to a point mapping. It is hard to conceive that such a mapping contains any information.

The difficulty is resolved by noting that in the circuit of Figure IV.3-1 the terms approaching infinity involve R_1^2 whereas in the reflection coefficient R_1 is involved only to the first power. Further, from the discussion of Section III.2, it is clear that any realizable immittance may be considered to involve only simple poles limited in location to the axis of imaginaries. Hence, it follows that if R_1 is defined as:

$$R_1 = \lim_{S \to S_k} \frac{1}{S \longrightarrow S_k}$$

that the circuit will simplify as required, while the mapping will not be reduced to a point mapping. The situation that makes both these events possible simultaneously is the fact that in the first case ratios of the form:

$$\frac{R_1^2}{Z(s)}$$

are involved while, in the second, the ratios are of the form:

$$\frac{R_1}{Z(s)}$$

Miyata Synthesis:

The outline of Miyata synthesis which follows will be only as complete as is required for present purposes. A complete description is already available in the literature.[24] The purpose here is a re-examination of the method as a special application of the Schwarz' lemma.

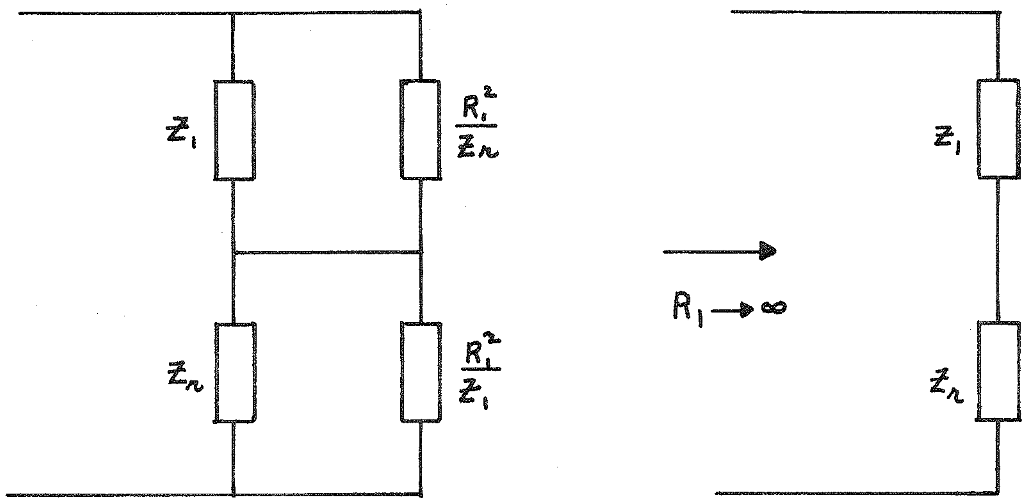


FIGURE IV.3-1

The Miyata contribution establishes the following:

a) For a function U(s) satisfying the conditions that it is an even rational function of s with real coefficients and with no poles on the imaginary axis and if $U(j\omega) \geq 0$, then there exists an impedance function Z(s) which has U(s) for its even part and which is uniquely determined when Z(s) has no poles on the imaginary axis.

b) The function Z(s) of "a" above may, in many important cases be written as a sum of separately realizable terms which have zeros of the even part at the origin or infinity.

c) Impedances with zero even part at the origin or infinity may be reduced in a repetitive manner similar to that of the Cauer process for reactive functions.

The formulas expressing the relationships between the even and odd parts of an immittance and its reciprocal are presented in Table IV.3-1. Listed in Table IV.3-2 are a few elementary deductions as to the characteristics of these functions when

Function	Form	Even Part	Odd Part
$Z(s)$	$\dfrac{N_1 + sN_2}{D_1 + sD_2}$	$\dfrac{N_1 D_1 - s^2 N_2 D_2}{D_1^2 - s^2 D_2^2}$	$\dfrac{s[N_2 D_1 - N_1 D_2]}{D_1^2 - s^2 D_2^2}$
$Y(s) = [Z(s)]^{-1}$	$\dfrac{D_1 + sD_2}{N_1 + sN_2}$	$\dfrac{N_1 D_1 - s^2 N_2 D_2}{N_1^2 - s^2 N_2^2}$	$\dfrac{s[N_1 D_2 - N_2 D_1]}{N_1^2 - s^2 N_2^2}$

TABLE IV.3-1

Z(s) has zeros of its odd and even parts at the origin. From the conclusions presented in this latter table, it is evident, as mentioned, that with a zero of the odd and even part of a p.r. function at the origin, the reciprocal function has an even part with a zero of lower order and an odd with a pole at that point.

Observation	Conclusion
$D_1 + SD_2$ is Hurwitz	D_1 & D_2 have no zeros at origin
Y(s) is p.r.	$N_1 + SN_2$ has a simple zero at origin ${N_1}^2 - S^2{N_2}^2$ has a factor S^2 ${N_1}^2$ has a factor S^2 N_2 has no zeros at the origin

TABLE IV.3-2

The reciprocal function then fits the conditions for application of the simpler form of Schwarz' lemma. The lemma is applied with $R_1 \longrightarrow \infty$ and a reactive element, a simple series inductance or parallel capacitance for the cases of impedance or immittance respectively, is removed. Miyata shows that if the original even part zero was of higher order than the first power in s, that the remainder function, in turn, has a zero even and odd part at the origin so that the process may be repeated. The repetitive process involving reciprocal functions at each new stage results in a ladder network of inductances and capacitances terminated finally in a resistance.

Kuh, in a reference which is discussed in more detail in Section V, develops the conditions on the locations of the original zeros of the even part of Z(s) in order that it may be developed into the series of additive terms which have zero even parts all at the origin or infinity. In terms of the Schwarz' lemma, the Miyata synthesis is a method whereby the simplified form of the lemma may be applied in an iterative manner.

It is worthwhile therefore to show a simple example to illustrate a single step in the use of the simplified Schwarz' lemma. The example taken is that of a simple resistor and inductance in parallel. The reflection coefficient using the form of Figure II.1-2 is:

$$\rho(s) = \frac{G_1 - \frac{R+Ls}{RLs}}{G_1 + \frac{R+Ls}{R+Ls}}$$

$$= \frac{RLs - R/G - \frac{Ls}{G}}{RLs + R/G + \frac{Ls}{G}}$$

Consider the reflection coefficients of the two elements separately

$$\rho_R(s) = \frac{1 - \frac{1}{RG_1}}{1 + \frac{1}{RG_1}} \qquad \rho_L(s) = \frac{1 - \frac{1}{GLs}}{1 + \frac{1}{GLs}}$$

and the product

$$\rho_R(s)\rho_L(s) = \frac{RLs - R/G - Ls/G + R/G^2}{RLs + R/G + Ls/G + R/G^2}$$

so that

$$\rho(s) = \rho_L(s)\,\rho_R(s)$$

in the limit as $G_1 \longrightarrow \infty$ to first order terms in G_1.

To re-iterate, Miyata synthesis is made up of two distinct features. The first part is the method of decomposing Z(s) into additive terms which have even part zeros at the s-plane origin or infinity. That technique has not been discussed here. The second contribution of Miyata is recognition of the fact that simple reactive elements can be removed in successive steps from an immittance function when it has even part zeros at the origin. In terms of the present discussion, Miyata recognized that the simple form of Schwarz' lemma of equation IV.3-2 may be applied on an iterative basis to the additive terms of Z(s) described above. The treatment when the zeros of the even part are at infinity is related to the treatment described here by the usual $^1/S$ mapping.

An example of Miyata synthesis applied to one of the additive terms of an immittance function is given in Figure IV.3-1. The material just discussed is by no means a complete analysis of Miyata synthesis. As mentioned the goal has been to demonstrate the method as an application of the Schwarz' lemma. Some of the limitations of the Miyata technique are discussed in a critique presented in Part V. For more details see the referenced article.[24]

Function	Even Part	Odd Part	Program
$Z_1 = \frac{5s^3+16s^2+8s}{s^3+3s^2+8s+4}$	$\frac{-5s^6}{-s^6-7s^4-40s^2+16}$	$\frac{s[-s^4-84s^2+40]}{-s^6-7s^4-40s^2+16}$	This is one term of a Miyata expansion of a prescribed immittance function. Has Zero of even and odd part at origin. Invert.
$Y = 1/Z_1$	$\frac{-5s^6}{-25s^6+76s^4-64s^2}$	$\frac{s[s^4+84s^2-32]}{-25s^6+76s^4-64s^2}$	Zero of even part, pole of odd part at origin. Remove pole by applying simple form of Schwarz' lemma (in concept practically speaking remove pole in usual way by parallel inductor.)
$Y_2 = \frac{2s^2+s}{10s^2+32s+16}$	$\frac{20s^4}{-924s^4+320s^2+256}$	$\frac{s[-22s^2+16]}{-924s^4+320s^2+256}$	Zero of even and odd part at origin. Invert.
$Z_2 = 1/Y_2$	$\frac{20s^4}{4s^4-s^2}$	$\frac{s[54s^2+16]}{4s^4-s^2}$	Zero of even part, pole of odd part at origin. Same as Y_1. Remove pole by series capacitor.
$Z_3 = 5+10s$	5	$10s$	Realize by inspection.

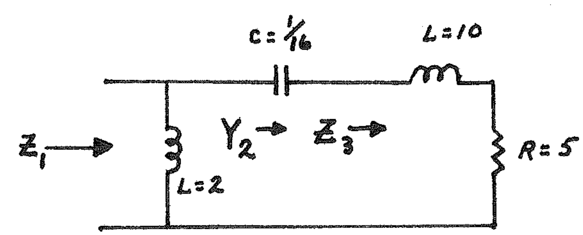


FIGURE IV.3-1

Brune Synthesis

Brune synthesis represents a peculiar extension of the simplified Schwarz' lemma form of equation IV.3-1 and 2. A necessary preamble to it is the creation of a zero even part on the s-plane axis of imaginaries by resistance minimization which need not be discussed. Further, for simplicity, the situation will be considered where the initial reactive element removed is a series inductance creating a zero of impedance of the s-plane axis of imaginaries. Once this zero has been created, the simplified Schwarz' lemma may be applied, in the usual Foster procedure, to remove a resonant section. Brune's contribution was the step required to make the Foster procedure possible.

Brune's thought processes, in terms of the Schwarz' lemma, might be described as follows: The Schwarz' lemma is written:

$$\rho(P) = E(P)\,\rho_1(P) \qquad \text{(IV.1-1)}$$

The success of other synthesis methods depended on selection of a unit function such that the remaining $\rho_1(P)$ was realizable and simpler in some way than was the original $\rho(P)$. Brune notes, however, that it is not really necessary that $\rho_1(P)$ be realizable by itself, since the realizability of $\rho(P)$ guarantees the realizability of the product. Further, referring to item 2(a) of Table II.2-2, he notes that the behavior of Z(s) at infinity can only be that of a simple pole, a simple zero, or a constant. In the immittances with which

Brune is concerned the behavior at infinity must be that of a constant. This is due to the Foster preamble which removed the other two possibilities. In this case if, by use of the Schwarz' lemma, a pole at infinity - which isn't there - is factored out, the $\rho_1(P)$ term in equation IV.1-1 above is necessarily unrealizable. The manner in which it is unrealizable, however, affords hope for a practical synthesis technique.

The pole or zero behavior at infinity is primarily due to the presence of reactive elements in the network. Factoring out a non-existent unit function thus created a non-existent reactive element and, in addition, created in $\rho_1(P)$ a reactive element of the opposite sign. Essentially Brune wrote the Schwarz' lemma as:

$$\rho(s) = \left[\frac{R_1 - Ls}{R_1 + Ls}\right] \left\{ \left[\frac{R_1 + Ls}{R_1 - Ls}\right] \rho(s) \right\} \qquad \text{(IV.3-3)}$$

$$= E(s) \cdot \rho_1(s)$$

The second reflection coefficient involving inductance is recognized as that of a negative inductance. Brune then essentially realized that he would have a configuration of reactive elements some of which would not be realizable individually but that collectively they were realizable. As is well known, the gain made by this procedure is involved in the fact that the inductance selected to be removed is of such a value that a zero of the impedance is created on the axis of imaginaries making a Foster reduction possible. Further, the fact that $\rho_1(s)$ is not realizable but the product $E(s)\,\rho_1(s)$ is, results in a circuit where the elements representing these two terms are combined in a unity coupled transformer.

As a matter of interest an example of Brune synthesis taken from the Tuttle text[9] has been worked in Figure IV.3-2. The Foster part of the procedure was carried through in terms of the simplified form of Schwarz' lemma. Since admittances are involved G_1 rather than R_1 is used in the reflection coefficient expression and as G_1 grows without bounds, in accordance with equations IV.3-1 and 2, its reciprocal is used and higher order terms of the reciprocal are discarded. The procedure while illustrative of the validity of the concept is obviously cumbersome compared to the usual manipulations in the Z(s) plane.

As is pointed out in Tuttle's text, the Brune process may be extended to the case of zeros of the even part on the s-plane real frequency axis. The same procedure is carried out as previously described, the non-existent pole at infinity is factored out leaving the same situation that led to the unity coupled transformer before. In this case of zeros on the real frequency axis, an RC or RL network is removed rather than the resonant network of the preceding situation.

Further, Tuttle points out that the extension of Brune's method to apply to zeros of the even part located generally in the s-plane, that is located neither on the real nor the imaginary axis, is identical with Darlington's synthesis procedure. The Darlington procedure is most simply explained in terms of four-terminal rather than two-terminal synthesis. Developing it as a special case of Brune's procedure is a complicated procedure useful principally for its conceptual importance.

<table>
<tr><th>Immittance</th><th>Characteristic & Action</th></tr>
<tr><td>$$Z_a = \frac{s^2 + 3.6s + 1.6}{s^2 + s + 10}$$</td><td>$$Ev[Z_a] = \frac{(s^2+4)^2}{s^4 + 19s^2 + 100}$$
Even part zeros at s = $\pm$ j2
Remove series inductance
$$Ls = Z(s)\Big|_{s=j2}$$
$$Z_b = Z_a - Ls \qquad L = \tfrac{1}{2} \text{ henry}$$</td></tr>
<tr><td>$$Z_b = \frac{-0.6s^3 + 0.4s^2 - 2.4s + 1.6}{s^2 + s + 10}$$</td><td>$$Even\,[Z_1]_{s=j2} = odd\,[Z_1]_{s=j2} = 0$$
Invert & apply simplified Schwarz' lemma at s = j2</td></tr>
<tr><td colspan="2">$$\rho_{Y_b}(s) = \frac{G_1 - Y_b(s)}{G_1 + Y_b(s)}$$
$$= \frac{-0.6G_1 s^3 + [0.4G_1 - 1]s^2 - [2.4G_1 + 1]s + [1.6G_1 - 10]}{-0.6G_1 s^3 + [0.4G_1 + 1]s^2 - [2.4G_1 - 1]s + [1.6G_1 + 10]}$$
$$= \frac{[s^2 - 2.5G_1^{-1}s + 4]\,[0.6s + 2.5G_1^{-1} - 0.4]}{[s^2 + 2.5G_1^{-1}s + 4]\,[0.6s - 2.5G_1^{-1} - 0.4]} \quad \text{(see text)}$$
$$= \rho_r(s) \quad \rho_1(s)$$</td></tr>
<tr><td colspan="2">$$Y_r(s) = G_1 \cdot \frac{1 - \rho_r(s)}{1 + \rho_r(s)} = \frac{\tfrac{5}{2}s}{s^2 + 4} \quad \text{(realize by inspection)}$$
$$Y_1(s) = G_1 \cdot \frac{1 - \rho_1(s)}{1 + \rho_1(s)} = \frac{-\tfrac{25}{6}}{s - \tfrac{2}{3}} \quad \text{(realize by inspection)}$$</td></tr>
</table>

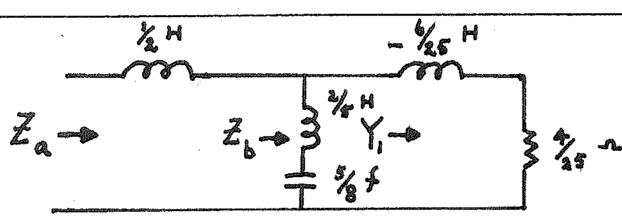


FIGURE IV.3-2

The Simplified Schwarz' Lemma - Summary

It has been pointed out that Foster and Cauer synthesis, the removal of resonant section in series or parallel or in a ladder network, may be looked upon as special cases of Miyata synthesis and that all three techniques are simply applications of the simplified form of Schwarz' lemma. The significant feature of Miyata synthesis is that, by virtue of being performed at the origin where the odd part of the immittance necessarily has a zero or a pole, the process is iterative and may be continued until the immittance is reduced to a simple resistance.

To create an iterative procedure for zeros of the even part that occur on the axis of imaginaries but not at the origin, requires use of the Brune procedure with a resistance minimization taking place between each Brune realization. As a purely intellectual achievement, Brune synthesis is remarkable in that in the process the individual elements are not required to be independently realizable, the mathematical manipulations being so contrived that the sub-circuit as a unit is realizable. Just as Foster and Cauer synthesis may be extended to zeros of the even part not on the s-plane axis of imaginaries by positive real mappings, so Brune synthesis may be extended in the same way. In particular, Darlington synthesis is a special case of the Brune procedure.

Many of the important results mentioned in the present section have been adequately treated in the literature and hence have merely been mentioned without the proofs being repeated. The emphasis has been on relating these topics to the simplified form of the Schwarz' lemma.

PART V

SUMMATION AND CONCLUSION

PART V

SUMMATION AND CONCLUSION

Summary

It was proposed at the beginning of this work to re-examine the developments in two-terminal network synthesis beginning with Brune's work to see whether those developments, when written in reflection coefficient terms could not be expressed in terms of a single unifying approach.

Upon investigating the properties of the reflection coefficient it was found that its most prominent feature, for realizable networks, is the unimodular bounded characteristic. This corresponds generally to the positive real concept for immittance functions which derives directly from the linear, passive, causality conditions.[25] Meixner[26,27,28] develops both concepts for thermodynamic systems for small departure from linearity and shows that the causality condition can be replaced by a statement of the second law of thermodynamics. The references serve to indicate the fundamental nature of both approaches to the subject.

Using the reflection coefficient, a basic synthesis method, E-function synthesis, was developed by direct application of some of the mathematical properties of unimodular bounded functions. E-function synthesis realizes a driving point immittance as a circuit made up of an infinite number of purely reactive components. Since it is the lossy elements which lead to the infinite number of elements in the circuit, it was clear that a method was necessary for handling dissipative elements. The Schwarz' lemma was shown to be a measure of the dissipation represented in a driving point function and proved to be the desired tool.

Finally, it was shown or indicated that all the various techniques for the exact synthesis of driving point immittance functions may be demonstrated to be special cases of the Schwarz' lemma. The resulted are tabulated in Table V-1.

In view of the fact that the two concepts are equally fundamental it can be expected, in general, that results obtained by use of the impedance concept may be duplicated by use of the reflection coefficient concept and vice versa. It does not follow, however, that parallel operations can be carried out with equal ease or yield equally worthwhile results in both frameworks. The present work has several examples of this statement. In Section IV.1 the usefulness of Pick's interpretation of Schwarz' lemma is apparent when the results are expressed in p-plane form; in the s-plane the results are merely a curious and somewhat awkward relationship between the angle of an immittance function and the angle of the independent variable in a restricted domain. In Section IV.2, it is seen that Fialkow and Gerst were able to generalize the Richards and Bott-Duffin work by reference to the p-plane. It would be awkward, but not impossible, to relate Cauer, Foster, Miyata, Brune, and Darlington synthesis as is done in Section IV.3 without use of the reflection coefficient. Indeed, the basic problem of synthesizing a rational function in a network consisting of a finite number of elements involves

Method	Relationship	Reference
Fialkow-Gerst	Direct Schwarz' Lemma	Section IV.2
Bott-Duffin	Special Case of Fialkow-Gerst	Section IV.2
Miyata	Simplified Form of Schwarz' Lemma Applied at Origin	Section IV.3
Foster	Simplified Form of Schwarz' Lemma Applied on Imag. Axis	Section IV.3
Cauer	Simplified Form of Schwarz' Lemma Applied on Imag. Axis	Section IV.3
Brune	Forced Foster Form by Artificial Form of Schwarz' Lemma	Equation IV.3-3
Darlington	Special Case of Brune	Not Discussed (See Ref. 9)

TABLE V-1

the method of handling dissipative elements and this feature became obvious early through use of reflection coefficients as was discussed in Sections III.2 and III.3.

If it is characteristic of the reflection coefficient notation that it presents the basic concepts clearly and hence acts as a guide in synthesis, it is also obvious that the immittance notation is the more convenient one in which to carry out the actual manipulations. There is simply no point in becoming involved in multiple mappings of complicated functions when such steps can be avoided. Even the simple removal of a resonant section becomes a very complicated procedure when it is required that the operation be carried out in terms of the reflection coefficient as is illustrated in Figure IV.3-2.

The Competitive Aspects of Synthesis Techniques:

In seeking to determine which synthesis technique to use for a particular problem, the fact that all practical techniques are just applications of the Schwarz' lemma may be used in the decision making process. That is, the basic topology of the network and the type of elements involved is determined by the choice of synthesis method. Thus, the direct application of the Schwarz' lemma in the Richards, Bott-Duffin, Fialkow-Gerst type procedure of Section IV.1 leads inevitably to the balanced bridge configuration. Similarly, the Brune procedure involves the use of unity coupled transformers.

There is a general feeling, and not an unjustified one, that the Miyata procedure is superior for general synthesis purposes. It avoids the large number of elements and the balanced bridge and its inherent sensitivity to variation in element value which result from the direct application of Schwarz' lemma. Although the Brune method uses the simplified Schwarz' lemma and so uses fewer elements it involves transformers which are less desirable than are the other passive elements.

Kuh, in a critique of Miyata synthesis,[29] shows that zeros of the even part which occur on the axis of imaginaries in the s-plane cannot be realized by the Miyata technique and that such singularities located in the sector:

$$\left| \tan^{-1} \frac{\operatorname{Im} Z(s)}{\operatorname{Re} Z(s)} \right| \leq \pi/4 \qquad \text{(V-1)}$$

may always be so realized. He then states the following:

> "The basic concept behind this technique is to realize that the minimum resistance, or looking at it another way, the zeros of the real part in the region described by equation V-1 represent the resources of a given immittance function. Advantage should be taken of the resources in the realization. The reason that Brune had close couplings and Bott-Duffin required a large number of elements is simply that they removed the minimum resistances to start with in each cycle."

Kuh's comments quoted above - and they have been paraphrased slightly to fit into the present text - are important in that they direct attention to the importance of the location of the even part zeros in synthesis. However, in the light of the developments presented here those remarks require extension. Brune and Bott-

Duffin synthesis have the defects mentioned by Kuh because those defects are inherent in the way the Schwarz' lemma is applied. They exist regardless of whether a resistance minimization has been performed or not and, indeed, since these techniques may be applied at even part zeros which are not on the axis of imaginaries, resistance minimization is not required.

The writer is not inclined to consider one synthesis technique as inherently superior to another. The implications of such a position are that one network topology is superior to another. If, as may be expected, an important future application of network theory is to be in the development of models of physical processes, then the model to be most useful must approximate the process as closely as possible. Meixner's fundamental approach already discussed indicates the reasons for the usefulness of electrical networks in such applications. Obviously it cannot be stated categorically that in all such applications one topology will always be superior to others.

There are many linear approximations to the transistor, for example, based simply on application of the g, h, y, z, or chain matrices to a four-terminal black box. Linvill[30], however, has developed a network model which promises to be truly useful in that the passive elements each represent an actual independent physical phenomenon. The model as used by Carver Mead[31] has proven to be capable of easily handling phenomena which were awkward to handle with previous models. In cases of this type, transformers or balanced bridge networks might be most appropriate to the problem at hand. In the case of models of nerve networks, for example, one might desire to use transformers as most closely approximating the actual process.

The Case for New Synthesis Procedures:

The statement can be made that all the practical synthesis techniques are merely applications of the Schwarz' lemma. It is probably true that the same situation will hold for any new techniques to be developed - although predicting the future is always a dangerous pastime. Accepting this restriction as reasonable, however, it would appear that there are two main sources of new techniques. The first involves broadening and extending the present procedures as was done, for example, by Fialkow and Gerst in connection with Bott-Duffin synthesis. The second involves going back to fundamentals to develop completely new approaches.

One of the most prolific sources of new techniques derived by extending known methods is through use of multiple mappings based on the property of realizable immittances given in item d of Table II.3-1 and its reflection coefficient analog. The property is that of $Z_1(s)$ and $Z_2(s)$ are realiable then $Z_1\left[Z_2(s)\right]$ is also realiable. Using essentially this property Reza[32] has very ably extended Foster synthesis to cover a wide class of special cases. Presumably the same thing could be done with the balanced bridge technique so that, for example, a single balanced structure would result rather than the bridges within bridges that are realized through formal iteration of the process. Somewhat similar results may possibly obtain in regard to Brune synthesis and it should be interesting to observe how complex networks, one of which is separately not realizable, combine to give a realizable configuration.

These types of synthesis techniques are more a matter of manipulation than of fundamental investigation. They have the characteristics that one has some assurance of success beforehand; the work required to accomplish the task is sufficient to establish a level of competence and the results are often useful and interesting.

A more worthwhile approach, however, would appear to be one which would develop methods for examining the zeros and poles of an immittance function to determine when such mappings should be applied. Guillemin in his synthesis text develops criteria for determining when complex poles may be removed in a Foster-like procedure. It is probable that the last word has not been said on this subject and that the topic could be placed on a more convenient basis if developed in terms of the various root loci discussed in Section II.3. Were it planned to extend the present paper, this topic would be the next topic for investigation.

The Brune use of the Schwarz' lemma, where the realizability condition is no longer applied to the individual factors in the product, is very stimulating to the imagination. It implies a useful approach - the factoring out of any expression whatsoever in order to so shape the second term in the product thatit will fall into a convenient class for realization. Since the product itself is realizable it is presumed that the circuit represented by the product is realizable as a whole. Such an approach might be applied to the synthesis method of Section IV.1 to extend the possible variations of the balanced bridge realization.

The foregoing are possibilities suggested by the theme of this investigation. All of them obviously are subject to further study. The test of the utility of a basic concept is that it does suggest new approaches to a subject. These foregoing topics are offered as evidence that the Schwarz' lemma unifying principle does meet that criterion.

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