THE STABILITY OF UNIFORM PLASMAS

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ABSTRACT

This thesis deals with the instabilities which can exist in infinite, uniform, collisionless plasmas having non-Maxwellian distributions of particle velocities. The instabilities are treated in terms of exponentially growing linearized plane waves in the plasma. The existence and properties of these waves can be determined from certain "dispersion relations", or equations relating the frequency of the waves to their wavelength. These dispersion relations are exhibited for all classes of linearized plane waves, and a formal solution of the stability problem is given. A new analogy to electrostatic potential theory, and a classical method, the Nyquist diagram, are used separately and in combination to reduce the formal solution to practicable techniques in several important restricted cases. For example, solutions are obtained to the problems of stability with respect to longitudinal waves in the absence of a D.C. magnetic field, and of stability with respect to longitudinal and to transverse waves propagating along a D.C. magnetic field.

In the course of the analysis it is found that a deficit of particles at a certain velocity tends to produce growing longitudinal oscillations of that phase velocity, while an excess tends to produce growing transverse waves. An arbitrarily small deficit or excess can still produce instability if it involves abrupt enough variations of the particle densities in velocity space.

Two examples are presented which are important in astrophysics for understanding the formation of shock fronts, and one example is given which may be of value in explaining D.C. plasma resistivity at high temperatures, when binary collisions are negligible. Certain instabilities in counterstreaming plasmas that have been used in the literature to determine hydromagnetic shock thicknesses, and to explain the presence of abnormally high energy electrons in the outer Van Allen belt or belts, are found to be absent unless the initial temperatures of the plasmas are extremely low.

CONTENTS

I. INTRODUCTION

Collective Motion in Plasmas

Just as it is useful to distinguish collective motion, such as sound waves, from random particle motion and fluctuations in an ordinary gas, it is desirable to characterize collective motions in a plasma, or fully ionized gas. Numerous perceptive analyses with varying degrees of rigor (1,2,3,4) have provided a sound theoretical basis for discussing such collective motions; on the whole (except possibly for unusually cold, dense plasmas) they support the validity of an equation first proposed by Vlasov (1) for determining the behavior of a plasma. Vlasov's equation is the Boltzmann equation (5) with the collision term proper omitted, but with interparticle electromagnetic forces taken into account in a smoothed-out way in the acceleration density term $\underline{F} \cdot (\partial f/\partial \underline{v})$. In the Lorentz acceleration $\underline{F} = (q/m)(\underline{E} + c^{-1}\underline{v} \times \underline{B})$, $\underline{E}$ and $\underline{B}$ are determined from the plasma charge density ρ and current $\underline{j}$ by Maxwell's equations. Various authors regard this technique as the neglect of collisions (1), "close collisions" (6), or certain "correlations" (4). Certainly binary collisions involving appreciable momentum transfer are rare at high temperatures (7), so that we should expect the description of interparticle forces in terms of a collective, macroscopic electromagnetic field to provide a good representation of the facts.

This thesis will consider only plasmas which fill all of space and are nearly uniform, having been slightly perturbed from a perfectly uniform state. Clearly, many of the results hold in bounded or non-uniform plasmas provided they possess large, nearly uniform regions. The undisturbed state will consist of an equal number n_o per unit volume of electrons and protons. The generalization to more species of particle is simple but will be omitted here.

Many kinds of organized motion have been predicted and observed in such a plasma. These can be divided broadly into three groups: the electromagnetic, hydromagnetic and electrostatic, or plasma oscillations. The first kind are very much like ordinary electromagnetic waves in a dielectric, with unusual resonance phenomena at the particle cyclotron frequencies (8). The second kind, hydromagnetic waves, may be divided into two subgroups: longitudinal ones, which are essentially sound waves with magnetic pressure added to the fluid pressure (8), and transverse, or Alfvén waves (8,9), which consist of disturbances propagating along magnetic lines of force much as waves travel along stretched strings. Finally, the electrostatic oscillations are longitudinal waves where the electrostatic forces act as restoring forces upon electrons and ions that have suffered displacements (6). As they depend on charge separation, they have no analogue in un-ionized media. A qualitative discussion of them will be given presently.

Instabilities in Uniform Plasmas

While various boundary instabilities in plasmas are of great importance in confinement schemes (10), this thesis is limited to the discussion of volume instabilities due to non-Maxwellian particle velocity distributions, and to the case of infinite, nearly uniform plasmas. It will be shown in Part II that all such instabilities can be described in terms of exponentially growing linearized infinite plane waves in the plasma. For example, when a uniform beam of electrons of well-defined velocity traverses a plasma, an instability with respect to longitudinal oscillations exists. Langmuir and Tonks (11) first used such oscillations to explain Langmuir's anomalously large measurement of electron scattering in plasmas (12). Careful attempts to observe the oscillations (13) and further theoretical studies (6,14) verified their presence under suitable boundary conditions. Such oscillations can also be used for amplification in traveling-wave tubes (15) and will be referred to repeatedly here in discussing additional implications of plasma instabilities. Therefore, a brief qualitative discussion of them is in order.

Consider a uniform, perfectly cold electron gas (assuming the protons only provide a smoothed-out neutralizing background) filling all of space. Suppose that a layer of electrons, say between $x = 0$ and $x = x_o$, is displaced a small amount Δx in the plus x direction. If the electrons have charge $-e$, mass m, and number density n_o/cm^3 this produces two charged layers with charge density $\pm n_o e \Delta x/cm^2$ at $x = 0$ and $x = x_o$. Then a restoring field $E_x = +4\pi e n_o \Delta x$ is established, which provides a force $(4\pi e n_o \Delta x)(n_o e x_o)/cm^2$ on the mass $n_o m x_o/cm^2$ between $x = 0$ and $x = x_o$. Thus simple harmonic oscillation is established at angular frequency $\omega_p = (4\pi n_o e^2/m)^{1/2}$, called the plasma frequency.

Now suppose the plasma consists of two interpenetrating cold streams of electrons with mean velocities $\pm U$. A small periodic perturbation density in one stream will produce electrostatic forces that set both streams to oscillating at the plasma frequency. Charge density variations in the second stream will be carried, oscillating, back along the first. These can in turn modulate the first stream. Thus a kind of feedback is established, and the system becomes unstable if the scale of the original disturbance is so adjusted that reinforcement occurs between the fed-back and original perturbations. Instabilities of this type are referred to as the double stream instability. Their use has been considered for producing electromagnetic waves, since inhomogeneities can couple electrostatic to electromagnetic waves, while it is thought that instability with respect to transverse oscillations does not occur when a beam traverses a plasma (16).

The double stream instability may also be quite important in determining the thickness of some kinds of hydromagnetic shocks. There has been considerable disagreement in the literature (17,18,19) as to the possible scale of shocks in plasmas, arguments being advanced for the collision mean free path, for the electron and proton Larmor radii, and for the distance in which a spatially growing plasma oscillation increases by some factor, such as e. The resolution of these divergent opinions appears to depend on a better understanding of processes that increase entropy, i.e., convert ordered to disordered motion in a plasma. At the present time most experiments on shocks in plasmas involve a shock with plasma on the trailing side but un-ionized gas ahead. The case of greatest interest in astrophysics, however, is a shock with ionized gas on both sides, and it is here that the disagreement on shock scale and structure is the greatest. The scale is particularly important in cosmic ray theory, as a shock of small thickness may serve to scatter cosmic ray particles into orbits where the Fermi mechanism can continue to increase their energy (20).

Davis, Lüst, and Schlüter consider nonlinear hydromagnetic waves as a possible component of a shock; in particular they find nonlinear solitary waves of unchanging form propagating in a cold plasma in a magnetic field (21). Since these waves contain regions where perfectly cold masses of protons and electrons move through each other, the double stream instability may exist. Such an instability could amplify small fluctuations initially present in the plasma so as to produce a wake of random plasma oscillations behind the wave. This would be a structure more like that of a classical shock. The waves of Davis et al are all symmetric about their maxima and hence cannot be regarded as shock waves. The research presented in this thesis was begun in an attempt to treat such effects, but this problem has still not been solved.

Another approach to shock structure is adopted by Parker (19) and by Kahn (22). Parker considers the shock as the region of interpenetration of two colliding bodies of plasma which interpenetrate for some time before instabilities become important. Then the distance over which a preferentially growing wave increases by some factor such as e or 2 in the region common to both plasmas gives some measure of the size to which this region can grow before plasma oscillations convert very much of the kinetic energy of motion into oscillations. When this finally happens, there is a sort of disordered region which can be called the shock front. The wavelength of oscillations that grow preferentially in time could be used as a measure of shock thickness just as well as the distance required for spatially growing ones to increase significantly. This idea will be discussed in detail in later sections.

Parker also shows (23) under certain assumptions, to be examined later, that the growing plasma oscillations produced when plasmas collide can transfer some of the ion kinetic energy into electron oscillatory energy, suggesting that the high energy electrons in the earth's outer magnetic field may have been produced by impinging plasma from the sun.

By providing a drag force, i.e., a means for converting the kinetic energy of relative motion of ions and electrons into random oscillatory energy, plasma oscillations are thought to contribute to the D.C. resistivity of a fully ionized gas. Buneman (24) has carried such a model beyond the linearized theory of instability, and has found that even an initially sinusoidal growing wave breaks later, forming a sort of disordered motion resembling thermal motion.

Methods proposed for obtaining thermonuclear power from fusion in plasmas (10) usually involve radically non-equilibrium states. For example, large currents often are made to flow in the plasma (stellarator; pinch) or a beam of particles is injected into a plasma (DCX machine) (10). While boundary instabilities are very important in such proposed configurations, volume instabilities due to the unusual distributions of particle velocities cannot be ignored.

New Results Concerning Instabilities

Most of the theoretical work just described was based on rather restrictive assumptions about the plasma. Ions were often assumed to be at rest (fixed) or to constitute simply a smeared out charge density maintaining, on the average, charge neutrality. The electrons were often taken to consist of several interpenetrating streams, each at zero temperature. Sometimes the entire plasma was assumed to be nearly in a state of thermal equilibrium. At high temperatures, when collisions are infrequent, the particles in a plasma might have a variety of velocity distributions differing markedly from the Maxwellian one, and in general interpenetrating streams of particles will be hot, not cold. We shall consider the equations for linearized waves of all types in an infinite spatially uniform hydrogen plasma having an almost arbitrary velocity distribution, and shall obtain a formal solution of the stability problem for such a plasma. This solution will be put in a form amenable to actual use in a variety of important cases. For example, the determination of whether a plasma can support growing longitudinal electrostatic oscillations when there is no D.C. magnetic field will be reduced to the evaluation of a few definite integrals. Furthermore, an analogy will be given to a simple electrostatics problem which will extend greatly the use of physical intuition in ascertaining what types of plasma are unstable, which groups of particles (i.e., particles of which species or velocities) participate the most in a growing oscillation, and other properties of the waves. While several examples of physical interest will be considered in detail, this paper is largely methodological. A detailed study of the stability of two hot interpenetrating neutral plasma streams, using a different method, has recently been given by Kellogg and Liemohn (25), who consider only longitudinal oscillations.

The method of stability analysis to be presented here will provide useful information on generalizations of Parker's (19,23) and Kahn's (22) shock model. Some strange instabilities will be found which can exist in plasmas arbitrarily close to thermal equilibrium (i.e., having initial velocity distributions differing in the mean or mean square sense arbitrarily little from the Maxwellian one.) We shall find that sometimes the removal of all particles within some range of velocities from a plasma renders it capable of supporting growing oscillations whose phase velocity lies in this range, a result which suggests modification of the idea (6) that the trapping of particles in the potential trough of a wave is necessary for the existence of growing electrostatic oscillations.

II. PLANE WAVES IN A UNIFORM PLASMA

Fundamental Equations

For a plasma which is sufficiently hot and tenuous, the collision term in the Boltzmann equation may be neglected to a good approximation, giving the Vlasov equations for the electron and ion coordinate velocity distributions, $f(\underline{r},\underline{v},t)$ and $F(\underline{r},\underline{v},t)$, $\underline{r}$ and $\underline{v}$ being position and velocity vectors respectively:

$$(\partial f/\partial t) + \underline{v} \cdot (\partial f/\partial \underline{r}) - (e/m)\ (\underline{E} + c^{-1}\ \underline{v}\times\underline{B}) \cdot (\partial f/\partial \underline{v}) = 0 \qquad (1a)$$

and

$$(\partial F/\partial t) + \underline{v} \cdot (\partial F/\partial \underline{r}) + (e/M)\ (\underline{E} + c^{-1}\ \underline{v}\times\underline{B}) \cdot (\partial F/\partial \underline{v}) = 0 \qquad (1b)$$

where e = charge of the proton, M = mass of the proton, and m = mass of the electron. $\underline{E}$ and $\underline{B}$ are due both to external sources and to the particles themselves. We shall assume the externally imposed magnetic field $\underline{B}_o$ is uniform and unchanging in time, and the externally imposed electric field is zero. A steady, uniform electric field with a component along $\underline{B}_o$ or with a component larger than B_o perpendicular to $\underline{B}_o$ would produce unbounded acceleration of the particles. This is quite a different problem as the plasma is not initially in an equilibrium state; a special case of the problem (which entails great mathematical difficulty) has been attacked, but no firm results obtained (26). A steady, uniform electric field orthogonal to $\underline{B}_o$ and smaller than B_o may be eliminated by a coordinate transformation.

Maxwell's equations determine the fields $\underline{E}$ and $\underline{B}_1$ produced by the plasma, viz.:

$$\operatorname{div} \underline{E} = 4\pi\rho \qquad c\nabla\times\underline{E} = -\,\partial\underline{B}_1/\partial t \qquad (2a,b)$$

$$\operatorname{div} \underline{B}_1 = 0 \qquad c\nabla\times\underline{B}_1 = 4\pi\,\underline{j} + \partial\underline{E}/\partial t \qquad (2c,d)$$

where

$$\rho = e\int (F - f)\,d\underline{v} \qquad \text{and} \qquad \underline{j} = e\int \underline{v}\,(F - f)\,d\underline{v} \qquad (3)$$

Equations 1 through 3 form a system of nonlinear partial differential and integral equations. While some non-trivial solutions of these are known which are not uniform in space (21,27), stability theory is most easily developed for perturbations about the uniform state $f(\underline{r},\underline{v},t) = n_o f_o(\underline{v})$, $F(\underline{r},\underline{v},t) = n_o F_o(\underline{v})$, n_o being the mean number of either kind of particle per unit volume. Assume that the plasma is very close to such a state, in the sense that

$$\begin{aligned} f(\underline{r},\underline{v},t) &= n_o f_o(\underline{v}) + f_1(\underline{r},\underline{v},t) \\ F(\underline{r},\underline{v},t) &= n_o F_o(\underline{v}) + F_1(\underline{r},\underline{v},t) \\ \underline{B}(\underline{r},t) &= \underline{B}_o + \underline{B}_1(\underline{r},t) \\ \underline{j}(\underline{r},t) &= \underline{j}_o + \underline{j}_1(\underline{r},t) \end{aligned} \qquad (4)$$

where $\int F_o\,d\underline{v} = \int f_o\,d\underline{v} = 1$

and powers greater than the first of the first order quantities f_1, F_1, $\underline{B}_1$, $\underline{E}$, ρ, and $\underline{j}_1$ can be neglected. This procedure is justified as a means for investigating stability since if the plasma is unstable there is by definition some kind of arbitrarily small perturbation of the plasma which will produce large effects at later times. It is conceivable that a plasma might be metastable, however, so that some small perturbations would produce large effects at later times, but the size of perturbation required would have a lower bound. Our analysis would not uncover such a situation.

Naturally, f_o, F_o and $\underline{B}_o$ must satisfy equations 1 through 3. If $B_o \neq 0$, this implies that the component of $\underline{v} \times (\partial f_o/\partial \underline{v})$ along $\underline{B}_o$ must vanish, and similarly for F_o. If $\underline{B}_o$ is to be uniform in space and $\underline{E}$ zero in the unperturbed state, equations 2d and 3 show that $\underline{j}_o$, the steady-state current, must be zero. There are several ways to relax this condition. One can consider the limit $c \rightarrow \infty$, as many authors have done (24,26,31). An alternative is to regard a portion of the infinite plasma as a model for a finite plasma, and to assume either that the system is sufficiently small in directions perpendicular to $\underline{j}_o$ that the variation in magnetic field over the plasma due to the steady state current is small, or that charge accumulates on some distant plane surfaces so as to produce a $\partial \underline{E}/\partial t$ satisfying equation 2d, although at the moment when we look at the plasma, there is no externally imposed electric field.

Consider an infinite plane wave in the plasma:

$$
\begin{aligned}
f_1(\underline{r},\underline{v},t) &= \hat{f}_1(v) \exp i(\underline{k}\cdot\underline{r} - \omega t) \\
F_1(\underline{r},\underline{v},t) &= \hat{F}_1(v) \exp i(\underline{k}\cdot\underline{r} - \omega t) \\
E(\underline{r},t) &= \hat{\underline{E}} \exp i(\underline{k}\cdot\underline{r} - \omega t) \qquad (5)
\end{aligned}
$$

$$\underline{B}_1(\underline{r},t) = \hat{\underline{B}}_1 \exp i(\underline{k}\cdot\underline{r} - \omega t)$$

$$\underline{j}_1(\underline{r},t) = \hat{\underline{j}}_1 \exp i(\underline{k}\cdot\underline{r} - \omega t)$$

$$\rho(\underline{r},t) = \hat{\rho} \exp i(\underline{k}\cdot\underline{r} - \omega t) \qquad (5)$$

Assuming equations 5 hold (it will later be shown that such waves give a complete stability analysis), the substitutions $(\partial/\partial\underline{r}) \rightarrow i\,\underline{k}$ and $(\partial/\partial t) \rightarrow -i\omega$ may be used to simplify the linearized Boltzmann and Maxwell equations obtained from equations 2 through 4 neglecting higher order terms. The resulting forms are

$$i(\underline{k}\cdot\underline{v} - \omega)\,\hat{f}_1 - (e/mc)(\underline{v}\times\underline{B}_o)\cdot(\partial\hat{f}_1/\partial\underline{v}) = (e/m)(\hat{\underline{E}} + c^{-1}\,\underline{v}\times\hat{\underline{B}}_1)\cdot n_o(\partial f_o/\partial\underline{v}) \qquad (6a)$$

$$i(\underline{k}\cdot\underline{v} - \omega)\,\hat{F}_1 + (e/Mc)(\underline{v}\times\underline{B}_o)\cdot(\partial\hat{F}_1/\partial\underline{v}) = -(e/M)(\hat{\underline{E}} + c^{-1}\,\underline{v}\times\hat{\underline{B}}_1)\cdot n_o(\partial F_o/\partial\underline{v}) \qquad (6b)$$

$$i\underline{k}\cdot\underline{E} = 4\pi\hat{\rho} \qquad c\underline{k}\times\underline{E} = \omega\underline{B}_1 \qquad (7a,b)$$

$$\underline{k}\cdot\hat{\underline{B}}_1 = 0 \quad \text{and} \quad ic\underline{k}\times\hat{\underline{B}}_1 = 4\pi\hat{\underline{j}} - i\omega\hat{\underline{E}} \qquad (7c,d)$$

where $\hat{\rho} = e\int(\hat{F}_1 - \hat{f}_1)d\underline{v}$ and $\hat{\underline{j}}_1 = e\int \underline{v}(\hat{F}_1 - \hat{f}_1)d\underline{v}$ (8)

Counting a vector equation as three equations and a vector as three unknowns, equations 6 through 8 are fourteen equations in the twelve unknowns $\hat{f}_1$, $\hat{F}_1$, $\hat{\rho}_1$, $\hat{j}_1$, $\hat{\underline{E}}$, and $\hat{\underline{B}}$. (ω and $\underline{k}$ are constants and

$\underline{v}$ is the independent variable.) But the divergence of equation 7b is equation 7c, and the equation of continuity, a consequence of equation 8, can be used to transform the divergence of equation 7d into equation 7a. Thus there are really only twelve independent equations present.

Since equations 6 through 8 constitute a homogeneous set, some relation $D(\underline{k},\omega) = 0$ (analogous to the secular equation for linear algebraic equations) will have to be satisfied in order for nontrivial solutions to exist. This relation is called the "dispersion relation" for waves in the plasma. The precise form of $D(\underline{k},\omega)$ will be determined presently, but first we should understand why waves of the form of equation 5 constitute a description of any possible instability.

Relationship between the Wave and Initial Value Problems

Certainly if waves of the form of equation 5 can exist in the plasma with $\mathrm{Im}(\omega) > 0$, the plasma is unstable. In order to verify the converse, it is necessary either to show that waves of the form of equation 5 comprise a "complete set" of solutions to equations 6 through 8, as Case (28) has done in the limit $c \to \infty$, or to investigate the initial value problem. Instead of extending Case's results to transverse waves, we shall observe briefly how the dispersion relation for plane waves enters into the solution of the linearized initial value problem for equations 1 through 4 .

Bernstein has solved this initial value problem by Laplace transforms (29). He has shown that $f(\underline{r},\underline{v},t)$ and $F(\underline{r},\underline{v},t)$ can be found from $f_1(\underline{r},\underline{v},0)$ and $F_1(\underline{r},\underline{v},0)$ once the electric field $\underline{E}(\underline{r},\underline{v},t)$ is known, and that

$$\underline{E}(\underline{r},\underline{v},t) = \int e^{i\underline{k}\cdot\underline{r}} \, d\underline{k} \int_{iA-\infty}^{iA+\infty} d\omega \, \frac{\underline{\Phi}(\underline{k},\omega)}{D(\underline{k},\omega)} \, e^{-i\omega t} \tag{9}$$

where A is real and exceeds the imaginary part of any pole of the integrand, D is the function set equal to zero in the dispersion relation for plane waves, and $\underline{\Phi}$ depends on the initial perturbation, being analytic for smooth initial perturbations in any case having no poles in the upper half plane. (Equation 9 is adapted from Bernstein's equation 4, corrected for a misprint of d^3v for d^3k and his equations 25 and 26, with $s = -i\omega$ and $|R| = D$.) The contour for the ω integral can be taken as the real axis if D has no zeros in the upper half plane, but "hangs up" on the zeros of D otherwise as shown in Fig. 1 (dashed curve). With this contour $\underline{E}$ takes the form

$$\underline{E}(\underline{r},\underline{v},t) = \int e^{i\underline{k}\cdot\underline{r}} \, d\underline{k} \left[\sum_{\ell=1}^{N(\underline{k})} \underline{E}_\ell(\underline{k}) \, e^{-i\omega_\ell(\underline{k})t} + \int_{-\infty}^{\infty} \underline{E}'(\underline{k},\omega) e^{-i\omega t} d\omega \right] \tag{10}$$

where $\omega_\ell(\underline{k})$ are the N zeros of $D(\underline{k},\omega)$ in the upper half ω plane (for fixed $\underline{k}$), and $\underline{E}_\ell(\underline{k})$ the residues of $\underline{\Phi}(\underline{k},\omega)/(D(\underline{k},\omega)$ there, the sum vanishing if $N = 0$. The second term of equation 10 is bounded for $t > 0$, by Riemann's lemma (30), but the first evidently becomes infinite as $t \to +\infty$ unless it vanishes, i.e., unless $N \equiv 0$ for all $\underline{k}$. Thus the plasma is stable if and only if there are no roots $\omega_\ell(\underline{k})$ of $D(\underline{k},\omega) = 0$ in the upper half plane for any $\underline{k}$.

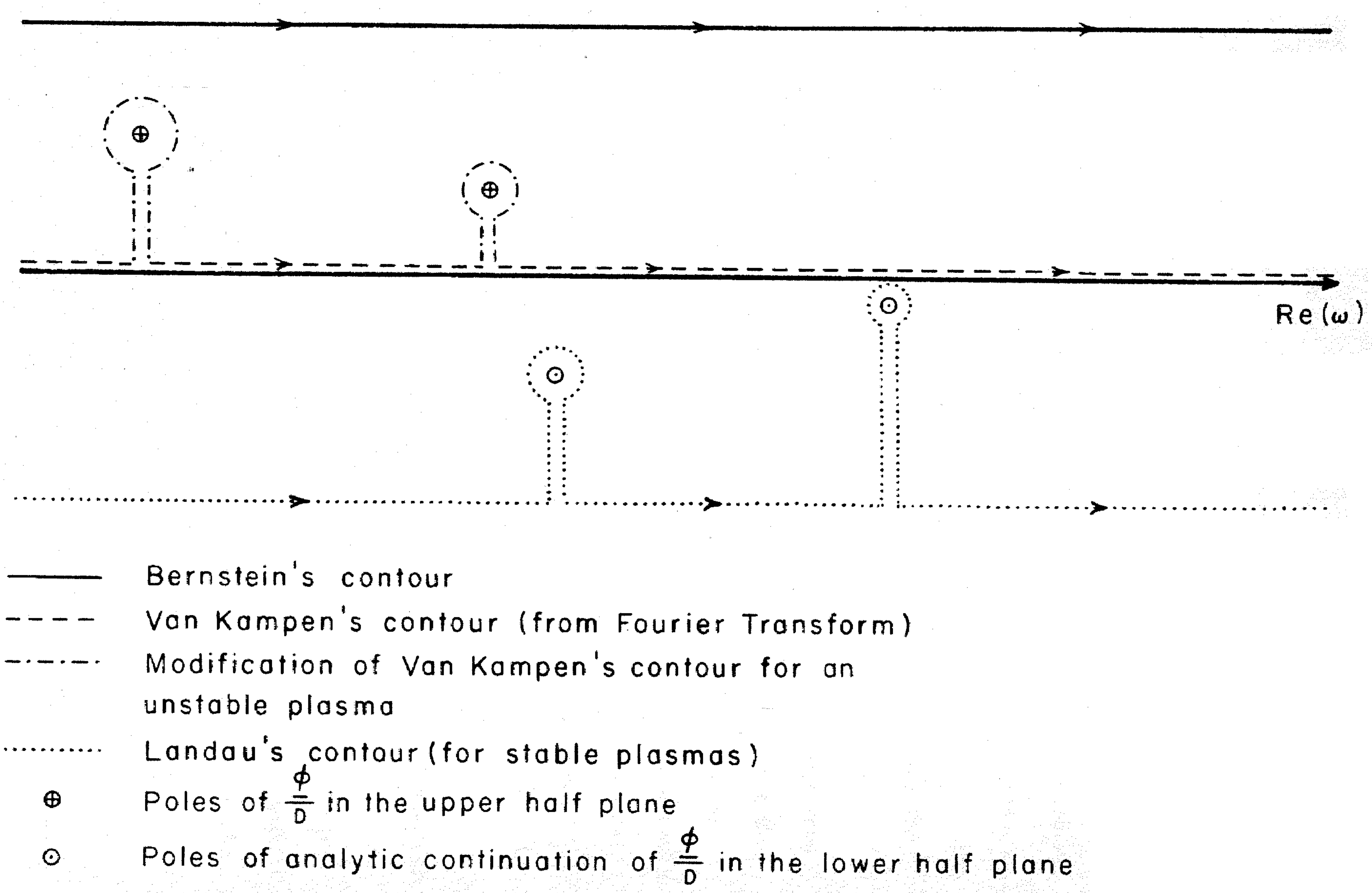


Fig. I Contour for the Inversion of the Laplace Transform in the ω Plane.

If the plasma is stable, the second term of equation 10 in the limit $c \to \infty$ gives a result in the same form as Van Kampen's (31); Landau's form (32) is obtained by deforming the contour as shown by the dotted curve in Fig. 1. In order to obtain Landau's representation, it is necessary to define D (and also $\underline{\Phi}$) below the real ω axis by analytic continuation from the upper half plane. This procedure yields a different function for D in the lower half plane than that obtained by a simple substitution analysis, using equations 5. The forms obtained for D in this thesis will be the proper ones for equation 9 above the real ω axis and thus will be adequate for stability analysis. On the basis of Landau's and Bernstein's form of the solution of the initial value problem, the roots of the dispersion relations obtained here in the lower half ω plane do not correspond to damped waves excited by a sensible initial perturbation in the plasma. With some effort it can also be shown that Case's solutions of the initial value problem (28) support this view.

The Form of the Dispersion Relation

Having established that the stability problem can be treated in terms of growing waves satisfying the dispersion relation (see also Sturrock (33) for an excellent discussion of the philosophy of this approach), we shall now find the exact form of $D(\underline{k},\omega)$. The carets in equations 5 through 8 will be dropped where no confusion results. From equations 7b and 7d we have

$$c^2\, \underline{k} \times (\underline{k} \times \underline{E}) = -4\pi i \omega \underline{j} - \omega^2 \underline{E}$$

or

$$E_{||} = (-4\pi i j_{||})/\omega \tag{11a}$$

and

$$\underline{E}_{\perp} = (4\pi i \omega \underline{j}_{\perp})/(k^2c^2 - \omega^2) \tag{11b}$$

where the subscripts "$||$" and "$\perp$" indicate components along and orthogonal to $\underline{k}$. B_1 can be eliminated from equation 6 by using equation 7b, and equation 11a can be simplified by using the equation of charge continuity to give

$$E_{||} = -4\pi i \rho / k \tag{11c}$$

which can be obtained also from equation 7a.

In solving equations 6 for the perturbed distributions, $\underline{E}$ will be taken as a given quantity. Then equations 8 and 11 will be used to obtain the dispersion relation, which will appear as the condition that the determinant of coefficients in three linear homogeneous algebraic equations for the components of $\underline{E}$ vanish.

Because the solution of equations 6 is tedious unless $B_o = B_1 = 0$ (longitudinal waves in a plasma not subjected to a D.C. magnetic field) and because stability theory is completely soluble in this simple case, this case will be done first. In discussing more general kinds of oscillations, the necessary conditions for the existence of purely longitudinal plasma oscillations will be examined in detail; here a formal justification can be given by taking the limit of equations 6 and 7 as $c \to \infty$.

For longitudinal electrostatic oscillations, equations 6, 8 and 11c read (omitting carets):

$$i(\underline{k}\cdot\underline{v} - \omega)f_1 + \left(\frac{n_o e}{m}\right)\left(\frac{4\pi i e}{k}\right)\left(\frac{\partial f_o}{\partial \underline{v}}\cdot\frac{\underline{k}}{k}\right)\int (F_1 - f_1)d\underline{v} = 0 \qquad (12a)$$

and

$$i(\underline{k}\cdot\underline{v} - \omega)F_1 - \left(\frac{n_o e}{M}\right)\left(\frac{4\pi i e}{k}\right)\left(\frac{\partial F_o}{\partial \underline{v}}\cdot\frac{\underline{k}}{k}\right)\int (F_1 - f_1)d\underline{v} = 0 \qquad (12b)$$

Let $I_i = \int F_1 d\underline{v}$, $I_e = \int f_1 d\underline{v}$, $\omega_e^2 = 4\pi n_o e^2/m$, and $\omega_i^2 = 4\pi n_o e^2/M$. Dividing equations 12 by $i(\underline{k}\cdot\underline{v} - \omega)$ [this division is allowable since $\mathrm{Im}(\omega)$ is assumed to be positive and $\underline{k}$ is real] and integrating over $\underline{v}$, one obtains

$$I_e + (I_i - I_e)\int \frac{\omega_e^2}{k}\cdot\frac{1}{\underline{k}\cdot\underline{v} - \omega}\,\frac{\partial f_o}{\partial \underline{v}}\cdot\frac{\underline{k}}{k}\,d\underline{v} = 0$$

and

$$I_i - (I_i - I_e)\int \frac{\omega_i^2}{k}\cdot\frac{1}{\underline{k}\cdot\underline{v} - \omega}\,\frac{\partial F_o}{\partial \underline{v}}\cdot\frac{\underline{k}}{k}\,d\underline{v} = 0$$

Defining

$$J_i = \frac{\omega_i^2}{k}\int \frac{1}{\underline{k}\cdot\underline{v} - \omega}\,\frac{\partial F_o}{\partial \underline{v}}\cdot\frac{\underline{k}}{k}\,d\underline{v}$$

and

$$J_e = \frac{\omega_e^2}{k}\int \frac{1}{\underline{k}\cdot\underline{v} - \omega}\,\frac{\partial f_o}{\partial \underline{v}}\cdot\frac{\underline{k}}{k}\,d\underline{v}$$

we find for I_i and I_e the linear algebraic equations

$$I_e + J_e(I_i - I_e) = 0$$

and

$$I_i - J_i(I_i - I_e) = 0 .$$

For these to have a nontrivial solution, the determinant $1 - J_e - J_i$ must vanish, i.e.,

$$\frac{\omega_e^2}{k^2}\int \underline{k}\cdot\left(\frac{\partial f_o}{\partial \underline{v}} + \frac{m}{M}\frac{\partial F_o}{\partial \underline{v}}\right)\frac{1}{\underline{k}\cdot\underline{v}-\omega}\,d\underline{v} \;=\; 1 \tag{13}$$

Using the previous notation of $||$ and $\perp$ to indicate components along and orthogonal to $\underline{k}$ allows integration over two components of $\underline{v}$ in equation 13:

$$\int \frac{\underline{k}\cdot[\partial f_o(\underline{v})/\partial \underline{v}]}{\underline{k}\cdot\underline{v}-\omega}\,d\underline{v} = \int_{-\infty}^{\infty}\frac{k}{kv_{||}-\omega}\left(\int\!\!\int\left[\frac{\partial f_o(\underline{v})}{\partial \underline{v}}\right]_{||} dv_{\perp}\right) dv_{||}$$

$$= k\int_{-\infty}^{\infty}\frac{df_o(v_{||})/dv_{||}}{kv_{||}-\omega}\,dv_{||}$$

where $f_o(v_{||}) \equiv \int f_o(\underline{v})d\underline{v}_{\perp}$. Similarly, $F_o(v_{||})$ will be defined by $F_o(v_{||}) \equiv \; F_o(\underline{v})dv_{\perp}$. In order to avoid excessive use of the subscript "$||$", the symbol v with no subscript will henceforth stand for $v_{||}$, and the symbols $|\underline{v}|$ and $\underline{v}^2$ will be used for the magnitude of the vector velocity and its square, respectively. With the foregoing abbreviations, the dispersion relation, equation 13, reads

$$\frac{\omega_e^2}{k^2}\int_{-\infty}^{\infty}\frac{[df_o(v)/dv] + (m/M)\,[dF_o(v)/dv]}{v-u}\,dv \;=\; 1 \tag{14}$$

where $u = \omega/k$ is the phase velocity of the wave. The total derivative sign will help as a reminder that $f_o(v)$ and $F_o(v)$ stand for functions of only one velocity component, $v_{||}$, obtained by integrating the three-dimensional distributions $f_o(\underline{v})$ and $F_o(\underline{v})$ over $\underline{v}_{\perp}$.

Equation 14 may be written most conveniently in the standard dispersion relation form $D(\underline{k},\omega) = 0$ by transposing all terms to the left hand side and multiplying through by k^2/ω_e^2. It is referred to as the Vlasov dispersion relation after its discoverer in the limit $M \to \infty$ (1). Since the integral in equation 14 is invariant under complex conjugation of ω, growing and damped roots occur in pairs. As emphasized before, the roots in the lower half ω plane do not represent waves which can be excited by most physically reasonable perturbations; they are definitely not the damped waves of Landau (32). In constructing Landau's analytic continuation of $D(\underline{k},\omega)$ into the lower half plane, the discontinuity of magnitude $\pi[(df_o/dv) + (m/M)(dF_o/dv)]$ in the imaginary part of the integral in equation 14 at the real axis must be eliminated by redefining D for points below the real axis. Roots of the dispersion relation in the lower half plane are irrelevant to stability, however, and will not be discussed further. The Vlasov dispersion relation is analyzed for unstable roots in Part III.

Klimontovich (34) has derived the relativistic Vlasov equations for the electron and ion coordinate-momentum distributions $f(\underline{r},\underline{p},t)$ and $F(\underline{r},\underline{p},t)$, where, for example, $f(\underline{r},\underline{p},t)d\underline{r}\, d\underline{p}$ is the number of particles in the volume of phase space $d\underline{r}\, d\underline{p}$ at $\underline{r},\underline{p}$. As usual, $\underline{p} = m\underline{v}\,\gamma \equiv m\underline{v}(1 - \underline{v}^2/c^2)^{-1/2}$. Klimontovich found

$$(\partial f/\partial t) + \underline{v} \cdot (\partial f/\partial \underline{r}) - e(\underline{E} + c^{-1}\, \underline{v} \times \underline{B}) \cdot (\partial f/\partial \underline{p}) = 0$$

and

$$(\partial F/\partial t) + \underline{v} \cdot (\partial F/\partial \underline{r}) + e(\underline{E} + c^{-1}\, \underline{v} \times \underline{B}) \cdot (\partial F/\partial \underline{p}) = 0$$

A derivation like the foregoing one for longitudinal electrostatic waves gives the dispersion relation

$$\int \underline{k} \cdot \frac{\omega_e^2\, m^4 (\partial f_o/\partial \underline{p}) + \omega_i^2\, M^4 (\partial F_o/2\underline{p})}{\underline{k} \cdot \underline{v} - \omega}\, \gamma^5\, d\underline{v} = 1 \qquad (15)$$

In equation 15 $\partial f_o/\partial \underline{p}$ and $\partial F_o/\partial \underline{p}$ must be expressed as functions of $\underline{v}$ using $\underline{p} = m\underline{v}\gamma$; since γ involves p it may not be brought through the differentiation sign. The domain of integration may be taken as all of $\underline{v}$ space, but the integrand is zero for $|\underline{v}| \geq c$. The normalization of, say, f_o is $\int f_o(\underline{p})d\underline{p} = \int m^3\gamma^5 f_o(m\underline{v}\,\gamma)d\underline{v} = 1.$ Thus f_o contains three inverse powers of m and $(\partial f_o/\partial \underline{p})$ contains four. The linear combination of $(\partial f_o/\partial \underline{p})$ and $(\partial F_o/\partial \underline{p})$ in the numerator of equation 15 is then really the same sort of combination as in equation 13. The integrations over $\underline{v}_\perp$ can be formally done in equation 15, just as was done in equation 13 to obtain equation 14. This operation would necessarily introduce more notation, e.g., the factor γ^5 would have to be written as a function of $v_{||}$ and $\underline{v}_\perp$. Since it is clear that these steps may be performed with no difficulty, and since no remarkable differences seem to exist in the relativistic calculations, the matter will be dropped here. Any analysis of the Vlasov dispersion relation may be carried over to the relativistic one.

While the Vlasov dispersion relation can be obtained from a one-dimensional analysis (that is, for example, by assuming $\underline{v}$ is along the x axis for all particles, and $\underline{E}$ and $\underline{k}$ are along the x axis), its relativistic analogue cannot; the factor γ^5 in equation 15 is the

Jacobian $(\partial \underline{p}/\partial \underline{v})$ in three dimensions and does not equal (dp_x/dv_x) when $v_y = v_z = 0$. The physical explanation is that particles with relativistic velocities in the y direction, say, are more resistant to changes of their x velocities than if they were at rest--their "transverse masses" increase.

In dealing with arbitrary kinds of linearized waves (i.e., $B_1 \neq 0$), the calculation of $D(\underline{k},\omega)$ is much simpler if $B_o = 0$ than otherwise. Furthermore, Bernstein has found (29) that as $B_o \to 0$, a certain class of waves can propagate only in directions more and more nearly perpendicular to $\underline{B}_o$. When $B_o = 0$ this class of waves cannot exist physically in the plasma, but they would be found in taking the limit of $D(\underline{k},\omega)$ for a plasma as $B_o \to 0$. This suggests again considering the case $B_o = 0$ separately. Very little extra labor is involved in carrying along $\underline{B}_o$ for a while; since some duplication of equations can be avoided in this manner this will be done. Assume without loss of generality $\underline{B}_o = \underline{e}_z B_o$, where $\underline{e}_x$, $\underline{e}_y$ and $\underline{e}_z$ are the unit triad. Let the abbreviations

$$\Omega = eB_o/m \quad , \quad (\partial f_1/\partial \underline{v}) = (p,q,r) \quad , \quad (\partial f_o/\partial \underline{v}) = (p_o,q_o,r_o) \; ,$$

$$(\partial F_1/\partial \underline{v}) = (P,Q,R) \quad , \quad (\partial F_o/\partial \underline{v}) = (P_o,Q_o,R_o)$$

be used whenever they shorten the equations. Once equation 6a has been solved for f_1, the substitutions $-e \to +e$, $(\partial f_o/\partial \underline{v}) \to (\partial F_o/\partial \underline{v})$, $m \to M$, and $\Omega \to -(m/M)\Omega$ will convert the result into a solution of 6b, i.e., F_1. Using equations 7b and 6a we find

$$i(\underline{k}\cdot\underline{v} - \omega)\, f_1 - \Omega(v_y p - v_x q) =$$

$$(en_o/m)[\underline{E} + \omega^{-1}\underline{v} \times (\underline{k} \times \underline{E})] \cdot (\underline{e}_x p_o + \underline{e}_y q_o + \underline{e}_z r_o) \qquad (16)$$

In the case $B_o = 0$ the term in Ω simply drops out of equation 16.

We can assume the coordinates chosen so that $\underline{k} = \underline{e}_z k$ when $B_o = 0$, as then $\underline{B}_o$ does not define a preferred direction. It should be remembered, however, that a rotation of coordinates has been made to bring $\underline{k}$ into coincidence with the z axis. (There is no convenient way to avoid choosing a specific coordinate system in this problem, on account of the distinction between the "parallel" and "perpendicular" components of $\underline{E}$ in equations 11.) The solution of equation 16 with $B_o = \Omega = 0$ and $\underline{k} = k\underline{e}_z$ is

$$f_1 = -i\, e\, n_o m^{-1}(kv_z - \omega)^{-1} \Big\{[1 - \omega^{-1}kv_z][E_x p_o + E_y q_o] + + [\omega^{-1}k(v_x E_x + v_y E_y) + E_z]\, r_o\Big\} \tag{17}$$

To find ρ and $\underline{j}_\perp$, f_1 as well as $v_x f_1$ and $v_y f_1$ are integrated first over v_x and v_y and finally over v_z. Certain terms such as $\iint p_o\, dv_x dv_y$ drop out, and others, such as $\iint q_o v_y\, dv_x dv_y$ can be integrated by parts to give terms involving $\iint f_o\, dv_x dv_y$. The dispersion relation has a simpler appearance if one imagines such integrations performed. In this spirit, define for any function $\psi(\underline{v})$, the function $\langle \psi \rangle = \iint \psi(\underline{v})\, dv_x dv_y$, which is then a function of v_z only. With this notation, the charge and "orthogonal current" $j_\perp$ due to the electrons alone are

$$\rho_{electron} = i\, e^2 n_o m^{-1} \int_{-\infty}^{\infty} (kv_z - \omega)^{-1}\, dv_z \times$$

$$\times\Big[\omega^{-1}k(\langle v_x r_o\rangle E_x + \langle v_y r_o\rangle E_y) + E_z \langle r_o\rangle\Big]$$

and

$$\begin{pmatrix} j_x \\ \\ j_y \end{pmatrix}_{\text{electron}} = ie^2 n_o m^{-1} \int_{-\infty}^{\infty} (kv_z - \omega)^{-1} dv_z \begin{pmatrix} -(1-\omega^{-1}kv_z) < f_o > E_x + \\ \\ -(1-\omega^{-1}kv_z) < f_o > E_y + \end{pmatrix.$$

$$\left. \begin{matrix} + \omega^{-1}k(< v_x^2 r_o > E_x + < v_x v_y r_o > E_y) + < v_x r_o > E_z \\ + \omega^{-1}k(< v_x v_y r_o > E_x + < v_y^2 r_o > E_y) + < v_y r_o > E_z \end{matrix} \right)$$

The proton terms have the same form, with $m \rightarrow M$, $f_o \rightarrow F_o$, and $r_o \rightarrow R_o$, leading again to a dispersion relation in which f_o and F_o and their derivatives and moments appear only in the linear combination $f_o + (m/M)F_o \equiv \tilde{f}_o$. Similarly, $\tilde{r}_o$, defined as $r_o + (m/M)R_o$, is $(\partial\tilde{f}_o/\partial z)$, etc. As the tildes serve only to remind us that the ion terms are absorbed into the electron terms, they will be dropped when no confusion results, and a tilde placed over the equation number.

Equations 11 now take the form

$$E_z - \omega_e^2 k^{-2} \int_{-\infty}^{\infty} (v_z - u)^{-1}[u^{-1}(< v_x r_o > E_x + < v_y r_o > E_y) + < r_o > E_z]dv_z = 0 \tag{$\widetilde{18a}$}$$

$$E_x + \omega_e^2(c^2 - u^2)^{-1}k^{-2} \int_{-\infty}^{\infty} (v_z - u)^{-1} [(v_z - u) < f_o > E_x + < v_x^2 r_o > E_x + + < v_x v_y r_o > E_y + u < v_x r_o > E_z] dv_z = 0 \tag{$\widetilde{18b}$}$$

and

$$E_y + \omega_e^2(c^2 - u^2)^{-1}k^{-2}\int_{-\infty}^{\infty}(v_z - u)^{-1}\,[(v_z - u)<f_o>E_y + v_x v_y r_o\; E_x +$$

$$+ <v_y^2 r_o> E_y + u<v_y r_o>E_z]dv_z = 0 \qquad (\widetilde{18c})$$

where $u = \omega/k$.

Equation 18a reduces to the Vlasov dispersion relation, equation 14, if $<v_x\tilde{r}_o>$ and $<v_y\tilde{r}_o>$ are zero. This is evidently the necessary condition for longitudinal electrostatic oscillations to be uncoupled from all others when $B_o = 0$, as then also the E_z terms disappear from equations 18b and 18c . Equations 18b and c can be put in a more convenient form by more manipulations, such as

$$\int_{-\infty}^{\infty}(v_z - u)^{-1}\,u<v_x r_o>dv_z =$$

$$\int_{-\infty}^{\infty}(v_z - u)^{-1}\,[(u - v_z) + v_z]<v_x r_o>dv_z =$$

$$\int_{-\infty}^{\infty}(v_z - u)^{-1}\,v_z<v_x r_o>dv_z$$

and

$$\int_{-\infty}^{\infty}(v_z - u)^{-1}\,(v_z - u)<\tilde{f}_o>dv_z = 1 + m/M$$

In this manner we can obtain for equations 18 the form

$$k^2\omega_e^{-2}E_z - \int_{-\infty}^{\infty}(v_z - u)^{-1}\,[u^{-1}(<v_x r_o>E_x + <v_y r_o>E_y) + <r_o>E_z]dv_z = 0$$

$$(\widetilde{19a})$$

$$k^2\omega_e^{-2}E_x + (c^2 - u^2)^{-1} \left\{[1 + (m/M)]E_x + \int_{-\infty}^{\infty} (v_z - u)^{-1} [<v_x^2 r_o> E_x + \right.$$

$$\left. + <v_x v_y r_o> E_y + v_z <v_x r_o> E_z] dv_z \right\} = 0 \qquad (\widetilde{19b})$$

and

$$k^2\omega_e^{-2}E_y + (c^2 - u^2)^{-1} \left\{[1 + (m/M)]E_y + \int_{-\infty}^{\infty} (v_z - u)^{-1} [<v_x v_y r_o> E_x + \right.$$

$$\left. + <v_y^2 r_o> E_y + v_z <v_y r_o> E_z] dv_z \right\} = 0 \qquad (\widetilde{19c})$$

The determinant of coefficients of E_x, E_y and E_z in equations 19 assumes the convenient form $D(\underline{k},\omega) = \Lambda(k^2, u)$, where Λ is analytic in the upper half u plane. Under special circumstances Λ may factor into two or into three factors each containing at worst sums of products of two integrals or just sums of integrals, respectively. For instance, if $<v_x \tilde{r}_o>$ and $<v_y \tilde{r}_o>$ are zero, the Vlasov dispersion relation splits off and a two by two determinant is left for equations 19b and 19c. This two by two determinant factors again if $<v_y^2 \tilde{r}_o> = <v_x^2 \tilde{r}_o>$, for then it has the form

$$\begin{vmatrix} A & B \\ B & A \end{vmatrix} = A^2 - B^2 = (A + B)(A - B)$$

The factors $A \pm B$ contain only sums of integrals like those in the Vlasov dispersion relation multiplied by coefficients involving u and real numbers only. This makes the stability problem completely soluble.

Since $\langle v_x \tilde{r}_o \rangle = \frac{d}{dv_z} \langle v_x \tilde{f}_o \rangle$, and $\langle v_x^2 \tilde{r}_o \rangle = \frac{d}{dv_z} \langle v_x^2 \tilde{f}_o \rangle$, etc., relations such as $\langle v_x \tilde{r}_o \rangle = 0$ or $\langle v_y^2 \tilde{r}_o \rangle = \langle v_x^2 \tilde{r}_o \rangle$ imply certain conditions on the total electron and ion momenta and energy. Since these relations, obtained by integrating--e.g., $\langle v_x \tilde{r}_o \rangle$ over v_z --are not sufficient to guarantee $\langle v_x \tilde{r}_o \rangle \equiv 0$ as a function of v_z, and are not very simply related to the momenta, they will not be discussed in detail.

To continue with the case $\underline{B}_o \neq 0$, let us substitute $\underline{v} = \underline{v}_r + \underline{e}_z v_z$ where $\underline{v}_r = v_x \underline{e}_x + v_y \underline{e}_y$, into equation 16, obtaining

$$v_x p - v_y q - i\Omega^{-1} (\underline{k} \cdot \underline{v} - \omega) f_1 =$$

$$-n_o \omega^{-1} B_o^{-1} [\omega \underline{E}_1 \cdot (\partial f_o / \partial \underline{v}) + \underline{v}_r \times (\underline{k} \times \underline{E}) \cdot (\underline{e}_x p_o + \underline{e}_y q_o) +$$

$$+ v_z \underline{e}_z \times (\underline{k} \times \underline{E}) \cdot (\underline{e}_x p_o + \underline{e}_y q_o) + \underline{v}_r \times (\underline{k} \times \underline{E}) \cdot \underline{e}_z r_o] \quad (20)$$

where a triple scalar product containing the two parallel vectors $v_z \underline{e}_z$ and $r_o \underline{e}_z$ has been omitted, as it must be zero. As has been noted previously, f_o must be a solution of the Boltzmann equation, implying $v_y p_o - v_x q_o = 0$. This means $\underline{v}_r$ and $\underline{e}_x p_o + \underline{e}_y q_o$ are parallel, and hence also that f_o is a function only of v_r and v_z. The second term in the brackets in equation 20 is thus zero.

Since $f_o(\underline{v}) = f_o(v_r, v_z)$, p_o and q_o may both be represented in terms of a single function $s_o = (\partial f_o / \partial v_r)$, viz.: $p_o = v_x v_r^{-1} s_o \equiv s_o \cos \Theta$ and $q_o = v_y v_r^{-1} s_o \equiv s_o \sin \Theta$, where

$\Theta = \tan^{-1}(v_y/v_x)$. Then $\underline{v}_r = v_r(\underline{e}_x \cos\Theta + \underline{e}_y \sin\Theta)$.

In terms of v_r, s_o and Θ, equation 20 reads

$$v_y p - v_x q - i\Omega^{-1}(\underline{k}\cdot\underline{v} - \omega)\, f_1 = -n_o \omega^{-1} B_o^{-1}\, [\omega\, \underline{E}\cdot(\partial f_o/\partial \underline{v}) +$$

$$+ \underline{e}_z \times (\underline{k}\times\underline{E})\cdot(\underline{e}_x \cos\Theta + \underline{e}_y \sin\Theta)(v_z s_o - r_o v_r) \equiv \Gamma \qquad (21)$$

where Γ is an abbreviation for the right hand side. This is a first order partial differential equation in $f_1(v_x, v_y, v_z)$. It may be solved by the method of characteristics (35), viz.: If $u_i(v_x, v_y, v_z) = c_i$, $i = 1,2,3$, are three independent solutions of the auxiliary equations

$$\frac{dv_x}{v_y} = -\frac{dv_y}{v_x} = \frac{dv_z}{0} = \frac{df_1}{i\Omega^{-1}(\underline{k}\cdot\underline{v} - \omega)f_1 + \Gamma} \qquad (22)$$

(the solution u_i must involve an arbitrary constant c_i), then the most general solution of equation 20 is $\phi(u_1, u_2, u_3) = 0$ where ϕ is an arbitrary function. By inspection, two independent integrals of equations 22 are $v_z = c_1$ and $v_r = c_2$ (from the equations $dv_z = 0$ and $v_x dv_x + v_y dv_y = 0$, respectively.) In terms of the representation

$$v_x = v_r \cos\Theta \quad , \qquad v_y = v_r \sin\Theta \qquad (23)$$

The remaining independent integral of equations 22 can be found by observing that $(dv_x/v_y) = (-dv_y/v_x) = -d\Theta$. Then equation 22 becomes

$$(df_1/d\Theta) + i\Omega^{-1}(\underline{k}\cdot\underline{v} - \omega)\, f_1 + \Gamma = 0 \qquad (24)$$

whose solution is

$$f_1 = -\left\{\exp\left[-i\Omega^{-1}\int_0^{\Theta}(\underline{k}\cdot\underline{v}-\omega)d\Theta\right]\right\}\left\{\int_0^{\Theta}\Gamma(\Theta)\exp\left[i\Omega^{-1}\int_0^{\Theta}(\underline{k}\cdot\underline{v}-\omega)d\Theta\right]d\Theta - C\right\}$$

$$\equiv f_{11}[v_r v_z,\Theta;\underline{k},\underline{E},\omega,(\partial f_o/\partial\underline{v})] + C\exp\left[-i\Omega^{-1}\int_0^{\Theta}(\underline{k}\cdot\underline{v}-\omega)\,d\Theta\right] \qquad (25)$$

which defines f_{11} . For f_1 to be single-valued in velocity space (that is, invariant under $\Theta \rightarrow \Theta \pm 2n\pi$), C must be zero. The formal machinery used in solving equation 21 cannot proceed properly, however, if C is set equal to zero at once; it must be retained as an "arbitrary" constant for a while.

The general solution of equation 21 is then

$$\phi\left([f_1 - f_{11}]\exp\left[i\Omega^{-1}\int_0^{\Theta}(\underline{k}\cdot\underline{v}-\omega)d\Theta\right],\ v_r,\ v_z\right) = 0 \quad ,$$

where ϕ is arbitrary. Solving for $f_1 - f_{11}$, this can be expressed in the form

$$f_1 = f_{11} + \Phi(v_r,v_z)\exp\left[-i\Omega^{-1}\int_0^{\Theta}(\underline{k}\cdot\underline{v}-\omega)d\Theta\right] \qquad (26)$$

where Φ is an arbitrary function of v_r and v_z . Now f_{11} must be invariant under $\Theta \rightarrow \Theta \pm 2n\pi$, but $\exp[-i\Omega^{-1}\int_0^{\Theta}(\underline{k}\cdot\underline{v}-\omega)d\Theta]$ is not. Therefore, the only physically meaningful solution of equation 21 is obtained by setting $\Phi \equiv 0$ in equation 25, i.e., $f_1 = f_{11}$.

The integrations in equation 25 can be performed, but unless $\underline{k}$ is along $\underline{B}_o$, only at the expense of introducing an infinite series of Bessel functions. The series of Bessel functions is likely to be of more use than the integral in specific examples, and permits the stability problem to be formally solved; hence the series method will be used. In the case $\underline{k} = \underline{e}_z k_z$, however, the integrals are elementary. The dispersion relation is so much simpler in this case that it will be derived first as a separate example.

From equation 21 , .

$$\left.\begin{aligned} &\Gamma = -(n_o \omega^{-1} B_o^{-1})(\Gamma_1 + \Gamma_2 \cos \Theta + \Gamma_3 \sin \Theta) \\ \text{where} \quad &\Gamma_1 = \omega r_o E_z \quad , \quad \Gamma_2 = \omega s_o E_x + (k_x E_z - k_z E_x)(v_z s_o - r_o v_r) \;, \\ \text{and} \quad &\Gamma_3 = \omega s_o E_y + (k_y E_z - k_z E_y)(v_z s_o - r_o v_r) \end{aligned}\right\} \quad (27)$$

If $k_x = k_y = 0$, this becomes

$$\Gamma_1 = \omega r_o E_z \quad , \quad \Gamma_2 = (\omega - k_z v_z) s_o E_x + r_o v_r k_z E_x \quad ,$$

$$\text{and} \qquad \Gamma_3 = (\omega - k_z v_z) s_o E_y + r_o v_r k_z E_y$$

When $\underline{k}$ is along $\underline{e}_z$, the integrating factor in equation 25 is $\exp[i\Omega^{-1}(k_z v_z - \omega)\Theta]$ and f_{11} becomes

$$\begin{aligned} f_{11} = -n_o \omega^{-1} B_o^{-1} \Big\{ &i\omega\Omega \, r_o (k_z v_z - \omega)^{-1} E_z \; + \\ &+ [(k_z v_z - \omega) s_o - k_z r_o v_r][i\Omega^{-1}(k_z v_z - \omega)\cos\Theta + \sin\Theta][1 - \Omega^{-2}(k_z v_z - \omega)^2] E_x \; + \\ &+ [(k_z v_z - \omega) s_o - k r_o v_r][i\Omega^{-1}(k_z v_z - \omega)\sin\Theta - \cos\Theta][1 - \Omega^{-2}(k_z v_z - \omega)^2] E_y \Big\} \end{aligned}$$

The subscript is retained on k_z because k_z may equal either $+k$ or $-k$. Again, F_{11} is obtained by replacing Ω by $-(m/M)\Omega$ and $(\partial f_o/\partial \underline{v})$ by $(\partial F_o/\partial \underline{v})$ in f_{11} . When equations 11b and 11c are used, the longitudinal and transverse oscillations are found to be uncoupled; the longitudinal ones obey the Vlasov dispersion relation. For the transverse oscillations equation 11b takes the form (with $\underline{dv} = 2\pi v_r dv_r dv_z$)

$$\underline{E}_\perp = (4\pi^2 n_o e B_o^{-1})(\omega^2 - k^2 c^2)^{-1} \int_0^\infty v_r^2 dv_r \int_{-\infty}^{\infty} dv_z \times$$

$$\times \left\{ \left[\frac{[(k_z v_z - \omega)s_o - k_z r_o v_r]\Omega^{-1}(k_z v_z - \omega)}{1 - \Omega^{-2}(k_z v_z - \omega)^2} + \frac{(M/m)[(k_z v_z - \omega)S_o - k_z R_o v_r]\Omega^{-1}(k_z v_z - \omega)}{1 - (M/m)^2\,\Omega^{-2}(k_z v_z - \omega)^2} \right] \underline{E}_\perp + \right.$$

$$\left. +i \left[\frac{(k_z v_z - \omega)s_o - k r_o' v_r}{1 - \Omega^{-2}(k_z v_z - \omega)^2} - \frac{(k_z v_z - \omega)S_o - k_z R_o v_r}{1 - (M/m)^2\,\Omega^{-2}(k_z v_z - \omega)^2} \right] \underline{e}_z \times \underline{E}_\perp \right\}$$

This can be written as

$$\underline{E}_\perp = A\underline{E}_\perp + B\,\underline{e}_z \times \underline{E}_\perp$$

where A and B are scalars, or as

$$(1 - A)E_x - BE_y = 0$$

and

$$B\,E_x + (1 - A)E_y = 0$$

The determinant of coefficients is

$$(1 - A)^2 + B^2 = (1 - A - iB)(1 - A + iB)$$

which is zero when either of the two factors vanishes. The dispersion relation $D(\underline{k},\omega) = 0$ then factors into $D_+(\underline{k},\omega)D_-(\underline{k},\omega) = 0$, where

$$D_\pm = 1 - \frac{\omega_e^2\,\pi}{k^2c^2 - \omega^2} \int_0^\infty v_r^2 dv_r \int_{-\infty}^{\infty} dv_z \left\{ \frac{(k_z v_z - \omega)s_o - k_z r_o v_r}{k_z v_z - \omega \pm \Omega} + \right.$$

$$+ \frac{m}{M} \frac{(k_z v_z - \omega) S_o - k_z R_o v_r}{k_z v_z - \omega \mp (m/M)\,\Omega} \Bigg\} \tag{28}$$

When allowances are made for different units and sign conventions, this reduces to Weibel's dispersion relation (36) in the limit $(M/m) \rightarrow \infty$.

Dropping the assumption that $\underline{k}$ is along $\underline{B}_o$, let us return to equations 25 and 27 to determine the form of f_{11} . The integrating factor $\exp\left[i\Omega^{-1} \int_0^{\Theta} (\underline{k} \cdot \underline{v} - \omega) d\Theta \right]$ is, up to a constant phase factor,

$$\exp \left\{ i\, \Omega^{-1} [k_x v_r \sin\Theta - k_y v_r \cos\Theta + (k_z v_z - \omega] \right\}$$

$$= \exp \left\{ i\, \Omega^{-1} [k_r v_r \sin\phi + (k_z v_z - \omega)\Theta] \right\}$$

where

$$k_r^2 = k_x^2 + k_y^2 , \quad \phi = \Theta - \psi \quad , \quad \text{and} \quad \psi = \tan^{-1}(k_y/k_x) \tag{29}$$

whence

$$f_{11} = (n_o \omega^{-1} B_o^{-1}) \exp \left\{ -i\, \Omega^{-1} [k_r v_r \sin\phi + (k_z v_z - \omega)] \right\} \times$$

$$\times \int \exp \left\{ i\,\Omega^{-1} [k_x v_r \sin\Theta - k_y v_r \cos\Theta + (k_z v_z - \omega)\Theta] \right\} \times$$

$$\left[\Gamma_1 + \Gamma_2 \cos\Theta + \Gamma_3 \sin\Theta \right] d\Theta \tag{30}$$

or

$$f_{11} \equiv \Gamma_1 f_{11}^{(1)} + \Gamma_2 f_{11}^{(2)} + \Gamma_3 f_{11}^{(3)} \quad .$$

Clearly the $f_{11}^{(2)}$ and $f_{11}^{(3)}$ terms can be found from the first by differentiating the integral in equation 30 with respect to k_x and k_y (before substituting the values of the Γ's). From equations 29 and the relation (37)

$$\exp(i\Omega^{-1}k_r v_r \sin \phi) = \sum_{n=-\infty}^{\infty} e^{in\phi} J_n(\Omega^{-1}k_r v_r) \quad ,$$

$f_{11}^{(1)}$ can be obtained in the form

$$f_{11}^{(1)} = -i(n_o\omega^{-1}B_o^{-1}) \exp\left\{-i\Omega^{-1}[v_r k_r \sin\phi + (k_z v_z - \omega)\phi]\right\} \times$$

$$\times \sum_{n=-\infty}^{\infty} [\Omega^{-1}(k_z v_z - \omega) + n)]^{-1} \exp\left\{i[\Omega^{-1}(k_z v_z - \omega) + n]\phi\right\} J_n(\Omega^{-1}k_r v_r) \tag{31}$$

Expressing ϕ in terms of Θ, substituting the appropriate expressions for $f_{11}^{(2)}$ and $f_{11}^{(3)}$, and inserting the Γ's from equations 27 we find after considerable manipulation:

$$f_{11} = -i(n_o\omega^{-1} B_o^{-1}) \exp[-i\Omega^{-1}(k_x v_r \sin\Theta - k_y v_r \cos\Theta)] \times$$

$$\sum_{n=-\infty}^{\infty} [\Omega^{-1}(k_z v_z - \omega) + n]^{-1} \exp\left\{in[\Theta - \tan^{-1}(k_y/k_x)]\right\} \times \Lambda_n$$

where

$$\Lambda_n = [\omega r_o + (v_z s_o - r_o v_r) n\Omega v_r^{-1}] J_n(\Omega^{-1}k_r v_r) E_z \quad +$$

$$+ [\omega s_o - k_z(v_z s_o - r_o v_r)][n\Omega v_r^{-1} k_r^{-2}(k_x E_x + k_y E_y) J_n(\Omega^{-1}k_r v_r) +$$

$$+ ik_r^{-1}(k_y E_x - k_x E_y) J_n'(\Omega^{-1}k_r v_r)]$$

The calculation of f_{11} was simplest in a coordinate system whose z

axis lay along $\underline{B}_o$; however, the application of equations 11 is most easily performed in a system with one axis along $\underline{k}$. Transforming to such a system would be most laborious, as not only $\underline{E}$ and $\underline{j}$, but $\underline{v}$ and hence $v_r, v_z, r_o, s_o \cdots$, etc. would haveto be expressed in the new coordinates. The alternative is to express $\underline{j}_\perp$ (ρ presents no difficulty at all) in the present coordinates, e.g. as $\underline{j}_\perp = \underline{j} - k^{-2}\underline{k}(\underline{k}\cdot\underline{j})$ where $\underline{j} = e\int(F_1 - f_1)\underline{v}\, d\underline{v}$. Then equations 11 will be four equations in E_x, E_y and E_z (one for ρ , one for each component of $\underline{j}$) , which must be redundant, as the equation of charge continuity holds. Except for special directions of polarization of the waves, it should make little difference which equation is dropped. Omitting the ρ equation leaves the remaining ones more symmetrical, but omitting one of the components of $j_\perp$ shortens the equations more. The equation for the z component of $\underline{j}_\perp$ should not be dropped, for if $\underline{k}$ is along $\underline{e}_x$ or $\underline{e}_y$, one of the other equations vanishes identically. If, however, the y component of $\underline{j}_\perp$ is dropped, there is no harm if $\underline{k}$ is along $\underline{e}_z$ since then the dispersion relation given by equation 28 may be used.

Since the details of a given problem may dictate which equation may most profitably be omitted (e.g., one may be looking for nearly longitudinal or transverse waves, in the limit of small Ω), all four will be given. Equation 11c reads:

$$i(4\pi\omega)^{-1}(k_xE_x + k_yE_y + k_zE_z) = 2\pi i\, n_o\omega^{-1}B_o^{-1}\int_{-\infty}^{\infty}dv_z\int_0^{\infty} v_r dv_r \sum_{n=-\infty}^{\infty} eJ_n(\Omega^{-1}k_rv_r) \times$$

$$\times\left[\Omega^{-1}(k_zv_z - \omega) + n\right]^{-1}\Lambda_n + \text{ion terms} \qquad (32a)$$

The ion terms are as usual obtained by letting $-e \rightarrow +e$, $\Omega \rightarrow -(m/M)\Omega$, $r_o \rightarrow R_o$, $s_o \rightarrow S_o$, etc. The x, y, and z components of equation 11b are, respectively:

$$i(4\pi\omega)^{-1}(\omega^2 - k^2c^2)\,[E_x - k_x k^{-2}(k_x E_x + k_y E_y + k_z E_z)] =$$

$$2\pi i n_o \omega^{-1} B_o^{-1} \int_{-\infty}^{\infty} dv_z \int_0^{\infty} v_r dv_r \sum_{n=-\infty}^{\infty} e[n\Omega\, k_x(k_r^{-2} - k^{-2})J_n(\Omega^{-1}k_r v_r) -$$

$$-i v_r k_r^{-1} k_y J_n'(\Omega^{-1}k_r v_r) - k_x k^{-2} v_z k_z J_n(\Omega^{-1}k_r v_r)][\Omega^{-1}(k_z v_z - \omega) + n]^{-1}\Lambda_n +$$

$$+ \text{ ion terms} \qquad (32b)$$

$$i(4\pi\omega)^{-1}(\omega^2 - k^2c^2)[E_y - k_y k^{-2}(k_x E_x + k_y E_y + k_z E_z)] = 2\pi i\, n_o \omega^{-1} B_o^{-1}$$

$$\int_{-\infty}^{\infty} dv_z \int_0^{\infty} v_r dv_r \sum_{n=-\infty}^{\infty} e[n\Omega k_y(k_r^{-2} - k^{-2})J_n(\Omega^{-1}k_r v_r) + i\, v_r k_x k_r^{-1} J_n'(\Omega^{-1}k_r v_r) -$$

$$-k_y k_z v_z k^{-2} J_n(\Omega^{-1}k_r v_r)][\Omega^{-1}(k_z v_z - \omega) + n]^{-1}\Lambda_n + \text{ ion terms} \qquad (32c)$$

and

$$i(4\pi\omega)^{-1}(\omega^2 - k^2c^2)[E_z - k_z k^{-2}(k_x E_x + k_y E_y + k_z E_z)] = 2\pi i\, n_o \omega^{-1} B_o^{-1}$$

$$\int_{-\infty}^{\infty} dv_z \int_0^{\infty} v_r dv_r \sum_{n=-\infty}^{\infty} e[v_z J_n(\Omega^{-1}k_r v_r) - k_z k^{-2}(n\Omega + v_z k_z)J_n(\Omega^{-1}k_r v_r)]$$

$$[\Omega^{-1}(k_z v_z - \omega) + n]^{-1}\Lambda_n + \text{ ion terms} \qquad (32d)$$

The dispersion relation $D(\underline{k},\omega) = 0$ may be obtained as described above by equating to zero the determinant of three of these four linear algebraic equations in E_x, E_y, and E_z. The components of $\underline{E}$ of course appear in Λ_n.

III. THE EXISTENCE AND PROPERTIES OF GROWING WAVES

Some of the dispersion relations presented in the last section will now be analyzed for growing waves in considerable detail. A formal solution of the stability problem in each case is simply to determine the zeros of $D(\underline{k},\omega)$, the function set equal to zero in the dispersion relation, by evaluating it at each point in the upper half of the ω plane for each value of the vector $\underline{k}$. The search in the ω plane may be simplified by the use of Nyquist's criterion (38): If $D(\underline{k},\omega)$ is analytic in the upper half ω plane and tends to zero as $\omega \to \infty$ there (which is the case for all dispersion relations considered here), then the equation $D(\underline{k},\omega) = 0$ has roots in the upper half ω plane if and only if the path $\Pi(\text{Re}\,\omega)$ traced out by D as ω traverses the real axis from $-\infty$ to $+\infty$ encloses the origin in the D plane. The Nyquist diagram (the path Π in the D plane) would still have to be constructed for each vector value of $\underline{k}$. There are obvious simplifications if only some components of $\underline{k}$, or k^2, etc. appear in $D(\underline{k},\omega)$. It will be shown here that one further major reduction in the labor is usually possible: Instead of having to construct the full Nyquist diagram, one may have only to find the points at which the path Π crosses the real axis, and in fact only some of these. In addition, these points are usually easy to locate, as $\text{Im}[D(\underline{k},\omega)]$ is often very easy to evaluate in the limit of real ω.

For the Vlasov dispersion relation and its relativistic analogue (equations 14 and 15), the problem of plasma stability can then be reduced to the calculation of a few definite integrals for each direction of $\underline{k}$. For the other dispersion relations given in part II, substantial reductions in the labor of stability analysis may be achieved. Because the method to be described here was first developed in ignorance of Nyquist's criterion, and has many worthwhile features of its own, it will be presented separately and later correlated with the better known Nyquist method.

Stability Theorem for Longitudinal Oscillations

Under the assumptions stated in parts I and II, a necessary and sufficient condition that growing linearized plane longitudinal oscillations may exist in an infinite, uniform plasma with angular frequency ω and wave number vector $\underline{k}$ is that ω and $\underline{k}$ fulfill the Vlasov dispersion relation, equation 14. The assumption that the waves grow in time but not in space means that $\mathrm{Im}(\omega) > 0$ and $\mathrm{Im}\,\underline{k} = 0$. Since this dispersion relation contains f_o and F_o only in the linear combination $f_o + (m/M)F_o \equiv \tilde{f}_o$, the discussion will be carried on in terms of $\tilde{f}$ with the tilde usually dropped when no confusion results and tildes placed over equation numbers. It should be remembered that $f_o(v)$ is defined as the integral of $f_o(\underline{v})$ over directions orthogonal to $\underline{k}$, with $v = k^{-1}(\underline{v} \cdot \underline{k})$; thus the form of $f_o(v)$ depends on the direction of $\underline{k}$. There is usually a good physical reason for believing one of a few directions is most likely to give instability. In order to carry through this simple case with considerable care, some properties of f_o must be noted and some restrictive assumptions about f_o must be made. The following ones are physically justifiable and sufficient for the analysis:

(a) Let $f_o'(v) \equiv (df_o/dv)$ exist and be differentiable at all but a finite number of points, where it has jump discontinuities.

(b) Let $f_o(v)$ and $f_o'(v)$ be small at least of order v^{-4} as $v \to \infty$.

(c) Note that $f_o \geq 0$.

(d) Note that $\int_{-\infty}^{\infty} f_o(v)dv = \int f_o(\underline{v})d\underline{v} = 1$

(e) Let f_o have a finite number of extrema and points of inflection.

Assumption (d) is actually a slightly incorrect normalization condition when applied to f_o , as $\int_{-\infty}^{\infty} f_o(v)dv = 1 + (m/M)$. This small discrepancy may be eliminated if ω_e^2 is replaced by $\omega_e^2[1+(m/M)]$ and f_o by $f_o[1+(m/M)]^{-1}$ in the Vlasov dispersion relation, but the physical effect of such replacement is only a slight change in the wave frequencies. Assumption (c) holds for any distribution function, as probabilities are positive, and (b) is just the requirement that the energy density be finite. In the present notation, equation 14 becomes

$$\int_{-\infty}^{\infty} \frac{f_o'(v)dv}{v-u} = \frac{k^2}{\omega_e^2} \qquad (\widetilde{33})$$

where again $u = \omega/k$ is the phase velocity of the wave, and it is assumed that $\text{Im}(\omega)$ [and hence $\text{Im}(u)$] is positive. Let $u = u_1 + iu_2$ and $\omega = \omega_1 + i\omega_1$. It will soon appear that the discontinuity of the imaginary part of the integral in equation 33 can be interpreted as the usual multi-valuedness of the imaginary part (stream function) of a complex potential in the neighborhood of charges; for definiteness, the imaginary part of the integral on the real u axis will be defined as its limit, $\pi f_o'(u_1)$, when the real axis is approached from above, whenever that limit exists (i.e., when $f_o'(v)$ is continuous at $v = u_1$).

Under the foregoing assumptions (a) and (b), equation 33 may be integrated by parts to give

$$-\int_{-\infty}^{\infty} \ln(v-u)\, f_o''(v)dv \equiv W(u) = k^2/\omega_e^2 \qquad (\widetilde{34})$$

which defines $W(u) = U(u) + iV(u)$. Any discontinuities in f_o' are to be interpreted as δ functions in f_o'' . $W(u)$ is the complex potential of a line charge distribution along the real axis of the u plane of strength $\frac{1}{2} f_o''(u_1)$. The δ functions in f_o'' correspond to true line charges (logarithmic singularities) and the remainder to a charged sheet. Henceforth, $f_o(v)$ and its derivatives will be imbedded in the u plane by regarding v as identical to u_1 whenever this is convenient, and the symbol u will be reserved for points in the open upper half plane. W is defined also by the integral in equation 33.

We shall take advantage of the large body of knowledge extant about the complex electrostatic potential by discussing the stability problem in terms of the properties of the "charge density" $\frac{1}{2} f_o''(v)$ and its "complex potential" $W(u)$ in the upper half u plane. [This analogy was discovered independently of a similar one discovered earlier (39), in which U and -V are the x and y components of the electric field of a line charge distribution $\frac{1}{2} f_o'(v)$, and which seems less fruitful, because the x and y components of an electric field do not possess as simple relations to charge distributions as do the potential and stream function.] The plasma is unstable if and only if there exists a point u such that

$$U(u) > 0 , \quad V(u) = 0 , \quad Im(u) > 0 \qquad (35)$$

for then a real number k can be chosen to fulfill equation 33 . The lines V = const are lines of force and the lines U = const are equipotentials. Some of the most important properties of the charge distribution and potential are listed below.

(i) By integrating f_o'' from minus infinity to v , we see that the total charge to the left of v is $\frac{1}{2} f_o'(v)$ and the total charge of the distribution is zero.

(ii) For this distribution the total dipole moment vanishes but the quadrupole moment is 1 . Asymptotically then, $W(u) \sim 1/u^2$ at infinity. Thus there is always a $V = 0$ line of force which tends to infinity asymptotically parallel to the imaginary axis. Any other $V = 0$ lines in the open upper half plane which tend to infinity would have to be asymptotically parallel to the real axis. It can be shown (with some effort) that there are no such lines; but if there were, U would be decreasing toward infinity on them and from the discussion it will be clear that they would have no effect on any of the conclusions. The asymptotic form shows that there are two $U = 0$ lines in the upper half plane which tend to infinity at angles $\pm 45^o$ with the imaginary axis. For large $|u|$, U is negative between them, and positive between them and the real axis.

(iii) Just above a point v on the real axis V is nearly $\pi f_o'(v)$ whenever the latter exists, since this is just half the flux from the charge $\frac{1}{2} f_o'(v)$ to the left of v* . When f_o' has a jump

*This can also be seen from $1/(v - u - i\epsilon) = P[1/(v - u)] + i\pi\delta(v - u)$.

discontinuity at v, say from value α to β, a succession of flux lines radiates from the discontinuity (a line charge) with V values ranging from $\pi\alpha$ to $\pi\beta$. Thus a $V = 0$ line meets the real axis at v if and only if f_o' changes sign there. This is just the condition that f_o have on extremum at v. If $f_o' = 0$ at a point v but $f_o'' \neq 0$ there, it can be seen by looking at the variation of V just above the real axis near v that a single $V = 0$ line meets the real axis there; again f_o has an extremum at v. If f_o'' is also zero at v, or if f_o' is zero in a whole neighborhood of v, several $V = 0$ lines may meet the real axis there. Points or intervals where $f_o' = f_o'' = 0$ and f_o' is of opposite sign on either side will be called "horizontal places of inflection" of f_o. Since these, along with extrema where $f_o'' = 0$, introduce qualifications into some of the following arguments, it will be assumed at first that they do not occur, but the extension of the method for them will be stated later. The particularly simple case where f_o is zero or an absolute maximum and $f_o' = 0$ will also be treated explicitly.

It has been found that lines $V = 0$ meet the real axis at places where f_o' changes sign (extrema of f_o) but one other possibility should be considered. If $f_o \equiv 0$, say, for $v > A$, a $V = 0$ line runs along the real axis from A to ∞. (For example, this must occur in the relativistic case, with $A \leq c$.) Is it possible for other $V = 0$ lines to meet the real axis in this region? If so, there would be a neutral point $\left(\frac{dW}{du} = \frac{\partial U}{\partial v} = 0\right)$ there, but differentiating the formula

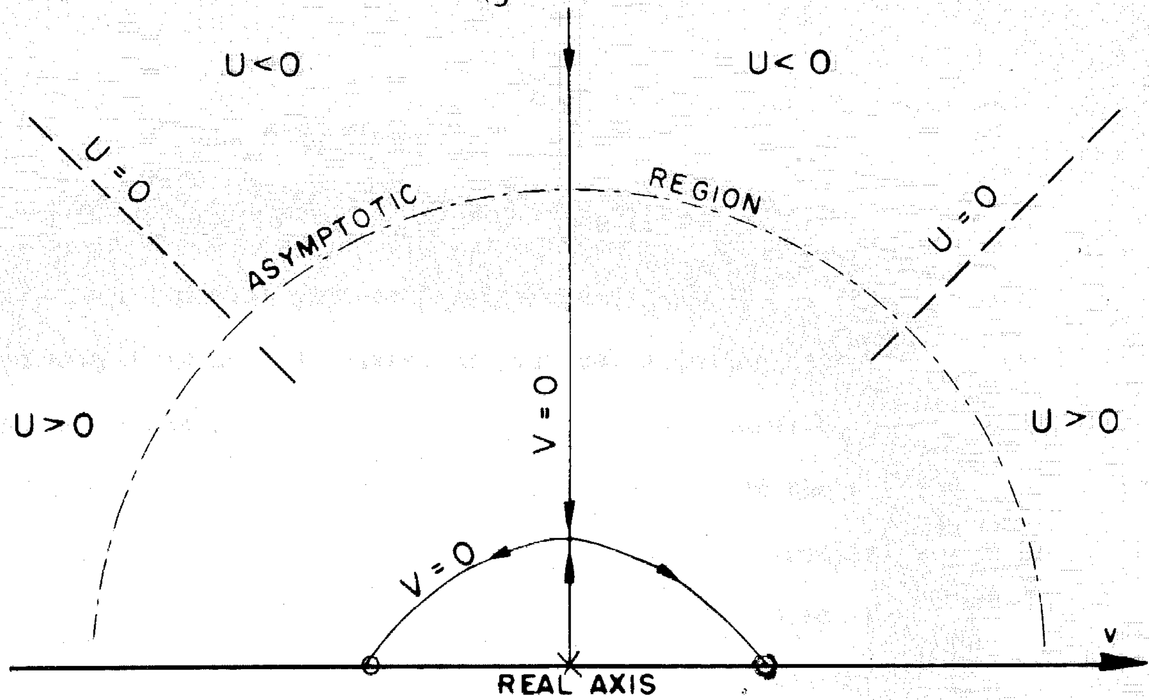
$$U(v) = \int_{-\infty}^{\infty} \frac{f_o'(p)dp}{p - v}$$

with respect to v and integrating by parts (for a point v where $f_o = f_o' = f_o'' = 0$), one obtains

$$\frac{\partial U}{\partial v} = \int_{-\infty}^{\infty} \frac{f_o(p)dp}{(p - v)^3}$$

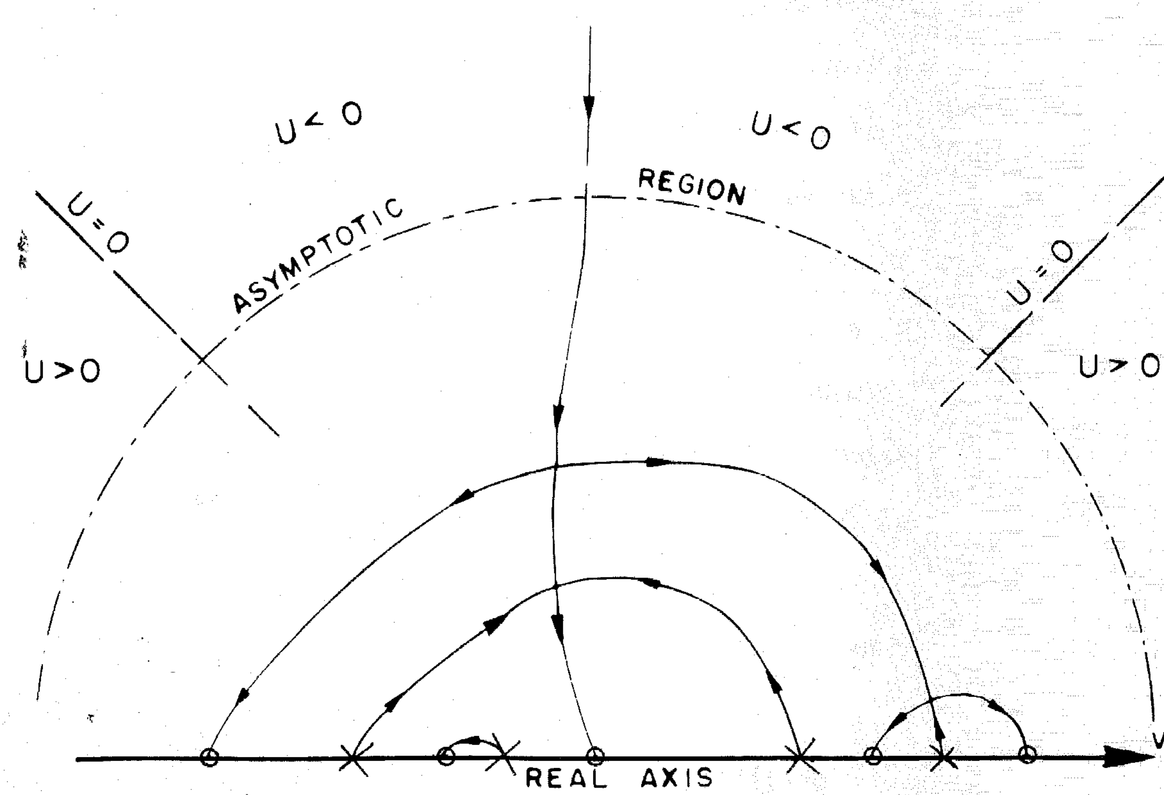
which is of constant sign by assumption (c) from A to infinity. Thus no $V = 0$ lines can meet the real axis entirely outside the region where $f_o \neq 0$.

In view of equation 35 the lines of flux on which $V = 0$ are particularly important, as the plasma is stable if and only if $U < 0$ everywhere on them. Since there are no charges in the open upper half plane all such lines must terminate on the real axis or tend to infinity; it has been shown that precisely one tends to $+i\infty$, but $U < 0$ on it near infinity [(ii) above]. Since U varies monotonically along lines of force except at neutral points, we can imagine each $V = 0$ line to be marked with an arrow in the direction of the electric field, i.e., the direction of decreasing U , illustrated in Figs. 2 and 3. The problem of checking the sign of U at each point on the net of $V = 0$ lines can now be reduced to the following simple procedure: By assumption (e) and property (iii) above, the lines $V = 0$ meet the real axis at a finite number of points, $v_1, v_2, \cdots v_n$ where f_o' changes sign. Starting at a point v_i we can follow a $V = 0$ line into the upper half plane, and by taking



O = maximum of f_o X = minimum of f_o

FIGURE 2 NET OF V=0 LINES IN THE COMPLEX u-PLANE, FOR A CASE WHERE $f_o(v)$ HAS TWO SYMMETRICAL PEAKS.



O = maximum of f_o X = minimum of f_o

FIGURE 3 NET OF V=0 LINES IN THE COMPLEX u-PLANE FOR A CASE WHERE $f_o(v)$ HAS FIVE MAXIMA the line immediately to our right away from any neutral point we encounter, return to the real axis at some point v_j (with the one exception that we may end on the line tending to $i\infty$). Since the arrows on lines at a neutral point are directed alternately toward and away from it, U varies monotonically during such a traversal and hence has its extrema on the ends of the lines, being larger at the end where the electric field points away from the axis. Since the normal component of the electric field at the real axis is $\pi f_o''(v)$, the points where U is the largest on those $V = 0$ lines which are traversed during the above process are the minima of f_o on the real axis. From the asymptotic form for W, the arrow on the line tending to $i\infty$ is toward the origin, so $U < 0$ on it up to the first neutral point. By a well-known theorem of potential theory, the $V = 0$ lines cannot enclose any region of the open upper half plane; therefore we traverse the entire net of $V = 0$ lines if we repeat the above process for all i, $1 \leq i \leq n$. This proves the theorem: <u>The plasma is stable of and only if $U < 0$ at each minimum of f_o.</u> (We have temporarily assumed that there are no places where $f_o' = f_o'' = 0$.) In generalizing this method to other dispersion relations, and in comparing it with Nyquist's criterion, it will be worth while to remember that the minima of $f_o(v)$ are those places where $V(v)$ changes sign and is increasing toward the right [see (iii) above].

In practice, the potential U at a point v on the real axis is calculated from

$$U(v) = \mathrm{Re} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{f_o'(p)\,dp}{p - v - i\epsilon} = P \int_{-\infty}^{\infty} \frac{f_o'(p)\,dp}{p - v} \qquad (\tilde{36})$$

The principal value sign is not needed if $f_o'(v) = 0$. If f_o' is discontinuous at v , a line charge is located there and $U \to \pm \infty$ at v according to the sign of the jump discontinuity in f_o' . If v is a nondifferentiable minimum the sign is "+" and the plasma is unstable, but a nondifferentiable maximum gives the opposite sign and introduces no instability. (See the analysis of transverse waves later in this part, however.)

Since the depth of the minimum and the discontinuity of $f_o'(v)$ may be arbitrarily small and still produce instability, there are unstable velocity distributions arbitrarily close to stable ones and even to the Maxwell-Boltzmann distribution in the sense of most of the usual metrics of function space.

If f_o has any nondifferentiable minima it is now known to represent an unstable plasma. If f_o is zero at one of its differentiable minima, the plasma must again be unstable: When $f_o'(v) = 0$, equation 36 may be integrated by parts to obtain:

$$U(v) = \int_{-\infty}^{\infty} \frac{f_o(p) - f_o(v)}{(p - v)^2} \, dp \qquad [\text{when } f_o'(v) = 0] \qquad (37)$$

which shows that $U(v)$ is positive wherever $f_o(v)$ is zero.

Suppose that there are no particles at all with a particular vector velocity $\underline{v}_2$ in the plasma, but that there are particles moving in the same direction with speeds both greater and less than v_2 . Then when $\underline{k}$ is taken along $\underline{v}_2$, $f_o(v_2)$ is zero, so that the plasma can support growing waves with phase velocity very close to $\underline{v}_2$.

Where $f_o' = f_o'' = 0$, several lines $V = 0$ or none may meet the real axis. At a horizontal place of inflection the sign of V just above the real axis is the same on either side, so an even number of $V = 0$ lines meet the axis there. Then if there are any such lines, there is at least one on which U increases away from the real axis. Following this line (as in the procedure outlined above for traversing lines), one reaches another point on the real axis where U is surely higher; if this is again a horizontal place of inflection one can continue following the line. By induction and by assumption (e), one eventually reaches an extremum of f_o where U is greater than at any of the horizontal places of inflection encountered, so that it is unnecessary to check the sign of U at horizontal places of inflection.

If f''_o is positive on each side of a place where $f'_o = f''_o = 0$ (making it a minimum), at least one $V = 0$ line must meet the real axis there, as just above the real axis V changes sign. Furthermore, consideration of the normal component of the electric field right above the real axis shows that there must be at least one such line on which U increases toward the real axis. Hence one must check the sign of U where such lines meet the axis. If $f'_o = f''_o = 0$ only at an isolated point, U may simply be evaluated there by means of equations 36 or 37, but if this holds in a whole interval, the situation becomes more complicated. By studying what patterns of lines of force are possible, one can show that if $(\partial U/\partial v)$ has no zero in the interval, one should check the end where U is largest, but otherwise at the zero of $(\partial U/\partial v)$. Similarly at an isolated maximum point of f_o there must be at least one $V = 0$ line meeting the real axis on which U increases away from the axis, and one need not check the sign of U there. At a maximum, however, where $f'_o = 0$

throughout a whole interval, one must check the sign of U at any neutral point $(\frac{\partial U}{\partial v} = 0)$ therein. In the above discussion, reference was made to finding the zeros of $(\partial U/\partial v)$ in an interval where $f_o' = 0$, a cumbersome procedure for most functions $f_o(v)$. If f_o is an absolute maximum or is zero there, equation 37 shows U is negative or positive throughout the interval, respectively, making it unnecessary to know just where the $V = 0$ line comes in.

From the preceding arguments it is clear that a single-peaked distribution is stable: the one extremum is an absolute maximum where either $f_o' = 0$ and equation 37 applies or f_o' is discontinuous and the first form of the stability theorem shows the plasma is stable.

Properties of the Longitudinal Waves

It has been shown that to each growing wave there corresponds a point u in the upper half of the complex phase velocity plane where equation 35 holds. From the foregoing discussion of the $V = 0$ lines and the variation of U along them, it is clear that these points comprise portions of the $V = 0$ lines which are connected to the real axis at those of the points v_i where $U(v_i) > 0$. Assume that $f_o'' \neq 0$ at each of these points, so that a single $V = 0$ meets the real axis at each of them and the normal component of the electric field is non-zero. Consider a point, say $\bar{v}$, where f_o is a minimum and $U > 0$. The electric field points away from the real axis at v, and by following the $V = 0$ line away from $\bar{v}$ one will eventually come to a point $\bar{u}$ where $W = 0$, or will return to the real axis at some point v_j (using the right-hand turn rule at neutral points). In the first case equation 34 shows that unstable solutions occur for values of k fulfilling

$$0 < k < k_{max}$$

$$k_{max} = \omega_e [U(\bar{v})]^{1/2} \tag{38}$$

and in the second case for

$$k_{min} < k < k_{max}$$

$$k_{min} = \omega_e [U(v_j)]^{1/2} \quad ; \quad k_{max} = \omega_e [U(\bar{v})]^{1/2} \tag{39}$$

The rate of growth of a wave is given by

$$\omega_2 = ku_2 = \omega_e u_2 [U(u)]^{1/2} \tag{40}$$

Thus ω_2 is zero at $\bar{u}$ and at $\bar{v}$; in most cases it will have only one maximum in between. This is surely true when $\bar{u}$ and $\bar{v}$ are close together, as when the distribution differs but little from a stable one, i.e., if $U(\bar{v})$ is small.

From equations 38 and 39, if U is bounded near $\bar{v}$ growing waves will be possible only for wavelengths λ longer than $\lambda_{min} = 2\pi/k_{max}$. If, however, f_o' is discontinuous at $\bar{v}$, producing a logarithmic singularity in U there, $k_{max} \to \infty$ and growing waves of arbitrarily short wavelength can occur. One might distrust this result since the derivation of equation 33 is valid only for sufficiently long wavelengths, when collective motion dominates individual particle effects (3). If $f_o(\underline{v})$ is nearly Maxwellian, equation 33 is valid for $\lambda \gtrsim \lambda_D = (kT/4\pi n_o e^2)^{1/2}$, but if not, one could consider using the cut-off distance λ_D' obtained by evaluating λ_D for a Maxwellian plasma with the same particle and energy densities as the one in question. Since an instability usually persists at long wavelengths $(k \to 0)$, the conclusions on instability will not be affected in most cases by the introduction of a minimum wavelength. The use of the concept of Debye shielding in discussing unstable plasma oscillations will be criticized in part IV.

Qualitatively, one can see from equations 38 through 40 that the sharper the minimum in $f_o(v)$ and the steeper its sides, the larger the maximum values of k and ω_2 will be for growing waves, as $U(\overline{v})$ will be greater.

There are no instabilities for $|u|$ sufficiently large since $W \sim 1/u^2$ asymptotically. On the other hand, U has at worst logarithmic singularities which are all on the real axis. Therefore the right hand side of equation 40 is bounded, and infinite rates of growth do not occur.

Since the $V = 0$ lines are lines of flux, the electric field $\underline{E} = -(\frac{dW}{du})^*$ is tangential to them. Having found a point $\overline{v}$ on the real axis where a $V = 0$ line meets it and $U > 0$, one could in principle trace the instabilities into the upper half plane by integrating the equation

$$\frac{du}{ds} = \frac{(\frac{dW}{du})^*}{\left|\frac{dW}{du}\right|}$$

where s is arc length, or simply

$$\frac{du}{dp} = \left(\frac{dW}{du}\right)^* \tag{41}$$

where p is a parameter. This would yield a parametric form of the $\omega - k$ relation for non-real ω. If $\bar{u}$ and $\bar{v}$ are close together, this can be done approximately by expanding W in a series about $\bar{v}$, provided f_o is sufficiently smooth there. The derivatives of W defined by

$$\left.\frac{d^n W^+}{du^n}\right|_v = \lim_{\epsilon \to 0+} \left(\frac{d^n W}{du^n}\right)_{u = v + i\epsilon}$$

can be used to form a Taylor series which gives W in the upper half plane and an analytic continuation of it below the real axis. For points near $\bar{v}$, write

$$k^2/\omega_e^2 = W(u) \approx W(\bar{v}) + (u - \bar{v})\left(\frac{dW^+}{du}\right)_{\bar{v}}$$

$$\approx U(\bar{v}) + (u_1 + iu_2 - \bar{v})\left[\left(\frac{\partial U}{\partial v}\right)_{\bar{v}} + i\left(\frac{\partial V}{\partial v}\right)_{\bar{v}}\right] \tag{42}$$

Taking real and imaginary parts of equation 42 yields

$$\omega_2 = ku_2 \approx \omega_e u_2 \left[U - u_2 \, |\nabla U|^2 / \left(\frac{\partial V}{\partial v}\right)\right]^{1/2} \tag{43}$$

where all quantities (except u_2) are to be evaluated at $\bar{v}$. For the maximum value of ω_2 between $\bar{v}$ and $\bar{u}$ we get

$$\omega_{2_{max}} \approx \omega_e \frac{2}{\sqrt{3}} \left[\frac{\left(\frac{\partial V}{\partial v}\right)}{|\nabla U|^2} \; U^{3/2}\right]_{u = \bar{v}} \tag{44}$$

This is valid if $U(\overline{v})$ is small but fails if $(\frac{\partial V}{\partial v})_{\overline{v}} = f_o''(\overline{v}) = 0$. Equation 44 shows that at the threshold of instability, when $U(\overline{v}) \approx 0$, the rates of growth increase slowly, like $U^{3/2}$ as U increases.

Since the instabilities lie on portions of the $V = 0$ lines emanating from the minima of $f_o(v)$, they may be divided into groups, a group consisting of the instabilities near a given minimum. Generally, a group of instabilities can be thought of as a sort of double stream instability between the beams representing the peaks on either side of the minimum. The behavior of W (and hence of the waves) near a given minimum, $v = \overline{v}$, is affected most strongly by the values of $f_o(v)$ near $\overline{v}$, i.e., by the distribution of particle velocities near the phase velocities of the waves. Also the phase velocity of growing waves near a given minimum is closest to those of the particles in nearby peaks of f_o, implying a larger wave-particle interaction. These ways of looking at which families of particles participate most in a growing wave usually agree with more quantitative methods, such as evaluating the mean velocity or energy changes of these families when a growing wave is present in the plasma. Qualitative methods such as looking at the general location of instabilities are a valuable complement to mathematical analysis of the problems.

Relationship to Nyquist's Criterion

Assume at first that $f_o'(v)$ is continuous for all v . It has been shown (see p.44) that the plasma is unstable if and only if $U = Re(W)$ is positive at one of those places on the real axis of the u plane where $V = \text{Im}(W)$ is zero and increasing to the right. This may be interpreted in terms of the crossings of the Nyquist path with the positive real axis in the Nyquist diagram plane (see p.36). Let $D(k,u) = D_1(k,u) + i\,D_2(k,u) = \omega_e^2\,W(u) - k^2 = [\omega_e^2\,U(u) - k^2] + i\omega_e^2\,V(u)$, where W is defined in equation 34 or by the integral in equation 33. The Nyquist diagram for finding roots of the equation $D = 0$ in the upper half u plane consists of the path Π_k traced out in the D plane by $D(k,v)$ as v traverses the real u axis from $-\infty$ to $+\infty$. From the asymptotic form $W \sim 1/u^2$ it can be seen that $D(k,-\infty) = D(k,+\infty) = -k^2$. The path Π_k will sometimes be said to start at $D = -k^2$ for $v = -\infty$, proceed into the complex D plane as v increases, and end back at $-k^2$ again as $v \to +\infty$, an arrow being affixed to Π_k in the sense corresponding to increasing v. Since $V(v) = \pi f_o'(v)$ and f_o tends to zero monotonically from above as $v \to \pm\infty$, $f_o'(v)$ and $D_2(k,v)$ tend to 0^+ as $v \to -\infty$ and to 0^- as $v \to +\infty$. From the asymptotic form for W, $D_1(k,v)$ tends to $-k^2$ from above as $v \to \pm\infty$. A little study of the behavior of U and V along the real axis shows that for a Gaussian or similar distribution, the Nyquist diagram looks like Fig. 4A. As long as $f_o(v)$ obeys assumptions (b), (c), and (e), the form of Π near its "ends" at $-k^2$ is about the same. It is because $D(k,u)$ maps the upper half u plane onto the interior of Π_k that the plasma is unstable when Π_k encloses the origin, for then the origin is the image under the mapping D of some point u_i in the upper half u plane, i.e., $D(k,u_i) = 0$. The $V = 0$ lines in the upper half u plane all map onto portions of the real D axis interior to $\Pi_{k=0} \equiv \Pi_o$. But Π_k is simply Π_o

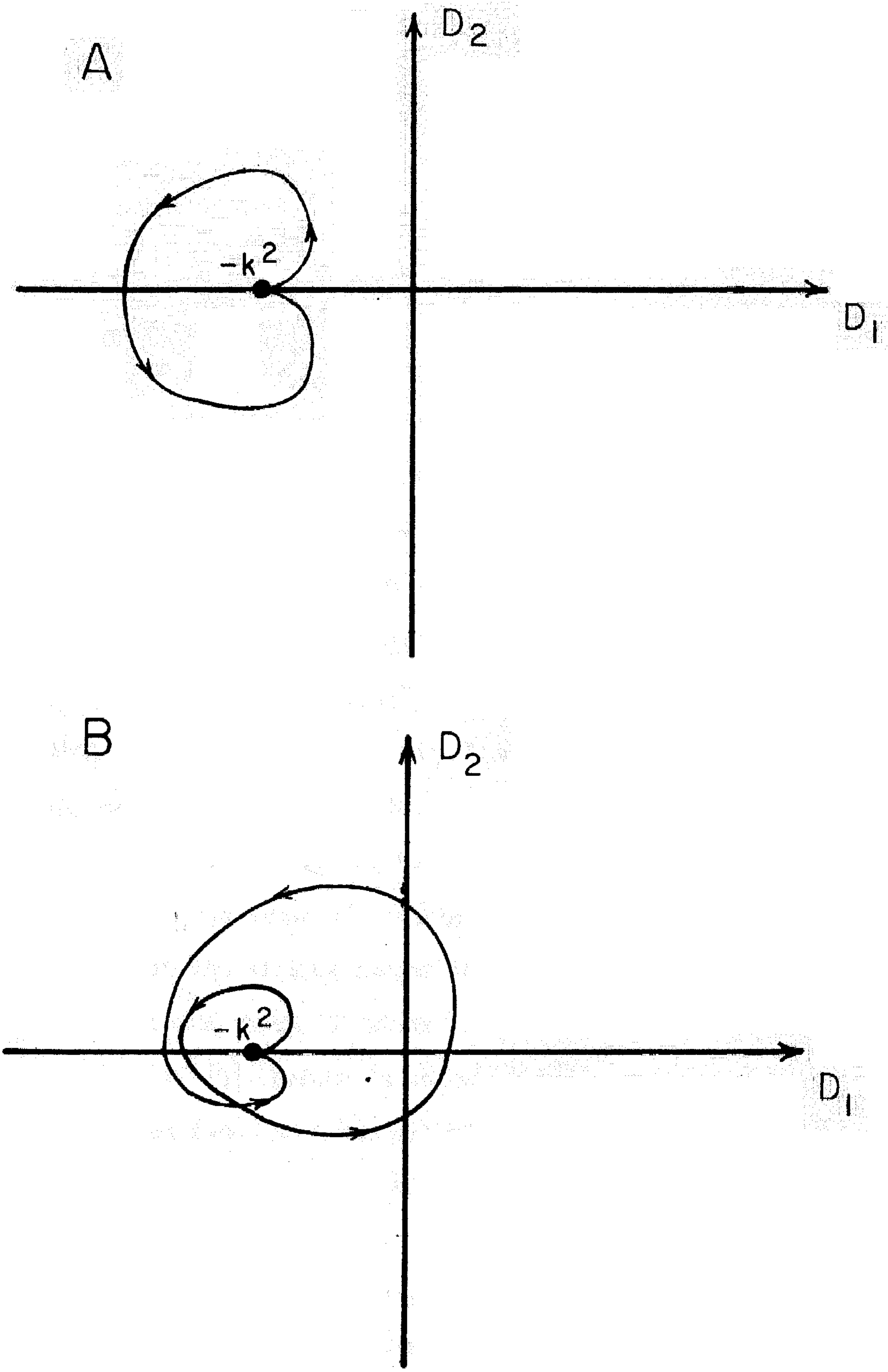


Fig. 4 Nyquist Diagrams for a Maxwellian Plasma (A) and for an Unstable Plasma (B)

translated to the left a distance k^2. Therefore the plasma is unstable if and only if Π_o encloses some point on the positive D_1 axis.

The Nyquist diagram for a typical unstable plasma represented by a velocity distribution $f_o(v)$ having two peaks is shown in Fig. 4B. In this example, instability exists for values of k^2 between zero and an upper limit k^2_{max}, given by the value of D_1 where Π_o crosses the positive real axis of the D plane. This crossing point is the image under $D(0,u)$ of the minimum $\bar{v}$ of $f_o(v)$ on the real axis of the u plane, since $V(v) = \omega_e^{-2} D_2(0,v) = \pi f_o'(v)$. Equation 38 can be obtained by setting $k^2_{max} = \omega_e^2 D_1(0,\bar{v})$. The crossing from below to above is characteristic of a minimum in $f_o(v)$ at $\bar{v}$.

Other initial distributions $f_o(v)$ produce more complicated Nyquist diagrams. Because Π is just translated as k is varied, instability exists, however, for precisely those values of k^2 such that the point $D = k^2$ is in the interior of Π_o. "Interior" is, of course, defined in the sense of winding number (40): A point is interior to Π if the winding number of Π about that point is not zero. A point around which Π winds n times is the image under the mapping D of $|n|$ points in the upper half u plane.

It has been found that the net of $V = 0$ lines in the u plane maps onto those portions of the real axis of the D plane interior to the Nyquist contour Π. Because this mapping may be many → one (not one-to-one), it is difficult to distinguish instabilities as being due to one or another portion of $f_o(v)$, or one or another family of particles, as can be done in the u plane. On the other hand, it is particularly easy to ascertain for what values of k^2 growing waves can occur. $D(0,u)$ maps the extrema of f_o and the point ∞ onto the points $P_1 \cdots P_n$ where Π_o crosses the real D axis. It is easy to find the direction of the arrow on Π_o at P_i ; when $1 \leq i \leq n-1$, the arrow is "up" when P_i is the image of a minimum of $f_o(v)$ and "down" when P_i is the image of a maximum of $f_o(v)$. The arrow on P_n, the image of "∞" , is always "up". It is usually preferable to imagine $P_1 \cdots P_n$ rearranged in order of increasing D_1. Define a function $w(D_1)$ for any point on the real D axis which is the number of points P_i to the right of D_1 with the arrow on Π_o "up", minus the number of points P_i to the right of D_1 with the arrow "down". Clearly, $w(D_1)$ is the winding number of Π_o with respect to D_1. Then if $w(k^2) \neq 0$, the plasma will support growing waves of wave number k . (Actually $2\pi/k$ is the wave number as usually defined). If Π_o passes through one of the P_i more than once, an arrow in the appropriate direction is of course included in the count for each passage.

Horizontal places of inflection of $f_o(v)$ correspond to places in the D plane where Π_o is tangent to the D_1 axis but does not cross it. Extended maxima and minima correspond to intervals where Π_o runs along the D_1 axis and does eventually cross it. The foregoing discussion of Nyquist's criterion may be modified for these cases, but seems to offer no advantage over the electrostatic analogue.

Generalizations to Other Waves

In principle, much of the foregoing analysis can be extended to any dispersion relation $D(\underline{k},\omega) = D_1 + iD_2 = 0$ such that D is analytic in the upper half ω plane and is bounded as $|\omega| \to \infty$ there. For example, the crossings of the Nyquist path Π_k with the real positive D axis (or any other straight line through the origin) may be used to find the winding number of Π_k about the origin. The diagram of crossings of Π_k with the real axis together with the directions of crossing will be called the "arrow diagram". Its use reduces the labor of constructing the Nyquist diagram whenever the zeros of D_2 for ω real are easily determined. These diagrams must usually be formed for each $\underline{k}$ value. When D is invariant to certain operations on $\underline{k}$, or transforms very simply under certain operations on $\underline{k}$, the labor can again be reduced. If D can be written as $D(\underline{k},\omega) = A(\omega) - k^2$ or $B(\omega/k) - k^2$ the dependence on k^2 can be handled as before by observing that Π_k encircles the origin if and only if Π_o encircles the point k^2. The charge analogy method gives an equivalent solution. If D can be written in the form $D(\underline{k},\omega) = Q(k_z,\omega) - k^2$, one can again avoid much of the labor of searching $\underline{k}$ space for instabilities. Fixing k_z, one can determine for what values of k^2 D has zeros in the upper half plane by finding what portions of the D_1 axis are interior to the Nyquist contour of Q. If these portions include any points to the right of $D_1 = k_z^2$, k_x and k_y can be chosen so that $D(\underline{k},\omega) = 0$; then the plasma is unstable. This would have to be done for each k_z, but the work of searching over k^2 is avoided. Other hypothetical examples can be generated ad.lib., but let us see how far the Nyquist and charge analogy methods can proceed for the dispersion relations of part II.

The dispersion relation $D(\underline{k},u) = 0$ for arbitrary waves (from equations 19) when $B_o = 0$ is generally cubic in k^2, and really involves the direction of $\underline{k}$ too, since a rotation was made to put $\underline{k}$ along $\underline{e}_z$. In addition, Im(D) is not easily evaluated along the real u axis since the real parts of the integrals come into the imaginary part of the determinant. None of the methods given can help appreciably in making the stability analysis manageable--the Nyquist diagram would have to be constructed for each vector $\underline{k}$ unless physical arguments suggested obvious choices. When $f_o(\underline{v})$ is such that D splits into the factors $A \pm B$, as described on p.25, the charge analogy method or the arrow diagram method may be used to solve the stability problem. The charge analogy method will be exhibited.

The precise form of $D(\underline{k},u)$ in this case is

$$\frac{k^2}{\omega_e^2} = \frac{1}{u^2 - c^2}\left[1 + (m/M) + \int_{-\infty}^{\infty} \frac{\frac{d}{dv_z}\left(< v_x^2 f_o > \pm < v_x v_y f_o >\right)}{v_z - u}\, dv_z\right]$$

$$\equiv (u^2 - c^2)^{-1}\left[1 + (m/M) + \overline{U}_\pm(u) + i\overline{V}_\pm(u)\right] \equiv G_\pm(u) + iH_\pm = T_\pm(u) \tag{45}$$

which defines $\overline{U}$, $\overline{V}$, G, H, and T. Let $\overline{U}_\pm + \overline{V}_\pm = \overline{W}_\pm$, and denote the numerator in the integral by $\phi'_\pm(v_z)$. Then $\phi_\pm(v_z) = < v_x^2 f_o > \pm < v_x v_y f_o >$. $\phi''_\pm(v_z)$ is the "charge density" for the "complex potential" $\overline{W}_\pm$, and $(u^2 - c^2)^{-1}\, \phi''_\pm(v_z)$ is the "charge density" for the "complex potential" $T_\pm$. If $f_o = f_o(v_r, v_z)$, $v_r^2 = v_x^2 + v_y^2$, then $< v_x v_y f_o > = 0$ and equation 45 reduces to equation 28 in the limit $\Omega \to 0$. The $\pm$ signs will be omitted from now on whenever no confusion results. In terms of ϕ equation 45 is, in part,

$$\frac{k^2}{\omega_e^2} = \frac{1}{u^2 - c^2}\left[1 + (m/M) + \int_{-\infty}^{\infty} \frac{\phi'(v_z)}{v_z - u}\, dv_z\right] = T(u) \qquad (46)$$

The plasma is unstable with respect to transverse waves if there are points in the upper half u plane where $G > 0$, $H = 0$. Since $\overline{W} \sim 1/u^2$ at ∞, just like W for the Vlasov dispersion relation, $T \sim \frac{1}{u^2}(1 + \frac{c^2}{u^2} + \frac{c^4}{u^4} + \cdots)\ [1 + (m/M) + 1/u^2] = \frac{1}{u^2}\ [1 + (m/M)] + \frac{c^2}{u^4}[2 + (m/M)]$. The second term will not be used here but is included because it is easy to find and may be useful in other connections. Since T is analytic in the upper half u plane, most of the discussion of the network of $V = 0$ lines and the variation of U along them carries over to H and G . The charge distribution for T now has two dipole sources at $u = \pm c$. It is these which make it more difficult to use Nyquist's criterion, as they make the Nyquist contour go to infinity when $v_z \to \pm c$. This can be handled by letting u move from $-\infty + i\epsilon$ to $\infty + i\epsilon$, $\epsilon \gtrsim 0$, in forming the diagram, but the electric dipole concept is of great value in determining the behavior of T(u) near $u = \pm c$. Again the plasma is unstable if at any point where an $H = 0$ line meets the real axis, but again we need check only those points where the "electric field" $\underline{E} = -(dT/du)^*$ points into the upper half plane. The $H = 0$ lines tending to infinity are of no importance, as before. The points where $H = 0$ lines meet the real axis are those where $\overline{V} = 0$ [i.e., $\phi(v_z)$ changes sign] and possibly the points

$v_z = \pm c$. If either of the point dipoles at $\pm c$ points toward the upper half plane, an $H = 0$ line with $G \to +\infty$ near the dipole will emanate upwards from it, implying instability. For other orientations no $H = 0$ lines with $\underline{E}$ directed into the upper half plane emanate from the dipoles. The phase of $1 + (m/M) + \overline{W}(\pm c)$ determines the dipole orientations. Since $u + c > 0$ at $u = c$ and $u - c < 0$ at $u = -c$, the dipoles both point directly away from the origin if $\overline{W}(\pm c)$ is real. The derivation of equation 45 involved a nonrelativistic form of the Boltzmann equation, so one should assume most particle velocities are much less than c. Then $\phi'(v_z)$ and $\overline{W}(v_z)$ are both small for $|v_z| \approx c$. Then the condition for no unstable roots near $u = \pm c$ is that $\lim_{\epsilon \to 0} \overline{V}(c + i\epsilon) = \phi'(c) \leq 0$, and similarly $\phi'(-c) \geq 0$. These are fulfilled if f_o is assumed to be zero in some neighborhood of $|v_z| = c$, say for $|v_z| > c - \epsilon$, $0 < \epsilon \ll c$. This small restriction on f_o seems desirable, since combining Maxwell's equations with nonrelativistic mechanics (as exemplified by equation 1) can be expected to give wrong results when there are particles moving at or arbitrarily close to the speed of light. It is sufficient, however, to have ϕ a decreasing function of $|v_z|$ at $|v_z| = c$. It can be shown that if f_o is zero for all $|v_z|$ greater than some constant, neutral points do not occur on that portion of the real u axis; only zeros of ϕ' interior to the region where $f_o \neq 0$ are intersections of $H = 0$ lines with the real axis. Because the factor $(u^2 - c^2)^{-1}$ is negative for real $u < c$, the analogue $\underline{E}$ is into the upper half plane at the maxima rather than at the minima of

$\phi(v_z)$. Assuming that f_1 has at the worst a small tail for $v_z > c$, the stability criterion is then as follows: The plasma is stable if and only if the quantity $G(u)$ defined in equation 45 is negative at each maximum of $\phi(v_z)$.

From the electrostatic analogy, it is evident that a nondifferentiable maximum in $\phi(v)$ produces instability with respect to transverse waves.

While the integral in equation 45 is invariant to Galilean transformations along the z axis, the factor in front is not. If the transformation is $\underline{v} \to \underline{v} + \underline{e}_z \Delta v$, $u \to u + \Delta v$, then $c^2 T$ is changed by an amount $\approx u\Delta v/c^2$ if $\Delta v \ll c$ and u is not close to c . If there are few particles with velocities near c , the dipole singularities dominate the behavior of $T(u)$ near $\pm c$; there are then no instabilities with $u \approx c$, and the second condition (u not close to c) is fulfilled. The small discrepancy $u\Delta v/c^2$ is another result of combining nonrelativistic mechanics with Maxwell's equations. Since there are no instabilities near $u = \pm c$, the stability of the plasma can be found in a variety of reference frames with no disagreement; only small changes in the properties of the growing waves will occur.

The dispersion relation for transverse waves propagating along $\underline{B}_o$, given in equation 28, can be analyzed by an "arrow diagram". If the $(k_z v_z \mp \omega)$ in the numerator of equation 28 is written as $(k_z v_z - \omega \pm \Omega) \mp \Omega$ for the electrons and $(k_z v_z - \omega \mp [m/M]\Omega) \pm (m/M)\Omega$ for the ions, ω can be eliminated from the numerators by the relation

$$1 = 2\pi \int_0^\infty dv_r \int_{-\infty}^{\infty} dv_z (v_r f_o) = \pi \int_0^\infty dv_r \int_{-\infty}^{\infty} dv_z (v_r^2 s_o)$$

and a similar relation for ions. (This is just the normalization condition and an integration by parts.) The resulting form for equation 28 is

$$D_{\pm}(k_z,\omega) \equiv 1 + \frac{\omega_e^2}{k^2c^2-\omega^2}\left\{1 + \frac{m}{M} + \pi \int_0^{\infty} v_r^2 dv_r \int_{-\infty}^{\infty} dv_z \left[\frac{\mp\Omega\, s_o - k_z v_r r_o}{k_z v_z - \omega \pm \Omega} + \frac{m}{M}\,\frac{\pm (m/M)\Omega\, S_o - k_z v_r R_o}{k_z v_z - \omega \mp (m/M)\,\Omega}\right]\right\} = 0 \qquad (47)$$

Using the previous notation $\iint \psi \, dv_x dv_y = \langle\psi\rangle$ and the equation $dv_x dv_y = 2\pi\, v_r dv_r$, one obtains

$$D_{\pm}(k_z,\omega) = 1 + \frac{\omega_e^2}{k^2c^2-\omega^2}\left\{1 + \frac{m}{M} + \frac{1}{2} \int_{-\infty}^{\infty} \left[\frac{\mp\Omega \langle v_r s_o\rangle - k_z \langle v_r^2 r_o\rangle}{k_z v_z - \omega \pm \Omega} + \frac{m}{M}\,\frac{\pm (m/M)\Omega \langle v_r S_o\rangle - k_z \langle v_r^2 R_o\rangle}{k_z v_z - \omega \mp (m/M)\,\Omega}\right] dv_z\right\} = 0 \qquad (48)$$

Again, let $\omega = \omega_1 + i\omega_2$ and let $D_{\pm} = D_{1\pm} + iD_{2\pm} = 1 - \omega_e^2(k^2c^2 - \omega^2)^{-1}$ $[1 + (m/M) + W_{\pm}(k_z,\omega)]$ where $W_{\pm}$ stands for the integral in equation 48. For fixed k_z , the Nyquist contour Π_{k_z} is the limit as $\epsilon \to 0$ of the path traversed by $D_{\pm}(k_z,\omega_1 + i\epsilon)$ when ω_1 goes from $-\infty$ to $+\infty$. Because of the factor $(k^2c^2 - \omega^2)^{-1}$, it is helpful to think of ϵ as not quite zero, so that the path Π is bounded. The electrostatic analogue will be useful in understanding the behavior of Π when $\omega \approx \pm kc$; unless $1 + (m/M) + W$ happens to be zero at $\omega = \pm kc$, D is singular at these places, and the singularities are again most easily studied by interpreting D near $\omega = \pm kc$ as the complex potential of a dipole whose orientation is related to the phase of $1 + (m/M) + W$ there. A study of the signs of the various factors shows that if $-[1 + (m/M) + W]$ is real and positive at both $\omega = kc$ and $\omega = -kc$, the dipoles both point away from the origin. Otherwise, each dipole is rotated counterclockwise an amount equal to the phase of $-[1 + (m/M) + W]$. Therefore, unless $-[1 + (m/M) + W]$ is real and positive at $\omega = \pm kc$, the dipoles produce crossings of Π_{k_z} with the real axis. For example, if the left hand dipole points "up", Π crosses the D_1 axis at $+\infty$, going from the lower to upper half plane as $\omega_1 + i\epsilon$ passes $-kc$. If it points "down", Π crosses the D_1 axis at $-\infty$ from the upper to the lower half of the D plane.

The remainder of the real axis crossings of Π come from zeros of $\text{Im}\, W$ as ω traverses the real axis. These may be found from

$$\lim_{\epsilon \to 0^+} \text{Im} \int_{-\infty}^{\infty} \frac{\kappa(v)\,dv}{kv - \omega - i\epsilon \pm \Omega} = \frac{\pi}{k}\;\kappa\left(\frac{\omega \mp \Omega}{k}\right) \tag{49}$$

While it is possible for $1 + (m/M) + W$ to have a zero at $\omega = \pm kc$, this can happen only for a discrete set of k values; according to the order of the zero (simple, double, etc) Π is affected in different ways. A simple zero just gets rid of one of the dipoles, while a double zero produces a crossing of Π with the real axis, etc. This matter must be dealt with on an individual basis in any given example. Such a situation can arise only when the numerators of the integrals in equation 48 are sizeable for $v_z \approx c \pm (\Omega/k)$; they represent some sort of resonance between particles and the wave where the frequency kv_z with which a particle crosses successive wave crests minus the frequency

kc with which a light signal does so is equal to $\pm\Omega$, the particle cyclotron frequency.

As $\omega_1 \to \pm\infty$, $D \to 1$. This is another crossing of Π with the real axis. The direction of the arrow may be found from the factor $(k^2c^2 - \omega^2)^{-1}$ alone if the integral goes to zero rapidly enough, and is "up" in that case. In most physical cases, $\langle v_r s_o \rangle$ and $\langle v_r^2 r_o \rangle$ tend to zero at least as fast as $\exp(-v_z^2/\nu^2)$ for some constant ν, which is sufficient to justify the neglect of the integral as $\omega \to \infty$. In summary, the places where Π crosses the real axis are as follows:

1. Crossings at $D_1 = \pm\infty$ due to the "dipoles at $\omega = \pm kc$, unless W is real there
2. A crossing in the upward direction at $D_1 = 1$.
3. Crossings due to zeros of $Im(W)$, easily found by means of equation 49.

While physical applications have been reserved for part IV, the practicability of the methods just given can best be shown by an example. Therefore some of the work will be carried through for the electron distribution function of Weibel (36), with $(M/m) \to \infty$. Weibel's form for f_o is

$$f_o(\underline{v}) = \frac{1}{a_r^2 a_z (2\pi)^{3/2}} \exp\left[-\frac{v_r^2}{2a_r^2} - \frac{v_z^2}{2a_z^2} \right] \tag{50}$$

Then $\langle v_r s_o \rangle = -2a_z^{-1}(2\pi)^{-1/2} \exp(-v_z^2/2\, a_z^2)$

and $\langle v_r^2 r_o \rangle = (a_r^2 v_z/a_z^2) \langle v_r s_o \rangle$.

Along the real ω axis, equation 49 gives

$$\text{Im}\,[W_{\pm}(k_z,\omega_1)] = \frac{-\sqrt{2\pi}\,\exp[-(\omega_1 \mp \Omega)^2/k^2 a_z^2]}{k_z a_z} \times$$

$$\times \left[\pm \left(\frac{a_r^2}{a_z^2} - 1\right)\Omega - \omega_1 \frac{a_r^2}{a_z^2} \right] \tag{51}$$

This has zeros at $\omega_1 = \pm\,(1 - \frac{a_z^2}{a_r^2})\Omega$ and the arrow on Π is up or down according to whether $(k^2c^2 - \omega_1^2)k_z$ is negative or positive. The phase of $1 + W_{\pm}$ at $\omega_1 = \pm\,kc$ must be found by integration, unless assumptions are made on Ω and k such that $|\text{Re}\,W|$ is certain to be less than 1 there. In the latter case, Im(W), which is exhibited in equation 51, is sufficient to determine whether the dipoles point into the upper or lower half of the ω plane. The integrals for Re(W) can be transformed from "principal part" integrals into forms involving the error function of imaginary argument, which is tabulated. The necessary formulas and references are given in the next part. Having found the real axis crossings of Π and the directions of crossing, one then finds the winding number of Π with respect to the origin by counting, say the number of arrows up minus the number down on the positive real D axis. If this number is not zero, the plasma is unstable with respect to waves with the value of k_z used in constructing the diagram. Since the calculations needed to complete the analysis are tedious (e.g. even for fixed a_r/a_z many graphs of functions of Ω/ka_z would have to be prepared) and since Weibel's treatment seems to be as detailed as is warranted by the physical importance of the example, it will be dropped here.

No further simplifications of the dispersion relation for arbitrary waves propagating at an arbitrary angle to $\underline{B}_o$, implicit in equations 32, seem likely to be found unless physical reasons suggest a preferred value for $\underline{k}$ and some likely values for the other parameters. It is hoped that in specific examples, certain directions or magnitudes of $\underline{k}$ will be suggested by physical arguments, or that some physical limit, such as $\Omega \to \infty$ or pressure transverse to $B_o \to 0$ can be taken, and that the equations will then become amenable to analysis by one of the condensed methods just described.

IV. EXAMPLES

Counterstreaming Electrons and Ions

If magnetic fields are neglected (see p.11), instability in a hot plasma carrying a current may be investigated via the Vlasov dispersion relation. If the plasma is first in thermal equilibrium and then an external uniform electric field is turned on and off, a relative drift velocity $\underline{v}_1$ will be established between the electrons and ions. The electrons and ions can then be described by displaced Maxwellian distributions at the same temperature T, and the combined effective velocity distribution in a frame moving with the ions is

$$f_o(\underline{v}) = (m/2\pi KT)^{3/2} \exp[-m(\underline{v} - \underline{v}_1)^2/2KT] + (m/M)(M/2\pi KT)^{3/2} \exp[-M\underline{v}^2/2KT] \quad .$$

The direction for $\underline{k}$ most likely to yield growing waves is along $\underline{v}_1$. Integration over $\underline{v}$ and differentiation gives

$$f_o'(v) - \left(\frac{m}{2\pi KT}\right)^{1/2} \left\{ \frac{m(v - v_1)}{KT} \exp\left[- \frac{m(v - v_1)^2}{2KT} \right] + \left[\frac{m}{M}\right]^{1/2} \frac{Mv}{KT} \exp\left[- \frac{Mv^2}{2KT} \right] \right\} .$$

The minimum $\overline{v}$ of f_o is where the arguments of the exponentials are equal, that is, where

$$(v_1 - \overline{v})(m/KT)^{1/2} = \overline{v}(M/KT)^{1/2} \equiv \xi \qquad (52a)$$

In terms of ξ

$$U(\overline{v}) = -2(m/KT)\, h(\xi) \qquad (52b)$$

where

$$h(\xi) = \frac{1}{\sqrt{2\pi}} \; P \int_{-\infty}^{\infty} \frac{x \exp\left(-\frac{1}{2}x^2\right)}{x - \xi} \, dx$$

$$= 1 - \xi \exp\left(-\frac{1}{2}\xi^2\right) \int_0^{\xi} e^{\frac{1}{2}t^2} \, dt \qquad (53)$$

The function $\int_0^{\xi} e^{\frac{1}{2}t^2} dt$ is tabulated in Jahnke and Emde (41), and $h(\xi\sqrt{2})$ is tabulated in Unsöld (42). More accurate tables, for complex argument, are available in Russian (43). The relations

$$\left.\frac{\partial U}{\partial v}\right|_{v=\bar{v}} = (m/KT)^{3/2} \left[1 + (M/m)^{1/2}\right] h'(\xi) \qquad (54)$$

and

$$h'(\xi) = -\xi - \xi^{-1}(1 - \xi^2)\left[1 - h(\xi)\right] \qquad (55)$$

are of interest if one wishes to study the properties of growing waves near the threshold of instability by the use of equation 44. From the tables, one finds that $h(\xi)$ has only one sign change, from positive to negative, at $\xi = 1.32$. Thus equation 52b shows the plasma is stable only if ξ is less than 1.32, which implies, through equation 52a, $v_1 < 1.35(KT/m)^{1/2}$. This corresponds to an electron translational energy in the center of mass frame of .87 KT, recovering the results of Jackson (44) and Buneman (24). As the relative velocity is increased beyond the threshold of instability, the growing waves occur first at very long wavelengths, since $U(\bar{v})$ is small. This suggests that it may be difficult to achieve Buneman's initial conditions where the relative velocity greatly exceeds the thermal velocity, so that the hydromagnetic approximation may be used to get the wavelength L of the most rapidly amplified waves, but there are as yet no waves larger than thermal fluctuations.

Colliding Plasmas and Shock Fronts

The shock model of Parker and of Kahn mentioned in the introduction will now be discussed more fully. Suppose that two identical uniform bodies of plasma, with mean velocities $\pm \underline{v}_1$, collide. It is reasonable to assume that each has had time beforehand to reach thermal equilibrium, but that collisions may be neglected for some time after they meet. When the region of interpenetration becomes large enough, the double stream instability may disrupt the motion there. It would be interesting to know the temperature range in which instability exists, and the particles (electrons or ions) that participate the most in the oscillations. If the value of n_o used to determine ω_e^2 is the total number of electrons per unit volume (from both plasmas), the appropriate combined electron and ion distribution in the region of interpenetration is

$$f_o(\underline{v}) = \frac{1}{2}\,(m/2\pi KT)^{3/2}\left\{\exp\left[-m\,\underline{v}^2/2\,KT\right] + \right.$$
$$\left. + \exp\left[-m(\underline{v}-\underline{v}_1)^2/2KT\right]\right\} +$$
$$+ (m/M)(M/2\pi KT)^{3/2}\left\{\exp\left[-M\underline{v}^2/2KT\right] + \right.$$
$$\left. + \exp\left[-M(\underline{v}-\underline{v}_1)^2/2KT\right]\right\}$$

Again taking $\underline{k}$ along $\underline{v}_1$, integrating over $\underline{v}_\perp$, and differentiating, one obtains

$$f_o'(v) = \frac{1}{2}\,(m/2\pi KT)^{1/2}\left[S(v) + S(v - v_1)\right]$$

where

$$S(v) = -(vm/KT)\ \exp(-mv^2/2KT) + (m/M)^{1/2}(vM/KT)\ \exp(-Mv^2/2KT)$$

The minimum (if any) of $f_o(v)$ must be at $v = \frac{1}{2} v_1$, where

$$U = -(m/KT)\ \left[h(\tfrac{1}{2}\xi) + h(\tfrac{1}{2}\xi\{M/m\}^{1/2})\right] \quad . \tag{56}$$

Again ξ means $v_1(m/KT)^{1/2}$ and h is defined in equation 53. The only zero of U as given by equation 56 is at $\xi = 2.64$, where the second (ion) term is negligible. This supports the view that the electrons come to equilibrium first, followed more slowly by the ions (19,22). For $v_1 < 2.64(KT/m)^{1/2}$ the plasma is stable and collisions are the principal thermalizing process. The transition to the unstable case $v_1 > 2.64(KT/m)^{1/2}$ where plasma oscillations are important is smooth, however, for the quantity $U(\overline{v})$ in equation 44 increases smoothly with increasing v_1, implying very small rates of growth at the threshold of instability.

An experiment has been performed in an attempt to verify Kahn's and Parker's predictions on the formation of shock fronts when two plasmas collide. Two identical beams of deuterium plasma each containing about 10^{12} particles/cc were directed at each other along a uniform magnetic field. The relative velocity of translation was 6×10^7 cm/sec, the ion temperature approximately 45 e.v., and the electron temperature was known to be somewhat higher. These conditions correspond, however, to a single peaked combined velocity distribution $\tilde{f}_o(v)$, and indeed no shocks or other strong interactions of the beams were observed. The electron thermal velocity $(KT/m)^{1/2}$ was at least

$2x10^8$ cm/sec , ten times the allowable value for instability to exist when $v_1 = 6x10^7$ cm/sec.

Later Phases of Shock Front Formation

Equations 12 may be solved for f_1 and F_1, the perturbations of the electron and ion distribution functions associated with a growing wave. It is possible to verify from the form of these functions that when the phase velocity u of a growing wave is near $\bar{v}$, the minimum of f_o , the ions are perturbed very little, as far as the linearized theory can determine. Certainly u is near $\bar{v}$, since U(u) can be positive only on the V = 0 line from $\bar{v}$ to the neutral point shown in Fig. 2. If v_1 appreciably exceeds the threshold value, electron oscillations triggered by inhomogeneities and fluctuations will grow large in a few electron plasma periods and bring the electron streams to a halt, while the ion streams remain relatively unaffected. This gives the Kahn-Parker shock model described previously (22,23). Kahn, however, assumed that the electrons and both ion streams were perfectly cold, while Parker allowed only the electrons to be hot. Unless the ions also are allowed to have non-zero temperature, there is always instability. Since Buneman (24) has shown that similar kinds of oscillations ought to produce a disordered state resembling thermal motion, it is not unreasonable to assume that the electrons convert their kinetic energy into heat, while the ions, unaffected as yet because of their greater mass, still constitute two streams of mean velocity $\pm v_1$, with some small initial temperature T_i . It will be shown that if the ion thermal energy exceeds $4.2x10^{-4}$ times the ion translational kinetic energy, there is no instability. This condition for the ion instability is much more stringent than the one for the original electron-electron two stream instability. The possible wavelengths for growing oscillations turn out to be bounded both below and above. This simplifies the estimation of a hydromagnetic shock thickness, but the low ion temperature required for instability to exist weakens the position of plasma oscillations in explaining shock fronts and "suprathermal electrons". (23)

If the initial ion and electron temperatures are both small, the mean electron thermal energy $\frac{3}{2} KT_e$ after the electrons have come to equilibrium should equal $\frac{1}{2} mv_1^2$. In the usual notation, the initial velocity distribution integrated over $v_\perp$ is

$$f_o(v) = 2\left(\frac{3}{2\pi v_1^2}\right)^{1/2} \exp\left(\frac{-3v^2}{2v_1^2}\right) + \left(\frac{m}{M}\right)\left(\frac{M}{2\pi KT_i}\right)^{1/2} \exp\left(\frac{-M(v-v_1)^2}{2KT_i}\right) + $$

$$+ \left(\frac{m}{M}\right)\left(\frac{M}{2\pi KT_i}\right)^{1/2} \exp\left(\frac{-M(v+v_1)^2}{2KT_i}\right) \qquad (57)$$

For sufficiently small T_i this function has two minima, say at $v = \pm v_o$, $v_o \lesssim v_1$. Growing waves can exist with phase velocities u near $\pm v_o$ if

$$U(u) = \mathrm{Re} \int_{-\infty}^{\infty} \frac{f_o'(v)\,dv}{v-u}$$

is positive at $u = \pm v_o$, otherwise not. Such waves are clearly recognizable as <u>electron-ion</u> waves since their phase velocities lie near the minima of $f_o(v)$ in the u-plane and these minima fall between electron and ion peaks. Moreover, one ion term in f_o is extremely small near the farther minimum. Modifying one ion peak, e.g., so as to destroy the instability there, would not appreciably affect the state of affairs near the other minimum. This means that any instability is an electron-ion interaction. It also suggests an obvious generalization of the results presented here to cases where the ion streams have different (small) initial temperatures, and suggests halving the computational work by considering, say, only the neighborhood of $+v_o$, dropping the last term of equation 57. It is easily verified, at the end of the calculation, that this approximation is justified.

The calculations are simplest using the dimensionless quantities

$$\alpha = (KT_i/M)^{1/2}\, v_1^{-1}$$

$$\zeta = \xi + i\eta = u/v_1 \quad , \quad \xi = v/v_1 \quad , \quad \xi_o = v_o/v_1 \quad ,$$

$$\bar{f}_o(\xi) = v_1 f_o(v_1 \xi) \quad , \text{ and}$$

$$\overline{W}(\zeta) = \overline{U} + i\overline{V} = \int_{-\infty}^{\infty} \frac{\bar{f}_o'(x)\,dx}{x - \zeta} = v_1^2\, W(v_1\zeta)$$

where W is the usual analogue electrostatic potential.

In terms of these variables the dispersion relation for growing waves of frequency $\omega_e\zeta$ and wave number k is $\overline{W}(\zeta) = k^2 v_1^2/\omega_e^2$, and $\bar{f}_o$ takes the form

$$\bar{f}_o(\xi) = 2\left(\frac{3}{2\pi}\right)^{1/2} \exp\left(-\frac{3\xi^2}{2}\right) + \frac{m}{M}\frac{1}{\alpha}\left(\frac{1}{2\pi}\right)^{1/2} \exp\left(-\frac{(\xi - 1)^2}{2\alpha^2}\right) \quad .$$

Thus $(M/m)(2\pi)^{1/2}\, \bar{f}_o'(\xi) = (1-\xi)\alpha^{-3} \exp\left(-\frac{1}{2}[1-\xi]^2\alpha^{-2}\right) -$

$$- 19080.3\, \xi \exp(-1.5\, \xi^2)$$

if (M/m) is taken as 1836.00 . The function $\overline{f}_o$ has two peaks if and only if $\alpha < \alpha_{cr}$, where $\alpha_{cr} = .011079$. When $\alpha = \alpha_{cr}$, the ion maximum ξ_i merges with the minimum ξ_o to form a horizontal point of inflection $(\overline{f}'_o = \overline{f}''_o = 0)$, at a value of ξ designated by $\xi = \xi_{cr} = .98182$. For $\alpha < \alpha_{cr}$ the ion maximum of $\overline{f}_o$ will be designated as ξ_i and the electron one as ξ_e .

It might be expected that if α were increased from a very small value up to the critical value α_{cr}, the plasma would first become stable [$\overline{U}(\xi_o) < 0$] and then the second peak would disappear $(\alpha \geq \alpha_{cr})$. Instead, careful investigation shows that $\overline{U}(\xi_o)$ is positive even when α is very nearly α_{cr} , and indeed $\overline{U}(\xi_{cr})$ is positive, being numerically equal to .2131 . This implies, by continuity, that $\overline{U}(\xi_o)$ is positive at least in some neighborhood of α_{cr}, say $\alpha_1 \leq \alpha \leq \alpha_{cr}$, so that instability persists as long as $f_o(v)$ has two peaks. The exact structure of the $\overline{V} = 0$ lines in the $\overline{W}$ plane for $\alpha \lessapprox \alpha_{cr}$ can be deduced from the charge analogy. The $\overline{V} = 0$ line from $+i\infty$ "starts" with $\overline{U} = 0$ at $+i\infty$, and, from the asymptotic form $W \sim 1/u^2$, $\overline{U}$ must decrease along it as it comes down toward the central part of the plane. By continuity, $\overline{U}$ is positive at both ξ_o and ξ_i for α just a little less than α_{cr}; therefore, the $\overline{V} = 0$ line from $i\infty$ cannot end at ξ_o or ξ_i and cannot cross any $\overline{V} = 0$ line connected to them. Thus the $\overline{V} = 0$ line from $+i\infty$ must simply curve down and end at ξ_e, and another $\overline{V} = 0$ line must run from ξ_o to ξ_i , as shown in Fig. 5.

The structure which has been established for the $\overline{V} = 0$ line connecting ξ_o and ξ_i has important consequences. When α is close

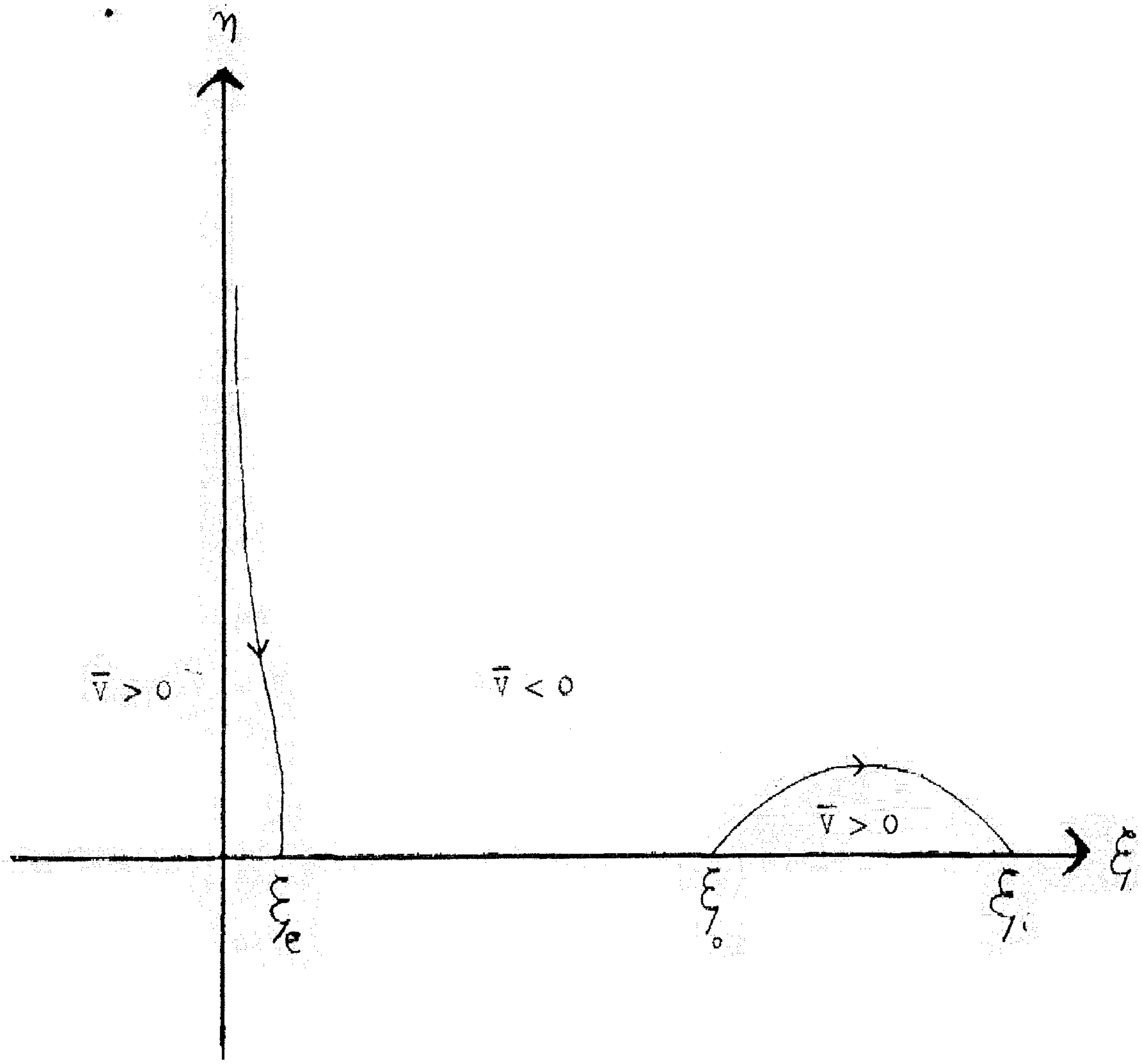


Fig. 5. Structure of the $\bar{V} = 0$ Lines in the ζ Plane for $\alpha \lesssim \alpha_{cr}$.

to α_{cr}, $\overline{U}$ is positive at both ξ_o and ξ_i . This means $\overline{U}$ is positive all along this $\overline{V} = 0$ line, or that any point on the line represents the phase velocity of a wave which can grow in the plasma. The relation $k = \omega_e (U)^{1/2} = \omega_e v_1^{-1} (\overline{U})^{1/2}$ shows then, that self-excited waves occur for wave numbers k in the range $\omega_e v_1^{-1} [\overline{U}(\xi_i)]^{1/2} \leq k \leq \omega_e v_1^{-1} [\overline{U}(\xi_o)]^{1/2}$, rather than the more usual case where k is bounded only above. For α close to α_{cr} , this condition restricts k to be approximately $\omega_e v_1^{-1} [\overline{U}(\xi_{cr})]^{1/2} = .4616 \omega_e / v_1$. This gives $\lambda = 2\pi/k = 13.6 \; v_1/\omega_e$. Since ω_e is here the electron plasma frequency $4\pi n_o e^2/m$ for one stream, this suggests a shock thickness on a scale of length $10 v_1/\omega_e$, rather less than Parker's value, v_1/ω_i .

The calculation above was for waves propagating along $\underline{v}_1$. Other directions of propagation give different $f_o(v)$, but certainly near threshold, at least, the fastest growing waves must be along $\underline{v}_1$. It can be shown that the sole effect of choosing $\underline{k}$ not along $\underline{v}_1$ is to replace v_1 in the last two terms of equation 57 by $(\underline{v}_1 \cdot \underline{k})/k$. The effect of this replacement is clearly to reduce $\overline{U}$ at the minimum of f_o or eliminate the minimum completely, which means more slowly growing waves or none at all along $\underline{k}$. It seems that these other waves will not substantially change the conclusions about shock thickness, and they certainly do not relax the stringent requirement of small T_i for instability.

As a numerical example, consider Parker's case of solar corpuscular radiation moving at 500 $km.sec^{-1}$ impinging on the outer geomagnetically trapped plasma. Parker suggests a temperature of 10^4 °K (19), or 10^5 °K

(23) which gives $\alpha = .018$ or $\alpha = .057$. These values are respectively twice and six times α_{cr}, so that instability of the ions would be absent once the electrons had thermalized. For obtaining "suprathermal" electrons, whose mean energy is close to the original ion translational energy, the ion instability is essential.

When α is indeed less than α_{cr} so that electron-ion oscillations grow, it is possible to verify Parker's assertion that the electrons receive a good deal of the ion translational energy in the form of oscillatory energy. The changes in mean electron velocity and mean ion velocity may be found from

$$\Delta v_e = v_1 \left[\int \xi \, f_1 \, d\xi \right] \left[\int f_1 d\xi \right]^{-1}$$

and

$$\Delta v_i = v_1 \left[\int (1 - \xi) F_1 d\xi \right] \left[\int F_1 \, d\xi \right]^{-1}$$

where $f_1(\xi)$ and $F_1(\xi)$ are the perturbations of f_o and F_1 expressed in the dimensionless units. If ζ is fairly near ξ_o, it may be shown that $\Delta v_e / \Delta v_i \approx 85$, or $m(\Delta v_e)^2 \approx 4M(\Delta v_i)^2$. Thus, within the domain of linearized theory, the electrons receive about four times as much oscillatory energy as do the ions.

Concerning the Debye Length

It was found in part III that in certain kinds of plasmas, the theory based on Vlasov's equations permits growing oscillations of all wavelengths. The theory is certainly not valid when the wavelength approaches an interparticle spacing, but I feel that it is not wholly correct to simply cut off the oscillations when $\lambda < \lambda_D$ on the basis of Landau damping (32). Landau considered a nearly Maxwellian plasma; furthermore a Debye length is not even defined for other plasmas. When several streams of particles are present, it is particularly imprudent to assume those of one stream can shield those of another. For example, suppose T_i in the preceding example is very small, but the electrons are heated to several times their original temperature. One might conjecture that the electrons can no longer shield the ion streams from each other, since $\lambda_{D_{electron}}$ is greater than the wavelength v_1/ω_i of preferentially growing ion-ion oscillations. Then ion-ion oscillations would grow. Any instabilities, however, must be near the electron-ion minima of $f_o(v)$ in the u plane, and hence are clearly electron-ion interactions. For the electron peak to become so broad and low that the minima are near the origin and the picture looks like an ion-ion two stream instability, it is necessary that the ratio of electron to ion thermal velocities

$$u_e/u_i = (KT_e/m)^{1/2} \Big/ (KT_i/M)^{1/2}$$

is nearly

$$u_e/u_i = (M/m)^{1/3} \left[(v_1^2/u_i^2) - 1 \right]^{-1/3} \exp(v_1^2/6u_i^2)$$

If, for example, $\alpha = .01$, this means $T_e/T_i > e^{3000}$, which is surely unattainable. This uncommonly large number results from the rapid falling off of the ion distribution function $F_o(v)$ away from the ion peaks at $v = \pm v_1$. Higher ion temperatures result in more modest figures.

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