THE GRAVITATIONAL FIELD OF A BODY WITH ROTATIONAL SYMMETRY IN EINSTEIN'S THEORY OF GRAVITATION

THESIS

by

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(ABSTRACT)

Einstein's set of field equations in vaccuo

$$G_{\mu\nu} = 0$$

is reduced to such a form that simple problems like the sphere (Schwarzschild's solution), the infinite plane and the infinite cylinder can be solved. The fundamental quadratic differential forms for the latter two cases are respectively

$$ds^2 = -\,[(1+4\pi\sigma z)^{-1}dz^2 + d\rho^2 + \rho^2 d\varphi^2] + (1+4\pi\sigma z)dt^2,$$

$$ds^2 = -\,c_4^2\rho^{-2}[(1+4m\log\rho)^{-1}d\rho^2 + \rho^2 d\varphi^2] - dz^2 + (1+4m\log\rho)dt^2,$$

where σ is the surface density of matter on the plane, $z = 0$; m the linear density of matter on the cylinder, $\rho = \text{const.}$; (ρ, z, φ) the cylindrical coordinates; c_4 an indeterminate constant and the velocity of light is unity. Setting g_{44} = the Newtonian potential + const., we can get the solution of the general gravitational problem for a body whose mass is distributed symmetrically about an axis provided we can solve

$$2\frac{\partial}{\partial\psi}\left[(1-2M\psi)\frac{\partial n}{\partial\psi}\right] + \frac{\partial^2}{\partial\theta^2}e^{2n} = 0 \qquad (M = \text{mass of the body}).$$

The gravitational field of an oblate spheroidal homoeoid is characterized by

$$ds^2 = -\,\psi^{-4}(1-2M\psi)^{-1}d\psi^2 - \psi^{-2}d\xi^2 - \psi^{-2}\cos^2\xi\, d\varphi^2 + (1-2M\psi)dt^2,$$

where $\psi = \kappa^{-1}\cot^{-1}(\sinh\eta)$, M = mass of the homoeoid whose equation is $c^2\rho^2+a^2z^2 = a^2c^2$, $\kappa^2 = a^2-c^2$ and ξ, η are related to the cylindrical coordinates (ρ, z, φ) by $\rho+iz = \kappa\cos(\xi+i\eta)$. Analogous expressions for a prolate spheroidal homoeoid are obtainable. The oblateness of the homoeoid causes a slight increase in the advance of the perihelion of a planet's orbit derived from Schwarzschild's solution.

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INTRODUCTION

The present paper is an attempt to solve rigorously the problem of the static gravitational field of a body whose mass is distributed symmetrically around an axis in Einstein's theory of gravitation. In §1 Einstein's field equations in vaccuo[1]

$$G_{\mu\nu} = 0 \tag{0.1}$$

are set up and reduced in §2 to a form such that simple problems like the sphere (§5), the plane (§6) and the infinite cylinder (§7) can be solved. In the general problem there is a fundamental difficulty which will be avoided by the introduction of the Newtonian potential (§8). The solution of the whole problem then depends upon the solution of a partial differential equation of the second order which is non-linear. Finally the gravitational fields of spheroidal homoeoids (§10, §11) are given as illustrations of the present investigation and the motion of a particle in the field of an oblate spheroidal homoeoid is discussed (§12). The paper also contains a critical examination of earlier works upon the problem notably those of Prof.'s H.Weyl and T.Levi-Civita (§3, §4).

I EINSTEIN'S LAW OF GRAVITATION

1. The field equations. Consider the static gravitational field outside of a body whose mass is distributed symmetrically about an axis. Hence the $g_{\mu\nu}$'s do not vary with respect to time. The most general fundamental quadratic differential form in such a field appears to be

$$ds^2 = -(g_{11}dx_1^2 + 2g_{12}dx_1dx_2 + g_{22}dx_2^2) - g_{33}dx_3^2 + g_{44}dx_4^2 \tag{1.1}$$

where x_1, x_2 are any two coordinates in the merideonal plane containing the z-axis, $x_3 = \varphi$, the azimuthal angle, $x_4 = t$, the time coordinate, the unit of time being so chosen that the velocity of light in vaccuo is unity. The $g_{\mu\nu}$'s in (1.1) are functions of x_1 and x_2 only.

[1] A. S. Eddington, "The Mathematical Theory of Relativity," 2nd Ed. (1924), p. 81. Eddington's notation with slight modifications will be followed throughout the present paper.

We assume that the values of the $g_{\mu\nu}$'s exist. From a well-known theorem[1] on positive definite quadratic differential forms of two variables in the parenthesis of (1.1), it is always possible when g_{11}, g_{22}, g_{12} are explicitly given, to make a real single-valued, continuous transformation from x_1 and x_2 to u and v by

$$(1.2) \qquad x_1 = x_1(u,v), \quad x_2 = x_2(u,v), \quad \text{where} \quad J \equiv \frac{\partial(x_1,x_2)}{\partial(u,v)} \neq 0$$

such that the following identity is true,

$$(1.3) \qquad g_{11}dx_1^2 + 2g_{12}dx_1dx_2 + g_{22}dx_2^2 \equiv e^{2m}(du^2 + dv^2).$$

Hence (1.1) becomes

$$(1.4) \qquad ds^2 = -e^{2m}(du^2 + dv^2) - e^{2n}dx_3^2 + e^{2\nu}dx_4^2,$$

where m, n, ν are functions of u and v to be determined. Let

$$u = x_1, \quad v = x_2.$$

Then

$$g_{11} = g_{22} = -e^{2m}, \quad g_{33} = -e^{2n}, \quad g_{44} = e^{2\nu},$$

$$(1.5) \qquad g = g_{11}g_{22}g_{33}g_{44} = -e^{4m+2n+2\nu},$$

$$g^{11} = g^{22} = -e^{-2m}, \quad g^{33} = -e^{-2n}, \quad g^{44} = e^{-2\nu}.$$

Now (1.4) is an orthogonal quadratic differential form. The general expressions of the Christoffel symbols of the second kind for such forms are well known.[2] In the present problem the non-vanishing symbols are

$$(1.6) \qquad \begin{aligned}
\{11,1\} &= m_u & \{11,2\} &= -m_v \\
\{12,1\} &= m_v & \{12,2\} &= m_u \\
\{22,1\} &= -m_u & \{22,2\} &= m_v \\
\{33,1\} &= -e^{2n-2m}n_u & \{33,2\} &= -e^{2n-2m}n_v \\
\{44,1\} &= e^{2\nu-2m}\nu_u & \{44,2\} &= e^{2\nu-2m}\nu_v \\
\{13,3\} &= n_u & \{14,4\} &= \nu_u \\
\{23,3\} &= n_v & \{24,4\} &= \nu_v
\end{aligned}$$

1 L. Bianchi - Lukat, "Vorlesungen über Differentialgeometrie", pp. 69 (1910).

2 A. S. Eddington, loc. cit., pp. 83.

where for simplicity we denote partial differentiations by subscripts.

Written out in full Einstein's field equations in vaccuo are

(1.7)
$$G_{\mu\nu} \equiv -\frac{\partial}{\partial x_\alpha}\{\mu\nu,\alpha\} + \{\mu\alpha,\beta\}\{\nu\beta,\alpha\} + \frac{\partial^2}{\partial x_\mu \partial x_\nu}\log(-g)^{1/2} - \{\mu\nu,\alpha\}\frac{\partial}{\partial x_\alpha}\log(-g)^{1/2} = 0$$

of which the following five components are not identically zero:

$$G_{11} \equiv -\frac{\partial}{\partial x_1}\{11,1\} - \frac{\partial}{\partial x_2}\{11,2\} + \{11,1\}\{11,1\} + 2\{11,2\}\{12,1\} + \{12,2\}\{12,2\} + \{13,3\}\{13,3\} + \{14,4\}\{14,4\} + \frac{\partial^2}{\partial x_1^2}\log(-g)^{1/2} - \{11,1\}\frac{\partial}{\partial x_1}\log(-g)^{1/2} - \{11,2\}\frac{\partial}{\partial x_2}\log(-g)^{1/2},$$

$$G_{12} \equiv -\frac{\partial}{\partial x_1}\{12,1\} - \frac{\partial}{\partial x_2}\{12,2\} + \{11,1\}\{21,1\} + \{11,2\}\{22,1\} + \{12,1\}\{12,2\} + \{12,2\}\{22,2\} + \{13,3\}\{23,3\} + \{14,4\}\{24,4\} + \frac{\partial^2}{\partial x_1 \partial x_2}\log(-g)^{1/2} - \{12,1\}\frac{\partial}{\partial x_1}\log(-g)^{1/2} - \{12,2\}\frac{\partial}{\partial x_2}\log(-g)^{1/2},$$

(1.8)
$$G_{22} \equiv -\frac{\partial}{\partial x_1}\{22,1\} - \frac{\partial}{\partial x_2}\{22,2\} + \{21,1\}\{21,1\} + 2\{21,2\}\{22,1\} + \{22,2\}\{22,2\} + \{23,3\}\{23,3\} + \{24,4\}\{24,4\} + \frac{\partial^2}{\partial x_2^2}\log(-g)^{1/2} - \{22,1\}\frac{\partial}{\partial x_1}\log(-g)^{1/2} - \{22,2\}\frac{\partial}{\partial x_2}\log(-g)^{1/2},$$

$$G_{33} \equiv -\frac{\partial}{\partial x_1}\{33,1\} - \frac{\partial}{\partial x_2}\{33,2\} + 2\{31,3\}\{33,1\} + 2\{32,3\}\{33,2\} - \{33,1\}\frac{\partial}{\partial x_1}\log(-g)^{1/2} - \{33,2\}\frac{\partial}{\partial x_2}\log(-g)^{1/2},$$

$$G_{44} \equiv -\frac{\partial}{\partial x_1}\{44,1\} - \frac{\partial}{\partial x_2}\{44,2\} + 2\{41,4\}\{44,1\} + 2\{42,4\}\{44,2\} - \{44,1\}\frac{\partial}{\partial x_1}\log(-g)^{1/2} - \{44,2\}\frac{\partial}{\partial x_2}\log(-g)^{1/2}.$$

If we substitute the three-index Christoffel symbols of the second kind from their values (1.6) into (1.8), we obtain

(1.9) $$G_{11} \equiv m_{uu} + m_{vv} + n_{uu} + \nu_{uu} + n_u^2 + \nu_u^2 - m_u(n_u+\nu_u) + m_v(n_v+\nu_v) = 0,$$

(1.10) $$G_{12} \equiv n_{uv} + \nu_{uv} + n_u n_v + \nu_u \nu_v - m_v(n_u+\nu_u) - m_u(n_v+\nu_v) = 0,$$

(1.11) $$G_{22} \equiv m_{uu} + m_{vv} + n_{vv} + \nu_{vv} + n_v^2 + \nu_v^2 + m_u(n_u+\nu_u) - m_v(n_v+\nu_v) = 0,$$

(1.12) $$G_{33} \equiv e^{2n-2m}[n_{uu} + n_{vv} + n_u(n_u+\nu_u) + n_v(n_v+\nu_v)] = 0,$$

(1.13) $$G_{44} \equiv - e^{2\nu-2m}[\nu_{uu} + \nu_{vv} + \nu_u(n_u+\nu_u) + \nu_v(n_v+\nu_v)] = 0.$$

2. Reduction of the field equations. By putting

(2.1) $$\chi = n + \nu,$$

and adding the expressions in the square brackets of (1.12) and (1.13) we get

(2.2) $$\chi_{uu} + \chi_{vv} + \chi_u{}^2 + \chi_v{}^2 = 0,$$

which becomes Laplace's equation in the uv-plane,

(2.3) $$\Phi_{uu} + \Phi_{vv} = 0, \quad \text{on setting} \quad \Phi = e^{\chi} = e^{n+\nu}.$$

It is well known that the solution of (2.3) is unique, if the boundary value of Φ be given in the uv-plane. Then $G_{44} = 0$ becomes

(2.4) $$\nu_{uu} + \nu_{vv} + \chi_u \nu_u + \chi_v \nu_v = 0,$$

which determines ν. We obtain n by (2.1).

To get the unknown function, m, we use (1.9), (1.10) and (1.11). Write

$$\text{(2.5)}\quad \begin{aligned} G_{12} = 0: &\quad \chi_v m_u + \chi_u m_v = \chi_{uv} + n_u n_v + v_u v_v \equiv A, \\ G_{11} - G_{22} = 0: &\quad -\chi_u m_u + \chi_v m_v = \frac{1}{2}[-\chi_{uu} + \chi_{vv} - n_u^2 - v_u^2 + n_v^2 + v_v^2] \\ &\qquad \equiv B. \end{aligned}$$

Then by solving m_u and m_v simultaneously from (2.5) we get

$$\text{(2.6)}\quad m_u = [\chi_u^2 + \chi_v^2]^{-1}[\chi_v A - \chi_u B], \quad m_v = [\chi_u^2 + \chi_v^2]^{-1}[\chi_u A + \chi_v B].$$

It will now be shown that

$$\text{(2.7)}\quad dm \equiv m_u du + m_v dv,$$

where m_u and m_v are given in (2.6) is an exact differential and secondly that m must satisfy (1.9) and (1.11). By (1.12), (1.13) and (2.2), we note that

$$\text{(2.8)}\quad \begin{aligned} A_u + B_v &= -\chi_u A - \chi_v(\chi_{vv} + n_v^2 + v_v^2), \\ A_v - B_u &= -\chi_u(\chi_{uu} + n_u^2 + v_u^2) - \chi_v A, \end{aligned}$$

from which we obtain immediately

$$\text{(2.9)}\quad \chi_v(A_v - B_u) - \chi_u(A_u + B_v) = (\chi_u^2 - \chi_v^2)A + 2\chi_u\chi_v B,$$

$$\text{(2.10)}\quad \begin{aligned} \chi_u(A_v - B_u) + \chi_v(A_u + B_v) = &- 2\chi_u\chi_v A - \chi_u^2(\chi_{uu} + n_u^2 + v_u^2) \\ &- \chi_v^2(\chi_{vv} + n_v^2 + v_v^2). \end{aligned}$$

By differentiating m_u with respect to v we get

$$\text{(2.11)}\quad \begin{aligned} \frac{\partial}{\partial v} m_u = [\chi_u^2 + \chi_v^2]^{-2}\Big[&(\chi_u^2 + \chi_v^2)\{\chi_v A_v - \chi_u B_v\} + \{\chi_{vv}(\chi_u^2 - \chi_v^2) - 2\chi_u\chi_v\chi_{uv}\}A \\ &+ \{\chi_{uv}(\chi_u^2 - \chi_v^2) + 2\chi_u\chi_v\chi_{vv}\}B\Big], \end{aligned}$$

which, by (2.2) and (2.9), becomes

$$\begin{aligned} &[\chi_u^2 + \chi_v^2]^{-2}\Big[(\chi_u^2 + \chi_v^2)\{\chi_u A_u + \chi_v B_u\} - \{\chi_{uu}(\chi_u^2 - \chi_v^2) + 2\chi_u\chi_v\chi_{vu}\}A \\ &\qquad + \{\chi_{vu}(\chi_u^2 - \chi_v^2) - 2\chi_u\chi_v\chi_{uu}\}B\Big] \\ &= \frac{\partial}{\partial u} m_v. \end{aligned}$$

Differentiating m_u and m_v with respect to u and v respectively we have

(2.12) $$m_{uu} = [\chi_u^2 + \chi_v^2]^{-2}[(\chi_u^2+\chi_v^2)\{\chi_v A_u - \chi_u B_u\} + \{\chi_{vu}(\chi_u^2-\chi_v^2) - 2\chi_u\chi_v\chi_{uu}\}A + \{\chi_{uu}(\chi_u^2-\chi_v^2) + 2\chi_u\chi_v\chi_{uv}\}B],$$

(2.13) $$m_{vv} = [\chi_u^2 + \chi_v^2]^{-2}[(\chi_u^2+\chi_v^2)\{\chi_u A_v + \chi_v B_v\} - \{\chi_{uv}(\chi_u^2-\chi_v^2) + 2\chi_u\chi_v\chi_{vv}\}A + \{\chi_{vv}(\chi_u^2-\chi_v^2) - 2\chi_u\chi_v\chi_{uv}\}B].$$

On adding (2.12) and (2.13) and by (2.2) and (2.10), we obtain

(2.14) $$m_{uu} + m_{vv} = n_u n_v + v_u v_v.$$

By (2.14) and (2.2), it is easily seen that (1.9) and (1.11) are satisfied. This completes the proof that the functions, m, n, v, thus obtained satisfy every component of Einstein's field equations (1.7). For future reference we collect the following independent equations:

(2.3) $$\Phi_{uu} + \Phi_{vv} = 0 \quad \text{where} \quad \Phi = e^{\chi} = e^{n+v},$$

(2.4) $$v_{uu} + v_{vv} + \chi_u v_u + \chi_v v_v = 0,$$

(2.6) $$m_u = [\chi_u^2 + \chi_v^2]^{-1}[\chi_v A - \chi_u B], \quad m_v = [\chi_u^2 + \chi_v^2]^{-1}[\chi_u A + \chi_v B],$$

$$A \equiv \chi_{uv} + n_u n_v + v_u v_v,$$

$$B \equiv \frac{1}{2}[-\chi_{uu} + \chi_{vv} - n_u^2 - v_u^2 + n_v^2 + v_v^2],$$

(1.4) $$ds^2 = -e^{2m}(du^2 + dv^2) - e^{2n}dx_3^2 + e^{2v}dt^2.$$

II REMARKS ON EARLIER DISCUSSIONS OF THE PROBLEM

3. On Weyl's original solution. The problem under consideration was first attacked by H.Weyl[1]. His result was sebsequently criticised by T.Levi-Civita[2] as being incomplete. The latter started with (0.1) and gave a complete though restricted set of solutions (for reasons see §4 below). The incompleteness of Weyl's procedure seems to lie in the fact that the formal statement of (0.1) by the variational principle fails when certain gravitational potentials, $g_{\mu\nu}$, vanish identically and some of them are equal to each other. It is well known that[3]

$$(3.1)\qquad \delta A = \frac{1}{8\pi}\,\delta \int G(-g)^{1/2}d\tau = \int T^{\mu\nu}\delta g_{\mu\nu}(-g)^{1/2}d\tau = 0 \quad (d\tau = dx_1\cdots dx_4)$$

gives the field equations

$$(3.2)\qquad T^{\mu\nu} \equiv -\frac{1}{8\pi}(G^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}G) = 0,$$

provided the $\delta g_{\mu\nu}$'s are entirely arbitrary. If, however, (to fix our ideas) $g_{12} = 0$ and $g_{11} = g_{22}$, it is unlikely that we have the right to conclude from (3.1) $T^{12} = 0$, $T^{11} = 0$, $T^{22} = 0$.

In Weyl's work the fundamental quadratic differential form for a body with axial symmetry is taken to be (1.4) in which $g_{11} = g_{22}$, and g_{11}, g_{33}, g_{44} are all functions of x_1 and x_2. He then constructes $G(-g)^{1/2}$ from (1.4) and varies the action integral, A, with respect to g_{11}, g_{33} and g_{44}, which are the only $g_{\mu\nu}$'s present in the integrand. The three Eulerian equations (3.2) thus obtained are indeed

$$(3.3)\qquad T^{11} + T^{22} = 0, \quad T^{33} = 0, \quad T^{44} = 0;$$

while the equation, $T^{12} = 0$, which is the only component of T^{ik} $(i\neq k)$ in the present problem not identically zero, is entirely neglected. Moreover, it is obvious that the solution of (3.2) satisfies (3.3). But the converse is in general false.

After Levi-Civita's criticism Weyl[4] replied in Annalen der Physik expecting to justify his previous result by assuming the relation

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1) H. Weyl, "Annalen der Physik", Bd. 54, pp. 134 (1918).
2) T. Levi-Civita, "Rend. Accad. dei Lincei", Vol. 28, i, pp. 10 (1919).
3) A. S. Eddington, loc. cit. p. 139.
4) H. Weyl, "Annalen der Physik", Bd. 59, pp. 185–188 (1919).

$$(3.4) \qquad \mathfrak{T}_1^1 + \mathfrak{T}_2^2 = 0$$

between the components of the mixed energy-momentum tensor density, $\mathfrak{T}_\mu^\nu$, which, in his opinion, keeps the body in equilibrium. This assumption, however, is inconsequential, because from the physical considerations every component of $\mathfrak{T}_\mu^\nu$ should vanish in the gravitational field outside of the body, and mathematically (3.4) in the present case is merely the first equation of (3.3). In short (3.4) does not lead us beyond the set (3.3).

A plausible mathematical justification of Weyl's result seems to lie in the four Bianchi relations[4],

$$(3.5) \qquad (T^{\mu\nu})_\nu = 0 \qquad (\mu = 1,2,3,4)$$

which reduce to the following two in the present case:

$$(3.6) \qquad \begin{aligned} (T^{11})_1 + (T^{12})_2 \equiv \frac{\partial T^{11}}{\partial x_1} + \frac{\partial T^{12}}{\partial x_2} + \left[\{11,1\} + \frac{\partial}{\partial x_1}\log(-g)^{1/2}\right]T^{11} + \{22,1\}T^{22} + \{33,1\}T^{33} \\ + \{44,1\}T^{44} + \left[2\{12,1\} + \frac{\partial}{\partial x_2}\log(-g)^{1/2}\right]T^{12} = 0, \\ (T^{21})_1 + (T^{22})_2 \equiv \frac{\partial T^{21}}{\partial x_1} + \frac{\partial T^{22}}{\partial x_2} + \left[\{22,2\} + \frac{\partial}{\partial x_2}\log(-g)^{1/2}\right]T^{22} + \{11,2\}T^{11} + \{33,2\}T^{33} \\ + \{44,2\}T^{44} + \left[2\{21,2\} + \frac{\partial}{\partial x_1}\log(-g)^{1/2}\right]T^{12} = 0. \end{aligned}$$

If equations (3.3) are satisfied, these relations become

$$(3.7) \qquad \begin{aligned} \frac{\partial T^{11}}{\partial x_1} + \frac{\partial T^{12}}{\partial x_2} + [\{11,1\} + \frac{\partial}{\partial x_1}\log(-g)^{1/2} - \{22,1\}]T^{11} \\ + [2\{12,1\} + \frac{\partial}{\partial x_2}\log(-g)^{1/2}]T^{12} = 0, \\ \frac{\partial T^{12}}{\partial x_1} - \frac{\partial T^{11}}{\partial x_2} + [\{11,2\} - \frac{\partial}{\partial x_2}\log(-g)^{1/2} - \{22,2\}]T^{11} \\ + [2\{12,2\} + \frac{\partial}{\partial x_1}\log(-g)^{1/2}]T^{12} = 0, \end{aligned}$$

where the coefficients of T^{11} and T^{12} are functions of x_1 and x_2. The general solutions of (3.7) are not $T^{11} \equiv T^{12} \equiv 0$ which, in fact, is only a very trivial case. This shows that even if (3.5) is introduced here, solutions of Weyl's equations (3.3) do not necessarily satisfy

[4] A. S. Eddington, loc. cit., p. 119.

every component of Einstein's field equations (3.2). Consequently the condition on the covariance of physical laws as required by the fundamental postulate of the general theory of relativity will not be satisfied generally.

4. <u>On Weyl-Levi-Civita's solution</u>. Levi-Civita's solution of (0.1) is a special case of the results we obtained in §2. Consider (2.3), and set $\Phi = \rho$. Let z be the conjugate function of ρ. Then

$$(4.1) \qquad \rho + iz = f(u + iv)$$

where $f(u+iv)$ is analytic in $u+iv$. From this it follows that

$$(4.2) \qquad d\rho^2 + dz^2 = f'(u+iv)f'(u-iv)(du^2 + dv^2);$$

namely, one set of coordinates (u,v) is conformally transformed into the other (ρ, z). In order to avoid cumbersome mathematical manipulations in §2, both Weyl and Levi-Civita assume initially that

$$(4.3) \qquad \rho = u, \quad z = v. \quad \text{Then} \quad e^{2n} = \rho^2 e^{-2v} \quad \text{and}$$

$$(4.4) \qquad ds^2 = -\,e^{2m}(d\rho^2 + dz^2) - \rho^2 e^{-2v} d\varphi^2 + e^{2v} dt^2.$$

Moreover, $G_{44} = 0$ and dm[1] become respectively

$$(4.5) \qquad \frac{\partial^2 v}{\partial \rho^2} + \frac{\partial^2 v}{\partial z^2} + \frac{1}{\rho}\frac{\partial v}{\partial \rho} = 0,$$

$$(4.6) \qquad dm = -\,dv + \rho\left[\left(\frac{\partial v}{\partial \rho}\right)^2 - \left(\frac{\partial v}{\partial z}\right)^2\right]d\rho + 2\rho\,\frac{\partial v}{\partial \rho}\frac{\partial v}{\partial z}\,dz.$$

We recognize (4.5) as Laplace's equation in cylindrical coordinates (ρ, z, φ) independent of φ. Weyl calls ρ, z in (4.3) the "canonical cylindrical coordinates"[2] which are apparently different from the ordinary cylindrical coordinates used in solving Newtonian potential problems. He then emphasizes the fact[3] that if the distribution of mass of a given body in our space-time manifold is known in terms of this set of configurational canonical coordinates, the problem is reduced to the solution

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1 T. Levi-Civita, "Rend. Accad. dei Lincei", Vol. 28, i, p. 9 (1919).

2 H. Weyl, "Ann. der Phys." Bd. 54, p. 137 (1918).

3 H. Weyl, "Raum, Zeit, Materie", 5th Ed. (1923), p. 266.

of (4.5). He shows[1] that Schwarzschild's solution in isotropic coordinates of a body with mass m having spherical symmetry, corresponds to that of a finite line segment of length 2m, with constant linear density, lying on the z-axis of the configurational canonical space-time manifold. But he does not make clear that it is almost impossible to know the corresponding distribution of mass in this canonical coordinate system when the distribution of mass in our space-time coordinates is given. This difficulty is clearly brought out by the following argument.

When we carry out the transformation from (x_1, x_2, φ, t) to (ρ, z, φ, t) by (1.2) and (4.3) we assume only the existence of the values of g_{11}, g_{12}, g_{22} in (1.1) so that the transformation is possible, but their explicit forms are not given *a priori* and consequently (1.2) is not explicitly known. Although we know the boundary values of $g_{\mu\nu}$ in the original (x_1, x_2, φ, t) system, we do not know the corresponding boundary conditions in the (u, v, φ, t) system on account of the uncertainty of (1.2). Since (4.5) has an infinite number of solutions if the boundary value of v is not specified, the solution obtainable from (4.5) and (4.6) will not be unique, and consequently it is indeterminate.

The same difficulty arises even if we do not assume the solution of Φ in (4.3). Here we do not know which solution of (2.3) we should take in order to solve (2.4). The complexity of the situation is further enhanced by the uncertainty of the boundary conditions of v in the uv-coordinates.

An alternative procedure to get a solution for the original physical problem from (4.5) and (4.6) is to choose a solution v of (4.5) in terms of the canonical coordinates first and then try to interpret it in the (x_1, x_2, φ, t) system by a transformation (1.2). The $g_{\mu\nu}$'s thus obtained must satisfy the original boundary conditions in terms of (x_1, x_2, φ, t) given initially. The question whether this procedure will lead to a unique transformation (1.2) needs further investigation. It appears not to have been considered in the literature.

Following Weyl and Levi-Civita's investigations, several authors[2] have given solutions of problems in which distribution of masses in

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1) H. Weyl, "Ann. der Phys." Bd. 54, p.140 (1918). Schwarzschild's solution is not necessarily limited to a particle. It can be applied equally well to a spherical shell (cf. J. P. Combridge, "Phil. mag." (7) Vol. 1. pp. 276 (1926)).

2) T. Levi-Civita, "Rend. Accad. dei Lincei," Vol. 28, i, pp. 101 (1919);
R. Bach, "Math. Zeitsch." Bd. 13, pp. 134 (1922);
G. Beck "Zeit. f. Phys." Bd. 33, pp. 713 (1925).

terms of the canonical coordinates is given. The corresponding mass distribution in our space-time manifold has been never discussed. Hence these solutions are academic and without physical significance.

II FIELDS OF SPHERE, PLANE AND CYLINDER

5. Schwarzschild's solution. As the first application of the results in §2 let us consider Schwarzschild's solution. The arc element in the gravitational field outside of a body with spherical symmetry is

(5.1) $$ds^2 = -e^{2\lambda}dr^2 - e^{2\mu}(r^2d\theta^2 + r^2\sin^2\theta d\varphi^2) + e^{2\nu}dt^2$$

where (r,θ,φ) denote spherical polar coordinates and λ,μ,ν are functions of r only. (5.1) may be put in the form of (1.4),

(5.2) $$ds^2 = -e^{2m}[du^2 + dv^2] - e^{2n}d\varphi^2 + e^{2\nu}dt^2, \quad \text{where}$$

(5.3) $$du = r^{-1}e^{\lambda-\mu}dr, \quad v = \theta, \quad e^m = re^{\mu}, \quad e^n = r\sin\theta e^{\mu},$$

m,λ,ν being then functions of u. $\Phi = e^{n+\nu} \equiv R\sin v$ (say). Then (2.3) is

(5.4) $$\frac{d^2R}{du^2} - R = 0.$$

Integrating (5.4), we obtain

(5.5) $$R = re^{\mu+\nu} = c_1\sinh(u+u_0) = c_1\sinh u$$ (by fixing u properly in (5.3)).

Hence $$\chi = \log\Phi = \log c_1 + \log\sinh u + \log\sin v.$$

Then (2.4) becomes $\dfrac{d^2v}{du^2} + \coth u\dfrac{dv}{du} = 0.$ Hence

(5.6) $$\frac{dv}{du} = c_2\operatorname{csch} u,$$

(5.7) $$\exp(v/c_2) = c_3[\coth u - \operatorname{csch} u].$$

Eliminating u between (5.5) and (5.7) we obtain

(5.8) $$\exp(v/c_2) = c_3 r^{-1} e^{-(\mu+v)}[(c_1^2 + r^2 e^{2(\mu+v)})^{1/2} - c_1].$$

By using the boundary condition on μ and v that both of them tend toward zero as r increases indefinitely we get $c_3 = 1$. Solving v from (5.8) we get

(5.9) $$\exp(2v/c_2) = 1 - 2c_1 r^{-1} e^{-(\mu+v)} \exp(v/c_2), \quad \text{or}$$
$$\sinh(v/c_2) = - c_1 r^{-1} e^{-(\mu+v)} = - \operatorname{csch} u.$$

From (5.3) we see that m is a function of u only. In (2.6) we must have $m_v = 0$ which gives

(5.10) $$\coth^2 u - 1 - v_u^2 = 0.$$

Relations (5.6) and (5.10) determine

(5.11) $$c_2 = \pm 1.$$

Take $c_2 = 1$. Then from (5.9) we have

(5.12) $$e^{2v} = 1 - \frac{2c_1}{r} e^{-\mu}, \quad \text{or} \ \sinh v = - \frac{c_1}{r} e^{-(\mu+v)}.$$

Eliminating u between (5.3) and (5.6) we obtain

(5.13) $$e^{\lambda} = - r \operatorname{csch} v \frac{dv}{dr} e^{\mu}.$$

The second case $c_2 = -1$ only changes c_1 to $-c_1$.

Relations (5.12) and (5.13) connect the three unknown functions, λ, μ, v. Consequently an infinite number of solutions arises. To obtain Schwarzschild's solution we set $\mu = 0$. Then (5.12) becomes

(5.14) $$g_{44} \equiv e^{2v} = 1 - \frac{2c_1}{r},$$

where c_1 may be identified as the mass of the body from Newton's theory. From (5.12) and (5.13) it follows that $\lambda = - v$. The same result can be also obtained by assuming that g_{44} is $1-2V$ to start with where V is the Newtonian potential of the body.

A second solution of interest is the one in isotropic coordinates where the velocity of light is independent of direction. Putting $\lambda = \mu$ in (5.13) and integrating, we get

(5.15) $$\sinh\nu = 2c_4 r[r^2 - c_4{}^2]^{-1}.$$

To determine the constant of integration, c_4, we use (5.12) and let r tend toward infinity. This gives

(5.16) $$2c_4 = - c_1.$$

Solving for e^ν and rejecting the negative root of e^ν which is essentially positive from (5.15), we get

(5.17) $$e^{2\nu} \equiv g_{44} = (2r - c_1)^2/(2r + c_1)^2 \quad \text{and} \quad e^{2\mu} = (1 + c_1/2r)^4.$$

This result was also obtained by a transformation of r in Schwarzschild's solution[1].

6. Infinite plane. Let the xy-plane be the given plane. From symmetry considerations around any line parallel to the z-axis the most general fundamental quadratic differential form appears to be

(6.1) $$ds^2 = - e^{2\lambda}(d\rho^2 + \rho^2 d\varphi^2) - e^{2\mu}dz^2 + e^{2\nu}dt^2$$

where λ, μ, ν are functions of z only. (6.1) can be put in the form (1.4),

(6.2) $$ds^2 = - e^{2m}[du^2 + dv^2] - e^{2n}d\varphi^2 + e^{2\nu}dt^2,$$ where

(6.3) $$du = e^{\mu-\lambda}dz, \quad \rho = v, \quad m = \lambda, \quad e^{2n} = \rho e^{\lambda}.$$

In the present case $\Phi = \rho e^{\lambda+\nu} \equiv \rho R$ and (2.3) becomes

(6.4) $$\frac{d^2R}{du^2} = 0.$$ Hence

(6.5) $$R = e^{\lambda+\nu} = c_1(u + u_0) = c_1 u$$ (by setting $u_0 = 0$).

Then (2.4) becomes

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[1] A. S. Eddington, loc. cit., p. 93.

(6.6) $$\frac{d^2v}{du^2} + \frac{1}{u}\frac{dv}{du} = 0$$ which gives

(6.7) $$v = \log c_3 + c_2 \log u \quad (c_2, c_3 = \text{constants of integration}).$$

From (6.3) and $n = \chi - v$, we find

(6.8) $$\lambda = (1-c_2)\log u + \log c_4, \quad \text{where} \quad c_4 = c_1/c_3.$$

By (2.6), $m_V = \lambda_V = 0$, we get

(6.9) $$c_2 = \pm 1.$$

Consider $c_2 = 1$. If we choose the unit of length properly, $c_3 = c_1$ and

(6.10) $$e^{\lambda} = 1.$$

Then (6.5) becomes

(6.11) $$e^{v} = c_1 u.$$

Differentiating (6.11) and on using (6.3) we find

(6.12) $$e^{v}\frac{dv}{dz} = c_1 e^{\mu}.$$

Here we have an infinite number of solutions of v and μ. To avoid this indeterminateness we use Newton's theory. By setting

(6.13) $$g_{44} \equiv e^{2v} = 1 - 2V = 1 + 4\pi\sigma z$$

and identifying c_1 as $2\pi\sigma$ where σ is the surface density of matter on the given plane, we get $\mu + v = 0$ and the final form of (6.1) is

(6.14) $$ds^2 = -[1 + 4\pi\sigma z]^{-1}dz^2 - (d\rho^2 + \rho^2 d\varphi^2) + [1 + 4\pi\sigma z]dt^2.$$

The additive constant in (6.13) is chosen to be unity. Here we are dealing with a body whose mass extends to an infinite distance and ds^2 is not Galilean at infinity. The latter condition, however, can be replaced by the one that space surrounding the plane is flat if the density of matter on the plane vanishes. This is satisfied by (6.14).

The solution (6.14) can be regarded as the limiting case of Schwarzschild's solution of a spherical shell when the radius of the shell becomes infinitely great (neglecting the infinite constant obtained in this limiting process). In fact Whittaker[1] uses this method to obtain his "quasi-uniform" gravitational field which, as we see in the present discussion, is the field outside an infinite material plane. The case $c_2 = -1$ and hence $e^{\lambda} = c_4 u^2$ has been treated by Levi-Civita[2] and the result extended to the gravitational field of a charged plane by Kar[3].

7. Infinite cylinder. Take z as the axis around which the mass of the cylinder is symmetrically distributed. The most possible fundamental quadratic differential form of such a field in rectangular coordinates is

$$(7.1) \qquad ds^2 = -e^{2\mu}(dx^2 + dy^2) - e^{2n}dz^2 - \frac{h^2}{\rho^2}(xdx + ydy)^2 + e^{2\nu}dt^2$$

where μ, n, h, ν are all functions of ρ. By using polar coordinates (ρ, φ) in the xy-plane, (7.1) becomes

$$(7.2) \qquad ds^2 = -e^{2\lambda}d\rho^2 - \rho^2 e^{2\mu}d\varphi^2 - e^{2n}dz^2 + e^{2\nu}dt^2$$

where $\qquad e^{2\lambda} = e^{2\mu} + h^2$. (7.2) can be put in the form,

$$(7.3) \qquad ds^2 = -e^{2m}[du^2 + dv^2] - e^{2n}dx_3^2 + e^{2\nu}dt^2 \quad \text{with}$$

$$du = \rho^{-1}e^{\lambda-\mu}d\rho, \quad v = \varphi, \quad e^{2m} = \rho^2 e^{2\mu}, \quad x_3 = z.$$

Now m, n, ν in (7.3) are functions of u only. In the present case (2.3) is

$$(7.4) \qquad \frac{d^2\Phi}{du^2} = 0 \qquad \text{and}$$

$$(7.5) \qquad \Phi = c_1(u + u_0) = c_1 u.$$

Then (2.4) becomes

$$(7.6) \qquad \frac{d^2\nu}{du^2} + \frac{1}{u}\frac{d\nu}{du} = 0. \qquad \text{Hence}$$

1) E. T. Whittaker, "Proc. Roy. Soc." (A), Vol. 116, p. 722 (1927).
2) T. Levi-Civita, "Accad. dei Lincei", Vol. 27, ii, pp. 240 (1918).
3) S. C. Kar, "Phys. Zeit." Vol. 27. pp. 208 (1926).

(7.7) $$\nu = \log c_3 + c_2 \log u.$$

From (2.6) we have $dm \equiv d\mu + \rho^{-1}d\rho = -c_2u^{-1}(1-c_2)du$ and hence

(7.8) $$\mu + \log\rho = \log c_4 + c_2(c_2-1)\log u.$$

From (7.5), (7.7), (7.8) and $du = \rho^{-1}e^{\lambda-\mu}d\rho$ we can eliminate the auxiliary variable u and obtain three relations between the four functions, λ, μ, n, ν. Consider the case $n = 0$. Then comparing (7.5) and (7.7) we have

(7.9) $$c_1 = c_3, \quad c_2 = 1 \quad \text{and from (7.8)} \quad \rho e^{\mu} = c_4.$$

There is still one degree of arbitrariness in the present solution. To avoid this difficulty we resort to Newton's theory once more. Let

(7.10) $$g_{44} \equiv e^{2\nu} = 1 - 2V = 1 + 4m\log\rho = c_1{}^2u^2,$$

where m is the mass per unit length of the cylinder. Identify c_1 as 2m. Then from (7.3) and (7.10)

(7.11) $$e^{2\lambda} = c_4{}^2\rho^{-2}[1 + 4m\log\rho]^{-1}.$$

The final form of ds^2 in the gravitational field outside an infinite cylinder is consequently

(7.12) $$ds^2 = -\rho^{-2}c_4{}^2[(1 + 4m\log\rho)^{-1}d\rho^2 + \rho^2d\varphi^2] - dz^2 + (1 + 4m\log\rho)dt^2.$$

Aside from the indeterminate constant c_4 the above solution is quite similar to that of an infinite plane, or of a body with spherical symmetry.

IV GENERAL SOLUTION OF THE PROBLEM

8. Transformation of the fundamental quadratic differential form. The foregoing three special cases are solvable from (2.3), (2.4) and (2.6).

This is because (2.3) degenerates into an ordinary differential equation in all these cases. In reality when Φ is a general function of u and v, on account of the uncertainty of the boundary conditions of Φ in the (u,v,φ,t) manifold as we have pointed out in §4, the problem can be hardly solvable. In the following section we shall avoid this difficulty by introducing the Newtonian potential into the present problem. As we shall see presently, the problem of the general static gravitational field of a finite body with rotational symmetry can be solved provided we can solve a non-linear partial differential equation of the second order.

We start with the cylindrical coordinates (ρ,z,φ), the z-axis being the axis of symmetry of the given body which is finite in extent. Consider the merideonal plane containing the z-axis. Choose in this plane as in ordinary potential theory a more general set (ξ,η) which is conformally mapped upon (ρ,z) by

(8.1) $$z + i\rho = F(\xi + i\eta)$$

where $F(\xi+i\eta)$ is a monogenic function of $\xi+i\eta$, so that

(8.2) $$dz^2 + d\rho^2 = h^2(d\xi^2 + d\eta^2), \quad h^2 = F'(\xi+i\eta)F'(\xi-i\eta).$$

Let $\psi(\xi,\eta) = \text{const.}$, $\theta(\xi,\eta) = \text{const.}$ be two orthogonal (in the Euclidean sense) families of curves in the plane, to be determined. Denote partial differentiations by subscripts as in §1. Then

(8.3) $$d\psi = \psi_\xi d\xi + \psi_\eta d\eta, \quad d\theta = \theta_\xi d\xi + \theta_\eta d\eta \quad \text{where}$$

(8.4) $$\psi_\xi\theta_\xi + \psi_\eta\theta_\eta = 0.$$

Choose the Jacobian of transformation of (8.3) to be

(8.5) $$J \equiv \frac{\partial(\psi,\theta)}{\partial(\xi,\eta)} = \psi_\xi\theta_\eta - \psi_\eta\theta_\xi = e^f(\psi_\xi^2 + \psi_\eta^2), \quad \text{where}$$

(8.6) $$e^f = \rho.$$

Solving θ_ξ, θ_η from (8.4) and (8.5) simultaneously we obtain

(8.7) $$\theta_\xi = -e^f\psi_\eta, \quad \theta_\eta = e^f\psi_\xi.$$

Since $d\theta$ is an exact differential, (8.7) must satisfy the necessary and sufficient condition,

$$\frac{\partial}{\partial \xi}\theta_\eta = \frac{\partial}{\partial \eta}\theta_\xi, \quad \text{giving} \tag{8.8}$$

$$\psi_{\xi\xi} + \psi_{\eta\eta} + f_\xi \psi_\xi + f_\eta \psi_\eta = 0. \tag{8.9}$$

By (8.1) and (8.6), f is a known function of ξ and η. Simple verification shows that (8.9) is Laplace's equation in the (ξ,η,φ) coordinates independent of φ.

Now solve (8.3) simultaneously for $d\xi$ and $d\eta$ in terms of $d\psi$ and $d\theta$ and put the results in (8.2) which then becomes

$$dz^2 + d\rho^2 = h^2(\psi_\xi^2 + \psi_\eta^2)^{-1}[d\psi^2 + \rho^{-2}d\theta^2]. \tag{8.10}$$

Consequently the fundamental quadratic differential form for a flat space-time continuum in the present (ψ,θ,φ,t) variables is

$$ds^2 = -\, h^2(\psi_\xi^2 + \psi_\eta^2)^{-1}[d\psi^2 + \rho^{-2}d\theta^2] - \rho^2 d\varphi^2 + dt^2. \tag{8.11}$$

When matter is present, ds^2 is no more Galilean. We suppose that in such cases (8.11) is replaced by

$$ds^2 = -\, e^{-2H}[e^{2\lambda}d\psi^2 + \rho^{-2}e^{2\mu}d\theta^2] - \rho^2 e^{2\gamma}d\varphi^2 + e^{2\nu}dt^2 \tag{8.12}$$

where we write

$$e^{-2H} = h^2(\psi_\xi^2 + \psi_\eta^2)^{-1}, \tag{8.13}$$

and λ,μ,γ,ν are functions of ψ and θ to be determined according to Einstein's law of gravitation, with the condition that at infinite distances from the body all four approach zero as a limit.

9. Introduction of the Newtonian potential. Next transform (2.3), (2.4), (2.6) into the (ψ,θ,φ,t) system. Consider the following expressions from (8.12),

(9.1) $$d\psi^2 + \rho^{-2}e^{-2\lambda+2\mu}d\theta^2 \equiv d\psi^2 + e^{-2g}d\theta^2 = (d\psi + ie^{-g}d\theta)(d\psi - ie^{-g}d\theta)$$

where we put $e^g = \rho e^{\lambda-\mu}$. Let $(\alpha+i\beta)^{-1} \neq 0$ where both α and β are real be an integrating factor of $d\psi + ie^{-g}d\theta$, so that

(9.2) $$d\psi + ie^{-g}d\theta = (\alpha + i\beta)(du + idv),$$ and (8.12) becomes

(9.3) $$ds^2 = -e^{-2H}e^{2\lambda}(\alpha^2+\beta^2)(du^2 + dv^2) - \rho^2e^{2\gamma}d\varphi^2 + e^{2\nu}dt^2.$$

Comparing (9.3) and (1.4) we obtain

(9.4) $$e^{2m} = e^{-2H}e^{2\lambda}(\alpha^2 + \beta^2).$$

Equating real and imaginary parts in (9.2) we get

(9.5) $$d\psi = \alpha du - \beta dv, \quad d\theta = e^g(\beta du + \alpha dv),$$

from which the conditions of integrability for $d\psi$ and $d\theta$ give

(9.6) $$\alpha_v + \beta_u = 0, \quad \alpha_u - \beta_v = \beta g_v - \alpha g_u,$$ and furthermore

(9.7) $$\begin{aligned} &\psi_u = \alpha, \quad \psi_v = -\beta; \quad \psi_{uu} = \alpha_u, \quad \psi_{vv} = -\beta_v; \\ &\theta_u = e^g\beta, \quad \theta_v = e^g\alpha; \quad \theta_{uu} = e^g(\beta_u+\beta g_u), \quad \theta_{vv} = e^g(\alpha_v+\alpha g_v). \end{aligned}$$

Hence by (9.6) and (9.7)

(9.8) $$\begin{aligned} \psi_{uu} + \psi_{vv} &= -(\alpha^2+\beta^2)\frac{\partial g}{\partial \psi}, \\ \theta_{uu} + \theta_{vv} &= (\alpha^2+\beta^2)e^{2g}\frac{\partial g}{\partial \theta}. \end{aligned}$$

By (9.5), (9.7) and (9.8), equation (2.3) becomes

(9.9) $$\frac{\partial^2\Phi}{\partial\psi^2} + e^{2g}\frac{\partial^2\Phi}{\partial\theta^2} - \frac{\partial g}{\partial\psi}\frac{\partial\Phi}{\partial\psi} + e^{2g}\frac{\partial g}{\partial\theta}\frac{\partial\Phi}{\partial\theta} = 0.$$

In the like manner we get (2.4) in the form,

(9.10) $$\frac{\partial^2\nu}{\partial\psi^2} + e^{2g}\frac{\partial^2\nu}{\partial\theta^2} - \frac{\partial g}{\partial\psi}\frac{\partial\nu}{\partial\psi} + e^{2g}\frac{\partial g}{\partial\theta}\frac{\partial\nu}{\partial\theta} + \frac{\partial\nu}{\partial\psi}\frac{\partial\chi}{\partial\psi} + e^{2g}\frac{\partial\nu}{\partial\theta}\frac{\partial\chi}{\partial\theta} = 0.$$

Save the intrduction of λ,μ,γ,ν into (8.12) the foregoing dicussions in §8 and §9 have been purely mathematical. We now use Newton's potential function. Silberstein[ψ] has shown that

(9.11) $$g_{44} \equiv e^{2\nu} = 1 - 2M\psi$$

where M is the mass of the body and ψ the Newtonian potential per unit mass, holds rigorously in local or geodesic coordinates and approximately in all other coordinate systems at distances large when compared with the dimensions of the gravitational body under investigation. Here we assume that (9.11) holds rigorously not only for geodesic coordinates but also for all other systems of reference. (9.10) then becomes

(9.12) $$\frac{\partial n}{\partial \psi} - \frac{\partial g}{\partial \psi} - \frac{\partial \nu}{\partial \psi} = 0, \qquad \text{giving}$$

(9.13) $$e^{n} = e^{\nu+g}\cdot\Theta(\theta), \quad \text{or} \quad e^{\gamma} = e^{\nu+\lambda-\mu}\cdot\Theta(\theta)$$

where $\Theta(\theta)$ is an arbitrary function. At infinity when $\psi = 0$, $\lambda = \mu = \gamma = \nu = 0$ for all values of θ. Hence we have $\Theta(\theta) \equiv 1$ and (9.13) can be rewritten in the form,

(9.14) $$\lambda + \nu = \gamma + \mu.$$

This condition is evidently satisfied by Schwarzschild's solution and by those of the infinite plane (6.14) and the cylinder (7.12).

By (9.11) and (9.12) and since $\Phi = e^{n+\nu}$, (9.9) becomes

(9.15) $$\frac{\partial^2 n}{\partial \psi^2} + 2\frac{\partial \nu}{\partial \psi}\frac{\partial n}{\partial \psi} + \frac{1}{2}e^{-2\nu}\frac{\partial^2}{\partial \theta^2}e^{2n} = 0 \quad \text{or}$$

(9.16) $$2\frac{\partial}{\partial \psi}[(1 - 2M\psi)\frac{\partial n}{\partial \psi}] + \frac{\partial^2}{\partial \theta^2}e^{2n} = 0.$$

Equation (9.15) is also obtainable by transforming (1.12), as we did with (1.13) in (9.10), and utilizing (9.11).

Consider (2.6). By (2.2), the exact differential

(9.17) $$dm = m_u du + m_v dv$$

can be integrated into the form,

ψ L. Silberstein, "Theory of Relativity", 2nd Ed. (1924), p. 372

(9.18) $$2m = \log(\chi_u^2 + \chi_v^2) + \chi + 2 \int P du + Q dv, \quad \text{where}$$

(9.19) $$P = \chi_v C + \chi_u D, \qquad Q = \chi_u C - \chi_v D, \quad \text{and}$$

(9.20) $$C = [\chi_u^2 + \chi_v^2]^{-1}(n_u n_v + \nu_u \nu_v),$$
$$D = \frac{1}{2}[\chi_u^2 + \chi_v^2]^{-1}(n_u^2 + \nu_u^2 - n_v^2 - \nu_v^2).$$

Expression (9.18) containds only first partial derivatives and is simpler than (2.6). By (9.5) and the inverse relations, (9.18) becomes

(9.21) $$2m = \log(\alpha^2 + \beta^2) + \log(\chi_\psi^2 + e^{2g}\chi_\theta^2) + \chi + 2 \int P' d\psi + Q' e^{-g} d\theta$$

where we define

(9.22) $$P' = e^g \chi_\theta C' + \chi_\psi D', \qquad Q' = \chi_\psi C' - e^g \chi_\theta D';$$
$$C' = [\chi_\psi^2 + e^{2g}\chi_\theta^2]^{-1} e^g (n_\psi n_\theta + \nu_\psi \nu_\theta),$$
$$D' = \frac{1}{2}[\chi_\psi^2 + e^{2g}\chi_\theta^2]^{-1}(n_\psi^2 + \nu_\psi^2 - e^{2g}[n_\theta^2 + \nu_\theta^2]).$$

Between (9.4) and (9.21) we can eliminate the auxiliary functions m and $\alpha^2+\beta^2$. A recapitulation of results gives

(9.23) $$2\lambda = \log(\chi_\psi^2 + e^{2g}\chi_\theta^2) + 2H + \chi + 2 \int P' d\psi + Q' e^{-g} d\theta,$$

(9.11) $$g_{44} \equiv e^{2\nu} = 1 - 2M\psi,$$

(5.9) $$\nabla^2 \psi = 0,$$

(9.14) $$\lambda + \nu = \gamma + \mu,$$

(9.16) $$2\frac{\partial}{\partial\psi}[(1 - 2M\psi)\frac{\partial n}{\partial\psi}] + \frac{\partial^2}{\partial\theta^2} e^{2n} = 0,$$

$$e^n = \rho e^\gamma, \quad \chi = n + \nu, \quad e^g = \rho e^{\lambda-\mu} = \rho e^{\gamma-\nu},$$

(8.12) $$ds^2 = e^{-2H}[e^{2\lambda} d\psi^2 + \rho^{-2} e^{2\mu} d\theta^2] - \rho^2 e^{2\gamma} d\varphi^2 + e^{2\nu} dt^2,$$

(8.13) $$e^{-2H} = h^2(\psi_\xi^2 + \psi_\eta^2)^{-1}.$$

From the aobve list we see immediately tnat the solution of the whole problem depends upon the solution of the non-linear equation (9.16) and a quadrature.

Instead of using λ in (9.23) it is sometimes more convenient to deal with its differential forms corresponding to (2.6). The following expressions are obtained from differentiating (9.23) and further simplified by (9.9):

$$\text{(9.24)}\qquad \begin{aligned} \frac{\partial\lambda}{\partial\psi} &= \frac{\partial H}{\partial\psi} + (\chi_\psi^2 + e^{2g}\chi_\theta^2)^{-1}[e^g\chi_\theta A' - \chi_\psi B'], \\ \frac{\partial\lambda}{\partial\theta} &= \frac{\partial H}{\partial\theta} + (\chi_\psi^2 + e^{2g}\chi_\theta^2)^{-1}e^{-g}[\chi_\psi A' + e^g\chi_\theta B'], \end{aligned} \qquad \text{where}$$

$$\text{(9.25)}\qquad \begin{aligned} A' &= e^g[\chi_{\psi\theta} + \frac{\partial g}{\partial\psi}\chi_\theta + n_\psi n_\theta + \nu_\psi\nu_\theta], \\ B' &= \frac{1}{2}\Big[-\chi_{\psi\psi} - \frac{\partial g}{\partial\psi}\chi_\psi + e^{2g}(\chi_{\theta\theta} + \frac{\partial g}{\partial\theta}\chi_\theta) - n_\psi^2 - \nu_\psi^2 + e^{2g}(n_\theta^2 + \nu_\theta^2)\Big]. \end{aligned}$$

V FIELDS OF SPHEROIDAL HOMOEOIDS

<u>10. Oblate spheroidal homoeoid.</u> Let the equation of the homoeoid be

$$\text{(10.1)}\qquad \frac{\rho^2}{a^2} + \frac{z^2}{c^2} = 1, \quad \text{where} \quad \rho^2 = x^2 + y^2, \quad a^2 > c^2.$$

Use spheroidal coordinates, ξ, η, defined by

$$\text{(10.2)}\qquad \rho + iz = \kappa\cos(\xi + i\eta) \quad \text{or} \quad \rho = \kappa\cos\xi\cosh\eta, \quad z = \kappa\sin\xi\sinh\eta,$$
$$\kappa^2 = a^2 - c^2.$$

Then η = cosnt. represents a family of oblate spheroids confocal with (10.1), which is $\kappa\cosh\eta = a$ in the family and ξ = const.,- a family of hyperboloids of one sheet confocal with and orthogonal to the spheroids.

The Newtonian potential for an oblate spheroidal homoeoid with unit mass is

$$\text{(10.3)}\qquad \psi = \kappa^{-1}\cot^{-1}(\sinh\eta); \quad \text{hence} \quad \sinh\eta = \cot\kappa\psi.$$

The function, θ, defined by (5.7) may be taken as

(10.4) $\theta = \sin\xi.$

From (10.2), (10.3) and (10.4),

(10.5) $\rho^2 = \kappa^2(1 - \theta^2)\csc^2\kappa\psi.$

In the present case (8.12) is

(10.6) $ds^2 = - e^{-2H}[e^{2\lambda}d\psi^2 + \rho^{-2}e^{2\mu}d\theta^2] - \rho^2 e^{2\gamma}d\varphi^2 + e^{2\nu}dt^2,$

where $e^{-2H} = \kappa^4\cosh^2\eta(\sinh^2\eta + \sin^2\xi),$

and λ, μ, γ are to be determined, ν being given by (8.11).

The equation (8.16) that γ must satisfy becomes

(10.7) $\frac{d}{d\psi}[(1 - 2M\psi)(\frac{d\gamma}{d\psi} - \kappa\cot\kappa\psi)] - \kappa^2 e^{2\gamma}\csc^2\kappa\psi = 0,$

in which we assume that γ is a function of ψ alone. If we write

$y = \kappa e^{\gamma}\csc\kappa\psi,$ and $d\sigma = (1 - 2M\psi)^{-1}d\psi,$

(10.7) becomes $\frac{d^2y}{d\sigma^2} = \frac{1}{y}(\frac{dy}{d\sigma})^2 + e^{-2M\sigma}y^3,$

which is a particular case of one of Painlevé's irreducible types[1]. But (10.7) is solvable in the present case by the following changes of variables,

(10.8) $R = e^{\gamma}(1 - 2M\psi)^{1/2}\csc\kappa\psi, \quad du = - \csc\kappa\psi\cdot e^{\gamma}\cdot(1 - 2M\psi)^{-1/2}\kappa d\psi,$

where the negative sign in du is chosen to make u tend toward positive infinity as ψ approaches positive zero. (10.7) then becomes

(10.9) $\frac{d^2R}{du^2} - R = 0,$ giving the solution

(10.10) $R = c_1\sinh(u + u_0) = c_1\sinh u, \quad (u_0 = 0).$

By (10.8) and (10.10), we have

[1] E. L. Ince, "Ordinary Differential Equations," (1927), p. 335 type XIII[1].

(10.11) $c_2 \sinh u = e^{\gamma}(1 - 2M\psi)^{1/2} \csc\kappa\psi$

from which, and by the choice of du in (10.8), we see that c_1 must be a positive constant. Eliminating e^{γ} between du in (10.8) and (10.11) we get

(10.12) $- c_1\kappa(1 - 2M\psi)^{-1}d\psi = \operatorname{csch}u \cdot du$, which gives

(10.13) $\frac{c_1\kappa}{2M}\log(1 - 2M\psi) = \log(\coth u - \operatorname{csch} u) + c_2.$

Eliminating u between (10.11) and (10.13) we find

(10.14) $$\gamma = -\frac{1}{2}\left(\frac{c_1\kappa}{M} + 1\right)\log(1 - 2M\psi) + \log\left\{[e^{2\gamma}(1-2M\psi) + c_1^2\sin^2\kappa\psi]^{1/2} - c_1\sin\kappa\psi\right\} + c_2,$$

the general solution of (10.7), involving the two arbitrary constants, c_1 and c_2. To determine c_2 let ψ approach zero. Then γ tends toward zero and $c_2 = 0$. The constant c_1 can be identified with M/κ, which is also a constant of integration in Newton's theory. Putting

(10.15) $c_1 = M/\kappa$

in (10.14) and solving for e^{γ}, we obtain

(10.16) $e^{\gamma} = \kappa^{-1}\psi^{-1}\sin\kappa\psi.$

Obviously (10.16) approaches unity as ψ tends toward zero.

Last, we must obtain λ. Knowing λ we can get μ by (9.14) and (10.16). In order to avoid cumbersome differentiations in integrating (9.23) directly we use the transformation of ψ in (10.8), and furthermore set

(10.17) $dv = d\xi.$

By (9.14), (10.3), (10.5), (10.8) and (10.16), (10.6) can be written in the form

(10.18) $$ds^2 = -\kappa^{-2}e^{-2H}e^{2\mu}\sin^2\kappa\psi[du^2 + dv^2] - \psi^{-2}\cos^2 v d\varphi^2 + e^{2\nu}dt^2,$$

which has the same form as (1.4), provided

(10.19) $$e^{2m} = \kappa^2 e^{2\mu}(\cot^2\kappa\psi + \sin^2 v).$$

By (10.8) and (10.16), we see that ψ can be expressed as an explicit function of u, and (2.6), that must be satisfied by μ, can be computed with the aid of R in (10.10). The quadrature in terms of the u, v variables is quite simple. Coupled with the condition that at infinite distances from the body μ must vanish, μ is found to be

(10.20) $$e^{2\mu} = \kappa^{-2}\psi^{-2}(\sinh^2\eta + \sin^2\xi)^{-1}.$$

From (9.14) and (10.20), λ is given by

(10.21) $$e^{2\lambda} = \kappa^{-4}\psi^{-4}(\sinh^2\eta + \sin^2\xi)^{-1}\operatorname{sech}^2\eta(1 - 2M\psi)^{-1}.$$

Again, ds^2 in (10.6) becomes

(10.22) $$ds^2 = -\kappa^{-2}\psi^{-2}[\psi^{-2}(1-2M\psi)^{-1}\operatorname{sech}^2\eta d\eta^2 + \kappa^2 d\xi^2 + \kappa^2\cos^2\xi d\varphi^2] + (1-2M\psi)dt^2.$$

Solving for $\sinh\eta$ and $\sin\xi$ from (10.2) we get

(10.23) $$\begin{aligned} 2\kappa^2\sinh^2\eta &= r^2 - \kappa^2 + [r^4 - 2\kappa^2(\rho^2-z^2) + \kappa^4]^{1/2}, \\ 2\kappa^2\sin^2\xi &= -(r^2-\kappa^2) + [r^4 - 2\kappa^2(\rho^2-z^2) + \kappa^4]^{1/2}, \end{aligned}$$

where $r^2 = \rho^2 + z^2$. When κ is small, these expressions can be expanded in the following forms:

(10.24) $$\begin{aligned} \kappa^2\sinh^2\eta &= r^2[1 - \frac{\rho^2}{r^2}\kappa^2 + \frac{(1-\omega^2)}{4r^4}\kappa^4 + \cdots\cdots], \\ \sin^2\xi &= \frac{z^2}{r^2} + \frac{1-\omega^2}{4r^4}\kappa^2 + \frac{\omega(1-\omega^2)}{4r^4}\kappa^4 + \cdots\cdots, \\ \omega &= (\rho^2 - z^2)/(\rho^2 + z^2). \end{aligned}$$

It is interesting to observe from (10.23) that when κ approaches zero, namely, when the spheroidal homoeoid tends toward a spherical shell as a limit, the line element (10.22) becomes Schwarzschild's solution. Furthermore, (10.22) is also the solution of an infinitely thin material disc with mass M and radius κ.

11. Prolate spheroidal homoeoid. The treatment of the prolate spheroidal homoeoid is analogous to the preceding problem. Here the equation of the surface of the body is given by

$$\frac{\rho^2}{a^2} + \frac{z^2}{c^2} = 1, \quad \text{with} \quad c^2 > a^2. \tag{11.1}$$

The spheroidal coordinates ξ, η used are defined by

$$\begin{aligned} z + i\rho &= \kappa\cos(\xi + i\eta), \quad \text{giving} \\ z &= \kappa\cos\xi\cosh\eta, \quad \rho = \kappa\sin\xi\sinh\eta. \end{aligned} \qquad (\kappa^2 = c^2 - a^2) \tag{11.2}$$

The Newtonian potential for (11.1) with unit mass is

$$\psi = \frac{1}{2\kappa}\log\frac{\cosh\eta + 1}{\cosh\eta - 1}. \tag{11.3}$$

Solving for η in terms of ψ from (11.3) we get

$$\sinh\eta = \operatorname{csch}\kappa\psi. \tag{11.4}$$

The function, θ, defined by (8.7) becomes

$$\theta = -\cos\xi \tag{11.5}$$

and ds^2 given in (8.12) is then

$$ds^2 = e^{-2H}[e^{2\lambda}d\psi^2 + \rho^{-2}e^{2\mu}d\theta^2] - \rho^2 e^{2\gamma}d\varphi^2 + e^{2\nu}dt^2 \tag{11.6}$$

where $e^{-2H} = \kappa^4\sinh^2\eta(\sinh^2\eta + \sin^2\xi)$,

$e^{2\nu} = 1 - 2M\psi, \quad M = \text{mass of } (11.1),$

and γ is assumed to be a function of ψ alone while both λ and μ are functions of ψ and θ. Between $\lambda, \mu, \gamma, \nu$ we have the relation, (9.14), namely, (11.7) $\lambda + \nu = \gamma + \mu.$

The equation (8.16) for γ in the present case is similar to (10.7), so the remaining analysis will be similar. Hence (11.6) is

(11.8) $$ds^2 = -\kappa^2(\sinh^2\eta + \sin^2\xi)[e^{2\lambda}d\eta^2 + e^{2\mu}d\xi^2] - \rho^2 e^{2\gamma}d\varphi^2 + e^{2\nu}dt^2,$$

where $$e^{2\lambda} = \kappa^{-4}\psi^{-4}\operatorname{csch}^2\eta(1-2M\psi)^{-1}[\sinh^2\eta + \sin^2\xi]^{-1},$$

$$e^{2\mu} = \kappa^{-2}\psi^{-2}[\sinh^2\eta + \sin^2\xi]^{-1},$$

$$e^{2\gamma} = \kappa^{-2}\psi^{-2}\operatorname{csch}^2\eta.$$

We note that (11.8) is also the solution for a rod of length κ and mass M lying on the z-axis. Similarly when the prolate spheroidal homoeoid approaches a spherical shell as a limit, (11.8) degenerates to Schwarzschild's solution.

12. Motion of a particle in the field of an oblate spheroidal homoeoid.

The fundamental quadratic differential form (10.22) for an oblate spheroidal homoeoid also can be written in the form

(12.1) $$ds^2 = -\psi^{-4}(1-2M\psi)^{-1}d\psi^2 - \psi^{-2}d\xi^2 - \psi^{-2}\cos^2\xi d\varphi^2 + (1-2M\psi)dt^2.$$

If, for convenience, we put

(12.2) $$\psi = 1/r, \quad \xi = \theta - \pi/2,$$

where it must be remembered that r and θ are not the r, θ used in previous sections, then (12.1) becomes

(12.3) $$ds^2 = -(1-2M/r)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2 + (1-2M/r)dt^2,$$

which has the same form as Schwarzschild's solution. The results worked out in the latter case are immediately applicable to the present problem, provided we interpret the symbols in (12.3) appropriately.

The four differential equations,

(12.4) $$\frac{d^2x_\alpha}{ds^2} + \{\mu\nu,\alpha\}\frac{dx_\mu}{ds}\frac{dx_\nu}{ds} = 0,$$

defining the motion of an infinitesimal particle in the four dimensional continuum characterized by (12.3) are[♆]

(12.5) $$\frac{d^2r}{ds^2} + \lambda'\left(\frac{dr}{ds}\right)^2 - re^{-2\lambda}\left(\frac{d\theta}{ds}\right)^2 - r\sin^2\theta e^{-2\lambda}\left(\frac{d\varphi}{ds}\right)^2 + e^{2\nu-2\lambda}\nu'\left(\frac{dt}{ds}\right)^2 = 0,$$

(12.6) $$\frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} - \sin\theta\cos\theta\left(\frac{d\varphi}{ds}\right)^2 = 0,$$

(12.7) $$\frac{d^2\varphi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\varphi}{ds} + 2\cot\theta\frac{d\theta}{ds}\frac{d\varphi}{ds} = 0,$$

(12.8) $$\frac{d^2t}{ds^2} + 2\nu'\frac{dr}{ds}\frac{dt}{ds} = 0,$$

where $e^{2\nu} = 1 - 2M/r, \quad \lambda + \nu = 0, \quad \nu' = \frac{d\nu}{dr}.$

Instead of using (12.5) we can take (12.3), which can be written as

(12.9) $$e^{-2\nu}\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{ds}\right)^2 - e^{2\nu}\left(\frac{dt}{ds}\right)^2 = -1.$$

Equations (12.7) and (12.8) are immediately integrable, giving respectively

(12.10) $$r^2\sin^2\theta\frac{d\varphi}{ds} = c_2,$$

(12.11) $$\frac{dt}{ds} = c_1 e^{-2\nu}.$$

Let the constants of integration c_1 and c_2 be positive.

Eliminating $d\varphi/ds$ between (12.6) and (12.10) we get

(12.12) $$\frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} - \frac{c_2^2}{r^4}\cos\theta\csc^3\theta = 0,$$ giving

(12.13) $$r^4\left(\frac{d\theta}{ds}\right)^2 + c_2^2\csc^2\theta = c_3^2.$$ (Take $c_3 > 0$)

Eliminating ds from (12.10) and (12.13) we find

(12.14) $$d\varphi = -c_2[(c_3^2 - c_2^2) - c_2^2\cot^2\theta]^{-1/2}\csc^2\theta d\theta.$$

We choose the negative sign in (12.14) to make θ decrease when φ in-

♆ A. S. Eddington, loc. cit., pp. 85.

creases. Put

(12.15) $p = (c_3^2 - c_2^2)^{1/2}/c_2.$

By (12.13) since r, θ, s are all real, we see that $c_3^2 \geqq c_2^2$ and consequently p is real. Integrating (12.14) we get

(12.16) $\cot\theta = p\sin(\varphi - \Omega)$ (Ω = const. of integration),

where Ω is the node, and θ is taken to be $\pi/2$ when $\varphi = \Omega$. The geometrical meaning of $\theta = \pi/2$ is that $z = 0$ where the particle crosses the equatorial plane of the oblate homoeoid (cf.(12.2) and (10.2)).

By using (12.9), (12.10), (12.11), (12.13) and (12.16), we obtain the following relation between ψ (or r) and φ,

(12.17) $c_2[2Mf(\psi)]^{-1/2}d\psi = - c_3[1 + p^2\sin^2(\varphi-\Omega)]^{-1}d\varphi,$ where

(12.18) $f(\psi) \equiv \psi^3 - \frac{1}{2M}\psi^2 + \frac{1}{c_3^2}\psi + \frac{c_1^2-1}{2Mc_3^2};$

the negative sign in (12.17) will be explained presently.

The right hand side of (12.17) is immediately integrable in terms of circular functions. The rigorous integration of the left hand side in terms of elliptic functions has been discussed by Forsyth[1] and subsequently by others. Let α, β, γ ($\alpha > \beta > \gamma$) be the three roots of $f(\psi)$. Then ψ can lie only within the interval $\beta \geqq \psi \geqq \gamma$. When $\psi = \beta$, we have the analogous "perihelion" and when $\psi = \gamma$, the "aphelion". Let $\varphi = \varphi_0$, when $\psi = \beta$. Integrating, we have

(12.19) $c_2 \int_\beta^\psi [2Mf(\psi)]^{-1/2}d\psi = - c_3 \int_{\varphi_0}^{\varphi} [1 + p^2\sin^2(\varphi-\Omega)]^{-1}d\varphi.$

Here we see that since ψ decreases after $\psi = \beta$, but that φ continues to increase after $\varphi = \varphi_0$, the negative sign in (12.17) must be taken.

From (12.19) we obtain

(12.20) $\psi = \gamma + (\beta - \gamma)\frac{1 + cn\mu}{1 + dn\mu},$

where μ is defined by the equation,

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[1] A. R. Forsyth, "Proc. Roy. Soc." (A) Vol. 97, pp. 145 (1920).

$$\mu = \frac{1}{P}\{\tan^{-1}[\sigma\tan(\varphi-\Omega)] - \tan^{-1}[\sigma\tan(\varphi_0-\Omega)]\}, \tag{12.21}$$

in which $\sigma = c_3/c_2$, $P = [2M(\alpha - \gamma)]^{-1/2}$. Let ~~and~~ K = the complete elliptic integral of the first kind with modulus k given by

$$k^2 = (\beta - \gamma)/(\alpha - \gamma). \tag{12.22}$$

From (10.2), (10.3), (12.2), (12.16), (12.20) and (12.21), we obtain the equations of the orbit of the particle in the following forms,

$$\begin{aligned} \rho &= \kappa[1 + p^2\sin^2(\varphi-\Omega)]^{-1/2}\csc\kappa\psi, \\ z &= \kappa p\sin(\varphi-\Omega)[1 + p^2\sin^2(\varphi-\Omega)]^{-1/2}\cot\kappa\psi. \end{aligned} \tag{12.23}$$

The equation θ = const. (cf.(12.2) and (10.2)) represents the family of hyperboloids of one sheet orthogonal to the family of spheroids ψ = const. Then (12.16) shows that the maximum and minimum latitudes of the particle in its orbit are invariable for given initial conditions.

The function, ψ, in (12.20) is a Jacobian elliptic function of φ. Hence the analogous "line of apsides" of the orbit precesses about the z-axis. The amount of this precession for the particle to prescribe the orbit once can be calculated in the following manner:[1] In (12.20) we have so chosen ψ, φ that at perihelion $\psi = \beta$, $\varphi = \varphi_0$. Then at aphelion $\psi = \gamma$, let $\varphi = \varphi_1$. From (12.20) and (12.21),

$$\tan^{-1}[\sigma\tan(\varphi_1 - \Omega)] - \tan^{-1}[\sigma\tan(\varphi_0 - \Omega)] = 2PK. \tag{12.24}$$

At the next perihelion let $\varphi = \varphi_2$. The relation analogous to (12.24) is

$$\tan^{-1}[\sigma\tan(\varphi_2 - \Omega)] - \tan^{-1}[\sigma\tan(\varphi_1 - \Omega)] = 2PK. \tag{12.25}$$

Adding (12.24) and (12.25) we get

$$\tan^{-1}[\sigma\tan(\varphi_2 - \Omega)] - \tan^{-1}[\sigma\tan(\varphi_0 - \Omega)] = 4PK. \tag{12.26}$$

The precession is given by

$$\Delta = \varphi_2 - \varphi_0 - 2\pi. \tag{12.27}$$

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[1] A. R. Forsyth, loc. cit., p. 148.

Solving φ_2 from (12.26) we get

(12.28) $$\Delta = \tan^{-1}\frac{1}{\sigma}\left[\frac{\tan 4PK + \sigma\tan(\varphi_0 - \mathcal{S})}{1 - \sigma\tan(\varphi_0 - \mathcal{S})\tan 4PK}\right] - (\varphi_0 - \mathcal{S}) - 2\pi.$$

It is interesting to note from (12.13), (12.15) and (12.23) that if the particle lies initially in the equatorial plane of the homoeoid, i.e. $d\theta/ds = 0$ when $\theta = \pi/2$, then subsequently $\theta = \pi/2$ and the particle will continually lie there. The approximate formula for Δ in this case can be calculated as follows: Regard (r,φ) as configurational polar coordinates of the particle. Then (12.3) shows that the motion of the particle in these coordinates is the same as the motion of a cooresponding particle in Schwarzschild's solution. Hence the constants c_1 and c_2 in (12.11) and (12.10) are given by[4]

(12.29) $$c_2^2 = r_0(1 - e^2)M, \quad c_1^2 - 1 = -M/r_0, \quad e^2 = (r_0^2 - r_1^2)/r_0^2,$$

where M is the mass of the homoeoid, r_0 the semi-major axis, r_1 the semi-minor axis, and e the eccentricity of the orbit in the configurational coordinate system. The advance of the perihelion, Δ, is given approximately by

(12.30) $$\Delta = 2\pi\cdot\frac{3M}{r_0(1 - e^2)}.$$

From (10.3), (10.23) with $z = 0$, and (12.2), we obtain

(12.31) $$\frac{1}{r} = \psi = \frac{1}{\kappa}\cot^{-1}\left[\frac{1}{\kappa}(\rho^2 - \kappa^2)^{1/2}\right].$$

When $\rho^2 - \kappa^2 > \kappa^2$, which is obviously satisfied by large values of ρ, we can expand (12.31) in ascending powers of κ/ρ in the form

(12.32) $$\frac{1}{r} = \frac{1}{\rho}\left[1 + \frac{1}{6}\frac{\kappa^2}{\rho^2} + \frac{3}{40}\frac{\kappa^4}{\rho^4} + \cdots\right].$$

Equation (12.31) shows that ρ is a monotonic function of r, and consequently the value of Δ in (12.30), which is primarily for the orbit in the (r,φ) configurational coordinates, will hold also in the (ρ,φ) system. Knowing the "semi-major" and "semi-minor" axes, ρ_0, and ρ_1, of the particle's orbit in the latter system we can compute the corresponding values of r_0 and r_1 by (12.32). Then (12.30) shows that the oblate-

[4] A. R. Forsyth, loc. cit., p. 145.

ness of the central body causes a small increase in the advance of the perihelion of the orbit predicted from Schwarzschild's solution. This increases vanishes when $\kappa = 0$, namely, when the oblate spheroidal homoeoid degenerates into a spherical shell.

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THE END