A SYSTEMATIC PRESENTATION OF THE THEORY OF THIN AIRFOILS IN NON-UNIFORM MOTION

Thesis

by

William Rees Sears

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ABSTRACT

The basic conceptions of the circulation theory of airfoils are reviewed briefly, and the mechanism by which a "wake" of vorticity is produced by an airfoil in non-uniform motion is pointed out. After a calculation of the induction effects of a wake vortex, it is shown how the lift and moment acting upon an airfoil in the two-dimensional case may be calculated directly from simple physical considerations of momentum and moment of momentum. Formulae for the lift and moment are then obtained which are applicable to all cases of motion of a two-dimensional thin airfoil in which the wake produced is approximately flat; i.e., in which the movement of the airfoil normal to its mean path is small.

The general results are applied first to the case of an oscillating airfoil, and vector diagrams giving the magnitudes and phase angles of the lift and moment are obtained. The results of a sudden change of angle of attack are then determined, and a general method for handling transient cases is set up. This method is applied to the calculation of the lift and moment acting on an airfoil entering sharp-edged and graded gusts. The case of a series of sinusoidal gusts is also considered.

A method of calculating the distribution of forces over the airfoil chord is then shown, and it is applied to the steady-state oscillation. The paper concludes with a discussion of the applicability of certain results to the explanation of observed phenomena beyond the stall.

I. INTRODUCTION

The theory of airfoils in non-uniform motion has several practically important applications, especially in connection with problems of wing flutter and of aircraft flying through gusts. It has been developed by a number of writers (see Bibliography); however, many of their works suffer from a certain lack of clarity, especially since the physical principles underlying the mathematical treatments have not always been pointed out. The present paper is the result of an attempt to obtain the more important results of the theory by the application of fundamental physical principles and to present them in forms suitable for direct application to certain flutter and gust problems.

It is advisable to review briefly the fundamental concepts of the circulation theory of airfoils in the case of two-dimensional motion, i.e., of infinite aspect ratio. The airfoil, when initially put into motion relative to the fluid, creates a vortex at its trailing edge due to the presence of a sharp corner there. According to the principle of conservation of angular momentum, an equal and opposite circulation develops around the airfoil. As the airfoil continues its motion, the "starting vortex" is left behind in the fluid. If the relatively slow displacement of this vortex in a direction perpendicular to the direction of flight is neglected, it can be assumed that this vortex remains stationary at the place where it was created.

If the subsequent motion of the airfoil is uniform, i.e., if its velocity and angle of attack remain constant, the effects of the starting vortex on the flow at the airfoil become very small and can be neglected after the airfoil has travelled a great distance from the starting point. However, if the motion of the airfoil relative to the fluid is variable, a continuous succession of starting vortices will be shed at its trailing edge, and the effects of this "wake" of continuously distributed vorticity must be accounted for in calculating the forces and moments acting on the airfoil. These effects can be evaluated by the use of the results of the following Section.

II. CALCULATION OF THE EFFECT OF THE WAKE VORTICES

The effects of the wake vortices are here calculated for the simple, two-dimensional case illustrated in Fig. 1. In accordance with the usual theory of thin airfoils the airfoil is considered to be made up of a vortex sheet, i.e., a series of infinitesimal vortex lines lying in the direction of the span, with a continuous distribution of vortex strength, or "vorticity", across the chord. The chord of the airfoil is taken equal to 2, so that all lengths are measured in half-chords. It is assumed, moreover,

(a) that the vertical displacement of any point of the airfoil from the mean flight path is small, so that the airfoil and the trail of wake vortices which it leaves behind may be considered to lie upon the X-axis;

(b) that the theory of thin airfoils may be applied to the calculation of the forces; in particular that the total circulation about the airfoil at any instant is such as to produce tangential flow at the trailing edge.

The effect of an element of the wake vorticity, Γ', located at a distance ξ from the center of the airfoil may be calculated with the aid of the conformal transformation pictured in Fig. 2. The transformation relating the two planes is

$$2z = z' + 1/z' \tag{1}$$

Since the airfoil lies on the X-axis between x = -1 and x = 1 in the z-plane, it is transformed into the unit circle in the z'-plane. In the z'-plane the vortex $-\Gamma'$ is placed at $x' = 1/\eta$ to make the unit circle a streamline of the flow, by the usual method of "images". This means that the resultant velocity induced by the two vortices is tangential at all points on the circle. Its magnitude is given by

$$v_{\theta_1} = \frac{\Gamma'}{2\pi} \left| \frac{1}{z'-\eta} - \frac{1}{z'-1/\eta} \right|_{z'=e^{i\theta}}$$

$$= \frac{\Gamma'}{2\pi} \left| \frac{\eta - 1/\eta}{e^{2i\theta} - (\eta + 1/\eta)e^{i\theta} + 1} \right| \tag{2}$$

From the equation of the transformation, (1), $\eta + 1/\eta = 2\xi$ and $\eta - 1/\eta = 2\sqrt{\xi^2 - 1}$. Using these relations, the magnitude of the tangential velocity becomes

$$v_{\theta_1} = \frac{\Gamma'}{2\pi} \left| \frac{2\sqrt{\xi^2-1}}{e^{2i\theta} - 2\xi e^{i\theta} + 1} \right|$$

$$= \frac{\Gamma'}{2\pi} \frac{\sqrt{\xi^2-1}}{\xi - \cos\theta} \tag{3}$$

In particular, the velocity at the trailing edge, where $\cos\theta = 1$, is equal to $\frac{\Gamma'}{2\pi}\sqrt{\frac{\xi+1}{\xi-1}}$. In accordance with assumption (b) above, a circulation arises which is just great enough to cancel this velocity. Hence a second, uniform velocity, $v_{\theta_2} = -\frac{\Gamma'}{2\pi}\sqrt{\frac{\xi+1}{\xi-1}}$, is added to v_{θ_1}. Then the total tangential velocity becomes

$$v_\theta = v_{\theta_1} + v_{\theta_2}$$

$$= \frac{\Gamma'}{2\pi}\left\{\frac{\sqrt{\xi^2-1}}{\xi-\cos\theta} - \sqrt{\frac{\xi+1}{\xi-1}}\right\} = \frac{\Gamma'}{2\pi}\sqrt{\frac{\xi+1}{\xi-1}}\left\{\frac{\xi-1}{\xi-\cos\theta} - 1\right\}$$

$$= -\frac{\Gamma'}{2\pi}\sqrt{\frac{\xi+1}{\xi-1}}\,\frac{1-\cos\theta}{\xi-\cos\theta} \tag{4}$$

The relation between the velocity v_θ and the vorticity distribution over the airfoil, $\gamma(x)$, is given by the formula $\gamma(x) = -2v_\theta/\sin\theta$.* Thus, from (4) it follows that

$$\gamma(x) = \frac{\Gamma'}{\pi}\,\frac{1}{\sin\theta}\sqrt{\frac{\xi+1}{\xi-1}}\,\frac{1-\cos\theta}{\xi-\cos\theta}$$

or, since $\cos\theta = x$ and $\sin\theta = \sqrt{1-x^2}$,

$$\gamma(x) = \frac{1}{\pi}\,\frac{\Gamma'}{\xi-x}\sqrt{\frac{1-x}{1+x}}\sqrt{\frac{\xi+1}{\xi-1}} \tag{5}$$

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*cf. von Kármán and Burgers, Ref. 1, p. 46, noting that their transformation differs by a factor 2 from the one given in equation (1).

This vorticity distribution on the airfoil is plotted in Fig. 3 in terms of the wake vortex strength, Γ', for several values of ξ. It is seen that a wake vortex located one half-chord length or more behind the trailing edge induces a vorticity distribution which is similar to the well-known one produced by a small angle of attack, while a vortex placed very close to the airfoil induces a much stronger vorticity over the chord, with a definite peak near the trailing edge in addition to that at the leading edge.

The total circulation about the airfoil due to the wake vortex is obtained by integration of (5), and is

$$\Gamma = \int_{-1}^{1} \gamma(x)\,dx = \frac{\Gamma'}{\pi}\sqrt{\frac{\xi+1}{\xi-1}}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\,\frac{dx}{\xi-x}$$

$$= \frac{\Gamma'}{\pi}\sqrt{\frac{\xi+1}{\xi-1}}\left\{\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}\,(\xi-x)} - \int_{-1}^{1}\frac{x\,dx}{\sqrt{1-x^2}\,(\xi-x)}\right\}$$

$$= \Gamma'\sqrt{\frac{\xi+1}{\xi-1}}\left\{\frac{1}{\sqrt{\xi^2-1}} + 1 - \frac{\xi}{\sqrt{\xi^2-1}}\right\}$$

$$= \Gamma'\left\{\sqrt{\frac{\xi+1}{\xi-1}} - 1\right\} \tag{6*}$$

If the wake behind the airfoil consists of a continuous distribution of small vortices whose strength is given by the function $\gamma(\xi)$, the effects of the part of the wake lying between ξ and

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*The evaluation of the definite integrals involved in this reduction is given in Appendix #1 at the end of the paper.

$\xi + d\xi$ can be calculated by replacing Γ' in equations (5) and (6) by $\gamma(\xi)d\xi$. Then, since the wake extends from the trailing edge ($\xi = 1$) to some value of ξ which corresponds to the beginning of the motion, and beyond which $\gamma(\xi) = 0$, the vorticity and circulation induced on the airfoil by the entire wake can be obtained by integration, i.e.,

$$\gamma(x) = \frac{1}{\pi}\sqrt{\frac{1-x}{1+x}}\int_1^\infty \frac{\gamma(\xi)}{\xi - x}\sqrt{\frac{\xi+1}{\xi-1}}\,d\xi \qquad (7)$$

and

$$\Gamma = \int_1^\infty \gamma(\xi)\left\{\sqrt{\frac{\xi+1}{\xi-1}} - 1\right\} d\xi \qquad (8)$$

III. GENERAL FORMULAE FOR THE LIFT AND MOMENT

In this Section, general formulae for the lift and moment acting on a thin airfoil in non-uniform motion are derived from the simple physical conceptions of momentum and moment of momentum. The values of these quantities are first determined for a two-dimensional system of vortex pairs.

The momentum of a vortex pair is given by the product of the fluid density, the circulation, and the distance between the vortices. The total momentum of a system of vortex pairs is equal to the sum of the momentums of the pairs which constitute the system. Thus, if all the constituent vortices can be assumed to lie along the X-axis, and if the strength of a particular vortex is denoted by Γ_i, its X-coordinate measured from an arbitrary origin by x_i, and the density of the fluid by ρ, the total momentum of the system is

$$I = \rho \sum \Gamma_i x_i \qquad (9)^*$$

Because of the symmetry of the individual vortex pairs, this momentum is directed perpendicular to the X-axis. The condition $\sum \Gamma_i = 0$ expresses the fact that the total circulation of the system does not change. The rate of change of the total momentum at any instant is equal to the force being exerted on the fluid (e.g. by an airfoil).

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* cf. Ref. 1, p.325

In a similar manner the total moment of momentum of the fluid with respect to a suitably chosen point may be expressed. If the strengths of the two vortices of a particular vortex pair are denoted by $\pm\Gamma$, and the X-coordinates of the two vortices by x_2 and x_1, then the momentum is $\rho\Gamma(x_2 - x_1)$, and the line of action of the momentum, due to symmetry, is given by $x = (x_1 + x_2)/2$. Consequently the moment of momentum with respect to the origin of the coordinate system is $\rho\Gamma(x_2^2 - x_1^2)/2$, and it is seen that the total moment of momentum of the system of vortex pairs is given by

$$M_m = \frac{1}{2}\rho\sum\Gamma_i x_i^2 \tag{10}$$

The rate of change of this quantity at any instant gives the moment acting on the fluid, referred to the origin of coordinates.

Hence the two equations

$$L = -\rho\frac{d}{dt}\sum\Gamma_i x_i \tag{11}$$

and

$$M = -\frac{1}{2}\rho\frac{d}{dt}\sum\Gamma_i x_i^2 \tag{12}$$

determine the lift and moment acting on the airfoil.

These results can now be applied to the case of any thin airfoil of infinite aspect ratio with a wake consisting of a plane vortex sheet. The chord of the airfoil is again taken as 2, so that all distances are measured relative to the half-chord length. All forces are calculated for a unit length in the spanwise direction. The symbol x is used for the X-coordinate between the leading edge ($x = -1$) and the trailing edge ($x = 1$), and the symbol ξ is used in the wake. Hence the vorticity bound to the airfoil is denoted by $\gamma(x)$ and that in the wake by $\gamma(\xi)$. The vorticity $\gamma(x)$ is composed of two parts:

a) the vorticity, $\gamma_o(x)$, which would be produced, according to the thin airfoil theory, by the motion of the airfoil or the given velocity distribution (gust) in the air, if the wake had no effect. $\gamma_o(x)$ is called the "quasi-steady" vorticity distribution;

b) the vorticity, $\gamma_1(x)$, which is induced by the wake, as calculated in the preceding Section.

The circulation resulting from a) is denoted by Γ_o, and that from b) by Γ_1; the total circulation about the airfoil is then $\Gamma = \Gamma_o + \Gamma_1$. According to the basic conceptions explained above, the total circulation of the whole system must be zero, hence

$$\Gamma + \int_1^\infty \gamma(\xi)\, d\xi = 0 \tag{13}$$

The circulation, Γ_1, induced by the wake, is just the circulation calculated in equation (8). Hence the total circulation about the airfoil at any instant is given by

$$\Gamma = \Gamma_0 + \int_1^\infty \gamma(\xi) \left\{ \sqrt{\frac{\xi+1}{\xi-1}} - 1 \right\} d\xi \tag{14}$$

From equation (13) it is apparent that the following relation exists between the vorticity in the wake and the quasi-steady circulation:

$$\Gamma_0 + \int_1^\infty \gamma(\xi) \sqrt{\frac{\xi+1}{\xi-1}}\, d\xi = 0 \tag{15}$$

This relation will be used later in the paper.

Since the total circulation of the system consisting of the airfoil and wake is zero, according to equation (13), the system can be considered as being composed of vortex pairs, and, since the wake is supposed to lie entirely along the X-axis, equations (11) and (12) can be applied to the calculation of the lift and moment. According to the principles described above and the result given in equation (9), the total momentum per unit span of the system of continuously distributed vortices is

$$I = \rho\int_{-1}^{1}\gamma(x)\,x\,dx + \rho\int_{1}^{\infty}\gamma(\xi)\,\xi\,d\xi \tag{16}$$

Putting $\gamma(x) = \gamma_0(x) + \gamma_1(x)$ and using formula (7) for $\gamma_1(x)$,

$$\int_{-1}^{1}\gamma(x)\,x\,dx = \int_{-1}^{1}\gamma_0(x)\,x\,dx + \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\,x\,dx\int_{1}^{\infty}\frac{\gamma(\xi)}{\xi - x}\sqrt{\frac{\xi+1}{\xi-1}}\,d\xi \tag{17}$$

If the integration with respect to x in the last term is carried out*, the term is reduced to

$$\int_{1}^{\infty}\left\{\frac{\xi}{\sqrt{\xi^2-1}} - 1 + \xi - \frac{\xi^2}{\sqrt{\xi^2-1}}\right\}\sqrt{\frac{\xi+1}{\xi-1}}\,\gamma(\xi)\,d\xi$$

$$= \int_{1}^{\infty}\gamma(\xi)\left(\sqrt{\xi^2-1} - \xi\right)d\xi$$

and therefore, putting the result into equation (16), the momentum becomes

$$I = \rho\int_{-1}^{1}\gamma_0(x)\,x\,dx + \rho\int_{1}^{\infty}\gamma(\xi)\sqrt{\xi^2-1}\,d\xi \tag{18}$$

Now it is desired to differentiate this expression to obtain the lift, but since $\partial\gamma(\xi)/\partial\xi$ may be discontinuous at certain points in the wake (e.g., the case considered in

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*see Appendix #1.

Sections V - VII of this paper), the use of an integration by parts (as employed in Ref. 1, p. 301) is not allowable in the evaluation of the second term. Since a similar problem is encountered later in calculating the moment, it is desirable to consider a general integral of the form

$$A = \int_{1}^{\infty} \gamma(\xi) f(\xi)\, d\xi$$

The wake vorticity, γ, according to the assumptions already made, is stationary relative to the fluid. Hence if X is the distance of an arbitrary wake vortex from a fixed origin, say from the location of the center of the airfoil at the instant $t = t$, then γ is a function of X only. The integral considered can therefore be written

$$A = \int_{1}^{\infty} \gamma(X)\, f(\xi)\, d\xi$$

If A is the value of this integral at the time t and $A + \Delta A$ the value at the time $t + \Delta t$, and if account is taken of the fact that the airfoil has moved through a distance $U \cdot \Delta t$ during this interval, where U is the velocity of flight, so that $\xi = X + U \cdot \Delta t$ (cf. Fig. 4), it is seen that

$$A + \Delta A = \int_{1-U\Delta t}^{\infty} \gamma(X)\, f(X + U\Delta t)\, dX$$

Neglecting terms of second order and higher,

$$\Delta A = \int_{1-U\Delta t}^{1} \gamma(X) f(X)\,dX + U\Delta t \int_{1}^{\infty} \gamma(X) f'(X)\,dX$$

Now if $\gamma(X)$ is finite in the interval and if $f(1) = 0$, then in the limit $\Delta t \rightarrow 0$ the first term vanishes, and, replacing X by ξ in the second term,

$$\frac{dA}{dt} = U \int_{1}^{\infty} \gamma(\xi) f'(\xi)\,d\xi \tag{19}$$

Applying this result to the differentiation of the second term of (18), the lift becomes

$$L = -\frac{dI}{dt} = -\rho \frac{d}{dt} \int_{-1}^{1} \gamma_0(x)\,x\,dx - \rho U \int_{1}^{\infty} \gamma(\xi) \frac{\xi\,d\xi}{\sqrt{\xi^2 - 1}} \tag{20}$$

Using the relation (15), the last integral may be related to the quasi-steady circulation:

$$\int_{1}^{\infty} \gamma(\xi) \frac{\xi\,d\xi}{\sqrt{\xi^2-1}} = \int_{1}^{\infty} \gamma(\xi) \left\{ \sqrt{\frac{\xi+1}{\xi-1}} - \frac{1}{\sqrt{\xi^2-1}} \right\} d\xi$$

$$= -\Gamma_0 - \int_{1}^{\infty} \gamma(\xi) \frac{d\xi}{\sqrt{\xi^2-1}}$$

Therefore the lift may be written as

$$L = -\rho \frac{d}{dt} \int_{-1}^{1} \gamma_0(x)\,x\,dx + \rho U \Gamma_0 + \rho U \int_{1}^{\infty} \gamma(\xi) \frac{d\xi}{\sqrt{\xi^2 - 1}} \tag{21}$$

Thus the lift consists of three parts:

(a) $L_1 = -\rho \frac{d}{dt} \int_{-1}^{1} \gamma_0(x) x \, dx$, which will be called the contribution of the apparent mass, for reasons to be explained later;

(b) $L_0 = \rho U \Gamma_0$, the quasi-steady lift;

(c) $L_2 = \rho U \int_{1}^{\infty} \gamma(\xi) \frac{d\xi}{\sqrt{\xi^2 - 1}}$. This is the only contribution which depends explicitly on $\gamma(\xi)$, the vorticity distribution in the wake, and it will be shown later that this portion of lift actually represents the entire effect of the wake.

In a similar manner, the moment can be calculated from the moment of momentum, referred to a fixed axis. If the center of the airfoil is imagined to be at a distance s from the fixed axis, the moment of momentum (cf. equation (10)) is

$$M_m = \frac{1}{2}\rho \int_{-1}^{1} \gamma(x)(x+s)^2 dx + \frac{1}{2}\rho \int_{1}^{\infty} \gamma(\xi)(\xi + s)^2 d\xi \tag{22}$$

where x and ξ are again measured from the center of the airfoil. The moment, M, acting on the airfoil, referred to its midpoint, is then given by the value of dM_m/dt for $s = 0$. Carrying out the differentiation and taking into account that $ds/dt = -U$, where U is the velocity of flight of the airfoil, this becomes

$$M = -\frac{1}{2}\rho\frac{d}{dt}\left\{\int_{-1}^{1}\gamma(x)x^2dx + \int_{1}^{\infty}\gamma(\xi)\xi^2 d\xi\right\}$$
$$-\rho\frac{ds}{dt}\left\{\int_{-1}^{1}\gamma(x)x\,dx + \int_{1}^{\infty}\gamma(\xi)\xi\,d\xi\right\}$$
$$= -\frac{1}{2}\rho\frac{d}{dt}\left\{\int_{-1}^{1}\gamma(x)x^2dx + \int_{1}^{\infty}\gamma(\xi)\xi^2 d\xi\right\} + UI \qquad (23)$$

where I is the total momentum as calculated in equation (18). A diving moment is here considered positive.

Now substituting again $\gamma(x) = \gamma_o(x) + \gamma_1(x)$, and using equation (7) for $\gamma_1(x)$, the moment is

$$M = -\frac{1}{2}\rho\frac{d}{dt}\left\{\int_{-1}^{1}\gamma_o(x)x^2dx + \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\,\frac{x^2dx}{\xi-x}\int_{1}^{\infty}\gamma(\xi)\sqrt{\frac{\xi+1}{\xi-1}}\,d\xi\right.$$
$$\left. + \int_{1}^{\infty}\gamma(\xi)\xi^2 d\xi\right\} + UI$$
$$= -\frac{1}{2}\rho\frac{d}{dt}\left\{\int_{-1}^{1}\gamma_o(x)x^2dx + \int_{1}^{\infty}\gamma(\xi)\left\{\frac{\xi^2}{\sqrt{\xi^2-1}} - \xi + \frac{1}{2} + \xi^2 - \frac{\xi^3}{\sqrt{\xi^2-1}}\right\}\sqrt{\frac{\xi+1}{\xi-1}}\,d\xi\right.$$
$$\left. + \int_{1}^{\infty}\gamma(\xi)\xi^2 d\xi\right\} + UI \qquad (24a)^*$$
$$= -\frac{1}{2}\rho\frac{d}{dt}\left\{\int_{-1}^{1}\gamma_o(x)x^2dx + \frac{1}{2}\int_{1}^{\infty}\gamma(\xi)\sqrt{\frac{\xi+1}{\xi-1}}\,d\xi\right.$$
$$\left. + \int_{1}^{\infty}\gamma(\xi)\xi\sqrt{\xi^2-1}\,d\xi\right\} + UI \qquad (24)$$

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*The integration with respect to x is again carried out with the aid of Appendix #1.

Now the second integral in the bracket is equal to $-\Gamma_0/2$ or $-\frac{1}{2}\int_{-1}^{1}\gamma_0(x)dx$ by equation (15), and the differentiation of the third integral can be carried out by the method of equation (19). Then, substituting for I from equation (18), the moment becomes

$$M = -\frac{1}{2}\rho\frac{d}{dt}\left\{\int_{-1}^{1}\gamma_0(x)x^2dx - \frac{1}{2}\int_{-1}^{1}\gamma_0(x)dx\right\}$$
$$-\frac{1}{2}\rho U\int_{1}^{\infty}\gamma(\xi)\left\{\sqrt{\xi^2-1}+\frac{\xi^2}{\sqrt{\xi^2-1}}\right\}d\xi + \rho U\left\{\int_{-1}^{1}\gamma_0(x)x\,dx + \int_{1}^{\infty}\gamma(\xi)\sqrt{\xi^2-1}\,d\xi\right\}$$
$$= -\frac{1}{2}\rho\frac{d}{dt}\int_{-1}^{1}\gamma_0(x)\left(x^2-\frac{1}{2}\right)dx - \frac{1}{2}\rho U\int_{1}^{\infty}\gamma(\xi)\left\{2\sqrt{\xi^2-1}+\frac{1}{\sqrt{\xi^2-1}}\right\}d\xi$$
$$+\rho U\int_{-1}^{1}\gamma_0(x)x\,dx + \rho U\int_{1}^{\infty}\gamma(\xi)\sqrt{\xi^2-1}\,d\xi$$
$$= -\frac{1}{2}\rho\frac{d}{dt}\int_{-1}^{1}\gamma_0(x)\left(x^2-\frac{1}{2}\right)dx + \rho U\int_{-1}^{1}\gamma_0(x)x\,dx$$
$$-\frac{1}{2}\rho U\int_{1}^{\infty}\gamma(\xi)\frac{d\xi}{\sqrt{\xi^2-1}} \qquad (25)$$

Therefore the moment also consists of three parts:

(a) $M_1 = -\frac{1}{2}\rho\frac{d}{dt}\int_{-1}^{1}\gamma_0(x)\left(x^2-\frac{1}{2}\right)dx$, analogous to L_1. This will be called the apparent-mass contribution to the moment.

(b) $M_0 = \rho U\int_{-1}^{1}\gamma_0(x)x\,dx$, the quasi-steady moment;

(c) $M_2 = -\frac{1}{2}\rho U \int_1^\infty \gamma(\xi) \frac{d\xi}{\sqrt{\xi^2-1}}$. By comparison

with equation (21), it is seen that $M_2 = -L_2/2$, i.e., the lift L_2 produced by the wake always acts through the quarter-chord point of the airfoil ($x = -1/2$).

The physical significance of the three parts of the lift, as given in equation (21), and of the moment, equation (25), will now be explained briefly.

Considering first the lift, let it be assumed that the airfoil carries out its motion without producing circulation. Then the quasi-steady lift, L_1, is zero, and, because obviously no wake is produced, the part called L_2 also vanishes. It is known from general principles that in such a case the only forces acting on a body moving in an ideal fluid are those corresponding to the apparent mass of the body. These can be obtained by integrating over the surface of the airfoil the so-called "impulsive pressures", $\rho\, \partial\varphi/\partial t$, where φ is the velocity potential of the circulationless flow. Hence, if C indicates a path of integration starting at some point A on the airfoil and going completely around the airfoil profile back to A, the lift is

$$L = \rho \int_C \frac{\partial \varphi}{\partial t}\, ds = \rho \frac{\partial}{\partial t} \int_C \varphi(s)\, ds$$

where s represents the distance along the surface. Now the velocity potential, φ, can be taken equal to zero at an arbitrary point, say at the trailing edge, $x = 1$. Then its value at any point x is given by $\varphi(x) = \int_1^x u(s)ds$, where u is the velocity of the fluid along the surface. Since the velocities on the upper and lower surfaces of a plane airfoil are equal and opposite, it is seen that $\varphi(x)$ also has equal and opposite values on the two surfaces at any point. Therefore the integration over C can be transformed into an integration over the chord, i.e.,

$$L = \rho \frac{\partial}{\partial t} \int_C \varphi(s)\, ds = 2\rho \frac{\partial}{\partial t} \int_{-1}^{1} \varphi(x)\, dx$$

$$= 2\rho \frac{\partial}{\partial t} \left\{ \varphi \cdot x \Big|_{-1}^{1} - \int_{-1}^{1} \frac{\partial \varphi}{\partial x}\, x\, dx \right\}$$

The first term obtained in the integration by parts vanishes because if $\varphi(1) = 0$, as assumed above, $\varphi(-1)$ must also be equal to zero, since there is no circulation, and therefore the values of φ on the upper and lower surfaces at the leading edge must be the same. Also $\partial\varphi/\partial x$, which is the velocity along the surface, is equal to $\gamma_o(x)/2$, because $\gamma_o(x)$, the vorticity at x, is the difference between the upper and lower surface velocities. Hence, the lift in this case is simply

$$L = -\rho \frac{d}{dt}\int_{-1}^{1} \gamma_o(x)\, x\, dx$$

which is exactly the term called L_1 above (equation (21)). Therefore, the term L_1 gives correctly the lift due to the apparent mass in the case of the airfoil without circulation. Now the addition of circulation increases the function $\gamma_o(x)$ by a term equal to $\frac{1}{\pi}\frac{\Gamma_o}{\sqrt{1-x^2}}$, which is the vorticity distribution of a pure circulation about a plane airfoil. It is seen by considerations of symmetry that this term does not contribute to the integral L_1, i.e., L_1 is again the lift due to the apparent mass.

Turning now to the corresponding term, M_1, in the expression for the moment, in the case of flow without circulation the moment can be calculated by integrating over the surface of the airfoil the moments of the impulsive pressures, and again the contour integration can immediately be written as an integration across the chord, i.e.,

$$M = \rho\int_C \frac{\partial\varphi}{\partial t}\, x\, ds = \rho\frac{\partial}{\partial t}\int_C \varphi(s)\, x\, ds = 2\rho\frac{\partial}{\partial t}\int_{-1}^{1}\varphi(x)\, x\, dx$$

and, again integrating by parts,

$$M = 2\rho \frac{\partial}{\partial t}\left\{ \frac{\varphi \cdot x^2}{2}\Big|_{-1}^{1} - \frac{1}{2}\int_{-1}^{1} \frac{\partial \varphi}{\partial x} x^2 dx \right\}$$

$$= -\frac{1}{2}\rho \frac{d}{dt}\int_{-1}^{1} \gamma_0(x) x^2 dx$$

The rest of the integral called M_1 in equation (25) vanishes because $\int_{-1}^{1} \gamma_0(x)dx = 0$. Again the addition of the circulation term to $\gamma_0(x)$ does not change the value of the whole integral, for this term is $\frac{1}{\pi}\frac{\Gamma_0}{\sqrt{1-x^2}}$, and

$$\frac{\Gamma_0}{\pi}\int_{-1}^{1} \frac{x^2 - 1/2}{\sqrt{1-x^2}} dx = 0 .$$

Hence it is found that the contributions L_1 and M_1 are equal to the force and moment which the airfoil would encounter in a flow without circulation, due to the reaction of the accelerated fluid masses. We call these terms in both cases the "apparent mass contributions". It should be noted that the determination of $\gamma_0(x)$ involves only the solution of steady-flow problems and can be done in any given case by the use of known formulae of the stationary airfoil theory.

The second terms of equations (21) and (25), the lift L_0 and moment M_0, are easily interpreted. They represent the force and moment which would be produced if the instantaneous velocity and angle of attack of the airfoil were permanently maintained. The calculation of L_0 and M_0 also requires only the solution of steady-flow problems by the usual methods.

The third contributions, L_2 and M_2, represent the influence of the wake. Their interpretation is simplified by considering a case in which quasi-steady lift and moment (i.e., the angle of attack or speed) undergo a sudden change at the instant $t = 0$ and are kept constant for $t > 0$. In this case $L_1 = M_1 = 0$ for $t > 0$, and the lift and moment are given by $L_0 + L_2$ and $M_0 + M_2$. For $t = \infty$ the final values of lift and moment will be L_0 and M_0 because the conditions of the "stationary" case are approached. It is seen that L_2 and M_2 give the difference between the transient and final values of lift and moment. Hence $-L_2$ and $-M_2$ can be called the "deficiencies" due to the non-uniformity of the motion of the airfoil or of the wind velocity encountered by it.

Before proceeding to the next Section, it should be pointed out that the general formula, (21) and (25), developed in this Section can be applied to the case of any thin airfoil with arbitrary shape, performing an arbitrary (accelerated, oscillatory, or uniform) motion, provided only that its deviation from a straight path is small, so that the assumption of a wake distributed along a straight line is justified.

IV. APPLICATION TO THE CASE OF STEADY STATE OSCILLATIONS

The theory of steady-state oscillations of an airfoil is closely related to the practical problem of wing flutter. In the earliest flutter theories it was assumed that the actual forces and moments on the wing could be approximately replaced by the portions referred to in the preceding Section as quasi-steady forces plus some damping forces applied rather arbitrarily on the basis of a few wind tunnel experiments. However, the difference in phase between the actual forces and the quasi-steady components was not accounted for, and the effects of the apparent mass were also neglected. The theory of the oscillating airfoil now opens the way to a more systematic analysis of the problem. To be sure, the assumption of infinite aspect ratio restricts the accuracy of purely theoretical predictions regarding the flutter of wings of finite span, but in any case the results obtained from the theory aid in scientific analysis of experimental data obtained in the laboratory and with actual airplanes in flight.

Experimental work done by Küssner and others has shown that the so-called "reduced frequency" (i.e., the product of half-chord and vibration frequency divided by the flying speed) is the most adequate parameter for the discussion of flutter data. The following analysis, based on the results of the preceding Section, leads to simple formulae and diagrams showing how the magnitude and the phase of the actual lift and moment depend on the reduced frequency.

If the motion of the airfoil is periodic, the resulting quasi-steady circulation is also periodic and, using the complex variable notation, may be written

$$\Gamma_0 = G_0 e^{i\nu t} \tag{26}$$

where G_0 is a constant.

If the motion has been occurring so long that transient phenomena have disappeared, it may be assumed that the vortex strength in the wake can be expressed as

$$\gamma(\xi) = g\, e^{i\nu(t-\xi/U)} \tag{27}$$

where g is also a constant and U is the mean horizontal velocity, which is assumed to be constant in this case.

Then the total circulation about the airfoil is given (from equation (14)) by

$$\Gamma = e^{i\nu t}\left\{G_0 + g\int_1^\infty\left(\sqrt{\frac{\xi+1}{\xi-1}}-1\right)e^{-i\nu\xi/U}\,d\xi\right\} \tag{28}$$

Hence Γ is also a periodic function of the time, and, because the wake vorticity is produced by the changes of circulation of the airfoil, the increment of circulation, $(d\Gamma/dt)dt$, must be equal and opposite to the circulation in the wake between $\xi = 1$ and $\xi = 1 + Udt$. Consequently $(d\Gamma/dt)dt = -\gamma(1)Udt$. By differentiation of (28) and substitution of $d\Gamma/dt = -\gamma(1)U$, a relation between G_o and g is obtained:

$$U g e^{i\nu(t-1/U)} = i\nu e^{i\nu t}\left\{G_o + g\int_1^\infty\left(\sqrt{\frac{\xi+1}{\xi-1}} - 1\right)e^{-i\nu\xi/U}d\xi\right\}$$

or

$$-\frac{G_o}{g} = \int_1^\infty\left(\sqrt{\frac{\xi+1}{\xi-1}} - 1\right)e^{-i\nu\xi/U}d\xi + \frac{U}{i\nu}e^{-i\nu/U} \tag{29}$$

The right side of (29) can be expressed as the sum of two modified Bessel functions of the second kind of the argument $iz = i\nu/U$, namely $K_o(iz) = \int_1^\infty e^{-iz\xi}\frac{d\xi}{\sqrt{\xi^2-1}}$ and

$K_1(iz) = -K_o'(iz)$ (cf. Ref. 2, p. 50, eq. (29) and p. 22, eq. (19)). This identification can be accomplished without encountering convergence difficulties by the following method: from equation (29),

$$-\frac{G_o}{g} = \int_1^\infty\left(\frac{1}{\sqrt{\xi^2-1}} + \frac{\xi}{\sqrt{\xi^2-1}} - 1\right)e^{-iz\xi}d\xi + \frac{e^{-iz}}{iz}$$

$$= K_o(iz) + \int_1^\infty\left(\frac{\xi}{\sqrt{\xi^2-1}} - 1\right)e^{-iz\xi}d\xi + \frac{e^{-iz}}{iz} \tag{30}$$

Now $K_0(iz)$ can be written in the form

$$K_0(iz) = \int_1^\infty e^{-iz\xi} \frac{d\xi}{\sqrt{\xi^2-1}} - \int_1^\infty e^{-iz\xi} \frac{d\xi}{\xi} + \int_z^\infty e^{-it} \frac{dt}{t}$$

because the sum of the last two terms vanishes identically. Differentiating with respect to iz, then,

$$K_1(iz) = K_0'(iz)$$
$$= \int_1^\infty \left(\frac{\xi}{\sqrt{\xi^2-1}} - 1 \right) e^{-iz\xi} d\xi + \frac{e^{-iz}}{iz}$$

This identifies the second and third terms of (30) and the relation between the quasi-steady circulation and the wake vorticity becomes, in this case,

$$-\frac{G_0}{g} = K_0\left(\frac{i\nu}{U}\right) + K_1\left(\frac{i\nu}{U}\right)$$

or
$$g = \frac{-G_0}{K_0\left(\frac{i\nu}{U}\right) + K_1\left(\frac{i\nu}{U}\right)} \tag{31}$$

In any given case of periodic motion, $\gamma_0(x)$ and Γ_0 can be easily calculated, and they determine directly the first two terms of the expressions for lift and moment, equations (21) and (25). After substitution of $\gamma(\xi)$ from (27) and g from (31), the third term of (21) becomes

$$L_2 = -\rho U \frac{G_0 e^{i\nu t}}{K_0\left(\frac{i\nu}{U}\right) + K_1\left(\frac{i\nu}{U}\right)} \int_1^\infty e^{-i\nu\xi/U} \frac{d\xi}{\sqrt{\xi^2-1}}$$

or

$$L_2 = -\rho U \Gamma_o \frac{K_o(\frac{i\nu}{U})}{K_o(\frac{i\nu}{U}) + K_1(\frac{i\nu}{U})} \tag{32}$$

The corresponding moment is $M_2 = -L_2/2$.

These results will now be applied to the case of an airfoil performing (1) translatory oscillations normal to the flight direction, and (2) rotational oscillations around its midpoint.

Case 1: Translatory Oscillation

For this case the vertical velocity of every point of the airfoil can be written as

$$w = A_o U e^{i\nu t} \tag{33}$$

where A_o is a constant and w is taken as positive downward. The quasi-steady portion of the vorticity depends only on the instantaneous relative velocity of the air and the airfoil, and therefore the quasi-steady quantities can be calculated by the formulae of Ref. 1, Chapter 2, by replacing v_y by $-w$. For the present case (from Ref. 1, p. 38, equations (7.8) and (7.9), putting $c = 2$)

$$\Gamma_o = 2\pi U A_o e^{i\nu t}$$

$$\gamma_o = 2UA_o e^{i\nu t} \frac{1-\cos\theta}{\sin\theta} \tag{34}$$

The three parts of the lift, as in equation (21), are therefore

$$L_0 = 2\pi\rho U^2 A_0 e^{i\nu t}$$

$$L_1 = -2\rho U A_0 i\nu e^{i\nu t}\int_0^{\pi}(1-\cos\theta)\cos\theta \, d\theta$$

$$= \pi\rho U A_0 i\nu e^{i\nu t}$$

and, from (32),

$$L_2 = -2\pi\rho U^2 A_0 e^{i\nu t}\frac{K_0(\frac{i\nu}{U})}{K_0(\frac{i\nu}{U}) + K_1(\frac{i\nu}{U})}$$

The total lift is therefore

$$L = 2\pi\rho U^2 A_0 e^{i\nu t}\left\{\frac{K_1(\frac{i\nu}{U})}{K_0(\frac{i\nu}{U}) + K_1(\frac{i\nu}{U})} + \frac{i\nu}{2U}\right\} \tag{35}$$

It is seen that L_1 is equal to the product of the acceleration, $A_0 U i\nu e^{i\nu t}$, and the apparent mass of a flat plate of length = 2, which is equal to $\pi\rho$.

The apparent-mass contribution to the moment, M_1, vanishes in this case, since the motion is purely translatory and the apparent-mass lift acts through the center of the airfoil. This can be verified by substituting $\gamma_0(x)$ from (34) into the first term of (25) and integrating. The quasi-steady lift in this case acts at the quarter-chord point ($x = -1/2$), as does L_2. The total moment is therefore

$$M = -\frac{1}{2}(L_o + L_2)$$

$$= -\pi \rho U^2 A_o e^{i\nu t} \frac{K_1\left(\frac{i\nu}{U}\right)}{K_o\left(\frac{i\nu}{U}\right) + K_1\left(\frac{i\nu}{U}\right)} \tag{36}$$

Case 2: Rotational Oscillation

In this case the vertical velocity of any point of the airfoil at any instant is proportional to the distance, x, of the point from the midpoint. Since $x = \cos\theta$, the vertical velocity may be written

$$w(\theta) = 2UA_1 e^{i\nu t} \cos\theta \tag{37}$$

where A_1 is a constant. In this case the formulae of Ref. 1 lead to the following expressions for the quasi-steady quantities:

$$\Gamma_o = 2\pi U A_1 e^{i\nu t}$$
$$\gamma_o = 4UA_1 e^{i\nu t} \sin\theta \tag{38}^*$$

By substitution into (21) and (32), then,

$$L_1 = -4\rho U A_1 i\nu e^{i\nu t} \int_0^{\pi} \sin^2\theta \cos\theta \, d\theta = 0$$

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*The details of the calculations leading to equations (38) are given in Appendix #2.

$$L_0 = 2\pi\rho U^2 A_1 e^{i\nu t}$$

$$L_2 = -2\pi\rho U^2 A_1 e^{i\nu t} \frac{K_0\left(\frac{i\nu}{U}\right)}{K_0\left(\frac{i\nu}{U}\right) + K_1\left(\frac{i\nu}{U}\right)}$$

so the total lift in this case becomes

$$L = 2\pi\rho U^2 A_1 e^{i\nu t} \frac{K_1\left(\frac{i\nu}{U}\right)}{K_0\left(\frac{i\nu}{U}\right) + K_1\left(\frac{i\nu}{U}\right)} \tag{39}$$

Also, by substitution into (25),

$$M_1 = -2\rho U A_1 i\nu e^{i\nu t} \int_0^\pi \sin^2\theta \left(\cos^2\theta - \tfrac{1}{2}\right) d\theta$$

$$= -2\rho U A_1 i\nu e^{i\nu t} \left\{\frac{\pi}{8} - \frac{\pi}{4}\right\}$$

$$= \frac{\pi}{4} \rho U A_1 i\nu e^{i\nu t}$$

$$M_0 = 4\rho U^2 A_1 e^{i\nu t} \int_0^\pi \sin^2\theta \cos\theta \, d\theta = 0$$

$$M_2 = -\tfrac{1}{2} L_2$$

Therefore, the total moment is

$$M = \pi\rho U^2 A_1 e^{i\nu t} \left\{ \frac{K_0\left(\frac{i\nu}{U}\right)}{K_0\left(\frac{i\nu}{U}\right) + K_1\left(\frac{i\nu}{U}\right)} + \frac{i\nu}{4U} \right\} \tag{40}$$

In a similar manner, the lift and moment on the airfoil may be calculated for the cases of periodic deformations represented by higher values of n in the general expression for the vertical velocity of any point of the airfoil:*

$$w(\theta) = U e^{i\nu t}\left\{A_0 + 2\sum_{n=1}^{\infty} A_n \cos n\theta\right\} \tag{41}$$

The results, including those of Cases 1 and 2 above, agree with those obtained by Küssner in Ref. 3.**

The physical significance of the complex forms of (35), (36), (39), and (40) may be clarified by means of "vector diagrams" which show the phase relationships of the quantities involved as well as their magnitudes. Each of these results may be abbreviated as

$$f(t) = F e^{i\nu t}\left\{f_1\left(\frac{\nu}{U}\right) + i\, f_2\left(\frac{\nu}{U}\right)\right\} \tag{42}$$

where f(t) represents the lift or moment; F is a constant involving only the dynamic pressure, $\rho U^2/2$, and the amplitude of the

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*These calculations are carried out in Appendix #5.

**In comparing Küssner's results, it should be noted that

$$\pi H_n^{(2)}(z) = 2\, i^{(n+1)} K_n(iz)$$

Also Küssner's $(-i\omega)$ is the same as (ν/U) here, and, because of the difference in definition of θ, his P_n is $(-)^n A_n$ of the present paper.

oscillation; and f_1 and f_2 are real functions. The real part of this expression (which is the actual force or moment) may be written as

$$\begin{aligned} \mathcal{R}[f(t)] &= F \cdot \left\{ f_1 \cos \nu t - f_2 \sin \nu t \right\} \\ &= F \cdot \sqrt{f_1^2 + f_2^2} \ \cos(\nu t + \varphi) \end{aligned} \tag{43}$$

where $\varphi = \tan^{-1}(f_2/f_1)$. Thus, in vector representation, the lift or moment vector has the magnitude $F \cdot \sqrt{f_1^2 + f_2^2}$ and leads the vector of the oscillating velocity, w, by a phase angle, φ. In Fig. 5 is given an example, taken from Case 1, above, which shows schematically how the total lift vector is composed of the vectors L_o, L_1, and L_2 for a certain value of ν/U. The quasi-steady part, L_o, being in phase with the velocity, appears as a horizontal vector, while the vector, L_2, tends to diminish the lift and cause it to lag behind the velocity. The apparent-mass lift, L_1, being proportional to the acceleration, is directed vertically, i.e., leads the velocity by 90°. The total lift, L, is the sum of these three vectors, and has the phase angle φ.

In Fig. 6 are plotted "vector diagrams" which give the magnitudes of the lift and moment together with their phase angles for various values of the reduced frequency ν/U (or $c\nu/2U$ for an airfoil of chord $= c$).* In these diagrams the length of the vector drawn from the origin to the appropriate value of $c\nu/2U$ on the curve gives the maximum value of the total lift or moment (referred to that of the corresponding quasi-steady quantity, L_0 or M_0), and its angle with the horizontal axis gives the phase angle relative to w. It is seen that as the frequency of the translatory oscillation (Case 1) is increased from $c\nu/2U = 0$ (uniform motion) the maximum value or amplitude of the lift at first steadily decreases, and the lift vector lags slightly behind the vertical-velocity vector, w. These effects are produced by the wake contribution, L_2. With further increase of the frequency, however, the apparent-mass contribution, L_1, which is proportional to the acceleration, becomes very large, and the lift vector leads the velocity vector. In the limit $c\nu/2U \rightarrow \infty$ the sum $(L_0 + L_2)$ is equal to half of L_0, but $L_1 \rightarrow \infty$, and the lift vector leads w by 90°. Since the apparent-mass lift, L_1, acts through the midpoint of the airfoil in this case, the limiting value of the moment is also half of its

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*Küssner (Ref. 3, p. 416), gives a table of Bessel functions involved in the calculation of Fig. 6.

steady-motion value. In the rotational case (Case 2) the total lift behaves exactly like the moment of Case 1 as the frequency is increased, while the apparent-mass moment contribution, M_1, increases proportionally to the frequency.

It is believed that the method of representation of lift and moment by vector diagrams will be useful in the discussion of flutter problems because both the elastic restoring forces and the inertia forces of the wing can be introduced in such diagrams.

The results presented in Fig. 6 are applicable to cases of bending-torsional wing flutter. The calculation of aileron or rudder flutter requires the expression of the non-steady forces acting on the aileron or rudder in a similar way. The aileron or rudder constitutes one portion of a wing or a fin, while the equations presented in this Section give only the lift and moment acting on the wing as a whole. However, by determining the vorticity distribution produced by the wake, similar equations can be deduced for the non-steady normal force on the aileron and for the non-steady hinge moment.

V. THE FUNDAMENTAL TRANSIENT CASE; THE LIFT AND MOMENT RESULTING FROM A SUDDEN CHANGE OF ANGLE OF ATTACK

There are many cases in which knowledge of the forces produced by a transient phenomenon is of practical interest. Examples of such cases are the reaction of an airplane to certain control operations (aileron or rudder deflection, etc.), and the behavior of an airplane encountering gusts. In the second case, an estimate of the forces acting on the wing is of importance, as well as the reaction of the airplane as a whole, in view of strength requirements. The results of the preceding Section, which was concerned only with steady-state problems, can be used to calculate the lift and moment in the fundamental transient case; i.e., the case of a sudden increment in Γ_o, the quasi-steady circulation. Since the quasi-steady circulation, as given by the stationary airfoil theory, is proportional to the product of the velocity and the angle of attack of the airfoil, the increment applied to Γ_o may be supposed to represent either a sudden increase of angle of attack while the airfoil moves at a constant velocity or a sudden change of velocity with the attitude being unchanged.

Since the quasi-steady quantity Γ_o and the wake vorticity function $\gamma(\xi)$ are related by equation (15) (page 11), the present problem might be attacked directly by attempting to solve this integral equation with the specified behavior of Γ_o. This, in fact, is the method used by Wagner (Ref. 4), who obtains an approximate solution which is valid for small values of s, the distance travelled by the airfoil after the sudden disturbance, and is also the method used in Section VI and Appendix #3 of the present paper. The method used here is equivalent to that used by Küssner (Ref. 3), although there is a certain difference in interpretation, as will be pointed out later.

The chord of the airfoil is again taken equal to 2, and it is supposed that the velocity of flight, U, is constant. The angle of attack is increased suddenly, at the instant t = 0, from zero to a constant value; i.e., if w is the normal velocity of the airfoil relative to the fluid, as in the preceding Section, then w = 0 for t < 0, and $w = w_0$ (a constant) for $t > 0$. This discontinuous function of the time can be expressed by means of the Dirichlet "discontinuous factor" in the form

$$w(t) = w_0 \left\{ \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{e^{i\nu t}}{i\nu} d\nu \right\} \tag{44}$$

In this expression the complex-variable notation has been employed, as in the preceding Section, i.e., the real part of the integral represents the desired discontinuous angle-of-attack function, and the imaginary part will be discarded in the final results.

In equation (44) the function w(t) has been expressed in a form which is equivalent to the superposition of a constant normal velocity, $w_0/2$, and an infinite number of sinusoidal velocities of the form $w(t) = \frac{w_0}{\pi i \nu} e^{i\nu t}$, with all frequencies between 0 and ∞. Hence the results of the case of translatory oscillation in the last Section can be applied to the calculation of the lift and the moment. By comparison of equations (33) (page 27) and (44), it is seen that in this case the total lift, from (35), becomes in this case

$$L = 2\pi\rho U w_0 \left\{\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{e^{i\nu t}}{i\nu}\left(\frac{K_1(\frac{i\nu}{U})}{K_0(\frac{i\nu}{U})+K_1(\frac{i\nu}{U})} - \frac{i\nu}{2U}\right)d\nu\right\} \tag{45}$$

This can be written

$$L = L_0\left\{\frac{1}{2} + F(t)\right\} + L_1(t) \tag{46}$$

where $L_0 = 2\pi\rho U w_0$, the quasi-steady lift.

$$F(t) = \frac{1}{\pi}\int_0^\infty \frac{e^{i\nu t}}{i\nu}\left(\frac{K_1(\frac{i\nu}{U})}{K_0(\frac{i\nu}{U})+K_1(\frac{i\nu}{U})}\right)d\nu$$

$$L_1(t) = -\frac{1}{\pi}\int_0^\infty \frac{e^{i\nu t}}{i\nu}\,\frac{i\nu}{2U}\,d\nu$$, the lift arising from the apparent mass.

It is apparent that $L_1(t)$ is indeterminate in the form above, but it can easily be shown that this portion of the lift vanishes for $t > 0$ by consideration of the corresponding part of the total momentum, which is given by the first term of the right side in equation (18) (page 12). This contribution is equal to $I_1 = \rho\int_{-1}^{1}\gamma_0(x)x\,dx$, and, substituting γ_0 for the case being considered (cf. equation

(34)) this becomes

$$I_1 = \rho \int_{-1}^{1} \gamma_0(x)\, x\, dx = \frac{2}{\pi} w_0 \rho \int_0^\infty \frac{e^{i\nu t}}{i\nu} d\nu \int_0^\pi (1-\cos\theta)\cos\theta\, d\theta$$

$$= -w_0 \rho \int_0^\infty \frac{e^{i\nu t}}{i\nu} d\nu \tag{47a}$$

If t > 0, the real part of this expression is simply

$$\mathcal{R}(I_1) = -\frac{\pi}{2}\rho w_0 \tag{47}$$

which is independent of the time, and therefore $L_1 = 0$.

Hence there remains only the problem of evaluating the function F(t) in equation (46). Introducing Ut = s, the distance travelled after the change of angle, and then replacing ν/U by z, this becomes

$$F(s) = \frac{1}{\pi} \int_0^\infty \frac{e^{isz}}{iz} \left(\frac{K_1(iz)}{K_0(iz) + K_1(iz)} \right) dz \tag{48}$$

Following Küssner, we may write

$$\frac{K_1(iz) - K_0(iz)}{K_1(iz) + K_0(iz)} = T(z) = T'(z) + i T''(z) \tag{49}$$

where T'(z) and T"(z) are real functions. If this substitution is made, the function F(s) becomes

$$F(s) = \frac{1}{2\pi}\int_0^{\infty}\frac{e^{isz}}{iz}\left[1+T'(z)+iT''(z)\right]dz \tag{50}$$

and the real part, which is to be determined, is

$$\mathcal{R}\left[F(s)\right] = \frac{1}{2\pi}\int_0^{\infty}\left\{\left[1+T'(z)\right]\sin sz + T''(z)\cos sz\right\}\frac{dz}{z}$$

$$= \frac{1}{4} + \frac{1}{2\pi}\int_0^{\infty}\left\{T'(z)\sin sz + T''(z)\cos sz\right\}\frac{dz}{z} \tag{51}$$

This infinite integral is found to be rapidly convergent, and has been evaluated approximately for several values of s by planimetry*, using the values of T' and T" tabulated in Ref. 3. The resulting curve for the total lift, L, as in equation (46), is presented in Fig. 7.

In Fig. 7 it is seen that the lift attains half of its final, or quasi-steady, value instantaneously, and then gradually approaches the final value as the distance s increases. When s = 10, i.e., when the airfoil has

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*Küssner (loc. cit.) has evaluated an infinite integral analogous to that in (51) above by a very elegant series expansion.

progressed 5 chord lengths after the sudden disturbance, the lift has attained 87% of the quasi-steady value. The lift curve is in good agreement with the corresponding result in Wagner's classical paper (loc. cit.) and with the function obtained by Küssner (loc. cit.) for the lift due to circulation. Küssner does not consider the apparent-mass contributions, which have been shown here to vanish for $s > 0$.

Since $L_1 = 0$ for $s > 0$, the entire lift acts at the quarter-chord point of the airfoil in this case (cf. pp. 28 and 29), and the moment about the center of the airfoil is given by $M = -L/2$ where L is the lift as calculated above.

VI. A GENERAL METHOD FOR TRANSIENT CASES

A general method of attacking transient problems may be developed once the effects of a sudden disturbance have been calculated. Although the results obtained in Section V will be employed ultimately, it is desirable at this point to consider the sudden-disturbance problem from the standpoint of the integral equation, (15). If the velocity of flight, U, is constant, and the quasi-steady circulation Γ_o is suddenly increased at the instant $t = 0$ from zero to unity and then held constant, the wake extends only from $\xi = 1$ to $\xi = 1 + Ut$, and equation (15) becomes

$$\Gamma_o = 1 = -\int_1^{1+Ut} \gamma(\xi)\sqrt{\frac{\xi+1}{\xi-1}}\,d\xi \tag{52}$$

The vortex strength in the wake is a function of the distance s from the endpoint of the wake $s = 1 + Ut - \xi$, hence it may be written as $\gamma(\xi) = \mu\,(1 + Ut - \xi)$. Equation (52) represents an integral equation for μ in the form

$$-\int_{1}^{1+Ut} \mu(1+Ut-\xi)\sqrt{\frac{\xi+1}{\xi-1}}\, d\xi = 1$$

or, introducing the variable s,

$$-\int_{0}^{Ut} \mu(s)\frac{\sqrt{2+Ut-s}}{\sqrt{Ut-s}}\, ds = 1 \tag{53}$$

The function $\mu(s)$ has been determined by Wagner (Ref. 4).

For the following applications, the main problem is the calculation of L_2, the contribution of the wake. For the case being considered, L_2 is equal to (from equation (21))

$$\rho U \int_{1}^{1+Ut} \gamma(\xi)\frac{d\xi}{\sqrt{\xi^2-1}} = \rho U \int_{1}^{1+Ut} \mu(1+Ut-\xi)\frac{d\xi}{\sqrt{\xi^2-1}} \tag{54}$$

where the function $\mu(s)$ is again the solution of the integral equation (53). The function

$$\Phi(Ut) = -\int_{1}^{1+Ut} \mu(1+Ut-\xi)\frac{d\xi}{\sqrt{\xi^2-1}} \tag{55}$$

will be called the lift deficiency function. Since $\rho U \Phi$ represents the difference between the instantaneous and final values of the lift in the case of sudden unit increase of the quasi-steady circulation, Φ is obviously the difference between unity and the function $L/2\pi\rho U w_o$ plotted in Fig. 7. It is a function of the distance Ut travelled by the airfoil since the change of circulation took place.

It is evident that the function Φ can be used to calculate the lift acting on an airfoil which is subjected to an arbitrary transient variation of the quasi-steady circulation Γ_o. Assuming that Γ_o changes at the instant τ by the increment $\Delta\Gamma_o = \Gamma_o'(\tau)\Delta\tau$, the deficiency in lift at t = t, i.e., after the airfoil has travelled a distance $U(t - \tau)$, will be $\Gamma_o'(\tau)\cdot\Phi[U(t-\tau)]\cdot\Delta\tau$, and the total deficiency in lift will be given by

$$-L_2 = \rho U \int_0^t \Gamma_o'(\tau)\cdot\Phi[U(t-\tau)]\,d\tau \qquad (56)^*$$

In this equation it is assumed that $\Gamma_o(0) = 0$. If, at the instant t = 0, Γ_o is suddenly increased from 0 to $\Gamma_o(0)$, a term equal to $\Gamma_o(0)\cdot\Phi(Ut)$ is to be added to the right-hand side of equation (56).

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*This is an example of the so-called DuHamel "superposition" integral.

Since the elementary case of a sudden increment in Γ_o has already been treated, and the function $1 - \Phi$ is known, the lift and moment can be calculated by (21) and (25), for a given $\gamma_o(x,t)$ distribution, because L_o, L_1, M_o, M_1 are determined by γ_o and its time derivative, L_2 is given by (56), and $M_2 = -L_2/2$.

The function $\Phi(Ut)$ has not been obtained in analytical form; however, it can be fairly closely approximated by the following relatively simple formulae which are chosen so as to facilitate subsequent calculations:

a) for $0 \leqq \sigma \leqq 2$ the following power series can be used:*

$$\Phi(\sigma) = \frac{1}{2} - \frac{\sigma}{8} + \frac{\sigma^2}{32} - 0.00554\,\sigma^3 \qquad (57)$$

b) for $0 \leqq \sigma \leqq 10$

$$\Phi(\sigma) = \frac{1}{4} e^{-\sigma/2} + \frac{1}{4}\left(1 + 0.185\,\sigma\right) e^{-0.185\,\sigma} \qquad (58)$$

For the actual calculations $1 - \Phi$ is used, and since $\Phi(\sigma)$ is very small for $\sigma > 10$, equation (58) can be applied for $0 \leqq \sigma \leqq \infty$ without introducing appreciable errors.

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*Formula (57) has been obtained by an approximate solution of equation (53). The details are presented in Appendix #3. It will be noted that the form of (58) has been chosen to agree with (57) in initial slope; i.e., $\Phi'(0) = -1/8$ (which is found in Appendix #3 to be the correct value).

In Table 1 the approximate values for $\Phi(\sigma)$ are shown in comparison with those given by Wagner (Ref. 4), which are assertedly correct in the first four digits for $0 \leqq \sigma \leqq 10$.

TABLE 1

Approximations to Φ

σ	Wagner	Eq. (57)	Eq. (58)
0	1/2	1/2	1/2
1/2	0.4443	0.4446	0.444
1	0.3994	0.4008	0.398
2	0.3307	0.3307	0.329
4	0.2418	--	0.242
10	0.1255	--	0.114
20	0.0679	--	0.029

The approximations given in (57) and (58) will be used in the application of the next Section.

VII. APPLICATION TO THE CASE OF A SHARP-EDGED GUST

The results of the preceding Section may be applied to the problem of a flat-plate airfoil entering a sharp-edged vertical gust. The flow in this case is not a potential one because the gust boundary itself constitutes a vortex sheet. It will be assumed that in spite of this fact the thin-airfoil theory can be applied to the calculation of the quasi-steady quantities $\gamma_0(x)$ and Γ_0. This corresponds to the assumption that the principle of superposition of flow patterns is applicable. Strictly speaking, this is only the case for potential motions; however, the method yields results which are probably sufficiently exact in the present case, provided that all additional velocities are small so that the deformation of the vortex sheet can be neglected.

Suppose that the leading edge of the airfoil reaches the gust boundary at the instant $t = 0$. Then at the time t, the relative transverse velocity is equal to V (cf. Fig. 8) between $x = -1$ and $x = -1 + Ut$, and it is equal to zero for $x > -1 + Ut$. The vorticity distribution $\gamma_0(x)$ produced by these velocities is the same as that of the broken-line airfoil represented in Fig. 8 in a parallel stream. Applying the equations of the thin-airfoil theory to this case, the following formulae are obtained:*

$$\gamma_o(x,t) = \frac{\Gamma_o}{\pi \sin\theta} + \sum_{k=1}^{\infty} a_k \frac{\cos k\theta}{\sin\theta} \tag{59}$$

where $\cos\theta = x$ and

$$a_k = -\frac{4V}{\pi}\int_{\cos^{-1}(Ut-1)}^{\pi} \sin\theta \sin k\theta \, d\theta \tag{60}$$

and

$$\Gamma_o(t) = 2V\left\{\cos^{-1}(1-Ut) - \sqrt{2Ut - U^2t^2}\right\} \tag{61}$$

These formulae apply to the interval $0 \leqq Ut \leqq 2$ during which the airfoil crosses the boundary of the gust. For $Ut > 2$ the airfoil is entirely within the gust, the transverse velocity is constant, and $\Gamma_o = 2\pi V$ and $\gamma_o(x) = \frac{\Gamma_o}{\pi}\sqrt{\frac{1-x}{1+x}}$, both independent of the time. The

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*The details of the calculations leading to equations (59)-(61) are included in Appendix #2.

two ranges are now considered separately:

a) $0 \leqq Ut \leqq 2$

The apparent-mass terms are readily obtained using (59)-(61). The integrals are

$$\int_{-1}^{1} \gamma_0(x)\, x\, dx = \frac{\Gamma_0}{\pi}\int_0^{\pi}\cos\theta\, d\theta + \sum_{k=1}^{\infty} a_k \int_0^{\pi}\cos k\theta \cos\theta\, d\theta$$

$$= \frac{\pi}{2} a_1 = -2V\int_{\cos^{-1}(Ut-1)}^{\pi} \sin^2\theta\, d\theta$$

$$= -V\left\{\pi - \cos^{-1}(Ut-1) + (Ut-1)\sqrt{2Ut-U^2t^2}\right\} \qquad (62)$$

$$\int_{-1}^{1} \gamma_0(x)(x^2-\tfrac{1}{2})\, dx = \frac{1}{2}\int_0^{\pi} \gamma_0(\theta)\cos 2\theta\, d\theta$$

$$= \frac{\Gamma_0}{2\pi}\int_0^{\pi}\cos 2\theta\, d\theta + \frac{1}{2}\sum_{k=1}^{\infty} a_k \int_0^{\pi}\cos k\theta\cos 2\theta\, d\theta$$

$$= \frac{\pi}{4} a_2 = -V\int_{\cos^{-1}(Ut-1)}^{\pi} \sin\theta \sin 2\theta\, d\theta$$

$$= \frac{2}{3}\left(2Ut - U^2t^2\right)^{3/2} \qquad (63)$$

Hence, from (61) and (62), the sum of the first two terms of the lift, equation (21), is

$$L_0 + L_1 = -\rho\frac{d}{dt}\int_{-1}^{1}\gamma_0(x)\, x\, dx + \rho U \Gamma_0$$

$$= \rho V U\left\{\frac{1}{\sqrt{2Ut-U^2t^2}} + \sqrt{2Ut-U^2t^2} - \frac{(Ut-1)^2}{\sqrt{2Ut-U^2t^2}} + 2\cos^{-1}(1-Ut) - 2\sqrt{2Ut-U^2t^2}\right\}$$

$$= 2\rho UV \cos^{-1}(1-Ut) \tag{64}$$

and, from (62) and (63), the sum of the first two terms of the moment, equation (25), is

$$M_1+M_0 = -\frac{1}{2}\rho\frac{d}{dt}\int_{-1}^{1}\gamma_0(x)(x^2-\tfrac{1}{2})\,dx + \rho U\int_{-1}^{1}\gamma_0(x)\,x\,dx$$

$$= \rho UV\Big\{(Ut-1)\sqrt{2Ut-U^2t^2} - \cos^{-1}(1-Ut)$$

$$- (Ut-1)\sqrt{2Ut-U^2t^2}\Big\}$$

$$= -\rho UV\cos^{-1}(1-Ut) \tag{65}$$

It is seen immediately that $M_0 + M_1 = -(L_0 + L_1)/2$, and since it has already been proved that $M_2 = -L_2/2$, this means that the total lift acts at the forward quarter-chord ($x = -1/2$) at every instant. This result was predicted by Küssner (Ref. 3) and verified experimentally by him (Ref. 5), but he was unable to prove it theoretically because of an error in his fundamental equation for waves progressing over the airfoil. Because of an error in sign (Ref. 3, p. 420, eq. 60), these disturbances move over his airfoil from rear to front, which, of course, confuses the results.

The calculation of L_2 is carried out by introducing Φ from (57) and $\Gamma_o(t)$ from (61) into (56). This leads to the elementary integrals

$$\int_0^{Ut} \sqrt{\frac{\tau}{2-\tau}}\, \tau^n \, d\tau \qquad \text{for } n = 0, 1, 2, \text{ and } 3.$$

Hence L_2 is easily calculated,* and when the result is combined with $L_o + L_1$ from (64) the total lift becomes

$$L = 2\rho UV \left\{ \left[0.2103 + 0.2603(Ut) - 0.0562(Ut)^2 + 0.0055(Ut)^3\right] \cos^{-1}(1 - Ut) + \left[0.7897 - 0.1637(Ut) + 0.0247(Ut)^2 - 0.0014(Ut)^3\right] \sqrt{2Ut - U^2t^2} \right\} \tag{66}$$

where $0 \leq \cos^{-1}(1 - Ut) \leq \pi$. The result, (66), is plotted in Fig. 9.

b) $Ut > 2$

In this regime, since the airfoil is subjected to a constant transverse velocity, V, it is seen immediately that $L_1 = M_1 = 0$, $L_o = 2\pi\rho UV$, and $M_o = -L_o/2$. Hence the lift again acts at the quarter-chord point. In the calculation of L_2 by means of equation (56) the function Φ is taken from

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*The details of the calculation are given in Appendix #4.

(58) and the value of $\Gamma_o'(\tau)$ is that obtained from (61) for $0 \leqq Ut \leqq 2$ and is equal to zero for greater values of Ut. Hence the integrals which arise in this case are the following:

$$\int_0^2 \sqrt{\frac{\tau}{2-\tau}}\, e^{a\tau} d\tau = \pi e^a \left\{ I_o(a) + I_1(a) \right\}$$

and
$$\int_0^2 \sqrt{\frac{\tau}{2-\tau}}\, e^{a\tau} \tau\, d\tau = \pi e^a \left\{ 2I_o(a) + \left(2 - \frac{1}{a}\right) I_1(a) \right\}$$

where $I_n(a)$ is a modified Bessel function of the first kind (Ref. 2, p. 46). The total lift in this range is $L_o + L_2$, which finally becomes

$$L = 2\pi\rho UV \left\{ 1 - 0.3304\ e^{-(Ut-1)/2} - (0.1917 + 0.0510\ t) e^{-0.185(Ut-1)} \right\} \quad (67)$$

This result is also plotted in Fig. 9. It is seen that the lift on the airfoil increases rapidly after the entrance of the leading edge into the gust (Ut = 0), and is equal to 55% of its final value when the trailing edge reaches the gust boundary (Ut = 2). The increase then becomes progressively slower, and when the leading edge has progressed five chord-lengths into the gust (Ut = 10) the lift is 86% of its final value. Thus, for a wing of chord equal to 20 ft., flying at 200 m.p.h., the lift would reach 55% of its final value in 0.07 sec. and 86% of its final value in 0.34 sec. It should be noted that for $Ut > 10$ the lift is in error due to the approximation involved in the expression used for Φ, equation (58). The dotted curve in Fig. 9, which will be explained in Section IX, below, may be used for large values of Ut.

It is known that the vertical gusts which actually occur in the atmosphere are not exactly sharp edged, but consist of a smooth, although rapid, transition of vertical velocity. The rate of increase of lift on an airfoil entering such a smoothly-graded gust can easily be calculated by means of another "superposition" integral using the curve of Fig. 9 for the response of the wing to a sudden disturbance. An example of such a calculation is given in the following Section.

VIII. EXTENSION TO THE CASE OF A GRADED GUST

In this Section the results obtained in the preceding pages will be employed in the calculation of the lift produced by an airfoil flying through the boundary of a gust which more closely resembles actual atmospheric gusts, i.e., in which infinite velocity gradients do not appear.

The graded gust may be considered to be composed of a continuous succession of small sharp-edged gusts. Making use of the "superposition" integral again, the lift may be written in the form

$$L(s) = \int_0^s F'(\sigma)\, \ell(s-\sigma)\, d\sigma \tag{68}$$

where s = the distance travelled by the airfoil after the leading edge reached the boundary of the gust. The velocity of flight, U, is assumed to be constant, so that s = Ut.

F(s) = the "gust velocity profile"; i.e., the function giving the vertical gust velocity at s. It is assumed that the gust velocity at any point is small compared to the velocity of flight, U.

$\ell(s)$ = the lift resulting from the entrance of the airfoil into a unit sharp-edged gust.

It is obvious that, if the chord is again taken equal to 2, the function $\ell(s)$ is exactly L(Ut) of formulae (66) and (67), as plotted in Fig. 9, for the gust velocity V = 1. It has been suggested by Küssner (Ref. 3) that for the gust function F(s) the results of Tollmien (Ref. 6) be used. These results give the velocity profile in the "mixing region" between a uniform jet and the surrounding stationary fluid according to Prandtl's theories of turbulent mixing.* It is believed that gusts of this type constitute the most logical available approximation to actual atmospheric gusts. Atmospheric gust profiles have been observed (e.g., Fig. 1 of Ref. 7) which are definitely similar to Tollmien's velocity profiles.

Tollmien's results are given by Küssner in a form equivalent to the following, which is convenient for the present application: the gust profile is

$$F(s) = w_0 \, f\left(\frac{s}{b}\right) \tag{69}$$

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*Calculations of the lift for such gusts have been carried out by Küssner (loc. cit.), but they are based on the erroneous results for the sharp-edged gust mentioned on page 50.

where w_o is the maximum vertical gust velocity, i.e., the jet velocity, f is a function such that $f(0) = 0$ and $f(1) = 1$, and b is the breadth of the mixing region. The gust profile, F(s), has been plotted in Fig. 10 for $b = 1$ and $b = 2$.

By differentiation of equation (69) one obtains

$$F'(s) = \frac{1}{b} w_o f'\left(\frac{s}{b}\right) = \frac{1}{b} w_o G\left(\frac{s}{b}\right) \tag{70}$$

where $f' = G$, the function tabulated in Table 4 of Kussner's paper (Ref. 3). The lift, from equation (68) is then

$$L(s) = \frac{w_o}{b} \int_0^s G\left(\frac{\sigma}{b}\right) \ell(s-\sigma)\, d\sigma \tag{71}$$

This expression has been evaluated by graphical integration for several values of s for two values of b. The results are presented in Fig. 10. It is seen that the initial rate of growth of the lift after the entrance of the leading edge into the gust is considerably less than in the case of the sharp-edged gust, but that as s increases the curves for the graded gusts are similar to the curves for the sharp-edged gust, the only appreciable difference being a slight displacement along the scale of abscissae.

Küssner (Ref. 5) has published experimental results obtained by dropping a model wing into the jet of a small wind tunnel and observing photographically its resulting motion. An attempt was made to verify the theoretical results for the rate of growth of the lift by comparing experimental and theoretical values of the radius of curvature of the path of the model airfoil. As has already been mentioned, Küssner's theoretical calculations are erroneous, and it is interesting to compare his experimental results with the results obtained here. In the experiments the value of b for the jet-boundary profile was 1.4. The theoretical values of $L/\pi \rho U w_0$ are obtained from Fig. 10 by interpolation, and Küssner's calculated radii of curvature (Table 4, Ref. 5) are increased in the ratio of Küssner's $g_2(\zeta s)$ to $L/\pi \rho U w_0$. The results are presented in Table 2 below. It is seen that the theoretical radii are less than those observed by an amount (11%-18%), which is probably attributable to effects of the finite aspect ratio of the model.

TABLE 2

s, measured in half-chords	1.05	1.61	2.98
$g_2(\zeta s)$ (Küssner)	0.41	0.83	1.28
Theo. radius of curv. (Küssner), cm.	28.4	14.25	9.57
$L/\pi\rho U w_o$ (Fig. 10, interpolated for b = 1.4)	0.35	0.72	1.15
Theo. radius of curv. (Sears), $R_{theo.}$, cm.	33.3	16.4	10.6
Experimental radius of curv., $R_{exp.}$, cm.	37.5	19.3	12.9
$R_{theo.}/R_{exp.}$	0.89	0.85	0.82

Since it was found in Section VII that all the lift produced by a sharp-edged gust acts at the quarter-chord point of the airfoil, it has been unnecessary to consider the moment acting on the airfoil in the present Section. It is obvious from the reasoning which led to equation (68) that the total lift will act at the quarter-chord point, regardless of the gust profile encountered.

With regard to the applicability of these results to actual aeronautical problems, it should be pointed out that the most important effects which have been neglected here, in addition to the effects of finite span, are those resulting from the elastic deflections of the wing and the motion of the airplane as a whole after entering the gust. Calculations including both effects, but using Wagner's elementary function, $1 - \Phi$, for the rate of build-up of lift (i.e., neglecting the fact that the entire chord does not strike the gust boundary at once), have been published in Ref. 8. The results of the present paper now provide a better foundation for such calculations.

IX. THE LIFT AND MOMENT PRODUCED BY A SERIES OF SINUSOIDAL GUSTS; APPLICATION TO THE CASE OF A SHARP-EDGED GUST

A calculation of the lift and moment on an airfoil flying through a sinusoidal gust pattern has two possibilities of interest: first, there is the possibility that such a series of upward and downward gusts, occurring with the proper frequency, may produce large forces on the airfoil; and second, it leads to an alternative method of calculating the effects of sharp-edged and graded gusts. The sinusoidal gust pattern is again not a possible potential motion, hence the assumption made at the beginning of Section VII (cf. p. 47), i.e., that the stationary airfoil theory is still applicable to the calculation of quasi-steady quantities, must also be made for this case.

If x is the coordinate along the flight path, measured in the direction of the trailing edge from an origin fixed in the center of the airfoil, and U is the flying speed, supposed to be constant, the vertical gust velocity may be written, in complex-variable notation, in the form

$$v(x,t) = We^{i\nu(t-x/U)} \tag{72}$$

This equation expresses the fact that the sinusoidal gust pattern, with maximum up or down velocity equal to W (a constant), moves past the airfoil with the speed of flight, U. If the wave length of the gusts is L_W (measured in half-chords), the frequency, ν, with which the sinusoidal waves pass any point of the airfoil is

$$\nu = 2\pi U/L_W \tag{73}$$

For $-1 < x < 1$, i.e., on the airfoil, the coordinate x may be replaced by cos θ, as before. Then, if the gust velocity, v(x,t), is positive upward, the relative vertical velocity of any point of the airfoil, measured positive downward, is

$$w(\theta) = We^{i\nu t}e^{-\frac{i\nu}{U}\cos\theta} \tag{74}$$

Now this may be put into the general form

$$w(\theta) = Ue^{i\nu t}\left\{A_0 + 2\sum_{n=1}^{\infty} A_n \cos n\theta\right\}$$ (41)
(cf. p. 31)

for which the lift and moment have been calculated in Appendix #5, for

$$e^{iz\cos\theta} = J_0(z) + 2\sum_{n=1}^{\infty} i^n J_n(z)\cos n\theta \tag{75}$$ *

and therefore, putting $z = -\nu/U$, equation (74) becomes

$$w(\theta) = We^{i\nu t}\left\{J_0\left(-\frac{\nu}{U}\right) + 2\sum_{n=1}^{\infty} i^n J_n\left(-\frac{\nu}{U}\right)\cos n\theta\right\}$$

or, since $J_n(-z) = (-)^n J_n(z)$,**

$$w(\theta) = We^{i\nu t}\left\{J_0\left(\frac{\nu}{U}\right) + 2\sum_{n=1}^{\infty}(-i)^n J_n\left(\frac{\nu}{U}\right)\cos n\theta\right\} \tag{76}$$

The formulae for the lift and moment for $w(\theta)$ as given by equation (41), above, are (from Appendix #5)

$$L = 2\pi\rho U^2 e^{i\nu t}\left\{\frac{K_1\left(\frac{i\nu}{U}\right)}{K_0\left(\frac{i\nu}{U}\right)+K_1\left(\frac{i\nu}{U}\right)}(A_0+A_1) + \frac{i\nu}{2U}(A_0-A_2)\right\} \tag{77}$$

$$M = -\pi\rho U^2 e^{i\nu t}\left\{\frac{A_0K_1\left(\frac{i\nu}{U}\right)-A_1K_0\left(\frac{i\nu}{U}\right)}{K_1\left(\frac{i\nu}{U}\right)+K_0\left(\frac{i\nu}{U}\right)} - \frac{i\nu}{4U}(A_1-A_3) - A_2\right\} \tag{78}$$

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*cf. Ref. 2, p. 32, eq. (6).

**This is apparent from the definition of J_n given in Ref. 2, p. 14, eq. (16).

By comparison of equations (41) and (76), it is seen that the lift and moment in the present case are

$$L = 2\pi\rho U W e^{i\nu t}\left\{\frac{K_1\left(\frac{i\nu}{U}\right)}{K_0\left(\frac{i\nu}{U}\right)+K_1\left(\frac{i\nu}{U}\right)}\left[J_0\left(\frac{\nu}{U}\right)-i\,J_1\left(\frac{\nu}{U}\right)\right] + \frac{i\nu}{2U}\left[J_0\left(\frac{\nu}{U}\right)+J_2\left(\frac{\nu}{U}\right)\right]\right\} \tag{79}$$

and

$$M = -\pi\rho U W e^{i\nu t}\left\{\frac{J_0\left(\frac{\nu}{U}\right)K_1\left(\frac{i\nu}{U}\right)+i\,J_1\left(\frac{\nu}{U}\right)K_0\left(\frac{i\nu}{U}\right)}{K_1\left(\frac{i\nu}{U}\right)+K_0\left(\frac{i\nu}{U}\right)} + \frac{i\nu}{4U}\left[i\,J_1\left(\frac{\nu}{U}\right)+i\,J_3\left(\frac{\nu}{U}\right)\right]+J_2\left(\frac{\nu}{U}\right)\right\} \tag{80}$$

Making use of the recurrence formula for Bessel functions,

$$\frac{2n}{z}J_n(z) = J_{n-1}(z) + J_{n+1}(z) \tag{81}$$*

one has

$$\frac{i\nu}{2U}\left[J_0\left(\frac{\nu}{U}\right)+J_2\left(\frac{\nu}{U}\right)\right] = i\,J_1\left(\frac{\nu}{U}\right) \tag{82}$$

and

$$\frac{i\nu}{4U}\left[i\,J_1\left(\frac{\nu}{U}\right)+i\,J_3\left(\frac{\nu}{U}\right)\right] = -J_2\left(\frac{\nu}{U}\right) \tag{83}$$

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*Ref. 2, p. 16, eq. (26).

Hence (79) and (80) can easily be reduced to

$$L = 2\pi\rho U W e^{i\nu t}\left\{\frac{J_0(\frac{\nu}{U})K_1(\frac{i\nu}{U})+iJ_1(\frac{\nu}{U})K_0(\frac{i\nu}{U})}{K_1(\frac{i\nu}{U})+K_0(\frac{i\nu}{U})}\right\} \tag{84}$$

and

$$M = -\frac{1}{2}L \tag{85}$$

which means that the lift acts through the quarter-chord point (x = -1/2) at all times.

The lift, as expressed in formula (84), can be presented in the form of another vector diagram, analogous to those employed in Section IV (cf. pp. 31-33 and Fig. 6). The necessary numerical calculations have been carried out, and the results are given in Fig. 11. It is seen that as the reduced frequency ν/U (or $c\nu/2U$ for a wing of chord = c) is increased from zero (i.e., as the wave length of the gusts becomes progressively shorter), the magnitude of the lift vector decreases continuously, while the lift vector leads the gust velocity vector by a progressively greater phase angle φ. In fact, for $c\nu/2U > 2$, the phase angle is approximately proportional to the reduced frequency. It is obvious that there exists no "critical frequency" which produces abnormally large forces on the airfoil.

As an alternative to the method used in Section VII, the lift produced by a sharp-edged gust may be calculated from the result given in equation (84). Employing again the Dirichlet "discontinuous factor", the gust velocity may be written in the form

$$v(x,t) = V\left\{\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{e^{i\nu(t-x/U)}}{i\nu}\,d\nu\right\} \tag{86}$$

Then, by comparison of (86) and (72), the lift can immediately be obtained in the form of (84):

$$L = 2\pi\rho UV\left\{\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{J_0(\frac{\nu}{U})K_1(\frac{i\nu}{U}) + iJ_1(\frac{\nu}{U})K_0(\frac{i\nu}{U})}{K_1(\frac{i\nu}{U}) + K_0(\frac{i\nu}{U})}\cdot\frac{e^{i\nu t}}{i\nu}\,d\nu\right\} \tag{87}$$

Introducing again s = Ut, which in this case is the distance of the midpoint of the airfoil past the gust boundary, replacing ν/U by z, and substituting for the K's in terms of T' and T" as in equation (49), the real part of this formula becomes, after a little calculation,

$$\mathcal{R}(L) = 2\pi\rho UV\left\{\frac{1}{2} + \frac{1}{2\pi}\int_0^\infty\left[J_0(z)\{1+T'(z)\} + J_1(z)T''(z)\right]\frac{\sin sz}{z}\,dz\right.$$
$$\left. + \frac{1}{2\pi}\int_0^\infty\left[J_1(z)\{1-T'(z)\} + J_0(z)T''(z)\right]\frac{\cos sz}{z}\,dz\right\} \tag{88}$$

But

$$\int_0^\infty J_0(z)\frac{\sin sz}{z}\,dz = \sin^{-1}s \quad \text{for } s \leqq 1$$

$$= \frac{\pi}{2} \quad \text{for } s \geqq 1 \qquad (89a)^*$$

and

$$\int_0^\infty J_1(z)\frac{\cos sz}{z}\,dz = \cos(\sin^{-1}s) = \sqrt{1-s^2} \quad \text{for } s \leqq 1$$

$$= 0 \quad \text{for } s \geqq 1 \qquad (89b)^*$$

Hence the lift, equation (88), can be evaluated if the following integrals can be determined:

$$\int_0^\infty \left[J_0(z)T'(z) + J_1(z)T''(z)\right]\frac{\sin sz}{z}\,dz$$

and

$$\int_0^\infty \left[J_0(z)T''(z) - J_1(z)T'(z)\right]\frac{\cos sz}{z}\,dz$$

The values of these integrals have been determined for a few values of s, taking T' and T" from Ref. 3. The integrals are rapidly convergent, and have been evaluated graphically. Accounting for the difference in definition of t in the two cases, the lift from equation (88) is found to agree closely with that obtained from equations (66) and

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*Ref. 2, pp. 77, 78.

(67) in Section VII, excepting for large values of Ut, for which the approximation involved in equation (67) is not close. The values obtained from equation (88) for this region are shown by the dotted line in Fig. 9.

It can be stated from the result of equation (85), without further calculation, that the entire lift acts at the quarter-chord point for a gust of any arbitrary profile.

X. CALCULATION OF THE PRESSURE DISTRIBUTION OVER THE AIRFOIL

In certain problems of non-uniform motion it becomes necessary to calculate the distribution of the load over the chord of the airfoil. An example is the problem of wing-aileron flutter, in which the non-uniform hinge moment must be determined.

For the plane airfoil of infinite span which is being considered here, the force per unit area is given by the sum of the force arising from the vorticity, which is $\rho U \gamma(x)$, and the difference between the "impulsive pressures", $\rho \partial \varphi / \partial t$, on the upper and lower surfaces, where φ is again the velocity potential of the motion. Hence the total upward load per unit area may be written as

$$\rho U \gamma(x) + \rho \frac{\partial}{\partial t}(\varphi_2 - \varphi_1) = \rho U \gamma_{eff.} \quad , \text{ say},$$

where $\gamma(x)$ = the vorticity at x

φ_2 = the value of φ on the lower surface at x

φ_1 = the value of φ on the upper surface at x

and γ_{eff} = the "effective vorticity distribution", which is defined as

$$\gamma_{eff} = \gamma(x) + \frac{1}{U}\frac{\partial}{\partial t}(\varphi_2 - \varphi_1) \tag{90}$$

Since φ may be taken equal to zero at any desired point, suppose this point to be the leading edge, where $x = -1$. Then $\varphi_2 = \int_{-1}^{x} u_2\, dx$ and $\varphi_1 = \int_{-1}^{x} u_1\, dx$, where u_2 and u_1 are the velocities along the lower and upper surfaces of the wing, respectively. Hence, since $u_2 - u_1$ is equal to the vorticity, $\gamma(x)$, one can write

$$\varphi_2 - \varphi_1 = \int_{-1}^{x} (u_2 - u_1)\, dx = \int_{-1}^{x} \gamma(x)\, dx \tag{91}$$

The effective vorticity at x is, therefore,

$$\gamma_{eff} = \gamma(x) + \frac{1}{U}\frac{d}{dt}\int_{-1}^{x} \gamma(x)\, dx \tag{92}$$

The vorticity $\gamma(x)$ in this expression can now be written as the sum of $\gamma_0(x)$ and $\gamma_1(x)$, as before, and $\gamma_1(x)$ can be obtained from equation (7); then

$$\gamma_{eff} = \gamma_0(x) + \frac{1}{U}\frac{d}{dt}\int_{-1}^{x} \gamma_0(x)\, dx + \gamma_1(x) + \frac{1}{U}\frac{d}{dt}\int_{-1}^{x} \gamma_1(x)\, dx \tag{93}$$

where $$\gamma_1(x) = \frac{1}{\pi}\sqrt{\frac{1-x}{1+x}} \int_{1}^{\infty} \frac{\gamma(\xi)}{\xi - x}\sqrt{\frac{\xi+1}{\xi-1}}\, d\xi \tag{7}$$

As an example, the case of any steady-state oscillation will be treated here. In this case (cf. Section IV), one has

$$\Gamma_0 = G_0 e^{i\nu t} \tag{26}$$

$$\gamma(\xi) = g \cdot e^{i\nu(t-\xi/U)} \tag{27}$$

and

$$g = \frac{-G_0}{K_0\left(\frac{i\nu}{U}\right) + K_1\left(\frac{i\nu}{U}\right)} \tag{31}$$

It is also apparent that the quasi-steady vorticity is periodic and can be written as

$$\gamma_0(x) = g_0(x)\, e^{i\nu t} \tag{94}$$

These quantities are to be introduced into equations (93) and (7). The analysis is simplified by the use of the variables $\theta = \cos^{-1} x$ and $\alpha = \cosh^{-1} \xi$. Introduction of (27) into (7) then gives the following expression for the induced vorticity:

$$\gamma_1(\theta) = \frac{g\, e^{i\nu t}}{\pi} \frac{1-\cos\theta}{\sin\theta} \int_0^\infty \frac{\cosh\alpha + 1}{\cosh\alpha - \cos\theta} e^{-\frac{i\nu}{U}\cosh\alpha} d\alpha \tag{95}$$

In Appendix #6, by means of a contour integration, it is shown that

$$\frac{(\cosh\alpha+1)(1-\cos\theta)}{(\cosh\alpha-\cos\theta)\sin\theta} = \frac{1-\cos\theta}{\sin\theta} + 2\sum_{n=1}^{\infty} e^{-n\alpha}\sin n\theta \qquad (96)$$

Therefore, equation (95) can be written in the form

$$\gamma_1(\theta) = \frac{ge^{i\nu t}}{\pi}\left\{Q_o \frac{1-\cos\theta}{\sin\theta} + 2\sum_{n=1}^{\infty} Q_n \sin n\theta\right\} \qquad (97)$$

where Q_n is used as an abbreviation for the integral

$$Q_n = \int_0^{\infty} e^{-\frac{i\nu}{U}\cosh\alpha - n\alpha}\, d\alpha \qquad (98)$$

It will be found later that only the integrals Q_o and Q_1 need be evaluated.

The integral in equation (93) involving the induced vorticity becomes, by application of (97),

$$\int_{-1}^{x}\gamma_1(x)dx = \frac{ge^{i\nu t}}{\pi}\left\{Q_o\int_{\theta}^{\pi}(1-\cos\theta)\,d\theta + \pi Q_1 + 2\sum_{n=2}^{\infty} Q_n \int_{\theta}^{\pi}\sin\theta\sin n\theta\, d\theta\right\}$$

$$= \frac{ge^{i\nu t}}{\pi}\left\{Q_o(\pi-\theta+\sin\theta) + \pi Q_1 + \sum_{n=1}^{\infty} Q_n\left[\frac{\sin(n+1)\theta}{n+1} - \frac{\sin(n-1)\theta}{n-1}\right]\right\}$$

$$= \frac{ge^{i\nu t}}{\pi}\left\{(Q_o+Q_1)(\pi-\theta) + \sum_{n=1}^{\infty}(Q_{n-1}-Q_{n+1})\frac{\sin n\theta}{n}\right\} \qquad (99)$$

In Appendix #7 a recurrence formula for the integrals Q_n

is found to be

$$Q_{n-1} - Q_{n+1} = -\frac{2U}{i\nu}\left(n Q_n - e^{-i\nu/U}\right) \tag{100}$$

Hence (99) is

$$\int_{-1}^{x} \gamma_1(x)\,dx = \frac{g e^{i\nu t}}{\pi}\left\{(Q_0+Q_1)(\pi-\theta) - \frac{2U}{i\nu}\sum_{n=1}^{\infty}\left(nQ_n - e^{-i\nu/U}\right)\frac{\sin n\theta}{n}\right\}$$

or, since $2\sum_{1}^{\infty}\frac{\sin n\theta}{n} = \pi-\theta,$

$$\int_{-1}^{x} \gamma_1(x)\,dx = \frac{g e^{i\nu t}}{\pi}\left\{(Q_0+Q_1)(\pi-\theta) - \frac{2U}{i\nu}\sum_{n=1}^{\infty} Q_n \sin n\theta + \frac{U}{i\nu}e^{-i\nu/U}(\pi-\theta)\right\} \tag{101}$$

Now, substituting $\gamma_1(\theta)$ from (97) and using (94) and (101), the effective vorticity, equation (93) becomes

$$\gamma_{eff} = \gamma_0(\theta) + \frac{i\nu}{U}\int_{\theta}^{\pi}\gamma_0(\theta)\sin\theta\,d\theta + \frac{g e^{i\nu t}}{\pi}\left\{Q_0\frac{1-\cos\theta}{\sin\theta} + \frac{i\nu}{U}(\pi-\theta)\left[Q_0+Q_1+\frac{U}{i\nu}e^{-i\nu/U}\right]\right\} \tag{102}$$

But, referring to the definition of the integrals Q_n in (98) and letting $\nu/U = z$, one can write

$$Q_0 + Q_1 + \frac{U}{i\nu} e^{-i\nu/U} = \int_0^\infty e^{-iz\cosh\alpha}(1+e^{-\alpha})\,d\alpha + \frac{e^{-iz}}{z}$$

This expression can now be identified in terms of Bessel functions, for, putting $\cosh\alpha = \xi$ and noting that $e^{-\alpha} = \xi - \sqrt{\xi^2 - 1}$, it can be written as

$$Q_0 + Q_1 + \frac{U}{i\nu} e^{-i\nu/U} = \int_1^\infty e^{-iz\xi}\left(\sqrt{\frac{\xi+1}{\xi-1}} - 1\right) d\xi + \frac{e^{-iz}}{z}$$

$$= K_0(iz) + K_1(iz) \qquad (103)$$

by comparison with equations (29) and (31). It is obvious, from the definition given on page 25, that $Q_0 = K_n(i\nu/U)$.

Applying these results in (102), and substituting for g from (31), above, the final expression for the effective vorticity distribution becomes

$$\gamma_{eff.} = \gamma_0(\theta) + \frac{i\nu}{U}\int_\theta^\pi \gamma_0(\theta)\sin\theta\,d\theta - \frac{\Gamma_0}{\pi}\left\{\frac{K_0\left(\frac{i\nu}{U}\right)}{K_0\left(\frac{i\nu}{U}\right)+K_1\left(\frac{i\nu}{U}\right)}\,\frac{1-\cos\theta}{\sin\theta} + \frac{i\nu}{U}(\pi-\theta)\right\} \qquad (104)$$

It has been verified by calculation of the integrals

$$L = \rho U\int_{-1}^{1}\gamma_{eff}\,dx \qquad M = \rho U\int_{-1}^{1}\gamma_{eff}\,x\,dx \qquad (105)$$

that equation (104) leads to the same results for the lift and moment on the airfoil as have already been obtained in Section IV and Appendix #5.

The final expression for γ_{eff} may be put into a somewhat more convenient form for practical application by expressing the quasi-steady vorticity $\gamma_o(\theta)$ in series form. In Ref. 1, page 37, eq. (6.16), $\gamma_o(\theta)$ is given in the form

$$\gamma_o(\theta) = \frac{\Gamma_o}{\pi \sin\theta} + \sum_{k=1}^{\infty} C_k \frac{\cos k\theta}{\sin\theta} \tag{106}$$*

Hence

$$\int_{\theta}^{\pi} \gamma_o(\theta) \sin\theta \, d\theta = \frac{\Gamma_o}{\pi}(\pi-\theta) - \sum_{k=1}^{\infty} C_k \frac{\sin k\theta}{k} \tag{107}$$

and the effective vorticity, from (104) can be written in the form

$$\begin{aligned}\gamma_{eff} &= \frac{\Gamma_o}{\pi \sin\theta} + \sum_{k=1}^{\infty} C_k \frac{\cos k\theta}{\sin\theta} - \frac{i\nu}{U}\sum_{k=1}^{\infty} C_k \frac{\sin k\theta}{k} \\ &\quad - \frac{\Gamma_o}{\pi} \frac{K_o\left(\frac{i\nu}{U}\right)}{K_1\left(\frac{i\nu}{U}\right) + K_o\left(\frac{i\nu}{U}\right)} \frac{1-\cos\theta}{\sin\theta} \\ &= \frac{\Gamma_o}{\pi \sin\theta} \frac{K_1\left(\frac{i\nu}{U}\right) + K_o\left(\frac{i\nu}{U}\right)\cdot\cos\theta}{K_1\left(\frac{i\nu}{U}\right) + K_o\left(\frac{i\nu}{U}\right)} \\ &\quad + \sum_{k=1}^{\infty} C_k \frac{\cos k\theta}{\sin\theta} - \frac{i\nu}{U}\sum_{k=1}^{\infty} C_k \frac{\sin k\theta}{k}\end{aligned} \tag{108}$$

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*The changes in notation from that used in Ref. 1 are obvious.

In order to satisfy the condition of finite velocities at the trailing edge, $\gamma_0(\theta)$ must be equal to zero for $\theta = 0$; hence, from (106),

$$\frac{\Gamma_0}{\pi} = -\sum_{k=1}^{\infty} C_k \tag{109}$$

The results given in this Section provide the method for the calculation of the non-steady aileron hinge-moment for the steady-state case of aileron flutter. The performance of the actual calculations is left for a subsequent paper.

XI. APPLICATION TO CONDITIONS BEYOND THE STALL

The airfoil theory for non-uniform motion suggests a possibility of explaining various phenomena relating to the lift of airfoils above the stall; i.e., after the rate of change of lift with respect to angle of attack becomes negative. Wieselsberger, in Ref. 9, first proposed that the ordinary theory of thin airfoils be applied above the stall. Although some of the assumptions upon which the theory is based are certainly violated in this regime, it is believed that the results provide at least a crude first approximation to the true conditions. The general method of attack upon this problem is simply to replace the lift slope, which is 2π in the case of an infinite thin airfoil, by the value $2\pi\lambda$, where λ is to have negative values. The value assumed for the factor λ must, of course, be obtained from experimental observations.

If the lift slope is altered in this manner, it is apparent that equation (8), which gives the total circulation induced by the wake, must be replaced by the following:

$$\Gamma = \lambda \int_1^\infty \gamma(\xi) \left\{ \sqrt{\frac{\xi+1}{\xi-1}} - 1 \right\} d\xi \tag{110}$$

The relation analogous to (14) is then

$$\Gamma = \Gamma_0 + \lambda \int_1^\infty \gamma(\xi) \left\{ \sqrt{\frac{\xi+1}{\xi-1}} - 1 \right\} d\xi \tag{111}$$

and since one may again assume that the total circulation in the fluid is always zero, this becomes

$$\Gamma_0 + \lambda \int_1^\infty \gamma(\xi) \left\{ \sqrt{\frac{\xi+1}{\xi-1}} - 1 \right\} d\xi + \int_1^\infty \gamma(\xi) d\xi = 0 \tag{112}$$

Now suppose, for example, that the airfoil is moving with constant velocity and at a constant angle of attack; i.e., that Γ_0 is constant. It is interesting to investigate the possibility of a wake effect which would produce changes of the total circulation while Γ_0 remains constant. It is sufficient to consider the case of $\Gamma_0 = 0$, since the results can be superimposed upon a constant quasi-steady circulation. It will also be assumed that $\gamma(\xi)$ can be expressed in the form

$$\gamma(\xi) = g\, e^{\alpha(t - \xi/U)} \tag{113}$$

where α may be complex. This form includes the various possibilities of a steady oscillation, a decreasing oscillation, and an exponential subsidence for the function $\gamma(\xi)$. These would indicate a steady oscillation, an increasing oscillation, or a divergence, respectively, as the behavior of the total circulation about the airfoil. Then, with $\Gamma_0 = 0$, (112) becomes

$$ge^{\alpha t}\lambda\int_1^\infty e^{-\frac{\alpha}{U}\xi}\left\{\sqrt{\frac{\xi+1}{\xi-1}}-1\right\}d\xi + ge^{\alpha t}\int_1^\infty e^{-\frac{\alpha}{U}\xi}d\xi = 0 \tag{114}$$

If $\alpha > 0$, this may be simplified according to the results of equations (29) to (31); i.e.,

$$\lambda\left[K_0\left(\frac{\alpha}{U}\right)+K_1\left(\frac{\alpha}{U}\right)-\frac{U}{\alpha}e^{-\alpha/U}\right]+\frac{U}{\alpha}e^{-\alpha/U} = 0$$

or $$\lambda\left[K_0\left(\frac{\alpha}{U}\right)+K_1\left(\frac{\alpha}{U}\right)\right]+(1-\lambda)\frac{U}{\alpha}e^{-\alpha/U} = 0 \tag{115}$$

This provides a relation between the lift-slope parameter, λ, and the exponent α. It has been determined that (115) cannot be satisfied for $\lambda < 0$ for any purely imaginary values of α. It is satisfied, however, for α purely real.

For example,

$$\lambda = -1.49 \quad \leftrightarrow \quad a/U = 0.10$$

$$\lambda = -0.89 \quad \leftrightarrow \quad a/U = 0.50$$

$$\lambda = -0.56 \quad \leftrightarrow \quad a/U = 1.00$$

This means that the negative lift slope results in a divergent increase of the total circulation.

The mechanism which produces this behavior is easily visualized. The condition $\lambda < 0$ means that the downward velocities induced over the wing by a counter-clockwise wake vortex (supposing the airfoil to be moving from right to left in our view) produce an increase of circulation about the wing. This is accompanied by the shedding of more counter-clockwise vortices at the trailing edge, which induce more circulation, etc. Although the integrals in (114) do not converge for $a < 0$, this physical reasoning shows clearly that a subsidence of the lift must also be possible, because the presence of a clockwise wake vortex would initiate an opposite sequence of events.

In an actual case the slope $2\pi\lambda$ would maintain over only a small range of angles of attack. Thus the divergence or subsidence of circulation would progress only until the lift slope became positive. It is possible that the rapid fluctuations of lift often observed above the stall are produced by an alternate increasing and decreasing of the circulation over a region of negative slope.

One of the questions suggested by this discussion is that of the significance of an experimentally observed negative lift slope. Since the wing is necessarily tested in the presence of a wake, any such effects as have been indicated above must occur during the observations. Consequently the true lift slope (if such exists) must be obscured by wake effects.

APPENDIX #1

EVALUATION OF THE INTEGRALS $\int_{-1}^{1} \frac{x^n\,dx}{(\xi - x)\sqrt{1-x^2}}$

1) n = 0: $$\int_{-1}^{1} \frac{dx}{(\xi - x)\sqrt{1-x^2}} = \frac{1}{\xi}\int_{0}^{\pi} \frac{d\theta}{1+a\cos\theta} = \frac{1}{2\xi}\int_{0}^{2\pi} \frac{d\theta}{1+a\cos\theta}$$

where $a = -\frac{1}{\xi}$ and $a^2 < 1$ for $\xi > 1$. This is given in Ref. 10, formula 859.2. The result is

$$\int_{-1}^{1} \frac{dx}{(\xi - x)\sqrt{1-x^2}} = \frac{\pi}{\xi}\frac{1}{\sqrt{1-a^2}} = \frac{\pi}{\sqrt{\xi^2-1}} \tag{116}$$

2) n = 1: $$\int_{-1}^{1} \frac{x\,dx}{(\xi - x)\sqrt{1-x^2}} = -\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} + \xi\int_{-1}^{1} \frac{dx}{(\xi - x)\sqrt{1-x^2}}$$

$$= \pi\left\{-1 + \frac{\xi}{\sqrt{\xi^2-1}}\right\} \tag{117}$$

3) n = 2: $$\int_{-1}^{1} \frac{x^2\,dx}{(\xi - x)\sqrt{1-x^2}} = \int_{-1}^{1} \frac{x\,dx}{\sqrt{1-x^2}} - \xi\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} + \xi^2\int_{-1}^{1} \frac{dx}{(\xi - x)\sqrt{1-x^2}}$$

$$= \pi\left\{-\xi + \frac{\xi^2}{\sqrt{\xi^2-1}}\right\} \tag{118}$$

4) n = 3: $$\int_{-1}^{1}\frac{x^3dx}{(\xi-x)\sqrt{1-x^2}} = -\int_{-1}^{1}\frac{x^2dx}{\sqrt{1-x^2}} - \xi\int_{-1}^{1}\frac{xdx}{\sqrt{1-x^2}} - \xi^2\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}} + \xi^3\int_{-1}^{1}\frac{dx}{(\xi-x)\sqrt{1-x^2}}$$

$$= \pi\left\{-\frac{1}{2} - \xi^2 + \frac{\xi^3}{\sqrt{\xi^2-1}}\right\} \qquad (119)$$

APPENDIX #2

CALCULATION OF THE QUASI-STEADY QUANTITIES

1. Rotational Oscillation (cf. page 29)

The vertical velocity, taken positive when downward, is given by

$$w(\theta) = 2UA_1 e^{i\nu t} \cos\theta \tag{120}$$

In Ref. 1, p. 37, the vorticity distribution is expressed as

$$\gamma_0(\theta) = \frac{\Gamma_0}{\pi \sin\theta} + \sum_{k=1}^{\infty} a_k \frac{\cos k\theta}{\sin\theta} \tag{121*}$$

where the upward velocity of the airfoil relative to the fluid is

$$v_y = -w(\theta) = \sum_{k=1}^{\infty} a_k \frac{\sin k\theta}{2\sin\theta} \tag{122}$$

Equating w(θ) from (120) and (122), one has

$$\sum_{k=1}^{\infty} a_k \sin k\theta = -4UA_1 e^{i\nu t} \cos\theta \sin\theta$$

$$= -2UA_1 e^{i\nu t} \sin 2\theta$$

Hence $a_2 = -2UA_1\ e^{i\nu t}$ and $a_k = 0$ for $k \neq 2$.

---

*Here a is put = 1/2 since, in Ref. 1, θ is defined by $x = 2a\cos\theta$.

The vorticity distribution becomes, therefore,

$$\gamma_o(\theta) = \frac{\Gamma_o}{\pi \sin\theta} - 2UA_1 e^{i\nu t} \frac{\cos 2\theta}{\sin\theta} \qquad (123)$$

In order to satisfy the condition of tangential flow at the trailing edge ($\theta = 0$), $\gamma_o(0)$ must vanish; hence

$$\frac{\Gamma_o}{\pi} - 2UA_1 e^{i\nu t} = 0$$

and $$\Gamma_o = 2\pi UA_1 e^{i\nu t} \qquad (124)$$

Putting this into (123), the final expression for the vorticity is

$$\gamma_o(\theta) = 2UA_1 e^{i\nu t} \frac{1-\cos 2\theta}{\sin\theta}$$

$$= 4UA_1 e^{i\nu t} \sin\theta \qquad (125)$$

2. <u>Airfoil Passing Boundary of Sharp-Edged Gust</u>
(cf. p. 48 and Fig. 8)

In this case

$$\begin{aligned} w(\theta) &= V \text{ for } -1 < x < Ut - 1 \\ w(\theta) &= 0 \text{ for } Ut - 1 < x < 1 \end{aligned} \qquad (126)$$

Applying (122) again, one has

$$-2w(\theta)\sin\theta = \sum_{k=1}^{\infty} a_k \sin k\theta \qquad (127)$$

By Fourier's theorem,

$$a_k = -\frac{2}{\pi}\int_0^{\pi} 2w(\theta)\sin\theta\sin k\theta\, d\theta$$

$$= -\frac{4V}{\pi}\int_{\cos^{-1}(Ut-1)}^{\pi}\sin\theta\sin k\theta\, d\theta \qquad (128)$$

Also, putting $\gamma_0(0) = 0$, from (121), the total circulation is obtained:

$$\Gamma_0 = -\pi\sum_{k=1}^{\infty} a_k$$

$$= 4V\sum_{k=1}^{\infty}\int_{\cos^{-1}(Ut-1)}^{\pi}\sin\theta\sin k\theta\, d\theta$$

$$= 2V\left\{\pi - \sum_{k=1}^{\infty}\frac{\sin(k-1)\alpha}{k-1} + \sum_{k=1}^{\infty}\frac{\sin(k+1)\alpha}{k+1}\right\}$$

(where $\alpha = \cos^{-1}(Ut - 1)$)

$$= 2V\left\{\pi - \alpha - \sin\alpha\right\}$$

$$= 2V\left\{\pi - \cos^{-1}(Ut-1) - \sqrt{2Ut - U^2t^2}\right\}$$

$$= 2V\left\{\cos^{-1}(1-Ut) - \sqrt{2Ut - U^2t^2}\right\} \qquad (129)$$

APPENDIX #3

APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION FOR THE CASE OF A SUDDEN CHANGE OF ANGLE

For the case of a sudden unit increment in Γ_0 at the time $t = 0$, the contribution of the wake to the lift is, from equations (53) to (55),

$$L_2(\sigma) = -\rho U \Phi(\sigma) \tag{130}$$

where $$\Phi(\sigma) = -\int_0^\sigma \mu(s) \frac{ds}{\sqrt{\xi^2 - 1}} \tag{131}$$

$\sigma = Ut$

$s = 1 + \sigma - \xi$

and $\mu(s)$ is the solution of the integral equation

$$1 = -\int_0^\sigma \mu(s) \sqrt{\frac{\xi + 1}{\xi - 1}}\, ds \tag{132}$$

Equation (131) can be written in the form

$$\Phi(\sigma) = -\frac{1}{2}\int_0^\sigma \mu(s) \left\{ \sqrt{\frac{\xi+1}{\xi-1}} - \sqrt{\frac{\xi-1}{\xi+1}} \right\} ds$$

$$= \frac{1}{2} + \frac{1}{2}\int_0^\sigma \mu(s) \sqrt{\frac{\xi-1}{\xi+1}}\, ds \tag{133}$$

by the use of the relation (132).

Now it is permissible to differentiate with respect to σ under the integral sign in the second term of (133) since this integral is not improper*. Therefore

$$\Phi'(\sigma) = \frac{1}{4}\int_0^\sigma \mu(s)\left\{\frac{1}{\sqrt{\xi^2-1}} - \frac{\sqrt{\xi-1}}{(\xi+1)^{3/2}}\right\}ds$$

$$= \frac{1}{4}\int_0^\sigma \mu(s)\left\{\frac{1}{2}\sqrt{\frac{\xi+1}{\xi-1}} - \frac{1}{2}\sqrt{\frac{\xi-1}{\xi+1}} - \frac{\sqrt{\xi-1}}{(\xi+1)^{3/2}}\right\}ds$$

$$= -\frac{1}{8} - \frac{1}{8}\int_0^\sigma \mu(s)\sqrt{\frac{\xi-1}{\xi+1}}\,ds - \frac{1}{4}\int_0^\sigma \mu(s)\frac{\sqrt{\xi-1}}{(\xi+1)^{3/2}}\,ds \qquad (134)$$

where (132) is again applied in the last step. Differentiating again,

$$\Phi''(\sigma) = -\frac{1}{16}\int_0^\sigma \mu(s)\left\{\frac{1}{\sqrt{\xi^2-1}} - \frac{\sqrt{\xi-1}}{(\xi+1)^{3/2}}\right\}ds$$

$$-\frac{1}{8}\int_0^\sigma \mu(s)\left\{\frac{1}{(\xi+1)\sqrt{\xi^2-1}} - \frac{3\sqrt{\xi-1}}{(\xi+1)^{5/2}}\right\}ds$$

$$= -\frac{1}{16}\int_0^\sigma \mu(s)\left\{\frac{1}{2}\sqrt{\frac{\xi+1}{\xi-1}} - \frac{1}{2}\sqrt{\frac{\xi-1}{\xi+1}} - \frac{\sqrt{\xi-1}}{(\xi+1)^{3/2}}\right\}ds$$

$$-\frac{1}{8}\int_0^\sigma \mu(s)\left\{\frac{1}{4}\sqrt{\frac{\xi+1}{\xi-1}} - \frac{1}{4}\sqrt{\frac{\xi-1}{\xi+1}} - \frac{1}{2}\frac{\sqrt{\xi-1}}{(\xi+1)^{3/2}} - \frac{3\sqrt{\xi-1}}{(\xi+1)^{5/2}}\right\}ds$$

---

* $\sqrt{\xi - 1} = 0$ when $s = \sigma$.

$$= \frac{1}{16} + \frac{1}{16}\int_0^{\sigma}\mu(s)\sqrt{\frac{\xi-1}{\xi+1}}\,ds + \frac{1}{8}\int_0^{\sigma}\mu(s)\frac{\sqrt{\xi-1}}{(\xi+1)^{3/2}}\,ds$$
$$+ \frac{3}{8}\int_0^{\sigma}\mu(s)\frac{\sqrt{\xi-1}}{(\xi+1)^{5/2}}\,ds \tag{135}$$

From equations (133), (134), (135), noting that $\sqrt{\xi - 1} = 0$ when $\sigma = 0$, one has

$$\Phi(0) = \frac{1}{2}$$
$$\Phi'(0) = -\frac{1}{8}$$
$$\Phi''(0) = \frac{1}{16}$$

Hence, forming a Taylor's series,

$$\Phi(\sigma) = \frac{1}{2} - \frac{1}{8}\sigma + \frac{1}{32}\sigma^2 + \cdots \tag{136}$$

The next term in the series (136) can be determined by a procedure analogous to that used above. The result obtained is $\Phi'''(0) = -7/128$. However, it is preferable to adjust the coefficient of σ^3 to make the approximate expression for $\Phi(\sigma)$ agree with Wagner's value (cf. Table 1, page 46) for $\sigma = 2$. When this is done, the expression for $\Phi(\sigma)$ to be applied in the range $0 \leqq \sigma \leqq 2$ becomes that given in equation (57).

APPENDIX #4

CALCULATION OF L_2 FOR THE CASE OF A SHARP-EDGED GUST

The expression for L_2 in terms of the "lift deficiency" function Φ and the quasi-steady circulation Γ_0 is given in equation (56). The two ranges of Ut are now considered separately.

a) $0 \leqq Ut \leqq 2$

Introducing Φ from (57) and Γ_0 from (61) into (56), one has

$$-L_2 = \rho U \int_0^t \Gamma_0'(\tau)\cdot \Phi\left[U(t-\tau)\right]\cdot d\tau$$

$$= 2\rho U^2 V \int_0^t \frac{U\tau}{\sqrt{2U\tau - U^2\tau^2}} \left\{\frac{1}{2} - \frac{U(t-\tau)}{8} + \frac{U^2(t-\tau)^2}{32} - 0.00554\, U^3(t-\tau)^3\right\} d\tau$$

or, putting Ut = s and Uτ = σ

$$-L_2 = 2\rho U V \left\{\frac{1}{2}\int_0^s \frac{\sigma}{\sqrt{2\sigma-\sigma^2}}\, d\sigma - \frac{1}{8}\int_0^s \frac{(s-\sigma)\sigma}{\sqrt{2\sigma-\sigma^2}}\, d\sigma + \frac{1}{32}\int_0^s \frac{(s-\sigma)^2\sigma}{\sqrt{2\sigma-\sigma^2}}\, d\sigma - 0.00554\int_0^s \frac{(s-\sigma)^3\sigma}{\sqrt{2\sigma-\sigma^2}}\, d\sigma\right\}$$

$$= 2\rho U V \left\{\left(\frac{1}{2} - \frac{s}{8} + \frac{s^2}{32} - 0.00554\, s^3\right)\int_0^s \sqrt{\frac{\sigma}{2-\sigma}}\, d\sigma + \left(\frac{1}{8} - \frac{s}{16} + 0.01662\, s^2\right)\int_0^s \sqrt{\frac{\sigma}{2-\sigma}}\,\sigma\, d\sigma + \left(\frac{1}{32} - 0.01662\, s\right)\int_0^s \sqrt{\frac{\sigma}{2-\sigma}}\,\sigma^2 d\sigma + 0.00554\int_0^s \sqrt{\frac{\sigma}{2-\sigma}}\,\sigma^3 d\sigma\right\} \quad (137)$$

Now
$$\int_0^s \sqrt{\frac{\sigma}{2-\sigma}}\, d\sigma = \cos^{-1}(1-s) - \sqrt{2s-s^2}$$

$$\int_0^s \sqrt{\frac{\sigma}{2-\sigma}}\, \sigma\, d\sigma = \frac{3}{2}\cos^{-1}(1-s) - \left(\frac{3}{2} + \frac{s}{2}\right)\sqrt{2s-s^2}$$

$$\int_0^s \sqrt{\frac{\sigma}{2-\sigma}}\, \sigma^2 d\sigma = \frac{5}{2}\cos^{-1}(1-s) - \left(\frac{5}{2} + \frac{5}{6}s + \frac{s^2}{3}\right)\sqrt{2s-s^2}$$

$$\int_0^s \sqrt{\frac{\sigma}{2-\sigma}}\, \sigma^3 d\sigma = \frac{35}{8}\cos^{-1}(1-s) - \left(\frac{35}{8} + \frac{35}{24}s + \frac{7}{12}s^2 + \frac{s^3}{4}\right)\sqrt{2s-s^2}$$

If these formulae are employed in (137) and terms are collected, a result for L_2 is obtained which, upon combining with $L_0 + L_1$ from (64), gives the formula (66).

b) <u>$2 \leqq Ut \leqq \infty$</u>

In this regime the formula (58) is used for Φ, and $\Gamma_0'(U\tau)$ is given by

$$\Gamma_0'(U\tau) = 2UV \frac{U\tau}{\sqrt{2U\tau - U^2\tau^2}} \quad \text{for } 0 \leqq U\tau \leqq 2$$
$$= 0 \text{ for } U\tau > 2$$

Hence, from equation (56), introducing again the variables s and σ as above,

$$
\begin{aligned}
-L_2 = 2\rho UV \Big\{ &\frac{1}{4} e^{-s/2} \int_0^2 \sqrt{\frac{\sigma}{2-\sigma}}\, e^{\sigma/2}\, d\sigma \\
&+ \frac{1}{4}(1+0.185\, s) e^{-0.185 s} \int_0^2 \sqrt{\frac{\sigma}{2-\sigma}}\, e^{0.185\sigma}\, d\sigma \\
&- \frac{1}{4}\cdot 0.185\, e^{-0.185 s} \int_0^2 \sqrt{\frac{\sigma}{2-\sigma}}\, e^{0.185\sigma}\, \sigma\, d\sigma \Big\}
\end{aligned}
\tag{138}
$$

The values of the integrals involved are given on page 52. Their evaluation, with the aid of the tables of Bessel functions in Ref. 2, leads to a result for L_2, which, added to the constant lift $L_o = 2\pi\rho UV$, gives the result stated in (67).

APPENDIX #5

CALCULATION OF LIFT AND MOMENT FOR PERIODIC DEFORMATIONS OF THE AIRFOIL

The general expression for the vertical velocity, $w(\theta)$, of any point of the airfoil is given in equation (41). The calculations of the lift and moment for the cases where A_0 and A_1, respectively, are different from zero, all other A_n's being equal to zero, are carried out in Section IV. The lift and moment for higher values of n are needed in Section IX. The quasi-steady circulation and vorticity in each case are determined from the relation (from Ref. 1, p. 37) (cf. also eq. (106), Section X of the present paper)

$$\gamma_0(\theta) = \frac{\Gamma_0}{\pi \sin\theta} + \sum_{k=1}^{\infty} C_k \frac{\cos k\theta}{\sin\theta} \tag{139}$$

where
$$-w(\theta) = v_y = \sum_{k=1}^{\infty} C_k \frac{\sin k\theta}{2\sin\theta} \tag{140}$$

and Γ_0 is determined by the condition $\gamma_0(0) = 0$ in (139). Consider the general case

$$w(\theta) = 2A_n U e^{i\nu t} \cos n\theta \tag{141}$$

where $n > 1$. Combining (140) and (141),

$$\sum_{k=1}^{\infty} C_k \sin k\theta = -4A_n U e^{i\nu t} \cos n\theta \sin\theta$$

$$= -2A_n U e^{i\nu t}\left[\sin(n+1)\theta - \sin(n-1)\theta\right]$$

Hence

$$C_{n-1} = 2A_n U e^{i\nu t}$$

$$C_{n+1} = -2A_n U e^{i\nu t}$$

$$C_k = 0 \quad \text{for} \quad k \neq n-1 \text{ or } n+1$$

Putting these into (139), one has

$$\gamma_0(\theta) = \frac{\Gamma_0}{\pi \sin\theta} + 2A_n U e^{i\nu t}\,\frac{\cos(n-1)\theta - \cos(n+1)\theta}{\sin\theta} \tag{142}$$

and, applying the condition $\gamma_0(0) = 0$,

$$\Gamma_0 = 0 \tag{143}$$

The three parts of the lift, from equations (21) and (32), are, therefore,

$$L_1 = -\rho \frac{d}{dt}\int_0^{\pi} \gamma_0(\theta)\cos\theta \sin\theta \, d\theta$$

$$= -2\rho A_n U i\nu e^{i\nu t}\int_0^{\pi}\left\{\cos(n-1)\theta - \cos(n+1)\theta\right\}\cos\theta \, d\theta$$

$$= -\pi\rho A_2 U i\nu e^{i\nu t} \qquad \text{if } n = 2$$

$$= 0 \quad \text{if } n = 3, 4, 5, \ldots$$

$$L_0 = 0$$

$$L_2 = 0$$

The three portions of the moment, from (25), are, similarly,

$$M_1 = -\tfrac{1}{2}\rho \frac{d}{dt}\int_0^{\pi} \gamma_0(\theta)(\cos^2\theta - \tfrac{1}{2}) \sin\theta \, d\theta$$

$$= -\rho A_n U i\nu e^{i\nu t} \int_0^{\pi} \{\cos(n-1)\theta - \cos(n+1)\theta\}(\cos^2\theta - \tfrac{1}{2}) \, d\theta$$

$$= -\frac{\pi}{4}\rho A_3 U i\nu e^{i\nu t} \quad \text{if } n = 3$$

$$= 0 \quad \text{if } n = 2, 4, 5, 6, \ldots$$

$$M_0 = \rho U \int_0^{\pi} \gamma_0(\theta) \cos\theta \sin\theta \, d\theta$$

$$= \pi\rho A_2 U^2 e^{i\nu t} \quad \text{if } n = 2$$

$$\doteq 0 \quad \text{if } n = 3, 4, 5, \ldots$$

Hence the total lift and moment for all values of $n > 1$ in equation (41) are

$$L = -\pi\rho A_2 U i\nu e^{i\nu t}$$
$$M = \pi\rho U^2 e^{i\nu t}\left(A_2 - \frac{i\nu}{4U} A_3\right) \qquad (144)$$

If these are combined with the lift and moment for n = 0 and 1, from formulae (35), (36), (39), and (40), the total lift and moment for the general oscillation represented by $w(\theta)$ in (41) are seen to be

$$L = 2\pi\rho U^2 e^{i\nu t}\left\{\frac{K_1\left(\frac{i\nu}{U}\right)}{K_0\left(\frac{i\nu}{U}\right)+K_1\left(\frac{i\nu}{U}\right)}\left(A_0+A_1\right)+\frac{i\nu}{2U}\left(A_0-A_2\right)\right\} \tag{145}$$

$$M = -\pi\rho U^2 e^{i\nu t}\left\{\frac{A_0 K_1\left(\frac{i\nu}{U}\right)-A_1 K_0\left(\frac{i\nu}{U}\right)}{K_1\left(\frac{i\nu}{U}\right)+K_0\left(\frac{i\nu}{U}\right)} - \frac{i\nu}{4U}\left(A_1-A_3\right)-A_2\right\} \tag{146}$$

APPENDIX #6

SERIES EXPANSION OF $\dfrac{(\cosh\alpha + 1)(1 - \cos\theta)}{(\cosh\alpha - \cos\theta)\sin\theta}$

The expression can first be put into a somewhat simpler form for expansion, for

$$\frac{(\cosh\alpha+1)(1-\cos\theta)}{(\cosh\alpha-\cos\theta)\sin\theta} = \frac{1-\cos\theta}{\sin\theta}\left\{1+\frac{\cos\theta+1}{\cosh\alpha-\cos\theta}\right\}$$

$$= \frac{1-\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cosh\alpha-\cos\theta}$$

$$= \frac{1-\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} a_n \sin n\theta \qquad (147)$$

where, by Fourier's theorem,

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\sin\theta\sin n\theta}{\cosh\alpha-\cos\theta}\,d\theta \qquad (148)$$

By use of the substitution $e^{i\theta} = z$, this becomes

$$a_n = -\frac{1}{2\pi}\int_C \frac{(z-z^{-1})(z^n-z^{-n})}{(e^{\alpha}+e^{-\alpha}-z-z^{-1})}\frac{dz}{iz}$$

$$= \frac{1}{2\pi i}\int_C \frac{(z^2-1)(z^{2n}-1)}{(z-e^{\alpha})(z-e^{-\alpha})}\frac{dz}{z^{n+1}} \qquad (149)$$

where the contour C consists of a complete circuit of the unit circle in the z-plane. The integrand is regular at all points of the unit circle except z = 0 and $z = e^{-\alpha}$, where it has poles of order (n + 1) and (1) respectively. Hence the value of the contour integral is $2\pi i \sum$ (Residues), or

$$a_n = \mathrm{Res.}\, f(z)\Big|_{z=0} + \mathrm{Res.}\, f(z)\Big|_{z=e^{-\alpha}} \tag{150}$$

where $$f(z) = \frac{(z^2-1)(z^{2n}-1)}{(z-e^{\alpha})(z-e^{-\alpha})\, z^{n+1}} \tag{151}$$

The residue at the simple pole is easily evaluated. It is

$$\mathrm{Res.}\, f(z)\Big|_{z=e^{-\alpha}} = \lim_{z\to e^{-\alpha}} \frac{(z^2-1)(z^{2n}-1)}{(z-e^{\alpha})\, z^{n+1}}$$

$$= e^{-n\alpha} - e^{n\alpha} \tag{152}$$

The residue at z = 0 is found by expanding f(z) in a power series. One has

$$f(z) = z^{-n-1}(z^2-1)(z^{2n}-1)\left(1-\frac{z}{e^{\alpha}}\right)^{-1}\left(1-\frac{z}{e^{-\alpha}}\right)^{-1}$$

$$= z^{-n-1}\left(z^{2(n+1)} - z^{2n} - z^2 + 1\right)\sum_{m=0}^{\infty}\left(\frac{z}{e^{\alpha}}\right)^m \cdot \sum_{r=0}^{\infty}\left(\frac{z}{e^{-\alpha}}\right)^r$$

$$= \left(z^{n+1} - z^{n-1} - z^{-n+1} + z^{-n-1}\right)\sum_{p=0}^{\infty}\sum_{m=0}^{p} e^{(p-2m)\alpha}\, z^p$$

Since n $\geqq$ 1, the coefficient of z^{-1} in this expansion is

$$Res.\, f(z)\Big|_{z=0} = -\sum_{m=0}^{n-2} e^{(n-2-2m)\alpha} + \sum_{m=0}^{n} e^{(n-2m)\alpha}$$

Putting m + 1 = q in the first summation, one has

$$Res.\, f(z)\Big|_{z=0} = -\sum_{q=1}^{n-1} e^{(n-2q)\alpha} + \sum_{m=0}^{n} e^{(n-2m)\alpha}$$

$$= e^{n\alpha} + e^{-n\alpha} \qquad (153)$$

By substitution of (152) and (153) into (150), the final value of a_n becomes

$$a_n \doteq 2e^{-n\alpha} \qquad (154)$$

This result, applied in (147) gives the series expansion employed in Section X, equation (96).

APPENDIX #7

RECURRENCE FORMULA FOR THE INTEGRALS Q_n

In Section X, equation (98), the integrals Q_n are defined as

$$Q_n = \int_0^{\infty} e^{-\frac{i\nu}{U}\cosh\alpha - n\alpha}\, d\alpha \tag{155}$$

$$(n = 1, 2, 3, \ldots)$$

Making the substitution $t = e^{\alpha}$, this can be written in the form

$$Q_n = \int_1^{\infty} e^{-\frac{i\nu}{2U}(t+t^{-1})}\, t^{-n-1}\, dt \tag{156}$$

Now consider the integral

$$\int_1^{\infty} \frac{d}{dt}\left\{ e^{-\frac{i\nu}{2U}(t+t^{-1})}\, t^{-n}\right\} dt = -e^{-i\nu/U}$$

or, performing the differentiation indicated,

$$\int_1^{\infty}\left\{-\frac{i\nu}{2U}(1-t^{-2})t^{-n} - n\, t^{-n-1}\right\} e^{-\frac{i\nu}{2U}(t+t^{-1})}\, dt = -e^{-i\nu/U}$$

By comparison with (156), this is seen to be

$$-\frac{i\nu}{2U}\left(Q_{n-1} - Q_{n+1}\right) - n\,Q_n = -e^{-i\nu/U} \tag{157}$$

which is the same recurrence formula as equation (100) in Section X.

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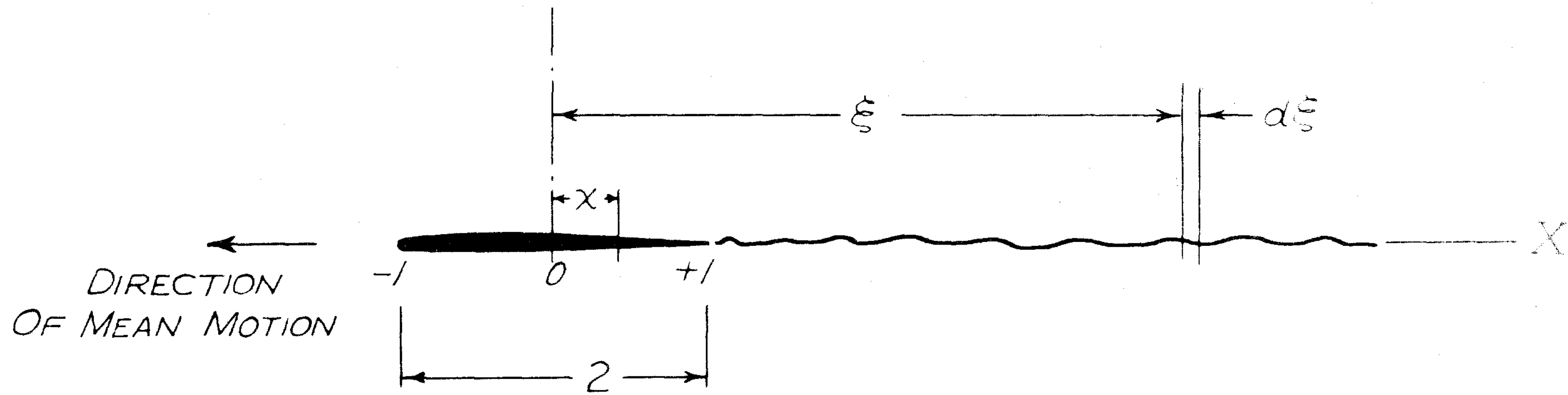


FIGURE 1

DIAGRAM SHOWING NOTATION EMPLOYED

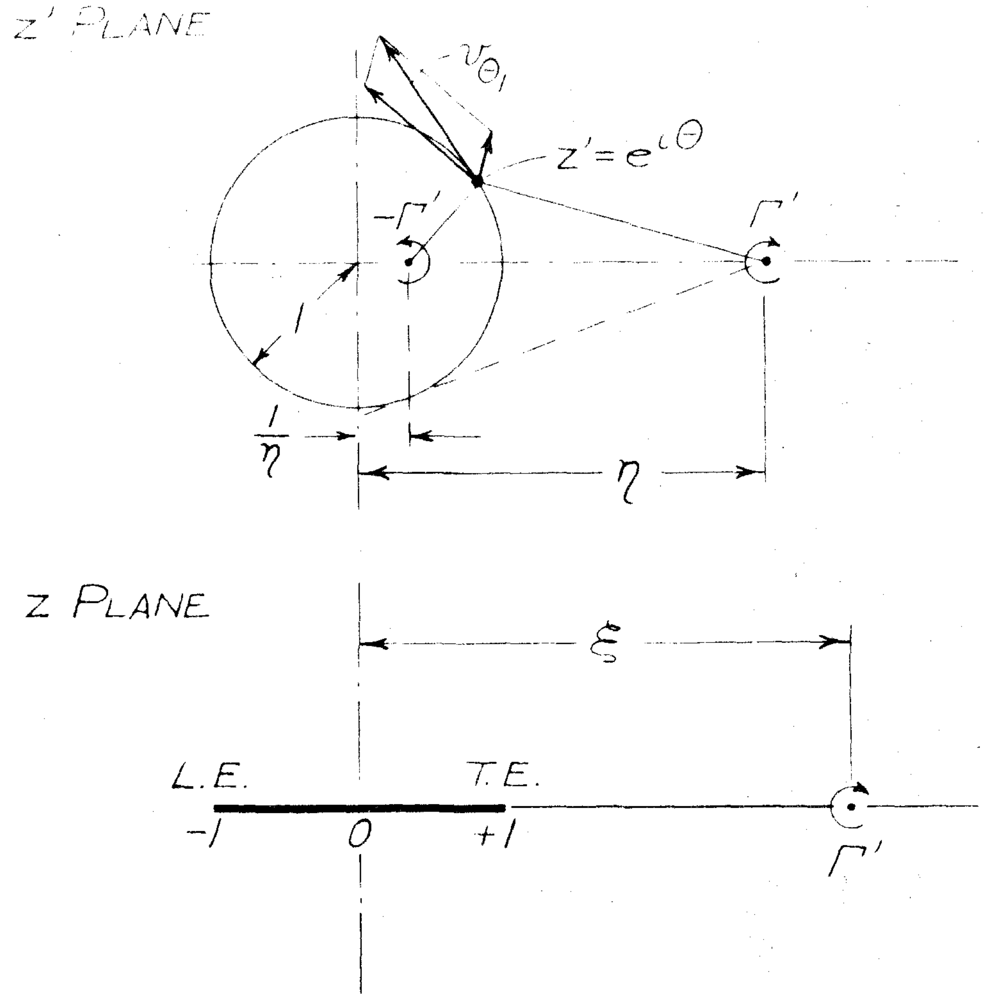


FIGURE 2

CONFORMAL REPRESENTATION OF THE AIRFOIL AND A WAKE VORTEX

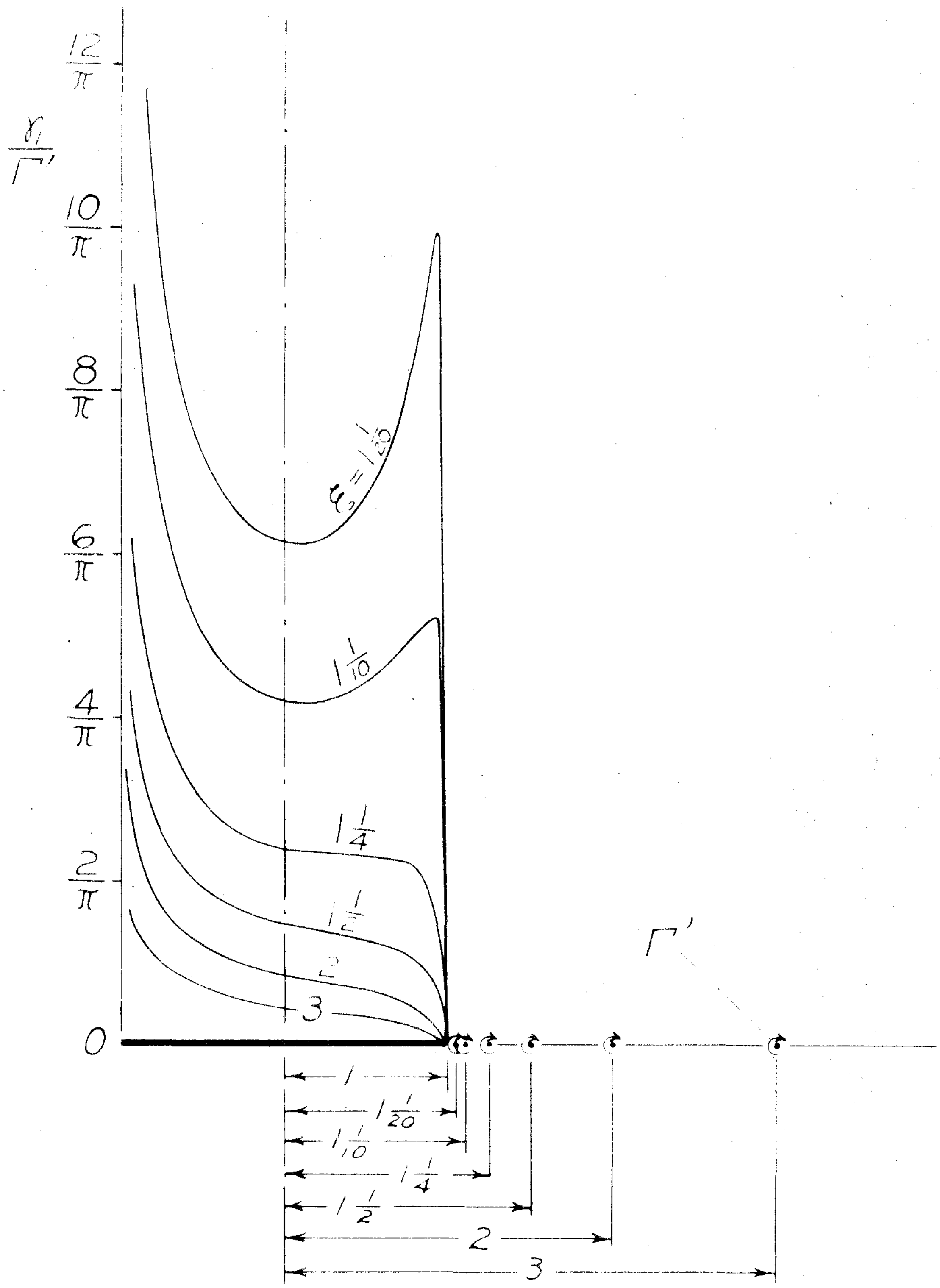


FIGURE 3

VORTICITY DISTRIBUTIONS INDUCED BY A WAKE VORTEX
AT VARIOUS DISTANCES FROM THE MIDPOINT OF THE AIRFOIL

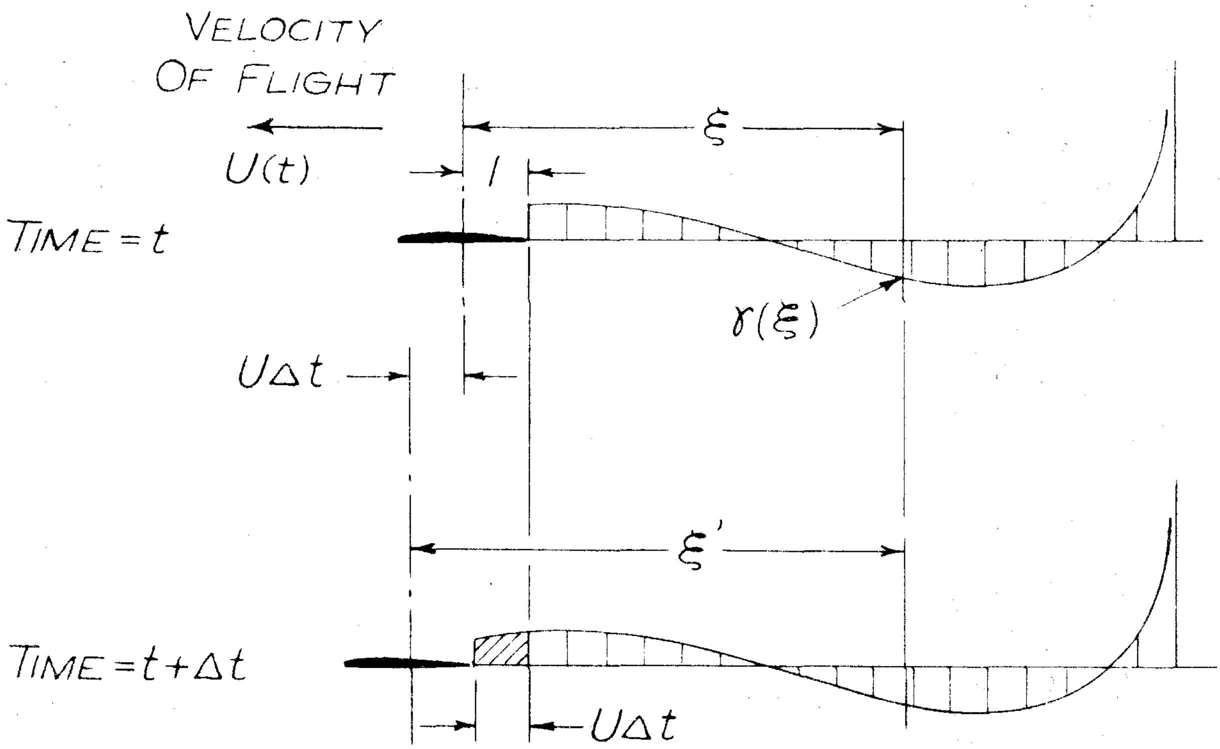


FIGURE 4

AUXILIARY DIAGRAM USED IN THE CALCULATION OF THE

TIME DERIVATIVES OF INTEGRALS OVER THE WAKE

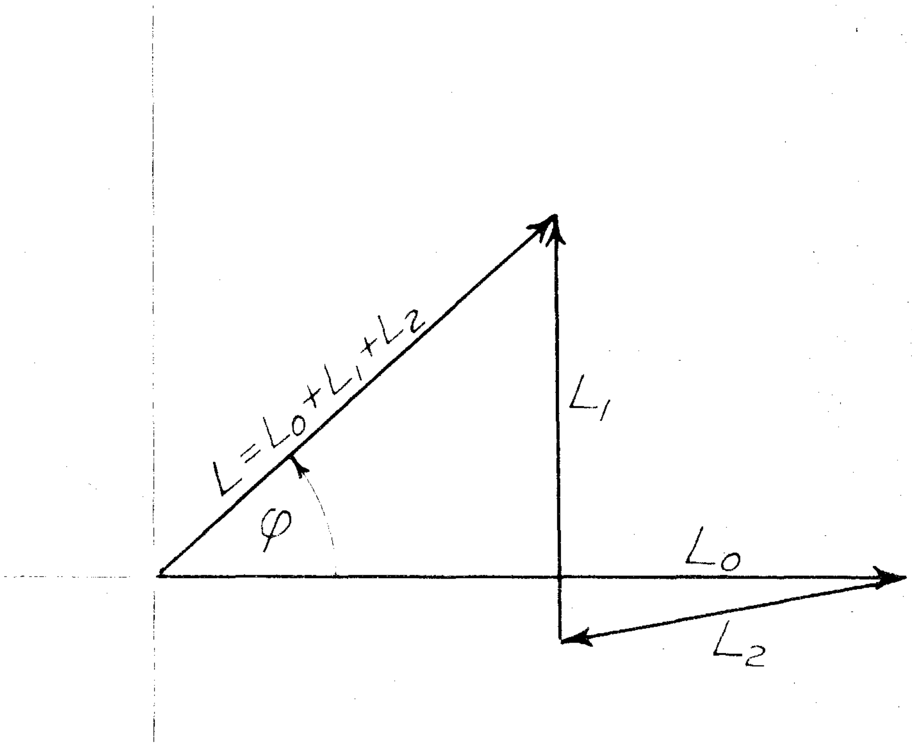
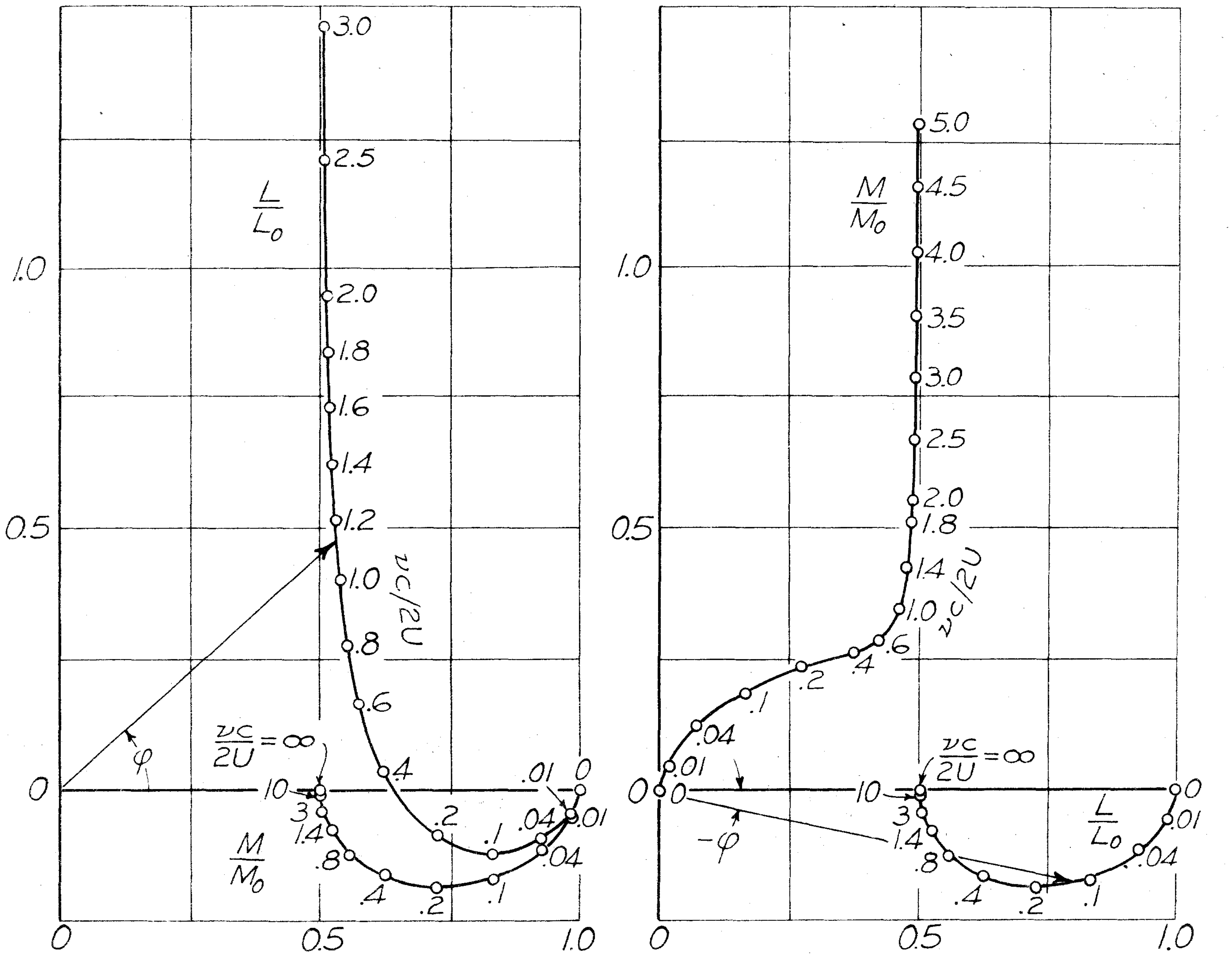


FIGURE 5

TYPICAL VECTOR DIAGRAM FOR THE LIFT OF AN OSCILLATING AIRFOIL



CASE 1: TRANSLATORY OSCILLATION   CASE 2: ROTATIONAL OSCILLATION

FIGURE 6

VECTOR DIAGRAMS FOR THE LIFT AND MOMENT OF OSCILLATING AIRFOILS, AS FUNCTIONS OF THE REDUCED FREQUENCY.

L_0 and M_0 are the respective quasi-steady values.

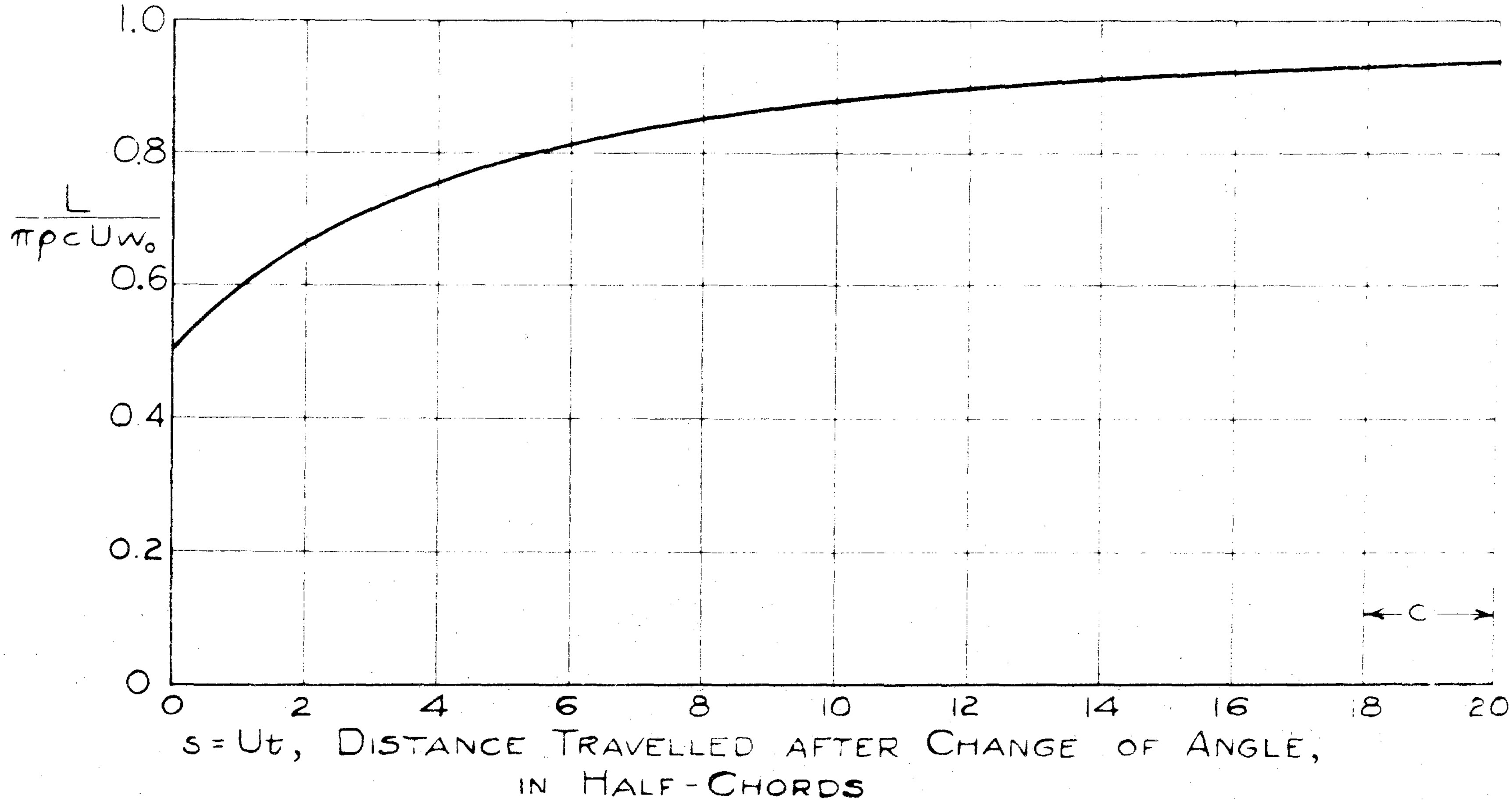


FIGURE 7

THE LIFT ON AN AIRFOIL FOLLOWING A SUDDEN
CHANGE OF ITS ANGLE OF ATTACK
(Airfoil chord = c)

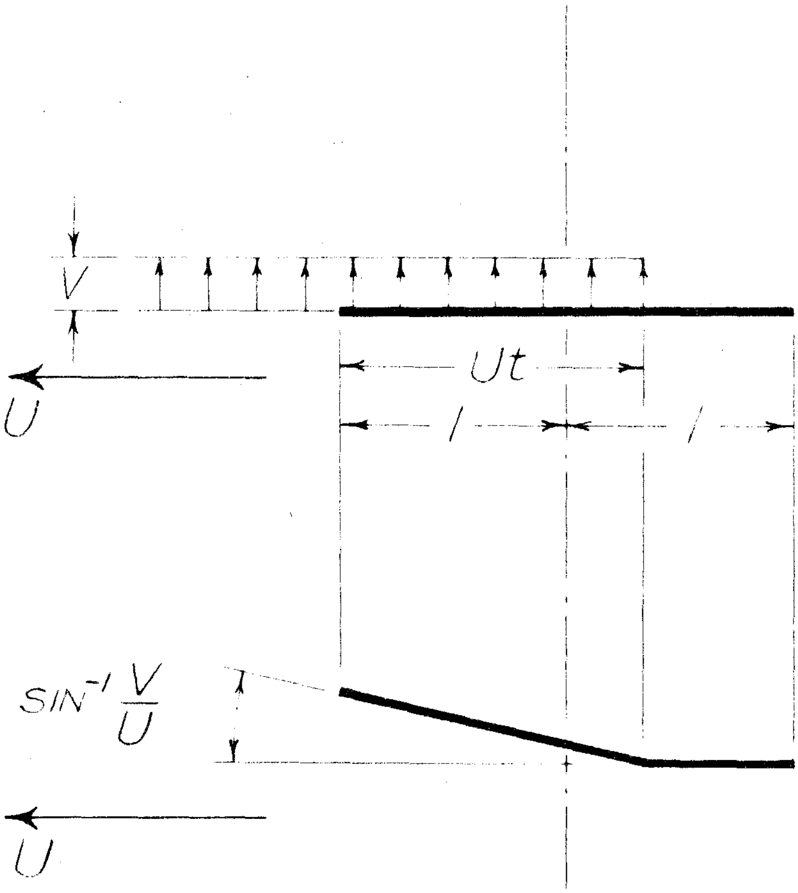


FIGURE 8

THE ANALOGY BETWEEN AN AIRFOIL ENTERING A
SHARP-EDGED GUST AND A BROKEN-LINE AIRFOIL

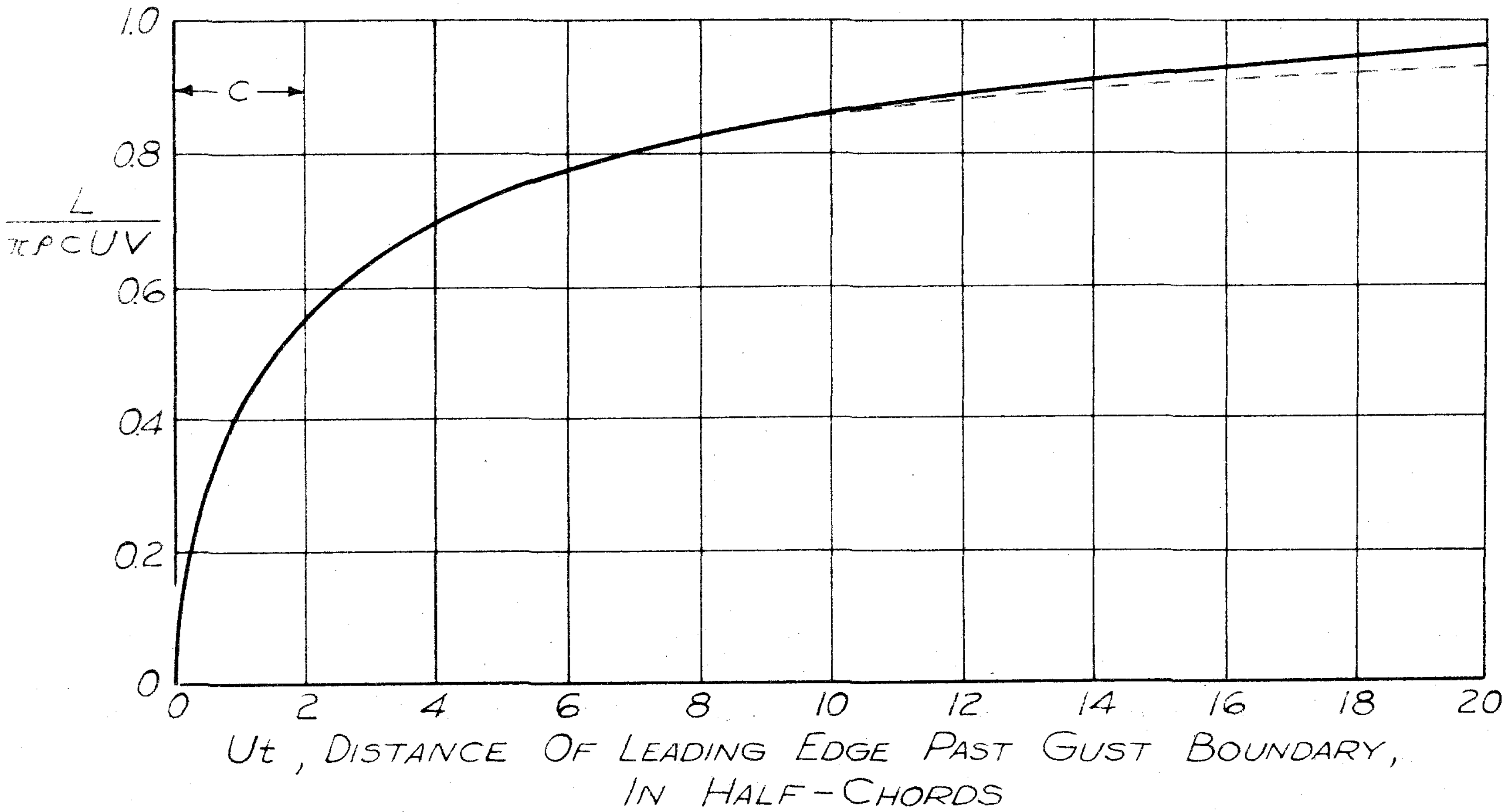


FIGURE 9

THE LIFT ON AN AIRFOIL DURING AND FOLLOWING
ITS ENTRANCE INTO A SHARP-EDGED GUST
(Airfoil chord = c)
The solid curve is obtained by the method of Section VII; the dotted curve by the method of Section IX.

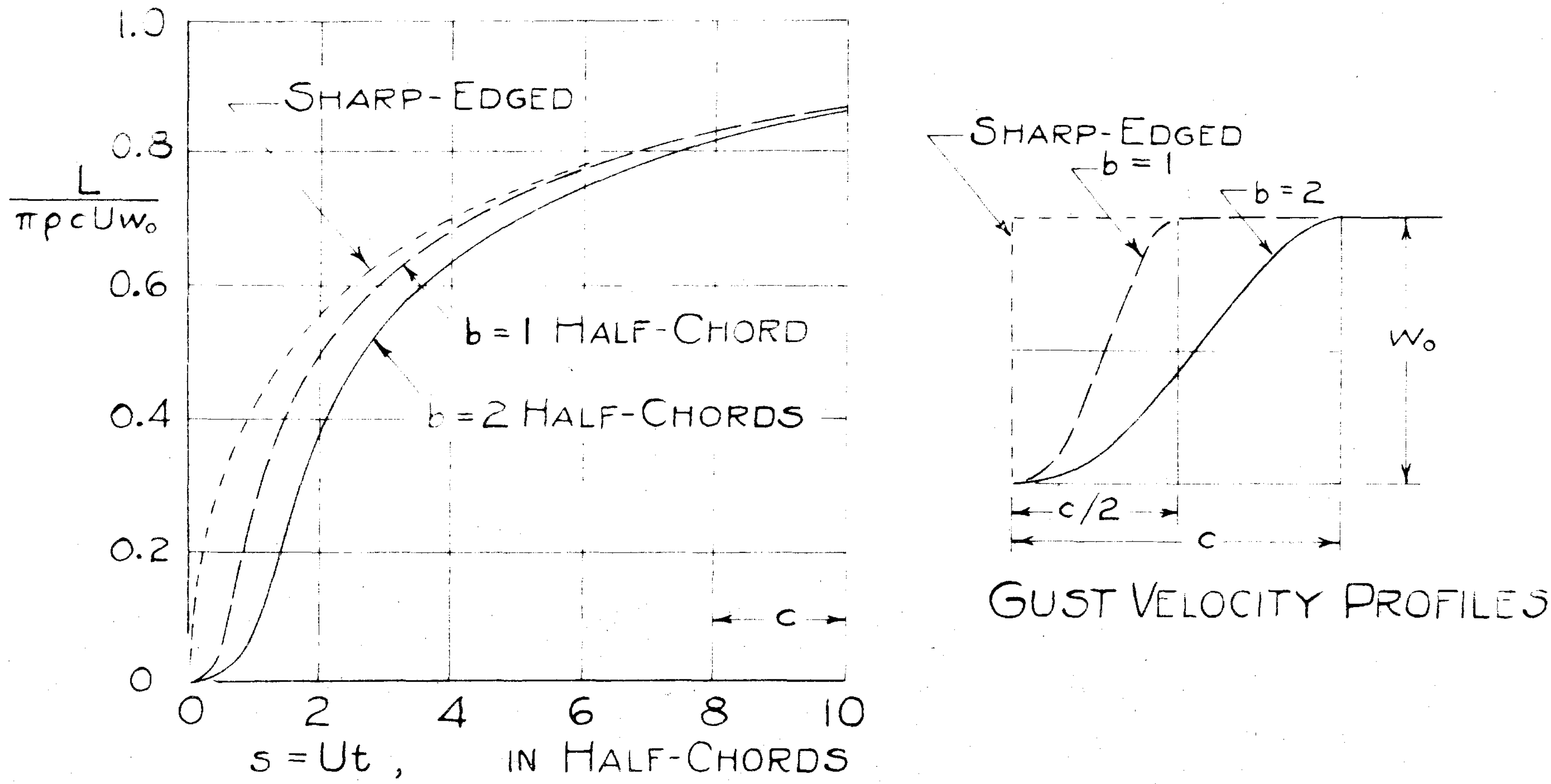


FIGURE 10

THE LIFT ON AN AIRFOIL DURING AND FOLLOWING ITS ENTRANCE INTO A GRADED GUST, FOR TWO VALUES OF b, THE WIDTH OF THE MIXING REGION

(Airfoil chord = c)

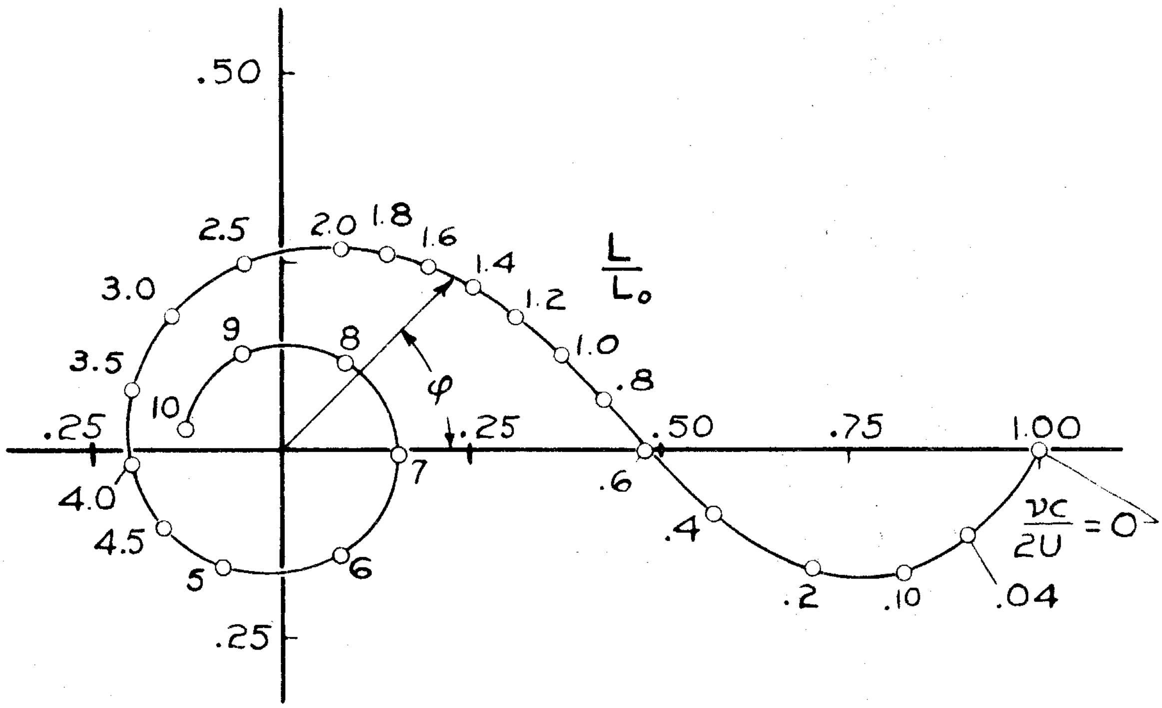


FIGURE 11

VECTOR DIAGRAM FOR THE LIFT ON AN AIRFOIL FLYING THROUGH A SERIES OF SINUSOIDAL GUSTS, AS A FUNCTION OF THE REDUCED FREQUENCY.

L_o is the corresponding quasi-steady lift.

(Airfoil chord = c, wave length of gusts = L_w, $\nu = 2\pi U/L_w$, thus $\nu c/2U = \pi c/L$)