Effects of Actuator Limits in Bifurcation Control with Applications to Active Control of Fluid Instabilities in Turbomachinery

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Abstract

Feedback stabilization is one of the most dominant issues in modern control theory. The validity of linear control design is based on the assumption that the system is stabilizable. With rapid broadening of control applications to complex systems during the past two decades, the attainability of linear stabilizability sometimes has to compromise with system constraints and affordability of distributed actuation. The goal of this thesis is to tackle some of the problems in control of nonequilibrium behavior and to apply the theory to active control of fluid instabilities in gas turbine engines.

We consider two of the simplest nontrivial scenarios in local smooth feedback stabilization: the steady-state case, when the linearly unstabilizable eigenvalue is zero; and the Hopf case, when the unstabilizable eigenvalues are a pair of pure imaginary numbers. Under certain nondegeneracy conditions, we give explicit algebraic conditions for stabilizability. And when the system is stabilizable, the stabilizing feedback can be explicitly constructed.

The problem of local smooth feedback stabilization for systems with critical unstabilizable modes is closely related to bifurcation control. Under certain nondegeneracy conditions, a steady-state/Hopf bifurcation can be turned into a supercritical pitchfork/Hopf bifurcation if and only if the system is locally stabilizable at the bifurcation point. Algebraic necessary and sufficient conditions are derived under which the criticality of a simple steady-state or Hopf bifurcation can be changed to supercritical by a smooth feedback. The effects of magnitude saturation, bandwidth, and rate limits are important issues in control engineering. We give qualitative estimates of the region of attraction to the stabilized bifurcating equilibrium/periodic orbits under these constraints.

We apply the above theoretical results to the Moore-Greitzer model in active control of rotating stall and surge in gas turbine engines. Though linear stabiliz-

ability can be achieved using distributed actuation, it limits the practical usefulness due to considerations of affordability and reliability. On the other hand, simple but practically promising actuation schemes such as outlet bleed valves, a couple of air injectors, and magnetic bearings will make the system loss of linear stabilizability, thus the control design becomes a challenging task. The above mentioned results in bifurcation stabilization can be applied to these cases. We analyze the effects of magnitude and rate saturations in active stall and surge control using bleed valves and magnetic bearings using the Moore-Greitzer model. The analytical formulas for bleed valve actuation give good qualitative predictions when compared with experiments. Our conclusion is that these constraints are serious limiting factors in stall control and must be addressed in practical implementation to the aircraft engines.

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Nomenclature

Symbols in Chapter 2, 3 and 4:

A	matrix of linearization for controllable (stabilizable) state
$\bar{A}$	matrix of linearization: full system
B	matrix of linearization from control input to stabilizable state
$\bar{B}$	matrix of linearization from control input to full state
c_{ijk}	cubic terms: unstabilizable subsystem
$\bar{c}$	cubic terms: full system
C	coefficient of the cubic term in the dynamics on the center manifold
C_0	constant term in $P_H(K, K^*)$
$\mathbb{C}$	the set of complex numbers
d	sensitivity of the (real part) of the critical eigenvalue: $d = \frac{d\,\mathrm{Re}\,\lambda}{d\mu}(0)$
D_i	$(i = 1, 2)$ coefficients of linear terms in $P_H(K, K^*)$
E_{ij}	$(i, j = 1, 2)$, coefficients of quadratic terms in $P_H(K, K^*)$
F_{ijk}	$(i, j, k) = (1, 1, 2)$ or $(i, j, k) = (1, 2, 2)$, coefficients of cubic terms in $P_H(K, K^*)$
$G(K, K^*)$	$G(K, K^*) = \mathrm{Re}\,\tilde{G}(K, K^*)$
$\tilde{G}(K, K^*)$	$\tilde{G}(K, K^*) = H(K, K^*) + \tilde{P}_H(K, K^*)$
$H(K, K^*)$	a function defined in (2.112) and (3.78)
I_j	$j = 1, \cdots, 4$, index sets in multi-input steady-state and Hopf cases
I_{jr}	$j = 1, \cdots, 4$, index sets in the multi-input steady-state case
K	steady-state case: $K = \left[I + K_1 A^{-1} B\right]^{-1} K_2$ Hopf case: $K = \left[I - K_1(i\omega - A)^{-1} B\right]^{-1} K_2 := K_R + iK_I$
$K_\flat$	feedback gain in the Hopf case $K_\flat := \begin{bmatrix} K_R \\ K_I \end{bmatrix}$
K_i	$(i = 1, \cdots, 9)$, feedback gains
l	left eigenvector of the critical mode $i\omega$
$l_\perp$	real orthogonal compliment of l

$\mathfrak{m}(s)$ $\mathfrak{m}(s) = \begin{bmatrix} s^{n-1} & \cdots & s & 1 \end{bmatrix}^T, s \in \mathbb{C}$

$\mathfrak{m}_j(s)$ $\mathfrak{m}_j(s) = \begin{bmatrix} s^{j-1} & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}^T, j = 1, \cdots, n.$

$p(s)$ characteristic polynomial: $p(s) = \det[sI - A]$

$\bar{p}(s)$ characteristic polynomial: $\bar{p}(s) = \det[sI - (A + BK_1)]$

P_S steady-state case: $P_S(K) = \alpha_0 + \alpha_1 K + K^T \alpha_2 K + \alpha_3(K, K, K)$

P_H Hopf case: $P_H(K_\flat) := \operatorname{Re} \tilde{P}_H(K, K^*)$

$= \alpha_0 + \alpha_1 K_\flat + K_\flat^T \alpha_2 K_\flat + \alpha_3(K_\flat, K_\flat, K_\flat)$

$\tilde{P}_H$ Hopf case: $\tilde{P}_H(K, K^*) = C_0 + D_1 K + D_2 K^* + E_{11} K^2 + E_{12}|K|^2$

$+ E_{22} K^{*2} + F_{112}|K|^2 K + F_{122}|K|^2 K^*$

q_{ij} $(i, j = 1, 2)$, quadratic terms: unstabilizable subsystem

$\bar{q}_{ij}$ $(i, j = 1, 2)$, quadratic terms: full system

$\hat{q}_{11}$ multi-input steady-state case: $\hat{q}_{11} = q_{11} - \frac{1}{4}\Upsilon_1 \Upsilon_2^{-1} \Upsilon_1^T$ when

Υ_2 is (semi)-definite and $\operatorname{rank} \begin{bmatrix} \Upsilon_1^T & \Upsilon_2 \end{bmatrix} = \operatorname{rank} \Upsilon_2$

$\tilde{q}_{ij}$ quadratic terms: stabilizable subsystem

Q coefficient of the quadratic term in the dynamics on the center manifold: steady-state case

Q_i $(i = 1, 2, 3)$, coefficient of the quadratic terms in the dynamics on the center manifold: Hopf case

r right eigenvector of the critical mode $i\omega$

$r_\perp$ real orthogonal compliment of r

$\mathbb{R}$ the set of real numbers

s a complex number, $s \in \mathbb{C}$

T orthonormal transformation in Chapter 2 and 3

u control input

u_{mag} controller magnitude limit

U multi-input steady-state case: an $m \times m$ orthonormal matrix such that $\hat{\Upsilon}_2 := U^T \Upsilon_2 U$ is diagonal

multi-input Hopf case: an $2m \times 2m$ orthonormal matrix such that $\hat{\alpha}_2 := U^T \alpha_2 U$ is diagonal

V	multi-input steady-state case: an $l \times l$ orthonormal matrix such that $\hat{\alpha}_{2r} := V^T \alpha_{2r} V$ is diagonal
W_i	$(i = 1, 2, 3)$, Hopf case: different parts of $\tilde{\alpha}$: $\tilde{\alpha} = W_1 + W_2 + W_3 - \frac{Q_1 Q_2}{i\omega}$
x	unstabilizable state: steady-state case
$\tilde{x}$	stabilizable state
y	full state in both cases
z	unstabilizable state: Hopf case
α	Lyapunov coefficient, $\alpha = \operatorname{Re} \tilde{\alpha}$
α_i	constant i^{th} $(i = 0, \cdots, 3)$ order tensors in the expression of α or C
α_{ir}	constant i^{th} $(i = 0, \cdots, 3)$ order tensors in the expression of C in steady-state case
$\tilde{\alpha}$	coefficient of the cubic term in the normal form: Hopf case
$\hat{\alpha}_0$	Hopf case: $\hat{\alpha}_0 := \alpha_0 - \frac{1}{4}\alpha_1 \alpha_2^{-1} \alpha_1^T$ when α_2 is positive (semi)definite and rank $\left[\alpha_1^T \;\; \alpha_2\right] = \operatorname{rank} \alpha_2$
$\hat{\alpha}_{0r}$	multi-input steady-state case: $\hat{\alpha}_{0r} := \alpha_{0r} - \frac{1}{4}\alpha_{1r} \alpha_{2r}^{-1} \alpha_{1r}^T$ when α_{2r} is positive (semi)definite and rank $\left[\alpha_{1r}^T \;\; \alpha_{2r}\right] = \operatorname{rank} \alpha_{2r}$
β_i	$(i = 1, \cdots, 5)$, coefficients in Taylor expansion of center manifold
Δ_α	a constant in the single-input steady-state case, $\Delta_\alpha = \alpha_1^2 - 4\alpha_2\alpha_0$
Δ_Υ	a constant in the single-input steady-state case, $\Delta_\Upsilon = \Upsilon_1^2 - 4q_{11}\Upsilon_2$
ϵ	a small real number
ζ	coordinate of the normal form of the dynamics on the center manifold
Θ_i	$(i = 1, \cdots, 4)$, constants determining stabilizability in the Hopf case
$\lambda(\mu)$	critical unstabilizable eigenvalue: $\operatorname{Re} \lambda(0) = 0$
μ	bifurcation parameter
Π_i^H	$(i = 1, \cdots, 4)$, constant vectors in the single-input Hopf case
Π_i^S	$(i = 1, 2)$, constant vectors in the single-input steady-state case
ρ	a constant in the single-input steady-state case, $\rho = -\frac{\Upsilon_1}{2\Upsilon_2}$
$\Sigma(s_1, s_2)$	matrix in Corollary 3.2 whose rank determines stabilizability
Υ_1, Υ_2	Constants in the expression of Q for the closed loop system
$\Phi_i(s)$	$(i = 1, 2)$, functions used in the Hopf case

$\tilde{\Phi}_i(s)$	$(i = 1, 2)$, functions used in the Hopf case
$\Psi(s_1, s_2)$	a function used in the Hopf case
$\tilde{\Psi}(s_1, s_2)$	a function used in the Hopf case
ω	frequency of the Hopf bifurcating mode
$\hat{}$	tensors expressions after orthonormal transformation
$\tilde{}$	coefficients of controllable (stabilizable) state
$*$	denotes complex conjugate

Symbols in Chapter 5 and 6:

a_1	$a_1(\xi) = Ae^{i\vartheta_1(\xi)}$, first Fourier mode of stall
a_2	$a_2(\xi) = A_2e^{i\vartheta_2(\xi)}$, second Fourier mode of stall
A, A_1	amplitude of first Fourier mode of stall
B	Greitzer B-parameter for surge
c_0	sensitivity of pressure rise to tip clearance at the peak of compressor characteristic: $c_0 = -\frac{\partial\psi_c}{\partial\epsilon}(\Phi_0, 0)$
$\mathcal{D}_b$	region in u-J plane constrained by actuator bandwidth
$\mathcal{D}_m$	region in u-J plane constrained by actuator magnitude saturation
$\mathcal{D}_r$	region in u-J plane constrained by actuator rate limits
e	noise (initial conditions of J)
$\mathcal{E}$	noise set (set of initial conditions)
$\mathrm{erfc}(x)$	complementary error function
J	$J = A^2$
J_0, J_2	energy of surge mode and second stall mode in the high B case
K, K_j	$(j = 0, \cdots, 4)$, feedback gains
l_c, m, μ	compressor parameters
u	control input
u_{mag}	actuator magnitude limit
u_{rate}	actuator rate limit
$W^s(P)$	stable manifold of the saddle point P
$\tilde{\alpha}$	cubic term in the normal form

α_i	$(i = 1, 2, \cdots)$, coefficients of the reduced system for the low B case
α_{ij}	$(i, j = 1, 2)$, coefficients of the nonlinear terms in the normal form for high B case
β_{ij}	coefficient in the Taylor series expansion of the center manifold
γ	throttle coefficient
$\Gamma(\alpha, x)$	incomplete Γ-function in Lemma 5.1
δ	deviation of throttle coefficient from the peak: $\delta = \gamma - \gamma_0$
$\hat{\delta}$	throttle position at which Hopf bifurcation to surge occurs
Δ	operability enhancement defined in Section 6.2.3
ϵ	noise level (maximum initial condition of J)
$\epsilon(\theta)$	tip clearance
θ	circumferential coordinate in compression systems
ν_i	$(i = 1, 2)$, coefficients of the linear terms in the normal form of high B case
ξ	nondimensional time
ξ_1	time delay caused by sensing and actuation
σ, η, λ	nondimensional parameters in Section 6.3.3
τ	time constant
ϕ, ψ	$\phi = \Phi - \Phi_0$, $\psi = \Psi - \Psi_0$
Φ	nondimensional mean axial velocity
Φ_0, Ψ_0, γ_0	quantities evaluated at the peak of the compressor characteristic
$\Phi_T(\Psi)$	throttle characteristic
$\psi_c(\cdot)$	compressor characteristic
ψ_c'', ψ_c'''	second and third derivatives of compressor characteristic at the peak
Ψ	nondimensional pressure rise coefficient
ω_1, ω_2	rotating frequency of first and second stall mode

Chapter 1 Introduction

1.1 Background

1.1.1 A motivating example

Control theory has been applied to increasingly complex engineering systems during the last two decades, partly due to the rapid development of cheap and powerful computational hardware and software in information technology. Another reason is that some engineering systems have been highly optimized in design over many years so that higher performance is harder to achieve without using feedback mechanisms. For instance, aircraft engines are very complex and high performance engineering systems that have been developed through many decades, yet there are many limitations on performance (see Figure 1.1). Rotating stall and surge are aerodynamics instabilities that occur near the operating conditions of maximum engine power output. Rotating stall is a nonaxisymmetric flow pattern in the fan or the compressor stage rotating at a fraction of the rotor speed, while surge is an axial oscillation of flow and pressure throughout the engine. The inception of rotating stall and surge is typically abrupt with a hysteresis and fully developed instabilities can damage the engine. Once rotating stall and surge develop, the engine cannot recover from the instabilities and engine shutdown and restart are necessary. Due to the detrimental effect of rotating stall and surge, there is a safety margin between the designed operation points and the stall/surge line.

Active control implemented on both laboratory compressor rigs and full scale engines in the past decade has demonstrated as a possible technology to mitigate the instabilities and enhance the performance by enlarging the safety margin so that compressors can be redesigned to operate closer to the peak of pressure rise. Due to the linear decoupling of nonaxisymmetric and axisymmetric flow disturbances, actuation

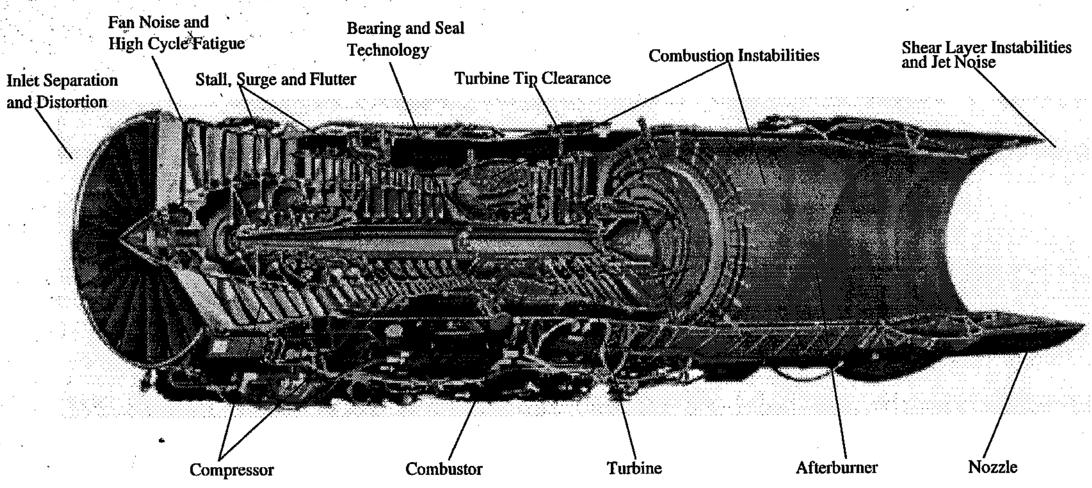


Figure 1.1: Performance limitations in aircraft engines (Pratt & Whitney F100 Engine).

that only changes the axisymmetric flow and pressure will make the nonaxisymmetric flow disturbances linearly uncontrollable. The tricky part of using such actuators is to change the bifurcation characteristic near the stability boundary from subcritical to supercritical, whereby the hysteresis associated with rotating stall and surge inception is eliminated and the amplitude of the limit cycles are reduced. An example of this type of actuation is a bleed valve downstream of the compressor.

To stabilize the steady axisymmetric flow against nonaxisymmetric flow disturbances, many actuators are needed, distributed around the annulus of the compressor. An example of this type of actuation is an array of air injectors upstream of compressor stages. Clearly simple actuation schemes such as a bleed valve have advantage over distributed actuation schemes such as an array of air injectors in terms of cost, reliability, and complexity. But the design of control laws for simple actuation is challenging since nonlinear couplings between the axisymmetric flow disturbance and nonaxisymmetric flow disturbances have to be exploited in order to change the bifurcation characteristic. Moreover, it is even more challenging to analyze the effects of bandwidth and magnitude and rate saturations of such actuation in that linear analysis is not applicable.

The goal of this thesis is to develop a framework of designing controllers that change the bifurcation characteristic near the stability boundary for a general class of finite dimensional nonlinear systems whose bifurcating modes are unstabilizable. Furthermore, a framework for analyzing the effects of actuator limits in control of bifurcations is established. The theory is then applied to nonlinear design and analysis of active control of rotating stall and surge in gas turbine engines.

1.1.2 Preliminaries on feedback stabilization

Many systems in engineering applications can be modeled as ordinary differential equations (ODEs). Often it is necessary to understand the dynamics near a specific solution. Consider the following autonomous ODE:

$$\dot{x} = f(x), \quad f(0) = 0, \quad x \in \mathbb{R}^n. \tag{1.1}$$

For any initial condition $x_0 \in \mathbb{R}^n$, we call $x(x_0, t)$ a solution of (1.1) if it satisfies the equation with $x(x_0, 0) = x_0$. We call $x = 0$ an equilibrium. The equilibrium $x = 0$ is locally asymptotically stable in the Lyapunov sense if for any $\epsilon > 0$, there exists a $\delta > 0$, such that for any $x_0 \in \mathbb{R}^n$ satisfying $\|x_0\| < \delta$, we have $\|x(x_0, t)\| < \epsilon$ for any $t \geq 0$, and $\lim_{t\to\infty} x(x_0, t) = 0$. Specifically, we say the equilibrium $x = 0$ is globally asymptotically stable if $\lim_{t\to\infty} x(x_0, t) = 0$ for every $x_0 \in \mathbb{R}^n$. In this thesis we only consider local stability.

The asymptotic stability of (1.1) is given by the Second Theorem of Lyapunov: the equilibrium $x = 0$ is asymptotically stable if and only if there exists a positive definite Lyapunov function V: $\mathbb{R}^n \longrightarrow \mathbb{R}$, $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$, such that

$$\dot{V} := \frac{\partial V}{\partial x} \cdot f(x) := \mathcal{L}_f V < 0. \tag{1.2}$$

where $\mathcal{L}_f V$ denotes the Lie derivative of $V(x)$ along the vector field $f(x)$. There are no general analytical solutions for the partial differential inequality (1.2), that is, there is no unified analytical procedure to generate a Lyapunov function for a nonlinear system. In many cases, the local stability can be determined by the linearization around the equilibrium. The First Theorem of Lyapunov says that if all the eigenvalues of $Df(0)$ have negative real parts, then the system (1.1) is locally asymptotically stable; if $Df(0)$ has an eigenvalue whose real part is positive, then the system (1.1) is locally unstable.

A more complicated situation occurs when some eigenvalues of $Df(0)$ have zero real parts and all other eigenvalues have negative real parts. In this case, general algebraic criteria of stability are not available. Nevertheless, there are two simple cases that are algebraically solvable. In the first case, there is a simple zero eigenvalue and all other eigenvalues have negative real parts. Let l and r be the left and right eigenvectors of $Df(0)$ corresponding to the eigenvalue 0, i.e., $lDf(0) = 0$, $Df(0)r = 0$. Define

$$\begin{aligned} Q &= lD^2f(0)(r,r), && (1.3)\\ C &= \frac{1}{3}lD^3f(0)(r,r,r) + \\ & \quad lD^2f(0)\left(-\left(Df(0)^TDf(0)+l^Tl\right)^{-1}Df(0)^TD^2f(0)(r,r),r\right). && (1.4)\end{aligned}$$

Then the system is asymptotically stable if $Q = 0$ and $C < 0$ (see [42]); the system is unstable if either $Q \neq 0$, or $Q = 0$, but $C > 0$. We call the case when $Q = C = 0$ degenerate, since the stability has to be determined by 4^{th} and 5^{th} order terms.

In the second case there are only a simple pair of pure imaginary eigenvalues of $Df(0)$ and all other eigenvalues have negative real parts. Let l and r be the left and right eigenvectors of $Df(0)$ corresponding to the eigenvalue $i\omega$, i.e., $lDf(0) = i\omega l$, and $Df(0)r = i\omega r$. Define $\alpha = \operatorname{Re}\tilde{\alpha}$, where $\tilde{\alpha}$ is given by (see [40])

$$\begin{aligned} \tilde{\alpha} &= \frac{1}{2}lD^3f(0)(r,r,r^*) + lD^2f(0)\left((-Df(0))^{-1}D^2f(0)(r,r^*),r\right) \\ & \quad +\frac{1}{2}lD^2f(0)\left((2\omega i - Df(0))^{-1}D^2f(0)(r,r),r^*\right), && (1.5)\end{aligned}$$

where r^* denotes the complex conjugate of r. Then if $\alpha < 0$, the equilibrium $x = 0$ is asymptotically stable; if $\alpha > 0$, then equilibrium $x = 0$ is unstable; if $\alpha = 0$, then the system is degenerate and the stability is determined by 4^{th} and 5^{th} order derivatives of the vector field.

Consider the following system with control

$$\dot{x} = f(x,u), \;\; f(0,0) = 0, \;\; x \in \mathbb{R}^n, \;\; u \in \mathbb{R}^m, \tag{1.6}$$

where u is the control input. Roughly speaking, the system is said to be *controllable* if for each point in the initial set in $\mathbb{R}^n$, there exists a control $u(t) \in \mathbb{R}^m$, such that the trajectory reaches the target set for some t satisfying $0 \leq t \leq \infty$. The concept of controllability can be refined by specifying the initial set, the target set, and the set of functions for allowable control inputs. For instance, for *local asymptotic controllability,* the initial set is a neighborhood of 0, and the target set is the equilibrium $x = 0$. Presently, no necessary and sufficient conditions have been obtained to test controllability. It is still an active area of research to find more necessary or sufficient conditions of controllability. *Local stabilizability* via continuous feedback is defined as the existence of a continuous feedback $u = k(x)$ with $k(0) = 0$ such that $x = 0$ is asymptotically stable for the closed loop system $\dot{x} = f(x, k(x))$. For a control affine system

$$\dot{x} = f(x) + g(x)u, \;\; f(0) = 0, \tag{1.7}$$

where $g(x) = [g_1(x) \;\; g_2(x) \;\; \cdots \;\; g_m(x)]$ and $u = [u_1 \;\; u_2 \;\; \cdots \;\; u_m]^T$, Artstein's Theorem [12] states that the system (1.7) is stabilizable via continuous feedback if and only if there exists a $\mathbf{C}^1$ control Lyapunov function V: $\mathbb{R}^n \longrightarrow \mathbb{R}$, satisfying $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$, such that for some open neighborhood $\mathcal{O} \in \mathbb{R}^m$ of $u = 0$, and each $x \neq 0$, the following inequality holds:

$$\inf_{u \in \mathcal{O}} \frac{\partial V}{\partial x} \cdot (f(x) + g(x)u) = \inf_{u \in \mathcal{O}} (\mathcal{L}_f V + u_1 \mathcal{L}_{g_1} V + \cdots + u_m \mathcal{L}_{g_m} V) < 0. \tag{1.8}$$

Presently no analytic solutions are available to the parameterized partial differential inequality (1.8). This implies analytical construction of control Lyapunov functions is still not known in general. For a general nonlinear control system (1.6), even the existence of a smooth control Lyapunov function $V(x)$ such that

$$\inf_{u \in \mathcal{O}} \frac{\partial V}{\partial x} \cdot f(x,u) < 0$$

does not guarantee a continuous stabilizing feedback. We remark here that asymptotic controllability to zero does not mean stabilizability via continuous feedback because the pointwise controllers cannot be patched together in a smooth way.

Consider the linearization of (1.6) around $(x,u) = (0,0)$,

$$\dot{x} = Ax + Bu, \tag{1.9}$$

where $A = \frac{\partial f}{\partial x}(0,0)$, $B = \frac{\partial f}{\partial u}(0,0)$. Define

$$\mathcal{C}(A,B) = \text{span}\left\{B, AB, \cdots, A^{n-1}B\right\}.$$

The linear system (1.9) is controllable if and only if $\mathcal{C}(A,B) = \mathbb{R}^n$. It is stabilizable if and only if $\mathcal{C}(A,B)$ contains all the eigenspaces corresponding to the eigenvalues with non-negative real parts. Clearly according to the First Theorem of Lyapunov, if (A,B) is stabilizable, then the full system (1.6) is also locally stabilizable; if (A,B) is unstabilizable and there exists an unstabilizable eigenvalue with positive real part, then the full system (1.6) is also locally unstabilizable. So one of the challenging problems of local feedback stabilization is when (A,B) is unstabilizable and all the unstabilizable eigenvalues have zero real parts. One of the major contributions of this thesis is to give a constructive solution to the following two cases: the steady state case, when the uncontrollable eigenvalue is 0; and the Hopf case, when the controllable eigenvalues are a pair of pure imaginary numbers.

One of the important applications of local feedback stabilization of nonlinear systems with linearly uncontrollable critical modes is in bifurcation control. Consider the following nonlinear system:

$$\dot{x} = f_\mu(x), \;\; f_\mu(0) = 0, \;\; x \in \mathbb{R}^n. \tag{1.10}$$

Here $\mu \in [-\epsilon, \epsilon] \subset \mathbb{R}$ is the bifurcation parameter. Define $A_\mu = \frac{\partial f_\mu}{\partial x}(0)$, and assume

S-1 A_μ has a real eigenvalue $\lambda(\mu)$ satisfying $\lambda(0) = 0$, $\lambda'(0) \neq 0$;

S-2 all other eigenvalues of A_μ have negative real parts for $\mu \in [-\epsilon, \epsilon]$.

Then there are two scenarios. The first is called a transcritical bifurcation, where the equilibrium branch $x(\mu) = 0$ intersects with another equilibrium branch at $\mu = 0$ and exchanges stability (see Figure 1.2 (a)). The second one is called a pitchfork bifurcation, where the bifurcating equilibrium branch is either stable or unstable. We call the bifurcation a subcritical pitchfork bifurcation if the bifurcating branch is unstable (see Figure 1.2 (b)); we call it a supercritical pitchfork bifurcation if the bifurcating branch is stable (see Figure 1.2 (c)). By evaluating (1.10) at $\mu = 0$, and defining Q and C according to (1.3) and (1.4), then we have

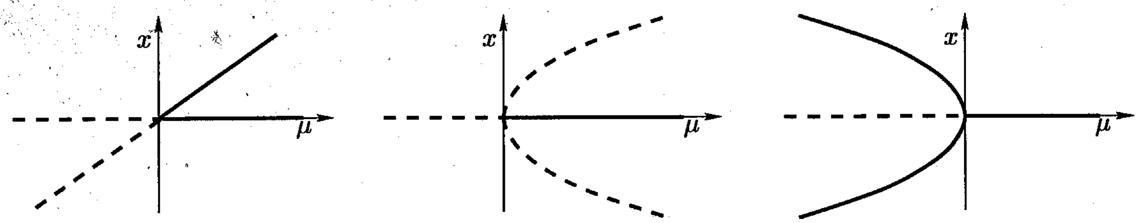
1. if $Q \neq 0$, then the bifurcation is a transcritical bifurcation (see Figure 1.2 (a));
2. if $Q = 0$ and $C > 0$, then the bifurcation is a subcritical pitchfork bifurcation (see Figure 1.2 (b));
3. if $Q = 0$ and $C < 0$, then the bifurcation is a supercritical pitchfork bifurcation (see Figure 1.2 (c));
4. if $Q = C = 0$, then the bifurcation is degenerate and has to be determined by higher order terms in the Taylor series expansion.

It should be pointed out that for the transcritical bifurcation and the subcritical pitchfork bifurcation, the equilibrium $x = 0$ at the bifurcation point is unstable, while it is stable for the supercritical pitchfork bifurcation.

The other case is called the Hopf bifurcation. Assume A_μ satisfies

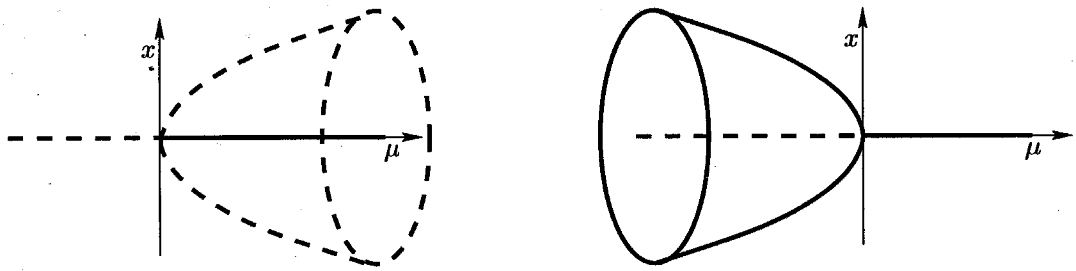
H-1 A_μ has a pair eigenvalues $\lambda(\mu) \pm i\omega(\mu)$ satisfying $\lambda(0) = 0$, $\lambda'(0) \neq 0$, and $\omega(0) \neq 0$;

H-2 all other eigenvalues of A_μ have negative real parts for $\mu \in [-\epsilon, \epsilon]$,



(a) Transcritical Bifurcation (b) Subcritical Pitchfork Bifurcation (c) Supercritical Pitchfork Bifurcation

Figure 1.2: Steady-state bifurcations.



(a) Subcritical Hopf Bifurcation (b) Supercritical Hopf Bifurcation

Figure 1.3: Hopf bifurcations.

then the system (1.10) undergoes a Hopf bifurcation at $\mu = 0$. Namely, a continuum family of periodic orbits emerge as μ deviates from 0. If the periodic orbits are unstable, we call it a subcritical Hopf bifurcation (see Figure 1.3 (a)); if the periodic orbits are stable, we call it a supercritical Hopf bifurcation (see Figure 1.3 (b)). By evaluating (1.10) at $\mu = 0$ and defining $\alpha := \operatorname{Re} \tilde{\alpha}$, where $\tilde{\alpha}$ is given by (1.5), then we have

1. if $\alpha > 0$, then the bifurcation is a subcritical Hopf bifurcation (see Figure 1.3 (a));

2. if $\alpha < 0$, then the bifurcation is a supercritical Hopf bifurcation (see Figure 1.3 (b));

3. if $\alpha = 0$, then the Hopf bifurcation is degenerate and has to be determined by higher order terms in the Taylor series expansion.

It should be pointed out that for the subcritical Hopf bifurcation, the equilibrium $x = 0$ is unstable at the bifurcation point, but it is stable for the supercritical Hopf bifurcation.

In some engineering applications, such as active control of rotating stall, surge, and combustion instabilities in gas turbine engines, the bifurcations especially the subcritical ones are undesirable because the limit cycle oscillations resulting from the Hopf bifurcations add unsteady loading to the engine parts and limit the engine performance. So tools for feedback control of bifurcations are needed to eliminate the bifurcations, or when elimination is unachievable, to change the bifurcation characteristic in order to reduce the amplitude of limit cycles. Consider the following system

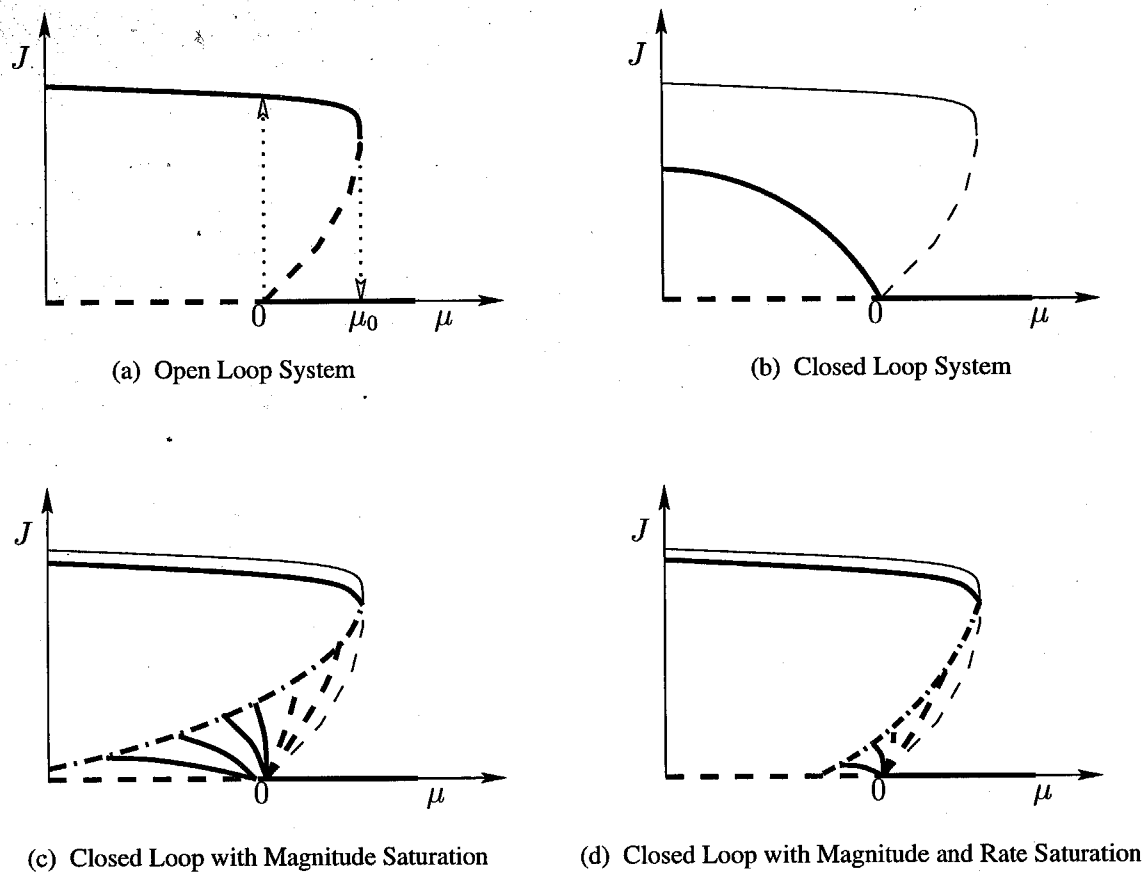
$$\dot{x} = f_\mu(x,u), \quad f_\mu(0,0) = 0, \tag{1.11}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and $\mu \in [-\epsilon, \epsilon] \subset \mathbb{R}$ is a bifurcation parameter. We assume here that when $u = 0$, the uncontrolled system (1.11) has either a steady state bifurcation or a Hopf bifurcation at $\mu = 0$. The linearization of (1.11) around $(x,u) = (0,0)$ is given by

$$\dot{x} = A_\mu x + B_\mu u,$$

where $A_\mu := \frac{\partial f_\mu}{\partial x}(0,0)$, $B_\mu := \frac{\partial f_\mu}{\partial u}(0,0)$. If (A_μ, B_μ) is stabilizable for $\mu \in [-\epsilon, \epsilon]$, then the nominal equilibria $x_0(\mu) = 0$ can be stabilized by a linear feedback and the bifurcation is removed. If the bifurcating eigenvalues are uncontrollable and all other eigenvalues are stabilizable, then the bifurcation cannot be eliminated and it is necessary to design control laws to change the bifurcation characteristic.

To motivate the benefit of bifurcation control and the effects of actuator limits, assume the uncontrolled system of (1.11) has a subcritical Hopf bifurcation at $\mu = 0$ and the bifurcating modes are linearly uncontrollable. Also suppose there is a hysteresis associated with the subcritical bifurcation. Figure 1.4 (a) shows the open loop bifurcation behavior. Suppose the initial conditions are small, then as μ decreases passing $\mu = 0$, any nonzero initial conditions will lock on to the large amplitude limit cycle. Thus the instability inception is abrupt. Due to the hysteresis, the bifurcation



(a) Open Loop System

(b) Closed Loop System

(c) Closed Loop with Magnitude Saturation

(d) Closed Loop with Magnitude and Rate Saturation

Figure 1.4: Open loop and closed loop bifurcation behavior. x-axis is the bifurcation parameter, y-axis is the square of limit cycle amplitude A, i.e., $J = A^2$. Solid line: stable equilibria. Dashed line: unstable equilibria. Dashdot line: saturation envelope formed by unstable equilibria.

parameter μ has to be increased so that the system will go back to the nominal equilibrium branch $\mu = 0$. Now suppose there is a control law $u = K(x)$ such that the Hopf bifurcation for the closed loop system is supercritical (see Figure 1.4 (b)). It is clear that as μ decreases passing $\mu = 0$, the amplitude of the limit cycles increases steadily from zero, and the hysteresis has been eliminated. If the controller has a magnitude saturation limit, then a secondary bifurcation occurs on the stabilized branch of limit cycles. All the saturation points for different gains in Figure 1.4 (c) are denoted by the dash-dot line, which we call the saturation envelope. For each fixed control gain, any small disturbances will converge to the upper branch of stable limit cycles if the system operates to the left of the intersection between the stabilized branch and the saturation envelope. If we also include the bandwidth and rate saturation limit, the region of stabilized branches becomes even smaller (see Figure 1.4 (d)).

One of the goals of this thesis is to investigate under what conditions a transcritical bifurcation/subcritical pitchfork bifurcation can be turned into a supercritical pitchfork bifurcation using a feedback, and under what conditions a subcritical Hopf bifurcation can be turned into a supercritical Hopf bifurcation using a feedback. Under some nondegenerate conditions, this is equivalent to stabilize the nominal equilibrium at the bifurcation point. Although the existence of control Lyapunov functions is a necessary condition for the existence of a stabilizing feedback, it is too general for solving our problem. Instead, we use dynamical systems theory to derive explicit algebraic necessary and sufficient conditions of stabilizability. And we give a constructive procedure of designing stabilizing feedback laws. We also consider the qualitative effects of magnitude saturation, bandwidth, and rate limits on the extension of the stabilized equilibria (limit cycles) by quantifying Figure 1.4 (c) and (d).

1.2 Previous Work

The goal of this thesis is to deepen understanding of local smooth feedback stabilization of nonlinear systems, the role of feedback in bifurcation control, and the effect of magnitude saturation and rate limits of controllers in control of bifurcations. The application of the theory is in active control of rotating stall and surge in gas turbine engines.

1.2.1 Local feedback stabilization

The geometric structure of stabilizability of finite dimensional linear time invariant (LTI) systems has been well known (see [44, 80]). Many techniques, such as pole placement and solving linear algebraic Lyapunov functions, can be used to design a stabilizing feedback. For finite dimensional nonlinear systems, the question of feedback stabilization becomes much more difficult and subtle. There is a huge amount of literature on feedback stabilization (see [61, 62] and the references therein). There are two basic ideas in designing stabilizing control laws, both of which are originated from Lyapunov stability theory.

The first idea is based on Lyapunov second method, i.e., using control Lyapunov functions (see [12, 18, 47, 60, 62, 64]). The advantage of this technique is that global stabilizability can be tackled. The drawback is that the method is too general and solving for control Lyapunov functions can be difficult for many systems. For control affine systems, Artstein's Theorem [12] states that the necessary and sufficient condition for smooth feedback stabilization is the existence of a smooth control Lyapunov function. The stabilizing control law can be constructed by Sontag's Formula [60]. For a general nonlinear control system, however, existence of smooth control Lyapunov functions does not guarantee a continuous feedback [47], but usually yields a discontinuous feedback. If the feedback is allowed to be time-varying, then existence of smooth Lyapunov functions guarantees a continuous time-varying feedback [19]. In his seminal 1983 paper [16], R. W. Brockett pointed out stabilizability via continuous feedback is not guaranteed even if for each point in a neighborhood of the equilibrium, there exists a smooth control law steering the system to the equilibrium. He gave an example of nonholonomic integrator to show that a necessary condition for the existence of a continuous feedback is that the function $f(x,u)$: $\mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is onto. Samson [59] in 1990 used a time-varying feedback to stabilize the mobile robot to a point. Many publications subsequently appeared on stabilization via time-varying feedback (see [53] and the references therein).

The second idea is based on the Lyapunov direct method and Poincaré normal form theory. The basic observation is that if the linearized system is stabilizable, then the full nonlinear system is also stabilizable by the same linear feedback. If the linearization is unstabilizable and there is an uncontrollable eigenvalue with positive real part, then the full nonlinear system is not stabilizable via a continuous feedback. So the only challenging problem in local feedback stabilization is the critical case: all the linearly unstabilizable eigenvalues are on the imaginary axis. In this case, the system can be reduced to the local center manifold which is tangent to the eigenspace associated with the unstabilizable critical eigenvalues. Normal form transformation is then used to analyze the stability of the reduced system. The advantage of this method is that local stabilizability for a fairly large class of systems can be analyzed. The drawback is that it is a local technique so it is not applicable to issues of global stabilizability.

Aeyels [7] used center manifold theory to study the stabilizability problem for single input systems expressed by $\dot{x} = f(x) + bu$ in a neighborhood of an equilibrium point, but sufficient conditions for the stabilizability were given only for a class of three-dimensional systems whose linearization has a pair of pure imaginary eigenvalues. Further, a very simple example was given to show that in some cases, linear feedbacks are not sufficient to stabilize a system and nonlinear feedback are needed. Abed and Fu [4] gave sufficient conditions for the existence of a state feedback controller with vanishing linear part to alter the criticality of the Hopf bifurcation with the bifurcated mode being linearly uncontrollable. The main assumption is that all other modes are linearly stable. Abed and Fu [5] gave some sufficient conditions for simple steady-state bifurcations. In [10, 32] the effects of linear feedbacks are addressed, where sufficient conditions for the linear feedback to stabilize the critical mode were obtained. Also, the major assumption is that the controllable subsystem is stable and the controller set is restricted to a ball in $\mathbb{R}^n$.

Behtash and Sastry [15] studied a class of systems with two uncontrollable modes: double zero with nondegenerate Jordan form, a zero and a pair of pure imaginary eigenvalues, and two pairs of pure imaginary eigenvalues without resonance, and some sufficient conditions of stabilizability via smooth feedback with vanishing linear part were derived. Abichou and D'Andréa Novel [6] considers a class of mechanical systems with a degenerate pitchfork bifurcation and stabilizing feedback is designed using Taylor series expansion up to the 5^{th} order terms. In Sontag and Sussmann [63], stabilizing feedback is derived for a three-dimensional system describing a rigid body that has two equal moments of inertia along the principle axis. The linearly uncontrollable part of the system has a double zero with degenerate Jordan forms. In [41, 50], planar systems are considered with only one uncontrollable zero eigenvalue. In this case, conditions of stabilizability can be established for higher order degeneracy in the Taylor series expansion due to the simplicity of the uncontrollable mode. Zaslavsky [86] studied the case when multiple complex conjugate pure imaginary eigenvalues are in the linearly uncontrollable subspace. Under the assumption that there are no low order resonances between the critical eigenvalues, sufficient conditions were obtained for the existence of linear state feedbacks such that the closed loop system is asymptotically stable.

The local feedback stabilization is closely related to control of bifurcations in that if the system is locally stabilizable at the bifurcation point, then the criticality of the bifurcation for the closed loop system is supercritical. But changing criticality is not the only subject in control of bifurcations. Mehra [54] addressed the problem of globally removing the steady-state bifurcations. Kang [45] studied the effects of linear state feedbacks on the steady-state bifurcations with a single linearly uncontrollable mode using the the technique of normal forms in control systems. Conditions for different types of bifurcations are derived in terms of nonlinear invariants.

1.2.2 Active control of rotating stall and surge

One of the limiting factors of aeroengine performance is aerodynamic instability in the compression system at high loading, typically in the form of rotating stall and surge (see [26, 27, 34, 35, 66] for detailed descriptions). Rotating stall occurs when a nonaxisymmetric flow disturbance develops around the annulus of the compressor and rotates at a fraction of the rotor speed [24, 21, 25, 55]. Surge is axisymmetric relaxation oscillation of flow and pressure throughout the whole compression system [33]. Fully developed rotating stall and surge have detrimental effects on engine components thus must be avoided.

In 1989, Epstein *et al.* first proposed the idea of damping out aerodynamic instabilities in gas turbine engines using active control [28]. Active control of rotating stall refers to either stabilization of the steady axisymmetric flow past the point of peak pressure rise, which typically requires many actuators assembled around the annulus [22, 23, 38, 58, 79], or operability enhancement, which implies alteration of the criticality of the bifurcation and elimination of the hysteresis associated with rotating stall [20, 29, 30, 49, 84]. With active control and the same stall margin, the compressor may operate stably near the peak of the compressor characteristic, resulting improvement of both operating range and performance. Experimentally there are essentially three basic types of actuation techniques that have been used in active control of rotating stall, namely inlet guide vanes (IGVs) [58], air injectors [20, 23, 38, 79] and outlet bleed valve [13, 29, 30, 84]. In [31], sleeve valves were used to recirculate air over the engine for active stall control, which is essentially a combination of bleed valves and injectors.

On the theoretical end, the design and analysis of many controllers for rotating stall and surge control are based on the model developed by Moore and Greitzer in their seminal papers [36, 56]. The model is a set of nonlinear partial differential equations (PDEs) for pressure rise, averaged and disturbed values of flow coefficient as functions of time and circumferential position around the compressor. The Moore-Greitzer model captures most of the dynamic behavior of stall and surge and is sufficient for design and analysis of active controllers [38, 48, 51, 58]. The three state Moore-Greitzer model is the Galerkin projection of the PDEs to the zeroth (axisymmetric) and the first spatial harmonics. One of the attractive features of the three state Moore-Greitzer model is that it captures the qualitative dynamic behavior of both surge and rotating stall, and is simple enough for designing active controllers [29, 30, 46, 49, 52].

One of the major considerations in practical implementation of active stall control is the complexity of sensing and actuation [1, 2, 3]. Since the rotating stall modes are rotating around the annulus of the compressor, distributed actuators are required to achieve controllability [38, 58, 79]. For example, in [79], 12 air injectors are used to stabilize the first and second harmonic. As the number of actuators is reduced, the rotating stall modes may become linearly uncontrollable. For instance, in [20], three pulsed air injectors are used and the first rotating stall mode is unstabilizable. For bleed valve actuators located downstream, the effects of actuation is essentially axisymmetric (nonaxisymmetric effects decay exponentially in the axial direction), and the rotating stall modes are linearly uncontrollable [29, 30, 49]. In order to reduce the complexity of the actuators, bifurcation control of rotating stall is one of the main design tools [49]. Research has been done on the effectiveness of different actuation schemes. In [39], gain and phase margins of different controllers based on the linearized Moore-Greitzer model are used to evaluate the effectiveness of different actuation schemes. In [46], effects of bandwidth were analyzed for a class of backstepping controllers.

1.3 Overview of Thesis

The main goal of this thesis is to understand the role of controller bandwidth, rate and magnitude saturation limits in control of bifurcations. To achieve this, first we have to understand when a subcritical bifurcation can be changed to supercritical by a smooth state feedback, which is equivalent to the local feedback stabilization of the system at the bifurcation point. The local feedback stabilization becomes nontrivial when the only unstabilizable eigenvalues of the linearization have zero real parts. We consider two of the simplest scenarios in local smooth feedback stabilization: the steady-state case, when the linearly unstabilizable eigenvalue is zero; and the Hopf case, when the unstabilizable eigenvalues are a pair of pure imaginary numbers. Under certain nondegeneracy conditions, we give explicit algebraic conditions for stabilizability. And when the system is stabilizable, the stabilizing feedback can be explicitly constructed.

Since a steady-state/Hopf bifurcation can be turned into a supercritical pitchfork/Hopf bifurcation if and only if the system is locally stabilizable at the bifurcation point, solving local feedback stabilization problem implies that we have derived algebraic necessary and sufficient conditions under which the criticality of a simple steady-state or Hopf bifurcation can be changed to supercritical by a smooth feedback. In applications, various constraints of actuation such as magnitude saturation, bandwidth, and rate limits have to take into account for control performance. We give qualitative estimates of the region of attraction to the stabilized bifurcating equilibria/periodic orbits under these constraints. It turns out the stability boundary of the stabilized equilibria/periodic orbits are formed by the unstable equilibrium/periodic orbits after the controller saturates.

We apply the above theoretical results to active control of rotating stall and surge in gas turbine engines. Although distributed actuators can achieve linear stabilizability, their drawback is the question of affordability and reliability. On the other hand, simple actuation schemes such as outlet bleed valve and magnetic bearings will make the system loss of linear controllability. We design control laws for magnetic bearing actuators that can change the criticality of the Hopf bifurcations of rotating stall and surge inception using the Moore-Greitzer model. We analyze the effects of magnitude and rate saturations in active stall and surge control using bleed valves by deriving analytical formulas of which the bandwidth requirement is a function of the systems parameters and the shape of compressor characteristics. These formulas give good qualitative predictions when compared with experimental results done by Simon Yeung. Our conclusion is that these constraints are serious limiting factors that have to be considered in practical implementation to the aircraft engines.

The outline of the thesis is as follows. In Chapter 2, we provide the classification of stabilizability of steady-state/Hopf bifurcations in single input nonlinear systems. We make use of the controller canonical form for the controllable subsystem of linearization. In Chapter 3, we classify the stabilizability of steady-state/Hopf bifurcations in multi-input systems. We relax the linear controllability assumption in Chapter 2 to linear stabilizability, and we provide a technique without using canonical forms. In Chapter 4, we analyze the effects of controller magnitude saturation by estimating the region of attraction to the stabilized equilibria/periodic orbits. In Chapter 5, we analyze the effects of bandwidth, rate and magnitude saturation of bleed valve actuator in active control of rotating stall in axial compression systems. The analytical results are compared with experiments. In Chapter 6, we design control laws that change the bifurcation behavior of rotating stall and surge for magnetic bearings that modulate the rotor tip clearance. The linearly unstabilizable modes are the second stall mode and the surge mode. And finally, we give conclusions and point out some possible future directions in Chapter 7.

Chapter 2 Feedback Stabilization: the Single Input Case

We consider single input nonlinear systems with steady-state or Hopf bifurcations for which the bifurcating modes are linearly unstabilizable. The goal is to find sufficiently smooth state feedbacks such that the bifurcation for the closed loop system is supercritical, and at the same time, the dynamics on the linearly controllable subspace is asymptotically stable. We solve this problem by giving algebraic necessary and sufficient conditions under the assumption that certain nondegeneracy conditions are satisfied. We also give explicit construction of stabilizing feedbacks and illustrate the theory using simple examples. This chapter is based on the paper [74].

Previous work on this problem can be found in Abed and Fu [4, 5], in which some sufficient conditions of stabilizability were given. Those conditions are some of the special cases described in this chapter. In contrast, the conditions in this chapter are not only sufficient but also necessary, and the procedure for design of stabilizing controllers is constructive.

2.1 Stabilizability of Steady-State Bifurcations

In this section we consider the case when a nonlinear system undergoes a steady-state bifurcation, with the bifurcating mode being linearly unstabilizable. Under certain nondegeneracy conditions, we derive necessary and sufficient conditions for the existence of a sufficiently smooth state feedback such that the dynamics on the linearly controllable subspace is asymptotically stable and at the same time, the bifurcation for the closed loop system is a supercritical pitchfork bifurcation. We also give explicit construction of the stabilizing controllers.

Consider the following single-input system

$$\dot{y} = f_\mu(y, u), \tag{2.1}$$

where $y \in \mathbb{R}^{n+1} (n \geq 1)$ is the state variable, $\mu \in \mathbb{R}$ is a bifurcation parameter, and $u \in \mathbb{R}$ is the control input. We omit the discussion of the trivial case when $n = 0$. Throughout this section we assume all the assumptions are valid for μ in the region $[-\bar{\mu}, \bar{\mu}]$. We make the following assumptions:

AS-1 *$f_\mu(y, u)$ is at least $\mathbf{C}^4$ with respect to (y, u) and $\mathbf{C}^2$ with respect to μ.*

AS-2 *For $u = 0$, there exists a nominal equilibrium solution $y = y_0(\mu)$ such that $f_\mu(y_0(\mu), 0) = 0$.*

AS-3 *$\lambda(\mu)$ is a simple real eigenvalue of $\frac{\partial f_\mu}{\partial y}(y_0(\mu), 0)$ and satisfies $\lambda(0) = 0$, $\frac{d\lambda}{d\mu}(0) \neq 0$.*

AS-4 *The eigenspace associated with $\lambda(\mu)$ is linearly uncontrollable, and all other eigenspaces are linearly controllable.*

Under these assumptions, we transform the system (2.1) into a standard form by the following procedure. First expand $f_\mu(y, u)$ into Taylor series around $(y_0(\mu), 0)$, and use a linear transformation to linearly decouple the uncontrollable eigenspace with the controllable eigenspaces. Then transform the linearly controllable subsystem into the controller canonical form using linear transformation. Finally, evaluate all the terms except the bifurcating eigenvalue at $\mu = 0$. The resulting normal form is given by

$$\begin{aligned} \dot{x} &= d\mu x + q_{11}x^2 + q_{12}\tilde{x}x + q_{13}xu + \tilde{x}^T q_{22}\tilde{x} + q_{23}\tilde{x}u + q_{33}u^2 \\ &\quad + c_{111}x^3 + c_{112}\tilde{x}x^2 + c_{113}x^2u + \tilde{x}^T c_{122}\tilde{x}x + c_{123}\tilde{x}ux + c_{133}xu^2 \\ &\quad + c_{222}(\tilde{x}, \tilde{x}, \tilde{x}) + \tilde{x}^T c_{223}\tilde{x}u + c_{233}\tilde{x}u^2 + c_{333}u^3 + \text{h.o.t.}, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + Bu + \tilde{q}_{11}x^2 + \tilde{q}_{12}\tilde{x}x + \tilde{q}_{13}xu + \tilde{q}_{22}(\tilde{x}, \tilde{x}) + \tilde{q}_{23}\tilde{x}u + \tilde{q}_{33}u^2 \\ &\quad + \text{h.o.t.}, \end{aligned} \tag{2.3}$$

where h.o.t denotes "higher order terms", $x \in \mathbb{R}$, $\tilde{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times 1}$ are given by

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{2.4}$$

and the dimensions of other coefficients are given by

$$\begin{aligned} & d, \mu, q_{11}, q_{13}, q_{33}, c_{111}, c_{113}, c_{133}, c_{333} \in \mathbb{R}, \\ & q_{12}, c_{112}, c_{123}, c_{233} \in \mathbb{R}^{1\times n}, \\ & \tilde{q}_{11}, \tilde{q}_{13}, \tilde{q}_{33} \in \mathbb{R}^{n\times 1}, \\ & q_{22} = q_{22}^T, c_{122} = c_{122}^T, c_{223} = c_{223}^T, \tilde{q}_{12}, \tilde{q}_{23} \in \mathbb{R}^{n\times n}, \\ & c_{222}, \tilde{q}_{22} \in \mathbb{R}^{n\times n\times n}. \end{aligned}$$

The tensors c_{222} and $\tilde{q}_{22}$ satisfy

$$c_{222}(u,v,w) = \sum_{j,k,l=1}^{n} c_{222}^{jkl} u^j v^k w^l, \quad \tilde{q}_{22}(u,v) = \sum_{j,k=1}^{n} \tilde{q}_{22}^{jk} u^j v^k, \quad c_{222}^{jkl} = c_{222}^{\hat{j}\hat{k}\hat{l}}, \quad \tilde{q}_{22}^{jk} = \tilde{q}_{22}^{kj},$$

where $j, k, l = 1, \cdots, n$, and $\hat{j}, \hat{k}, \hat{l}$ is any permutation of j, k, l.

The goal is to find a sufficiently smooth (possibly nonlinear) feedback such that the linearly controllable subspace is asymptotically stable, and at the same time, the equilibrium $(0,0)$ for the closed loop system is asymptotically stable at the bifurcation point, i.e., at $\mu = 0$. It can be shown that if the bifurcation for the closed loop system is a nondegenerate supercritical pitchfork bifurcation, then $(0,0)$ is asymptotically stable at $\mu = 0$; the converse is also true if certain nondegenerate conditions are satisfied.

Let $\mathfrak{m}(s) = [\ s^{n-1}\ \cdots\ s\ \ 1\ ]^T$. In the following, we denote

$$q_{12}^n = q_{12}\mathfrak{m}(0),\ q_{22}^{nn} = \mathfrak{m}(0)^T q_{22}\mathfrak{m}(0),\ \tilde{q}_{12}^n = \tilde{q}_{12}\mathfrak{m}(0),\ c_{222}^{nnn} = c_{222}(\mathfrak{m}(0), \mathfrak{m}(0), \mathfrak{m}(0)),\ \text{etc.}$$

Define

$$\Upsilon_1 = q_{12}^n + q_{13}a_n, \tag{2.5}$$

$$\Upsilon_2 = q_{22}^{nn} + q_{23}^n a_n + q_{33}a_n^2, \tag{2.6}$$

$$\Pi_1^S = [\ 0\ \ q_{12}^1\ \ \cdots\ \ q_{12}^{n-1}\ ] + q_{13}[\ 1\ \ a_1\ \ \cdots\ \ a_{n-1}\ ], \tag{2.7}$$

$$\begin{aligned}\Pi_2^S &= 2[\ 0\ \ q_{22}^{n1}\ \ \cdots\ \ q_{22}^{n,n-1}\ ] + a_n[\ 0\ \ q_{23}^1\ \ \cdots\ \ q_{23}^{n-1}\ ] \\ &\quad +(q_{23}^n + 2q_{33}a_n)[\ 1\ \ a_1\ \ \cdots\ \ a_{n-1}\ ],\end{aligned} \tag{2.8}$$

$$\alpha_0 = c_{111} - \Pi_1^S\tilde{q}_{11}, \tag{2.9}$$

$$\alpha_1 = c_{112}^n + c_{113}a_n - \Pi_1^S(\tilde{q}_{12}^n + \tilde{q}_{13}a_n) - \Pi_2^S\tilde{q}_{11}, \tag{2.10}$$

$$\begin{aligned}\alpha_2 &= c_{122}^{nn} + c_{123}^n a_n + c_{133}a_n^2 - \Pi_1^S(\tilde{q}_{22}^{nn} + \tilde{q}_{23}^n a_n + \tilde{q}_{33}a_n^2) \\ &\quad -\Pi_2^S(\tilde{q}_{12}^n + \tilde{q}_{13}a_n),\end{aligned} \tag{2.11}$$

$$\alpha_3 = c_{222}^{nnn} + c_{223}^{nn}a_n + c_{233}^n a_n^2 + c_{333}a_n^3 - \Pi_2^S(\tilde{q}_{22}^{nn} + \tilde{q}_{23}^n a_n + \tilde{q}_{33}a_n^2), \tag{2.12}$$

$$\Delta_\Upsilon = \Upsilon_1^2 - 4q_{11}\Upsilon_2,$$

$$\Delta_\alpha = \alpha_1^2 - 4\alpha_2\alpha_0,$$

$$\rho = -\frac{\Upsilon_1}{2\Upsilon_2},$$

where $p(s) = \det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$.

With these definitions, we define the degeneracy conditions as

SD-1 $q_{11} = \Upsilon_1 = \Upsilon_2 = \alpha_3 = \alpha_2 = \alpha_1 = \alpha_0 = 0$,

SD-2 $q_{11} = \Upsilon_1 = \Upsilon_2 = \alpha_3 = \Delta_\alpha = 0,\ \alpha_2 > 0,\ \alpha_0 \geq 0$,

SD-3 $q_{11} = \Upsilon_1 = \alpha_0 = 0,\ \Upsilon_2 \neq 0$,

SD-4 $q_{11} \neq 0$, $\Upsilon_2 \neq 0$, $\Delta_\Upsilon = 0$, $P_S(\rho) = 0$, *where* $P_S(\cdot)$ *is defined as*

$$P_S(K) = \alpha_0 + \alpha_1 K + \alpha_2 K^2 + \alpha_3 K^3, \tag{2.13}$$

where $K \in \mathbb{R}$.

Theorem 2.1 *Except for the degenerate cases, there does not exist a sufficiently smooth feedback*

$$u = K_1\tilde{x} + K_2 x + K_3 x^2 + K_4 \tilde{x} x + \tilde{x}^T K_5 \tilde{x} + h.o.t.,$$

with

$$K_1, K_4 \in \mathbb{R}^{1\times n}, \quad K_2, K_3 \in \mathbb{R}, \quad K_5 = K_5^T \in \mathbb{R}^{n\times n},$$

such that

(i) $A + BK_1$ *is Hurwitz,*

(ii) At $\mu = 0$, *the equilibrium is asymptotically stable,*

if and only if one of the following conditions holds

SU-1 $q_{11} = \Upsilon_1 = \Upsilon_2 = \alpha_3 = \alpha_2 = \alpha_1 = 0$, $\alpha_0 > 0$,

SU-2 $q_{11} = \Upsilon_1 = \Upsilon_2 = \alpha_3 = 0$, $\alpha_2 > 0$, $\alpha_0 > 0$, $\Delta_\alpha < 0$,

SU-3 $q_{11} = \Upsilon_1 = 0$, $\Upsilon_2 \neq 0$, $\alpha_0 > 0$,

SU-4 $q_{11} \neq 0$, $\Upsilon_1 = \Upsilon_2 = 0$,

SU-5 $q_{11} \neq 0$, $\Upsilon_2 \neq 0$, $\Delta_\Upsilon < 0$,

SU-6 $q_{11} \neq 0$, $\Upsilon_2 \neq 0$, $\Delta_\Upsilon = 0$, $P_S(\rho) > 0$, *where* $P_S(\cdot)$ *is given by (2.13).*

A proof of the theorem is in Section 2.4.

Classification of stabilizability for Steady-State Bifurcations

Cases		q_{11}	Υ_1	Υ_2	Δ_Υ	α_3	α_2	α_1	α_0	Δ_α	$P_S(\rho)$
Stabilizable	**SS-1**	$=0$							<0		
	SS-2	$=0$	$\neq 0$						≥ 0		
	SS-3	$=0$	$=0$	$=0$		$\neq 0$			≥ 0		
	SS-4	$=0$	$=0$	$=0$		$=0$	<0		≥ 0		
	SS-5	$=0$	$=0$	$=0$		$=0$	$=0$	$\neq 0$	≥ 0		
	SS-6	$=0$	$=0$	$=0$		$=0$	>0		≥ 0	>0	
	SS-7	$\neq 0$	$\neq 0$	$=0$							
	SS-8	$\neq 0$		$\neq 0$	>0						
	SS-9	$\neq 0$		$\neq 0$	$=0$						<0
Degeneracy Conditions	**SD-1**	$=0$	$=0$	$=0$		$=0$	$=0$	$=0$	$=0$		
	SD-2	$=0$	$=0$	$=0$		$=0$	>0		≥ 0	$=0$	
	SD-3	$=0$	$=0$	$\neq 0$					$=0$		
	SD-4	$\neq 0$		$\neq 0$	$=0$						$=0$
Not Stabilizable	**SU-1**	$=0$	$=0$	$=0$		$=0$	$=0$	$=0$	>0		
	SU-2	$=0$	$=0$	$=0$		$=0$	>0		>0	<0	
	SU-3	$=0$	$=0$	$\neq 0$					>0		
	SU-4	$\neq 0$	$=0$	$=0$							
	SU-5	$\neq 0$		$\neq 0$	<0						
	SU-6	$\neq 0$		$\neq 0$	$=0$						>0

Table 2.1: Classification of stabilizability for steady-state bifurcations.

In summary, the classification of stabilizability of steady-state bifurcations is given in Table 2.1. The blank spaces imply that there are no explicit constraints on the corresponding parameters, but they may be implicitly constrained by other columns in the same row. Cases **SU-1** to **SU-6** are such that there does not exist a sufficiently smooth state feedback such that $(0,0)$ is asymptotically stable for the closed loop system at $\mu = 0$; cases **SD-1** to **SD-4** are degenerate, i.e., we have to resort to higher (4^{th} and 5^{th}) order terms to determine the stability of the closed loop system; cases **SS-1** to **SS-9** are such that $(0,0)$ can be stabilized by a smooth state feedback for $\mu = 0$.

For the cases from **SS-1** to **SS-9**, we can construct feedback laws such that the closed loop system satisfies (i) and (ii) in Theorem 2.1. First, we select K_1 such that

$A + BK_1$ is Hurwitz, and define

$$\bar{p}(s) := \det[sI - (A + BK_1)] = s^n + \bar{a}_1 s^{n-1} + \cdots + \bar{a}_{n-1}s + \bar{a}_n.$$

The bifurcation of the closed loop system is determined by the dynamics on the center manifold near the bifurcation point. The dynamics on the center manifold is given by

$$\dot{x} = d\mu x + Qx^2 + Cx^3 + \text{h.o.t.},$$

where

$$Q = q_{11} + \Upsilon_1 K + \Upsilon_2 K^2, \tag{2.14}$$

$$C = \frac{1}{\bar{a}_n}(\Upsilon_1 + 2\Upsilon_2 K)\Big(K_3 + K_4^n K + K_5^{nn} K^2 + \begin{bmatrix} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1} \end{bmatrix} \tilde{Q}\Big) + P_S(K). \tag{2.15}$$

$$\tilde{Q} = \tilde{q}_{11} + (\tilde{q}_{12}^n + \tilde{q}_{13} a_n)K + (\tilde{q}_{22}^{nn} + \tilde{q}_{23}^n a_n + \tilde{q}_{33} a_n^2)K^2, \tag{2.16}$$

where $K := \frac{K_2}{\bar{a}_n}$, and Υ_1, Υ_2, and $P_S(K)$ are given by (2.5), (2.6), and (2.13), respectively. The goal is to find feedback gains such that $Q = 0$ and $C < 0$. The construction is as follows. Let $K_4 = K_5 = 0$, and select K_2 and K_3 according to the following different cases.

SS-1 Let $K = 0$ and $K_3 = -[\,1 \quad \bar{a}_1 \quad \cdots \quad \bar{a}_{n-1}\,]\tilde{q}_{11}$, then we have $Q = 0$ and $C = \alpha_0 < 0$.

SS-2 Let $K = 0$, then $Q = 0$. Select K_3 such that

$$\alpha_0 + \frac{\Upsilon_1}{\bar{a}_n}\Big(K_3 + [\,1 \quad \bar{a}_1 \quad \cdots \quad \bar{a}_{n-1}\,]\tilde{q}_{11}\Big) < 0.$$

SS-3 We have $Q = 0$ for any $K \in \mathbb{R}$. Let $K_3 = 0$ and select K such that

$$C = P_S(K) = \alpha_0 + \alpha_1 K + \alpha_2 K^2 + \alpha_3 K^3 < 0.$$

Note this can always be achieved by letting $|K|$ large enough because $\alpha_3 \neq 0$.

SS-4 We have $Q = 0$ for any $K \in \mathbb{R}$. Let $K_3 = 0$ and select K such that

$$C = P_S(K) = \alpha_0 + \alpha_1 K + \alpha_2 K^2 < 0.$$

Note this can always be achieved by letting $|K|$ large enough because $\alpha_2 < 0$.

SS-5 We have $Q = 0$ for any $K \in \mathbb{R}$. Let $K_3 = 0$ and select K such that $C = \alpha_0 + \alpha_1 K < 0$. This can be achieved because $\alpha_1 \neq 0$.

SS-6 We have $Q = 0$ for any $K \in \mathbb{R}$. For any $K_3 \in \mathbb{R}$, $C = P_S(K)$ has a minimum at $K = K_m := -\frac{\alpha_1}{2\alpha_2}$. So we have

$$C = P_S(K_m) = \alpha_0 + \alpha_1 K_m + \alpha_2 K_m^2 = -\frac{\Delta_\alpha}{4\alpha_2} < 0.$$

SS-7 We have $Q = 0$ if and only if $K = -\frac{q_{11}}{\Upsilon_1}$. Let $K = -\frac{q_{11}}{\Upsilon_1}$ and select K_3 such that

$$C = \Upsilon_1 \frac{K_3}{\bar{a}_n} + \tilde{\alpha}_0 + P_S(K) < 0,$$

where

$$\tilde{\alpha}_0 = \frac{\Upsilon_1}{\bar{a}_n} \begin{bmatrix} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1} \end{bmatrix} \tilde{Q}(K).$$

and $\tilde{Q}$ is given by (2.16).

SS-8 We have $Q = 0$ if and only if $K = \frac{1}{2\Upsilon_2}\left(-\Upsilon_1 \pm \sqrt{\Delta_\Upsilon}\right)$. So we choose K such that

$$K = \frac{1}{2\Upsilon_2}\left(-\Upsilon_1 + \sqrt{\Delta_\Upsilon}\right),$$

and select K_3 such that

$$C = \frac{K_3}{\bar{a}_n}\sqrt{\Delta_\Upsilon} + \hat{\alpha}_0 + P_S(K) < 0,$$

Construction of Stabilizing Feedback for Steady-State Bifurcations		
	$K := \frac{K_2}{\bar{a}_n}$	K_3
SS-1	0	$-[\ 1\ \ \bar{a}_1\ \ \cdots\ \ \bar{a}_{n-1}\ ]\tilde{q}_{11}$
SS-2	0	$\alpha_0 + \frac{\Upsilon_1}{\bar{a}_n}\left(K_3 + [\ 1\ \ \bar{a}_1\ \ \cdots\ \ \bar{a}_{n-1}\ ]\tilde{q}_{11}\right) < 0$
SS-3	$P_S(K) < 0$	0
SS-4	$P_S(K) < 0$	0
SS-5	$P_S(K) < 0$	0
SS-6	$-\frac{\alpha_1}{2\alpha_2}$	0
SS-7	$-\frac{q_{11}}{\Upsilon_1}$	$\Upsilon_1\frac{K_3}{\bar{a}_n} + \tilde{\alpha}_0 + P_S(K) < 0$
SS-8	$\frac{1}{2\Upsilon_2}\left(-\Upsilon_1 + \sqrt{\Delta_\Upsilon}\right)$	$\frac{K_3}{\bar{a}_n}\sqrt{\Delta_\Upsilon} + \hat{\alpha}_0 + P_S(K) < 0$
SS-9	$-\frac{\Upsilon_1}{2\Upsilon_2}$	0

Table 2.2: Construction of stabilizing feedbacks for steady-state bifurcations.

where

$$\hat{\alpha}_0 = \frac{1}{\bar{a}_n}\sqrt{\Delta_\Upsilon}\left[\ 1\ \ \bar{a}_1\ \ \cdots\ \ \bar{a}_{n-1}\ \right]\tilde{Q}(K).$$

SS-9 W have $Q = 0$ if and only if $K = \rho := -\frac{\Upsilon_1}{2\Upsilon_2}$. Let $K_3 = 0$ and $K = \rho$, then $C = P_S(\rho) < 0$.

In summary, the construction of feedbacks for each case is given in Table 2.1. It should be pointed out that these conditions are at most third order scalar algebraic inequalities that can be trivially solved.

Example 2.1 Consider the following system

$$\begin{aligned}\dot{x}_1 &= x_1^2 + x_1x_2 + \gamma x_2^2,\\ \dot{x}_2 &= u,\end{aligned}$$

where $x_1, x_2, u, \gamma \in \mathbb{R}$, and γ is a parameter.

It is easy to compute by the definitions in (2.5)– (2.12), that

$$q_{11} = 1,\quad \Upsilon_1 = 1,\quad \Upsilon_2 = \gamma,\quad \alpha_j = 0\,(j = 0, \cdots, 3),\quad \tilde{Q} = 0.$$

So we consider two cases.

1. $\gamma = 0$. In this case, the condition **SS-7** is satisfied. So by the construction procedure **SS-7**, we choose

$$K_1 = -1, \quad K_2 = -\tfrac{q_{11}}{\Upsilon_1}\bar{a}_1 = -1, \quad K_3 < 0,$$

i.e., the feedback is given by $u = -x_2 - x_1 + K_3 x_1^2$, where $K_3 < 0$. It is easy to check that the dynamics on the center manifold is given by

$$\dot{x}_1 = K_3 x_1^3 + \text{h.o.t.},$$

so the equilibrium $(0,0)$ is asymptotically stable.

2. $\gamma \neq 0$. We consider the following three cases:

   i. For $\gamma > \frac{1}{4}$, we have $\Upsilon_1^2 - 4q_{11}\Upsilon_2 < 0$. In this case, it is easy to check the system satisfies condition **SU-5** in Theorem 2.1, so there does not exist a sufficiently smooth state feedback such that $(0,0)$ is asymptotically stable. In fact, the dynamics on the center manifold is given by

$$\dot{x}_1 = Qx_1^2 + Cx_1^3 + \text{h.o.t.},$$

   where $Q = 1 + K + \gamma K^2$, and $K = \frac{K_2}{\bar{a}_1}$. So it is clear that there does not exist a $K \in \mathbb{R}$ such that $Q = 0$, and $(0,0)$ cannot be stabilized by a smooth feedback.

   ii. For $\gamma < \frac{1}{4}$ and $\gamma \neq 0$, we have $\Upsilon_1^2 - 4q_{11}\Upsilon_2 > 0$. It is easy to check that the condition **SS-8** is satisfied. By the construction procedure **SS-8**, we select

$$K_1 = -1, \quad K_2 = \tfrac{1}{2\gamma}\left(-1 + \sqrt{1 - 4\gamma}\right), \quad K_3 < 0,$$

i.e., the stabilizing feedback is given by

$$u = -x_2 + \frac{1}{2\gamma}\left(-1 + \sqrt{1-4\gamma}\right) x_1 + K_3 x_1^2.$$

In fact, the dynamics on the center manifold is given by

$$\dot{x}_1 = K_3 x_1^3 + \text{h.o.t.},$$

so $(0,0)$ is asymptotically stable.

iii. For $\gamma = \frac{1}{4}$, then the system satisfies the degeneracy condition **SD-4** and it is inconclusive by Theorem 2.1.

2.2 Stabilizability of Hopf Bifurcations

In this section we consider the case when a nonlinear system undergoes a Hopf bifurcation with bifurcating modes being linearly unstabilizable. In this case, we have to use a normal form reduction in addition to the center manifold reduction. Under certain nondegeneracy conditions, we obtain necessary and sufficient conditions for the existence of smooth state feedbacks such that there exists a state feedback such that the dynamics on the linearly controllable subspace is stable and at the same time, the Hopf bifurcation for the closed loop system is supercritical. We also give explicit construction of the stabilizing controllers.

Consider the following single-input system

$$\dot{y} = f_\mu(y, u), \tag{2.17}$$

where $y \in \mathbb{R}^{n+2} (n \geq 1)$ is the state variable, $\mu \in \mathbb{R}$ is a bifurcation parameter, and $u \in \mathbb{R}$ is the control input. We omit the discussion of the trivial case when $n = 0$. Throughout this section we assume all the assumptions are valid for μ in the region $[-\bar{\mu}, \bar{\mu}]$. We make the following assumptions:

AH-1 *$f_\mu(y, u)$ is at least $\mathbf{C}^4$ with respect to (y, u) and $\mathbf{C}^2$ with respect to μ.*

AH-2 *For $u = 0$, there exists a steady-state solution $y = y_0(\mu)$ such that $f_\mu(y_0(\mu), 0) = 0$.*

AH-3 *$\lambda_{1,2}(\mu) = \sigma(\mu) \pm i\omega(\mu)$ are a simple pair of eigenvalues of $\frac{\partial f_\mu}{\partial y}(y_0(\mu), 0)$ and satisfy $\sigma(0) = 0$, $\frac{d\sigma}{d\mu}(0) \neq 0$, and $\omega(0) \neq 0$.*

AH-4 *The eigenspaces associated with $\lambda_{1,2}(\mu)$ are linearly uncontrollable, and all other eigenspaces are linearly controllable.*

Under these assumptions, we transform the system (2.1) into the standard form by the following procedure. First we expand $f_\mu(y, u)$ into Taylor series around $(y_0(\mu), 0)$. Then we use a linear transformation to linearly decouple the uncontrollable eigenspaces with all other controllable eigenspaces. Then we transform the linearly controllable subsystem into the controller canonical form using a linear transformation. Finally, we evaluate all the terms except the bifurcating eigenvalues at $\mu = 0$. The resulting normal form is given by

$$
\begin{aligned}
\dot{z} &= (d\mu + i\omega)z + q_{11}z^2 + q_{12}|z|^2 + q_{13}\tilde{x}z + q_{14}zu + q_{22}z^{*2} + q_{23}\tilde{x}z^* \\
&\quad + q_{24}z^*u + \tilde{x}^T q_{33}\tilde{x} + q_{34}\tilde{x}u + q_{44}u^2 + c_{111}z^3 + c_{112}|z|^2 z + c_{113}\tilde{x}z^2 \\
&\quad + c_{114}z^2u + c_{122}|z|^2 z^* + c_{123}\tilde{x}|z|^2 + c_{124}|z|^2 u + \tilde{x}^T c_{133}\tilde{x}z + c_{134}\tilde{x}zu \\
&\quad + c_{144}zu^2 + c_{222}z^{*3} + c_{223}\tilde{x}z^{*2} + c_{224}z^{*2}u + \tilde{x}^T c_{233}\tilde{x}z^* + c_{234}\tilde{x}z^*u \\
&\quad + c_{244}z^*u^2 + c_{333}(\tilde{x}, \tilde{x}, \tilde{x}) + \tilde{x}^T c_{334}\tilde{x}u + c_{344}\tilde{x}u^2 + c_{444}u^3 + \text{h.o.t.}, \qquad (2.18) \\
\dot{\tilde{x}} &= A\tilde{x} + Bu + \tilde{q}_{11}z^2 + \tilde{q}_{12}|z|^2 + \tilde{q}_{13}\tilde{x}z + \tilde{q}_{14}zu + \tilde{q}_{22}z^{*2} + \tilde{q}_{23}\tilde{x}z^* \\
&\quad + \tilde{q}_{24}z^*u + \tilde{q}_{33}(\tilde{x}, \tilde{x}) + \tilde{q}_{34}\tilde{x}u + \tilde{q}_{44}u^2 + \text{h.o.t.}, \qquad (2.19)
\end{aligned}
$$

where $z \in \mathbb{C}$, $\tilde{x} \in \mathbb{R}^n$, (A, B) is in controller canonical form given by (2.4), and

$$
\begin{aligned}
& d, \mu, \omega \in \mathbb{R},\ \omega > 0, \\
& q_{11}, q_{12}, q_{14}, q_{22}, q_{24}, q_{44}, c_{111}, c_{112}, c_{114}, c_{122}, c_{124}, c_{144}, c_{222}, c_{224}, c_{244}, c_{444} \in \mathbb{C}, \\
& q_{13}, q_{23}, q_{34}, c_{113}, c_{123}, c_{223}, c_{234}, c_{344} \in \mathbb{C}^{1\times n}, \\
& \tilde{q}_{11}^* = \tilde{q}_{22}, \tilde{q}_{14}^* = \tilde{q}_{24} \in \mathbb{C}^{n\times 1},\ \ \tilde{q}_{12}, \tilde{q}_{44} \in \mathbb{R}^{n\times 1},\ \ \tilde{q}_{34} \in \mathbb{R}^{n\times n}, \\
& q_{33} = q_{33}^T, c_{133} = c_{133}^T, c_{233} = c_{233}^T, c_{334} = c_{334}^T, \tilde{q}_{13} = \tilde{q}_{23}^* \in \mathbb{C}^{n\times n},
\end{aligned}
$$

$$c_{333} \in \mathbb{C}^{n\times n\times n}, c_{333}(u,v,w) = c_{333}(\hat{u},\hat{v},\hat{w}), \quad \tilde{q}_{33} \in \mathbb{R}^{n\times n\times n}, \tilde{q}_{33}(u,v) = \tilde{q}_{33}(v,u),$$

where $(\hat{u},\hat{v},\hat{w})$ is any permutation of (u,v,w).

The goal is to find a sufficiently smooth (possibly nonlinear) feedback such that the dynamics on the linearly controllable subspace is asymptotically stable, and at the same time, the equilibrium $(0,0)$ for the closed loop system is asymptotically stable at the bifurcation point, i.e., at $\mu = 0$. It can be shown that if the bifurcation for the closed loop system is a nondegenerate supercritical Hopf bifurcation, then $(0,0)$ is asymptotically stable at $\mu = 0$; the converse is also true if certain nondegenerate conditions are satisfied.

Define

$$\mathfrak{m}_1(s) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathfrak{m}_2(s) = \begin{bmatrix} s \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathfrak{m}_{n-1}(s) = \begin{bmatrix} s^{n-2} \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \quad \mathfrak{m}_n(s) = \mathfrak{m}(s) = \begin{bmatrix} s^{n-1} \\ \vdots \\ s \\ 1 \end{bmatrix}.$$

For a row vector $v = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \in \mathbb{C}^{1\times n}$, we define the polynomials $v_j(s)$ by $v_j(s) = v\mathfrak{m}_j(s)$, where $(j = 1, \cdots, n)$. We also define

$$v(s) := v_n(s) = v_1 s^{n-1} + v_2 s^{n-2} + \cdots + v_n.$$

For a symmetric matrix $U = U^T \in \mathbb{C}^{n\times n}$ and $s_1, s_2 \in \mathbb{C}$, we define the polynomial $U(s_1,s_2)$ by $U(s_1,s_2) = \mathfrak{m}(s_1)^T U \mathfrak{m}(s_2)$. We also define

$$\begin{aligned} p(s) := p_n(s) &= \det[sI - A] = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n, \\ p_{n-1}(s) &= s^{n-1} + a_1 s^{n-2} + \cdots + a_{n-2}s + a_{n-1}, \\ \cdots \quad \cdots &\quad \cdots \quad \cdots \quad \cdots \\ p_1(s) &= s + a_1. \end{aligned} \tag{2.20}$$

Suppose $s, s_1, s_2 \in \mathbb{C}$, we define the following polynomials:

$$
\begin{aligned}
\Phi_1(s) &= q_{13}(s) + q_{14}p(s),\\
\tilde{\Phi}_1(s) &= \tilde{q}_{13}(s) + \tilde{q}_{14}p(s),\\
\Phi_2(s) &= q_{23}(s) + q_{24}p(s),\\
\tilde{\Phi}_2(s) &= \tilde{q}_{23}(s) + \tilde{q}_{24}p(s), \qquad (2.21)\\
\Psi(s_1, s_2) &= 2q_{33}(s_1, s_2) + p(s_1)q_{34}(s_2) + q_{34}(s_1)p(s_2) + 2q_{44}p(s_1)p(s_2),\\
\tilde{\Psi}(s_1, s_2) &= 2\tilde{q}_{33}(s_1, s_2) + p(s_1)\tilde{q}_{34}(s_2) + \tilde{q}_{34}(s_1)p(s_2) + 2\tilde{q}_{44}p(s_1)p(s_2),
\end{aligned}
$$

where $p(s)$ is defined as $p(s) = \det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$, and we have used the following notation:

$$\tilde{q}_{23}(s) := \tilde{q}_{23}\mathfrak{m}(s),\ \ \tilde{q}_{33}(s_1, s_2) := \tilde{q}_{33}(\mathfrak{m}(s_1), \mathfrak{m}(s_2)),\ \ \tilde{q}_{34}(s) := \tilde{q}_{34}\mathfrak{m}(s).$$

We define

$$
\begin{aligned}
\Pi_1^H &= \begin{bmatrix} 0 & q_{13}^1 & \cdots & q_{13}^{n-1} \end{bmatrix} + q_{14}\begin{bmatrix} 1 & a_1 & \cdots & a_{n-1} \end{bmatrix}, \qquad (2.22)\\
\Pi_2^H &= \begin{bmatrix} 0 & q_{23,1}(2\omega i) & \cdots & q_{23,n-1}(2\omega i) \end{bmatrix}\\
&\quad + q_{24}\begin{bmatrix} 1 & p_1(2\omega i) & \cdots & p_{n-1}(2\omega i) \end{bmatrix}, \qquad (2.23)\\
\Pi_3^H &= 2\mathfrak{m}(i\omega)^T \begin{bmatrix} 0 & q_{33}^1 & \cdots & q_{33}^{n-1} \end{bmatrix} + p(i\omega)\begin{bmatrix} 0 & q_{34}^1 & \cdots & q_{34}^{n-1} \end{bmatrix}\\
&\quad + [q_{34}(i\omega) + 2q_{44}p(i\omega)]\begin{bmatrix} 1 & a_1 & \cdots & a_{n-1} \end{bmatrix}. \qquad (2.24)\\
\Pi_4^H &= 2\mathfrak{m}(-i\omega)^T \begin{bmatrix} 0 & q_{33,1}(2\omega i) & \cdots & q_{33,n-1}(2\omega i) \end{bmatrix}\\
&\quad + p(-i\omega)\begin{bmatrix} 0 & q_{34,1}(2\omega i) & \cdots & q_{34,n-1}(2\omega i) \end{bmatrix}\\
&\quad + [q_{34}(-i\omega) + 2q_{44}p(-i\omega)]\begin{bmatrix} 1 & p_1(2\omega i) & \cdots & p_{n-1}(2\omega i) \end{bmatrix}, \qquad (2.25)\\
C_0 &= c_{112} - \frac{1}{i\omega}q_{11}q_{12} - \Pi_1^H\tilde{q}_{12} - \Pi_2^H\tilde{q}_{11}, \qquad (2.26)\\
D_1 &= c_{123}(i\omega) + c_{124}p(i\omega) - \Pi_1^H\tilde{\Phi}_2(i\omega) - \Pi_2^H\tilde{\Phi}_1(i\omega)\\
&\quad - \Pi_3^H\tilde{q}_{12} - \frac{2}{i\omega}q_{11}\Phi_2(i\omega), \qquad (2.27)
\end{aligned}
$$

$$\begin{aligned}
D_2 &= c_{113}(-i\omega) + c_{114}p(-i\omega) - \Pi_1^H \tilde{\Phi}_1(-i\omega) - \Pi_4^H \tilde{q}_{11} \\
&\quad -\frac{1}{i\omega}(q_{11} + q_{12}^*)\Phi_1(-i\omega) - \frac{1}{3\omega i} q_{22}^* \Phi_2(-i\omega), && (2.28) \\
E_{11} &= 2c_{233}(i\omega, i\omega) + c_{234}(i\omega)p(i\omega) + c_{244}p(i\omega)^2 - \frac{1}{2}\Pi_2^H \tilde{\Psi}(i\omega, i\omega) \\
&\quad -\Pi_3^H \tilde{\Phi}_2(i\omega) + \frac{1}{2\omega i}\left[q_{12}\Psi(i\omega, i\omega) - 2\Phi_1(i\omega)\Phi_2(i\omega)\right], && (2.29) \\
E_{12} &= 2c_{133}(i\omega, -i\omega) + c_{134}(i\omega)p(-i\omega) + c_{134}(-i\omega)p(i\omega) \\
&\quad +2c_{144}|p(i\omega)|^2 - \Pi_1^H \tilde{\Psi}(i\omega, -i\omega) - \Pi_3^H \tilde{\Phi}_1(-i\omega) \\
&\quad -\Pi_4^H \tilde{\Phi}_1(i\omega) - \frac{1}{i\omega}(2q_{11} + q_{12}^*)\Psi(i\omega, -i\omega), && (2.30) \\
E_{22} &= -\frac{1}{i\omega}\Phi_1(-i\omega)\Phi_2^*(-i\omega) - \frac{1}{3\omega i} q_{22}^* \Psi(-i\omega, -i\omega), && (2.31) \\
F_{112} &= 3c_{333}(i\omega, i\omega, -i\omega) + c_{334}(i\omega, i\omega)p(-i\omega) + 3c_{444}|p(i\omega)|^2 p(i\omega) \\
&\quad +2c_{334}(i\omega, -i\omega)p(i\omega) + 2c_{344}(i\omega)|p(i\omega)|^2 + c_{344}(-i\omega)p(i\omega)^2 \\
&\quad -\Pi_3^H \tilde{\Psi}(i\omega, -i\omega) - \frac{1}{2}\Pi_4^H \tilde{\Psi}(i\omega, i\omega) + \frac{1}{2\omega i}\Phi_1(-i\omega)\Psi(i\omega, i\omega) \\
&\quad -\frac{1}{i\omega}\left[\Phi_1(i\omega) + \Phi_1^*(i\omega)\right]\Psi(i\omega, -i\omega) - \frac{1}{6\omega i}\Phi_2(-i\omega)\Psi^*(i\omega, i\omega), && (2.32) \\
F_{122} &= -\frac{1}{i\omega}\Phi_2^*(-i\omega)\Psi(i\omega, -i\omega) - \frac{1}{i\omega}\Phi_1(-i\omega)\Psi^*(-i\omega, i\omega) \\
&\quad -\frac{1}{3\omega i}\Phi_2^*(i\omega)\Psi(-i\omega, -i\omega), && (2.33)
\end{aligned}$$

where $\Phi_j(s)(j = 1, 2)$, $\tilde{\Phi}_j(s)(j = 1, 2)$, $\Psi(s_1, s_2)$ and $\tilde{\Psi}(s_1, s_2)$ are defined in (2.21).

Define

$$\begin{aligned}
&\alpha_3^1 = \mathrm{Re}(F_{112} + F_{122}), && \alpha_3^2 = \mathrm{Re}(F_{122} - F_{112}), \\
&\alpha_0 = \mathrm{Re}\, C_0, && \alpha_2^{11} = \mathrm{Re}(E_{11} + E_{12} + E_{22}), \\
&\alpha_1^1 = \mathrm{Re}(D_1 + D_2), && \alpha_2^{12} = \mathrm{Im}(E_{22} - E_{11}), \\
&\alpha_1^2 = \mathrm{Im}(D_2 - D_1), && \alpha_2^{22} = \mathrm{Re}(E_{12} - E_{11} - E_{22}),
\end{aligned} \tag{2.34}$$

and

$$\alpha_1 = \begin{bmatrix} \alpha_1^1 & \alpha_1^2 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} \alpha_2^{11} & \alpha_2^{12} \\ \alpha_2^{12} & \alpha_2^{22} \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} \alpha_3^1 & \alpha_3^2 \end{bmatrix}. \tag{2.35}$$

Let $T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$ be an orthonormal matrix, i.e., $T^{-1} = T^T$, such that

$$\hat{\alpha}_1 := \begin{bmatrix} \hat{\alpha}_1^1 & \hat{\alpha}_1^2 \end{bmatrix} = \begin{bmatrix} \alpha_1^1 & \alpha_1^2 \end{bmatrix} T, \quad \hat{\alpha}_2 := T^T \alpha_2 T = \begin{bmatrix} \hat{\alpha}_2^1 & 0 \\ 0 & \hat{\alpha}_2^2 \end{bmatrix}, \tag{2.36}$$

where T and $\hat{\alpha}_2$ can be explicitly calculated as

$$T = \begin{bmatrix} \dfrac{\alpha_2^{12}}{\sqrt{(\alpha_2^{12})^2 + (\hat{\alpha}_2^1 - \alpha_2^{11})^2}} & \dfrac{\alpha_2^{12}}{\sqrt{(\alpha_2^{12})^2 + (\hat{\alpha}_2^2 - \alpha_2^{11})^2}} \\ \dfrac{\hat{\alpha}_2^1 - \alpha_2^{11}}{\sqrt{(\alpha_2^{12})^2 + (\hat{\alpha}_2^1 - \alpha_2^{11})^2}} & \dfrac{\hat{\alpha}_2^2 - \alpha_2^{11}}{\sqrt{(\alpha_2^{12})^2 + (\hat{\alpha}_2^2 - \alpha_2^{11})^2}} \end{bmatrix}, \tag{2.37}$$

$$\hat{\alpha}_2^1 = \frac{\alpha_2^{11} + \alpha_2^{22}}{2} + \sqrt{\left(\frac{\alpha_2^{11} - \alpha_2^{22}}{2}\right)^2 + (\alpha_2^{12})^2}, \tag{2.38}$$

$$\hat{\alpha}_2^2 = \frac{\alpha_2^{11} + \alpha_2^{22}}{2} - \sqrt{\left(\frac{\alpha_2^{11} - \alpha_2^{22}}{2}\right)^2 + (\alpha_2^{12})^2}. \tag{2.39}$$

Define

$$\hat{\alpha}_{1m} = \frac{(\hat{\alpha}_1^1)^2}{4\hat{\alpha}_2^1}, \quad \hat{\alpha}_{2m} = \frac{(\hat{\alpha}_1^2)^2}{4\hat{\alpha}_2^2}, \quad \hat{\alpha}_m = \hat{\alpha}_{1m} + \hat{\alpha}_{2m}. \tag{2.40}$$

Define

$$\begin{aligned} &\Theta_1 = \operatorname{Re}\Phi_1(0), \quad \Theta_3 = \Psi(i\omega, 0), \qquad \Theta_5 = F_{112} + F_{122}^*, \\ &\Theta_2 = \Phi_2(2\omega i), \quad \Theta_4 = \Psi(-i\omega, 2\omega i), \quad \Theta = \textstyle\sum_{j=1}^5 |\Theta_j|, \end{aligned}$$

F_{112} and F_{122} are given by (2.32) and (2.33). It is clear that $\Theta = 0$ if and only if $\Theta_j = 0$ $(j = 1, \cdots, 5)$.

The degenerate cases are defined as those satisfying the following conditions

HD-1 $\Theta = \hat{\alpha}_2^1 = \hat{\alpha}_2^2 = \hat{\alpha}_1^1 = \hat{\alpha}_1^2 = \alpha_0 = 0$,

HD-2 $\Theta = \hat{\alpha}_2^1 = \hat{\alpha}_1^1 = 0,\ \hat{\alpha}_2^2 > 0,\ \alpha_0 = \hat{\alpha}_{2m} \geq 0,$

HD-3 $\Theta = \hat{\alpha}_2^2 = \hat{\alpha}_1^2 = 0,\ \hat{\alpha}_2^1 > 0,\ \alpha_0 = \hat{\alpha}_{1m} \geq 0,$

HD-4 $\Theta = 0,\ \hat{\alpha}_2^1 > 0,\ \hat{\alpha}_2^2 > 0,\ \alpha_0 = \hat{\alpha}_m \geq 0.$

Note condition **HD-1** implies $\alpha_j = 0$ $(j = 0, 1, 2, 3)$.

Theorem 2.2 *Except for the degenerate cases, there does not exist a sufficiently smooth state feedback*

$$\begin{aligned} u \;=\; & K_1\tilde{x} + K_2 z + K_3 z^* + K_4 z^2 + K_5|z|^2 + K_6 z^{*2} \\ & + K_7\tilde{x}z + K_8\tilde{x}z^* + \tilde{x}^T K_9 \tilde{x} + h.o.t. \end{aligned} \tag{2.41}$$

with

$$\begin{aligned} &K_1 \in \mathbb{R}^{1\times n}, \quad K_2 = K_3^* \in \mathbb{C}, \qquad K_4 = K_6^* \in \mathbb{C}, \\ &K_5 \in \mathbb{R}, \qquad\;\; K_7 = K_8^* \in \mathbb{C}^{1\times n}, \quad K_9 = K_9^T \in \mathbb{R}^{n\times n}, \end{aligned}$$

such that

(i) $A + BK_1$ *is Hurwitz,*

(ii) At $\mu = 0$*, the equilibrium is asymptotically stable,*

if and only if one of the following conditions holds.

HU-1 $\Theta = \hat{\alpha}_2^1 = \hat{\alpha}_2^2 = \hat{\alpha}_1^1 = \hat{\alpha}_1^2 = 0,\ \alpha_0 > 0,$

HU-2 $\Theta = \hat{\alpha}_2^1 = \hat{\alpha}_1^1 = 0,\ \hat{\alpha}_2^2 > 0,\ \alpha_0 > \hat{\alpha}_{2m} \geq 0,$

HU-3 $\Theta = \hat{\alpha}_2^2 = \hat{\alpha}_1^2 = 0,\ \hat{\alpha}_2^1 > 0,\ \alpha_0 > \hat{\alpha}_{1m} \geq 0,$

HU-4 $\Theta = 0,\ \hat{\alpha}_2^1 > 0,\ \hat{\alpha}_2^2 > 0,\ \alpha_0 > \hat{\alpha}_m \geq 0.$

where $\hat{\alpha}_2^1$ *and* $\hat{\alpha}_2^2$ *are given by (2.38) and (2.39), respectively.* $\hat{\alpha}_1^k$ $(k = 1, 2)$ *is given by (2.36), and* $\hat{\alpha}_{1m}$*,* $\hat{\alpha}_{2m}$*, and* $\hat{\alpha}_m$ *are given in (2.40).*

Classification of Stabilizability for Hopf Bifurcations

Cases		Θ_1	Θ_2	Θ_3	Θ_4	Θ_5	$\hat{\alpha}_2^1$	$\hat{\alpha}_2^2$	$\hat{\alpha}_1^1$	$\hat{\alpha}_1^2$	α_0
Stabilizable	**HS-1**										< 0
	HS-2	$\neq 0$									≥ 0
	HS-3	0	$\neq 0$								≥ 0
	HS-4	0	0	$\neq 0$							≥ 0
	HS-5	0	0	0	$\neq 0$						≥ 0
	HS-6	0	0	0	0	$\neq 0$					≥ 0
	HS-7	0	0	0	0	0	> 0	0		0	$0 \leq \alpha_0 < \hat{\alpha}_{1m}$
	HS-8	0	0	0	0	0	0	> 0	0		$0 \leq \alpha_0 < \hat{\alpha}_{2m}$
	HS-9	0	0	0	0	0	> 0	> 0			$0 \leq \alpha_0 < \hat{\alpha}_m$
	HS-10	0	0	0	0	0	< 0				≥ 0
	HS-11	0	0	0	0	0		< 0			≥ 0
	HS-12	0	0	0	0	0	0	0	$\neq 0$		≥ 0
	HS-13	0	0	0	0	0	0	0		$\neq 0$	≥ 0
Degeneracy Conditions	**HD-1**	0	0	0	0	0	0	0	0	0	0
	HD-2	0	0	0	0	0	0	> 0	0	≥ 0	$= \hat{\alpha}_{2m} \geq 0$
	HD-3	0	0	0	0	0	> 0	0	≥ 0	0	$= \hat{\alpha}_{1m} \geq 0$
	HD-4	0	0	0	0	0	> 0	> 0			$= \hat{\alpha}_m \geq 0$
Not Stabilizable	**HU-1**	0	0	0	0	0	0	0	0	0	> 0
	HU-2	0	0	0	0	0	0	> 0	0	≥ 0	$> \hat{\alpha}_{2m} \geq 0$
	HU-3	0	0	0	0	0	> 0	0	≥ 0	0	$> \hat{\alpha}_{1m} \geq 0$
	HU-4	0	0	0	0	0	> 0	> 0			$> \hat{\alpha}_m \geq 0$

Table 2.3: Classification of stabilizability for Hopf bifurcations.

A proof of the theorem is given in Section 2.5.

Classification of different cases is given in Table 2.3. As before, the blank spaces in each case imply that there are no explicit constraints on the corresponding parameters, but they may be implicitly constrained by other columns in the same row. Cases **HU-1** to **HU-4** are such that there does not exist a sufficiently smooth state feedback such that $(0,0)$ is asymptotically stable for the closed loop system for $\mu = 0$; cases from **HD-1** to **HD-4** are degenerate, i.e., we have to resort to higher (4^{th} and 5^{th}) order terms to determine the stability of the closed loop system; cases **HS-1** to **HS-13** are such that $(0,0)$ can be stabilized by state feedbacks at $\mu = 0$. For the stabilizable cases from **HS-1** to **HS-13**, we could construct the feedback explicitly such that

$A + BK_1$ is Hurwitz and $(0,0)$ is asymptotically stable. We define

$$\begin{aligned}\bar{p}(s) := \bar{p}_n(s) &= \det[sI - (A + BK_1)] = s^n + \bar{a}_1 s^{n-1} + \cdots + \bar{a}_{n-1}s + \bar{a}_n,\\ \bar{p}_{n-1}(s) &= s^{n-1} + \bar{a}_1 s^{n-2} + \cdots + \bar{a}_{n-2}s + \bar{a}_{n-1},\\ \cdots\ \cdots & \cdots\ \cdots\ \cdots\\ \bar{p}_1(s) &= s + \bar{a}_1.\end{aligned} \tag{2.42}$$

Define $K = \dfrac{K_2}{\bar{p}(i\omega)} = K_R + iK_I$, $K_\flat := \begin{bmatrix} K_R \\ K_I \end{bmatrix}$, and

$$P_H(K_\flat) = \alpha_0 + \alpha_1 K_\flat + K_\flat^T \alpha_2 K_\flat + \alpha_3 K_\flat \|K_\flat\|^2, \tag{2.43}$$

$$Q_1 = q_{11} + \Phi_1(i\omega)K + \frac{1}{2}\Psi(i\omega, i\omega)K^2, \tag{2.44}$$

$$Q_2 = q_{12} + \Phi_2(i\omega)K + \Phi_1(-i\omega)K^* + \Psi(i\omega, -i\omega)|K|^2, \tag{2.45}$$

$$Q_3 = q_{22} + \Phi_2(-i\omega)K^* + \frac{1}{2}\Psi(-i\omega, -i\omega)K^{*2}, \tag{2.46}$$

$$\tilde{Q}_1 = \tilde{q}_{11} + \tilde{\Phi}_1(i\omega)K + \frac{1}{2}\tilde{\Psi}(i\omega, i\omega)K^2, \tag{2.47}$$

$$\tilde{Q}_2 = \tilde{q}_{12} + \tilde{\Phi}_2(i\omega)K + \tilde{\Phi}_1(-i\omega)K^* + \tilde{\Psi}(i\omega, -i\omega)|K|^2, \tag{2.48}$$

where $\Phi_j(s)$ $(j = 1, 2)$, $\tilde{\Phi}_j(s)$ $(j = 1, 2)$, $\Psi(s_1, s_2)$ and $\tilde{\Psi}(s_1, s_2)$ are given in (2.21), and α_j $(j = 0, \cdots, 3)$ are given in (2.34) and (2.35). Then from Section 2.5 the normal form of the dynamics on the center manifold is given by

$$\dot{\zeta} = (d\mu + i\omega)\zeta + \tilde{\alpha}|\zeta|^2\zeta + \text{h.o.t.},$$

where $\alpha := \operatorname{Re}\tilde{\alpha}$ is given by

$$\begin{aligned}\alpha = & \frac{\Theta_1 + \operatorname{Re}\{K\Theta_3\}}{\bar{p}(0)} \Big\{ K_5 + K_7(-i\omega)K^* + K_7^*(i\omega)K + 2K_9(i\omega, -i\omega)|K|^2 \\ & + \begin{bmatrix} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1} \end{bmatrix} \left(\tilde{Q}_2 - \mathfrak{m}(i\omega)Q_2K - \mathfrak{m}(-i\omega)Q_2^*K^*\right) \\ & + \frac{\bar{p}(0)}{i\omega}\left(Q_2^*K^* - Q_2K\right) \Big\} +\end{aligned}$$

$$+\mathrm{Re}\left\{\frac{\Theta_2+K^*\Theta_4}{\bar{p}(2\omega i)}\left[K_4+K_7(i\omega)K+K_9(i\omega,i\omega)K^2+\right.\right.$$
$$\left[\begin{array}{cccc}1 & \bar{p}_1(2\omega i) & \cdots & \bar{p}_{n-1}(2\omega i)\end{array}\right]\left(\tilde{Q}_1-\mathfrak{m}(i\omega)Q_1K-\mathfrak{m}(-i\omega)Q_3^*K^*\right)$$
$$\left.\left.+\frac{\bar{p}(2\omega i)}{i\omega}\left(Q_1K+\frac{1}{3}Q_3^*K^*\right)\right]\right\}$$
$$+\alpha_0+\alpha_1K_\flat+K_\flat^T\alpha_2K_\flat+\alpha_3K_\flat\|K_\flat\|^2. \tag{2.49}$$

Since the cases when $\alpha<0$, $\alpha>0$ and $\alpha=0$ are corresponding to the supercritical, subcritical, and degenerate Hopf bifurcation of the closed loop system, the goal of designing a stabilizing feedback is equivalent to finding feedback gains $K_j\ (j=1,\cdots,9)$ such that $\alpha<0$. When such gains do not exist, then the system is either unstabilizable or degenerate.

In the following, we give explicit construction of stabilizing controllers. The goal is to choose feedback gains $K_1,\cdots,K_9$ such that $\alpha:=\mathrm{Re}\,\tilde{\alpha}<0$. First, we select K_1 such that $A+BK_1$ is Hurwitz. The construction of stabilizing feedback for the cases **HS-1** to **HS-13** in Table 2.3 is given as follows.

HS-1 Let $K_j=0\ (j=2,3,7,8,9)$. Select $K_5=-\left[\begin{array}{cccc}1 & \bar{a}_1 & \cdots & \bar{a}_{n-1}\end{array}\right]\tilde{q}_{12}$, and $K_4=-\left[\begin{array}{cccc}1 & \bar{p}_1(2\omega i) & \cdots & \bar{p}_{n-1}(2\omega i)\end{array}\right]\tilde{q}_{11}$. Then we have $\alpha=\alpha_0<0$.

HS-2 Let $K_j=0\ (j=2,\cdots,9,\ j\neq 5)$, and select K_5 such that

$$\alpha=\alpha_0+\frac{\Theta_1}{\bar{p}(0)}\left(K_5+\left[\begin{array}{cccc}1 & \bar{a}_1 & \cdots & \bar{a}_{n-1}\end{array}\right]\tilde{q}_{12}\right)<0.$$

HS-3 Let $K_j=0\ (j=2,\cdots,9,\ j\neq 4,6)$, and select $K_4=K_6^*$ such that

$$\alpha=\alpha_0+\mathrm{Re}\left\{\frac{\Theta_2}{\bar{p}(2\omega i)}\left(K_4+\left[\begin{array}{cccc}1 & \bar{p}_1(2\omega i) & \cdots & \bar{p}_{n-1}(2\omega i)\end{array}\right]\tilde{q}_{11}\right)\right\}<0.$$

HS-4 Let $K_j=0\ (j=4,\cdots,9,\ j\neq 5)$, for any $K_2=K_3^*\neq 0$, select K_5 such that

$$\begin{aligned}\alpha &= \frac{\mathrm{Re}\{\Theta_3K\}}{\bar{p}(0)}\left\{K_5+\left[\begin{array}{cccc}1 & \bar{a}_1 & \cdots & \bar{a}_{n-1}\end{array}\right]\left(\tilde{Q}_2-\mathfrak{m}(i\omega)Q_2K\right.\right.\\ &\quad \left.\left.-\mathfrak{m}(-i\omega)Q_2^*K^*\right)+\frac{\bar{p}(0)}{i\omega}\left(Q_2^*K^*-Q_2K\right)\right\}+P_H(K_\flat)<0,\end{aligned}$$

where $P_H(K_\flat)$ is given by (2.43) which is independent of K_5.

HS-5 Let $K_j = 0$ ($j = 5, \cdots, 9,\ j \neq 6$), for any $K_2 = K_3^* \neq 0$, select $K_4 = K_6^*$ such that

$$\begin{aligned}\alpha &= \mathrm{Re}\left\{\frac{\Theta_4 K^*}{\bar{p}(2\omega i)}\left[K_4 + \begin{bmatrix} 1 & \bar{p}_1(2\omega i) & \cdots & \bar{p}_{n-1}(2\omega i)\end{bmatrix}\cdot\right.\right.\\ &\left(\tilde{Q}_1 - \mathfrak{m}(i\omega)Q_1K - \mathfrak{m}(-i\omega)Q_3^*K^*\right)\\ &\left.\left.+\frac{\bar{p}(2\omega i)}{i\omega}\left(Q_1K + \frac{1}{3}Q_3^*K^*\right)\right]\right\} + P_H(K_\flat) < 0,\end{aligned}$$

where $P_H(K_\flat)$ is given by (2.43) which is independent of K_4.

HS-6 Let $K_j = 0$ ($j = 1, \cdots, 9, j \neq 2$), and select $K_2 = K_3^*$ with $|K_2|$ large enough such that

$$\alpha = \alpha_0 + \alpha_1 K_\flat + K_\flat^T \alpha_2 K_\flat + \alpha_3 K_\flat \|K_\flat\|^2 < 0,$$

where $K_\flat := \begin{bmatrix} K_R \\ K_I \end{bmatrix}$, $K = \dfrac{K_2}{\bar{p}(i\omega)} = K_R + iK_I$, and α_j ($j = 0, \cdots, 3$) are given in (2.34) and (2.35).

HS-7 to HS-11 Letting $K_\flat = T\begin{bmatrix} \xi \\ \eta \end{bmatrix}$, where $T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$ is given by (2.37) such that

$$\begin{bmatrix} \alpha_1^1 & \alpha_1^2 \end{bmatrix} T = \begin{bmatrix} \hat{\alpha}_1^1 & \hat{\alpha}_1^2 \end{bmatrix}, \quad T^T \alpha_2 T = \begin{bmatrix} \hat{\alpha}_2^1 & 0 \\ 0 & \hat{\alpha}_2^2 \end{bmatrix} := \hat{\alpha}_2.$$

So we have

$$\begin{aligned}\alpha &= \alpha_0 + \alpha_1 K_\flat + K_\flat^T \alpha_2 K_\flat,\\ &= \alpha_0 + \hat{\alpha}_1^1 \xi + \hat{\alpha}_1^2 \eta + \hat{\alpha}_2^1 \xi^2 + \hat{\alpha}_2^2 \eta^2.\end{aligned}$$

HS-7 Let $K_j = 0$ ($j = 1, \cdots, 9, j \neq 2$), and $K_R = -\dfrac{\hat{\alpha}_1^1}{2\hat{\alpha}_2^1} t_{11}$, $K_I = -\dfrac{\hat{\alpha}_1^1}{2\hat{\alpha}_2^1} t_{21}$, where

t_{11} and t_{21} are given by (2.37). Then the minimum of

$$\alpha = \alpha_0 + \alpha_1 K_\flat + K_\flat^T \alpha_2 K_\flat = \alpha_0 + \hat{\alpha}_1^1 \xi + \hat{\alpha}_2^1 \xi^2$$

is $\hat{\alpha}_0 = \alpha_0 - \hat{\alpha}_{1m} < 0$ when $\xi = -\dfrac{\hat{\alpha}_1^1}{2\hat{\alpha}_2^1}$ and $\eta = 0$, where $\hat{\alpha}_{1m} := \frac{(\hat{\alpha}_1^1)^2}{4\hat{\alpha}_2^1}$.

HS-8 Let $K_j = 0$ $(j = 1, \cdots, 9, j \neq 2)$, and $K_R = -\dfrac{\hat{\alpha}_1^2}{2\hat{\alpha}_2^2} t_{12}$, $K_I = -\dfrac{\hat{\alpha}_1^2}{2\hat{\alpha}_2^2} t_{22}$, where t_{12} and t_{22} are given by (2.37). Then the minimum of

$$\alpha = \alpha_0 + \alpha_1 K_\flat + K_\flat^T \alpha_2 K_\flat = \alpha_0 + \hat{\alpha}_1^2 \eta + \hat{\alpha}_2^2 \eta^2$$

is $\hat{\alpha}_0 = \alpha_0 - \hat{\alpha}_{2m} < 0$ when $\xi = 0$ and $\eta = -\dfrac{\hat{\alpha}_1^2}{2\hat{\alpha}_2^2}$, where $\hat{\alpha}_{2m} := \frac{(\hat{\alpha}_1^2)^2}{4\hat{\alpha}_2^2}$.

HS-9 Let $K_j = 0$ $(j = 1, \cdots, 9, j \neq 2)$, and

$$\begin{bmatrix} K_R \\ K_I \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} -\frac{\hat{\alpha}_1^1}{2\hat{\alpha}_2^1} \\ -\frac{\hat{\alpha}_1^2}{2\hat{\alpha}_2^2} \end{bmatrix},$$

where t_{jk} $(j, k = 1, 2)$ are given by (2.37). Then the minimum of

$$\alpha = \alpha_0 + \alpha_1 K_\flat + K_\flat^T \alpha_2 K_\flat = \alpha_0 + \hat{\alpha}_1^1 \xi + \hat{\alpha}_1^2 \eta + \hat{\alpha}_2^1 \xi^2 + \hat{\alpha}_2^2 \eta^2$$

is $\hat{\alpha}_0 = \alpha_0 - \hat{\alpha}_m < 0$ when $\xi = -\dfrac{\hat{\alpha}_1^1}{2\hat{\alpha}_2^1}$ and $\eta = -\dfrac{\hat{\alpha}_1^2}{2\hat{\alpha}_2^2}$, where $\hat{\alpha}_m = \hat{\alpha}_{1m} + \hat{\alpha}_{2m}$.

HS-10 Let $K_j = 0$ $(j = 1, \cdots, 9, j \neq 2)$, and $K_R = t_{11}\xi$, $K_I = t_{21}\xi$, where ξ satisfies

$$\alpha = \alpha_0 + \alpha_1 K_\flat + K_\flat^T \alpha_2 K_\flat = \alpha_0 + \hat{\alpha}_1^1 \xi + \hat{\alpha}_2^1 \xi^2 < 0.$$

Note this inequality can always be satisfied because $\hat{\alpha}_2^1 < 0$.

HS-11 Let $K_j = 0$ $(j = 1, \cdots, 9, j \neq 2)$, and $K_R = t_{12}\eta$, $K_I = t_{22}\eta$, where η satisfies

$$\alpha = \alpha_0 + \alpha_1 K_\flat + K_\flat^T \alpha_2 K_\flat = \alpha_0 + \hat{\alpha}_1^2 \eta + \hat{\alpha}_2^2 \eta^2 < 0.$$

Note this inequality can always be satisfied because $\hat{\alpha}_2^2 < 0$.

HS-12 Let $K_j = 0$ $(j = 1, \cdots, 9, j \neq 2)$, $K_I = 0$, and K_R satisfy

$$\alpha = \alpha_0 + \hat{\alpha}_1^1 K_R < 0.$$

HS-13 Let $K_j = 0$ $(j = 1, \cdots, 9, j \neq 2)$, $K_R = 0$, and K_R satisfy

$$\alpha = \alpha_0 + \hat{\alpha}_1^2 K_I < 0.$$

The above construction procedure is given in Table 2.2. We select K_1 such that $A + BK_1$ is Hurwitz, and let $K_7 = K_8^* = 0$, $K_9 = 0$. We choose $K_2 = K_3^*$, $K_4 = K_6^*$, and K_5 to satisfy the conditions in Table 2.2. The notations in the table are given by

$$\begin{aligned}
\bar{\mathfrak{p}}(0) &= \begin{bmatrix} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1} \end{bmatrix}, \\
\bar{\mathfrak{p}}(2\omega i) &= \begin{bmatrix} 1 & \bar{p}_1(2\omega i) & \cdots & \bar{p}_{n-1}(2\omega i) \end{bmatrix}, \\
L_1(K_5) &= \alpha_0 + \frac{\Theta_1}{\bar{p}(0)} \left[K_5 + \bar{\mathfrak{p}}(0)\tilde{q}_{11}\right], \\
L_2(K_4) &= \alpha_0 + \mathrm{Re}\left\{ \frac{\Theta_2}{\bar{p}(2\omega i)} \left[K_4 + \bar{\mathfrak{p}}(2\omega i)\tilde{q}_{12}\right] \right\}, \\
L_3(K_5) &= \frac{\mathrm{Re}\left\{\Theta_3 K\right\}}{\bar{p}(0)} \left\{ K_5 + \begin{bmatrix} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1} \end{bmatrix} \cdot \right. \\
&\quad \left(\tilde{Q}_2 - \mathfrak{m}(i\omega) Q_2 K - \mathfrak{m}(-i\omega) Q_2^* K^*\right) \\
&\quad \left. + \frac{\bar{p}(0)}{i\omega} \left(Q_2^* K^* - Q_2 K\right) \right\} + P_H(K_\flat), \\
L_4(K_4) &= \mathrm{Re}\left\{ \frac{\Theta_4 K^*}{\bar{p}(2\omega i)} \left[K_4 + \begin{bmatrix} 1 & \bar{p}_1(2\omega i) & \cdots & \bar{p}_{n-1}(2\omega i) \end{bmatrix} \cdot \right. \right. \\
&\quad \left(\tilde{Q}_1 - \mathfrak{m}(i\omega) Q_1 K - \mathfrak{m}(-i\omega) Q_3^* K^*\right) \\
&\quad \left. \left. + \frac{\bar{p}(2\omega i)}{i\omega} \left(Q_1 K + \frac{1}{3} Q_3^* K^* \right) \right] \right\} + P_H(K_\flat), \\
P_H(K_\flat) &= \alpha_0 + \alpha_1 K_\flat + K_\flat^T \alpha_2 K_\flat + \alpha_3 K_\flat \|K_\flat\|^2.
\end{aligned}$$

It is clear that $L_1(K_5)$ and $L_3(K_5)$ are linear functions of K_5, and $L_2(K_4)$ and $L_4(K_4)$

Construction of Stabilizing Feedback for Hopf Bifurcations			
	$K := \frac{K_2}{\bar{a}_n} = K_R + iK_I$	$K_4 = K_6^*$	K_5
HS-1	0	$-\bar{\mathfrak{p}}(2\omega i)\tilde{q}_{11}$	$-\bar{\mathfrak{p}}(0)\tilde{q}_{12}$
HS-2	0	0	$L_1(K_5) < 0$
HS-3	0	$L_2(K_4) < 0$	0
HS-4	$K \neq 0$	0	$L_3(K_5) < 0$
HS-5	$K \neq 0$	$L_4(K_4) < 0$	0
HS-6	$\Vert K_\flat \Vert$ large enough, $P_H(K_\flat) < 0$	0	0
HS-7	$-\frac{\hat{\alpha}_1^1}{2\hat{\alpha}_2^1}(t_{11} + it_{21})$	0	0
HS-8	$-\frac{\hat{\alpha}_1^2}{2\hat{\alpha}_2^2}(t_{12} + it_{22})$	0	0
HS-9	$\begin{bmatrix} K_R \\ K_I \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} -\frac{\hat{\alpha}_1^1}{2\hat{\alpha}_2^1} \\ -\frac{\hat{\alpha}_1^2}{2\hat{\alpha}_2^2} \end{bmatrix}$	0	0
HS-10	$K_R = t_{11}\xi,\ K_I = t_{21}\xi$	0	0
HS-11	$K_R = t_{12}\eta,\ K_I = t_{22}\eta$	0	0
HS-12	$K_I = 0,\ \alpha_0 + \alpha_1^1 K_R < 0$	0	0
HS-13	$K_R = 0,\ \alpha_0 + \alpha_1^2 K_I < 0$	0	0

Table 2.4: Construction of stabilizing feedbacks for Hopf bifurcations.

are linear functions of K_4. Hence the constructions **HS-1** to **HS-13** are explicit in that the stabilizing controllers can be solved explicitly.

Example 2.2 Consider the following system

$$\begin{aligned} \dot{z} &= i\omega z + \gamma x^2, \\ \dot{x} &= u, \end{aligned}$$

where $z, \gamma \in \mathbb{C}$, $x, u, \omega\ (\omega \neq 0) \in \mathbb{R}$. γ is a parameter. It is easy to check that for $\gamma \neq 0$, then

$$\begin{aligned} &\theta_1 = 0, \quad \Theta_2 = 0, \quad \Theta_3 = \gamma, \quad \Theta_4 = \gamma, \ \Pi_j^H = 0\ (j = 1, \cdots, 4), \\ &Q_1 = \gamma K^2, \ Q_2 = 2\gamma K^2, \ Q_3 = \gamma K^{*2}, \ \tilde{Q}_1 = 0, \ \tilde{Q}_2 = 0. \end{aligned}$$

So the condition **HS-4** is satisfied. By the construction procedure **HS-4**, we first select $K_1 = -1$ and let $K_4 = K_6^* = 0$, $K_7 = K_8^* = 0$, and $K_9 = 0$. Then by substituting all the above values to (2.49), we get

$$\begin{aligned}\tilde{\alpha} &= 2\gamma K K_5 - 4\gamma |K|^2 K\left(\gamma K + \gamma^* K^*\right) \\ &\quad -\frac{2\gamma|K|^2}{2\omega i + 1}\left(\gamma K^2 + \gamma^*|K|^2\right) - \frac{2\gamma^2|K|^2K^2}{i\omega}.\end{aligned}$$

By selecting $K = \gamma^*$ and letting $\mathrm{Re}\,\tilde{\alpha} < 0$, we get

$$K_5 < \left(4 + \frac{2}{1+4\omega^2}\right)|\gamma|^4. \tag{2.50}$$

Hence a stabilizing feedback is given by

$$u = -x + \gamma^* z + \gamma z^* + K_5|z|^2,$$

where $K_5 \in \mathbb{R}$ satisfies (2.50).

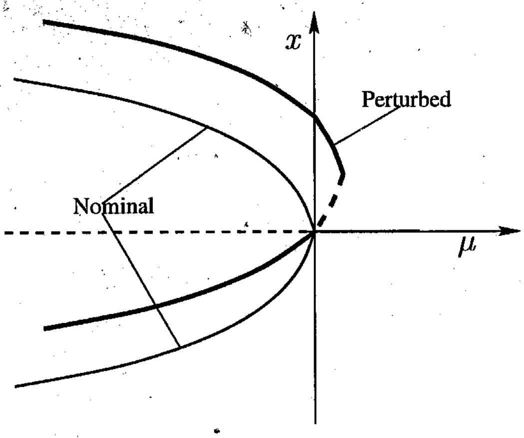
If $\gamma = 0$, then the system satisfies the degeneracy condition **HD-1** and it is inconclusive by Theorem 2.2.

2.3 Robustness Issues

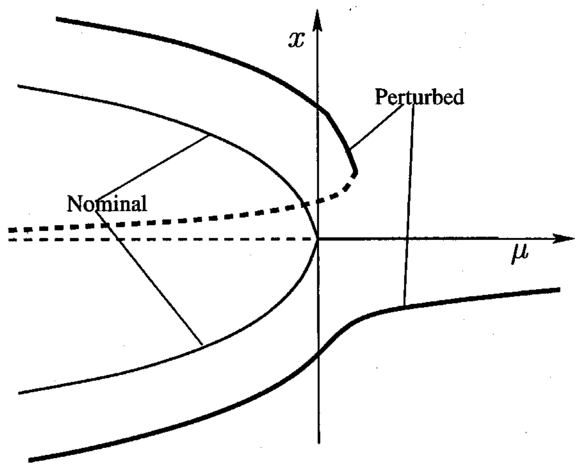
Robustness is one of the most important issues in control design. In this section we discuss the sensitivity of closed loop bifurcation diagrams to control gain variations and parametric perturbations. We also examine the robustness of stabilizability with respect to parametric variations. We will discuss in later chapters another important issue of robustness: the closed loop system behavior in the presence of actuator magnitude saturation, bandwidth and rate limits.

For the steady-state bifurcation case, it is clear from Section 2.1 that under nondegeneracy conditions, the bifurcation is stabilizable if and only if there exists a control law

$$u = K_1\tilde{x} + K_2 x + K_3 x^2 + K_4 \tilde{x} x + \tilde{x}^T K_5 \tilde{x} + \text{h.o.t.},$$



(a) Perturbation of Control Gains



(b) Perturbation of System Parameters

Figure 2.1: Bifurcation diagram of the perturbed system in the steady-state case.

such that $Q = 0$ and $C < 0$, where Q and C are given by

$$\begin{aligned} Q &= q_{11} + \Upsilon_1 K + \Upsilon_2 K^2, \\ C &= \frac{1}{\bar{a}_n}(\Upsilon_1 + 2\Upsilon_2 K)\left(K_3 + K_4^n K + K_5^{nn} K^2 + \begin{bmatrix} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1} \end{bmatrix} \tilde{Q}\right) \\ &\quad + \left[\alpha_0 + \alpha_1 K + \alpha_2 K^2 + \alpha_3 K^3\right], \end{aligned}$$

where $K := \frac{K_2}{\bar{a}_n}$. Meanwhile, the dynamics on the center manifold is given by

$$\dot{x} = d\mu x + Qx^2 + Cx^3 + \text{h.o.t.}.$$

Suppose K_j $(j = 1, \cdots, 5)$ are selected such that $Q = 0$ and $C < 0$. Now by perturbing K_j to $K_j + \epsilon_j$, where ϵ_j are small, then generically we get $Q \neq 0$ and $C < 0$. This implies that under generic perturbations of the controller gains, the local bifurcation of the closed loop system is a transcritical bifurcation instead of the supercritical pitchfork bifurcation of the unperturbed system. The bifurcations of the nominal and perturbed system are given by Figure 2.1 (a). It can be seen that a saddle-node bifurcation occurs on the bifurcated equilibrium branch in addition to the transcritical bifurcation at $\mu = 0$.

If we fix the controller gains and allow generic parametric perturbations to the systems parameters, then the nominal equilibrium branch $x(\mu) = 0$ will be perturbed away and a generic bifurcation diagram for the closed loop system is given by Figure 2.1 (b). It can be seen that the transcritical bifurcation is broken and ceases to exist, but a saddle-node bifurcation pops up on one of the equilibrium branches.

We mention here that in the above discussions we only consider generic perturbations. Certainly there are perturbations that do not change the qualitative local bifurcation behavior of the closed loop system, but only change the amplitude of the bifurcation equilibrium. In practice the perturbations might not be mathematically generic so we have to quantify the uncertainties in order to analyze bifurcation of the closed loop system under those perturbations. Also we note here that if the perturbations are small enough, then the perturbed bifurcation equilibria and the nominal bifurcation equilibria are close so the perturbations might not have huge impact in some practical applications, although the local qualitative bifurcation behavior is altered by the perturbations.

We point out an important fact that stabilizability is non-robust to parametric variations. For example, the case of **SS-1** with $\Upsilon_1 = \Upsilon_2 = 0$ can be perturbed into case **SU-4** by perturbing $q_{11} = 0$ to $q_{11} = \epsilon \neq 0$. This is in contrast to the linear time invariant systems whose stabilizability is robust to parameter variations. Apparently, the degenerate cases are non-robust: any arbitrarily small perturbations will perturb the system into a stabilizable one or an unstabilizable one. Also, the concept of unstabilizability is nonrobust to parametric variations. For example, the case **SU-1** will be perturbed into **SS-3** by perturbing $\alpha_3 = 0$ to $\alpha_3 = \epsilon \neq 0$ and leaving other parameters unchanged. This is also in contrast to the linear time invariant case where unstabilizability is robust to parameter variations if the real part of one of the uncontrollable eigenvalues is positive.

In the case of Hopf bifurcations, from Section 2.2, there exists a smooth control law

$$\begin{aligned} u &= F(\tilde{x}, z, z^*) \\ &= K_1\tilde{x} + K_2 z + K_3^* z^* + K_4 z^2 + K_5|z|^2 + K_6 z^{*2} + K_7\tilde{x}z + K_8\tilde{x}z^* + K_9(\tilde{x},\tilde{x}) + \text{h.o.t.}, \end{aligned}$$

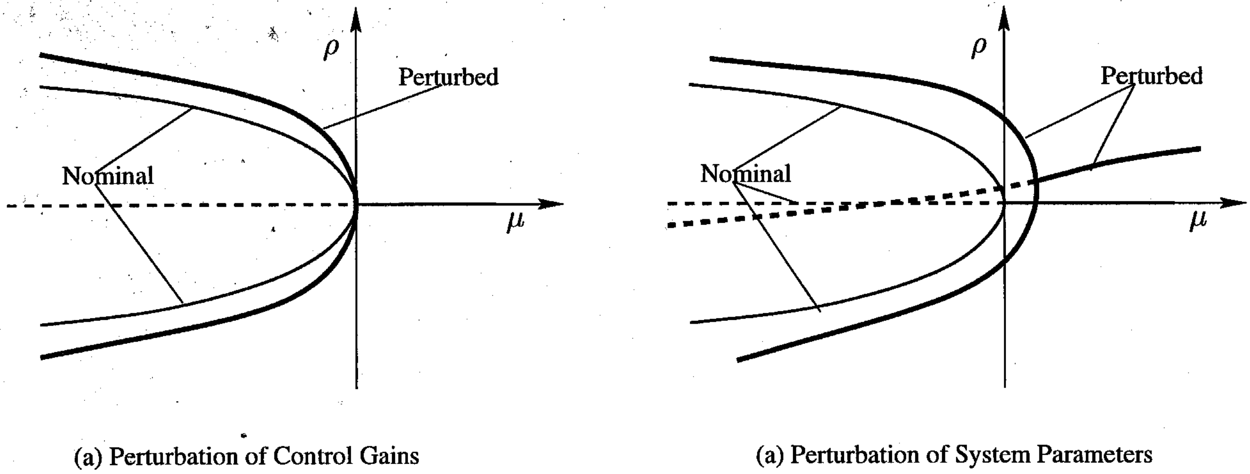


Figure 2.2: Bifurcation diagram of the perturbed system in the Hopf case. The y-axis ρ is the amplitude of the periodic orbits.

such that the bifurcation for the closed loop system is supercritical if and only if $\alpha < 0$, where α is given by

$$\begin{aligned}\alpha \;=\; & \frac{\Theta_1 + \mathrm{Re}\{K\Theta_3\}}{\bar{p}(0)}\Big\{ K_5 + K_7(-i\omega)K^* + K_7^*(i\omega)K + 2K_9(i\omega,-i\omega)|K|^2 + \\ & \begin{bmatrix} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1}\end{bmatrix}\left(\tilde{Q}_2 - \mathfrak{m}(i\omega)Q_2K - \mathfrak{m}(-i\omega)Q_2^*K^*\right) \\ & + \frac{\bar{p}(0)}{i\omega}\left(Q_2^*K^* - Q_2K\right)\Big\} \\ & + \mathrm{Re}\Big\{\frac{\Theta_2 + K^*\Theta_4}{\bar{p}(2\omega i)}\Big[K_4 + K_7(i\omega)K + K_9(i\omega,i\omega)K^2 + \\ & \begin{bmatrix} 1 & \bar{p}_1(2\omega i) & \cdots & \bar{p}_{n-1}(2\omega i)\end{bmatrix}\left(\tilde{Q}_1 - \mathfrak{m}(i\omega)Q_1K - \mathfrak{m}(-i\omega)Q_3^*K^*\right) \\ & + \frac{\bar{p}(2\omega i)}{i\omega}\left(Q_1K + \frac{1}{3}Q_3^*K^*\right)\Big]\Big\} + P_H(K_\flat),\end{aligned}$$

where $K := \dfrac{K_2}{\bar{p}(i\omega)} = K_R + iK_I$, $K_\flat = \begin{bmatrix} K_R \\ K_I \end{bmatrix}$, Q_1, Q_2, Q_3, $\tilde{Q}_1$, and $\tilde{Q}_2$ are given by (2.44), (2.45), (2.46), (2.47), and (2.48), respectively. $P_H(K_\flat)$ is given by

$$P_H(K_\flat) = \alpha_0 + \alpha_1 K_\flat + K_\flat^T \alpha_2 K_\flat + \alpha_3 K_\flat \|K_\flat\|^2,$$

where α_j $(j = 0, \cdots, 3)$ are given in (2.34) and (2.35).

Suppose K_j $(j = 1, \cdots, 9)$ have been selected such that $\alpha < 0$, i.e., the Hopf bifurcation for the closed loop system is supercritical. Now we perturb K_j to $K_j + \epsilon_j$. If ϵ_j is small enough, then α will remain to be negative for the perturbed system, i.e., the Hopf bifurcation for the perturbed system is also supercritical. Thus the stabilizing control law in the Hopf bifurcation case is robust to gain variations. The bifurcation diagrams for the nominal and perturbed system are given by Figure 2.2 (a).

If we fix the gains K_j ($j = 1, \cdots, 9$), and consider a generic parametric perturbation to the vector field of the closed loop system, then the nominal equilibrium branch $y(\mu) = 0$ is perturbed away to $y(\mu) = y_d(\mu)$. But there is still a supercritical Hopf bifurcation on the perturbed equilibrium branch if the perturbations are small enough. Therefore, the stabilizing controllers are robust to generic parametric variations in the system. The bifurcation diagrams for the nominal and the perturbed system are shown in Figure 2.2 (b). We point out here that although the qualitative local bifurcation behavior for the closed loop system is robust to parametric uncertainties, the location of the bifurcation, the center manifold, the nominal equilibria and the periodic orbits are all perturbed away. So in practice we have to quantify the parametric uncertainties in order to determine the bifurcation for the perturbed system.

We mention here that if the system is unstabilizale, then the Hopf bifurcation for the closed loop system is also robust, i.e., the bifurcation remains subcritical under perturbations in the control gains and the parametric variations.

We point out here that the stabilizability in the Hopf case is robust to parametric variations, which implies that if the system is stabilizable, then the perturbed system is also stabilizable provided the perturbations are small enough. The degeneracy is non-robust: any generic parametric perturbations will make the system either stabilizable or unstabilizable. The unstabilizability is also non-robust to parametric variations. For example, all the unstabilizable cases **HU-1** to **HU-4** can become stabilizable by perturbing $\Theta_1 = 0$ to $\Theta_1 = \epsilon \neq 0$. This is in contrast to the linear time invariant case where the unstabilizability is robust to parameter variations if there is an uncontrollable eigenvalue whose real part is positive.

The robustness of the closed loop system with respect to controller magnitude saturation limits, bandwidth and rate limits is also an important issue in any control design in engineering applications. In control of bifurcations, these issues become crucial in that the region of attraction to the stabilized equilibria/periodic orbits can be seriously affected by these constraints. In Chapter 4 and Chapter 5 we will present a framework to quantify the effects of controller bandwidth and saturation limits.

2.4 Proof of Theorem 2.1

Consider the following feedback

$$u = K_1\tilde{x} + K_2 x + K_3 x^2 + K_4\tilde{x}x + \tilde{x}^T K_5 \tilde{x} + \text{h.o.t.}, \tag{2.51}$$

with

$$K_1, K_4 \in \mathbb{R}^{1\times n}, \quad K_2, K_3 \in \mathbb{R}, \quad K_5 = K_5^T \in \mathbb{R}^{n\times n}.$$

First we consider the case when $K_2 = 0$. The center manifold around $\mu = 0$ is given by the Taylor series expansion

$$\tilde{x} = \beta_1 x^2 + \text{h.o.t.}, \tag{2.52}$$

By differentiating (2.52), we get

$$\dot{\tilde{x}} = 2\beta_1 x\dot{x} + \text{h.o.t.},$$

and substituting the system (2.2), (2.3) and the feedback (2.51) into the above equation yields

$$A\beta_1 x^2 + B(K_1\beta_1 + K_3)x^2 + \cdots + \tilde{q}_{11}x^2 + \cdots \quad = \quad 0 + \cdots.$$

So we get

$$\beta_1 = -(A+BK_1)^{-1}(BK_3+\tilde{q}_{11}). \tag{2.53}$$

By substituting (2.52) into (2.2), we get the dynamics on the center manifold

$$\dot{x} = d\mu x + q_{11}x^2 + Cx^3 + \text{h.o.t.},$$

where

$$C = c_{111} + (q_{12}+q_{13}K_1)\beta_1 + q_{13}K_3. \tag{2.54}$$

Proposition 2.1 *From (2.54) and (2.53), we have*

$$C = c_{111} - \Pi_1^S\tilde{q}_{11} + \frac{\Upsilon_1}{\bar{a}_n}\left(K_3 + [\begin{array}{cccc} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1}\end{array}]\tilde{q}_{11}\right),$$

where

$$\begin{aligned} \Upsilon_1 &= q_{12}^n + q_{13}a_n, & (2.55)\\ \Pi_1^S &= [\begin{array}{cccc} 0 & q_{12}^1 & \cdots & q_{12}^{n-1}\end{array}] + q_{13}[\begin{array}{cccc} 1 & a_1 & \cdots & a_{n-1}\end{array}], & (2.56)\end{aligned}$$

and $\bar{a}_j (j=1,\cdots,n)$ are given by

$$\bar{p}(s) = \det[sI-(A+BK_1)] = s^n + \bar{a}_1 s^{n-1} + \cdots + \bar{a}_{n-1}s + \bar{a}_n,$$

where $\bar{a}_j = a_j - K_1^j$ $(j=1,\cdots,n)$, and $K_1 = [\begin{array}{cccc} K_1^1 & K_1^2 & \cdots & K_1^n\end{array}]$.

Proof:

$$\begin{aligned} C &= c_{111} + (q_{12}+q_{13}K_1)\beta_1 + q_{13}K_3 \\ &= c_{111} - (q_{12}+q_{13}K_1)(A+BK_1)^{-1}(BK_3+\tilde{q}_{11}) + q_{13}K_3 \end{aligned}$$

$$
\begin{aligned}
&= c_{111} - (q_{12} + q_{13}K_1) \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 \\ -\frac{1}{\bar{a}_n} & -\frac{\bar{a}_1}{\bar{a}_n} & \cdots & -\frac{\bar{a}_{n-1}}{\bar{a}_n} \end{bmatrix} (BK_3 + \tilde{q}_{11}) + q_{13}K_3 \\
&= c_{111} - (q_{12} + q_{13}K_1) \left(-\frac{1}{\bar{a}_n} \right) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} K_3 \\
&\quad - \left[\begin{array}{cccc} 0 & q_{12}^1 + q_{13}K_1^1 & \cdots & q_{12}^{n-1} + q_{13}K_1^{n-1} \end{array} \right] \tilde{q}_{11} \\
&\quad + \frac{1}{\bar{a}_n}(q_{12}^n + q_{13}K_1^n)[\begin{array}{cccc} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1} \end{array}]\tilde{q}_{11} + q_{13}K_3 \\
&= c_{111} + \frac{q_{12}^n + q_{13}K_1^n}{\bar{a}_n} K_3 + q_{13}K_3 + \frac{q_{12}^n + q_{13}K_1^n}{\bar{a}_n} [\begin{array}{cccc} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1} \end{array}]\tilde{q}_{11} \\
&\quad + q_{13}[\begin{array}{cccc} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1} \end{array}]\tilde{q}_{11} - q_{13}[\begin{array}{cccc} 1 & a_1 & \cdots & a_{n-1} \end{array}]\tilde{q}_{11} \\
&\quad - [\begin{array}{cccc} 0 & q_{12}^1 & \cdots & q_{12}^{n-1} \end{array}]\tilde{q}_{11} \\
&= c_{111} - \Pi_1^S \tilde{q}_{11} + \frac{\Upsilon_1}{\bar{a}_n} \left(K_3 + [\begin{array}{cccc} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1} \end{array}]\tilde{q}_{11} \right).
\end{aligned}
$$

It should be noted that in the above calculations we have used that fact that $K_1^j = a_j - \bar{a}_j$, for $j = 1, \cdots, n$. ■

From Proposition 2.1 it is easy to get the following corollary.

Corollary 2.1 *Except for the case $q_{11} = \Upsilon_1 = \alpha_0 = 0$, there exists a state feedback (2.51) with $K_2 = 0$ such that $(0,0)$ is asymptotically stable at $\mu = 0$ if and only if either one of the following conditions is satisfied:*

*(i) (**SS-1**) $q_{11} = \Upsilon_1 = 0$, and $\alpha_0 < 0$.*

*(ii) (**SS-2**) $q_{11} = 0$ and $\Upsilon_1 \neq 0$,*

Now consider the cases when feedbacks with $K_2 = 0$ fail to stabilize $(0,0)$ at $\mu = 0$, i.e., we consider the cases when $q_{11} \neq 0$, or $q_{11} = \Upsilon_1 = 0$, $\alpha_0 \geq 0$. In this case, we must assume $K_2 \neq 0$ in the state feedback (2.51). After substituting (2.51) into (2.2) and (2.3), we suppose that the center manifold is expressed by the following Taylor series:

$$\tilde{x} = \beta_1 x + \beta_2 x^2 + \text{h.o.t..} \tag{2.57}$$

By using the same procedure as for the $K_2 = 0$ case, we get

$$\begin{aligned}
\beta_1 &= -(A+BK_1)^{-1}BK_2 = K\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^T = \mathfrak{m}(0)K, \\
\beta_2 &= -(A+BK_1)^{-1}\left[B(K_3 + K_4\beta_1 + \beta_1^T K_5 \beta_1) + \tilde{Q} - \beta_1 Q\right] \\
&= \frac{1}{\bar{a}_n}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}(K_3 + K_4^n K + K_5^{nn} K^2) - \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\frac{1}{\bar{a}_n} & -\frac{\bar{a}_1}{\bar{a}_n} & \cdots & -\frac{\bar{a}_{n-1}}{\bar{a}_n} \end{bmatrix}\tilde{Q} \\
&\quad + \begin{bmatrix} 0 \\ \vdots \\ 1 \\ -\frac{\bar{a}_{n-1}}{\bar{a}_n} \end{bmatrix} KQ,
\end{aligned}$$

where $K := \frac{K_2}{\bar{a}_n}$, and

$$\begin{aligned}
Q &= q_{11} + q_{12}\beta_1 + q_{13}(K_1\beta_1 + K_2) + \beta_1^T q_{22}\beta_1 + q_{23}\beta_1(K_1\beta_1 + K_2) \\
&\quad + q_{33}(K_1\beta_1 + K_2)^2, \\
\tilde{Q} &= \tilde{q}_{11} + \tilde{q}_{12}\beta_1 + \tilde{q}_{13}(K_1\beta_1 + K_2) + \tilde{q}_{22}(\beta_1, \beta_1) + \tilde{q}_{23}\beta_1(K_1\beta_1 + K_2) \\
&\quad + \tilde{q}_{33}(K_1\beta_1 + K_2)^2.
\end{aligned}$$

It is easy to show that $K_1\beta_1 + K_2 = a_n K$, and it is straightforward to show that

$$Q = q_{11} + \Upsilon_1 K + \Upsilon_2 K^2, \tag{2.58}$$

$$\tilde{Q} = \tilde{q}_{11} + (\tilde{q}_{12}^n + \tilde{q}_{13}a_n)K + (\tilde{q}_{22}^{nn} + \tilde{q}_{23}^n a_n + \tilde{q}_{33}a_n^2)K^2, \tag{2.59}$$

where Υ_1 is given by (2.55), and Υ_2 is given by

$$\Upsilon_2 = q_{22}^{nn} + q_{23}^{n} a_n + q_{33} a_n^2. \tag{2.60}$$

By substituting (2.57) into (2.2), we get the dynamics on the center manifold

$$\dot{x} = d\mu x + Qx^2 + Cx^3 + \text{h.o.t.},$$

where $C = C_1 + C_2$, and

$$\begin{aligned}
C_1 &= (q_{12} + q_{13}K_1)\beta_2 + q_{13}(K_3 + K_4\beta_1 + \beta_1^T K_5 \beta_1) \\
&\quad + 2\beta_1^T q_{22}\beta_2 + q_{23}(K_1\beta_1 + K_2)\beta_2 \\
&\quad + [q_{23}\beta_1 + 2q_{33}(K_1\beta_1 + K_2)]\,(K_1\beta_2 + K_3 + K_4\beta_1 + \beta_1^T K_5\beta_1), \\
C_2 &= c_{111} + c_{112}\beta_1 + c_{113}(K_1\beta_1 + K_2) + \beta_1^T c_{122}\beta_1 + c_{123}\beta_1(K_1\beta_1 + K_2) \\
&\quad + c_{133}(K_1\beta_1 + K_2)^2 + c_{222}(\beta_1, \beta_1, \beta_1) + \beta_1^T c_{223}\beta_1(K_1\beta_1 + K_2) \\
&\quad + c_{233}\beta_1(K_1\beta_1 + K_2)^2 + c_{333}(K_1\beta_1 + K_2)^3 \\
&= c_{111} + (c_{112}^{n} + c_{113}a_n)K + (c_{122}^{nn} + c_{123}^{n}a_n + c_{133}a_n^2)K^2 \\
&\quad + (c_{222}^{nnn} + c_{223}^{nn}a_n + c_{233}^{n}a_n^2 + c_{333}a_n^3)K^3.
\end{aligned} \tag{2.61}$$

Suppose we have selected $K \in \mathbb{R}$ such that $Q = 0$, then it can be calculated that

$$\begin{aligned}
C_1 &= \left\{q_{12} + 2\beta_1^T q_{22} + q_{23}(K_1\beta_1 + K_2) + [q_{13} + q_{23}\beta_1 + 2q_{33}(K_1\beta_1 + K_2)]\,K_1\right\}\beta_2 \\
&\quad + [q_{13} + q_{23}\beta_1 + 2q_{33}(K_1\beta_1 + K_2)](K_3 + K_4\beta_1 + \beta_1^T K_5\beta_1) \\
&= \left\{q_{12} + (2\mathrm{m}(0)^T q_{22} + q_{23}a_n)K + [q_{13} + (q_{23}^{n} + 2q_{33}a_n)K]\,K_1\right\} \cdot \\
&\quad \left(\frac{1}{\bar{a}_n} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} (K_3 + K_4^{n}K + K_5^{nn}K^2) - \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\frac{1}{\bar{a}_n} & -\frac{\bar{a}_1}{\bar{a}_n} & \cdots & -\frac{\bar{a}_{n-1}}{\bar{a}_n} \end{bmatrix} \tilde{Q} \right) \\
&\quad + [q_{13} + (q_{23}^{n} + 2q_{33}a_n)K]\,(K_3 + K_4^{n}K + K_5^{nn}K^2)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\bar{a}_n}(\Upsilon_1 + 2\Upsilon_2 K)\left(K_3 + K_4^n K + K_5^{nn} K^2\right) - \left[\begin{array}{cccc} 0 & q_{12}^1 & \cdots & q_{12}^{n-1} \end{array}\right]\tilde{Q} \\
&\quad -\left[\begin{array}{cccc} 0 & 2q_{22}^{n1} + q_{23}^1 a_n & \cdots & 2q_{22}^{n,n-1} + q_{23}^{n-1} a_n \end{array}\right] K\tilde{Q} \\
&\quad -(q_{13} + (q_{23}^n + 2q_{33}a_n)K)\left[\begin{array}{cccc} 0 & K_1^1 & \cdots & K_1^{n-1} \end{array}\right]\tilde{Q} \\
&\quad +\frac{1}{\bar{a}_n}\left\{q_{12}^n + (2q_{22}^{nn} + q_{23}^n a_n)K + [q_{13} + (q_{23}^n + 2q_{33}a_n)K]\, K_1^n\right\} \cdot \\
&\quad \left[\begin{array}{cccc} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1} \end{array}\right]\tilde{Q} \\
&= \frac{1}{\bar{a}_n}(\Upsilon_1 + 2\Upsilon_2 K)\Big(K_3 + K_4^n K + K_5^{nn} K^2 \\
&\quad + \left[\begin{array}{cccc} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1} \end{array}\right]\tilde{Q}\Big) - (\Pi_1^S + \Pi_2^S K)\tilde{Q}, \qquad (2.62)
\end{aligned}
$$

where Π_1^S is given by (2.56) and Π_2^S is given by

$$
\begin{aligned}
\Pi_2^S &= 2[\begin{array}{cccc} 0 & q_{22}^{n1} & \cdots & q_{22}^{n,n-1} \end{array}] + a_n[\begin{array}{cccc} 0 & q_{23}^1 & \cdots & q_{23}^{n-1} \end{array}] \\
&\quad +(q_{23}^n + 2q_{33}a_n)[\begin{array}{cccc} 1 & a_1 & \cdots & a_{n-1} \end{array}]. \qquad (2.63)
\end{aligned}
$$

From (2.61) and (2.62), we get

$$
C = \frac{1}{\bar{a}_n}(\Upsilon_1 + 2\Upsilon_2 K)\left(K_3 + K_4^n K + K_5^{nn} K^2 + \left[\begin{array}{cccc} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1} \end{array}\right]\tilde{Q}\right) + P_S(K),
$$

where $P_S(K)$ is given by

$$
P_S(K) = \alpha_0 + \alpha_1 K + \alpha_2 K^2 + \alpha_3 K^3, \qquad (2.64)
$$

and

$$
\begin{aligned}
\alpha_0 &= c_{111} - \Pi_1^S \tilde{q}_{11}, \qquad (2.65) \\
\alpha_1 &= c_{112}^n + c_{113}a_n - \Pi_1^S(\tilde{q}_{12}^n + \tilde{q}_{13}a_n) - \Pi_2^S \tilde{q}_{11}, \qquad (2.66) \\
\alpha_2 &= c_{122}^{nn} + c_{123}^n a_n + c_{133}a_n^2 - \Pi_1^S(\tilde{q}_{22}^{nn} + \tilde{q}_{23}^n a_n + \tilde{q}_{33}a_n^2) \\
&\quad -\Pi_2^S(\tilde{q}_{12}^n + \tilde{q}_{13}a_n), \qquad (2.67) \\
\alpha_3 &= c_{222}^{nnn} + c_{223}^{nn}a_n + c_{233}^n a_n^2 + c_{333}a_n^3 - \Pi_2^S(\tilde{q}_{22}^{nn} + \tilde{q}_{23}^n a_n + \tilde{q}_{33}a_n^2), \qquad (2.68)
\end{aligned}
$$

where $p(s) = \det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$.

The goal is to find the feedback gains such that $Q = 0$ and $C < 0$, where

$$
\begin{aligned}
Q &= q_{11} + \Upsilon_1 K + \Upsilon_2 K^2, \\
C &= \frac{1}{\bar{a}_n}(\Upsilon_1 + 2\Upsilon_2 K)\Big(K_3 + K_4^n K + K_5^{nn} K^2 \\
&\quad + \begin{bmatrix} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1} \end{bmatrix} \tilde{Q}\Big) + P_S(K),
\end{aligned}
$$

and $P_S(K)$ is given by (2.64). By letting $K_4^n = K_5^{nn} = 0$, we have the following cases:

I. $q_{11} \neq 0$,

I-1. **(SU-4)** $\Upsilon_2 = \Upsilon_1 = 0$, then there does not exist a $K \in \mathbb{R}$ such that $Q = 0$, so the equilibrium 0 at $\mu = 0$ cannot be stabilized by sufficiently smooth feedbacks.

I-2. **(SS-7)** $\Upsilon_2 = 0$, $\Upsilon_1 \neq 0$, then $Q = 0$ if and only if $K = -\frac{q_{11}}{\Upsilon_1}$. Let $K_4 = 0$, $K_5 = 0$, and select $K_3 \in \mathbb{R}$, such that

$$
C = \frac{\Upsilon_1}{\bar{a}_n}\left\{K_3 + \begin{bmatrix} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1} \end{bmatrix} \tilde{Q}\left(-\frac{q_{11}}{\Upsilon_1}\right)\right\} + P_S\left(-\frac{q_{11}}{\Upsilon_1}\right) < 0.
$$

I-3. **(SS-8)** $\Upsilon_2 \neq 0$, $\Delta_\Upsilon > 0$, then for $Q = 0$ if and only if $K^{1,2} = \frac{1}{2\Upsilon_2}\left(-\Upsilon_1 \pm \sqrt{\Delta_\Upsilon}\right)$. Select $K_3 \in \mathbb{R}$, such that

$$
C = \frac{\sqrt{\Delta_\Upsilon}}{\bar{a}_n}\left\{K_3 + \begin{bmatrix} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1} \end{bmatrix} \tilde{Q}\left(K^1\right)\right\} + P_S\left(K^1\right) < 0,
$$

where $P_S(K)$ is given by (2.64).

I-4. $\Upsilon_2 \neq 0$, $\Delta_\Upsilon = 0$, then $Q = 0$ if and only if $K = \rho := -\frac{\Upsilon_1}{2\Upsilon_2}$. In this case, we have $C = P_S(K)$. So we should check the sign of $P_S(\rho)$.

(i) **(SS-9)** If $P_S(\rho) < 0$, then the pitchfork bifurcation for the closed loop system is supercritical.

(ii) **(SU-6)** If $P_S(\rho) > 0$, then the pitchfork bifurcation for the closed loop system is subcritical.

(iii) (**SD-4**) If $P_S(\rho) = 0$, this is the degeneracy condition **SD-4**. In this case, the pitchfork bifurcation for the closed loop system is degenerate, i.e., we need to refer to the higher order terms to determine the criticality.

I-5. (**SU-5**) $\Upsilon_2 \neq 0$, $\Delta_\Upsilon < 0$, then there does not exist a $K \in \mathbb{R}$ such that $Q = 0$. So the bifurcation for the closed loop system is a transcritical bifurcation, and the equilibrium 0 is at $\mu = 0$ cannot be stabilized by sufficiently smooth feedback.

II. $q_{11} = 0$, $\Upsilon_1 = 0$, and $\alpha_0 \geq 0$.

II-1. $\Upsilon_2 = 0$, then for any $K \in \mathbb{R}$, $Q = 0$. In this case, we have $C = P_S(K)$. So we need to select K such that $P_S(K) < 0$, where $P_S(K)$ is given by (2.64).

(i) (**SS-3**) If $\alpha_3 \neq 0$, then we could select K with $|K|$ large enough, such that $P_S(K) < 0$.

(ii) (**SS-4**) If $\alpha_3 = 0$ and $\alpha_2 < 0$, then we could select K with $|K|$ is large enough, such that $P_S(K) < 0$.

(iii) (**SU-2**) If $\alpha_3 = 0$, $\alpha_2 > 0$, and $\Delta_\alpha < 0$, then for any $K \in \mathbb{R}$, we have $C = P_S(K) > 0$, i.e., the pitchfork bifurcation for the closed loop system is subcritical.

(iv) (**SS-6**) If $\alpha_3 = 0$, $\alpha_2 > 0$, and $\Delta_\alpha > 0$, then by selecting K such that

$$\frac{-\alpha_1 - \sqrt{\Delta_\alpha}}{2\alpha_2} < K < \frac{-\alpha_1 + \sqrt{\Delta_\alpha}}{2\alpha_2},$$

we have $C = P_S(K) < 0$.

(v) (**SD-2**) If $\alpha_3 = 0$, $\alpha_2 > 0$, and $\Delta_\alpha = 0$, the system is degenerate. In this case, we have to refer to higher order terms to determine the criticality of the closed loop system.

(vi) (**SS-5**) If $\alpha_3 = \alpha_2 = 0$, but $\alpha_1 \neq 0$, then we select K such that $C = \alpha_0 + \alpha_1 K < 0$.

(vii) **(SU-1)** If $\alpha_3 = \alpha_2 = \alpha_1 = 0$, and $\alpha_0 > 0$, then the pitchfork bifurcation for the closed loop system is subcritical.

(viii) **(SD-1)** If $\alpha_3 = \alpha_2 = \alpha_1 = \alpha_0 = 0$, the system is degenerate. In this case, we have to refer to higher order terms to determine the criticality of the closed loop system.

II-2. $\Upsilon_2 \neq 0$, then $Q = \Upsilon_2 K^2$. So $Q = 0$ if and only if $K = 0$.

(i) **(SD-3)** If $\alpha_0 = 0$, the system is degenerate. In this case, we have to refer to higher order terms to determine the criticality of the closed loop system.

(ii) **(SU-3)** If $\alpha_0 > 0$, then by Corollary 2.1, the criticality cannot be changed.

The classification of stabilizability in Table 2.1 and the construction of stabilizing control laws in Table 2.2 are the summary of the above discussions.

2.5 Proof of Theorem 2.2

The following lemma gives the normal form for a two-dimensional system with Hopf bifurcation.

Lemma 2.1 *[37] Consider the following system*

$$\begin{aligned} \dot{z} &= (d\mu + i\omega)z + Q_1 z^2 + Q_2|z|^2 + Q_3 z^{*2} \\ &\quad + C_{111}z^3 + C_{112}|z|^2 z + C_{122}|z|^2 z^* + C_{222}z^{*3} + h.o.t., \end{aligned} \tag{2.69}$$

where $z \in \mathbb{C}$. Then a normal form of (2.69) is given by

$$\dot{\zeta} = (d\mu + i\omega)\zeta + \tilde{\alpha}|\zeta|^2\zeta + h.o.t., \tag{2.70}$$

where

$$\tilde{\alpha} = C_{112} - \frac{Q_1 Q_2}{i\omega} + \frac{|Q_2|^2}{i\omega} + \frac{2|Q_3|^2}{3i\omega}.$$

The coordinate transformation is given by

$$z = \zeta + \sigma_1 \zeta^2 + \sigma_2 |\zeta|^2 + \sigma_3 \zeta^{*2} + h.o.t.,$$

where

$$\sigma_1 = \frac{Q_1}{i\omega}, \quad \sigma_2 = -\frac{Q_2}{i\omega}, \quad \sigma_3 = -\frac{Q_3}{3i\omega}.$$

The proof of this lemma is straightforward and is omitted. The criticality of the Hopf bifurcation in (2.69) is determined by $\alpha = \operatorname{Re} \tilde{\alpha}$. If $\alpha > 0$, then the Hopf bifurcation is subcritical and the periodic orbits bifurcating from the equilibria are unstable; If $\alpha < 0$, then the Hopf bifurcation is supercritical and the periodic orbits bifurcated from the equilibria are stable; If $\alpha = 0$, then the Hopf bifurcation is degenerate and the criticality is determined by higher order terms.

Consider the normal form (2.18) and (2.19) with the state feedback given by (2.41). First we consider the case when $K_2 = 0$. By substituting (2.41) into (2.18) and (2.19), and letting the center manifold be

$$\tilde{x} = \beta_1 z^2 + \beta_2 |z|^2 + \beta_3 z^{*2} + \text{h.o.t.}, \tag{2.71}$$

we get the dynamics on the center manifold

$$\dot{z} = (d\mu + i\omega) z + q_{11} z^2 + q_{12} |z|^2 + q_{13} z^{*2} + C|z|^2 z + \text{o.c.t.} + \text{h.o.t.}, \tag{2.72}$$

where o.c.t. denotes "other cubic terms."

$$C = c_{112} + (q_{13} + q_{14} K_1)\beta_2 + q_{14} K_5 + (q_{23} + q_{24} K_1)\beta_1 + q_{24} K_4. \tag{2.73}$$

A normal form of (2.72) is given by

$$\dot{\zeta} = (d\mu + i\omega)\zeta + \tilde{\alpha}|\zeta|^2\zeta + \text{h.o.t.},$$

where

$$\tilde{\alpha} = C - \frac{q_{11}q_{12}}{i\omega} + \text{p.i.t.}, \tag{2.74}$$

where p.i.t. denotes "pure imaginary terms." The goal is to calculate $\tilde{\alpha}$ as a function of the feedback gains. We first calculate β_1 and β_2. By differentiating the center manifold expansion (2.71), we get

$$\dot{x} = 2\beta_1 z\dot{z} + \beta_2(\dot{z}z^* + z\dot{z}^*) + 2\beta_3 z^*\dot{z}^* + \text{h.o.t.}. \tag{2.75}$$

By substituting (2.18) and (2.19) into (2.75) and letting the coefficients of z^2 and $|z|^2$ be zero, we get

$$\begin{aligned} \beta_1 &= [2\omega i - (A + BK_1)]^{-1}(BK_4 + \tilde{q}_{11}), & (2.76)\\ \beta_2 &= -(A + BK_1)^{-1}(BK_5 + \tilde{q}_{12}). & (2.77) \end{aligned}$$

Proposition 2.2 *From (2.73), (2.74), (2.76), and (2.77), α is given by*

$$\begin{aligned} \tilde{\alpha} &= C_0 + \frac{\Theta_1}{\bar{p}(0)}\left(K_5 + \begin{bmatrix} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1} \end{bmatrix}\tilde{q}_{12}\right) \\ &\quad + \frac{\Theta_2}{\bar{p}(2\omega i)}\left(K_4 + \begin{bmatrix} 1 & \bar{p}_1(2\omega i) & \cdots & \bar{p}_{n-1}(2\omega i) \end{bmatrix}\tilde{q}_{11}\right) + p.i.t., \end{aligned}$$

where $\Theta_1 = \operatorname{Re}\Phi_1(0)$, $\Theta_2 = \Phi_2(2\omega i)$, and

$$\begin{aligned} \Phi_1(s) &= q_{13}(s) + q_{14}p(s), & (2.78)\\ \Phi_2(s) &= q_{23}(s) + q_{24}p(s), & (2.79) \end{aligned}$$

$$C_0 = c_{112} - \frac{q_{11}q_{12}}{i\omega} - \Pi_1^H \tilde{q}_{12} - \Pi_2^H \tilde{q}_{11}, \tag{2.80}$$

$$\Pi_1^H = [\ 0 \ \ q_{13}^1 \ \ \cdots \ \ q_{13}^{n-1}\ ] + q_{14}[\ 1 \ \ a_1 \ \ \cdots \ \ a_{n-1}\ ], \tag{2.81}$$

$$\begin{aligned}\Pi_2^H &= [\ 0 \ \ q_{23,1}(2\omega i) \ \ \cdots \ \ q_{23,n-1}(2\omega i)\ ] \\ &\quad + q_{24}[\ 1 \ \ p_1(2\omega i) \ \ \cdots \ \ p_{n-1}(2\omega i)\ ],\end{aligned} \tag{2.82}$$

where $q_{13} = \begin{bmatrix} q_{13}^1 & q_{13}^2 & \cdots & q_{13}^n \end{bmatrix}$. $p_j(s)$ $(j = 1, \cdots, n)$ *is defined as*

$$\begin{aligned} p(s) := p_n(s) &= \det[sI - A] = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n, \\ p_{n-1}(s) &= s^{n-1} + a_1 s^{n-2} + \cdots + a_{n-2}s + a_{n-1}, \\ \cdots & \cdots \ \cdots \ \cdots \ \cdots \\ p_1(s) &= s + a_1. \end{aligned} \tag{2.83}$$

$\bar{p}_j(s)$ $(j = 1, \cdots, n)$ *is defined as*

$$\begin{aligned} \bar{p}(s) := \bar{p}_n(s) &= \det[sI - (A + BK_1)] = s^n + \bar{a}_1 s^{n-1} + \cdots + \bar{a}_{n-1}s + \bar{a}_n, \\ \bar{p}_{n-1}(s) &= s^{n-1} + \bar{a}_1 s^{n-2} + \cdots + \bar{a}_{n-2}s + \bar{a}_{n-1}, \\ \cdots & \cdots \ \cdots \ \cdots \ \cdots \\ \bar{p}_1(s) &= s + \bar{a}_1, \end{aligned}$$

where $\bar{a}_j = a_j - K_1^j$ *for* $j = 1, \cdots, n$. *And* $q_{23,j}(s)$ *is defined as* $q_{23,j}(s) := q_{23}\mathfrak{m}_j(s)$ $(j = 1, \cdots, n-1)$, *where*

$$\mathfrak{m}_1(s) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathfrak{m}_2(s) = \begin{bmatrix} s \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathfrak{m}_{n-1}(s) = \begin{bmatrix} s^{n-2} \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \quad \mathfrak{m}_n(s) = \mathfrak{m}(s) = \begin{bmatrix} s^{n-1} \\ \vdots \\ s \\ 1 \end{bmatrix}.$$

Note: It is easy to see that we have $\bar{p}_j(s) = p_j(s) - K_{1,j}(s)$ $(j = 1, \cdots, n)$, where $K_{1,j}(s) := K_1\mathfrak{m}_j(s)$, for $j = 1, \cdots, n$.

Proof: We prove

$$
\begin{aligned}
(q_{23}+q_{24}K_1)\beta_1 + q_{24}K_4 &= -\Pi_2^H \tilde{q}_{11} + \frac{q_{23}(2\omega i)+q_{24}p(2\omega i)}{\bar{p}(2\omega i)}\Big(K_4 + \\
&\quad \begin{bmatrix} 1 & \bar{p}_1(2\omega i) & \cdots & \bar{p}_{n-1}(2\omega i) \end{bmatrix} \tilde{q}_{11}\Big), \qquad (2.84) \\
(q_{13}+q_{14}K_1)\beta_2 + q_{14}K_5 &= -\Pi_1^H \tilde{q}_{12} + \frac{q_{13}(0)+q_{14}p(0)}{\bar{p}(0)} \cdot \\
&\quad \Big(K_5 + \begin{bmatrix} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1} \end{bmatrix} \tilde{q}_{12}\Big). \qquad (2.85)
\end{aligned}
$$

First we show

$$
\begin{aligned}
&(q_{23}+q_{24}K_1)\left[sI-(A+BK_1)\right]^{-1}(BK_4+\tilde{q}_{11}) + q_{24}K_4 \\
= &-\begin{bmatrix} 0 & q_{23,1}(s) & \cdots & q_{23,n-1}(s) \end{bmatrix} - q_{24}\begin{bmatrix} 1 & p_1(s) & \cdots & p_{n-1}(s) \end{bmatrix} \\
&+\frac{q_{23}(s)+q_{24}p(s)}{\bar{p}(s)}\Big(K_4 + \begin{bmatrix} 1 & \bar{p}_1(s) & \cdots & \bar{p}_{n-1}(s) \end{bmatrix}\tilde{q}_{11}\Big). \qquad (2.86)
\end{aligned}
$$

Then the identities (2.84) and (2.85) are direct conclusions of (2.86).

From page 660 of [44], if (A, B) is in controller canonical form, then

$$
\begin{aligned}
(sI-A)^{-1} &= \frac{1}{p(s)}\begin{bmatrix} s^{n-1} \\ \vdots \\ s \\ 1 \end{bmatrix}\begin{bmatrix} 1 & s & \cdots & s^{n-1} \end{bmatrix}\begin{bmatrix} 1 & a_1 & \cdots & a_{n-1} \\ 0 & 1 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \\
&\quad - \begin{bmatrix} 0 & 1 & s & \cdots & s^{n-2} \\ 0 & 0 & 1 & \cdots & s^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.
\end{aligned}
$$

So

$$
\begin{aligned}
& (q_{23}+q_{24}K_1)\left[sI-(A+BK_1)\right]^{-1}(BK_4+\tilde{q}_{11})+q_{24}K_4 \\
= \; & \frac{q_{23}(s)+q_{24}K_1(s)}{\bar{p}(s)}\begin{bmatrix} 1 & s & \cdots & s^{n-1}\end{bmatrix}\begin{bmatrix} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1} \\ 0 & 1 & \cdots & \bar{a}_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1\end{bmatrix}\left(\begin{bmatrix} K_4 \\ 0 \\ \vdots \\ 0\end{bmatrix}+\tilde{q}_{11}\right) \\
& -(q_{23}+q_{24}K_1)\begin{bmatrix} 0 & 1 & s & \cdots & s^{n-2} \\ 0 & 0 & 1 & \cdots & s^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}\left(\begin{bmatrix} K_4 \\ 0 \\ \vdots \\ 0\end{bmatrix}+\tilde{q}_{11}\right)+q_{24}K_4 \\
= \; & \frac{q_{23}(s)+q_{24}K_1(s)}{\bar{p}(s)}\begin{bmatrix} 1 & \bar{p}_1(s) & \cdots & \bar{p}_{n-1}(s)\end{bmatrix}\left(\begin{bmatrix} K_4 \\ 0 \\ \vdots \\ 0\end{bmatrix}+\tilde{q}_{11}\right)- \\
& \begin{bmatrix} 0 & q_{23,1}(s)+q_{24}K_{1,1}(s) & \cdots & q_{23,n-1}(s)+q_{24}K_{1,n-1}(s)\end{bmatrix}\left(\begin{bmatrix} K_4 \\ 0 \\ \vdots \\ 0\end{bmatrix}+\tilde{q}_{11}\right) \\
& +q_{24}K_4 \\
= \; & \left(\frac{q_{23}(s)+q_{24}K_1(s)}{\bar{p}(s)}+q_{24}\right)K_4+\frac{q_{23}(s)+q_{24}K_1(s)}{\bar{p}(s)}\begin{bmatrix} 1 & \bar{p}_1(s) & \cdots & \bar{p}_{n-1}(s)\end{bmatrix}\tilde{q}_{11} \\
& -\begin{bmatrix} 0 & q_{23,1}(s) & \cdots & q_{23,n-1}(s)\end{bmatrix}\tilde{q}_{11}-q_{24}\begin{bmatrix} 0 & K_{1,1}(s) & \cdots & K_{1,n-1}(s)\end{bmatrix}\tilde{q}_{11} \\
= \; & \frac{q_{23}(s)+q_{24}p(s)}{\bar{p}(s)}K_4+\left(\frac{q_{23}(s)+q_{24}K_1(s)}{\bar{p}(s)}+q_{24}\right)\begin{bmatrix} 1 & \bar{p}_1(s) & \cdots & \bar{p}_{n-1}(s)\end{bmatrix}\tilde{q}_{11} \\
& -\begin{bmatrix} 0 & q_{23,1}(s) & \cdots & q_{23,n-1}(s)\end{bmatrix}\tilde{q}_{11}-q_{24}\begin{bmatrix} 1 & p_1(s) & \cdots & p_{n-1}(s)\end{bmatrix}\tilde{q}_{11} \\
= \; & -\left(\begin{bmatrix} 0 & q_{23,1}(s) & \cdots & q_{23,n-1}(s)\end{bmatrix}+q_{24}\begin{bmatrix} 1 & p_1(s) & \cdots & p_{n-1}(s)\end{bmatrix}\right)\tilde{q}_{11} \\
& +\frac{q_{23}(s)+q_{24}p(s)}{\bar{p}(s)}\left(K_4+\begin{bmatrix} 1 & \bar{p}_1(s) & \cdots & \bar{p}_{n-1}(s)\end{bmatrix}\tilde{q}_{11}\right).
\end{aligned}
$$

The formula (2.86) implies (2.84) and (2.85). ■

The criticality of the closed loop system is determined by the sign of $\operatorname{Re}\tilde{\alpha}$: if $\operatorname{Re}\tilde{\alpha} > 0$, then the periodic orbits bifurcated from the steady-state solution are unstable and the Hopf bifurcation for the closed loop system is subcritical; if $\operatorname{Re}\tilde{\alpha} < 0$, then periodic orbits bifurcated from the steady-state solution are stable and the Hopf bifurcation for the closed loop system is supercritical; if $\operatorname{Re}\tilde{\alpha} = 0$, then the bifurcation is degenerate and the criticality is determined by higher order terms. So the goal is to find feedback gains such that $\operatorname{Re}\tilde{\alpha} < 0$. The following results are direct conclusions of Proposition 2.2.

Corollary 2.2 *Except for the degenerate condition $\Theta_1 = \Theta_2 = \operatorname{Re} C_0 = 0$, there exists a state feedback (2.41) with $K_2 = 0$ such that the Hopf bifurcation of the controlled system (2.18) and (2.19) is supercritical if and only if one of the following three conditions holds:*

(i) **HS-1** $\alpha_0 = \operatorname{Re} C_0 < 0$,

(ii) **HS-2** $\Theta_1 \neq 0$, $\alpha_0 = \operatorname{Re} C_0 \geq 0$,

(iii) **HS-3** $\Theta_2 \neq 0$, $\alpha_0 = \operatorname{Re} C_0 \geq 0$.

The construction procedures **HS-1, HS-2** and **HS-3** are direct results from Proposition 2.2 and Corollary 2.2.

In the following we consider the cases when smooth state feedbacks with $K_2 = 0$ fails to alter the criticality of the bifurcation, i.e., $\Theta_1 = \Theta_2 = 0$ but $\alpha_0 \geq 0$. In this case, we have to select a nonzero K_2. We substitute the state feedback (2.41) into (2.18) and (2.19), and let the center manifold be given by

$$\tilde{x} = \beta_1 z + \beta_2 z^* + \beta_3 z^2 + \beta_4 |z|^2 + \beta_5 z^{*2} + \text{h.o.t.}. \tag{2.87}$$

So the dynamics on the center manifold are given by

$$\dot{z} = (d\mu + i\omega)z + Q_1 z^2 + Q_2 |z|^2 + Q_3 z^{*2} + C|z|^2 z + \text{o.c.t.} + \text{h.o.t.}, \tag{2.88}$$

where $C = W_1 + W_2 + W_3$, and

$$
\begin{aligned}
Q_1 &= q_{11} + q_{13}\beta_1 + q_{14}(K_1\beta_1 + K_2) + \beta_1^T q_{33}\beta_1 + q_{34}(K_1\beta_1 + K_2)\beta_1 \\
&\quad + q_{44}(K_1\beta_1 + K_2)^2, \qquad (2.89)\\
Q_2 &= q_{12} + q_{13}\beta_2 + q_{14}(K_1\beta_2 + K_3) + q_{23}\beta_1 + q_{24}(K_1\beta_1 + K_2) \\
&\quad + 2\beta_1^T q_{33}\beta_2 + q_{34}\left[(K_1\beta_2 + K_3)\beta_1 + (K_1\beta_1 + K_2)\beta_2\right] \\
&\quad + 2q_{44}(K_1\beta_1 + K_2)(K_1\beta_2 + K_3), \qquad (2.90)\\
Q_3 &= q_{22} + q_{23}\beta_2 + q_{24}(K_1\beta_2 + K_3) + \beta_2^T q_{33}\beta_2 + q_{34}(K_1\beta_2 + K_3)\beta_2 \\
&\quad + q_{44}(K_1\beta_2 + K_3)^2, \qquad (2.91)\\
W_1 &= (q_{13} + q_{14}K_1)\beta_4 + 2\beta_1^T q_{33}\beta_4 + q_{34}\beta_1 K_1\beta_4 \\
&\quad + q_{34}\beta_4(K_1\beta_1 + K_2) + 2q_{44}(K_1\beta_1 + K_2)K_1\beta_4 \\
&\quad + \left[q_{14} + q_{34}\beta_1 + 2q_{44}(K_1\beta_1 + K_2)\right](K_5 + K_7\beta_2 + K_8\beta_1 + 2\beta_1^T K_9\beta_2) \\
&= \Big\{ \left[q_{13} + 2\beta_1^T q_{33} + q_{34}(K_1\beta_1 + K_2)\right] \\
&\quad + \left[q_{14} + q_{34}\beta_1 + 2q_{44}(K_1\beta_1 + K_2)\right] K_1 \Big\}\beta_4 \\
&\quad + \left[q_{14} + q_{34}\beta_1 + 2q_{44}(K_1\beta_1 + K_2)\right](K_5 + K_7\beta_2 + K_8\beta_1 + 2\beta_1^T K_9\beta_2) \\
W_2 &= (q_{23} + q_{24}K_1)\beta_3 + 2\beta_2^T q_{33}\beta_3 \\
&\quad + q_{34}\beta_2 K_1\beta_3 + q_{34}\beta_3(K_1\beta_2 + K_3) + 2q_{44}(K_1\beta_2 + K_3)K_1\beta_3 \\
&\quad + \left[q_{24} + q_{34}\beta_2 + 2q_{44}(K_1\beta_2 + K_3)\right](K_4 + K_7\beta_1 + \beta_1^T K_9\beta_1) \\
&= \Big\{ \left[q_{23} + 2\beta_2^T q_{33} + q_{34}(K_1\beta_2 + K_3)\right] \\
&\quad + \left[q_{24} + q_{34}\beta_2 + 2q_{44}(K_1\beta_2 + K_3)\right] K_1 \Big\}\beta_3 \\
&\quad + \left[q_{24} + q_{34}\beta_2 + 2q_{44}(K_1\beta_2 + K_3)\right](K_4 + K_7\beta_1 + \beta_1^T K_9\beta_1) \\
W_3 &= c_{112} + c_{113}\beta_2 + c_{114}(K_1\beta_2 + K_3) + c_{123}\beta_1 + c_{124}(K_1\beta_1 + K_2) + 2\beta_1^T c_{133}\beta_2 \\
&\quad + c_{134}\left[\beta_1(K_1\beta_2 + K_3) + \beta_2(K_1\beta_1 + K_2)\right] + 2c_{144}(K_1\beta_1 + K_2)(K_1\beta_2 + K_3) \\
&\quad + \beta_1 c_{233}\beta_1 + c_{234}\beta_1(K_1\beta_1 + K_2) + c_{244}(K_1\beta_1 + K_2)^2 + 3c_{333}(\beta_1, \beta_1, \beta_2) \\
&\quad + \beta_1^T c_{334}\beta_1(K_1\beta_2 + K_3) + 2\beta_1^T c_{334}\beta_2(K_1\beta_1 + K_2) + c_{344}[2\beta_1(K_1\beta_1 + K_2)\cdot \\
&\quad (K_1\beta_2 + K_3) + \beta_2(K_1\beta_1 + K_2)^2] + 3c_{444}(K_1\beta_1 + K_2)^2(K_1\beta_2 + K_3).
\end{aligned}
$$

A normal form of (2.72) is given by

$$\dot{\zeta} = (d\mu + i\omega)\zeta + \tilde{\alpha}|\zeta|\zeta + \text{h.o.t.},$$

where

$$\tilde{\alpha} = C - \frac{Q_1 Q_2}{i\omega} + \text{p.i.t.} = W_1 + W_2 + W_3 - \frac{Q_1 Q_2}{i\omega} + \text{p.i.t.} \tag{2.92}$$

Since the sign of $\alpha = \operatorname{Re}\tilde{\alpha}$ determines the criticality of the bifurcation, the goal is to find α the feedback gains such that $\alpha < 0$. The main task in the following is to derive the algebraic relations between α and K_j $(j = 1, \cdots, 9)$.

Now, in order to find the expressions for $\beta_k (k = 1, \cdots, 4)$, we differentiate (2.87) and utilize (2.18) and (2.19), and we get

$$\begin{aligned}
\beta_1 &= [i\omega - (A + BK_1)]^{-1} BK_2 = K\mathfrak{m}(i\omega), \\
\beta_2 &= [-i\omega - (A + BK_1)]^{-1} BK_3 = K^*\mathfrak{m}(-i\omega), \\
\beta_3 &= [2\omega i - (A + BK_1)]^{-1} \left[B(K_4 + K_7\beta_1 + \beta_1^T K_9 \beta_1) + \tilde{Q}_1 - \beta_1 Q_2 - \beta_2 Q_3^*\right], \\
\beta_4 &= -(A + BK_1)^{-1} \left[B(K_5 + K_7\beta_2 + K_8\beta_1 + 2\beta_1^T K_9 \beta_2) + \tilde{Q}_2 - \beta_1 Q_2 - \beta_2 Q_2^*\right],
\end{aligned}$$

where $K = \dfrac{K_2}{\bar{p}(i\omega)}$, $\mathfrak{m}(s) = \begin{bmatrix} s^{n-1} & \cdots & s & 1 \end{bmatrix}^T$, Q_1, Q_2 and Q_3 are given by (2.89), (2.90), and (2.91), respectively, and

$$\begin{aligned}
\tilde{Q}_1 &= \tilde{q}_{11} + \tilde{q}_{13}\beta_1 + \tilde{q}_{14}(K_1\beta_1 + K_2) + \tilde{q}_{33}(\beta_1, \beta_1) \\
&\quad + \tilde{q}_{34}\beta_1(K_1\beta_1 + K_2) + \tilde{q}_{44}(K_1\beta_1 + K_2)^2, \\
\tilde{Q}_2 &= \tilde{q}_{12} + \tilde{q}_{13}\beta_2 + \tilde{q}_{14}(K_1\beta_2 + K_3) + \tilde{q}_{23}\beta_1 + \tilde{q}_{24}(K_1\beta_1 + K_2) \\
&\quad + 2\tilde{q}_{33}(\beta_1, \beta_2) + \tilde{q}_{34}\left[\beta_1(K_1\beta_2 + K_3) + \beta_2(K_1\beta_1 + K_2)\right] \\
&\quad + 2\tilde{q}_{44}(K_1\beta_1 + K_2)(K_1\beta_2 + K_3).
\end{aligned}$$

Since $\beta_1 = K\mathfrak{m}(i\omega)$, $\beta_2 = K^*\mathfrak{m}(-i\omega)$, it is easy to show that $K_1\beta_1 + K_2 = Kp(i\omega)$, and $K_1\beta_2 + K_2 = K^*p(-i\omega)$.

Now, Q_j $(j = 1,2,3)$, $\tilde{Q}_j$ $(j = 1,2)$ and W_3 can be readily calculated as

$$Q_1 = q_{11} + \Phi_1(i\omega)K + \frac{1}{2}\Psi(i\omega, i\omega)K^2, \tag{2.93}$$

$$Q_2 = q_{12} + \Phi_2(i\omega)K + \Phi_1(-i\omega)K^* + \Psi(i\omega, -i\omega)|K|^2, \tag{2.94}$$

$$Q_3 = q_{22} + \Phi_2(-i\omega)K^* + \frac{1}{2}\Psi(-i\omega, -i\omega)K^{*2}, \tag{2.95}$$

$$\tilde{Q}_1 = \tilde{q}_{11} + \tilde{\Phi}_1(i\omega)K + \frac{1}{2}\tilde{\Psi}(i\omega, i\omega)K^2, \tag{2.96}$$

$$\tilde{Q}_2 = \tilde{q}_{12} + \tilde{\Phi}_2(i\omega)K + \tilde{\Phi}_1(-i\omega)K^* + \tilde{\Psi}(i\omega, -i\omega)|K|^2, \tag{2.97}$$

$$\begin{aligned} W_3 = {} & c_{112} + \left[c_{123}(i\omega) + c_{124}p(i\omega)\right] K \\ & + \left[c_{113}(-i\omega) + c_{114}p(-i\omega)\right] K^* \\ & + \left[c_{233}(i\omega, i\omega) + c_{234}(i\omega)p(i\omega) + c_{244}p(i\omega)^2\right] K^2 \\ & + [2c_{133}(i\omega, -i\omega) + c_{134}(i\omega)p(-i\omega) + \\ & c_{134}(-i\omega)p(i\omega) + 2c_{144}|p(i\omega)|^2]\ |K|^2 \\ & + \{3c_{333}(i\omega, i\omega, -i\omega) + c_{334}(i\omega, i\omega)p(-i\omega) + \\ & 2c_{334}(i\omega, -i\omega)p(i\omega) + 2c_{344}(i\omega)|p(i\omega)|^2 + \\ & c_{344}(-i\omega)p(i\omega)^2 + 3c_{444}|p(i\omega)|^2 p(i\omega)\}|K|^2 K, \end{aligned} \tag{2.98}$$

where $\Phi_1(s)$, $\Phi_2(s)$ are given by (2.78) and (2.79), respectively. $\tilde{\Phi}_1(s)$, $\tilde{\Phi}_2(s)$, $\Psi(s_1, s_2)$, and $\tilde{\Psi}(s_1, s_2)$ are defined as

$$\tilde{\Phi}_1(s) = \tilde{q}_{13}(s) + \tilde{q}_{14}p(s), \tag{2.99}$$

$$\tilde{\Phi}_2(s) = \tilde{q}_{23}(s) + \tilde{q}_{24}p(s), \tag{2.100}$$

$$\Psi(s_1, s_2) = 2q_{33}(s_1, s_2) + p(s_1)q_{34}(s_2) + q_{34}(s_1)p(s_2) + 2q_{44}p(s_1)p(s_2), \tag{2.101}$$

$$\tilde{\Psi}(s_1, s_2) = 2\tilde{q}_{33}(s_1, s_2) + p(s_1)\tilde{q}_{34}(s_2) + \tilde{q}_{34}(s_1)p(s_2) + 2\tilde{q}_{44}p(s_1)p(s_2). \tag{2.102}$$

In the expression of W_3 we have used the following notation:

$$c_{133}(i\omega, -i\omega) := \mathfrak{m}(i\omega)^T c_{133}\mathfrak{m}(-i\omega),$$

$$c_{333}(i\omega, i\omega, -i\omega) := c_{333}(\mathfrak{m}(i\omega), \mathfrak{m}(i\omega), \mathfrak{m}(-i\omega)), \quad \cdots$$

Now we calculate W_1.

$$
\begin{aligned}
W_1 &= \Big\{ \left[q_{13} + 2\beta_1^T q_{33} + q_{34}(K_1\beta_1 + K_2)\right] \\
&\quad + \left[q_{14} + q_{34}\beta_1 + 2q_{44}(K_1\beta_1 + K_2)\right] K_1 \Big\} \beta_4 \\
&\quad + \left[q_{14} + q_{34}\beta_1 + 2q_{44}(K_1\beta_1 + K_2)\right] (K_5 + K_7\beta_2 + K_8\beta_1 + 2\beta_1^T K_9 \beta_2) \\
&= \Big\{ \left[q_{13} + \left(2\mathrm{m}(i\omega)^T q_{33} + q_{34} p(i\omega)\right) K\right] + \left[q_{14} + \left(q_{34}(i\omega) + 2q_{44} p(i\omega)\right) K\right] K_1 \Big\} \cdot \\
&\quad \left[-(A + BK_1)\right]^{-1} \cdot \\
&\quad \Big\{ B\left[K_5 + K_7(-i\omega)K^* + K_7^*(i\omega)K + 2K_9(i\omega, -i\omega)|K|^2\right] \\
&\quad + \tilde{Q}_2 - \beta_1 Q_2 - \beta_2 Q_2^* \Big\} + \left[q_{14} + \left(q_{34}(i\omega) + 2q_{44} p(i\omega)\right) K\right] \cdot \\
&\quad \left[K_5 + K_7(-i\omega)K^* + K_7^*(i\omega)K + 2K_9(i\omega, -i\omega)|K|^2\right] \\
&= \frac{1}{\bar{p}(0)} \Big\{ \left[q_{13}(0) + \left(2\mathrm{m}(i\omega)^T q_{33} \mathrm{m}(0) + q_{34}(0) p(i\omega)\right) K\right] \\
&\quad + \left[q_{14} + \left(q_{34}(i\omega) + 2q_{44} p(i\omega)\right) K\right] (K_1(0) + \bar{p}(0)) \Big\} \cdot \\
&\quad \left[K_5 + K_7(-i\omega)K^* + K_7^*(i\omega)K + 2K_9(i\omega, -i\omega)|K|^2\right] + \\
&\quad \Big\{ \left[q_{13} + \left(2\mathrm{m}(i\omega)^T q_{33} + q_{34} p(i\omega)\right) K\right] + \left[q_{14} + \left(q_{34}(i\omega) + 2q_{44} p(i\omega)\right) K\right] K_1 \Big\} \cdot \\
&\quad \left(- \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\frac{1}{\bar{a}_n} & -\frac{\bar{a}_1}{\bar{a}_n} & \cdots & -\frac{\bar{a}_{n-1}}{\bar{a}_n} \end{bmatrix} \right) \left(\tilde{Q}_2 - \beta_1 Q_2 - \beta_2 Q_2^*\right) \\
&= \left[\Phi_1(0) + K\Psi(i\omega, 0)\right] \left[K_5 + K_7(-i\omega)K^* + K_7^*(i\omega)K + 2K_9(i\omega, -i\omega)|K|^2\right] \\
&\quad \Big\{ \frac{1}{\bar{p}(0)} \left[q_{13}(0) + (2q_{33}(i\omega, 0) + q_{34}(0) p(i\omega))K\right. \\
&\quad \left. + (q_{14} + (q_{34}(i\omega) + 2q_{44} p(i\omega))K) K_1(0)\right] \begin{bmatrix} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1} \end{bmatrix} \\
&\quad - \begin{bmatrix} 0 & q_{13}^1 & \cdots & q_{13}^{n-1} \end{bmatrix} - 2\mathrm{m}(i\omega)^T \begin{bmatrix} 0 & q_{33}^1 & \cdots & q_{33}^{n-1} \end{bmatrix} K \\
&\quad - p(i\omega) \begin{bmatrix} 0 & q_{34}^1 & \cdots & q_{34}^{n-1} \end{bmatrix} K - \left[q_{14} + (q_{34}(i\omega) + 2q_{44} p(i\omega))K\right] \cdot \\
&\quad \begin{bmatrix} 0 & K_1^1 & \cdots & K_1^{n-1} \end{bmatrix} \Big\} \left(\tilde{Q}_2 - \beta_1 Q_2 - \beta_2 Q_2^*\right) \\
&= \frac{\Theta_1 + K\Theta_3}{\bar{p}(0)} \left[K_5 + K_7(-i\omega)K^* + K_7^*(i\omega)K + 2K_9(i\omega, -i\omega)|K|^2\right]
\end{aligned}
$$

$$
\begin{aligned}
& +\left\{\frac{\Theta_1 + K\Theta_3}{\bar{p}(0)}\left[\begin{array}{cccc} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1}\end{array}\right]\right. \\
& -\left[q_{14} + (q_{34}(i\omega) + 2q_{44}p(i\omega))K\right]\left[\begin{array}{cccc} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1}\end{array}\right] \\
& -\left[\begin{array}{cccc} 0 & q_{13}^1 & \cdots & q_{13}^{n-1}\end{array}\right] - 2\mathfrak{m}(i\omega)^T\left[\begin{array}{cccc} 0 & q_{33}^1 & \cdots & q_{33}^{n-1}\end{array}\right]K \\
& -p(i\omega)\left[\begin{array}{cccc} 0 & q_{34}^1 & \cdots & q_{34}^{n-1}\end{array}\right]K \\
& \left.-\left[q_{14} + (q_{34}(i\omega) + 2q_{44}p(i\omega))K\right]\left[\begin{array}{cccc} 0 & K_1^1 & \cdots & K_1^{n-1}\end{array}\right]\right\}\cdot \\
& \left(\tilde{Q}_2 - \beta_1 Q_2 - \beta_2 Q_2^*\right) + \text{p.i.t} \\
= & \frac{\Theta_1 + K\Theta_3}{\bar{p}(0)}\left[K_5 + K_7(-i\omega)K^* + K_7^*(i\omega)K + 2K_9(i\omega, -i\omega)|K|^2\right] \\
& +\left\{\frac{\Theta_1 + K\Theta_3}{\bar{p}(0)}\left[\begin{array}{cccc} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1}\end{array}\right]\right. \\
& -\left[q_{14} + (q_{34}(i\omega) + 2q_{44}p(i\omega))K\right]\left[\begin{array}{cccc} 1 & a_1 & \cdots & a_{n-1}\end{array}\right] \\
& -\left[\begin{array}{cccc} 0 & q_{13}^1 & \cdots & q_{13}^{n-1}\end{array}\right] - 2\mathfrak{m}(i\omega)^T\left[\begin{array}{cccc} 0 & q_{33}^1 & \cdots & q_{33}^{n-1}\end{array}\right]K \\
& \left.-p(i\omega)\left[\begin{array}{cccc} 0 & q_{34}^1 & \cdots & q_{34}^{n-1}\end{array}\right]K\right\}\left(\tilde{Q}_2 - \beta_1 Q_2 - \beta_2 Q_2^*\right) + \text{p.i.t} \\
= & \frac{\Theta_1 + K\Theta_3}{\bar{p}(0)}\Big[K_5 + K_7(-i\omega)K^* + K_7^*(i\omega)K + 2K_9(i\omega, -i\omega)|K|^2 \\
& +\left[\begin{array}{cccc} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1}\end{array}\right]\left(\tilde{Q}_2 - \mathfrak{m}(i\omega)Q_2K - \mathfrak{m}(-i\omega)Q_2^*K^*\right)\Big] \\
& -\left(\Pi_1^H + \Pi_3^H K\right)\left(\tilde{Q}_2 - \mathfrak{m}(i\omega)Q_2K - \mathfrak{m}(-i\omega)Q_2^*K^*\right) + \text{p.i.t},
\end{aligned}
$$

where $\Theta_1 = \operatorname{Re}\Phi_1(0)$, $\Theta_3 = \Psi(i\omega, 0)$, Π_1^H is given by (2.81), and Π_3^H is given by

$$
\begin{aligned}
\Pi_3^H = & \; 2\mathfrak{m}(i\omega)^T\left[\begin{array}{cccc} 0 & q_{33}^1 & \cdots & q_{33}^{n-1}\end{array}\right] + p(i\omega)\left[\begin{array}{cccc} 0 & q_{34}^1 & \cdots & q_{34}^{n-1}\end{array}\right] \\
& + (q_{34}(i\omega) + 2q_{44}p(i\omega))\left[\begin{array}{cccc} 1 & a_1 & \cdots & a_{n-1}\end{array}\right].
\end{aligned} \tag{2.103}
$$

So we get

$$
\begin{aligned}
W_1 = & \frac{\Theta_1 + K\Theta_3}{\bar{p}(0)}\Big[K_5 + K_7(-i\omega)K^* + K_7^*(i\omega)K + 2K_9(i\omega, -i\omega)|K|^2 \\
& +\left[\begin{array}{cccc} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1}\end{array}\right]\left(\tilde{Q}_2 - \mathfrak{m}(i\omega)Q_2K - \mathfrak{m}(-i\omega)Q_2^*K^*\right)\Big] \\
& -\left(\Pi_1^H + \Pi_3^H K\right)\left(\tilde{Q}_2 - \mathfrak{m}(i\omega)Q_2K - \mathfrak{m}(-i\omega)Q_2^*K^*\right) + \text{p.i.t.}
\end{aligned} \tag{2.104}
$$

Now we calculate W_2.

$$\begin{aligned}
W_2 &= \Big\{ \left[q_{23} + 2\beta_2^T q_{33} + q_{34}(K_1\beta_2 + K_3)\right] \\
&\quad + \left[q_{24} + q_{34}\beta_2 + 2q_{44}(K_1\beta_2 + K_3)\right] K_1 \Big\}\beta_3 \\
&\quad + \left[q_{24} + q_{34}\beta_2 + 2q_{44}(K_1\beta_2 + K_3)\right] (K_4 + K_7\beta_1 + \beta_1^T K_9 \beta_1) \\
&= \Big\{ \left[q_{23} + \left(2\mathrm{m}(-i\omega)^T q_{33} + q_{34}p(-i\omega)\right) K^*\right] \\
&\quad + \left[q_{24} + (q_{34}(-i\omega) + 2q_{44}(-i\omega))K^*\right] K_1 \Big\} \cdot \\
&\quad [2\omega i - (A + BK_1)]^{-1} \left[B(K_4 + K_7\beta_1 + \beta_1^T K_9\beta_1) + \tilde{Q}_1 - \beta_1 Q_1 - \beta_2 Q_3^*\right] \\
&\quad + \left[q_{24} + (q_{34}(-i\omega) + 2q_{44}p(-i\omega))K^*\right] (K_4 + K_7\beta_1 + \beta_1^T K_9\beta_1) \\
&= \frac{1}{\bar{p}(2\omega i)} \Big\{ q_{23}(2\omega i) + \left(2\mathrm{m}(-i\omega)^T q_{33}\mathrm{m}(2\omega i) + q_{34}(2\omega i)p(-i\omega)\right) K^* \\
&\quad + \left[q_{24} + (q_{34}(-i\omega) + 2q_{44}p(-i\omega))K^*\right] (K_1(2\omega i) + \bar{p}(2\omega i)) \Big\} \cdot \\
&\quad \left[K_4 + K_7(i\omega)K + K_9(i\omega, i\omega)K^2\right] \\
&\quad + \Big\{ \left[q_{23} + \left(2\mathrm{m}(-i\omega)^T q_{33} + q_{34}p(-i\omega)\right) K^*\right] \\
&\quad + \left[q_{24} + (q_{34}(-i\omega) + 2q_{44}(-i\omega))K^*\right] K_1 \Big\} \cdot \\
&\quad \left\{ \frac{1}{\bar{p}(2\omega i)} \begin{bmatrix} (2\omega i)^{n-1} \\ \vdots \\ 2\omega i \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2\omega i & \cdots & (2\omega i)^{n-1} \end{bmatrix} \begin{bmatrix} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1} \\ 0 & 1 & \cdots & \bar{a}_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \right. \\
&\quad \left. - \begin{bmatrix} 0 & 1 & \cdots & (2\omega i)^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right\} \left(\tilde{Q}_1 - \beta_1 Q_1 - \beta_2 Q_3^*\right) \\
&= \frac{1}{\bar{p}(2\omega i)} \left[\Phi_2(2\omega i) + \Psi(-i\omega, 2\omega i)K^*\right] \left[K_4 + K_7(i\omega)K + K_9(i\omega, i\omega)K^2\right] \\
&\quad + \left\{ \frac{1}{\bar{p}(2\omega i)} [q_{23}(2\omega i) + (2q_{33}(-i\omega, 2\omega i) + q_{34}(2\omega i)p(-i\omega))K^* \right. \\
&\quad + (q_{24} + (q_{34}(-i\omega) + 2q_{44}p(-i\omega))K^*)K_1(2\omega i)] \cdot \\
&\quad \begin{bmatrix} 1 & \bar{p}_1(2\omega i) & \cdots & \bar{p}_{n-1}(2\omega i) \end{bmatrix} - \begin{bmatrix} 0 & q_{23,1}(2\omega i) & \cdots & q_{23,n-1}(2\omega i) \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
&-2\mathfrak{m}(-i\omega)^T \begin{bmatrix} 0 & q_{33,1}(2\omega i) & \cdots & q_{33,n-1}(2\omega i) \end{bmatrix} K^* \\
&-p(-i\omega) \begin{bmatrix} 0 & q_{34,1}(2\omega i) & \cdots & q_{34,n-1}(2\omega i) \end{bmatrix} K^* \\
&- [q_{24} + (q_{34}(-i\omega) + 2q_{44}p(-i\omega))K^*] \cdot \\
&\left.\begin{bmatrix} 0 & K_{1,1}(2\omega i) & \cdots K_{1,n-1}(2\omega i) \end{bmatrix} \right\} \left(\tilde{Q}_1 - \beta_1 Q_1 - \beta_2 Q_3^*\right) \\
= \; & \frac{\Theta_2 + K^*\Theta_4}{\bar{p}(2\omega i)} \left[K_4 + K_7(i\omega)K + K_9(i\omega, i\omega)K^2\right] \\
&+\left\{ \frac{\Theta_2 + K^*\Theta_4}{\bar{p}(2\omega i)} \begin{bmatrix} 1 & \bar{p}_1(2\omega i) & \cdots & \bar{p}_{n-1}(2\omega i) \end{bmatrix} \right. \\
&- [q_{24} + (q_{34}(-i\omega) + 2q_{44}p(-i\omega))K^*] \cdot \\
&\left(\begin{bmatrix} 1 & \bar{p}_1(2\omega i) & \cdots & \bar{p}_{n-1}(2\omega i) \end{bmatrix} \right. \\
&\left. + \begin{bmatrix} 0 & K_{1,1}(2\omega i) & \cdots K_{1,n-1}(2\omega i) \end{bmatrix} \right) \\
&- \begin{bmatrix} 0 & q_{23,1}(2\omega i) & \cdots & q_{23,n-1}(2\omega i) \end{bmatrix} \\
&-2\mathfrak{m}(-i\omega)^T \begin{bmatrix} 0 & q_{33,1}(2\omega i) & \cdots & q_{33,n-1}(2\omega i) \end{bmatrix} K^* \\
&\left. -p(-i\omega) \begin{bmatrix} 0 & q_{34,1}(2\omega i) & \cdots & q_{34,n-1}(2\omega i) \end{bmatrix} K^* \right\} \cdot \\
&\left(\tilde{Q}_1 - \beta_1 Q_1 - \beta_2 Q_3^*\right) \\
= \; & \frac{\Theta_2 + K^*\Theta_4}{\bar{p}(2\omega i)} \Big[K_4 + K_7(i\omega)K + K_9(i\omega, i\omega)K^2 + \\
&\begin{bmatrix} 1 & \bar{p}_1(2\omega i) & \cdots & \bar{p}_{n-1}(2\omega i) \end{bmatrix} \cdot \\
&\left(\tilde{Q}_1 - \mathfrak{m}(i\omega)Q_1 K - \mathfrak{m}(-i\omega)Q_3^* K^*\right) \Big] \\
&- \left(\Pi_2^H + \Pi_4^H K^*\right) \left(\tilde{Q}_1 - \mathfrak{m}(i\omega)Q_1 K - \mathfrak{m}(-i\omega)Q_3^* K^*\right),
\end{aligned}
$$

where $\Theta_2 = \Phi_2(0)$, $\Theta_4 = \Psi(-i\omega, 2\omega i)$, Π_2^H is given by (2.82), and Π_4^H is given by

$$
\begin{aligned}
\Pi_4^H = \; & 2\mathfrak{m}(-i\omega)^T \begin{bmatrix} 0 & q_{33,1}(2\omega i) & \cdots & q_{33,n-1}(2\omega i) \end{bmatrix} \\
&+p(-i\omega) \begin{bmatrix} 0 & q_{34,1}(2\omega i) & \cdots & q_{34,n-1}(2\omega i) \end{bmatrix} \\
&+ [q_{34}(-i\omega) + 2q_{44}p(-i\omega)] \begin{bmatrix} 1 & p_1(2\omega i) & \cdots & p_{n-1}(2\omega i) \end{bmatrix}.
\end{aligned} \tag{2.105}
$$

So we get

$$
\begin{aligned}
W_2 &= \frac{\Theta_2 + K^*\Theta_4}{\bar{p}(2\omega i)} \Big[K_4 + K_7(i\omega)K + K_9(i\omega, i\omega)K^2 \\
&\quad + \begin{bmatrix} 1 & \bar{p}_1(2\omega i) & \cdots & \bar{p}_{n-1}(2\omega i) \end{bmatrix} \left(\tilde{Q}_1 - \mathfrak{m}(i\omega)Q_1K - \mathfrak{m}(-i\omega)Q_3^*K^* \right) \Big] \\
&\quad - (\Pi_2^H + \Pi_4^H K^*) \left(\tilde{Q}_1 - \mathfrak{m}(i\omega)Q_1K - \mathfrak{m}(-i\omega)Q_3^*K^* \right). \qquad (2.106)
\end{aligned}
$$

Define $\tilde{G}_H(K, K^*)$ and $G_H(K, K^*)$ as

$$
\begin{aligned}
\tilde{G}_H(K, K^*) &= -(\Pi_1^H + \Pi_3^H K) \left(\tilde{Q}_2 - \mathfrak{m}(i\omega)Q_2K - \mathfrak{m}(-i\omega)Q_2^*K^* \right) \\
&\quad - (\Pi_2^H + \Pi_4^H K^*) \left(\tilde{Q}_1 - \mathfrak{m}(i\omega)Q_1K - \mathfrak{m}(-i\omega)Q_3^*K^* \right) \\
&\quad - \frac{Q_1Q_2}{i\omega} + W_3, \qquad (2.107) \\
G_H(K, K^*) &= \operatorname{Re} \tilde{G}_H(K, K^*), \qquad (2.108)
\end{aligned}
$$

where Q_1, Q_2, Q_3, $\tilde{Q}_1$, $\tilde{Q}_2$ and W_3 are given by (2.93), (2.94), (2.95), (2.96), (2.97), and (2.98), respectively. From (2.92), (2.98), (2.104), (2.106), and (2.107), we get

$$
\begin{aligned}
\tilde{\alpha} &= W_1 + W_2 + W_3 - \frac{Q_1Q_2}{i\omega} + \text{p.i.t.} \\
&= \frac{\Theta_1 + K\Theta_3}{\bar{p}(0)} \Big[K_5 + K_7(-i\omega)K^* + K_7^*(i\omega)K + 2K_9(i\omega, -i\omega)|K|^2 \\
&\quad + \begin{bmatrix} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1} \end{bmatrix} \left(\tilde{Q}_2 - \mathfrak{m}(i\omega)Q_2K - \mathfrak{m}(-i\omega)Q_2^*K^* \right) \Big] \\
&\quad + \frac{\Theta_2 + K^*\Theta_4}{\bar{p}(2\omega i)} \Big[K_4 + K_7(i\omega)K + K_9(i\omega, i\omega)K^2 \\
&\quad + \begin{bmatrix} 1 & \bar{p}_1(2\omega i) & \cdots & \bar{p}_{n-1}(2\omega i) \end{bmatrix} \cdot \\
&\quad \left(\tilde{Q}_1 - \mathfrak{m}(i\omega)Q_1K - \mathfrak{m}(-i\omega)Q_3^*K^* \right) \Big] + \tilde{G}_H(K, K^*) + \text{p.i.t.} \qquad (2.109)
\end{aligned}
$$

Now the goal is to find feedback gains such that $\alpha = \operatorname{Re} \tilde{\alpha} < 0$. The construction procedure **HS-4** and **HS-5** in the above section is clear from (2.109). If $\Theta_j = 0$ ($j = 1, \cdots, 4$), then we have

$$
\alpha = \operatorname{Re} \tilde{\alpha} = G_H(K, K^*). \qquad (2.110)
$$

where $G_H(K, K^*)$ is given by (2.108). The interesting feature of the expression is that $\tilde{\alpha}$ is only a function of K and K^* if $\Theta_j = 0$ $(j = 1, \cdots, 4)$. The goal in the following is to calculate the explicit expressions of $\tilde{G}_H(K, K^*)$ as a function of K and K^*. We claim that the following identities are true:

$$\begin{aligned}
\Pi_1^H \mathrm{m}(i\omega) &= \frac{1}{i\omega}\left[\Phi_1(i\omega) - i\,\mathrm{Im}\,\Phi_1(0) - \Theta_1\right],\\
\Pi_1^H \mathrm{m}(-i\omega) &= -\frac{1}{i\omega}\left[\Phi_1(-i\omega) - i\,\mathrm{Im}\,\Phi_1(0) - \Theta_1\right],\\
\Pi_2^H \mathrm{m}(i\omega) &= -\frac{1}{i\omega}\left[\Phi_2(i\omega) - \Theta_2\right],\\
\Pi_2^H \mathrm{m}(-i\omega) &= -\frac{1}{3\omega i}\left[\Phi_2(-i\omega) - \Theta_2\right],\\
\Pi_3^H \mathrm{m}(i\omega) &= \frac{1}{i\omega}\left[\Psi(i\omega, i\omega) - \Theta_3\right],\\
\Pi_3^H \mathrm{m}(-i\omega) &= -\frac{1}{i\omega}\left[\Psi(i\omega, -i\omega) - \Theta_3\right],\\
\Pi_4^H \mathrm{m}(i\omega) &= -\frac{1}{i\omega}\left[\Psi(i\omega, -i\omega) - \Theta_4\right],\\
\Pi_4^H \mathrm{m}(-i\omega) &= -\frac{1}{3\omega i}\left[\Psi(-i\omega, -i\omega) - \Theta_4\right],
\end{aligned}$$

where $\Phi_j(s)(j = 1, 2)$, $\tilde{\Phi}_j(s)(j = 1, 2)$, $\Psi(s_1, s_2)$ and $\tilde{\Psi}(s_1, s_2)$ are given in (2.21).

Proof: We calculate $\Pi_1^H \mathrm{m}(i\omega)$.

$$\begin{aligned}
\Pi_1^H \mathrm{m}(i\omega) &= \left\{\begin{bmatrix} 0 & q_{13}^1 & \cdots & q_{13}^{n-1} \end{bmatrix} + q_{14}\begin{bmatrix} 1 & a_1 & \cdots & a_{n-1} \end{bmatrix}\right\}\begin{bmatrix} (i\omega)^{n-1} \\ \vdots \\ i\omega \\ 1 \end{bmatrix}\\
&= q_{13,n-1}(i\omega) + q_{14}p_{n-1}(i\omega)\\
&= \frac{1}{i\omega}\left\{\left[q_{13}(i\omega) + q_{14}p(i\omega)\right] - \left[q_{13}(0) + q_{14}p(0)\right]\right\}\\
&= \frac{1}{i\omega}\left[\Phi_1(i\omega) - i\,\mathrm{Im}\,\Phi_1(0) - \Theta_1\right].
\end{aligned}$$

Note we have used $\Theta_1 = \mathrm{Re}\,\Phi_1(0)$ in the last step. The case for $\Pi_1^H \mathrm{m}(-i\omega)$ is the same.

Now we calculate $\Pi_2^H \mathrm{m}(i\omega)$.

$$\begin{aligned}
\Pi_2^H \mathrm{m}(i\omega) &= \left\{ \begin{bmatrix} 0 & q_{23,1}(2\omega i) & \cdots & q_{23,n-1}(2\omega i) \end{bmatrix} \right. \\
&\quad \left. + q_{24} \begin{bmatrix} 1 & p_1(2\omega i) & \cdots & p_{n-1}(2\omega i) \end{bmatrix} \right\} \begin{bmatrix} (i\omega)^{n-1} \\ \vdots \\ i\omega \\ 1 \end{bmatrix} \\
&= \left[(i\omega)^{n-2} q_{23,1}(2\omega i) + \cdots + (i\omega) q_{23,n-2}(2\omega i) + q_{23,n-1}(2\omega i) \right] \\
&\quad + q_{24} \left[(i\omega)^{n-1} + (i\omega)^{n-2} p_1(2\omega i) + \cdots + (i\omega) p_{n-2}(2\omega i) + p_{n-1}(2\omega i) \right] \\
&= q_{23}^1 \left[(i\omega)^{n-2} + (2\omega i)(i\omega)^{n-3} + \cdots + (2\omega i)^{n-2} \right] + \cdots + q_{23}^{n-1} \\
&\quad + q_{24} \left\{ \left[(i\omega)^{n-1} + (2\omega i)(i\omega)^{n-2} + \cdots + (2\omega i)^{n-1} \right] \right. \\
&\quad \left. + a_1 \left[(i\omega)^{n-2} + (2\omega i)(i\omega)^{n-3} + \cdots + (2\omega i)^{n-2} \right] + \cdots + a_{n-1} \right\} \\
&= q_{23}^1 \left[2(2\omega i)^{n-2} - (i\omega)^{n-2} \right] + \cdots + q_{23}^{n-1} \\
&\quad + q_{24} \left\{ \left[2(2\omega i)^{n-1} - (i\omega)^{n-1} \right] + a_1 \left[2(2\omega i)^{n-2} - (i\omega)^{n-2} \right] + \cdots + a_{n-1} \right\} \\
&= 2 q_{23,n-1}(2\omega i) - q_{23,n-1}(i\omega) + q_{24}[2 p_{n-1}(2\omega i) - p_{n-1}(i\omega)] \\
&= 2 \cdot \frac{q_{23}(2\omega i) - q_{23}(0)}{2\omega i} - \frac{q_{23}(i\omega) - q_{23}(0)}{i\omega} \\
&\quad + q_{24} \left(2 \cdot \frac{p(2\omega i) - p(0)}{2\omega i} - \frac{p(i\omega) - p(0)}{i\omega} \right) \\
&= \frac{1}{i\omega} [\Phi_2(2\omega i) - \Phi_2(i\omega)] = -\frac{1}{i\omega} [\Phi_2(i\omega) - \Theta_2].
\end{aligned}$$

Note we have used the fact that $\Theta_2 = \Phi_2(2\omega i)$ in the last step.

Now we calculate $\Pi_2^H \mathfrak{m}(-i\omega)$.

$$\begin{aligned}
\Pi_2^H \mathfrak{m}(-i\omega) &= \left\{ \begin{bmatrix} 0 & q_{23,1}(2\omega i) & \cdots & q_{23,n-1}(2\omega i) \end{bmatrix} \right. \\
&\quad \left. + q_{24} \begin{bmatrix} 1 & p_1(2\omega i) & \cdots & p_{n-1}(2\omega i) \end{bmatrix} \right\} \begin{bmatrix} (-i\omega)^{n-1} \\ \vdots \\ -i\omega \\ 1 \end{bmatrix} \\
&= \Big[(-i\omega)^{n-2} q_{23,1}(2\omega i) + \cdots + (-i\omega) q_{23,n-2}(2\omega i) + q_{23,n-1}(2\omega i)\Big] \\
&\quad + q_{24}\Big[(-i\omega)^{n-1} + (-i\omega)^{n-2} p_1(2\omega i) + \cdots + (-i\omega) p_{n-2}(2\omega i) \\
&\quad + p_{n-1}(2\omega i)\Big] \\
&= q_{23}^1 \Big[(-i\omega)^{n-2} + (2\omega i)(-i\omega)^{n-3} + \cdots + (2\omega i)^{n-2}\Big] + \cdots + q_{23}^{n-1} \\
&\quad + q_{24} \Big\{ \Big[(-i\omega)^{n-1} + (2\omega i)(-i\omega)^{n-2} + \cdots + (2\omega i)^{n-1}\Big] \\
&\quad + a_1 \Big[(-i\omega)^{n-2} + (2\omega i)(-i\omega)^{n-3} + \cdots + (2\omega i)^{n-2}\Big] + \cdots + a_{n-1} \Big\} \\
&= q_{23}^1 \left[\frac{2}{3}(2\omega i)^{n-2} + \frac{1}{3}(-i\omega)^{n-2}\right] + \cdots + q_{23}^{n-1} + q_{24} \cdot \\
&\quad \left\{ \left[\frac{2}{3}(2\omega i)^{n-1} + \frac{1}{3}(-i\omega)^{n-1}\right] + a_1 \left[\frac{2}{3}(2\omega i)^{n-2} + \frac{1}{3}(-i\omega)^{n-2}\right] \right. \\
&\quad \left. + \cdots + a_{n-1} \right\} \\
&= \frac{2}{3} q_{23,n-1}(2\omega i) + \frac{1}{3} q_{23,n-1}(-i\omega) + q_{34} \left[\frac{2}{3} p_{n-1}(2\omega i) + \frac{1}{3} p_{n-1}(-i\omega)\right] \\
&= \frac{1}{3}\left(2 \cdot \frac{q_{23}(2\omega i) - q_{23}(0)}{2\omega i} + \frac{q_{23}(-i\omega) - q_{23}(0)}{-i\omega}\right) \\
&\quad + \frac{1}{3} q_{24} \left(2 \cdot \frac{p(2\omega i) - p(0)}{2\omega i} + \frac{p(-i\omega) - p(0)}{-i\omega}\right) \\
&= \frac{1}{3\omega i}[\Phi_2(2\omega i) - \Phi_2(-i\omega)] = -\frac{1}{3\omega i}[\Phi_2(-i\omega) - \Theta_2].
\end{aligned}$$

Note that we have used the definition $\Theta_2 = \Phi_1(2\omega i)$.

Now we calculate $\Pi_3^H \mathrm{m}(i\omega)$.

$$\begin{aligned}
\Pi_3^H \mathrm{m}(i\omega) &= \Big\{ 2\mathrm{m}(i\omega)^T \begin{bmatrix} 0 & q_{33}^1 & \cdots & q_{33}^{n-1} \end{bmatrix} + q_{34}p(i\omega) \begin{bmatrix} 0 & q_{34}^1 & \cdots & q_{34}^{n-1} \end{bmatrix} \\
&\quad + (q_{34}(i\omega) + 2q_{44}p(i\omega)) \begin{bmatrix} 1 & a_1 & \cdots & a_{n-1} \end{bmatrix} \Big\} \begin{bmatrix} (i\omega)^{n-1} \\ \vdots \\ i\omega \\ 1 \end{bmatrix} \\
&\doteq \frac{1}{i\omega} \{ [q_{33}(i\omega, i\omega) - q_{33}(i\omega, 0)] + q_{34}p(i\omega) [q_{34}(i\omega) - q_{34}(0)] \} \\
&\quad + \frac{1}{i\omega} (q_{34}(i\omega) + 2q_{44}p(i\omega)) [p(i\omega) - p(0)] \\
&= \frac{1}{i\omega} [\Psi(i\omega, i\omega) - \Psi(i\omega, 0)] \\
&= \frac{1}{i\omega} [\Psi(i\omega, i\omega) - \Theta_3] .
\end{aligned}$$

Note we have used the fact that $\Theta_3 = \Psi(i\omega, 0)$ in the last step. Calculating $\Pi_3^H \mathrm{m}(-i\omega)$ is the same. Now we calculate $\Pi_4^H \mathrm{m}(i\omega)$.

$$\begin{aligned}
\Pi_4^H \mathrm{m}(i\omega) &= \Big\{ 2\mathrm{m}(-i\omega)^T \begin{bmatrix} 0 & q_{33,1}(2\omega i) & \cdots & q_{33,n-1}(2\omega i) \end{bmatrix} \\
&\quad + p(-i\omega) \begin{bmatrix} 0 & q_{34,1}(2\omega i) & \cdots & q_{34,n-1}(2\omega i) \end{bmatrix} \\
&\quad + [q_{34}(-i\omega) + 2q_{44}p(-i\omega)] \begin{bmatrix} 1 & p_1(2\omega i) & \cdots & p_{n-1}(2\omega i) \end{bmatrix} \Big\} \begin{bmatrix} (i\omega)^{n-1} \\ \vdots \\ i\omega \\ 1 \end{bmatrix} \\
&= \left[2\mathrm{m}(-i\omega)^T q_{33} + p(-i\omega) q_{34} \right] \left(2 \begin{bmatrix} (2\omega i)^{n-2} \\ \vdots \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} (i\omega)^{n-2} \\ \vdots \\ 1 \\ 0 \end{bmatrix} \right) \\
&\quad + [q_{34}(-i\omega) + 2q_{44}p(-i\omega)] [2p_{n-1}(2\omega i) - p_{n-1}(i\omega)]
\end{aligned}$$

$$
\begin{aligned}
&= 2\cdot\frac{1}{2\omega i}\{[2q_{33}(-i\omega,2\omega i)+p(-i\omega)q_{34}(2\omega i)]-[2q_{33}(-i\omega,0)+p(-i\omega)q_{34}(0)]\}\\
&\quad -\frac{1}{i\omega}\{[2q_{33}(-i\omega,i\omega)+p(-i\omega)q_{34}(i\omega)]-[2q_{33}(-i\omega,0)+p(-i\omega)q_{34}(0)]\}\\
&\quad +[q_{34}(-i\omega)+2q_{44}p(-i\omega)]\left(2\cdot\frac{p(2\omega i)-p(0)}{2\omega i}-\frac{p(i\omega)-p(0)}{i\omega}\right)\\
&= \frac{1}{i\omega}\left[\Psi(-i\omega,2\omega i)-\Psi(-i\omega,i\omega)\right]=-\frac{1}{i\omega}\left[\Psi(-i\omega,i\omega)-\Theta_4\right].
\end{aligned}
$$

Now we calculate $\Pi_4^H \mathrm{m}(-i\omega)$.

$$
\begin{aligned}
\Pi_4^H\mathrm{m}(-i\omega) &= \Big\{2\mathrm{m}(-i\omega)^T\begin{bmatrix}0 & q_{33,1}(2\omega i) & \cdots & q_{33,n-1}(2\omega i)\end{bmatrix}\\
&\quad +p(-i\omega)\begin{bmatrix}0 & q_{34,1}(2\omega i) & \cdots & q_{34,n-1}(2\omega i)\end{bmatrix}\\
&\quad +[q_{34}(-i\omega)+2q_{44}p(-i\omega)]\begin{bmatrix}1 & p_1(2\omega i) & \cdots & p_{n-1}(2\omega i)\end{bmatrix}\Big\}\cdot\\
&\quad \begin{bmatrix}(-i\omega)^{n-1}\\ \vdots\\ -i\omega\\ 1\end{bmatrix}\\
&= \left[2\mathrm{m}(-i\omega)^T q_{33}+p(-i\omega)q_{34}\right]\left(\frac{2}{3}\begin{bmatrix}(2\omega i)^{n-2}\\ \vdots\\ 1\\ 0\end{bmatrix}+\frac{1}{3}\begin{bmatrix}(-i\omega)^{n-2}\\ \vdots\\ 1\\ 0\end{bmatrix}\right)\\
&\quad +[q_{34}(-i\omega)+2q_{44}p(-i\omega)]\left[\frac{2}{3}p_{n-1}(2\omega i)+\frac{1}{3}p_{n-1}(-i\omega)\right]\\
&= \frac{2}{3}\cdot\frac{1}{2\omega i}\{[2q_{33}(-i\omega,2\omega i)+p(-i\omega)q_{34}(2\omega i)]\\
&\quad -2q_{33}(-i\omega,0)+p(-i\omega)q_{34}(0)]\}\\
&\quad +\frac{1}{3}\frac{1}{-i\omega}\{[2q_{33}(-i\omega,-i\omega)+p(-i\omega)q_{34}(-i\omega)]\\
&\quad -[2q_{33}(-i\omega,0)+p(-i\omega)q_{34}(0)]\}\\
&\quad +\frac{1}{3}\left[q_{34}(-i\omega)+2q_{44}p(-i\omega)\right]\left(2\cdot\frac{p(2\omega i)-p(0)}{2\omega i}+\frac{p(-i\omega)-p(0)}{-i\omega}\right)\\
&= \frac{1}{3\omega i}\left[\Psi(-i\omega,2\omega i)-\Psi(-i\omega,-i\omega)\right]=-\frac{1}{3\omega i}\left[\Psi(-i\omega,-i\omega)-\Theta_4\right].
\end{aligned}
$$

■

Now we calculate $\tilde{G}_H(K, K^*)$.

$$
\begin{aligned}
\tilde{G}_H(K,K^*) &= -\left(\Pi_1^H + \Pi_3^H K\right)\left(\tilde{Q}_2 - \mathfrak{m}(i\omega)Q_2K - \mathfrak{m}(-i\omega)Q_2^*K^*\right) \\
&\quad -\left(\Pi_2^H + \Pi_4^H K^*\right)\left(\tilde{Q}_1 - \mathfrak{m}(i\omega)Q_1K - \mathfrak{m}(-i\omega)Q_3^*K^*\right) \\
&\quad -\frac{Q_1Q_2}{i\omega} + W_3 + \text{p.i.t.} \\
&= -\frac{Q_1Q_2}{i\omega} + \Pi_1^H\mathfrak{m}(i\omega)KQ_2 + \Pi_1^H\mathfrak{m}(-i\omega)K^*Q_2^* + \Pi_2^H\mathfrak{m}(i\omega)KQ_1 \\
&\quad +\Pi_2^H\mathfrak{m}(-i\omega)K^*Q_3^* + \Pi_3^H\mathfrak{m}(i\omega)K^2Q_2 + \Pi_3^H\mathfrak{m}(-i\omega)|K|^2Q_2^* \\
&\quad +\Pi_4^H\mathfrak{m}(i\omega)|K|^2Q_2 + \Pi_4^H\mathfrak{m}(-i\omega)K^{*2}Q_3^* - \left(\Pi_1^H + \Pi_3^H K\right)\tilde{Q}_2 \\
&\quad -\left(\Pi_2^H + \Pi_4^H K^*\right)\tilde{Q}_1 + W_3 + \text{p.i.t.} \\
&= H(K,K^*) - \frac{1}{i\omega}\left[q_{11} + \Phi_1(i\omega)K + \frac{1}{2}\Psi(i\omega,i\omega)K^2\right]\cdot \\
&\quad \left[q_{12} + \Phi_2(i\omega)K + \Phi_1(-i\omega)K^* + \Psi(i\omega,-i\omega)|K|^2\right] \\
&\quad +\frac{1}{i\omega}\Phi_1(i\omega)K\left[q_{12} + \Phi_2(i\omega)K + \Phi_1(-i\omega)K^* + \Psi(i\omega,-i\omega)|K|^2\right] \\
&\quad -\frac{1}{i\omega}\Phi_1(-i\omega)K^*\left[q_{12}^* + \Phi_2^*(-i\omega)K^* + \Phi_1^*(i\omega)K + \Psi^*(-i\omega,i\omega)|K|^2\right] \\
&\quad -\frac{1}{i\omega}\Phi_2(i\omega)K\left[q_{11} + \Phi_1(i\omega)K + \frac{1}{2}\Psi(i\omega,i\omega)K^2\right] \\
&\quad -\frac{1}{3\omega i}\Phi_2(-i\omega)K^*\left[q_{22}^* + \Phi_2^*(i\omega)K + \frac{1}{2}\Psi^*(i\omega,i\omega)K^2\right] \\
&\quad +\frac{1}{i\omega}\Psi(i\omega,i\omega)K^2\left[q_{12} + \Phi_2(i\omega)K + \Phi_1(-i\omega)K^* + \Psi(i\omega,-i\omega)|K|^2\right] \\
&\quad -\frac{1}{i\omega}\Psi(i\omega,-i\omega)|K|^2\cdot \\
&\quad \left[q_{12}^* + \Phi_2^*(-i\omega)K^* + \Phi_1^*(i\omega)K + \Psi^*(-i\omega,i\omega)|K|^2\right] \\
&\quad -\frac{1}{i\omega}\Psi(i\omega,-i\omega)|K|^2\left[q_{11} + \Phi_1(i\omega)K + \frac{1}{2}\Psi(i\omega,i\omega)K^2\right] \\
&\quad -\frac{1}{3\omega i}\Psi(-i\omega,-i\omega)K^{*2}\left[q_{22}^* + \Phi_2^*(i\omega)K + \frac{1}{2}\Psi^*(i\omega,i\omega)K^2\right] \\
&\quad -(\Pi_1^H + \Pi_3^H K)\left[\tilde{q}_{12} + \tilde{\Phi}_2(i\omega)K + \tilde{\Phi}_1(-i\omega)K^* + \tilde{\Psi}(i\omega,-i\omega)|K|^2\right] \\
&\quad -(\Pi_2^H + \Pi_4^H K^*)\left[\tilde{q}_{11} + \tilde{\Phi}_1(i\omega)K + \frac{1}{2}\tilde{\Psi}(i\omega,i\omega)K^2\right] + W_3 + \text{p.i.t.} \\
&= H(K,K^*) + C_0 + D_1K + D_2K^* + E_{11}K^2 + E_{12}|K|^2 + E_{22}K^{*2} \\
&\quad +F_{112}|K|^2K + F_{122}|K|^2K^* + \text{p.i.t.},
\end{aligned}
$$

i.e., we have

$$\tilde{G}_H(K,K^*) \quad = \quad H(K,K^*) + \tilde{P}_H(K,K^*) \tag{2.111}$$

where

$$\begin{aligned} H(K,K^*) &= \frac{1}{i\omega}(\Theta_1+\Theta_3 K)\,(Q_2^* K^* - Q_2 K) \\ &\quad +\frac{1}{i\omega}\,(\Theta_2+\Theta_4 K^*)\left(Q_1 K + \frac{1}{3}Q_3^* K^*\right), \end{aligned} \tag{2.112}$$

$$\begin{aligned} \tilde{P}_H(K,K^*) &= C_0 + D_1 K + D_2 K^* + E_{11}K^2 + E_{12}|K|^2 \\ &\quad + E_{22}K^{*2} + F_{112}|K|^2 K + F_{122}|K|^2 K^* + \text{p.i.t.}, \end{aligned} \tag{2.113}$$

and

$$\begin{aligned} C_0 &= c_{112} - \frac{1}{i\omega}q_{11}q_{12} - \Pi_1^H \tilde{q}_{12} - \Pi_2^H \tilde{q}_{11}, \\ D_1 &= c_{123}(i\omega) + c_{124}p(i\omega) - \Pi_1^H \tilde{\Phi}_2(i\omega) - \Pi_2^H \tilde{\Phi}_1(i\omega) \\ &\quad -\Pi_3^H \tilde{q}_{12} - \frac{2}{i\omega}q_{11}\Phi_2(i\omega), \\ D_2 &= c_{113}(-i\omega) + c_{114}p(-i\omega) - \Pi_1^H \tilde{\Phi}_1(-i\omega) - \Pi_4^H \tilde{q}_{11} \\ &\quad -\frac{1}{i\omega}(q_{11}+q_{12}^*)\Phi_1(-i\omega) - \frac{1}{3\omega i}q_{22}^*\Phi_2(-i\omega), \\ E_{11} &= 2c_{233}(i\omega,i\omega) + c_{234}(i\omega)p(i\omega) + c_{244}p(i\omega)^2 - \frac{1}{2}\Pi_2^H \tilde{\Psi}(i\omega,i\omega) \\ &\quad -\Pi_3^H \tilde{\Phi}_2(i\omega) + \frac{1}{2\omega i}\left[q_{12}\Psi(i\omega,i\omega) - 2\Phi_1(i\omega)\Phi_2(i\omega)\right], \\ E_{12} &= 2c_{133}(i\omega,-i\omega) + c_{134}(i\omega)p(-i\omega) + c_{134}(-i\omega)p(i\omega) \\ &\quad +2c_{144}|p(i\omega)|^2 - \Pi_1^H \tilde{\Psi}(i\omega,-i\omega) - \Pi_3^H \tilde{\Phi}_1(-i\omega) \\ &\quad -\Pi_4^H \tilde{\Phi}_1(i\omega) - \frac{1}{i\omega}(2q_{11}+q_{12}^*)\Psi(i\omega,-i\omega), \\ E_{22} &= -\frac{1}{i\omega}\Phi_1(-i\omega)\Phi_2^*(-i\omega) - \frac{1}{3\omega i}q_{22}^*\Psi(-i\omega,-i\omega), \end{aligned}$$

$$
\begin{aligned}
F_{112} &= 3c_{333}(i\omega, i\omega, -i\omega) + c_{334}(i\omega, i\omega)p(-i\omega) + 3c_{444}|p(i\omega)|^2 p(i\omega) \\
&\quad +2c_{334}(i\omega, -i\omega)p(i\omega) + 2c_{344}(i\omega)|p(i\omega)|^2 + c_{344}(-i\omega)p(i\omega)^2 \\
&\quad -\Pi_3^H \tilde{\Psi}(i\omega, -i\omega) - \frac{1}{2}\Pi_4^H \tilde{\Psi}(i\omega, i\omega) + \frac{1}{2\omega i}\Phi_1(-i\omega)\Psi(i\omega, i\omega) \\
&\quad -\frac{1}{i\omega}\left[\Phi_1(i\omega) + \Phi_1^*(i\omega)\right]\Psi(i\omega, -i\omega) \\
&\quad -\frac{1}{6\omega i}\Phi_2(-i\omega)\Psi^*(i\omega, i\omega), \qquad (2.114) \\
F_{122} &= -\frac{1}{i\omega}\Phi_2^*(-i\omega)\Psi(i\omega, -i\omega) - \frac{1}{i\omega}\Phi_1(-i\omega)\Psi^*(-i\omega, i\omega) \\
&\quad -\frac{1}{3\omega i}\Phi_2^*(i\omega)\Psi(-i\omega, -i\omega), \qquad (2.115)
\end{aligned}
$$

where $\Phi_j(s)(j = 1,2)$, $\tilde{\Phi}_j(s)(j = 1,2)$, $\Psi(s_1, s_2)$ and $\tilde{\Psi}(s_1, s_2)$ are given in (2.21). Letting $P_H(K, K^*) := \operatorname{Re}\tilde{P}(K, K^*)$, and $K_\flat = \begin{bmatrix} K_R \\ K_I \end{bmatrix}$, then we get

$$
\begin{aligned}
P_H(K_\flat) &= \alpha_0 + \alpha_1^1 K_R + \alpha_1^2 K_I + \alpha_2^{11} K_R^2 + 2\alpha_2^{12} K_R K_I \\
&\quad +\alpha_2^{22} K_I^2 + (\alpha_3^1 K_R + \alpha_3^2 K_I)(K_R^2 + K_I^2) \\
&= \alpha_0 + \alpha_1 K_\flat + K_\flat^T \alpha_2 K_\flat + \alpha_3 K_\flat \|K_\flat\|^2. \qquad (2.116)
\end{aligned}
$$

where Define

$$
\begin{aligned}
&\alpha_3^1 = \operatorname{Re}(F_{112} + F_{122}), && \alpha_3^2 = \operatorname{Re}(F_{122} - F_{112}), \\
&\alpha_0 = \operatorname{Re} C_0, && \alpha_2^{11} = \operatorname{Re}(E_{11} + E_{12} + E_{22}), \\
&\alpha_1^1 = \operatorname{Re}(D_1 + D_2), && \alpha_2^{12} = \operatorname{Im}(E_{22} - E_{11}), \\
&\alpha_1^2 = \operatorname{Im}(D_2 - D_1), && \alpha_2^{22} = \operatorname{Re}(E_{12} - E_{11} - E_{22}),
\end{aligned}
\qquad (2.117)
$$

and

$$
\alpha_1 = \begin{bmatrix} \alpha_1^1 & \alpha_1^2 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} \alpha_2^{11} & \alpha_2^{12} \\ \alpha_2^{12} & \alpha_2^{22} \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} \alpha_3^1 & \alpha_3^2 \end{bmatrix}. \qquad (2.118)
$$

By combining (2.109), (2.111) and (2.112), we get

$$\begin{aligned}\tilde{\alpha} \;=\; & \frac{\Theta_1 + K\Theta_3}{\bar{p}(0)}\Big[K_5 + K_7(-i\omega)K^* + K_7^*(i\omega)K + 2K_9(i\omega,-i\omega)|K|^2 + \\ & \begin{bmatrix} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1}\end{bmatrix}\Big(\tilde{Q}_2 - \mathfrak{m}(i\omega)Q_2K - \mathfrak{m}(-i\omega)Q_2^*K^*\Big) \\ & + \frac{\bar{p}(0)}{i\omega}\left(Q_2^*K^* - Q_2K\right)\Big] \\ & + \frac{\Theta_2 + K^*\Theta_4}{\bar{p}(2\omega i)}\Big[K_4 + K_7(i\omega)K + K_9(i\omega,i\omega)K^2 + \\ & \begin{bmatrix} 1 & \bar{p}_1(2\omega i) & \cdots & \bar{p}_{n-1}(2\omega i)\end{bmatrix}\Big(\tilde{Q}_1 - \mathfrak{m}(i\omega)Q_1K - \mathfrak{m}(-i\omega)Q_3^*K^*\Big) \\ & + \frac{\bar{p}(2\omega i)}{i\omega}\left(Q_1K + \frac{1}{3}Q_3^*K^*\right)\Big] \\ & + \alpha_0 + \alpha_1K_\flat + K_\flat^T\alpha_2K_\flat + \alpha_3K_\flat\|K_\flat\|^2 + \text{p.i.t.}\end{aligned} \tag{2.119}$$

Define $\alpha := \operatorname{Re}\tilde{\alpha}$, then we have

$$\begin{aligned}\alpha \;=\; & \frac{\Theta_1 + \operatorname{Re}\{K\Theta_3\}}{\bar{p}(0)}\Big\{K_5 + K_7(-i\omega)K^* + K_7^*(i\omega)K + 2K_9(i\omega,-i\omega)|K|^2 + \\ & \begin{bmatrix} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1}\end{bmatrix}\Big(\tilde{Q}_2 - \mathfrak{m}(i\omega)Q_2K - \mathfrak{m}(-i\omega)Q_2^*K^*\Big) \\ & + \frac{\bar{p}(0)}{i\omega}\left(Q_2^*K^* - Q_2K\right)\Big\} \\ & + \operatorname{Re}\Big\{\frac{\Theta_2 + K^*\Theta_4}{\bar{p}(2\omega i)}\Big[K_4 + K_7(i\omega)K + K_9(i\omega,i\omega)K^2 + \\ & \begin{bmatrix} 1 & \bar{p}_1(2\omega i) & \cdots & \bar{p}_{n-1}(2\omega i)\end{bmatrix}\Big(\tilde{Q}_1 - \mathfrak{m}(i\omega)Q_1K - \mathfrak{m}(-i\omega)Q_3^*K^*\Big) \\ & + \frac{\bar{p}(2\omega i)}{i\omega}\left(Q_1K + \frac{1}{3}Q_3^*K^*\right)\Big]\Big\} \\ & + \alpha_0 + \alpha_1K_\flat + K_\flat^T\alpha_2K_\flat + \alpha_3K_\flat\|K_\flat\|^2.\end{aligned} \tag{2.120}$$

The goal is to find control gains such that $\alpha < 0$. We have already discussed the cases when $\Theta_j \neq 0$ $(j = 1, \cdots, 4)$. In the following we assume $\Theta_j = 0$ for $j = 1, \cdots, 4$. In this case, we have

$$\alpha = P_H(K_\flat) = \alpha_0 + \alpha_1K_\flat + K_\flat^T\alpha_2K_\flat + \alpha_3K_\flat\|K_\flat\|^2.$$

If $\alpha_3 \neq 0$, we have the following two cases:

- if $\alpha_3^1 \neq 0$, then let $K_I = 0$ and $K_R = -\kappa \operatorname{sgn} \alpha_3^1$, we have $\alpha < 0$ for κ large enough.

- if $\alpha_3^2 \neq 0$, then let $K_R = 0$ and $K_I = -\kappa \operatorname{sgn} \alpha_3^2$, we have $\alpha < 0$ for κ large enough.

This explains the feedback construction **HS-6** in the previous section.

Now suppose $\alpha_3 = 0$, then let $T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$ be an orthonormal matrix, i.e., $T^{-1} = T^T$, such that

$$\hat{\alpha}_1 := \begin{bmatrix} \hat{\alpha}_1^1 & \hat{\alpha}_1^2 \end{bmatrix} = \begin{bmatrix} \alpha_1^1 & \alpha_1^2 \end{bmatrix} T, \quad \hat{\alpha}_2 := T^T \alpha_2 T = \begin{bmatrix} \hat{\alpha}_2^1 & 0 \\ 0 & \hat{\alpha}_2^2 \end{bmatrix}, \tag{2.121}$$

where T can be explicitly calculated as

$$T = \begin{bmatrix} \dfrac{\alpha_2^{12}}{\sqrt{(\alpha_2^{12})^2 + (\hat{\alpha}_2^1 - \alpha_2^{11})^2}} & \dfrac{\alpha_2^{12}}{\sqrt{(\alpha_2^{12})^2 + (\hat{\alpha}_2^2 - \alpha_2^{11})^2}} \\ \\ \dfrac{\hat{\alpha}_2^1 - \alpha_2^{11}}{\sqrt{(\alpha_2^{12})^2 + (\hat{\alpha}_2^1 - \alpha_2^{11})^2}} & \dfrac{\hat{\alpha}_2^2 - \alpha_2^{11}}{\sqrt{(\alpha_2^{12})^2 + (\hat{\alpha}_2^2 - \alpha_2^{11})^2}} \end{bmatrix}, \tag{2.122}$$

$$\hat{\alpha}_2^1 = \frac{\alpha_2^{11} + \alpha_2^{22}}{2} + \sqrt{\left(\frac{\alpha_2^{11} - \alpha_2^{22}}{2}\right)^2 + (\alpha_2^{12})^2}, \tag{2.123}$$

$$\hat{\alpha}_2^2 = \frac{\alpha_2^{11} + \alpha_2^{22}}{2} - \sqrt{\left(\frac{\alpha_2^{11} - \alpha_2^{22}}{2}\right)^2 + (\alpha_2^{12})^2}. \tag{2.124}$$

Define

$$\hat{\alpha}_{1m} = \frac{(\hat{\alpha}_1^1)^2}{4\hat{\alpha}_2^1}, \quad \hat{\alpha}_{2m} = \frac{(\hat{\alpha}_1^2)^2}{4\hat{\alpha}_2^2}, \quad \hat{\alpha}_m = \hat{\alpha}_{1m} + \hat{\alpha}_{2m}. \tag{2.125}$$

Letting $K_\flat = T \begin{bmatrix} \xi \\ \eta \end{bmatrix}$, where T is an orthogonal transformation satisfying (2.121), then the expression of α in (2.116) becomes

$$\begin{aligned} \alpha &= \alpha_0 + \alpha_1 K_\flat + K_\flat^T \alpha_2 K_\flat + \alpha_3 K_\flat \|K_\flat\|^2 \\ &= \alpha_0 + \hat{\alpha}_1^1 \xi + \hat{\alpha}_1^2 \eta + \hat{\alpha}_2^1 \xi^2 + \hat{\alpha}_2^2 \eta^2. \end{aligned} \tag{2.126}$$

So the construction procedures **HS-7** to **HS-13** in the previous section are clear from (2.126).

Chapter 3 Feedback Stabilization: the Multi-Input Case

Classification of stabilizability is obtained for multi-input nonlinear systems possessing a simple steady-state or Hopf bifurcation with the critical mode being linearly uncontrollable. Stabilizability is defined as the existence of a sufficiently smooth state feedback such that the bifurcation for the closed loop system is supercritical, and in the meantime, the linearly stabilizable modes are locally asymptotically stable. Necessary and sufficient conditions of stabilizability are derived under certain nondegeneracy conditions. Explicit construction of stabilizing feedbacks is obtained for the cases when the system is stabilizable. This chapter is based on the paper [75]. The derivation in this chapter does not depend on the controller canonical form as in the previous chapter. Also, in this chapter, we relax the assumptions in the previous chapter by allowing uncontrollable but stable eigenvalues in the linearization.

This chapter is a generalization of the results in the previous chapter. No previous work has been done on stabilization of bifurcations in multi-input systems. So all the results in this chapter are new.

3.1 Stabilizability of Steady-State Bifurcations

In this section we consider the case when a nonlinear system undergoes a simple steady-state bifurcation, with the critical mode being linearly unstabilizable. We classify the stabilizability of the bifurcation by providing necessary and sufficient conditions.

Consider the following nonlinear multi-input system

$$\dot{y} = f_\mu(y, u), \tag{3.1}$$

where $y \in \mathbb{R}^{n+1} (n \geq 1)$ is the state variable, $\mu \in \mathbb{R}$ is a bifurcation parameter, and $u \in \mathbb{R}^m$ $(m \geq 1)$ is the control input. We assume $n \geq 1$ in this paper since feedback stabilization for the case when $n = 0$ is trivial. Throughout this section we assume all the assumptions are valid for μ in the region $[-\bar{\mu}, \bar{\mu}]$. We make the following assumptions:

AS-1 *$f_\mu(y, u)$ is at least $\mathbf{C}^4$ with respect to (y, u) and $\mathbf{C}^2$ with respect to μ.*

AS-2 *For $u = 0$, there exists a nominal equilibrium solution $y = y_0(\mu)$ such that $f_\mu(y_0(\mu), 0) = 0$.*

AS-3 *$\lambda(\mu)$ is a simple real eigenvalue of $\frac{\partial f_\mu}{\partial y}(y_0(\mu), 0)$ and satisfies $\lambda(0) = 0$, $\frac{d\lambda}{d\mu}(0) \neq 0$.*

AS-4 *The eigenspace associated with $\lambda(\mu)$ is linearly uncontrollable, and all other eigenspaces are linearly stabilizable.*

Under these assumptions, we transform the system (3.1) into a standard form by the following procedure. First expand $f_\mu(y, u)$ into Taylor series around $(y_0(\mu), 0)$, and use a linear transformation to linearly decouple the unstabilizable eigenspace from the stabilizable eigenspaces. Then we evaluate all the terms except the bifurcating eigenvalue at $\mu = 0$. The resulting system is given by

$$\begin{aligned} \dot{x} &= d\mu x + q_{11}x^2 + q_{12}\tilde{x}x + q_{13}ux + \tilde{x}^T q_{22}\tilde{x} + \tilde{x}^T q_{23}u + u^T q_{33}u + c_{111}x^3 \\ &\quad + c_{112}\tilde{x}x^2 + c_{113}ux^2 + \tilde{x}^T c_{122}\tilde{x}x + \tilde{x}^T c_{123}ux + u^T c_{133}ux + c_{222}(\tilde{x}, \tilde{x}, \tilde{x}) \\ &\quad + c_{223}(\tilde{x}, \tilde{x}, u) + c_{233}(\tilde{x}, u, u) + c_{333}(u, u, u) + \text{h.o.t.}, \qquad (3.2) \\ \dot{\tilde{x}} &= A\tilde{x} + Bu + \tilde{q}_{11}x^2 + \tilde{q}_{12}\tilde{x}x + \tilde{q}_{13}ux + \tilde{q}_{22}(\tilde{x}, \tilde{x}) + \tilde{q}_{23}(\tilde{x}, u) + \tilde{q}_{33}(u, u) \\ &\quad + \text{h.o.t.}, \qquad (3.3) \end{aligned}$$

where $x \in \mathbb{R}$, $\tilde{x} \in \mathbb{R}^n$, (A, B) is stabilizable, and all the coefficients are real tensors with appropriate dimensions. We assume the tensors have symmetric properties if two or three subscripts are the same, for example, $q_{22} = q_{22}^T$, $c_{223}^{ijk}\tilde{x}_i\tilde{x}_j u_k = c_{223}^{jik}\tilde{x}_i\tilde{x}_j u_k$, for any $\tilde{x} \in \mathbb{R}^n$, $u, \in \mathbb{R}^m$, $i, j \in \{1, \cdots, n\}$, and $k \in \{1, \cdots, m\}$.

The goal is to find a sufficiently smooth feedback with Taylor series expansion

$$u = K_1\tilde{x} + K_2 x + K_3 x^2 + K_4 \tilde{x} x + K_5(\tilde{x}, \tilde{x}) + \text{h.o.t.},$$

and

$$K_1, K_4 \in \mathbb{R}^{m\times n}, \quad K_2, K_3 \in \mathbb{R}^{m\times 1}, \quad K_5 \in \mathbb{R}^{m\times n\times n},$$

such that the dynamics on the linearly stabilizable subspace is asymptotically stable, and at the same time, the equilibrium $(0,0)$ for the closed loop system is asymptotically stable at the bifurcation point. For the simple steady-state bifurcation, this is equivalent to the supercriticality of the bifurcation. We claim that the stabilizability of the bifurcation is not changed by a state feedback.

Claim 3.1 *The stabilizability of the bifurcation cannot be changed by sufficiently smooth state feedbacks that vanish on the nominal equilibria.*

Proof: Suppose the system (3.1) is stabilizable, then there exists a sufficiently smooth state feedback $u = H_0(y)$ satisfying $H_0(y_0(\mu)) = 0$, such that the equilibrium $y = y_0(0)$ of the system

$$\dot{y} = f_\mu(y, H_0(y))$$

is locally asymptotically stable at the bifurcation point $\mu = 0$. Let $H_1(y)$ by any sufficiently smooth map $\mathbb{R}^m \longrightarrow \mathbb{R}^{n+1}$ satisfying $H_1(y_0(\mu)) = 0$, then for the system

$$\dot{y} = f_\mu(y, v + H_1(y)),$$

the feedback $v = H(y) := H_0(y) - H_1(y)$ is a stabilizing feedback.

On the other hand, if we assume the system (3.1) is not stabilizable, then for any sufficiently smooth state feedback, say $u = H_0(y)$ with $H_0(y_0(\mu)) = 0$, the equilibrium

$y = y_0(0)$ of the system

$$\dot{y} = f_\mu(y, H_0(y))$$

is not locally asymptotically stable at the bifurcation point $\mu = 0$. Now suppose there is a sufficiently smooth feedback $H_1(y) : \mathbb{R}^m \longrightarrow \mathbb{R}^{n+1}$ that changes the stabilizability of (3.1), i.e., the system

$$\dot{y} = f_\mu(y, v + H_1(y))$$

is stabilizable, that is, there exists $H(y) : \mathbb{R}^m \longrightarrow \mathbb{R}^{n+1}$ such that the system

$$\dot{y} = f_\mu(y, H(y) + H_1(y))$$

is locally asymptotically stable at the bifurcation point $\mu = 0$. But this implies the original system (3.1) is stabilizable with the feedback $u = H(y) + H_1(y)$, which contradicts the assumption. ■

Since the system (3.1) is linearly unstabilizable, the linearization of the dynamics near the nominal equilibrium at the bifurcation point is not asymptotically stable by any state feedback. So we must consider the nonlinear stability of the system. If the vector field is sufficiently smooth and all the hyperbolic eigenvalues are stable, then Theorem 3.2.2 in [37] states that the system is locally asymptotically stable if and only if the dynamics on the center manifold is locally asymptotically stable. Our approach of classification of the stabilizability relies on the this theorem. First, we use a linear feedback $u = K_1\tilde{x}$ such that $A + BK_1$ is Hurwitz, then we reduce the system on the center manifold. We choose other feedback gains such that the dynamics on the center manifold of the closed loop system is asymptotically stable. If the dynamics on the center manifold truncated after the 3^{rd} order terms is unstable for all possible feedback gains, then we conclude that the system is unstabilizable. If the stability has to be determined through higher (4^{th} and 5^{th}) order terms, then we say the system is degenerate. In the rest of this section we first define constants that are needed for the classification of stabilizability, then we state the main theorem. The derivation of the main result and feedback design are in Section 3.3.

Without loss of generality, we assume that A^{-1} exists. If not, then we use a feedback $u = v + K_0\tilde{x}$ such that $A + B\bar{K}_1$ is invertible, and the stabilizability does not change by Claim 3.1. Define

$$\begin{aligned}
\Upsilon_1 &= q_{13} + q_{12}(-A)^{-1}B, && (3.4)\\
\Upsilon_2 &= B^T(-A)^{-T}q_{22}(-A)^{-1}B + q_{33}\\
&\quad +\frac{1}{2}\left[B^T(-A)^{-T}q_{23} + q_{23}^T(-A)^{-1}B\right], && (3.5)\\
\alpha_0 &= c_{111} + q_{12}(-A)^{-1}\tilde{q}_{11}. && (3.6)
\end{aligned}$$

Since $\Upsilon_2 = \Upsilon_2^T \in \mathbb{R}^{m\times m}$, there exists an orthonormal matrix $U \in \mathbb{R}^{m\times m}$ such that

$$\begin{aligned}
\hat{\Upsilon}_2 &:= U^T\Upsilon_1 U = \text{Diag}\left[\begin{array}{cccc}\hat{\Upsilon}_2^1, & \hat{\Upsilon}_2^2, & \cdots, & \hat{\Upsilon}_2^m\end{array}\right], && (3.7)\\
\hat{\Upsilon}_1 &:= \Upsilon_1 U = \left[\begin{array}{cccc}\hat{\Upsilon}_1^1 & \hat{\Upsilon}_1^2 & \cdots & \hat{\Upsilon}_1^m\end{array}\right], && (3.8)
\end{aligned}$$

Denote $\underline{\text{m}} = \{1, 2, \cdots, m\}$, and define index set I_j $(j = 1, \cdots, 4) \in \underline{\text{m}}$ as

$$\begin{aligned}
I_1 &= \left\{ i \in \underline{\text{m}};\ \hat{\Upsilon}_2^i > 0 \right\}, && (3.9)\\
I_2 &= \left\{ i \in \underline{\text{m}};\ \hat{\Upsilon}_2^i < 0 \right\}, && (3.10)\\
I_3 &= \left\{ i \in \underline{\text{m}};\ \hat{\Upsilon}_2^i = 0,\ \hat{\Upsilon}_1^i \neq 0 \right\}, && (3.11)\\
I_4 &= \left\{ i \in \underline{\text{m}};\ \hat{\Upsilon}_2^i = \hat{\Upsilon}_1^i = 0 \right\}. && (3.12)
\end{aligned}$$

It is clear that $I_i \cap I_j = \emptyset$ $(i \neq j,\ i, j = 1, \cdots, 4)$, and $I_1 \cup I_2 \cup I_3 \cup I_4 = \underline{\text{m}}$. Define

$$\begin{aligned}
K_a &= U\left[\begin{array}{cccc} -\frac{\hat{\Upsilon}_1^1}{2\hat{\Upsilon}_2^1} & -\frac{\hat{\Upsilon}_1^2}{2\hat{\Upsilon}_2^2} & \cdots & -\frac{\hat{\Upsilon}_1^m}{2\hat{\Upsilon}_2^m}\end{array}\right]^T, \quad \text{if } I_3 = I_4 = \emptyset, && (3.13)\\
\hat{q}_{11} &= q_{11} - \sum_{i\in I_1\cup I_2} \frac{\left(\hat{\Upsilon}_1^i\right)^2}{4\hat{\Upsilon}_2^i}, && (3.14)\\
P_S(K) &= \alpha_0 + \alpha_1 K + K^T\alpha_2 K + \alpha_3(K, K, K), && (3.15)
\end{aligned}$$

where

$$
\begin{aligned}
\alpha_1 &= c_{112}(-A)^{-1}B + c_{113} + q_{12}(-A)^{-1}\tilde{\Upsilon}_1 \\
&\quad + \tilde{q}_{11}^T(-A)^{-T}\left[2q_{22}(-A)^{-1}B + q_{23}\right], \\
\alpha_2 &= \frac{1}{2}\left(\alpha_{2t} + \alpha_{2t}^T\right), \\
\alpha_{2t} &= B^T(-A)^{-T}c_{122}(-A)^{-1}B + B^T(-A)^{-T}c_{123} + c_{133} \\
&\quad + q_{12}(-A)^{-1}\cdot\tilde{\Upsilon}_2 + \left[2B^T(-A)^{-T}q_{22} + q_{23}^T\right](-A)^{-1}\tilde{\Upsilon}_1, \\
\alpha_3(X,Y,Z) &= \frac{1}{6}\left[\alpha_{3t}(X,Y,Z) + \alpha_{3t}(X,Z,Y) + \alpha_{3t}(Y,X,Z)\right. \\
&\quad \left. + \alpha_{3t}(Y,Z,X) + \alpha_{3t}(Z,X,Y) + \alpha_{3t}(Z,Y,X)\right], \\
\alpha_{3t}(X,Y,Z) &= c_{222}((-A)^{-1}BX, (-A)^{-1}BY, (-A)^{-1}BZ) \\
&\quad + c_{223}((-A)^{-1}BX, (-A)^{-1}BY, Z) \\
&\quad + c_{233}((-A)^{-1}BX, Y, Z) + c_{333}(X,Y,Z) \\
&\quad + X^T\left[2B^T(-A)^{-T}q_{22} + q_{23}^T\right](-A)^{-1}\tilde{\Upsilon}_2(Y,Z), \\
\tilde{\Upsilon}_1 &= \tilde{q}_{13} + \tilde{q}_{12}(-A)^{-1}B, \qquad (3.16) \\
\tilde{\Upsilon}_2(X,Y) &= \tilde{q}_{22}((-A)^{-1}BX, (-A)^{-1}BY) + \tilde{q}_{23}((-A)^{-1}BX, Y) \\
&\quad + \tilde{q}_{33}(X,Y), \qquad (3.17)
\end{aligned}
$$

where $X, Y, Z \in \mathbb{R}^m$.

If $q_{11} \neq 0$, $I_2 = I_3 = \emptyset$, but $I_1 \neq \emptyset$, then define

$$
K_b = U\xi_b, \quad \xi_b \in \mathbb{R}^{m\times 1}, \quad \xi_b^i = \begin{cases} -\frac{\hat{\Upsilon}_1^i}{2\hat{\Upsilon}_2^i}, & i \in I_1, \\ 0, & i \in I_4. \end{cases} \qquad (3.18)
$$

If $q_{11} \neq 0$, $I_1 = I_3 = \emptyset$, but $I_2 \neq \emptyset$, then define

$$
K_b = U\xi_b, \quad \xi_b \in \mathbb{R}^{m\times 1}, \quad \xi_b^i = \begin{cases} -\frac{\hat{\Upsilon}_1^i}{2\hat{\Upsilon}_2^i}, & i \in I_2, \\ 0, & i \in I_4. \end{cases} \qquad (3.19)
$$

If $q_{11} = 0$, and $I_1 = \emptyset$ or $I_2 = \emptyset$, then we define $K_b = 0$.

Now let $f_1, \cdots, f_l$ be a set of basis of Ker $\begin{bmatrix} \Upsilon_1 \\ \Upsilon_2 \end{bmatrix}$, then any $K_e \in$ Ker $\begin{bmatrix} \Upsilon_1 \\ \Upsilon_2 \end{bmatrix}$, we have $K_e = FX$, where $F = \begin{bmatrix} f_1 & f_2 & \cdots & f_l \end{bmatrix} \in \mathbb{R}^{m \times l}$ and $X \in \mathbb{R}^{l \times 1}$. Define

$$\begin{aligned}
\alpha_{0r} &= \alpha_0 + \alpha_1 K_b + K_b^T \alpha_2 K_b + \alpha_3(K_b, K_b, K_b) \in \mathbb{R}, && (3.20)\\
\alpha_{1r} &= \left[\alpha_1 + 2K_b^T \alpha_2 + 3\alpha_3(K_b, K_b)\right] F \in \mathbb{R}^{1 \times l}, && (3.21)\\
\alpha_{2r} &= F^T \left[\alpha_2 + 3\alpha_3(K_b)\right] F \in \mathbb{R}^{l \times l}, && (3.22)\\
\alpha_{3r}(X, X, X) &= \alpha_3(FX, FX, FX) \in \mathbb{R}^{l \times l \times l}. && (3.23)
\end{aligned}$$

Let $V \in \mathbb{R}^{l \times l}$ be an orthonormal matrix such that

$$\begin{aligned}
\hat{\alpha}_{1r} &:= \alpha_{1r} V = \begin{bmatrix} \hat{\alpha}_{1r}^1 & \hat{\alpha}_{1r}^2 & \cdots & \hat{\alpha}_{1r}^l \end{bmatrix}, && (3.24)\\
\hat{\alpha}_{2r} &:= V^T \alpha_{2r} V = \text{Diag}\begin{bmatrix} \hat{\alpha}_{2r}^1, & \hat{\alpha}_{2r}^2, & \cdots, & \hat{\alpha}_{2r}^l \end{bmatrix}, && (3.25)
\end{aligned}$$

Let $\underline{l} = \{1, 2, \cdots, l\}$, and define index sets I_{jr} $(j = 1, \cdots, 4) \in \underline{l}$ as

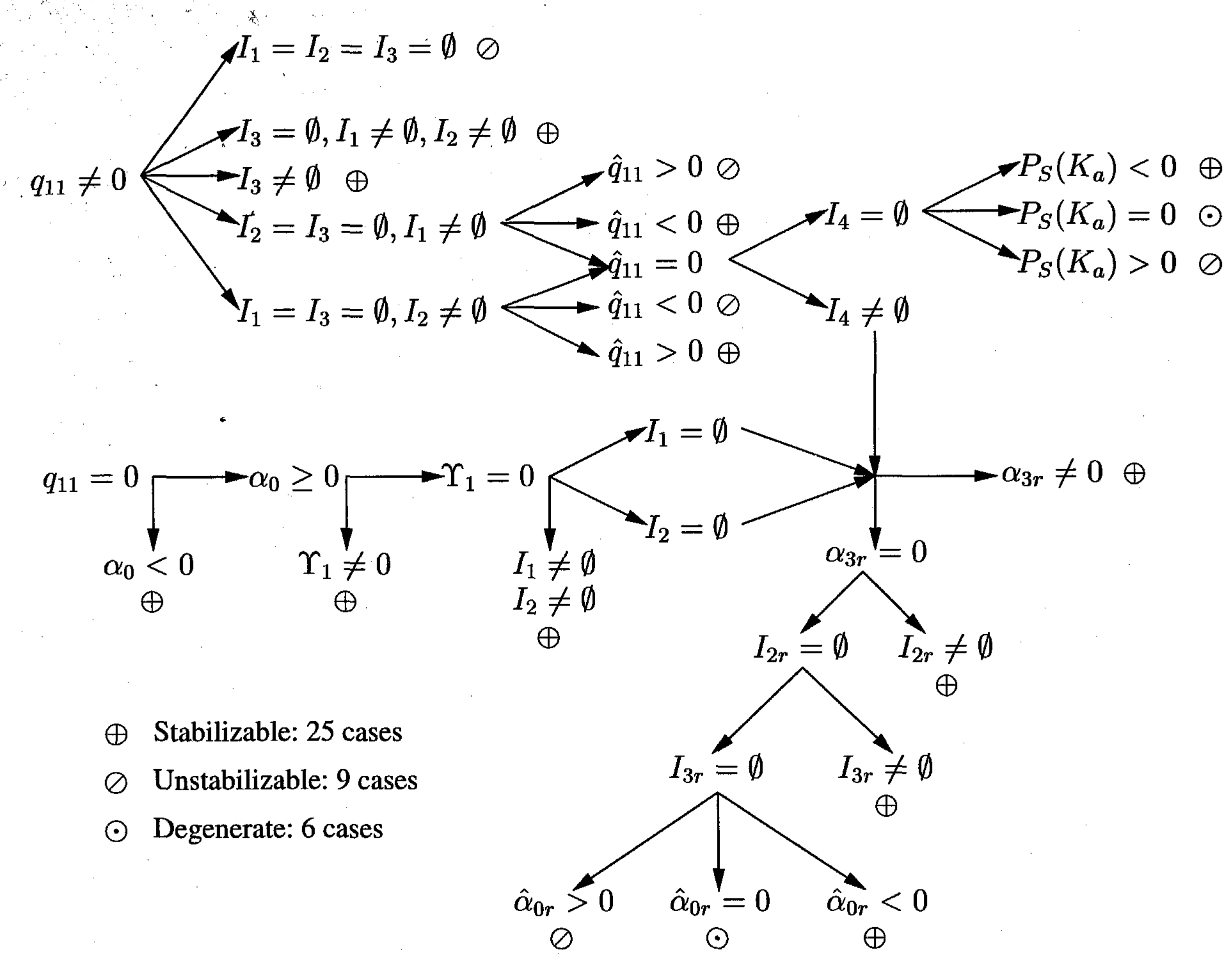
$$\begin{aligned}
I_{1r} &= \left\{ i \in \underline{l};\ \hat{\alpha}_{2r}^i > 0 \right\}, && (3.26)\\
I_{2r} &= \left\{ i \in \underline{l};\ \hat{\alpha}_{2r}^i < 0 \right\}, && (3.27)\\
I_{2r} &= \left\{ i \in \underline{l};\ \hat{\alpha}_{2r}^i = 0,\ \ \hat{\alpha}_{1r}^i \neq 0 \right\}, && (3.28)\\
I_{4r} &= \left\{ i \in \underline{l};\ \hat{\alpha}_{2r}^i = \hat{\alpha}_{1r}^i = 0 \right\}. && (3.29)
\end{aligned}$$

Define

$$\hat{\alpha}_{0r} := \alpha_{0r} - \sum_{i \in I_{1r}} \frac{(\hat{\alpha}_{1r}^i)^2}{4\hat{\alpha}_{2r}^i}. \qquad (3.30)$$

With above definitions, the classification of stabilizability for steady-state bifurcations is given by the following theorem.

Theorem 3.1 *A complete classification of stabilizability of simple steady-state bifurcation is given in the diagram below. Each of the cases is a full path on the tree in the diagram.*



A proof of this theorem is given in Section 3.3. It should be noted that the classification of stabilizability here for the multi-input case is more complicated than that for a single input system in the previous section. Theorem 3.1 gives a complete classification of stabilizability if stability can be determined through cubic terms in the dynamics on the center manifold. The degenerate cases imply that the stabilizability has to be determined by the 4^{th}, 5^{th}, or even higher order terms in the differential equation describing the dynamics on the center manifold.

The results in Theorem 3.1 can be written in a more compact form.

Corollary 3.1 *The system is stabilizable if and only if there exist* $K, K_3 \in \mathbb{R}^m$, *where*

$K = [I + K_1 A^{-1} B]^{-1} K_2$, *such that*

$$Q(K) := K^T \Upsilon_2 K + \Upsilon_1 K + q_{11} = 0,$$
$$C(K, K_3) := \frac{dQ}{dK} K_3 + \alpha_3(K, K, K) + K^T \alpha_2 K + \alpha_1 K + \alpha_0 < 0.$$

More specifically, we have the following cases.

1. $\forall K \in \mathbb{R}^m$, *we have* $Q(K) \neq 0$, *then the following three cases are unstabilizable.*

   (i) $\Upsilon_2 = 0$, $\Upsilon_1 = 0$, $q_{11} \neq 0$.

   (ii) Υ_2 *is positive (semi)-definite,* $\mathrm{rank}[\Upsilon_1^T \ \ \Upsilon_2] = \mathrm{rank}\, \Upsilon_2$, *and* $\hat{q}_{11} := q_{11} - \frac{1}{4}\Upsilon_1 \Upsilon_2^{-1} \Upsilon_1^T > 0$.

   (iii) Υ_2 *is negative (semi)-definite,* $\mathrm{rank}[\Upsilon_1^T \ \ \Upsilon_2] = \mathrm{rank}\, \Upsilon_2$, *and* $\hat{q}_{11} < 0$.

2. $\exists K \in \mathbb{R}^m$, *such that* $Q(K) = 0$, *but* $R(K) \neq 0$, *then the following four cases are stabilizable.*

   (i) Υ_2 *is indefinite.*

   (ii) Υ_2 *is (semi)-definite,* $\mathrm{rank}[\Upsilon_1^T \ \ \Upsilon_2] = \mathrm{rank}\, \Upsilon_2 + 1$.

   (iii) Υ_2 *is positive (semi)-definite,* $\mathrm{rank}[\Upsilon_1^T \ \ \Upsilon_2] = \mathrm{rank}\, \Upsilon_2$, *and* $\hat{q}_{11} < 0$.

   (iv) Υ_2 *is negative (semi)-definite,* $\mathrm{rank}[\Upsilon_1^T \ \ \Upsilon_2] = \mathrm{rank}\, \Upsilon_2$, *and* $\hat{q}_{11} > 0$.

3. *Any* $K \in \mathbb{R}^m$ *satisfying* $Q(K) = 0$ *also satisfies* $R(K) = 0$. *This case is true if and only if* Υ_2 *is (semi)-definite,* $\mathrm{rank}[\Upsilon_1^T \ \ \Upsilon_2] = \mathrm{rank}\, \Upsilon_2$, *and* $\hat{q}_{11} = 0$. *So* $Q(K) = 0$ *iff* $K = K_b + F\xi$, *where* $K_b = -\frac{1}{2}\Upsilon_2^{-1}\Upsilon_1^T$, *and* $V = [f_1 \ \ f_2 \ \ \cdots \ \ f_r]$ *is a set of basis of* $\mathrm{Ker}\, \Upsilon_2$, *and* $\xi \in \mathbb{R}^r$, *so in the* ξ*-coordinate,* $\alpha(K)$ *is given by*

$$\alpha(\xi) = \alpha_{3r}(\xi, \xi, \xi) + \xi^T \alpha_{2r} \xi + \alpha_{1r} \xi + \alpha_{0r} < 0,$$

   where α_{jr} $(r = 0, \cdots, 3)$ *are given by (3.20), (3.21), (3.22), (3.23), respectively. We have the following cases:*

   (i) $\alpha_{0r} < 0$, *then stabilizable.*

(ii) $\alpha_{0r} \geq 0$, $\alpha_{3r} \neq 0$, *then stabilizable.*

(iii) $\alpha_{0r} \geq 0$, $\alpha_{3r} = 0$, $\alpha_{2r} \neq 0$ *is indefinite or negative (semi)-definite, then stabilizable.*

(iv) $\alpha_{0r} \geq 0$, $\alpha_{3r} = 0$, α_{2r} *is positive (semi)-definite, then*

(a) $\mathrm{rank}[\alpha_{1r}^T \ \ \alpha_{2r}] = \mathrm{rank}\,\alpha_{2r} + 1$, *then stabilizable.*

(b) $\mathrm{rank}[\alpha_{1r}^T \ \ \alpha_{2r}] = \mathrm{rank}\,\alpha_{2r}$, *define* $\hat{\alpha}_{0r} := \alpha_{0r} - \frac{1}{4}\alpha_{1r}\alpha_{2r}^{-1}\alpha_{1r}^T$, *then*

- *If* $\hat{\alpha}_{0r} < 0$, *then stabilizable.*
- *If* $\hat{\alpha}_{0r} > 0$, *then unstabilizable.*
- *If* $\hat{\alpha}_{0r} = 0$, *then degenerate.*

This corollary is a direct result from Theorem 3.1.

3.2 Stabilizability of Hopf Bifurcations

In this section we consider the case when a nonlinear system undergoes a Hopf bifurcation with the critical modes being linearly unstabilizable. We classify the stabilizability of Hopf bifurcations by giving necessary and sufficient conditions.

Consider the following multi-input system

$$\dot{y} = f_\mu(y, u), \tag{3.31}$$

where $y \in \mathbb{R}^{n+2} (n \geq 1)$ is the state variable, $\mu \in \mathbb{R}$ is a bifurcation parameter, and $u \in \mathbb{R}^m$ $(m \geq 1)$ is the control input. We assume $n \geq 1$ since feedback stabilization is trivial if $n = 0$. Throughout this section we assume all the assumptions are valid for μ in the region $[-\bar{\mu}, \bar{\mu}]$. We make the following assumptions.

AH-1 $f_\mu(y, u)$ *is at least* $\mathbf{C}^4$ *with respect to* (y, u) *and* $\mathbf{C}^2$ *with respect to* μ.

AH-2 *For* $u = 0$, *there exists a steady-state solution* $y = y_0(\mu)$ *such that* $f_\mu(y_0(\mu), 0) = 0$.

AH-3 $\lambda_{1,2}(\mu) = \sigma(\mu) \pm i\omega(\mu)$ *are a simple pair of eigenvalues of* $\frac{\partial f_\mu}{\partial y}(y_0(\mu), 0)$ *and satisfy* $\sigma(0) = 0$, $\frac{d\sigma}{d\mu}(0) \neq 0$, *and* $\omega(0) \neq 0$.

AH-4 *The eigenspaces associated with* $\lambda_{1,2}(\mu)$ *are linearly uncontrollable, and all other eigenspaces are linearly stabilizable.*

Under these assumptions, we transform the system (3.1) into the standard form by the same procedure as the steady-state case. The resulting normal form is given by

$$\begin{aligned}
\dot{z} &= (d\mu + i\omega)z + q_{11}z^2 + q_{12}|z|^2 + q_{13}\tilde{x}z + q_{14}uz + q_{22}z^{*2} + q_{23}\tilde{x}z^* + q_{24}uz^* \\
&\quad + \tilde{x}^T q_{33}\tilde{x} + \tilde{x}^T q_{34}u + u^T q_{44}u + c_{111}z^3 + c_{112}|z|^2 z + c_{113}\tilde{x}z^2 + c_{114}uz^2 \\
&\quad + c_{122}|z|^2 z^* + c_{123}\tilde{x}|z|^2 + c_{124}u|z|^2 + \tilde{x}^T c_{133}\tilde{x}z + \tilde{x}^T c_{134}uz + u^T c_{144}uz \\
&\quad + c_{222}z^{*3} + c_{223}\tilde{x}z^{*2} + c_{224}uz^{*2} + \tilde{x}^T c_{233}\tilde{x}z^* + \tilde{x}^T c_{234}uz^* + u^T c_{244}uz^* \\
&\quad + c_{333}(\tilde{x}, \tilde{x}, \tilde{x}) + c_{334}(\tilde{x}, \tilde{x}, u) + c_{344}(\tilde{x}, u, u) + c_{444}(u, u, u) + \text{h.o.t.}, \qquad (3.32) \\
\dot{\tilde{x}} &= A\tilde{x} + Bu + \tilde{q}_{11}z^2 + \tilde{q}_{12}|z|^2 + \tilde{q}_{13}\tilde{x}z + \tilde{q}_{14}uz + \tilde{q}_{22}z^{*2} + \tilde{q}_{23}\tilde{x}z^* \\
&\quad + \tilde{q}_{24}uz^* + \tilde{q}_{33}(\tilde{x}, \tilde{x}) + \tilde{q}_{34}(\tilde{x}, u) + \tilde{q}_{44}(u, u) + \text{h.o.t.}, \qquad (3.33)
\end{aligned}$$

where $z \in \mathbb{C}$, $\tilde{x} \in \mathbb{R}^n$, and other coefficients are real or complex tensors with appropriate dimensions. As in the steady-state bifurcation case, we also assume the tensors have symmetry properties if they have two or more identical subscripts.

The goal is to find a sufficiently smooth feedback with Taylor series expansion

$$\begin{aligned}
u &= K_1\tilde{x} + K_2 z + K_3 z^* + K_4 z^2 + K_5|z|^2 + K_6 z^{*2} + K_7\tilde{x}z + K_8\tilde{x}z^* \\
&\quad + K_9(\tilde{x}, \tilde{x}) + \text{h.o.t.}, \qquad (3.34)
\end{aligned}$$

and with

$$\begin{aligned}
&K_1 \in \mathbb{R}^{m\times n}, \quad K_2 = K_3^* \in \mathbb{C}^m, \quad K_4 = K_6^* \in \mathbb{C}^m, \\
&K_5 \in \mathbb{R}^m, \quad K_7 = K_8^* \in \mathbb{C}^{m\times n}, \quad K_9 \in \mathbb{R}^{m\times n\times n},
\end{aligned}$$

such that the dynamics on the linearly stabilizable subsystem is asymptotically stable, and at the same time, the equilibrium $(0,0)$ for the closed loop system is asymptotically stable at the bifurcation point.

Similar to the steady-state bifurcation case, a state feedback does not affect stabilizability of the system. We first select K_1 such that $A+BK_1$ is Hurwitz. Then we reduce the dynamics of the closed loop system to the center manifold. Finally, we use a diffeomorphism to transform the dynamics on the center manifold to a normal form. We select feedback gains such that the normal form is locally asymptotically stable at the bifurcation point. In the following we define constants that will be needed for the statement of the main theorem. Without loss of generality, we assume $(sI-A)^{-1}$ exists for $s=0$, $\pm i\omega$, and $\pm 2\omega i$. If not, we use a feedback $u=K_0\tilde{x}+v$ to move away the resonance eigenvalues of A and the stabilizability will not be affected.

Letting $s, s_1, s_2 \in \mathbb{C}$, define

$$
\begin{aligned}
\Phi_1(s) &= q_{14}+q_{13}(sI-A)^{-1}B, \\
\Phi_2(s) &= q_{24}+q_{23}(sI-A)^{-1}B, \\
\tilde{\Phi}_1(s) &= \tilde{q}_{14}+\tilde{q}_{13}(sI-A)^{-1}B, \\
\tilde{\Phi}_2(s) &= \tilde{q}_{24}+\tilde{q}_{23}(sI-A)^{-1}B, \\
\Psi(s_1,s_2) &= 2B^T(s_1I-A)^{-T}q_{33}(s_2I-A)^{-1}B+q_{34}^T(s_2I-A)^{-1}B \\
&\quad +B^T(s_1I-A)^{-T}q_{34}+2q_{44}, \\
\tilde{\Psi}(s_1,s_2)(X,Y) &= 2\tilde{q}_{33}((s_1I-A)^{-1}BX,(s_2I-A)^{-1}BY) \\
&\quad +\tilde{q}_{34}((s_2I-A)^{-1}BX,Y)+\tilde{q}_{34}((s_1I-A)^{-1}BY,X)+2\tilde{q}_{44}(X,Y).
\end{aligned}
\tag{3.35}
$$

We define

$$
\Theta_1=\operatorname{Re}\Phi_1(0), \quad \Theta_2=\Phi_2(2\omega i), \quad \Theta_3=\Psi(i\omega,0), \quad \Theta_4=\Psi(-i\omega,2\omega i). \tag{3.36}
$$

Letting $K=\left[I-K_1(i\omega-A)^{-1}B\right]^{-1}K_2=K_R+iK_I$, we define

$$
C_0 = c_{112}-\frac{1}{i\omega}q_{11}q_{12}+q_{13}(-A)^{-1}\tilde{q}_{12}+q_{23}(2\omega i-A)^{-1}\tilde{q}_{11},
$$

$$
\begin{aligned}
D_1 &= c_{123}(i\omega - A)^{-1}B + c_{124} + q_{13}(-A)^{-1}\tilde{\Phi}_2(i\omega) \\
&\quad + q_{23}(2\omega i - A)^{-1}\Phi_1(i\omega) + \\
&\quad \tilde{q}_{12}^T(-A)^{-T}\left[2q_{33}(i\omega - A)^{-1}B + q_{34}\right] - \frac{2}{i\omega}q_{11}\Phi_2(i\omega), \\
D_2 &= c_{113}(-i\omega - A)^{-1}B + c_{114} + q_{13}(-A)^{-1}\tilde{\Phi}_1(i\omega) + \\
&\quad \tilde{q}_{11}^T(2\omega i - A)^{-T}\left[2q_{33}(-i\omega - A)^{-1}B + q_{34}\right] - \\
&\quad \frac{1}{i\omega}\left(q_{11} + q_{12}^*\right)\Phi_1(-i\omega) - \frac{1}{3\omega i}q_{22}^*\Phi_2(-i\omega), \\
E_{11} &= B^T(i\omega - A)^{-T}c_{233}(i\omega - A)^{-1}B + B^T(i\omega - A)^{-T}c_{234} \\
&\quad + c_{244} + \left[2B^T(i\omega - A)^{-T}q_{33} + q_{34}^T\right](-A)^{-1}\tilde{\Phi}_2(i\omega) \\
&\quad + \frac{1}{2}q_{23}(2\omega i - A)^{-1}\tilde{\Psi}(i\omega, i\omega) + \\
&\quad \frac{1}{2\omega i}\left[q_{12}\Psi(i\omega, i\omega) - 2\Phi_1^T(i\omega)\Phi_2(i\omega)\right], \\
E_{12} &= 2B^T(-i\omega - A)^{-T}c_{133}(i\omega - A)^{-1}B + c_{134}^T(i\omega - A)^{-1}B \\
&\quad + B^T(-i\omega - A)^{-T}c_{134} + 2c_{144} \\
&\quad + \tilde{\Phi}_1^T(-i\omega)(-A)^{-T}\left[2q_{33}(i\omega - A)^{-1}B + q_{34}\right] + \\
&\quad \left[2B^T(-i\omega - A)^{-T}q_{33} + q_{34}^T\right](2\omega i - A)^{-1}\tilde{\Phi}_1(i\omega) + \\
&\quad q_{13}(-A)^{-1}\tilde{\Psi}(i\omega, -i\omega) - \frac{1}{i\omega}(2q_{11} + q_{12}^*)\Psi(-i\omega, i\omega), \\
E_{22} &= -\frac{1}{i\omega}\Phi_1^T(-i\omega)\Phi_2^*(-i\omega) - \frac{1}{3\omega i}q_{22}^*\Psi(-i\omega, -i\omega), \\
F_{112}(K, K, K^*) &= 3c_{333}((i\omega - A)^{-1}BK, (i\omega - A)^{-1}BK, (-i\omega - A)^{-1}BK^*) \\
&\quad + c_{334}((i\omega - A)^{-1}BK, (i\omega - A)^{-1}BK, K^*) \\
&\quad + 2c_{334}((i\omega - A)^{-1}BK, (-i\omega - A)^{-1}BK^*, K) \\
&\quad + 2c_{344}((i\omega - A)^{-1}BK, K, K^*) \\
&\quad + c_{344}((-i\omega - A)^{-1}BK^*, K, K) + 3c_{444}(K, K, K^*) \\
&\quad + K^T\left[2B^T(i\omega - A)^{-T}q_{33} + q_{34}^T\right](-A)^{-1} \cdot \\
&\quad \tilde{\Psi}(i\omega, -i\omega)(K, K^*) \\
&\quad + \frac{1}{2}K^H\left[2B^T(-i\omega - A)^{-T}q_{33} + q_{34}^T\right](2\omega i - A)^{-1} \cdot \\
&\quad \tilde{\Psi}(i\omega, i\omega)(K, K) \\
&\quad + \frac{1}{2\omega i}\Phi_1(-i\omega)K^* \cdot K^T\Psi(i\omega, i\omega)K
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{i\omega}\left[\Phi_1(i\omega)+\Phi_1^*(i\omega)\right]K\cdot K^H\Psi(-i\omega,i\omega)K \\
& -\frac{1}{6\omega i}\Phi_2(-i\omega)K^*\cdot K^T\Psi^*(i\omega,i\omega)K, \\
F_{122}(K,K^*,K^*) \quad = \quad & -\frac{1}{i\omega}\Phi_1(-i\omega)K^*\cdot K^H\Psi^*(-i\omega,i\omega)K \\
& -\frac{1}{i\omega}\Phi_2^*(-i\omega)K^*\cdot K^H\Psi(-i\omega,i\omega)K \\
& -\frac{1}{3\omega i}\Phi_2^*(i\omega)K\cdot K^H\Psi(-i\omega,-i\omega)K^*,
\end{aligned}
$$

Define

$$
\begin{aligned}
\alpha_0 &= \operatorname{Re} C_0. && (3.37)\\
\alpha_1^1 &= \operatorname{Re}\{D_1+D_2\}, \\
\alpha_1^2 &= \operatorname{Im}\{D_2-D_1\}, \\
\alpha_2^{11} &= \frac{1}{2}\left[\operatorname{Re}\{E_{11}+E_{12}+E_{22}\}+\operatorname{Re}\{(E_{11}+E_{12}+E_{22})^T\}\right], \\
\alpha_2^{12} &= \frac{1}{2}\left[\operatorname{Im}\{E_{22}-E_{11}\}+\operatorname{Im}\{(E_{22}-E_{11})^T\}\right], \\
\alpha_2^{22} &= \frac{1}{2}\left[\operatorname{Re}\{E_{12}-E_{11}-E_{22}\}+\operatorname{Re}\{(E_{12}-E_{11}-E_{22})^T\}\right], \\
\alpha_{3a}^{111}(K_R,K_R,K_R) &= \operatorname{Re}\{F_{112}+F_{122}\}(K_R,K_R,K_R), \\
\alpha_{3a}^{112}(K_R,K_R,K_I) &= \operatorname{Im}\{F_{112}+F_{122}\}(K_R,K_R,K_I) \\
&\quad -\operatorname{Im}\{F_{112}-F_{122}\}(K_R,K_I,K_R) \\
&\quad -\operatorname{Im}\{F_{112}+F_{122}\}(K_I,K_R,K_R), \\
\alpha_{3a}^{122}(K_R,K_I,K_I) &= \operatorname{Re}\{F_{112}+F_{122}\}(K_R,K_I,K_I) \\
&\quad +\operatorname{Re}\{F_{112}-F_{122}\}(K_I,K_R,K_I) \\
&\quad -\operatorname{Re}\{F_{112}+F_{122}\}(K_I,K_I,K_R), \\
\alpha_{3a}^{222}(K_I,K_I,K_I) &= \operatorname{Im}\{F_{122}-F_{112}\}(K_I,K_I,K_I),
\end{aligned}
$$

and

$$
\alpha_1=\begin{bmatrix}\alpha_1^1 & \alpha_1^2\end{bmatrix},\quad \alpha_2=\begin{bmatrix}\alpha_2^{11} & \alpha_2^{12}\\ \alpha_2^{12} & \alpha_2^{22}\end{bmatrix}. \tag{3.38}
$$

Define $K_\flat = \begin{bmatrix} K_R \\ K_I \end{bmatrix}$, and

$$\begin{aligned} \alpha_{3a}(K_\flat, K_\flat, K_\flat) &= \alpha_{3a}^{111}(K_R, K_R, K_R) + \alpha_{3a}^{112}(K_R, K_R, K_I) + \\ &\quad \alpha_{3a}^{122}(K_R, K_I, K_I) + \alpha_{3a}^{222}(K_I, K_I, K_I), \end{aligned}$$

and define α_3 as the symmetrization of α_{3a}, i.e., for any $i, j, k \in (1, \cdots, 2m)$, we have

$$\alpha_3^{ijk} = \frac{1}{6}\left(\alpha_{3a}^{ijk} + \alpha_{3a}^{ikj} + \alpha_{3a}^{jik} + \alpha_{3a}^{jki} + \alpha_{3a}^{kij} + \alpha_{3a}^{kji}\right). \tag{3.39}$$

Define the norm of α_3 as the infinity norm $\|\alpha_3\| := \max_{i,j,k\in\underline{2\mathrm{m}}} \left|\alpha_3^{ijk}\right|$, and $\|\Theta_j\|$ $(j = 1, \cdots, 4)$ are defined similarly. Define

$$\Theta = \|\Theta_1\| + \|\Theta_2\| + \|\Theta_3\| + \|\Theta_4\| + \|\alpha_3\|. \tag{3.40}$$

From Section 3.4, the normal form of the dynamics on the center manifold is given by

$$\dot{\zeta} = (d\mu + i\omega)\zeta + \tilde{\alpha}|\zeta|^2\zeta + \text{h.o.t.},$$

where $\alpha := \operatorname{Re}\tilde{\alpha}$ is given by

$$\begin{aligned} \alpha &= \left(\Theta_1 + \operatorname{Re}\left\{K^T\Theta_3\right\}\right)\left[I + K_1A^{-1}B\right]^{-1}\Big[K_5 + K_7\beta_2 + K_8\beta_1 + 2K_9(\beta_1, \beta_2) \\ &\quad + K_1A^{-1}B\left(\tilde{Q}_2 - \beta_1Q_2 - \beta_2Q_2^*\right) + \frac{1}{i\omega}\left(I + K_1A^{-1}B\right)\left(K^*Q_2^* - KQ_2\right)\Big] \\ &\quad + \operatorname{Re}\Big\{\left(\Theta_2 + K^H\Theta_4\right)\left[I - K_1(2\omega i - A)^{-1}B\right]^{-1}\Big[K_4 + K_7\beta_1 \\ &\quad + K_9(\beta_1, \beta_1) - K_1(2\omega i - A)^{-1}B\left(\tilde{Q}_1 - \beta_1Q_1 - \beta_2Q_3^*\right) \\ &\quad + \frac{1}{i\omega}\left[I - K_1(2\omega i - A)^{-1}B\right]\left(KQ_1 + \frac{1}{3}K^*Q_3^*\right)\Big]\Big\} \\ &\quad + \left[\alpha_0 + \alpha_1K_\flat + K_\flat^T\alpha_2K_\flat + \alpha_3(K_\flat, K_\flat, K_\flat)\right], \end{aligned} \tag{3.41}$$

where Θ_j $(j = 1, \cdots, 4)$ are given by (3.36), and α_j $(j = 0, \cdots, 3)$ are given by (3.37), (3.38), and (3.39). Here K, $K_\flat$, β_j $(j = 1, 2)$, Q_j $(j = 1, 2, 3)$, and $\tilde{Q}_j$ $(j = 1, 2)$ are given by

$$
\begin{aligned}
K &= \left[I - K_1(i\omega - A)^{-1}B\right]^{-1} K_2 = K_R + iK_I, \\
K_\flat &= \begin{bmatrix} K_R \\ K_I \end{bmatrix}, \\
\beta_1 &= (i\omega - A)^{-1}K, \\
\beta_2 &= (-i\omega - A)^{-1}K^* = \beta_1^*, \\
Q_1 &= q_{11} + \Phi_1(i\omega)K + \frac{1}{2}K^T\Psi(i\omega, i\omega)K, \\
Q_2 &= q_{12} + \Phi_2(i\omega)K + \Phi_1(-i\omega)K^* + K^H\Psi(-i\omega, i\omega)K, \\
Q_3 &= q_{22} + \Phi_2(-i\omega)K^* + \frac{1}{2}K^H\Psi(-i\omega, -i\omega)K^*, \\
\tilde{Q}_1 &= \tilde{q}_{11} + \tilde{\Phi}_1(i\omega)K + \frac{1}{2}\tilde{\Psi}(i\omega, i\omega)(K, K), \\
\tilde{Q}_2 &= \tilde{q}_{12} + \tilde{\Phi}_2(i\omega)K + \tilde{\Phi}_1(-i\omega)K^* + \frac{1}{2}\tilde{\Psi}(i\omega, -i\omega)(K, K^*).
\end{aligned}
$$

where $\Phi_j(s)$ $(j = 1, 2)$, $\tilde{\Phi}_j(s)$ $(j = 1, 2)$, $\Psi(s_1, s_2)$, and $\Psi(s_1, s_2)$ are defined in (3.35).

Since the Hopf bifurcation for the closed loop system is supercritical if and only if $\alpha < 0$ except the degenerate case when $\alpha = 0$, the goal is to find feedback gains such that $\alpha < 0$. From (3.41) it is clear that if any of Θ_j $(j = 1, \cdots, 4)$ is nonzero, then feedback gains can be selected such that $\alpha < 0$. If all of Θ_j $(j = 1, \cdots, 4)$ are zero, then α is given by

$$\alpha = \alpha_0 + \alpha_1 K_\flat + K_\flat^T \alpha_2 K_\flat + \alpha_3(K_\flat, K_\flat, K_\flat).$$

Apparently α can be made negative if $\alpha_3 \neq 0$. In the following we consider the case when $\alpha_3 = 0$.

Let $U \in \mathbb{R}^{2m\times 2m}$ be an orthonormal matrix, i.e., $U^{-1} = U^T$, such that α_2 is diagonalized, i.e.,

$$\begin{aligned}\hat{\alpha}_1 &:= \alpha_1 U = \left[\begin{array}{cccc} \hat{\alpha}_1^1 & \hat{\alpha}_1^2 & \cdots & \hat{\alpha}_1^{2m}\end{array}\right], \\ \hat{\alpha}_2 &:= U^T\alpha_2 U = \mathrm{Diag}\left[\begin{array}{cccc} \hat{\alpha}_2^1, & \hat{\alpha}_2^2, & \cdots, & \hat{\alpha}_2^{2m}\end{array}\right],\end{aligned}$$

Define $\underline{2\mathrm{m}} = \{1, 2, \cdots, 2m\}$, and

$$I_1 = \{j \in \underline{2\mathrm{m}}\,;\ \hat{\alpha}_2^j > 0\}, \tag{3.42}$$

$$I_2 = \{j \in \underline{2\mathrm{m}}\,;\ \hat{\alpha}_2^j < 0\}, \tag{3.43}$$

$$I_3 = \{j \in \underline{2\mathrm{m}}\,;\ \hat{\alpha}_2^j = 0,\ \hat{\alpha}_1^j \neq 0\}, \tag{3.44}$$

$$I_4 = \{j \in \underline{2\mathrm{m}}\,;\ \hat{\alpha}_2^j = 0,\ \hat{\alpha}_1^j = 0\}, \tag{3.45}$$

$$\hat{\alpha}_0 = \alpha_0 - \sum_{j\in I_1} \frac{\left(\hat{\alpha}_1^j\right)^2}{4\hat{\alpha}_2^j}. \tag{3.46}$$

Let $K_\flat = U\xi$, then α can be written as

$$\begin{aligned}\alpha &= \alpha_0 + \alpha_1 K_\flat + K_\flat^T \alpha_2 K_\flat \\ &= \alpha_0 + \hat{\alpha}_1\xi + \xi^T\hat{\alpha}_2\xi \\ &= \hat{\alpha}_0 + \sum_{j\in I_2\cup I_3} \hat{\alpha}_1^j \xi_j + \sum_{j\in I_2} \hat{\alpha}_2^j\xi_j^2 + \sum_{j\in I_1} \hat{\alpha}_2^j \left(\xi_j - \frac{\hat{\alpha}_1^j}{2\hat{\alpha}_2^j}\right)^2.\end{aligned}$$

It is apparent that α can be made negative if $I_2 \cup I_3 \neq \emptyset$. If $I_2 \cup I_3 = \emptyset$, then the minimum of α is $\hat{\alpha}_0$. In this case system is stabilizable, unstabilizable, and degenerate if and only if $\hat{\alpha}_0 < 0$, $\hat{\alpha}_0 > 0$, and $\hat{\alpha}_0 = 0$, respectively.

In summary of the above discussions, the following theorem gives a complete classification of stabilizability for a simple Hope bifurcation except for the degenerate case.

Theorem 3.2 *A complete classification of stabilizability for a simple Hopf bifurcation is given in the following table, where α_0, Θ, I_2, I_3, and $\hat{\alpha}_0$ are given by (3.37), (3.40), (3.43), (3.44), and (3.46), respectively.*

Classification of Stabilizability of Hopf Bifurcations					
Cases		α_0	Θ	$I_2 \cup I_3$	$\hat{\alpha}_0$
Stabilizable	**HS-1**	< 0			
	HS-2	≥ 0	$\neq 0$		
	HS-3	≥ 0	$= 0$	$\neq \emptyset$	
	HS-4	≥ 0	$= 0$	$= \emptyset$	< 0
Degenerate	**HD**	≥ 0	$= 0$	$= \emptyset$	$= 0$
Unstabilizable	**HU**	≥ 0	$= 0$	$= \emptyset$	> 0

A proof of Theorem 3.2 is given in Section 3.4, in which the main task is to derive the formula (3.41). A more compact form of Theorem 3.2 can be given by the following corollary.

The Taylor series expansion of (3.31) is given by

$$\dot{y} = \bar{A}y + \bar{B}u + \bar{q}_{11}(y, y) + 2\bar{q}_{12}(y, u) + \bar{q}_{22}(u, u) + \bar{c}(w, w, w) + \text{h.o.t.},$$

where $\bar{A} \in \mathbb{R}^{(n+2)\times(n+2)}$, $\bar{B} \in \mathbb{R}^{(n+2)\times m}$, and $w := \begin{bmatrix} y \\ u \end{bmatrix}$. Define

$$\begin{aligned} Q_{11} &= l \cdot \bar{q}_{11} - \frac{1}{2}\left[(rl)^T(l \cdot \bar{q}_{11})(r_\perp l_\perp) + (r_\perp l_\perp)^T(l \cdot \bar{q}_{11})(rl)\right] \\ &\quad + \frac{1}{2}\left[(r^*l)^T(l^* \cdot \bar{q}_{11})(r_\perp l_\perp) + (r_\perp l_\perp)^T(l^* \cdot \bar{q}_{11})(r^*l)\right], \\ Q_{12} &= l \cdot \bar{q}_{12} - \frac{1}{2}(rl)^T(l \cdot \bar{q}_{12}) + \frac{1}{2}(r^*l)^T(l^* \cdot \bar{q}_{12}), \\ Q_{22} &= l \cdot \bar{q}_{22}, \end{aligned}$$

where l, $l_\perp$, r, and $r_\perp$, satisfy

$$\begin{bmatrix} l \\ l^* \\ l_\perp \end{bmatrix} \begin{bmatrix} r & r^* & r_\perp \end{bmatrix} = I, \quad \begin{bmatrix} l \\ l^* \\ l_\perp \end{bmatrix} \bar{A} \begin{bmatrix} r & r^* & r_\perp \end{bmatrix} = \begin{bmatrix} i\omega & 0 & 0 \\ 0 & -i\omega & 0 \\ 0 & 0 & A \end{bmatrix}.$$

Define

$$\Sigma(\lambda_1,\lambda_2) := \begin{bmatrix} 0 & \bar{A}-\lambda_1 I & \bar{B} \\ (\bar{A}-\lambda_2 I)^T & Q_{11} & Q_{12} \\ \bar{B}^T & Q_{12}^T & Q_{22} \end{bmatrix}$$

Corollary 3.2 *The classification of stabilizability of system is given by the following:*

1. *If* $\operatorname{rank}\Sigma(0,i\omega) \geq 2(n+2)$*, or* $\operatorname{rank}\Sigma(2\omega i,-i\omega) \geq 2(n+2)$*, then the system is stabilizable.*

2. *If* $\operatorname{rank}\Sigma(0,i\omega) < 2(n+2)$*, and* $\operatorname{rank}\Sigma(2\omega i,-i\omega) < 2(n+2)$*, then the system is stabilizable if and only if there exists a* $K_\flat \in \mathbb{R}^{2m}$*, where* $K_\flat = \begin{bmatrix} K_R \\ K_I \end{bmatrix}$*, and*

$$K_R + iK_I = K := \left[I - K_1(i\omega - A)^{-1}B\right]^{-1} K_2,$$

such that

$$\alpha(K_\flat) = a_3(K_\flat,K_\flat,K_\flat) + K_\flat^T\alpha_2 K_\flat + \alpha_1 K_\flat + \alpha_0 < 0,$$

or more specifically,

(i) $\alpha_0 < 0$*, the system is stabilizable,*

(ii) $\alpha_0 \geq 0$*,* $\alpha_3 \neq 0$*, the system is stabilizable,*

(iii) $\alpha_0 \geq 0$*,* $\alpha_3 = 0$*,* $\alpha_2 \neq 0$ *is indefinite or negative (semi)-definite, the system is stabilizable,*

(iv) $\alpha_0 \geq 0$*,* $\alpha_3 = 0$*,* α_2 *is positive (semi)-definite, then*

(a) $\operatorname{rank}[\alpha_1^T \ \ \alpha_2] = \operatorname{rank}\alpha_2 + 1$*, then the system is stabilizable.*

(b) $\operatorname{rank}[\alpha_1^T \ \ \alpha_2] = \operatorname{rank}\alpha_2$*, then define*

$$\hat{\alpha}_0 := \alpha_0 - \frac{1}{4}\alpha_1\alpha_2^{-1}\alpha_1^T,$$

- $\hat{\alpha}_0 < 0$, *the system is stabilizable,*
- $\hat{\alpha}_0 > 0$, *the system is unstabilizable,*
- $\hat{\alpha}_0 = 0$, *the system is degenerate (stability is determined by higher order terms in the normal form).*

A proof of this corollary is also given in Section 3.4.

3.3 Proof of Theorem 3.1

First, consider the case when $K_2 = 0$. Choose K_1 such that $A + BK_1$ is Hurwitz. Then at $\mu = 0$, the center manifold is given by

$$\tilde{x} = \beta_1 x^2 + \text{h.o.t.}$$

By differentiation and using the system dynamics, we get

$$\beta_1 = [-(A + BK_1)]^{-1} (BK_3 + \tilde{q}_{11}).$$

The dynamics on the center manifold is given by

$$\dot{x} = d\mu x + q_{11}x^2 + Cx^3 + \text{h.o.t.},$$

where

$$\begin{aligned}
C &= c_{111} + (q_{12} + q_{13}K_1)\beta_1 + q_{13}K_3 \\
&= c_{111} + (q_{12} + q_{13}K_1)[-(A + BK_1)]^{-1}(BK_3 + \tilde{q}_{11}) + q_{13}K_3 \\
&= c_{111} + \left\{q_{13} + (q_{12} + q_{13}K_1)[-(A + BK_1)]^{-1}B\right\} K_3 \\
&\quad +(q_{12} + q_{13}K_1)[-(A + BK_1)]^{-1}\tilde{q}_{11} \\
&= c_{111} + \left[q_{13} + q_{12}(-A)^{-1}B\right] \left(I + K_1A^{-1}B\right)^{-1} K_3 + q_{12}(-A)^{-1}\tilde{q}_{11} \\
&\quad + \left[q_{13} + q_{12}(-A)^{-1}B\right] K_1[-(A + BK_1)]^{-1}\tilde{q}_{11}
\end{aligned}$$

$$= c_{111} + q_{12}(-A)^{-1}\tilde{q}_{11} + \left[q_{13} + q_{12}(-A)^{-1}B\right]\left(I + K_1A^{-1}B\right)^{-1} \cdot$$
$$\left\{K_3 + \left(I + K_1A^{-1}B\right)K_1\left[-(A+BK_1)\right]^{-1}\tilde{q}_{11}\right\}$$
$$= c_{111} + q_{12}(-A)^{-1}\tilde{q}_{11} + \left[q_{13} + q_{12}(-A)^{-1}B\right]\left(I + K_1A^{-1}B\right)^{-1} \cdot$$
$$\left[K_3 + K_1(-A)^{-1}\tilde{q}_{11}\right]$$

Here we have used the equalities $[I + K_1A^{-1}B]\,K_1 = K_1\,[I + A^{-1}BK_1]$, and

$$(q_{12} + q_{13}K_1)[sI - (A+BK_1)]^{-1}B + q_{13}$$
$$= \left[q_{13} + q_{12}(sI - A)^{-1}B\right]\left[I - K_1(sI-A)^{-1}B\right]^{-1}.$$

It should be noted that we have assumed A is invertible. If not, then we use a feedback K_0 such that $A + BK_0$ is invertible. Using the notations in (3.4), (3.5), and (3.6), we have

$$C = \alpha_0 + \Upsilon_1\left[I + K_1A^{-1}B\right]^{-1}\left[K_3 + K_1(-A)^{-1}\tilde{q}_{11}\right]. \qquad (3.47)$$

Using this relation, it is easy to know

I. $q_{11} \neq 0$, then the system is not stabilizable with $K_2 = 0$.

II. $q_{11} = 0$, $\Upsilon_1 = 0$, and $\alpha_0 \geq 0$, then the system is not stabilizable with $K_2 = 0$.

III. $q_{11} = 0$, and $\alpha_0 < 0$, the system is stabilizable with $K_2 = 0$. In fact, the controller is given by $u = K_1\tilde{x} + K_3x^2$ where $A + BK_1$ is Hurwitz, and $K_3 = K_1A^{-1}\tilde{q}_{11}$.

IV. $q_{11} = 0$, $\alpha_0 \geq 0$, and $\Upsilon_1 \neq 0$, then the system is stabilizable with $K_2 = 0$. The stabilizing controller is given by $u = K_1\tilde{x} + K_3x^2$ with $A + BK_1$ being Hurwitz and K_3 satisfying $C < 0$, where C is giving by (3.47).

For the cases I. and II., we must consider the state feedback with $K_2 \neq 0$. In this case, the center manifold is given by

$$\tilde{x} = \beta_1 x + \beta_2 x^2 + \text{h.o.t.}$$

The dynamics on the center manifold is given by

$$\dot{x} = d\mu x + Qx^2 + Cx^3 + \text{h.o.t.},$$

where $C = C_1 + C_2$, and

$$\begin{aligned}
Q &= q_{11} + q_{12}\beta_1 + q_{13}(K_1\beta_1 + K_2) + \beta_1^T q_{22}\beta_1 + \beta_1^T q_{23}(K_1\beta_1 + K_2) \\
&\quad +(K_1\beta_1 + K_2)^T q_{33}(K_1\beta_1 + K_2), \\
C_1 &= \left\{ \left[q_{12} + 2\beta_1^T q_{22} + (K_1\beta_1 + K_2)^T q_{23}^T\right] \right. \\
&\quad \left. + \left[q_{13} + \beta_1^T q_{23} + 2(K_1\beta_1 + K_2)^T q_{33}\right] K_1 \right\} \beta_2 \\
&\quad + \left[q_{13} + \beta_1^T q_{23} + 2(K_1\beta_1 + K_2)^T q_{33}\right] \left[K_3 + K_4\beta_1 + K_5(\beta_1, \beta_1)\right], \\
C_2 &= c_{111} + c_{112}\beta_1 + c_{113}(K_1\beta_1 + K_2) + \beta_1^T c_{122}\beta_1 + \beta_1^T c_{123}(K_1\beta_1 + K_2) \\
&\quad +(K_1\beta_1 + K_2)^T c_{133}(K_1\beta_1 + K_2) + c_{222}(\beta_1, \beta_1, \beta_1) \\
&\quad +c_{223}(\beta_1, \beta_1, K_1\beta_1 + K_2) + c_{223}(\beta_1, K_1\beta_1 + K_2, K_1\beta_1 + K_2) \\
&\quad +c_{333}(K_1\beta_1 + K_2, K_1\beta_1 + K_2, K_1\beta_1 + K_2).
\end{aligned}$$

Now we calculate β_1 and β_2. By differentiating the center manifold expansion and using the system dynamics, we obtain

$$\begin{aligned}
\beta_1 &= [-(A + BK_1)]^{-1} BK_2 = (-A)^{-1} BK \\
\beta_2 &= [-(A + BK_1)]^{-1} \left\{ B\left[K_3 + K_4\beta_1 + K_5(\beta_1, \beta_1)\right] + \tilde{Q} - \beta_1 Q \right\}, \\
\tilde{Q} &= \tilde{q}_{11} + \tilde{q}_{12}\beta_1 + \tilde{q}_{13}(K_1\beta_1 + K_2) + \tilde{q}_{22}(\beta_1, \beta_1) + \tilde{q}_{23}(\beta_1, K_1\beta_1 + K_2) \\
&\quad +\tilde{q}_{33}(K_1\beta_1 + K_2, K_1\beta_1 + K_2),
\end{aligned}$$

where $K := \left[I + K_1 A^{-1} B\right]^{-1} K_2$.

Now, using the expressions of β_1 and β_2, and the identities $K_1\beta_1 + K_2 = K$ and $I + K_1[-(A + BK_1)]^{-1}B = \left[I + K_1 A^{-1} B\right]^{-1}$, we calculate Q, $\tilde{Q}$, C_1 and C_2 in terms of feedback gains by straightforward matrix manipulations.

$$
\begin{aligned}
Q &= q_{11} + \Upsilon_1 K + K^T \Upsilon_2 K, && (3.48)\\
\tilde{Q} &= \tilde{q}_{11} + \tilde{\Upsilon}_1 K + \tilde{\Upsilon}_2(K,K), && (3.49)\\
C_1 &= (\Upsilon_1 + 2K^T\Upsilon_2)\left[I + K_1 A^{-1} B\right]^{-1} \cdot \\
&\quad \left[K_3 + K_4\beta_1 + K_5(\beta_1,\beta_1) + \tilde{Q} - \beta_1 Q\right] \\
&\quad + \left[q_{13} + K^T\left(B^T(-A)^{-T} q_{23} + 2q_{33}\right)\right]\left(\tilde{Q} - \beta_1 Q\right), && (3.50)\\
C_2(K) &= c_{111} + \left[c_{112}(-A)^{-1}B + c_{113}\right] K \\
&\quad + K^T\left[B^T(-A)^{-T} c_{122}(-A)^{-1}B + B^T(-A)^{-T}c_{123} + c_{133}\right] K \\
&\quad + c_{222}((-A)^{-1}BK, (-A)^{-1}BK, (-A)^{-1}BK) + \\
&\quad c_{223}((-A)^{-1}BK, (-A)^{-1}BK, K) \\
&\quad + c_{233}((-A)^{-1}BK, K, K) + c_{233}(K,K,K), && (3.51)
\end{aligned}
$$

where

$$
\begin{aligned}
\Upsilon_1 &= q_{13} + q_{12}(-A)^{-1}B, && (3.52)\\
\Upsilon_2 &= B^T(-A)^{-T} q_{22}(-A)^{-1}B + \frac{1}{2}\left[B^T(-A)^{-T}q_{23} + q_{23}^T(-A)^{-1}B\right] \\
&\quad + q_{33}, && (3.53)\\
\tilde{\Upsilon}_1 &= \tilde{q}_{13} + \tilde{q}_{12}(-A)^{-1}B. && (3.54)
\end{aligned}
$$

So we have

$$
\begin{aligned}
C &= C_1 + C_2 \\
&= (\Upsilon_1 + 2K^T\Upsilon_2)\left[I + K_1 A^{-1}B\right]^{-1} \cdot \\
&\quad \left[K_3 + K_4\beta_1 + K_5(\beta_1,\beta_1) + \tilde{Q} - \beta_1 Q\right] + P_S(K), && (3.55)
\end{aligned}
$$

where

$$
P_S(K) = C_2(K) + \left[q_{13} + K^T\left(B^T(-A)^{-T}q_{23} + 2q_{33}\right)\right]\left(\tilde{Q} - \beta_1 Q\right), \qquad (3.56)
$$

where $C_2(K)$ is given by (3.51). When $Q=0$, $P_S(K)$ can be calculated as

$$P_S(K) = \alpha_0 + \alpha_1 K + K^T \alpha_2 K + \alpha_3(K,K,K), \tag{3.57}$$

where

$$\begin{aligned}
\alpha_1 &= c_{112}(-A)^{-1}B + c_{113} + q_{12}(-A)^{-1}\tilde{\Upsilon}_1 \\
&\quad + \tilde{q}_{11}^T(-A)^{-T}\left[2q_{22}(-A)^{-1}B + q_{23}\right], \\
\alpha_2 &= \frac{1}{2}\left(\alpha_{2t} + \alpha_{2t}^T\right), \\
\alpha_{2t} &= B^T(-A)^{-T}c_{122}(-A)^{-1}B + B^T(-A)^{-T}c_{123} + c_{133} \\
&\quad + q_{12}(-A)^{-1}\cdot\tilde{\Upsilon}_2 + \left[2B^T(-A)^{-T}q_{22} + q_{23}^T\right](-A)^{-1}\tilde{\Upsilon}_1, \\
\alpha_3(X,Y,Z) &= \frac{1}{6}\left[\alpha_{3t}(X,Y,Z) + \alpha_{3t}(X,Z,Y) + \alpha_{3t}(Y,X,Z)\right. \\
&\quad \left. + \alpha_{3t}(Y,Z,X) + \alpha_{3t}(Z,X,Y) + \alpha_{3t}(Z,Y,X)\right], \\
\alpha_{3t}(X,Y,Z) &= c_{222}((-A)^{-1}BX,(-A)^{-1}BY,(-A)^{-1}BZ) \\
&\quad + c_{223}((-A)^{-1}BX,(-A)^{-1}BY,Z) \\
&\quad + c_{233}((-A)^{-1}BX,Y,Z) + c_{333}(X,Y,Z) \\
&\quad + X^T\left[2B^T(-A)^{-T}q_{22} + q_{23}^T\right](-A)^{-1}\tilde{\Upsilon}_2(Y,Z), \\
\tilde{\Upsilon}_2(X,Y) &= \tilde{q}_{22}((-A)^{-1}BX,(-A)^{-1}BY) + \tilde{q}_{23}((-A)^{-1}BX,Y) \\
&\quad + \tilde{q}_{33}(X,Y),
\end{aligned} \tag{3.58}$$

where $X, Y, Z \in \mathbb{R}^m$.

Now the goal is to find feedback gains K_1, K, K_3, K_4, and K_5 such that $Q=0$ and $C<0$, where Q and C are given by (3.48) and (3.55), respectively. Letting $K = U\xi$, where $U^{-1} = U^T \in \mathbb{R}^{m\times m}$ is an orthonormal matrix such that $U^T\Upsilon_2 U$ is diagonalized, i.e., we have

$$\hat{\Upsilon}_2 := U^T\Upsilon_2 U = \mathrm{Diag}\left[\begin{array}{cccc} \hat{\Upsilon}_2^1, & \hat{\Upsilon}_2^2, & \cdots, & \hat{\Upsilon}_2^m \end{array}\right], \tag{3.59}$$

$$\hat{\Upsilon}_1 := \Upsilon_1 U = \left[\begin{array}{cccc} \hat{\Upsilon}_1^1 & \hat{\Upsilon}_1^2 & \cdots & \hat{\Upsilon}_1^m \end{array}\right], \tag{3.60}$$

Denote $\underline{m} = \{1, 2, \cdots, m\}$, and define index set $I_j\ (j = 1, \cdots, 4) \in \underline{m}$ as

$$I_1 = \left\{ i \in \underline{m};\ \hat{\Upsilon}_2^i > 0 \right\}, \qquad (3.61)$$
$$I_2 = \left\{ i \in \underline{m};\ \hat{\Upsilon}_2^i < 0 \right\}, \qquad (3.62)$$
$$I_3 = \left\{ i \in \underline{m};\ \hat{\Upsilon}_2^i = 0,\ \hat{\Upsilon}_1^i \neq 0 \right\}, \qquad (3.63)$$

$$I_4 = \left\{ i \in \underline{m};\ \hat{\Upsilon}_2^i = \hat{\Upsilon}_1^i = 0 \right\}. \qquad (3.64)$$

It is clear that $I_i \cap I_j = \emptyset\ (i \neq j,\ i, j = 1, \cdots, 4)$, and $I_1 \cup I_2 \cup I_3 \cup I_4 = \underline{m}$.

Now in the ξ-coordinate, Q can be expressed as

$$Q = \hat{q}_{11} + \sum_{I \in I_3} \hat{\Upsilon}_1^i \xi_i + \sum_{i \in I_1 \cup I_2} \hat{\Upsilon}_2^i \left(\xi_i + \frac{\hat{\Upsilon}_1^i}{2\hat{\Upsilon}_2^i} \right)^2,$$

where $\hat{q}_{11}$ is given by (3.14). Now we consider different cases.

I. Consider the case when $q_{11} \neq 0$.

(I) $I_2 = I_3 = \emptyset$, and $I_1 \neq \emptyset$. In this case,

$$Q = \hat{q}_{11} + \sum_{i \in I_1} \hat{\Upsilon}_2^i \left(\xi_i + \frac{\hat{\Upsilon}_1^i}{2\hat{\Upsilon}_2^i} \right)^2.$$

1. If $\hat{q}_{11} > 0$, there does not exist an $\xi \in \mathbb{R}^{m\times 1}$ such that $Q = 0$, i.e., the system is not stabilizable.
2. If $\hat{q}_{11} < 0$, then the bifurcation is stabilizable. We construct the stabilizing controllers as follows. Let

$$\xi_i = \begin{cases} -\frac{\hat{\Upsilon}_1^i}{2\hat{\Upsilon}_2^i} + \frac{\epsilon_i}{\sqrt{\hat{\Upsilon}_2^i}}, & i \in I_1, \\ 0, & i \notin I_1, \end{cases}$$

where $\epsilon_i \in \mathbb{R}$ $(i \in I_1)$ are selected such that

$$Q = \hat{q}_{11} + \sum_{i \in I_1} \epsilon_i^2 = 0. \tag{3.65}$$

Define $\hat{E} = \begin{bmatrix} e_1 & e_2 & \cdots & e_m \end{bmatrix}$, where

$$e_i = \begin{cases} 2\sqrt{\hat{\Upsilon}_2^i}\epsilon_i, & i \in I_1 \\ 0, & i \notin I_1, \end{cases}$$

From (3.65), and $\hat{q}_{11} < 0$, we have $\hat{E} \neq 0$. From the above construction it is easy to see that $2\xi^T \hat{\Upsilon}_2 + \hat{\Upsilon}_1 = \hat{E}$. So we have

$$2K^T \Upsilon_2 + \Upsilon_1 = E,$$

where $E = \hat{E}U^T \neq 0$, and $K = U\xi$. Now from (3.48) and (3.55), we have $Q = 0$, and

$$C = E\left[I + K_1 A^{-1} B\right]^{-1} [K_3 + K_4\beta_1 + K_5(\beta_1, \beta_1)] + P_S(K).$$

Since $E \neq 0$, we fix K_1 and $K = U\xi$, and let $K_4 = 0$, $K_5 = 0$. Then it is easy to choose K_3 such that $C < 0$.

3. $\hat{q}_{11} = 0$, then we have the following cases.

   (1) If $I_4 = \emptyset$, it is clear that $Q = 0$ if and only if $\xi_i = -\frac{\hat{\Upsilon}_1^i}{2\hat{\Upsilon}_2^i} (i \in I_1 = \underline{\mathrm{m}})$, and if and only if $K = K_a$, where K_a is given by (3.13). In this case, we have $2K_a^T \Upsilon_2 + \Upsilon_1 = 0$. So

$$C = P_S(K_a),$$

   where $P_S(\cdot)$ is given by (3.15). We will treat this case later in this section.

(2) If $I_4 \neq \emptyset$, then $Q = 0$ if and only if

$$\xi_i = \begin{cases} -\frac{\hat{\Upsilon}_1^i}{2\hat{\Upsilon}_2^i}, & i \in I_1, \\ \in \mathbb{R}, & i \in I_4, \end{cases}$$

i.e., if and only if $K = K_b + K_e$, where

$$K_e \in \text{Ker} \begin{bmatrix} \Upsilon_1 \\ \Upsilon_2 \end{bmatrix}, \quad K_b = U\xi_b, \quad \xi_b = \begin{cases} -\frac{\hat{\Upsilon}_1^i}{2\hat{\Upsilon}_2^i}, & i \in I_1, \\ 0, & i \in I_4. \end{cases}$$

In this case we have $2K^T\Upsilon_2 + \Upsilon_1 = 0$, and

$$C = P_S(K_b + K_e),$$

where $P_S(\cdot)$ is given by (3.15). We will elaborate this case later in this section.

(II) $I_1 = I_3 = \emptyset$, but $I_2 \neq \emptyset$. We have

$$Q = \hat{q}_{11} + \sum_{i \in I_2} \hat{\Upsilon}_2^i \left(\xi_i + \frac{\hat{\Upsilon}_1^i}{2\hat{\Upsilon}_2^i} \right)^2,$$

and the situation is similar to the previous case.

1. If $\hat{q}_{11} < 0$, then there does not exist $K \in \mathbb{R}^{m\times 1}$ such that $Q(K) = 0$. So the system is unstabilizable.

2. If $\hat{q}_{11} > 0$, then the bifurcation is stabilizable. We construct the stabilizing controllers as follows. Let

$$\xi_i = \begin{cases} -\frac{\hat{\Upsilon}_1^i}{2\hat{\Upsilon}_2^i} + \frac{\epsilon_i}{\sqrt{-\hat{\Upsilon}_2^i}}, & i \in I_2, \\ 0, & i \notin I_2, \end{cases}$$

where $\epsilon_i \in \mathbb{R}$ $(i \in I_2)$ are selected such that

$$Q = \hat{q}_{11} - \sum_{i \in I_2} \epsilon_i^2 = 0. \tag{3.66}$$

Define $\hat{E} = \begin{bmatrix} e_1 & e_2 & \cdots & e_m \end{bmatrix}$, where

$$e_i = \begin{cases} 2\sqrt{-\hat{\Upsilon}_2^i}\epsilon_i, & i \in I_2 \\ 0, & i \notin I_2. \end{cases}$$

From (3.66) and $\hat{q}_{11} > 0$, we have $\hat{E} \neq 0$. From the above construction it is easy to see that $2\xi^T\hat{\Upsilon}_2 + \hat{\Upsilon}_1 = \hat{E}$. So we have

$$2K^T\Upsilon_2 + \Upsilon_1 = E,$$

where $E = \hat{E}U^T \neq 0$, and $K = U\xi$. Now from (3.48) and (3.55), we have $Q = 0$, and

$$C = E\left[I + K_1A^{-1}B\right]^{-1}\left[K_3 + K_4\beta_1 + K_5(\beta_1, \beta_1)\right] + P_S(K).$$

Since $E \neq 0$, we fix K_1 and $K = U\xi$, and let $K_4 = 0$, $K_5 = 0$. Then it is easy to choose K_3 such that $C < 0$.

3. $\hat{q}_{11} = 0$, then we have the following cases.

   (1) If $I_4 = \emptyset$, it is clear that $Q = 0$ if and only if $\xi_i = -\frac{\hat{\Upsilon}_1^i}{2\hat{\Upsilon}_2^i}$ $(i \in I_2 = \underline{m})$, and if and only if $K = K_a$, where K_a is given by (3.13). In this case, we have $2K_a^T\Upsilon_2 + \Upsilon_1 = 0$. We have

   $$C = P_S(K_a),$$

   where $P_S(\cdot)$ is given by (3.15). We will discuss this case later in this section.

(2) If $I_4 \neq \emptyset$, then $Q = 0$ if and only if

$$\xi_i = \begin{cases} -\frac{\hat{\Upsilon}_1^i}{2\hat{\Upsilon}_2^i}, & i \in I_2, \\ \in \mathbb{R}, & i \in I_4, \end{cases}$$

i.e., if and only if $K = K_b + K_e$, where

$$K_e \in \operatorname{Ker} \begin{bmatrix} \Upsilon_1 \\ \Upsilon_2 \end{bmatrix}, \quad K_b = U\xi_b, \quad \xi_b = \begin{cases} -\frac{\hat{\Upsilon}_1^i}{2\hat{\Upsilon}_2^i}, & i \in I_2, \\ 0, & i \in I_4. \end{cases}$$

In this case we have $2K^T\Upsilon_2 + \Upsilon_1 = 0$, and

$$C = P_S(K_b + K_e).$$

We will elaborate this case later in this section.

(III) $I_3 = \emptyset$, $I_1 \neq \emptyset$, and $I_2 \neq \emptyset$, then

$$Q = \hat{q}_{11} + \sum_{i \in I_1 \cup I_2} \hat{\Upsilon}_2^i \left(\xi_i + \frac{\hat{\Upsilon}_1^i}{2\hat{\Upsilon}_2^i} \right)^2.$$

Let

$$\xi_i = \begin{cases} -\frac{\hat{\Upsilon}_1^i}{2\hat{\Upsilon}_2^i} + \frac{\epsilon_i}{\sqrt{\hat{\Upsilon}_2^i}}, & i \in I_1, \\ -\frac{\hat{\Upsilon}_1^i}{2\hat{\Upsilon}_2^i} + \frac{\epsilon_i}{\sqrt{-\hat{\Upsilon}_2^i}}, & i \in I_2, \\ 0, & i \notin I_1 \cup I_2, \end{cases}$$

such that

$$Q = \hat{q}_{11} + \sum_{i \in I_1} \epsilon_i^2 - \sum_{j \in I_2} \epsilon_j^2 = 0,$$

and $\hat{E} = \begin{bmatrix} e_1 & e_2 & \cdots & e_m \end{bmatrix} \neq 0$, where

$$e_i = \begin{cases} 2\sqrt{\hat{\Upsilon}_2^i}\epsilon_i, & i \in I_1 \\ 2\sqrt{-\hat{\Upsilon}_2^i}\epsilon_i, & i \in I_2 \\ 0, & i \notin I_1 \cup I_2, \end{cases}$$

then we have $2K^T\Upsilon_2 + \Upsilon_1 = E$, where $E = \hat{E}U^T$. Now we have

$$C = E\left[I + K_1A^{-1}B\right]^{-1}\left[K_3 + K_4\beta_1 + K_5(\beta_1, \beta_1)\right] + P_S(K).$$

Since $E \neq 0$, we fix K_1 and $K = U\xi$, and let $K_4 = 0$, $K_5 = 0$. Then it is easy to choose K_3 such that $C < 0$.

(IV) $I_1 = I_2 = I_3 = \emptyset$, then we have $\Upsilon_1 = 0$, and $\Upsilon_2 = 0$. In this case, $Q = q_{11} \neq 0$. So the system is not stabilizable.

(V) $I_3 \neq \emptyset$, then

$$Q = \hat{q}_{11} + \sum_{I \in I_3} \hat{\Upsilon}_1^i \xi_i + \sum_{i \in I_1 \cup I_2} \hat{\Upsilon}_2^i \left(\xi_i + \frac{\hat{\Upsilon}_1^i}{2\hat{\Upsilon}_2^i}\right)^2,$$

and the system is stabilizable. The argument is as follows. Let

$$\xi_i = \begin{cases} -\frac{\hat{\Upsilon}_1^i + \epsilon_i}{2\hat{\Upsilon}_2^i}, & i \in I_1 \cup I_2, \\ \epsilon_i, & i \in I_3, \\ 0, & i \in I_4, \end{cases}$$

where ϵ_i $(i \in I_1 \cup I_2 \cup I_3)$ are chosen such that $Q = 0$ and

$$\hat{E} := \begin{bmatrix} \epsilon_1 & \epsilon_2 & \cdots & \epsilon_m \end{bmatrix} \neq 0.$$

Note this can always be achieved. If there are more than one number of elements in I_3, then we fix one to be nonzero and choose another such that $Q = 0$. Suppose there is only one element in I_3, say $\hat{\Upsilon}_1^1 \neq 0$. If $I_1 \cup I_2 \neq \emptyset$, then we select $\epsilon_j \neq 0$ for some $j \in I_1 \cup I_2$ and choose ϵ_1 such that $Q = 0$. If $I_1 \cup I_2 = \emptyset$, then $Q = 0$ if and only if $\epsilon_1 = -\frac{\hat{q}_{11}}{\hat{\Upsilon}_1^1} = -\frac{q_{11}}{\hat{\Upsilon}_1^1} \neq 0$. Since $\hat{E} \neq 0$, we have $2K^T\Upsilon_2 + \Upsilon_1 = E = \hat{E}U^T \neq 0$. Now

$$C = E\left[I + K_1A^{-1}B\right]^{-1}[K_3 + K_4\beta_1 + K_5(\beta_1, \beta_1)] + P_S(K).$$

Since $E \neq 0$, we fix K_1 and $K = U\xi$, and let $K_4 = 0$, $K_5 = 0$. Then it is easy to choose K_3 such that $C < 0$.

II. Consider the case when $q_{11} = 0$, $\Upsilon_1 = 0$, and $\alpha_0 \geq 0$. Then

$$Q = q_{11} + K^T\Upsilon_1 + K^T\Upsilon_2K = K^T\Upsilon_2K.$$

Let $K = U\xi$, where $U \in \mathbb{R}^{m\times m}$ is an orthonormal matrix such that $U^T\Upsilon_2U$ is diagonal. Then we define the index sets I_j $(j = 1, \cdots, 4)$ as in (3.9), (3.10), (3.11), and (3.12), respectively. Since $\Upsilon_1 = 0$, we have $I_3 = \emptyset$. Now we consider different cases.

(I) If $I_1 = \emptyset$, or $I_2 = \emptyset$, then $Q = \xi^T\hat{\Upsilon}_2\xi = 0$ if and only if $\xi^T\hat{\Upsilon}_2 = 0$, if and only if $K^T\Upsilon_2 = 0$, if and only if $K \in \text{Ker}\begin{bmatrix}\Upsilon_1\\ \Upsilon_2\end{bmatrix}$ (since $\Upsilon_1 = 0$), if and only if $\Upsilon_1 + 2K^T\Upsilon_2 = 0$. So

$$C = P_S(K),$$

where $P_S(\cdot)$ is given by (3.15). We will further discuss this case later.

(II) $I_1 \neq \emptyset$, $I_2 \neq \emptyset$. We choose $\xi \in \mathbb{R}^{m\times 1}$ such that $Q = \xi^T\hat{\Upsilon}_2\xi = 0$ but $\xi^T\hat{\Upsilon}_2 \neq 0$, i.e., $K^T\Upsilon_2 \neq 0$. Since $\Upsilon_1 = 0$, we have

$$C = 2K^T\Upsilon_2\left[I + K_1A^{-1}B\right]^{-1}[K_3 + K_4\beta_1 + K_5(\beta_1, \beta_1)] + P_S(K).$$

Since $K^T\Upsilon_2 \neq 0$, we fix K_1 and $K = U\xi$, and let $K_4 = 0$, $K_5 = 0$. Then it is easy to choose K_3 such that $C < 0$.

Now we consider the cases that have not been settled above.

i. For the cases I-(I)-3-(1) and I-(II)-3-(1), we have $Q = 0$ if and only if $K = K_a$, where K_a is given by (3.13). In this case, $C = P_S(K_a)$.

   (i) If $P_S(K_a) < 0$, then the system is stabilizable.

   (ii) If $P_S(K_a) = 0$, then the system is degenerate, i.e., we have to resort to the 4^{th}, 5^{th}, and higher order terms to determine the stabilizability.

   (iii) If $P_S(K_a) > 0$, then the system is not stabilizable.

ii. For the cases I-(I)-3-(2), I-(II)-3-(2), and II-(I), we have $Q = 0$ if and only if $K = K_b + K_e$, where K_a is given by (3.13). In this case, $C = P_S(K_b + K_e)$, where K_b is a fixed vector given by (3.18) and (3.19) for the cases I-(I)-3-(2) and I-(II)-3-(2), respectively; and $K_b = 0$ for the case II-(I). Here $K_e \in \text{Ker}\begin{bmatrix}\Upsilon_1 \\ \Upsilon_2\end{bmatrix}$. By substituting $K = K_b + K_e$ into (3.15), we get

$$C = P_b(K_e) = \alpha_{0b} + \alpha_{1b}K_e + K_e^T\alpha_{2b}K_e + \alpha_{3b}(K_e, K_e, K_e),$$

where

$$\begin{aligned}
\alpha_{0b} &= \alpha_0 + \alpha_1 K_b + K_b^T\alpha_2 K_b + \alpha_3(K_b, K_b, K_b) \in \mathbb{R}, \\
\alpha_{1b} &= \left[\alpha_1 + 2K_b^T\alpha_2 + 3\alpha_3(K_b, K_b)\right] F \in \mathbb{R}^{l\times m}, \\
\alpha_{2b} &= F^T\left[\alpha_2 + 3\alpha_3(K_b)\right] F \in \mathbb{R}^{l\times m}, \\
\alpha_{3b} &= \alpha_3 \in \mathbb{R}^{l\times l\times m}.
\end{aligned}$$

Let $f_1, \cdots, f_l$ be a set of basis of $\text{Ker}\begin{bmatrix}\Upsilon_1 \\ \Upsilon_2\end{bmatrix}$, then any $K_e \in \text{Ker}\begin{bmatrix}\Upsilon_1 \\ \Upsilon_2\end{bmatrix}$, we have $K_e = FX$, where $F = \begin{bmatrix} f_1 & f_2 & \cdots & f_l \end{bmatrix} \in \mathbb{R}^{m\times l}$ and $X \in \mathbb{R}^{l\times 1}$. Define

$$\begin{aligned}
\alpha_{0r} &= \alpha_{0b} \in \mathbb{R}, \\
\alpha_{1r} &= \alpha_{1b}F \in \mathbb{R}^{1\times l}, \\
\alpha_{2r} &= F^T\alpha_{2b}F \in \mathbb{R}^{l\times l}, \\
\alpha_{3r}(X,X,X) &= \alpha_{3b}(FX,FX,FX) \in \mathbb{R}^{l\times l\times l}.
\end{aligned}$$

Here α_{jr} $(j = 1,2,3)$ are the restriction of the tensors α_{jb} $(j = 1,2,3)$ to the linear space $\mathrm{Ker}\begin{bmatrix} \Upsilon_1 \\ \Upsilon_2 \end{bmatrix} \subset \mathbb{R}^m$. Now we have

$$C = \alpha_{0r} + \alpha_{1r}X + X^T\alpha_{2r}X + \alpha_{3r}(X,X,X).$$

The goal is to find $X \in \mathbb{R}^{l\times 1}$ such that $C < 0$.

a. If $\alpha_{3r} \neq 0$, then $C < 0$ can be achieved by letting X large enough.

b. If $\alpha_{3r} = 0$, then

$$\begin{aligned}
C &= \alpha_{0r} + \alpha_{1r}X + X^T\alpha_{2r}X \\
&= \alpha_{0r} + \hat{\alpha}_{1r}\eta + \eta^T\hat{\alpha}_{2r}\eta,
\end{aligned}$$

where $X = V\eta$, $V^{-1} = V^T \in \mathbb{R}^{l\times l}$, and $\hat{\alpha}_{1r}$ and $\hat{\alpha}_{2r}$ are given by (3.24) and (3.25), respectively. Let I_{jr} $(j = 1,\cdots,4)$ be defined as from (3.26) to (3.29), then

$$C = \hat{\alpha}_{0r} + \sum_{i\in I_{3r}\cup I_{2r}} \hat{\alpha}^i_{1r}\eta_i + \sum_{i\in I_{2r}} \hat{\alpha}^i_{2r}\eta_i^2 + \sum_{i\in I_{1r}} \hat{\alpha}^i_{2r}\left(\eta_i + \frac{\hat{\alpha}^i_{1r}}{2\hat{\alpha}^i_{2r}}\right)^2.$$

(a) If $I_{2r} \neq \emptyset$, then by letting $\eta_i\,(i \in I_{2r})$ large enough, we have $C < 0$. So the bifurcation is stabilizable.

(b) If $I_{2r} = \emptyset$, but $I_{3r} \neq \emptyset$, then by fixing $\eta_i\,(i \in I_{1r})$, and letting $\eta_i\,(i \in I_{3r})$ large enough, we have $C < 0$. So the bifurcation is stabilizable.

(c) If $I_{2r} = I_{3r} = \emptyset$, then

$$C = \hat{\alpha}_{0r} + \sum_{i \in I_{1r}} \hat{\alpha}^i_{2r} \left(\eta_i + \frac{\hat{\alpha}^i_{1r}}{2\hat{\alpha}^i_{2r}} \right)^2 .$$

a) If $\hat{\alpha}_{0r} < 0$, then choose η as

$$\eta_i = \begin{cases} -\frac{\hat{\alpha}^i_{1r}}{2\hat{\alpha}^i_{2r}}, & i \in I_{1r}, \\ 0, & i \in I_{4r}, \end{cases}$$

we have $C < 0$. The bifurcation is stabilizable.

b) If $\hat{\alpha}_{0r} > 0$, then for any $\eta \in \mathbb{R}^{l \times 1}$, $C > 0$. So the bifurcation is not stabilizable.

c) If $\hat{\alpha}_{0r} = 0$, then the minimum of C for all $\eta \in \mathbb{R}^{l \times 1}$ is zero. This case is degenerate, i.e., we have to resort to the higher (4^{th} and 5^{th}) order terms to determine the stabilizability.

By combining all the cases, we obtain the diagram in Theorem 3.1. It should be noted if $m = 1$, then both the cases and the control laws here are the same as those obtained in Chapter 2. Also in Chapter 2, we use the controller canonical forms of (A, B) so that the inverse of A can be explicitly obtained. Hence the control laws in the single input case are more simple than the multi-input case in the sense that they do not need to calculate the inverse of matrices. But one of the advantages of the procedure in this section is that it does not rely on the controller canonical forms.

3.4 Proof of Theorem 3.2 and Corollary 3.2

Theorem 3.2

Consider the feedback

$$\begin{aligned} u &= F(\tilde{x}, z, z^*) \\ &= K_1\tilde{x} + K_2 z + K_3^* z^* + K_4 z^2 + K_5|z|^2 + K_6 z^{*2} + K_7\tilde{x}z + K_8\tilde{x}z^* + K_9(\tilde{x},\tilde{x}) \\ &\quad +\text{h.o.t.}, \end{aligned}$$

where

$$\begin{aligned} &K_1 \in \mathbb{R}^{m\times n}, \quad K_2 = K_3^* \in \mathbb{C}^m, \quad\quad K_4 = K_6^* \in \mathbb{C}^{m\times 1}, \\ &K_5 \in \mathbb{R}^{m\times 1}, \quad K_7 = K_8^* \in \mathbb{C}^{m\times n}, \quad K_9 \in \mathbb{R}^{m\times n\times n}. \end{aligned}$$

We select K_1 such that $A + BK_1$ is Hurwitz. Without loss of generality we also assume A does not have eigenvalues at 0 and $\pm 2\omega i$. If it does, we can use a linear feedback to move away those eigenvalues. We first consider the case when $K_2 = 0$, then the center manifold is given by

$$\tilde{x} = \beta_1 z^2 + \beta_2|z|^2 + \beta_3 z^{*2} + \text{h.o.t.}$$

Then the dynamics on the center manifold is given by

$$\dot{z} = i\omega z + q_{11}z^2 + q_{12}|z|^2 + q_{13}z^{*2} + C|z|^2 z + \text{o.c.t.} + \text{h.o.t.},$$

where

$$C = c_{112} + (q_{13} + q_{14}K_1)\beta_2 + (q_{23} + q_{24}K_1)\beta_1 + q_{14}K_5 + q_{24}K_4.$$

Now, a normal form is given by

$$\dot{\zeta} = i\omega\zeta + \tilde{\alpha}|\zeta|^2\zeta + 5^{th} \text{ order terms},$$

where

$$\tilde{\alpha} = -\frac{q_{11}q_{12}}{i\omega} + C + \text{p.i.t.}$$

As in the single-input case, β_1 and β_2 can be calculated as

$$\begin{aligned}
\beta_1 &= [2\omega i - (A + BK_1)]^{-1}(BK_4 + \tilde{q}_{11}), \\
\beta_2 &= [-(A + BK_1)]^{-1}(BK_5 + \tilde{q}_{12}),
\end{aligned}$$

So

$$\begin{aligned}
\tilde{\alpha} &= -\frac{q_{11}q_{12}}{i\omega} + C + \text{p.i.t.} \\
&= -\frac{q_{11}q_{12}}{i\omega} + c_{112} + \left\{q_{24} + (q_{23} + q_{24}K_1)[2\omega i - (A + BK_1)]^{-1}B\right\} K_4 \\
&\quad + \left\{q_{14} + (q_{13} + q_{14}K_1)[-(A + BK_1)]^{-1}B\right\} K_5 \\
&\quad +(q_{23} + q_{24}K_1)[2\omega i - (A + BK_1)]^{-1}\tilde{q}_{11} + (q_{13} + q_{14}K_1)[-(A + BK_1)]^{-1}\tilde{q}_{12} \\
&\quad +\text{p.i.t.} \\
&= -\frac{q_{11}q_{12}}{i\omega} + q_{23}(2\omega i - A)^{-1}\tilde{q}_{11} + q_{13}(-A)^{-1}\tilde{q}_{12} + c_{112} \\
&\quad +[q_{24} + q_{23}(2\omega i - A)^{-1}B] \left[I - K_1(2\omega i - A)^{-1}B\right]^{-1} \left[K_4 + K_1(2\omega i - A)^{-1}\tilde{q}_{11}\right] \\
&\quad +[q_{14} + q_{13}(-A)^{-1}B] \left[I - K_1(-A)^{-1}B\right]^{-1} \left[K_5 + K_1(-A)^{-1}\tilde{q}_{12}\right] + \text{p.i.t.}
\end{aligned}$$

Note in the derivations we have used the following identities:

$$\begin{aligned}
&q_{24} + (q_{23} + q_{24}K_1)[sI - (A + BK_1)]^{-1}B \\
&\qquad = \left[q_{24} + q_{23}(sI - A)^{-1}B\right] \left[I - K_1(sI - A)^{-1}B\right]^{-1},
\end{aligned}$$

$$\begin{aligned}
&(q_{23} + q_{24}K_1)[sI - (A + BK_1)]^{-1}\tilde{q}_{11} \\
&\qquad = q_{23}(sI - A)^{-1}\tilde{q}_{11} + \left[q_{24} + q_{23}(sI - A)^{-1}B\right] K_1[sI - (A + BK_1)]^{-1}\tilde{q}_{11},
\end{aligned}$$

$$\left[I - K_1(sI - A)^{-1}B\right] K_1[sI - (A + BK_1)]^{-1} = K_1(sI - A)^{-1}.$$

The third identity is trivial. A proof of the first two identities is as follows. For the first identity,

$$\begin{aligned} X &= q_{24} + (q_{23} + q_{24}K_1)[sI - (A + BK_1)]^{-1}B \\ &= q_{24} + (q_{23} + q_{24}K_1)[I + (sI - A - BK_1)^{-1}BK_1](sI - A)^{-1}B \\ &= q_{24} + (q_{23} + q_{24}K_1)(sI - A - BK_1)^{-1}BK_1(sI - A)^{-1}B \\ &\quad + (q_{23} + q_{24}K_1)(sI - A)^{-1}B \\ &= q_{24} + (X - q_{24})K_1(sI - A)^{-1}B + (q_{23} + q_{24}K_1)(sI - A)^{-1}B \\ &= q_{24} + XK_1(sI - A)^{-1}B + q_{23}(sI - A)^{-1}B, \end{aligned}$$

so we have

$$X = \left[q_{24} + q_{23}(sI - A)^{-1}B\right]\left[I - K_1(sI - A)^{-1}B\right]^{-1}.$$

For the second identity, we have

$$\begin{aligned} & (q_{23} + q_{24}K_1)[sI - (A + BK_1)]^{-1}\tilde{q}_{11} \\ = & \left[q_{23} + \left(q_{24} + q_{23}(sI - A)^{-1}B\right)K_1 - q_{23}(sI - A)^{-1}BK_1\right] \cdot \\ & [sI - (A + BK_1)]^{-1}\tilde{q}_{11} \\ = & \; q_{23}\left[I - (sI - A)^{-1}BK_1\right][sI - (A + BK_1)]^{-1}\tilde{q}_{11} \\ & + \left(q_{24} + q_{23}(sI - A)^{-1}B\right)K_1[sI - (A + BK_1)]^{-1}\tilde{q}_{11} \\ = & \; q_{23}(sI - A)^{-1}\tilde{q}_{11} + \left[q_{24} + q_{23}(sI - A)^{-1}B\right]K_1[sI - (A + BK_1)]^{-1}\tilde{q}_{11}. \end{aligned}$$

Letting $\alpha = \operatorname{Re}\tilde{\alpha}$, and

$$\begin{aligned} \Theta_1 &= \operatorname{Re}\left\{q_{14} + q_{13}(-A)^{-1}B\right\}, \\ \Theta_2 &= q_{24} + q_{23}(2\omega i - A)^{-1}B, \\ \alpha_0 &= \operatorname{Re}\left\{c_{112} - \frac{q_{11}q_{12}}{i\omega} + q_{23}(2\omega i - A)^{-1}\tilde{q}_{11} + q_{13}(-A)^{-1}\tilde{q}_{12}\right\}, \end{aligned}$$

then

$$\begin{aligned}\alpha &= \alpha_0 + \Theta_1 \left[I - K_1(-A)^{-1}B\right]^{-1} \left[K_5 + K_1(-A)^{-1}\tilde{q}_{12}\right] \\ &\quad + \mathrm{Re}\left\{\Theta_2 \left[I - K_1(2\omega i - A)^{-1}B\right]^{-1} \left[K_4 + K_1(2\omega i - A)^{-1}\tilde{q}_{11}\right]\right\}.\end{aligned}$$

It is easy to see that for $K_2 = 0$, there exists K_1, K_4 and K_5 such that $\alpha < 0$ if and only if

(1) $\alpha_0 < 0$,

(2) $\alpha_0 \geq 0$, $\Theta_1 \neq 0$ or $\Theta_2 \neq 0$.

Now we consider the case when $\alpha_0 \geq 0$, and $\Theta_1 = \Theta_2 = 0$. In this case we must have $K_2 \neq 0$. The center manifold is given by

$$\tilde{x} = \beta_1 z + \beta_2 z^* + \beta_3 z^2 + \beta_4 |z|^2 + \beta_5 z^{*2} + \text{h.o.t.}$$

The dynamics on the center manifold are given by

$$\dot{z} = i\omega z + Q_1 z^2 + Q_2 |z|^2 + Q_3 z^{*2} + \text{other } 3^{rd} \text{ order terms} + \text{h.o.t.}$$

where

$$\begin{aligned}Q_1 &= q_{11} + q_{13}\beta_1 + q_{14}(K_1\beta_1 + K_2) + \beta_1^T q_{33}\beta_1 + \beta_1^T q_{34}(K_1\beta_1 + K_2) \\ &\quad + (K_1\beta_1 + K_2)^T q_{44}(K_1\beta_1 + K_2), \\ Q_2 &= q_{12} + q_{13}\beta_2 + q_{14}(K_1\beta_2 + K_3) + q_{23}\beta_1 + q_{24}(K_1\beta_1 + K_2) + 2\beta_1^T q_{33}\beta_2 \\ &\quad + \beta_1^T q_{34}(K_1\beta_2 + K_3) + \beta_2^T q_{34}(K_1\beta_1 + K_2) \\ &\quad + 2(K_1\beta_1 + K_2)^T q_{44}(K_1\beta_2 + K_3), \\ Q_3 &= q_{22} + q_{23}\beta_2 + q_{24}(K_1\beta_2 + K_3) + \beta_2^T q_{33}\beta_2 + \beta_2^T q_{34}(K_1\beta_2 + K_3) \\ &\quad + (K_1\beta_2 + K_3)^T q_{44}(K_1\beta_2 + K_3),\end{aligned}$$

and $C = W_1 + W_2 + W_3$, where

$$\begin{aligned} W_1 &= \left\{ \left[q_{13} + 2\beta_1^T q_{33} + (K_1\beta_1 + K_2)^T q_{34}^T \right] \right. \\ &\quad \left. + \left[q_{14} + \beta_1^T q_{34} + 2(K_1\beta_1 + K_2)^T q_{44} \right] K_1 \right\} \beta_4 \\ &\quad + \left[q_{14} + \beta_1^T q_{34} + 2(K_1\beta_1 + K_2)^T q_{44} \right] \cdot \\ &\quad [K_5 + K_7\beta_2 + K_8\beta_1 + K_9(\beta_1, \beta_2)], \end{aligned} \tag{3.67}$$

$$\begin{aligned} W_2 &= \left\{ \left[q_{23} + 2\beta_2^T q_{33} + (K_1\beta_2 + K_3)^T q_{34}^T \right] \right. \\ &\quad \left. + \left[q_{24} + \beta_2^T q_{34} + 2(K_1\beta_2 + K_3)^T q_{44} \right] K_1 \right\} \beta_3 \\ &\quad + \left[q_{24} + \beta_2^T q_{34} + 2(K_1\beta_2 + K_3)^T q_{44} \right] \cdot \\ &\quad [K_4 + K_7\beta_1 + K_9(\beta_1, \beta_1)], \end{aligned} \tag{3.68}$$

$$\begin{aligned} W_3 &= c_{112} + c_{113}\beta_2 + c_{114}(K_1\beta_2 + K_3) + c_{123}\beta_1 + c_{134}(K_1\beta_1 + K_2) \\ &\quad + 2\beta_1^T c_{133}\beta_2 + \beta_1^T c_{134}(K_1\beta_2 + K_3) + \beta_2^T c_{134}(K_1\beta_1 + K_2) \\ &\quad + 2(K_1\beta_1 + K_2)^T c_{144}(K_1\beta_2 + K_3) \\ &\quad + \beta_1^T c_{233}\beta_1 + \beta_2^T c_{134}(K_1\beta_1 + K_2) \\ &\quad + c_{334}(\beta_1, \beta_1, K_1\beta_2 + K_3) + 2c_{334}(\beta_1, \beta_2, K_1\beta_1 + K_2) \\ &\quad + 2c_{344}(\beta_1, K_1\beta_1 + K_2, K_1\beta_2 + K_3) \\ &\quad + c_{344}(\beta_2, K_1\beta_1 + K_2, K_1\beta_1 + K_2) \\ &\quad + 3c_{444}(K_1\beta_1 + K_2, K_1\beta_1 + K_2, K_1\beta_1 + K_2). \end{aligned} \tag{3.69}$$

Now β_j $(j = 1, \cdots, 4)$ can be calculated as

$$\beta_1 = [i\omega - (A + BK_1)]^{-1} BK_2 = (i\omega - A)^{-1} BK, \tag{3.70}$$

$$\beta_2 = [-i\omega - (A + BK_1)]^{-1} BK_2 = (-i\omega - A)^{-1} BK^* = \beta_1^*, \tag{3.71}$$

$$\begin{aligned} \beta_3 &= [2i\omega - (A + BK_1)]^{-1} \Big\{ [K_4 + K_7\beta_1 + K_9(\beta_1, \beta_1)] \\ &\quad + \tilde{Q}_1 - \beta_1 Q_1 - \beta_2 Q_3^* \Big\}, \end{aligned} \tag{3.72}$$

$$\begin{aligned} \beta_4 &= [-(A + BK_1)]^{-1} \Big\{ B[K_5 + K_7\beta_2 + K_8\beta_1 + 2K_9(\beta_1, \beta_2)] \\ &\quad + \tilde{Q}_2 - \beta_1 Q_2 - \beta_2 Q_2^* \Big\}, \end{aligned} \tag{3.73}$$

where

$$\begin{aligned}
K &= \left[I - K_1(i\omega - A)^{-1}B\right]^{-1} K_2, \\
\tilde{Q}_1 &= \tilde{q}_{11} + \tilde{q}_{13}\beta_1 + \tilde{q}_{14}(K_1\beta_1 + K_2) + \tilde{q}_{33}(\beta_1, \beta_1) \\
&\quad + \tilde{q}_{34}(\beta_1, K_1\beta_1 + K_2) + \tilde{q}_{44}(K_1\beta_1 + K_2, K_1\beta_1 + K_2), \\
\tilde{Q}_2 &= \tilde{q}_{12} + \tilde{q}_{13}\beta_2 + \tilde{q}_{14}(K_1\beta_2 + K_3) + \tilde{q}_{23}\beta_1 + \tilde{q}_{24}(K_1\beta_1 + K_2) \\
&\quad + \tilde{q}_{33}(\beta_1, \beta_2) + \tilde{q}_{34}(\beta_1, K_1\beta_2 + K_3) + \tilde{q}_{34}(\beta_2, K_1\beta_1 + K_2) \\
&\quad + 2\tilde{q}_{44}(K_1\beta_1 + K_2, K_1\beta_2 + K_3).
\end{aligned}$$

Define

$$\begin{aligned}
\Phi_1(s) &= q_{13}(sI - A)^{-1}B + q_{14}, \\
\Phi_2(s) &= q_{23}(sI - A)^{-1}B + q_{24}, \\
\tilde{\Phi}_1(s) &= \tilde{q}_{13}(sI - A)^{-1}B + \tilde{q}_{14}, \\
\tilde{\Phi}_2(s) &= \tilde{q}_{23}(sI - A)^{-1}B + \tilde{q}_{24}, \\
\Psi(s_1, s_2) &= 2B^T(s_1I - A)^{-T}q_{33}(s_2I - A)^{-1}B + q_{34}^T(s_2 - A)^{-1}B \\
&\quad + B^T(s_1I - A)^{-T}q_{34} + 2q_{44}, \\
\tilde{\Psi}(s_1, s_2)(X, Y) &= 2\tilde{q}_{33}((s_2I - A)^{-1}BX, (s_2I - A)^{-1}BY) + 2\tilde{q}_{44}(X, Y) \\
&\quad + \tilde{q}_{34}((s_1 - A)^{-1}BX, Y) + \tilde{q}_{34}((s_2I - A)^{-1}BY, X).
\end{aligned}$$

Using the fact that $K_1\beta_1 + K_2 = K$, we get

$$\begin{aligned}
Q_1 &= q_{11} + \Phi_1(i\omega)K + \frac{1}{2}K^T\Psi(i\omega, i\omega)K, \\
Q_2 &= q_{12} + \Phi_2(i\omega)K + \Phi_1(-i\omega)K^* + K^H\Psi(-i\omega, i\omega)K, \\
Q_3 &= q_{22} + \Phi_2(-i\omega)K^* + \frac{1}{2}K^H\Psi(-i\omega, -i\omega)K^*, \\
\tilde{Q}_1 &= \tilde{q}_{11} + \tilde{\Phi}_1(i\omega)K + \frac{1}{2}\tilde{\Psi}(i\omega, i\omega)(K, K), \\
\tilde{Q}_2 &= \tilde{q}_{12} + \tilde{\Phi}_2(i\omega)K + \tilde{\Phi}_1(-i\omega)K^* + \frac{1}{2}\tilde{\Psi}(i\omega, -i\omega)(K, K^*),
\end{aligned}$$

$$
\begin{aligned}
W_3 &= c_{112} + \left[c_{113}(-i\omega - A)^{-1}B + c_{114}\right]K^* + \left[c_{123}(i\omega - A)^{-1}B + c_{124}\right]K \\
&\quad + K^H\left[2B^T(-i\omega - A)^{-T}c_{133}(i\omega - A)^{-1}B + c_{134}^T(i\omega - A)^{-1}B\right. \\
&\quad \left. + B^T(-i\omega - A)^{-T}c_{134} + 2c_{144}\right]K \\
&\quad + K^T\left[B^T(i\omega - A)^{-T}c_{233}(i\omega - A)^{-1}B + B^T(i\omega - A)^{-T}c_{234} + c_{244}\right]K \\
&\quad + 3c_{333}\left((i\omega - A)^{-1}BK, (i\omega - A)^{-1}BK, (-i\omega - A)^{-1}BK^*\right) \\
&\quad + c_{334}\left((i\omega - A)^{-1}BK, (i\omega - A)^{-1}BK, K^*\right) \\
&\quad + 2c_{334}\left((i\omega - A)^{-1}BK, (-i\omega - A)^{-1}BK^*, K\right) \\
&\quad + 2c_{344}\left((i\omega - A)^{-1}BK, K, K^*\right) + c_{344}\left((-i\omega - A)^{-1}K^*, K, K\right) \\
&\quad + 3c_{444}(K, K, K^*).
\end{aligned}
$$

Now we calculate W_1. Define $\Theta_1 = \operatorname{Re}\Phi_1(0)$, and $\Theta_3 = \Psi_1(i\omega, 0)$. By substituting (3.70) into (3.67), and using the following identities:

$$K_1\beta_1 + K_2 = K,$$

$$K_1[sI - (A + BK_1)]^{-1} + I = \left[I - K_1(sI - A)^{-1}B\right]^{-1},$$

$$[sI - (A + BK_1)]^{-1}B = (sI - A)^{-1}B\left[I - K_1(sI - A)^{-1}B\right]^{-1},$$

$$q_{14} = \Theta_1 - q_{13}(-A)^{-1}B + \text{p.i.t.},$$

$$B^T(i\omega - A)^{-T}q_{34} + 2q_{44} = \Theta_3 - \left[2B^T(i\omega - A)^{-T}q_{33} + q_{34}^T\right](-A)^{-1}B,$$

we get

$$
\begin{aligned}
W_1 &= \left[\operatorname{Re}\Phi_1(0) + K^T\Psi(i\omega, 0)\right]\left[I + K_1A^{-1}B\right]^{-1}\cdot \\
&\quad [K_5 + K_7\beta_2 + K_8\beta_1 + 2K_9(\beta_1, \beta_2)] \\
&\quad + \left\{\left[q_{13} + 2K^TB^T(i\omega - A)^{-T}q_{33} + K^Tq_{34}^T\right]\right. \\
&\quad \left. + \left[q_{14} + K^TB^T(i\omega - A)^{-T}q_{34} + 2K^Tq_{44}\right]K_1\right\}\cdot \\
&\quad [-(A + BK_1)]^{-1}\left(\tilde{Q}_2 - \beta_1Q_2 - \beta_2Q_2^*\right) + \text{p.i.t.}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\Theta_1 + K^T\Theta_3\right)\left[I + K_1A^{-1}B\right]^{-1}\left[K_5 + K_7\beta_2 + K_8\beta_1 + 2K_9(\beta_1,\beta_2)\right] \\
&\quad +\Big\{\left(\Theta_1 + K^T\Theta_3\right)K_1 + \left[q_{13} + 2K^TB^T(i\omega - A)^{-T}q_{33} + K^Tq_{34}^T\right]\cdot \\
&\quad \left[I - (-A)^{-1}BK_1\right]\Big\}[-(A+BK_1)]^{-1}\left(\tilde{Q}_2 - \beta_1Q_2 - \beta_2Q_2^*\right) + \text{p.i.t.}, \\
&= \left(\Theta_1 + K^T\Theta_3\right)\left[I + K_1A^{-1}B\right]^{-1}\left[K_5 + K_7\beta_2 + K_8\beta_1 + 2K_9(\beta_1,\beta_2)\right] \\
&\quad + \left(\Theta_1 + K^T\Theta_3\right)\left(\left[I + K_1A^{-1}B\right]^{-1} - I\right)\left(\tilde{Q}_2 - \beta_1Q_2 - \beta_2Q_2^*\right) \\
&\quad + \left[q_{13} + 2K^TB^T(i\omega - A)^{-T}q_{33} + K^Tq_{34}^T\right](-A)^{-1}\left(\tilde{Q}_2 - \beta_1Q_2 - \beta_2Q_2^*\right) \\
&\quad +\text{p.i.t.} \\
&= \left(\Theta_1 + K^T\Theta_3\right)\left[I + K_1A^{-1}B\right]^{-1}\cdot \\
&\quad \left[K_5 + K_7\beta_2 + K_8\beta_1 + 2K_9(\beta_1,\beta_2) + K_1A^{-1}B\left(\tilde{Q}_2 - \beta_1Q_2 - \beta_2Q_2^*\right)\right] \\
&\quad + \left[q_{13} + 2K^TB^T(i\omega - A)^{-T}q_{33} + K^Tq_{34}^T\right](-A)^{-1}\left(\tilde{Q}_2 - \beta_1Q_2 - \beta_2Q_2^*\right) \\
&\quad +\text{p.i.t.},
\end{aligned}
$$

Similarly, define $\Theta_2 = \Phi_2(2\omega i)$, and $\Theta_4 = \Psi(-i\omega, 2\omega i)$, so we have

$$q_{24} = \Theta_2 - q_{23}(2\omega i - A)^{-1}B,$$

$$B^T(-i\omega - A)^{-T}q_{34} + 2q_{44} = \Theta_4 - \left[2B^T(-i\omega - A)^{-T}q_{33} + q_{34}^T\right](2\omega i - A)^{-1}B,$$

and W_2 can be calculated as

$$
\begin{aligned}
W_2 &= \left[\Phi_2(2\omega i) + K^H\Psi(-i\omega, 2\omega i)\right]\left[I - K_1(2\omega i - A)^{-1}B\right]^{-1}\cdot \\
&\quad \left[K_4 + K_7\beta_1 + K_9(\beta_1,\beta_1)\right] \\
&\quad +\Big\{\left[q_{23} + 2K^HB^T(-i\omega - A)^{-T}q_{33} + K^Hq_{34}^T\right] \\
&\quad + \left[q_{24} + K^HB^T(-i\omega - A)^{-T}q_{34} + 2K^Hq_{44}\right]K_1\Big\}\cdot \\
&\quad [2\omega i - (A+BK_1)]^{-1}\left(\tilde{Q}_1 - \beta_1Q_1 - \beta_2Q_3^*\right) \\
&= \left(\Theta_2 + K^H\Theta_4\right)\left[I - K_1(2\omega i - A)^{-1}B\right]^{-1}\left[K_4 + K_7\beta_1 + K_9(\beta_1,\beta_1)\right] \\
&\quad +\Big\{\left(\Theta_2 + K^H\Theta_4\right)K_1 + \left[q_{23} + 2K^HB^T(-i\omega - A)^{-T}q_{33} + K^Hq_{34}^T\right]\cdot \\
&\quad \left[I - (2\omega i - A)^{-1}BK_1\right]\Big\}\cdot[2\omega i - (A+BK_1)]^{-1}\left(\tilde{Q}_1 - \beta_1Q_1 - \beta_2Q_3^*\right)
\end{aligned}
$$

$$
\begin{aligned}
&= (\Theta_2 + K^H\Theta_4)\left[I - K_1(2\omega i - A)^{-1}B\right]^{-1}\left[K_4 + K_7\beta_1 + K_9(\beta_1,\beta_1)\right] \\
&\quad + (\Theta_2 + K^H\Theta_4)\left(\left[I - K_1(2\omega i - A)^{-1}B\right]^{-1} - I\right)\left(\tilde{Q}_1 - \beta_1 Q_1 - \beta_2 Q_3^*\right) \\
&\quad + \left[q_{23} + 2K^H B^T(-i\omega - A)^{-T}q_{33} + K^H q_{34}^T\right](2\omega i - A)^{-1}\left(\tilde{Q}_1 - \beta_1 Q_1 - \beta_2 Q_3^*\right) \\
&= (\Theta_2 + K^H\Theta_4)\left[I - K_1(2\omega i - A)^{-1}B\right]^{-1}\cdot \\
&\quad + \left[K_4 + K_7\beta_1 + K_9(\beta_1,\beta_1) - K_1(2\omega i - A)^{-1}B\left(\tilde{Q}_1 - \beta_1 Q_1 - \beta_2 Q_3^*\right)\right] \\
&\quad + \left[q_{23} + 2K^H B^T(-i\omega - A)^{-T}q_{33} + K^H q_{34}^T\right](2\omega i - A)^{-1}\left(\tilde{Q}_1 - \beta_1 Q_1 - \beta_2 Q_3^*\right),
\end{aligned}
$$

Define

$$
\begin{aligned}
\tilde{G}_H(K,K^*) &= \left[q_{13} + 2K^T B^T(i\omega - A)^{-T}q_{33} + K^T q_{34}^T\right](-A)^{-1}\left(\tilde{Q}_2 - \beta_1 Q_2 - \beta_2 Q_2^*\right) \\
&\quad + \left[q_{23} + 2K^H B^T(-i\omega - A)^{-T}q_{33} + K^H q_{34}^T\right](2\omega i - A)^{-1}\cdot \\
&\quad \left(\tilde{Q}_1 - \beta_1 Q_1 - \beta_2 Q_3^*\right) - \frac{Q_1 Q_2}{i\omega} + W_3(K,K^*), &(3.74) \\
G_H(K,K^*) &= \operatorname{Re}\tilde{G}_H(K,K^*), &(3.75)
\end{aligned}
$$

then we have

$$
\begin{aligned}
\alpha &:= \operatorname{Re}\tilde{\alpha} = \operatorname{Re}\left\{W_1 + W_2 + W_3 - \frac{Q_1 Q_2}{i\omega}\right\} \\
&= \left(\Theta_1 + \operatorname{Re}\left\{K^T\Theta_3\right\}\right)\left[I + K_1 A^{-1}B\right]^{-1}\cdot \\
&\quad \left[K_5 + K_7\beta_2 + K_8\beta_1 + 2K_9(\beta_1,\beta_2) + K_1 A^{-1}B\left(\tilde{Q}_2 - \beta_1 Q_2 - \beta_2 Q_2^*\right)\right] \\
&\quad + \operatorname{Re}\left\{(\Theta_2 + K^H\Theta_4)\left[I - K_1(2\omega i - A)^{-1}B\right]^{-1}\cdot\right. \\
&\quad \left.\left[K_4 + K_7\beta_1 + K_9(\beta_1,\beta_1) - K_1(2\omega i - A)^{-1}B\left(\tilde{Q}_1 - \beta_1 Q_1 - \beta_2 Q_3^*\right)\right]\right\} \\
&\quad + G_H(K,K^*). &(3.76)
\end{aligned}
$$

The cases when $\alpha < 0$, $\alpha > 0$, and $\alpha = 0$ correspond to supercritical, subcritical, and degenerate Hopf bifurcations of the closed loop system. Thus designing a stabilizing feedback is equivalent to finding $K_j\,(j = 1, \cdots, 9)$ such that $\alpha = \operatorname{Re}\tilde{\alpha} < 0$. The cases when $\Theta_1 \neq 0$ or $\Theta_2 \neq 0$ have been discussed. We consider the following two cases:

(1) If $\Theta_3 \neq 0$, then let $K_4 = 0$, $K_7 = K_8 = 0$, $K_9 = 0$, and fix K such that

$K^T\Theta_3 \neq 0$. By selecting $K_5 = -\kappa\left[I + K_1 A^{-1} B\right]^{-T} \Theta_3^T K$ and letting $\kappa > 0$ large enough, we have $\alpha < 0$.

(2) If $\Theta_4 \neq 0$, then let $K_5 = 0$, $K_7 = K_8 = 0$, $K_9 = 0$, and fix K such that $K^T\Theta_4 \neq 0$. By selecting $K_5 = -\kappa\left[I - K_1(2\omega i - A)^{-1} B\right]^{-H} \Theta_4^H K$ and letting $\kappa > 0$ large enough, we have $\alpha < 0$.

(3) If $\Theta_3 = \Theta_4 = 0$, then $\alpha = G_H(K, K^*)$ is independent of K_j $(j = 4, \cdots, 9)$.

If $\Theta_3 = \Theta_4 = 0$ in addition to $\Theta_1 = \Theta_2 = 0$, then we have

$$\begin{aligned}
\tilde{\alpha} &= \tilde{G}_H(K, K^*) \\
&= \left[q_{13} + K^T\left(2B^T(i\omega - A)^{-T} q_{33} + q_{34}^T\right)\right](-A)^{-1}\left(\tilde{Q}_2 - \beta_1 Q_2 - \beta_2 Q_2^*\right) \\
&\quad + \left[q_{23} + K^H\left(2B^T(-i\omega - A)^{-T} q_{33} + q_{34}^T\right)\right]\left[2\omega i - A\right]^{-1}\left(\tilde{Q}_1 - \beta_1 Q_1 - \beta_2 Q_3^*\right) \\
&\quad - \frac{Q_1 Q_2}{i\omega} + W_3(K, K^*) + \text{p.i.t.}
\end{aligned}$$

It is clear that $\alpha = \operatorname{Re}\tilde{\alpha}$ is only a function of K and K^* since β_j, Q_j and $\tilde{Q}_j$, $j = 1, 2$, only depend on K and K^*. In the following we show that α is a third order polynomial of K and K^*.

By using the fact that $\Theta_j = 0$ $(j = 1, \cdots, 4)$, and the following equalities

$$\begin{aligned}
(-A)^{-1}(i\omega - A)^{-1} &= -\frac{1}{i\omega}(i\omega - A)^{-1} + \frac{1}{i\omega}(-A)^{-1}, \\
(-A)^{-1}(-i\omega - A)^{-1} &= \frac{1}{i\omega}(-i\omega - A)^{-1} - \frac{1}{i\omega}(-A)^{-1}, \\
(2\omega i - A)^{-1}(i\omega - A)^{-1} &= \frac{1}{i\omega}(i\omega - A)^{-1} - \frac{1}{i\omega}(2\omega i - A)^{-1}, \\
(2\omega i - A)^{-1}(-i\omega - A)^{-1} &= \frac{1}{3\omega i}(i\omega - A)^{-1} - \frac{1}{3\omega i}(2\omega i - A)^{-1},
\end{aligned}$$

it is straightforward to show that

$$\begin{aligned}
&\left[q_{13} + K^T\left(2B^T(i\omega - A)^{-T} q_{33} + q_{34}^T\right)\right](-A)^{-1}\beta_1 \\
&\quad = -\frac{1}{i\omega}\left[\Phi_1(i\omega)K + K^T\Psi(i\omega, i\omega)K\right] + \frac{\operatorname{Im}\Phi_1(0)}{\omega}K + \frac{1}{i\omega}\left[\Theta_1 + K^T\Theta_3\right]K,
\end{aligned}$$

$$\left[q_{13} + K^T \left(2B^T(i\omega - A)^{-T} q_{33} + q_{34}^T\right)\right] (-A)^{-1}\beta_2$$
$$= \frac{1}{i\omega}\left[\Phi_1(-i\omega)K^* + K^T\Psi(i\omega, -i\omega)K^*\right] - \frac{\operatorname{Im}\Phi_1(0)}{\omega}K^* - \frac{1}{i\omega}\left[\Theta_1 + K^T\Theta_3\right]K^*,$$

$$\left[q_{23} + K^H \left(2B^T(-i\omega - A)^{-T} q_{33} + q_{34}^T\right)\right] [2\omega i - A]^{-1}\beta_1$$
$$= \frac{1}{i\omega}\left[\Phi_2(i\omega)K + K^H\Psi(-i\omega, i\omega)K\right] - \frac{1}{i\omega}\left[\Theta_2 + K^H\Theta_4\right]K,$$

$$\left[q_{23} + K^H \left(2B^T(-i\omega - A)^{-T} q_{33} + q_{34}^T\right)\right] [2\omega i - A]^{-1}\beta_2$$
$$= \frac{1}{3\omega i}\left\{\Phi_2(-i\omega)K^* + K^H\Psi(-i\omega, -i\omega)K^*\right\} - \frac{1}{3\omega i}\left[\Theta_2 + K^H\Theta_4\right]K^*.$$

Now $\tilde{G}_H(K, K^*)$ can be calculated as

$$\tilde{G}_H(K, K^*) = H(K, K^*) + \tilde{P}(K, K^*), \tag{3.77}$$

where

$$\begin{aligned} H(K, K^*) &= \frac{1}{i\omega}\left(\Theta_1 + K^T\Theta_3\right)\left(K^*Q_2^* - KQ_2\right) + \\ &\quad \frac{1}{i\omega}\left(\Theta_2 + K^H\Theta_4\right)\left(KQ_1 + \frac{1}{3}K^*Q_3^*\right), \qquad (3.78) \\ \tilde{P}(K, K^*) &= C_0 + D_1K + D_2K^* + K^TE_{11}K + K^HE_{12}K + K^HE_{22}K^* \\ &\quad + F_{112}(K, K, K^*) + F_{122}(K, K^*, K^*) + \text{p.i.t.}, \qquad (3.79) \end{aligned}$$

and

$$\begin{aligned} C_0 &= c_{112} - \frac{q_{11}q_{12}}{i\omega} + q_{23}(2\omega i - A)^{-1}\tilde{q}_{11} + q_{13}(-A)^{-1}\tilde{q}_{12}, \\ D_1 &= c_{123}(i\omega - A)^{-1}B + c_{124} + q_{13}(-A)^{-1}\tilde{\Phi}_2(i\omega) + q_{23}(2\omega i - A)^{-1} \cdot \\ &\quad \tilde{\Phi}_1(i\omega) + \tilde{q}_{12}^T(-A)^{-T}\left[2q_{33}(i\omega - A)^{-1}B + q_{34}\right] - \frac{2}{i\omega}q_{11}\Phi_2(i\omega), \end{aligned}$$

$$
\begin{aligned}
D_2 &= c_{113}(-i\omega - A)^{-1}B + c_{114} + q_{13}(-A)^{-1}\tilde{\Phi}_1(i\omega) + \\
&\quad \tilde{q}_{11}^T(2\omega i - A)^{-T}\left[2q_{33}(-i\omega - A)^{-1}B + q_{34}\right] - \\
&\quad \frac{1}{i\omega}\left(q_{11} + q_{12}^*\right)\Phi_1(-i\omega) - \frac{1}{3\omega i}q_{22}^*\Phi_2(-i\omega), \\
E_{11} &= B^T(i\omega - A)^{-T}c_{233}(i\omega - A)^{-1}B + B^T(i\omega - A)^{-T}c_{234} \\
&\quad + c_{244} + \left[2B^T(i\omega - A)^{-T}q_{33} + q_{34}^T\right](-A)^{-1}\tilde{\Phi}_2(i\omega) + \\
&\quad \frac{1}{2}q_{23}(2\omega i - A)^{-1}\tilde{\Psi}(i\omega, i\omega) + \\
&\quad \frac{1}{2\omega i}\left[q_{12}\Psi(i\omega, i\omega) - 2\Phi_1^T(i\omega)\Phi_2(i\omega)\right], \\
E_{12} &= 2B^T(-i\omega - A)^{-T}c_{133}(i\omega - A)^{-1}B + \\
&\quad c_{134}^T(i\omega - A)^{-1}B + B^T(-i\omega - A)^{-T}c_{134} + 2c_{144} \\
&\quad + \tilde{\Phi}_1^T(-i\omega)(-A)^{-T}\left[2q_{33}(i\omega - A)^{-1}B + q_{34}\right] + \\
&\quad \left[2B^T(-i\omega - A)^{-T}q_{33} + q_{34}^T\right](2\omega i - A)^{-1}\tilde{\Phi}_1(i\omega) + \\
&\quad q_{13}(-A)^{-1}\tilde{\Psi}(i\omega, -i\omega) - \frac{1}{i\omega}(2q_{11} + q_{12}^*)\Psi(-i\omega, i\omega), \\
E_{22} &= -\frac{1}{i\omega}\Phi_1^T(-i\omega)\Phi_2^*(-i\omega) - \frac{1}{3\omega i}q_{22}^*\Psi(-i\omega, -i\omega), \\
F_{112}(K, K, K^*) &= 3c_{333}((i\omega - A)^{-1}BK, (i\omega - A)^{-1}BK, (-i\omega - A)^{-1}BK^*) \\
&\quad + c_{334}((i\omega - A)^{-1}BK, (i\omega - A)^{-1}BK, K^*) \\
&\quad + 2c_{334}((i\omega - A)^{-1}BK, (-i\omega - A)^{-1}BK^*, K) \\
&\quad + 2c_{344}((i\omega - A)^{-1}BK, K, K^*) \\
&\quad + c_{344}((-i\omega - A)^{-1}BK^*, K, K) + 3c_{444}(K, K, K^*) + \\
&\quad K^T\left[2B^T(i\omega - A)^{-T}q_{33} + q_{34}^T\right](-A)^{-1}\cdot\tilde{\Psi}(i\omega, -i\omega)(K, K^*) \\
&\quad + \frac{1}{2}K^H\left[2B^T(-i\omega - A)^{-T}q_{33} + q_{34}^T\right](2\omega i - A)^{-1}\cdot \\
&\quad \tilde{\Psi}(i\omega, i\omega)(K, K) + \frac{1}{2\omega i}\Phi_1(-i\omega)K^*\cdot K^T\Psi(i\omega, i\omega)K \\
&\quad - \frac{1}{i\omega}\left[\Phi_1(i\omega) + \Phi_1^*(i\omega)\right]K\cdot K^H\Psi(-i\omega, i\omega)K \\
&\quad - \frac{1}{6\omega i}\Phi_2(-i\omega)K^*\cdot K^T\Psi^*(i\omega, i\omega)K,
\end{aligned}
$$

$$
\begin{aligned}
F_{122}(K, K^*, K^*) &= -\frac{1}{i\omega}\Phi_1(-i\omega)K^* \cdot K^H\Psi^*(-i\omega, i\omega)K \\
&\quad -\frac{1}{i\omega}\Phi_2^*(-i\omega)K^* \cdot K^H\Psi(-i\omega, i\omega)K \\
&\quad -\frac{1}{3\omega i}\Phi_2^*(i\omega)K \cdot K^H\Psi(-i\omega, -i\omega)K^*,
\end{aligned}
$$

Define

$$
\begin{aligned}
\alpha_0 &= \operatorname{Re} C_0. \qquad (3.80) \\
\alpha_1^1 &= \operatorname{Re}\{D_1 + D_2\}, \\
\alpha_1^2 &= \operatorname{Im}\{D_2 - D_1\}, \\
\alpha_2^{11} &= \frac{1}{2}\left[\operatorname{Re}\{E_{11} + E_{12} + E_{22}\} + \operatorname{Re}\{(E_{11} + E_{12} + E_{22})^T\}\right], \\
\alpha_2^{12} &= \frac{1}{2}\left[\operatorname{Im}\{E_{22} - E_{11}\} + \operatorname{Im}\{(E_{22} - E_{11})^T\}\right], \\
\alpha_2^{22} &= \frac{1}{2}\left[\operatorname{Re}\{E_{12} - E_{11} - E_{22}\} + \operatorname{Re}\{(E_{12} - E_{11} - E_{22})^T\}\right], \\
\alpha_{3a}^{111}(K_R, K_R, K_R) &= \operatorname{Re}\{F_{112} + F_{122}\}(K_R, K_R, K_R), \\
\alpha_{3a}^{112}(K_R, K_R, K_I) &= \operatorname{Im}\{F_{112} + F_{122}\}(K_R, K_R, K_I) \\
&\quad - \operatorname{Im}\{F_{112} - F_{122}\}(K_R, K_I, K_R) \\
&\quad - \operatorname{Im}\{F_{112} + F_{122}\}(K_I, K_R, K_R), \\
\alpha_{3a}^{122}(K_R, K_I, K_I) &= \operatorname{Re}\{F_{112} + F_{122}\}(K_R, K_I, K_I) \\
&\quad + \operatorname{Re}\{F_{112} - F_{122}\}(K_I, K_R, K_I) \\
&\quad - \operatorname{Re}\{F_{112} + F_{122}\}(K_I, K_I, K_R), \\
\alpha_{3a}^{222}(K_I, K_I, K_I) &= \operatorname{Im}\{F_{122} - F_{112}\}(K_I, K_I, K_I),
\end{aligned}
$$

and

$$
\alpha_1 = \begin{bmatrix} \alpha_1^1 & \alpha_1^2 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} \alpha_2^{11} & \alpha_2^{12} \\ \alpha_2^{12} & \alpha_2^{22} \end{bmatrix}. \qquad (3.81)
$$

Define $K_\flat = \begin{bmatrix} K_R \\ K_I \end{bmatrix}$, and

$$\begin{aligned}\alpha_{3a}(K_\flat, K_\flat, K_\flat) &= \alpha_{3a}^{111}(K_R, K_R, K_R) + \alpha_{3a}^{112}(K_R, K_R, K_I) + \\ &\quad \alpha_{3a}^{122}(K_R, K_I, K_I) + \alpha_{3a}^{222}(K_I, K_I, K_I).\end{aligned}$$

Let α_3 as the symmetrization of α_{3a}, i.e., for any $i, j, k \in (1, \cdots, 2m)$, we have

$$\alpha_3^{ijk} = \frac{1}{6}\left(\alpha_{3a}^{ijk} + \alpha_{3a}^{ikj} + \alpha_{3a}^{jik} + \alpha_{3a}^{jki} + \alpha_{3a}^{kij} + \alpha_{3a}^{kji}\right).$$

Define $P_H(K_\flat) := \operatorname{Re} \tilde{P}_H(K, K^*)$, then we get

$$P_H(K_\flat) = \alpha_0 + \alpha_1 K_\flat + K_\flat^T \alpha_2 K_\flat + \alpha_3(K_\flat, K_\flat, K_\flat). \tag{3.82}$$

Define $\alpha = \operatorname{Re} \tilde{\alpha}$, then from (3.74), (3.75), (3.76), (3.77), (3.78), (3.79), we get

$$\begin{aligned}\alpha &= \left(\Theta_1 + \operatorname{Re}\left\{K^T\Theta_3\right\}\right)\left[I + K_1A^{-1}B\right]^{-1}\Big[K_5 + K_7\beta_2 + K_8\beta_1 + 2K_9(\beta_1, \beta_2) \\ &\quad + K_1A^{-1}B\left(\tilde{Q}_2 - \beta_1 Q_2 - \beta_2 Q_2^*\right) + \frac{1}{i\omega}\left(I + K_1A^{-1}B\right)\left(K^*Q_2^* - KQ_2\right)\Big] \\ &\quad + \operatorname{Re}\Big\{\left(\Theta_2 + K^H\Theta_4\right)\left[I - K_1(2\omega i - A)^{-1}B\right]^{-1}\Big[K_4 + K_7\beta_1 \\ &\quad + K_9(\beta_1, \beta_1) - K_1(2\omega i - A)^{-1}B\left(\tilde{Q}_1 - \beta_1 Q_1 - \beta_2 Q_3^*\right) \\ &\quad + \frac{1}{i\omega}\left[I - K_1(2\omega i - A)^{-1}B\right]\left(KQ_1 + \frac{1}{3}K^*Q_3^*\right)\Big]\Big\} \\ &\quad + \left[\alpha_0 + \alpha_1 K_\flat + K_\flat^T\alpha_2 K_\flat + \alpha_3(K_\flat, K_\flat, K_\flat)\right],\end{aligned} \tag{3.83}$$

The objective is to find feedback gains such that $\alpha < 0$. We have discussed the cases when $\Theta_j \neq 0$ for $j = 1, \cdots, 4$. When $\Theta_j = 0$ $(j = 1, \cdots, 4)$, then α is expressed as

$$\alpha = P_H(K_\flat) = \alpha_0 + \alpha_1 K_\flat + K_\flat^T\alpha_2 K_\flat + \alpha_3(K_\flat, K_\flat, K_\flat). \tag{3.84}$$

$K_\flat \in \mathbb{R}^{2m}$, such that $\alpha < 0$ in (3.76). The goal is to select $K_\flat$ such that $\alpha < 0$, we have the following cases:

(1) If $\alpha_3 \neq 0$, then there is a $K_\flat^0$ such that $\alpha_3(K_\flat^0, K_\flat^0, K_\flat^0) \neq 0$. Letting

$$K_\flat = -\kappa \operatorname{sgn}\left\{\alpha_3(K_\flat^0, K_\flat^0, K_\flat^0)\right\},$$

where $\kappa \in \mathbb{R}$ and $\kappa > 0$, then

$$\alpha = \alpha_0 - \kappa \alpha_1 K_\flat^0 \operatorname{sgn}\left\{\alpha_3(K_\flat^0, K_\flat^0, K_\flat^0)\right\} + \kappa^2 K_\flat^{0T} \alpha_2 K_\flat^0 - \kappa^3 \left|\alpha_3(K_\flat^0, K_\flat^0, K_\flat^0)\right|.$$

Apparently, α can be made negative when κ increases.

(2) If $\alpha_3 < 0$, then

$$\alpha = \alpha_0 + \alpha_1 K_\flat + K_\flat^T \alpha_2 K_\flat.$$

Letting $K_\flat = U\xi$, where $U \in \mathbb{R}^{2m \times 2m}$ is an orthonormal matrix such that, $U^T \alpha_2 U$ is diagonal. Define

$$\begin{aligned} \hat{\alpha}_1 &:= \alpha_1 U = \left[\begin{array}{cccc} \hat{\alpha}_1^1 & \hat{\alpha}_1^2 & \cdots & \hat{\alpha}_1^{2m} \end{array}\right], \\ \hat{\alpha}_2 &:= U^T \alpha_2 U = \operatorname{Diag}\left[\begin{array}{cccc} \hat{\alpha}_2^1, & \hat{\alpha}_2^2, & \cdots, & \hat{\alpha}_2^{2m} \end{array}\right], \end{aligned}$$

Define $\underline{2\mathrm{m}} = \{1, 2, \cdots, 2m\}$, and

$$I_1 = \{j \in \underline{2\mathrm{m}}\,;\ \hat{\alpha}_2^j > 0\}, \tag{3.85}$$

$$I_2 = \{j \in \underline{2\mathrm{m}}\,;\ \hat{\alpha}_2^j < 0\}, \tag{3.86}$$

$$I_3 = \{j \in \underline{2\mathrm{m}}\,;\ \hat{\alpha}_2^j = 0,\ \hat{\alpha}_1^j \neq 0\}, \tag{3.87}$$

$$I_4 = \{j \in \underline{2\mathrm{m}}\,;\ \hat{\alpha}_2^j = 0,\ \hat{\alpha}_1^j = 0\}, \tag{3.88}$$

$$\hat{\alpha}_0 = \alpha_0 - \sum_{j \in I_1} \frac{\left(\hat{\alpha}_1^j\right)^2}{4\hat{\alpha}_2^j}. \tag{3.89}$$

Then

$$\begin{aligned}
\alpha &= \alpha_0 + \alpha_1 K_\flat + K_\flat^T \alpha_2 K_\flat \\
&= \alpha_0 + \hat{\alpha}_1 \xi + \xi^T \hat{\alpha}_2 \xi \\
&= \alpha_0 + \sum_{j=0}^{2m} \hat{\alpha}_1^j \xi_j + \sum_{j=0}^{2m} \hat{\alpha}_2^j \xi_j^2 \\
&= \hat{\alpha}_0 + \sum_{j \in I_1} \hat{\alpha}_2^j \left(xi_j + \frac{\hat{\alpha}_1^j}{2\hat{\alpha}_2^j} \right)^2 + \sum_{j \in I_2} (\hat{\alpha}_1^j \xi_j + \hat{\alpha}_2^j \xi_j^2) + \sum_{j \in I_3} \hat{\alpha}_1^j \xi_j,
\end{aligned}$$

where

$$\hat{\alpha}_0 = \alpha_0 - \sum_{j \in I_1} \frac{\left(\hat{\alpha}_1^j\right)^2}{4\hat{\alpha}_2^j}.$$

We have the following cases:

(i) If $I_2 \cup I_3 \neq \emptyset$, then α can be made negative by letting $|\xi_j|$ large enough for $j \in I_2 \cup I_3$.

(ii) If $I_2 \cup I_3 = \emptyset$, then we have

$$\min_{\xi \in \mathbb{R}^{2m}} \alpha = \hat{\alpha}_0,$$

where the minimum is achieved when $\xi_j = -\frac{\hat{\alpha}_1^j}{2\hat{\alpha}_2^j}$ for $j \in I_1$. It is clear that when $\hat{\alpha}_0 < 0$, the system is stabilizable; when $\hat{\alpha}_0 > 0$, the system is unstabilizable; when $\hat{\alpha}_0 = 0$, the system is degenerate.

Corollary 3.2

Now we proceed to prove the corollary. The Taylor series expansion of (3.31) around $(y, u) = (0, 0)$ at $\mu = 0$ is given by

$$\dot{y} = \bar{A}y + \bar{B}u + \bar{q}_{11}(y, y) + 2\bar{q}_{12}(y, u) + \bar{q}_{22}(u, u) + \bar{c}(w, w, w) + \text{h.o.t.}, \qquad (3.90)$$

where $w := \begin{bmatrix} y \\ u \end{bmatrix}$, and

$$\bar{A} := D_1 f(0,0), \qquad \bar{B} := D_2 f(0,0), \qquad \bar{q}_{11} = \frac{1}{2} D_{11}^2 f(0,0),$$

$$\bar{q}_{12} = \frac{1}{2} D_{12}^2 f(0,0), \quad \bar{q}_{22} = \frac{1}{2} D_{22}^2 f(0,0), \quad \bar{c} = \frac{1}{6} D^3 f(0,0),$$

where D_1 is differentiation with respect to y, D_2 is differentiation with respect to u, and D is differentiation with respect to w. Now let

$$y = T \begin{bmatrix} z \\ z^* \\ \tilde{x} \end{bmatrix},$$

where

$$T = [r \;\; r^* \;\; r_\perp], \quad T^{-1} = \begin{bmatrix} l \\ l^* \\ l_\perp \end{bmatrix},$$

satisfying

$$T^{-1}\bar{A}T = \begin{bmatrix} i\omega & 0 & 0 \\ 0 & -i\omega & 0 \\ 0 & 0 & A \end{bmatrix}, \qquad T\bar{B} = \begin{bmatrix} 0 \\ 0 \\ B \end{bmatrix}, \tag{3.91}$$

and $l_\perp \in \mathbb{R}^{n\times(n+2)}$, $r_\perp \in \mathbb{R}^{(n+2)\times n}$, and $A \in \mathbb{R}^{n\times n}$. With the linear transformation, (3.90) is transformed into the following equations:

$$\begin{aligned} \dot{z} &= i\omega z + (l\cdot\bar{q}_{11})(y,y) + (l\cdot\bar{q}_{12})(y,u) + (l\cdot\bar{q}_{22})(u,u) + l\cdot\bar{c}(w,w,w) + \text{h.o.t.} \\ \dot{\tilde{x}} &= A\tilde{x} + Bu + (l_\perp\cdot\bar{q}_{11})(y,y) + (l_\perp\cdot\bar{q}_{12})(y,u) + (l_\perp\cdot\bar{q}_{22})(u,u) + l_\perp\bar{c}(w,w,w) \\ &\quad +\text{h.o.t.} \end{aligned}$$

By comparing with system (3.32) and (3.33), we have

$$z = ly, \quad z^* = l^* y, \quad \tilde{x} = l_\perp y \quad y = rz + r^* z + r_\perp z,$$

and

$$\begin{array}{lll}
q_{11} = r^T (l \cdot \bar{q}_{11}) r, & q_{12} = 2r^T (l \cdot \bar{q}_{11}) r^*, & q_{13} = 2r^T (l \cdot \bar{q}_{11}) r_\perp, \\
q_{14} = 2r^T (l \cdot \bar{q}_{12}), & q_{22} = r^H (l \cdot \bar{q}_{11}) r^*, & q_{23} = 2r^H (l \cdot \bar{q}_{11}) r_\perp, \\
q_{24} = 2r^H (l \cdot \bar{q}_{12}), & q_{33} = r_\perp^T (l \cdot \bar{q}_{11}) r_\perp, & q_{34} = 2r_\perp^T (l \cdot \bar{q}_{12}), \\
q_{44} = l \cdot \bar{q}_{12}, & \tilde{q}_{11} = l_\perp \cdot \bar{q}_{11}(r, r), & \tilde{q}_{12} = 2l_\perp \cdot \bar{q}_{11}(r, r^*), \\
\tilde{q}_{13} = 2l_\perp \cdot \bar{q}_{11}(r, r_\perp), & \tilde{q}_{14} = 2l_\perp \cdot \bar{q}_{12}(r, Id), & \tilde{q}_{22} = l_\perp \cdot \bar{q}_{11}(r^*, r^*), \\
\tilde{q}_{23} = 2l_\perp \cdot \bar{q}_{11}(r^*, r_\perp), & \tilde{q}_{24} = 2l_\perp \cdot \bar{q}_{12}(r, Id), & \tilde{q}_{33} = l_\perp \cdot \bar{q}_{11}(r_\perp, r_\perp), \\
\tilde{q}_{34} = 2l_\perp \cdot \bar{q}_{12}(r_\perp, Id), & \tilde{q}_{44} = 2l_\perp \cdot \bar{q}_{12}, & \text{etc},
\end{array}$$

where Id is the m by m identity matrix. Define

$$\Sigma(\lambda_1, \lambda_2) := \begin{bmatrix} 0 & \bar{A} - \lambda_1 I & \bar{B} \\ (\bar{A} - \lambda_2 I)^T & Q_{11} & Q_{12} \\ \bar{B}^T & Q_{12}^T & Q_{22} \end{bmatrix}$$

where

$$\begin{aligned}
Q_{11} &= l \cdot \bar{q}_{11} - \frac{1}{2}\left[(rl)^T (l \cdot \bar{q}_{11})(r_\perp l_\perp) + (r_\perp l_\perp)^T (l \cdot \bar{q}_{11})(rl)\right] \\
&\quad + \frac{1}{2}\left[(r^* l)^T (l^* \cdot \bar{q}_{11})(r_\perp l_\perp) + (r_\perp l_\perp)^T (l^* \cdot \bar{q}_{11})(r^* l)\right], \\
Q_{12} &= l \cdot \bar{q}_{12} - \frac{1}{2}(rl)^T (l \cdot \bar{q}_{12}) + \frac{1}{2}(r^* l)^T (l^* \cdot \bar{q}_{12}), \\
Q_{22} &= l \cdot \bar{q}_{22},
\end{aligned}$$

Using the relations in (3.91), it is straightforward to check in the $(z, z^*, \tilde{x}, u)$ coordinate, $\Sigma(\lambda_1, \lambda_2)$ is given by

$$\left[\begin{array}{ccc|cccc} 0 & 0 & 0 & i\omega-\lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i\omega-\lambda_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A-\lambda_1 I & B \\ \hline i\omega-\lambda_2 & 0 & 0 & q_{11} & \frac{1}{2}q_{12} & \frac{1}{2}\operatorname{Re} q_{13} & \frac{1}{2}\operatorname{Re} q_{14} \\ 0 & -i\omega-\lambda_2 & 0 & \frac{1}{2}q_{12} & q_{22} & \frac{1}{2}q_{23} & \frac{1}{2}q_{24} \\ 0 & 0 & (A-\lambda_2 I)^T & \frac{1}{2}\operatorname{Re} q_{13}^T & \frac{1}{2}q_{23}^T & q_{33} & \frac{1}{2}q_{34} \\ 0 & 0 & B^T & \frac{1}{2}\operatorname{Re} q_{14}^T & \frac{1}{2}q_{24}^T & \frac{1}{2}q_{34}^T & \frac{1}{2}q_{44} \end{array}\right] \tag{3.92}$$

Now by substituting $\lambda_1 = 0$, $\lambda_2 = i\omega$ into (3.92), and applying elementary transforms to rows and columns, we get

$$\hat{\Sigma}(0, i\omega) = \left[\begin{array}{ccc|cccc} 0 & 0 & 0 & i\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \Theta_1 \\ 0 & -2\omega i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (A-i\omega)^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Theta_3 \end{array}\right],$$

where

$$\begin{aligned} \Theta_1 &= \operatorname{Re}\left\{q_{14} + q_{13}(-A)^{-1}B\right\}, \\ \Theta_3 &= 2B^T(i\omega - A)^{-T}q_{33}(-A)^{-1}B + q_{34}^T(-A)^{-1}B + B^T(i\omega - A)^{-T}q_{34} + 2q_{44}. \end{aligned}$$

It can be easily seen that rank $\hat{\Sigma}(0, i\omega) \geq 2(n+2)$ if and only if $\Theta_1 \neq 0$ or $\Theta_3 \neq 0$.

Similarly, by substituting $\lambda_1 = -i\omega$, $\lambda_2 = 2\omega i$ into (3.92), and applying elementary transforms to rows and columns, we get

$$\hat{\Sigma}(-i\omega, 2\omega i) = \left[\begin{array}{ccc|cccc} 0 & 0 & 0 & 2\omega i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A + i\omega & 0 \\ \hline -i\omega & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\omega i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (A - 2\omega i)^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Theta_2^T & 0 & \Theta_4^T \end{array}\right],$$

where

$$\begin{aligned} \Theta_2 &= q_{14} + q_{13}(2\omega i - A)^{-1}B, \\ \Theta_4 &= 2B^T(-i\omega - A)^{-T}q_{33}(2\omega i - A)^{-1}B + q_{34}^T(2\omega i - A)^{-1}B \\ &\quad + B^T(-i\omega - A)^{-T}q_{34} + 2q_{44}. \end{aligned}$$

It can be easily seen that rank $\hat{\Sigma}(-i\omega, 2\omega i) \geq 2(n+2)$ if and only if $\Theta_2 \neq 0$ or $\Theta_4 \neq 0$.

Chapter 4 Effects of Magnitude Saturation in Control of Bifurcations

Motivated by problems such as active control of rotating stall in compression systems, an analysis of the effects of controller magnitude saturation in feedback stabilization of steady-state bifurcations is performed. In particular the region of attraction to the stabilized bifurcated equilibria is solved for feedback controllers with magnitude saturation limits using the technique of center manifold reduction and bifurcation analysis. It has been shown that the stability boundary is the saturation envelope formed by the unstable equilibria for the closed loop system when the controllers saturate. The framework allows the design of feedback control laws to achieve a desirable size of the region of attraction when the noise is modeled as a closed set of initial conditions in the phase space. Modulo the phase of the limit cycles, the qualitative behavior in the Hopf bifurcation case is the same as some cases in the steady-state bifurcations. This chapter is based on the paper [71].

Although much research has been done on bifurcation control, no previous research has addressed the role of controller magnitude and rate saturation. The results in this chapter is the first to give qualitative analysis on the effects of magnitude saturation in bifurcation control.

4.1 Steady-State Bifurcations

In this section we use the same notation as Section 2.1. Consider the following single-input system

$$\dot{y} = f_\mu(y, u), \tag{4.1}$$

where $y \in \mathbb{R}^{n+1} (n \geq 1)$ is the state variable, $\mu \in \mathbb{R}$ is a bifurcation parameter, and $u \in \mathbb{R}$ is the control input. We assume that the vector field is sufficiently smooth and the system has a steady-state solution $y_0(\mu)$. $\lambda(\mu)$ is a simple real eigenvalue of the linearization around $y_0(\mu)$ and the eigenspace associated with $\lambda(\mu)$ is the only linearly uncontrollable mode. Under these assumptions, we use Taylor series expansion followed by a linear transformation to decouple the uncontrollable mode from the controllable modes. The resulting standard form is given by

$$\begin{aligned}
\dot{x} &= d\mu x + q_{11}x^2 + q_{12}\tilde{x}x + q_{13}xu + \tilde{x}^T q_{22}\tilde{x} + q_{23}\tilde{x}u + q_{33}u^2 \\
&\quad + c_{111}x^3 + c_{112}\tilde{x}x^2 + c_{113}x^2u + \tilde{x}^T c_{122}\tilde{x}x + c_{123}\tilde{x}ux + c_{133}xu^2 \\
&\quad + c_{222}(\tilde{x},\tilde{x},\tilde{x}) + \tilde{x}^T c_{223}\tilde{x}u + c_{233}\tilde{x}u^2 + c_{333}u^3 + \cdots, \qquad (4.2) \\
\dot{\tilde{x}} &= A\tilde{x} + Bu + \tilde{q}_{11}x^2 + \tilde{q}_{12}\tilde{x}x + \tilde{q}_{13}xu + \tilde{q}_{22}(\tilde{x},\tilde{x}) + \tilde{q}_{23}\tilde{x}u + \tilde{q}_{33}u^2 + \cdots, \qquad (4.3)
\end{aligned}$$

where $x \in \mathbb{R}$, $\tilde{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times 1}$, and (A, B) is in controller canonical form. The state x represents the uncontrollable mode and the vector $\tilde{x}$ represents the controllable modes.

Let $e_n = [\begin{array}{cccc} 0 & \cdots & 0 & 1 \end{array}]^T \in \mathbb{R}^n$. In the following, we denote $q_{12}^n = q_{12}e_n$, $q_{22}^{nn} = e_n^T q_{22} e_n$, etc. Define

$$\begin{aligned}
\Upsilon_1 &= q_{12}^n + q_{13}a_n, \qquad (4.4) \\
\Upsilon_2 &= q_{22}^{nn} + q_{23}^n a_n + q_{33}a_n^2, \qquad (4.5) \\
\Pi_1^S &= [\begin{array}{cccc} 0 & q_{12}^1 & \cdots & q_{12}^{n-1} \end{array}] + q_{13}[\begin{array}{cccc} 1 & a_1 & \cdots & a_{n-1} \end{array}], \qquad (4.6) \\
\Pi_2^S &= 2[\begin{array}{cccc} 0 & q_{22}^{n1} & \cdots & q_{22}^{n,n-1} \end{array}] + a_n[\begin{array}{cccc} 0 & q_{23}^1 & \cdots & q_{23}^{n-1} \end{array}] \\
&\quad + (q_{23}^n + 2q_{33}a_n)[\begin{array}{cccc} 1 & a_1 & \cdots & a_{n-1} \end{array}], \qquad (4.7) \\
\alpha_0 &= c_{111} - \Pi_1^S \tilde{q}_{11}, \qquad (4.8) \\
\alpha_1 &= c_{112}^n + c_{113}a_n - \Pi_1^S(\tilde{q}_{12}^n + \tilde{q}_{13}a_n) - \Pi_2^S \tilde{q}_{11}, \qquad (4.9) \\
\alpha_2 &= c_{122}^{nn} + c_{123}^n a_n + c_{133}a_n^2 - \Pi_1^S(\tilde{q}_{22}^{nn} + \tilde{q}_{23}^n a_n + \tilde{q}_{33}a_n^2) \\
&\quad - \Pi_2^S(\tilde{q}_{12}^n + \tilde{q}_{13}a_n), \qquad (4.10) \\
\alpha_3 &= c_{222}^{nnn} + c_{223}^{nn}a_n + c_{233}^n a_n^2 + c_{333}a_n^3 - \Pi_2^S(\tilde{q}_{22}^{nn} + \tilde{q}_{23}^n a_n + \tilde{q}_{33}a_n^2), \qquad (4.11)
\end{aligned}$$

	q_{11}	Υ_1	Υ_2	Δ_Υ	α_3	α_2	α_1	α_0	Δ_α	$P_S(\rho)$
SS-1	$=0$							<0		
SS-2	$=0$	$\neq 0$						≥ 0		
SS-3	$=0$	$=0$	$=0$		$\neq 0$			≥ 0		
SS-4	$=0$	$=0$	$=0$		$=0$	<0		≥ 0		
SS-5	$=0$	$=0$	$=0$		$=0$	$=0$	$\neq 0$	≥ 0		
SS-6	$=0$	$=0$	$=0$		$=0$	>0		≥ 0	>0	
SS-7	$\neq 0$	$\neq 0$	$=0$							
SS-8	$\neq 0$		$\neq 0$	>0						
SS-9	$\neq 0$		$\neq 0$	$=0$						<0

Table 4.1: Different cases when steady-state bifurcations are stabilizable.

	K	K_3
SS-1	0	$-[\,1 \quad \bar{a}_1 \quad \cdots \quad \bar{a}_{n-1}\,]\tilde{q}_{11}$
SS-2	0	$\alpha_0 + \frac{\Upsilon_1}{\bar{a}_n}\left(K_3 + [\,1 \quad \bar{a}_1 \quad \cdots \quad \bar{a}_{n-1}\,]\tilde{q}_{11}\right) < 0$
SS-3	$P_S(K) < 0$	0
SS-4	$P_S(K) < 0$	0
SS-5	$P_S(K) < 0$	0
SS-6	$-\frac{\alpha_1}{2\alpha_2}$	0
SS-7	$-\frac{q_{11}}{\Upsilon_1}$	$\Upsilon_1\frac{K_3}{\bar{a}_n} + \tilde{\alpha}_0 + P_S(K) < 0$
SS-8	$\frac{1}{2\Upsilon_2}\left(-\Upsilon_1 + \sqrt{\Delta_\Upsilon}\right)$	$\frac{K_3}{\bar{a}_n}\sqrt{\Delta_\Upsilon} + \hat{\alpha}_0 + P_S(K) < 0$
SS-9	$-\frac{\Upsilon_1}{2\Upsilon_2}$	0

Table 4.2: Construction of stabilizing feedback for steady-state bifurcations.

$$\begin{aligned} P_S(K) &= \alpha_0 + \alpha_1 K + \alpha_2 K^2 + \alpha_3 K^3. \qquad (4.12)\\ \Delta_\Upsilon &= \Upsilon_1^2 - 4q_{11}\Upsilon_2,\\ \Delta_\alpha &= \alpha_1^2 - 4\alpha_2\alpha_0,\\ \rho &= -\frac{\Upsilon_1}{2\Upsilon_2}, \end{aligned}$$

where $p(s) = \det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$.

Under some nondegeneracy conditions, there exists a sufficiently smooth feedback

$$u = F(\tilde{x}, x) = K_1\tilde{x} + K_2 x + K_3 x^2 + K_4\tilde{x}x + \tilde{x}^T K_5 \tilde{x} + \text{h.o.t.}, \qquad (4.13)$$

such that $(0,0)$ is asymptotically stable if and only if one of the conditions in Table 4.1 is satisfied. The construction of the stabilizing feedbacks is given in Table 4.2, where $K = \frac{K_2}{\bar{a}_n}$, and

$$\begin{aligned}
\tilde{\alpha}_0 &= \frac{\Upsilon_1}{\bar{a}_n}\left[\begin{array}{cccc} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1} \end{array}\right]\tilde{Q}(K), \\
\hat{\alpha}_0 &= \frac{1}{\bar{a}_n}\sqrt{\Delta_\Upsilon}\left[\begin{array}{cccc} 1 & \bar{a}_1 & \cdots & \bar{a}_{n-1} \end{array}\right]\tilde{Q}(K), \\
\tilde{Q}(K) &= \tilde{q}_{11} + (\tilde{q}_{12}^n + \tilde{q}_{13}a_n)K + (\tilde{q}_{22}^{nn} + \tilde{q}_{23}^n a_n + \tilde{q}_{33}a_n^2)K^2,
\end{aligned} \tag{4.14}$$

where $\det[sI-(A+BK_1)] = s^n+\bar{a}_1 s^{n-1}+\cdots+\bar{a}_{n-1}s+\bar{a}_n$. The construction procedure consists of the following two steps: First, choose K_1 such that $A+BK_1$ is Hurwitz, then let $K_4 = 0$, $K_5 = 0$, and choose K_2, K_3 as shown in Table 4.2.

We consider the system (4.1) with state feedback $u = \alpha(y)$ constrained by controller magnitude limits:

$$u = \sigma(\alpha(y)) := \begin{cases} u_{mag}, & \text{if } \alpha(y) \geq u_{mag}, \\ \alpha(y), & \text{if } |\alpha(y)| < u_{mag}, \\ -u_{mag}, & \text{if } \alpha(y) \leq -u_{mag}. \end{cases}$$

By the same procedure in the previous section, we transform the system into a standard form (4.2) and (4.3) with $u = \sigma(F(\tilde{x}, x))$, where $F(\tilde{x}, x)$ is given by (4.13). For the simplicity of discussion, we make the following additional assumption in addition to **AS-1** to **AS-4** in Chapter 2:

AS-5 A is Hurwitz for all values of μ in a neighborhood of $\mu = 0$.

From the construction procedure in Chapter 2 it is sufficient to only consider the state feedback on the linearly uncontrollable mode x, i.e., $K_1 = 0$, $\bar{a}_j = a_j$ $(j = 1, \cdots, n)$, and $u = \sigma(F(x))$.

The goal in this section is to analyze the qualitative bifurcation behavior of the closed loop system under the constraints of controller magnitude saturation. We approach this problem in the following way: first, we derive the dynamics on the center manifold for the closed loop system without controller magnitude saturation,

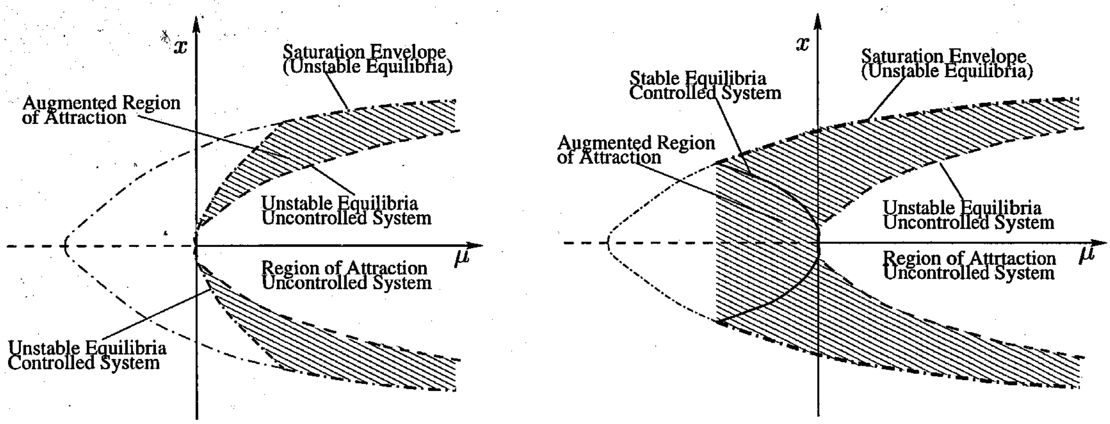


Figure 4.1: Bifurcation diagrams of case **SS-2** for two different gains.

then we add saturation limits to the dynamics on the center manifold. By doing so, we could estimate the region of attraction for the stabilized equilibria. The details of the center manifold reduction can be found in Chapter 2. We analyze the effects of magnitude limits case by case. For **SS-1**, the bifurcation for the uncontrolled system is already supercritical. For **SS-2**, the dynamics on the center manifold is given by

$$\begin{aligned} \dot{x} &= d\mu x + b_1 x u + b_2 x^3, \\ u &= \sigma\left(K_3 x^2\right) \end{aligned}$$

where $b_1 = \frac{\Upsilon_1}{a_n}$, and

$$b_2 = c_{111} + \frac{1}{a_n}\left(q_{12}^n \begin{bmatrix} 1 & a_1 & \cdots & a_{n-1} \end{bmatrix} - a_n \begin{bmatrix} 0 & q_{12}^1 & \cdots & q_{12}^{n-1} \end{bmatrix}\right) \tilde{q}_{11}.$$

Without loss of generality we assume $d < 0$ and $b_1 < 0$, then the bifurcation for the open loop system is a subcritical pitchfork bifurcation since $b_2 > 0$ (see Figure 4.1 (a)). If $K_3 > 0$ is small, then the bifurcation for closed loop system is given in Figure 4.1 (a). The bifurcation for the closed loop system is still a subcritical bifurcation, but the region of attraction to the nominal equilibria $x = 0$ is enlarged. The enlarged region is given by the shaded area in Figure 4.1 (a). If we increase K_3, then the bifurcation for the closed loop system is a supercritical pitchfork bifurcation (see Figure 4.1 (b)). Also, the region enlarged from control is given by the shaded region. The intersection between the stabilized equilibria and the saturation envelope signifies the operating limit in terms of the bifurcation parameter, i.e., any trajectory from a nonzero initial condition to the left of the intersection will go to infinity. As the gain K_3 goes to infinity, the saturation envelope is a closed curve shown in Figure 4.1 (b). The equation of the saturation envelope can be solved by calculating the equilibria when the controller saturates, i.e., when $u = u_{mag}$. The saturation envelope is given by

$$\mu = \frac{1}{d}\left(|b_1|u_{mag} + b_2x^2\right).$$

It is clear that any trajectory from a nonzero initial conditions will go to infinity if the system operates at $\mu < \mu^* := \frac{|b_1|}{d}u_{mag}$. In other words, the region between the saturation envelope and the unstable bifurcated equilibria of the uncontrolled system is the maximal augmented region of attraction due to the magnitude saturation limit.

For **SS-3**, the dynamics on the center manifold is given by

$$\begin{aligned} \dot{x} &= d\mu x + \alpha_0 x^3 + \alpha_1 x^2 u + \alpha_2 x u^2 + \alpha_3 u^3, \\ u &= \sigma(Kx) \end{aligned}$$

where α_j $(j = 0, \cdots, 3)$ are given by (4.8), (4.9), (4.10), and (4.11). The saturation envelope is given by

$$\mu = -\frac{1}{d}\left(\alpha_0 x^2 \pm \alpha_1 u_{mag} x + \alpha_2 u_{mag}^2 \pm \frac{\alpha_3 u_{mag}^3}{x}\right).$$

The bifurcation diagrams could be characterized by two cases and they are given in Figure 4.2. The interpretations to Figure 4.2 for the region of attraction are the same as the previous case. In the first figure one branch of the saturation envelope has at most one point at which $\frac{d\mu}{dx} = 0$, whereas in the second figure there are two points on the saturation envelope at which $\frac{d\mu}{dx} = 0$. The solid line parts on the saturation envelope in the second figure are stable equilibria. If $|K|$ is sufficiently large, this stable equilibrium branch attracts the initial conditions between the upper and lower

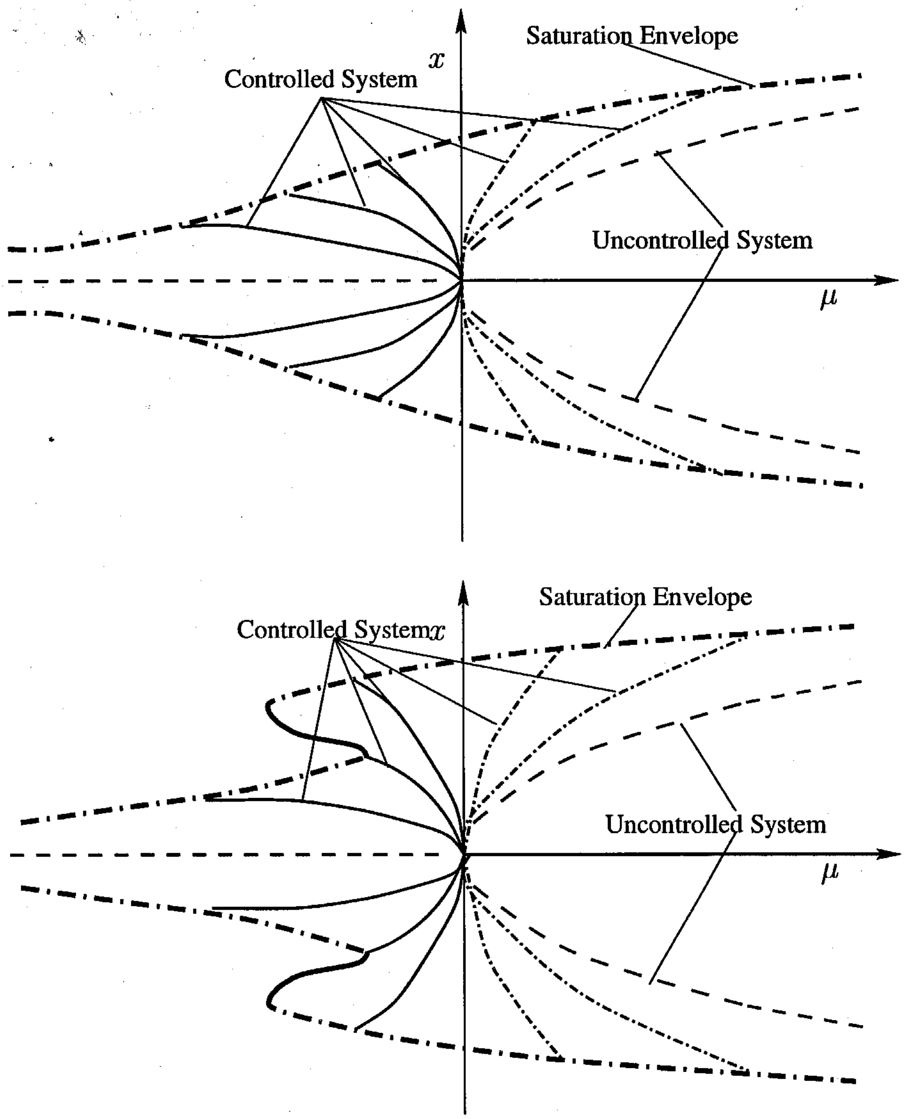


Figure 4.2: Bifurcation diagrams of case **SS-3** with different values of gain.

branch of the unstable equilibria on the saturation envelope. The difference between the saturation envelopes of **SS-2** and **SS-3** is that the saturation envelope of **SS-2** is closed but the saturation envelope of **SS-3** is open and goes off along the μ-axis. This implies that if the noise (i.e., initial conditions) is infinitely small, then **SS-3** will give larger region of stable operating range.

For **SS-4**, the dynamics on the center manifold is given by

$$\begin{aligned} \dot{x} &= d\mu x + \alpha_0 x^3 + \alpha_1 x^2 u + \alpha_2 x u^2, \\ u &= \sigma(Kx) \end{aligned}$$

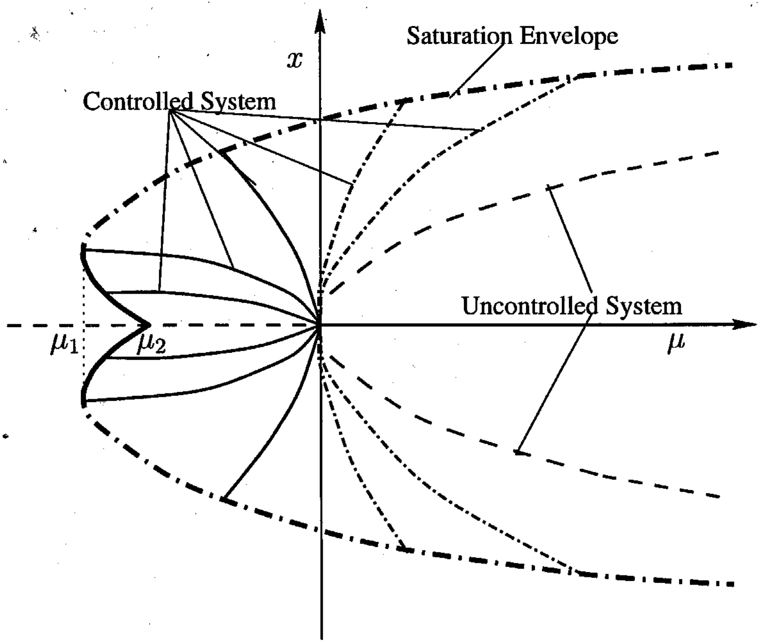


Figure 4.3: Bifurcation diagram of case **SS-4** with different values of gain.

where α_j $(j = 0, 1, 2)$ are given by (4.8), (4.9), and (4.10). The saturation envelope is given by

$$\mu = -\frac{1}{d}\left(\alpha_0 x^2 \pm \alpha_1 u_{mag} x + \alpha_2 u_{mag}^2\right).$$

The bifurcation diagram is given in Figure 4.3. The saturation envelope is a closed curve. μ_1 is the maximum of operating range that can be achieved, i.e., any trajectory from a nonzero initial condition of x diverges to infinity. Also the stable equilibria for $\mu \in [\mu_1, \mu_2]$ are determined by the controller saturation limit, so increasing gain will not make them arbitrarily closed to the nominal equilibria $x = 0$.

For **SS-5**, the dynamics on the center manifold is given by

$$\begin{aligned} \dot{x} &= d\mu x + \alpha_0 x^3 + \alpha_1 x^2 u, \\ u &= \sigma(Kx), \end{aligned}$$

where α_0 and α_1 are given by (4.8) and (4.9), respectively. The saturation envelope

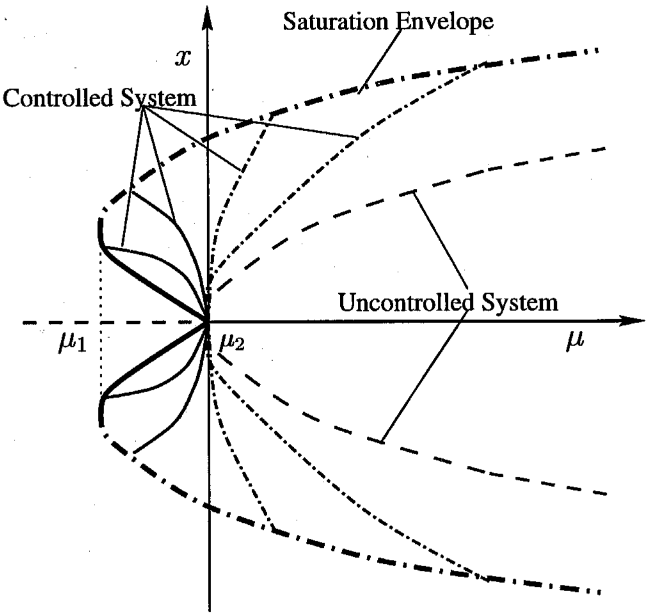


Figure 4.4: Bifurcation diagram of case **SS-5** with different values of gain.

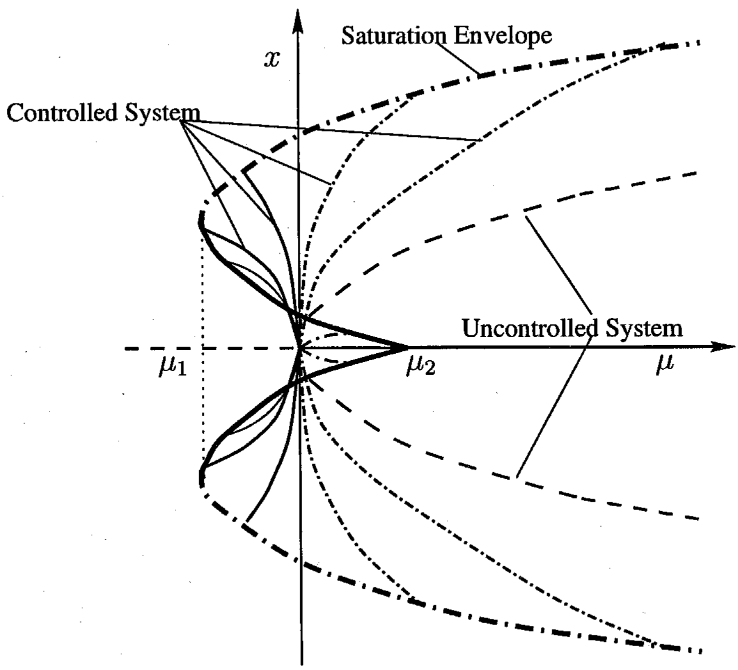


Figure 4.5: Bifurcation diagram of case **SS-6** with different values of gain.

is given by

$$\mu = -\frac{1}{d}\left(\alpha_0 x^2 \pm \alpha_1 u_{mag} x\right).$$

The bifurcation diagram is given by Figure 4.4. In this case, we have $\mu_1 < 0 < \mu_2$, so the bifurcation for the closed loop system is subcritical if the gain is very large, but the region of attraction to the stable part of the saturation envelope is larger than that to the nominal equilibria for the open loop system.

For **SS-6**, the dynamics on the center manifold is given by

$$\begin{aligned} \dot{x} &= d\mu x + \alpha_0 x^3 + \alpha_1 x^2 u + \alpha_2 x u^2, \\ u &= \sigma(Kx), \end{aligned}$$

where α_j $(j = 0, 1, 2)$ are given by (4.8), (4.9), and (4.10). The saturation envelope is given by

$$\mu = -\frac{1}{d}\left(\alpha_0 x^2 \pm \alpha_1 u_{mag} x + \alpha_2 u_{mag}^2\right).$$

The bifurcation diagram is given by Figure 4.5. It can be seen that the saturation envelope intersects the bifurcation point, so as the gain increases, the stabilized bifurcation branch will intersects the stable part of the saturation envelope. In this case, we have $\mu_2 = 0$, so it is impossible to bring the stable equilibria arbitrarily close to the nominal equilibria for $\mu \in [\mu_1, 0)$.

For **SS-7**, the dynamics on the center manifold is given by

$$\begin{aligned} \dot{x} &= d\mu x + \left[b_1 K_3 + \tilde{\alpha}_0 + P\left(-\frac{q_{11}}{\Upsilon_1}\right)\right] x^3, \\ u &= \sigma\left(-\frac{q_{11}}{\Upsilon_1} x + K_3 x^2\right), \end{aligned}$$

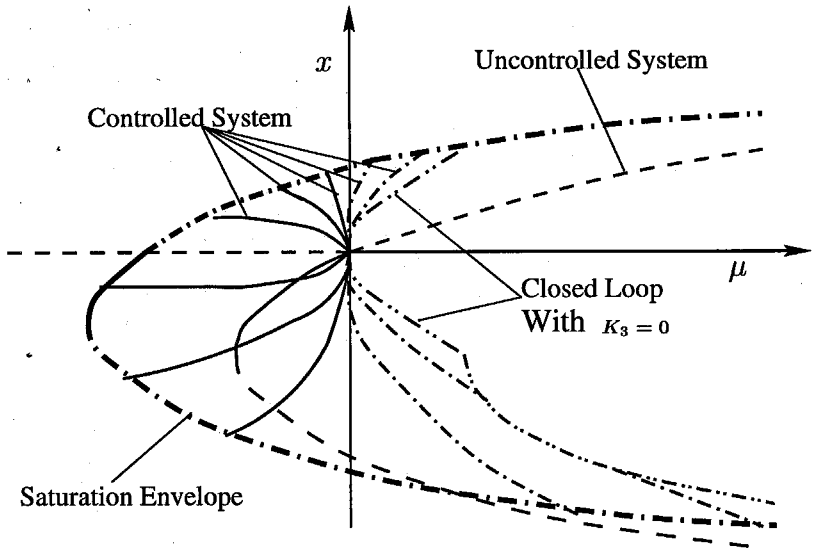


Figure 4.6: Bifurcation diagram of case **SS-7** and **SS-8** with different values of gain.

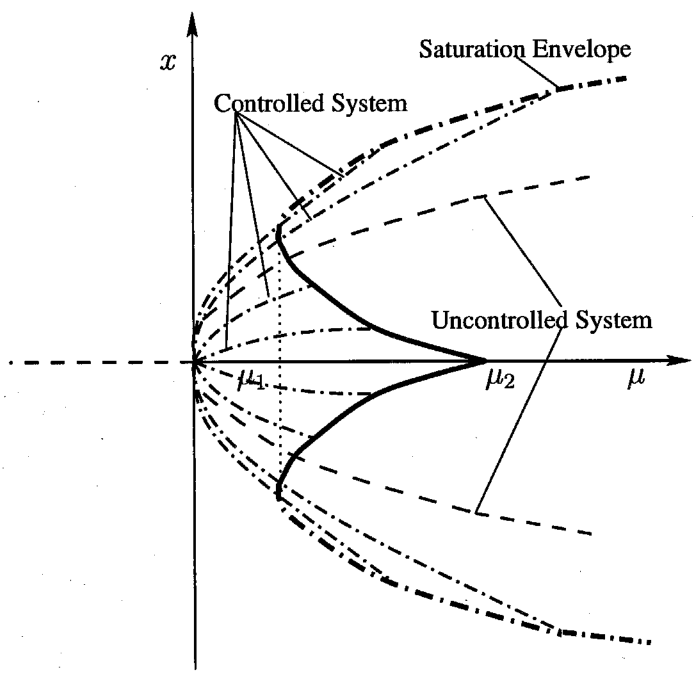


Figure 4.7: Bifurcation diagram of case **SU-2** with different values of gain. Note the system is unstabilizable.

where

$$\tilde{\alpha}_0 = \frac{\Upsilon_1}{a_n}\left[\begin{array}{cccc} 1 & a_1 & \cdots & a_{n-1} \end{array}\right]\tilde{Q}\left(-\frac{q_{11}}{\Upsilon_1}\right),$$

$P(\cdot)$ is given by (4.12), and $\tilde{Q}(\cdot)$ is given by (4.14). The saturation envelope is given by

$$\begin{aligned} \mu &= -\frac{\Upsilon_1}{da_n}\left(\pm u_{mag} + \frac{q_{11}}{\Upsilon_1}x\right) - \frac{1}{d}\left[\tilde{\alpha}_0 + P\left(-\frac{q_{11}}{\Upsilon_1}\right)\right]x^2, \\ \pm u_{mag} &= -\frac{q_{11}}{\Upsilon_1}x + K_3x^2. \end{aligned}$$

A bifurcation diagram is illustrated in Figure 4.6. Since the bifurcation for the open loop system is not symmetric about the x-axis, the bifurcation for the closed loop system is also asymmetric about the x-axis. In Figure 4.6, the region of attraction to the stabilized equilibria for $x > 0$ is smaller than that for $x < 0$. The case **SS-9** can be similarly analyzed, and the qualitatively picture is similar to that of Figure 4.6.

Now we consider the case **SU-2** for which the system is unstabilizable. The conditions for this case is given by Table 2.1 in Chapter 2, i.e., $q_{11} = \Upsilon_1 = \Upsilon_2 = \alpha_3 = 0$, $\alpha_2 > 0$, $\alpha_0 > 0$, but $\Delta_\alpha < 0$. The dynamics on the center manifold is given by

$$\begin{aligned} \dot{x} &= d\mu x + \alpha_0 x^3 + \alpha_1 x^2 u + \alpha_2 x u^2, \\ u &= \sigma(Kx), \end{aligned}$$

The shape of the saturation envelope is given by Figure 4.7. It should be noted that the saturation envelope is in the region $\mu > 0$, and the bifurcation for the closed loop system is always subcritical for any control gain. But still, the region of attraction to the nominal equilibria and the stable equilibria on the saturation envelope can be enlarged by choosing appropriate values of the control gain.

The above is a complete discussion of the effects of magnitude saturation in feedback stabilization of steady state bifurcations. The most important feature is that the boundary of the region of attraction to the stabilized equilibria for all the gains is the saturation envelope formed by the unstable equilibria for the closed loop system when the controllers saturate. We remark here that for the cases when there does not exist a sufficiently smooth feedback such that the bifurcation for the closed loop system is supercritical, we can design feedbacks to enlarge the region of attraction. As in the stabilizable cases discussed above, the boundary for the enlarged region of attraction is formed by the saturation envelope.

4.2 Hopf Bifurcations

Similar to the steady-state bifurcation, we have classified the stabilizability for Hopf bifurcations in Chapter 2. We have also given the construction of the stabilizing feedback controllers. We remark here that the techniques for analyzing the region of attraction to the stabilized equilibria in steady-state bifurcations here apply to the Hopf bifurcations. The only difference is that in the Hopf bifurcation case, we have to take into account the normal form transformation in addition to the center manifold reduction. The qualitative behavior of the feedback controllers with magnitude saturation for the Hopf bifurcation case is the similar to the steady-state bifurcations discussed in the steady-sate bifurcations.

In addition to the assumptions **AH-1** to **AH-4** in Chapter 2, we make the following assumption

AH-5 A is Hurwitz for all the values of μ in a neighborhood of $\mu = 0$.

Now we consider the case **HS-2**, i.e.,

$$\Theta_1 \;=\; \text{Re}\left\{q_{13}^n + q_{14}a_n\right\} \neq 0,$$

where

$$\begin{aligned} C_0 &= c_{112} - \frac{q_{11}q_{12}}{i\omega} - \Pi_1^H \tilde{q}_{12} - \Pi_2^H \tilde{q}_{11}, \\ \Pi_1^H &= \left[\, 0 \;\; q_{13}^1 \;\; \cdots \;\; q_{13}^{n-1} \,\right] + q_{14}\left[\, 1 \;\; a_1 \;\; \cdots \;\; a_{n-1} \,\right], \end{aligned}$$

$$\Pi_2^H = [\ 0 \quad q_{23,1}(2\omega i) \quad \cdots \quad q_{23,n-1}(2\omega i)\ ] + q_{24}[\ 1 \quad p_1(2\omega i) \quad \cdots \quad p_{n-1}(2\omega i)\ ].$$

We choose the control law as $u = \sigma(K|z|^2)$ with $K \in \mathbb{R}$ and $\sigma(\cdot)$ is the saturation function. The normal form of the dynamics on the center manifold of the closed loop system is given by

$$\dot{\zeta} = (d\mu + i\omega)\zeta + \tilde{\alpha}|\zeta|^2\zeta,$$

where

$$\begin{aligned}
\tilde{\alpha} &= a_0 + b_0 K + \text{p.i.t.}, \\
a_0 &= \alpha_0 + \frac{\Theta_1}{a_n}[1 \quad a_1 \quad \cdots \quad a_{n-1}]\tilde{q}_{12} + \mathrm{Re}\left\{\frac{\Theta_2}{a_n}[1 \quad p_1(2\omega i) \quad \cdots \quad p_{n-1}(2\omega i)]\tilde{q}_{11}\right\}, \\
b_0 &= \frac{\Theta_1}{a_n}, \\
z &= \zeta + \gamma_{11}\zeta^2 + \gamma_{12}|\zeta|^2 + \gamma_{11}\zeta^{*2} + \text{h.o.t.},
\end{aligned}$$

where γ_{11}, γ_{12}, and γ_{22} can be calculated explicitly. Here p.i.t stands for pure imaginary terms. Letting $z = \rho e^{i\phi}$, and $\zeta = re^{i\theta}$, then then normal form is given by

$$\dot{r} = d\mu r + (a_0 + b_0 K)r^2.$$

Now

$$\begin{aligned}
u &= \sigma(K|z|^2) \\
&= \sigma(K|\zeta + \gamma_{11}\zeta^2 + \gamma_{12}|\zeta|^2 + \gamma_{11}\zeta^{*2}|^2) \\
&\approx \sigma(Kr^2).
\end{aligned}$$

Now if the controller is not saturated, then the normal form is given by

$$\begin{aligned}
\dot{r} &= d\mu r + (a_0 + b_0 K)r^2 \\
&= d\mu r + a_0 r^2 + b_0 u r.
\end{aligned}$$

Here we have used the fact that $u = Kr^2$. If the controller has magnitude saturation limit, we use the following system to approximate the normal form dynamics on the center manifold:

$$\begin{aligned} \dot{r} &= d\mu r + a_0 r^2 + b_0 ur, \\ u &= \sigma(Kr^2). \end{aligned}$$

Note the equations are the same as the case **SS-2** in the steady-state bifurcations. So the bifurcation diagram is the same as Figure 4.1, except that the y-axis is now replaced by r. By fixing K such that the bifurcation for the closed loop system is supercritical, the bifurcation diagram and the phase portraits are shown in Figure 4.8. Phase portrait (c) shows that if $\mu > 0$, then the nominal equilibrium is stable and the region of attraction of the equilibrium is bounded by an unstable limit cycle which is on the saturation envelope. Let μ_K be the bifurcation parameter at which the stabilized periodic orbits intersect the saturation envelope. If $\mu_K < \mu < 0$, then the phase portrait is given by (b). The nominal equilibrium becomes unstable due to the supercritical Hopf bifurcation at $\mu = 0$. The region of attraction is bounded by the unstable limit cycle. At $\mu = \mu_K$, the stabilized limit cycle merges with the unstable periodic orbit on the stability envelope and the region of attraction of the nominal equilibria becomes unbounded when $\mu < \mu_K$.

The bifurcation and the saturation envelope for other cases in the Hopf bifurcations can be analyzed in the same way and they are the qualitatively the same as **SS-3** to **SS-6** in the steady-state bifurcations except that the equilibria in the bifurcation diagram are replaced by limit cycles. We also point out that the shape of the periodic orbit on the stability boundary is not necessarily circular in the normal from coordinates for other cases. We also point out here that the bifurcation diagram Figure 4.6 does not show up in the Hopf bifurcations since the dynamics on the center manifold for the Hopf bifurcations can always be transformed into a normal form in which the periodic orbits are circles.

If the controllers also have bandwidth and rate limits, then bifurcation analysis

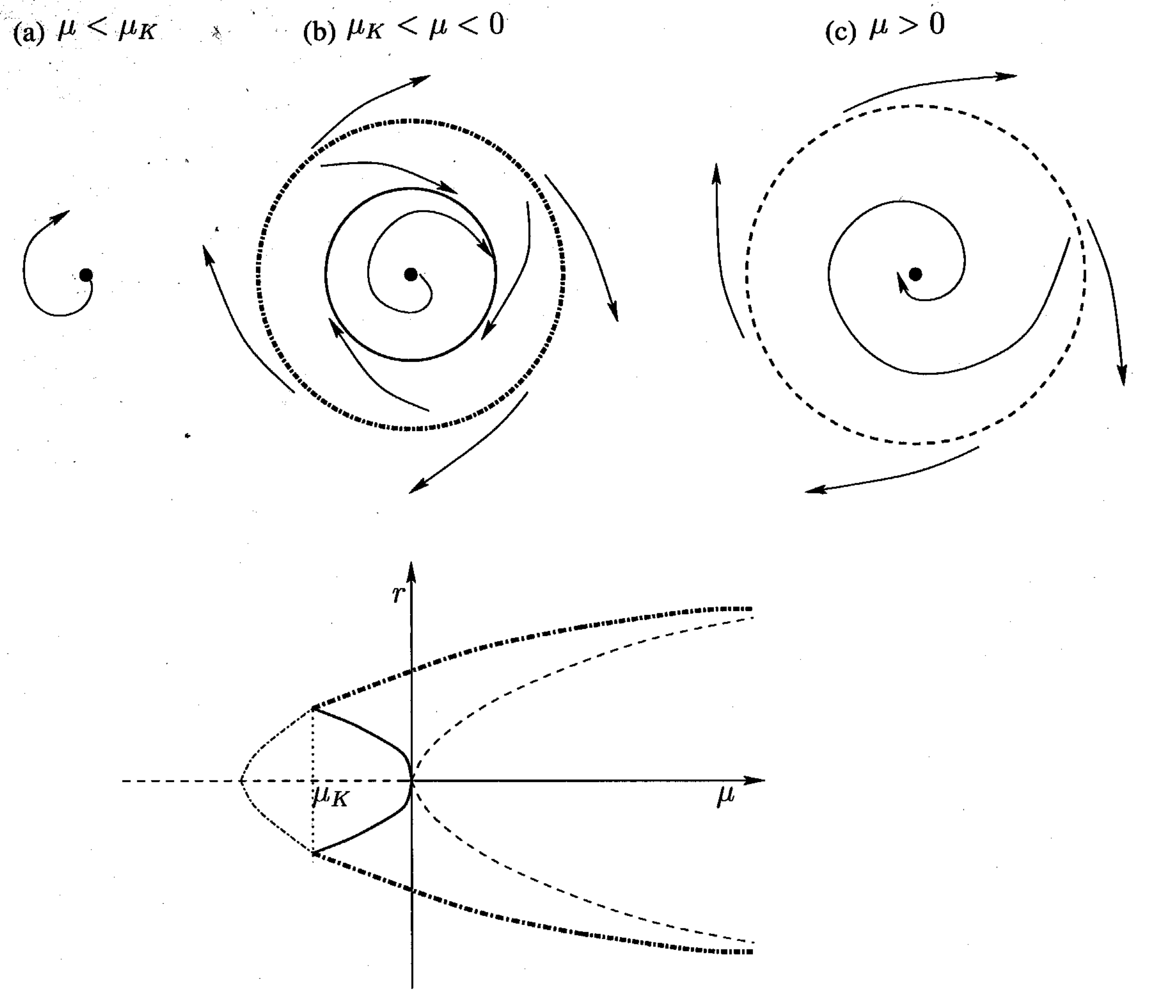


Figure 4.8: Bifurcation diagram and phase portraits of case **HS-2** with a fixed gain. Dashed: unstable periodic orbits for the open loop system. Solid: stable periodic orbits for the closed loop system. Dashdot: saturation envelope composed of unstable periodic orbits when the controller saturates.

can also be carried out. We omit this part since the general idea of analysis will be illustrated in the next chapter by analyzing the Moore-Greitzer model for rotating stall and surge in axial compression systems.

Chapter 5 Application to Active Control of Rotating Stall Using Bleed Valves with Magnitude and Rate Limits

In this chapter, nonlinear qualitative analysis is performed on a reduced order model of compression system instabilities to evaluate the tradeoff of fluid noise, actuator magnitude saturation, bandwidth, rate limits, and the shape of compressor characteristics in active control of rotating stall in axial compressors with bleed valve actuators. Further model order reduction is achieved by approximating the dynamics on the invariant manifold that captures the bifurcations and instabilities. Bifurcations and qualitative dynamics are obtained by analyzing the reduced system. The operability enhancement is defined as the extension of operating range for which fully developed rotating stall is avoided. Analytic formulas are derived for the operability enhancement as a function of noise level, actuator saturation limits, and the shape of the compressor characteristic, which is the major nonlinearity in the model. The shape of the compressor characteristic, especially the unstable part, is critical to the rate required for robust operability near the peak for the closed loop system. Experiments are carried out on a single-stage low-speed axial compressor using different level of steady air injections to generate different compressor characteristics. The theoretical formulas give good qualitative estimates to experimental data. This chapter is based on the papers [68, 70, 72, 73].

Previous work on the effects of actuator bandwidth are given in [39, 46], but the technique is linear stability analysis and the effects of magnitude saturation and rate limits cannot be cast into a linear framework. The results in this chapter are new.

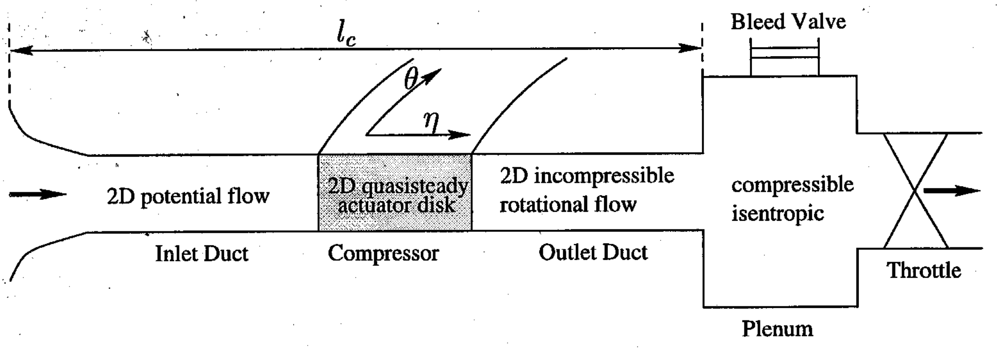


Figure 5.1: The composition of an axial compression system and assumptions on the flow.

5.1 Background and Motivation

In this section we describe the Moore-Greitzer model and the dynamics and bifurcations associated with rotating stall and surge. Then we discuss the benefit of bifurcation control. Finally, we motivate the effects of actuator limits from bifurcation theoretic point of view.

5.1.1 Background

The Moore-Greitzer model for an axial compression system

An axial compression system is composed of an inlet duct, a compressor, an outlet duct, a plenum, and a throttle (see Figure 5.1). The important assumptions of the Moore-Greitzer model are shown in Figure 5.1 (see [56]). Also shown in the figure is the bleed valve actuator on the plenum. The flow in the inlet duct is assumed to be two-dimensional potential flow, where η is in the axial direction and θ is in the circumferential direction. The compressor is modeled as two-dimensional quasi-steady actuator disc, which implies that the pressure-rise across the compressor is given by

$$\frac{\Delta P}{\frac{1}{2}\rho U^2} = F(\varphi) - \tau\frac{d\varphi}{dt},$$

where U is the wheel speed at mean diameter, $\varphi = \varphi(\theta, \xi)$ is the local, unsteady axial velocity coefficient at the compressor face, $F(\varphi)$ is the axisymmetric steady performance of the blade row, τ is the coefficient of the pressure-rise lag. The flow in the outlet duct is modeled as linearized steady two-dimensional rotational flow. The flow in the plenum is modeled as compressible, isentropic, and quiescent. Using these assumptions, Moore and Greitzer derived a set of partial differential equations (PDEs) for which the unknowns are the flow coefficient $\varphi(\theta, \xi)$ evaluated as the compressor face and the annulus-averaged pressure-rise coefficient $\Psi(\xi)$ across the compressor [56]. By projecting the PDEs onto the zeroth and the first spatial harmonic, Moore and Greitzer derived a three state model given by

$$\begin{aligned} \frac{d\Phi}{d\xi} &= \frac{1}{l_c}\left(\frac{1}{2\pi}\int_0^{2\pi} \psi_c(\Phi + A\sin\zeta)\, d\zeta - \Psi\right), \\ \frac{d\Psi}{d\xi} &= \frac{1}{4B^2 l_c}\left(\Phi - \Phi_T(\Psi)\right), \\ \frac{dA}{d\xi} &= \frac{1}{\pi}\frac{1}{m+\mu}\int_0^{2\pi} \psi_c(\Phi + A\sin\zeta)\sin\zeta\, d\zeta, \end{aligned} \tag{5.1}$$

where $\Phi(\xi) := \int_0^{2\pi} \varphi(\theta, \xi) d\theta$ is the annulus-averaged flow coefficient, A is the amplitude of the first harmonic of the axial flow disturbances around the compressor annulus, $\psi_c(\cdot)$ is the compressor characteristic, $\Phi_T(\cdot)$ is the throttle characteristic, l_c is the effective length of the compression system, B is the Greitzer B-parameter, m is the duct parameter, μ is the inertia parameter, ξ is the nondimensional time in rotor radians, and ζ is the circumferential angle around the compressor annulus. Suppose the compressor characteristic is analytic and the throttle characteristic is $\Phi_T(\Psi) = (\gamma + u)\sqrt{\Psi}$, where u is the control input of the bleed valve and γ is the throttle coefficient denoting the opening of the throttle ($\gamma = 0$ means the throttle is fully closed). γ and B are the bifurcation parameters of the model [52]. The operating condition is determined by γ and rotating stall occurs when the throttle characteristic intersects the peak of the compressor characteristic. B is proportional to the size of the plenum and determines the axisymmetric dynamics Φ and Ψ. When B is large, the eigenvalues of the axisymmetric dynamics tends to become unstable

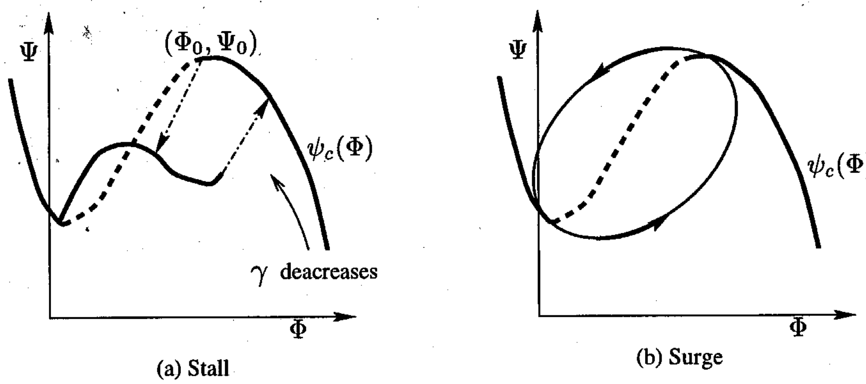


Figure 5.2: Behavior of rotating stall and surge in the Φ-Ψ plane.

near the peak of the compressor characteristic.

Properties of the dynamics of the model

We first discuss the dynamics and bifurcations for the uncontrolled system, i.e. $u = 0$. We treat the throttle coefficient γ as the bifurcation parameter. The shape of a typical compressor characteristic is shown in Figure 5.2.

The unstalled equilibria of the system are the axisymmetric equilibria given by

$$\Phi = \Phi_e(\gamma), \quad \Psi = \Psi_e(\gamma) = \psi_c(\Phi_e(\gamma)), \quad J = 0.$$

where $J = A^2$. The axisymmetric equilibria denote the axisymmetric flow condition. There are two types of bifurcations of the axisymmetric equilibria.

- At the peak of the compressor characteristic $\gamma = \gamma_0$, a transcritical bifurcation to rotating stall occurs. The rotating stall equilibria denote the unsteady non-axisymmetric flow which is a first modal traveling wave rotating around the annulus of the compressor at a fraction of the rotor speed.

- If B is large, then at a point on the positive slope of the characteristic $\gamma = \gamma_{sg} < \gamma_0$, a Hopf bifurcation to surge occurs. The surge limit cycle denotes the axial unsteady axisymmetric flow and pressure oscillation across the compression system.

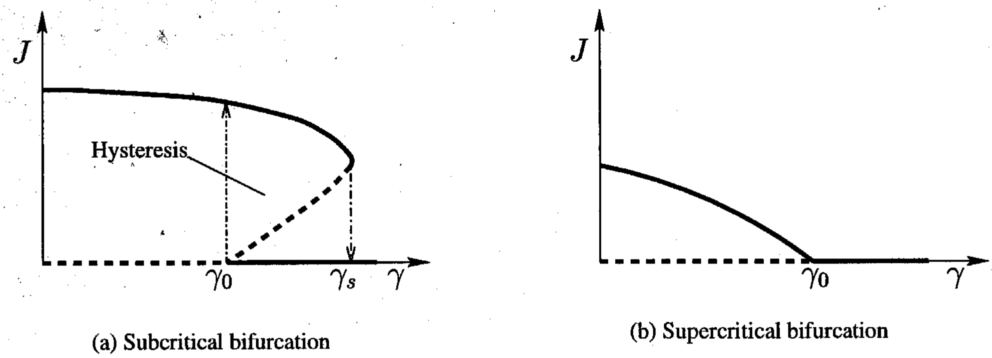


Figure 5.3: Open and closed loop bifurcation behavior with feedback $u = KJ$.

A complete numerical bifurcation analysis is in [52]. Throughout this chapter we assume B is small such that there is no Hopf bifurcation to surge. The transcritical bifurcation to rotating stall is typically subcritical, and there is a saddle-node bifurcation on the stalled branch (see Figure 5.3 (a)). The stall inception is catastrophic because of the hysteresis associated with the subcritical bifurcation. It is easy to show that the stall mode A is linearly uncontrollable with bleed valve actuators. So it is impossible to design a state feedback to stabilize the axisymmetric equilibria beyond the peak of the compressor characteristic. Nevertheless, Liaw and Abed [49] designed a feedback control law $u = KJ$ such that the bifurcation of the closed loop system is supercritical. By changing the criticality of the bifurcation, the stall inception becomes progressive and more benign (see Figure 5.3 (b)).

5.1.2 Motivation from bifurcation theoretic point of view

Due to actuator magnitude and rate saturation, the actual domains of attraction of the stable operating points could be small. In this section we motivate the effects of actuator saturation limits from bifurcation theoretic point of view.

Consider the a feedback control law with magnitude and rate saturation constraint and first order dynamics

$$\dot{u} = \chi(\Phi, \Psi, J; u_{mag}, u_{rate}, \tau), \tag{5.2}$$

where u_{mag} and u_{rate} are magnitude and rate saturation limits, i.e., $0 \leq u \leq u_{mag}$, and $-u_{rate} \leq \dot{u} \leq u_{rate}$, where τ is the time constant associated with the actuator. Especially, we consider the feedback control law $u = KJ$ with first order actuator dynamics constrained by magnitude and rate saturations. The full closed loop system is given by

$$\begin{aligned}
\frac{d\Phi}{d\xi} &= \frac{1}{l_c}\left(\frac{1}{2\pi}\int_0^{2\pi}\psi_c(\Phi + A\sin\zeta)\,d\zeta - \Psi\right), \\
\frac{d\Psi}{d\xi} &= \frac{1}{4B^2 l_c}\left(\Phi - (\gamma + u)\sqrt{\Psi}\right), \\
\frac{dA}{d\xi} &= \frac{1}{\pi}\frac{1}{m+\mu}\int_0^{2\pi}\psi_c(\Phi + A\sin\zeta)\sin\zeta\,d\zeta, \\
\frac{du}{d\xi} &= \begin{cases} -u_{rate} & \text{if } \dfrac{u_{des} - u}{\tau} \leq -u_{rate}, \\ \dfrac{u_{des} - u}{\tau} & \text{if } \left|\dfrac{u_{des} - u}{\tau}\right| < u_{rate}, \\ u_{rate} & \text{if } \dfrac{u_{des} - u}{\tau} \geq u_{rate}, \end{cases} \\
u_{des} &= \begin{cases} 0 & \text{if } KJ \leq 0, \\ KJ & \text{if } 0 < KJ < u_{mag}, \\ u_{mag} & \text{if } KJ \geq u_{mag}, \end{cases}
\end{aligned} \tag{5.3}$$

together with the magnitude constraint $0 \leq u \leq u_{mag}$.

The effects of actuator limits (magnitude saturation, bandwidth and rate limits) and fluid noise on rotating stall control can be demonstrated by analyzing the bifurcation diagrams of open and closed loop systems. A typical bifurcation diagram for the uncontrolled three state Moore-Greitzer model is in Figure 5.4 (a). The region of attraction of the stable axisymmetric equilibrium is the shaded region in Figure 5.4 (a). Now consider the controller with infinite magnitude saturation limit and infinite bandwidth, i.e., the bleed valve can bleed out as much air as we want and is infinitely fast, then the bifurcation diagram for the closed loop system is shown in Figure 5.4 (b). The bifurcation for the stall equilibria is supercritical and the stall inception is progressive. The stall branch can be arbitrarily close to the axisymmetric equilibrium by increasing K. The region stabilized by control is shown in Figure 5.4 (b).

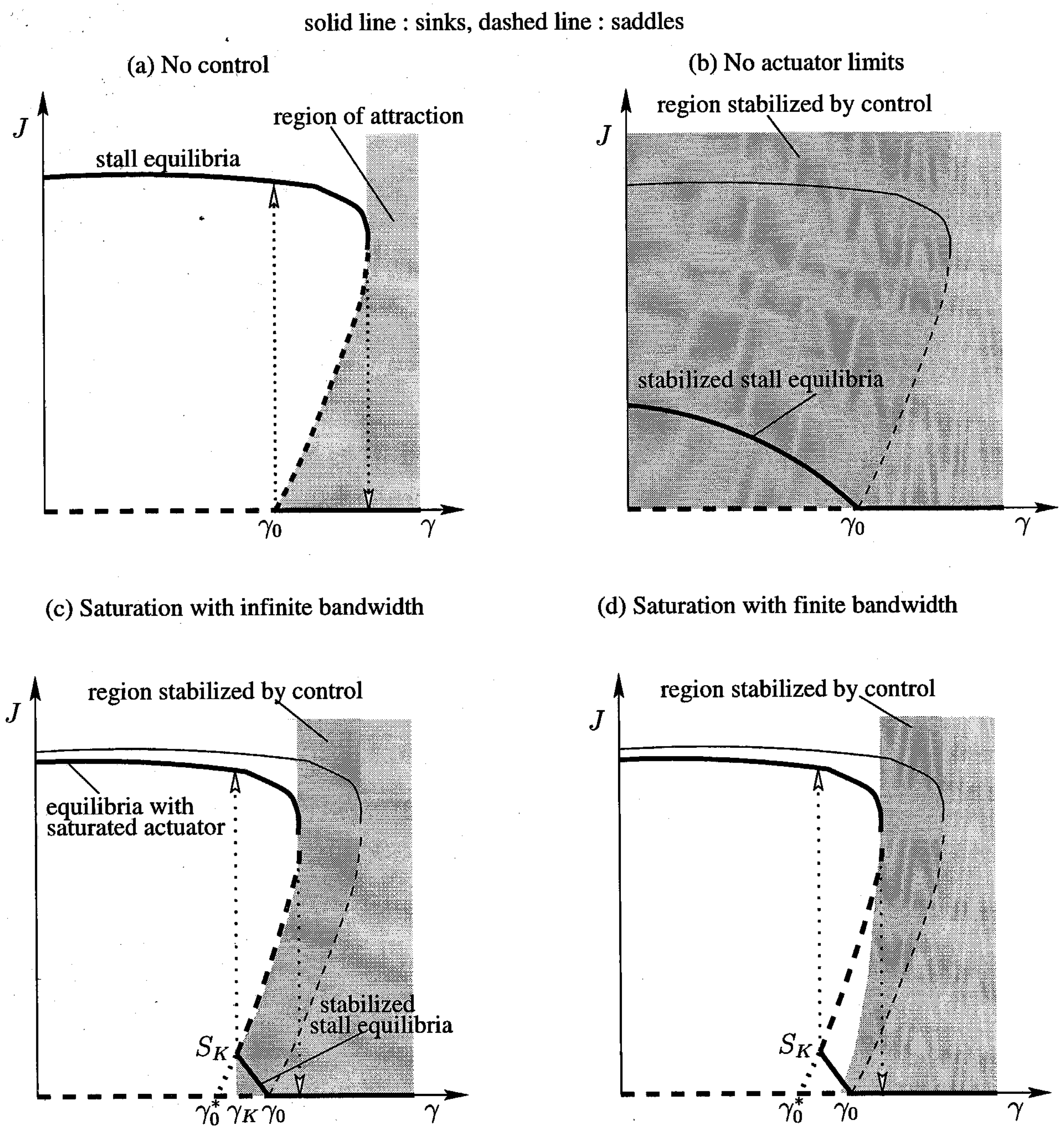


Figure 5.4: Bifurcation diagrams for the effects of actuator limits.

Now suppose the magnitude saturation for the actuator is finite but the actuator bandwidth is infinite, i.e., the valve is infinitely fast but it can only bleed out a certain amount of air. The bifurcation diagram for the closed loop system is shown in Figure 5.4 (c). The thick solid and dash lines denote the equilibria for the closed loop system. It can be seen from this figure that there is a new saddle-node like bifurcation point S_K due to the actuator magnitude saturation. This saddle-node like bifurcation point defines γ_K such that any arbitrarily small noise of J will drive the system to the fully developed rotating stall equilibrium if the throttle operates at a position such that $\gamma < \gamma_K$. Also we have

$$\lim_{K \to +\infty} \gamma_K = \gamma_0^* = \gamma_0 - u_{mag},$$

where γ_0^* is the throttle coefficient of the stall inception point when the bleed valve is fully open. It is clear from Figure 5.4 (c) that the bleed valve controller cannot eliminate the hysteresis loop if its magnitude saturation is finite. Furthermore, any arbitrary small noise of J will grow to fully developed rotating stall no matter how large the controller gain is when the throttle operates at a position such that $\gamma < \gamma_0^*$. It can also be seen that the region stabilized by control is much smaller than the previous case if u_{mag} is small. This implies it is impossible to build a valve with small air bleed to achieve a large extension of operable range.

Now, in addition to the finite magnitude saturation assumption, suppose the bandwidth and rate limits are finite, i.e., the bleed valve opens and closes with a finite speed. Also assume the bleed valve is initially closed, i.e., $u(0) = 0$. Then the region stabilized by control is shown in Figure 5.4 (d). By comparing (c) and (d), it is clear that the region of attraction from control is further restrained by the actuator bandwidth and rate limits. It can be proved in later sections that if the rate limits goes to zero, then the region of attraction from control becomes arbitrarily small.

The noise level for a real compression system is not arbitrarily small due to disturbances such as inlet distortion and inlet turbulence level. We model the noise as a closed set of initial conditions of J. When the noise level is of finite amplitude, the open loop system goes to rotating stall at a throttle coefficient larger than the nominal stall inception point γ_0. The extension of operable range becomes even smaller if the noise level is of finite amplitude.

It should be noted that in the above discussions are valid only if the initial conditions for Φ and Ψ are close to the axisymmetric equilibrium for a fixed throttle coefficient and B is small enough such that there no surge dynamics. In the following sections we will systematically develop the ideas in the above discussions and obtain analytical estimates for the controller performance in the presence of noise, actuator magnitude saturation and rate limits.

5.2 Control Analysis

In this section we reduce the four-dimensional system (5.3) into a two-dimensional system. We give qualitative phase portraits for the reduced system for different throttle settings. It turns out that the stable manifold of a saddle equilibrium in the system is the boundary between two regions: one with the trajectories converging to the stabilized stall equilibrium, the other with the trajectories converging to the fully developed stall equilibrium.

5.2.1 System reduction and approximation

Numerical bifurcation analysis by McCaughan [52] confirmed the typical qualitative phase portraits of the open loop system for different γ's look like Figure 5.5 if the Greitzer B-parameter is small enough. Let Φ_0, Ψ_0 and γ_0 be the axial flow coefficient, the pressure rise coefficient, and throttle coefficient at the peak of the compressor characteristic, respectively. Let γ_s be the throttle coefficient at which the unstable stall equilibria merge with the stable stall equilibria. It is clear that the saddle-sink connections in Figure 5.5 are attracting (stable) and they captures the instabilities (rotating stall) of the Moore-Greitzer model. The main effort of this section is to approximate the dynamics on the saddle-sink connection.

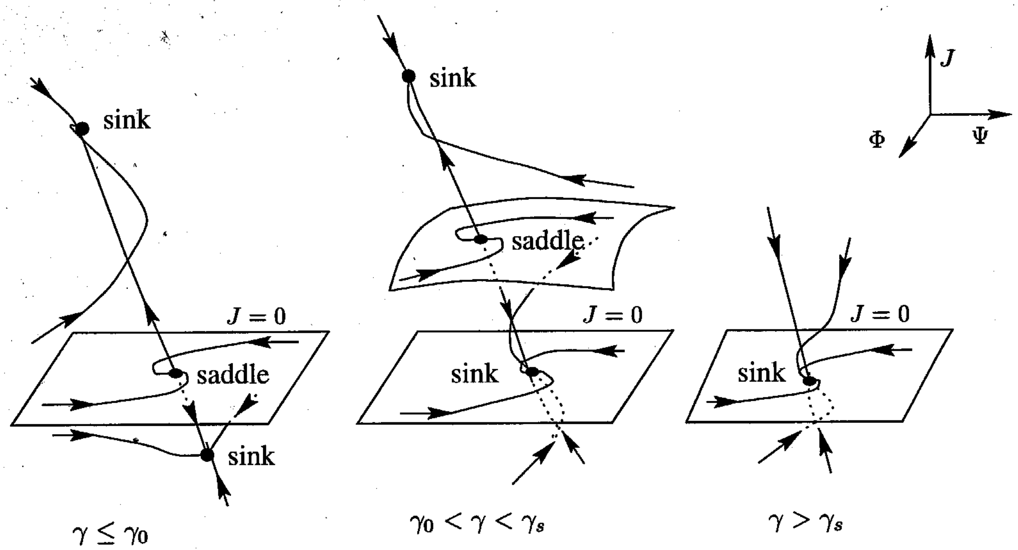


Figure 5.5: Phase portraits for the three state Moore-Greitzer model at different throttle coefficients.

The following proposition gives the center manifold reduction of the controlled system. We use the convention $\mathcal{O}(x^n)$ to represent terms of the order $\|x\|^n$ and higher, where $x \in \mathbb{R}^n$.

Proposition 5.1 *For sufficiently small B, the dynamics of (5.3) near the transcritical bifurcation point $(\Phi_0, \Psi_0, 0)$ can be approximated by the dynamics on the center manifold and can be approximated by the following two-dimensional system*

$$\frac{dJ}{d\xi} = \alpha_1(\delta + u)J + \alpha_2 J^2 + \mathcal{O}\left(J[\delta \;\; u \;\; J]^2\right), \tag{5.4}$$

$$\frac{du}{d\xi} = \begin{cases} -u_{rate} & \text{if } \dfrac{u_{des} - u}{\tau} \leq -u_{rate}, \\ \dfrac{u_{des} - u}{\tau} & \text{if } \left|\dfrac{u_{des} - u}{\tau}\right| < u_{rate}, \\ u_{rate} & \text{if } \dfrac{u_{des} - u}{\tau} \geq u_{rate}, \end{cases} \tag{5.5}$$

$$u_{des} = \begin{cases} 0 & \text{if } KJ \leq 0, \\ KJ & \text{if } 0 < KJ < u_{mag}, \\ u_{mag} & \text{if } KJ \geq u_{mag}, \end{cases}$$

where $J = A^2$, $\delta = \gamma - \gamma_0$,

$$\alpha_1 = \frac{2\sqrt{\Psi_0}\psi_c''}{m+\mu}, \tag{5.6}$$

$$\alpha_2 = \frac{1}{4(m+\mu)}\left(\psi_c''' + \frac{\gamma_0\psi_c''^2}{\sqrt{\Psi_0}}\right), \tag{5.7}$$

and all the derivatives of ψ_c *are evaluated at* Φ_0.

Proof of Proposition 5.1: The proof of Proposition 5.1 is the standard center manifold reduction (see [37]), and is given as follows. We assume the compressor characteristic $\psi_c(\cdot)$ is an analytic function, i.e., the Taylor series of expansion of $\psi_c(\Phi + x)$ is convergent for any Φ. So

$$\psi_c(\Phi + x) = \sum_{k=0}^{\infty} \frac{\psi_c^{(k)}(\Phi)}{k!} x^k.$$

Now, by substituting the Taylor series expansion into the three state Moore-Greitzer model (5.1), we get

$$\begin{aligned}
\frac{d\Phi}{d\xi} &= \frac{1}{l_c}\left(-\Psi + \psi_c(\Phi) + \sum_{k=1}^{\infty}\frac{\psi_c^{(2k)}(\Phi)}{[(2k)!!]^2}J^k\right) \\
\frac{d\Psi}{d\xi} &= \frac{1}{4B^2 l_c}\left(\Phi - (\gamma+u)\sqrt{\Psi}\right) \\
\frac{dJ}{d\xi} &= \frac{4}{m+\mu}\sum_{k=1}^{\infty}\frac{2k\psi_c^{(2k-1)}(\Phi)}{[(2k)!!]^2}J^k
\end{aligned} \tag{5.8}$$

where $J := A^2$. From the actuator dynamics (5.3) we know in different regions of the u-J plane the control law has different forms:

(1) The bandwidth limit form: $u = \frac{KJ - u}{\tau}$, and $u = \frac{u_{mag} - u}{\tau}$.

(2) The rate limit form $u = \pm u_{rate}$.

(3) The magnitude limit form $u = u_{mag}$.

We first consider the case when $\dot{u} = \frac{KJ-u}{\tau}$. Letting $\phi = \Phi - \Phi_0$, $\psi = \Psi - \Psi_0$, $v = u - KJ$, and $\delta = \gamma - \gamma_0$, then by using Taylor series expansion, the system becomes

$$
\begin{aligned}
\dot{J} &= \frac{1}{m+\mu}\left(2\psi_c''\phi J + \frac{\psi_c'''}{4}J^2\right) + \text{h.o.t.} \\
\dot{\phi} &= \frac{\psi_c''}{4l_c}J - \frac{1}{l_c}\psi + \frac{1}{l_c}\left(\frac{\psi_c''}{2}\phi^2 + \frac{\psi_c'''}{4}\phi J + \frac{\psi_c^{(4)}}{64}J^2\right) + \text{h.o.t.} \\
\dot{\psi} &= -\frac{\sqrt{\Psi_0}K}{4B^2l_c}J + \frac{1}{4B^2l_c}\phi - \frac{\gamma_0}{8B^2l_c\sqrt{\Psi_0}}\psi - \frac{\sqrt{\Psi_0}}{4B^2l_c}v - \frac{\sqrt{\Psi_0}}{4B^2l_c}\delta \\
&\quad - \frac{1}{8B^2l_c\sqrt{\Psi_0}}\left(\psi v + K\psi J - \frac{\gamma_0}{4\Psi_0}\psi^2\right) + \text{h.o.t.} \\
\dot{v} &= -\frac{1}{\tau}v + \frac{K}{m+\mu}\left(2\psi_c''\phi J + \frac{\psi_c'''}{4}J^2\right) + \text{h.o.t.}
\end{aligned}
$$

Since $\tau > 0$ and B is small, the system possesses a three-dimensional stable manifold and a one-dimensional center manifold. Let

$$
\begin{aligned}
\phi &= \beta_{11}\delta + \beta_{12}J + \mathcal{O}\left([\delta \;\; J]^2\right), \\
\psi &= \beta_{21}\delta + \beta_{22}J + \mathcal{O}\left([\delta \;\; J]^2\right), \\
v &= \beta_{31}\delta + \beta_{32}J + \mathcal{O}\left([\delta \;\; J]^2\right).
\end{aligned}
$$

By differentiating these equalities and using the systems equations, we get

$$
\begin{aligned}
&\beta_{11} = \sqrt{\Psi_0}, && \beta_{21} = 0, && \beta_{31} = 0, \\
&\beta_{12} = \sqrt{\Psi_0}\,K + \tfrac{\gamma_0\psi_c''}{8\sqrt{\Psi_0}}, && \beta_{22} = \tfrac{\psi_c''}{4}, && \beta_{32} = 0.
\end{aligned}
$$

So the dynamics on the center manifold is given by

$$
\begin{aligned}
\dot{J} &= \frac{1}{m+\mu}\left(2\psi_c''\phi J + \frac{\psi_c'''}{4}J^2\right) + \text{h.o.t.} \\
&= \alpha_1(\delta + KJ)J + \alpha_2 J^2 + \text{h.o.t.} \\
&= \alpha_1(\delta + u)J + \alpha_2 J^2 + \text{h.o.t.},
\end{aligned}
$$

where α_1 and α_2 are given by (5.6) and (5.7), respectively. The last equality holds because $u = v + KJ$ and $v = \mathcal{O}\left(\delta^2, \delta J, J^2\right)$. So the reduced system is given by

$$\begin{aligned} \dot{J} &= \alpha_1(\delta + u)J + \alpha_2 J^2 + \text{h.o.t.}, \\ \dot{u} &= \frac{KJ - u}{\tau}, \end{aligned} \tag{5.9}$$

where α_1 and α_2 are given by (5.6) and (5.7).

For other types of control laws, similar procedure could be followed. The resulting reduced system is the same as (5.9), together with the actuator dynamics. It should be mentioned that for the case when $\dot{u} = \pm u_{rate}$, the reduction is no longer a center manifold reduction, but rather an approximation of the ϕ and ψ variables using J and δ. The proof is done by combining all these cases. ■

Here are several observations from Proposition 5.1.

(1) If $u = 0$, the coefficients α_1 and α_2 in the center manifold equation determine the bifurcation characteristics of the uncontrolled Moore-Greitzer model in the neighborhood of γ_0.

(2) For a typical compressor characteristic we have $\psi_c'' < 0$, so $\alpha_1 < 0$. The sensitivity of the eigenvalue of the stall mode to variations of the bifurcation parameter γ is given by α_1. It is easy to see that if $|\psi_c''|$ and Ψ_0 are large, which is typical of the high speed compressor of an aircraft engine, then $|\alpha_1|$ is large and the system quickly goes linearly unstable beyond the bifurcation point.

(3) If $\alpha_2 > 0$, then the transcritical bifurcation to rotating stall is subcritical; if $\alpha_2 < 0$, then the transcritical bifurcation to rotating stall is supercritical. The subcritical bifurcation is typical of axial compression systems and is more detrimental in that there is a hysteresis and the stall inception is catastrophic. The positivity of α_2 implies nonlinear instability which accelerates the growth rate of rotating stall once it is out of the linearly dominated region. Also, the positivity of α_2 is the main reason why active control of rotating stall requires high bandwidth actuators. This point will become more clear in the later sections.

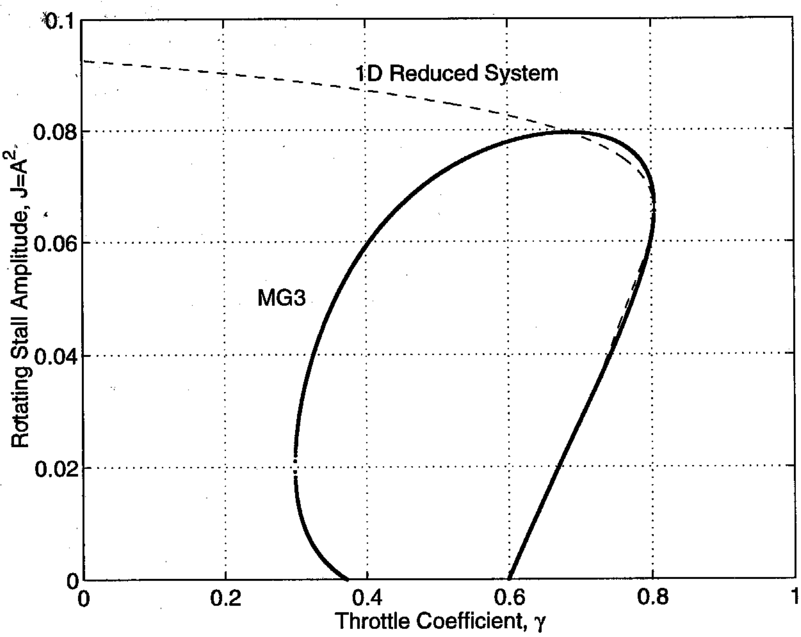


Figure 5.6: The stall equilibria of the MG3 and the approximate 1D system for the Caltech rig.

Note that α_2 increases quadratically with ψ_c'', and increases linearly with ψ_c'''. If the unstable (left) part of the characteristic becomes steeper, then $\psi_c''' > 0$ and α_2 become larger, and the nonlinear instability becomes more severe. Later in the experiment section the unstable part of the characteristic is leveled by upstream steady air injections and the rate requirement of bleed valve actuator is drastically reduced.

(4) The minimum gain for feedback control law $u = KJ$ to make the closed loop system be supercritical is $K^* = -\frac{\alpha_2}{\alpha_1}$. It is clear that K^* increases as $|\psi_c''|$ and ψ_c''' increase. This is also true for the rate requirement and will be shown in later sections.

Since the equation (5.4) only captures the local behavior near the bifurcation point, we use the following system to approximate the dynamics on the attracting saddle-sink connections to capture the hysteresis behavior (see Figure 5.5):

$$\dot{J} = \alpha J \left(\alpha_1(\delta + u) + \sum_{k=2}^{n} \alpha_k J^{k-1} \right), \tag{5.10}$$

where u is the control input given by (5.5), and the α's are selected such that the equilibria of the ordinary differential equation (5.10) fit the rotating stall equilibria in γ-J plane in the following way:

(1) α_1 and α_2 are taken from the center manifold equation (5.6) and (5.7).

(2) α_k $(k = 3, \cdots, n)$ are selected such that the equilibria of the one-dimensional system (5.10) fit the equilibria of the full system in the hysteresis region.

(3) α is selected such that the growth rate of J for the approximate system (5.10) matches that for the full system at $\gamma = \gamma_0$.

This choice is designed to give an approximation that not only matches the local bifurcation characteristics, but also matches the magnitude and growth rate of fully developed rotating stall. Figure 5.6 shows the stall equilibria for the three state Moore-Greitzer model for the Caltech rig (see section 5.4.2) with a fourth order polynomial compressor characteristic and the stall equilibria for the reduced system with $n = 7$.

Throughout this chapter we make the following assumptions on the shape of the stall equilibrium branch:

(A1) The bifurcation of the open loop system is subcritical, i.e., $\alpha_2 > 0$.

(A2) There is only one saddle-node bifurcation on the stall equilibrium branch. The bifurcation point is at $\delta_s := \gamma_s - \gamma_0$.

We remark here that assumption (A2) may not be true for δ's for the full model (see Figure 5.6), but it is true in a neighborhood of $\delta = 0$, which is sufficient for our analysis since the magnitude saturation limit u_{mag} is small, and the Moore-Greitzer model is not an accurate model when the throttle operates far away to the left of the peak of the characteristic.

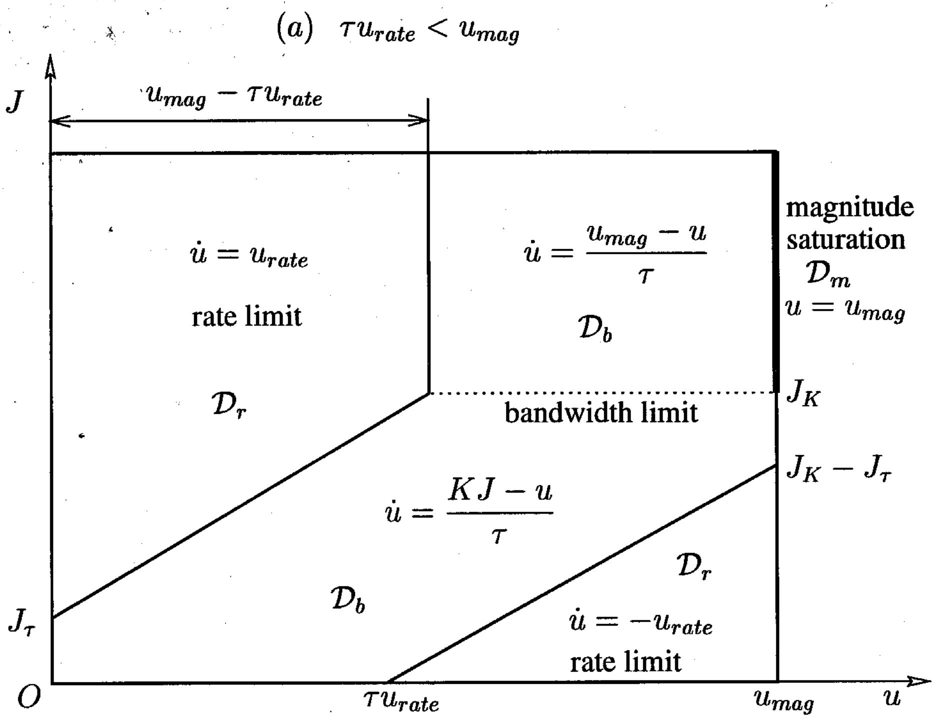


Figure 5.7: Controller dynamics in u-J plane when $\tau u_{rate} < u_{mag}$.

5.2.2 Bifurcations and phase portraits of the reduced system

In the following we describe the qualitative dynamics of the reduced two-dimensional system (5.5) and (5.10) by providing the phase portraits for different throttle coefficients.

Due to the magnitude saturation and the rate limits, different regions in the u-J phase plane are governed by different control laws. Let

$$J_K = \frac{u_{mag}}{K}, \quad J_\tau = \frac{\tau u_{rate}}{K},$$

Then we have the following two cases. If $\tau u_{rate} < u_{mag}$, then there are three regions in the phase space: the bandwidth limited region $\mathcal{D}_b$, the rate limited region $\mathcal{D}_r$, and the magnitude saturation part $\mathcal{D}_m = \{(u, J) |\, u = u_{mag}, J > J_K\}$ (see Figure 5.7). If $\tau u_{rate} > u_{mag}$, then there are two regions: the bandwidth limited region $\mathcal{D}_b$, and the magnitude saturation part $\mathcal{D}_m$ (see Figure 5.8). Due to the magnitude saturation constraint, the phase space is only limited to the region $0 \le u \le u_{mag}$, and $J \ge 0$.

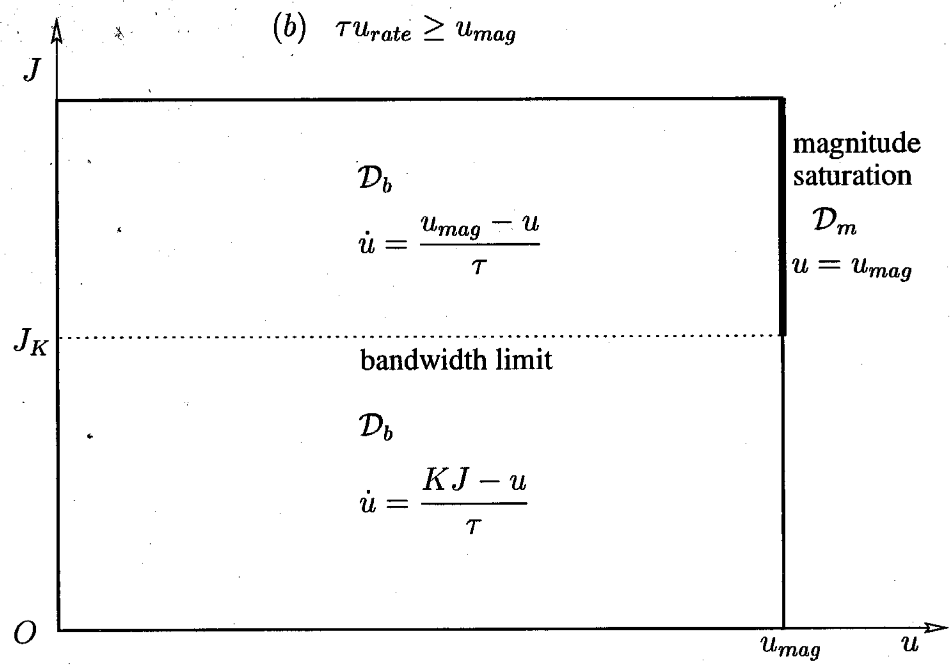


Figure 5.8: Controller dynamics in u-J plane when $\tau u_{rate} \geq u_{mag}$.

Let $\gamma_s^* = \gamma_s - u_{mag}$, and

$$\gamma_H \approx \gamma_0 - \frac{\alpha_1 K + \alpha_2}{\alpha\alpha_1\alpha_2\tau}, \quad \gamma_K = \gamma_0 - u_{mag} - \frac{1}{\alpha_1} h\left(\tfrac{u_{mag}}{K}\right), \quad \tau_H \approx \frac{K}{\alpha\alpha_2 u_{mag}},$$

where $h(J) := \sum_{k=2}^{n} \alpha_k J^{k-1}$. Assuming $u_{mag} < \gamma_s - \gamma_0$, the qualitative phase portraits for the reduced two-dimensional system (5.10) and (5.5) at different operating throttle positions are shown in Figure 5.9. The sequence of the figures is when the throttle is closing beginning from the stable side of the compressor characteristic, i.e., when the throttle coefficient γ is decreasing.

(1) Figure 5.9 (a) shows that if the throttle is operated at a throttle position γ such that $\gamma > \gamma_s^*$, then the axisymmetric equilibria O is the only equilibrium and it is globally stable.

(2) Figure 5.9 (b) shows that if the throttle is operated at γ such that $\gamma = \gamma_s^*$, then a saddle-node bifurcation occurs. The equilibrium splits into two as γ decreases.

(3) Figure 5.9 (c) shows that if the throttle is operated at γ such that $\gamma_0 < \gamma < \gamma_s^*$, then two equilibria are born from the saddle node bifurcation. S_1 is a saddle

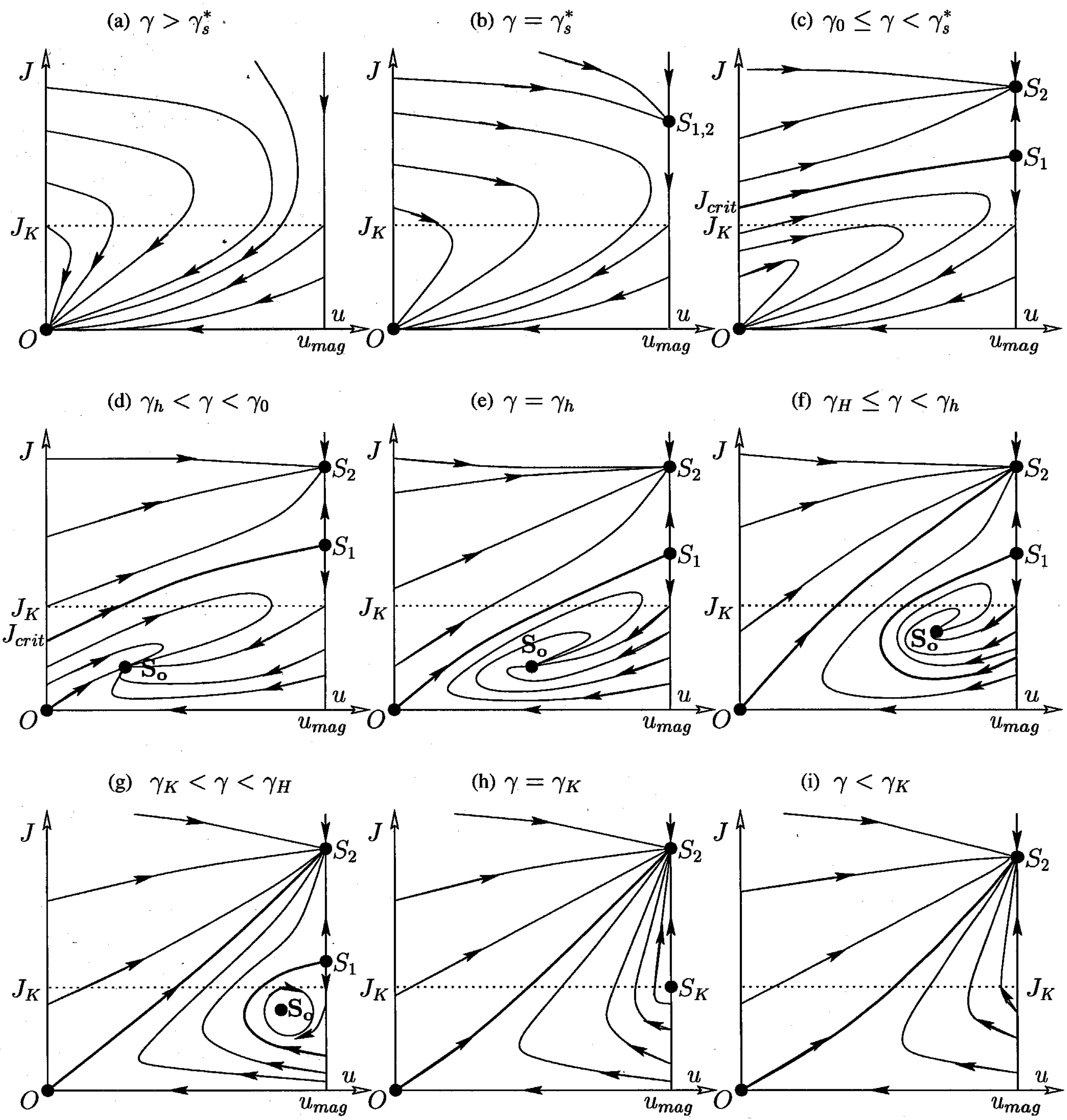


Figure 5.9: Qualitative phase portraits of the reduced system.

denoting the unstable stall equilibrium when the actuator is fully open. S_2 is a sink denoting the stable stall equilibrium when the actuator is fully open. There are two different regions in the u-J plane divided by the stable manifold of the saddle S_1, denoted by $W^s(S_1)$: the trajectories above $W^s(S_1)$ converge to the fully developed rotating stall equilibrium S_2, while the trajectories below $W^s(S_1)$ converge to the stable axisymmetric equilibrium O.

(4) Figure 5.9 (c) shows at $\gamma = \gamma_0$, i.e., the throttle is operated at the peak of the compressor characteristic, a transcritical bifurcation occurs on the axisymmetric equilibrium O.

(5) Figure 5.9 (d) shows that if the throttle is operated at γ such that $\gamma_h < \gamma < \gamma_0$, then the stabilized stall equilibrium S_o is born from the axisymmetric equilibrium O. At the same time, O becomes a saddle. The trajectories below $W^s(S_1)$ converge to the stabilized stall equilibrium S_o.

(6) Figure 5.9 (e) shows that when $\gamma = \gamma_h$, heteroclinic connection occurs between the unstable axisymmetric equilibrium O and the unstable stall equilibrium S_1, and the heteroclinic orbit is structurally unstable. This implies that if the bleed valve actuator is initially closed, i.e., $u(0) = 0$, then any nonzero initial condition of J would go to fully developed rotating stall and at the same time the actuator saturates. γ_h can be found numerically by integrating a point near S_1 backward in time at different throttle coefficients.

(7) Figure 5.9 (f) shows that if the throttle is operated at γ such that $\gamma_H < \gamma \leq \gamma_h$, then the heteroclinic connection is broken and $W^s(S_1)$ is connecting $u = u_{mag}$ instead of $u = 0$. An interesting point is that if the initial condition of J is small, then for any initial condition of u, the trajectory will go to the fully developed stall equilibrium S_2; while if the initial condition of J is not too small, then there exist initial conditions of u such that the trajectory goes to the stabilized stall equilibrium S_o.

(8) Suppose $\tau > \tau_H$, then at $\gamma = \gamma_H$, a Hopf bifurcation occurs on the stabilized

stall branch S_o: S_o becomes unstable and stable limit cycles are born (see Figure 5.9 (g)). If $\tau \leq \tau_H$, then there is no Hopf bifurcation on the stabilized stall branch S_o. Here we assume $\gamma_h > \gamma_H$.

(9) Figure 5.9 (h) shows that at $\gamma = \gamma_K$, the source S_o, the stable limit cycle, and the saddle S_1 are all collapsed together through a degenerate saddle-node bifurcation.

(10) Figure 5.9 (i) shows that if the throttle is operated at γ such that $\gamma < \gamma_K$, then the degenerate node S_K disappears and every trajectory except $u = 0$ converges to the fully developed stall equilibrium S_2.

If $u_{mag} > \gamma_s - \gamma_0$, then we have $\gamma_s^* < \gamma_0$. This implies the saddle-node bifurcation of the stalled branch is at a lower throttle coefficient than that of the peak of the compressor characteristic.

A typical phase portrait for the reduced system for the case (d) in Figure 5.9 is given by Figure 5.10. The main new feature of the phase portrait is that there is strongly attracting manifold whose physical meaning is that the fully developed stall occurs and the bleed valve opens until it saturates.

5.2.3 Control analysis

In this section we give the problem statement of control analysis. We omit the solution here for the control law (5.5), but we sketch the ideas of how to solve the problem. In the next section we given a solution to the "bang-on" control, which is a limiting case of the control law (5.5).

The noise in the flow through the compressor is a complicated issue. The turbulence level in the inlet flow may be the source of localized disturbances. But other disturbances such as inlet swirl and inlet distortion may have larger amplitude. We model the noise as a closed set of initial conditions for J in the system, i.e.,

$$\mathcal{E} = \{J \,|\, 0 \leq J \leq \epsilon\},$$

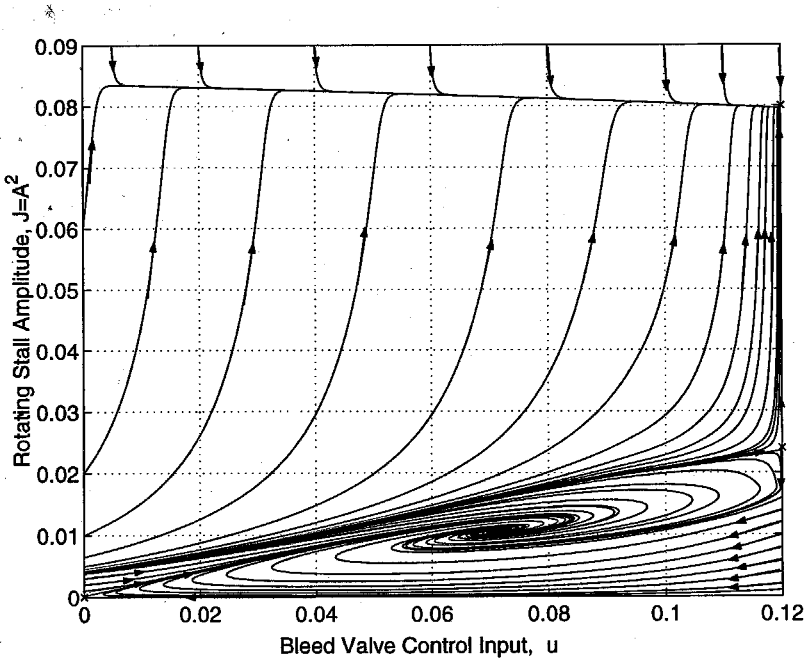


Figure 5.10: A phase portrait from numerical simulation of the reduced system (case (d) in Figure 5.9). The main throttle is operated at 4 percent of the stall flow to the left of the peak. The values of the parameters are: $u_{mag} = 0.12$, $u_{rate} = 0.0385$, $\tau = 0.6293$, $\delta = -0.036$, $K = 2K^* = 6.7257$. Here $K^* := -\frac{\alpha_2}{\alpha_1}$ is the minimum gain such that the bifurcation for the closed loop system is supercritical. The equilibria are denoted by "x."

where ϵ is the largest initial condition for J and is called the noise level. We remark here that in practice ϵ can be increasing as the throttle coefficient γ decreases. Here we assume ϵ is a constant.

The operability boundary $\delta(u(\cdot), \mathcal{E})$ for the reduced system (5.10) and (5.5) is defined as the minimum throttle coefficient δ at which none of the trajectories with initial conditions $u(0) = 0$ and $J(0) \in \mathcal{E}$ reaches the fully developed rotating stall equilibrium. The operability enhancement is defined as the difference between the stability of the uncontrolled system and that of the controlled system, i.e.,

$$\Delta(u(\cdot), \mathcal{E}) := \delta(0, \mathcal{E}) - \delta(u(\cdot), \mathcal{E}).$$

The control analysis problem is: given all the parameters in the control law (5.5) and determine the operability enhancement $\Delta(u(\cdot), \mathcal{E})$.

It is trivial to show that the operability boundary for the uncontrolled system is given by $\delta(0,\mathcal{E}) = -\frac{1}{\alpha_1}\sum_{k=2}^{n}\alpha_k\epsilon^{k-1} \approx -\frac{\alpha_2\epsilon}{\alpha_1} > 0$. This implies that rotating stall occurs to the right of the peak of the characteristic. Now from Figure 5.9 it is clear that at the operability boundary of the controlled system, we have $J_{crit} = \epsilon$, where J_{crit} is the y-coordinate of the intersection between $W^s(S_1)$, the stable manifold of S_1, and the straight line $u = 0$. J_{crit} can be determined by integrating backward in time of the local stable manifold of S_1. Exact analytic solution of $W^s(S_1)$ can be difficult to find since the system is nonlinear and $W^s(S_1)$ might travel through different regions with different control laws (see Figure 5.7 and 5.8). But we can obtain approximate solution of $W^s(S_1)$ assuming the shape and matching the boundary conditions at $u = 0$ and $u = u_{mag}$. We omit the details here, but we will illustrate the idea in the next section for the bang-on control law. The stable manifold $W^s(S_1)$ can also be found by numerical integration of the reduced system with an initial condition near the saddle S_1 backward in time. The stability boundary can be found by binary search on γ until $J_{crit} = \epsilon$ is satisfied.

5.3 Control Synthesis

In this section we claim the "bang-on" control law is optimal in the sense that it maximizes the operability enhancement when there are magnitude and rate limits on the actuators. We derive approximate formulas for the operability enhancement as a function of noise and actuator limits. The formulas will be used in the next section to compare with experiments.

5.3.1 System reduction and bang-on control law

In the following we consider the control input u in the function set $\mathcal{U}$ defined as

$$\mathcal{U} := \{u \mid 0 \le u \le u_{mag}, |\dot{u}| \le u_{rate}, u \text{ is piecewise smooth}\}.$$

The goal is to find the control laws to maximize the operability enhancement. As for the control analysis problem, we first reduce the order of the Moore-Greitzer model with control $u(\xi) \in \mathcal{U}$.

Proposition 5.2 *If B is small, $\psi_c(\cdot)$ is analytic, and the axisymmetric disturbances are small, then the Moore-Greitzer model (5.1) with control input $u \in \mathcal{U}$ can be approximated by the following one dimensional system:*

$$\dot{J} \;=\; \alpha J\left(\alpha_1(\delta+u)+\sum_{k=2}^{n}\alpha_k J^{k-1}\right), \tag{5.11}$$

where δ_1 and α_2 are given by (5.6) and (5.7), respectively, and other α's are selected in the same way as in Section 5.2.1.

Sketch of proof of Proposition 5.2: The reduction is the similar to Proposition 5.1. Let

$$\begin{aligned}\phi &= \beta_{11}\delta+\beta_{12}J+\beta_{13}u+\text{h.o.t.},\\ \psi &= \beta_{21}\delta+\beta_{22}J+\beta_{23}u+\text{h.o.t..}\end{aligned}$$

By differentiating these equalities and use the system dynamics, we get

$$\begin{array}{lll}\beta_{11}=\sqrt{\Psi_0}, & \beta_{12}=\sqrt{\Psi_0}\,K+\frac{\gamma_0\psi_c''}{8\sqrt{\Psi_0}}, & \beta_{13}=\sqrt{\Psi_0},\\ \beta_{21}=0, & \beta_{22}=\frac{\psi_c''}{4}, & \beta_{23}=0.\end{array}$$

So the dynamics of the reduced system is given by

$$\dot{J}=\alpha_1(\delta+u)J+\alpha_2 J^2+\text{h.o.t.},$$

where α_1 and α_2 are given by (5.6) and (5.7), respectively. The other coefficients in (5.11) are used to match the hysteresis loop and the growth rate of stall at the bifurcation point. ■

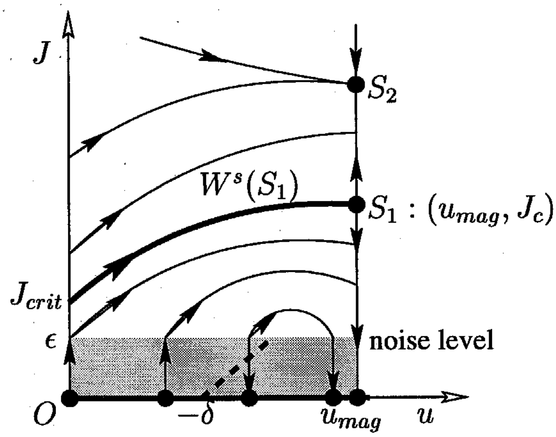


Figure 5.11: A phase portrait for bang-on control with threshold. The dashed line denotes the unstable stall equilibria, and the shaded region is the noise set.

The "bang-on" controller is defined as

$$\dot{u} = \begin{cases} 0 & \text{if } J \leq J_{thresh}, \text{ or } J > J_{thresh} \text{ and } u = u_{mag}, \\ u_{rate} & \text{if } J > J_{thresh} \text{ and } 0 \leq u < u_{mag}, \end{cases} \tag{5.12}$$

where ξ is the nondimensional time, and J_{thresh} is the threshold. Usually J_{thresh} is set to be above the noise level, i.e., $J_{thresh} > \epsilon$. The "bang-on" control law can be obtained from the control law (5.5) by increasing K and decreasing τ. Then the bandwidth limit region $\mathcal{D}_b$ shrinks to the empty set and the whole phase region is rate limited. A phase portrait for the reduced system (5.11) with the bang on control law is given by Figure 5.11. The line segment $\{(u, J) \mid 0 \leq u \leq u_{mag}, J = 0\}$ forms a continuum of equilibria, with $(u, 0)$ being stable for $-\delta < u \leq u_{mag}$ and unstable for $0 \leq u \leq -\delta$. It should be mentioned that for the bang-on control law (5.12) the actuator is not actuated if the stall disturbance is in the noise level. If the actuator is actuated when the disturbances are under the noise level, induced oscillations might occur. Consider the following control law:

$$\dot{u} = \begin{cases} -u_{rate} & \text{if } J \leq J_{thresh}, \\ 0 & \text{if } J > J_{thresh} \text{ and } u = u_{mag}, \\ u_{rate} & \text{if } J > J_{thresh} \text{ and } 0 \leq u < u_{mag}. \end{cases} \tag{5.13}$$

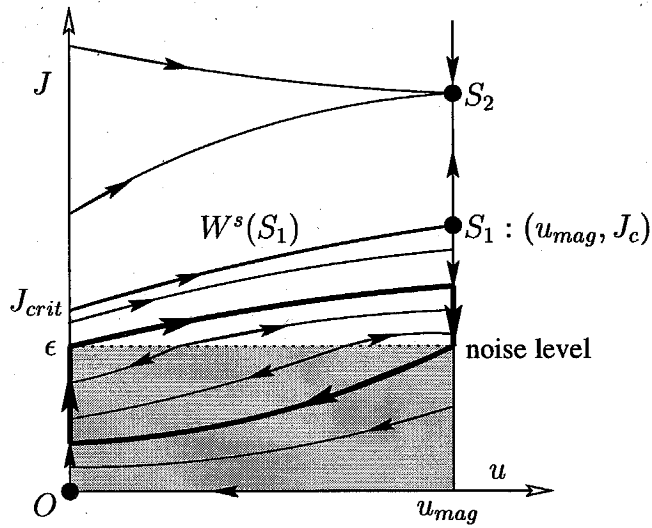


Figure 5.12: A phase portrait for bang-on control without a threshold.

The actuator would close if the disturbance is under the noise level. A phase portrait for the reduced system (5.11) with control law (5.13) is shown in Figure 5.12. It can be seen that a stable limit cycle exists, and the actuator experiences magnitude and rate saturations during the cycle. In practice, this would lead to "chattering" of the actuator, and the system may become unstable if there are sensing uncertainties and delays.

5.3.2 Problem formulation

Let $\mathcal{A}_\delta = \{J \mid 0 \leq J \leq a(\delta)\}$, where $a(\delta)$ denotes the maximum amount of rotating stall that can be tolerated. Usually $a(\delta)$ is chosen to be larger than the noise level and smaller than the magnitude of fully developed stall. The noise set is given by $\mathcal{E} = \{J \mid 0 \leq J \leq \epsilon\}$. In the following we assume $a(\delta) = J_c(\delta)$, where $J_c(\delta)$ is the y-coordinate of the saddle S_1 (see Figure 5.11). Let $u(0)$ be an initial condition for u. We assume $u(0) = 0$, which means that the bleed valve is initially closed. Let $(J_\delta(\xi; e), u_\delta(\xi; e))$ be the trajectory of (5.11) from the initial condition $(J(0), u(0)) = (e, 0)$. We say the initial condition e is $\mathcal{A}_\delta$ attractive if $\limsup_{\xi\to\infty} J_\delta(\xi; e) \leq a(\delta)$. We define the operability boundary for the initial condition e and the control $u(\xi)$ as $\delta_c(u, e) := \min\{\delta \mid e \text{ is } \mathcal{A}_\delta\text{-attractive}\}$. Define $\delta_c(u, \mathcal{E}) := \max_{e\in\mathcal{E}} \delta_c(u, e)$ as the operability boundary for the noise level $\mathcal{E}$ with control $u(\xi)$, and $\delta_o(\mathcal{E}) := \delta_c(0,\mathcal{E})$ as operability boundary for the uncontrolled system. In other words, $\delta_c(u,\mathcal{E})$ is the minimum throttle coefficient such that none of the trajectories for (5.11) and control $u(\xi)$ with initial conditions $u(0) = 0$ and $J(0) \in \mathcal{E}$ goes to the fully developed rotating stall. We define the operability enhancement as $\Delta(u,\mathcal{E}) := \delta_c(0,\mathcal{E}) - \delta_c(u,\mathcal{E})$. The optimal control problem is a minimax problem defined as finding a control $u^* \in \mathcal{U}$, such that the operability enhancement is maximized, i.e.,

$$\Delta(u^*,\mathcal{E}) := \max_{u\in\mathcal{U}} \Delta(u,\mathcal{E}) = \delta_o(\mathcal{E}) - \min_{u\in\mathcal{U}} \max_{e\in\mathcal{E}} \delta_c(u,e).$$

Theorem 5.1 *Suppose the assumptions (A1) and (A2) are true, then for any $e \in \mathcal{E}$, we have*

$$\begin{aligned}
\delta_c(u^*,e) &= \min_{u\in\mathcal{U}} \delta_c(u,e), \\
\delta_c(u^*,e^*) &= \min_{u\in\mathcal{U}} \max_{e\in\mathcal{E}} \delta_c(u,e), \\
u^* &= \begin{cases} u_{rate}\xi & \text{if } 0 \le \xi < \xi_2, \text{ and } J > \epsilon, \\ u_{mag} & \text{if } \xi \ge \xi_2, \text{ and } J > \epsilon. \end{cases}
\end{aligned}$$

where $e^* = \epsilon$, $\xi_2 := \frac{u_{mag}}{u_{rate}}$.

The idea behind the proof of Theorem 5.1 is the simple fact that for the system (5.11) and (5.14), the stable manifold of the saddle S_1, denoted as $W^s(S_1)$, is monotonously increasing as u increases from 0 to u_{mag}. Also, any trajectory for the system (5.11) with a controller $u \in \mathcal{U}$ cannot intersect $W^s(S_1)$ transversally from above. The detail proof is given as follows.

Proof of Theorem 5.1: The key figure for the proof is Figure 5.13. The reduced system with bang-on control is given by

$$\dot{J} = \alpha\left[\alpha_1(\delta + u) + h(J)\right] J,$$

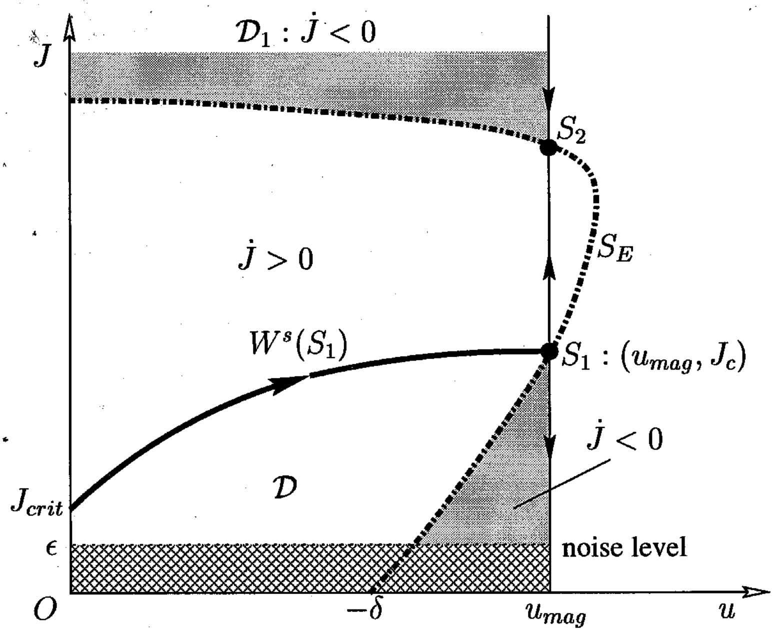


Figure 5.13: Key figure for the proof of Theorem 5.1.

$$\dot{u} = \begin{cases} 0 & \text{if } J \leq J_{thresh}, \text{ or } J > J_{thresh} \text{ and } u = u_{mag}, \\ u_{rate} & \text{if } J > J_{thresh} \text{ and } 0 \leq u < u_{mag}, \end{cases}$$

where $h(J) = \sum_{k=2}^{n} \alpha_k J^{k-1}$, and $J_{thresh} = \epsilon$. It is easy to see that the curve $\dot{J} = 0$ with $J \neq 0$ is given by the curve S_E:

$$S_E : u = -\delta - \frac{1}{\alpha_1} h(J).$$

From the assumptions (A1) and (A2), the shape and location of $\dot{J} = 0$ is the dash-dot line in Figure 5.13. Assume $u_{mag} > -\delta$, then the curve S_E intersects the saturation line $u = u_{mag}$ at two points S_1 and S_2: S_1 is a saddle and S_2 is a sink. The shaded regions in Figure 5.13 satisfy $\dot{J} < 0$. Let $W^s(S_1)$ be the stable manifold of S_1. It is easy to prove that $W^s(S_1)$ must be in the region $\dot{J} > 0$. Or in other words, on $W^s(S_1)$, we have $\dot{J} > 0$. It is also clear that the trajectories under $W^s(S_1)$ denoted by $\mathcal{D}$ will remain in $\mathcal{D}$. Let $\mathcal{A}_\delta = \{J \mid 0 \leq J \leq J_c(\delta)\}$, it is clear that the region $\mathcal{D}$ is $\mathcal{A}_\delta$ attractive, i.e., all the trajectories in $\mathcal{D}$ satisfy $J(\xi) \leq J_c(\delta)$ for any $\xi \geq 0$. Since the noise set is $\mathcal{E} = \{J \mid 0 \leq J \leq \epsilon\}$, the operability boundary for the bang-on control denoted by $\Delta(u(\cdot), \mathcal{E})$ satisfies $J_{crit}(\Delta) = \epsilon$, where J_{crit} is the intersection of $W^s(S_1)$ and $u = 0$.

Now fix $\delta = \Delta(u, \mathcal{E})$, and let

$$v \in \mathcal{U} := \{u \mid 0 \le u \le u_{mag}, |\dot{u}| \le u_{rate}, u \text{ is piecewise smooth}\}$$

be any control law with magnitude limit u_{mag} and rate limits $\pm u_{rate}$. Let (u_1, J_1) be a point on $W^s(S_1)$, then for the reduced system we have

$$\dot{J}\Big|_v = \dot{J}\Big|_u, \quad \dot{v} \le \dot{u} = u_{rate}, \tag{5.14}$$

where $\dot{J}\Big|_v$ is $\dot{J}$ with control input $v(\xi)$, and $\dot{J}\Big|_u$ is $\dot{J}$ with control input $v(\xi)$. Relations in (5.14) implies that the vector field for control input $v(\xi)$ on $W^s(S_1)$ point to the outside of $\mathcal{D}$. In other words, each trajectory for control $v(\xi)$ with initial condition $(0, J_{crit})$ satisfying $J_{crit} > \epsilon$ will not enter region $\mathcal{D}$, and will eventually intersects with S_E since the trajectory is either in the region $\dot{J} > 0$, or in the region $\mathcal{D}_1$ with $\dot{J} < 0$ (see Figure 5.13). So the operability boundary for control input $v(\xi)$ denoted by $\Delta(v(\cdot), \mathcal{E})$ satisfies

$$\Delta(v, \mathcal{E}) \ge \Delta(u, \mathcal{E}).$$

The fact that ϵ is the worst noise is clear since if a trajectory for the bang-on control input u with initial condition $(u, J) = (0, J_0)$ $(J_0 \le \epsilon)$ converges to the fully developed stall equilibrium S_2, i.e., $(0, J_0)$ is not $\mathcal{A}_\delta$ attractive, then we must have $J_0 > J_{crit}$ (see Figure 5.13). So $\epsilon \ge J > J_{crit}$, and the trajectory with initial condition $(u, J) = (0, \epsilon)$ will also converge to S_2, i.e., $(0, \epsilon)$ is not $\mathcal{A}_\delta$ attractive.

We remark here that control laws in $\mathcal{U}$ might create equilibria or limit cycles for the controlled system. But Theorem 5.1 only claims that it is the bang-on control law that prevents more trajectories going to the fully developed stall equilibrium. ■

The types of control laws for the disturbances in the noise level do not affect the operability enhancement. To avoid the chattering effects, we set $\dot{u} = 0$ for $J \leq \epsilon$.

5.3.3 Calculation of operability enhancement

Now we consider the reduced order system with the optimal control u^* and the worst noise e^*. The goal is to solve for $W^s(S_1)$. Specifically, we consider the following system

$$\dot{J} = \alpha_1(\delta + u)J + \alpha_2 J^2, \tag{5.15}$$

$$\dot{u} = \begin{cases} 0 & \text{if } J \leq \epsilon, \text{ or } J > \epsilon \text{ and } u = u_{mag}, \\ u_{rate} & \text{if } J > \epsilon \text{ and } 0 \leq u < u_{mag}, \end{cases} \tag{5.16}$$

with initial conditions and final conditions

$$\begin{aligned} &u(0) = 0, \quad u(\xi_2) = u_{mag}, \\ &J(0) = \epsilon, \quad J(\xi_2) = -\tfrac{\alpha_1}{\alpha_2}(u_{mag} + \delta), \end{aligned} \tag{5.17}$$

From Theorem 5.1 we have

$$\Delta(u^*, \mathcal{E}) = \delta_o - \delta_c(u^*, e^*),$$

where δ_o can be calculated as $-\frac{\alpha_2\epsilon}{\alpha_1}$, and $\delta_c(u^*, e^*)$ can be found by solving the system (5.15) and (5.16) with with the boundary conditions (5.17). In the following we solve for δ. Let

$$\Delta^* = 1 + \frac{\alpha_1\delta}{\alpha_2\epsilon}, \quad \eta = \alpha_2\epsilon\xi, \quad f = \frac{J}{\epsilon}, \quad \lambda = \frac{\sigma}{\eta_2},$$

$$\sigma = \frac{-\alpha_1 u_{mag}}{\alpha_2\epsilon}, \quad \eta_2 = \alpha_2\epsilon\xi_2, \quad f' = \frac{df}{d\eta}, \quad J(\xi_2) = \epsilon(\sigma + 1 - \Delta^*).$$

Then the equations (5.15) and (5.16) with initial and final conditions (5.17) can be written as follows:

$$\begin{aligned} f' &= (\Delta^* - 1 - \lambda\eta)f + f^2, && (5.18)\\ f(0) &= 1, && (5.19)\\ f(\eta_2) &= 1 + \sigma - \Delta^*. && (5.20) \end{aligned}$$

The exact solution of (5.18) with initial condition (5.19) and final condition (5.20) is given by the following integral-algebraic equation

$$1 - \frac{e^{-(\frac{\sigma}{2}+1-\Delta^*)\eta_2}}{1+\sigma-\Delta^*} = \int_0^1 \eta_2 e^{-(\frac{\sigma}{2}\zeta^2+(1-\Delta^*)\zeta)\eta_2} d\zeta. \qquad (5.21)$$

Suppose the solution to (5.21) is $\Delta^*(\sigma, \eta_2)$. Then it is easy to see that if $\eta_2 = 0$, then $\Delta^*(\sigma, \eta_2) = \sigma$. This implies that the operability enhancement is σ if the bleed valve actuator is infinitely fast. The following proposition gives the operability enhancement when the bleed valve is slow.

Proposition 5.3 *If $\eta_2 \gg \sigma$, then $\Delta^*(\sigma, \eta_2) = \dfrac{\sigma}{\eta_2} + \mathcal{O}\left(\dfrac{\sigma^2}{\eta_2^2}\right)$.*

To prove the proposition, we need the following lemma on asymptotic properties of the incomplete Γ-function.

Lemma 5.1 *Define the incomplete Γ-function as*

$$\Gamma(\alpha, x) = \int_x^\infty e^{-t} t^{\alpha-1} dt, \quad (\alpha > 0)$$

then as $x \to \infty$, $\Gamma(\alpha, x)$ can be expressed as the following series

$$\Gamma(\alpha, x) = x^{\alpha-1} e^{-x} \left[\sum_{m=0}^{M-1} \frac{(-1)^m \Gamma(1-\alpha+m)}{x^m \Gamma(1-\alpha)} + \mathcal{O}\left(x^{-M}\right) \right],$$

where the Γ-function $\Gamma(\alpha)$ is defined as

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt.$$

Proof of Proposition 5.3:

Define the *complementary error function* as

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt = 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

It is trivial to show

$$\operatorname{erfc}(x) = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}, x^2\right).$$

By Lemma 5.1, we know for x very large, we can write

$$\operatorname{erfc}(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} \left(\frac{1}{x} - \frac{1}{2x^3}\right) + \mathcal{O}\left(\frac{e^{-x^2}}{x^5}\right). \tag{5.22}$$

Consider the integral in the right hand side of the integral-algebraic equation (5.21), it can be show by straightforward calculation that

$$\begin{aligned} RHS &= \int_0^1 \eta_2 e^{-\left(\frac{\sigma}{2}\zeta^2 + (1-\Delta)\zeta\right)\eta_2} d\zeta \\ &= \sqrt{\frac{\pi\eta_2}{2\sigma}} e^{\frac{(1-\Delta)^2}{2\sigma}\eta_2} \left[\operatorname{erfc}\left(\sqrt{\frac{\eta_2}{2\sigma}}(1-\Delta)\right) - \operatorname{erfc}\left(\sqrt{\frac{\eta_2}{2\sigma}}(1-\Delta+\sigma)\right)\right] \\ &= \frac{1}{1-\Delta} - \frac{\sigma}{\eta_2(1-\Delta)^3} - \frac{e^{-\left(\frac{\sigma}{2}+1-\Delta\right)\eta_2}}{1+\sigma-\Delta} + \mathcal{O}\left(\frac{1}{{\eta_2}^3}\right). \end{aligned}$$

We have utilized (5.22) in the last equality. Now by equating this to the left hand side of (5.21), we get

$$\Delta(1-\Delta)^2 = \frac{\sigma}{\eta_2} + \mathcal{O}\left(\frac{1}{{\eta_2}^3}\right). \tag{5.23}$$

Suppose $\Delta^*(\sigma, \eta_2)$ is the solution to this equation. Letting $\eta_2 \to \infty$, we have $\Delta^*(\sigma, \eta_2) \to 0$ or $\Delta^*(\sigma, \eta_2) \to 1$.

Now we show that

$$\lim_{\eta_2 \to \infty} \Delta^*(\sigma, \eta_2) \neq 1.$$

By substituting $\Delta^*(\sigma, \eta_2) = 1$ into the integral-algebraic equation (5.21), we get

$$LHS = 1 - \frac{1}{\sigma} e^{-\frac{1}{2}\sigma\eta_2},$$

$$RHS = \sqrt{\frac{2\eta_2}{\sigma}} \int_0^{\sqrt{\frac{\sigma\eta_2}{2}}} e^{-\zeta^2} d\zeta = \sqrt{\frac{\pi\eta_2}{2\sigma}} + \mathcal{O}(1), \quad (\eta_2 \approx \infty).$$

Hence, when $\eta_2 \approx \infty$, $LHS \neq RHS$. So we have

$$\lim_{\eta_2 \to \infty} \Delta^*(\sigma, \eta_2) = 0.$$

By solving (5.23), we get

$$\Delta^*(\sigma, \eta_2) = \frac{\sigma}{\eta_2} + \mathcal{O}\left(\frac{1}{\eta_2{}^2}\right).$$

■

The implication of Proposition 5.3 is that when $\eta_2 \gg \sigma$, i.e., the bleed valve is very slow, the noise is large, or the compressor characteristic is very steep near the peak, then the operability enhancement is given by

$$\Delta(u^*, \mathcal{E}) = \delta_o(\mathcal{E}) - \delta = \frac{\alpha_2\epsilon}{-\alpha_1}\Delta^* \approx \frac{4(m+\mu)u_{rate}}{\epsilon\left(\psi_c''' + \frac{\gamma_0\psi_c''^2}{\sqrt{\Psi_0}}\right)}. \tag{5.24}$$

This implies that the operability enhancement decreases if either the noise level ϵ, or $|\psi_c''(\Phi_0)|$, or $\psi_c'''(\Phi_0)$ increases. If the rate limit u_{rate} decreases, then the operability enhancement decreases. In the multi-stage high speed compressor in aircraft engines, $|\psi_c''(\Phi_0)|$ increases sharply as the engine speed increases, so the operability enhancement is expected to reduce drastically when engine speeds up (see [35]). The third derivative ψ_c''' denotes the relative curvature on each side of the peak. If the left (unstable) side is steeper (or shallower) than the right (stable) side, then $\psi_c''' > 0$ (or $\psi_c''' < 0$), then formula (5.24) implies that the operability enhancement decreases as the unstable part of the compressor characteristic becomes steeper. This result is interesting because the effectiveness of the controller is determined by the unstable part of the characteristic which cannot be directly measured in experiments. The experiment in the next section proves this point, i.e., the rate requirement for the actuator reduced drastically when the unstable part of the characteristic is made shallower by steady air injection upstream of the rotor.

It might be difficult to solve $\Delta^*(\sigma, \eta_2)$ explicitly from (5.21). In the following we solve $\Delta^*(\sigma, \eta_2)$ by approximating the stable manifold of the saddle. Suppose $W^s(S_1)$ is the stable manifold of the saddle S_1 in the (η, f) plane, and $W^s(S_1)$ is parameterized by $f = f(\eta)$ in the (η, f) plane. Then from (5.18), (5.19) and (5.20) we get the boundary conditions of $W^s(S_1)$

$$\begin{aligned} &f(0) = 1, \quad f(\eta_2) = 1 + \sigma - \Delta^*, \\ &f'(0) = \Delta^*, \quad f'(\eta_2) = 0. \end{aligned} \tag{5.25}$$

We assume $f(\eta)$ has the following form

$$\begin{aligned} f(\eta) &= \bar{f}\left(1 - C\frac{2\sigma}{\pi(1+\sigma)}g(\eta)\right), \\ g(\eta) &= \arctan\left(\frac{\pi\lambda}{4\sigma}(1+\sigma)(\eta-\eta_2)^2\right), \end{aligned} \tag{5.26}$$

where the unknowns $\bar{f}$ and C are to be determined by boundary conditions. Letting $f(\eta)$ satisfy the boundary conditions (5.25), and considering that $\sigma \gg 1$ in practice, we get

$$\Delta_1^*(\sigma, \eta_2) = \frac{\sigma}{1 + \frac{1}{8}\pi\sigma{\eta_2}^2 \arctan\left(\frac{1}{4}\pi\sigma\eta_2\right)}. \tag{5.27}$$

If we use the form of (5.26) to satisfy the following boundary conditions of $W^s(S_1)$

$$\begin{aligned} &f(0) = 1, \quad f(\eta_2) = 1 + \sigma - \Delta^*, \\ &f'(\eta_2) = 0, \quad f''(\eta_2) = -\lambda(1 + \sigma - \Delta^*), \end{aligned} \tag{5.28}$$

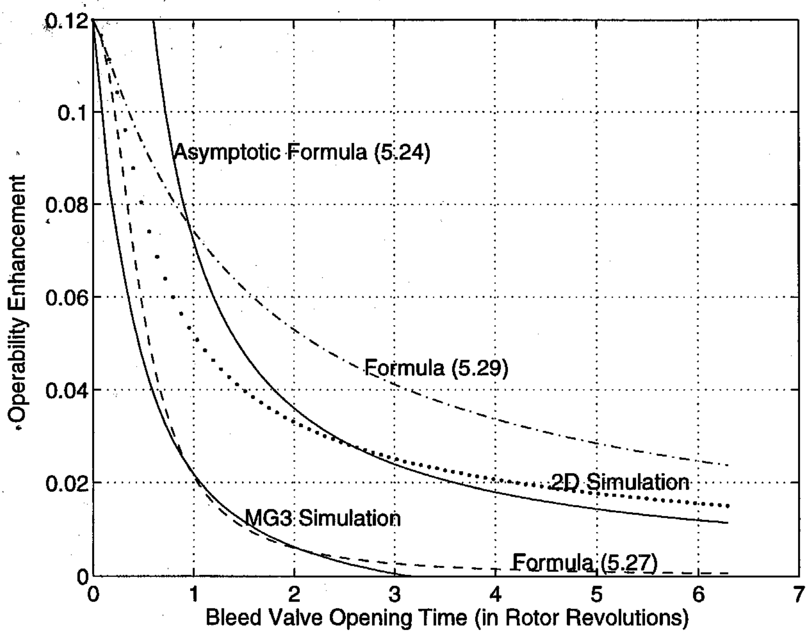


Figure 5.14: Comparison of operability enhancement for the bang-on control law obtained through simulations of the full Moore-Greitzer model and the reduced 2D model, and the predictions of the formulas (5.27), (5.29) and the asymptotic formula (5.24). The y-axis is the operability enhancement normalized by the throttle coefficient at the peak of the characteristic, γ_0.

then we get

$$\Delta_2^*(\sigma, \eta_2) = \sigma \frac{1 - \frac{2}{\pi} \arctan\left(\frac{\pi}{4}(1+\sigma)\eta_2\right)}{1 - \frac{2}{\pi}\frac{\sigma}{1+\sigma} \arctan\left(\frac{\pi}{4}(1+\sigma)\eta_2\right)}. \tag{5.29}$$

The comparisons for predictions of the formulas (5.24), (5.27), and (5.29), and numerical simulation of the full Moore-Greitzer model and the reduced two-dimensional system is given by Figure 5.14. Although the formulas do not match the simulations qualitatively, they give the same trend as the bleed valve rate limit is reduced. The formulas (5.27) and (5.29) will be used in the next section to compare with the experimental results a low speed compressor.

5.3.4 Effects of time delay

In this section, we present the effects of time delay on the operability enhancement for the bang-on control. Finite bandwidths of sensors, the filters, and the computer hardware cause time delay between the actual flow signal at the sensor position and the control input in the control logic embedded in the compressor. Spatial separation between the downstream bleed valve and the stall disturbances within the compressor is another source of time delay due to the compressibility and the inertia of air. The main point of this section is that the operability enhancement decreases *exponentially* with time delay, noise level, and the second and third derivatives of the compressor characteristic.

By considering the time delays, the bang-on control law is give by

$$\dot{u}^* = \begin{cases} 0 & \text{if } 0 \leq \xi < \xi_1, \\ 0 & \text{if } \xi \geq \xi_1,\ J \leq \epsilon; \text{ or } \xi \geq \xi_1\ J > \epsilon,\ u = u_{mag}, \\ u_{rate} & \text{if } \xi \geq \xi_1, \text{ and } J > \epsilon, \end{cases} \tag{5.30}$$

with initial condition $u(0) = 0$. Here ξ_1 is the time delay. Assuming $\xi_1 \ll \frac{u_{mag}}{u_{rate}}$ and u_{rate} is small, then by similar asymptotic analysis as in the rate limits case, the operability enhancement is given by

$$\Delta_d(u^*, \mathcal{E}) \approx \frac{\alpha_2 \epsilon}{-\alpha_1} \cdot \frac{\sigma}{\bar{\eta} e^{\eta_1}} = \Delta_r e^{-\frac{\epsilon \xi_1}{4(m+\mu)}\left(\psi_c''' + \frac{\gamma_0 \psi_c''^2}{\sqrt{\Psi_0}}\right)}, \tag{5.31}$$

where

$$\sigma = \frac{-\alpha_1 u_{mag}}{\alpha_2 \epsilon}, \quad \eta_1 = \alpha_2 \epsilon \xi_1, \quad \bar{\eta} = \frac{\alpha_2 \epsilon u_{mag}}{u_{rate}}, \quad \Delta_r = \frac{4(m+\mu) u_{rate}}{\epsilon \left(\psi_c''' + \frac{\gamma_0 \psi_c''^2}{\sqrt{\Psi_0}}\right)}.$$

Here Δ_r is the operability enhancement for the pure rate limit case. It can be seen that the time delay is more detrimental than rate limit since the operability enhancement decreases exponentially with the time delay ξ_1, the noise level ϵ, and the second and third derivatives of the compressor characteristic at the peak, while in the pure rate limit case, the operability enhancement depends on the reciprocal of these parameters. For high speed multi-stage compressors, the curvature of the compressor characteristic near the peak is large, so the operability enhancement could be very sensitive to time delays.

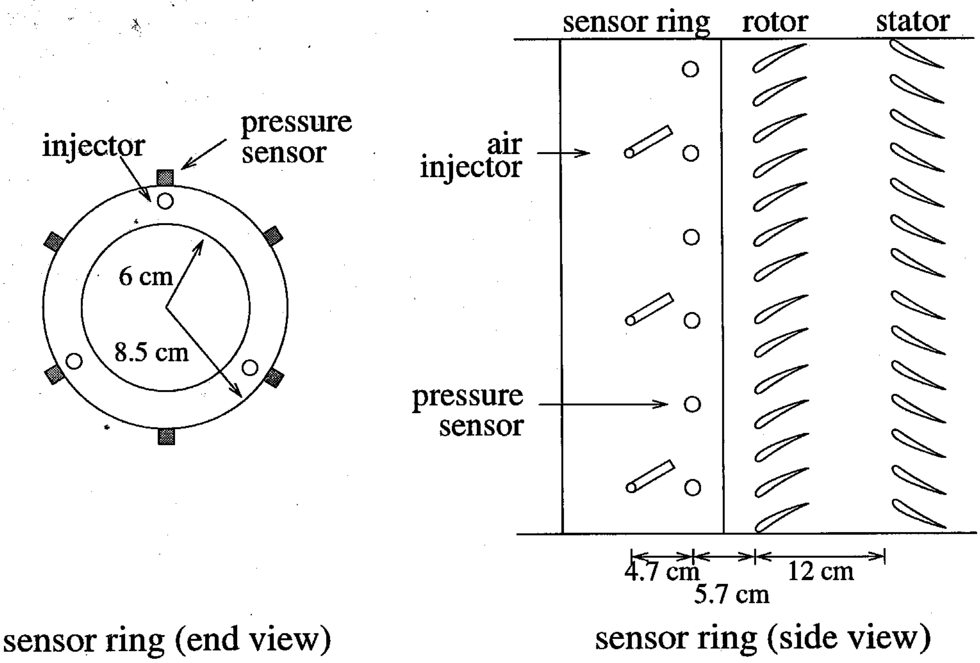


Figure 5.15: Sensor and injection actuator ring (courtesy Simon Yeung).

5.4 Comparison with Experiments

In this section we describe experiments to measure rate requirement for different compressor characteristics actuated by different level of steady air injection upstream. We compare the predictions of rate requirement from the theoretic formulas derived in the previous section and the experiments. The experiments were done by Simon Yeung [81, 82, 83, 84, 85]. In the following we only sketch the experimental setup and procedures, details can be found in [81].

5.4.1 Experimental setup and procedures

The Caltech compressor rig is a single-stage, low-speed, axial compressor. Figure 5.15 shows the sensor and injection actuator ring and Figure 5.16 shows the entire rig. Experiments are run with a rotor frequency of 100 Hz, and under this condition the frequency of rotating stall is 65 Hz . Experimental data taken for a stall transition that the stall cell grows from the noise level to its fully developed size in approximately 30 msec (3 rotor revolutions).

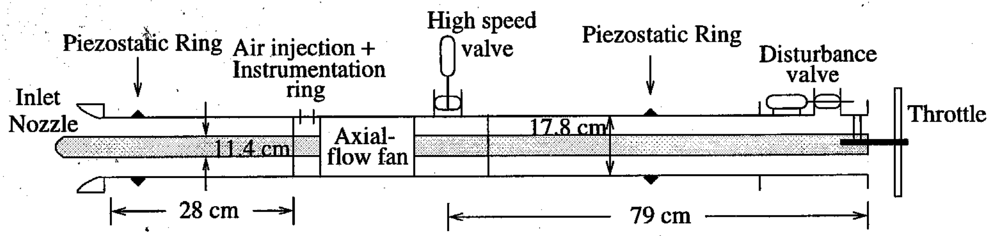


Figure 5.16: Experimental setup (courtesy Simon Yeung).

The sensors are composed of six static pressure transducers with 1000 Hz bandwidth that are evenly distributed along the annulus of the compressor. The amplitude and phase of the first and second mode of the stall cell can be obtained using discrete Fourier transform. A high speed valve that is capable of bleeding 12% of the flow at the stall inception point is used for control of rotating stall. A low speed valve is used to move the operating points, i.e., to change the throttle coefficient γ) of the system. Three air injectors are used to change the shape of the compressor characteristic $\psi_c(\cdot)$.

In the experiments, the injector angle with respect to the axial flow direction is varied between 27° and 40° in the opposite direction of the rotor rotation, and the back pressure of the injectors is varied between 40 to 60 psi. There are 17 different scenarios with different angles of injection and different back pressure of injector. At the various injection settings, experiments are carried out to obtain the rate values required for peak stabilization. Peak stabilization is achieved if the conditions $\Phi \geq 0.9\Phi_0$ and $A \leq 0.5A_{\text{nom}}$ are met during the experiment, where Φ is the nondimensional axial velocity, A the amplitude of the first Fourier mode, Φ_0 the nondimensional axial velocity at stall inception, and A_{nom} the amplitude of fully developed stall without bleed valve control. It should be noted that Φ_0 and A_{nom} are different for each of the air injector settings. The experiment procedure is as follows. First set the throttle at the stable side of the characteristic to make sure there is no stall. By turning off the controller and closing the throttle, stall point of the open loop system is recorded. Then reset the throttle to the stable side of the characteristic, increase the rate and gain, and close the throttle to the open loop stall point. This procedure is repeated by increasing the rate until the conditions of peak stabilization are met. Among the 17 injection settings, peak stabilization is achieved in 11 cases and the

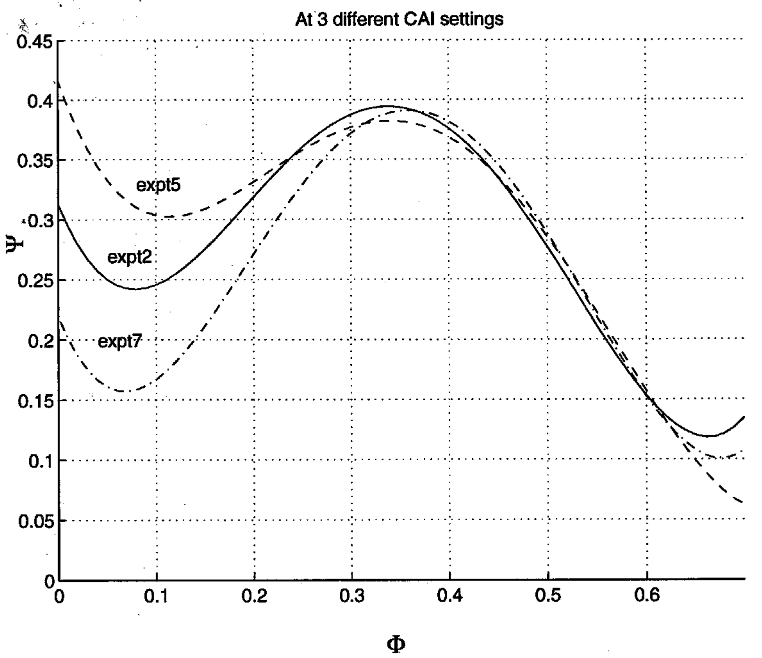


Figure 5.17: Identified compressor characteristics at three different continuous air injection settings (courtesy Simon Yeung).

nominally stable side of the compressor characteristic is experimentally recorded at each of the 11 settings. The unstable sides for each of these cases are identified by using surge cycles data with an algorithm proposed by Behnken [14]. For this study, a fourth order polynomial is used to approximate the piecewise continuous curve for each case. The polynomial compressor characteristics are to be used in the theoretical formulas (5.27) and (5.29). The fitted compressor characteristics for three injector settings are shown in Figure 5.17.

5.4.2 Comparisons of theory and experiments

Based on the functional dependence of the analytical relations for the minimum rate requirement on $\psi_c''(\Phi_0)$ and $\psi_c'''(\Phi_0)$, an examination of Figure 5.17 would indicate that expt5 should require the least rate while expt7 should require the most. With the back pressure to the air injectors being 55 psi, Figure 5.18 shows the open loop and closed loop behavior in the Φ-Ψ plane and the γ-J plane. The subcritical bifurcation of the open loop system associated with stall inception is changed to supercritical

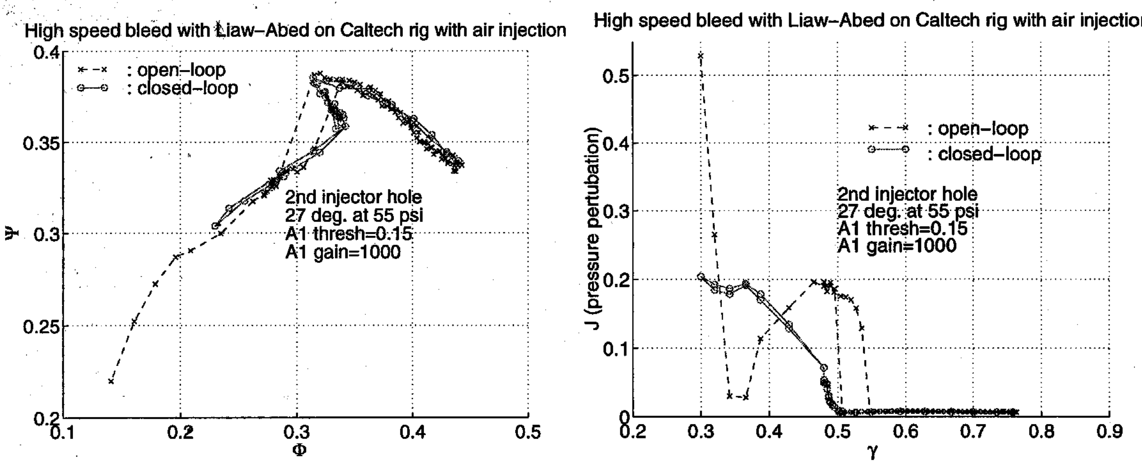


Figure 5.18: Open- and closed-loop behavior of system on Φ-Ψ plane for control with bleed valve and continuous air injection at 55 psi injector back pressure (courtesy Simon Yeung).

for the closed loop system. An interesting phenomenon is that the amplitude of the first stall mode becomes smaller when the throttle is operated around 0.37. This is because at these throttle values, the second mode dominates the compressor stall. Another interesting behavior of the closed loop system in the Φ-Ψ plane is that the stabilized stall branch tilts to the right of the peak before folding back to the left. The phenomenon is different from the closed loop system behavior in the experiments of Eveker *et al.* (1995, 1998): the stabilized stall branch tilts to the left of the peak. The back-tracking stall branch can be explained from the three state Moore-Greitzer model as follows. From the center manifold analysis in the proof of Proposition 5.1, we know that for the control law $u = KJ$, the center manifold is given by

$$\phi = \sqrt{\Psi_0}\delta + \frac{\gamma_0 \psi_c''}{8\sqrt{\Psi_0}} J + \text{h.o.t.} \tag{5.32}$$

$$\psi = \frac{\psi_c''}{4} J + \text{h.o.t.} \tag{5.33}$$

where $\phi = \Phi - \Phi_0$, $\psi = \Psi - \Psi_0$. On the other hand, for the control law $u = KJ$, the dynamics on the center manifold is given by

$$\dot{J} = \alpha_1(\delta + KJ)J + \alpha_2 J^2 + \text{h.o.t.}$$

So on the stabilized stall branch, we have

$$\delta = -\left(K + \frac{\alpha_2}{\alpha_1}\right) J, \tag{5.34}$$

where α_1 and α_2 are given by (5.6) and (5.7), respectively. Now, by substituting (5.34) into (5.32) and (5.33), we get

$$\phi = -\frac{\psi_c'''}{8\psi_c''} J + \text{h.o.t.} \tag{5.35}$$

Equation (5.33) implies that on the stabilized stall branch we must have $\psi < 0$, since $\psi_c'' < 0$ and $J > 0$, i.e., the stabilized stall branch always drops below the peak. If $\psi_c''' > 0$, i.e., the unstable (left) part of the characteristic is steeper than the stable (right) part of the characteristic, then we have $\phi > 0$ by equation (5.35), which means the stabilized stall branch tilts to the right of the characteristic. This back-tracking phenomenon is observed in our experiments (see Figure 5.18). On the other hand if $\psi_c''' < 0$, i.e., the unstable (left) part of the characteristic is shallower than the stable (right) part of the characteristic, then we have $\phi < 0$ by equation (5.35), which means the stabilized stall branch tilts to the left of the characteristic. This agrees the phenomena observed by Eveker *et al* (1995, 1998). It should be mentioned that the case $\psi_c''' > 0$ requires faster bleed valves than the case when $\psi_c''' < 0$ provided all other conditions are the same.

In Figure 5.19, the dashed line represents the one-to-one line between the theory-predicted and experimentally obtained rate values. The uncertainties associated with disturbance valve in the experiments is about three percent of throttle coefficient at the peak of the characteristic. So in the two formulas (5.27) and (5.29) we use three percent operability enhancement and solve for the rate. The rate obtained from the formulas are plotted against the rate obtained on the experiments in Figure 5.19. Although the rate predicted by formulas is quantitatively different from that of the experiment, they give qualitatively the same trend. For example, the rate requirement order of expt5, expt2 and expt7 is consistent for both the experiment and the theoret-

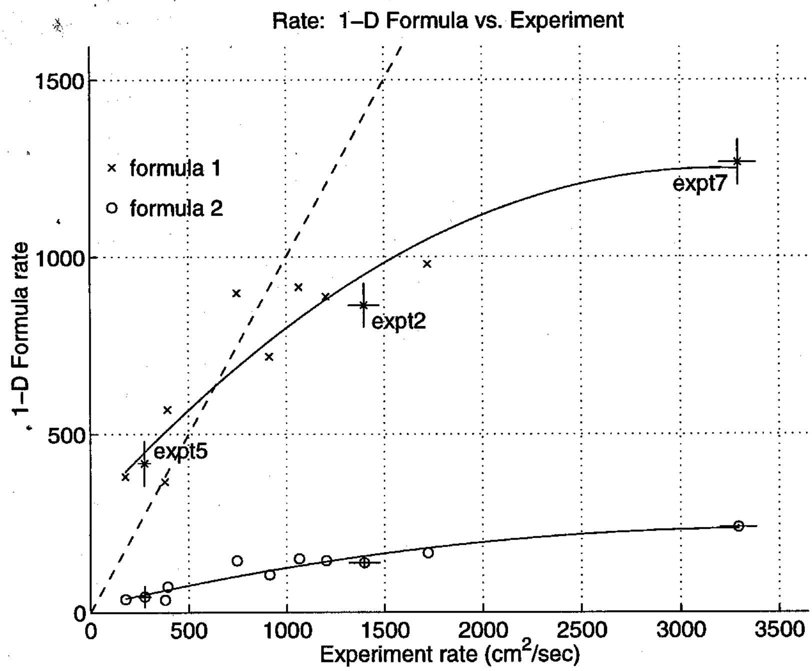


Figure 5.19: Comparison of rate predicted by theory and experiments (courtesy Simon Yeung). The bars in the x-direction indicate the standard deviation associated with the fluid- and human-induced error in computing the theoretical rate values (through identifying the unstable of $\Psi_c(\Phi)$ using surge cycle data), and the bars in the y-direction the standard deviation in the experimental values.

ical formulas (see Figure 5.19). For the comparison results, it should be pointed out that the experiments show that the rate requirement for peak stabilization is reduced from approximately 3300 cm^2/sec to below 200 cm^2/sec are reported by varying the amount of compressor characteristic actuation. This is equivalent to a reduction of bandwidth from about 145 Hz to below 10 Hz. Regarding the theoretical tools used for prediction, one can see from the figure that formula (5.27) seems to predict the rate requirement more accurately than formula (5.29). The main difference between the two expressions originates from the different ways an approximation to the solution to the stable manifold of the saddle S_1. The difference between the predictions of the two formulas has also been shown in Figure 5.14. Also, the difference between rate predictions by the formulas and the experiments becomes larger as the rate increases. The reason is that in the experiments, the rate was set as a constraint so that the valve would not open or close faster than that. As the constraint relaxes (the rate increases), the valve hit the rate constraint during only part of the cycle, and is limited by bandwidth during the other part of the cycle. But the theoretical formulas assume the valve hits the rate limit all the time, and thus they predicted lower rate requirement. Nevertheless, both the formulas and experiments have predicted the same trend: the steeper the unstable part of the compressor characteristic near the peak, the more rate it requires. This can be seen from Figure 5.17 and Figure 5.19.

Chapter 6 Application to Active Control of Rotating Stall and Surge Using Magnetic Bearings

A set of magnetic bearings supporting the compressor rotor is a potential actuator for active control of rotating stall and surge. Based on a first-principles model we show that using this type of actuation, the first harmonic mode of rotating stall is linearly controllable, but the second harmonic mode and the surge mode are linearly uncontrollable. We then give an explicit procedure for designing feedback laws such that the first mode is linearly stabilized and the criticality of the Hopf bifurcations of the second mode and the surge mode are supercritical. We also investigate the effects of magnitude saturation on the regions of attraction. We demonstrate the theoretical results by numerical simulations of a model for a transonic compressor at the NASA Glenn Research Center. This chapter is based on the paper [76].

In [65], extensive analytic and numeric analysis has been done on modeling the effects of tip clearance and its potential applications to active control. But the technique is linear control design. The results in this chapter address issues of nonlinear control law design so that in addition to stabilizing the first stall mode, the criticality of Hopf bifurcations to the second stall mode and the surge mode is changed to supercritical.

6.1 Problem Statement

In the following we model the effects of tip clearance into the Moore-Greitzer model. Suppose the rotor position is controlled by the magnetic bearing actuator to create a nonaxisymmetric tip clearance (see Figure 6.1). Then, to first order, the tip clearance

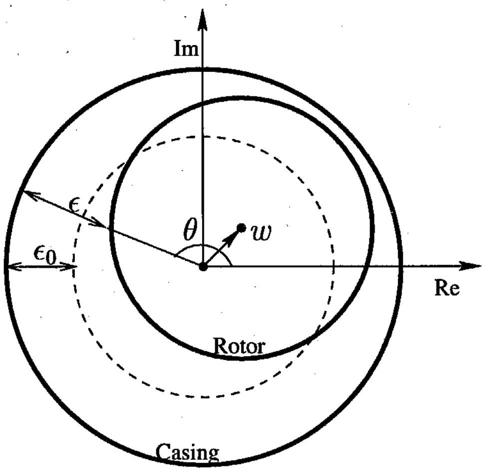


Figure 6.1: Geometric relation between rotor position and tip clearance.

ϵ is sinusoidal in the circumferential coordinate, θ:

$$\epsilon(\theta) = \epsilon_0 - \frac{1}{2}\left(we^{-i\theta} + w^* e^{i\theta}\right),$$

where ϵ_0 is the nominal tip clearance and $w = X + iY$ is the rotor position. The local (in θ) compressor pressure rise depends strongly on this tip clearance variation. Consistent with our current understanding, we express the pressure rise $\tilde{\psi}_c$ as a nonlinear function of local flow rate $\tilde{\Phi}(\theta, \xi)$ and a linear function of ϵ:

$$\tilde{\psi}_c(\tilde{\Phi}, \epsilon) = \tilde{\psi}_c(\tilde{\Phi}, \epsilon_0) + \frac{\partial \tilde{\psi}_c}{\partial \epsilon}(\tilde{\Phi}, \epsilon_0)(\epsilon - \epsilon_0) = \psi_c(\tilde{\Phi}) - c(\tilde{\Phi})(\epsilon - \epsilon_0),$$

where $\psi_c(\tilde{\Phi}) = \tilde{\psi}_c(\tilde{\Phi}, \epsilon_0)$ is nominal compressor map and $c(\tilde{\Phi}) = -\frac{\partial \tilde{\psi}_c}{\partial \epsilon}(\tilde{\Phi}, \epsilon_0)$ is the sensitivity of the axisymmetric pressure rise to the tip clearance variation.

Suppose also that at any given time the axial flow at the compressor face can be expressed as:

$$\tilde{\Phi}(\theta) = \Phi + a_1 e^{i\theta} + a_2 e^{i2\theta} + a_1^* e^{-i\theta} + a_2^* e^{-i2\theta},$$

i.e., we only consider flow disturbances in the zeroth mode (axisymmetric disturbance) plus first and the second modal waves representing stall disturbances. Then the Moore-Greitzer model (see [8, 9]) with magnetic bearing actuators truncated at the fourth order is given by

$$
\begin{aligned}
\dot{\Phi} &= \frac{1}{l_c}\left[-\Psi + \psi_c(\Phi) + \psi_c''(\Phi)\left(|a_1|^2 + |a_2|^2\right) + \frac{1}{2}\psi_c'''(\Phi)\left(a_1^2 a_2^* + a_1^{*2} a_2\right)\right] \\
\dot{\Psi} &= \frac{1}{4B^2 l_c}\left[\Phi - \gamma\sqrt{\Psi}\right], \qquad (6.1) \\
\dot{a}_1 &= \frac{1}{m+\mu}\left[\left(\psi_c'(\Phi) - i\lambda\right) a_1 + \psi_c''(\Phi) a_2 a_1^* + \psi_c'''(\Phi)\left(\frac{1}{2}|a_1|^2 + |a_2|^2\right) a_1 + u\right] \\
\dot{a}_2 &= \frac{1}{\frac{m}{2}+\mu}\left[\left(\psi_c'(\Phi) - i2\lambda\right) a_2 + \frac{1}{2}\psi_c''(\Phi) a_1^2 + \psi_c'''(\Phi)\left(|a_1|^2 + \frac{1}{2}|a_2|^2\right) a_2\right]
\end{aligned}
$$

where $a_1(\xi) = A_1(\xi)e^{i\vartheta_1(\xi)}$ and $a_2(\xi) = A_2(\xi)e^{i\vartheta_2(\xi)}$ are first and second mode of nonaxisymmetric flow disturbance. Here u is the control input:

$$u = \frac{1}{2}c_0 w,$$

where $c_0 = -\frac{\partial \psi_c}{\partial \epsilon}(\Phi_0, 0)$ is the sensitivity of pressure rise to tip clearance variations at the peak of the compressor characteristic, Φ_0. Note in the above model we have omitted nonlinear effects of the tip clearance on pressure rise.

To make our overall goals clear, we first discuss the dynamics and bifurcations for the uncontrolled system, i.e., $u = 0$. As in the previous chapter, we treat the throttle coefficient γ as the bifurcation parameter.

The unstalled equilibria of the system are

$$\Phi = \Phi_e(\gamma), \quad \Psi = \Psi_e(\gamma) = \psi_c(\Phi_e(\gamma)), \quad a_1 = a_2 = 0,$$

which we call the axisymmetric equilibria. The bifurcations of the axisymmetric equilibria are:

- At the peak of the compressor characteristic $\gamma = \gamma_0$, double Hopf bifurcations occur for the first and second mode of rotating stall.

- If B is large, then at a positive slope of the characteristic $\gamma = \gamma_s < \gamma_0$, Hopf bifurcation to surge occurs.

It is clear from controlled system (6.1) that the first stall mode (a_1) is linearly controllable, the second mode (a_2) and the surge mode (Φ, Ψ) are linearly uncontrollable. To further elucidate the bifurcation characteristics, consider the case where $u = K_1 a_1$, with $\operatorname{Re} K_1 < 0$. This feedback stabilizes the first mode; the bifurcation properties of the remaining modes depend on the shape of the compressor characteristic as follows:

- if $\psi_c''' + \frac{\gamma_0 \psi_c''^2}{\sqrt{\Psi_0}} > 0$, then the Hopf bifurcation of the second mode is subcritical;
- if $\psi_c''' + \frac{\gamma_0 \psi_c''^2}{\sqrt{\Psi_0}} < 0$, then the Hopf bifurcation of the second mode is supercritical,

where all the derivatives of ψ_c are evaluated at Φ_0.

If B is large enough, then from [72], we know that

- if $\psi_c'''(\Phi_0) > 0$, then the Hopf bifurcation to surge is subcritical;
- if $\psi_c'''(\Phi_0) < 0$, then the Hopf bifurcation to surge is supercritical.

6.2 Low B Case

In this section we consider the case when B is small enough such that the axisymmetric dynamics are stable. We design feedback control laws such that the bifurcation to the second mode stall is supercritical. The results are given by the following proposition.

Proposition 6.1 *The following choice of feedback makes the closed loop system satisfy*

- *the first stall mode is stabilized,*
- *the Hopf bifurcation to the second mode is supercritical.*

(1) If $\psi_c''' < -\frac{\gamma_0 \psi_c''^2}{\sqrt{\Psi_0}}$*, then we choose* $u = K_1 a_1$ *with* $\operatorname{Re} K_1 < 0$.

Feedback chosen: N	$\|a_2\|^2$	$a_2 a_1^*$	$\|a_1\|^2$	ϕ	ψ
Gain required: K_3	K	$\beta_{31}^{*-1}K$	$\|\beta_{31}\|^{-2}K$	$\beta_{12}^{-1}K$	$\beta_{22}^{-1}K$

Figure 6.2: Feedback gains for different state variables.

(2) If $-\frac{\gamma_0\psi_c''^2}{\sqrt{\Psi_0}} \leq \psi_c''' < -\frac{\gamma_0\psi_c''^2}{2\sqrt{\Psi_0}}$, *then we choose* $u = K_1 a_1 + K_2 a_2$, $K_1, K_2 \in \mathbb{R}$, *such that*

$$\begin{aligned} K_1 &< 0, \\ \frac{K_1}{K_1^2+\omega_1^2} &> (m+\mu)\left(\frac{\psi_c'''}{\psi_c''^2} + \frac{\gamma_0}{2\sqrt{\Psi_0}}\right) \\ |K_2| &> K_{2m} \end{aligned}$$

where

$$K_{2m} = \sqrt{\frac{\frac{1}{2}\left[K_1^2 + (\omega_2-\omega_1)^2\right]\left(\psi_c''' + \frac{\gamma_0\psi_c''^2}{\sqrt{\Psi_0}}\right)}{-\left(\psi_c''' + \frac{\gamma_0\psi_c''^2}{2\sqrt{\Psi_0}} + \frac{\psi_c''^2}{m+\mu}\frac{-K_1}{K_1^2+\omega_1^2}\right)}}, \quad \omega_1 = \frac{\lambda}{m+\mu}, \quad \omega_2 = \frac{2\lambda}{\frac{m}{2}+\mu}.$$

(3) If $\psi_c''' \geq -\frac{\gamma_0\psi_c''^2}{2\sqrt{\Psi_0}}$, *then we choose* $u = K_1 a_1 + K_2 a_2 + K_3 N$. *The closed loop system has a supercritical bifurcation only when* N *is chosen to be* $|a_2|^2$, $a_2 a_1^*$, $|a_1|^2$, ϕ, *or* ψ, *where* $\phi = \Phi - \Phi_e(\gamma)$, $\psi = \Psi - \Psi_e(\gamma)$. K_3 *is chosen as in Table 6.2, where* $K_1, K_2, K \in \mathbb{C}$ *satisfy* $\operatorname{Re} K_1 < 0$, $K_2 \neq 0$,

$$\operatorname{Re}\left(\frac{\psi_c''\beta_{31}K}{K_1 - i\omega_1}\right) > \frac{1}{2}\left(\psi_c''' + \frac{\gamma_0\psi_c''^2}{\sqrt{\Psi_0}}\right) + |\beta_{31}|^2\left(\psi_c''' + \frac{\gamma_0\psi_c''^2}{2\sqrt{\Psi_0}} + \frac{\psi_c''^2}{m+\mu}\frac{-K_1}{K_1^2+\omega_1^2}\right),$$

where

$$\begin{aligned} \beta_{31} &= \frac{-K_2}{K_1 + i(\omega_2-\omega_1)}, \\ \beta_{12} &= \frac{\gamma_0\psi_c''}{2\sqrt{\Psi_0}}\left(1+|\beta_{31}|^2\right), \\ \beta_{22} &= \psi_c''\left(1+|\beta_{31}|^2\right). \end{aligned} \tag{6.2}$$

The proof of this proposition, using the center manifold theorem, is given in Section 6.5.

6.2.1 Implementation considerations

Having derived expressions for control laws that achieve the desired result, we turn now to some practical issues. First is the question of which feedback variables to use. A straight-forward (although not necessarily optimum) answer comes from sensing considerations: one must sense distributed pressure or velocity perturbations to linearly stabilize the first harmonic of rotating stall. Thus for the remainder of the chapter we concentrate on a control law which could utilize the same sensors for both first harmonic stabilization and bifurcation modification of the zeroth and second harmonics, i.e., $u = K_1 a_1 + K_2 a_2 + K_3 |a_2|^2$.

The next practical issue is that of gain selection. Because of the complexity of the expressions Proposition 6.1, the method by which gains should be chosen is not clear. To elucidate gain selection, consider the normal form for the dynamics on the center manifold for the closed loop system:

$$\dot{z} = (d_2\delta + i\omega_2)\delta z + \hat{\alpha}|z|^2 z + \text{h.o.t.} \tag{6.3}$$

where $\delta = \gamma - \gamma_0$, $d_2 = \dfrac{\sqrt{\Psi_0}\psi_c''}{\frac{m}{2} + \mu}$, and

$$\begin{aligned} \hat{\alpha} \;=\; & \frac{1}{\frac{m}{2} + \mu}\left[\frac{-\psi_c''\beta_{31}K_3}{K_1 - i\omega_1} + \frac{1}{2}\left(\psi_c''' + \frac{\gamma_0\psi_c''^2}{\sqrt{\Psi_0}}\right)\right. \\ & \left. +|\beta_{31}|^2\left(\psi_c''' + \frac{\gamma_0\psi_c''^2}{2\sqrt{\Psi_0}} - \frac{\psi_c''^2}{m+\mu}\frac{1}{K_1 - i\omega_1}\right)\right], \end{aligned} \tag{6.4}$$

where β_{31} is given by (6.2). If we define $\alpha := \operatorname{Re}\hat{\alpha}$, then α determines the bifurcation branch for the closed loop system (see the normal form (6.3)). If $\alpha > 0$ then the Hopf bifurcation for the closed loop system is subcritical; if $\alpha < 0$ then the Hopf bifurcation for the closed loop system is supercritical. The bifurcations for the open loop system and the closed loop system are given in Figure 6.3.

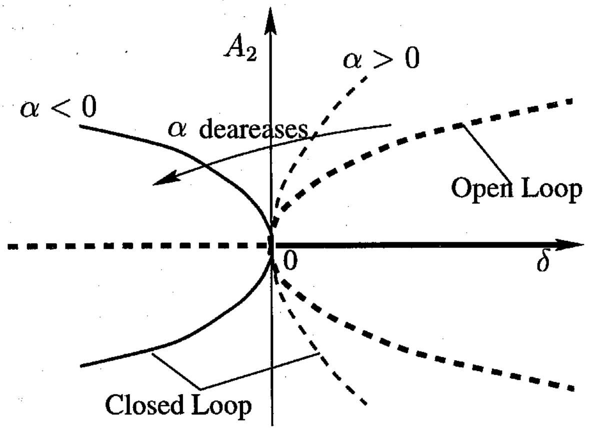


Figure 6.3: The bifurcations for the closed loop system by varying the control variable α.

The parameter α governs the "gain" of the system, and the resulting degree to which the bifurcation is "softened" by feedback. Thus α is an intuitive design parameter for trade studies, performed either experimentally or in higher fidelity simulations. Determining K_1, K_2, and K_3 from α requires a better understanding of the practical design issues. A brief discussion will therefore help to motivate our derivations.

Although the simple model used here does not predict it, the first harmonic mode is actually the first to go unstable in most compressors, and thus optimization of the linear feedback for this mode can be carried out independently of the other gains. The design of the linearized controller for first harmonic rotating stall typically involves system identification of first harmonic compressor dynamics, followed by robust linear controller design, as detailed in [78, 79]. It has already been shown experimentally that this type of control can extend the compressor's stable operating range to the point at which the second or zeroth harmonic go unstable (see [65]).

For our analysis purposes, it is consistent to view the first harmonic controller as a constant gain K_1 over the frequency range of interest. Experimental procedures to optimize this complex gain are well in hand. The question, then, is how to choose K_2 and K_3. First, we define

$$U = U(K_1) = \frac{\psi_c''}{(K_1 - i\omega_1)[K_1 + i(\omega_2 - \omega_1)]},$$

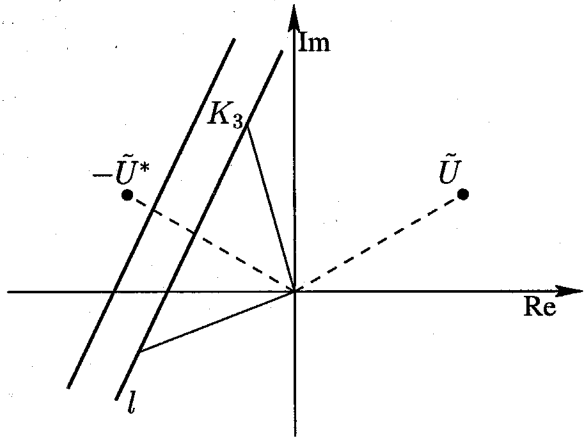


Figure 6.4: Design of nonlinear gains based on control variable α.

$$\begin{aligned} V &= V(K_1) = \frac{1}{|K_1 + i(\omega_2 - \omega_1)|^2}\left(\psi_c''' + \frac{\gamma_0 \psi_c''^2}{2\sqrt{\Psi_0}} - \frac{\psi_c''^2}{m+\mu}\mathrm{Re}\left\{\frac{1}{K_1 - i\omega_1}\right\}\right), \\ W &= \frac{1}{2}\left(\psi_c''' + \frac{\gamma_0 \psi_c''^2}{\sqrt{\Psi_0}}\right) - \left(\frac{m}{2} + \mu\right)\alpha. \end{aligned}$$

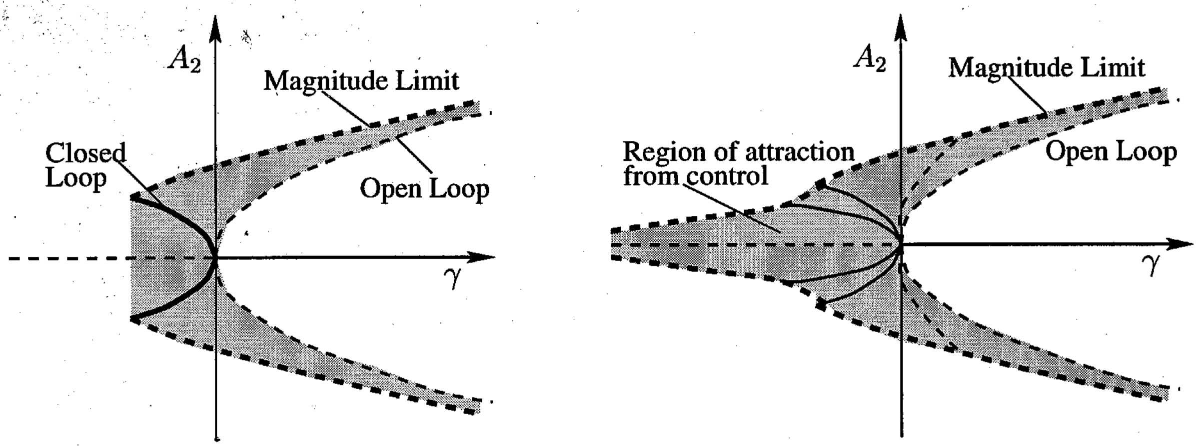
Then the real part of the equality (6.4) is given by

$$\mathrm{Re}\{UK_2K_3\} + V|K_2|^2 + W = 0.$$

Note also that α appears explicitly in the expression for W and that U and V depend only on K_1 and system parameters. We next consider case (3) in the proposition (i.e., modification of the second mode bifurcation from super- to sub-critical). If we set $\alpha < 0$ as indicated in Figure 6.3, then we know that $V > 0$ and $W > 0$. If we let $K_2 = R_2 e^{i\theta_2}$, choose R_2 as our design parameter for the second harmonic, and express $\tilde{U}$ as $Ue^{i\theta_2}$, we can write an expression that can be solved for K_3:

$$\mathrm{Re}\left\{\tilde{U}K_3\right\} + VR_2 + \frac{W}{R_2} = 0. \tag{6.5}$$

The geometric explanation is shown in Figure 6.4. Suppose K_1 and R_2 are chosen; then the set of K_3's that satisfy (6.5) is the straight line l which is perpendicular to



(a) Region of attraction for single α

(b) Maximum region of attraction by varying α

Figure 6.5: Effects of magnitude saturation limit.

the vector $-\tilde{U}^*$ and at a distance from the origin of $\frac{1}{|U|}\left(VR_2 + \frac{W}{R_2}\right)$. If we select different α and R_2, then the set K_3 is another straight line parallel to l, the phase of K_2, θ_2, would rotate the solution in the complex plane, but would have no real effect, due to the circumferential symmetry of the problem.

6.2.2 Magnitude saturation

Our design problem has been clarified somewhat, and reduced to the problem of choosing R_2 and α. We can further clarify the design process by considering the effects of magnitude saturation. Tip clearance modulation obviously carries with it a hard limit on actuation, one which cannot easily be increased. Thus the effect of magnitude saturation on the effectiveness of the control laws is critical. In Chapter 4 general results on the effects of magnitude saturation on bifurcation control have been derived that address this issue. We can apply these results here.

In the following we again assume that K_1 is fixed from the linear design. Assume now that the control magnitude is limited by $|u| \leq u_{mag}$, i.e.,

$$|K_1 a_1 + K_2 a_2 + K_3 |a_2|^2| \leq u_{mag}.$$

If this constraint is violated by the states, then the system will go unstable; this effect is illustrated in Figure 6.5. In Figure 6.5 (a), a single set of gains are chosen such that the bifurcation for the closed loop system is supercritical. The enlarged region of attraction to the stabilized stall equilibria and the unstalled equilibria is given by the shaded region, which is bounded by the saturation envelope composed of unstable equilibria for which the controller saturates. As the gains change such that α increase, then the amplitude of the stabilized limit cycle decreases, and the enlarged region of attraction becomes larger (see Figure 6.5 (b)). The shaded region is the region of attraction when α goes to infinity. Our goal is to choose K_2 and K_3 such that the length of the stabilized bifurcation branch is maximized under the constraint of magnitude saturation. The particular branch being maximized depends on α; the combination of α, K_2 and K_3 that maximizes the domain of attraction depends on the details of the dynamical system (i.e., the parameters in the model).

To derive expressions for K_2 and K_3, We consider the worst case, i.e.,

$$|K_1|A_1 + |K_2|A_2 + |K_3|A_2^2 \leq u_{mag}, \tag{6.6}$$

where A_1 and A_2 are the amplitudes of a_1 and a_2. From the center manifold relation in Section 6.5, we know $a_1 = \beta_{31}a_2 + \text{h.o.t.}$, so

$$A_1 \approx |\beta_{31}|A_2 = \frac{|K_2|}{|K_1 + i(\omega_2 - \omega_1)|}A_2,$$

substituting this into (6.6) and recalling that $|K_2| = R_2$, we get

$$|K_3|A_2^2 + \zeta R_2 A_2 - u_{mag} \leq 0,$$

where

$$\zeta = 1 + \left|\frac{K_1}{K_1 + i(\omega_2 - \omega_1)}\right|.$$

So we have $A_{2,1} \leq A_2 \leq A_{2,2}$, where

$$A_{2,2}(A_{2,1}) = \frac{2u_{mag}}{-\zeta R_2 \pm \sqrt{(\zeta R_2)^2 + 4|K_3|u_{mag}}}.$$

The goal is to select K_3 and R_2 such that $A_{2,2}$ is maximized. From (6.5), we have

$$|K_3| \geq \frac{1}{|U|}\left(VR_2 + \frac{W}{R_2}\right),$$

so we choose R_2 to minimize $f(R_2)$, where

$$f(R_2) := \zeta R_2 + \sqrt{(\zeta R_2)^2 + \frac{4u_{mag}}{|U|}\left(VR_2 + \frac{W}{R_2}\right)}.$$

Assuming that u_{mag} is small, we get an approximate solution

$$\begin{aligned} R_2 = R_m &= \left(\frac{Wu_{mag}}{2\zeta^2|U|}\right)^{\frac{1}{3}}, \\ A_2 = A_{2m} &= \frac{2\left(\frac{2|U|u_{mag}^2}{\zeta W}\right)^{\frac{1}{3}}}{1 + 3\sqrt{1 + \frac{4}{9}V\left(\frac{\sqrt{2}u_{mag}}{\zeta^2|U|}\right)^{\frac{2}{3}}}}. \end{aligned}$$

The optimal gains are thus given by

$$\begin{aligned} K_2 = K_{2m} &= R_m e^{i\theta_2}, \\ K_3 = K_m &= -\frac{U^* e^{-i\theta_2}}{|U|^2}\left(VR_2 + \frac{W}{R_2}\right). \end{aligned}$$

Based on this derivation, we can describe an experimental procedure for optimizing the gains of a particular system with magnetic bearings. After the linear controller has been designed and tested, α is chosen based on the desired shape of the bifurcated branch. Based on α and u_{mag}, optimal values for K_2 and K_3 can be computed. Next an experiment is performed to determine the range of mass flows over which bifurcation modification is successful, the limit cycle amplitudes, and (possibly) the domains of attraction of important operating points. Various values of α can then be compared. In addition, an experimentally optimized complex gain on the entire second harmonic feedback can be introduced, to account for the effects of unmodeled dynamics (actuator lags, unsteady losses, etc.). This additional phase shift is the spatial equivalent of a lead compensator.

6.3 High B Case

In this section we consider the case when B is large. In this case, the Hopf bifurcation to surge occurs very close to the peak of the characteristic. Since both of the surge mode and the second stall mode are not linearly controllable when the system operates beyond the bifurcation points, the nonlinear interactions of the unstable modes are strong. The goal is to design feedback control laws such that both the surge and stall bifurcations are supercritical.

Proposition 6.2 *If B is large enough, and $\psi_c''(\Phi_0) \neq 0$, then both Hopf bifurcations to the second mode stall and surge can be changed to supercritical via a smooth feedback.*

The proof of Proposition 6.2 is in Section 6.6 and the the rest of this section.

If B is large, it can be calculated that the Hopf bifurcation from the axisymmetric equilibria to surge occurs at $\gamma = \gamma_s \approx \gamma_0 + \hat{\delta}$ satisfying

$$\psi_c'(\Phi_e(\gamma_s)) = \frac{\gamma_s}{8B^2\sqrt{\Psi_e(\gamma_s)}}, \quad \hat{\delta} = \frac{\gamma_0}{8B^2\Psi_0\psi_c''},$$

where γ_0 is the throttle coefficient at the peak of the characteristic, and ψ_c'' is evaluated at the peak. Suppose the control feedback is given by

$$u = K_0 a_0 + K_1 a_1 + K_2 a_2 + K_3 |a_0|^2 + K_4 |a_2|^2,$$

where $a_0 = x + iy$ is the complex form of the surge mode given by

$$\begin{aligned} x &= \tilde{\phi}; \\ y &= -\frac{d_0}{\tilde{\omega}_0}\left(\delta + \hat{\delta}\right)\tilde{\phi} + \frac{1}{\tilde{\omega}_0 l_c}\tilde{\psi}, \\ \tilde{\phi} &= \phi - \frac{\gamma_0 \psi_c''}{2\sqrt{\Psi_0}}|a_2|^2 + \text{h.o.t.}, \\ \tilde{\psi} &= \psi - \psi_c''|a_2|^2 + \text{h.o.t.}, \end{aligned}$$

and

$$d_0 = \frac{\sqrt{\Psi_0}\psi_c''}{2l_c}, \quad \tilde{\omega}_0 = \omega_0\sqrt{1 - \frac{\gamma_0\psi_c''\left(\delta + \hat{\delta}\right)^2}{8\hat{\delta}}}, \quad \omega_0 = \frac{1}{2Bl_c}.$$

A normal form for the dynamics on the center manifold is given by

$$\dot{J}_0 = \nu_0\left(\delta - \hat{\delta}\right)J_0 + (\alpha_{11}J_0 + \alpha_{12}J_2)J_0 + \text{h.o.t.} \tag{6.7}$$

$$\dot{J}_2 = \nu_2\delta J_2 + (\alpha_{21}J_0 + \alpha_{122}J_2)J_2 + \text{h.o.t.} \tag{6.8}$$

where $J_0 \approx |a_0|^2$, $J_2 \approx |2a_2|^2$, $\nu_0 = \frac{\sqrt{\Psi_0}\psi_c''}{l_c}$, $\nu_2 = \frac{\sqrt{\Psi_0}\psi_c''}{m+2\mu}$, and α_{ij} $(i,j = 1,2)$ are functions of the feedback gains and are given in Section 6.6. Also in Section 6.6, we show that feedback gains could be selected such that the $\alpha_{ij} < 0$ $(i,j = 1,2)$. Now we discuss the dynamics of the normal form (6.7) and (6.8).

1. Types of equilibria

   (1) Axisymmetric equilibrium $(J_0, J_2) = (0,0)$.

   (2) Pure modes

      (i) Pure surge mode $(J_0, J_2) = \left(-\frac{\nu_0}{\alpha_{11}}\left(\delta - \hat{\delta}\right);0\right)$.

      (ii) Pure stall mode $(J_0, J_2) = \left(0, -\frac{\nu_0}{\alpha_{22}}\delta\right)$.

   (3) Mixed mode

$$(J_0, J_2) = \left(\frac{(\alpha_{12}\nu_2 - \alpha_{22}\nu_0)\delta + \alpha_{22}\nu_0\hat{\delta}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}, \frac{(\alpha_{21}\nu_0 - \alpha_{11}\nu_2)\delta - \alpha_{21}\nu_0\hat{\delta}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}\right).$$

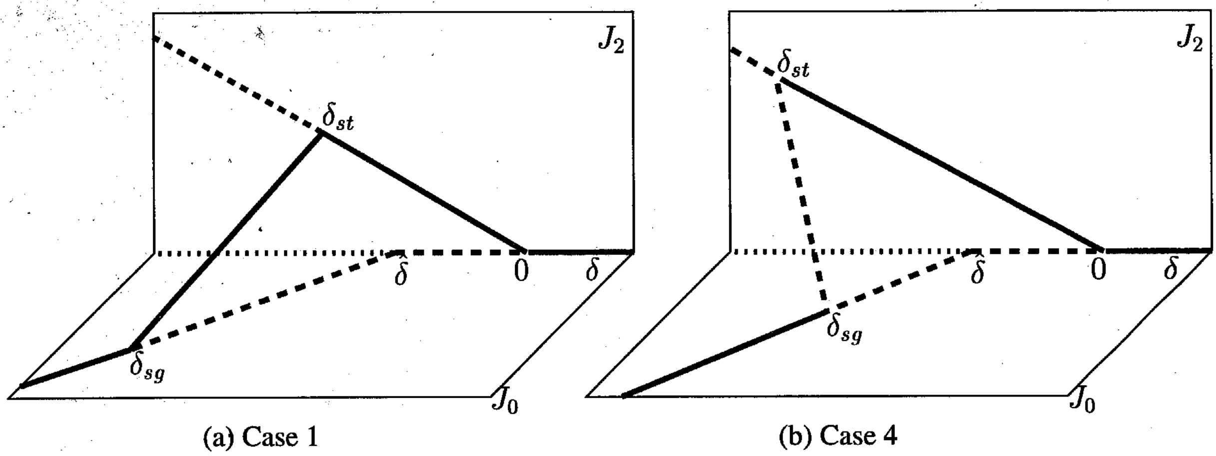


Figure 6.6: The bifurcations for closed loop system for the high B case. Solid: sinks, dashed: saddles, dotted: sources.

2. Bifurcation of the mode in the region $J_0 \geq 0$, $J_2 \geq 0$,

   (1) At $\delta = 0$, the pure stall mode merges with the axisymmetric equilibrium.

   (2) At $\delta = \hat{\delta}$, the pure surge mode merges with the axisymmetric equilibrium.

   (3) At

$$\delta = \delta_{st} = \frac{\hat{\delta}}{1 - \frac{\alpha_{12}\nu_2}{\alpha_{22}\nu_0}},$$

   the pure stall mode merges with the mixed mode.

   (4) At

$$\delta = \delta_{sg} = \frac{\hat{\delta}}{1 - \frac{\alpha_{11}\nu_2}{\alpha_{21}\nu_0}},$$

   the pure surge mode merges with the mixed mode.

There are six different cases of bifurcations for the closed loop system.

1. $\dfrac{\nu_0}{\nu_2} > \dfrac{\alpha_{11}}{\alpha_{21}} > \dfrac{\alpha_{12}}{\alpha_{22}}$, the bifurcations are shown in Figure 6.6 (a), and the phase portraits are in Figure 6.7.

2. $\dfrac{\alpha_{11}}{\alpha_{21}} > \dfrac{\nu_0}{\nu_2} > \dfrac{\alpha_{12}}{\alpha_{22}}$, the bifurcation is given by (a)-(d) in Figure 6.7 (in this case $\delta_{sg} > 0$).

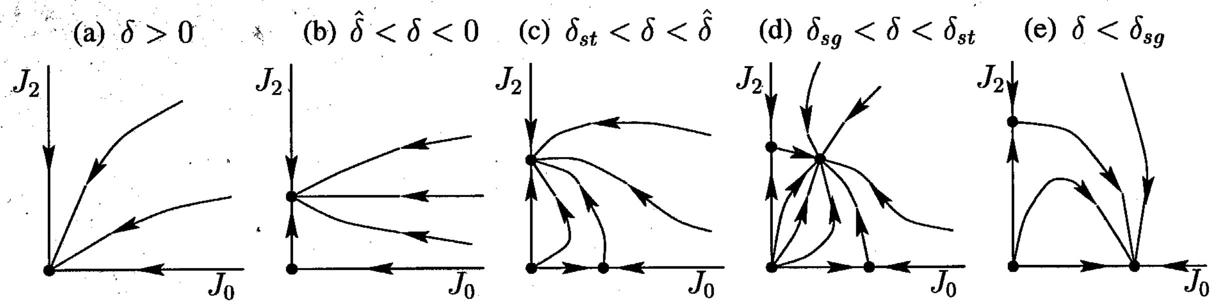


Figure 6.7: One possible phase portraits for the closed loop system in the high B case.

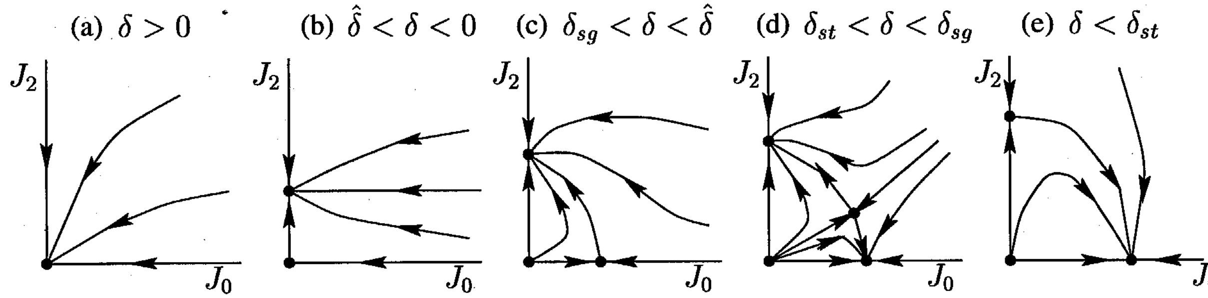


Figure 6.8: Another possible phase portraits for the closed loop system in the high B case.

3. $\frac{\alpha_{11}}{\alpha_{21}} > \frac{\alpha_{12}}{\alpha_{22}} > \frac{\nu_0}{\nu_2}$, the bifurcation is given by (a)-(c) in Figure 6.7 (in this case $\delta_{sg} > 0$, $\delta_{st} > 0$).

4. $\frac{\nu_0}{\nu_2} > \frac{\alpha_{12}}{\alpha_{22}} > \frac{\alpha_{11}}{\alpha_{21}}$, the bifurcations are shown in Figure 6.6 (b), and the phase portraits are in Figure 6.8.

5. $\frac{\alpha_{12}}{\alpha_{22}} > \frac{\nu_0}{\nu_2} > \frac{\alpha_{11}}{\alpha_{21}}$, the bifurcation is given by (a)-(d) in Figure 6.8 (in this case $\delta_{st} > 0$).

6. $\frac{\alpha_{12}}{\alpha_{22}} > \frac{\alpha_{11}}{\alpha_{21}} > \frac{\nu_0}{\nu_2}$, the bifurcation is given by (a)-(c) in Figure 6.8 (in this case $\delta_{st} > 0$, $\delta_{sg} > 0$).

6.4 Simulations for a Model of a Single-Stage Transonic Compressor

In this section we simulate the Moore-Greitzer model for a single stage transonic compressor at the NASA Glenn Research Center. The compressor characteristic we show here is a cubic polynomial fit to a more precise piecewise polynomial characteristic identified in [17]. The fitted polynomial is given by

$$\psi_c(\Phi) = -12.5381\Phi^3 + 7.8460\Phi^2 - 0.6498\Phi + 0.0495,$$

which is shown in Figure 6.9 (b). Other compressor parameters are also from [17], which are given by

$$\begin{aligned} m &= 1.0047, \quad \mu = 0.6040, \quad \lambda = 0.4044, \\ l_c &= 17.5500, \quad B = 0.3464, \quad u_{mag} = 0.0079. \end{aligned}$$

A more complete description of the compressor is can be found in [77] and [17].

The simulation results for the open loop system are in Figure 6.9 (a) and (b). It can be seen that there is a hysteresis since the stall inception point is at $\gamma_0 = 0.744$. Now we consider the controller is given by $u = K_1 a_1$, where $K_1 = -10$. Then the simulation results for this case are shown in Figure 6.9 (c) and (d). It can be seen that although the first mode is stabilized, the bifurcation for the second mode is still subcritical and there is a hysteresis associated with it. Also, the amplitude of the second mode in this case is much larger than the second mode for the open loop system. Now we choose the controller $u = K_1 a_1 + K_2 a_2 + K_3 |a_2|^2$, where $K_1 = -10$, $K_2 = 10$, and $K_3 = 4K_{3m} = 388.6$. The simulation results for the closed loop system are given by Figure 6.9 (e) and (f). In this case, the Hopf bifurcation to stall is supercritical and the stall inception is progressive rather than catastrophic as in the previous cases. It should be noted that the first mode also takes part in the limit cycle oscillation since the controller couples the first and the second mode. When the throttle is operated near $\gamma = 0.725$, both of the controlled first and second mode

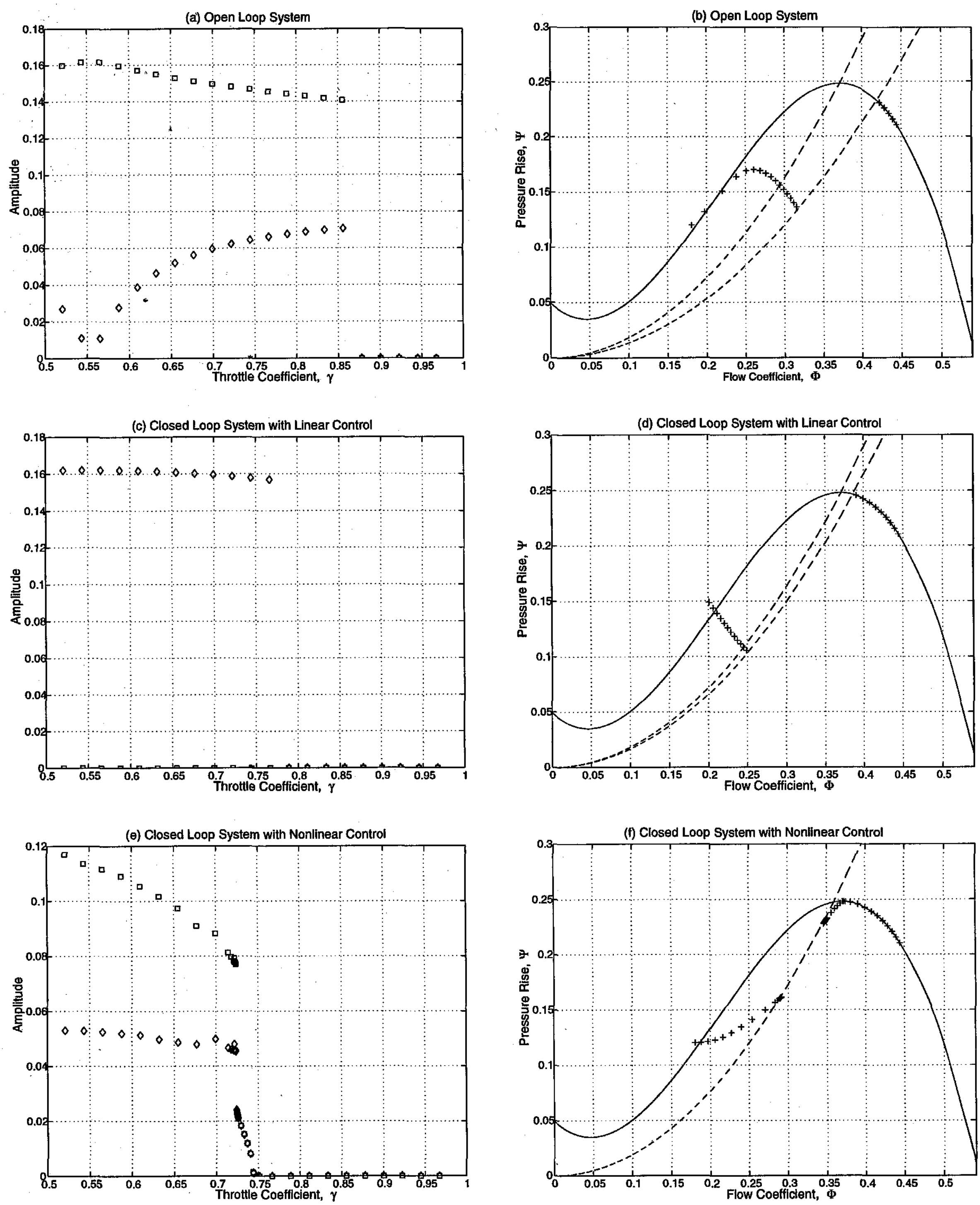


Figure 6.9: Bifurcation of uncontrolled and controlled systems from numerical simulations. Solid: compressor characteristic, dashed: throttle characteristic, square: first mode amplitude, diamond: second mode amplitude, star: stall inception point.

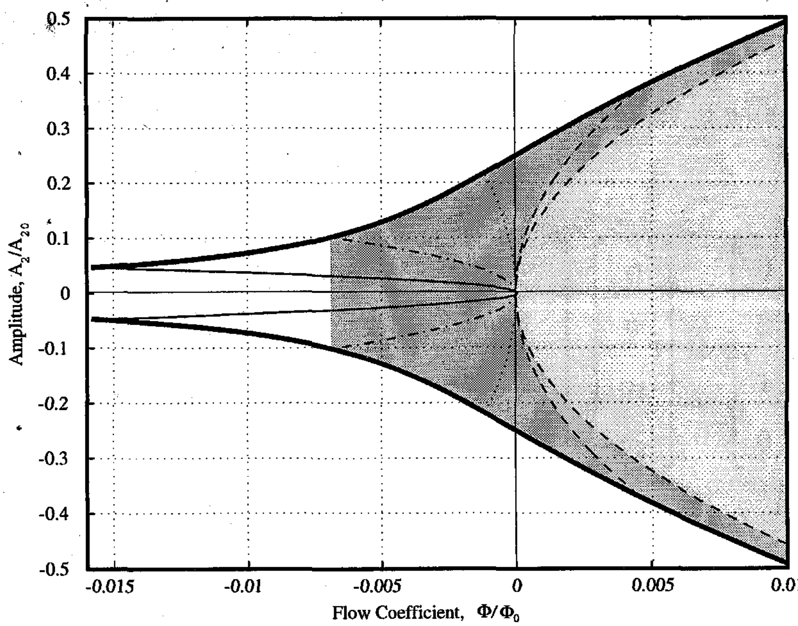


Figure 6.10: The bifurcations for the closed loop system with magnitude saturation.

increase drastically as γ decreases, but there is no evidence of hysteresis. The local bifurcation analysis in Section 6.2 did not predict this phenomenon, but it gives good predictions of controlled system behavior when $\gamma > 0.73$.

We remark here that our simulations also showed that as the the gain K_3 or K_2 increases, the oscillations of the limit cycles for the closed loop system also increase when the throttle operates father away from the stall inception point. This phenomenon is not predicted by the local bifurcation developed in Section 6.2.

The theoretical predictions of the control effectiveness for the magnetic bearing actuator with magnitude saturation is given by Figure 6.10. The x-axis is the throttle coefficient normalized by the throttle coefficient at the peak of the characteristic. The y-axis is the amplitude of the second mode normalized by the fully developed second mode amplitude. If the noise level is one percent, i.e., the ratio between the energy of the second modal content in the noise before stall inception is one percent of that for the fully developed stall, then the range extension is about 0.6 percent, which is so small that it might very difficult to observe in experiments. Hence the magnitude saturation seriously restricts the range of extension.

6.5 Proof of Proposition 6.1

Letting $\phi = \Phi - \Phi_e(\gamma)$, $\psi = \Psi - \Psi_e(\gamma)$, then the Taylor series expansion around the peak of the characteristic is given by

$$\begin{bmatrix} \dot{\phi} \\ \dot{\psi} \\ \dot{a}_1 \\ \dot{a}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{l_c} & 0 & 0 \\ \frac{1}{4B^2 l_c} & -\frac{\gamma_0}{8B^2 l_c \sqrt{\Psi_0}} & 0 & 0 \\ 0 & 0 & d_1\delta - i\omega_1 & 0 \\ 0 & 0 & 0 & d_2\delta - i\omega_2 \end{bmatrix} \begin{bmatrix} \phi \\ \psi \\ a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix},$$

where

$$\begin{aligned} f_1 &= \frac{\psi_c''}{l_c}\left(\frac{1}{2}\phi^2 + |a_1|^2 + |a_2|^2\right) + \frac{\psi_c'''}{l_c}\left[\frac{1}{6}\phi^3 + \left(|a_1|^2 + |a_2|^2\right)\phi + \frac{1}{2}\left(a_1^2 a_2^* + a_1^{*2} a_2\right)\right] \\ &\quad + \text{h.o.t.}, \\ f_2 &= \frac{\gamma_0}{32B^2 l_c \Psi_0^{\frac{5}{2}}}\left(\psi^2 - \frac{1}{2\Psi_0}\psi^3\right) + \text{h.o.t.}, \\ f_3 &= \frac{\psi_c''}{m+\mu}\left(\phi a_1 + a_2 a_1^*\right) + \frac{\psi_c'''}{m+\mu}\left[\frac{1}{2}\phi^2 a_1 + \phi a_2 a_1^* + \left(\frac{1}{2}|a_1|^2 + |a_2|^2\right) a_1\right] + \text{h.o.t.}, \\ f_4 &= \frac{\psi_c''}{\frac{m}{2}+\mu}\left(\phi a_2 + \frac{1}{2}a_1^2\right) + \frac{\psi_c'''}{\frac{m}{2}+\mu}\left[\frac{1}{2}\left(\phi a_2 + a_1^2\right)\phi + \left(|a_1|^2 + \frac{1}{2}|a_2|^2\right) a_2\right] + \text{h.o.t.} \end{aligned}$$

It should be noted here that the axisymmetric mode (ϕ, ψ) and the second stall mode a_2 are linearly uncontrollable, but the mode is stable if we assume B is small enough. Suppose the control law is given by

$$u = K_1 a_1 + K_2 a_2 + K_3 |a_2|^2,$$

and substitute the control law into the system. Since the axisymmetric dynamics is stable, the center manifold expansion is given by

$$\phi = \beta_{11} a_2^2 + \beta_{12}|a_2|^2 + \beta_{13} a_2^{*2} + \cdots, \tag{6.9}$$

$$\psi = \beta_{21} a_2^2 + \beta_{22}|a_2|^2 + \beta_{23} a_2^{*2} + \cdots, \tag{6.10}$$

$$a_1 = \beta_{31} a_2 + \beta_{32} a_2^* + \beta_{33} a_2^2 + \beta_{34}|a_2|^2 + \beta_{35} a_2^{*2} + \cdots. \tag{6.11}$$

The dynamics on the center manifold is given by

$$\dot{a}_2 = (d_2\delta - i\omega_2)a_2 + Q_1 a_2^2 + Q_2|a_2|^2 + Q_3 a_2^{*2} + C|a_2|^2 + \text{o.c.t.} + \text{h.o.t.}, \tag{6.12}$$

where o.c.t. denotes "other cubic terms," and

$$\begin{aligned}
Q_1 &= \frac{\frac{1}{2}\psi_c''}{\frac{m}{2}+\mu}\beta_{31}^2, \\
Q_2 &= \frac{\psi_c''}{\frac{m}{2}+\mu}\beta_{31}\beta_{32}, \\
Q_3 &= \frac{\frac{1}{2}\psi_c''}{\frac{m}{2}+\mu}\beta_{32}^2, \\
C &= \frac{\psi_c''}{\frac{m}{2}+\mu}(\beta_{12} + \beta_{31}\beta_{34} + \beta_{32}\beta_{33}) + \frac{\psi_c'''}{\frac{m}{2}+\mu}\left(|\beta_{31}|^2 + |\beta_{32}|^2 + \frac{1}{2}\right).
\end{aligned}$$

The normal form of (6.12) is given by

$$\dot{z} = (d_2\delta - i\omega_2)z + \hat{\alpha}|z|^2 z + \text{h.o.t.},$$

where

$$\hat{\alpha} = -\frac{Q_1 Q_2}{i\omega_2} + C + \text{p.i.t.}$$

Letting $\alpha = \operatorname{Re}\hat{\alpha}$, then the criticality of the Hopf bifurcation for the closed loop system is determined by α. If $\alpha > 0$, then the bifurcation is subcritical; if $\alpha < 0$, then the bifurcation is supercritical.

Now we calculated β_{ij}'s. This can be done by differentiating (6.9), (6.10), and (6.11), and using the system dynamics. The results are as follows:

$$\begin{aligned}
\beta_{31} &= -\frac{K_2}{K_1 - i(\omega_2 - \omega_1)}, \\
\beta_{32} &= 0, \\
\beta_{33} &= \frac{\beta_{31}Q_1}{K_1 + i(2\omega_2 - \omega_1)}, \\
\beta_{34} &= \frac{1}{K_1 - i\omega_1}\left(\beta_{31}Q_2 - \frac{\psi_c''}{m+\mu}\beta_{31}^* - K_3\right),
\end{aligned}$$

and $Q_2 = Q_3 = 0$. So we get

$$\hat{\alpha} = \frac{1}{\frac{m}{2}+\mu}\left[\frac{-\psi_c''\beta_{31}K}{K_1 - i\omega_1} + \frac{1}{2}\left(\psi_c''' + \frac{\gamma_0\psi_c''^2}{\sqrt{\Psi_0}}\right) + |\beta_{31}|^2\left(\psi_c''' + \frac{\gamma_0\psi_c''^2}{2\sqrt{\Psi_0}} - \frac{\psi_c''^2}{m+\mu}\frac{1}{K_1 - i\omega_1}\right)\right].$$

The first two claims in Proposition 6.1 can be proved from the expression of $\hat{\alpha}$. For the third claim, we could repeat this procedure for other feedbacks.

6.6 Proof of Proposition 6.2

For the high B case, the Hopf bifurcation to surge occurs very close to the peak of the characteristic. In this case, both the second mode stall and surge are linearly unstabilizable. By expanding the system around the peak of the characteristic, we get

$$\begin{bmatrix} \dot{\phi} \\ \dot{\psi} \\ \dot{a}_1 \\ \dot{a}_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{\Psi_0}\psi_c''}{l_c}\delta & -\frac{1}{l_c} & 0 & 0 \\ \frac{1}{4B^2 l_c} & -\frac{\sqrt{\Psi_0}\psi_c''}{l_c}\hat{\delta} & 0 & 0 \\ 0 & 0 & d_1\delta - i\omega_1 & 0 \\ 0 & 0 & 0 & d_2\delta - i\omega_2 \end{bmatrix} \begin{bmatrix} \phi \\ \psi \\ a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix},$$

where f_j $(j = 1, \cdots, 4)$ are the same as in the previous section.

In the small B case, the surge (axisymmetric) dynamics is stable. If we use the linear feedback $u = K_1 a_1(\operatorname{Re} K_1 < 0)$ to stabilize the first mode, then for $\gamma > \gamma_0$ near the peak, there are saddle-sink connections between the stable axisymmetric equilibria, the unstable stall equilibria and the stable fully developed stall equilibria; if $\gamma < \gamma_0$, then the saddle-sink connection is between the unstable axisymmetric equilibria and the fully developed stall equilibria. We postulate that these connections are persisted as B in creases, but the stability changes. Using the center manifold approximation as in the previous section to the low B case for $u = K_1 a_1$, the attractions are obtained by $(\tilde{\phi}, \tilde{\psi}) = (0,0)$, where

$$
\begin{aligned}
\tilde{\phi} &= \phi - \frac{\gamma_0 \psi_c''}{2\sqrt{\Psi_0}} |a_2|^2 + \text{h.o.t.}, \\
\tilde{\psi} &= \psi - \psi_c'' |a_2|^2 + \text{h.o.t.}
\end{aligned}
$$

Now we consider the coordinates transformation

$$
\begin{pmatrix} \phi & \psi & a_1 & a_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{\phi} & \tilde{\psi} & a_1 & a_2 \end{pmatrix},
$$

the system is transformed into

$$
\begin{bmatrix} \dot{\tilde{\phi}} \\ \dot{\tilde{\psi}} \\ \dot{a}_1 \\ \dot{a}_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{\Psi_0}\psi_c''}{l_c}\delta & -\frac{1}{l_c} & 0 & 0 \\ \frac{1}{4B^2 l_c} & -\frac{\sqrt{\Psi_0}\psi_c''}{l_c}\hat{\delta} & 0 & 0 \\ 0 & 0 & d_1\delta - i\omega_1 & 0 \\ 0 & 0 & 0 & d_2\delta - i\omega_2 \end{bmatrix} \begin{bmatrix} \tilde{\phi} \\ \tilde{\psi} \\ a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \\ \tilde{f}_4 \end{bmatrix},
$$

where

$$
\begin{aligned}
\tilde{f}_1 &= \frac{\psi_c''}{l_c}\left(\frac{1}{2}\tilde{\phi}^2 + |a_1|^2\right) + \frac{1}{l_c}\left[\psi_c''' + \left(\frac{1}{2} - \frac{l_c}{\frac{m}{2}+\mu}\right)\frac{\gamma_0\psi_c''^2}{\sqrt{\Psi_0}}\right]\tilde{\phi}|a_2|^2 \\
&\quad + \frac{1}{2l_c}\left(\psi_c''' - \frac{l_c}{m+2\mu}\frac{\gamma_0\psi_c''^2}{\sqrt{\Psi_0}}\right)\left(a_1^2 a_2^* + a_1^{*2} a_2\right) + \frac{\psi_c'''}{l_c}\left(\frac{1}{6}\tilde{\phi}^2 + |a_1|^2\right)\tilde{\phi} + \text{h.o.t.}, \\
\tilde{f}_2 &= \frac{\gamma_0}{32B^2 l_c \Psi_0^{\frac{5}{2}}}\left(\tilde{\psi}^2 + 2\psi_c''|a_2|^2\tilde{\psi} - \frac{1}{2\Psi_0}\tilde{\psi}^3\right) \\
&\quad - \frac{\psi_c''^2}{\frac{m}{2}+\mu}\left[2\tilde{\phi}|a_2|^2 + \frac{1}{2}\left(a_1^2 a_2^* + a_1^{*2} a_2\right)\right] + \text{h.o.t.}, \\
\tilde{f}_3 &= \frac{\psi_c''}{m+\mu}\left(\tilde{\phi}a_1 + a_2 a_1^*\right) + \frac{1}{m+\mu}\left(\psi_c''' + \frac{\gamma_0\psi_c''^2}{2\sqrt{\Psi_0}}\right)|a_2|^2 a_1 \\
&\quad + \frac{\psi_c'''}{m+\mu}\left[\frac{1}{2}\left(\tilde{\phi}^2 + |a_1|^2\right)a_1 + \tilde{\phi}a_2 a_1^*\right] + \text{h.o.t.}, \\
\tilde{f}_4 &= \frac{\psi_c''}{\frac{m}{2}+\mu}\left(\tilde{\phi}a_2 + \frac{1}{2}a_1^2\right) + \frac{1}{m+2\mu}\left(\psi_c''' + \frac{\gamma_0\psi_c''^2}{\sqrt{\Psi_0}}\right)|a_2|^2 a_2 \\
&\quad + \frac{\psi_c'''}{\frac{m}{2}+\mu}\left[\frac{1}{2}\left(\tilde{\phi}a_2 + a_1^2\right)\tilde{\phi} + |a_1|^2 a_2\right] + \text{h.o.t.}
\end{aligned}
$$

Let

$$\begin{bmatrix} \tilde{\phi} \\ \tilde{\psi} \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 \\ l_c d_0 \left(\delta + \hat{\delta}\right) & l_c \tilde{\omega}_0 \end{bmatrix},$$

where

$$\tilde{\omega}_0 = \omega_0 \sqrt{1 - \frac{\gamma_0 \psi_c'' \left(\delta + \hat{\delta}\right)^2}{8\hat{\delta}}}, \quad \omega_0 = \frac{1}{2Bl_c}.$$

And letting $a_0 = x + iy$, then the system is transformed into

$$\begin{bmatrix} \dot{a}_0 \\ \dot{a}_1 \\ \dot{a}_2 \end{bmatrix} = \begin{bmatrix} d_0\left(\delta - \hat{\delta}\right) + i\omega_0 & 0 & 0 \\ 0 & d_1\delta - i\omega_1 & 0 \\ 0 & 0 & d_2\delta - i\omega_2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} \tilde{g}_0 \\ \tilde{g}_1 \\ \tilde{g}_2 \end{bmatrix}, \tag{6.13}$$

where

$$\begin{aligned}
\tilde{g}_0 \;=\;& \frac{\psi_c''}{l_c}\left[\frac{1}{8}\left(a_0^2 + a_0^{*2}\right) + \frac{1}{4}|a_0|^2 + |a_1|^2\right] + \frac{1}{2l_c}\left(\psi_c''' - \frac{l_c}{m+2\mu}\frac{\gamma_0 \psi_c''^2}{\sqrt{\Psi_0}} - \frac{\psi_c''^2}{\left(\frac{m}{2}+\mu\right)\omega_0} i\right) \cdot \\
& \left(a_1^2 a_2^* + a_1^{*2} a_2\right) + \frac{1}{2l_c}\left[\psi_c''' + \left(\frac{1}{2} - \frac{l_c}{\frac{m}{2}+\mu}\right)\frac{\gamma_0 \psi_c''^2}{\sqrt{\Psi_0}} - \frac{2\psi_c''^2}{\left(\frac{m}{2}+\mu\right)\omega_0} i\right] \cdot \\
& \left(|a_1|^2 a_0 + |a_2|^2 a_0^*\right) + \frac{\psi_c'''}{2l_c}\left[\frac{1}{24}\left(a_0^2 + a_0^{*2}\right) + \frac{1}{12}|a_0|^2 + |a_1|^2\right]\left(a_0 + a_0^*\right) + \text{h.o.t.}, \\
\tilde{g}_1 \;=\;& \frac{\psi_c''}{m+\mu}\left[\frac{1}{2}\left(a_0 + a_0^*\right) a_1 + a_2 a_1^*\right] + \frac{1}{m+\mu}\left(\psi_c''' + \frac{\gamma_0 \psi_c''^2}{2\sqrt{\Psi_0}}\right)|a_2|^2 a_1 \\
& + \frac{\psi_c'''}{m+\mu}\left[\frac{1}{8}\left(a_0 + a_0^*\right)^2 a_1 + \frac{1}{2}|a_1|^2 a_1 + \frac{1}{2}\left(a_0 + a_0^*\right) a_2 a_1^*\right] + \text{h.o.t.}, \\
\tilde{g}_2 \;=\;& \frac{\psi_c''}{m+2\mu}\left[\left(a_0 + a_0^*\right) a_2 + a_1^2\right] + \frac{1}{m+2\mu}\left(\psi_c''' + \frac{\gamma_0 \psi_c''^2}{\sqrt{\Psi_0}}\right)|a_2|^2 a_2 \\
& + \frac{\psi_c'''}{\frac{m}{2}+\mu}\left[\frac{1}{8}\left(a_0^2 + a_0^{*2}\right) a_2 + \frac{1}{4}|a_0|^2 a_2 + \frac{1}{4}\left(a_0 + a_0^*\right) a_1^2 + |a_1|^2 a_2\right] + \text{h.o.t.}
\end{aligned}$$

It should be noted that we have dropped higher order terms associated with $\hat{\delta}$. Suppose the controller is given by the following form:

$$u = K_0 a_0 + K_1 a_1 + K_2 a_2 + K_3 |a_0|^2 + K_4 |a_2|^2.$$

By substituting the control law into the system and assume $\operatorname{Re} K_1 < 0$, the center manifold is given by the following Taylor series

$$\begin{aligned} a_1 &= \beta_1 a_0 + \beta_2 a_2 + \beta_3 a_0^* + \beta_4 a_2^* + \beta_5 a_0^2 + \beta_6 |a_0|^2 + \beta_7 a_0^{*2} + \beta_8 a_2^2 + \beta_9 |a_2|^2 \\ &\quad + \beta_{10} a_2^{*2} + \beta_{11} a_0 a_2 + \beta_{12} a_0 a_2^* + \beta_{13} a_0^* a_2 + \beta_{14} a_0^* a_2^* + \text{h.o.t.} \end{aligned} \tag{6.14}$$

Later in this section we will show that $\beta_3 = \beta_4 = 0$, so the dynamics on the center manifold is given by

$$\begin{bmatrix} \dot{a}_0 \\ \dot{a}_2 \end{bmatrix} = \begin{bmatrix} d_0 \left(\delta - \hat{\delta}\right) + i\omega_0 & 0 \\ 0 & d_2 \delta - i\omega_2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_2 \end{bmatrix} + \begin{bmatrix} g_0 \\ g_2 \end{bmatrix}, \tag{6.15}$$

where

$$\begin{aligned} g_0 &= \frac{\psi_c''}{l_c} \left[\frac{1}{8} a_0^2 + \frac{1}{8} a_0^{*2} + \left(\frac{1}{4} + |\beta_1|^2 \right) |a_0|^2 + |\beta_2|^2 |a_2|^2 + \beta_1 \beta_2^* a_0 a_2^* + \beta_0^* \beta_2 a_0^* a_2 \right] \\ &\quad + \hat{\alpha}_{11} |a_0|^2 a_0 + \hat{\alpha}_{12} |a_2|^2 a_0 + \text{o.c.t.} + \text{h.o.t.}, \\ g_2 &= \frac{\psi_c''}{m + 2\mu} \left[\beta_1^2 a_0^2 + (1 + 2\beta_1 \beta_2) a_0 a_2 + \beta_2^2 a_2^2 + a_0^* a_2 \right] \\ &\quad + \hat{\alpha}_{21} |a_0|^2 a_2 + \hat{\alpha}_{22} |a_2|^2 a_2 + \text{o.c.t.} + \text{h.o.t.}, \end{aligned}$$

where

$$\begin{aligned} \hat{\alpha}_{11} &= \frac{\psi_c''}{l_c} \left(\beta_1 \beta_6^* + \beta_1^* \beta_5 \right) + \frac{\psi_c'''}{2 l_c} \left(\frac{1}{8} + |\beta_1|^2 \right), \\ \hat{\alpha}_{12} &= \frac{\psi_c''}{l_c} \left(\beta_1 \beta_9^* + \beta_2 \beta_{13}^* + \beta_2^* \beta_{11} \right) + \frac{1}{2 l_c} \left(1 + 2\beta_1 \beta_2 + |\beta_2|^2 \right) \psi_c''' \\ &\quad + \frac{1}{2 l_c} \left[\left(\frac{1}{2} - \frac{l_c (1 + \beta_1 \beta_2)}{\frac{m}{2} + \mu} \right) \frac{\gamma_0 \psi_c''^2}{\sqrt{\Psi_0}} - \frac{2 \psi_c''^2 (1 + \beta_1 \beta_2)}{\left(\frac{m}{2} + \mu \right) \omega_0} i \right], \end{aligned}$$

$$\hat{\alpha}_{21} \doteq \frac{\psi_c''}{\frac{m}{2}+\mu}(\beta_1\beta_{13}+\beta_2\beta_6)+\frac{\psi_c'''}{\frac{m}{2}+\mu}\left(\frac{1}{4}+\frac{1}{2}\beta_1\beta_2+|\beta_1|^2\right),$$

$$\hat{\alpha}_{22} = \frac{\psi_c''}{\frac{m}{2}+\mu}\beta_2\beta_9+\frac{1}{m+2\mu}\left[\left(1+2|\beta_2|^2\right)\psi_c'''+\frac{\gamma_0\psi_c''^2}{\sqrt{\Psi_0}}\right].$$

Letting $\delta = 0$ and $\hat{\delta} = 0$ in the system (6.15), we use the following coordinate transformation to eliminate the quadratic terms and "other cubic terms" in (6.15),

$$\begin{aligned} a_0 &= z_0 + q_0(z_0, z_2, z_0^*, z_2^*) + c_0(z_0, z_2, z_0^*, z_2^*), \\ a_2 &= z_2 + q_2(z_0, z_2, z_0^*, z_2^*) + c_2(z_0, z_2, z_0^*, z_2^*), \end{aligned}$$

where q_0 and q_2 are quadratic functions, c_0 and c_2 are cubic functions. The major assumptions are that there are no lower order resonance between the surge mode and the second stall mode, i.e.,

$$n_0\omega_0 + n_2\omega_2 \neq 0, \;\; |n_0|+|n_2| \leq 4, \;\; |n_0|+|n_2| \neq 0, \;\; n_0, n_2 \in \mathbb{Z}.$$

Using the coordinate transformation, a normal form of system (6.15) is given by

$$\begin{aligned} \dot{z}_0 &= \left[d_0\left(\delta-\hat{\delta}\right)+i\omega_0\right]z_0+\left(\tilde{\alpha}_{11}|z_0|^2+\tilde{\alpha}_{12}|z_2|^2\right)z_0+\text{h.o.t.}, \\ \dot{z}_2 &= \left(d_0\delta-i\omega_2\right)z_2+\left(\tilde{\alpha}_{21}|z_0|^2+\tilde{\alpha}_{22}|z_2|^2\right)z_2+\text{h.o.t.}, \end{aligned}$$

where

$$\begin{aligned} \tilde{\alpha}_{11} &= \frac{\psi_c''}{l_c}(\beta_1\beta_6^*+\beta_1^*\beta_5)+\frac{\psi_c''^2|\beta_1|^2\beta_1\beta_2}{l_c(m+2\mu)(\omega_2-2\omega_0)}i+\frac{\psi_c'''}{2l_c}\left(\frac{1}{8}+|\beta_1|^2\right)+\text{p.i.t.}, \\ \tilde{\alpha}_{12} &= \frac{\psi_c''}{l_c}(\beta_1\beta_9^*+\beta_2\beta_{13}^*+\beta_2^*\beta_{11})+\frac{2\psi_c''^2\beta_1\beta_2|\beta_2|^2}{l_c(m+2\mu)(2\omega_2-\omega_0)}i \\ &\quad +\frac{1}{2l_c}\left[\left(1+2\beta_1\beta_2+|\beta_2|^2\right)\psi_c'''+\left(\frac{1}{2}-\frac{l_c(1+\beta_1\beta_2)}{\frac{m}{2}+\mu}\right)\frac{\gamma_0\psi_c''^2}{\sqrt{\Psi_0}}\right]+\text{p.i.t.}, \end{aligned}$$

$$\begin{aligned}
\tilde{\alpha}_{21} &= \frac{\psi_c''}{\frac{m}{2}+\mu}(\beta_1\beta_{13}+\beta_2\beta_6)+\frac{\psi_c''^2\beta_1\beta_2}{\frac{m}{2}+\mu}\left[\frac{4\omega_2}{(m+2\mu)\left(4\omega_2^2-\omega_0^2\right)}+\right.\\
&\left.\frac{1}{\omega_0 l_c}\left(\frac{1}{4}+\frac{3\omega_0+\omega_2}{2\omega_0+\omega_2}|\beta_1|^2\right)\right]i+\frac{\psi_c'''}{\frac{m}{2}+\mu}\left(\frac{1}{4}+\frac{1}{2}\beta_1\beta_2+|\beta_1|^2\right)+\text{p.i.t.},\\
\tilde{\alpha}_{22} &= \frac{\psi_c''}{\frac{m}{2}+\mu}\beta_2\beta_9+\frac{\psi_c''^2|\beta_2|^2\beta_1\beta_2}{\left(\frac{m}{2}+\mu\right)\omega_0 l_c}i+\frac{1}{m+2\mu}\left[\left(1+2|\beta_2|^2\right)\psi_c'''+\frac{\gamma_0\psi_c''^2}{\sqrt{\Psi_0}}\right].
\end{aligned}$$

Let

$$\begin{aligned}
z_0(\xi) &= \rho_0(\xi)e^{i\varphi_0(\xi)}, \quad J_0(\xi)=z_0(\xi)^2,\\
z_2(\xi) &= \rho_2(\xi)e^{i\varphi_2(\xi)}, \quad J_2(\xi)=z_2(\xi)^2,
\end{aligned}$$

then the dynamics for J_0 and J_2 are given by

$$\begin{aligned}
\dot{J}_0 &= \nu_0\left(\delta-\hat{\delta}\right)J_0+\left(\alpha_{11}J_0+\alpha_{12}J_2\right)J_0+\text{h.o.t.},\\
\dot{J}_2 &= \nu_2\delta J_2+\left(\alpha_{21}J_0+\alpha_{122}J_2\right)J_2+\text{h.o.t.},
\end{aligned}$$

where

$$\begin{aligned}
\nu_0 = 2d_0 &= \frac{\sqrt{\Psi_0}\psi_c''}{l_c}, \quad \alpha_{11}=2\operatorname{Re}\tilde{\alpha}_{11}, \quad \alpha_{12}=\frac{1}{2}\operatorname{Re}\tilde{\alpha}_{12},\\
\nu_2 = 2d_0 &= \frac{\sqrt{\Psi_0}\psi_c''}{m+2\mu}, \quad \alpha_{21}=2\operatorname{Re}\tilde{\alpha}_{21}, \quad \alpha_{12}=\frac{1}{2}\operatorname{Re}\tilde{\alpha}_{22}.
\end{aligned}$$

Now we calculate β_j $(j=0,\cdots,14)$ by differentiating the center manifold expression (6.14) and using the dynamics of the system (6.13). The results are as follows:

$$\begin{aligned}
\beta_1 &= -\frac{K_0}{K_1-i(\omega_1+\omega_0)},\\
\beta_2 &= -\frac{K_2}{K_1-i(\omega_1-\omega_2)},\\
\beta_3 &= 0,\\
\beta_4 &= 0,\\
\beta_5 &= \frac{\psi_c''\beta_1}{K_1-i(\omega_1+2\omega_0)}\left[\frac{1}{8l_c}+\frac{\beta_1\beta_2}{m+2\mu}-\frac{1}{2(m+\mu)}\right],
\end{aligned}$$

$$\begin{aligned}
\beta_6 &= \frac{1}{K_1 - i\omega_1}\left[\frac{\psi_c''}{l_c}\left(\frac{1}{4}+|\beta_1|^2\right)\beta_1 - \frac{\psi_c''\beta_1}{2(m+\mu)} - K_3\right],\\
\beta_7 &= \frac{1}{K_1 - i(\omega_1 - 2\omega_0)}\frac{\psi_c''\beta_1}{8l_c},\\
\beta_8 &= \frac{1}{K_1 - i(\omega_1 - 2\omega_2)}\frac{\psi_c''\beta_2^3}{m+2\mu},\\
\beta_9 &= \frac{1}{K_1 - i\omega_1}\left(\frac{\psi_c''}{l_c}|\beta_2|^2\beta_1 + \frac{\psi_c''\beta_2^*}{m+\mu} - K_4\right),\\
\beta_{10} &= 0,\\
\beta_{11} &= \frac{\psi_c''\beta_2}{K_1 - i(\omega_0+\omega_1-\omega_2)}\left[\frac{1+2\beta_1\beta_2}{m+2\mu} - \frac{1}{2(m+\mu)}\right],\\
\beta_{12} &= \frac{\psi_c''}{K_1 - i(\omega_0+\omega_1+\omega_2)}\frac{\beta_1^2\beta_2^*}{l_c},\\
\beta_{13} &= \frac{\psi_c''}{K_1 + i(\omega_0-\omega_1+\omega_2)}\left[\frac{1}{l_c}|\beta_1|^2\beta_2 + \frac{\beta_2}{m+2\mu} - \frac{\beta_2+2\beta_1^*}{2(m+\mu)}\right],\\
\beta_{14} &= 0.
\end{aligned}$$

We have the following proposition.

Proposition 6.3 *If $\psi_c'' < 0$, then there exist K_j $(j = 0,\cdots,4)$ such that $\alpha_{ij} < 0$ $(i,j = 1,2)$.*

The proposition can be proved by observing that the map

$$\mathbb{C}\times\mathbb{C} \longrightarrow \mathbb{C}\times\mathbb{C} : (K_3, K_4) \longrightarrow (\beta_6, \beta_9)$$

is an isomorphic affine map, and α_{ij} $(i,j = 1,2)$ are given by

$$\begin{aligned}
\alpha_{11} &= \frac{2\psi_c''}{l_c}\operatorname{Re}\{\beta_1\beta_6^*\} + F_{11}(K_0,K_1,K_2),\\
\alpha_{12} &= \frac{\psi_c''}{2l_c}\operatorname{Re}\{\beta_1\beta_9^*\} + F_{12}(K_0,K_1,K_2),\\
\alpha_{21} &= \frac{2\psi_c''}{\frac{m}{2}+\mu}\operatorname{Re}\{\beta_2\beta_6\} + F_{21}(K_0,K_1,K_2),\\
\alpha_{22} &= \frac{\psi_c''}{m+2\mu}\operatorname{Re}\{\beta_2\beta_9\} + F_{22}(K_0,K_1,K_2).
\end{aligned}$$

In summary, we design the stabilizing control laws by the following procedure.

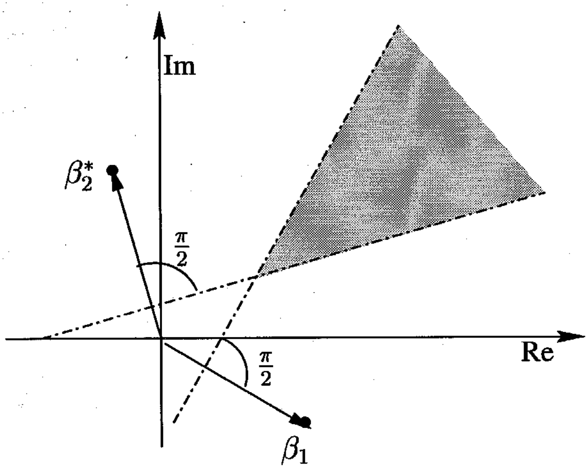


Figure 6.11: Geometric interpretation of constructing stabilizing feedbacks.

(1) Select $K_1 \in \mathbb{C}$ with $\operatorname{Re} K_1 < 0$,

(2) Select $K_0, K_2 \in \mathbb{C}$ such that $\beta_1 \neq -\kappa\beta_2^*$, where $\kappa \in \mathbb{R}^+$,

(3) Select $K_3, K_4 \in \mathbb{C}$ such that $\alpha_{ij} < 0$ $(i, j = 1, 2)$.

The geometric interpretation for β_6 and β_9 in the region in $\mathbb{C}$ in which $\alpha_{ij} < 0$ $(i, j = 1, 2)$ is given in the shaded region in Figure 6.11.

Remark 6.6.1:

1. If $\psi_c'' = 0$, then we have to use the fourth order derivative of ψ_c.

2. The dynamics of the closed loop system is in Section 6.3.

Chapter 7 Conclusions and Future Work

In this chapter we give a summary of the thesis and suggestions for future work.

7.1 Conclusions

One of the main goals of this thesis is to characterize the local smooth stabilizability of nonlinear systems near an equilibrium, under the assumption that the linearly unstabilizable eigenvalue is zero or a pair of pure imaginary numbers with algebraic multiplicity one. Explicit algebraic necessary and sufficient conditions of stabilizability are obtained up to the third order in the normal form of the dynamics on the center manifold. Explicit feedback laws are constructed when the system is stabilizable. These results not only reveal the complexity of local stabilization in non-hyperbolic systems, but also give insights into the geometric structure of stabilization.

One of the most important applications of local smooth stabilizability of nonlinear systems near an equilibrium is in bifurcation control theory. An important question in bifurcation control is how to design a feedback to change an open loop subcritical bifurcation to closed loop supercritical bifurcation, under the assumption that the bifurcating eigenvalues are linearly uncontrollable. For simple steady-state and Hopf bifurcations, this question is equivalent to design a control law to stabilize the system at the bifurcation point. Thus we have solved the problem of feedback stabilizability of a simple steady-state bifurcation or a Hopf bifurcation by providing explicit algebraic necessary and sufficient conditions. These results contribute to the understanding of feedback control of non-equilibrium behavior.

Since subcritical bifurcations are usually associated with nonlinear instability at the bifurcation point, controller bandwidth, rate limits and magnitude saturation have a significant effect on the region of attraction to the stabilized equilibria or periodic orbits. We evaluated the size and the shape of region of attraction for controllers with magnitude saturation limits. The boundary of the region of attraction to the stabilized equilibria/periodic orbits is formed by the equilibria/periodic orbits after the controller saturates. The effects of bandwidth and rate limits in control of bifurcations are demonstrated by analyzing the Moore-Greitzer model for compression systems. The analysis on the effects of controller bandwidth and rate limits can be carried over to bifurcation control of general nonlinear system in a straightforward way.

By applying the general results to the Moore-Greitzer model, the effects of actuator magnitude saturation, bandwidth, and rate limits in active control of rotating stall are analyzed for the bleed valve actuators. Analytical formulas are obtained to relate parameters and characteristic shape of a compression system to the rate and magnitude requirements of bleed valve actuators. The formulas give good qualitative predictions when comparing with the experimental results on a low speed compressor. The formulas have also shown that the actuator bandwidth and magnitude saturation limits are serious constraints that have to be considered for implementation of bleed valve actuators in an aircraft engine.

Magnetic bearings are potential actuators for active control of rotating stall and surge in compression systems. The second stall mode and the surge mode are linearly uncontrollable when magnetic bearings are used to actuate the tip clearance. Control laws are constructed to change the criticality of the Hopf bifurcation of the second stall mode when the surge mode are stable. The effects of magnitude saturation of magnetic bearings are analyzed. The magnitude saturation seriously affect the effectiveness of tip clearance modulation in active stall control. Control laws are also designed to simultaneously change the Hopf bifurcations of second stall mode and the surge mode for the high B case.

7.2 Future Research Areas

Nonlinear control theory has been developed rapidly in the past two decade based on differential geometry [43]. With the rapid development of computational power and information technology, the role of computation in nonlinear control design becomes more and more important. On the other hand, with the rapid widening of control technology in engineering applications, new problems will definitely arise and will require new tools in nonlinear controls.

Local feedback stabilization of bifurcations can be further pursued for systems which have multiple bifurcations which are near one another in the space of bifurcation parameters. One example is control of rotating stall and surge using magnetic bearings, where the Hopf bifurcations of second mode stall and surge occur near each other and both modes are linearly uncontrollable. For Hopf/Hopf bifurcations without low order resonance, the dynamics near the bifurcation point has been well known for the nondegenerate case [37]. Some sufficient conditions have been obtained for a class of systems with two pairs of uncontrollable pure imaginary eigenvalues [15], or for multiple critical modes without lower order resonance [86]. Necessary and sufficient conditions for a general multi-input systems with multiple pairs of uncontrollable pure imaginary eigenvalues remain to be solved. If the Hopf/Hopf bifurcation exhibits lower order resonance, then algebraic tests of stability property may not exist [11]. In this case, obtaining algebraic necessary and sufficient conditions of stabilizability is hopeless. So it is necessary to obtain necessary and sufficient conditions that are numerically tractable. The role of symmetry in control of bifurcations is an interesting problem. For systems whose critical modes have some symmetry (say $SO(2)$), it is worthwhile to study the role of symmetry preserving and symmetry breaking control schemes in local stabilization of nonlinear systems. The research in this direction not only will shed light on the geometry of local feedback stabilization, but also may have wide applications in engineering systems. It is necessary to direct more research effort in bifurcation control of dynamical systems described by functional differential equations. Some sufficient conditions of stabilizability of Hopf bifurcations in functional differential equations have been obtained in [67]. The results in this area will have implications in understanding the sensor/actuator allocation problems in control of distributed systems. One of the applications would be in active control of combustion instabilities [69], where the model is a retarded functional differential equation. The issue of robustness in bifurcation control is also worthwhile to investigate. Although a constructive procedure of designing stabilizing feedback laws is developed in this thesis, the geometric structure of stabilizability in these cases remains to be explored. Some preliminary results can be found in [57].

Control of nonequilibrium behavior in nonlinear systems is expected to receive more research effort. In some applications it might be undesirable or impossible to stabilize a local bifurcation. Control of homoclinic/heteroclinic cycles near their tangency will shed light on the role of feedback control on the global dynamic behavior which are crucial in some engineering applications. Much research have been done on control of chaos, but there is not a general framework. It is interesting to develop some tools to design feedback laws to regulate a chaotic attractor. The research in this area will heavily rely on the development in the theory of dynamical systems.

Much research has been done in the area of active control of rotating stall and surge based on analysis of the Moore-Greitzer model, which is an oversimplified model for an aircraft engine. Since actuator limits are crucial in active stall control, developing novel high authority, high bandwidth actuators is necessary for implementation on an engine. It is not clear how the active control schemes would affect other components in the engine, such as the downstream combustors, the turbines and the afterburners. The results in this thesis will not lead to understanding of these questions. More experiments and modeling have to be done to obtain a sensible model. Other trade-offs such as affordability, reliability, complexity of control architecture also remain to be investigated, though preliminary evaluations exist [1, 2, 3]. Most of the experiments on active stall control are on laboratory compressor rigs, and more tests on engines are necessary. In [31], active control of rotating stall is achieved on an engines using sleed valves to recirculate air. Robustness of active control technology is an important issue. For example, a systematic tool is needed to understand how the inlet distortions affect the controller performance in different actuation schemes.

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