STUDIES IN ACOUSTIC PULSE PROPAGATION

Thesis by

Terry Warren Spencer

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1956

A C K N O W L E D G M E N T S

Professor C. H. Dix has been a constant source of encouragement since the inception of this research and throughout my graduate work. I wish to thank Dr. Dix for introducing me to the work of Cagniard originally, and for the great amount of time which he has devoted to private conversations. These conversations have always been stimulating and have quite often led to fruitful study.

I am deeply indebted to the Lane Wells Company for the opportunities which their Fellowship has provided and to Dr. Dix for recommending that I be the recipient of this Fellowship. Without the financial benefits which this Fellowship has provided it would have been impossible for me to pursue work leading to the advanced degree. I have enjoyed the opportunity to meet and become acquainted with many of the officers of the Lane Wells organization, and to gain some insight into the problems which are encountered in well-logging.

I wish to thank my wife for the understanding and encouragement which she has never ceased to give, and for the many hours which she has spent in typing and checking this manuscript.

A B S T R A C T

A theoretical study is made of the transient response in two acoustic systems. Each system consists of an ideal fluid in contact with an elastic solid. In one case the interface is plane, in the other case it is cylindrical.

In the plane case it is found that an exact algebraic solution can be obtained on the axis of symmetry. The vertical displacement at axial points is composed of the acoustic, afterflow, and correction terms. In solids for which Poisson's ratio is greater than one third the initial variation of the correction is toward positive values (corresponding to motion directed toward the interface). In solids for which Poisson's ratio is less than one third the initial variation may be either positive or negative depending on the magnitude of the compressional velocity ratio. An interface wave is shown to exist regardless of the choice of elastic parameters. It is found that the reflected wave has a forerunner in the region of the fluid in which the refracted wave is the first arrival.

In the cylindrical case the initial pulse shape is distorted upon reflection. It is found that as the wave approaches the axis of the cylinder the leading edge steepens. If, at the source, the initial slope of the pressure-time curve is finite the amount of steepening is infinite. An exact expression for the transient response at points off the axis is obtained which can be evaluated by numerical means.

INTRODUCTION

The primary object of this research is to predict, on the basis of theoretical considerations, the transient behavior of two acoustic systems. Each system consists of an ideal fluid in contact with a perfectly elastic, isotropic solid. In one case the interface is plane, in the other it is cylindrical.

In many problems of this type the main source of difficulty arises in the evaluation of the formal integral representations for the response. The method to be used in any given case will therefore depend to a large extent upon one's ability to evaluate the integrals in the region of space where the response is required. At the present time it appears that, in the treatment of problems dealing with plane boundaries, the operational approach possesses certain advantages over the steady state approach in those cases where the response is required in the vicinity of the axis of symmetry. The operational approach in the form used here was introduced by L. Cagniard (1) in connection with his study of the buried point source problem.

A number of simplifying assumptions are made in deriving the usual form of the wave equation from the fundamental equations of hydrodynamics. These assumptions must be taken into consideration in interpreting the results of this investigation. The derivation of the equations of continuity and motion for an ideal fluid may be based upon the assumption that it is possible to follow a fluid element in its motion through space. This fluid element is enclosed within a hypothetical surface which permanently separates the fluid on the outside from the fluid on the inside. Although the shape of this surface and the volume enclosed by it change continuously, none of the initially enclosed mass ever crosses it. Implicit in this assumption is the requirement that the element contain sufficient mass to warrant our treating it in the same manner as matter in bulk. In order for such a formulation to be useful it is necessary to assume that the spatial derivatives of such quantities as the pressure and the velocity components exist and are finite at all points within the fluid at all instants of time. In an ideal fluid the equations of continuity and motion take the form

$$(1) \quad \frac{\partial \sigma}{\partial t} + \nabla\cdot(\sigma \vec{V}) = 0$$

and

$$(2) \quad \frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V} = \vec{F} - \frac{1}{\sigma}\nabla P \ ;$$

where σ is the fluid density; $\vec{V}$, the vector velocity; P, the total pressure; and $\vec{F}$, the force exerted on unit mass by any external fields which may be present.

We now consider the assumptions which must be made in deriving a wave equation from equations 1 and 2. If the vector velocity vanishes in the time interval preceding the introduction of acoustic energy, equations 1 and 2 reduce to the form

$$(3) \quad \frac{\partial \sigma_0}{\partial t} = 0$$

and

$$(4) \quad \sigma_0 \vec{F} = \nabla P_0 ,$$

where the subscript indicates that the density and pressure are to be determined in the undisturbed system. In the problems to be investigated it is assumed that the only external force acting is the force of gravity. The initial density distribution, σ_0, will, in general, reflect the influence of a number of factors acting simultaneously. For example, temperature and salinity as well as gravity exert an important influence on the density variation in the ocean.

In order to describe the propagation of sound in such a medium it is convenient to introduce the notation

$$\begin{aligned} \sigma &= \sigma_0 + \delta \\ P &= P_0 + P_\delta \,, \end{aligned} \tag{5}$$

where δ is the change in density and P_δ is the change in pressure caused by the disturbance. The equation of continuity may then be expressed in the form

$$\frac{d\delta}{dt} + \vec{V}\cdot\nabla\sigma_0 + \sigma_0\nabla\cdot\vec{V} + \vec{V}\cdot\nabla\delta + \delta\nabla\cdot\vec{V} = 0 \,. \tag{6}$$

It is interesting to note that, if the amplitude of the sound wave is finite, the terms which contain $\nabla\sigma_0$ and $\nabla\delta$ can be made to approach each other in magnitude by increasing the frequency of the vibration. In order to obtain the first order sound theory it is necessary to assume that the displacement amplitude is vanishingly small; then equations 1 and 2 reduce to the form

$$\frac{d\delta}{dt} + \vec{V}\cdot\nabla\sigma_0 + \sigma_0\nabla\cdot\vec{V} = 0 \tag{7}$$

and

$$\sigma_0\frac{d\vec{V}}{dt} = \vec{F}\delta - \nabla P_\delta \,. \tag{8}$$

A solution to this system of equations can be obtained if it is assumed that there exists a unique relationship between the pressure and the density within each fluid element. On the basis of this assumption it is possible to define an effective bulk modulus, K, by means of the relation

(9) $\frac{\partial P}{\partial t}+\vec{V}\cdot\nabla P=\frac{K}{\sigma}\left\{\frac{\partial\sigma}{\partial t}+\vec{V}\cdot\nabla\sigma\right\}.$

In general, K will depend not only on position but also on time. Equation 9 expresses the fact that, within the fluid element, the time rate of change of the density is related to the time rate of change of the pressure by the proportionality factor K/σ. In the limiting case, when the oscillations about the equilibrium position are small, K can be replaced by the adiabatic bulk modulus, K_o, and equation 9 can be reduced to the form

(10) $\frac{\partial P_\delta}{\partial t}+\vec{V}\cdot\nabla P_o=\frac{K_o}{\sigma_o}\left\{\frac{\partial\delta}{\partial t}+\vec{V}\cdot\nabla\sigma_o\right\}.$

P. G. Bergmann (2) has taken equations 7, 8, and 10 as the basic equations of the first order sound theory and has derived from them a linear, second order, partial differential equation for the differential pressure associated with a periodic disturbance. If the system has been completely specified this equation can be used to determine which factors can be neglected.

In the discussion which follows we will restrict our attention to systems which can be characterized by a velocity potential*; that is, to systems which are in adiabatic equilibrium. Equation 10 can be rewritten in the form

(11) $\frac{1}{K_o}\frac{\partial P_\delta}{\partial t}-\frac{1}{\sigma_o}\frac{\partial\delta}{\partial t}=\vec{V}\cdot\vec{G},$

where

(12) $\vec{G}=\frac{\nabla\sigma_o}{\sigma_o}-\frac{\nabla P_o}{K_o}.$

In the cases to be considered the vector $\vec{G}$ vanishes. This is a consequence of the fact that the magnitude of $\vec{G}$ is propor-

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* Sufficient conditions for the existence of a velocity potential are discussed in Lamb, reference (3), pages 17 and 18.

tional to the difference between the existing density gradient and that which would be present if the system were in a state of adiabatic equilibrium.

A partial differential equation in $\vec{V}$ can be obtained by differentiating the equation of motion, equation 8, with respect to time and by using equations 7 and 11 to eliminate $\frac{\partial \delta}{\partial t}$ and $\frac{\partial P_\delta}{\partial t}$. The resulting expression for $\vec{V}$ has the form

(13) $$\sigma_0 \frac{\partial^2 \vec{V}}{\partial t^2} = -\vec{F}\left\{\vec{V}\cdot\nabla\sigma_0 + \sigma_0\nabla\cdot\vec{V}\right\} + \nabla\left\{\frac{K_0}{\sigma_0}\left[\vec{V}\cdot\nabla\sigma_0 + \sigma_0\nabla\cdot\vec{V}\right]\right\}.$$

The introduction of a velocity potential Φ, defined by the relation

(14) $$\vec{V} = \nabla\Phi ,$$

reduces equation 13 to the form

(15) $$\nabla\left\{\sigma_0\frac{\partial^2\Phi}{\partial t^2} - \frac{K_0}{\sigma_0}\left[\nabla\Phi\cdot\nabla\sigma_0 + \sigma_0\nabla^2\Phi\right]\right\} = -\vec{F}\left\{\nabla\Phi\cdot\nabla\sigma_0 + \sigma_0\nabla^2\Phi\right\} + \frac{\partial^2\Phi}{\partial t^2}\nabla\sigma_0 .$$

This relation may be further simplified if we note that the vanishing of the vector $\vec{G}$ implies that

(16) $$\frac{\nabla\sigma_0}{\sigma_0} = \frac{\nabla P_0}{K_0} = \frac{\sigma_0\vec{F}}{K_0}.$$

Equation 15 can now be rewritten in the form

(17) $$\nabla\left\{\sigma_0\frac{\partial^2\Phi}{\partial t^2} - \frac{K_0}{\sigma_0}\left[\nabla\Phi\cdot\nabla\sigma_0 + \sigma_0\nabla^2\Phi\right]\right\} = (\nabla\sigma_0)\left\{\frac{\partial^2\Phi}{\partial t^2} - \frac{K_0}{\sigma_0^2}\nabla\Phi\cdot\nabla\sigma_0 - \frac{K_0}{\sigma_0}\nabla^2\Phi\right\}.$$

This relation indicates that

(18) $$\sigma_0\nabla\left\{\frac{\partial^2\Phi}{\partial t^2} - \frac{K_0}{\sigma_0^2}\left[\nabla\Phi\cdot\nabla\sigma_0 + \sigma_0\nabla^2\Phi\right]\right\} = 0 .$$

The quantity in parenthesis in equation 18 must vanish identically. It is clear that this quantity cannot be a function of the space coordinates—neither can it be a constant nor a function of time, for if this were the case the resulting expression would imply that Φ is variable in regions of space which have not been reached by the disturbance. The equation

(19) $$\frac{\partial^2\Phi}{\partial t^2} - \frac{K_0}{\sigma_0}\nabla^2\Phi - \frac{K_0}{\sigma_0^2}\nabla\Phi\cdot\nabla\sigma_0 = 0$$

describes the propagation of small amplitude sound waves in an ideal fluid which is in adiabatic equilibrium and which is acted on by an external, conservative force field. If the direction of the external force is constant throughout the fluid and the coordinate system is chosen in such a way that the direction of increasing Z coincides with the direction of $\vec{F}$, the equation for Φ can be reduced to the form

(20) $$\frac{\partial^2 \Phi}{\partial t^2} - \frac{K_0}{\sigma_0} \nabla^2 \Phi - F_z \frac{\partial \Phi}{\partial Z} = 0.$$

The equation which connects the differential pressure and the velocity potential can be found by introducing the expression for $\frac{\partial \delta}{\partial t}$, from the equation of continuity, in equation 11 and by replacing $\vec{V}$ in the resulting expression by $\nabla \Phi$; then

(21) $$\frac{\partial P_\delta}{\partial t} = -\frac{K_0}{\sigma_0} \nabla \Phi \cdot \nabla \sigma_0 - K_0 \nabla^2 \Phi.$$

It is evident that

(22) $$P_\delta = -\sigma_0 \frac{\partial \Phi}{\partial t}.$$

The analytical results are most easily interpreted if we focus our attention on the displacement field, $\vec{u}$. One of the chief consequences of our previous assumption concerning the smallness of the motion is that the velocity and displacement fields are simply related by the expression

(23) $$\vec{V} = \frac{\partial \vec{u}}{\partial t}.$$

This relation implies the existence of a scalar displacement potential, φ, which satisfies the equation

(24) $$\vec{u} = \nabla \varphi.$$

The fact that $\vec{u}$ vanishes in the region of the fluid which has not been reached by the disturbance indicates that φ must be independent of the spatial coordinates in that region. It is important to note that φ is not necessarily independent of time in the undisturbed region. The velocity potential may be expressed in terms of φ as follows:

(25) $\Phi = \frac{\partial \varphi}{\partial t} + f(t)$.

Throughout the subsequent discussion it will be assumed that the arbitrary function of time, $f(t)$, may be set equal to zero; then the expression for the differential pressure becomes

(26) $P_\delta = -\sigma_0 \frac{\partial^2 \varphi}{\partial t^2}$.

It is clear that if $f(t) \equiv 0$ the displacement potential must satisfy equation 19, which can now be rewritten in the form

(27) $\frac{\partial^2 \varphi}{\partial t^2} - V_L^2 \nabla^2 \varphi - g \frac{\partial \varphi}{\partial z} = 0$,

where $V_L = (K_0/\sigma_0)^{1/2}$ and $\vec{F} = g\vec{k}$. The significance of the individual terms in equation 27 is made apparent if this expression is first multiplied by $-\sigma_0$. The first term is then just the differential pressure acting at a point and the third term is the change in hydrostatic pressure associated with a given vertical displacement from that point. In a system of this type it is expected that the displacement amplitude will decrease as the frequency of the pressure oscillation is increased. For frequencies which are not too low the direct effect of gravity, expressed by the third term in equation 27, becomes negligible. The ultimate justification for neglecting this term can be obtained only after the solution is introduced into equation 27 and the relative importance of the individual terms ascertained.

Gravity also exerts an indirect influence because of the fact that it causes the density and incompressibility to be depth dependent—that is, it makes the medium dispersive. In our investigations it will be assumed that within the confines of the system this variation is so slight that it may be neglected. In this special case equation 27 reduces to the form

$$\frac{d^2\varphi}{dt^2} - V_L^2 \nabla^2 \varphi = 0. \quad (28)$$

Equations 24, 26, and 28 will be taken as the starting point in all of the subsequent discussion.

PART I

REFLECTION OF AN ACOUSTICAL PRESSURE PULSE FROM A FLUID-SOLID PLANE BOUNDARY

1.1 Introduction

The purpose of this investigation is to study the transient behavior of the fluid in the vicinity of a plane fluid-solid interface. In solving problems of this type the usual procedure consists in first determining a valid asymptotic solution to the steady state problem at great distances from the source, and then in using the Fourier integral to synthesize the transient response. This investigation will serve to illustrate a different approach to problems of this general type which was developed by L. Cagniard. Application of Cagniard's method leads directly to an expression for the transient response at any point in the system. One of the remarkable features of this method is that it gives an exact algebraic expression for the response at points located on the axis of symmetry.

I.2 Description of the Source

We will leave unspecified the physical characteristics of the source and assume only that it is capable of exerting a uniform pressure over the spherical surface $r=a_0$ and that the pressure-time dependence at this surface is known.

It is instructive to consider the case in which the differential pressure, at the spherical surface $r=a_0$, is given by the relations:

(29) $P_\delta(t,a_0)\equiv 0, \quad 0\leq t< a_0/V_L;$

(29) $P_\delta(t,a_0) = P_0 \sin\{K(t-a_0/V_L)\}, \quad a_0/V_L \le t \le a_0/V_L + T;$
$\equiv 0, \quad t > a_0/V_L + T;$

where $K=2\pi/T$. The displacement potential, φ, and the radial displacement, u_r, are readily obtained by expressing the wave equation, equation 26, and the source function in terms of the Laplace transform variables $\bar{P}_\delta$, $\bar{\varphi}$, and $\bar{u}_r$. If it is assumed that φ and $\frac{d\varphi}{dt}$ vanish at $t=0$ at all points in the fluid for which $r \ge a_0$, the transformed wave equation takes the form

(30) $\nabla^2\bar{\varphi} = (S^2/V_L^2)\,\bar{\varphi};$

which, in the special case being considered, reduces to

(31) $\frac{d^2(r\bar{\varphi})}{dr^2} = (S^2/V_L^2)(r\bar{\varphi}).$

The quantity S is the variable which appears in the Laplace transform. A solution to equation 31, which is appropriate for describing a spherical wave diverging from the point $r=0$, is

(32) $\bar{\varphi}(r,S) = (A/r)\,e^{-S(r/V_L)}.$

The constant, A, can be evaluated by requiring that the differential pressure, which is to be determined from the relation

(33) $\bar{P}_\delta(r,S) = -\sigma_0 S^2\bar{\varphi} = -\sigma_0 S^2 (A/r)\,e^{-S(r/V_L)},$

reduce to the correct form on the spherical surface $r=a_0$. The Laplace transform of equation 29 is

(34) $\bar{P}_\delta(a_0,S) = P_0(K/K^2+S^2)(1-e^{-ST})(e^{-S(a_0/V_L)}).$

A comparison of equations 33 and 34 indicates that the constant, A, has the form

(35) $A = -(a_0P_0/\sigma_0)\{K/[S^2(S^2+K^2)]\}(1-e^{-ST}).$

It may be readily verified that the result which is obtained by inverting equation 32 is:

(36) $$\begin{aligned}\Phi(r,t) &\equiv 0, \quad t < r/V_L; \\ &= -(a_0 P_0/\sigma_0)(1/rK)\{(t-r/V_L) - (1/K)\sin[K(t-r/V_L)]\}, \quad r/V_L \leq t \leq r/V_L+T; \\ &= -(a_0 P_0/\sigma_0)(T^2/2\pi r), \quad t > r/V_L+T.\end{aligned}$$

The radial displacement is:

(37) $$\begin{aligned}u_r(r,t) &\equiv 0, \quad t < r/V_L; \\ &= (a_0 P_0/\sigma_0)(1/rK)\{(1/V_L)[1-\cos K(t-r/V_L)] + (1/r)[(t-r/V_L) - (1/K)\sin K(t-r/V_L)]\}, \\ &\quad r/V_L \leq t \leq r/V_L+T; \\ &= (a_0 P_0/\sigma_0)(T^2/2\pi r^2), \quad t > r/V_L+T.\end{aligned}$$

The second expression in equation 37 (which describes the time variation of u_r in the interval $r/V_L \leq t \leq r/V_L+T$) is composed of an oscillatory part and a part which increases monotonically with time. In order for the amplitude of the oscillatory part to be large compared with the residual displacement it is necessary that

(38) $$2/V_L \gg T/a_0;$$

that is, the wave length of the oscillation must be small compared to the diameter of the source. This inequality is somewhat misleading. Actually T and a_0 are not independent. The relationship between these two quantities cannot be specified without a knowledge of the physical characteristics of the source. Satisfaction of the inequality and of the requirement that the area under the pressure time curve vanish are sufficient to define an "acoustic source" in this particular case. Cole (4) has shown that if the latter condition is not satisfied the fluid is left with a residual velocity. In such a case the source acts more like a source of fluid than a source of sound. It can also be shown that there is no residual displacement if the area under the pressure time curve vanishes and the condition

(39) $$\int_{a_0/V_L}^{a_0/V_L+T}\left\{\int_{a_0/V_L}^{t} P_\delta(\tau - a_0/V_L, a_0)\,d\tau\right\}dt = 0$$

is satisfied.

It is of interest to compare the relative amplitudes of the first and third terms in equation 27 for the special case in which Φ is determined from equation 36. The differential pressure at a point on the vertical axis through the source is

(40) $$P_\delta = -\sigma_0 \frac{d^2\Phi}{dt^2} = (a_0 P_0/z)\sin[K(t-z/V_L)]\,.$$

The fluid element, which was initially at z , experiences a change in hydrostatic pressure of amount

(41) $$\sigma_0 g\frac{d\Phi}{dz} = (a_0 P_0 g/Kz)\left\{(1/V_L)[1-\cos K(t-z/V_L)] + (1/z)[(t-z/V_L)-(1/K)\sin K(t-z/V_L)]\right\}.$$

The amplitude of the third term in equation 27 will be small compared to the amplitude of the first if the period satisfies the condition

(42) $$T \ll \pi V_L/g\,.$$

The quantity V_L/g enters in this equation as a natural period of the system. It is interesting to note that for a given T the inequality might be satisfied in water but not in air. This is due to the fact that a pressure pulse of given amplitude and period causes larger fluctuations in the hydrostatic pressure acting on an element of air than on an element of water.

We will next put in evidence a quite general expression for the Laplace transform of the source potential function. The most general expression for a diverging spherical wave is of the form:

(43) $$\Phi_0 \equiv 0, \quad t < r/V_L \ (r \geq a_0);$$
$$= (a_0/r)f(t-r/V_L), \quad t \geq r/V_L\,.$$

The Laplace transform of this function is

(44) $$\bar{\Phi}_0 = (a_0/r)\int_{r/V_L}^{\infty} e^{-st} f(t - r/V_L)\,dt.$$

Changing the integration variable to $\tau\{=t-r/V_L\}$ reduces $\bar{\Phi}_0$ to the form

(45) $$\bar{\Phi}_0 = (a_0/r)\int_0^{\infty} e^{-s(\tau + r/V_L)} f(\tau)\,d\tau$$

$$= (a_0/r)\,e^{-sr/V_L}\int_0^{\infty} e^{-s\tau} f(\tau)\,d\tau = [(a_0 K(s))/rs]\,e^{-sr/V_L},$$

where $K(s)$ is an arbitrary function of s to be determined from the particular form of the pressure-time relation at the surface $r=a_0$. This form of the source function is not of much value in problems where the boundary conditions are imposed on planes of constant z . A more suitable relationship can be obtained by making use of the integral transformation

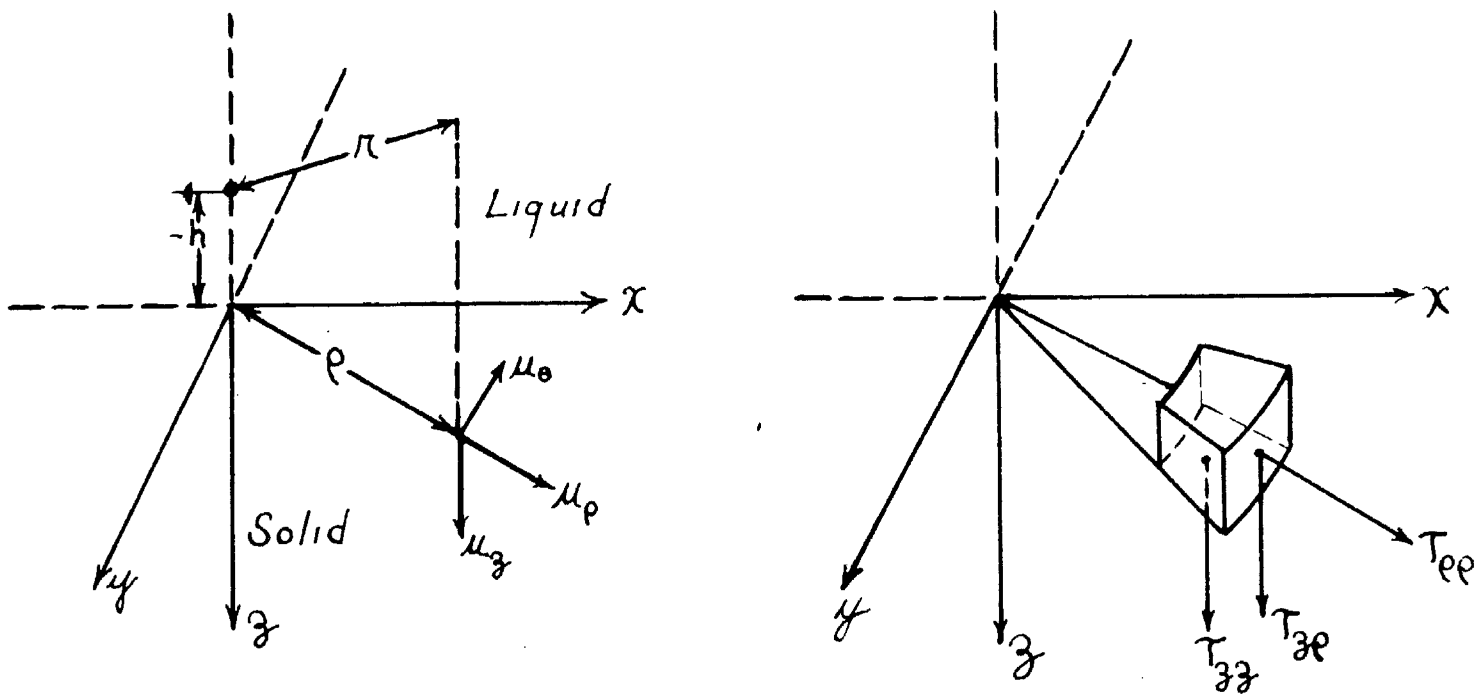
(46) $$(1/rs)\,e^{-sr/V_L} = \int_0^{\infty} (\lambda/s\alpha)\, J_0(\lambda\rho)\, e^{-\alpha|z-h|}\, d\lambda,$$

where $\alpha=(\lambda^2+s^2/V_L^2)^{1/2}$, $-h$ is the vertical distance between the point $r=0$ and the interface $(z=0)$, and $r=\{(z-h)^2+\rho^2\}^{1/2}$. The coordinate system is chosen in such a way that the xy plane coincides with the plane of the interface and positive z values are associated with points in the solid (fig. 1). This transformation is given in Watson (5).

I.3 Specification of the Boundary Conditions

It is of interest to consider the form which the boundary conditions take on a plane surface between two fluids. If the effect of capillary forces is neglected both the vertical displacement and the total pressure must be continuous across the surface of discontinuity. The pressure condition takes the form

(47) $${}^{a}P_0(d+\eta) + {}^{a}P_\delta(d+\eta) = {}^{w}P_0(d+\eta) + {}^{w}P_\delta(d+\eta),$$


Liquid
Solid
-h
π
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x
y
z
μ_θ
μ_ρ
μ_z
$\tau_{\rho\rho}$
τ_{zz}
$\tau_{z\rho}$

Figure 1

where $\eta = u_z|_{z=d}$ is the vertical displacement of a point on the interface and $z=d$ is the original position of that point. Expanding both sides of equation 47 in Taylor series and retaining only first order terms leads to the condition

$$(48)\quad \frac{\partial\, {}^aP_0}{\partial z}\Big|_{z=d}\eta + {}^aP_s(d) = \frac{\partial\, {}^wP_0}{\partial z}\Big|_{z=d}\eta + {}^wP_s(d)\ ,$$

which, in terms of the displacement potential, is

$$(49)\quad {}^a\sigma_0 g \frac{\partial\, {}^a\phi}{\partial z} - {}^a\sigma_0 \frac{\partial^2\, {}^a\phi}{\partial t^2} = {}^w\sigma_0 g \frac{\partial\, {}^w\phi}{\partial z} - {}^w\sigma_0 \frac{\partial^2\, {}^w\phi}{\partial t^2}.$$

The question concerning the form of the boundary condition at an air-water interface arises in many problems of practical interest—for example, in the study of acoustic pulse propagation in the ocean. In equation 49 let the superscript a refer to air and the superscript w refer to water. If the displacement of the interface takes place very gradually the movement produces almost no compression of the air in the vicinity of the interface. In this case the left side of equation 49 becomes quite small and the boundary condition reduces to

$$(50)\quad {}^w\sigma_0 \frac{\partial^2\, {}^w\phi}{\partial t^2} = {}^w\sigma_0 g \frac{\partial\, {}^w\phi}{\partial z}.$$

It is clear that in the low frequency limit the differential pressure at the original position of the interface is due entirely to the change in the hydrostatic pressure resulting from the elevation of the interface. This is also the reason that equation 50 is identical with the condition imposed at the free surface of an incompressible fluid. It is interesting to note that if the density and incompressibility of the air are decreased (but not so much that the water in the vicinity of the interface changes phase) the left side of equation 49 diminishes in value. In many papers on this subject it is assumed that the differential pressure at an air-water interface vanishes. The foregoing discussion seems to indicate that this is not a valid assumption. If the vertical displacement oscillates very rapidly, the correct boundary condition is obtained from equation 49 by neglecting the terms involving g . In the low frequency limit equation 50 must be satisfied. It appears that the differential pressure will vanish only if both the frequency and the amplitude of the vertical displacement go to zero.

It may be verified that a solution to the liquid layer problem, which satisfies equations 27 and 49, can be obtained by an analysis which is quite similar to the one carried out below if either the liquid layer is so shallow that the variation of V_L with depth can be neglected or V_L is constant. It is worth noting that if V_L is constant the density-depth relation can be determined directly from equation 17; it is

(51) $$\sigma_0 = {}^s\sigma_0 \, e^{(g/V_L^2)z} \sim \sigma_0^s(1+(g/V_L^2)z),$$

where σ_0^s is the density at the surface. It would be extremely interesting to compare this solution with existing solutions in which gravity has been neglected. Such a comparison would yield valuable information about the frequency dependence of the gravitational effect and the consequences of requiring that the differential pressure vanish at the air-water interface.

It is convenient to formulate the boundary conditions at the liquid-solid interface in terms of the potentials φ , ψ , and U . The displacement field in the solid can be derived from the relations

(52) $_s u_\rho = \frac{\partial\psi}{\partial\rho} - \frac{\partial U}{\partial z}$

and

(53) $_s u_z = \frac{\partial\psi}{\partial z} + \frac{\partial U}{\partial\rho} + \frac{U}{\rho}$.

The stress components can also be written in terms of the potentials if these expressions for $_s u_\rho$ and $_s u_z$ are substituted in the stress-strain relations. The relevant stress components have the form

(54) $T_{zz} = \lambda_s \nabla^2\psi + 2\mu_s \left\{ \frac{\partial^2\psi}{\partial z^2} + \frac{1}{\rho}\frac{\partial^2(\rho U)}{\partial\rho\, \partial z} \right\}$

and

(55) $T_{\rho z} = \mu_s \left\{ 2\frac{\partial^2\psi}{\partial\rho\, \partial z} + \frac{\partial^2 U}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial\rho} - \frac{U}{\rho^2} - \frac{\partial^2 U}{\partial z^2} \right\}$,

where λ_s and μ_s are the Lame constants appropriate to the solid. The equations

(56) $\nabla^2\psi = \frac{1}{V_s^2}\frac{\partial^2\psi}{\partial t^2}$

and

(57) $\nabla^2 U - \frac{U}{\rho^2} = \frac{1}{v_s^2}\frac{\partial^2 U}{\partial t^2}$

are obtained by introducing relations 52-55 in the equations of motion. V_s and v_s are the compressional and transverse velocities in the solid. In deriving equations 56 and 57 it has been assumed that the following conditions are fulfilled: (a) the solid is isotropic, perfectly elastic, and homogeneous; (b) the variation of the density and elastic parameters with depth can be neglected; (c) the stresses produced by the disturbance are sufficiently small to justify the use of the linear elasticity theory; and (d) the stresses vary so rapidly with time that the body force can be neglected.

At the liquid-solid interface we require that: (a) the vertical component of displacement be continuous, (b) the normal component of stress be continuous, and (c) the tangential component of stress in the solid vanish. We find that, in view of our assumptions, these conditions take the form:

(58a) $$\frac{\partial \Phi}{\partial z}\Big|_{z=0} = \left(\frac{\partial \psi}{\partial z} + \frac{\partial U}{\partial \rho} + \frac{U}{\rho}\right)\Big|_{z=0} \;,$$

(58b) $$-\sigma_0 \frac{\partial^2 \Phi}{\partial t^2} = -\left[\frac{\lambda_s}{V_s^2}\frac{\partial^2 \psi}{\partial t^2} + 2\mu_s\left\{\frac{\partial^2 \psi}{\partial z^2} + \frac{1}{\rho}\frac{\partial^2(\rho U)}{\partial \rho\, \partial z}\right\}\right]\Big|_{z=0} \;,$$

and

(58c) $$\left[2\frac{\partial^2 \psi}{\partial \rho \partial z} - 2\frac{\partial^2 U}{\partial z^2} + \frac{1}{V_s^2}\frac{\partial^2 U}{\partial t^2}\right]\Big|_{z=0} = 0 .$$

I.4 Formal Solutions for the Potentials

The Laplace transform of the displacement potential, Φ, can be expressed in the form

(59) $$\bar{\Phi}(s;\rho,z) = \int_0^\infty \left[(a_0 X(s)\lambda)/s\alpha\right] e^{-\alpha|z-h|} J_0(\lambda\rho)\, d\lambda + \int_0^\infty f(\lambda) e^{\alpha z} J_0(\lambda\rho)\, d\lambda .$$

The first term is just the source function. The second term describes the perturbing influence of the boundary. Similar expressions for $\bar{\Psi}$ and $\bar{U}$ can be obtained from equations 56 and 57 if we first take the Laplace transform of these relations. We find that

(60) $$\bar{\Psi}(s;\rho,z) = \int_0^\infty h(\lambda) J_0(\lambda\rho) e^{-\beta z} d\lambda$$

and

(61) $$\bar{U}(s;\rho,z) = \int_0^\infty j(\lambda) \frac{d J_0(\lambda\rho)}{d\rho} e^{-\Gamma z} d\lambda .$$

α, β, and Γ all have the same form. β differs from α only in the substitution of V_s for V_L; similarly, Γ is obtained from α by substituting v_s for V_L. The boundary conditions will be satisfied if $f(\lambda)$, $h(\lambda)$, and $j(\lambda)$ are determined from the equations:

(62a) $$\alpha f + \beta h + \lambda^2 j = (a_0 X(s)\lambda e^{\alpha h})/s \;,$$

(62b) $$-\sigma_0 s^2 f + 2\sigma_s v_s^2 \Omega h + 2\sigma_s v_s^2 \Gamma \lambda^2 j = \sigma_0 s a_0 X(s)\lambda e^{\alpha h}/\alpha \;,$$

and

(62c) $\beta h + \Omega j = 0$;

where $\Omega = \lambda^2 + S^2/2v_S^2$. The solution to this system of equations can be reduced to the form:

(63) $f(\lambda) = (a_0 \chi(S)/S)(\lambda/\alpha)\, e^{\alpha h}\,(G-H)/(G+H)$,

(64) $h(\lambda) = (a_0 \chi(S) S)(\sigma_0/\sigma_S v_S^2)(\lambda\Omega/\alpha)\, e^{\alpha h}/(G+H)$,

and

(65) $j(\lambda) = -(a_0 \chi(S) S)(\sigma_0/\sigma_S v_S^2)(\lambda\beta/\alpha)\, e^{\alpha h}/(G+H)$;

where

(66) $G(\lambda) = \Omega^2 - \lambda^2 \beta \Gamma$

and

(67) $H(\lambda) = (\sigma_0/\sigma_S)(S^4/4v_S^4)(\beta/\alpha)$.

In this discussion we will confine our attention to the response in the fluid and to that component of the response which describes the perturbing influence of the boundary. The second term in equation 59 can now be written in the form

(68) $\bar{\varphi}_p(S;\rho,z) = (a_0\chi(S)/S)\int_0^\infty (\lambda/\alpha)\, e^{-\alpha r_1} A(\lambda)\, J_0(\lambda\rho)\, d\lambda$,

where $A(\lambda) = (G-H)/(G+H)$ and $r_1 = -h - z$. It is important to note that r_1 is the vertical distance from the image source to the point of observation. It is possible to achieve a considerable degree of simplification in equation 68 by making the substitution $\lambda = Su$. This substitution makes both the function $A(\lambda)$ and the radicals independent of S and reduces equation 68 to the form

(69) $\bar{\varphi}_p(S;\rho,z) = a_0\chi(S)\int_0^\infty (u/a)\, A(u)\, J_0(Su\rho)\, e^{-Sar_1}\, du$.

The displacement components can be obtained directly from equation 69; they are

(70) $$\bar{u}_z(s;\rho,z) = a_0 s X(s) \int_0^\infty u\, A(u)\, J_0(su\rho) e^{-saT_1} du$$

and

(71) $$\bar{u}_\rho(s;\rho,z) = -a_0 s X(s) \int_0^\infty (u^2/a) A(u) J_1(su\rho) e^{-saT_1} du .$$

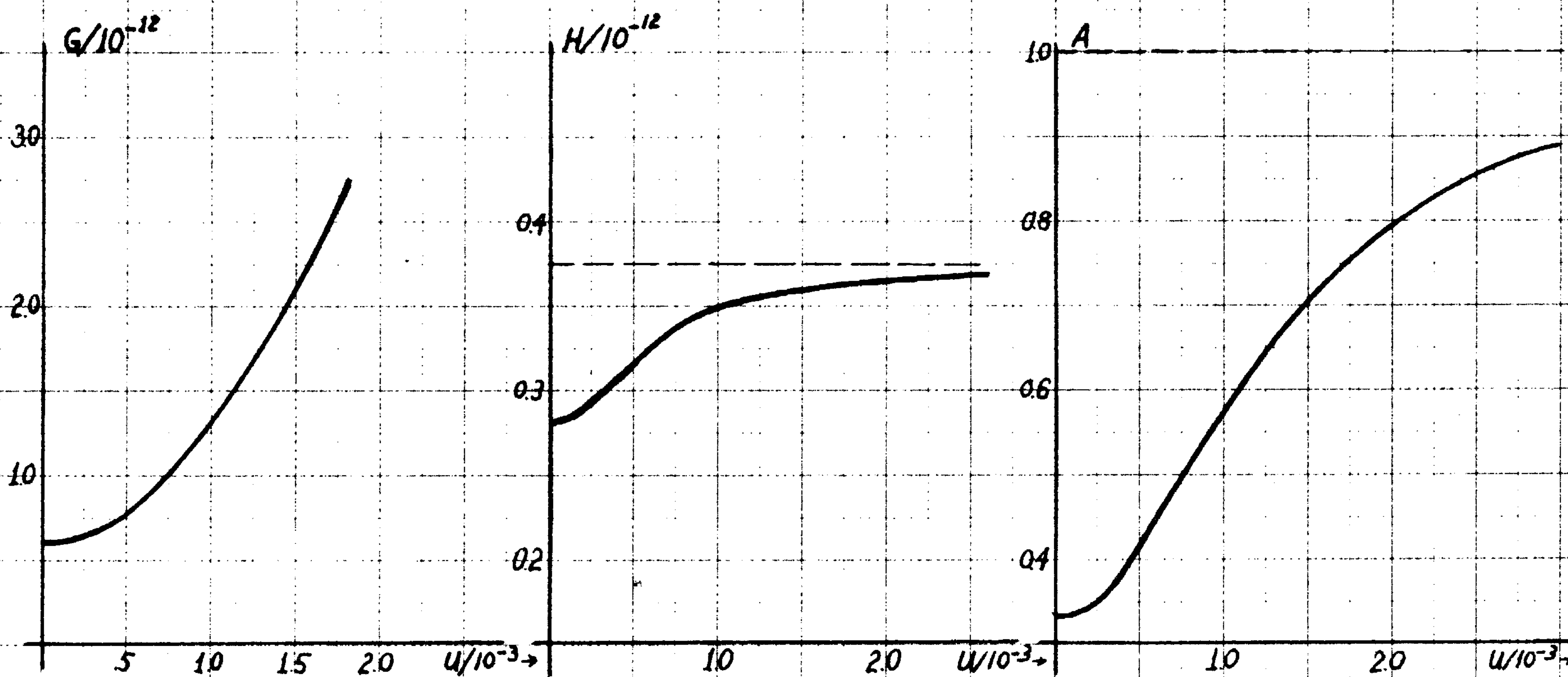
On the axis the transform of the radial displacement vanishes. This indicates that the radial displacement itself must be zero for all time at axial points.

I.5 Inversion of $\bar{u}_z$ at Axial Points

The behavior of the function $A(u)$, in the interval $0 \le u < \infty$, can be inferred from the fact that $G(u)$ is positive real in this interval. The value of $G(u)$ at $u=0$ is $S_3^4/4$ $(S_3 = 1/v_S)$. The fact that the compressional velocity is always greater than the transverse velocity implies that

(72) $$u^2(u^2+S_3^2)^{1/2}\{(u^2+S_3^2)^{1/2}-(u^2+S_2^2)^{1/2}\} \ge 0 ,$$

where $S_2 = 1/V_S$. If the multiplication is carried out and the expression $S_3^4/4$ is added to each side of the inequality, it becomes evident that $G(u)$ must be positive on the positive real u axis and greater than or equal to its value at $u=0$. Since $G(u)$ and $H(u)$ are positive, $A(u)$ must have a value between plus and minus one. It should be noted, however, that $A(u)$ approaches unity in the limit as u approaches infinity. Also, if $\sigma_0 V_L/\sigma_S V_S \ge 1$, $A(u)$ must pass through zero as u varies between zero and infinity. Graphs of $G(u)$, $H(u)$, and $A(u)$ are plotted in fig. 2 for a special selection of parameters. It can be verified that $A(u)$ reduces to unity if either the density ratio vanishes or the rigidity of the solid becomes infinite. In either case the solid acts like a perfect reflector.



Case I

Figure 2

The evaluation of the vertical displacement at axial points is straightforward. If $\rho=0$, $J_0(Su\rho)=1$ and equation 70 reduces to*

(73) $$\bar{u}_z(S;0,z) = a_0 S \int_0^\infty u\, A(u)\, e^{-Sa\Gamma_1} du .$$

The determining function can be obtained directly if we make the substitution $t=a\Gamma_1$, $du/dt=a/u\Gamma_1$; then

(74) $$\bar{u}_z(S;0,z) = -(a_0/\Gamma_1)\int_{S_1\Gamma_1}^\infty a A\{d(e^{-St})/dt\}\,dt$$
$$=(a_0/\Gamma_1)\left\{\left[(aA)_{t=S_1\Gamma_1}\right]e^{-S(S_1\Gamma_1)} + \int_{S_1\Gamma_1}^\infty e^{-St}\{d(aA)/dt\}\,dt\right\}.$$

It is convenient to make the substitution $A=1-B$ $(B=2H/G+H)$ in the integral. This is done to separate the terms which describe the afterflow and the transient behavior. The transform of the vertical displacement can then be written as follows:

(75) $$\bar{u}_z(S;0,z) = a_0\left\{(S_1/\Gamma_1)((1-\chi)/(1+\chi))e^{-S(S_1\Gamma_1)} + (1/\Gamma_1)\int_{S_1\Gamma_1}^\infty e^{-St}\{da/dt\}\,dt\right.$$
$$\left. -(1/\Gamma_1)\int_{S_1\Gamma_1}^\infty e^{-St}\{d(aB)/dt\}\,dt\right\},$$

where $\chi=\sigma_0 V_L/\sigma_S V_S$. The reason for making the substitution is now apparent. The first integral is simply the Laplace transform of the unit step function $1(t-S_1\Gamma_1)/\Gamma_1$. The second integral is the transform of the function $\xi(t;0,z)$, defined as follows:

(76) $$\xi(t;0,z)\equiv 0,\quad t<S_1\Gamma_1;$$
$$= -(a_0/\Gamma_1)\,d(aB)/dt = -(a_0/\Gamma_1^2)(a/u)\,d(aB)/du,\quad t\geq S_1\Gamma_1.$$

The first term in equation 75 is interpreted as the Laplace transform of the Dirac delta function, $\delta(t-S_1\Gamma_1)$. The vertical displacement itself can be written in the form

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* The consequences of setting $\chi(S)$ equal to unity are discussed on pages 28 and 29.

$$(77)\quad u_z(t;0,z) \equiv 0\,,\quad t < S_1\Gamma_1\,;$$

$$= a_0\Big[\{(1-\chi)/(1+\chi)\}(S_1/\Gamma_1)\,\delta(t-S_1\Gamma_1) + (1/\Gamma_1^2)\,1(t-S_1\Gamma_1) - (1/\Gamma_1^2)(a/u)\,d(aB)/du\Big]\,,\quad t \geq S_1\Gamma_1\,.$$

The first and second terms add nothing new to our understanding of the reflection process. These two terms are identical with the corresponding two terms in the expression for the incident wave except for the presence of the factor $(1-\chi)/(1+\chi)$. It is clear that this quantity is just the plane wave reflection coefficient for vertical incidence.

The third term in equation 77 describes the transient characteristics of the response. On the axis the amplitude of this term decreases inversely as the square of the distance from the image source. It therefore represents a correction to the afterflow. It is important to note that the effect of making the density ratio arbitrarily small is to cause the correction term to vanish while leaving the afterflow term unaffected.

A spreading of the correction term accompanies its propagation. The characteristics (e.g. - maximum and minimum) of the function ξ depend only on the value of u, while the time depends on the values of both u and Γ_1. This means that the time difference between two points of the correction curve, u_1 and u_2, increases linearly with increasing distance from the image source according to the relation

$$(78)\quad t_2 - t_1 = \Gamma_1\{(u_2^2+S_1^2)^{1/2} - (u_1^2+S_1^2)^{1/2}\}.$$

It can also be shown that the amount of spreading increases toward the tail of the wave.

I.5a Calculations

Detailed calculations of the time dependence of the function $-\Gamma_1^2 \xi / a_0$ were made for three choices of the elastic parameters.

Case	Poisson's Ratio	Compressional Velocity (mt./sec.)		Density Ratio D_L/D_S
		Solid	Liquid	
I	0.4	2,000	1,500	2/3
II	0.4	2,000	500	2/3
III	0.2	6,000	1,500	1/2

The choice of parameters in Case I should be appropriate for describing the wave which has been reflected from a moderately well compacted, easily deformed material. In Case III the solid has properties similar to those of granite.

The curves of fig. 3 have been drawn for the special case in which the source is located two meters above the interface and the reflected wave is observed at a point midway between the source and the interface. Each of these curves has a discontinuous beginning and each approaches zero asymptotically from negative values. The function $-(\Gamma_1^2/a_0)\xi$ has the asymptotic form

$$(79) \quad -(\Gamma_1^2/a_0)\xi \underset{u\to\infty}{\sim} -(\sigma_0/\sigma_S)(S_3^4/(S_3^2-S_2^2))(1/u^2).$$

It is noteworthy that this expression is independent of V_L. The effect of changing the liquid velocity can be studied by comparing Cases I and II. Case II is marked by a much larger discontinuity and a more rapid oscillation between its maximum and minimum values. Case III differs from the other two in three important ways: (a) the displacement is negative originally; (b) there is no oscillation in the displacement; and (c) instead of decaying the displacement increases slightly to a

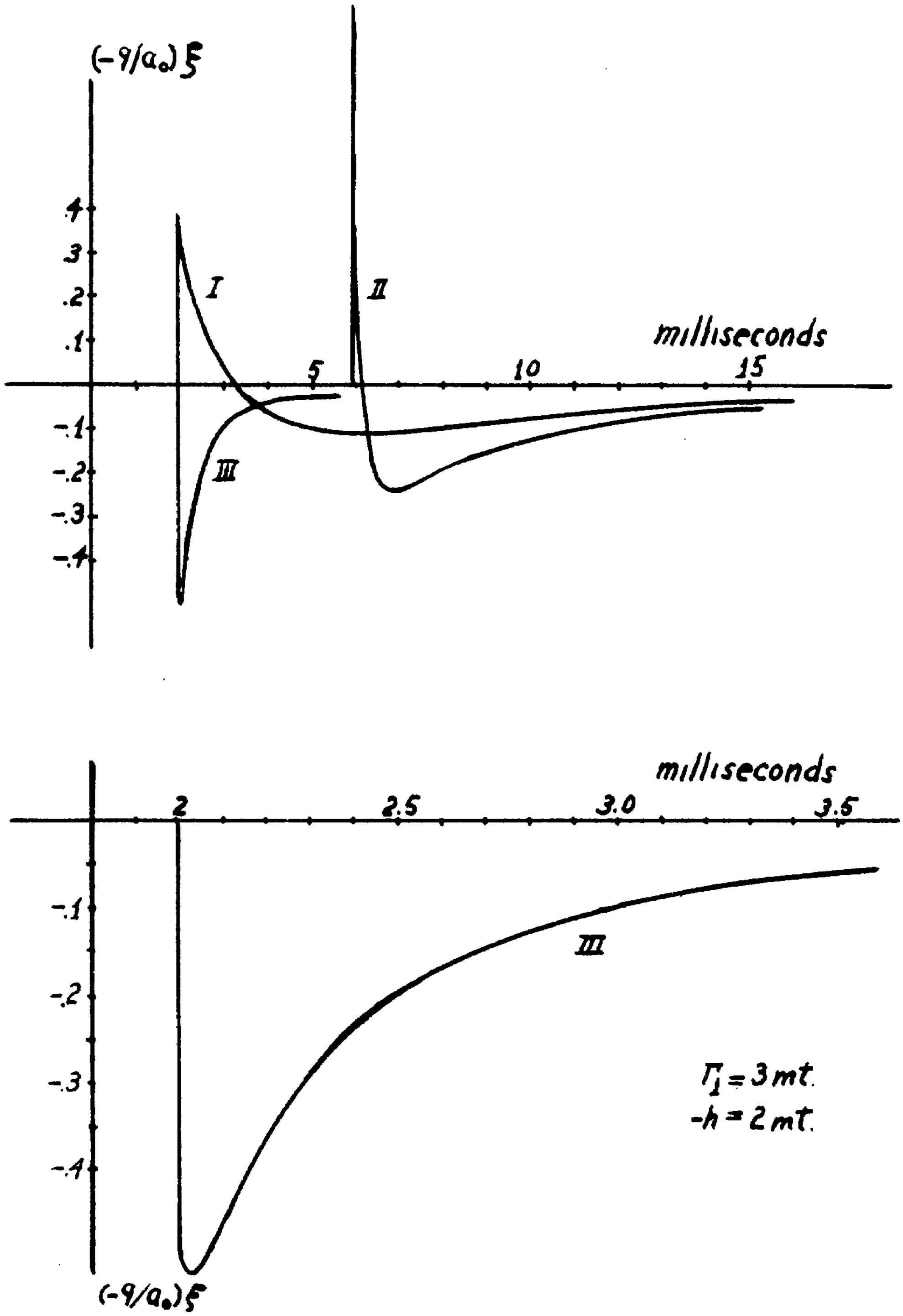
(-q/a₀)ξ
.4
.3
.2
.1
-.1
-.2
-.3
-.4
I
II
III
milliseconds
5
10
15
milliseconds
2
2.5
3.0
3.5
-.1
-.2
-.3
-.4
III
Γ₁ = 3 mt.
-h = 2 mt.
(-q/a₀)ξ

Figure 3

maximum negative value.

Fig. 4 was constructed in order to shed some light on the reasons for these differences. In this figure $-(\Gamma_1^2/2a_0)\,\xi(S_1\Gamma_1;0,Z)$ has been plotted as a function of the ratio of the compressional velocities, $\eta(=V_L/V_S)$, for different values of Poisson's ratio and for a fixed value of the density ratio. The expression from which these curves were obtained can be put in the form

(80) $$-(\Gamma_1^2/2a_0)\,\xi(S_1\Gamma_1;0,Z) = (\sigma_0/\sigma_S)(\eta/(1+(\sigma_0/\sigma_S)\eta))\cdot \left[(1/(1+(\sigma_0/\sigma_S)\eta))\,[(1/\eta^2)\{1-8(S_2^2/S_3^2)(1-S_2/S_3)\}-1]+1\right].$$

The dependence of the sign of this function upon the elastic parameters can be investigated by determining the values of Poisson's ratio for which there exists a value of η which causes the expression in the outer square brackets to vanish. Equating the quantity within the outer square brackets to zero leads to an equation from which the desired information can be extracted; namely,

(81) $$(\sigma_0/\sigma_S)\eta_0^3 = 8(S_2^2/S_3^2)(1-S_2/S_3)-1.$$

η_0 has a real positive value only if the term on the right is greater than or equal to zero. This means that the velocity ratio, v_S/V_S, must satisfy the relation

(82) $$(v_S/V_S)^2(1-v_S/V_S) \geq 1/8.$$

The positive roots of the equation are 0.500 and 0.809. In actual materials Poisson's ratio cannot be less than zero and the velocity ratio cannot exceed $2^{-1/2}$. This means that it is always possible to find a value of η which makes equation 80 vanish if Poisson's ratio lies between zero and one-third. It should be noted that η_0 has a maximum value for a Poisson's

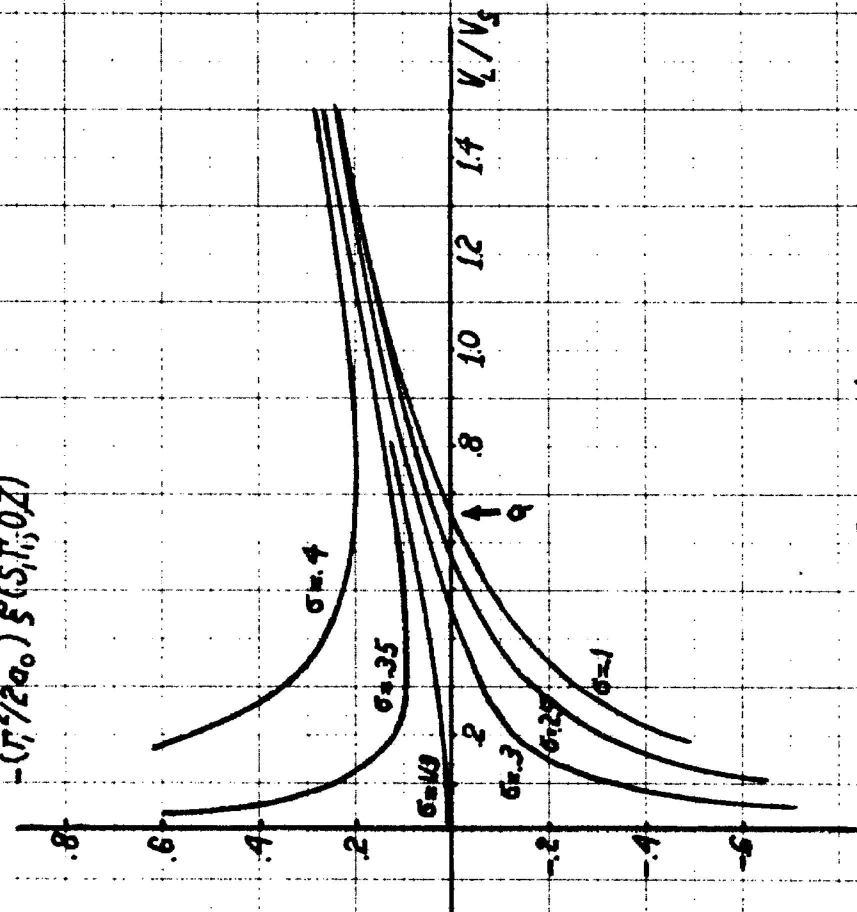

Figure 4

ratio of one-tenth. If curves corresponding to Poisson's ratios less than one-tenth had been included in fig. 4, it would have been found that they cross the abscissa slightly to the left of the point Q. How far to the left can be readily determined from equation 81. For a Poisson's ratio of zero and a density ratio of two thirds (which was used in obtaining all the curves of fig. 4) this distance is about .009 units.

I.5b Interpretation

The interpretation of fig. 4 and the response curves cannot be carried out in a satisfactory manner unless certain points of the discussion, which were passed over rather rapidly, are clarified at this point. First it is important to distinguish between the correction term and the expression which describes the actual motion of the interface at the axial point. The displacement of the "epicentral point" is obtained by adding the displacements produced by the direct and reflected waves; it is given by the expression

$$(83)\quad u_z(t;0,0) = -(a_0/-h)\Big[(1/V_L)(2\chi/(1+\chi))\,\delta(t-(-h)/V_L) + (1/-h)(a/u)\,d(aB)/du\Big].$$

The presence of the delta function indicates that the incident wave produces an instantaneous oscillation in the vertical displacement. The absence of an afterflow term indicates that the displacement at the axial point approaches zero asymptotically according to equation 79.

A second point which requires clarification is the role played by the Dirac delta function. This function enters at the stage of evaluation represented by equation 77. Its appearance is a direct consequence of setting $\mathcal{H}(S)$ equal to unity in equation 73. The function $\mathcal{H}(S)$ cannot have the constant value unity and at the same time represent a physically realizable source*. This means that the inverse of the function $\mathcal{H}(S)e^{-sS_1T_1}$ should have been found and not the inverse of the function $e^{-sS_1T_1}$. It is clear that if this had been done the result obtained would have been identical with the result which is obtained from using the superposition integral to generalize equation 77. We therefore do not attach any significance to the Dirac delta function when it appears by itself but only to the term obtained from it after superposition. If we bear this in mind it is possible to reach some interesting conclusions without actually carrying through the numerical integrations. These conclusions are not restricted to a specific form of the pressure-time relation, as they would be if the inversion had been carried out directly, but apply equally well to a broad class of physically realizable source conditions.

The response of the system to an arbitrary pressure variation, which can be derived from a displacement potential φ_g, can be found from the superposition integral. This integral relates the generalized response, R_g , to the response produced by an input step function in the displacement potential. It has the form

* The pressure variation corresponding to $\mathcal{H}(S) = 1$ can be determined directly from equations 33 and 45. It is given by minus the derivative of the Dirac delta function. The time variation of the displacement potential is given by the unit step function. It is interesting to note that these same results can be obtained by multiplying equations 29 and 36 by minus one and allowing the pressure amplitude to approach infinity and the period to approach zero in such a way that the quantity $P_oT^2/2\pi\sigma_o$ approaches unity.

(84) $$R_g(t;0,Z) = \varphi_g(0)\,u_z(t) + \int_0^{t-s_1T_1} \frac{d\varphi_g}{d\tau}\, u_z(t-\tau)\,d\tau.$$

Actually we shall only be interested in that part of the generalized response which depends on the correction term; accordingly, we replace $u_z(t-\tau)$ by $\xi(t-\tau)$ in equation 84. It can be readily verified that both the generalized response and its time derivative vanish at $t=s_1T_1$ if the function φ_g and its time derivative vanish at $\tau=0$. Furthermore, if these conditions are satisfied, the initial variation of the time derivative of the displacement potential must be toward negative values when the initial variation of the source pressure is toward positive values. This fact indicates that if the initial variation in the source pressure is toward positive values the initial variation in the sign of the function $-\xi$ will be preserved in the process of superposition. This fact is extremely important in interpreting fig. 4.

We are now in a position to interpret the response curves and fig. 4. Descent along a vertical line in fig. 4 corresponds to changing the elastic properties of the solid in such a way that the compressional velocity is unaffected while the transverse velocity is continuously increased. The relevant relationships among the elastic constants are:

(85a) $v_s^2 = (V_s^2/2)(1-2\sigma)/(1-\sigma)$,

(85b) $\mu_s = (\sigma_s V_s^2/2)(1-2\sigma)/(1-\sigma)$,

(85c) $k_s = (\sigma_s V_s^2/3)(1+\sigma)/(1-\sigma)$,

and

(85d) $E_s = (\sigma_s V_s^2)(1+\sigma)(1-2\sigma)/(1-\sigma)$.

Changing Poisson's ratio from .5 to 0 increases the transverse velocity, the rigidity, and Young's modulus and decreases the incompressibility. These facts indicate that if the behavior of two elastic solids, which have the same densities and compressional velocities, is studied during a short time interval following the arrival of the disturbance, it will be found that the solid which has the larger value of Poisson's ratio will undergo a larger distortion of the interface, a deeper indentation at the point of application, and a smaller over-all change in volume.

Sketches have been made in figs. 5A and 5B which illustrate these effects. These sketches are drawn for the particular case in which the velocity ratio, η , is small compared to unity. The part of the interface which is intersected by the spherical wave front is indicated by the line segment OP. The solid vertical lines indicate the radial distance traveled by the compressional wave in the solid. Equation 83 shows that there is a residual deformation of the interface even after the effect of the incident and reflected acoustic* waves has been removed. This residual deformation has an effect on the fluid which is expressed at axial points by the correction term. The part of the deformation which is produced by the direct and reflected acoustic waves is insensitive to changes in the rigidity and the Poisson's ratio as long as σ_s and V_s remain constant. The effect of such changes in the elastic properties of the solid must therefore appear in the correction term. The residual displacement is sketched in fig. 5A

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* The "acoustic" wave is to be associated with the delta function in equation 77.

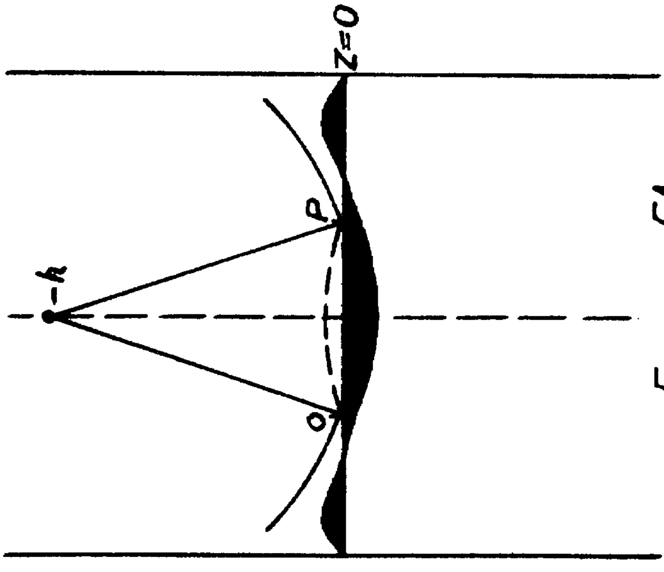

Figure 5A

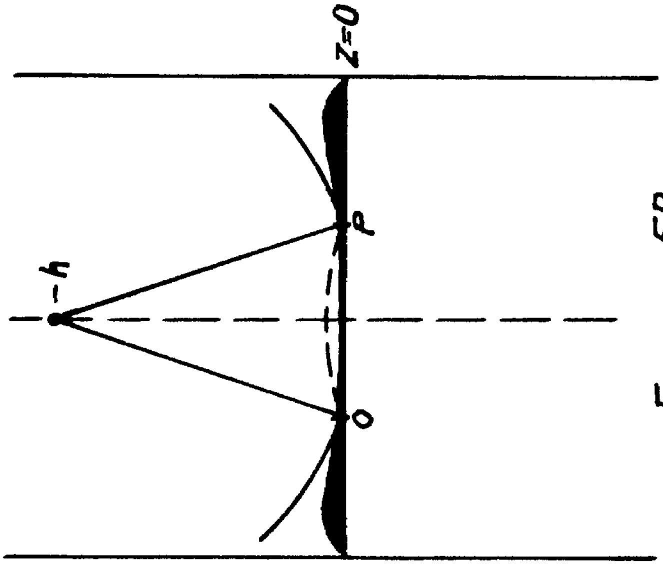

Figure 5B

for a solid with a Poisson's ratio close to one half, and in fig. 5B for a solid with a Poisson's ratio close to zero. Fig. 5A illustrates a situation in which the residual deformation causes the fluid at axial points to move toward the interface. Fig. 5B illustrates a case in which the motion is directed away from the interface. The response curves describe the effect of this residual deformation on the motion of the fluid. Cases I and II are to be associated with a situation of the type illustrated in fig. 5A. Case III is to be associated with fig. 5B. It is interesting to note that the curves of fig. 4 indicate that prescribed changes in the elastic properties of the solid at constant values of σ_s and V_s lead to a great amount of variability in the initial behavior of the correction term if η is small and to practically none if η is large.

I.6 Singularities of the Function $A(u)$

The investigation of the singularities of the function $A(u)$ is a necessary preliminary to the study of the response at points off the axis. It is evident that the function $G(u)$ has branch points at $u=\pm iS_2$ and at $u=\pm iS_3$. The function $H(u)$ has branch points at $u=\pm iS_2$ and at $u=\pm iS_1$. In order to select the proper sheet of the Riemann surface we require that the sign of the radicals be positive on the positive real u axis and join the pairs of branch points by cuts of finite length along the imaginary u axis (fig. 6).

The behavior of $G(u)$ and $H(u)$ for large values of u is given by the expressions

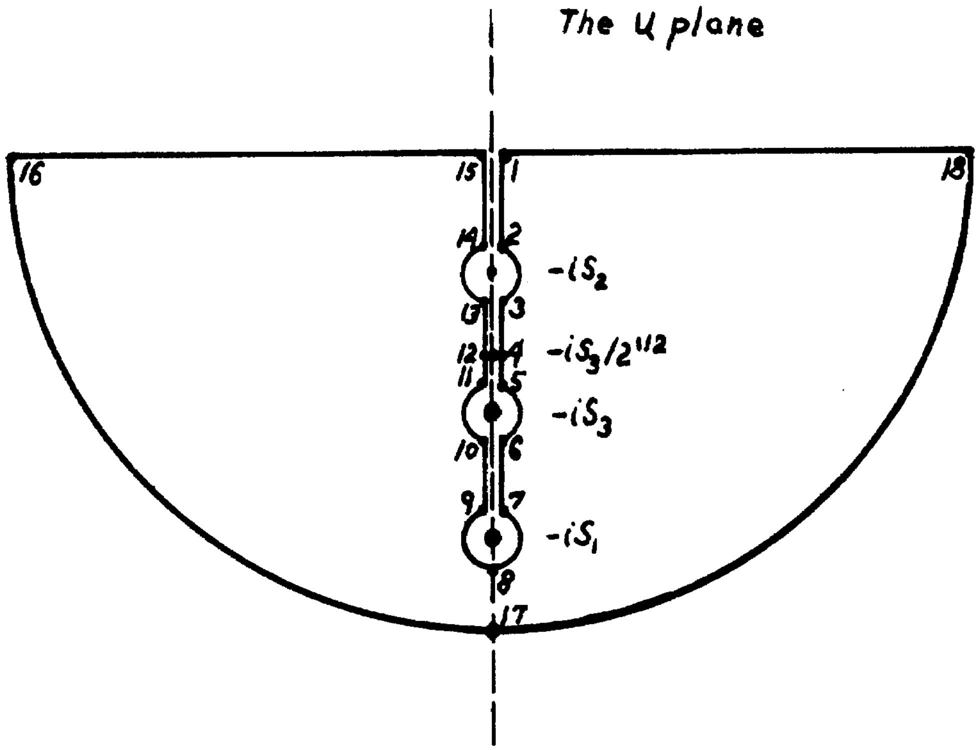
The u plane
16
15
1
18
14
2
$-iS_2$
13
3
12
4
$-iS_3/2^{1/2}$
11
5
$-iS_3$
10
6
9
7
$-iS_1$
8
17

Figure 6

(86) $G(u) \underset{u\to\infty}{\sim} (u^2/2)(S_3^2 - S_2^2)$

and

(87) $H(u) \underset{u\to\infty}{\sim} (\sigma_0 V_L/\sigma_s V_s)(S_3^4/4).$

It is clear that in this limit the function A approaches unity. At $u = \pm i S_1$ the function A has the value minus one. The problem therefore reduces to finding the points in the u plane where the function $G+H$ vanishes*. One such point can be found immediately. If $S_2 = S_3/2^{1/2}$ (corresponding to a Poisson's ratio of zero), the quantity $(u^2+S_2^2)^{1/2}$ can be factored out of the expression for $G+H$. In this special case the function $G+H$ vanishes at the branch points $\pm i S_2$.

We next show that the u contour indicated in fig. 6 encloses one and only one zero of the function $G+H$. In order to do this we must show that the mapping of this contour in the $G+H$ plane circles the origin once. Actually we will prove this result only for the case in which the reciprocal velocities satisfy the inequality $S_1 > S_3 > S_2$; that the result remains valid for any selection of the elastic parameters can be readily verified.

The procedure to be used can be divided into three parts. The first step consists in mapping the u contour into the H plane (fig. 7). Points on the negative imaginary axis between the origin and the point $-iS_2 + i\epsilon$ map into a finite segment of the positive real H axis. It can be shown that H decreases monotonically throughout this interval. The semi-circle about the branch point $-iS_2$ maps into a quarter circle about

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* A similar investigation is carried out in Cagniard, ref. (1) ch. 4.

The H plane

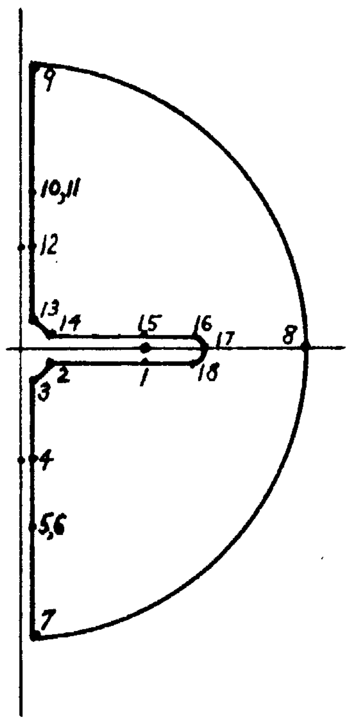


Figure 7

The G plane

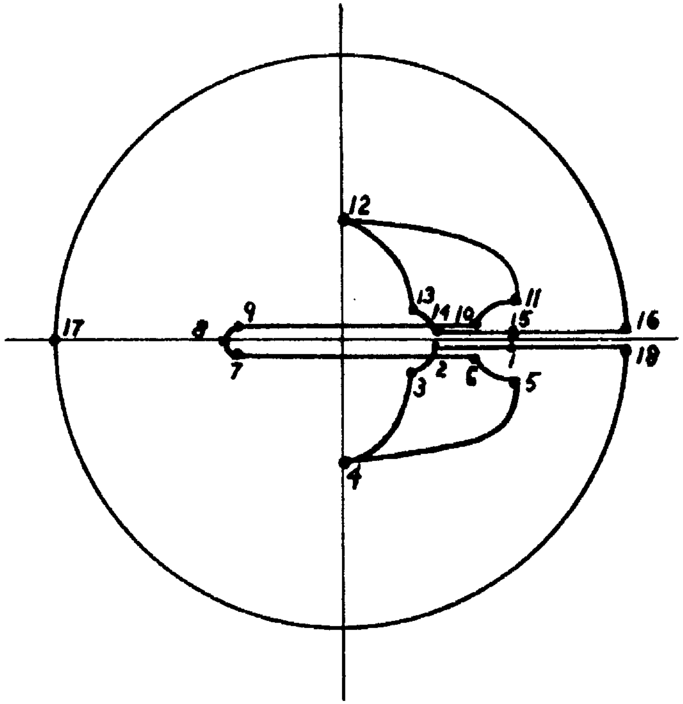


Figure 8

The G+H plane

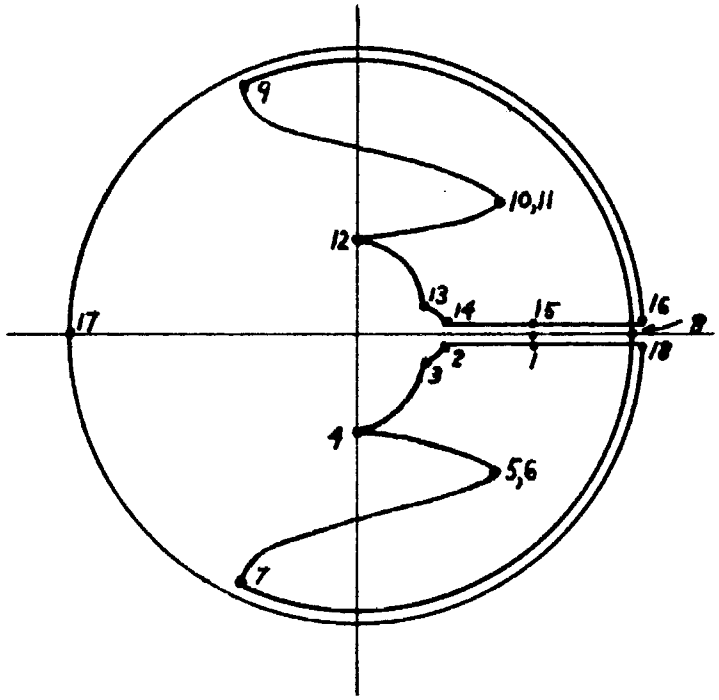


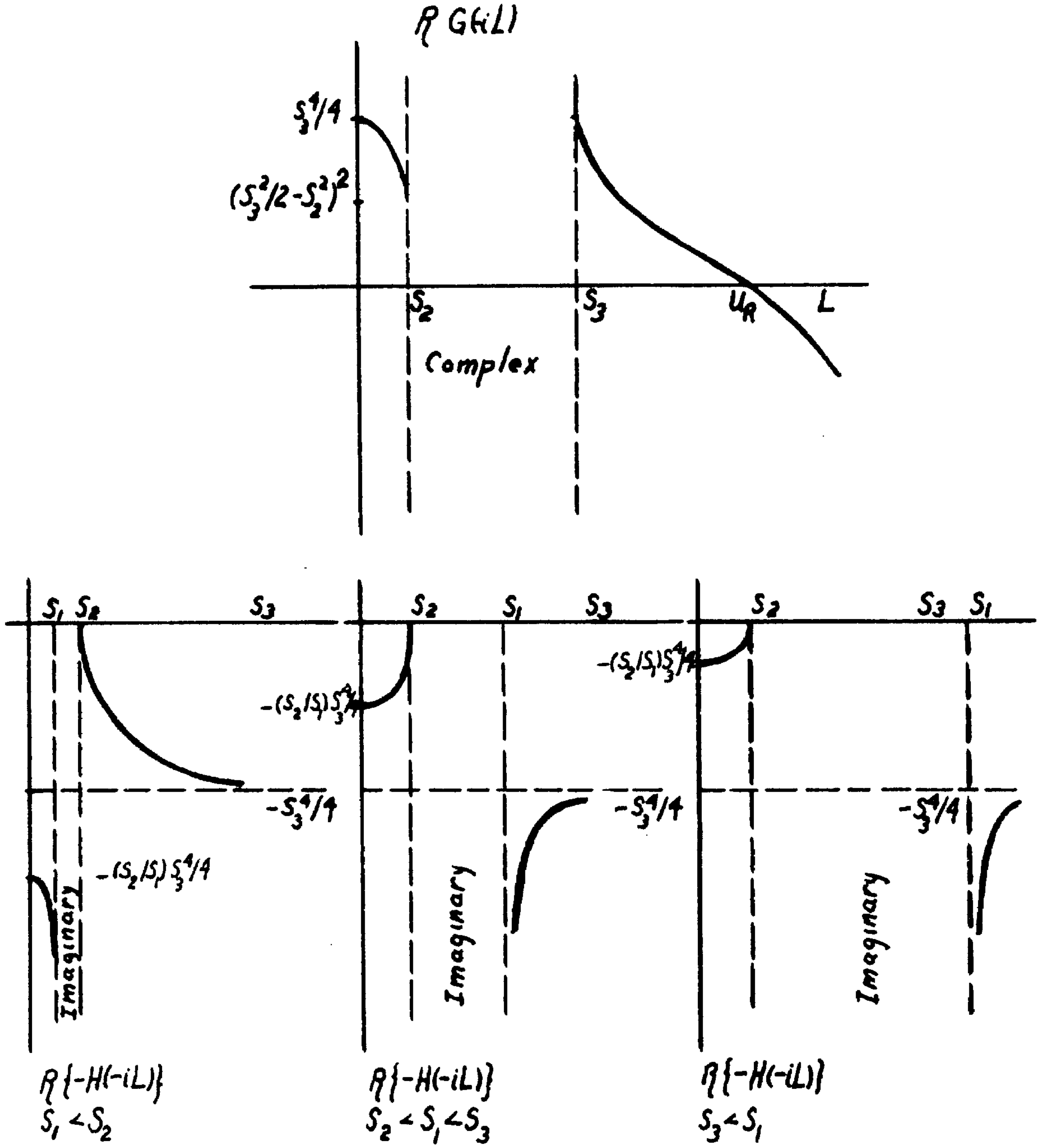
Figure 9

the origin of the H plane. The interval between the points $-iS_2-i\epsilon$ and $-iS_1+i\epsilon$ maps into the negative imaginary H axis. It is obvious that H decreases monotonically throughout this interval. The circle about the branch point $-iS_1$ maps into the infinite semi-circle to the right of the imaginary H axis. The rest of the mapping can be obtained directly from these results. It is important to note that the infinite semi-circle in the u plane maps into a single point in the H plane. Clearly there are no zeros of the function H within the contour.

The second step consists in using the procedure just outlined to obtain the mapping of the u contour in the G plane. The result is indicated in fig. 8. Points on the imaginary axis between the origin and the point $-iS_2+i\epsilon$ map into a finite segment of the positive real G axis. With one exception points on the imaginary axis between $-iS_2-i\epsilon$ and $-iS_3+i\epsilon$ map into the fourth quadrant of the G plane. The point $u=-iS_3/2^{1/2}$ maps into a point on the negative imaginary G axis. Points in the interval between $-iS_3-i\epsilon$ and $-iS_1-i\epsilon$ map into a finite segment of the real G axis. These points are confined entirely to the positive real G axis if S_1 does not exceed the Rayleigh pole, u_R. If S_1 exceeds the Rayleigh pole some of these points will map into the negative real G axis. The infinite semi-circle in the u plane maps into the infinite circle in the G plane. It is clear that if S_1 does not exceed the Rayleigh pole, the u contour encloses a zero of the function G ; if S_1 exceeds the value of the Rayleigh pole the contour encloses no zeros of the function G .

The mapping of the u contour in the $G+H$ plane can now be obtained by making a vector addition of the individual mappings. The result is indicated in fig. 9. The mapping circles the origin only once. This indicates that the function $G+H$ vanishes at only one point within the u contour. A similar result can be obtained for the upper half of the u plane. There still remains the possibility that $G+H$ vanishes at a point or points on the branch cut. It is not difficult to demonstrate that this is possible only if the Poisson's ratio of the solid vanishes, in which case the radical $(u^2+S_2^2)^{1/2}$ may be factored out of the expression for $G+H$.

The easiest way to locate the zeros is to actually plot the real part of G and of H as a function of $u=-iL\,(0\leq L<\infty)$. This has been done in fig. 10 for three cases. In the first case the liquid velocity exceeds the compressional velocity, V_S; in the second case the liquid velocity is intermediate between the compressional and transverse velocities; and in the third case the liquid velocity is less than the transverse velocity. The curves of fig. 10 show quite clearly that, in each case, the function $G+H$ vanishes at a point, $-iu_0$, which exceeds u_R in absolute value. In the subsequent discussion it will be shown that the existence of this singularity is connected with the presence of an interface or Stoneley wave. It is interesting to note that the velocity of the Stoneley wave and V_L approach each other in the limit as V_L approaches zero. It is also evident that the Stoneley wave velocity must always be less than the smallest body wave velocity in the system.



R G(iL)
$S_3^4/4$
$(S_3^2/2-S_2^2)^2$
S_2
S_3
U_R
L
Complex
S_1 S_2 S_3
$-S_3^4/4$
$-(S_2/S_1)S_3^4/4$
Imaginary
$R\{-H(-iL)\}$
$S_1<S_2$
S_2 S_1 S_3
$-(S_2/S_1)S_3^4/4$
$-S_3^4/4$
Imaginary
$R\{-H(-iL)\}$
$S_2<S_1<S_3$
S_2 S_3 S_1
$-(S_2/S_1)S_3^4/4$
$-S_3^4/4$
Imaginary
$R\{-H(-iL)\}$
$S_3<S_1$

Figure 10

I.7 Solution at Points off the Axis

In order to study many of the more interesting aspects of the response it is necessary to invert the transforms of the vertical and radial components of displacement at points off the axis. The steps leading to the inversion of $\bar{u}_z$ will now be considered.

The success of the Cagniard treatment rests, to a large extent, upon the fact that it is possible to express the dependence of the integrand in such a way that s appears only as a coefficient in an exponential. In the present case this is accomplished by introducing an integral representation for $J_0(su\rho)$ in equation 70. It is convenient to express the integral representation in the form (6)

$$J_0(su\rho) = \frac{2}{\pi} \mathcal{R} \int_0^{\pi/2} e^{-isu\rho\cos\omega} d\omega. \tag{88}$$

The expression for $\bar{u}_z$ reduces to

$$\bar{u}_z = a_0 s \chi(s)(2/\pi) \mathcal{R} \int_0^{\pi/2} \left\{ \int_0^{\infty} uA(u)\, e^{-s(a\Gamma_1 + iu\rho\cos\omega)} du \right\} d\omega. \tag{89}$$

The symbol $\mathcal{R}$ indicates that only the real part of the quantity on the right is to be considered. The interchange in the order of integration is legitimate due to the uniform convergence of the inner integral.

We now define the new variable t by the relation

$$t = a\Gamma_1 + iu\rho\cos\omega. \tag{90}$$

This transformation reduces the expression for $\bar{u}_z$ to the form

$$\bar{u}_z = a_0 s\chi(s)(2/\pi) \mathcal{R} \int_0^{\pi/2} \left\{ \int_{s_1 \Gamma_1}^{H_\omega} u(\omega,t) A[u(\omega,t)]\, e^{-st} \frac{du}{dt}\Big|_\omega dt \right\} d\omega, \tag{91}$$

where the subscript ω indicates that each point on the contour, H_ω, is dependent upon the value of ω. This contour is sketched in fig. 11.

Explicit expressions for u and $a(u)$, in terms of the independent variables t and ω, can be obtained from the defining relation for t; they have the form

(92) $u(\omega,t) = (-it\rho\cos\omega + \Gamma_1(t^2 - s_1^2\varsigma^2)^{1/2})/\varsigma^2$

and

(93) $a(\omega,t) = (t\Gamma_1 - i\rho\cos\omega\,(t^2 - s_1^2\varsigma^2)^{1/2})/\varsigma^2$,

where $\varsigma^2 = \Gamma_1^2 + \rho^2\cos^2\omega$. In order to keep u and $a(u)$ single-valued we cut the t plane along the real axis between the branch points $s_1\varsigma$ and $-s_1\varsigma$ and define the radical to be positive real on the real t axis to the right of $s_1\varsigma$. Equations 92 and 93 are useful in obtaining equivalent forms of the expression for the partial derivative of u with respect to t; namely

(94) $\left.\frac{\partial u}{\partial t}\right|_\omega = 1/(\Gamma_1 u/a + i\rho\cos\omega) = a/(t^2 - s_1^2\varsigma^2)^{1/2}$.

A study of equations 92, 93, and 94 indicates that the singularities of the function $u(\omega,t)\left.\frac{\partial u}{\partial t}\right|_\omega$ are located at the points $\pm s_1\varsigma$.

We next close the contour, H_ω, by adding to it the circular arc of infinite radius, C, and the portion of the positive real t axis which lies between the point $s_1\Gamma_1$ and infinity (fig. 11). In closing the contour we must be careful to stay above the branch cut. If we fail to do this, the mapping of the t contour in the u plane will not close on itself and we cannot apply Cauchy's integral theorem. This mapping is indicated in fig. 12. The branch point $s_1\varsigma$ maps into the point $-is_1\rho\cos\omega/\varsigma$. The segment of the real t axis which lies between the points $s_1\Gamma_1$ and $s_1\varsigma$ maps into a segment of the negative imaginary u axis which lies between the origin and the point $-is_1\rho\cos\omega/\varsigma$. We have already noted that the singularities of the function $A(u)$ are located at the branch points $\pm is_1$, $\pm is_2$, and $\pm is_3$ and

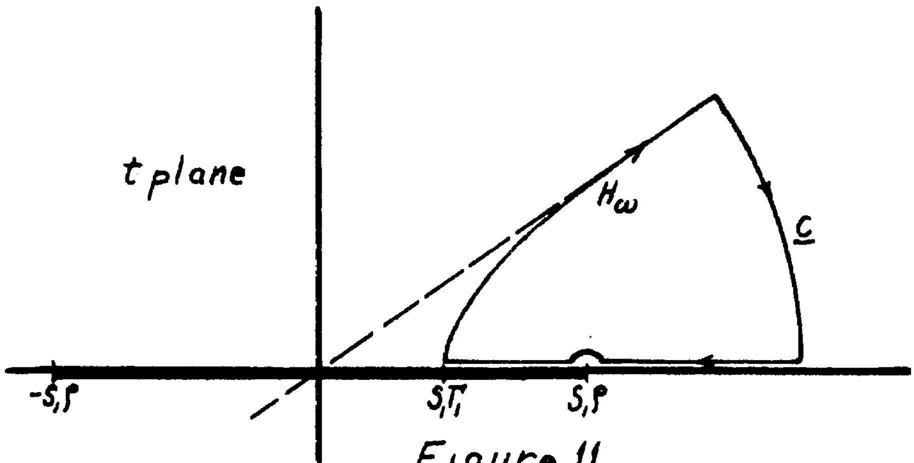


Figure 11

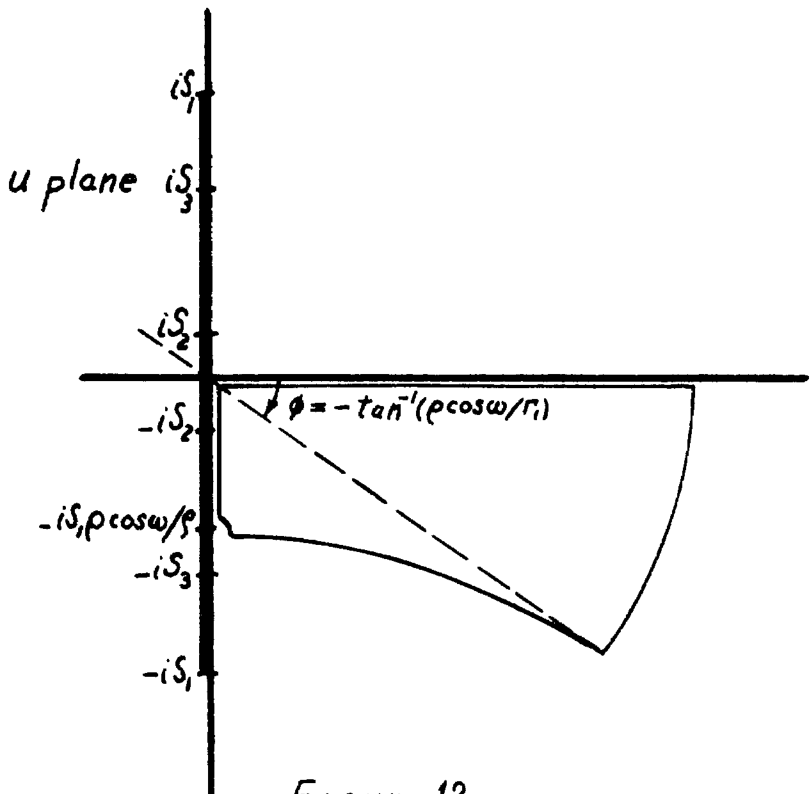


Figure 12

at the poles $\pm iu_0$. In the discussion which follows we will continue to assume the existence of finite cuts along the imaginary u axis between the branch points iS_1 and $-iS_1$, etc. The turning point, $-iS_1\rho\cos\omega/\beta$, is always less than S_1 in absolute value. This indicates that it must also be less than u_0. This statement follows from the fact that the Stoneley wave velocity cannot exceed the fluid velocity. These facts indicate that the closed contour encloses no singularities of the integrand and, therefore, the integral around it must vanish. We note that the presence of the exponential, e^{-st}, makes the result obtained from the integration along C arbitrarily small in the limit $\beta t \to \infty$. We therefore arrive at the conclusion that H_ω and the portion of the real t axis to the right of the point S_1T_1' and lying above the branch cut are equivalent contours. This fact enables us to rewrite equation 91 in the form

$$(95)\quad \bar{u}_z = a_0 S \mathcal{X}(S)(2/\pi)\beta \int_0^{\infty} \left\{ \int_{S_1T_1'+i\epsilon}^{\infty+i\epsilon} u(\omega,t) A[u(\omega,t)] e^{-st} \frac{du}{dt}\Big|_{\omega} dt \right\} d\omega,$$

where ϵ is a finite but small quantity which is inserted in equation 95 for the sole purpose of indicating which sign is to be associated with the radicals.

This expression for $\bar{u}_z$ can be put in the form of the direct Laplace transform by interchanging the order of integration. In order to justify the interchange we appeal to Fubini's theorem (7). This theorem requires that the double integral of the absolute value of the integrand exist. This is readily demonstrated if we recall that $A(u)$ is bounded everywhere except in the vicinity of the points $\pm iu_0$. In no case does the contour in the u plane approach these singular points. This fact permits us to replace $A(\omega)$ by a finite upper bound. Therefore, the result which is obtained from the integration of equation 95 cannot exceed

$$(96)\quad \left[\int_0^{\pi/2} d\omega \int_{S_1\Gamma_1}^{S_1\rho} e^{-St} \left| u(\omega,t) \frac{du}{dt}\Big|_\omega \right| dt + \int_0^{\pi/2} d\omega \int_{S_1\rho}^{\infty} e^{-St} \left| u(\omega,t) \frac{du}{dt}\Big|_\omega \right| dt \right] \text{Max.} |A(\omega)|$$

in absolute value. An investigation of the quantity $u(\omega,t)\frac{du}{dt}\Big|_\omega$ indicates that, in absolute value, it cannot exceed $(t^2\rho/\Gamma_1^3)|S_1^2\rho^2-t^2|^{1/2} + t^2r^2/\Gamma_1^4 + (\rho/\Gamma_1^3)/|S_1^2\rho^2-t^2|^{1/2}$. It is apparent that only the first term can lead to a divergent result. Proof of the existence of the double integral can now be obtained by integrating the expressions

$$(97)\quad \int_0^{\pi/2} d\omega \int_{S_1\Gamma_1}^{S_1\rho} e^{-St}\, t^2\, dt/(S_1^2\rho^2-t^2)^{1/2}$$

and

$$(98)\quad \int_0^{\pi/2} d\omega \int_{S_1\rho}^{\infty} e^{-St}\, t^2 dt/(t^2-S_1^2\rho^2)^{1/2}$$

by parts. Interchanging the order of integration in equation 95 leads to the expression

$$(99)\quad \overline{u}_z = a_0 S\chi(S)(2/\pi)\rho \int_{S_1\Gamma_1}^{\infty} e^{-St} \left\{ \int_0^{\pi/2} u(\omega,t) A[u(\omega,t)] \frac{du}{dt}\Big|_\omega d\omega \right\} dt .$$

A remarkable simplification can be obtained at this point by using the original transformation, equation 90, to replace the integration variable ω and the path along the real ω axis by a contour in the complex u plane. Equation 92 can be used to obtain the desired mapping. The point $\omega=0$ maps into the point

$$(100)\quad u_{\omega=0} = (-it\rho + \Gamma_1(t^2-S_1^2r^2)^{1/2})/r^2.$$

This point lies on the negative imaginary axis if $S_1\Gamma_1 \le t \le S_1 r$

and in the fourth quadrant if $t>s_1 r$. If $s_1\Pi_1 \leq t < s_1 r$, the interval, $0 \leq \omega \leq \cos^{-1}[(t^2 - s_1^2\Pi_1^2)^{1/2}/s_1\rho]$, maps into a finite segment of the negative imaginary u axis. The endpoint of this interval maps into the turning point

(101) $-i\,u_T = -i s_1 (1 - s_1^2\Pi_1^2/t^2)^{1/2}$.

It is evident that the imaginary part of u decreases monotonically throughout this interval. The interval, $\cos^{-1}[(t^2 - s_1^2\Pi_1^2)^{1/2}/s_1\rho] < \omega < \pi/2$, maps into the fourth quadrant of the u plane. Both the real and imaginary parts of u increase monotonically in this interval. If $t > s_1 r$, the interval, $0 \leq \omega < \pi/2$, maps into the fourth quadrant of the u plane. In this case both the real and imaginary parts of u increase monotonically throughout the interval. Regardless of the value of t the endpoint, $\omega = \pi/2$, maps into a point on the positive real u axis; namely

(102) $u_{\omega=\pi/2} = (t^2 - s_1^2\Pi_1^2)^{1/2}/\Pi_1$.

The mapping is sketched in fig. 13.

The desired transformation of the integrand can be obtained by substituting the partial derivative of ω with respect to u in equation 99. This partial derivative has the form

(103) $\left.\frac{\partial\omega}{\partial u}\right|_t = (\Pi_1 u/a + i\rho\cos\omega)/i\rho u\sin\omega = (\Pi_1 u/a + i\rho\cos\omega)/i((t-a\Pi_1)^2 + u^2\rho^2)^{1/2}$.

The product of the partial derivatives is

(104) $\left.\frac{\partial u}{\partial t}\right|_\omega \left.\frac{\partial\omega}{\partial u}\right|_t = 1/i((t-a\Pi_1)^2 + u^2\rho^2)^{1/2}$.

These relations reduce equation 99 to the form

(105) $\bar{u}_z = a_0 s\chi(s)(2/\pi)\beta \int_{s_1\Pi_1}^{\infty} e^{-st}\left\{ \int_{\underline{C}\; u_{\omega=0}}^{u_{\omega=\pi/2}} u A(u)\,du/i((t-a\Pi_1)^2 + u^2\rho^2)^{1/2} \right\} dt$,

where the contour, C, will depend on the particular instant

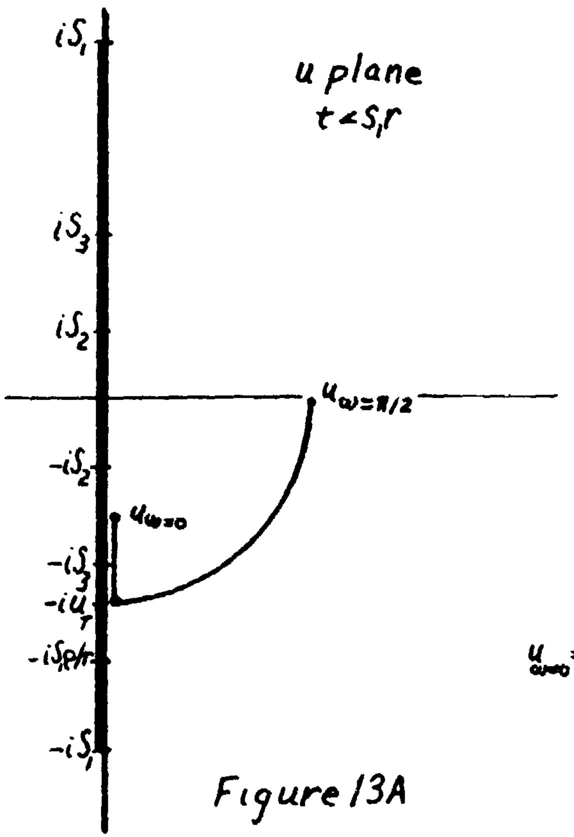


Figure 13A

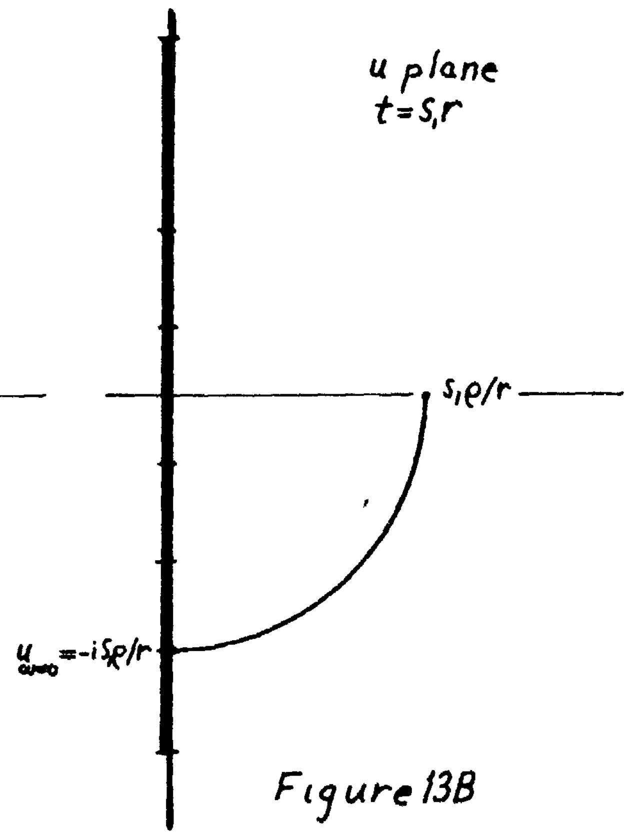


Figure 13B

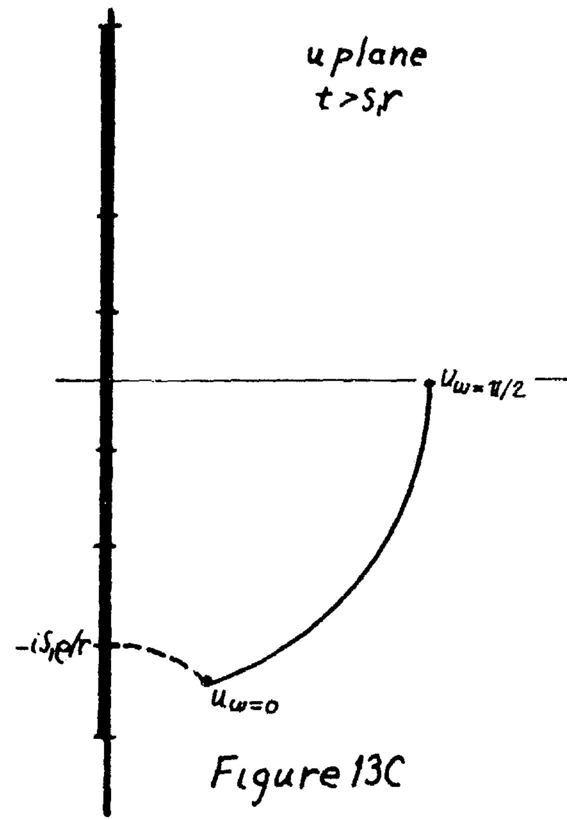


Figure 13C

at which the response is to be computed. If $\chi(s)=1/s$, the vertical displacement itself can be obtained directly from equation 105; it is

(106) $u_z(t;\rho,\Pi_1)\equiv 0, \quad t<S_1\Pi_1;$

$$= a_0(2/\pi)\iint_{\mathcal{C}}\int_{u_{\omega=0}}^{u_{\omega=\pi/2}} u\,A(u)\,du/i((t-a\Pi_1)^2+u^2\rho^2)^{1/2}, \quad t>S_1\Pi_1.$$

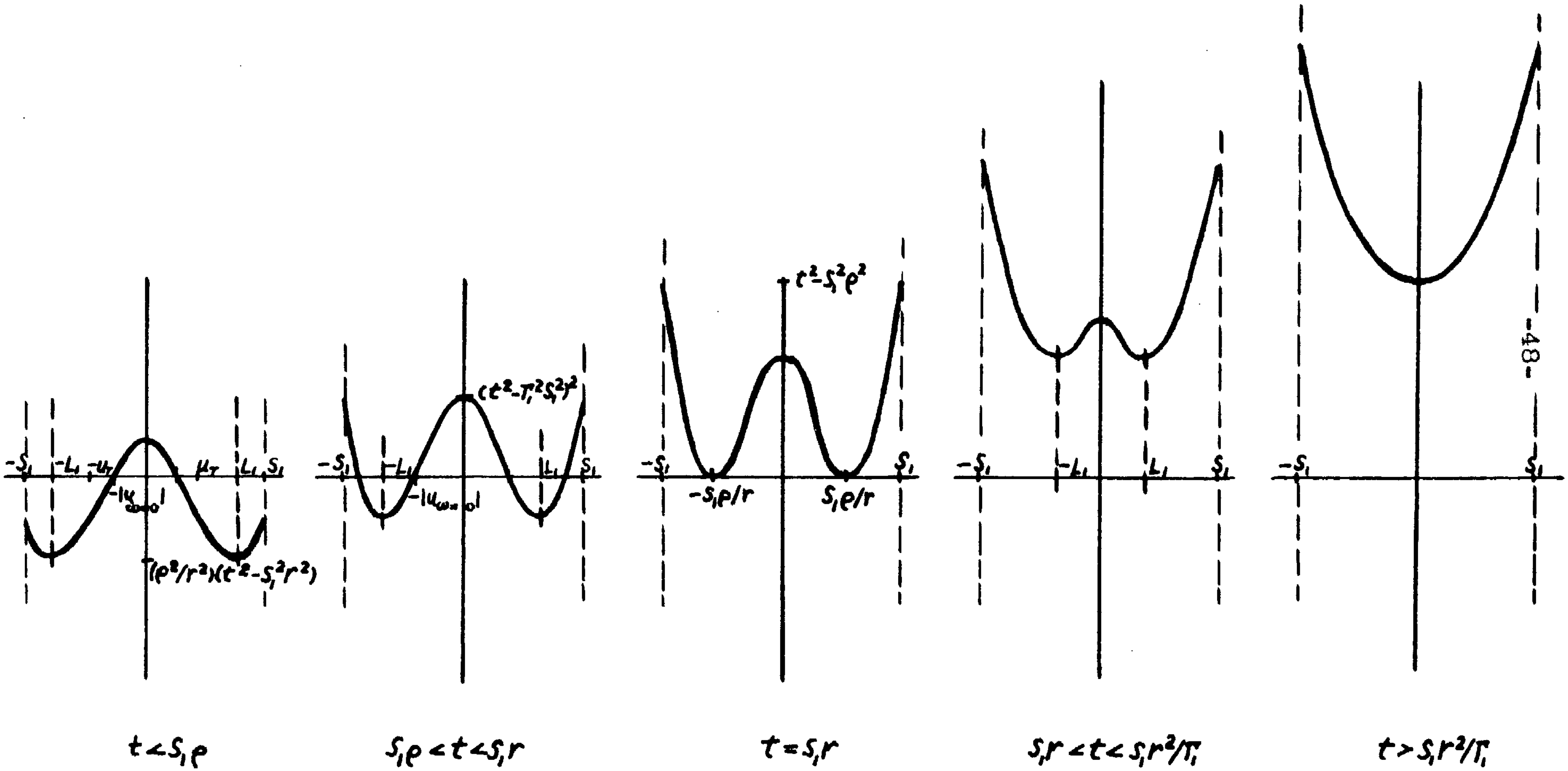
It must be kept in mind that this displacement field is to be associated with a source pressure variation which is given by minus the Dirac delta function.

I.7a Investigation of the Function $f(u)=(t-a\Pi_1)^2+u^2\rho^2$

We cannot proceed further without investigating the properties of the function $f(u)=(t-a\Pi_1)^2+u^2\rho^2$. First we will consider the behavior of $f(u)$ in the interval $-iS_1\leq u\leq iS_1$ at points which are located an infinitesimal distance to the right of the branch cut. The substitution $u=iL\,(-S_1\leq L\leq S_1)$ reduces $f(u)$ to the form

(107) $f(L)=(t-\Pi_1(S_1^2-L^2)^{1/2})^2-L^2\rho^2.$

This function has been sketched in fig. 14 for various values of the parameter, t. The following facts are readily established. If $t<S_1r^2/\Pi_1$, the first derivative of $f(L)$ vanishes at $L=0$ and at $L=\pm L_1=\pm(S_1^2-\Pi_1^2t^2/r^4)^{1/2}$. If $t\geq S_1r^2/\Pi_1$, the first derivative vanishes only at the origin. In the first case $f(L)$ has a relative maximum at $L=0$ ($f(0)=(t-S_1\Pi_1)^2$) and relative minima at $\pm L_1$. If $t\geq S_1r^2/\Pi_1$, the function has a relative minimum at the origin. $f(L)$ has the value $(\rho^2/r^2)(t^2-S_1^2r^2)$ at $L=\pm L_1$ and the value $t^2-S_1^2\rho^2$ at $L=\pm S_1$. These facts indicate that $f(L)$ vanishes at two points in the interval $-S_1\leq L\leq S_1$ if $t<S_1\rho$ or if $t=S_1r$ and at four points if $S_1\rho\leq t<S_1r$. It is important to note that if $t<S_1r$ the



$-S_1$
$-L_1$
$-\mu_T$
μ_T
L_1
S_1
$-|u_{\omega=0}|$
$(\rho^2/r^2)(t^2-S_1^2r^2)$
$t<S_1\rho$
$-S_1$
$-L_1$
L_1
S_1
$-|u_{\omega=0}|$
$(t^2-T_1^2S_1^2)^2$
$S_1\rho<t<S_1r$
$-S_1$
S_1
$t^2-S_1^2\rho^2$
$-S_1\rho/r$
$S_1\rho/r$
$t=S_1r$
$-S_1$
$-L_1$
L_1
S_1
$S_1r<t<S_1r^2/T_1$
$-S_1$
S_1
$t>S_1r^2/T_1$

Figure 14

point u_T lies in the interval $-L_1 < -u_T < -|u_{\omega=0}|$, while if $t = s_1 r$ the points $-L_1$, $-u_T$, and $-|u_{\omega=0}|$ coincide. These facts will be utilized many times in the investigation of the response in the time interval $s_1 r - \epsilon \le t \le s_1 r + \epsilon$.

The behavior of $f(u)$ at points which are located an infinitesimal distance to the left of the branch cut can be found in the same manner; the only difference being that the sign of the radical in equation 107 must now be positive. The facts concerning the behavior of $f(u)$ on the branch cut are summarized in the following remarks: (1) if $t < s_1 \rho$, the function vanishes at two points on the right side of the cut and at two points on the left side of the cut; (2) if $t = s_1 \rho$, the function vanishes at $\pm S_1$ and at two points on the right side of the cut; (3) if $s_1 \rho < t < s_1 r$, the function vanishes at four points on the right side of the cut; (4) if $t = s_1 r$, the function vanishes at two points on the right side of the cut; and (5) if $t > s_1 r$, the function does not vanish either on the right side or on the left side of the cut and is positive real on both sides of the cut. The effect of increasing t, in the interval $0 < t \le s_1 r$, is to cause the zeros of $f(u)$, which lie on the positive imaginary axis (and also those which lie on the negative imaginary axis), to migrate toward each other. Exactly at $t = s_1 r$ the two zeros coincide.

If $t > s_1 r$, the points $u_{\omega=0}$ and $u^*_{\omega=0}$ are the only zeros of $f(u)$. This fact can be established by mapping particular u contours in the f plane and by making use of the same procedures as were used previously in our study of the function $A(u)$. Each contour in the bottom row of fig. 15 is the mapping in the f plane of the u contour directly above it. A study of the first mapping on the left reveals that the associated u contour encloses two zeros of the function $f(\omega)$. It is found that if $t=s_1 r$ the mapping circles the origin twice in the counterclockwise direction (indicated by the paths 1-3-10-1 and 3-4-8-9) and twice in the clockwise direction (indicated by the paths 4-5-8 and 5-6-7). Therefore, in this case, the u contour encloses no zeros of the function $f(\omega)$. A study of the center mapping reveals that the associated u contour encloses a single zero of the function $f(u)$. It is found that if $t=s_1 r$ the mapping circles the origin once in the counterclockwise direction (indicated by the path 1-2-3-4) and once in the clockwise direction (indicated by the path 6-7-8-9). Therefore, in this case, just as in the previous case, the contour encloses no zeros of the function $f(u)$ if $t=s_1 r$. A study of the mapping on the right reveals that the u contour circles the origin once in the counterclockwise direction and once in the clockwise direction. This is true regardless of whether t is greater than or equal to $s_1 r$ and indicates that the u contour encloses no zeros of the function $f(u)$. We conclude that for $t<s_1 r$ the function $f(u)$ vanishes at four points in the complex u plane and for $t \geq s_1 r$ it vanishes at two points.

We will now determine whether the zeros of the function $f(u)$ are simple or multiple by examining the second term in the Taylor's series expansion of $f(u)$ about the point $u_{\omega=0}$. The first derivative of $f(u)$, evaluated at $u_{\omega=0}$, is

(108) $$\left.\frac{df}{du}\right|_{u=u_{\omega=0}} = 2u_{\omega=0}\left\{r^2 - \Pi_1 t/(u_{\omega=0}^2 + s_1^2)^{1/2}\right\}$$
$$= -2i\rho(t^2-s_1^2r^2)^{1/2}\left\{(-it\rho+\Pi_1(t^2-s_1^2r^2)^{1/2})/(t\Pi_1 - i\rho(t^2-s_1^2r^2)^{1/2})\right\}.$$

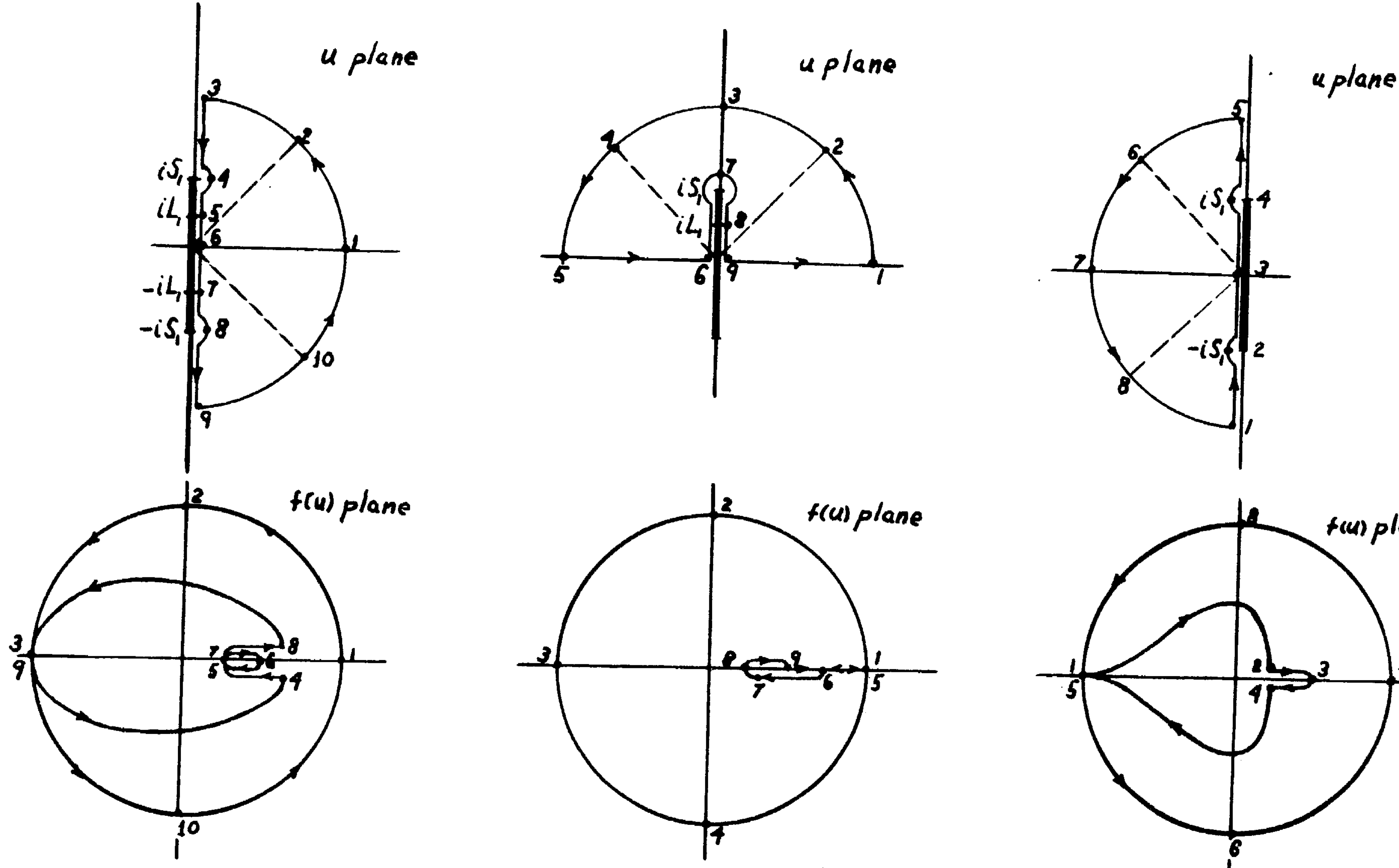

u plane
iS₁
iL₁
-iL₁
-iS₁
f(u) plane
u plane
iS₁
iL₁
f(u) plane
u plane
iS₁
-iS₁
f(u) plane

Figure 15

The second derivative of $f(u)$, evaluated at $u_{\omega=0}$, is

$$(109)\quad \left.\frac{d^2 f}{du^2}\right|_{u=u_{\omega=0}} = 2\left\{r^2 - T_1 S_1^2 t/(u_{\omega=0}^2 + S_1^2)^{3/2}\right\}.$$

It is clear that $u_{\omega=0}$ is a simple zero if $t \neq S_1 r$ and a double zero if $t = S_1 r$. An examination of the expansion of $f(\omega)$ about the point $u^*_{\omega=0}$ leads to the same conclusion.

I.7b The Refracted Wave

These facts concerning the behavior of $f(\omega)$ will be used to investigate the response in the time interval preceding the arrival of the reflected wave. If $S_1 T_1 < t < S_1 r$, the singular points, $u_{\omega=0}$ and $u^*_{\omega=0}$, lie on the imaginary u axis. The contour, C, has already been sketched in fig. 13A. This contour can be replaced by the one indicated in fig. 16A by the dashed lines. We have already seen that $f(\omega)$ is positive real along that part of the imaginary u axis which lies between the branch points $u_{\omega=0}$ and $u^*_{\omega=0}$. It can be easily established that this function cannot vanish on the real u axis unless $t = S_1 T_1$. Since we are only interested in the response at points off the vertical axis we need not consider this particular instant of time. We also recall that the function $A(u)$ is real when u is real and when u lies on the imaginary axis between the branch points iS_2 and $-iS_2$. We conclude that $u_z(t;\rho,T_1)$ vanishes identically if $|u_{\omega=0}| \leq S_2$. The location of the point $u_{\omega=0}$ depends on the time, the spatial coordinates, and the liquid velocity. In order for there to be a contribution to the response, in the interval $S_1 T_1 < t < S_1 r$, these parameters must be adjusted in such a way that the inequality $|u_{\omega=0}| > S_2$ holds. This means that

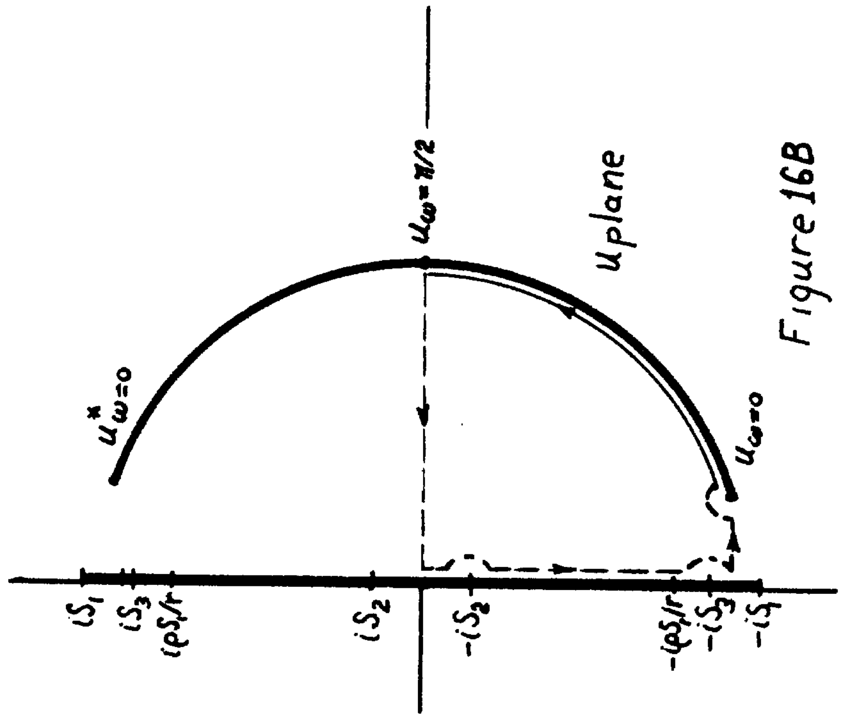


Figure 16B

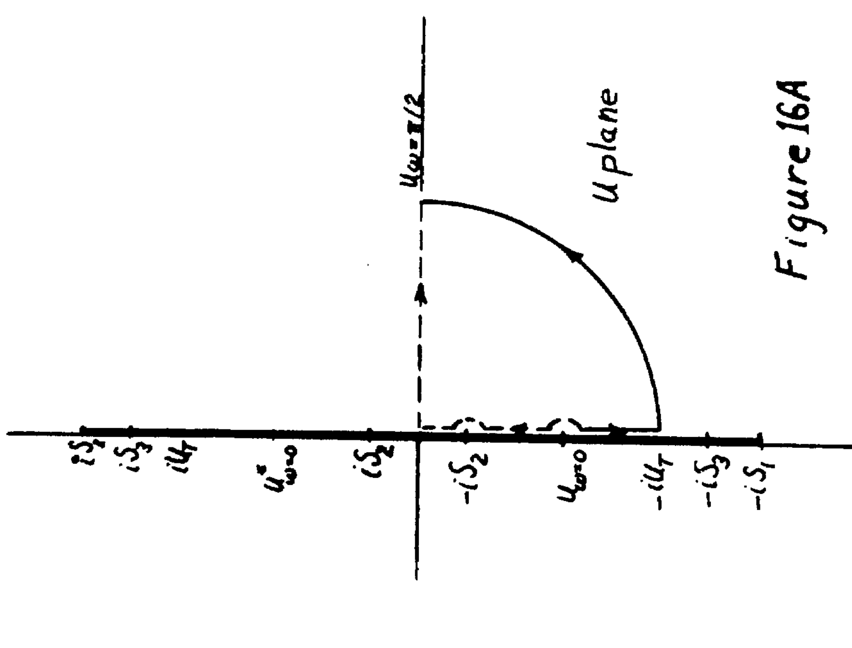


Figure 16A

(110) $(t\rho - \Pi_1(S_1^2r^2 - t^2)^{1/2})/r^2 > S_2$.

If the function $g(t)(=(t\rho - \Pi_1(S_1^2r^2 - t^2)^{1/2})/r^2 - S_2)$ is plotted as a function of t, it is found that equation 110 cannot be satisfied for any value of t, in the interval $S_1\Pi_1 < t < S_1 r$, if $(S_1\rho/r) - S_2 < 0$. This means that, in the time interval preceding the arrival of the reflected wave, $u_z(t;\rho,\Pi_1)$ can be different from zero only in that region of space for which $(S_1\rho/r) - S_2 > 0$. According to ray theory this is just the region of space in which the refracted wave is the first arrival (fig. 17). If $(S_1\rho/r) - S_2 > 0$, equation 110 is satisfied for all values of time which exceed $t_0 = \rho S_2 + \Pi_1(S_1^2 - S_2^2)^{1/2}$.

In fig. 17 we have divided space into the two regions I and II. In region I the reflected wave is the first arrival, in region II the refracted wave is the first arrival. In order to study the refracted wave we will confine our attention to region II and to the time interval $t_0 \leq t < S_1 r$. It is apparent that, in this time interval, $u_z(t;\rho,\Pi_1)$ is given by the expression

(111) $$u_z(t;\rho,\Pi_1) = a_0(2/\pi)\int_{u_{\omega=0}}^{-iS_2} (\mathcal{J}A(\omega))\,u\,du\Big/((t-a\Pi_1)^2 + u^2\rho^2)^{1/2}, \quad t_0 \leq t < S_1 r.$$

If the compressional velocity in the liquid is intermediate between the transverse and compressional velocities in the solid, the imaginary part of $A(u)$ has the form

(112) $$\mathcal{J}A(u) = (u^2 + S_3^2/2)^2 (S_3^2/2)(\sigma_0/\sigma_S)((-u^2 - S_2^2)/(u^2 + S_1^2))^{1/2} \cdot$$
$$\left\{1\Big/\left[(u^2+S_3^2/2)^4 + \{u^2(-u^2-S_2^2)^{1/2}(u^2+S_3^2)^{1/2} - (S_3^4/4)(\sigma_0/\sigma_S)((-u^2-S_2^2)/(u^2+S_1^2))^{1/2}\}^2\right]\right\}.$$

It is important to note that this expression is positive or zero for all values of u on the path of integration. The result which is obtained from integrating equation 111 must

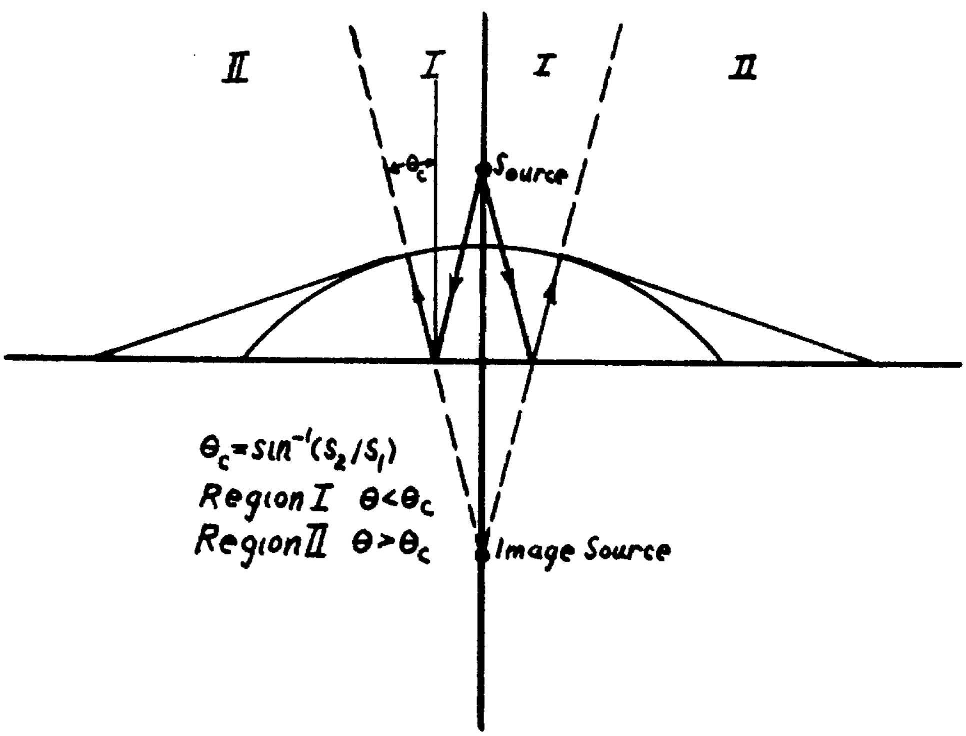

II
I
I
II
θc
Source
θc=sin⁻¹(S2/S1)
Region I θ<θc
Region II θ>θc
Image Source

Figure 17

therefore be a positive quantity. This fact indicates that the vertical motion at the point of observation must be directed towards the interface. This result can be interpreted if we recall that the source pressure variation, corresponding to $Y(s)=1/s$, is given by minus the Dirac delta function. This means that at the instant the acoustic wave reaches the interface there is a release of pressure which causes that part of the interface which is in contact with the negative pressure area to move upward. This upward motion in the region of interaction must be accompanied by a downward motion at the periphery. It is this downward motion which causes the fluid to move toward the interface.

If the compressional velocity in the liquid is less than the transverse velocity in the solid, the imaginary part of $A(u)$ is given by equation 112 when u lies between the branch points $-iS_2$ and $-iS_3$ and is given by

$$(113)\quad \mathcal{I}A(u) = \left\{(u^2+S_3^2/2)^2+u^2(-u^2-S_2^2)^{1/2}(-u^2-S_3^2)^{1/2}\right\}(S_3^4/2)(\sigma_0/\sigma_S)\left((-u^2-S_2^2)/(u^2+S_1^2)\right)^{1/2}\cdot$$
$$\left\{1/\left[\left\{(u^2+S_3^2/2)^2+u^2(-u^2-S_2^2)^{1/2}(-u^2-S_3^2)^{1/2}\right\}^2+\left\{(S_3^4/4)(\sigma_0/\sigma_S)\left((-u^2-S_2^2)/(u^2+S_1^2)\right)^{1/2}\right\}^2\right]\right\}$$

when u lies between $-iS_3$ and $u_{\omega=0}$. We have now reduced the original integral to a form which can be evaluated quite easily by numerical methods. We note that in every case the magnitude of the refracted arrival must be critically dependent upon the density ratio and the rigidity of the bottom.

I.7c The Reflected Wave

We next consider the response in region II in the time interval, $S_1 r-\epsilon \le t < S_1 r$, just preceding the arrival of the reflected wave. The change of variable, $s=a(u)$, reduces equation 111 to the form

(114) $$u_z(t;\rho,T_1)=(a_0/r)(2/\pi)\int_{(tT_1+\rho(S_1^2r^2-t^2)^{1/2})/r^2}^{(S_1^2-S_2^2)^{1/2}}(dA(\Lambda))\Lambda d\Lambda/\{(\Lambda-tT_1/r^2)^2-(\rho^2/r^4)(S_1^2r^2-t^2)\}^{1/2}.$$

This integral can be transformed into the Stieltjes form if we note that the part of the integrand which does not contain $dA(\Lambda)$ can be expressed as the differential of the function $Q(\Lambda)$, where

(115) $$Q(\Lambda)=\{(\Lambda-tT_1/r^2)^2-(\rho^2/r^4)(S_1^2r^2-t^2)\}^{1/2} + (tT_1/r^2)\,\mathrm{Log}\{(\Lambda-tT_1/r^2)+[(\Lambda-tT_1/r^2)^2-(\rho^2/r^4)(S_1^2r^2-t^2)]^{1/2}\}.$$

Sufficient conditions for the applicability of the first mean value theorem for Stieltjes integrals are that $dA(\Lambda)$ be a continuous function of Λ and that $Q(\Lambda)$ be monotonic in the interval $(tT_1+\rho(S_1^2r^2-t^2)^{1/2})/r^2\leq\Lambda\leq(S_1^2-S_2^2)^{1/2}$ (8). Application of the mean value theorem reduces equation 114 to

(116) $$u_z(t;\rho,T_1)=(a_0/r)(2/\pi)(dA(\Lambda'))\left\{Q((S_1^2-S_2^2)^{1/2})-(tT_1/r^2)\,\mathrm{Log}((\rho^2/r^2)(S_1^2r^2-t^2)^{1/2})\right\},$$

where $(tT_1+\rho(S_1^2r^2-t^2)^{1/2})/r^2\leq\Lambda'\leq(S_1^2-S_2^2)^{1/2}$. We now see that the magnitude of the response diverges logarithmically when t is allowed to approach S_1r. This is a rather remarkable result in that it indicates that the reflected wave has a forerunner in that region of space where the refracted wave is the first arrival.

We next consider the response in region II in the time interval, $S_1r<t\leq S_1r+\epsilon$, just following the arrival of the reflected wave. The branch points, $u_{\omega=0}$ and $u^*_{\omega=0}$, are now located in the fourth and first quadrants of the complex u plane. In order to keep $\{f(\omega)\}^{1/2}$ single-valued we join these branch points by the branch cut indicated in fig. 16B by the heavy dark line. The path of integration lies an infinitesimal distance to the left of that portion of the branch cut

which lies in the fourth quadrant. An equivalent contour is indicated in fig. 16B by the dashed line. The fact that $f(u)$ is positive real on the real u axis and on the imaginary u axis between the branch points $-iS_1$ and iS_1 enables us to rewrite equation 106 in the form

$$(117)\quad u_z(t;\rho,\Pi_1) = a_0(2/\pi)\int_{-it\rho/r^2}^{-iS_2} \frac{u(\Im(A(\omega)))\,du}{((t-a\Pi_1)^2+u^2\rho^2)^{1/2}} + a_0(2/\pi)\Re\int_{u_{\omega=0}}^{-it\rho/r^2} \frac{u\,A(u)\,du}{i\,((t-a\Pi_1)^2+u^2\rho^2)^{1/2}} .$$

The procedure which was used in studying the response at times just prior to the arrival of the reflected wave is also applicable here. The substitution $\Lambda = a(\omega)$ and the use of the first mean value theorem for Stieltjes integrals reduce the first integral in equation 117 to the form

$$(118)\quad a_0(2/\pi)\int_{-it\rho/r^2}^{-iS_2} u\,A(\omega)\,du/((t-a\Pi_1)^2+u^2\rho^2)^{1/2} = (a_0/r)(2/\pi)(\Im A(\Lambda'))\{Q((S_1^2-S_2^2)^{1/2}) - Q((S_1^2-t^2\rho^2/r^4)^{1/2})\},$$

where $(S_1^2-t^2\rho^2/r^4)^{1/2} \le \Lambda' \le (S_1^2-S_2^2)^{1/2}$. It can be readily verified that $Q((S_1^2-t^2\rho^2/r^4)^{1/2})$ diverges logarithmically in the limit $t \to S_1 r^+$. We have now shown that in region II the reflected wave has a forerunner and that the response diverges logarithmically when t approaches $S_1 r$ either from the right or from the left.

We next consider the response in region I in the time interval, $S_1 r < t \le S_1 r + \epsilon$, immediately following the arrival of the reflected wave. The singular point, $u_{\omega=0}$, lies in the fourth quadrant. The imaginary part of $u_{\omega=0}$ cannot exceed S_2 in absolute value if the values of t are restricted to a sufficiently limited interval following the arrival of the reflected wave. This fact indicates that the response can be obtained by integrating along a contour which is parallel to the positive real u axis and a distance $t\rho/r^2$ below it and which extends from the point $-it\rho/r^2$ to the point $u_{\omega=0}$. In order to evaluate this integral we make the substitution $t=s_1r+\epsilon$ and define the new variable η by the relation $\eta=is_1\rho/r+i\epsilon\rho/r^2+u$. The mean value theorem for Riemann integrals can then be used to reduce the integral appearing in equation 106 to the form

$$(119)\quad u_z(s_1r+\epsilon;\rho,\pi_1)=-(a_0/\pi)2R\Big\{(-i\rho s_1/r)A(-i\rho s_1/r)\cdot \int_0^{\pi_1(2\epsilon s_1 r)^{1/2}/r^2} d\eta/i\Big\{[s_1r+\epsilon-\pi_1\{(-i\rho s_1/r-i\rho\epsilon/r^2+\eta)^2+s_1^2\}^{1/2}]^2+\rho^2[-i\rho s_1/r-i\rho\epsilon/r^2+\eta]^2\Big\}^{1/2}.$$

The integration of this expression is involved algebraically but straightforward. We note that terms which contain the second power of η must be carried in the calculations since η^2 becomes comparable with ϵ at the upper limit. Equation 119 can be reduced to the form

$$(120)\quad u_z(s_1r^+;\rho,\pi_1)=-a_0(2/\pi)R\Big\{(-s_1\pi_1/r^2)A(-is_1\rho/r)\int_0^{\pi_1(2\epsilon s_1 r)^{1/2}/r^2} d\eta/(2\epsilon s_1\pi_1^2/r^2-\eta^2)^{1/2}\Big\}$$
$$=(a_0/2)(s_1\pi_1/r^2)\Big\{A(-i\rho s_1/r)+A(i\rho s_1/r)\Big\}.$$

The expression on the right is just the discontinuity in the response at the time s_1r. The fact that the discontinuity is finite indicates that the S which appears in equation 105 can be removed by integrating that expression by parts. The effect of performing this operation is to introduce the expression on the right side of equation 120 as a reflection coefficient. This same procedure was used previously to separate out the reflection coefficient at axial points. It is interesting to compare the result obtained by setting $\rho=0$ in equation 120 with the acoustic term in equations 75 and 77. In region II the factor S cannot be removed by integrating by parts. The reason for this lies in the fact that the response diverges logarithmically at $t=s_1r$. It is clear that in region II the usual conception of a reflection coefficient, in the sense of a quantity which measures the fraction of the energy in the incident acoustic pulse which is reflected in various directions, must be abandoned.

I.7d Evaluation of the Response in the Time Interval $t > S, r$

We next consider the response in the time interval following the arrival of the reflected wave. The singularities of the function $\{f(u)\}^{1/2}$ are located at the branch points $u_{\omega=0}$ and $u^*_{\omega=0}$. An expression for the real part of the integral appearing in equation 106 can be obtained by adding to it, its complex conjugate and by dividing the sum by two. The expression for the vertical displacement then reduces to the form

$$(121) \quad u_z(t;\rho,\Pi_1) = (a_0/\pi i)\int_{u_{\omega=0}}^{u^*_{\omega=0}} u\,A(u)\,du\,/\,((t-a\Pi_1)^2+u^2\rho^2)^{1/2}.$$

In order to keep the radical, $\{f(u)\}^{1/2}$, single-valued we cut the u plane along the dark line in fig. 18A connecting the branch points $u_{\omega=0}$ and $u^*_{\omega=0}$. The sign of the radical, $\{f(u)\}^{1/2}$, is then negative at points on the real u axis which lie to the right of the branch cut. The actual path of integration lies an infinitesimal distance to the left of the cut. We note that the integral along the contour which lies an infinitesimal distance to the right of the cut reduces to that given in equation 121, if the direction in which the integration is carried out is taken in the opposite sense. This is a consequence of the fact that the sign of the function, $\{f(u)\}^{1/2}$, changes if a complete loop is made about either of the branch points. It is with this fact in mind that we deform the original contour

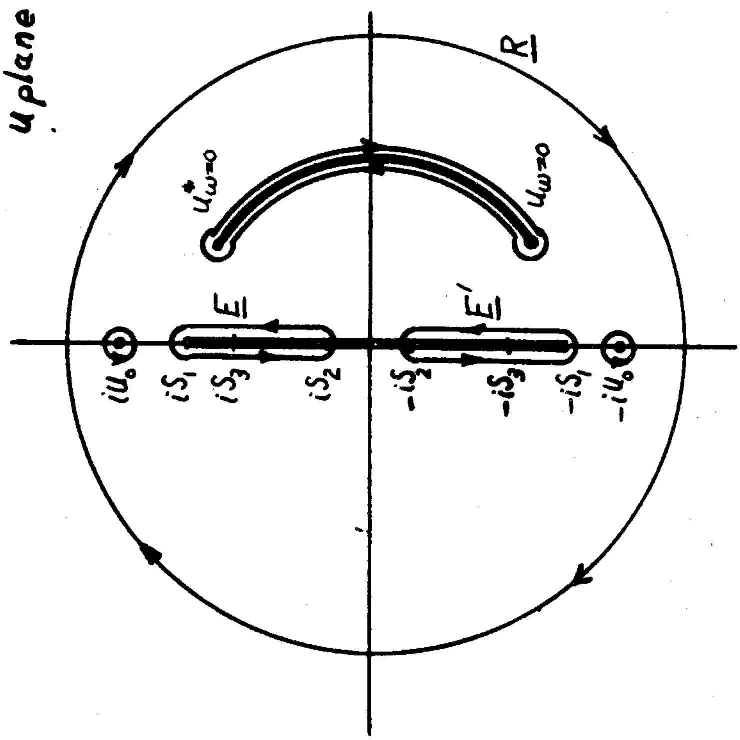


Figure 18B

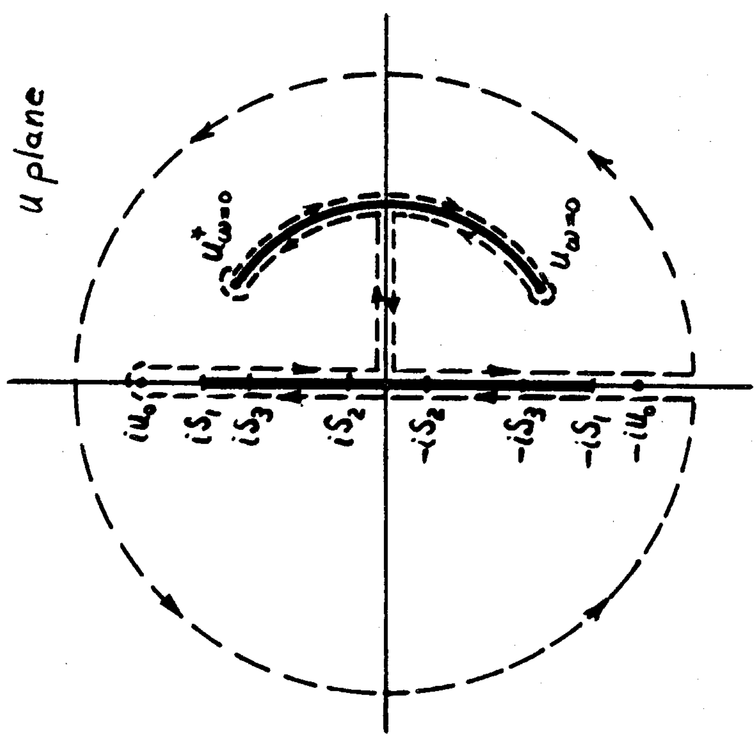


Figure 18A

into the one which is indicated in fig. 18A by the dashed lines.

It is clear that the closed contour encloses no singularities of the integrand. The integral around it must therefore vanish. The parts of the contour which parallel the real u axis are traversed in opposite directions and therefore cancel each other. The parts of the contour which parallel the imaginary axis in the intervals $iS_1 \leq u \leq iu_0 - i\epsilon$, $-iu_0 + i\epsilon \leq u \leq -iS_1$, and $-i\infty < u \leq -iu_0 - i\epsilon$ must also vanish and for the same reason. The integral along that portion of the imaginary axis which lies between iS_2 and $-iS_2$ must also vanish. This is a consequence of the fact that only the sign of u changes when the contour crosses the real u axis. These facts have been used to express equation 121 in the form

(122) $$(a_0/\pi i)\int_{u_{\omega=0}}^{u^*_{\omega=0}} \{\,\}du = (a_0/2\pi i)\left[\int_{E'}\{\,\}du + \int_{-iu_0}\{\,\}du + \int_{R}\{\,\}du + \int_{iu_0}\{\,\}du + \int_{E}\{\,\}du\right],$$

where $\{\,\} = uA(u)/((t-aT_1)^2+u^2\rho^2)^{1/2}$. The meaning of the subscripts is indicated in fig. 18B. The integral $\int_R$ can be evaluated by expanding the integrand in powers of u. The first two terms in the asymptotic expansions of $A(u)$ and $u/(f(u))^{1/2}$ are

(123) $$A(u) \underset{u\to\infty}{\sim} 1 - (\sigma_0/\sigma_S)(S_3^4/(S_3^2-S_2^2))(1/u^2)$$

and

(124) $$u/(f(u))^{1/2} \underset{u\to\infty}{\sim} 1/r + tT_1/ur^3.$$

The integral therefore has the value

(125) $$(a_0/2\pi i)\int_R uA(u)du/((t-aT_1)^2+u^2\rho^2)^{1/2} = -a_0 tT_1/r^3, \quad t > S_1 r.$$

The linear increase with time is a consequence of the fact that the area under the pressure-time curve, corresponding to

$X(s) = 1/S$, does not vanish.

The evaluation of the integrals around the poles is straightforward. The residue at the point $-iu_o$ is given by the expression

(126) $$(1/2\pi i)\int_{-iu_o}\{\}du = 2iu_o H(-iu_o)/[(t+iT_1(u_o^2-S_1^2)^{1/2})^2-u_o^2\rho^2]^{1/2}[\partial G/\partial u+\partial H/\partial u]_{u=-iu_o},$$

where

(127) $$\partial G/\partial u+\partial H/\partial u = u\{4(u^2+S_3^2/2)-u^2[((u^2+S_2^2)/(u^2+S_3^2))^{1/2}+((u^2+S_3^2)/(u^2+S_2^2))^{1/2}]$$
$$-2(u^2+S_2^2)^{1/2}(u^2+S_3^2)^{1/2}+(\sigma_o/\sigma_s)(S_3^4/4)((S_1^2-S_2^2)/(u^2+S_1^2)^2)((u^2+S_1^2)/(u^2+S_2^2))^{1/2}\}.$$

The sum of the residues is

(128) $$(1/2\pi i)\left[\int_{iu_o}\{\}du+\int_{-iu_o}\{\}du\right] = -(\sigma_o/\sigma_s)S_3^4((u_o^2-S_2^2)/(u_o^2-S_1^2))^{1/2}K(t;\rho,T_1)\cdot$$
$$\Big[1/\{4(-u_o^2+S_3^2/2)+u_o^2[((u_o^2-S_2^2)/(u_o^2-S_3^2))^{1/2}+((u_o^2-S_3^2)/(u_o^2-S_2^2))^{1/2}]+2(u_o^2-S_3^2)^{1/2}(u_o^2-S_2^2)^{1/2}$$
$$+(\sigma_o/\sigma_s)(S_3^4/4)((S_1^2-S_2^2)/(u_o^2-S_1^2)^2)((u_o^2-S_1^2)/(u_o^2-S_2^2))^{1/2}\}\Big]$$

The coefficient of $K(t;\rho,T_1)$ relates the amplitude of the Stoneley wave to the elastic properties of the liquid-solid system. The actual time variation of the response is given by the expression

(129) $$K(t;\rho,T_1) = \cos\{\pi/2-(1/2)\tan^{-1}[2tT_1(u_o^2-S_1^2)^{1/2}/(u_o^2r^2-T_1^2S_1^2-t^2)]\}\cdot$$
$$\{1/[(t^2+T_1^2S_1^2-u_o^2r^2)^2+4t^2T_1^2(u_o^2-S_1^2)]^{1/4}\},\ S_1r\leq t\leq(u_o^2r^2-T_1^2S_1^2)^{1/2};$$
$$= \cos\{(1/2)\tan^{-1}[2tT_1(u_o^2-S_1^2)^{1/2}/(t^2+T_1^2S_1^2-u_o^2r^2)]\}\cdot$$
$$\{1/[(t^2+T_1^2S_1^2-u_o^2r^2)^2+4t^2T_1^2(u_o^2-S_1^2)]^{1/4}\},\ t>(u_o^2r^2-T_1^2S_1^2)^{1/2}.$$

In order to obtain the response which is associated with an input step function in the displacement potential we convolute $K(t;\rho,T_1)$ with the function

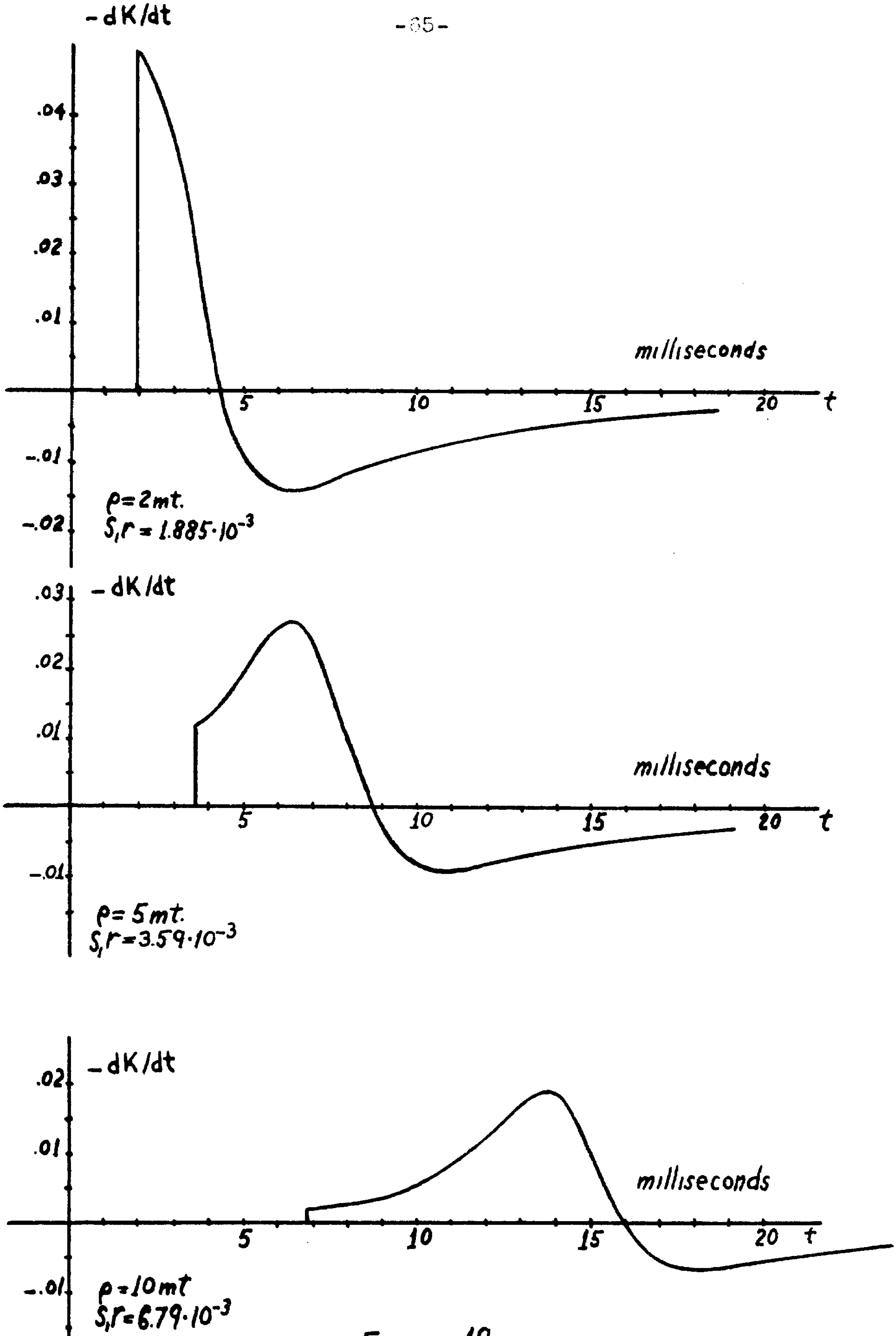
(130) $$F(t)\equiv 0,\ t<T,$$
$$= -(P_o/\sigma_o)\sin(2\pi/T)t,\ 0\leq t\leq T,$$

and require that the period, T, approach zero and the pressure amplitude, P_0, approach infinity in such a way that the product $P_0 T^2/2\pi\sigma_0$ approaches unity. This procedure amounts to taking the time derivative of $K(t;\rho,\Pi_1)$. In fig. 19 the time derivative of $K(t;\rho,\Pi_1)$ has been plotted as a function of time for various values of ρ. In each case both the source and receiver are located one meter above the interface. The elastic parameters which were used in making the calculations are identical with those used in Case I.

There is a very striking resemblance between the time variation of $-dK/dt$ at a radial distance of two meters and the time variation of the correction term at axial points (fig. 3). This fact can be used to argue that the formation of the phase, which we identify as the Stoneley wave at large values of ρ, begins at the instant the direct wave reaches the interface.

In fig. 20 we have plotted the amplitude of the response as a function of ρ on log-log graph paper. A straight line can be drawn through all the points for which ρ is greater than two meters. The slope of the straight line is minus one-half. The amplitude of the Stoneley wave therefore decays like $\rho^{-1/2}$. Such a result is to be expected for a wave that spreads in two dimensions.

A time-distance curve reveals that the point at which the response is a maximum travels with the velocity $1/u_0$. An investigation of the dependence of u_0 upon the elastic properties of the liquid-solid system can be used to predict how these



-dK/dt
.04
.03
.02
.01
-.01
-.02
milliseconds
5
10
15
20
t
ρ=2mt.
$S_1 r = 1.885 \cdot 10^{-3}$
-dK/dt
.03
.02
.01
-.01
milliseconds
5
10
15
20
t
ρ= 5mt.
$S_1 r = 3.59 \cdot 10^{-3}$
-dK/dt
.02
.01
-.01
milliseconds
5
10
15
20
t
ρ=10mt
$S_1 r = 6.79 \cdot 10^{-3}$

Figure 19

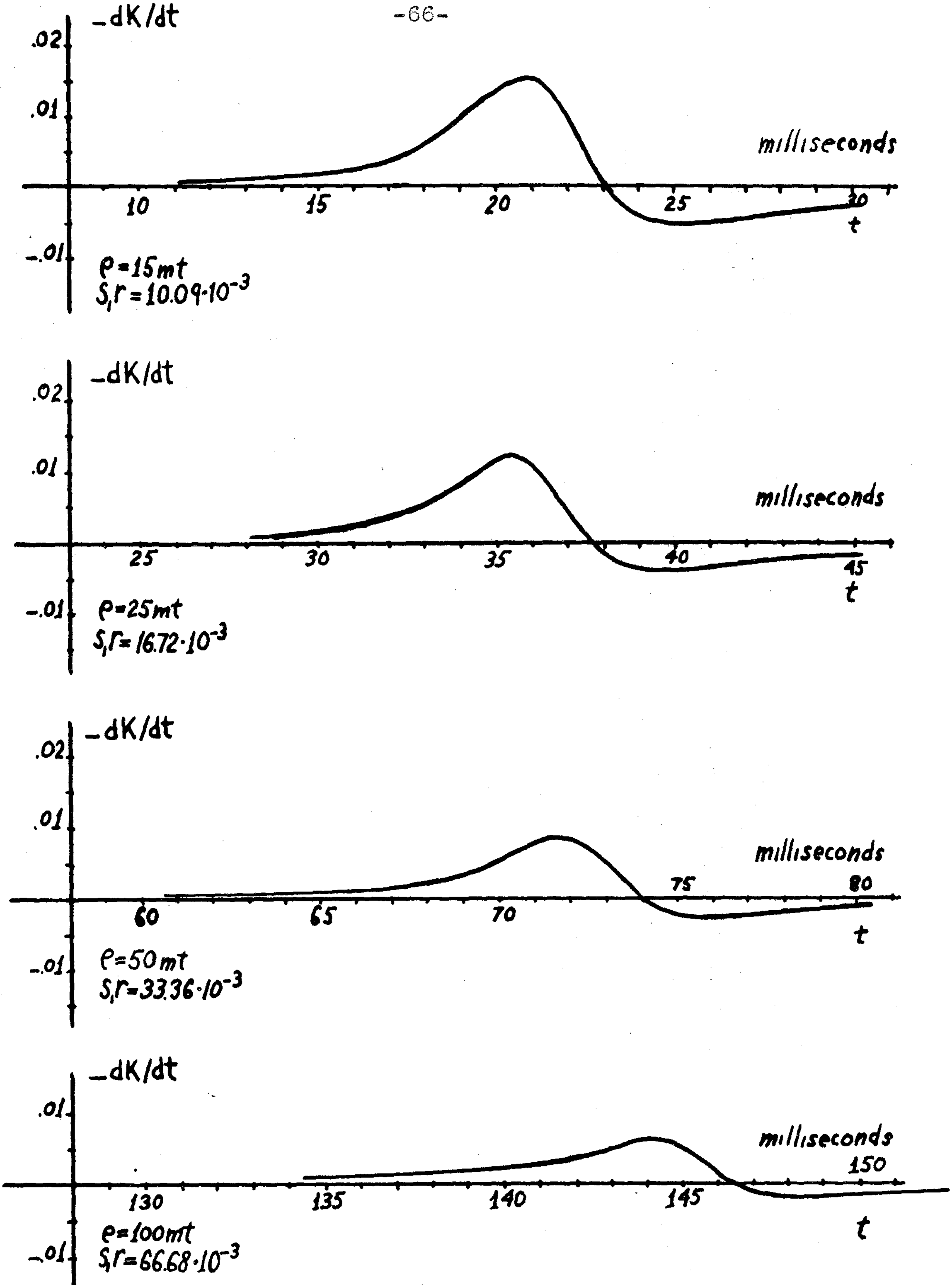

-dK/dt
.02
.01
-.01
milliseconds
10
15
20
25
30
t
ρ=15mt
$S_1 r = 10.09 \cdot 10^{-3}$
-dK/dt
.02
.01
-.01
milliseconds
25
30
35
40
45
t
ρ=25mt
$S_1 r = 16.72 \cdot 10^{-3}$
-dK/dt
.02
.01
-.01
milliseconds
60
65
70
75
80
t
ρ=50mt
$S_1 r = 33.36 \cdot 10^{-3}$
-dK/dt
.01
-.01
milliseconds
130
135
140
145
150
t
ρ=100mt
$S_1 r = 66.68 \cdot 10^{-3}$

Figure 19

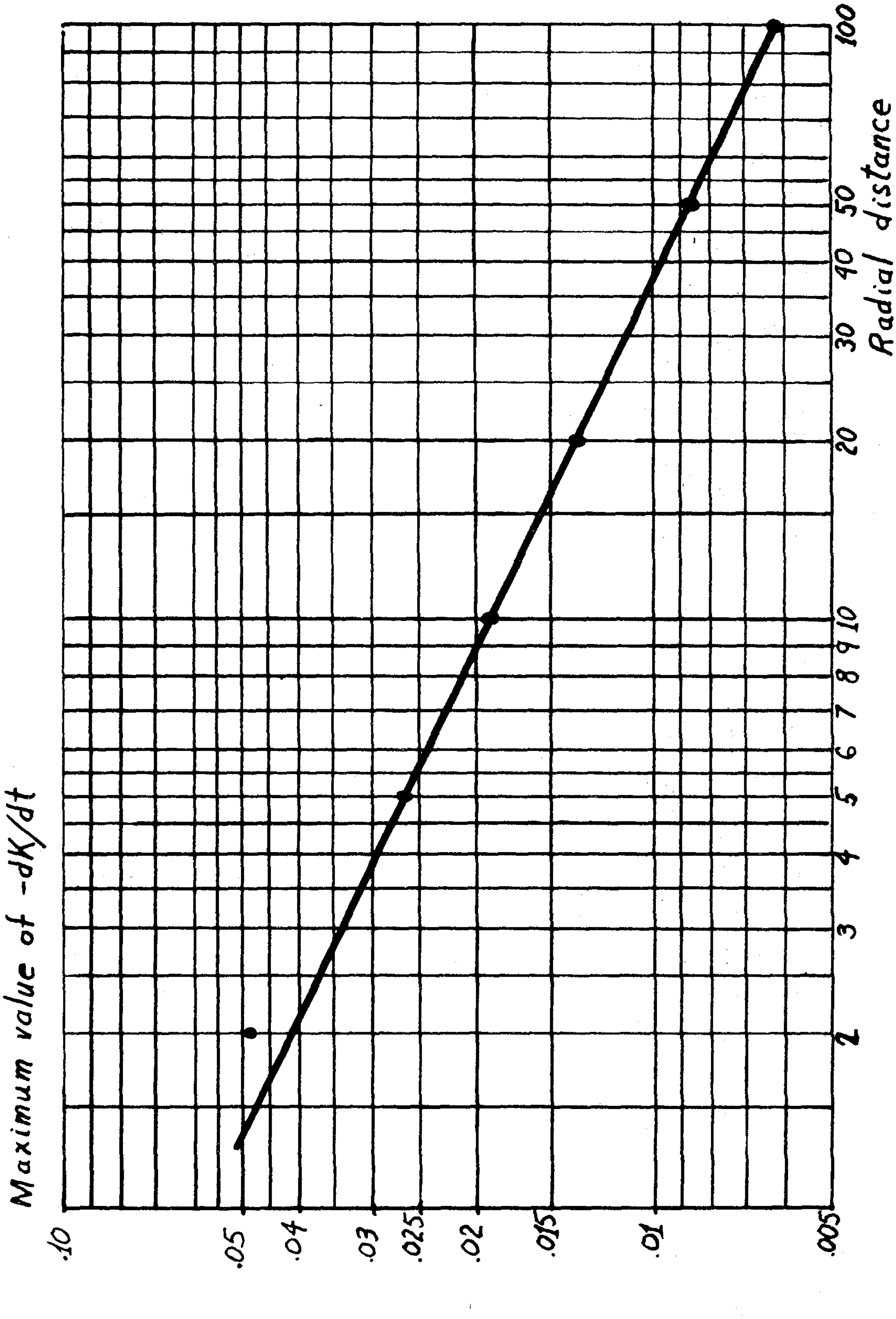

Maximum value of -dK/dt
.10
.05
.04
.03
.025
.02
.015
.01
.005
2
3
4
5
6
7
8
9
10
20
30
40
50
100
Radial distance

Figure 20

factors affect the Stoneley wave velocity. It is interesting to note that at distances of five meters or more we can characterize the Stoneley wave by a half period (defined as the time interval between the relative maximum and the relative minimum) and that this quantity does not seem to vary appreciably as ρ increases. The half period of the oscillation is approximately 4.3 milliseconds and the velocity is 689 meters per second. These two results can be combined to give a wavelength of 5.9 meters.

PART II

PULSE PROPAGATION IN A FLUID CYLINDER

II.1 Introduction

It is of interest to consider whether the Cagniard method can be extended to include regions which are bounded by cylindrical surfaces. In the problems which we will study the fluid is assumed to be enclosed in an infinitely long cylinder having a finite radius. The center of the source is located on the axis of the cylinder. We consider the following cases: an ideal fluid enclosed by perfectly rigid walls, a perfectly elastic solid cylinder in vacuum, and an ideal fluid surrounded by a perfectly elastic solid extending to infinity in all directions.

The coordinate system will be chosen in such a way that the z axis coincides with the axis of the cylinder, and the center of the source cavity coincides with the origin. By differentiating the source function (equation 46) with respect to z one obtains the vertical component of displacement. This function must vanish in the plane $z=0$ and be antisymmetric with respect to it. It is apparent that the result obtained from the differentiation of equation 46 cannot be used to define the vertical displacement in the plane $z=0$. However, a more suitable representation for the source can be obtained in which the z dependence enters in the form $\cos \lambda z$. This transformation takes the form (Watson, page 416)

$$(131) \quad \bar{\varphi}_0(s;\rho,z) = (1/sr)\, e^{-s(r/V_L)} = (2/\pi s) \int_0^\infty K_0\left(\rho(\omega^2 + s^2/V_L^2)^{1/2}\right) \cos \omega z \, d\omega,$$

with the restriction that ρ be non-zero.

II.2 Cylinder with Rigid Walls

We consider first the propagation of acoustic waves in an ideal fluid, which is contained in an infinitely long cylindrical tube having perfectly rigid walls. Several important questions arise immediately. In particular, one must consider whether there exist certain points in the fluid where the pressure does not remain finite. The absence of such points is certainly not obvious. If such points do exist they must result from a focusing effect of the boundaries, similar to what happens when a uniform pressure is applied to the surface of a perfect sphere. In that case it is found that as the wave approaches the center of the sphere the normal stress increases without limit, and the radial displacement approaches zero in such a way that the inward flow of energy remains constant. That such a result is physically unrealistic is obvious. It results from the assumption that the stress-strain relationship remains linear even when the applied stresses become quite large. In the problem of the sphere it is the advancing wave front itself which gives rise to the non-linear behavior. Effectively what happens is that the wave causes the medium to react as if its elastic properties varied continuously with the distance from the center. Evidently the second order terms should be taken into consideration in the basic equations. This non-linear behavior causes energy to be reflected as well as transmitted, and this division of energy must take place in such a way that no energy reaches the center of the sphere.

A complete solution to our problem should also answer such questions as the following: (a) does a spherically symmetric pulse develop a tail after reflection from a rigid cylindrical boundary? (b) in what way is the original wave shape distorted as the wave approaches the axis of the cylinder? and (c) at what rate is energy removed from the neighborhood of the source?

In what follows we attempt to answer some of these basic questions. The complete solution for the transform of the displacement potential will consist of a term which describes the source singularity, $\bar{\varphi}_0$, and a term which describes the perturbing effect of the boundary, $\bar{\varphi}_p$. $\bar{\varphi}_p$ is a particular solution of the transformed wave equation which is well behaved on the axis and symmetric with respect to the plane $z=0$. Such a solution is of the form

(132) $$\bar{\varphi}_p = \int_0^\infty f(\lambda)\, I_0\{\rho(\lambda^2+s^2/V_L^2)^{1/2}\} \cos\lambda Z \, d\lambda .$$

The total transform of the displacement potential is simply

(133) $$\bar{\varphi} = \int_0^\infty \cos\lambda Z \left\{ (2/\pi s)\chi(s) K_0(\rho\alpha) + f(\lambda) I_0(\rho\alpha) \right\} d\lambda .$$

The function, $f(\lambda)$, can be determined by requiring that the radial displacement, $\bar{u}_\rho$, vanish on the cylindrical surface $\rho=R$. $f(\lambda)$ is given by the relationship

(134) $$f(\lambda) = (2/\pi s)\chi(s)\, K_1(\alpha R)/I_1(\alpha R) .$$

The main problem centers around the inversion of the expression

(135) $$\bar{\varphi}_p = (2/\pi s) \int_0^\infty (K_1(\alpha R)/I_1(\alpha R))\, I_0(\alpha\rho) \cos\lambda Z \, d\lambda .$$

Probably the most natural way of proceeding is to attempt to find an expansion of equation 135 in which each term can be associated with a particular reflection event. On physical grounds this seems reasonable; for it is apparent that the original wave will undergo multiple reflections at the cylindrical wall. What is not apparent is whether or not the cylinder has associated with it a dispersive property which distorts the initial pulse shape. Should this be the case, the response at times which are large compared with the arrival time of the first energy might be expressed more naturally as a superposition of an infinite number of modes of oscillation, which are characteristic of the cavity itself when certain conditions are satisfied at the boundary. In what follows both approaches will be discussed.

We consider first the expansion of $\bar{\varphi}_p$ in an infinite geometric series. The substitution $\lambda = su$ reduces equation 135 to the form

(136) $$\bar{\varphi}_p = (2/\pi)\int_0^\infty \left(K_1(sRa)/I_1(sRa)\right) I_0(s\rho a)\cos suz\, du.$$

If both the numerator and denominator of the integrand are multiplied by the factor $K_0(sRa)$ and use is made of the relationship

(137) $$K_0(sRa)I_1(sRa) + K_1(sRa)I_0(sRa) = 1/sRa,$$

$\bar{\varphi}_p$ can be rewritten in the form

(138) $$\bar{\varphi}_p = (2/\pi)\int_0^\infty sRaK_1(sRa)K_0(sRa)I_0(s\rho a)\cos suz\, du \big/ \left(1 - sRaK_1(sRa)I_0(sRa)\right).$$

Direct calculations show that the function $T(\xi) = \xi K_1(\xi)I_0(\xi)$ has the value unity if $\xi = 0$, and has a value between zero and unity for positive, real, non-zero values of ξ. It is clear that, in the present case, ξ cannot vanish (sRs_1 is the minimum value of this quantity). $\bar{\varphi}_p$ can now be expressed in terms of the geometric series

(139) $\bar{\varphi}_p = (2/\pi)\sum_{k=0}^{\infty}\int_0^{\infty} Q(SRa)\, T^k(SRa)\, I_0(S\rho a)\cos SUZ\, dU,$

where

(140) $Q(SRa) = SRa\, K_1(SRa) K_0(SRa).$

That the individual terms in this expansion do not describe individual reflections can be seen from the following argument. Allow S to become arbitrarily large, then $T(SRa)$ approaches the value one-half and each term in the geometric series reduces to the same asymptotic form. This indicates that the term by term inversion of equation 139 leads to a series in which the time behavior of the individual terms is nearly identical in a small but finite time interval following the arrival of the first energy.

We also note that the asymptotic behavior of the integrand in equation 135 and of the integrand in the first term of the expansion are identical except for a factor one-half. Therefore, by inverting the first term in the expansion we can obtain information about the perturbation potential itself in a limited time interval following the arrival of the first energy.

Nicholson's (9) integral representation for the product, $K_1(SRa)K_0(SRa)$, has the form

(141) $K_1(SRa)K_0(SRa) = 2\int_0^{\infty} K_1(2SRa\cosh\xi)\cosh\xi\, d\xi$

$$= 2\int_0^{\infty} d\xi \int_0^{\infty} \cosh\xi \cosh\eta\, e^{-2RSa\cosh\xi\cosh\eta}\, d\eta.$$

This expression is useful in reducing the first term in the expansion to a more convenient form. On the axis of the cylinder the first term, $_1\bar{\varphi}_p$, can be written in the form

(142) $_1\bar{\varphi}_p = (4/\pi)SR\, \mathcal{R}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} a(u)\cosh\eta\cosh\xi\, e^{-S(2Ra\cosh\xi\cosh\eta - iuz)}\, d\xi\, d\eta\, du.$

In order to invert ${}_1\bar{\Phi}_P$ we make the change of variable

(143) $t = 2Ra\cosh\xi\cosh\eta - iuZ.$

For the present we consider this relationship to define u in terms of t, ξ, and η. The partial derivative of u with respect to t is

(144) $$\left.\frac{\partial u}{\partial t}\right|_{\xi,\eta} = a(u)/(2Ru\cosh\xi\cosh\eta - iZa(u)).$$

If ${}_1\bar{\Phi}_P$ is rewritten in terms of the new variable t, we obtain

(145) $${}_1\bar{\Phi}_P = (4/\pi)RS\beta\int_0^\infty\int_0^\infty\left\{\int_{S_1\Gamma_0}^{H_{\xi,\eta}}\frac{a^2(t,\xi,\eta)e^{-st}dt}{\Gamma_0 u(t,\xi,\eta) - iZa(t,\xi,\eta)}\right\}\cosh\xi\cosh\eta\, d\xi\, d\eta,$$

where

(146) $\Gamma_0 = 2R\cosh\xi\cosh\eta.$

The path of integration, $H_{\xi,\eta}$, now depends on the values of both ξ and η (fig. 21). What we would like to do is deform this path into one which lies along the real t axis. The defining relation for t can be used to obtain explicit expressions for u and $a(u)$ in terms of the variables t, ξ, and η; namely

(147) $u = \{itZ + \Gamma_0(t^2 - S_1^2(\Gamma_0^2+Z^2))^{1/2}\}/(\Gamma_0^2+Z^2)$

and

(148) $a(u) = \{t\Gamma_0 + iZ(t^2 - S_1^2(\Gamma_0^2+Z^2))^{1/2}\}/(\Gamma_0^2+Z^2).$

In order to keep the radical single-valued we cut the t plane along the portion of the real t axis which lies between the points $\pm S_1(\Gamma_0^2+Z^2)^{1/2}$ and require that the sign of the radical be positive on the real t axis to the right of the point $S_1(\Gamma_0^2+Z^2)^{1/2}$. The expression for ${}_1\bar{\Phi}_P$ reduces to the form

(149) $${}_1\bar{\Phi}_P = (4/\pi)RS\beta\int_0^\infty\int_0^\infty\left\{\int_{S_1\Gamma_0}^{H_{\xi,\eta}}\frac{a^2(t,\xi,\eta)e^{-st}dt}{(t^2 - S_1^2(\Gamma_0^2+Z^2))^{1/2}}\right\}\cosh\xi\cosh\eta\, d\xi\, d\eta.$$

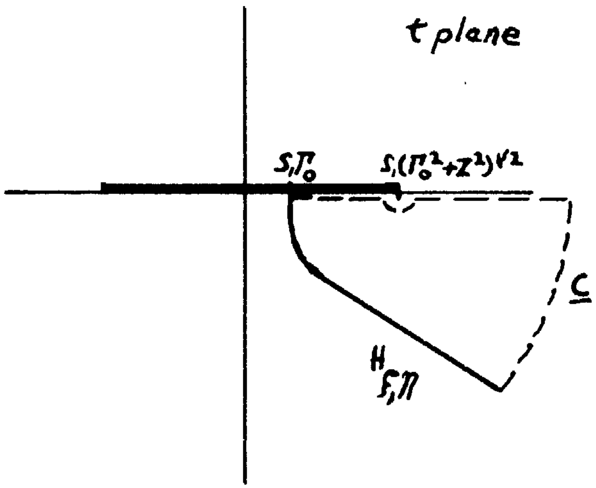


Figure 21

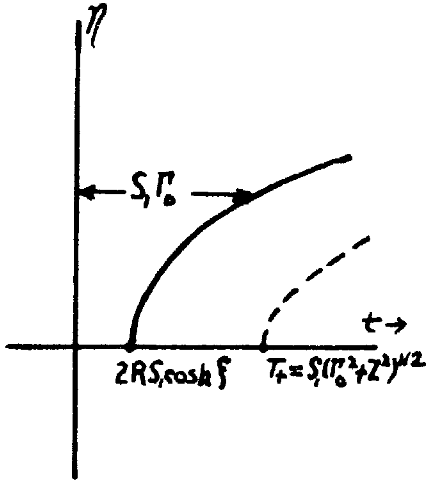


Figure 22

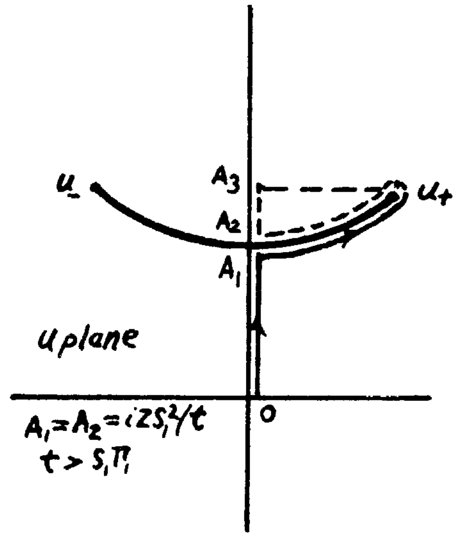


Figure 23

It is clear that the singularities of the integrand are located at the branch points $\pm S_1(\Pi_0^2+Z^2)^{1/2}$. We now close the contour, $H_{\xi,\eta}$, by the dotted path in fig. 21. The integral taken along the closed contour vanishes since it encloses no singularities of the integrand. The exponential dominates the behavior of the integrand for large values of t. This indicates that the result which is obtained by integrating along C must become vanishingly small in the limit $\Re t \to \infty$.

In order to proceed with the inversion it is necessary to change the order of integration in equation 149 in such a way that the integration is carried out first with respect to η. Reference to fig. 22 facilitates the inversion. The result is

$$(150)\quad {}_1\bar{\varphi}_p = (4/\pi)RS\,\Re\int_0^\infty \cosh\xi\, d\xi \int_{2RS_1\cosh\xi}^{\infty} e^{-st}dt \int_0^{\cosh^{-1}(t/2RS_1\cosh\xi)} a^2(t,\xi,\eta)\cosh\eta\, d\eta / (t^2 - S_1^2(\Pi_0^2+Z^2))^{1/2}.$$

A much simpler expression for ${}_1\bar{\varphi}_p$ can be obtained by using the original transformation to introduce the variable u in place of η. The partial derivative of η with respect to u is

$$(151)\quad \left.\frac{\partial\eta}{\partial u}\right|_{t,\xi} = -(1/a)\left\{(\Pi_0 u - iZa)/2Ra\cosh\xi\sinh\eta\right\}.$$

If use is made of the fact that

$$(152)\quad \left.\frac{\partial u}{\partial t}\right|_{\xi,\eta} = a/(\Pi_0 u - iZa) = a/(t^2 - S_1^2(\Pi_0^2+Z^2))^{1/2},$$

the integral over η, in equation 150, can be reduced to

$$(153)\quad I(t,\xi) = \Re\int_{u_{\eta=\cosh^{-1}(t/2RS_1\cosh\xi)}}^{u_{\eta=0}} a(u)\cosh\eta(t,\xi,u)\, du / 2Ra(u)\cosh\xi\sinh\eta(t,\xi,u).$$

The u contour is easily determined from equation 147 by allowing η to vary between its upper and lower limits for various values of t. The end points of the u contour are

(154) $u_{\eta=0} = [itz + 2R\cosh\xi\,\{t^2 - S_1^2(z^2+4R^2\cosh^2\xi)\}^{1/2}]/(4R^2\cosh^2\xi + Z^2) = u_+$

and

(155) $u_{\eta=\cosh^{-1}(t/2RS_1\cosh\xi)} = 0.$

We now distinguish between the two cases $t<S_1\Gamma_1\ (\Gamma_1=(z^2+4R^2\cosh^2\xi)^{1/2})$ and $t>S_1\Gamma_1$. In the first case the contour lies entirely on the positive imaginary axis. It is clear that in this case the function $I(t,\xi)$ vanishes. This fact enables us to replace the lower limit in the integral over t, in equation 150, by $S_1\Gamma_1$.

Next consider the case in which $t>S_1\Gamma_1$. The corresponding contour is indicated in fig. 23 by the path $OA_1u_{\eta=0}$. The investigation of the singularities of the integrand, in equation 153, is facillitated by expressing the denominator as an explicit function of the independent variables, namely

(156) $2Ra\cosh\xi\sinh\eta = \{(t+iuz)^2 - 4R^2a^2(u)\cosh^2\xi\}^{1/2} = \{f(t,u,\xi)\}^{1/2}.$

The function $f(t,u,\xi)$ is a quadratic in u with zeros located at the points

(157) $u_\pm = [itz \pm 2R\cosh\xi(t^2 - S_1^2\Gamma_1^2)^{1/2}]/\Gamma_1^2.$

Equation 143 can be used to express $a\cosh\eta$ in terms of the independent variables. The relationship has the form

(158) $a\cosh\eta = (t+iuz)/2R\cosh\xi.$

It is clear that $u_{\eta=0}$ is the only singularity of the integrand, in equation 153, which lies to the right of the imaginary axis. In what follows we consider the u plane to be cut along the heavy dark curve in fig. 23 which connects the branch points u_+ and u_- .

Now let us determine the range of t for which $f(u)$ is positive real when u is pure imaginary. The substitution $u=iL$

reduces $f(u)$ to the form

(159) $f(L) = (t-Lz)^2 - (S_1^2 - L^2)(4R^2\cosh^2\xi)$.

The first and second derivatives of f with respect to L are

(160) $\frac{\partial f}{\partial L} = 2(L\Gamma_1^2 - zt)$

and

(161) $\frac{\partial^2 f}{\partial L^2} = 2\Gamma_1^2$.

The function $f(u)$ therefore has a minimum value at

(162) $L_0 = zt/\Gamma_1^2$,

which is

(163) $f(L_0) = (t^2 - S_1^2\Gamma_1^2)(4R^2\cosh^2\xi)/\Gamma_1^2$.

Therefore, if $t > S_1\Gamma_1$, $f(u)$ is positive real at all points on the imaginary u axis.

The expression for $I(t,\xi)$ can now be rewritten in the form

(164) $I(t,\xi) = (1/2R\cosh\xi)\mathcal{R}\int_0^{u_{\eta=0}} (t+iuz)\,du \Big/ \{(t+iuz)^2 - 4R^2a^2\cosh^2\xi\}^{1/2}$.

The part of the u contour which lies on the imaginary axis contributes nothing to the final result since it is a pure imaginary quantity. The integral from A_1 to $u_{\eta=0}$ and the integral from $u_{\eta=0}$ to A_2 are identical since on the latter path the direction of integration is in the opposite sense and the sign of the radical is negative. It is clear that this latter contour can be replaced by the path $A_2A_3u_{\eta=0}$. The integral along the path A_2A_3 is pure imaginary. In order to evaluate the integral along the path $A_3u_{\eta=0}$ we make the substitution

(165) $u = itz/\Gamma_1^2 + k$

in equation 164. The integral, $I(t,\xi)$, then reduces to the form

(166)
$$I(t,\xi) = (1/2R\Gamma_1\cosh\xi)\mathcal{R}\int_0^{(t^2-S_1^2\Gamma_1^2)^{1/2}(2R\cosh\xi)/\Gamma_1^2} ((4R^2\cosh^2\xi/\Gamma_1^2)t + izk)\,dk \Big/ \{(4R^2\cosh^2\xi/\Gamma_1^4)(t^2-S_1^2\Gamma_1^2) - k^2\}^{1/2},$$

which, upon being integrated, yields the result

(167) $I(t,\xi) = (\pi/2)(1/2R\cosh\xi)(4R^2\cosh^2\xi/\Pi_1^3)t.$

$_{,}\bar{\varphi}_p$ can now be written in greatly simplified form by introducing this expression for $I(t,\xi)$ in equation 150. We find that

(168) $$_{,}\bar{\varphi}_p = 4R^2 S\int_0^\infty (\cosh^2\xi/\Pi_1^3)\left\{\int_{S_1\Pi_1}^\infty e^{-St}\,t\,dt\right\}d\xi.$$

If the order of integration is interchanged this expression for $_{,}\bar{\varphi}_p$ takes the form of the direct Laplace transform. The lower limit of the t variable is found to be $S_1(4R^2+Z^2)^{1/2}$. This is just the time required for the wave which has been reflected once from the walls of the cylinder to arrive at the axial point Z. Interchanging the order of integration reduces equation 168 to the form

(169) $$_{,}\bar{\varphi}_p = 4R^2 S\int_{S_1(4R^2+Z^2)^{1/2}}^\infty e^{-St}\,t\left\{\int_0^{\cosh^{-1}((t^2-S_1^2Z^2)^{1/2}/2RS_1)} (\cosh^2\xi/\Pi_1^3)\,d\xi\right\}dt.$$

The S, which appears in the coefficient, can be removed by integrating the expression

(170) $$_{,}\bar{\varphi}_p = -4R^2\int_{S_1(4R^2+Z^2)^{1/2}}^\infty t\,(de^{-St}/dt)\,g(t)\,dt$$

by parts. $g(t)$ has been used to denote the function in parenthesis in equation 169. The integration by parts leads to the expression

(171) $$_{,}\bar{\varphi}_p = -4R^2(e^{-St}t\,g(t))\Big|_{t=S_1(4R^2+Z^2)^{1/2}}^{t=\infty} + 4R^2\int_{S_1(4R^2+Z^2)^{1/2}}^\infty e^{-St}(d\,t g(t)/dt)\,dt.$$

The first term in equation 171 vanishes at infinity because of the presence of the exponential and at the lower limit because $g(t)$ is zero there. We obtain in this way the determining function which is the inverse transform of the first term in the series expansion of $\overline{\varphi}_p$; it is

(172) $_1\varphi_p(t,z) \equiv 0, \quad 0 \leq t \leq S_1(4R^2+Z^2)^{1/2} = T_0;$

$$= (S_1/t)\{(t^2-S_1^2Z^2)/(t^2-S_1^2(4R^2+Z^2))\}^{1/2} + 4R^2g(t), \quad t > T_0.$$

As indicated earlier the total displacement potential will be proportional to $_1\varphi_p$ in the time interval $T_0 < t < T_0+\epsilon$. In this time interval the behavior of $_1\varphi_p$ is governed by the first term in equation 172. We are therefore led to the rather surprising result that the wave arrives at axial points with an infinite discontinuity in the displacement potential.

The pressure response, $P(t)$, to an arbitrary time variation in the source displacement potential can be determined from the superposition integral. The resultant displacement potential at an axial point, ϕ, is related to the source function, ψ, and the response function, φ, as follows:

(173) $\phi = \int_0^{t-T_0} \frac{d\psi}{dT}\varphi(t-T)dT.$

The pressure is simply

(174) $P(t) = -\sigma_0 \frac{d^2\phi}{dt^2}.$

The quantities ϕ and P are a valid description of the response only in the time interval, ϵ, following the arrival of the first energy. The term which dominates the φ behavior in this interval can be obtained if we replace t by T_0 everywhere in equation 172 except in the denominator of the radical. We then find that

(175) $\varphi(t) \sim (2RS_1/(4R^2+Z^2)^{1/2})(1/(t^2-T_0^2)^{1/2}), \quad T_0 < t < T_0+\epsilon.$

If this result is introduced into the expression for ϕ and the resulting integral integrated by parts, ϕ is obtained in the form

(176) $$\phi = -(2RS_1/(4R^2+Z^2)^{1/2})\left\{ \frac{d\psi}{d\tau}\Big|_{\tau=t-T_0} \operatorname{Log} T_0 - \frac{d\psi}{d\tau}\Big|_{\tau=0} \operatorname{Log}(t+(t^2-T_0^2)^{1/2}) \right\}$$
$$+ (2RS_1/(4R^2+Z^2)^{1/2}) \int_0^{t-T_0} \operatorname{Log}\left[t-\tau+\{(t-\tau)^2-T_0^2\}^{1/2}\right] \frac{d^2\psi}{d\tau^2}\, d\tau .$$

It is evident that unless $\frac{d\psi}{d\tau}\big|_{\tau=0}$ vanishes the time derivative of ϕ becomes infinite at $t=T_0$. It can be shown in a similar manner that the pressure has an infinite discontinuity at T_0 unless the following conditions are satisfied:

(177) $$\psi\big|_{\tau=0^+} = 0, \quad \frac{d\psi}{d\tau}\Big|_{\tau=0^+} = 0, \quad \frac{d^2\psi}{d\tau^2}\Big|_{\tau=0^+} = 0.$$

If these conditions are satisfied the pressure can be determined from the relation

(178) $$P(t) = -\sigma_0\,(2RS_1/(4R^2+Z^2)^{1/2}) \int_0^{t-T_0} \{1/((t-\tau)^2-T_0^2)^{1/2}\}\, \frac{d^3\psi}{d\tau^3}\, d\tau$$
$$= (2RS_1/(4R^2+Z^2)^{1/2}) \int_0^{t-T_0} \{1/((t-\tau)^2-T_0^2)^{1/2}\}\, \frac{dP_0}{d\tau}\, d\tau,$$

where P_0 refers to the source pressure. A study of equations 177 and 178 reveals three very interesting facts: (1) a finite but instantaneous change in the source pressure leads to an infinite change in the pressure at axial points at the time T_0; (2) the form of the coefficient indicates that the initial rate of change of the pressure is proportional to the cosine of the angle of incidence, and (3) the initial wave shape is distorted. These effects have no counterpart in the case of plane boundaries and serve to illustrate the influence of curvature.

II.2a Study of the Reflected Wave at Points off the Axis

The objective in what follows is to determine how the discontinuity in pressure, across the leading edge of the reflected wave, changes as the wave approaches the axis of the cylinder.

We have just shown that if $\chi(S)=1/S^2$, corresponding to an input step function in pressure at the source, this discontinuity is infinite at axial points. We now determine the discontinuity at non-axial points.

The first term in the asymptotic expansion of the ratio $K_1(SRa)/I_1(SRa)$ is*

(179) $$K_1(SRa)/I_1(SRa) \underset{S\to\infty}{\sim} \pi e^{-2SaR}.$$

The displacement potential then takes the form

(180) $$\bar{\varphi}_p = 2\int_0^\infty e^{-2SaR} I_0(Sa\rho) \cos SUZ\, du.$$

In order to invert this expression we replace $I_0(Sa\rho)$ by the integral representation

(181) $$I_0(Sa\rho) = (2/\pi)\int_0^1 \cosh(Sa\rho\xi)\, d\xi/(1-\xi^2)^{1/2}.$$

The expression for $\bar{\varphi}_p$ is then

(182) $$\bar{\varphi}_p = (4/\pi)\int_0^\infty e^{-2SaR} \cos SUZ \left\{\int_0^1 \cosh Sa\rho\xi\, d\xi/(1-\xi^2)^{1/2}\right\} du.$$

If we interchange the order of integration and replace the hyperbolic cosine by the sum of a decaying and increasing exponential, equation 182 becomes

(183) $$\bar{\varphi}_p = (2/\pi)\int_0^1 (d\xi/(1-\xi^2)^{1/2})\left\{\int_0^\infty \cos SUZ\, e^{-2SaR}(e^{Sa\rho\xi} + e^{-Sa\rho\xi})du\right\}$$

$$= \mathcal{R}\,(2/\pi)\int_0^1 (d\xi/(1-\xi^2)^{1/2})\left\{\int_0^\infty (e^{-S[a(2R-\rho\xi)-iuZ]} + e^{-S[a(2R+\rho\xi)-iuZ]})du\right\}.$$

We introduce the notation

(184) $$\bar{I}_1(S;\rho,Z) = (2/\pi)\mathcal{R}\int_0^1 (d\xi/(1-\xi^2)^{1/2})\left\{\int_0^\infty e^{-S[a(2R-\rho\xi)-iuZ]}du\right\}$$

$$\bar{I}_2(S;\rho,Z) = (2/\pi)\mathcal{R}\int_0^1 (d\xi/(1-\xi^2)^{1/2})\left\{\int_0^\infty e^{-S[a(2R+\rho\xi)-iuZ]}du\right\},$$

and consider each of these integrals separately.

---

* The asymptotic expansions of the Bessel functions are derived in Watson, ch. 7.

The method previously used yields the result that the expression for $\overline{I}_2$ is the Laplace transform of a function which vanishes up to the time $T_0 = S_1(4R^2+Z^2)^{1/2}$. From geometrical considerations we know that the first energy arrives at $T_1 = S_1[(2R-\rho)^2+Z^2]^{1/2}$. Therefore, the expression for $\overline{I}_2$ cannot be used to determine the magnitude of the discontinuity in the displacement potential at the arrival time. In fact no meaning should be attached to $\overline{I}_2$ since the use of equation 179 automatically restricts the validity of our results to a small time interval following the first arrival.

The expression for $\overline{I}_1$ can be inverted by the methods previously demonstrated. First we make the change of variable

(185) $t = a(2R-\rho\xi)-iuz.$

This transformation maps the real u axis into a contour which lies in the fourth quadrant of the t plane. The contour begins on the positive real t axis and then moves to the right with monotonically increasing real part and monotonically decreasing imaginary part approaching the asymptote $\theta = -\tan^{-1}(z/(2R-\rho\xi))$. This contour can be deformed into one which coincides with the real t axis in the interval $S_1[(2R-\rho\xi)^2+z^2]^{1/2} < t < \infty$ and which is parallel to and an infinitesimal distance below the real axis in the interval $S_1(2R-\rho\xi) \leq t \leq S_1[(2R-\rho\xi)^2+z^2]^{1/2}$. The only singularities of the integrand occur at the branch points $T_\pm = \pm S_1[(2R-\rho\xi)^2+z^2]^{1/2}$. To keep the integrand single-valued we cut the t plane along the real axis between the two branch points. If we then interchange the order of integration in equation 184, $\overline{I}_1$ reduces to

(186) $$\bar{I}_1(S;\rho,z) = (2/\pi)\,\mathcal{R}\int_{S_1(2R-\rho)}^{2RS_1} e^{-St}dt\left\{\int_{(2RS_1-t)/S_1\rho}^{1}(du/dt)_\xi\, d\xi/(1-\xi^2)^{1/2}\right\}$$
$$+ (2/\pi)\,\mathcal{R}\int_{2RS_1}^{\infty} e^{-St}dt\left\{\int_0^1 (du/dt)_\xi\, d\xi/(1-\xi^2)^{1/2}\right\},$$

where

(187) $$\left.\frac{\partial u}{\partial t}\right|_\xi = a\{u(t,\xi)\}\Big/\{(2R-\rho\xi)u(t,\xi) - iz\,a\{u(t,\xi)\}\}$$
$$= a\{u(t,\xi)\}\Big/\{t^2 - S_1^2[(2R-\rho\xi)^2+z^2]\}^{1/2}$$

and

(188) $$u(t,\xi) = \{izt + (2R-\rho\xi)[t^2 - S_1^2((2R-\rho\xi)^2+z^2)]^{1/2}\}/(z^2+(2R-\rho\xi)^2).$$

The final step in the inversion procedure consists in using the original transformation, equation 185, to express ξ in terms of u and t. The partial derivative of ξ with respect to u is simply

(189) $$\left.\frac{\partial \xi}{\partial u}\right|_t = [(2R-\rho\xi)(u/a) - iz]/a\rho.$$

The product of the partial derivatives is

(190) $$\left.\frac{\partial u}{\partial t}\right|_\xi \left.\frac{\partial \xi}{\partial u}\right|_t = 1/a(u)\rho.$$

Now let us consider the mapping of the interval $(2RS_1-t)/S_1\rho \le \xi \le 1$ in the u plane when t is confined between the limits $S_1(2R-\rho) \le t \le 2RS_1$ (corresponding to the first term in equation 186). The upper and lower limits of u are

(191) $$u_{\xi=1} = \{izt + (2R-\rho)[t^2 - S_1^2((2R-\rho)^2+z^2)]^{1/2}\}/(z^2+(2R-\rho)^2)$$

and

(192) $$u_{\xi=(2RS_1-t)/S_1\rho} = 0.$$

We see from the first expression that if $t > S_1[(2R-\rho)^2+z^2]^{1/2}$ the point $u_{\xi=1}$ lies in the first quadrant, whereas, if $t \le S_1[(2R-\rho)^2+z^2]^{1/2}$ the contour is confined to the positive imaginary u axis. In either case the first term in equation 186 reduces to

(193) $${}_a\bar{I}_1(S;\rho,z) = (2/\pi)\Re\int_{S_1(2R-\rho)}^{2RS_1} e^{-st}dt\left\{\int_0^{u_{\xi=1}} du/a(u)\rho(1-\xi^2(u,t))^{1/2}\right\}.$$

We infer from equation 191 that if $t \leq S_1[(2R-\rho)^2+z^2]^{1/2}$, $u_{\xi=1}$ is a pure positive imaginary quantity which cannot exceed S_1 in absolute value. In this time interval the double integral is pure imaginary and contributes nothing to the final result. It also indicates that ${}_a\bar{I}_1$ vanishes identically in the region of the cylinder for which

(194) $$S_1[(2R-\rho)^2+z^2]^{1/2} > 2RS_1$$
$$z^2 > 4R\rho-\rho^2.$$

If $S_1[(2R-\rho)^2+z^2]^{1/2} < t \leq 2RS_1$, , the integral does not vanish but makes a contribution to the final result. For the time being we will consider the response in a region of the cylinder for which the inequality in equation 194 is satisfied. We therefore focus our entire attention on the second term in equation 186.

Equation 185 can be used to reduce the second term in the expression for $\bar{I}_1$ to the form

(195) $${}_b\bar{I}_1 = (2/\pi)\Re\int_{2RS_1}^{\infty} e^{-st}dt\left\{\int_{u_{\xi=0}}^{u_{\xi=1}} du/a(u)\rho(1-\xi^2(u,t))^{1/2}\right\},$$

where

(196) $$u_{\xi=0} = \left\{izt+2R(t^2-S_1^2(4R^2+z^2))^{1/2}\right\}/(4R^2+z^2).$$

In the interval $2RS_1 \leq t \leq S_1[(2R-\rho)^2+z^2]^{1/2}$ the path of integration is confined to the positive imaginary axis and the absolute value of $u_{\xi=1}$ cannot exceed S_1. This indicates that ${}_b\bar{I}_1 \equiv 0$ in this time interval. It is now clear that no energy arrives at any point in the cylinder before the time required for the wave to reach the point of observation from the ring source which has a radius $2R$ and which lies in the plane $Z=0$.

We are interested in times immediately following the arrival time. If t lies in the interval

(197) $2RS_1 < S_1[(2R-\rho)^2+Z^2]^{1/2} < t \leq S_1(4R^2+Z^2)^{1/2}$,

the contour in the u plane starts at $u_{\xi=0}$ on the positive imaginary axis and moves upward along it to the point u_T. It then leaves the imaginary axis and moves into the first quadrant (fig. 24). The point u_T is the mapping of the value of ξ which causes the radical in equation 188 to vanish, namely

(198) $\xi_T = 2R/\rho - (t^2-S_1^2Z^2)^{1/2}/\rho S_1$.

The value of u_T is

(199) $u_T = iZS_1^2/t$.

The fact that t satisfies equation 197 indicates that u_T cannot exceed S_1. This means that on the path $u_{\xi=0}\,u_T$ the integral is pure imaginary. Therefore, the entire contribution to the value of ${}_b\overline{I}_1$ comes from the integration along the curve $u_T\,u_{\xi=1}$.

The integrand in equation 195 can be written as an explicit function of u and t by substituting the expression $2R/\rho-(t+iuZ)/a\rho$ for ξ. ${}_b\overline{I}_1$ then reduces to the form

(200) $${}_b\overline{I}_1 = (2/\pi)\mathcal{R}\int_{2RS_1}^{\infty} e^{-st}dt\left\{\int_{u_T}^{u_{\xi=1}} du/[a^2\rho^2-(2Ra-(t+iuZ))^2]^{1/2}\right\}.$$

The radical is most easily investigated if it is expressed as the product of the two functions

(201a) $f_1(u) = -a(2R-\rho)+(t+iuZ)$

and

(201b) $f_2(u) = a(2R+\rho)-(t+iuZ)$.

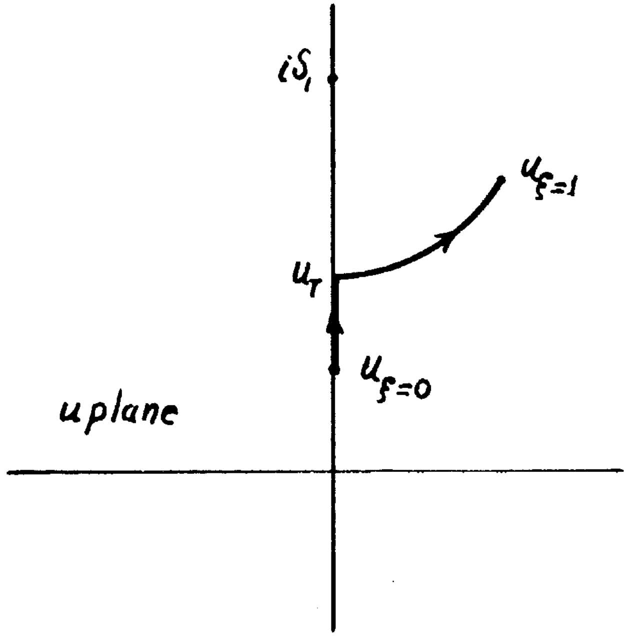


Figure 24

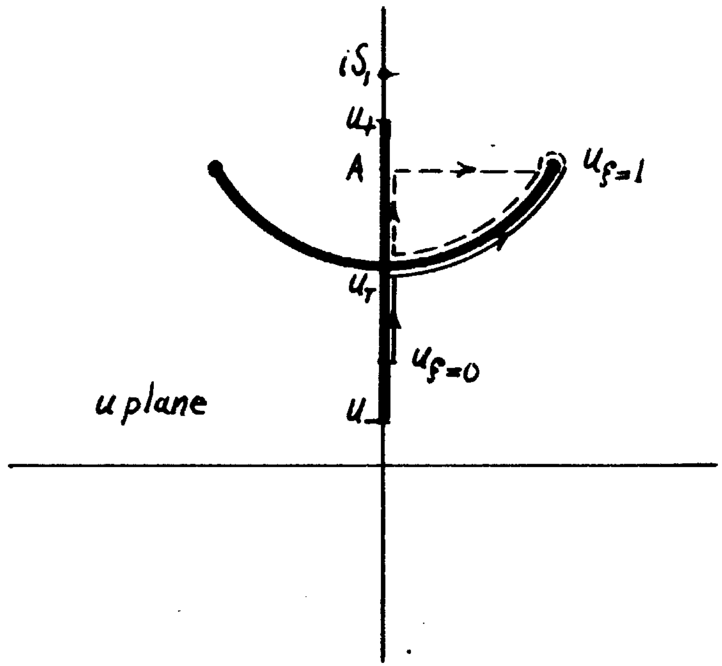


Figure 25

It can be readily verified that in the time interval given by equation 197 these functions have the following properties: (a) $f_1(u)$ vanishes at $u_{\rho=1}$ in the first quadrant and at a point in the second quadrant which is obtained from $u_{\rho=1}$ by changing the sign of the real part; (b) $f_1(u)$ is positive real in the interval $-iS_1 \leq u \leq iS_1$; (c) $f_2(u)$ vanishes at the points:

(202) $u_+ = \{itz + i(2R+\rho)[S_1^2(z^2+(2R+\rho)^2)-t^2]^{1/2}\}/(z^2+(2R+\rho)^2)$

and

(203) $u_- = \{itz - i(2R+\rho)[S_1^2(z^2+(2R+\rho)^2)-t^2]^{1/2}\}/(z^2+(2R+\rho)^2)$;

(d) u_+ is located on the imaginary axis between u_T and iS_1 and cannot approach u_T ; (e) u_- is located on the imaginary axis between $-iS_1$ and $u_{\rho=0}$; and (f) $f_2(u)$ is positive real in the interval $u_- < u < u_+$.

To make the square roots of $f_1(u)$ and $f_2(u)$ single-valued we make cuts in the u plane as shown in fig. 25. We now return to the determination of the discontinuity in the displacement potential at the arrival time. As mentioned earlier the entire contribution to the value of ${}_b\bar{I}_1$ must come from the integration along the curve $u_T u_{\rho=1}$. The result of integrating along this curve is identical with the result which is obtained by integrating from $u_{\rho=1}$ to u_T above the branch cut. This integral is in turn equal to the integral along the path $u_T A u_{\rho=1}$. Properties (b), (d), and (f) of the functions $f_1(u)$ and $f_2(u)$ indicate that the radical is real between u_T and A and, therefore, the integral is pure imaginary and contributes nothing. We have therefore reduced the original path of integration, $u_{\rho=0}\, u_{\rho=1}$, to the contour $A u_{\rho=1}$. By rewriting the inner integral in the expression for ${}_b\overline{I}_1$ we obtain

(204) $${}_bI_1(t;\rho,z) = (2/\pi)\,\mathcal{R}\int_A^{u_{\xi=1}} du/(f_1(u)f_2(u))^{1/2},$$

where $t = S_1[(2R-\rho)^2+z^2]^{1/2}+\epsilon = S_1r+\epsilon$.

To evaluate ${}_bI_1(t;\rho,z)$ we introduce the new variable δ by means of the relationship

(205) $$u = izS_1/r + iz\epsilon/r^2 + \delta = A + \delta.$$

The value of $u_{\xi=1}$, in terms of ϵ, is

(206) $$u_{\xi=1} = izS_1/r + iz\epsilon/r^2 + (2R-\rho)(2\epsilon S_1 r)^{1/2}/r^2 = A + (2R-\rho)(2\epsilon S_1 r)^{1/2}/r^2.$$

In equation 206 and throughout the evaluation of ${}_bI_1(t;\rho,z)$ we will not carry terms which involve ϵ raised to higher powers than the first.

The result of introducing equations 205 and 206 in the expression for ${}_1I_b(t;\rho,z)$ is

(207) $${}_1I_b(t;\rho,z) = (2/\pi)\,\mathcal{R}\int_0^{(2R-\rho)(2\epsilon S_1 r)^{1/2}/r^2} d\delta/[a^2(\delta)\rho^2 - \{2Ra(\delta) - [(S_1r+\epsilon)+iz(izS_1/r+iz\epsilon/r^2+\delta)]\}^2]^{1/2}.$$

We only note here that since we are carrying terms up to the first power in ϵ we must carry terms up to the second power in δ. If we do this we find that the coefficient of the term in the radical which involves δ to the first power vanishes and the integral reduces to

(208) $${}_1I_b(t;\rho,z) = (2/\pi)(1/r)((2R-\rho)/\rho)^{1/2}\cdot \int_0^{((2R-\rho)/r^2)(2\epsilon S_1 r)^{1/2}} d\delta/[2\epsilon S_1(2R-\rho)^2/r^3 - \delta^2]^{1/2}.$$

The actual discontinuity in the displacement potential is

(209) $$\varphi(S_1r^+) - \varphi(S_1r) = (1/r)((2R-\rho)/\rho)^{1/2}.$$

At $\rho = R$, the discontinuity is $1/r$. This is just the discontinuity in the source potential at the boundary. It is clear from equation 209 that as ρ approaches zero the discontinuity increases until, at $\rho=0$, it becomes infinite. This is in accord with the result already obtained, but now we are in a position to interpret this result. In virtue of our discussion concerning the superposition integral we see immediately that, if the source pressure is continuous, equation 209 predicts a steepening of the leading edge of the pressure pulse as the reflected wave approaches the axis of the cylinder.

II.2b Response in the Plane of a Ring Source

In the preceding discussion it was noted that the singly reflected wave arrived at the point of observation at the same time it would have arrived if energy had been introduced at $t=0$ along the circumference of a ring line source of radius $2R$ located in the plane $z=0$. It is interesting to determine if these two waves have other properties in common. A study of the exact transient response produced by a ring line source in an infinite fluid leads, in addition, to a better understanding of the effects produced by focusing in the cylinder.

In what follows the radius of the ring is designated by b and the height by $2h$. We divide space into the two regions (a) $0\le\rho\le b$ and (b) $b\le\rho<\infty$. In region (a) we will take a solution of the transformed wave equation of the form

$$(210)\quad \bar{\varphi}_a=\int_0^\infty f_a(\lambda)I_0(\rho\alpha)\cos\lambda Z\,d\lambda$$

and in region (b) a solution of the form

$$(211)\quad \bar{\varphi}_b=\int_0^\infty f_b(\lambda)K_0(\rho\alpha)\cos\lambda Z\,d\lambda\,.$$

The functions $f_a(\lambda)$ and $f_b(\lambda)$ are to be determined by requiring that the pressure be continuous across the cylindrical surface $\rho=b$ and the radial displacement have a discontinuity at $\rho=b$ which is given by the function $Q(z)$. This function is defined by the relations

$$(212)\quad (u_\rho|_{\rho=b})_b-(u_\rho|_{\rho=b})_a=Q(z)=1/h,\ |z|<h;$$
$$=1/2h,\ |z|=h;$$
$$=0,\ |z|>h.$$

It is convenient to write $Q(z)$ in terms of the Fourier cosine integral

$$(213)\quad Q(z)=(2/\pi)\int_0^\infty(\sin\lambda h/\lambda h)\cos\lambda Z\,d\lambda\,.$$

The boundary conditions will be satisfied if $f_a(\lambda)$ and $f_b(\lambda)$ are determined from the equations

$$(214)\quad f_a(\lambda)I_0(b\alpha)=f_b(\lambda)K_0(b\alpha)$$

and

$$(215)\quad \alpha f_a(\lambda)I_1(b\alpha)+\alpha f_b(\lambda)K_1(b\alpha)=-(2/\pi)\chi(S)\sin\lambda h/\lambda h,$$

where $\chi(S)$ designates the Laplace transform of a function which describes how the discontinuity in the radial displacement changes with time. The expressions for $f_a(\lambda)$ and $f_b(\lambda)$ can be simplified to a considerable degree by utilizing the formula for the Wronskian of the modified Bessel functions of the first and second kinds (equation 137). The expressions for the potentials then reduce to the form

$$(216)\quad \bar{\varphi}_a=-(2/\pi)\,b\chi(S)\int_0^\infty(\sin\lambda h/\lambda h)K_0(b\alpha)I_0(\rho\alpha)\cos\lambda Z\,d\lambda$$

and

$$(217)\quad \bar{\varphi}_b=-(2/\pi)\,b\chi(S)\int_0^\infty(\sin\lambda h/\lambda h)I_0(b\alpha)K_0(\rho\alpha)\cos\lambda Z\,d\lambda\,.$$

The potentials which describe the radiation emitted from a true ring line source are obtained from equations 216 and 217 by allowing h to approach zero.

Our primary objective in what follows will be to investigate the effects of focusing. We note immediately that, if ρ and h both vanish, equation 216 reduces to

$$(218)\quad \overline{\Phi}_a = -(2/\pi)\, b\,\chi(S) \int_0^\infty K_0(b\alpha) \cos \lambda Z \, d\lambda \, .$$

Therefore,

$$(219)\quad \overline{P}_a = \sigma_0 b s^2 \chi(S) (2/\pi) \int_0^\infty K_0(b\alpha) \cos \lambda Z \, d\lambda = \sigma_0 b s^2 \chi(S)\, e^{-srs_1}/r ,$$

where $r = (b^2+Z^2)^{1/2}$ and $\overline{P}_a$ is the Laplace transform of the pressure response in region (a). For the moment let us assume that $S^2\chi(S)$ is the Laplace transform of the function $f(t)$. The pressure response can then be expressed in the form

$$(220)\quad P_a(t;0,Z) \equiv 0 \, , \quad t < rS_1 ;$$

$$= (\sigma_0 b/r) f(t - rS_1) , \quad t > rS_1 .$$

It is clear that the pressure at axial points is directly proportional to the cosine of the angle of incidence, a fact which has been previously demonstrated for the wave reflected from a rigid cylindrical surface.

We now consider the problem of inverting $\overline{P}_a$ in the plane of the source $(Z=0)$. The expression for $\overline{P}_a$ reduces to the form

$$(221)\quad \overline{P}_a = (2/\pi)\sigma_0 b s^2 \chi(S) \int_0^\infty K_0(b\alpha) I_0(\rho\alpha)\, d\lambda = A(S)\, G(\rho, S) ,$$

where

$$(222)\quad G(\rho, S) = \int_0^\infty K_0(b\alpha) I_0(\rho\alpha)\, d\lambda \, .$$

In order to invert this expression for $G(\rho,S)$ we make the substitution $\lambda = Su$ and introduce the following relationship for the product of the Bessel functions (9, page 98):

$$(223)\quad K_0(Sba) I_0(S\rho a) = \int_0^\infty J_0\left(2Sa(b\rho)^{1/2} \sinh \xi\right) e^{-Sa(b-\rho)\cosh \xi}\, d\xi$$

$$(223) = \int_0^\infty \left\{ (2/\pi) \Re \int_0^{\pi/2} e^{2iSa(b\rho)^{1/2}\sinh\xi\cos\psi} d\psi \right\} e^{-Sa(b-\rho)\cosh\xi} d\xi .$$

The interchange of order of integration is clearly legitimate in equation 223 and reduces the expression for $G(\rho,S)$ to the form

$$(224)\quad G(\rho,S) = S(2/\pi)\Re \int_0^\infty du \int_0^{\pi/2} d\psi \int_0^\infty d\xi\, e^{-Sa\{(b-\rho)\cosh\xi - 2i(b\rho)^{1/2}\sinh\xi\cos\psi\}} .$$

Now make the transformation

$$(225)\quad t = a\{(b-\rho)\cosh\xi - 2i(b\rho)^{1/2}\sinh\xi\cos\psi\}$$

and consider this relation to define ξ in terms of the independent variables t, u, and ψ. The partial derivative of ξ with respect to t is simply

$$(226)\quad \left.\frac{d\xi}{dt}\right|_{u,\psi} = 1/a\{(b-\rho)\sinh\xi - 2i(b\rho)^{1/2}\cosh\xi\cos\psi\} .$$

The integral over ξ, in equation 224, is reduced by this transformation to the form

$$(227)\quad F(u,\psi) = \int_{a(b-\rho)}^{H_{u,\psi}} e^{-St} \left.\frac{d\xi}{dt}\right|_{u,\psi} dt ,$$

where the path $H_{u,\psi}$ is dependent on the values of both u and ψ. In order to investigate the singularities of the integrand we write $\left.\frac{d\xi}{dt}\right|_{u,\psi}$ as an explicit function of the independent variables. An expression for the $\cosh\xi$ can be obtained from the defining relation for t. It has the form

$$(228)\quad \cosh\xi = \{(b-\rho)t + 2(b\rho)^{1/2}\cos\psi\,(a^2Q^2 - t^2)^{1/2}\}/aQ^2 ,$$

where

$$(229)\quad Q^2 = (b-\rho)^2 + 4b\rho\cos^2\psi .$$

A one to one correspondence between points in the ξ and t planes can be obtained by cutting the t plane along the real t axis between the branch points $\pm aQ$ and by selecting the positive sign for the radical when $t > aQ$. The relation

(230) $\sinh^2\xi = \cosh^2\xi - 1$

can be used to obtain a similar expression for the $\sinh\xi$, namely

(231) $\sinh\xi = -(i/aQ^2)\{(b-\rho)(a^2Q^2-t^2)^{1/2} - 2t(b\rho)^{1/2}\cos\psi\}$.

These expressions for the hyperbolic sine and cosine reduce the expression for $\left.\frac{d\xi}{dt}\right|_{u,\psi}$ to the form

(232) $\left.\frac{d\xi}{dt}\right|_{u,\psi} = i/(a^2Q^2-t^2)^{1/2}$.

It is clear that the singularities of the integrand, in equation 227, are located at the branch points $\pm aQ$. This indicates that the contour, $H_{u,\psi}$, and the portion of the real t axis which extends from $a(b-\rho)$ to infinity and which lies an infinitesimal distance below the branch cut are equivalent contours. The expression for $G(\rho,S)$ can be reduced to the form

(233) $$G(\rho,S) = (2/\pi)S\,R\int_0^\infty du\int_0^{\pi/2} d\psi\int_{a(b-\rho)}^\infty e^{-St}\,i\,dt/(a^2Q^2-t^2)^{1/2}.$$

This result can be further reduced by noting that the real part of the integral over t vanishes if $a(b-\rho)\leq t\leq aQ$. Accordingly, we rewrite $G(\rho,S)$ in the form

(234) $$G(\rho,S) = (2/\pi)S\int_0^\infty du\int_0^{\pi/2} d\psi\int_{aQ}^\infty e^{-St}dt/(t^2-a^2Q^2)^{1/2}.$$

To proceed further we assume that the order in which the u and ψ integrations are carried out can be interchanged. It is then easy to justify the change in the order in which the u and t integrations are carried out. This double interchange reduces the expression for $G(\rho,S)$ to a form which can be readily integrated, namely

(235) $$G(\rho,S) = (2/\pi)S\int_0^{\pi/2} d\psi\int_{S_1Q}^\infty e^{-St}dt\int_0^{(t^2-S_1^2Q^2)^{1/2}/Q} du/(t^2-a^2Q^2)^{1/2}.$$

The fact that

(236) $$\int_0^{(t^2-S_1^2Q^2)^{1/2}/Q} du/(t^2-a^2Q^2)^{1/2} = (1/Q)\int_0^{(t^2-S_1^2Q^2)^{1/2}/Q} du/\{(t^2-S_1^2Q^2)/Q^2-u^2\}^{1/2} = \pi/2Q,$$

reduces the expression for $G(S,\rho)$ to the double integral

(237) $$G(S,\rho) = S\int_0^{\pi/2} d\psi \int_{S_1Q}^{\infty} e^{-St}dt/Q,$$

from which the final result can be obtained by interchanging the order of integration. The expression for $G(\rho,S)$ then reduces to

(238) $$G(\rho,S) = S\int_{S_1(b-\rho)}^{S_1(b+\rho)} e^{-St}dt \int_{\cos^{-1}[(t^2-S_1^2(b-\rho)^2)/4S_1^2b\rho]^{1/2}}^{\pi/2} d\psi/Q \quad + S\int_{S_1(b+\rho)}^{\infty} e^{-St}dt\int_0^{\pi/2} d\psi/Q.$$

The coefficient, S, can be removed from this expression by integrating the first term on the right by parts. We obtain the result that

(239) $$S\int_{S_1(b-\rho)}^{S_1(b+\rho)} e^{-St}f(t,\rho)dt = -e^{-St}f(t,\rho)\Big|_{t=S_1(b-\rho)}^{t=S_1(b+\rho)} + \int_{S_1(b-\rho)}^{S_1(b+\rho)} e^{-St}\frac{df}{dt}dt$$

$$= -e^{-SS_1(b+\rho)}f(S_1(b+\rho),\rho) + e^{-SS_1(b-\rho)}f(S_1(b-\rho),\rho) + \int_{S_1(b-\rho)}^{S_1(b+\rho)} e^{-St}\frac{df}{dt}dt,$$

where

(240) $$f(t,\rho) = \int_{\cos^{-1}[(t^2-S_1^2(b-\rho)^2)/4S_1^2b\rho]^{1/2}}^{\pi/2} d\psi/Q.$$

The fact that $f(S_1(b-\rho),\rho)\equiv 0$ and that the first term on the right of equation 239 and the second term on the right of equation 238 are equal and opposite in sign indicates that $G(\rho,S)$ is the Laplace transform of the function $q(t,\rho)$, where

(241) $$q(t,\rho)\equiv 0, \quad t<S_1(b-\rho);$$

$$= \frac{df}{dt} = S_1/\{(t^2-S_1^2(b-\rho)^2)^{1/2}(S_1^2(b+\rho)^2-t^2)^{1/2}\}, \quad S_1(b-\rho)<t<S_1(b+\rho);$$

$$\equiv 0, \quad t>S_1(b+\rho).$$

This result can be easily interpreted if $A(S)$ is specified by the relation

(242) $A(S)=(2/\pi)(\sigma_0 b/S_1^2)(K/(S^2+K^2))(1-e^{-ST})=\mathcal{L}\{A(t)\}$,

where $K=2\pi/T$. The determining function, $A(t)$, is then given by the relations

(243) $A(t)=(2/\pi)(\sigma_0 b/S_1^2)\sin Kt, \quad 0\leq t\leq T;$

$$\equiv 0, \quad t>T.$$

The final result can be obtained by convoluting $q(t,\rho)$ with $A(t)$. It is

(244) $P(t)\equiv 0, \quad t<S_1(b-\rho);$

$$=(2/\pi)(\sigma_0 b/S_1^2)\int_{S_1(b-\rho)}^{t}\sin K(t-u)q(u,\rho)du, \quad S_1(b-\rho)\leq t\leq S_1(b-\rho)+T;$$

$$=(2/\pi)(\sigma_0 b/S_1^2)\int_{t-T}^{t}\sin K(t-u)q(u,\rho)du, \quad S_1(b-\rho)+T\leq t\leq S_1(b+\rho);$$

$$=(2/\pi)(\sigma_0 b/S_1^2)\int_{t-T}^{S_1(b+\rho)}\sin K(t-u)q(u,\rho)du, \quad S_1(b+\rho)\leq t\leq S_1(b+\rho)+T;$$

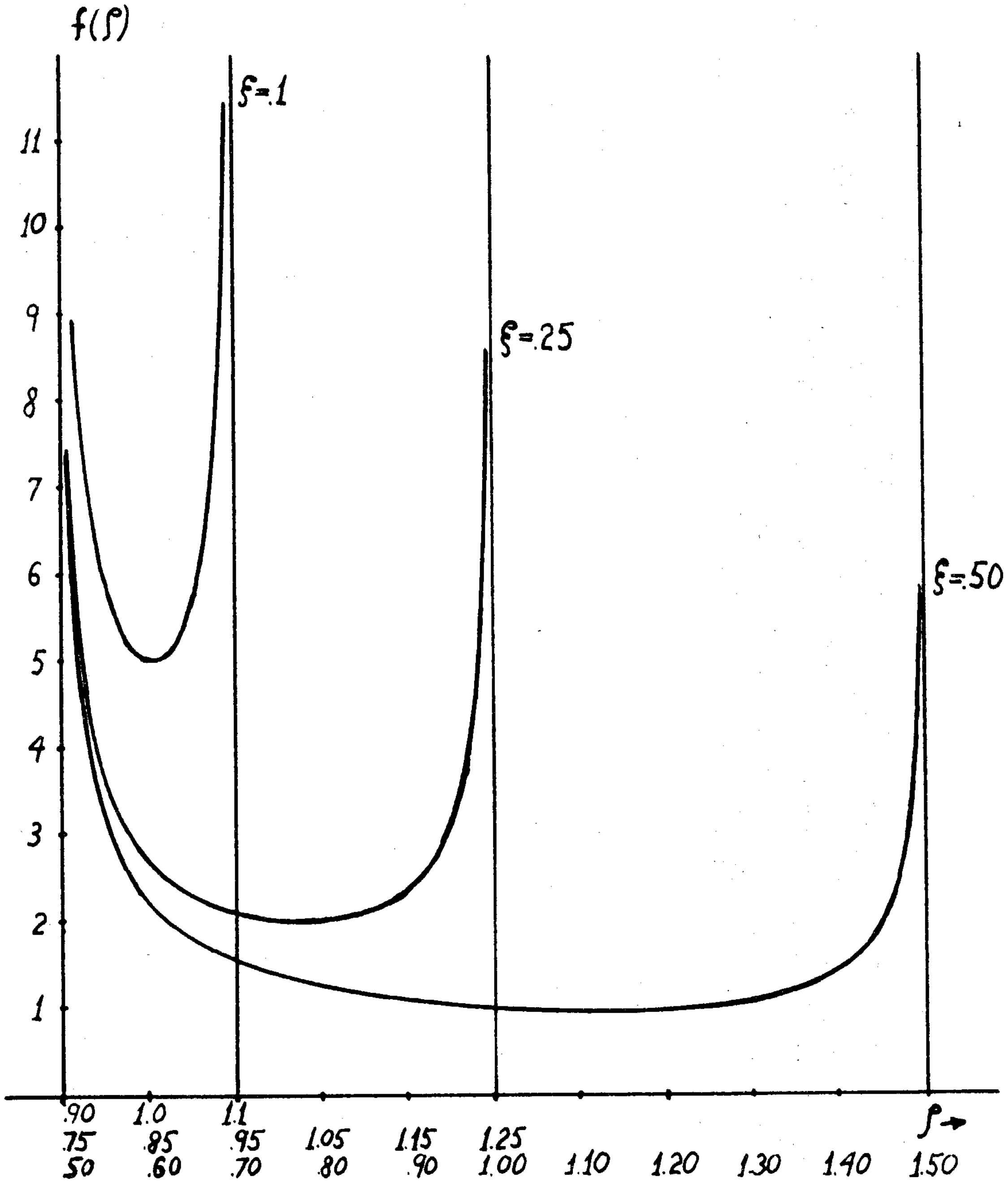
$$\equiv 0, \quad t>S_1(b+\rho)+T.$$

Many of the salient features of the pressure response can be obtained without actually carrying through the numerical integrations. This procedure is facilitated by rewriting $P(t)$ in terms of the dimensionless time $t/S_1 b=t/T_0=\int$ and the dimensionless distance $\rho/b=\xi$. In the time interval $S_1(b-\rho)+T\leq t\leq S_1(b+\rho)$, $P(t)$ can be expressed in the form

(245) $P(t)=(2/\pi)(\sigma_0/S_1^2)\int_{(t-T)/T_0}^{t/T_0}\sin[(2\pi T_0/T)(t/T_0-\int)]d\int/\{(\int^2-(1-\xi)^2)^{1/2}((1+\xi)^2-\int^2)^{1/2}\}$,

$$(1-\xi)+T/T_0\leq t/T_0\leq(1+\xi).$$

In fig. 26 we have plotted curves of the function $f(\int)=1/\{(\int^2-(1-\xi)^2)^{1/2}((1+\xi)^2-\int^2)^{1/2}\}$ for values of ξ equal to .1, .25, and .50. It is easily demonstrated that $f(\int)$ has a minimum value at $\int_0=(1+\xi^2)^{1/2}$, which is $f(\int_0)=1/2\xi$. The period of the sine term can be fixed by specifying the value of the quantity T/T_0. This quantity is just the ratio of the wave length of the sine term to the radius of the source. In order to investigate



f(ρ)
ξ=.1
ξ=.25
ξ=.50
11
10
9
8
7
6
5
4
3
2
1
.90
.75
.50
1.0
.85
.60
1.1
.95
.70
1.05
.80
1.15
.90
1.25
1.00
1.10
1.20
1.30
1.40
1.50
ρ→

Figure 26

the time behavior of the pressure response we must consider how $P(t)$ changes as the sine wave is displaced from left to right across the graph of the function $f(\rho)$. This is most easily accomplished if we take the ratio of the wave length, λ, and the radius, b, to be quite small, say .02, and if we introduce the quantities T_a^ξ, T_f^ξ, and T_m^ξ. T_a refers to the time required for the direct wave to reach the point of observation (or the time at which the first discontinuity in fig. 26 occurs), T_f refers to the time required for the wave from the opposite side of the ring source to reach this point (or the time at which the second discontinuity in fig. 26 occurs), T_m specifies the time at which the individual curves in fig. 26 take on their minimum values, and the superscript ξ designates the curve which is being considered.

By differentiating the second and fourth relations, in equation 244, we can show that the slope of the response curve vanishes at T_a and T_f+T. Comparison of the areas under the ξ curves, in the interval $T_a \leq t \leq T_a+T/2$, indicates that the pressure increases steadily as ξ varies from .5 to .1. If we focus our attention on a particular value of ξ, we find that, in the interval $T_a \leq t \leq T_a+T$, the pressure first increases to a maximum positive value, then decreases to a maximum negative value, and then approaches zero from negative values as the time increases. It is also noteworthy that the amplitude of the pressure maximum is greater than the amplitude of the pressure minimum. These remarks certainly remain valid as long as the time interval, T_f-T_a, is greater than about twice the period of the sine wave. Points which are so close to the axis of the cylinder that they violate this restriction require special attention.

This same type of reasoning can be used to investigate the behavior of the pressure response in the time interval $T_f - T \leq t \leq T_f + T$. Study of a specific ξ curve reveals that the reflected wave has a positive forerunner. The amplitude of the response increases with time until it attains a maximum positive value, it then decreases to a maximum negative value, and then returns to zero at $t = T_f + T$. It is interesting to note that the depth of the minimum is greater than the height of the maximum.

The characteristics of the response can be readily interpreted if we consider the ring to be composed of an infinite number of point sources. The distortion results from a progressive variation in the way in which the waves radiated from the individual point sources superpose themselves. The steepening results from the fact that, as ρ decreases, the waves arrive more nearly in phase and reinforce each other. The presence of a forerunner shows that it is possible for waves radiated by point sources on either side of the diametrically opposite point to reinforce each other. The response vanishes at $T_f + T$ because of the fact that the last energy which reaches the point of observation comes from the diametrically opposite point and that this point radiates only during the time interval $0 \leq t \leq T$.

We now see that the pulse which is radiated by a true ring source and the pulse which appears to be radiated by a ring source, of radius $b = 2R$, have at least two properties in common. In both cases the pulse pressure at axial points is directly proportional to the cosine of the angle of incidence and, in the plane $z=0$, the leading edge of the pressure pulse steepens as it approaches the axis. It also appears that we can now offer an explanation for the difficulty which is encountered in expanding equation 135 into a series of terms which represent multiple reflections. In the case of the ring source the positive forerunner results from a constructive interference of waves originating on either side of the diametrically opposite point—that is, the least time path is no longer the path from the point of observation to the diametrically opposite point. It seems quite reasonable to believe that such an effect should also be observed in the cylinder. If this is indeed the case, we should not expect to find distinct phases appearing at times which correspond to the arrival times of the multiply reflected waves.

II.2c The Pressure Response Within the Cylinder

We now derive an exact expression for the pressure response. We will use an approach which has certain features in common with the method used to find the steady state, normal mode solutions for systems having more than one plane boundary. The expression for the transform of the pressure response can be reduced to the form

$$(246)\quad \bar{P} = -(\sigma_0 S^2)(2/\pi S) S X(S) \int_0^\infty \cos SUZ \left[\{ K_0(S\rho a) I_1(SRa) + K_1(SRa) I_0(S\rho a) \} / I_1(SRa) \right] du.$$

This relation can be obtained by combining the source and perturbation terms in equation 133, multiplying the resulting expression by $-\sigma_0 S^2$, and making the substitution $\lambda = Su$. The presence of the radical, $a(u)$, indicates that the sign of the argument of the Bessel functions changes when one complete loop is made about either of the points $\pm iS_1$. We now show that the integrand is unchanged when u makes a complete circuit about the point iS_1. To do this we assume that the plane is cut along the imaginary axis between the points iS_1 and $-iS_1$ and that the sign of the radical, a, is positive on the real u axis. The study of the quantity in brackets, at points which are located on the left side of the cut, is facilitated by introducing the following relations:

(247) $$I_0(S\rho a e^{i\pi}) = I_0(S\rho a),$$

(248) $$I_1(SRa e^{i\pi}) = -I_1(SRa),$$

(249) $$K_0(S\rho a e^{i\pi}) = K_0(S\rho a) - \pi i I_0(S\rho a),$$

and

(250) $$K_1(SRa e^{i\pi}) = -K_1(SRa) - \pi i I_1(SRa).$$

On the left side of the cut the integrand has the form

$$\left[-(K_0(S\rho a) - \pi i I_0(S\rho a))I_1(SRa) - (K_1(SRa) + \pi i I_1(SRa))I_0(S\rho a)\right] / -I_1(SRa)$$

$$= \left[K_0(S\rho a)I_1(SRa) + K_1(SRa)I_0(S\rho a)\right] / I_1(SRa).$$

It is clear that neither iS_1 nor $-iS_1$ are branch points of the integral. Therefore, the assumed branch cut between these two points can be removed. The expression for $\bar{P}$ reduces to

(251) $$\bar{P} = -(\sigma_0 S^2)(2/\pi S)(S/2)\chi(S)\int_{-\infty}^{\infty} e^{iSuz}\left[(K_0(S\rho a)I_1(SRa) + K_1(SRa)I_0(S\rho a))/I_1(SRa)\right]du.$$

The behavior of the integrand, in the vicinity of the point iS_1, can be investigated by introducing the first few terms in the series expansions of the Bessel functions about zero, namely

(252) $$I_0(S\rho a) \sim 1,$$

(253) $I_1(SRa) \sim SRa/2,$

(254) $K_0(S\rho a) \sim -(\gamma + \mathrm{Log}\, S\rho a/2),$

and

(255) $K_1(SRa) \sim ((SRa)/2)(\gamma + \mathrm{Log}\, SRa/2) + (1/SRa) - (SRa/4).$

In the vicinity of the point iS_1 the integrand reduces to

$$(256)\quad e^{iSuz}_{u\to iS_1}[\,] \sim (e^{-SS_1z}/(SRa/2))\,[-(SRa/2)(\gamma + \mathrm{Log}\, S\rho a/2) + (SRa/2)(\gamma + \mathrm{Log}\, SRa/2) + 1/SRa]$$

$$= (e^{-SS_1z}/(SRa/2))\,(1/SRa + (SRa/2)\,\mathrm{Log}\, R/\rho)$$

$$= e^{-SS_1z}\,(2/S^2R^2(u^2+S_1^2) + \mathrm{Log}\, R/\rho).$$

The point iS_1 therefore behaves like a simple pole. The other singularities of the integrand are also simple poles and are located on the imaginary axis at points where $I_1(SRa)$ vanishes.

Next we consider whether the result which is obtained by integrating along the arc of the semi-circle, which lies in the first and second quadrants of the u plane, can be made to vanish when the radius of the semi-circle approaches infinity. In order to do this we make use of the asymptotic expansions of the Bessel functions. This means that from the beginning ρ must be assumed to be non-zero.

We make the substitution $u = \varrho e^{i\varphi}$ in the integrand of equation 251. ϱ is a positive quantity which can be made arbitrarily large. The first terms in the asymptotic expansions of the Bessel functions have the following form:

(257) $K_0(S\rho\varrho e^{i\varphi}) \sim e^{-S\rho\varrho e^{i\varphi}}/(2/\pi)^{1/2}(S\rho\varrho e^{i\varphi})^{1/2},$

(258) $K_1(SR\varrho e^{i\varphi}) \sim e^{-SR\varrho e^{i\varphi}}/(2/\pi)^{1/2}(SR\varrho e^{i\varphi})^{1/2},$

(259) $I_0(S\rho\varrho e^{i\varphi}) \sim (e^{S\rho\varrho e^{i\varphi}} + i\, e^{-S\rho\varrho e^{i\varphi}})/(2\pi S\rho\varrho e^{i\varphi})^{1/2},$

and

(260) $I_1(SR\varrho e^{i\varphi}) \sim (e^{SR\varrho e^{i\varphi}} - i\, e^{-SR\varrho e^{i\varphi}})/(2\pi SR\varrho e^{i\varphi})^{1/2}.$

If these relations are substituted in the expression in brackets in equation 251, it is found that

(261) $[\ ]\sim(\pi/2S\rho Q e^{i\varphi})^{1/2}\{(e^{SQ(R-\rho)e^{i\varphi}}+e^{-SQ(R-\rho)e^{i\varphi}})/(e^{SRQe^{i\varphi}}-ie^{-SRQe^{i\varphi}})\}$.

The result of rationalizing the denominator is

(262) $e^{2SRQ\cos\varphi}-2\sin(2SRQ\sin\varphi)+e^{-2SRQ\cos\varphi}$.

If $\varphi=\pi/2$, this expression reduces to $2(1-\sin 2SRQ)$. Therefore, the denominator vanishes if $SRQ=(m+1/4)\pi$. This means that we must choose the value of Q in such a way that this relation is not satisfied—any other value of Q is permissible. If $0\leq\varphi\leq\pi/2-\delta\ (\delta\neq 0)$, the first term in equation 262 increases exponentially. If $\pi/2+\delta\leq\varphi\leq\pi$, the third term increases exponentially. In the interval $\pi/2-\delta\leq\varphi\leq\pi/2+\delta$ we can always select a particular value of Q which will make the denominator greater than some positive quantity B. This fact is necessary to the subsequent discussion.

A little algebra is sufficient to show that, in the interval $0\leq\varphi\leq\pi$, the absolute value of the integrand cannot exceed

(263) $$|[\ ]||e^{isuz}|\leq(\pi Q/2S\rho)^{1/2}e^{-SzQ\sin\varphi}\cdot\{e^{2SRQ\cos\varphi-S\rho Q\cos\varphi}+e^{-S\rho Q\cos\varphi}+e^{S\rho Q\cos\varphi}+e^{-2SRQ\cos\varphi+S\rho Q\cos\varphi}\}/\{e^{2SRQ\cos\varphi}-2\sin(2SRQ\sin\varphi)+e^{-2SRQ\cos\varphi}\}.$$

In the interval $0\leq\varphi\leq\pi/2-\delta$ the denominator cannot be less than $(1/2)e^{2SRQ\cos\varphi}$. Similarly, the numerator is always less than $4e^{SQ(2R-\rho)\cos\varphi}$. We see that the absolute value of the integrand cannot exceed

(264) $8(\pi Q/2S\rho)^{1/2}e^{-SQ(z\sin\varphi+\rho\cos\varphi)}$

and that the integral is less than

(265) $8(\pi Q/2S\rho)^{1/2}\int_0^{\pi/2-\delta}e^{-SQ(z\sin\varphi+\rho\cos\varphi)}d\varphi$

If we note that $\cos\varphi \geq 1-(2/\pi)\varphi\ (0\leq\varphi\leq\pi/2-\delta)$, it is an easy matter to demonstrate that this integral must vanish when Q is allowed to approach infinity even if z is identically zero. A similar statement also holds in the interval $\pi/2+\delta\leq\varphi\leq\pi$.

We have now demonstrated that the result which is obtained by integrating along each of the circular arcs, $0\leq\varphi\leq\pi/2-\delta$ and $\pi/2+\delta\leq\varphi\leq\pi$, can be made to vanish in the limit. It remains to be shown that this result is also true if we integrate along the arc $\pi/2-\delta\leq\varphi\leq\pi/2+\delta$. We now make use of the fact that Q can always be chosen in such a way that the denominator is greater than B . This means that, in absolute value, the integral must be less than

(266) $(\pi Q/2S\rho)^{1/2}(8\delta/B)e^{-SzQ\cos\delta}e^{SQ(2R-\rho)\sin\delta}$.

Therefore, if ρ and z are non-zero, the result which is obtained by integrating along the semi-circle can be made arbitrarily small by selecting progressively larger values of Q.

We now see that $\bar{P}$ can be expressed in the form

(267) $$\bar{P}=A(S)\int_{-\infty}^{\infty}e^{iSuz}[\,]\,du=2\pi i A(S)\sum_{k=0}^{\infty}R_k ,$$

where the R_k are the residues of the integral at the points on the positive imaginary axis where $I_1(SRa)$ vanishes. The function $I_1(SRa)$ vanishes if u satisfies the condition

(268) $SRa(u_k)=i\omega_k$

or

(269) $u_k=(i/SR)(\omega_k^2+S^2S_1^2R^2)^{1/2}$,

where the ω_k are the positive zeros of $J_1(x)$. We can find the first term in the series expansion of the function $I_1(SRa)$ about the point u_k from Taylor's theorem. It is

(270) $I_1(SRa) = (SRu_k/a(u_k))\,I_0(SRa(u_k))(u-u_k) + \cdots$

The residue, at the point u_k, can be put in the form

(271) $$2\pi i R_k = (\pi^3 \omega_k^2/2SS_1R^2)\left(e^{-zS_1(\omega_k^2/R^2S_1^2+S^2)^{1/2}}/(\omega_k^2/R^2S_1^2+S^2)^{1/2}\right)\cdot J_0(\omega_k\rho/R)\,Y_1^2(\omega_k).$$

In deriving this expression we have used the fact that $K_1(i\omega_k) = (i\pi/2)Y_1(\omega_k)$ and the fact that the Wronskian of the Bessel functions of the first and second kinds, of zero order, reduces to

(272) $J_0(\omega_k)Y_1(\omega_k) = -2/\pi\omega_k$.

The residue at the point iS_1 is easily evaluated from equation 256. It is found to be

(273) $2\pi i R_0 = (2\pi/S_1)e^{-SS_1Z}/S^2R^2.$

The transform of the pressure response can now be put in the form

(274) $$\bar{P}(S;\rho,z) = -(\sigma_0 S^2)(2/\pi S)(S/2)\chi(S)(\pi/S_1R^2)\cdot\left\{(2/S^2)e^{-SS_1Z} + (\pi^2/2S)\sum_{k=1}^{\infty}\omega_k^2 Y_1^2(\omega_k)J_0(\rho\omega_k/R)\,e^{-2S_1(\omega_k^2/R^2S_1^2+S^2)^{1/2}}/(\omega_k^2/R^2S_1^2+S^2)^{1/2}\right\}.$$

It is clear that the Laplace transform variable, S, has been removed from the argument of the Bessel functions. This fact reduces to a considerable degree the labor required in inverting $\bar{P}$.

In order to invert the expression for $\bar{P}$ we make use of the result (10)

(275) $$e^{-zS_1(\omega_k^2/R^2S_1^2+S^2)^{1/2}}/(\omega_k^2/R^2S_1^2+S^2)^{1/2} = \mathcal{L}\{f_k(t;z,\omega_k)\},$$

where

(276) $$f_k(t;z,\omega_k) \equiv 0,\quad t<S_1Z;$$
$$= J_0\{(\omega_k/RS_1)(t^2-S_1^2Z^2)^{1/2}\},\quad t>S_1Z.$$

Now we consider the result of convoluting each term in the series expansion with the function whose Laplace transform is $s\chi(s)$. If we call this function $f(t)$, the expression for $\bar{P}$ can be rewritten in the form

$$(277)\quad \bar{P}(s;\rho,z) = -(2\sigma_0/s_1R^2)\int_{s_1z}^{\infty} e^{-st}\left[\int_{s_1z}^{t} f(t-u)\,1(u-s_1z)\,du\right]dt$$
$$-(\sigma_0\pi^2/2s_1R^2)\sum_{k=1}^{\infty}\omega_k^2 Y_1^2(\omega_k)J_0(\omega_k\rho/R)\int_{s_1z}^{\infty} e^{-st}\left[\int_{s_1z}^{t} f(t-u)J_0\{(\omega_k/Rs_1)(u^2-s_1^2z^2)^{1/2}\}du\right]dt.$$

It is clear that the expression for the actual pressure variation can be obtained from equation 277 by interchanging the order in which the operations of summation with respect to k and integration with respect to t are carried out. If we attempt to justify the interchange by appealing to Fubini's theorem we must show that either

$$(278)\quad \sum_{k=1}^{\infty}\int_{s_1z}^{\infty}|Q_k(t,s,\rho,z)|\,dt$$

or

$$(279)\quad \int_{s_1z}^{\infty}\left\{\sum_{k=1}^{\infty}|Q_k(t,s,\rho,z)|\right\}dt$$

exists. The function, Q_k, is defined by the relation

$$(280)\quad Q_k = \omega_k^2 Y_1^2(\omega_k)J_0(\omega_k\rho/R)\,e^{-st}\int_{s_1z}^{t} f(t-u)J_0\{(\omega_k/Rs_1)(u^2-s_1^2z^2)^{1/2}\}\,du.$$

The difficulty which is encountered in obtaining a good approximation to the integral stems in part from the fact that for large values of ω_k (and/or small values of the bore hole radius) the integrand is a rapidly oscillating function. It appears that the problem of justifying the interchange reduces to that of finding a particular time function for which the integral can be evaluated.

It is clear that each term in the infinite series expansion of the pressure response starts contributing at the time s_1z, independent of the value of ρ. This indicates that the individual terms of the expansion must interfere destructively in the time interval $S_1(\rho^2+Z^2)^{1/2}-S_1Z$.

The ρ dependence enters only in the combination $J_0(\omega_k \rho/R)$. This indicates that within a given mode there is an oscillation of the sign of the Bessel function when ρ is allowed to increase from zero to R and that the rate of oscillation increases as we go to higher modes. Also, if $\rho=R$, the sign of the Bessel function is alternatively positive and negative as k increases. Finally, we note that the pressure varies as the inverse square of the bore hole radius R . Such a result clearly has meaning only for those times which are large compared with the time required for the direct wave to reach the point of observation.

II.3 Pulse Propagation in an Elastic Rod

A closely related problem, which is of practical interest and which serves to illustrate the extreme complexity of the transformed functions in a less idealized situation, will now be formulated.

We will obtain transforms of the functions which describe the transient response in a perfectly elastic rod of infinite length. The source is considered to be spherical in shape. It's center is located on the axis of the rod in the plane $Z=0$. As in the preceding case the source function can be expressed in the form

(281) $$\bar{\Psi}_s=(2/\pi S)X(S)\int_0^\infty K_0(\rho\beta)\cos\lambda Z\,d\lambda,$$

where $\bar{\Psi}_s$ is the Laplace transform of that part of the scalar displacement potential which describes the effect of the source and $\beta=(\lambda^2+S^2/V_s^2)^{1/2}$. The transform of the displacement potential is

(282) $\bar{\Psi} = (2/\pi S)\chi(S)\int_0^\infty K_0(\rho\beta)\cos\lambda Z\,d\lambda + \int_0^\infty f(\lambda)I_0(\rho\beta)\cos\lambda Z\,d\lambda$.

If the solution for the transform of the vector potential is expressed in the form

(283) $\bar{U} = \int_0^\infty g(\lambda)I_1(\Gamma\rho)\sin\lambda Z\,d\lambda$

($\Gamma=(\lambda^2+S^2/v_s^2)^{1/2}$), it can be easily verified that the normal and tangential components of stress, and the vertical and radial components of displacement have the correct symmetry with respect to the plane $Z=0$. The functions $f(\lambda)$ and $g(\lambda)$ can be determined by requiring that the radial and tangential stress components vanish on the cylindrical surface $\rho=R$. These conditions are satisfied if

(284) $f(\lambda)=-(2/\pi S)\chi(S)(1/D(\lambda))\cdot$
$\{\Omega^2 I_1(R\Gamma)K_0(R\beta)+\lambda^2\beta\Gamma K_1(R\beta)I_0(R\Gamma)+(S^2/2v_s^2)(\beta/R)K_1(R\beta)I_1(R\Gamma)\}$

and

(285) $g(\lambda)=-(2/\pi S)\chi(S)(1/D(\lambda))(\lambda\Omega/R)$,

where

(286) $\Omega=\lambda^2+S^2/2v_s^2$

and

(287) $D(\lambda)=\Omega^2 I_1(R\Gamma)I_0(R\beta)-\lambda^2\beta\Gamma I_1(R\beta)I_0(R\Gamma)-(S^2/2v_s^2)(\beta/R)I_1(R\Gamma)I_1(R\beta)$.

It is noteworthy that the integrand, in the expressions for $\bar{\Psi}$ and $\bar{U}$, remains unchanged when a complete loop is made about either of the points where β or Γ vanishes. This is true even though the argument of the Bessel functions changes sign.

If we combine the source and perturbation terms and limit ourselves to a consideration of the response at the surface of the rod, $\bar{\Psi}$ reduces to the form

$$(288)\quad \bar{\psi} = -(2/\pi S)X(S)(1/R)\int_0^\infty (\cos\lambda Z\, d\lambda/D(\lambda))\left[\lambda^2\Gamma I_0(R\Gamma) + (S^2/2v_S^2)(1/R)I_1(R\Gamma)\right].$$

An alternate way of establishing a compressional source for this problem consists in specifying the normal stress at the surface of the rod in such a way that it is symmetric with respect to the plane $Z=0$. The expressions which are obtained for the potentials contain $D(\lambda)$ in the denominator and are not mathematically simpler than those already obtained for the response produced by a point source.

II.4 Pulse Propagation in a Fluid Cylinder with Elastic Walls

A related problem is concerned with predicting the response in a liquid-filled cylindrical cavity which is surrounded by a perfectly elastic, isotropic solid extending to infinity. This case is an idealization of the situation which occurs in acoustic well-logging, where the crystal source and detector, as well as the electronic equipment required for exciting the source and amplifying the received signal, are lowered into the bore hole in a compact unit having a cylindrical shape. The source is generally located at the lower end of the unit and the two detectors are separated from the source by prescribed distances. At least a part of the energy which is radiated by the source travels through the adjacent formation before reaching the detectors. In favorable situations one can determine, from an analysis of the records, the average longitudinal and transverse velocities in the adjacent formation. This information is generally used to obtain the velocity depth dependence. It can also be used to determine certain of the elastic quantities, such as the rigidity and incompressibility, and hence can provide additional parameters which may be useful in correlating between bore holes.

It is easily verified that expressions of the form

(289) $\bar{\varphi} = \int_0^\infty \left((2/\pi S)\chi(S) K_0(\rho\alpha) + f(\lambda) I_0(\rho\alpha) \right) \cos\lambda Z \, d\lambda,$

(290) $\bar{\Psi} = \int_0^\infty h(\lambda) K_0(\rho\beta) \cos\lambda Z \, d\lambda,$

and

(291) $\bar{u} = \int_0^\infty j(\lambda) K_1(\rho\Gamma) \sin\lambda Z \, d\lambda$

satisfy the transformed wave equations, the conditions at the source, and the symmetry conditions. The functions f, h, and j can be determined by requiring that the radial displacement and the radial stress be continuous and that the tangential stress vanish at the cylindrical boundary. Satisfaction of these conditions leads to the following relations for the functions f, h, and j:

(292) $f(\lambda) = (2/\pi S)\chi(S)(1/D(\lambda))(K_1(\alpha R) G(\lambda) - K_0(\alpha R) H(\lambda)),$

(293) $h(\lambda) = (2/\pi S)\chi(S)(1/D(\lambda))(\Omega/\alpha)(S^2/2v_S^2)(D_R/R) K_1(\Gamma R),$

and

(294) $j(\lambda) = -(2/\pi S)\chi(S)(1/D(\lambda))(S^2/2v_S^2)(D_R/R)(\lambda\beta/\alpha) K_1(\beta R).$

The quantity D_R has been substituted for the density ratio, σ_0/σ_S. The functions D, G, and H are given by relations

(295) $D(\lambda) = I_1(\alpha R) G(\lambda) + I_0(\alpha R) H(\lambda),$

(296) $G(\lambda) = \Omega^2 K_1(R\Gamma) K_0(R\beta) - \lambda^2 \beta\Gamma K_0(R\Gamma) K_1(R\beta) + (S^2/2v_S^2)(\beta/R) K_1(R\beta) K_1(R\Gamma),$

and

(297) $H(\lambda) = (S^4/4v_S^4) D_R (\beta/\alpha) K_1(R\beta) K_1(R\Gamma).$

If either the density ratio approaches zero or the rigidity approaches infinity the functions h and j vanish and f reduces to the form it has for a rigid boundary.

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