TRANSIENT MOTION MEASUREMENTS FOR STRUCTURAL RESPONSE ANALYSIS

Thesis by

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ABSTRACT

The theory of the design of instruments suitable for the measurement of the acceleration-time history of a transient motion is considered. It is assumed that the primary purpose of the acceleration-time record is the calculation of structural response. As a criterion of instrument behavior, it is proposed that the correct reproduction of the response spectrum of the transient motion be adopted. The application of this response spectrum criterion to some selected acceleration pulses indicates that the optimum instrument parameters for a transient measurement are essentially the same as for the measurement of periodic motions.

A general theory for the multiple degree of freedom seismic accelerometer characteristics is established. The particular case of a two degree of freedom accelerometer is studied and it is shown that the sensitivity to acceleration can be increased to about five times that of a single degree of freedom instrument having the same useful frequency range.

An instrument which can be used to obtain structural response information directly, without the necessity of an acceleration-time recording is studied. This instrument, the reed gage, gives directly information concerning the transient response of a single degree of freedom system or of individual modes of a complex structure. The total response of a complex structure can be determined from reed gage information by the superposition of the maximum responses in the individual modes. An analysis is made of the errors in total response inherent in this superposition process. It is found that in many practical problems the distribution of mode frequencies and the form of the excitation is such that the total error due to superposition is not a significant factor in the problem.

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GENERAL INTRODUCTION

Transient motions have been measured and studied for many years. The measurements have been made with various forms of instrumentation (1) and have been motivated by various objectives. These motions arise from such phenomena as earthquakes, explosions, collisions, and operation of machinery.

Any instrumentation system will involve, in addition to the motion sensing instrument, a complete system of amplification, transmission, and recording of the information. The specific choice of the system in any particular application depends on many factors. These are concerned with the accuracy of the result and the practicability of the entire process. This thesis will be concerned with the motion sensing instrument itself.

An important reason for measuring transient motions or forces is that from such measurements, calculations of structural response can be made. In terms of the form of the recorded information, there are two types of instrumentation systems that are useful for response analysis. The first of these types attempts to record the true time history of the excitation. This measured excitation is then applied to the structure as a forcing function either by a computational process or by electric analog simulation. An instrumentation system of the second type records structural response information directly. The first type is more frequently used, but there has recently been interest in the more direct measurement of system response afforded by the second type.

The first part of this thesis will be concerned with vibration pickups of the seismic type. Such vibration pickups consist of a mechanical seismic system and a transducer which converts the motion of the seismic mass relative to the instrument case into a signal which can be recorded. There are two distinct frequency ranges of operation of the seismic pick-up. For frequencies high in comparison to the natural frequency, the pickup operates in the vibrometer or amplitude measuring range. For frequencies low compared to the natural frequency, operation is in the accelerometer or acceleration measuring range.

The accelerometer is often the type of instrument used under field conditions. This is due in part to the great usefulness of the data in the form of load information. An additional advantage is that, because of its high frequency, the accelerometer can be fabricated in compact form of high mechanical strength. In accelerometers, viscous damping is intentionally introduced to increase the useful frequency range of operation. This is feasible for all except the very high frequency instruments (such as, for example, the piezoelectric type), for which adequate damping is difficult to attain.

Specifically, Part One of this thesis will treat the design of the damped mechanical spring-mass system as an accelerometer. The basis of the treatment will be the need for a recorded acceleration-time history, which, when applied as an excitation to structures, will have an effect equivalent to that of the original excitation.

The seismic pickup, in the past, has been intentionally designed as a single degree of freedom system. The first consideration in this thesis will be a determination of the optimum damping of a single degree of freedom accelerometer for transient measurements. It will be indicated that the steady-state characteristics are a complete description of the response of a linear system to transient as well as periodic motions. Hence it will be possible to describe the operation of a transient measuring accelerometer in terms of its steady-state characteristics.

After the optimum conditions for a single degree of freedom system have been established, a multi-degree of freedom seismic system will be investigated for use as an improved accelerometer. It will be shown that the pickup sensitivity can be significantly increased over that of a single degree of freedom instrument having the same useful frequency range.

In Part Two of this thesis the second general class of instruments will be considered. It is assumed that the prime reason for making the measurement is to obtain a forcing function from which structural responses can be calculated. One may ask if it is possible to record information concerning structural response without the intermediate step of recording the excitation itself. From this point of view the instrument should consist of a simple mechanical model of the structure. This is most completely accomplished by a relatively new instrument known as the reed gage (2). In this instrument the peak responses of essentially single degree of freedom structures called reeds are mechanically recorded*. These peak motions are proportional to the peak normal mode responses of an actual structure. Failures due to transient excitation are normally not of a fatigue type, and hence the interest in such a problem is in peak structural response. This makes the reed gage useful even though the information derived from it is not complete.

The advantages of the reed gage as a rugged, self-contained, peak-

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* Peak reading devices may also be considered to be instruments which give response information directly. To the extent that they are single degree of freedom linear mechanical systems, they actually serve the same purpose as does a single reed. These devices are usually intended to have a natural frequency high enough so that they respond in a static manner to the excitation. The peak instrument response is then proportional to the peak value of the measured acceleration.

reading device have been gained at the expense of lost time information. The relative times of occurrence of the peak reed responses are not known. For this reason, answers from reed gage data for other than single mode responses are necessarily approximate. As an example of the magnitude of the error which might be introduced by the lack of time data, an analysis is carried out for a typical single pulse shock.

Part One

INTRODUCTION

The design considerations to be treated below are applicable to any instrument governed by a second order linear differential equation with constant coefficients operating in the "accelerometer" frequency range. The pressure pickup and the galvanometer are examples, in addition to the vibration pickup, which is the primary consideration in this part of the thesis.

In general, the response of a linear system to an arbitrary excitation has a different shape than does the excitation. However, there exists one unique excitation, namely sinusoidal, for which the shape is not changed and only two parameters are required to express the relation of steady-state output to input. The response is a sine wave differing from the input only in amplitude and phase. These alterations give rise to the amplitude and phase shift characteristics normally referred to as the steady-state characteristics*.

The steady-state characteristics of an accelerometer depart from the ideal at frequencies above a certain fraction of the instrument natural frequency. The sensitivity of the instrument (mechanical motion per unit acceleration) is inversely proportional to the natural frequency squared. Hence as one increases the instrument natural frequency, it becomes capable of recording higher frequency motions, but its sensitivity rapidly decreases and the transducer becomes incapable of producing signals large enough for accurate measurement. Thus in general one designs the instrument

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* Various synonyms for the steady-state characteristics in the terminology of the electrical engineer are the filter characteristics, admittance, and transfer function of a linear network.

to be useful to as high a frequency as is consistent with the desired signal level.

The characteristics required of an accelerometer to measure periodic motions are well understood. For the single degree of freedom instrument, the steady-state characteristics have been derived and plotted in standard texts on vibrations. A more complete discussion is given by Weiss (3) and Draper, McKay, and Lees (4). Damping in the range of 60 to 70 percent of critical is found to result in an instrument whose characteristics depart least from the ideal (within reasonable engineering accuracy) over the largest frequency range. The Fourier components of a measured periodic motion lying in the useful frequency range of the instrument are reproduced without significant distortion.

The accelerometer design required for the measurement of a transient acceleration, however, is not so well understood. Through a limiting procedure of the periodic case, it can readily be shown that a transient motion can be represented by a continuous spectrum of harmonic components (5, pg. 388; 6, pg. 444). This consideration leads to a criterion of instrument design which requires that a significant portion of the harmonic content of a transient lies in the useful frequency range of the instrument. This is clearly a different criterion than one which requires that a specific parameter of the motion be recorded accurately. In the past, the peak value criterion has been used. The reason for this choice of criterion is quite apparent. It is the most susceptible to theoretical treatment. The error in the reproduction of the peak value of single pulses of acceleration has been determined as a function of the instrument damping and the ratio of instrument natural period and pulse duration. The results of such investigations indicate that a very wide range of damping is satisfactory. No attempt has been made to generalize the results to more complicated transients. It will be pointed out below that the peak value criterion is not generally useful if the measurement is to be used as a forcing function, and a different criterion for this purpose will be proposed.

Once the optimum design for the single degree of freedom system has been obtained, it is natural to ask if further improvement can be made by the use of a more complicated seismic system. Therefore, a multiple degree of freedom system will be treated with the purpose of optimizing the steady-state characteristics for use as an accelerometer. With the resulting increase in the number of instrument parameters at our disposal, it is expected that an improvement can be made. It then becomes a matter of balancing the possible improvements and the practicality of the design.

I. STEADY-STATE CHARACTERISTICS OF THE SINGLE DEGREE OF FREEDOM ACCELEROMETER

In this section the performance of the single degree of freedom accelerometer is analyzed. The analysis is a summary of the steady-state response theory which is included to serve as a basis for the extension to transient excitation analysis. It will also serve to establish a notation and a technique that will be used to analyze the multiple degree of freedom accelerometer.

The single degree of freedom seismic system is shown in figure 1. Resulting from a frame motion $y(t)$ is the absolute motion of the mass x and the relative motion z. The mechanical output is proportional to z. The system parameters are the mass m, the linear spring constant k, and the viscous damping coefficient c. The differential equation of motion is

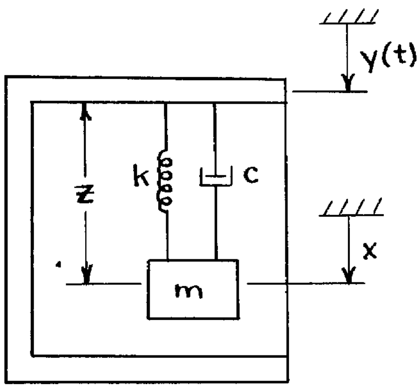

Figure 1. Single Degree of Freedom Seismic System

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x-y) = 0 .$$

Substituting the quantity $z = x - y$, we have

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y}$$

or

$$\ddot{z} + 2\beta p\dot{z} + p^2 z = -\ddot{y}, \tag{1}$$

where

$\beta = c/2\sqrt{km}$, fraction of critical damping

$p = \sqrt{k/m}$, natural frequency.

Thus the response is the same as if the base were fixed and the mass were excited by an acceleration $-\ddot{y}$ or a force $-m\ddot{y}$. If $\ddot{y}(t) = A \sin \omega t$, the steady-state solution may be written in the form

$$\frac{z}{A} = \frac{G(\omega)}{p^2} \sin\left[\omega t - \phi(\omega)\right], \qquad (2)$$

where

$$G(\omega) = \left\{\left[1-(\omega/p)^2\right]^2 + \left[2\beta\,\omega/p\right]^2\right\}^{-1/2}$$

$$\phi(\omega) = \tan^{-1} \frac{2\beta\,\omega/p}{1-(\omega/p)^2} .$$

A plot of $G(\omega)$ versus ω/p (fig. 2) is commonly referred to as the amplitude characteristic; $\phi(\omega)$ versus ω/p (fig. 3) is the phase shift characteristic. These two characteristics completely define the steady-state response to harmonic excitation.

For distortion-free superposition of any harmonic components of an input function, the amplitude characteristic must have a constant ordinate and the phase shift characteristic must be linear over the frequency range in question. To see this more clearly let us consider a periodic excitation which may be considered to be a superposition of a series of harmonic waves or Fourier components. The amplitude and phase shift effects brought about by the accelerometer on each component in the periodic excitation to be measured can be stated mathematically as follows: From the input acceleration component $A_i \sin\omega_i t$ results the output relative displacement component

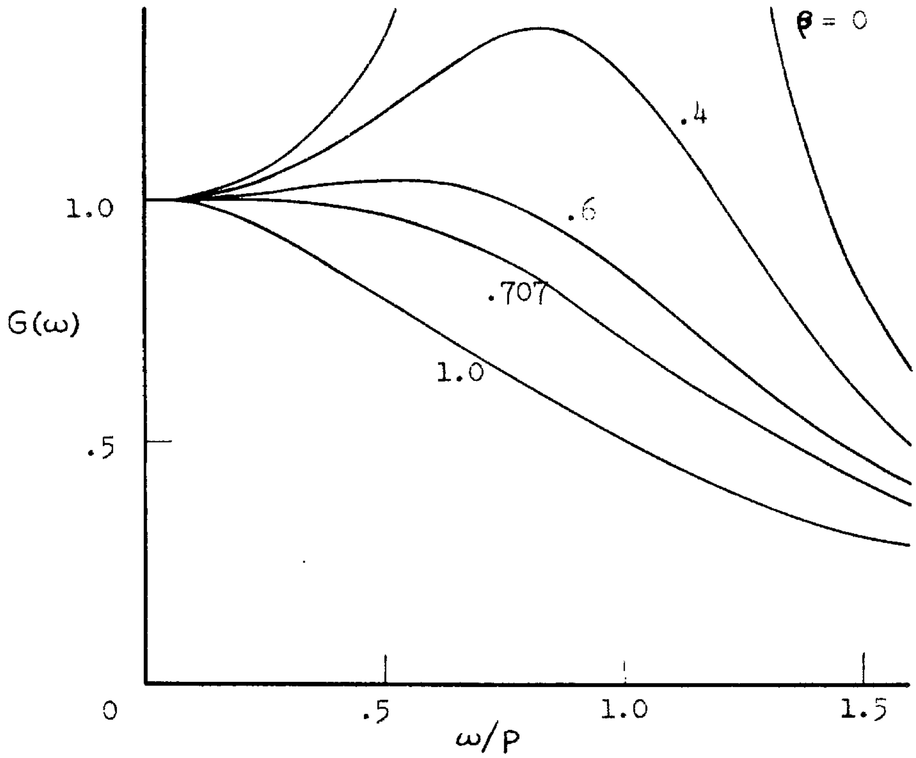

FIGURE 2. Single Degree of Freedom Amplitude Characteristic.

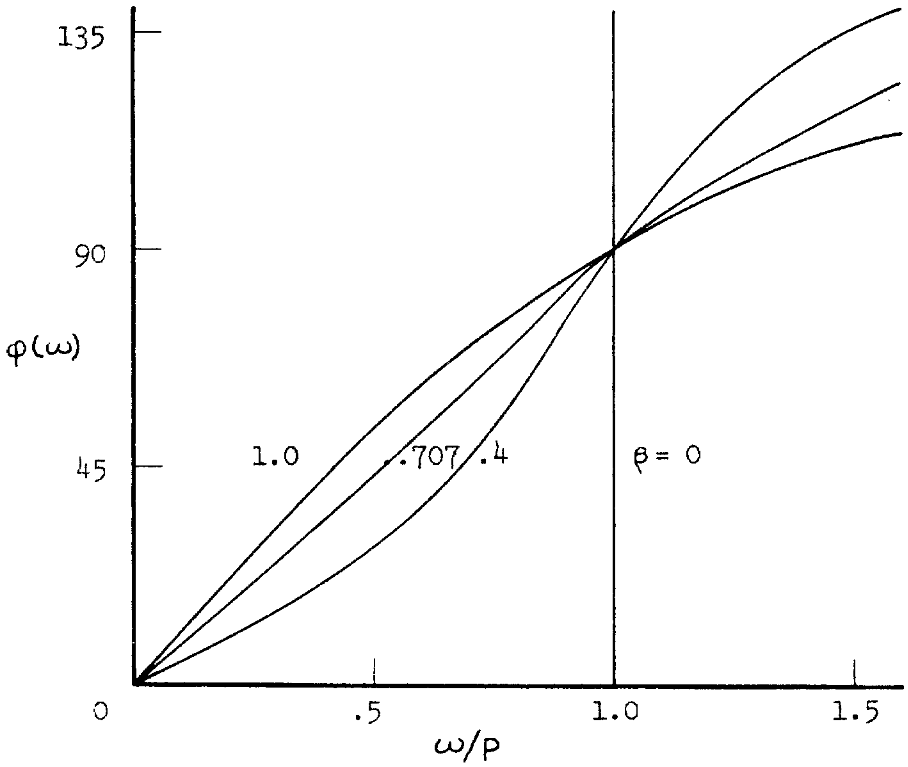

FIGURE 3. Single Degree of Freedom Phase Shift Characteristic.

$$\frac{G(\omega_i)}{p^2} A_i \sin\left[\omega_i t - \phi(\omega_i)\right] \quad \text{or} \quad \frac{G(\omega_i)}{p^2} A_i \sin \omega_i \left[t - \frac{\phi(\omega_i)}{\omega_i}\right]$$

where

ω_i = frequency of component under consideration

A_i = peak acceleration of the ω_i component

$\frac{G(\omega_i)A_i}{p^2}$ = peak relative amplitude of accelerometer mass due to A_i

$\phi(\omega_i)$ = phase lag between the response and input.

The quantity $\phi(\omega_i)/\omega_i$ represents the time lag of the i^{th} component through the accelerometer. Unless this quantity has a value independent of i, each of the components will be shifted in time by a different amount and their resultant will be a distorted version of the input wave form. Thus if $\phi(\omega_i)/\omega_i = t_0$ in a given frequency range, each input component in this range will be delayed t_0 seconds by the accelerometer. The combination of flat amplitude characteristic and linear phase shift will result in undistorted accelerometer response. The instrument only operates without distortion where both conditions are satisfied.

Figures 2 and 3 show that the characteristics in the range β = 0.6 to 0.7 provide a large useful range. For β = 0.707 the second derivative of the amplitude characteristic is zero at zero frequency, thus giving the "flattest" curve*. The phase shift for β = 0.707 is very nearly linear. A little less damping produces a slight hump in the amplitude characteristic giving a somewhat longer useful amplitude range if a few percent error is tolerated. The phase shift is not quite as linear

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* The odd derivatives are all zero at zero frequency.

as for β = 0.707, but is still satisfactory. Therefore, for the largest useful range with respect to the natural frequency, a single degree of freedom seismic accelerometer should be damped in the range β = 0.6 to 0.7.

II. OPTIMUM CHARACTERISTICS FOR TRANSIENT MEASUREMENTS

It is not obvious that the steady-state characteristics defined above have direct quantitative application to the measurement of transient accelerations. If a sufficiently large range of the frequency spectrum of the transient lies in the range of ideal steady-state characteristics, the output will closely duplicate the input. This can always be accomplished by using a sufficiently high natural frequency instrument. Two problems then arise. The first is how to build a very high frequency single degree of freedom system with sufficient damping. The second is to find a transducer with sufficient sensitivity. The mechanical system should be designed to alleviate these difficulties.

Mechanically the problem is to determine, given an accelerometer natural frequency and hence a sensitivity, what value of damping will give the "optimum" response. It is apparent that the definition of "optimum" might affect the result. For some definitions, the flat amplitude and linear phase shift characteristics are not the most satisfactory. As a matter of fact, the result depends on the particular transient chosen for study, as well as the criterion for optimum response.

Quantitative treatments of accelerometer design for transients have all been based on the reproducibility of the peak value of the transient (3, 7, 8). This has been the sole criterion for satisfactory response.

Single shock pulses of the half-sine, triangular, and square types have been treated. The results of these studies have not been generalized to produce design criteria for the measurement of more complex transient accelerations. The conclusion reached was: for five percent accuracy in the measurement of the peak value of acceleration pulses having the general character of triangular or sinusoidal pulses, the natural period of the accelerometer must be less than 1/3 of the pulse duration and the instrument damping should be 40 to 70 percent of the critical value (7).

The questionable usefulness of the peak value criterion and the inability to generalize the results suggest a closer look at the problem. As a first step, a more useful criterion for satisfactory response is necessary.

Given a tolerable error, each of the following criteria (in order of decreasing severity on the accelerometer design) is reasonable, depending on the use to which the measurement is to be put:

1. Undistorted reproduction of transient in both magnitude and time;
2. Undistorted reproduction in magnitude, allowing time delay due to instrument phase lag;
3. Response spectrum* of accelerometer output to be an undistorted version of the response spectrum of the transient;
4. Magnitude of peak accelerometer response to equal peak value of transient;
5. Accurate reproduction of the area under the acceleration-time history (velocity change for impulsive loading).

* The response spectrum of an acceleration-time history gives the relationship between some maximum response parameter of a single degree of freedom system, and the natural frequency of the system. In this discussion the ordinate is an equivalent static acceleration, which is that value of acceleration statically applied to deflect the system to the peak dynamic displacement.

Criterion 3 is suggested as the most useful one, in view of the very common use of transient measurements to calculate the response of structures to the excitation. If this be the case the peak value criterion 4, the only one considered previously, ignores all those systems whose natural frequencies lie outside the static region of the response spectrum (where the peak value is of prime importance). Actually the last two criteria are included as special cases of the third one. The static portion of the response spectrum is a function of the magnitude of the peak excitation, the constant slope portion for low frequency systems is a measure of the area under the acceleration-time history. Hence a study of criterion 3 will also give results for criteria 4 and 5.

In conclusion, for those cases where the transient measurement is required in order to predict the peak response of single degree of freedom systems (or those systems for which the single degree of freedom approximation is valid (9)), the response spectrum approach for qualifying the acceptability of the measurement is certainly the most logical one. Also, insofar as the response spectrum values can be applied to normal mode responses in multi-degree of freedom structures, this criterion is a useful one.

III. RESPONSE SPECTRUM CRITERION FOR QUALIFICATION OF ACCELEROMETER DESIGN

Instead of using the entire response spectrum in the qualification of accelerometer measurements, it is found that three principal parameters from the response spectrum of an acceleration pulse determine its essential features. These are represented by the symbols A , B , and C in figure 4.

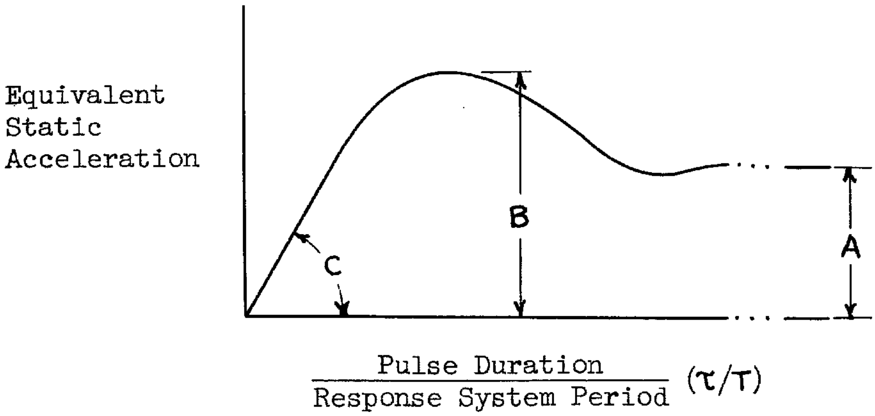

Figure 4. Response Spectrum for Single Pulse of Acceleration

This is the type of spectrum obtained for acceleration pulses not containing infinite rates of acceleration (10). The value A is the peak acceleration in the transient; B is the peak of the spectrum; C is the slope at τ/T equals zero.

To apply this criterion an electric analog was first used to obtain the accelerometer response to an input function*. The input function and the response were then spectrum analyzed (11). From these spectra the ratios of the quantities shown in figure 4 for the input and response were plotted. This was done as a function of τ/T for 40, 60, 70 and 100 percent of critical damping in the accelerometer for the half-sine and the triangular acceleration pulse. The results are shown in figures 5 - 10. The subscripts r and i refer to the accelerometer response and input respectively.

For the curves of A_r/A_i (figs. 5, 6) a number of interesting observations may be made. In the first place, the results of Levy and Kroll (7) may be plotted on these curves. These are the points denoted by X's obtained by numerical integration for T/τ values of 0.2, 0.3, and 1.0

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* See Appendix I.

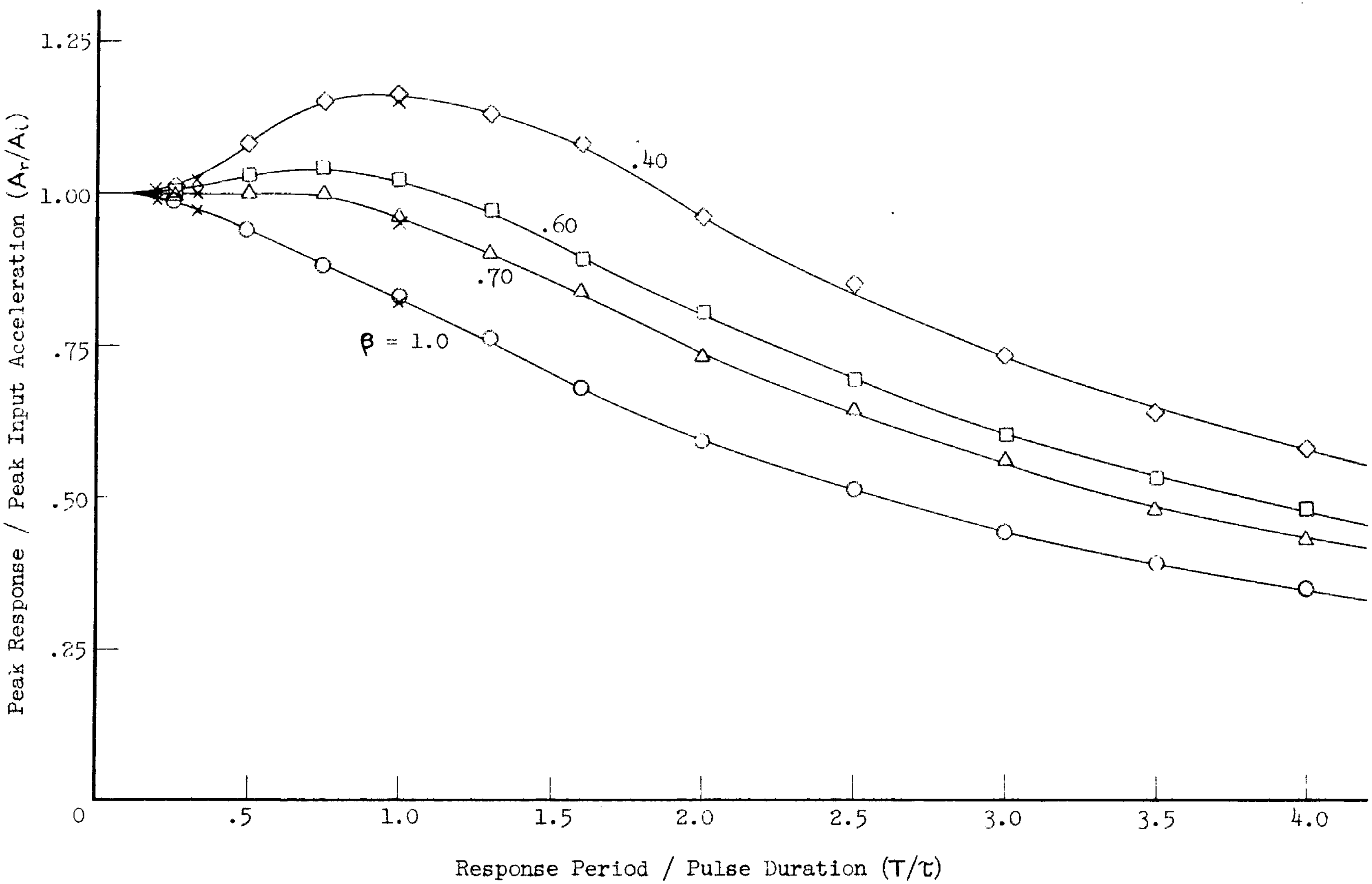

FIGURE 5. Reproduction of Static Level of Response Spectrum for Half Sine Input Acceleration.

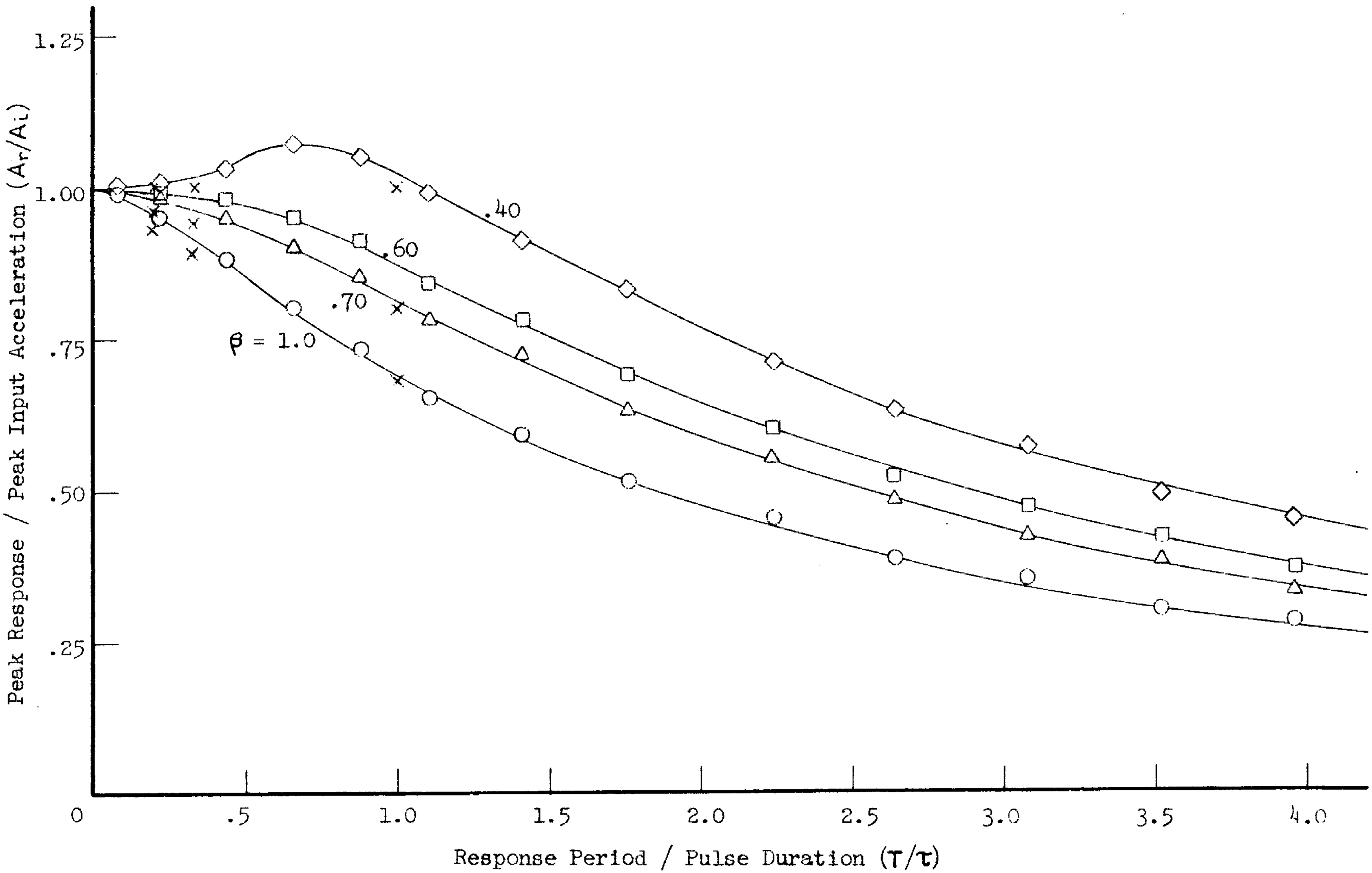

FIGURE 6. Reproduction of Static Level of Response Spectrum for Triangular Input Acceleration.

at 40, 70, and 100 percent of critical damping. The correspondence to the analog results is seen to be reasonably good. An interesting point is to notice the position of the points given by Levy and Kroll for the 40 percent damping with the triangular input. Their calculations were for points on either side of a hump in the curve and hence indicated a flatter frequency response than seems to exist.

The case of the half sine pulse shows a behavior of the A_r/A_i curves similar in appearance to the steady-state amplitude characteristics. The 70 percent damping curve has the longest flat range, to about 0.8 T/τ. The triangular input shows a different behavior with none of the damping values chosen giving a large flat region. Here is seen a reason for the statement made earlier concerning the fact that the conclusions as to acceptability of a measurement may differ for different transients. In the case of the triangular output, for example, 50 percent damping appears best for the A_r/A_i parameter. Qualitatively, this is due to the fact that some of the low frequency content is amplified, while the high frequency content is attenuated by the instrument, resulting in a good value of peak response. This is illustrated by one of the figures in Levy and Kroll's paper. In this paper figure 6, curve (2) shows an accurate peak value, but the high frequency content in the input does not appear in the output which has a nicely rounded peak. Thus, depending on the harmonic content of the transient, different values of accelerometer damping may prove best, if the peak value is the sole criterion.

The set of curves for B_r/B_i (figs. 7, 8) shows a more consistent behavior with regard to damping. Both inputs give a large rise (25 percent for triangle, 30 percent for half sine) in the value of B_r/B_i at

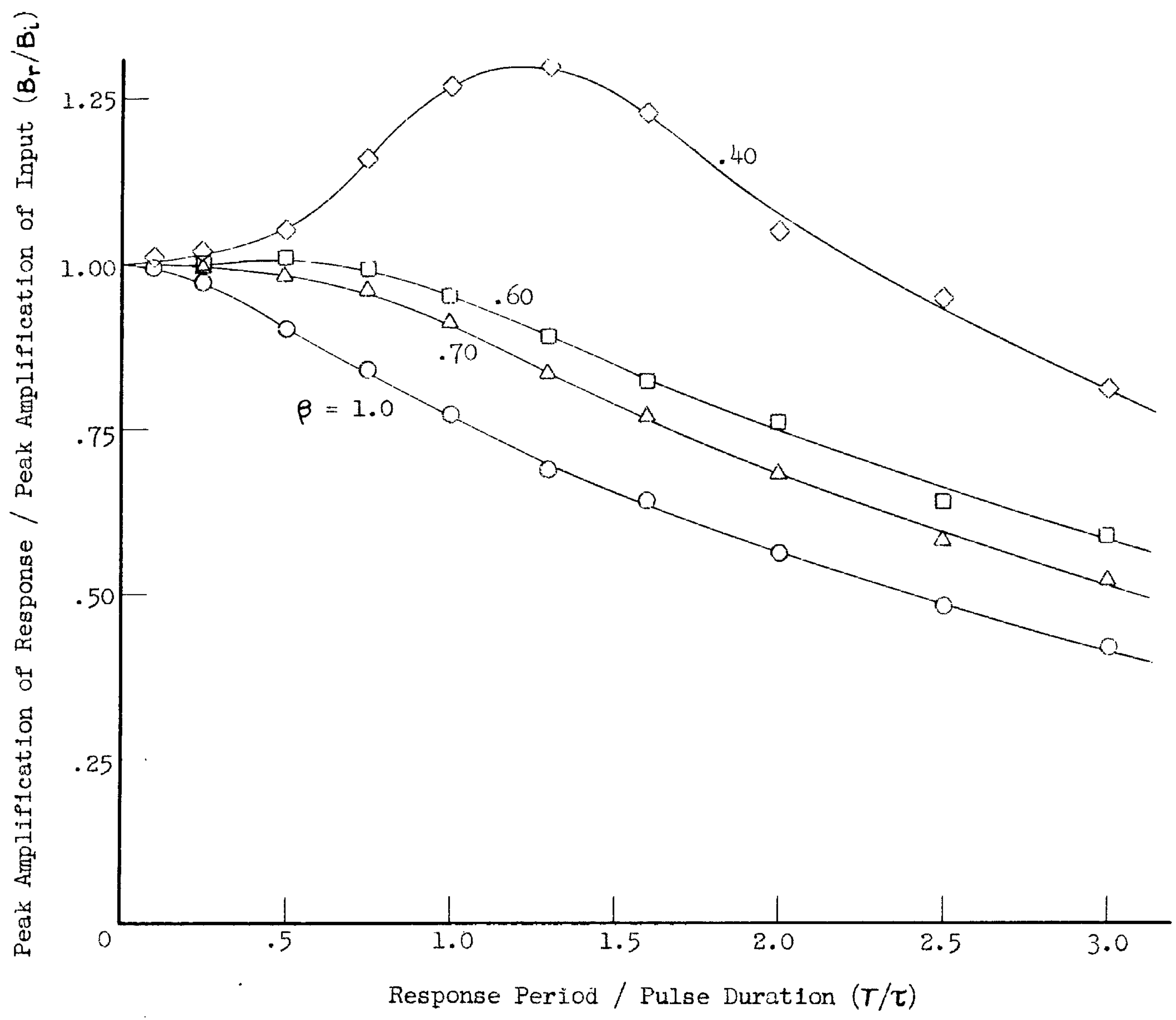

FIGURE 7. Reproduction of Peak of Response Spectrum for Half Sine Input Acceleration.

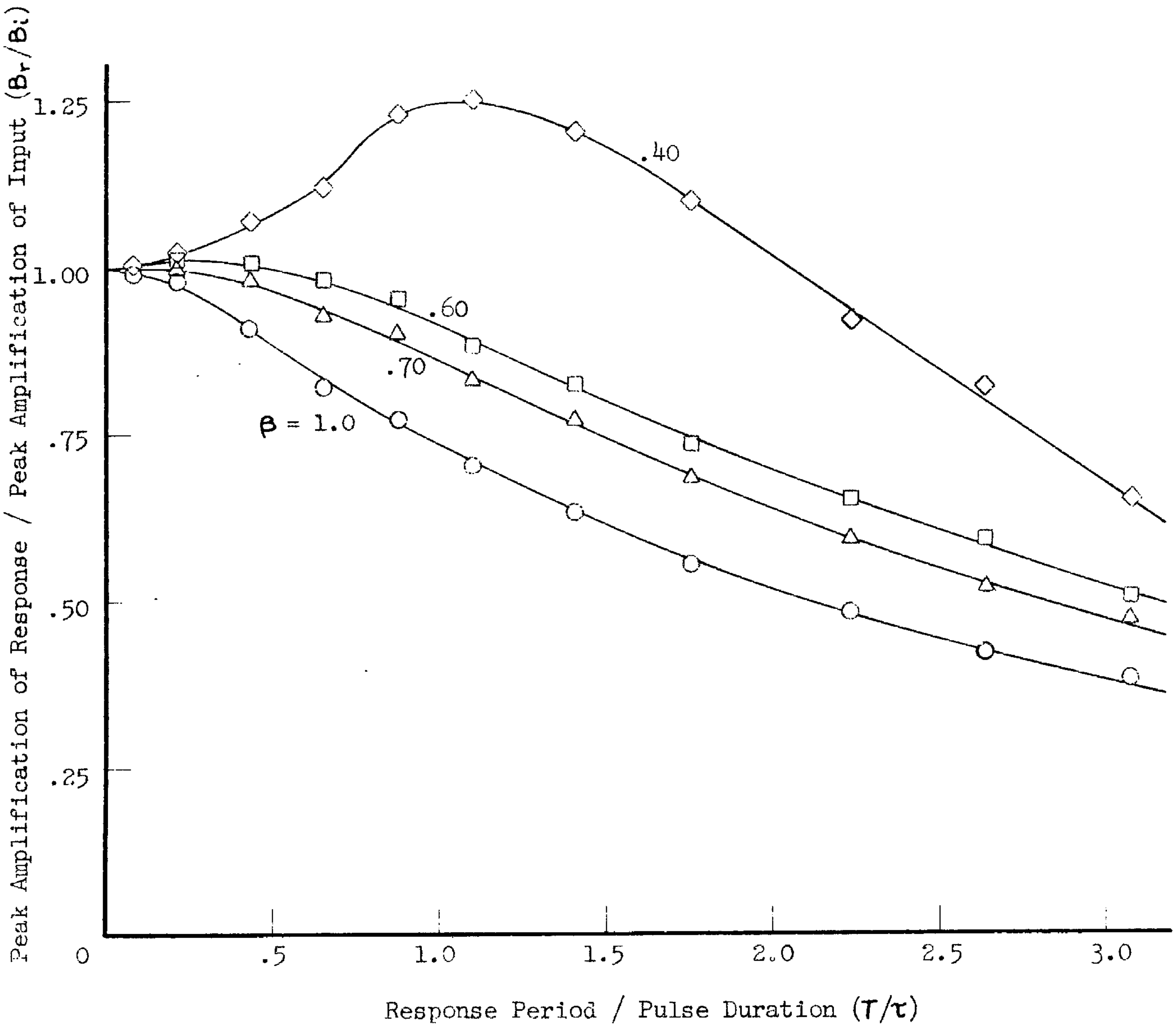

FIGURE 8. Reproduction of Peak of Response Spectrum for Triangular Input Acceleration.

40 percent damping between T/τ values of 1.0 and 1.4. The reason for this behavior is readily apparent when viewing the accelerometer response. At 40 percent damping, a decaying tail involving the damped natural frequency of the accelerometer appears. A spectrum analysis of such a response shows that systems having a natural frequency near the frequency in the tail are somewhat resonated. This then clearly indicates one reason why such a low value of damping is undesirable. Harmonic content near the accelerometer damped natural frequency is introduced into the response with sufficient magnitude to give highly erroneous results for a large range of response system periods. With 60 percent or more damping no trouble of this nature arises.

One technique of record improvement mentioned by Levy and Kroll involves fairing a line through oscillations in the record occurring near the accelerometer natural frequency. This requires that some information about the shape of the transient be known. Otherwise one cannot be sure that the slight wiggles are not actually in the input function. Secondly, it may be that for more complicated transients, these wiggles are completely indistinguishable from other features in the motion. Only in the case of a relatively high frequency accelerometer with almost no damping, wherein the response clearly oscillates about the input function, can this technique be generally employed. Hence for damping as high as 40 percent, one must expect not to be able to rid the record of oscillation at the damped natural frequency of the accelerometer. Consequently the record will give erroneous results of the type shown in figures 7 and 8.

Figures 9 and 10 showing the reproducibility of the impulsive character of the transient have considerable practical importance. This parameter is much less sensitive to damping than is A or B. Good results

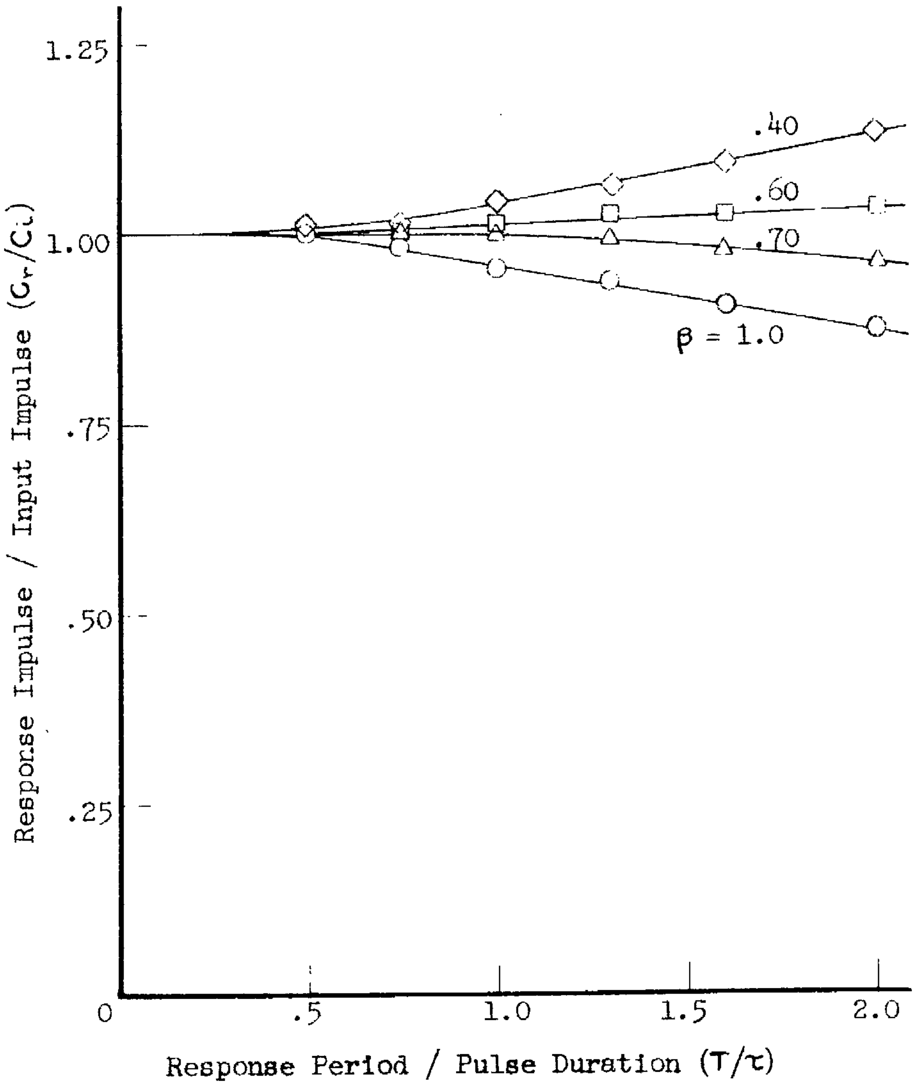

FIGURE 9. Reproduction of Impulsive Level of Response Spectrum for Half Sine Input Acceleration.

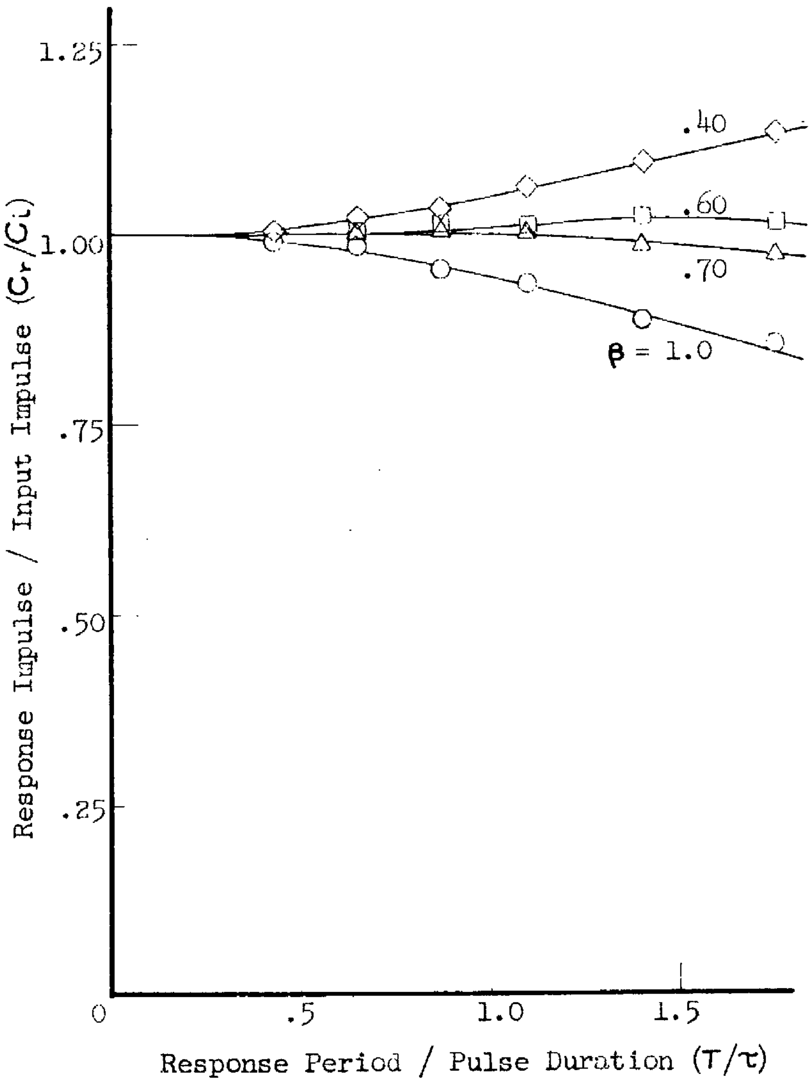

FIGURE 10. Reproduction of Impulsive Level of Response Spectrum for Triangular Input Acceleration.

are obtained over a large response period range. Therefore, if only the impulsive nature of the transient is desired, a much lower frequency accelerometer is required than any other criterion might indicate. The curves of C_r/C_i are limited to the T/τ values shown by the lowest frequency obtainable on the spectrum analyzer.

The purpose of studying these simple pulses is to point out the implications of the statements of other investigators concerning these cases. It is also important to generalize the results in order for them to be applicable to more complex transients. The only conclusion that can be drawn from the data presented is that if the reproducibility of the response spectrum is the criterion for satisfactory accelerometer response, the instrument damping must be such as to give flat amplitude and linear phase shift characteristics.

For the single degree of freedom accelerometer, damping at 70.7 percent of critical results in an amplitude characteristic having a zero second derivative at zero frequency. Since the amplitude characteristics always have a zero slope initially, this value of damping produces the "flattest" amplitude characteristic. Zero* or 86.6 percent damping results in a zero second derivative in the phase shift characteristic at zero frequency. These values of damping produce the most linear phase shift characteristic. If an estimate is made of the relative importance of maintaining the ideal conditions in the amplitude and phase shift characteristics, it is found that the instrument accuracy is more dependent

* The undamped accelerometer (such as the piezoelectric type) causes difficulty because of resonance of harmonics in the neighborhood of the natural frequency. This effect is minimized if either the input contains no harmonic content or the output is attenuated in this frequency range. This latter is accomplished by use of a low-pass filter if the accelerometer signal is electrical in nature.

on the amplitude characteristic. This can be seen by observing that the phase shift characteristic departs from its initial slope at 70.7 percent damping by only about 2.5 degrees at half the natural frequency. This corresponds to a time shift between a harmonic of low frequency and a harmonic at half the accelerometer natural frequency of 1.4 percent of the period of the higher frequency harmonic. The magnitude of the error thus introduced in the phase shift is a small price to pay for the useful frequency range gained in the amplitude characteristic (see fig. 2).

If we consider a complex transient as being composed of pulses, the sharpest pulse (highest important frequency components) will determine the natural frequency required for the accelerometer. If this pulse is satisfactorily reproduced, a significant portion of its harmonic content must have been in the useful range of the accelerometer. The other pulses, having less high frequency content, will have been reproduced even more accurately. Since the phase shift characteristic is linear in the useful range, no significant phase distortion will have been introduced between the pulses. Figures 5 to 10 show that the accelerometer period must be less than about 0.4 of the period of the sharpest pulse to obtain at least 5 percent accuracy in the principal features of the response spectrum of the measured transient.

As a point of interest it is noted that the requirement that there be no phase distortion between the pulses in the transient is not necessary for reproducing its impulsive nature. This is true since only the area under the acceleration-time history and not the specific shape is required. Therefore, the curves of C_r/C_i are only dependent on the amplitude characteristic,and 60 to 70 percent of critical damping must be the most desirable range.

In general, one does not know the shape of the transient to be measured. Hence one cannot know its harmonic content. Therefore it appears evident that any attempt which makes use of other than the essentially ideal steady-state accelerometer characteristics cannot have general applicability. The only general approach must attempt to reproduce a certain portion of the harmonic content of the transient with a small error. The extent of this range is determined by the use to which the measurement is to be put.

As previously discussed, if the measurement is to be used to predict the response of any structure, the entire response spectrum must be reproduced. The natural frequency of the accelerometer required is thus determined by the harmonic content in the transient and the accuracy desired. However, there is another important case that must be considered. It may be that the measurement is made for the response analysis of a specific structure. Harmonic content in any excitation in the frequency range where the amplitude characteristic of the structure is extremely small will not have a significant contribution to the structural response. For this situation the accelerometer natural frequency may be determined by the structure itself, since only harmonic content in the transient to which the structure will respond with significant amplitude need be recorded. In conclusion, either the harmonic content of the transient or the characteristics of the structure on which the transient will act may determine the required range of the accelerometer. The consideration which gives the smaller range would, of course, be the determining factor.

IV. MULTIPLE DEGREE OF FREEDOM ACCELEROMETER

It has been determined that the optimum steady-state accelerometer characteristics for transient as well as periodic motions are the flat amplitude and linear phase shift characteristics. This will insure that the motion will be reproduced sufficiently well to allow accurate determination of peak structural response from the recorded acceleration-time history. Having the optimum conditions for a single degree of freedom system, it is now logical to consider a more complex system to see if improvement in overall characteristics can be attained.

Taylor (12) and Manildi (13) have treated a specific example of a two degree of freedom system for use as an improved accelerometer. They considered an ordinary simple seismic system, spring coupled to a rotary mass. In both treatments only the rotary mass was damped and the instrument output was proportional to the angular position of the rotary mass.

Taylor's analysis aimed at increasing the rate of cutoff of the amplitude characteristic of the system to increase its effectiveness as low-pass filter. He considered both the amplitude and phase response of the system, varying the parameters until the best design was obtained.

Manildi showed three types of improvement over the conventional single degree of freedom design. The first was an increase of the "flat response" range with respect to the natural frequency of the non-rotary spring-mass system in the two degree of freedom accelerometer. Secondly the rate of cutoff was increased. Finally, the system was shown to be a better filter at high frequencies. It was indicated that the amplitude characteristic falls off as the fourth power of frequency of harmonic excitation, instead of the second power as it does for the single degree of freedom instrument.

These studies show that the additional parameters of a multiple degree of freedom system can be used to improve its characteristics as an accelerometer.

These previous studies confined themselves to a particular combination of a linear and a rotary element. It is the object of the present work to investigate a more general theory of the multiple degree of freedom system as an accelerometer. The single degree of freedom system has a sensitivity of $G(\omega)/p^2$ (displacement per unit acceleration), where $G(\omega) = 1$ at zero frequency. As shown above, the static sensitivity can be maintained over a large frequency range by the use of proper damping. In the multiple degree of freedom case $\Gamma(\omega)/p_1^2$ shall be used for the acceleration sensitivity. The static sensitivity is then Γ_0/p_1^2, where Γ_0 is $\Gamma(0)$ and p_1 is the first natural frequency. In the multiple degree of freedom case, as in the single degree of freedom system, damping will be introduced to extend the static sensitivity to as large a range as possible.

To gain an insight into the general features of the problem, an electric analog system of a two mass mechanical oscillator was set up and the damping coefficients were adjusted to maximize the region of flat response. It was experimentally observed that the obtainable length of the flat amplitude characteristic was principally determined by the first natural frequency. Thus a logical approach is to first maximize the static sensitivity Γ_0, treating an undamped system, and then to introduce damping into the system to maximize the region of flat amplitude and linear phase shift. This approach limits the number of parameters that must be considered at one time.

A. STATIC SENSITIVITY

Consider the spring-mass system shown in figure 11. The absolute displacement of the i^{th} mass is denoted by x_i. Applying Newton's Second Law of Motion, the equations of motion for the masses are

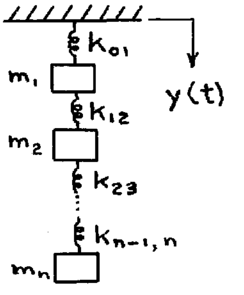

Figure 11. Multi-Degree of Freedom Seismic System.

$$m_1\ddot{x}_1 + k_{01}(x_1 - y) + k_{12}(x_1 - x_2) = 0$$

$$m_2\ddot{x}_2 + k_{12}(x_2 - x_1) + k_{23}(x_2 - x_3) = 0$$

$$\vdots$$

$$m_i\ddot{x}_i + k_{i-1,i}(x_i - x_{i-1}) + k_{i,i+1}(x_i - x_{i+1}) = 0$$

$$\vdots$$

$$m_n\ddot{x}_n + k_{n-1,n}(x_n - x_{n-1}) = 0.$$

If the relative displacements between the masses and the base are denoted by z, so that $z_i = x_i - y$, the equations become

$$\begin{aligned}
&m_1\ddot{z}_1 + k_{01}z_1 + k_{12}(z_1 - z_2) = -m_1\ddot{y} \\
&m_2\ddot{z}_2 + k_{12}(z_2 - z_1) + k_{23}(z_2 - z_3) = -m_2\ddot{y} \\
&\vdots \\
&m_i\ddot{z}_i + k_{i-1,i}(z_i - z_{i-1}) + k_{i,i+1}(z_i - z_{i+1}) = -m_i\ddot{y} \\
&\vdots \\
&m_n\ddot{z}_n + k_{n-1,n}(z_n - z_{n-1}) = -m_n\ddot{y}.
\end{aligned} \qquad (3)$$

Letting $y(t) = Y \sin \omega t$, $z_i = Z_i \sin \omega t$ is obtained under steady-state conditions. The following matrix equation describes this steady-state excitation.

$$\begin{bmatrix} k_{01}+k_{12}-m_1\omega^2 & -k_{12} & 0 & \cdots & 0 \\ -k_{12} & k_{12}+k_{23}-m_2\omega^2 & -k_{23} & \cdots & 0 \\ 0 & -k_{23} & k_{23}+k_{34}-m_3\omega^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & k_{n-1,n} \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{Bmatrix} = -\omega^2 Y \begin{Bmatrix} m_1 \\ m_2 \\ m_3 \\ \vdots \\ m_n \end{Bmatrix}. \qquad (4)$$

This equation may be written in the abbreviated form

$$\left([K] - [M]\,\omega^2\right)\{z\} = \{m\}A, \qquad (4\ )$$

where

$$[K] = \begin{bmatrix} k_{01}+k_{12} & -k_{12} & 0 & \cdots & 0 \\ -k_{12} & k_{12}+k_{23} & -k_{23} & \cdots & 0 \\ 0 & -k_{23} & k_{23}+k_{34} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & k_{n-1,n} \end{bmatrix}$$

$$[M] \begin{bmatrix} m_1 & 0 & 0 & \cdots & 0 \\ 0 & m_2 & 0 & \cdots & 0 \\ 0 & 0 & m_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & m_n \end{bmatrix}, \quad \{z\} = \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{Bmatrix}, \quad \{m\} = \begin{Bmatrix} m_1 \\ m_2 \\ m_3 \\ \vdots \\ m_n \end{Bmatrix}, \quad A = -\omega^2 Y.$$

The characteristic equation for the determination of the natural frequencies of the system is

$$\left| [K] - [M]\,\omega^2 \right| = 0. \qquad (5)$$

For cases of practical interest this determinant will have n distinct, positive solutions p_i^2 (i = 1, 2,...n). Hence it must be possible to write the determinant in the equivalent form

$$\left|[K]-[M]\omega^2\right| = a(\omega^2-p_1^2)(\omega^2-p_2^2)\cdots(\omega^2-p_n^2).$$

Since the ω^{2n} term in the expansion of equation 5 comes from the product of the diagonal terms, we see that $a = m_1 m_2 \cdots m_n$. Thus

$$\left|[K]-[M]\omega^2\right| = \prod_{i=1}^{n} m_i(\omega^2-p_i^2). \tag{6}$$

In particular for $\omega = 0$

$$|K| = (-1)^n \prod_{i=1}^{n} m_i p_i^2. \tag{7}$$

If this system is to be used as an accelerometer, taking the relative displacement of the n^{th} mass (z_n) as the response quantity, we must solve for the output per unit acceleration under static conditions. We have denoted this by Γ_0/p_1^2 which must equal z_n/A at $\omega = 0$. Solving equation 4 we have

$$|K|\frac{z_n}{A}\bigg|_{\omega=0} = |K|\frac{\Gamma_0}{p_1^2} = \begin{vmatrix} k_{01}+k_{12} & -k_{12} & 0 & \cdots & m_1 \\ -k_{12} & k_{12}+k_{23} & -k_{23} & \cdots & m_2 \\ 0 & -k_{23} & k_{23}+k_{34} & \cdots & m_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & m_n \end{vmatrix}. \tag{8}$$

It will now be shown that the determinant on the right hand side of equation 8 is exactly equal to the negative of the coefficient of ω^2 in the expansion of the characteristic equation 5. In a determinant, any column may be added to another column without changing the value of the determinant. If in the determinant of equation 5 each of the first n-1 columns is added to the n^{th} column, the equation has the form

$$\begin{vmatrix} k_{01}+k_{12}-m_1\omega^2 & -k_{12} & 0 & \cdots & -m_1\omega^2+k_{01} \\ -k_{12} & k_{12}+k_{23}-m_2\omega^2 & -k_{23} & \cdots & -m_2\omega^2 \\ 0 & -k_{23} & k_{23}+k_{34}-m_3\omega^2 & \cdots & -m_3\omega^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -m_n\omega^2 \end{vmatrix} = 0.$$

It is noted that all terms in the last column contain an ω^2 factor except for the k_{01} term in the first row. The coefficient of the ω^2 term in the expansion of this determinant by the n^{th} column without the k_{01} term is clearly seen to be equal to the negative of the right hand side of equation 8. It can be readily shown that the k_{01} term produces no ω^2 terms in the expansion. Thus it has been shown that $-|K|\Gamma_0/p_1^2$ equals the coefficient of ω^2 in the characteristic equation.

In the expansion of the characteristic equation in the form

$$\prod_{i=1}^{n} m_i(\omega^2 - p_i^2) = 0,$$

the coefficient of ω^2 is equal to $(-1)^{n+1}\prod_{i=1}^{n} m_i$ times the sum of ll possible products of the p_i^2 values taken n -1 at a time. Symboli- this coefficient is

$$(-1)^{n+1}\left(\prod_{k=1}^{n} m_k\right) \sum_{j=1}^{n}\left(\prod_{\substack{i=1,2,\dots\\ j-1,j+1,\\ \dots n}} p_i^2\right) \quad \text{for} \quad n > 1 .$$

A special case occurs at $n = 1$, where this is equal to $m_1 p_1^2$. For $n = 2$, this gives $-m_1 m_2 (p_1^2 + p_2^2)$; for $n = 3$, $m_1 m_2 m_3 (p_1^2 p_2^2 + p_1^2 p_3^2 + p_2^2 p_3^2)$; and so on. Equating this coefficient to $-|K|\Gamma_0/p_1^2$, and using equation 7

$$\frac{\Gamma_0}{p_1^2} = \frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} + \cdots + \frac{1}{p_n^2}$$

or

$$\Gamma_0 = 1 + \frac{p_1^2}{p_2^2} + \frac{p_1^2}{p_3^2} + \cdots + \frac{p_1^2}{p_n^2} . \tag{9}$$

Obviously the n^{th} order system shown in figure 11 is not the most general type. No springs between non-adjacent masses and no springs to the base (except for m_1) are included. It so happens that if any of these terms are included, negative quantities containing these spring constants will appear in the quantity Γ_0. However it is desired to maximize Γ_0 and these negative terms would decrease its value. From this consideration the choice of the form for the system shown in figure 11 as the optimum form of seismic system can be justified.

As an example of the form of these negative terms due to other springs, consider the system of figure 12. Using z_2 for the response quantity, the sensitivity is

$$\Gamma_0 = 1 + \frac{p_1^2}{p_2^2} - \frac{k_{02}}{m_2 p_2^2} .$$

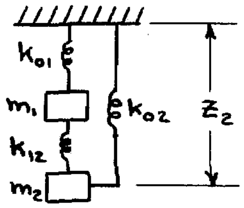

Figure 12

Thus if two systems have the same set of natural frequencies, one with the springs indicated in figure 11 and the other containing some other springs, the type of figure 11 will have the higher static sensitivity to acceleration.

The quantity Γ_0 is then a factor of merit for the accelerometer. The larger its value, the more output per unit acceleration will be obtained for equal first natural frequencies. Equation 2 shows that $\Gamma_0 = G(0) = 1$ for the one degree of freedom system. For a two degree of freedom system, the value of $\Gamma_0 = 2$ will be approached as $p_2^2 \to p_1^2$. It can be shown that to make p_2^2/p_1^2 have a value near one in a two degree of freedom system, the following should be true:

$$\frac{k_{01}}{m_1} = \frac{k_{12}}{m_2}, \quad m_2 << m_1.$$

A plot of how Γ_0 and p_2/p_1 vary with the parameter m_2/m_1, which equals k_{12}/k_{01}, is shown in figure 13.

The results indicated by this figure can be extended to the n degree of freedom system. In figure 14, if we let $\frac{k_{01}}{m_1} = \frac{k_{12}}{m_2} = \frac{k_{23}}{m_3} = \cdots = p^{*2}$ and $\frac{m_{i+1}}{m_i} << 1$, each of the $k_{i-1,i}/m_i$ systems has the same natural frequency p^*, and is only negligibly affected by the systems below it. The result is that the natural frequencies of the system taken as a whole are close together, which gives rise to a large value of Γ_0 (eq. 9).

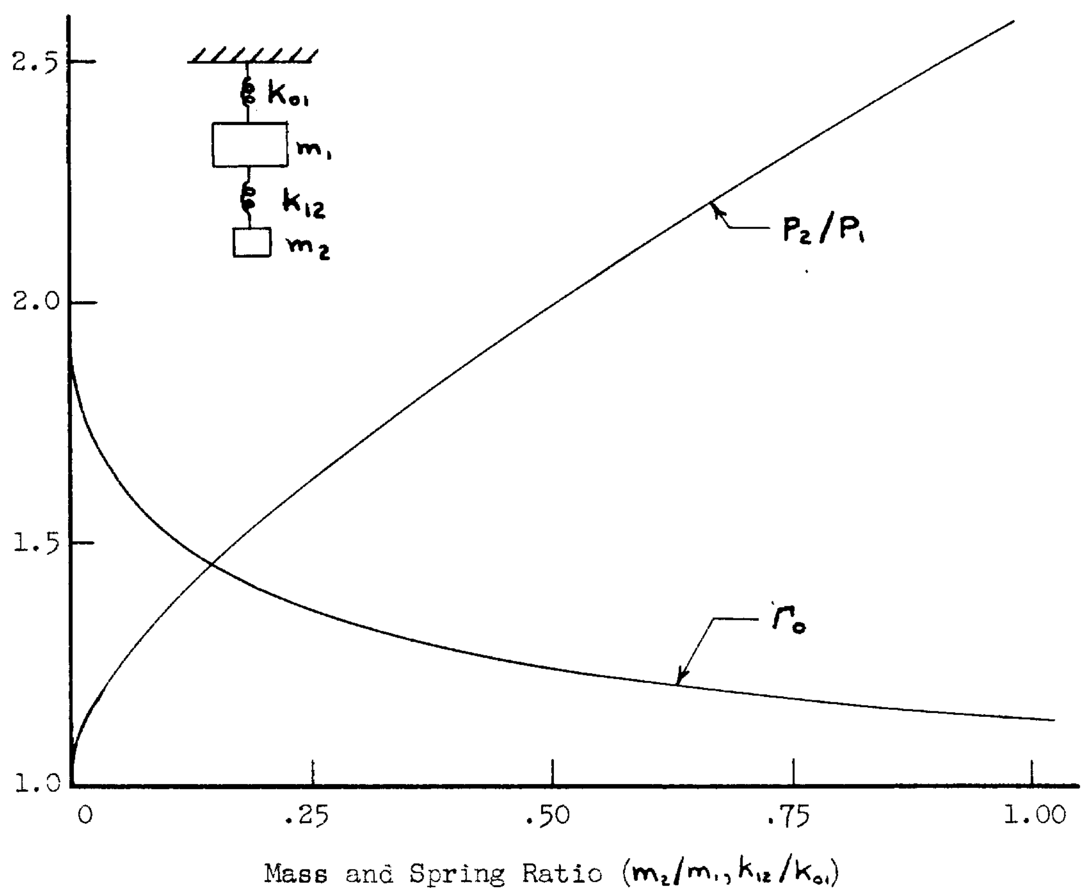

Figure 13. Static Level of Amplitude Characteristic and Natural Frequency Ratio for Two Degree of Freedom System.

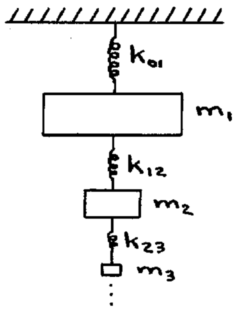

Figure 14. Tapered Multi-Degree of Freedom Seismic System

For example with $n = 2$, if

$$p^{*2} = \frac{k_{01}}{m_1} = \frac{k_{12}}{m_2} \quad \text{and} \quad \frac{m_2}{m_1} = \frac{k_{12}}{k_{01}} = \varepsilon,$$

the characteristic equation becomes

$$p^4 - (2+\varepsilon)p^{*2}p^2 + p^{*4} = 0. \tag{10}$$

The solution is

$$p^2 = \left[1 + \frac{\varepsilon}{2} \pm \left(\varepsilon + \frac{\varepsilon^2}{4}\right)^{1/2}\right]p^{*2}.$$

For $\varepsilon << 1$,

$$p^2 \approx (1 \pm \varepsilon^{1/2})p^{*2}$$

and hence

$$\frac{p_1^2}{p_2^2} \approx \frac{1-\varepsilon^{1/2}}{1+\varepsilon^{1/2}} \approx 1 - 2\varepsilon^{1/2}.$$

with the result that

$$\Gamma_o = 1 + \frac{p_1^2}{p_2^2} \approx 2(1 - \varepsilon^{1/2}), \qquad \varepsilon << 1 .$$

This same analysis could be applied to larger values of n, with the result that

$$\lim_{\frac{m_{i+1}}{m_i} \to 0} \Gamma_o = n , \quad \text{where} \quad \frac{k_{01}}{m_1} = \frac{k_{12}}{m_2} = \cdots .$$

A way of viewing this limiting value for Γ_o is as follows: knowing that a base acceleration A can be replaced by forces at the masses such that the force on m_i is $m_i A$, we can replace A with the loading shown in figure 15. Now if the sum of the forces acting below any mass m_i is negligible compared to the external force $m_i A$, the displacement

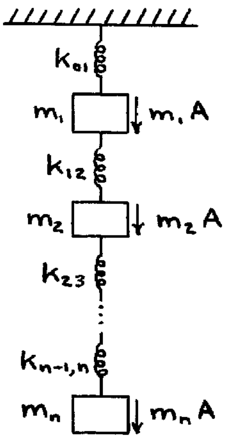

Figure 15. Multi-Degree of Freedom Seismic System

$$z_i - z_{i-1} = \frac{m_i A}{k_{i-1,i}} .$$

Then given that $\frac{k_{i-1,i}}{m_i} = p^{*2}$, the displacement between any two adjacent masses is A/p^{*2} and $p_1 = p^*$. These displacements add algebraically until a value nA/p^{*2} is attained for the displacement between m_n and the base. Thus the sensitivity is nA/p^{*2} and Γ_o is n.

In practice it is not possible to achieve the limit of $\Gamma_o = n$ since this requires an infinite rate of taper (ε = 0). For n = 2 the rate of approach to this limit is given in figure 13. It is seen that a significant gain can be obtained for a reasonable taper. For example, a gain of 53 percent is obtained for $\varepsilon = 0.1$. It may be of interest to consider the effect of a taper on Γ_0 for a distributed system. As an example consider the case of the bending on a cantilever beam with a constant height h_0 and a variable width $b(x) = b_0(x/\ell)^n$, loaded by a unit of acceleration (see fig. 16). The previous analysis was made for a system governed by a second order differential equation, whereas the first order beam deflection is the solution of a fourth order differential equation. However the concept of using a tapered system to bring the natural frequencies closer together in order to obtain greater values of Γ_0 still may be valid.

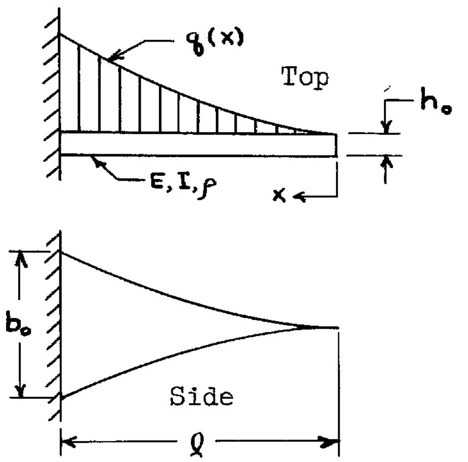

Figure 16. Tapered Cantilever Beam.

In this case Γ_0/p_1^2 is the static deflection per unit acceleration at the end of the beam, where p_1 is the first natural frequency. The differential equation for the beam displacement is

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2y}{dx^2}\right) = q(x) \,. \tag{11}$$

Now

$$q(x) = \rho\, b_0 h_0 (x/\ell)^n$$

$$I(x) = \frac{b_0 h_0^3}{12}(x/\ell)^n \,.$$

The solution for the deflection at the end is

$$y(0) = \frac{\rho \ell^4}{E h_o^2} \frac{3}{(n+1)(n+2)} \tag{12}$$

per unit acceleration.

Using the Rayleigh-Ritz or energy method of obtaining the lowest natural frequency (5, pg. 352), Γ_o may be computed as $y(0)\, p_1^2$. Figure 17 is a plot of Γ_o as a function of the parameter n . Theoretically large values of Γ_o could be attained for large n . However the practical limitation of an extremely thin section at the end of the beam limits the extent to which one may go. Other effects due to the inadequacy of the simple approach used in the calculations and the inadequacy of the differential equation will become significant for large n. Something between the linear tapered beam (n = 1) and the parabolic taper (n = 2) appears quite feasible, so that a value of Γ_o equal to approximately 2.5 is certainly attainable. The beam would then produce a displacement, under a unit static acceleration loading, of 2.5 times that of a one degree of freedom system under the same loading, and having a natural frequency equal to the lowest natural frequency of the beam.

B. USEFUL FREQUENCY RANGE

With the form of the undamped mechanical system required to maximize the static acceleration sensitivity determined, damping may now be inserted into this system to increase the useful range of accelerometer operation. From a practical aspect, the simplest design resulting in significant improvement is desired. Hence the two degree of freedom system shown in figure 18 will be considered.

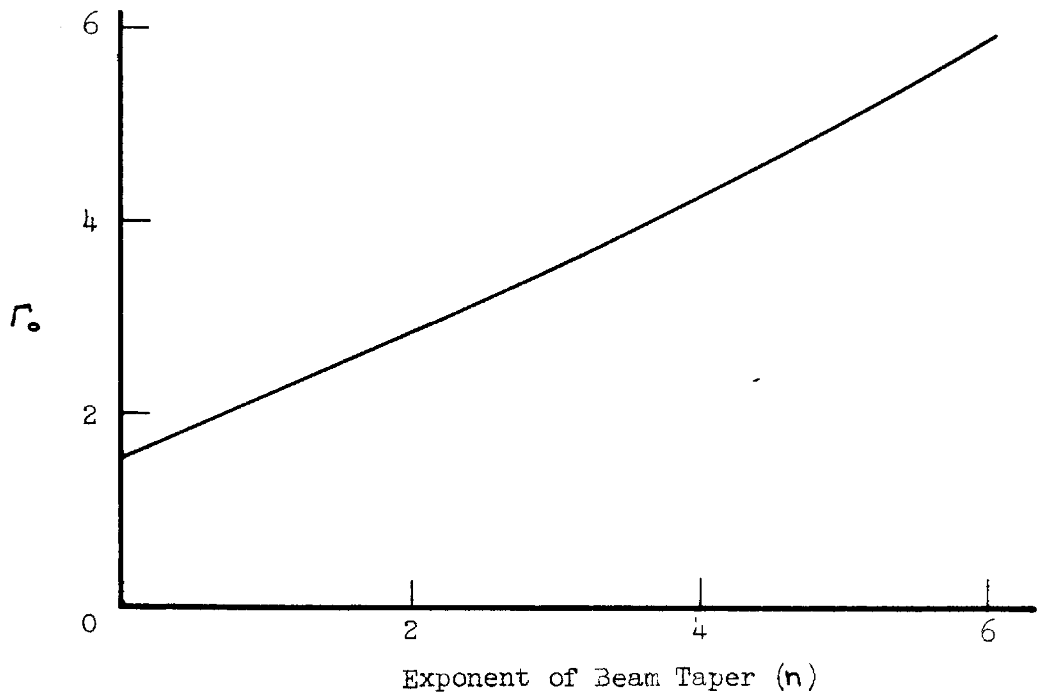

Figure 17. Static Level of Amplitude Characteristic for Tapered Cantilever Beams.

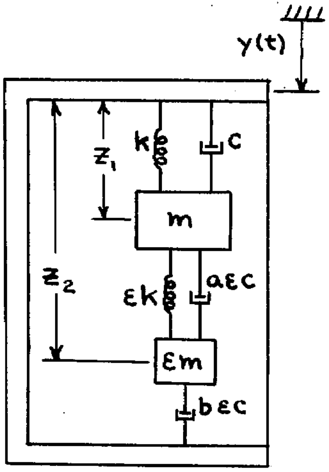

Figure 18. Two Degree of Freedom Accelerometer.

The equations of motion may be written in the form

$$\begin{aligned}
m\ddot{z}_1 + c\dot{z}_1 + kz_1 + a\varepsilon c(\dot{z}_1 - \dot{z}_2) + \varepsilon k(z_1 - z_2) &= -m_1\ddot{y} \\
\varepsilon m\ddot{z}_2 + b\varepsilon c\dot{z}_2 + a\varepsilon c(\dot{z}_2 - \dot{z}_1) + \varepsilon k(z_2 - z_1) &= -m_2\ddot{y}\,.
\end{aligned} \tag{13}$$

With an excitation of the form $-\ddot{y}(t) = A\exp(i\omega t)$, the steady-state solutions may be presented as

$$z_1 = Z_1 \exp[i(\omega t + \phi_1)],$$

$$z_2 = Z_2 \exp[i(\omega t + \phi_2)].$$

With these substitutions the accelerometer response is

$$Z_2 \exp(i\phi_2) = \frac{(2+\varepsilon-\eta^2) + i2\beta\eta(1+a+a\varepsilon)}{[(1-\eta^2)+i2\beta\eta(a+b)][(1+\varepsilon-\eta^2)+i2\beta\eta(1+a\varepsilon)] - \varepsilon[1+i2\beta\eta a]^2}\,\frac{1}{P_0^2}, \tag{14}$$

where

$$P_0^2 = \frac{k}{m}, \quad \beta = \frac{c}{2(km)^{1/2}}, \quad \eta = \frac{\omega}{P_0}\,.$$

The amplitude characteristic is given by $\Gamma(\omega) = Z_2(\omega)\,p_1^2$, hence

$$\Gamma = \left\{\frac{M^2 + N^2}{L^2 + Q^2}\right\}^{1/2} \frac{P_1^2}{P_2^2} \tag{15}$$

and the phase angle is

$$\phi_2 = \tan^{-1} \frac{LN - MQ}{LM + NQ}, \tag{16}$$

where

$$M = 2 + \varepsilon - \eta^2$$

$$N = 2\beta\eta(1 + a + a\varepsilon)$$

$$L = (1-\eta^2)^2 - 4\beta^2\eta^2(a + b + ab\varepsilon) - \varepsilon\eta^2$$

$$Q = 2\beta\eta[(1-\eta^2)(1 + a + b + ab\varepsilon) + \varepsilon(b - a).$$

It is intended that a large flat region in the amplitude characteristic be obtained. Hence, as a first step, the relation among a, b, and c to give a zero value for $d^2\Gamma/d\eta^2$ at $\eta = 0$ must be found. The odd derivatives at $\eta = 0$ do not exist. This procedure gave $\beta = 0.707$ for the single degree of freedom system. The corresponding value of β for the two degree of freedom system is

$$\beta^* = \left\{ \frac{(2+\varepsilon)[(2+\varepsilon)^2 - 1]}{2(2+\varepsilon)^2[(1 + a + b + ab\varepsilon)^2 - 2(a + b + ab\varepsilon)] - 2(1 + a + a\varepsilon)^2} \right\}^{1/2}. \tag{17}$$

For any choice of a and b, this relation yields the value of β or c resulting in an initially flat amplitude characteristic. If amplitude characteristics corresponding to various combination of a, b, and β^* are plotted, one finds both longer and shorter flat ranges than one obtains for the single degree of freedom case.

In order to find those systems having superior amplitude characteristics, one may also set $d^4\Gamma/d\eta^4 = 0$ at $\eta = 0$ for $\beta = \beta^*$. The condition for this to be true is

$$(2+\varepsilon)^2\left\{2+(2+\varepsilon)^2-8\beta^{*2}\left[1+(a+b)^2+(a^2+b^2)\varepsilon-2\beta^{*2}(a+b+ab\varepsilon)^2\right]\right\}=1. \tag{18}$$

The solutions of equations 17 and 18 may be given in the form of a family of curves, a^* versus b^*, with the parameter ε. This appears in figure 19. Any combination of a^* and b^* on one of these curves, with the corresponding value of β^*, results in an amplitude characteristic which may be expressed in a Maclaurin series in $\eta^* = \omega/p_1$

$$\Gamma(\eta^*)-\Gamma(0) = \frac{\eta^{*6}}{6!}\left.\frac{d^6\Gamma}{d\eta^{*6}}\right|_{\eta=0} + \frac{\eta^{*8}}{8!}\left.\frac{d^8\Gamma}{d\eta^{*8}}\right|_{\eta=0} + \cdots .$$

Among the possibilities described by the curves in figure 19, those having small values for the coefficient of η^{*6} in the above expression will have longer flat ranges. The values of this quantity appear in figure 19 as bracketed numbers near the points to which they correspond. The values for $a^* = 0$ are seen to be small, in general, compared to the values for the other points. Combining the notion that it is desirable from a practical viewpoint to have one damping parameter zero and the fact that one wishes the flattest amplitude characteristic, subsequent considerations will be confined to those points for which $a^* = 0$ and $b = b^*$.

Figures 20 and 21 are the amplitude and phase shift characteristics, corresponding to the values of ε in figure 19, for which $a^* = 0$. Positive values of the leading term in the expansion of the above series produce a hump in the curves for $\varepsilon = 0.1, 0.5, 1.0$. This is especially noticeable for $\varepsilon = 0.5$. The point for $\varepsilon = 1.444$ is especially inter-

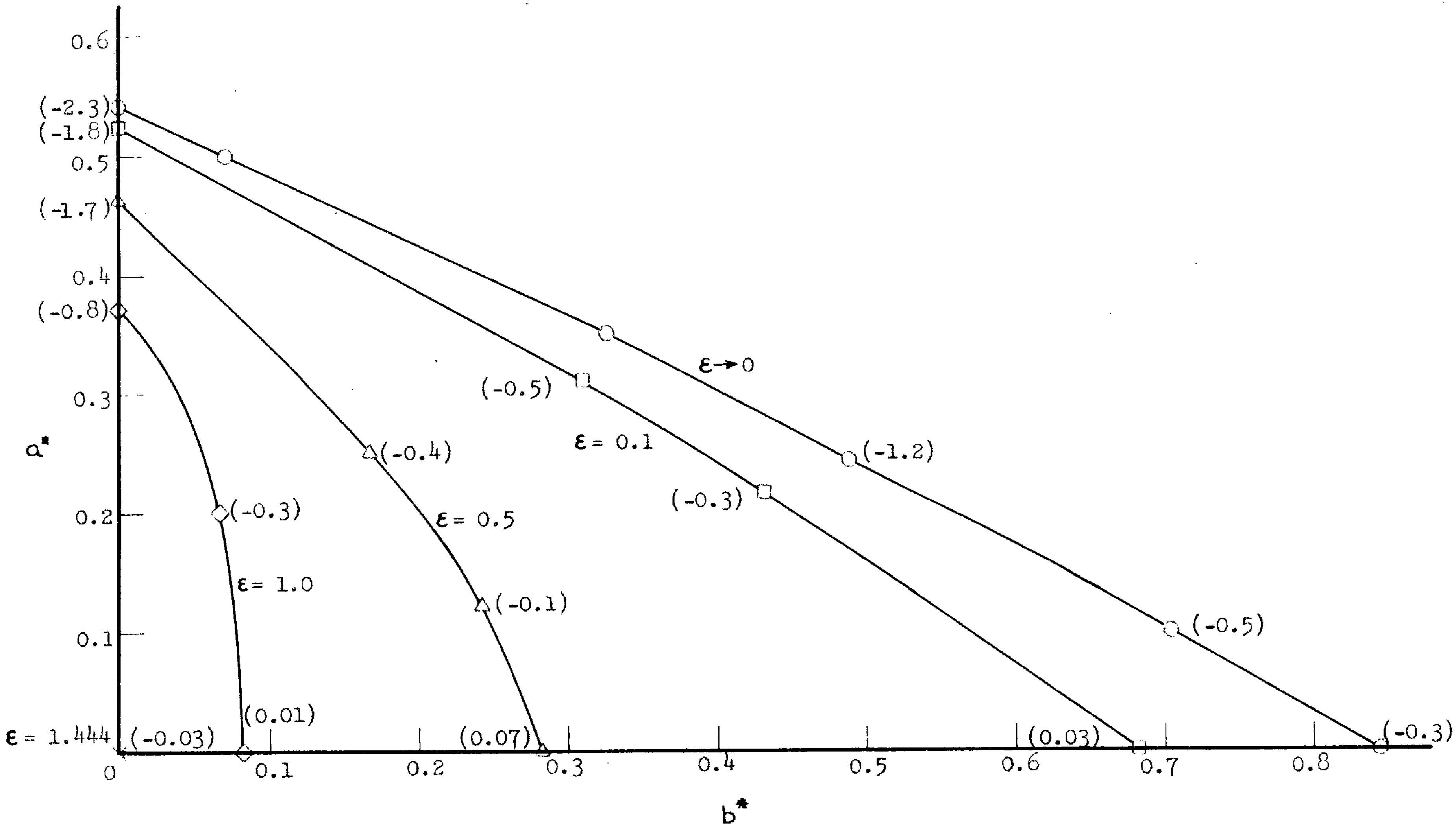

FIGURE 19. Damping Parameters for Two Degree of Freedom Accelerometer.

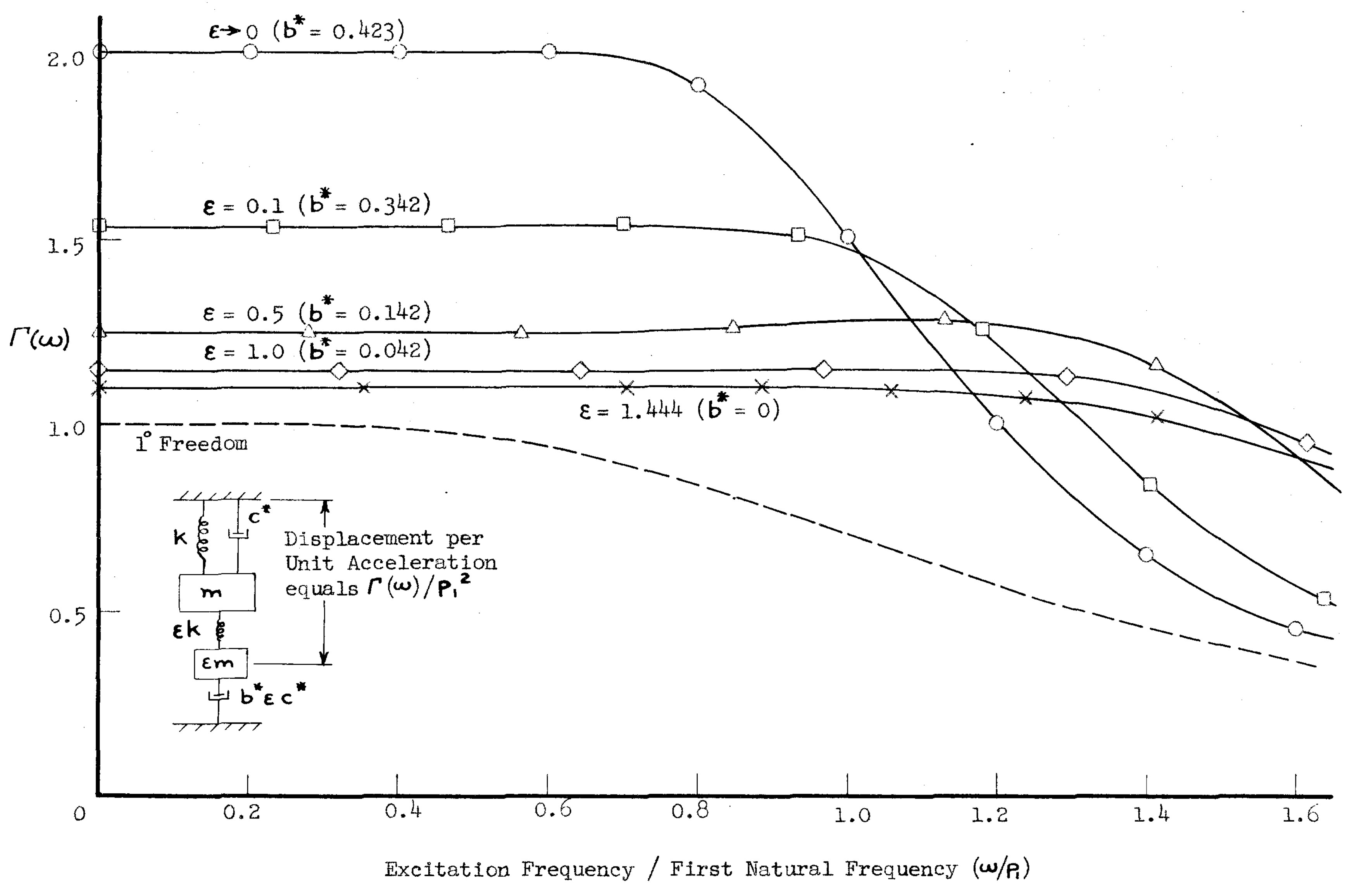

FIGURE 20. Amplitude Characteristics for Two Degree of Freedom Accelerometers.

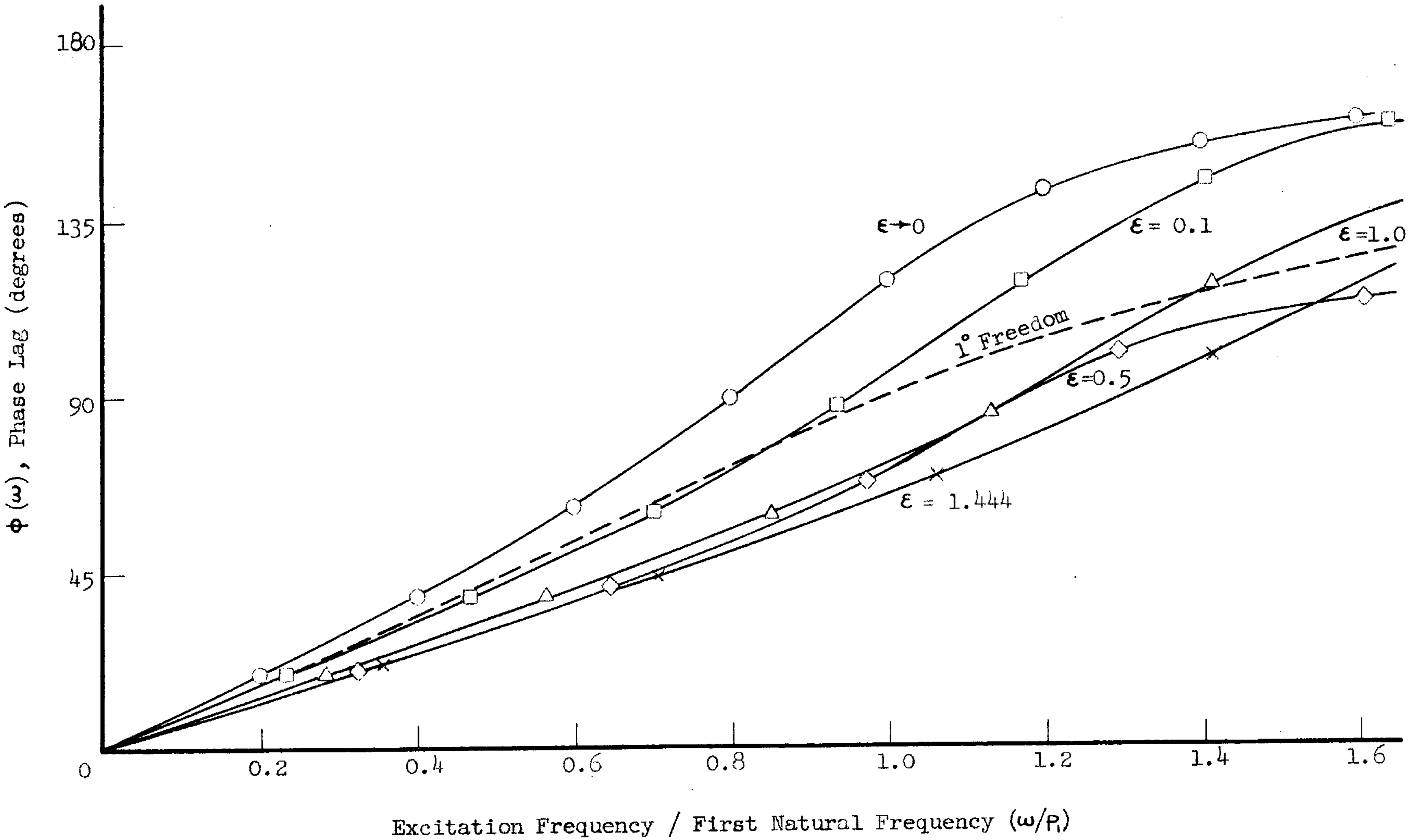

FIGURE 21. Phase Shift Characteristics For Two Degree of Freedom Accelerometers.

esting since both a^* and b^* are zero. Figure 22 shows the effect of slight changes in the parameters from β^* and b^*. The sensitivity to a change in β is shown. It is seen that a 5 percent decrease in the value of β from the value β for ε = 0.5 results in a maximum change in the amplitude characteristic of about 2 percent and slight modification to the phase shift in the region of interest. For ε = 0.5 the value of b was increased by 10 percent over b^* and the corresponding β^* calculated. The result is only a slight modification to the characteristics.

Let us now compare the characteristics of these two degree of freedom instruments with those obtainable in a single degree of freedom system. It is seen in figure 20 that the amplitude characteristics, especially for the larger values of ε, are far superior to those in figure 2. However the phase shift characteristics are more non-linear in figure 21 in the range where the amplitude characteristic is flat than those in figure 3. In order to evaluate the gain made by the use of the two degree of freedom system, we must consider the relative importance of maintaining the flat amplitude characteristic as opposed to maintaining the linear phase shift characteristic. This can be done very simply as follows: Consider the curve in figure 21 for ε = 1.0. This is the most non-linear in the range of flat amplitude characteristic, which in this case exists up to about 1.4 times the first natural frequency. If the superposition of any two harmonics in an excitation to be measured in this range is considered, the non-linearity of the phase shift characteristic will cause a relative shift of these harmonics by the accelerometer of about 20 degrees in the motion of the higher harmonic, in the worst case. This corresponds to a relative time shift of about 5 1/2 percent of the

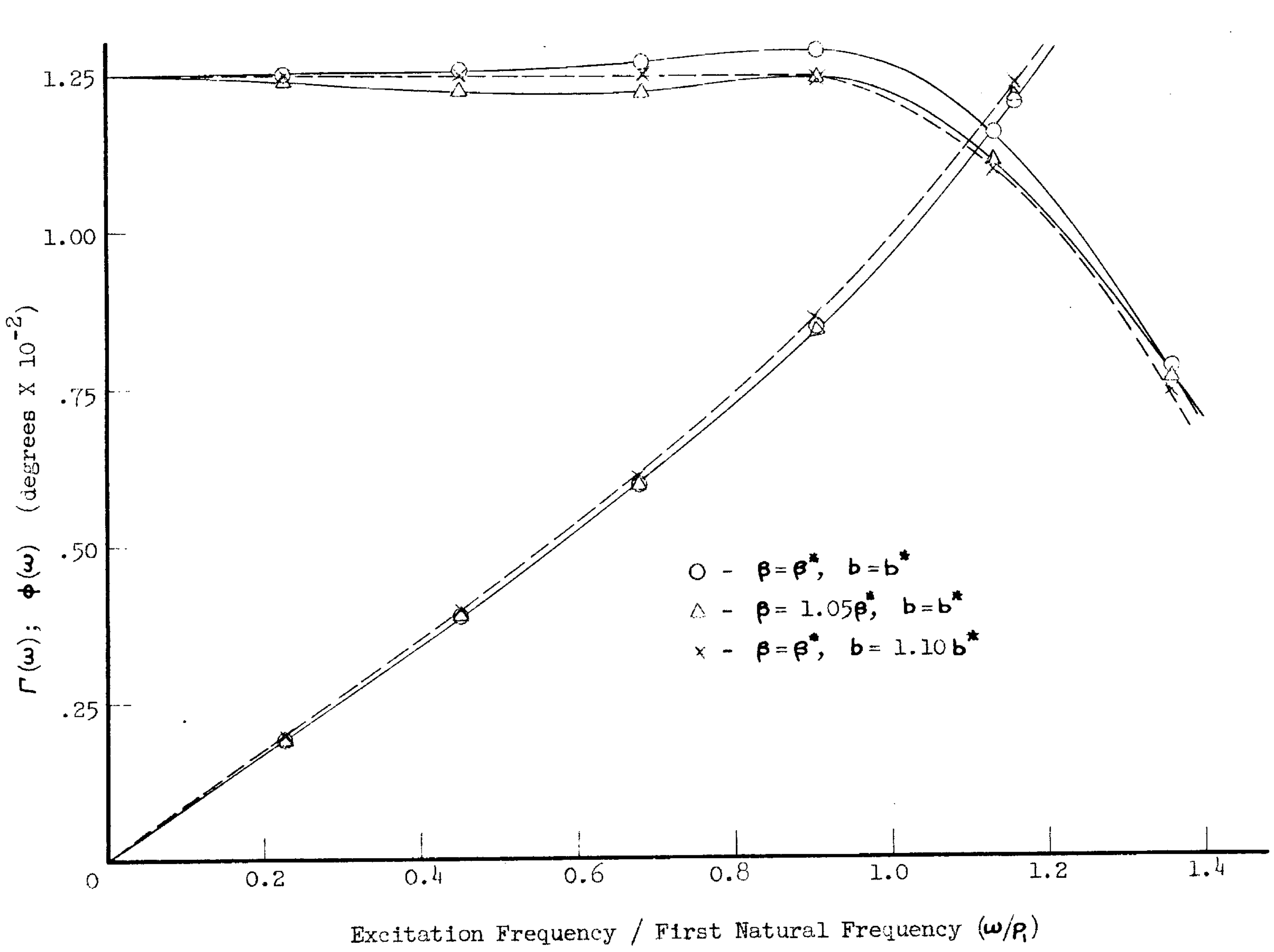

FIGURE 22. Sensitivity of Amplitude and Phase Shift Characteristics to Small Changes in Damping for $\varepsilon = 0.5$.

period of the higher frequency harmonic. In terms of usual engineering accuracies, deviations in linearity which are usually much smaller than this value of 20 degrees may be ignored. Conclusions as to usable frequency range can therefore be based on the amplitude characteristic alone.

Arbitrarily, the amplitude characteristic for 65 percent of critical damping in the single degree of freedom case will be used as a reference. Based on a total spread of the amplitude characteristic (see fig. 23) of 2.5 percent, the usable range is 0.62 of the natural frequency. If the damping is modified slightly for the curves if figure 20 to obtain about the same order and type of spread (+ 1.25 to -1.25 percent approximately), the usable ranges shown in Table I are obtained. As a basis for comparison, the sensitivity to acceleration for equal usable frequency ranges will be considered.

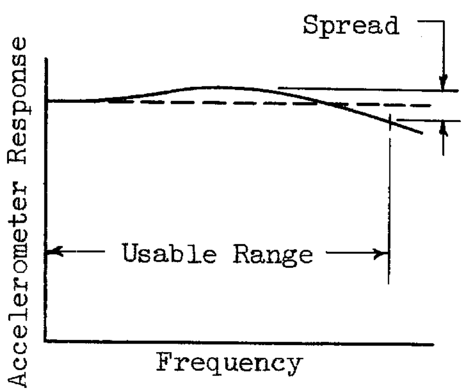

Figure 23. Usable Range of Accelerometer

For a single degree of freedom accelerometer to be usable over the same frequency range as a superior two degree of freedom instrument, its natural frequency p' must be higher than the first natural frequency of the two degree of freedom instrument. The sensitivity of the single degree of freedom instrument is then $1/p'^2$. The sensitivity of the two degree of freedom accelerometer is Γ_0/p_1^2. Thus if S is defined to be the gain in sensitivity, its value will be $\Gamma_0 p'^2/p_1^2$. Values of S also appear in Table I.

TABLE I

ε	Usable Range / First Natural Frequency	$\frac{p'}{p_1}$	Γ_0	S
0.1	0.97	1.57	1.5333	3.8
0.5	1.27	2.05	1.250	5.3
1.0	1.35	2.18	1.146	5.4
1.444	1.25	2.02	1.102	4.5

Hence it is possible to increase the accelerometer sensitivity from 3.8 to 5.4 times for the same useful frequency range by use of a two degree of freedom instrument.

Part One

SUMMARY AND CONCLUSIONS

Any transient can be considered as a superposition of a continuous distribution of harmonics. The steady-state characteristics of an accelerometer describe the manner in which each of these harmonics is altered by the instrument. An ideal accelerometer must have an amplitude characteristic which has a constant value, and a phase shift characteristic which has a constant slope, in the frequency range in which harmonics in the excitation exist.

The correct reproduction of the peak value of a transient has often in the past been used as a measure of the accuracy of an instrument. Application of this peak-value criterion to acceleration pulses shows that the "optimum" accelerometer design may not be the one whose characteristics closely approximate the ideal over the largest frequency range. The peak value criterion has merit only if the measurement is to be used to determine the peak response of a structure which responds statically to the excitation.

To the extent that one is interested in the peak response of single degree of freedom structures, or the peak response of normal modes of complex structures, the correct reproduction of the response spectrum of the excitation is a reasonable criterion for instrument design. The response spectrum criterion permits a quantitative evaluation of the degree to which the accelerometer characteristics approximate the ideal.

The required useful accelerometer frequency range need not always be based on the harmonic content of the excitation. The extent of this range may be, in some cases, modified by the sensitivity to harmonics of particular frequencies. Harmonics to which the structure will not significantly respond need not be recorded.

Treatment of a multiple degree of freedom seismic system leads to the result that significant improvement can be attained in the two degree of freedom case. The sensitivity to acceleration can be increased to about five times that of a single degree of freedom instrument having the same useful frequency range. It is shown for the systems treated that the phase shift characteristic is not as significant as the amplitude characteristic for determining the useful frequency range.

Part Two

INTRODUCTION

The theoretical foundation of the reed gage rests on the possibility of dynamic similarity between a complex linear system and a series of simple systems*. For a simple structure dynamic modeling is accomplished merely by employing a single degree of freedom system having the same natural frequency and damping. A complex structure can usually be considered in terms of its normal modes, each of which responds in a manner characterized by the mode frequency and damping. It is not always possible to consider a damped structure in this manner, but for most cases of practical interest, the damping is either small enough for the modal analysis to be approximately correct or the damping is of a special form for which the modal analysis is strictly applicable (14, pg. 130; 15).

The individual reeds of a reed gage should ideally be single degree of freedom systems. In any practical situation a reed is designed so that it responds predominantly in its first mode. The first consideration of Part Two of this thesis will be an investigation of the means of correcting the response of a reed to obtain the response of the ideal single degree of freedom system for which the reed is a model.

If one has complete knowledge of a particular structure whose peak response to an excitation is desired, the reeds can be constructed as models to the normal modes of the structure. In a more typical case, however, one wishes to investigate the peak responses of various structures to excitations. This includes an important use of reed gage information as a basis for the design of a structure to best withstand the excitation.

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* The word simple implies single degree of freedom.

In this regard it is more general to view the reeds, not as dynamic models, but as providing the data required to establish the character and magnitude of the response spectrum of the excitation in the frequency range of interest.

The usefulness of the response spectrum as obtained from the reed gage or from an acceleration-time record has only been recently appreciated. This type of information has been found so useful that special equipment has been constructed for the sole purpose of obtaining the response spectrum of a recorded acceleration-time history (11, 16). The response spectrum can be expressed in terms of various parameters. Displacement, velocity, and acceleration in terms of absolute or relative motions are all possibilities. The quantities which are most useful for reed gage application are the relative displacement or absolute acceleration due to prescribed base motions.

In the past there has been some uncertainty as to the quantitative usefulness of the reed gage and its general applicability. The analysis in this second part of the thesis will show the use of reed gage information to calculate an approximate peak response of a complex structure. Such results are necessarily conservative due to the fact that the peak responses in each of the modes are added algebraically to obtain the overall peak response, even though they do not occur at the same instant of time. An investigation of the magnitude of the error in such a process will be carried out for a typical form of shock excitation. The object will be to determine the relationship between the response system parameters and the errors, for cases of practical interest.

I. RESPONSE SPECTRUM FROM REED GAGE DATA

In this section the quantities which may be calculated from reed gage data will be given and the manner in which the response spectrum can be determined will be outlined. Consider the ideal simple system in figure 24.

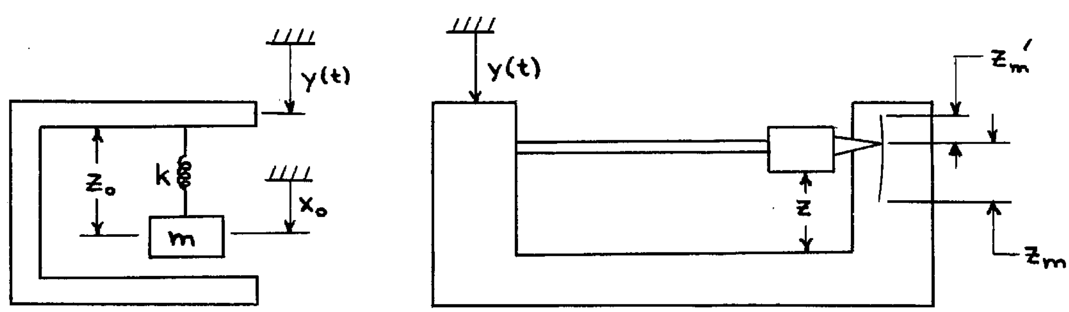

Figure 24. Reed and Ideal Simple System Configurations

Its motion is expressed by the equation

$$\ddot{z}_o + p^2 z_o = -\ddot{y} \tag{19}$$

where

$$p^2 = k/m .$$

The cantilever system in figure 24 is the basic reed configuration. It is designed with an end mass in order to make its response in all modes above the first insignificant compared to its response in the first mode for inertia loading. It is essentially a single degree of freedom system whose natural frequency is p_1 , the first mode frequency of the cantilever system. If $p_1 = p$ (and the damping in both systems is the same), both will respond in the same fashion. The magnitude of the response, however, will differ by a constant, $z(t) = \alpha z_o(t)$. This constant is a function of the geometry of the cantilever system. Since this constant is independent of the type of excitation, it may be determined from a static test. If δ is the displacement z per unit acceleration $\ddot{y}$ for the cantilever, $1/p^2$ is the corresponding quantity for the lumped system. The geometric factor is thus given by $\alpha = p^2 \delta$. With this definition of α , all information can be referred to the response of the lumped system.

For reed gage application, two quantities are found most useful to express the response of the simple system. Each may form the basis for a response spectrum to a motion $y(t)$. The first type of spectrum is formed from the extreme values of z_o. This can be termed the relative displacement spectrum for acceleration loading. The second possibility is based on the extreme values of $\ddot{x}_o$, which equal $-p^2 z_o$. These generate an absolute acceleration response spectrum for acceleration loading.

To use response spectrum data one would prefer some type of equivalent loading rather than a resulting displacement or acceleration. This can be accomplished by defining an equivalent static acceleration A_{eq}(17). This is that value of acceleration, statically applied, which would deflect the system to its peak dynamic displacement. A spectrum of this type is shown in figure 25. The solid line represents the peak motion in an assigned positive direction; the dotted line, the peak motion in the opposite or negative direction. Positive values are assigned to A_{eq} , negative to A_{eq}' . This is the spectrum for a half sine pulse of acceleration (10).

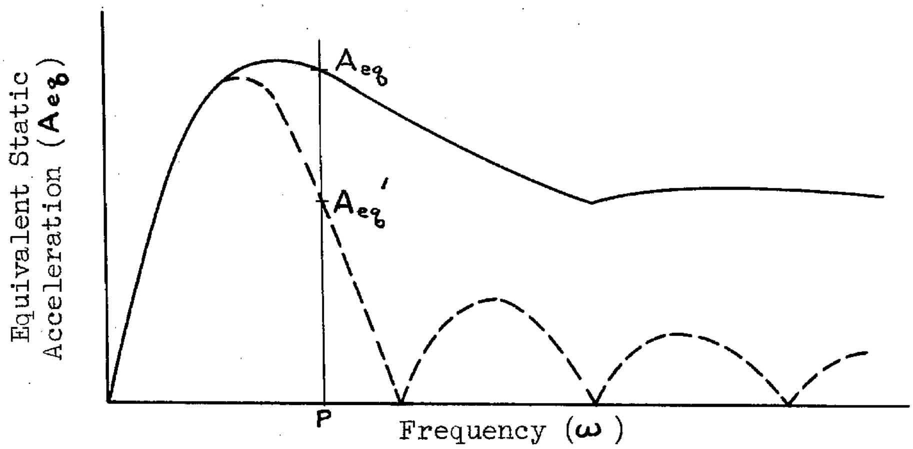

Figure 25. Equivalent Static Acceleration Spectrum For Half-Sine Acceleration Pulse

For example, assume a reed with a natural frequency p, having a static sensitivity to acceleration δ. That is, it deflects δ inches for each acceleration of gravity applied, at the position where the reed response is recorded. If a half-sine pulse of acceleration were applied to the reed, it would deflect z_m and z_m' in the positive and negative directions respectively. The spectrum values are thus $A_{eq} = z_m/\delta$ and $A_{eq}' = z_m'/\delta$, as shown in figure 25. A number of reeds, in the frequency range of interest, are used to define the spectrum in such a region.

The response quantities previously described are all derivable from the equivalent static acceleration. The relationship among these quantities is

$$A_{eq} = z_m/\delta = p^2 z_{om} = -\ddot{x}_{om} . \tag{20}$$

It should be noted that essentially similar curves are given different names in the literature. For example, amplification factor curves, maximax and residual displacement curves (10), and dynamic load factor curves (9) have been used to refer to the response spectrum.

II. STRUCTURAL RESPONSE BASED ON SPECTRUM DATA

The reed gage has as its primary function the recording of information concerning an excitation from which peak structural response to that excitation can be determined. Another possible function might be to obtain information concerning the excitation itself. Although one may imply from large reed resonances that strong harmonics exist in the excitation, it does not appear that any general statement can be made about the actual time history. An attempt to correlate the response spectrum with its corresponding single pulse excitation has been made with some success. However, the results are not applicable to other than those pulses treated (2).

To treat the problem of obtaining structural response information from the reed gage, a base acceleration $A(t)$ will be assumed. The reed gage will respond to this acceleration allowing the response spectrum of $A(t)$ to be obtained. The response of the normal modes of a structure to $A(t)$ will then be considered. In the previous section it was shown that the reed response to the excitation is proportional to the lumped simple system response, which is given by

$$z_o(t) = \frac{1}{p}\int_0^t A(\tau)\, e^{-\beta p(t-\tau)} \sin p(t-\tau)\, d\tau \, . \tag{21}$$

This is the form of the Duhamel integral for a single degree of freedom slightly damped oscillator having a natural frequency p and a fraction of critical damping β (5, pg. 403). A time dependent equivalent static acceleration may be defined as $A_{eq}(p,t) = p^2 z_o(t)$. The equivalent static acceleration A_{eq} equals the maximum value of this quantity, $A_{eq}(p) = p^2 z_{om}$. Thus

$$A_{eq}(p) = \left[p \int_0^t A(\tau)\, e^{-\beta p(t-\tau)} \sin p(t-\tau)\, d\tau \right]_{max.} \qquad (22)$$

The value of $A_{eq}(p)$ in the positive and negative directions, with respect to the reed gage block, are the ordinates of the response spectrum.

The first step toward obtaining structural response is to determine the mode frequencies and shapes of the structure. This can be done by various analytical or experimental methods. Next the influence function $\Delta_R^{(i)}(s)$ must be found. This is the value of the response quantity R at the position s in the i^{th} mode due to a unit static base acceleration. R can refer to any response quantity desired. Examples are displacement, stress, bending moment, shear force, and acceleration. This function can be analytically derived* or it can be experimentally determined by loading the structure with that portion of the inertia loading due to a unit base acceleration, which excites the mode i only.

At the position s the response can be expressed as a sum of normal mode responses

$$R(s,t) = \sum_i R^{(i)}(s,t) \,. \qquad (23)$$

Given the influence function at s , it is seen that

---

* See Appendix II.

$$R^{(i)}(s,t) = \Delta_R^{(i)}(s) \int_0^t p_i A(\tau) e^{-\beta_i p_i (t-\tau)} \sin p_i(t-\tau) d\tau . \qquad (24)$$

The maximum response in the i^{th} mode is thus

$$R_m^{(i)}(s) = \Delta_R^{(i)}(s) A_{eq}(p_i) \qquad (25)$$

where $A_{eq}(p_i)$ refers to either of the spectrum values at the frequency p_i . Up to this point the theory is exact. No approximations have been made to obtain the peak response in each of the normal modes at s . The peak response $R_m(s)$ is the desired end result. However, no information is available on the time of occurrence of the peak response in each mode. Hence only an estimate of peak response is possible. A conservative estimate would be to merely add the peak mode responses. Such an addition should be made in each of the response directions. The sign of the product of $\Delta_R^{(i)}(s)$ and $A_{eq}(p_i)$ defines the direction of response.

The final result in the two directions will be given by the spectral maximum values

$$R_{sm}(s,+) = \sum_i R_m^{(i)}(s,+) = \sum_i \Delta_R^{(i)}(s) A_{eq}(p_i) + \sum_j \Delta_R^{(j)}(s)' A_{eq}(p_j)'$$

or (26)

$$R_{sm}(s,-) = \sum_i R_m^{(i)}(s,-) = \sum_j \Delta_R^{(j)}(s)' A_{eq}(p_j) + \sum_i \Delta_R^{(i)}(s) A_{eq}(p_i)' ,$$

where the $+$ sign indicates the positive direction chosen arbitrarily and the $-$ sign indicates the negative or opposite direction. The $\Delta_R^{(i)}(s)$ are those with positive sign and the $\Delta_R^{(j)}(s)'$ are those with negative sign.

III. ERROR INVESTIGATION FOR HALF-SINE PULSE OF ACCELERATION

Admittedly the result of a peak response analysis from reed gage data is always conservative. In many cases of practical interest, however, the error in the analysis is well within engineering accuracy. For example, for base shear loads in a uniform five story structure (see Appendix II), the influence coefficient in the second mode is only about 10 percent of that in the first mode. Hence one would not expect much error in the spectral analysis. A rather complete analysis was carried out by Clough (17) on an actual building. The average results for the base shear load due to an earthquake excitation showed that the spectral maximum was of the order of 5 - 6 percent above an actual maximum.

As an example of the errors that might occur in the application of the above theory, a study of the errors in the spectral analysis for a half-sine pulse of acceleration will be made. The purpose of this study will be to indicate the magnitude of the errors and the regions in which large errors occur. It is assumed that two undamped modes contribute to the response. This minimizes the number of structural parameters that must be considered, while retaining the essential features of the approximations. The response is given by

$$
\begin{aligned}
R(s,t) &= R^{(1)}(s,t) + R^{(2)}(s,t) \\
&= \Delta_R^{(1)}(s) \int_0^t p_1 A(\tau) \sin p_1 (t-\tau)\, d\tau \\
&\quad + \Delta_R^{(2)}(s) \int_0^t p_2 A(\tau) \sin p_2 (t-\tau)\, d\tau \,, \qquad (27)
\end{aligned}
$$

where

$$A(t) = \sin \omega t \,, \qquad 0 < t < \pi/\omega$$
$$= 0 \qquad\quad , \qquad t > \pi/\omega \,.$$

The actual maximum response is given by the maximum of $R(s,t)$ in time or $R_m(s)$. The spectral maximum is given by

$$R_{sm}(s) = \Delta_R^{(1)}(s) A_{eq}(p_1) + \Delta_R^{(2)}(s) A_{eq}(p_2) \,. \tag{28}$$

The error in the analysis is $1 - [R_m(s)/R_{sm}(s)]$, where

$$\frac{R_m(s)}{R_{sm}(s)} = \frac{\left[\int_0^t p_1 A(\tau) \sin p_1(t-\tau)\,d\tau + \frac{\Delta_R^{(2)}(s)}{\Delta_R^{(1)}(s)} \int_0^t p_2 A(\tau) \sin p_2(t-\tau)\,d\tau\right]_{max}}{\left[\int_0^t p_1 A(\tau) \sin p_1(t-\tau)\,d\tau\right]_{max} + \frac{\Delta_R^{(2)}(s)}{\Delta_R^{(1)}(s)} \left[\int_0^t p_2 A(\tau) \sin p_2(t-\tau)\,d\tau\right]_{max}} \,. \tag{29}$$

Since $A(t)$ is given, the error is a function of three parameters. This may be taken to be p_1/ω , p_2/ω , and γ . Where γ is defined as $\Delta_R^{(2)}(s)/\Delta_R^{(1)}(s)$ and $p_2/p_1 > 1$. The results of this error study may be given in various forms. Perhaps the most revealing are plots of the percentage error contours for p_1/ω versus p_2/ω for constant values of γ (fig. 26 to fig. 30). Since the largest errors occur when the static contributions in each mode are of the same order, the values γ = 1/4, 1/2, 1, 2, 4 are considered. Negative values of γ are not considered since they imply a subtraction of mode responses for this type of excitation wherein the peak response always occurs in the direction of application of the pulse. This assumes that the main interest in this analysis lies in the estimation of the largest response of the structure and hence will occur where the modes add. The response spectrum of the half-sine pulse is drawn along the axes of the figures to aid in the orientation of the modes. Each spectrum is drawn in proportion to the static contribution of the mode corresponding to the axis on which it is drawn.

Since the modes are undamped, the lower limit to the peak response is the sum of the residual responses in the modes. These are the amplitudes of sinusoidal motion remaining after the pulse has acted. Eventually this motion in the two modes will be in phase, so that the peak displacement will at least be the sum of the residual values. Hence no error is obtained when both modes lie in the region where the peak responses in each motion are in the residual motion. This is true in the triangular region for $p_2/\omega < 1$ in figures 26 to 30. Also in the limiting case where the two modes have the same frequency ($p_2/p_1 = 1$), exact results are expected from the spectral analysis. In all other cases, conservative results are obtained.

The errors are largest when one mode is in the impulsive region and the other is in the static region of the response spectrum. In the former the peak motion occurs in the residual motion and the response at the time of peak excitation is small. For the static mode the reverse is true. This situation can only occur for relatively large values of p_2/p_1.

For large p_2/p_1 and $\gamma \leq 1$, the errors are largest when the peak responses in each mode have approximately the same magnitude and the residual response in the higher mode is small. This is especially pronounced near $p_2/p_1 = 3$. The zeroes in the residual amplitudes only

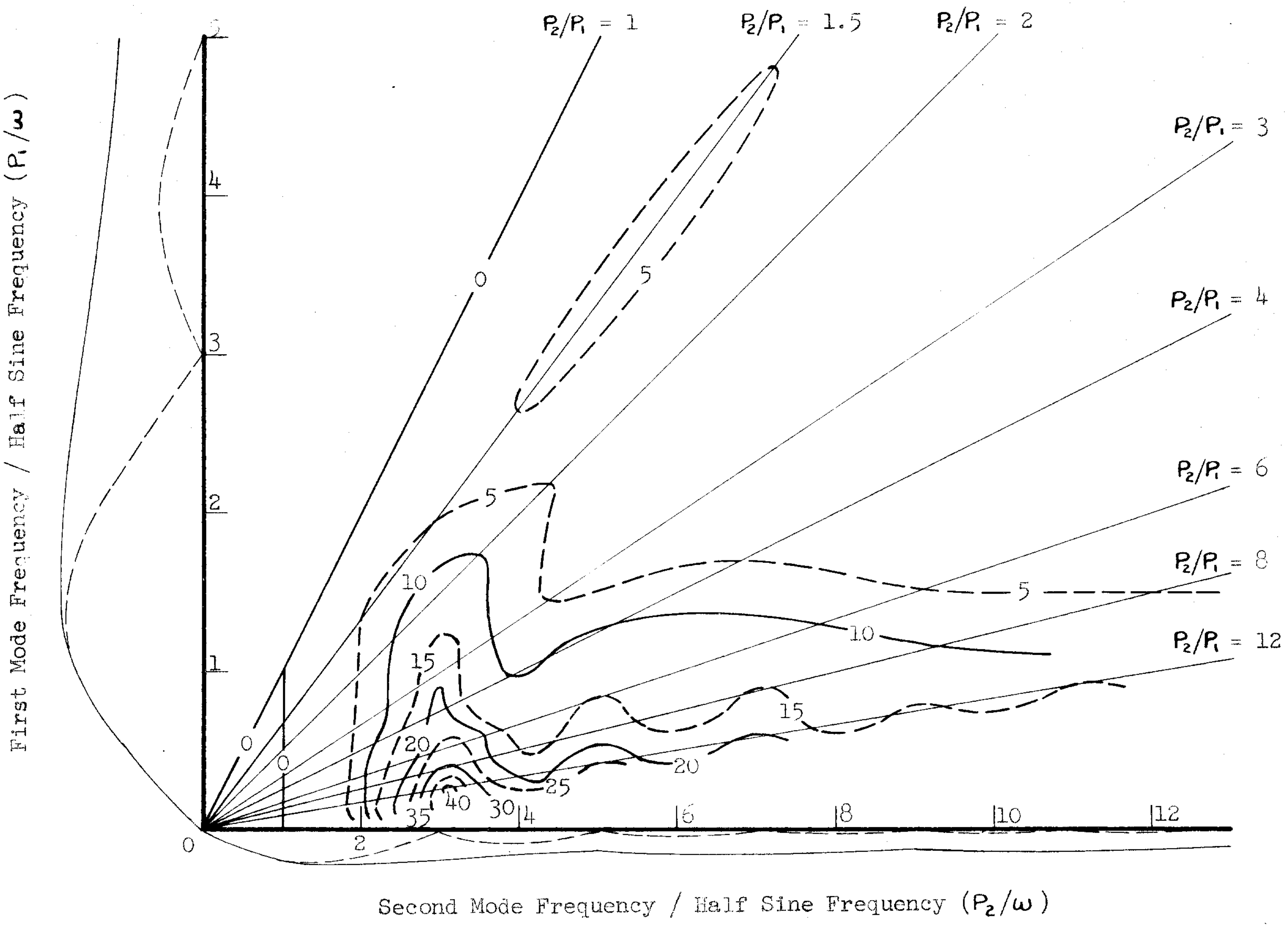

FIGURE 26. Error Contours for $\delta = 1/4$.

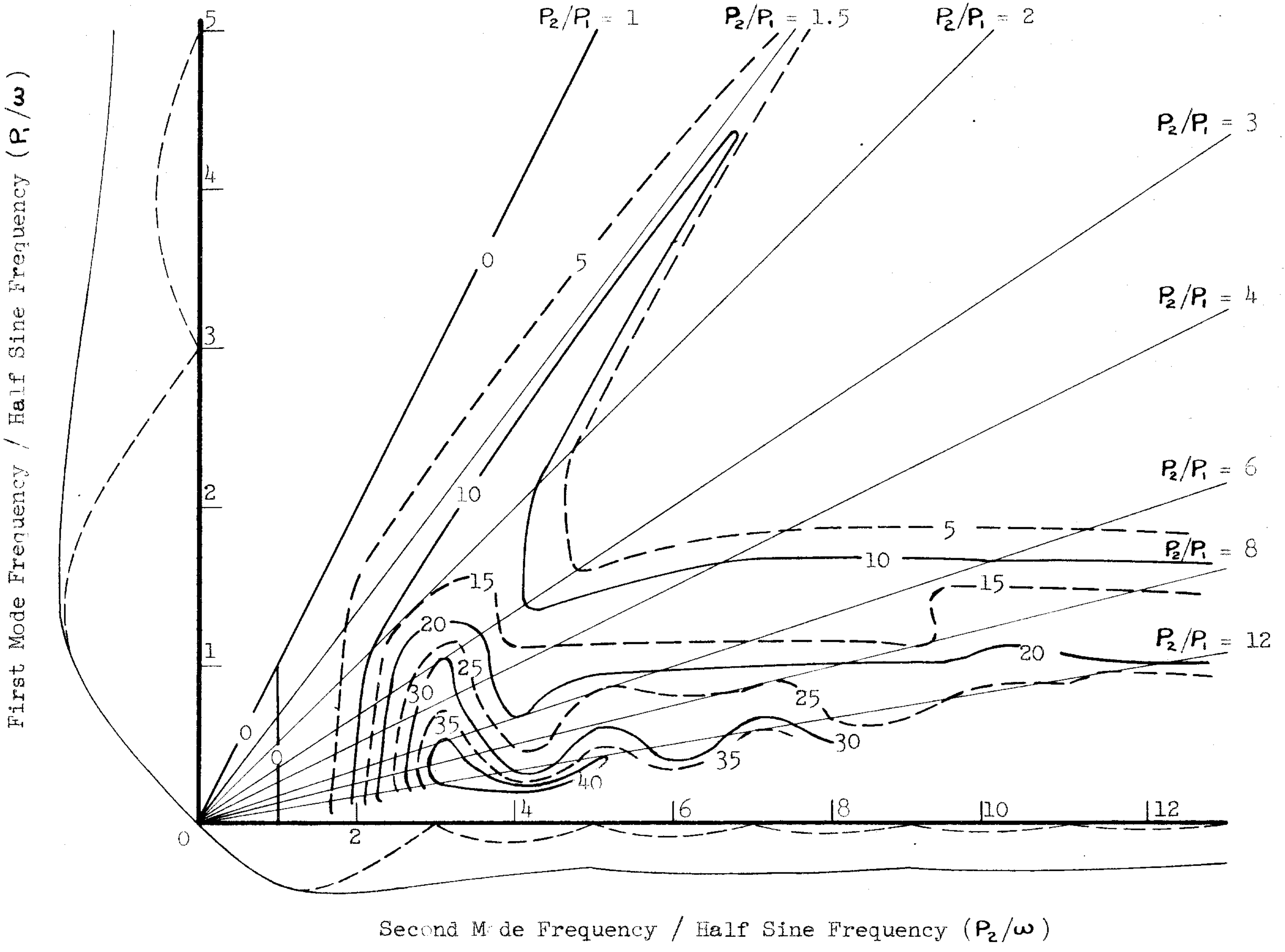

FIGURE 27. Error Contours for $\gamma = 1/2$.

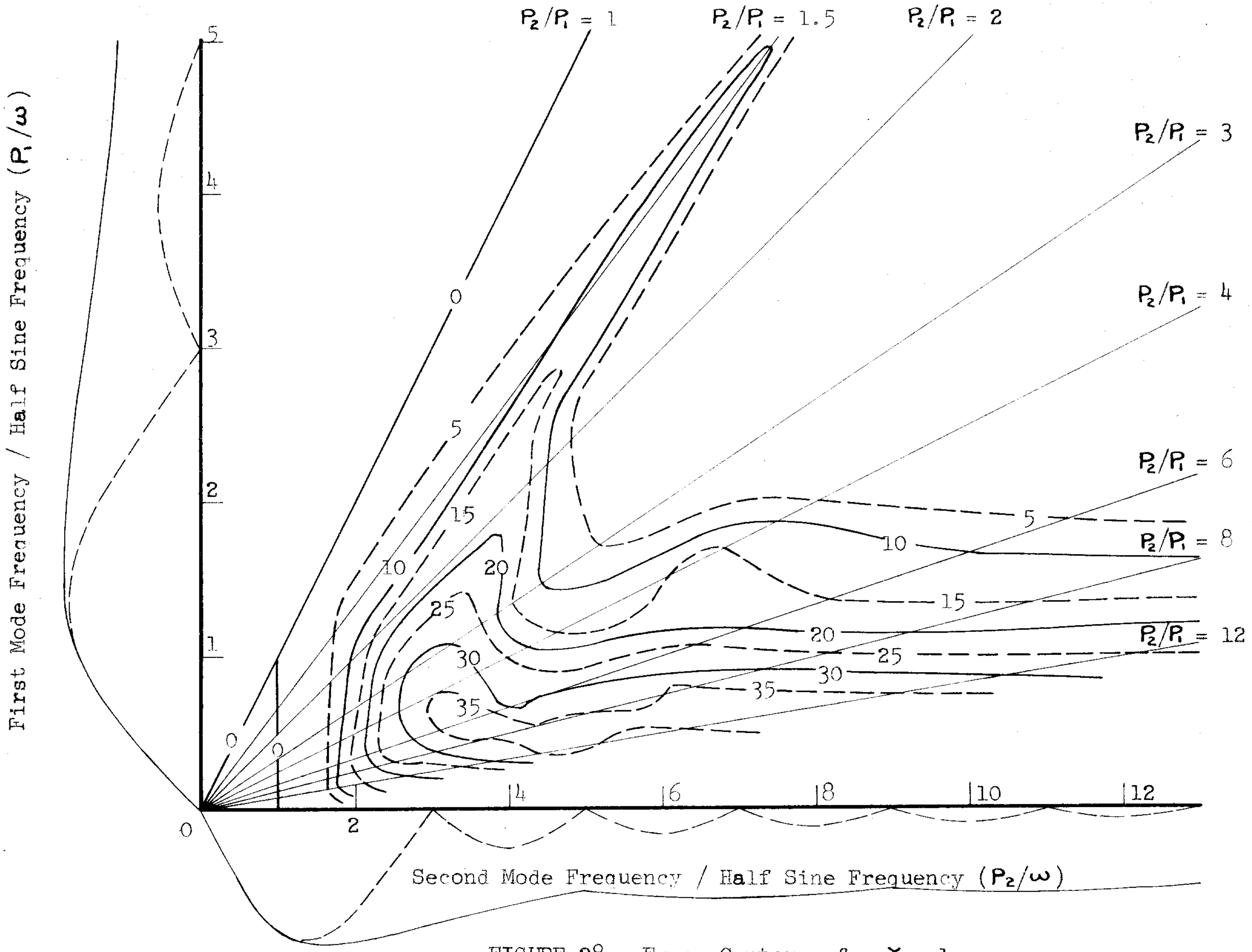

FIGURE 28. Error Contours for γ = 1.

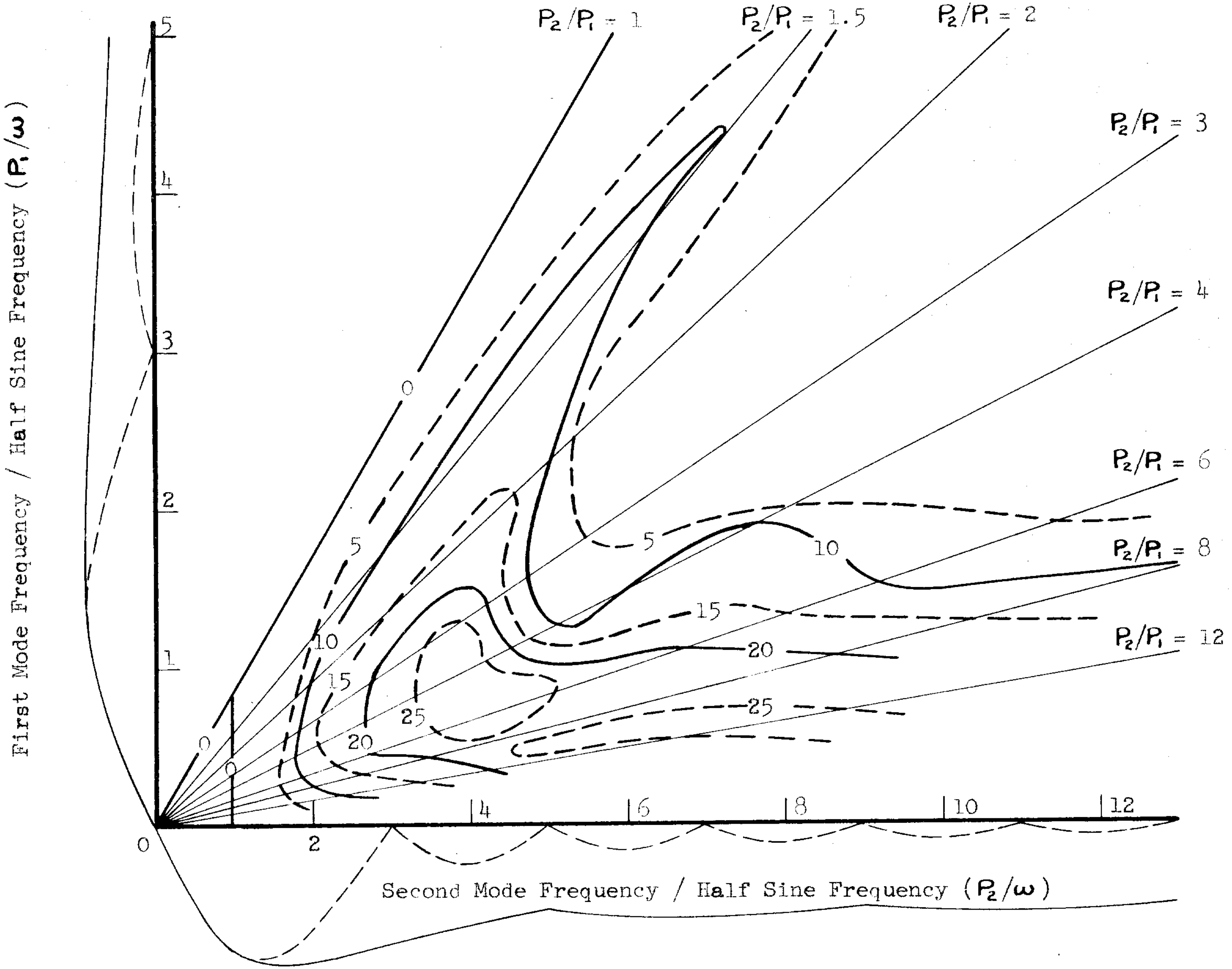

FIGURE 29. Error Contours for δ = 2.

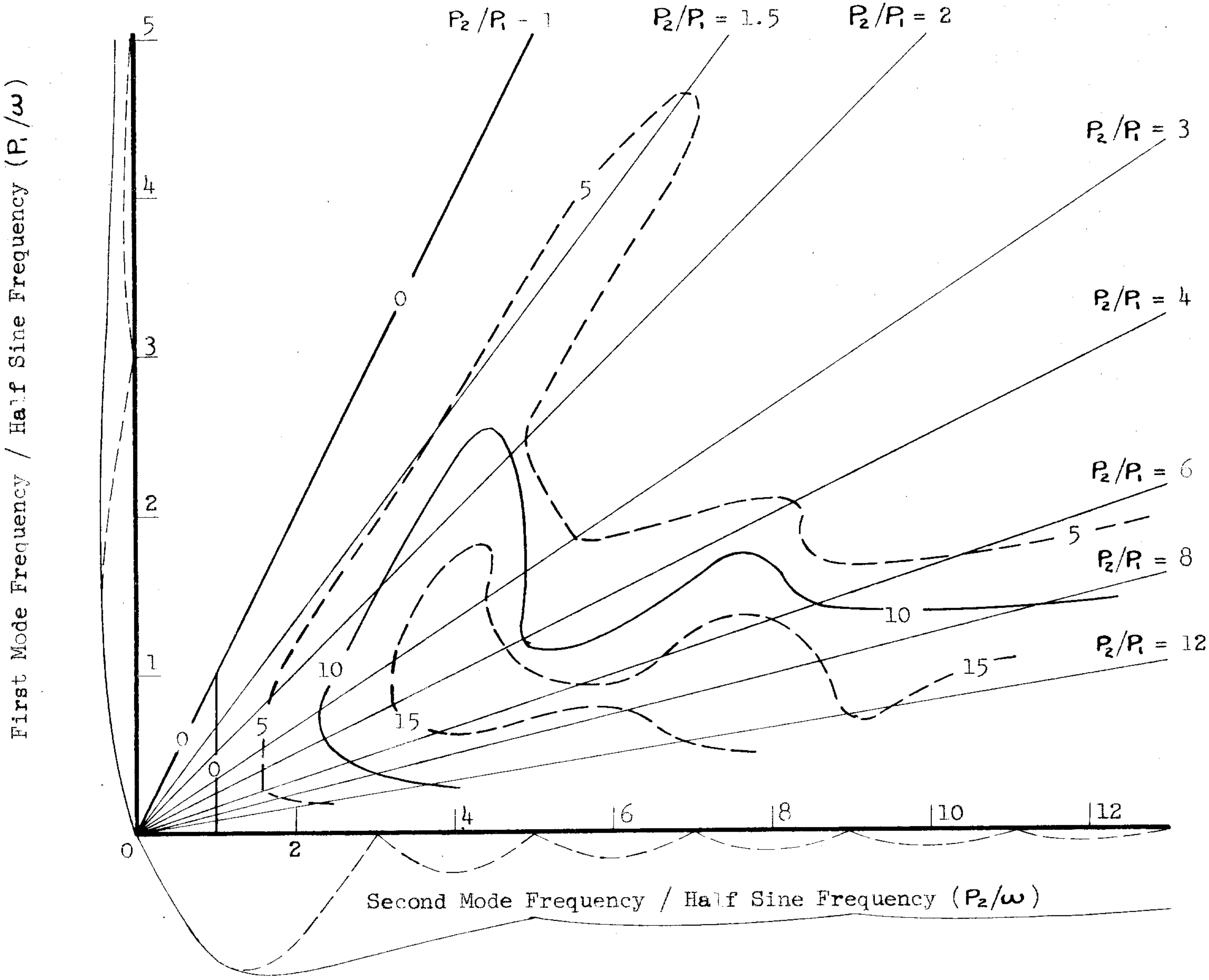

FIGURE 30. Error Contours for $\delta = 4$.

occur in the manner shown for symmetrical pulses (10). Any lack of symmetry in the pulse tends to smooth out the residual amplitude or peak negative response curve of the spectrum. The half-sine pulse is perhaps an unfortunate choice in this respect, but a qualitative statement can still be made for non-symmetrical cases.

Part Two

SUMMARY AND CONCLUSIONS

In the actual design of the reed gage the individual reeds will inevitably depart slightly from the ideal single degree of freedom system assumed in the theory of the instrument. A method of making a correction for this departure and hence arriving at an equivalent single degree of freedom system is described.

A method is outlined for treating a complex structure in terms of its normal modes in such a way that the calculation of peak response can be expressed as a superposition of peak single degree of freedom responses. This makes reed gage data applicable to the determination of the response of complex structures.

The calculation of the maximum response of a complex structure from reed gage data can only be approximate since the phase relationships between the peak responses in the various modes are unknown. It is shown that a direct superposition of the maximum motions in each mode will give an upper bound to the system's maximum response and hence as a design procedure, the approximation would always be on the conservative side. To investigate the errors in a specific case the response of a two degree of freedom undamped system to a half-sine pulse of acceleration is studied. The following conclusions are reached:

1. Errors are largest when the two modes lie in different regions of the response spectrum, e.g. one mode lies in the impulsive and the other in the static region of the response spectrum.

2. For a single pulse type excitation, it is unlikely that the most unfavorable error situations will often occur. In most practical problems

the distribution of mode frequencies and the form of the excitation is such that the total error due to superposition is probably not greater than ten percent. In view of the inaccuracies inherent in the reed gage, it appears that superposition errors are not usually significant.

3. The superposition errors mentioned in conclusion one above assume that nothing is known about the specific shape of the excitation function. If some information of this nature is available, corrections to the maximum response should be possible. There does not appear to be any general correction procedure that can be based on response spectrum information alone with an assurance of conservative results.

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Appendix I

ELECTRIC ANALOG STUDIES

A. Two Degree of Freedom Accelerometer

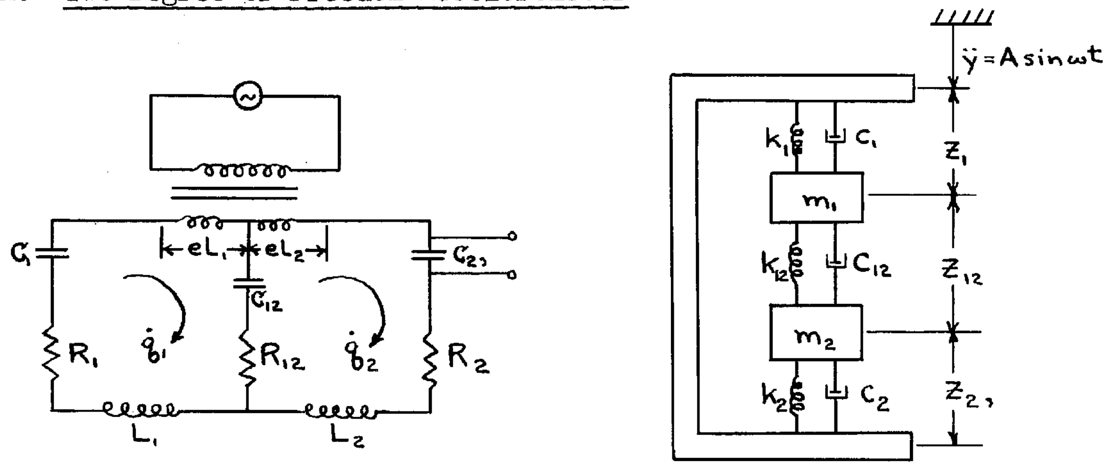

Figure 31

Electrical Equations:

$$L_1 \ddot{q}_1 + R_1 \dot{q}_1 + \frac{1}{C_1} q_1 + R_{12}(\dot{q}_1 - \dot{q}_2) + \frac{1}{C_{12}}(q_1 - q_2) = eL_1$$

$$L_2 \ddot{q}_2 + R_2 \dot{q}_2 + \frac{1}{C_2} q_2 + R_{12}(\dot{q}_2 - \dot{q}_1) + \frac{1}{C_{12}}(q_2 - q_1) = eL_2 .$$

Mechanical Equations:

$$m_1 \ddot{z}_1 + c_1 \dot{z}_1 + k_1 z_1 + c_{12}(\dot{z}_1 - \dot{z}_2) + k_{12}(z_1 - z_2) = -m_1 \ddot{y}$$

$$m_2 \ddot{z}_2 + c_2 \dot{z}_2 + k_2 z_2 + c_{12}(\dot{z}_2 - \dot{z}_1) + k_{12}(z_2 - z_1) = -m_2 \ddot{y} .$$

Comparison of the differential equations in the two systems shows the following relationships:

$$L \sim m \qquad R \sim c \qquad -e \sim \ddot{y} = A \sin \omega t.$$

$$\frac{1}{C} \sim k \qquad q \sim z$$

The voltage across C_2 is measured as being proportional to z_2, the accelerometer output.

B. Spectrum Analysis Comparison of Accelerometer Response and Input Excitation

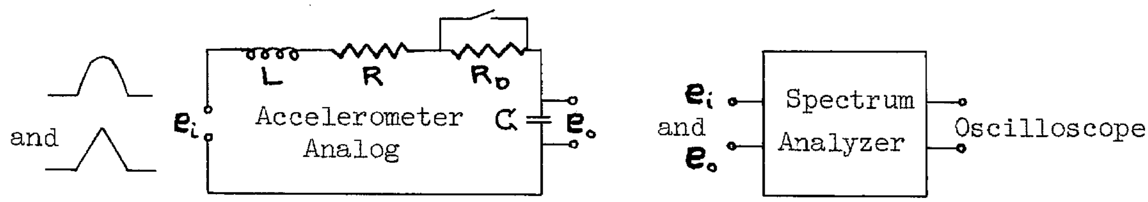

Figure 32

The response of a single degree of freedom accelerometer is found for half-sine and triangular pulses of acceleration by the loop analog shown in figure 32. The resistor, R_D, is used to damp out the response between repetitions of the input. The accelerometer response and the input excitation are then spectrum analyzed. The spectrum analyzer (11) is an electric analog of a single degree of freedom mechanical oscillator with provision for maintaining constant damping for a sequence of frequencies. Peak responses read off an oscilloscope face provide points on the response spectrum of the function being analyzed.

Appendix II

EXAMPLE OF CALCULATION OF INFLUENCE COEFFICIENTS

In this section the influence coefficients for a specific case are calculated to indicate the steps required in this type of analysis. A structure consisting of repeated identical elements is chosen to permit a concise presentation of the mode shapes and frequencies.

Consider the five mass system shown in figure 33. For use in the discussion of errors in a spectral analysis, this system is viewed as a model for a five story building deflecting in shear. The response quantity chosen for this computation is the base shear force ($k\,x_1$). This is a quantity of interest in an earthquake excitation of a building.

If p_i are the natural frequencies of the structure, the application of finite difference analysis gives (5,pg.453)

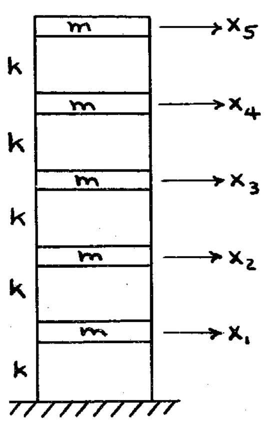

Figure 33

$$p_i = 2\sqrt{k/m}\,\sin\left[(i-1/2)\pi/11\right]. \tag{30}$$

The same analysis yields the mode shapes for the j^{th} mass and the i^{th} mode

$$a_j^{(i)} = \sin(ij\pi/11). \tag{31}$$

In matrix notation the relation between the mass displacements (x) and the normal modes (q) is

$$[x] = [a][q], \tag{32}$$

which expresses the equations

$$x_j = \sum_i a_j^{(i)} q_i .$$

The results of matrix theory applied to modal analysis (equivalence of work in the two coordinate systems) show that the generalized forces are given by

$$[Q] = [a]^T [f], \tag{33}$$

where $[a]^T$ is the transpose of $[a]$ and $[f]$ is the matrix of the forces acting on the structure at the mass positions.

Consider a one gravity (g) static acceleration applied to the base of the structure. This is equivalent to the set of forces mg acting at the masses. The matrix $[f]$ is then a unit column matrix times mg. The normal mode responses are given by

$$q_i = Q_i / M_i p_i^2 , \tag{34}$$

where M_i is the i^{th} generalized mass and is given by

$$M_i = \sum_j m \left[a_j^{(i)} \right]^2 . \tag{35}$$

The influence coefficients in this case are the base shear forces in each of the modes per g of base acceleration. That is

$$\Delta^{(i)} = x_1^{(i)} q_i \text{, for } [f] = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} mg . \tag{36}$$

The result of this computation is

$$\Delta^{(1)} = 4.399 \text{ mg}$$
$$\Delta^{(2)} = 0.436 \text{ mg}$$
$$\Delta^{(3)} = 0.121 \text{ mg}$$
$$\Delta^{(4)} = 0.038 \text{ mg}$$
$$\Delta^{(5)} = 0.008 \text{ mg}.$$

In this case all the Δ 's are positive, a result of the uniformity of the structure. A less uniform structure may give rise to Δ's of both signs.

NOMENCLATURE

A	peak acceleration of harmonic or transient motion (9, 15)*
A_{eq}	equivalent static acceleration (55)
B	peak of response spectrum (15)
C	slope of response spectrum at $\tau/T = 0$ (15)
$\mathcal{C}$	capacitance (73)
E	Young's Modulus (37)
$G(\omega)$	amplitude characteristic for single degree of freedom accelerometer (9)
$I(x)$	moment of inertia of tapered cantilever beam (37)
$[K]$	spring constant matrix (29)
L	inductance (73)
L, M, N, Q	algebraic quantities (41)
$[M]$	mass matrix (29)
M_i	i th generalized mass (76)
Q_i	i th generalized force (76)
R	resistance (73)
$R(s,t)$	response quantity (58)
$R_m(s)$	maximum response (59)
$R_{sm}(s)$	spectral maximum response (59)
S	sensitivity gain (48)
T	response system period (15)
Y	amplitude of harmonic frame motion (28)
Z	amplitude of harmonic relative displacement (28)
a, b	numerical factors (40)
a^*, b^*	values of a and b to make $d^4\Gamma/d\eta^4 = 0$ at $\eta = 0$ for $\beta = \beta^*$ (42)
$a_j^{(i)}$	mode shape for j th mass and i th mode (75)
$b(x)$	width of cantilever beam (37)
c	viscous damping coefficient (8)
c^*	value of c corresponding to β^* (40)

* Numbers in parentheses refer to the page on which the quantity first appears or is defined.

e	voltage (73)
$[f]$	force matrix (76)
g	acceleration of gravity (76)
$h(x)$	height of cantilever beam (37)
i	index (11), subscript denoting input (15)
j	index (32)
k	spring constant (8)
ℓ	length of cantilever beam (37)
m	mass (8)
$\{m\}$	mass column matrix (29)
n	index (28), exponent of beam taper (37)
p	natural frequency (8)
p_i	i th natural frequency (30)
p^*	$\sqrt{k_{i-1}/m_i}$ (33)
p'	natural frequency of single degree of freedom system having same useful frequency range as two degree of freedom system (48)
p_o	$\sqrt{k/m}$ for two degree of freedom system (40)
q	charge (73)
$q(x)$	distributed loading on cantilever beam (37)
$[q]$	normal mode displacement matrix (75)
r	subscript denoting response (15)
s	position (58)
t	time (8)
x	distance along cantilever (37)
$x(t)$	absolute displacement (8)
$[x]$	mass displacement matrix (75)
$y(x)$	deflection of cantilever (37)
$y(t)$	frame motion (8)
$z(t)$	relative displacement (8)
$\{z\}$	relative displacement column matrix (29)
$\Gamma(\omega)$	defined to be the amplitude characteristic of a multiple degree of freedom accelerometer, where the acceleration sensitivity is $\Gamma(\omega)/p_1^2$ (27)
Γ_o	value of $\Gamma(\omega)$ at $\omega = 0$, i.e. static sensitivity (27)
Δ	influence function (58)

α	constant of proportionality (55)
β	fraction of critical damping or $c/2\sqrt{km}$ (8, 40)
β^*	value of β making $d^2\Gamma/d\eta^2$ = 0 at η = 0 (41)
γ	ratio of mode contributions under static loading (61)
δ	displacement of reed per unit acceleration (55)
ε	ratio of masses and spring constants in two degree of freedom system (35)
η	ratio of excitation frequency to p_0 (40)
η^*	ratio of excitation frequency to p_1 (42)
ρ	density (37)
τ	time duration of pulse (15)
$\phi(\omega)$	phase shift characteristic (9, 41)
ω	frequency (9)