SIMILARITY SOLUTION FOR TRANSONIC FLOW PAST A CONE

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In memory of my Mother, my Brother,

and Szechun.

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ABSTRACT

By applying a transonic expansion procedure to a conical flow field, a system of approximate transonic equations, boundary conditions, and shock relations is derived. A similarity law for the pressure coefficient on the surface of slender cones is established. The surface pressure is computed by solving the approximate equations.

By use of similarity, the second order differential equations of the first two steps of the approximation scheme are reduced to first order equations. The solution of the first step is carried out numerically in great detail for different transonic parameters; the procedure for solving the latter is explained in the Appendix.

The results are compared with the exact solution, and a highly satisfactory agreement is reached.

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SYMBOLS

a	local speed of sound
a^*	critical speed of sound for $M = 1$
K	transonic similarity parameter = $\frac{1-M^2}{\delta^2}$
L	characteristic length
M	Mach number = $\frac{U}{a}$
P	pressure
$\vec{q}$	local velocity
q	local speed
q_x and q_r	axial and radial velocity respectively
x and r	dimensionless axial and radial coordinate respectively
s,t	transformed Cartesian coordinates
U	flight speed
β	shock wave angle
δ	tangent of semi-cone angle = $\tan\theta$
ζ,η	transformed Cartesian coordinates
θ	semi-cone angle
λ	parameter independent of $\delta = K$
π	$\frac{P}{\rho^\gamma}$
ρ	density
ϕ	velocity potential
$(\ )_i$	ith approximation
$(\ )_w$	value at shock wave
$(\ )_\infty$	free stream value

I. INTRODUCTION

The purpose of this investigation is to study the axial supersonic flow around slender cones in the transonic range by applying the expansion procedures and similarity laws for conical transonic flow. The derivation of the procedures and laws is based upon the techniques in Reference one. The investigation will thus serve to justify the usefulness of the expansion method and it will also determine what range of the transonic similarity parameter will give a good result from the present theory.

As is well known, the subject of axial supersonic flow around cones was first introduced in 1929 by Busemann (ref. 2). This same type of flow has since then been dealt with by G. I. Taylor and J. W. Maccoll (ref. 3), Z. Kopal (ref. 4) and several other authors. However the numerical as well as the graphical solution is carried out in a very laborious way. This paper presents a much simpler method for solving this type of problem in the transonic range.

The transonic equation has been derived in many different ways, but most of them lack a systematic procedure. The techniques adapted from Reference one make the derivation of the approximate equations for conical transonic flow part of a systematic expansion procedure. Thus it becomes possible to compute the higher terms of this approximation or at least to estimate errors.

The transonic differential equation thus derived can be simplified to a first-order differential equation by means of a transformation. The shock relations reduce to a single curve which we shall call the universal hedgehog and the axis of the cone reduces to a point at the origin in this new system. As the equation is of first order, the computation work is much less than that for a second order differential equation. Furthermore, by use of similarity, the solution of a flow problem represents the flow of a family of cones having the same transonic parameter, while the usual method of investigation requires a calculation for every cone angle.

Although transonic similarity laws for the pressure distribution around slender bodies have been derived by von Kármán (ref. 5), and Oswatitsch and Berndt (ref. 6) previously, the function of similarity parameter in the pressure formula has not been determined explicitly. In this paper, this function is found by numerical integration.

The results from the approximate solution agree with those from the exact solution (ref. 4) in a very satisfactory manner for a slender cone. However, as the cone angle and the transonic parameter become larger, the agreement becomes poorer.

Oswatitsch and Sjödin (ref. 7) have independently studied the same type of transonic flow over a cone in a quite different approach which requires much more computation work than that required in the present theory.

II. EXPANSION PROCEDURE FOR TRANSONIC EQUATIONS AND BOUNDARY CONDITIONS.

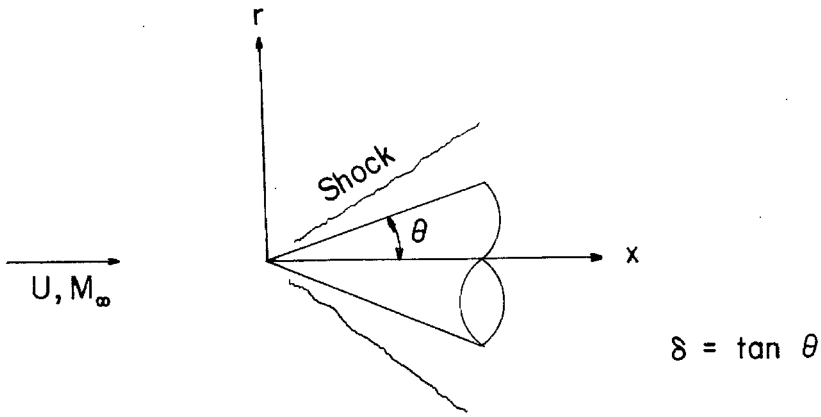


Fig. 1

Supersonic Flow Past a Cone.

The basic differential equations of flow (ref. 1) which apply behind the shock on a cone as shown in fig. 1 are:

Continuity: (Modified) $$a^2 \operatorname{div} \vec{q} = \vec{q} \cdot \nabla\left(\frac{q^2}{2}\right) \tag{2-1}$$

Irrotationality: $$\operatorname{curl} \vec{q} = 0 \tag{2-2}$$

Energy: $$\frac{q^2}{2} + \frac{a^2}{\gamma-1} = \frac{U^2}{2} + \frac{a_\infty^2}{\gamma-1} = \frac{1}{2}\frac{\gamma+1}{\gamma-1}\, a^{*2} \tag{2-3}$$

The shock relations (ref. 1) are:

Shock polar: $$\left(\frac{(q_r)_w}{U}\right)^2 = \left(1 - \frac{(q_x)_w}{U}\right)^2 \frac{\frac{(q_x)_w}{U} - \left(\frac{a^*}{U}\right)^2}{\left(\frac{a^*}{U}\right)^2 + \frac{2}{\gamma+1} - \frac{(q_x)_w}{U}} \tag{2-4}$$

Wave angle: $$\tan\beta = \frac{1 - \frac{(q_x)_w}{U}}{\frac{(q_r)_w}{U}} \tag{2-5}$$

Pressure: $$(p)_w = p_\infty + \rho_\infty U(U - (q_x)_w) \tag{2-6}$$

Density: $$\frac{\rho_\infty}{\rho} = \frac{(q_x)_w}{U} - \frac{(q_r)_w}{U\tan\beta} \tag{2-7}$$

The boundary condition on the body is

$$\frac{q_r}{U} = \frac{q_x}{U}\cdot\delta \tag{2-8}$$

It is known from the similarity of the problem that the velocity is a function only of $\frac{x}{r}$. An expansion procedure is now applied to the above system of equations. The following form of expansion which preserves similarity is assumed for the velocity components

$$\frac{q_x}{U} = 1 + \epsilon_1(\delta)u_1(\sigma;\lambda) + \epsilon_2(\delta)u_2(\sigma;\lambda) + \ldots\ldots\ldots$$

$$\frac{q_r}{U} = \nu_1(\delta)v_1(\sigma;\lambda) + \nu_2(\delta)v_2(\sigma;\lambda) + \ldots\ldots\ldots \tag{2-9}$$

where the ϵ_i, ν_i each form a decreasing sequence as $\delta \to 0$ and

$$\sigma = \frac{x}{\delta^a r} \tag{2-10}$$

λ = parameter independent of δ

The assumed form of expansion is now substituted into equation 2-1, and reasoning as in Reference one, we find the following expansions:

$$\frac{q_x}{U} = 1 + \delta^2 u_1(\sigma; K) + \delta^4 \log\delta\, u_2(\sigma; K) + \delta^4 u_3(\sigma; K) + \delta^6 \log^2\delta\, u_4(\sigma; K) + \delta^6 \log\delta\, u_5(\sigma; K) + \ldots\ldots$$

$$\frac{q_r}{U} = \delta^3 v_1(\sigma; K) + \delta^5 \log\delta\, v_2(\sigma; K) + \delta^5 v_3(\sigma; K) + \delta^7 \log^2\delta\, v_4(\sigma; K) + \delta^7 \log\delta\, v_5(\sigma; K) + \ldots\ldots \qquad (2\text{-}11)$$

where $\sigma = \dfrac{x}{\delta r}$ and $K = \dfrac{1 - M_\infty^2}{\delta^2}$.

The approximate equations which result are:

(a) First approximation.

$$\left[K - (\gamma + 1) u_1\right] \frac{\partial u_1}{\partial \sigma} + v_1 - \sigma \frac{\partial v_1}{\partial \sigma} = 0 \qquad (2\text{-}12a)$$

$$\frac{\partial v_1}{\partial \sigma} = -\sigma \frac{\partial u_1}{\partial \sigma} \qquad (2\text{-}12b)$$

Shock polar: $$(v_1)_w^2 = \frac{(u_1)_w^2\left[(\gamma+1)(u_1)_w - 2K\right]}{2} \qquad (2\text{-}13)$$

Wave angle: $$\tan\beta = -\frac{1}{\delta}\frac{(u_1)_w}{(v_1)_w}\left\{1 + \ldots\ldots\right\} \qquad (2\text{-}14)$$

Pressure: $$\frac{(p)_w}{p_\infty} = 1 - \delta^2 \gamma (u_1)_w + \ldots\ldots \qquad (2\text{-}15)$$

Density: $$\frac{\rho_\infty}{(\rho)_w} = 1 + \delta^2 (u_1)_w + \ldots\ldots \qquad (2\text{-}16)$$

(b) Second approximation

$$K\frac{\partial u_2}{\partial\sigma} - \sigma\frac{\partial v_2}{\partial\sigma} + v_2 = (\gamma+1)\frac{\partial u_1 u_2}{\partial\sigma} \tag{2-17a}$$

$$\frac{\partial v_2}{\partial\sigma} = -\sigma\frac{\partial u_2}{\partial\sigma} \tag{2-17b}$$

Shock polar:

$$\frac{2}{\gamma+1} v_1(\sigma_{1w})\left(v_2(\sigma_{1w}) + \sigma_{2w}\frac{\partial v_1(\sigma_{1w})}{\partial\sigma_{1w}}\right)$$
$$= u_1(\sigma_{1w})\left(\frac{3}{2}u_1(\sigma_{1w}) - \frac{2K}{\gamma+1}\right)\left(u_2(\sigma_{1w}) + \sigma_{2w}\frac{\partial u_1(\sigma_{1w})}{\partial\sigma_{1w}}\right) \tag{2-18}$$

Wave angle:

$$\tan\beta = \frac{1}{\delta}\frac{u_1(\sigma_{1w})}{v_1(\sigma_{1w})}\left\{1+\delta^2\log\delta\left[\frac{1}{u_1(\sigma_{1w})}\left(u_2(\sigma_{1w}) + \sigma_{2w}\frac{\partial u_1(\sigma_{1w})}{\partial\sigma_{1w}}\right)\right.\right.$$
$$\left.\left.-\frac{1}{v_1(\sigma_{1w})}\left(v_2(\sigma_{1w}) + \sigma_{2w}\frac{\partial v_1(\sigma_{1w})}{\partial\sigma_{1w}}\right)\right] + \ldots\ldots\right\} \tag{2-19}$$

Pressure:

$$\frac{(p)_w}{p_\infty} = 1 - \delta^2\gamma u_1(\sigma_{1w}) - \delta^4\log\delta\,\gamma\left(u_2(\sigma_{1w}) + \sigma_{2w}\frac{\partial u_1(\sigma_{1w})}{\partial\sigma_{1w}}\right) + \ldots\ldots \tag{2-20}$$

Density:

$$\frac{\rho_\infty}{(\rho)_w} = 1 + \delta^2 u_1(\sigma_{1w}) + \delta^4\log\delta\left(u_2(\sigma_{1w}) + \sigma_{2w}\frac{\partial u_1(\sigma_{1w})}{\partial\sigma_{1w}}\right) + \ldots\ldots \tag{2-21}$$

where $\sigma_{1w} + \delta^2 \log \delta\, \sigma_{2w} + \dots\dots\dots$ is the expansion for the shock location σ_w , and $u_1(\sigma_{1w})$, $v_1(\sigma_{1w})$ and σ_{1w} are results from the solution of the first approximation.

(c) Third approximation.

$$K\frac{\partial u_3}{\partial\sigma} - \sigma\frac{\partial v_3}{\partial\sigma} + v_3 = (\gamma+1)\frac{\partial u_1 u_3}{\partial\sigma} + \frac{1}{2}(2\gamma-1)(\gamma+1)u_1^2\frac{\partial u_1}{\partial\sigma}$$

$$- 2K\gamma u_1\frac{\partial u_1}{\partial\sigma} - 2v_1\sigma\frac{\partial u_1}{\partial\sigma} \qquad (2\text{-}22a)$$

$$\frac{\partial v_3}{\partial\sigma} = -\sigma\frac{\partial u_3}{\partial\sigma} \qquad (2\text{-}22b)$$

(d) Fourth approximation.

$$(K-(\gamma-1)u_1)\frac{\partial u_4}{\partial\sigma} - (\gamma-1)\frac{\partial u_1}{\partial\sigma} - \sigma\frac{\partial v_4}{\partial\sigma} + v_4 = (\gamma+1)u_2\frac{\partial u_2}{\partial\sigma} \qquad (2\text{-}27a)$$

$$\frac{\partial v_4}{\partial\sigma} = -\sigma\frac{\partial u_4}{\partial\sigma} \qquad (2\text{-}27b)$$

III. INTRODUCTION OF POTENTIAL FUNCTION AND ITS EXPANSION NEAR THE AXIS.

Since the flow is irrotational, there exists a velocity potential Φ which is represented by the following expansion (cf. equation 2-11)

$$\Phi = UL\left\{x + \delta^2 x\,\phi_1(\sigma; K) + \delta^4 \log\delta\, x\,\phi_2(\sigma; K) + \delta^4 x\,\phi_3(\sigma; K)\right.$$

$$\left. + \delta^6 \log^2\delta\,\phi_4(\sigma; K) + \ldots\ldots\right\} \qquad (3\text{-}1)$$

It can easily be shown from equations 2-11 and 3-1 that

$$u_i(\sigma; K) = \phi_i(\sigma; K) + \sigma \frac{\partial \phi_i}{\partial \sigma}$$

$$i = 1, 2, 3, 4 \ldots \qquad (3\text{-}2)$$

$$v_i(\sigma; K) = -\sigma^2 \frac{\partial \phi_i}{\partial \sigma}$$

and consequently

$$\frac{\partial u_i}{\partial \sigma} = 2\frac{\partial \phi_i}{\partial \sigma} + \sigma \frac{\partial^2 \phi_i}{\partial \sigma^2}$$

$$i = 1, 2, 3, 4 \ldots\ldots \qquad (3\text{-}3)$$

$$\frac{\partial v_i}{\partial \sigma} = -2\sigma \frac{\partial \phi_i}{\partial \sigma} - \sigma^2 \frac{\partial^2 \phi_i}{\partial \sigma^2}$$

With equations 3-2 and 3-3, the transonic equations 2-12a, 2-17a, 2-22a and 2-27a reduce to

$$\left[K - (\gamma+1)\left(\phi_1 + \sigma \frac{\partial \phi_1}{\partial \sigma}\right)\right]\left(2\frac{\partial \phi_1}{\partial \sigma} + \sigma \frac{\partial^2 \phi_1}{\partial \sigma^2}\right) + \sigma^2 \frac{\partial \phi_1}{\partial \sigma} + \sigma^3 \frac{\partial^2 \phi_1}{\partial \sigma^2} = 0 \qquad (3\text{-}4)$$

$$\left[K-(\gamma+1)u_1\right]\left(2\frac{\partial\phi_2}{\partial\sigma}+\sigma\frac{\partial^2\phi_2}{\partial\sigma^2}\right)-(\gamma+1)\frac{\partial u_1}{\partial\sigma}\left(\phi_2+\sigma\frac{\partial\phi_2}{\partial\sigma}\right)$$

$$+\sigma^2\frac{\partial\phi_2}{\partial\sigma}+\sigma^3\frac{\partial^2\phi_2}{\partial\sigma^2}=0 \qquad (3\text{-}5)$$

$$\left\{\sigma^3-\sigma\left[(\gamma+1)u_1-K\right]\right\}\frac{\partial^2\phi_3}{\partial\sigma^2}+\left\{\sigma^2-2\left[(\gamma+1)u_1-K\right]-(\gamma+1)\sigma\frac{\partial u_1}{\partial\sigma}\right\}\frac{\partial\phi_3}{\partial\sigma}$$

$$-(\gamma+1)\frac{\partial u_1}{\partial\sigma}\phi_3=-2v_1\sigma\frac{\partial u_1}{\partial\sigma}+\left[\frac{1}{2}(2\gamma-1)(\gamma+1)u_1-2K\gamma\right]u_1\frac{\partial u_1}{\partial\sigma} \qquad (3\text{-}6)$$

and

$$\sigma^3\frac{\partial^2\phi_4}{\partial\sigma^2}+\sigma^2\frac{\partial\phi_4}{\partial\sigma}=(\gamma+1)u_2\frac{\partial u_2}{\partial\sigma}+(\gamma-1)\frac{\partial u_1}{\partial\sigma}\left(\phi_4+\sigma\frac{\partial\phi_4}{\partial\sigma}\right)$$

$$-\left[K-(\gamma-1)u_1\right]\left(2\frac{\partial\phi_4}{\partial\sigma}+\sigma\frac{\partial^2\phi_4}{\partial\sigma^2}\right) \qquad (3\text{-}7)$$

respectively.

In the process of achieving the expansion form 2-11, the behavior of u_1, v_1 near the axis has been assumed from the same physical reasoning as in Reference one. After the introduction of the potential function, the expansions near the axis are now carried out in more detail.

Equation 3-4 can be rewritten as

$$\left[(\gamma+1)\frac{\partial}{\partial\sigma}(\sigma\phi_1)-K\right]\left[2\frac{\partial\phi_1}{\partial\sigma}+\sigma\frac{\partial^2\phi_1}{\partial\sigma^2}\right]=\sigma^2\frac{\partial}{\partial\sigma}\left(\sigma\frac{\partial\phi_1}{\partial\sigma}\right)$$

Consequently

$$\sigma \frac{\partial \phi_1}{\partial \sigma} = \int \frac{1}{\sigma^2}\left[(\gamma+1)\frac{\partial}{\partial \sigma}(\sigma\phi_1) - K\right]\left(2\frac{\partial \phi_1}{\partial \sigma} + \sigma\frac{\partial^2 \phi_1}{\partial \sigma^2}\right) d\sigma + C_1$$

or

$$\frac{\partial \phi_1}{\partial \sigma} = \frac{C_1}{\sigma} + \frac{1}{\sigma}\int \frac{1}{\sigma^2}\left[(\gamma+1)\frac{\partial}{\partial \sigma}(\sigma\phi_1) - K\right]\left(2\frac{\partial \phi_1}{\partial \sigma} + \sigma\frac{\partial^2 \phi_1}{\partial \sigma^2}\right) d\sigma$$

or

$$\phi_1(\sigma;K) = C_1 \log \sigma + C_2 + \int \frac{d\sigma}{\sigma}\int \frac{1}{\sigma^2}\left[(\gamma+1)\frac{\partial}{\partial \sigma}(\sigma\phi_1) - K\right]\left(2\frac{\partial \phi_1}{\partial \sigma} + \sigma\frac{\partial^2 \phi_1}{\partial \sigma^2}\right) d\sigma$$

The successive terms in the expansion can be obtained from iteration by using the first terms in the right hand side. The result is the following asymptotic expansion near the axis $(\sigma \to \infty)$

$$\phi_1(\sigma;K) = C_1 \log \sigma + C_2 + \frac{\gamma+1}{4} C_1^2 \frac{\log \sigma}{\sigma^2} - \frac{C\left[K - 2(\gamma+1)C_1 - (\gamma+1)C_2\right]}{4}\frac{1}{\sigma^2} + O\left(\frac{\log^2 \sigma}{\sigma^4}\right) \tag{3-8}$$

$$u_1(\sigma;K) = C_1 \log \sigma + C_1 + C_2 - \frac{\gamma+1}{4} C_1^2 \frac{\log \sigma}{\sigma^2} + \frac{C_1\left[K - (\gamma+1)C_1 - (\gamma+1)C_2\right]}{4}\frac{1}{\sigma^2} + O\left(\frac{\log^2 \sigma}{\sigma^4}\right) \tag{3-9}$$

$$v_1(\sigma;K) = -C_1\sigma + \frac{\gamma+1}{2} C_1^2 \frac{\log\sigma}{\sigma} - \frac{C_1\left[2K-3(\gamma+1)C_1-2(\gamma+1)C_2\right]}{4}\frac{1}{\sigma}$$

$$+ O\left(\frac{\log^2\sigma}{\sigma^3}\right) \qquad (3\text{-}10)$$

Also by applying the same reasoning to equations 3-5, 3-6, and 3-7, it has been found that

$$\phi_2(\sigma;K) = C_3\log\sigma + C_4 + \frac{(\gamma+1)C_1C_3}{2}\frac{\log\sigma}{\sigma^2}$$

$$-\frac{C_3K-(\gamma+1)(4C_1C_3+C_2C_3+C_1C_4)}{4}\frac{1}{\sigma^2} + O\left(\frac{\log^2\sigma}{\sigma^4}\right) \qquad (3\text{-}11)$$

$$u_2(\sigma;K) = C_3\log\sigma + C_3 + C_4 - \frac{(\gamma+1)}{2}C_1C_3\frac{\log\sigma}{\sigma^2}$$

$$+\frac{C_3K-(\gamma+1)(2C_1C_3+C_2C_3+C_1C_4)}{4}\frac{1}{\sigma^2} + O\left(\frac{\log^2\sigma}{\sigma^4}\right) \qquad (3\text{-}12)$$

$$v_2(\sigma;K) = -C_3\sigma + (\gamma+1)C_1C_3\frac{\log\sigma}{\sigma} - \frac{C_3K-(\gamma+1)(3C_1C_3+C_2C_3+C_1C_4)}{2}\frac{1}{\sigma}$$

$$+ O\left(\frac{\log^2\sigma}{\sigma^3}\right) \qquad (3\text{-}13)$$

$$\phi_3(\sigma;K) = C_1^2\log^2\sigma + C_5\log\sigma + C_6 + \frac{1}{4}(\gamma+1)\left(\frac{5}{2}+\gamma\right)C_1^3\frac{\log^2\sigma}{\sigma^2}$$

$$+(\gamma+1)\left\{\left(\gamma+\frac{5}{2}\right)C_1^3 + \frac{1}{4}(2\gamma+1)C_1^2C_2 + \frac{1}{2}C_1C_5 - \frac{1}{2}C_1^2K\right\}\frac{\log\sigma}{\sigma^2}$$

$$+ O\left(\frac{1}{\sigma^2}\right) \qquad (3\text{-}14)$$

$$u_3(\sigma;K) = C_1^2 \log^2\sigma + (2C_1^2 + C_5)\log\sigma + (C_5 + C_6) - \frac{1}{4}(\gamma+1)(\frac{5}{2}+\gamma)C_1^3 \frac{\log^2\sigma}{\sigma^2}$$

$$+(\gamma+1)\left\{-\frac{1}{2}(\gamma+\frac{5}{2})C_1^3 - \frac{1}{4}(2\gamma+1)C_1^2C_2 - \frac{1}{2}C_1C_5 + \frac{1}{2}C_1^2K\right\}\frac{\log\sigma}{\sigma^2}$$

$$+O\left(\frac{1}{\sigma^2}\right) \qquad (3\text{-}15)$$

$$v_3(\sigma;K) = -2C_1^2\sigma\log\sigma - C_5\sigma + \frac{1}{2}(\gamma+1)(\frac{5}{2}+\gamma)C_1^3\frac{\log^2\sigma}{\sigma}$$

$$+2(\gamma+1)\left\{\frac{3}{4}(\gamma+\frac{5}{2})C_1^3 + \frac{1}{4}(2\gamma+1)C_1^2C_2 + \frac{1}{2}C_1C_5 - \frac{1}{2}C_1^2K\right\}\frac{\log\sigma}{\sigma}$$

$$+O\left(\frac{1}{\sigma}\right) \qquad (3\text{-}16)$$

and

$$\phi_4(\sigma;K) = C_7\log\sigma + C_8 + \frac{1}{4}\left\{(\gamma+1)C_3^2 + 2(\gamma-1)C_1C_7\right\}\frac{\log\sigma}{\sigma^2}$$

$$+\frac{1}{4}\left\{(\gamma+1)(2C_3^2 + C_3C_4) + (\gamma-1)(4C_1C_7 + C_1C_8 + C_2C_7) - C_7K\right\}\frac{1}{\sigma^2}$$

$$+O\left(\frac{\log^2\sigma}{\sigma^4}\right) \qquad (3\text{-}17)$$

$$u_4(\sigma;K) = C_7\log\sigma + C_7 + C_8 - \frac{1}{4}\left\{(\gamma+1)C_3^2 + 2(\gamma-1)C_1C_7\right\}\frac{\log\sigma}{\sigma^2}$$

$$-\frac{1}{4}\left\{(\gamma+1)(C_3^2 + C_3C_4) + (\gamma-1)(2C_1C_7 + C_1C_8 + C_2C_7) - C_7K\right\}\frac{1}{\sigma^2}$$

$$+O\left(\frac{\log^2\sigma}{\sigma^4}\right) \qquad (3\text{-}18)$$

$$v_4(\sigma;K) = -C_7\sigma + \frac{1}{2}\left\{(\gamma+1)C_3^2 + 2(\gamma-1)C_1C_7\right\}\frac{\log\sigma}{\sigma}$$

$$+\frac{1}{4}\left\{(\gamma+1)(3C_3^2 + 2C_3C_4) + (\gamma-1)(6C_1C_7 + 2C_1C_8 + 2C_2C_7)\right.$$

$$\left. - 2C_7K\right\}\frac{1}{\sigma} + O\left(\frac{\log^2\sigma}{\sigma^3}\right) \qquad (3\text{-}19)$$

where the C_i's, independent of σ, are either functions of K or constants.

It will be shown in the next section that C_1, C_3, C_5,.... can be found from expansion at boundary while $C_2, C_4, C_6, \ldots$, being functions of K, can be obtained only by solving the transonic equations. The C_i's, as will be seen later, are important in determining the pressure on the cone surface.

IV. EXPANSION ON THE BOUNDARY.

In applying the reasoning in Reference one for deriving the transonic equations, use is made of the requirement for flow tangent to the body surface, that is, on $\sigma = \frac{1}{\delta^2}$

$$\frac{q_r}{U} = \frac{q_x}{U} \cdot \delta \tag{4-1}$$

By using the form of expansion 2-11, equation 4-1 becomes

$$\delta^3 v_1\left(\frac{1}{\delta^2}; K\right) + \delta^5 \log \delta\, v_2\left(\frac{1}{\delta^2}; K\right) + \delta^5 v_3\left(\frac{1}{\delta^2}; K\right) + \ldots\ldots$$

$$= \delta\left\{1 + \delta^2 u_1\left(\frac{1}{\delta^2}; K\right) + \delta^4 \log \delta\, u_2\left(\frac{1}{\delta^2}; K\right) + \delta^4 u_3\left(\frac{1}{\delta^2}; K\right) + \ldots\right\} \tag{4-2}$$

Equation 4-2, which holds on the boundary, gives the expansion on the boundary as $\delta \to 0$. By substituting the asymptotic expansions for u_i, v_i near the axis from previous section into equation 4-2 and collecting terms of the same order, it is found

$$\left.\begin{aligned} C_1 &= -1 \\ C_3 &= 2 \\ C_5 &= 1 - C_2 \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{4-3}$$

With equation 4-3, the following expansions at the boundary are obtained:

$$\frac{q_x}{U} = 1 + \delta^2 \log \delta\,(2) + \delta^2 (C_2 - 1) + \delta^4 \log^2 \delta\,(0) + \delta^4 \log \delta\,(2C_2 + C_4 - 4)$$

$$+ O(\delta^4) \tag{4-4}$$

$$\frac{q_r}{U} = \delta + \delta^3 \log \delta\,(-2) + \delta^3 (C_2 - 1) + O(\delta^5 \log^2 \delta) \tag{4-5}$$

and

$$\left(\frac{q}{U}\right)^2 = \left(\frac{q_x}{U}\right)^2 + \left(\frac{q_r}{U}\right)^2 = 1 + \delta^2 \log \delta\,(4) + \delta^2 (2C_2 - 1) + \delta^4 \log^2 \delta\,(4)$$

$$+ \delta^4 \log \delta\,(8C_2 + 2C_4 - 16) + O(\delta^4) \tag{4-6}$$

V. SURFACE PRESSURE.

It follows from the definition of speed of sound that

$$\left(\frac{a}{U}\right)^2 = \frac{\gamma p}{\rho U^2} = \left(\frac{\gamma p_\infty}{\rho_\infty U^2}\right)\frac{\frac{p}{p_\infty}}{\frac{\rho}{\rho_\infty}} = \frac{1}{M^2}\frac{\frac{p}{p_\infty}}{\frac{\rho}{\rho_\infty}}$$

or

$$\frac{p}{p_\infty} = \left(M_\infty^2 \frac{a^2}{U^2}\right)\frac{\rho}{\rho_\infty} \tag{5-1}$$

Defining $\pi = p/\rho^\gamma$ equation 5-1 reads

$$\frac{p}{p_\infty} = \left[(1-K\delta^2)\frac{a^2}{U^2}\right]^{\frac{\gamma}{\gamma-1}}\left(\frac{\pi}{\pi_\infty}\right)^{-\frac{1}{\gamma-1}} \tag{5-2}$$

and is valid at the body surface as well as away from the body.

Equation 2-3 can be rewritten as

$$\frac{a^2}{U^2} = \frac{1}{1-K\delta^2} + \frac{\gamma-1}{2}\left\{1-\left(\frac{q}{U}\right)^2\right\} \tag{5-3}$$

which, evaluated at the cone surface, reduces to

$$\left(\frac{a}{U}\right)^2 = 1 + \delta^2\log\delta\left[-2(\gamma-1)\right] + \delta^2\left[K - \frac{\gamma-1}{2}(2C_2-1)\right] + \delta^4\log^2\delta\left[-2(\gamma-1)\right]$$
$$+\delta^4\log\delta\left[(\gamma-1)(8-4C_2-C_4)\right] + O(\delta^4) \tag{5-4}$$

Now using the fact $\pi/\pi_\infty = 1 + O(\delta^6)$, that is, the entropy change is very small, equation 5-2 becomes:

$$\frac{p}{p_\infty} = \left\{1 + \delta^2 \log\delta\left[-2\gamma\right] + \delta^2\left[-\frac{\gamma}{2}(2C_2 - 1)\right] + \delta^4 \log^2\delta(0)\right.$$

$$\left. + \delta^4 \log\delta\left[\gamma(7 - 2C_2 - C_4 + 2K)\right] + O(\delta^4)\right. \tag{5-5}$$

By substituting this result into the expression for the pressure coefficient, it is found

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U^2} = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right) = \frac{2}{\gamma(1 - K\delta^2)}\left(\frac{p}{p_\infty} - 1\right)$$

$$= \delta^2 \log\delta(-4) + \delta^2(1 - 2C_2) + \delta^4 \log^2\delta\ (0)$$

$$+ \delta^4 \log\delta\ (14 - 4C_2 - 2C_4) + O(\delta^4) \tag{5-6}$$

where C_2 and C_4 are functions of K only and can only be found by solving the transonic equations.

By defining

$$C_p^+ = \frac{C_p}{\delta^2} + 4\log\delta - 1 = -2C_2(K) + O(\delta^2 \log\delta) \tag{5-7}$$

a similarity law for pressure coefficient is introduced; that is C_p^+ is the same function for flows over a family of cones provided that K for such flows is the same.

The similarity law thus defined is essentially the same as that derived by Oswatitsch and Berndt (ref. 6). However, they do not show the relation of equation 5-7 to an expansion procedure or estimate the error.

VI. SOLUTION FOR TRANSONIC EQUATION OF FIRST APPROXIMATION.

It has been pointed out in Section III that $C_2(K)$, $C_4(K)$, $C_6(K)$,...... can be found only by solving the transonic equations of the first, second, and third approximations........respectively. The method of finding $C_2(K)$ is explained as follows:

(A). Formulation of the boundary value problem.

The transonic equation of the first approximation is given by equation 2-12

$$\left[K-(\gamma+1)u_1\right]\frac{\partial u_1}{\partial \sigma}+v_1-\sigma\frac{\partial v_1}{\partial \sigma}=0$$

$$\frac{\partial v_1}{\partial \sigma}=-\sigma\frac{\partial u_1}{\partial \sigma} \tag{6-1}$$

By defining

$$u=(\gamma+1)u_1-K \tag{6-2}$$
$$v=(\gamma+1)v_1$$

Equation 6-1 reduces to

$$u\frac{\partial u}{\partial \sigma}+\sigma\frac{\partial v}{\partial \sigma}-v=0 \tag{6-3a}$$

$$\frac{\partial v}{\partial \sigma}=-\sigma\frac{\partial u}{\partial \sigma} \tag{6-3b}$$

Since the flow is conical,

$$u = f(v) \tag{6-4}$$

and consequently

$$\frac{du}{d\sigma} = \frac{du}{dv}\frac{dv}{d\sigma}$$

But from equation 6-3b

$$\frac{dv}{d\sigma} = -\sigma\frac{du}{d\sigma}$$

Therefore

$$\frac{du}{dv} = -\frac{1}{\sigma} \tag{6-5}$$

Differentiating equation 6-5 with respect to σ yields

$$\frac{dv}{d\sigma} = \frac{1}{\sigma^2 \frac{d^2u}{dv^2}} \tag{6-6}$$

and therefore

$$\frac{du}{d\sigma} = -\frac{1}{\sigma^3 \frac{d^2u}{dv^2}} \tag{6-7}$$

Substituting equations 6-5, 6-6, and 6-7 into equation 6-3a, the result is

$$v\frac{d^2u}{dv^2} + \frac{du}{dv} = u\left(\frac{du}{dv}\right)^3 \tag{6-8}$$

The following conditions are prescribed for equation 6-8.

(1) At the shock wave:

Substitution of equation 6-2 into equations 2-13 and 2-14 yields

$$v_w^{\,2} = (u_w + K)^2 \frac{u_w - K}{2} \tag{6-9}$$

and

$$\tan\beta = -\frac{1}{\delta}\,\frac{u_w + K}{v_w} + \ldots\ldots \tag{6-10}$$

Hence, from equation 6-5

$$\left(\frac{du}{dv}\right)_w = \left(\frac{u_w + K}{v_w} + \ldots\ldots\right) \tag{6-11}$$

(2) On the cone surface:

The boundary condition on the cone surface is replaced by the expansion near the axis as carried out in Section III. For u and v, the following expansions near the axis are obtained from equations 3-9, 3-10, and 6-2

$$u = (\gamma+1)u_1 - K = (\gamma+1)(-\log\sigma - 1 + C_2 - \frac{K}{\gamma+1} + \ldots\ldots) \tag{6-12}$$

$$v = (\gamma+1)v_1 = (\gamma+1)(\sigma + \ldots\ldots) \tag{6-13}$$

and consequently

$$\frac{du}{dv} = -\frac{1}{\sigma} \cong -\frac{\gamma+1}{v} + \ldots\ldots \tag{6-14}$$

(B) Reduction to first order differential equation.

Equation 6-8 is a non-linear second order differential equation. It has the property of scale invariance, that is, if a scale transformation is introduced to both u and v $(v = Av', u = Bu')$ it is invariant with a suitable choice of the scale factors for the u and v ($B = A^{\frac{2}{3}}$). This means that if $u = f(v)$ is a solution, $u = A^{\frac{2}{3}} f\left(\frac{v}{A}\right)$ is also a solution. For differential equations with this property, it is possible to reduce the order of the differential equation. Consequently, if the following transformation

$$s = v^{-\frac{2}{3}} u \tag{6-15a}$$

$$t = v^{\frac{1}{3}} \frac{du}{dv} \tag{6-15b}$$

is introduced, the second order equation 6-8 is reduced to the following first order differential equation

$$\frac{dt}{ds} = \frac{3st^2 - 2t}{3t - 2s} \tag{6-16}$$

Furthermore, from equation 6-15

$$\log v = 3\int \frac{ds}{3t - 2s} \tag{6-17}$$

If, now, equation 6-16 is solved so that t as a function of s is known, then equation 6-17 will give the value of v corresponding to a certain value of s. Then from equation 6-15, the values of u can be found. Therefore, the solution in the u, v -system can be obtained if the solution in the s-t system is known.

The boundary condition in the s-t system will be

(1) At the shock wave:

From equation 6-15

$$s_w = v_w^{-\frac{2}{3}} u_w, \quad t_w = v_w^{\frac{1}{3}} \left(\frac{du}{dv} \right)_w \tag{6-18}$$

Substitution of equations 6-9 and 6-11 into equation 6-18 yields

$$s_w = 2^{\frac{1}{3}} \frac{u_w}{K} \left(\frac{u_w}{K} - 1 \right)^{-\frac{1}{3}} \left(\frac{u_w}{K} + 1 \right)^{-\frac{2}{3}} \tag{6-19a}$$

$$t_w = 2^{\frac{1}{3}} \left(\frac{u_w}{K} - 1 \right)^{-\frac{1}{3}} \left(\frac{u_w}{K} + 1 \right)^{\frac{1}{3}} \tag{6-19b}$$

From equation 6-19b

$$\frac{u_w}{K} = \frac{t^3 + 3}{t^3 - 2} \tag{6-20}$$

By substituting equation 6-20 into equation 6-19a, it is found that

$$s_w = \frac{t_w{}^3 + 2}{2\, t_w^2} = \frac{t_w}{2} + \frac{1}{t_w^2} \tag{6-21}$$

which represents the shock relation. The curve corresponding to equation 6-21 shall be called the universal hedgehog.

(2) On the cone surface:

The boundary condition on the cone surface is replaced by the expansion near the axis as in equations 6-12, 6-13, and 6-14. In the $s-t$ system, they reduce to

$$s = v^{-\frac{2}{3}} u \cong (\gamma+1)^{\frac{1}{3}} \sigma^{-\frac{2}{3}} (-\log \sigma - 1 + C_2 - \frac{K}{\gamma+1} + \ldots\ldots) \qquad (6\text{-}22)$$

$$t = v^{\frac{1}{3}} \frac{du}{dv} \cong -(\gamma+1) v^{-\frac{2}{3}} + \ldots\ldots = -(\gamma+1)^{\frac{1}{3}} \sigma^{-\frac{2}{3}} \qquad (6\text{-}23)$$

Near the axis, σ approaches infinity, hence s and t approach zero.

Hence the axis of the cone in the $x-r$ system corresponds to the origin in the $s-t$ system.

Before carrying out the numerical integration of equation 6-16, the qualitative picture of the integral curves is investigated. The procedure is:

Firstly, the singular points of the differential equation in the system will be located. The structure of the integral curves at these points will be studied.

Secondly, isoclines will be drawn for the general orientation of the integral curves.

The singular points of the differential equation is found from the fact that the derivative $\frac{dt}{ds}$ is indeterminate at these points. For equation 6-16, the singular points in the finite $s-t$ plane are found to be

$$s = 0, \quad t = 0$$

$$s = \left(\frac{3}{2}\right)^{\frac{1}{3}}, \quad t = \left(\frac{2}{3}\right)^{\frac{2}{3}} \qquad (6\text{-}24)$$

Consider first the singularity at the point $s = 0$, $t = 0$. By following the usual procedure (ref. 8) for investigating the conditions near a singular point, the essential behavior of the integral curves will be described by the approximate equation

$$\frac{dt}{ds} \cong \frac{-2t}{-2s+3t} \tag{6-25}$$

The criterion of distinguishing the types of singularities (ref. 8) shows that this point is a node.

Equation 6-25, being a homogeneous equation, can be integrated directly to give

$$t = C e^{-\frac{2}{3}\left(\frac{s}{t}\right)} \tag{6-26}$$

where C is a constant of integration.

The structure of the integral curve is shown in fig. 2.

For the investigation of the condition near the singular point at

$$s_o = \left(\frac{3}{2}\right)^{\frac{1}{3}}, \quad t_o = \left(\frac{2}{3}\right)^{\frac{2}{3}}. \tag{6-27}$$

the following transformation

$$s = \zeta + s_0, \quad t = \eta + t_0 \tag{6-28}$$

is inserted into equation 6-16 which, after neglecting higher order terms, reduces to

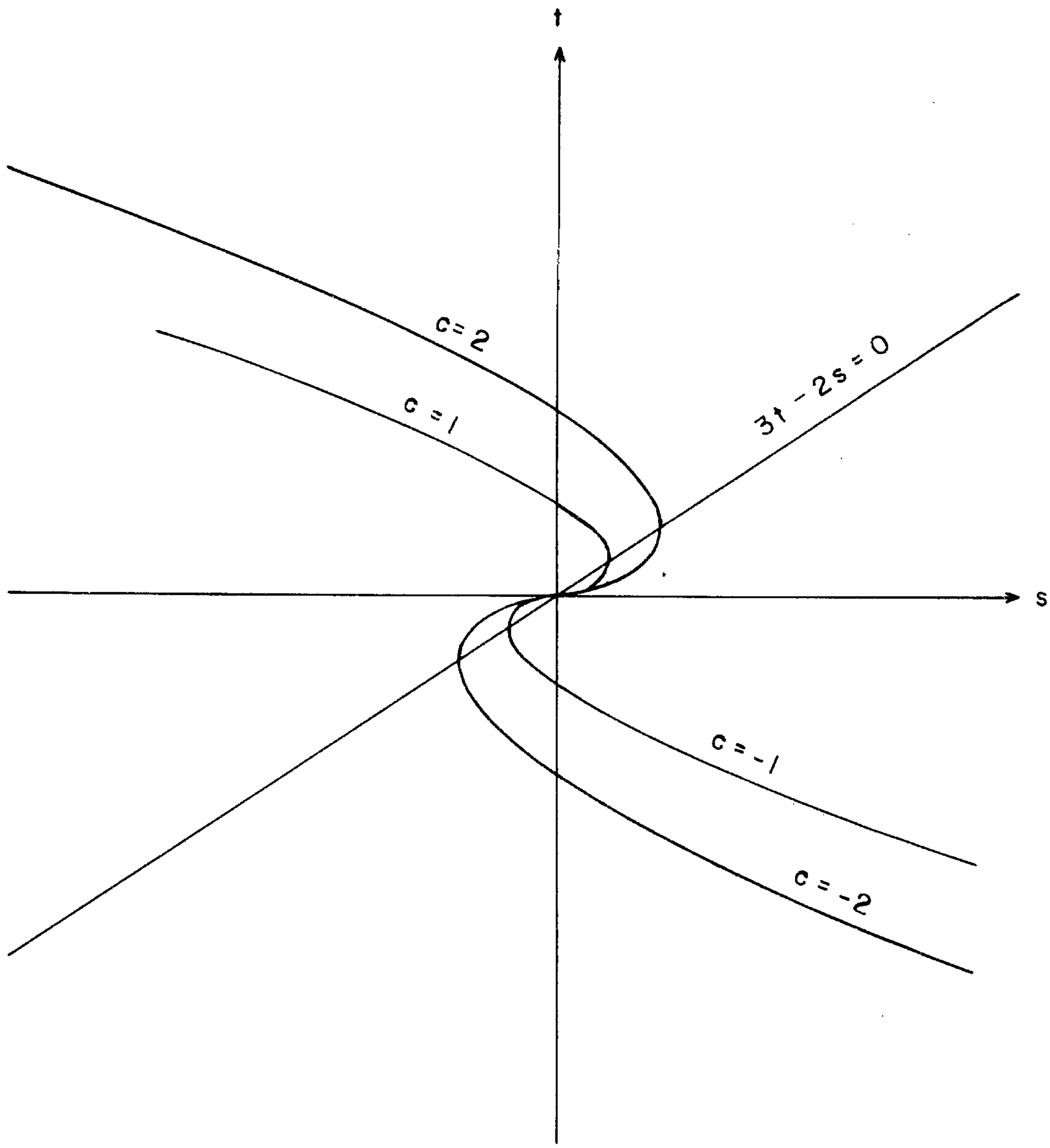


FIG. 2

STRUCTURE OF SINGULAR POINT AT s=0, t=0

$$\frac{d\eta}{d\zeta} \cong \frac{3t_o^3\zeta + (9s_o t_o^2 - 2)\eta}{3\eta - 2\zeta} \qquad (6\text{-}29)$$

With $s_o = \left(\frac{3}{2}\right)^{\frac{1}{3}}$, $t_o = \left(\frac{2}{3}\right)^{\frac{2}{3}}$ it becomes

$$\frac{d\eta}{d\zeta} \cong \frac{4\zeta - 12\eta}{-6\zeta + 9\eta} \qquad (6\text{-}30)$$

By following the criterion for classifying the types of singularities, it is found that this point is a node.

The integral curves approximated by (6-30) are

$$\frac{1}{C} = \left[3\eta + (1 - \sqrt{5})\zeta\right]^{\frac{1}{2} - \frac{3}{2\sqrt{5}}} \left[3\eta + (1 + \sqrt{5})\zeta\right]^{\frac{1}{2} + \frac{3}{2\sqrt{5}}} \qquad (6\text{-}31)$$

with the structure shown in fig. 3.

Now with the help of the isoclines $\frac{dt}{ds} = \infty$, $\frac{dt}{ds} = 0$, $\frac{dt}{ds} = 1$, and the universal hedgehog the integral curves are shown in fig. 4.

However, owing to the fact that $v > 0$, $\frac{du}{dv} = -\frac{1}{\sigma} < 0$ and $u > 0$ for supersonic flow, it follows from equation 6-15 that $t < 0$ and $s > 0$ Hence only the fourth quadrant of fig. 4 has physical meaning in supersonic flow.

(C) Determination of $C_2(K)$ by numerical integration.

It is apparent from the transformation by equation 6-15 that each point in the s, t system corresponds to a certain u and v in the physical plane. Since this is a conical flow, u and v are constant along a ray from the tip of the cone. Consequently each point in the

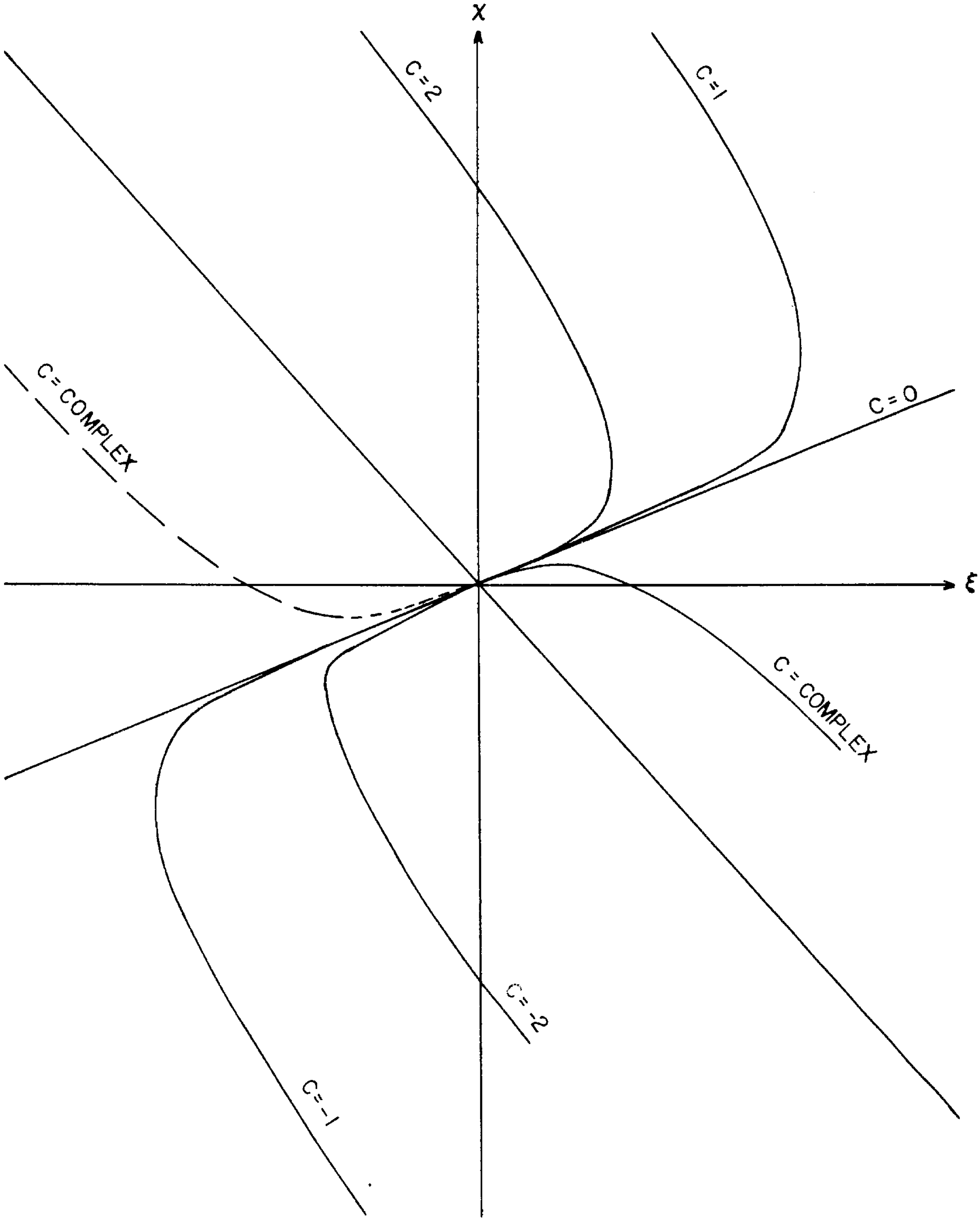


FIG. 3

STRUCTURE OF SINGULAR POINT AT $s = \left(\frac{3}{2}\right)^{\frac{1}{3}}$, $t = \left(\frac{2}{3}\right)^{\frac{2}{3}}$

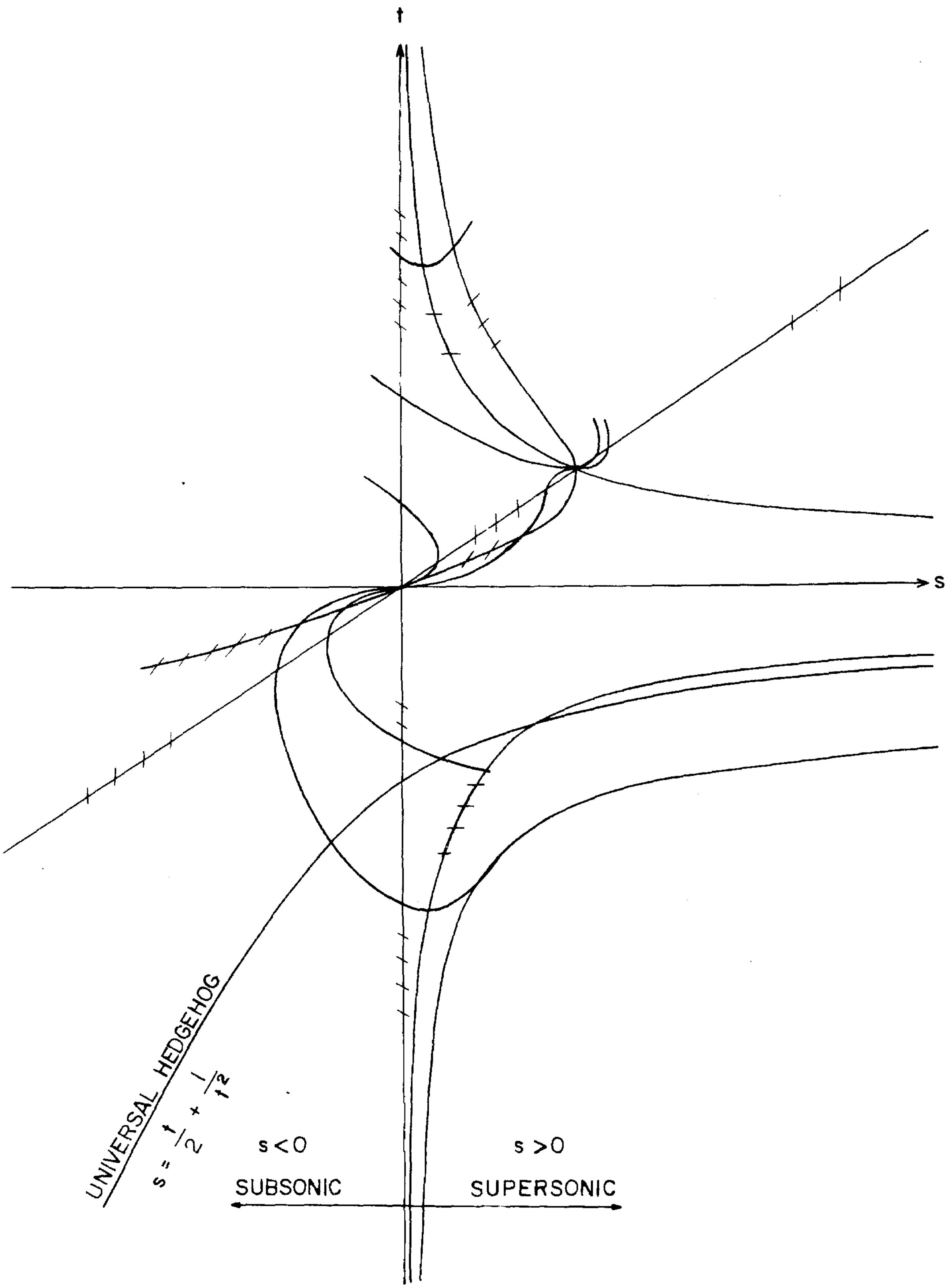


FIG. 4

INTEGRAL CURVES OF THE s-t SYSTEM

$s-t$ system corresponds to a ray in the $x-r$ system. The shock relation in the physical plane is thus represented by a point on universal hedgehog.

Furthermore, from equations 6-5, 6-9, and 6-11, it is found that

$$\left(\frac{1}{\sigma}\right)_w = -\left(\frac{du}{dv}\right)_w \cong -\frac{u_w+K}{v_w} + \ldots\ldots \cong -\sqrt{\frac{2}{|u_w-K|}} + \ldots\ldots \qquad (6\text{-}32)$$

Thus u, being a function of σ and K, is now only a function of K at the shock, that is

$$u_w = u_w(K) \qquad (6\text{-}33)$$

Accordingly each point on the universal hedgehog has a different K.

Equation 6-16, being a non-linear differential equation, can be solved only by numerical integration. The numerical integration is started by using the Runge Kutta method (ref. 9) with a certain $\frac{u_w}{K} = m$ which corresponds to a certain point on universal hedgehog, as initial condition and continued by the method of successive approximations (ref. 9) until the desirability for decreasing the interval of integration by applying the Runge-Kutta method again arises. The integration is carried out toward the origin until the singularity solution (equation 6-26) holds. The constant C of integration can thus be found. The scheme of numerical integration is shown as follows:

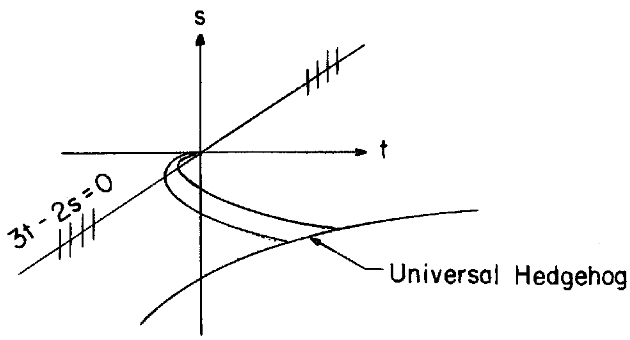


Fig. 5
Path of Integration.

Equation 6-23

$$t = -(\gamma + 1) v^{-\frac{2}{3}}$$

and equation 6-26

$$t = Ce^{-\frac{2}{3}\left(\frac{s}{t}\right)}$$

combine to give

$$v = \left(\frac{\gamma + 1}{-C}\right)^{\frac{3}{2}} e^{\frac{s}{t}} \qquad (6\text{-}34)$$

Then equation 6-17 can be used to find v_w , equation 6-15a will give u_w , and K is thus determined.

By substituting equations 6-22 and 6-23 into the singularity solution 6-26, it is found that

$$C_2(K) = \frac{1}{2}(\gamma+1) + 1 + \frac{K}{\gamma+1} - \frac{3}{2}\log(-C) \qquad (6-35)$$

where C, being constant of integration, is equal to $(-t)e^{\frac{2}{3}\frac{s}{t}}$

From equation 5-7

$$C_p^+(K) = -2\,C_2(K) + O(\delta^2 \log \delta)$$

C_p^+ is thus found.

(D) Results from numerical integration.

$C_2(K)$ has been computed by the method described above with four different values of $\frac{u_w}{K}$ as initial conditions. The results are:

$\frac{u_w}{K}$	v_w	u_w	K	$C_2(K)$	$C_p^+ = -2C_2(K)$
-.970	1.963	15.832	-16.322	1.613	-3.226
-.950	2.198	11.934	-12.562	1.438	-2.877
-.897	2.523	7.710	-8.594	1.239	-2.477
-.800	2.917	4.946	-6.182	.961	-1.923

VII. COMPARISON OF SIMILARITY SOLUTION WITH EXACT SOLUTION.

In order to show the accuracy of the first order transonic approximation from the similarity solution, the results computed from it are compared with those from Kopal's table (ref. 4).

The results from Kopal's table are as follows:

θ	δ	M	K	C_p	C_p^+
5^o	.087489	1.0152	-4.00169	.07482	-.97006
		1.0385	-10.25311	.06142	-2.72072
		1.0795	-21.59841	.05478	-3.58821
7.5^o	.131652	1.0484	-5.72010	.12522	-1.88573
		1.0737	-8.81760	.11378	-2.54577
		1.1144	-13.95596	.10440	-3.08695
		1.1902	-24.03478	.09456	-3.65468
10^o	.176327	1.0902	-6.06409	.18444	-2.00946
		1.1162	-7.90899	.17262	-2.38963
		1.2330	-16.73431	.14730	-3.20401
		1.4028	-31.12938	.13020	-3.75401
12.5^o	.221695	1.1381	-6.00771	.24754	-1.98923
		1.1646	-7.24925	.23504	-2.24356
		1.2825	-13.11962	.20476	-2.85965
		1.4552	-22.73944	.18228	-3.31704
15^o	.267949	1.1916	-5.84859	.31302	-1.90803
		1.2186	-6.75504	.29996	-2.08994
		1.3382	-11.01414	.26554	-2.56934
		1.5144	-18.01488	.23840	-2.94736
		1.7178	-27.17170	.21936	-3.21255

The results from the similarity solution are as follows:

θ	K	C_p^+	C_p	M
5^o	-6.182	-1.923	.0675	1.023
	-8.594	-2.477	.0633	1.032
	-12.562	-2.877	.0602	1.047
	-16.322	-3.226	.0576	1.061
7.5^o	-6.182	-1.923	.1246	1.052
	-8.594	-2.477	.1150	1.072
	-12.562	-2.877	.1080	1.104
	-16.322	-3.226	.1020	1.133
10^o	-6.182	-1.923	.1871	1.092
	-8.594	-2.477	.1699	1.126
	-12.562	-2.877	.1575	1.179
	-16.322	-3.226	.1466	1.228
12.5^o	-6.182	-1.923	.2508	1.142
	-8.594	-2.477	.2235	1.193
	-12.562	-2.877	.2039	1.272
	-16.322	-3.226	.1867	1.342
15^o	-6.182	-1.923	.3120	1.202
	-8.594	-2.477	.2721	1.272
	-12.562	-2.877	.2435	1.379
	-16.322	-3.226	.2184	1.474

The comparison between the exact solution and the similarity solution is shown in the following figures (fig. 6, fig. 7 and fig. 8).

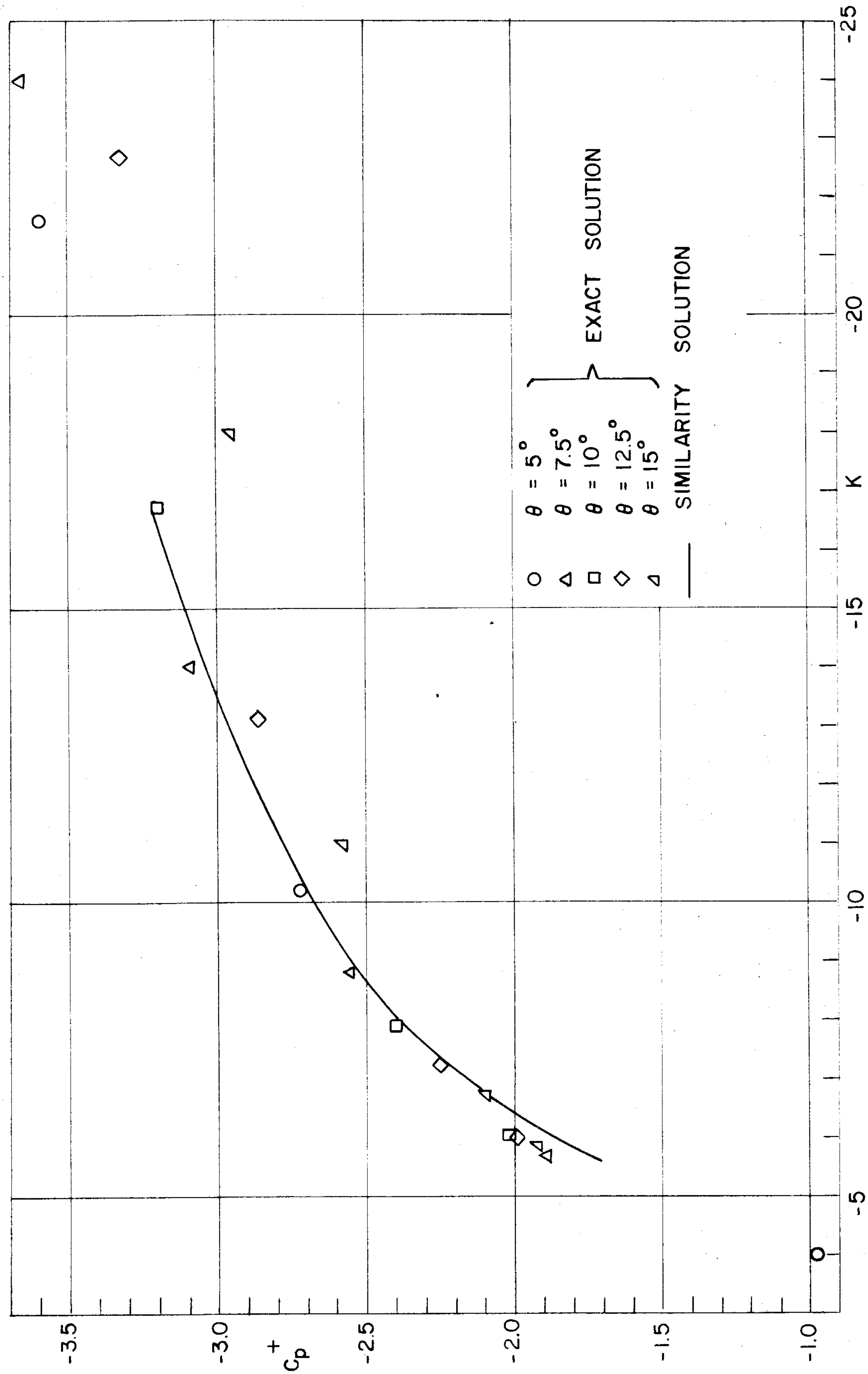


FIG. 6 C_p^+ VERSUS TRANSONIC SIMILARITY PARAMETER K

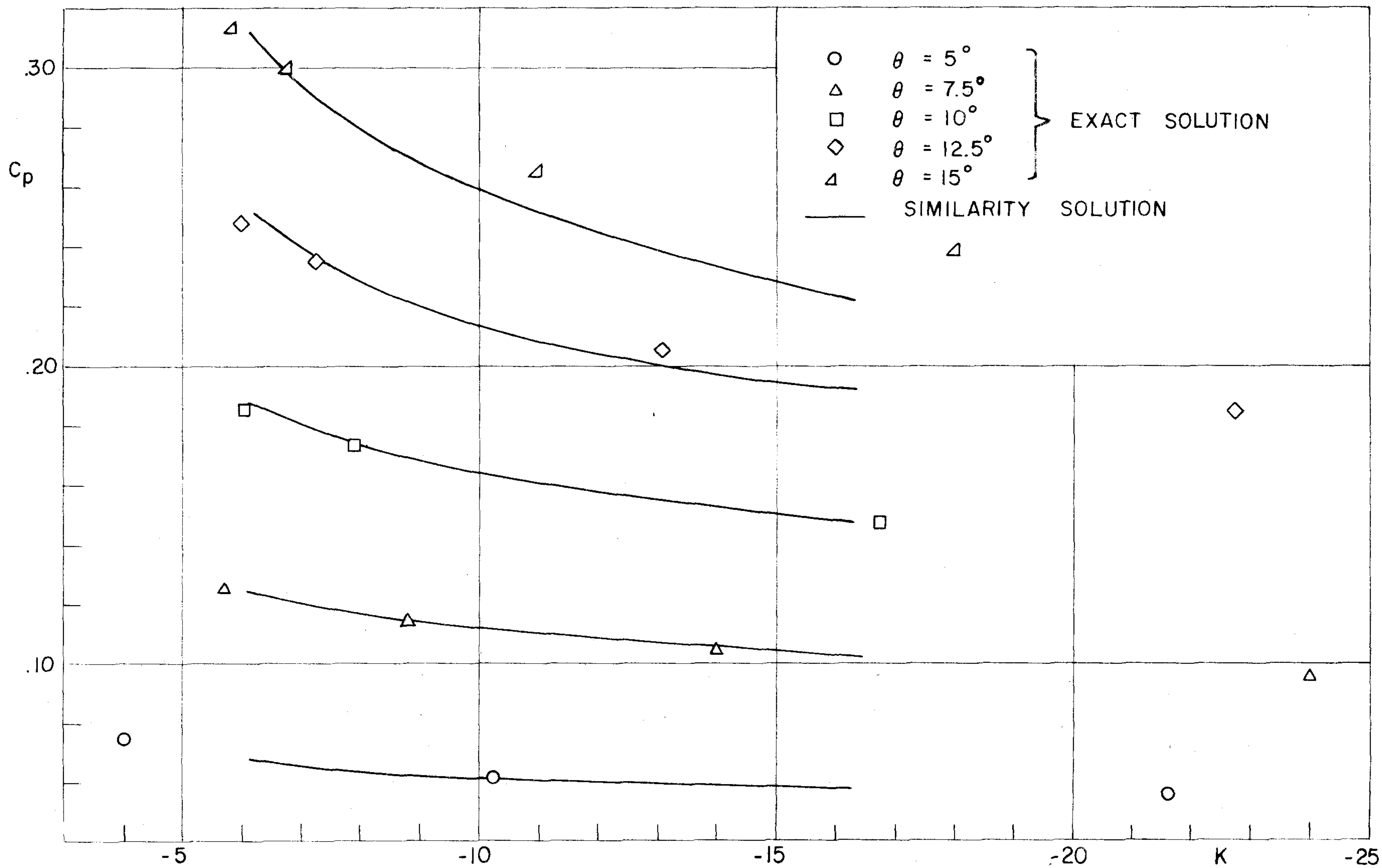


FIG. 7. PRESSURE C_p VERSUS TRANSONIC SIMILARITY PARAMETER K

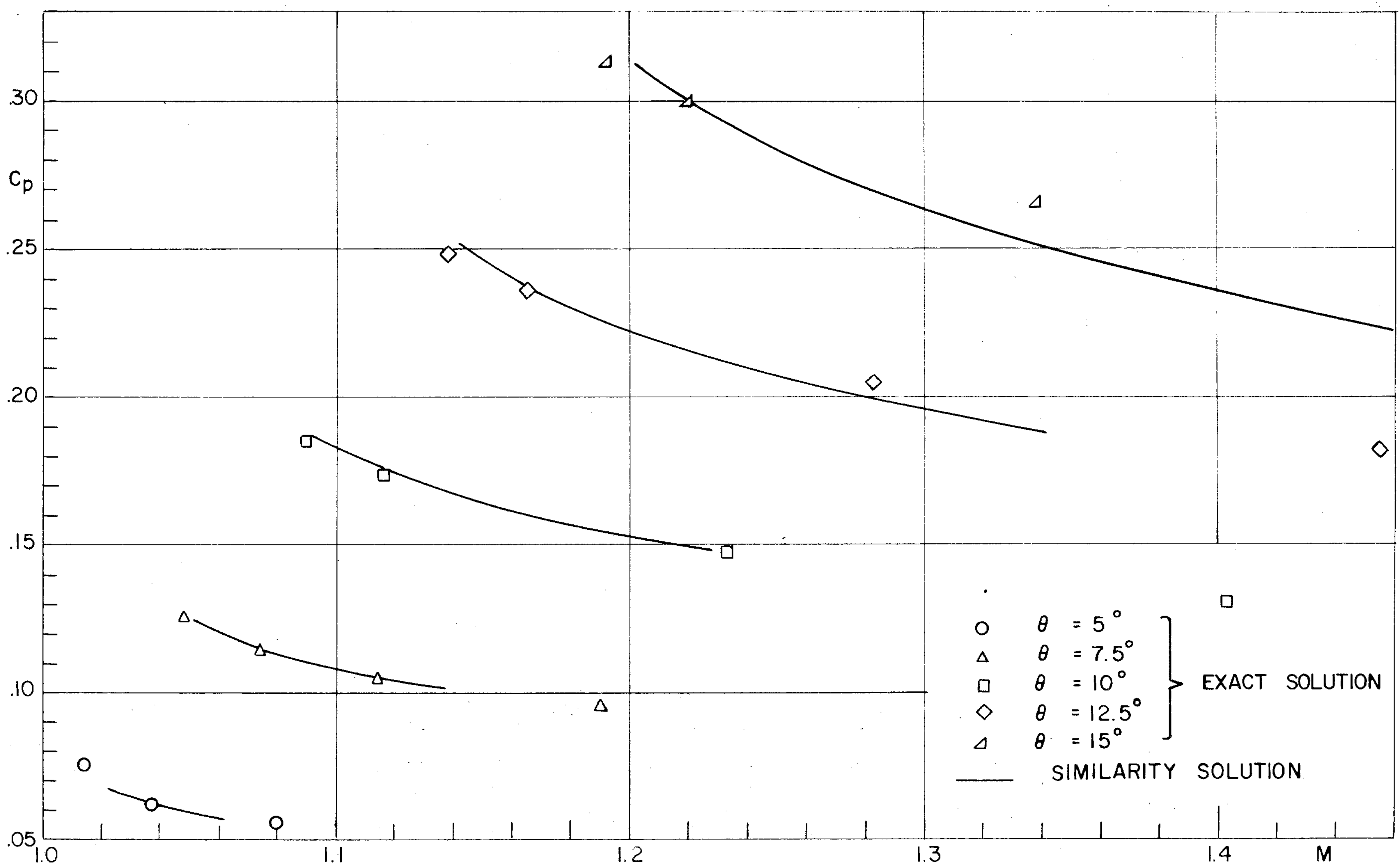


FIG. 8. PRESSURE C_p VERSUS MACH NUMBER M

VIII. CONCLUSIONS

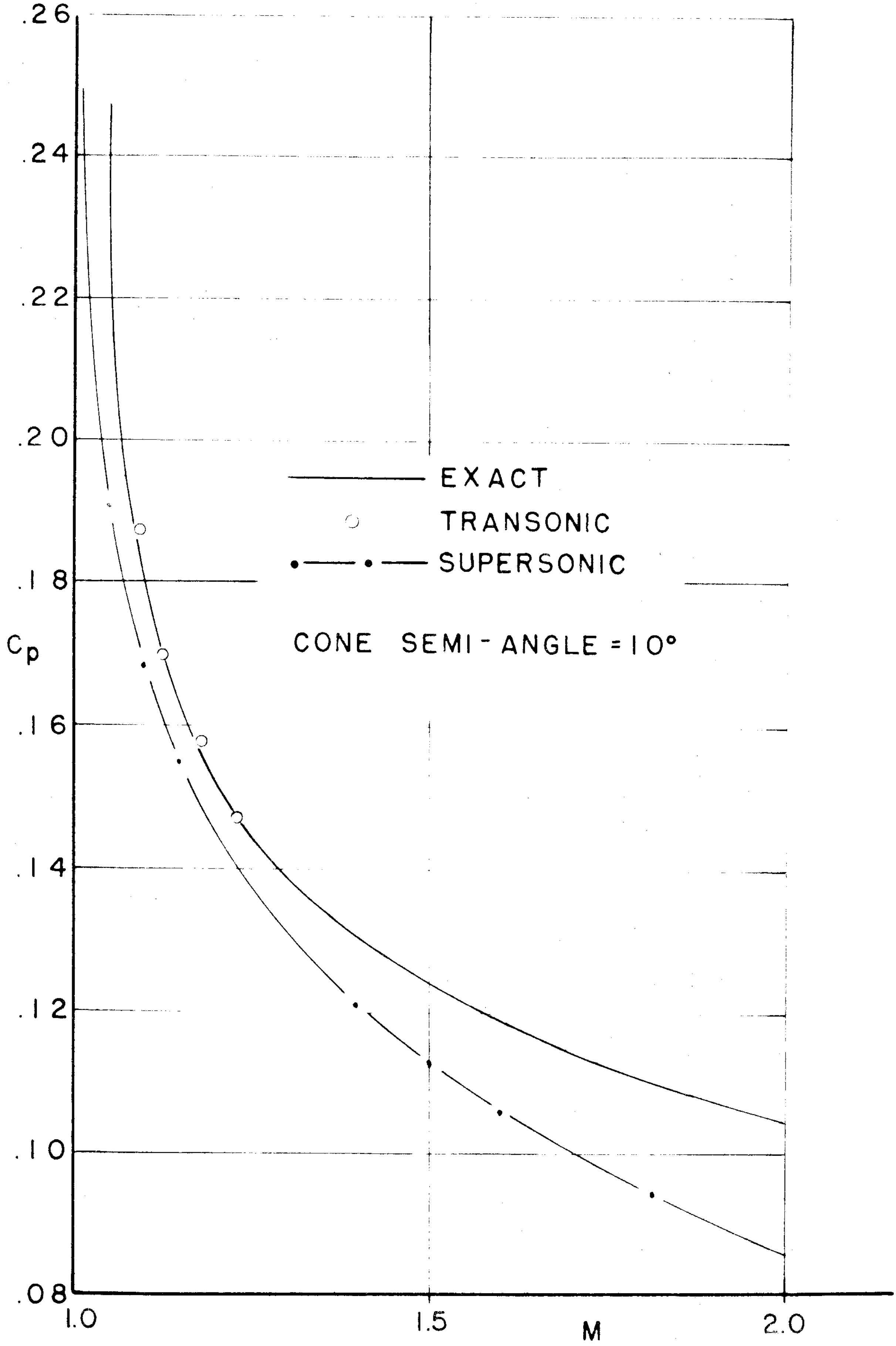
The transonic expansion procedures by Cole and Messiter (ref. 1) have been successfully applied to the problem of flow past a cone. The similarity law for the pressure coefficient on the cone surface is established by means of the expansion. By applying this law, a great deal of labor in the computation of pressures for different cones is saved. Because of the scale invariance property of the transonic equations of the first and second approximations, the problem has been reduced to integration of first order differential equations. Four different values of $\frac{u_w}{K}$ ($\frac{u_w}{K}$ = -.970, -.950, -.897, and -.800) have been used as initial conditions in the numerical integration. The results have been compared with those from exact solution (ref. 4) for semi-cone angles of 5^o, 7.5^o, 10^o, 12.5^o, and 15^o, and a very satisfactory agreement is noticed. However, the deviation becomes larger as $|K|$ becomes greater than 10 for a less slender cone (say $\theta = 12.5^o$ and 15^o). For a much more slender cone ($\theta = 5^o$, 7.5^o and 10^o), the theory holds very well even for $|K|$ greater than 10. Furthermore, the deviation will be less if higher order terms are included, the computation of which is possible under the present systematic expansion procedure.

The transonic approximation is found to be very good in the case of flow over a cone. The same technique should be expected to be also valid in the case of flow over any slender body of revolution.

The transonic theory is also compared with the linearized supersonic theory which gives a pressure coefficient

$$C_p = -2\delta^2 \log \frac{\sqrt{M_\infty^2 - 1}\,\delta}{2} - \delta^2$$

The comparison shown in fig. 9 for a 10° cone shows that transonic theory gives a much better result than supersonic theory in the transonic region.



C_p VS. MACH NUMBER M FOR A CONE OF SEMI-ANGLE = 10°

FIG. 9

APPENDIX

FORMULATION FOR TRANSONIC EQUATION OF SECOND APPROXIMATION.

In order to find C_4 which appears in the expression 5-6 for pressure coefficient, equation 2-17

$$K \frac{\partial u_2}{\partial \sigma} - \sigma \frac{\partial v_2}{\partial \sigma} + v_2 = (\gamma + 1) \frac{\partial u_1 u_2}{\partial \sigma} \tag{A-1a}$$

$$\frac{\partial v_2}{\partial \sigma} = - \sigma \frac{\partial u_2}{\partial \sigma} \tag{A-1b}$$

has to be solved with the shock relations specified in equation 2-18 and equation 2-19

$$\frac{2}{\gamma+1} v_1(\sigma_{1w}) \left(v_2(\sigma_{1w}) + \sigma_{2w} \frac{\partial v_1(\sigma_{1w})}{\partial \sigma_{1w}} \right)$$
$$= u_1(\sigma_{1w}) \left(\frac{3}{2} u_1(\sigma_{1w}) - \frac{2K}{\gamma+1} \right) \left(u_2(\sigma_{1w}) + \sigma_{2w} \frac{\partial u_1(\sigma_{1w})}{\partial \sigma_{1w}} \right) \tag{A-2}$$

$$\tan \beta = - \frac{1}{\delta} \frac{u_1(\sigma_{1w})}{v_1(\sigma_{1w})} \left\{ 1 + \delta^2 \log \delta \left[\frac{1}{u_1(\sigma_{1w})} \left(u_2(\sigma_{1w}) + \sigma_{2w} \frac{\partial u_1(\sigma_{1w})}{\partial \sigma_{1w}} \right) \right.\right.$$
$$\left.\left. - \frac{1}{v_1(\sigma_{1w})} \left(v_2(\sigma_{1w}) + \sigma_{2w} \frac{\partial v_1(\sigma_{1w})}{\partial \sigma_{1w}} \right) \right] + \ldots\ldots \right\} \tag{A-3}$$

and with the surface condition replaced by expansion near the axis in equations 3-12 and 3-13

$$u_2(\sigma; K) = C_3 \log \sigma + C_3 + C_4 - \frac{\gamma+1}{2} C_1 C_3 \frac{\log \sigma}{\sigma^2} + O\left(\frac{1}{\sigma^2}\right) \tag{A-4}$$

$$v_2(\sigma; K) = -C_3\sigma + (\gamma+1)C_1C_3\frac{\log\sigma}{\sigma} + O\left(\frac{1}{\sigma}\right) \tag{A-5}$$

where

$$C_1 = -1 \text{ and } C_3 = 2 \tag{A-6}$$

Since $u_2 = u_2(\sigma; K)$ and $v_2 = v_2(\sigma; K)$, hence $u_2 = f(v_2)$ provided that K is fixed. By applying the same reasoning as in Section 6, equation A-1 reduces to the following form

$$\left\{v_2 - \left[(\gamma+1)\frac{du_1}{d\sigma}\right]u_2\right\}\frac{d^2u_2}{dv_2^2} + \frac{du_2}{dv_2} = \left[(\gamma+1)u_1 - K\right]\left(\frac{du_2}{dv_2}\right)^3 \tag{A-7}$$

Again, as equation A-7 has the property of scale invariance, by the following transformation

$$\zeta = v_2^{-1}u_2 \qquad \eta = \frac{du_2}{dv_2} = -\frac{1}{\sigma} \tag{A-8}$$

it is reduced to

$$(\eta-\zeta)\frac{d\eta}{d\zeta} - \left[(\gamma+1)\frac{du_1}{d\sigma}\right]\zeta(\eta-\zeta)\frac{d\eta}{d\zeta} + \eta = \left[(\gamma+1)u_1 - K\right]\eta^3 \tag{A-9}$$

Furthermore, since

$$(\gamma+1)\frac{du_1}{d\sigma} = \frac{du}{d\sigma} = -\frac{1}{\sigma^3\frac{d^2u}{dv^2}}$$

and from equation 6-8

$$\frac{d^2u}{dv^2} = \frac{u}{v}\left(\frac{du}{dv}\right)^3 - \frac{1}{v}\left(\frac{du}{dv}\right)$$

it follows that

$$(\gamma+1)\frac{du_1}{d\sigma} = \frac{v}{u-\sigma^2} = \frac{v}{u-\frac{1}{\eta^2}} \tag{A-10}$$

Consequently equation A-9 finally becomes

$$\left[1-\frac{v}{u-\frac{1}{\eta^2}}\zeta\right](\eta-\zeta)\frac{d\eta}{d\zeta} + \eta = u\eta^3 \tag{A-11}$$

In addition, it follows from equation A-8 that

$$\log\ v_2 = \int \frac{d\zeta}{\eta-\zeta} \tag{A-12}$$

The boundary conditions in the $\eta-\zeta$ plane are:

(a) Near the axis

$$\zeta = v_2^{-1}u_2 \cong (-2\sigma+\ldots)^{-1}(2\log\sigma + 2 + C_4 + \ldots) = -\frac{\log\sigma}{\sigma} - \frac{1}{\sigma} - \frac{C_4}{2\sigma} + \ldots \tag{A-13a}$$

$$\eta = -\frac{1}{\sigma} \tag{A-13b}$$

Near the axis, $\sigma \rightarrow \infty$. Thus $\zeta \rightarrow 0$ and $\eta \rightarrow 0$.

(b) At the shock wave

$$\frac{2}{\gamma+1}v_1(\sigma_{1w})\left(v_2(\sigma_{1w}) + \sigma_{2w}\frac{\partial v_1(\sigma_{1w})}{\partial\sigma_{1w}}\right)$$

$$= u_1(\sigma_{1w})\left(\frac{3}{2}u_1(\sigma_{1w}) - \frac{2K}{\gamma+1}\right)\left(u_2(\sigma_{1w}) + \sigma_{2w}\frac{\partial u_1(\sigma_{1w})}{\partial\sigma_{1w}}\right) \tag{A-14}$$

$$\eta_w = -\frac{1}{\sigma_w} = -\delta \tan\beta \cong \frac{u_1(\sigma_{1w})}{v_1(\sigma_{1w})} \left\{ 1 + \delta^2 \log\delta \left[\frac{1}{u_1(\sigma_{1w})} \left(u_2(\sigma_{1w}) + \sigma_{2w} \frac{\partial u_1(\sigma_{1w})}{\partial \sigma_{1w}} \right) \right.\right.$$

$$\left.\left. - \frac{1}{v_1(\sigma_{1w})} \left(v_2(\sigma_{1w}) + \sigma_{2w} \frac{\partial u_1(\sigma_{1w})}{\partial \sigma_{1w}} \right) \right] + \dots \right\} = \eta(\sigma_{1w}) + \dots \quad \text{(A-15)}$$

Before starting the numerical integration, the singular points of equation A-11 are investigated. It is found that $\eta = 0$, $\zeta = 0$ is a singular point (node) and equation A-11 can be approximated near this singular point by

$$\frac{d\eta}{d\zeta} = \frac{\eta}{\zeta - \eta} \quad \text{(A-16)}$$

which, by integration, gives

$$\eta = Ce^{-\frac{\zeta}{\eta}} \quad \text{(A-17)}$$

where C is a constant of integration.

Equation A-11, being a non-linear differential equation, can be solved only by numerical integration. The procedure of numerical solution is:

(1) Choose K, then $u(\sigma_{1w})$ and $v(\sigma_{1w})$ are known from the previous solution in Section VI. Thus $\sigma_{1w} = -\frac{v_1(\sigma_{1w})}{u_1(\sigma_{1w})}$, $\frac{\partial v_1(\sigma_{1w})}{\partial \sigma_{1w}}$, and $\frac{\partial u_1(\sigma_{1w})}{\partial \sigma_{1w}}$ can easily be calculated.

(2) Assume δ, $u_2(\sigma_{1w})$, and $v_2(\sigma_{1w})$. Then $\zeta(\sigma_{1w})$, σ_{2w}, and $\eta(\sigma_{1w})$ can be calculated from equations A-8, A-14, and A-15.

(3) Using $\zeta(\sigma_{1w})$ and $\eta(\sigma_{1w})$ as initial conditions, the numerical integration of equation A-11 is carried out toward $\eta_0 = -\frac{1}{\sigma_0^2} = -\delta^2$.

(4) From the result of numerical integration, form the following integral

$$\log v_2 \Big|_{(v_2)_0}^{v_2(\sigma_{lw})} = \int_{\zeta_0}^{\zeta(\sigma_{lw})} \frac{d\zeta}{\eta - \zeta} \qquad \text{(A-18)}$$

where

$$(v_2)_0 = -\frac{2}{\delta^2} + 4(\gamma+1)\,\delta^2 \log \delta + \ldots \qquad \text{(A-19)}$$

$v_2(\sigma_{lw})$ is thus determined.

(5) If $v_2(\sigma_{lw})$ calculated from equation A-18 is different from the assumed value, then repeat the above procedures until they check satisfactorily.

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