Quadratic Differential Equations in Banach Spaces and Analytic Functionals

Thesis by

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Introduction and Summary

In Chapter I the preliminary concepts are introduced and applied to the quadratic differential equation, $\frac{dy}{dt} = yA(t)y$, in a complete normed linear ring. In Chapters II and V the functional equation, $y = f + K \odot y + y \circledast H$ is studied under suitable assumptions and shown to be related to the Fréchet differential of the above differential equation as a functional of $A(t)$. In Chapter IV the more general functional equation, $y = f + T(y)$, where T is an endomorphism, is treated.

The existence of the Fréchet differential with respect to $A(t)$ of $\frac{dy}{dt} = yA(t)y$ is shown in Chapter II by means of a power series for the solution of $y = f + Q(y, A, y)$ where Q is a trilinear function. The solution of $\frac{dy}{dt} = yA(t)y$ is obtained in Chapter VI in the terminology of Chapter IV. Moreover, the solution is shown to satisfy uniquely the differential system, $\delta y[A/t] = y(\int_{t_0}^{t} \delta A(s)\,ds)y$, $y[0/t] = y_0$ and to possess a generalized Taylor series expansion. The above equation is generalized with similar results to $\frac{dy}{dt} = T(y, A(t), y)$, where T is a trilinear function.

Chapter VII is concerned with examples of Chapter VI. For instance, the solution of the matrix differential equation, $\frac{dy^i_\alpha}{dt} = y^i_\beta A^\beta_j(t)\, y^j_\alpha$, $1 \le i \le n$, $1 \le \alpha \le m$, is treated both as a function of t and as an analytic functional of the nm continuous functions $A^\beta_j(t)$

CONTENTS

Chapter I

Ordinary Differential Equations Over a Banach Space

It is evident that a number of the properties of real valued functions carry over in part to the more general Banach space valued functions. In particular the calculus and the differential equation can be conceived in this more general space.

The derivative is defined in the usual manner. Let $f(t) \in B$, a (real) Banach space, such that for each number t in the closed real interval (a, b), there is an element expressed by the function, $f(t)$, in B. Then if there is an element $g \in B$ such that

$$\lim_{|h| \to 0} \left\| \frac{f(t+h) - f(t)}{h} - g \right\| \to 0$$

g is called the derivative of $f(t)$ with respect to t. As is customary one writes

$$g = f'(t) = \frac{df(t)}{dt}$$

For the notion of integration of functions of a real variable we will restrict ourselves to the Riemann-Graves integral. The fundamental properties are to be found in (1) and (2) and will be assumed throughout this work.

The concept of uniform convergence with respect to the norm is on the whole quite analagous to the real or complex valued function theory of the same. In fact in a great many cases the classical proofs may be followed step by step, replacing absolute values by norms throughout. Two examples of this useful for our purposes, are the evident generalizations of the theorems given in section 1.71 and 1.72 of (3), concerned with the term by term integration and differentiation of a series of Banach space valued functions of a real variable.

Throughout this work, we shall use freely the fundamental properties of Fréchet differentials of functions in Banach spaces. This subject was initiated in 1925 by Fréchet in (11), and since then it has been extensively developed in particular by Prof. A. D. Michal and his school (see (12), (13), and (14)). A theory of analytic functions of variables in Banach spaces as well as a theory of differential equations whose unknowns are functions (not necessarily analytic) of variables in Banach spaces was also initiated and developed. These researches were begun in 1931 by Prof. A. D. Michal. For a history of the development of these subjects refer to (7). (See also (10) for normed linear rings, and (12), (13), and (14) for further references).

It is now our purpose to discuss abstract ordinary differential equations of the type

$$(1.1) \qquad \frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0 \ , \quad |t - t_0| \leq h$$

where the independent variable t is a real variable and y is the dependent variable with values in the Banach space, B The particular example of (1.1) that we will consider is the abstract Riccati equation where the Banach space, B , is replaced by a complete normed linear ring, R . In a later chapter our methods will lead to the solution of some rather interesting matrix differential equations. For example we will be able to treat the rectangular matrix differential equation

$$\frac{dy^i_\alpha}{dt} = y^i_\beta A^\beta_j y^j_\alpha \ , \ y^i_\alpha(t_0) = y^i_{0\alpha} \ , \quad 1 \leq i \leq n, \text{ and } 1 \leq \alpha \leq m$$

where $A^\beta_j(t)$ are continuous real valued functions over interval $|t-t_0| \leq h$ and treat the solutions as analytic functionals of the nm functions $A^\beta_j(t)$.

To treat (1.1) in general it is necessary to first obtain some criteria for the existence of solutions. Such an existence theorem is embodied in the following theorem:

Theorem 1.1 If B is a Banach space and E is the space of real numbers, then let:

(i) $f(t, y)$ on EB to B be defined and continuous in each variable separately for domain D, defined by $|t-t_0| \leq h$, $\|y-y_0\| \leq b$ with h and b fixed.

(ii) $\|f(t,y)\| \leq M$ with M determined by $hM \leq b$ and $t, y \in D$.

(iii) $\|f(t,Y)-f(t,y)\| \leq K\|Y-y\|$ for $t, y, Y \in D$ and K is a fixed positive number.

If (i), (ii), and (iii) are satisfied, then there is one and only one solution $y = y(t)$ of the differential equation

$$\frac{dy}{dt} = f(t, y)$$

in $|t-t_0| \leq h$ such that $y(t_0) y_0$.

Proof: The classical method of successive approximations is used to prove this theorem (see (2) and (4)). For later use, however, we shall give the details of such a proof in part.

If a solution, $y(t)$, of (1.1) were known, which reduces to y_0 when $t = t_0$, then the solution will satisfy

$$(1.2) \qquad y(t) = y + \int_{t_0}^{t} f(s, y(s))\, ds$$

or an integral equation involving the dependent variable under the integral sign. Let the B-valued function, $f(t)$, now be regarded as unknown; the integral equation (1.2) may be solved then by the method of successive approximations.

Let t, the independent variable, lie in the interval (t_0, t_0+h) and consider the sequence of B-valued functions $y_1(t), y_2(t), \ldots, y_n(t)$ defined as follows:

$$(1.3) \qquad \begin{aligned} y_1(t) &= y_0 + \int_{t_0}^{t} f(s, y_0)\, ds \\ y_2(t) &= y_0 + \int_{t_0}^{t} f(s, y(s))\, ds \\ &\vdots \\ y_n(t) &= y_0 + \int_{t_0}^{t} f(s, y_{n-1}(s))\, ds \end{aligned}$$

We shall now prove that the limit function of $y_n(t)$ defined by (1.3), exists and is a continuous function of when $|t-t_0| \leq h$. First of all from (1.3) and (ii) we have

$$\| y_1(t) - y_0 \| \leq \int_{t_0}^{t} \| f(s, y_0) \|\, ds \leq M(t-t_0) \leq Mh \leq b$$

for $t_0 \leq t \leq t_0+h$. But this implies by (ii) that

$$(1.4) \qquad \| f(t, y_1(t)) \| \leq M$$

Since (1.4) is true, let us suppose for purposes of induction that $\| y_{n-1}(t) - y_0 \| \leq b$

Then we have $\| f(t, y_{n-1}(t)) \| \leq M$ and further that

$$\| y_n(t) - y_0 \| \leq \int_{t_0}^{t} \| f(t, y_{n-1}(t)) \| dt \leq Mh \leq b$$

Hence induction is complete and by (ii) it follows that

(1.5) $\| f(t, y_n(t)) \| \leq M$ when $t_0 \leq t \leq t_0 + h$, all integers n. With (1.5) true we can assume again for induction that

(1.6) $$\| y_{n-1}(t) - y_{n-2}(t) \| \leq \frac{MK^{n-2} |t-t_0|^{n-1}}{(n-1)!}$$

Then $$\| y_n(t) - y_{n-1}(t) \| \leq \int_{t_0}^{t} \| f(s, y_{n-1}(s)) - f(s, y_{n-2}(s)) \| ds$$

$$\leq \int_{t_0}^{t} K \| y_{n-1}(s) - y_{n-2}(s) \| ds \quad \text{by (iii)}$$

So that by (1.6)

(1.7) $$\| y_n(t) - y_{n-1}(t) \| \leq \frac{MK^{n-1}}{(n-1)!} \int_{t_0}^{t} |s-t_0|^{n-1} dt = \frac{MK^{n-1}}{n!} |t-t_0|^n$$

when $t_0 \leq t \leq t_0 + h$. Since (1.7) is true for $n=1$ and can be proved similarily for interval $t_0 - h \leq t \leq t_0$, the induction is complete and we have

(1.8) $\| y_n(t) - y_{n-1}(t) \| \leq \dfrac{MK^{n-1} |t-t_0|^n}{n!}$ for all n when $|t-t_0| \leq h$

But now it is clear that

$$y_n(t) = y_0 + \sum_{r=1}^{n} [y_r(t) - y_{r-1}(t)]$$

and further from (1.8) we have

$$\| y_r(t) - y_{r-1}(t) \| < \frac{MK^{r-1} h^r}{r!} = M_r \qquad \text{for } |t-t_0| \leq h$$

with $\sum_{r=1}^{\infty} M_r$ being convergent. Thus given ϵ there exists $N(\epsilon)$ such that $\sum_{r=m+1}^{n} M_r < \epsilon$ for $n, m > N(\epsilon)$

Also

$$\| y_n(t) - y_m(t) \| = \| \sum_{r=m+1}^{n} [y_r(t) - y_{r-1}(t)] \| < \sum_{r=m+1}^{n} M_r$$

Thus we have the statement of uniform convergence:

(1.9) Given ϵ, there exists $N(\epsilon)$ such that $\| y_n(t) - y_m(t) \| < \epsilon$ for $n, m > N(\epsilon)$, independently of t in closed interval defined by $|t-t_0| \leq h$

By the completeness of $\mathcal{B}$ and (1.9) there exists $y(t) = \lim_{n\to\infty} y_n(t)$ and:

(1.10) Given t, there exists $N'(\epsilon)$ such that $\|y(t)-y_m(t)\| < \frac{\epsilon}{3}$ for $m > N'$, independently of t where $|t-t_0| \leq h$.

Thus by (1.10) if t' is any value of t in interval defined by $|t-t_0| \leq h$ we have:

(1.11) Given ϵ, there exists $N'(\epsilon)$ such that $\|y(t')-y_m(t')\| < \frac{\epsilon}{3}$ for $m > N'$.

Further if m is fixed and $m > N'$, $y_m(t)$ is a continuous function of t, thus:

(1.12) Given ϵ, there exists $\delta(\epsilon)$ such that $\|y_m(t')-y_m(t)\| < \frac{\epsilon}{3}$ for $|t-t'| < \delta$.

Hence from (1.10), (1.11), and (1.12) we have:

Given ϵ, there exists $\delta(\epsilon)$ such that

$$\|y(t')-y(t)\| = \|y(t')-y_m(t')+y_m(t')-y_m(t)+y_m(t)-y(t)\|$$
$$\leq \|y(t')-y_m(t)\| + \|y_m(t')-y_m(t)\| + \|y(t)-y_m(t)\| < \epsilon$$

if $|t-t'| < \delta$

Thus the limit function, $y(t) = \lim_{n\to\infty} y_n(t)$ exists, and is a continuous function of t in the interval $|t-t_0| \leq h$

Now from (1.10) and (iii) we have

(1.13) $$\left\| \int_{t_0}^{t} [f(s,y(s)) - f(s,y_{n-1}(s))]\,ds \right\| \leq K \left| \int_{t_0}^{t} \|y(s)-y_{n-1}(s)\|\,ds \right|$$
$$\leq K\epsilon_n |t-t_0| < K\epsilon_n h$$

where ϵ_n is independent of t and tends to zero as n tends to infinity. Then by (1.13)

$$\lim_{n\to\infty} y_n(t) = y_0 + \lim_{n\to\infty} \int_{t_0}^{t} f(s,y_{n-1}(s))\,ds$$
$$= y_0 + \int_{t_0}^{t} \lim_{n\to\infty} f(s,y_{n-1}(s))\,ds$$

and consequently, $y(t)$ is a solution of the integral equation. (1.14)

By (i) $f(t, y(t))$ is continuous in the interval $|t-t_0| \leq h$, hence we have from the theory of integration for a B-valued function of areal variable that

(1.15) $$\frac{dy(t)}{dt} = \frac{d}{dt}\int_{t_0}^{t} f(s, y(s))\,ds = f(t, y(t)).$$

Consequently, by (1.14) and (1.15) $y(t)$ satisfies the integral equation, (1.14), and the differential equation and assumes the value, y_0, when $t = t_0$.

The uniqueness of the solution, $y(t)$, obtained in the preceding manner by the method of successive approximations, can be proved in the classical way. See for instance (2), pg. 17. Hence the theorem is established.

Before discussing the existence of a solution to the Riccati equation, mentioned in the fifth paragraph of this chapter, we will give the definition of a Complete Normed Linear Ring (10).

Definition 1.1. R is a Complete Normed Linear Ring if R is a ring as well as a Banach space and if, in addition for $x, y \in R$

$$\|xy\| \leq m\|x\|\|y\|$$

where m is the modulus of the product xy. It is a real or a complex complete normed linear ring according as Φ, the scalar field, is the real or complex number field.

It should be mentioned at this point that herein we will be restricted to the real scalar field, so when the terms Banach space or complete normed linear ring are used we will mean over the scalar field of real numbers.

The following theorem is an application of Theorem 1.1 to a Riccati type equation:

Theorem 1.2. If R is a complete normed linear ring and E is the space of real numbers, then $y\,A(t)\,y$ on ER to R is continuous in each variable, separately, for domain D, defined by $|t-t_0| \leq h$, $\|y-y_0\| \leq b$ with h and b fixed, under the conditions that $y \in R$ and $A(t)$ on interval, $|t-t_0| \leq h$, of E to R is continuous and consequently bounded or that $\|A(t)\| \leq N$ when $|t-t_0| \leq h$. Furthermore if

$$N = \frac{b}{m^2 h\,(\|y_0\|+b)^2}$$

where m is the modulus of the ring product, then the abstract Riccati equation

(1.16) $\quad \frac{dy}{dt} = y\,A(t)\,y$

has one and only one solution $y = y(t)$ in interval, $|t-t_0| \leq h$ such that $y(t_0) = y_0$.

Proof: The proof will follow from theorem 1.1 if we satisfy conditions (i), (ii), and (iii). The domain D in this theorem is the same as D of Theorem 1.1. Now since $\|y-y_0\| \leq b$ we have

(1.17) $\quad \|y\| \leq c\,,\ y \in D$ and $c = \|y_0\|+b$.

Also from the continuity of $A(t)$ in D or over the closed interval, defined by $|t-t_0| \leq h$, we have that:

(1.18) There exists a finite N such that $\|A(t)\| \leq N$ for $|t-t_0| \leq h$;

stating the above continuity:

(1.19) Given ϵ, there exists $\delta_1(\epsilon)$ such that

$$\|A(t)-A(t')\| < \frac{\epsilon}{3m^2c^2} \quad \text{for } |t-t'| < \delta\,,$$

and

(1.20) Given ϵ , there exists $\delta_2(\epsilon) = \frac{\epsilon}{3m^2cN}$ such that

$$\|y-y'\| < \frac{\epsilon}{3m^2cN} \quad for \quad \|y-y'\| < \delta_2 \ .$$

Now to satisfy (i) of Theorem 1.1 we have $f(t,y) = y\,A(t)\,y$ and by (1.17), (1.18), (1.19), and (1.20), that given ϵ , there exists $\delta(\epsilon) = min\,(\delta_1, \delta_2)$ such that

$$\begin{aligned}
&\|y'A(t')y' - yA(t)y\| \\
&\quad = \|y'A(t')y' - y'A(t')y + y'A(t')y - y'A(t)y + y'A(t)y - yA(t)y\| \\
&\quad \le \|y'A(t')y' - y'A(t')y\| + \|y'A(t')y - y'A(t)y\| + \|y'A(t)y - yA(t)y\| \\
&\quad \le m^2\left[\|y'\|\,\|A(t')\|\frac{\epsilon}{3m^2cN} + \|y'\|\,\|y\|\frac{\epsilon}{3m^2cN} + \|y\|\,\|A(t)\|\frac{\epsilon}{3m^2cN}\right] \\
&\quad < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \quad for\ |t-t'| < \delta,\ \|y'-y\| < \delta \ ,\ and
\end{aligned}$$

t' and y' varying independently in D . Thus (i) is satisfied.

For (ii) by hypothesis and (1.18),

$$\|A(t)\| \le N = \frac{b}{m^2h(\|y_0\|+b)^2}$$

hence by (1.17)

$$\begin{aligned}
\|f(t,y)\| &= \|y\,A(t)\,y\| \le m^2\|y\|^2\|A(t)\| \\
&\le m^2\|y\|^2\frac{b}{m^2h(\|y_0\|+b)^2} \le \frac{b}{h} = M .
\end{aligned}$$

Thus $M = \frac{b}{h}$ and (ii) is satisfied.

If now (t,y) and (t,Y) are within D , then

$$\begin{aligned}
&\|YA(t)Y - yA(t)y\| \\
&\quad = \|YA(t)Y - YA(t)y + YA(t)y - yA(t)y\| \\
&\quad \le \|YA(t)Y - YA(t)y\| + \|YA(t)y - yA(t)y\| \\
&\quad \le m^2\|A(t)\|\,(\|Y\| + \|y\|)\,\|Y-y\|
\end{aligned}$$

Hence by (1.18) and (1.19)

$$\|YA(t)Y - yA(t)y\| \le 2m^2Nc\,\|Y-y\|$$

or $$\|YA(t)Y - yA(t)y\| \le K\,\|Y-y\|$$

where $K = 2m^2NC$ and is a finite positive number. Thus condition (iii) of Theorem 1.1 is satisfied and Theorem is established.

In concluding this chapter it is relevent to say that we will be concerned with methods in the next few chapters of obtaining Fréchet differentials of the solutions of differential equations, such as (1.16). We will assume the definition and the elementary operational properties of the Fréchet differential.

Chapter II

A Functional Equation Related to the Fréchet Differential of the Solution of $\frac{dy}{dt} = yA(t)y$ as a Functional of $A(t)$

Under the assumption that the Fréchet differential with respect to $A(t)$ of the solution, $y(t)$, of (1.16) exists, it is not difficult to see (see (7)) that $\delta y[A/t]$ could be obtained from the integral equation,

$$(2.1)\quad \delta y[A/t] = \int_{t_0}^{t} y[A/s]\,\delta A(s)\,y[A/s]\,ds + \int_{t_0}^{t} \delta y[A/s]\,A(s)\,y[A/s]\,ds + \int_{t_0}^{t} y[A/s]\,A(s)\,\delta y[A/s]\,ds$$

if $y[A/t]$, the solution of (1.16) were known. (2.1) is of the form of the more general Volterra type integral equation

$$(2.2)\quad y(\tau) = f(\tau) + \int_a^{\tau} K(\tau,\xi)\,y(\xi)\,d\xi + \int_a^{\tau} y(\xi)\,H(\xi,\tau)\,d\xi \quad (a \le \xi \le \tau \le b)$$

where $K(t,s)$, $f(s)$ and $H(s,t)$ are continuous in the closed interval $a \le s,t \le b$ with values in a complete normed linear ring, R (not necessarily commutative).

In order to solve (2.2) and hence (2.1) let us consider the functional equation,

$$(2.3)\quad y = f + K \odot y + y \circledast H$$

with respect to which we make the following assumptions:

Assumption 2.1. $K \odot y$ and $y \otimes H$ are bilinear functions whose values and independent variable y are in a Banach space B, while the independent variable K ranges over a complete normed linear ring R_K and the independent variable H ranges over a complete normed linear ring R_H where the unit elements are not assumed to exist.

Assumption 2.2 $(K_1 K_2) \odot y = K_1 \odot (K_2 \odot y)$ for all $K_1, K_2 \in R_K$ and $y \in B$. $y \otimes [H_1 H_2] = [y \otimes H_1] \otimes H_2$ for all $H_1, H_2 \in R_H$ and $y \in B$.

Assumption 2.3 There exists a positive number M such that

$$\|K^i\| \leq \frac{M^{i-1}\|K\|^i}{(i-1)!}$$

$$\|H^i\| \leq \frac{M^{i-1}\|H\|^i}{(i-1)!} \qquad (i = 1, 2, 3, \dots)$$

Assumption 2.4 If $F(K_1, K_2, \dots K_m ; H_1, H_2, \dots H_n ; y)$ represents the multilinear function formed by iterating either $K_1 \odot y$ or $y \otimes H_1$ on y with respect to $K_2, K_3, \dots K_m$, $H_2, H_3, \dots H_n$ in any specified ordering, then with M given by Assumption 2.3

$$\| F(K_1, K_2, \dots K_m ; H_1, H_2, \dots H_n ; y) \| \leq$$
$$\leq M^{m+n}\Big(\prod_{i=1}^{n}\|K_i\|\Big)\Big(\prod_{j=1}^{n}\|H_j\|\Big)\Big/(m+n)! \quad (m, n = 1, 2, 3, \dots)$$

where $m = 0$ and $n = r > 0$ means F is independent of $K_i \in R_K$, $m = s > 0$, $n = 0$ means F is independent of $H_j \in R_H$ and $m = 0$, $n = 0$ means $F \equiv f$. An example in which the above four assumptions are all satisfied is given in Theorem 2.2.

We will now define the m-th alternate iterations of the two functions $K \odot y$ and $y \otimes H$.

Definition 2.1 The m-th alternate iterations of the two functions $K \odot y = \mathcal{K}(K, y)$ and $y \otimes H = \mathcal{H}(y, H)$ with respect to the same two functions are defined as follows:

$$\mathcal{K}^{(m)}(K, f) = \mathcal{K}(K, \mathcal{H}^{(m-1)}(f, H)) = K \odot (\mathcal{H}^{(m-1)}(f, H))$$

$$\mathcal{H}^{(m)}(f, H) = \mathcal{H}(\mathcal{K}^{(m-1)}(K, f), H) = [\mathcal{K}^{(m-1)}(K, f)] \otimes H$$

with $\mathcal{K}^{(1)}(K, f) = K \odot f$ and $\mathcal{H}^{(1)}(f, H) = f \otimes H$.

From Assumption 2.4 and Definition 2.1 the following lemma is clear:

Lemma 2.1. There exists a positive number M such that

$$\|\mathcal{K}^{(m)}(K,f)\| \leq \underset{x=(\|K\|,\|H\|)}{\mathrm{Max}} \frac{M^m x^m \|f\|}{m!}$$

$$\|\mathcal{H}^{(m)}(f,H)\| \leq \underset{x=(\|K\|,\|H\|)}{\mathrm{Max}} \frac{M^m x^m \|f\|}{m!} .$$

We are able now to prove the following theorem, concerning the solution of (2.3) for y. (Compare with Theorem 5.1)

<u>Theorem 2.1</u> The solution of (2.3) under A-ssumptions 2.1, 2.2, 2.3, and 2.4 and Definition 2.1 is an entire analytic function of K and H, separately, and is given uniquely by

(2.4) $$y = f + \sum_{i=1}^{\infty} \mathcal{K}^{(i)}(K,f) + \sum_{j=1}^{\infty} \mathcal{H}^{(j)}(f,h)$$
$$= f + k \circ f + k \circ (f \otimes h) + k \circ ([k \circ f] \otimes h) + \cdots$$
$$+ f \otimes h + [k \circ f] \otimes h + [k \circ (f \otimes h)] \otimes h + \cdots$$

where

(2.5) $$k = K + K^2 + K^3 + \cdots \quad \in R_K$$

and $$h = H + H^2 + H^3 + \cdots \quad \in R_H .$$

<u>Proof:</u> By assumption 2.3

$$\|K^i\| \leq \frac{M^{(i-1)} \|K\|^i}{(i-1)!} .$$

Hence

(2.6) $$\|k\| \leq \|K + K^2 + K^3 + \cdots\| \leq \|K\| + \|K^2\| + \cdots$$
$$\leq \|K\| e^{M\|K\|}$$

Thus the representation for k converges, and by the completeness of R_K, k is an element of R_K for all K in R_K as given by (2.5). Similarly

(2.7) $$\|h\| \leq \|H\| e^{M\|H\|} , \quad H \in R_H$$

and his representable by (2.5) in R_H. From (2.4), (2.6), and (2.7) and Lemma 2.1, we have

$$\|y\| < \underset{x=(\|K\|,\|H\|)}{\mathrm{Max}} 2\|f\| e^{Mx} e^{Mx}$$

so that (2.4) converges for all $K \in R_K$ and $H \in R_H$.
Thus by the completeness of the space B , y exists in B as defined by (2.4) as an entire analytic function of K and H , separately.

Now to show that (2.4) satisfies (2.3) we substitute (2.4) in the right side of (2.3) and obtain by using Assumptions 2.1 and 2.2, Definition 2.1, and (2.5),

$$f + K \odot f + K \odot (K \odot f) + \sum_{i=1}^{\infty} K \odot (K \odot \mathcal{H}^{(i)}(f,h)) + \sum_{i=1}^{\infty} K \odot (\mathcal{H}^{(i)}(f,h)) +$$

$$+ f \otimes H + \sum_{i=1}^{\infty} [\mathcal{K}^{(i)}(K,f)] \otimes H + [f \otimes h] \otimes H + \sum_{i=1}^{\infty} [[\mathcal{K}^{(i)}(K,f)] \otimes h] \otimes h =$$

$$= f + K \odot f + \{K \odot f - K \odot f\} + \{\sum_{i=2}^{\infty} \mathcal{K}^{(i)}(K,f) - \sum_{i=1}^{\infty} K \odot (\mathcal{H}^{(i)}(f,h))\} +$$

$$+ \sum_{i=1}^{\infty} K \odot (\mathcal{H}^{(i)}(f,h)) + f \otimes h + \sum_{i=1}^{\infty} [\mathcal{K}^{(i)}(K,f)] \otimes H + \{f \otimes h - f \otimes H\} +$$

$$+ \{\sum_{i=2}^{\infty} \mathcal{H}^{(i)}(f,h) - \sum_{i=1}^{\infty} [\mathcal{K}^{(i)}(K,f)] \otimes H\} = f + \sum_{i=2}^{\infty} \mathcal{K}^{(i)}(K,f) + \sum_{i=2}^{\infty} \mathcal{H}^{(i)}(f,h) + K \odot f + f \otimes h =$$

$$= f + \sum_{i=1}^{\infty} \mathcal{K}^{(i)}(K,f) + \sum_{i=1}^{\infty} \mathcal{H}^{(i)}(f,h)$$

which is the same as the left side of (2.3) as defined by (2.4). Thus (2.4) is a solution, satisfying (2.3).

To obtain the uniqueness suppose that y and y' are the solutions, satisfying (2.3). Then

$$y = f + K \odot y + y \otimes H$$

$$y' = f + K \odot y' + y' \otimes H$$

from which follows

$$(2.7) \qquad y - y' = K \odot (y - y') + [y - y'] \otimes H \quad .$$

If we iterate, successively, n times, $y - y'$ in (2.7), we obtain from Assumption 2.4 that

$$(2.8) \qquad \|y - y'\| \leq \frac{M^n (2x)^n}{n!} \|y - y'\| \quad , \quad x = Max(\|K\|, \|H\|)$$

Now suppose that y' were different from y , then $\|y - y'\| > 0$

and from (2.8)

$$1 \leq \frac{M^n (2x)^n}{n!} \qquad (n = 1, 2, 3, 4, \cdots) .$$

But this is a contradiction for there exists n_0 such that

$$\frac{M^{n_0} (2x)^{n_0}}{n_0 !} < 1$$

Hence $\|y - y'\| = 0$ or $y = y'$ and the solution of (2.3) is unique. Thus the theorem is proved.

Now let R be a complete normed linear ring (not necessarily commutative) with modulus m in which there are elements $K(t,s)$, $f(s)$, $H(s,t)$, continuous with respect to the real variables t and s, in the interval $a \leq s, t \leq b$. Let R_K and R_H be complete normed linear rings and B, the Banach space, as used in Theorem 2.1 with values of the functions in R for each t and s. The products in R_K and R_H will be Volterra integral compositions and

(2.9)
$$K \circ y = \int_a^x K(x,\xi)\, y(\xi)\, d\xi$$

$$y \otimes H = \int_a^x y(\xi)\, H(\xi, x)\, d\xi , \qquad (a \leq x \leq b) .$$

Moreover, the norms in R_K, R_H, and B are defined, respectively, by

(2.10)
$$\|K\| = \max_{a \leq t, s \leq b} \|K(t,s)\|_R$$
$$\|H\| = \max_{a \leq t, s \leq b} \|H(s,t)\|_R$$
$$\|f\| = \max_{a \leq s \leq b} \|f(s)\|_R$$

where $\| \ \|_R$ is the norm of R. We see now the possibility of the following theorem:

<u>Theorem 2.2</u> If $K(t,s)$, $f(s)$, and $H(s,t)$ are continuous in the closed interval $a \leq s, t \leq b$ with values in a complete normed linear ring R of modulus m, then the unique continuous solution $y(x)$ of

$$y(x) = f(x) + \int_a^x K(x,\xi)\, y(\xi)\, d\xi + \int_a^x y(\xi)\, H(\xi, x)\, d\xi , \quad (a \leq x \leq b),$$

is given by

$$y(x) = f(x) + \sum_{i=1}^{\infty} \mathcal{K}^{(m)}(K,f) + \sum_{i=1}^{\infty} \mathcal{H}^{(m)}(f,h)$$

where

$$\mathcal{K}^{(2n)}(K,f) = \int_a^x \int_\xi^x \int_\xi^{\rho_{2n-1}} \cdots \int_\xi^{\rho_3} \int_\xi^{\rho_2} K(x,\rho_{2n-1}) \cdots K(\rho_2,\rho_1) f(\xi) h(\xi,\rho_1) \cdots h(\rho_{2n-2},\rho_{2n-1}) \, d\rho_1 \cdots d\rho_{2n-1} \, d\xi$$

$$\mathcal{K}^{(2n+1)}(K,f) = \int_a^x \int_\xi^x \int_\xi^{\rho_{2n}} \cdots \int_\xi^{\rho_3} \int_\xi^{\rho_2} K(x,\rho_{2n}) \cdots K(\rho_1,\rho) f(\xi) h(\rho_1,\rho_2) \cdots h(\rho_{2n-1},\rho_{2n}) \, d\rho_1 \cdots d\rho_{2n} \, d\xi$$

$$\mathcal{H}^{(2n)}(f,K) = \int_a^x \int_\xi^x \int_\xi^{\rho_{2n-1}} \cdots \int_\xi^{\rho_3} \int_\xi^{\rho_2} K(\rho_{2n-1},\rho_{2n-2}) \cdots K(\rho_1,\xi) f(\xi) h(\rho_1,\rho_2) \cdots h(\rho_{2n-1},x) \, d\rho_1 \cdots d\rho_{2n-1} \, d\xi$$

$$\mathcal{H}^{(2n+1)}(f,K) = \int_a^x \int_\xi^x \int_\xi^{\rho_{2n}} \cdots \int_\xi^{\rho_3} \int_\xi^{\rho_2} K(\rho_{2n},\rho_{2n-1}) \cdots K(\rho_2,\rho_1) f(\xi) h(\xi,\rho_1) \cdots h(\rho_{2n},x) \, d\rho_1 \cdots d\rho_{2n} \, d\xi$$

for $n > 1$ and

$$\mathcal{K}^{(1)}(K,f) = \int_a^x K(x,\xi) f(\xi) \, d\xi \quad \text{and} \quad \mathcal{H}^{(1)}(f,h) = \int_a^x f(\xi) h(\xi,x) \, d\xi$$

and further

$$k(x,\xi) = K(x,\xi) + K^2(x,\xi) + K^3(x,\xi) + \cdots$$

$$h(\xi,x) = H(\xi,x) + H^2(\xi,x) + H^3(\xi,x) + \cdots$$

where

$$K^m(x,\xi) = \int_\xi^x \int_\xi^{\rho_{m-1}} \int_\xi^{\rho_{m-2}} \cdots \int_\xi^{\rho_2} K(x,\rho_{m-1}) K(\rho_{m-1},\rho_{m-2}) \cdots K(\rho_2,\rho_1) K(\rho_1,\xi) \, d\rho_1 \cdots d\rho_{m-1}$$

and

$$H^m(\xi,x) = \int_\xi^x \int_\xi^{\rho_{m-1}} \int_\xi^{\rho_{m-2}} \cdots \int_\xi^{\rho_2} H(\xi,\rho_1) H(\rho_1,\rho_2) \cdots H(\rho_{m-2},\rho_{m-1}) H(\rho_{m-1},x) \, d\rho_1 \cdots d\rho_{m-1} .$$

Proof: By (2.9) and (2.10) it is easy to verify that Assumptions 2.1, 2.2, 2.3, and 2.4 are satisfied by the hypothesis. Therefore if we apply Theorem 2.1, the theorem follows.

Theorem 2.2 provides a solution for (2.2) and hence for (2.1), thereby offering a means for computing the Fréchet differential of the solution to (1.16) with respect to $A(t)$, provided we knew it existed. In the next chapter we propose to show that this Fréchet differential does exist.

Chapter III

The Existence of the Fréchet Differential with Respect to $A(t)$ of the Solution to $\frac{dy}{dt} = yA(t)y$

Let us consider again the Riccati differential equation (1.16),

$$\frac{dy(t)}{dt} = y(t)A(t)y(t) \quad , \quad y(t_0) = y_0 ,$$

where $A(t) \in R$, a complete normed linear ring whose products have modulus m. Here, we shall designate the norm in R as $\| \|_R$. Let B denote the Banach space of R-valued continuous functions over $|t-t_0| \leq h$. The norm in B for such an element, $A(t)$, we shall take to be

(3.1) $$\|A\| = \max_{|t-t_0|\leq h} \|A(t)\|_R$$

A restatement of Theorem 1.2 in line with the purposes of this chapter is:

(3.2) The unique, continuous solution of (1.16) is given by $y(t) = \lim_{n\to\infty} y_n(t)$, uniformly, when $|t-t_0| \leq h$

where
$$y_1(t) = y_0 + \int_{t_0}^{t} y_0 A(s) y_0 \, ds$$
and
$$y_n(t) = y_0 + \int_{t_0}^{t} y_{n-1}(s) A(s) y_{n-1}(s) \, ds$$
under the conditions, that $A(t)$ is continuous and

$\|A(t)\|_R \leq \frac{b}{m^2 h(\|y_0\|+b)^2}$ when $|t-t_0| \leq h$ and $\|y-y_0\| \leq b$.

We will now state y_n in terms of two trilinear functions, $Q(y_1, \nu, y_2)$ and $T(y_1, \nu, y_2)$ (additive and contintinuous in each of the three variables) on BBB to B or R, respectively, as follows:

(3.3) $$y_n[A|t] = y_0 + Q(y_{n-1}[A|t], A(t), y_{n-1}[A|t])$$
$$= y_0 + \int_{t_0}^{t} T(y_{n-1}(A|s), A(s), y_{n-1}(A|s))\, ds$$

and $$y_1[A|t] = y_0 + Q(y_0, A(t), y_0) = y_0 + \int_{t_0}^{t} T(y_0, A(s), y_0)\, ds$$

In a more short hand notation (deleting the dependence on the real variable) we may rewrite (3.3) as

(3.4) $$y_n = y_0 + Q(y_{n-1}, A, y_{n-1}) = y_0 + \int_{t_0}^{t} (y_{n-1}, A, y_{n-1}/s))\, ds$$

and

$$y_1 = y_0 + Q(y_0, A, y_0) = y_0 + \int_{t_0}^{t} T(y_0, A, y_0/s)\, ds$$

In accordance with (3.2), (3.3), and (3.4) the definition of $Q(x_1, A_1, x_2)$ and $T(x_1, A_1, x_2/t)$ is explicitly as follows:

Definition 3.1 $Q(x_1, A_1, x_2)$ and $T(x_1, A_1, x_2/t)$ are trilinear functions (additive and continuous in each of the three variables) on BBB to B or R (to R for each value of t in interval $|t-t_0| \leq h$, the dependence on t of $Q(x_1, A_1, x_2)$ being understood), defined by

$$Q(x_1, A_1, x_2) = \int_{t_0}^{t} T(x_1, A_1, x_2/s)\,ds = \int_{t_0}^{t} x_1(s) A_1(s) x_2(s)\, ds$$

From Definition 3. 1 and (3.1) we have

(3.5) $$\|T(x_1, A_1, x_2/t)\|_R = \|x_1(t) A_1(t) x_2(t)\|_R \leq m^2 \|x_1(t)\|_R \|x_2(t)\|_R \|A(t)\|_R$$
$$\leq m^2 \|x_1(t)\|_R \|x_2(t)\|_R \|A\|$$

Moreover from (3.5),

(3.6) $$\|Q(x_1, A_1, x_2)\|_R \leq \left|\int_{t_0}^{t} \|T(x_1, A_1, x_2/s)\|_R ds\right| \leq m^2 \|A\| \left|\int_{t_0}^{t} \|x_1(s)\|_R \|x_2(s)\|_R\, ds\right|$$

and

(3.7) $$\|Q(x_1, A_1, x_2)\| \leq \max_{|t-t_0|\leq h} \|Q(x_1, A_1, x_2)\|_R$$
$$\leq m^2 \|A\| \max_{\sigma = \pm h} \left|\int_{t_0}^{t+\sigma} \|x_1(s)\|_R \|x_2(s)\|_R\, ds\right|$$

Now let us consider three lemmas which give the polynomial properties (as a polynomial in A) of y_n. With these lemmas we will be able to prove that $y(A/t)$ can be represented as an abstract power series of homogeneous polynomials of A with a definite radius of analyticity greater than zero. (See (5) for elementary properties of homogeneous polynomials).

Lemma 3.1 y_n is a polynomial in A of degree 2^n-1 which can be represented as a sum of homogeneous polynomials in the following manner:

$$y_n = \sum_{i=0}^{2^n-1} \Gamma_i^{(n)}(A)$$

where $\Gamma_i^{(n)}(A)$ is a homogeneous polynomial in A of degree i depending in general on n, and

$$\Gamma_0^{(n)}(A) = y_0 \qquad \text{all } n$$

Proof: From (3.4)

$$y_1 = y_0 + Q(y_0, A, y_0) .$$

Hence

$$y_1 = \sum_{i=0}^{2^1-1} \Gamma_i^{(1)}(A)$$

where $\Gamma_0^{(1)}(A) = y_0$ and $\Gamma_1'(A) = Q(y_0, A, y_0)$. Thus the lemma is true for $n=1$. Now assume for purposes of induction that the lemma is true for all integers n such that $n \leq m$, then

(3.8) $$y_m = \sum_{i=0}^{2^m-1} \Gamma_i^{(m)}(A) \quad \text{and} \quad \Gamma_0^{(m)}(A) = y_0 .$$

Then by (3.4) and (3.8) we have

(3.9) $$y_{m+1} = y_0 + Q(y_m, A, y_m)$$

$$= y_0 + Q\Big(\sum_{i=0}^{2^m-1} \Gamma_i^{(m)}(A), A, \sum_{i=0}^{2^m-1} \Gamma_i^{(m)}(A)\Big)$$

$$= y_0 + \sum_{j=0}^{(2^m-1)+(2^m-1)} \sum_{\substack{0 \leq r, s \leq 2^m-1 \\ r+s=j}} Q(\Gamma_r^{(m)}(A), A, \Gamma_s^{(m)}(A))$$

$$= y_0 + \sum_{j=0}^{2^{m+1}-2} \sum_{\substack{r+s=j \\ 0 \leq r, s \leq 2^m-1}} Q(\Gamma_r^{(m)}(A), (A), \Gamma_s^{(m)}(A)) .$$

But $$\sum_{\substack{r+s=j \\ 0 \leq r, s \leq 2^m-1}} Q(\Gamma_r^{(m)}(A), A, \Gamma_s^{(m)}(A))$$

is a homogeneous polynomial in A of degree, $j+1$, hence

$$\Gamma_i^{(m+1)}(A) = \sum_{\substack{r+s=i-1 \\ 0 \leq r, s \leq 2^m-1}} Q(\Gamma_r^{(m)}(A), A, \Gamma_s^{(m)}(A)) .$$

Thus (3.9) becomes

$$y_{m+1} = y_0 + \sum_{i=1}^{2^{m+1}-1} \Gamma_i^{(m+1)}(A)$$

where $\Gamma_0^{(m+1)}(A) = y_0$,

thereby completing induction. Hence lemma follows.

<u>Lemma 3.2</u> $\Gamma_m^{(m)}(A) = \Omega_m(A)$ is a unique homogeneous polynomial of degree m in A , unique in the sense that $\Gamma_m^{(n)} = \Omega_m(A)$ for all n, $n \geq m$,

and

$$\Omega_m(A) = Q(y_0, A, \Omega_{m-1}(A)) + Q(\Omega_{m-1}(A), A, y_0) + \sum_{\substack{i+j=m-1 \\ i,j\geq 1}} Q(\Omega_i(A), A, \Omega_j(A)), \quad \Omega_0(A) = y_0 . \tag{3.10}$$

<u>Proof</u>: By Lemma 3.1

$$\Gamma_0^{(n)}(A) = \Gamma_0^{(0)}(A) = \Omega_0(A) = y_0 \quad , \text{all } n ,$$

thus $\Gamma_0^{(0)}$ is unique and our lemma is satisfied for $m=0$

Now from Lemma 3.1 and (3.4)

$$y_1 = y_0 + Q(y_0, A, y_0) = y_0 + \Gamma_1^{(1)}(A) .$$

Thus $\Gamma_1^{(1)}(A) = \Omega_1(A) = Q(y_0, A, y_0)$. Now assume that $\Gamma_1^{(i)}(A) = \Omega_i(A)$ for $1 \leq i \leq j$. Then by Lemma 3.1

$$y_j = y_0 + \sum_{r=1}^{2^j-1} \Gamma_r^{(j)}(A)$$

where $\Gamma_1^{(j)}(A) = \Omega_1(A)$, and by (3.4)

$$y_{j+1} = y_0 + Q(y_j, A, y_j) = y_0 + Q\left(y_0 + \sum_{r=1}^{2^j-1}\Gamma_r^{(j)}(A), A, y_0 + \sum_{r=1}^{2^j-1}\Gamma_r^{(j)}(A)\right) =$$

$$= y_0 + Q(y_0, A, y_0) + \sum_{r=1}^{2^j-1}\{Q(y_0, A, \Gamma_r^{(j)}(A)) + Q(\Gamma_r^{(j)}(A), A, y_0)\} +$$

$$+ \sum_{r=2}^{2^{j+1}-2} \sum_{\substack{k+s=r \\ 1\leq k,s\leq 2^j-1}} Q(\Gamma_k^{(j)}(A), A, \Gamma_s^{(j)}(A)) .$$

Thus $\Gamma_1^{(j+1)}(A) = Q(y_0, A, y_0) = \Omega_1(A)$ and induction is complete, implying that $\Omega_1(A) = Q(y_0, A, y_0)$ is unique and that our lemma is true for $m=1$. Now that lemma is true for $m=0, 1$

let us assume it true for all $m \leq K$. Then
$\Gamma_m^{(m)}(A) = \Omega_m(A)$, uniquely, or
(3.11) $\Gamma_m^{(n)}(A) = \Omega_m(A)$ *for all* $n \geqslant m$ when $m \leq K$.

By Lemma 3.2

$$y_K = y_0 + \sum_{r=1}^{2^K-1} \Gamma_r^{(K)}(A),$$

and from (3.4)

$$y_{K+1} = y_0 + Q(y_K, A, y_K) =$$

$$= y_0 + Q(y_0, A, y_0) + \sum_{r=1}^{2^K-1} \{Q(y_0, A, \Gamma_r^{(K)}(A)) + Q(\Gamma_r^{(K)}(A), A, y_0)\} +$$

$$+ \sum_{r=2}^{2^{K-1}-2} \sum_{\substack{l+s=r \\ 1 \leq l, s \leq 2^K-1}} Q(\Gamma_l^{(K)}(A), A, \Gamma_s^{(K)}(A)).$$

Hence

$$\Omega_{K+1}(A) = \Gamma_{K+1}^{(K+1)}(A) = Q(y_0, A, \Gamma_K^{(K)}(A)) + Q(\Gamma_K^{(K)}(A), A, y_0) +$$

$$+ \sum_{\substack{l+s=K \\ 1 \leq l, s \leq K-1 < 2^K-1}} Q(\Gamma_l^{(K)}(A), A, \Gamma_s^{(K)}(A)),$$

and by (3.11)

(3.12) $$\Omega_{K+1}(A) = Q(y_0, A, \Omega_K(A)) + Q(\Omega_K(A), A, y_0)$$

$$+ \sum_{\substack{l+j=K \\ l,s \geqslant 1}} Q(\Omega_l(A), A, \Omega_s(A)).$$

Now assume that $\Gamma_{K+1}^{(n)} = \Omega_{K+1}(A)$ for $K+1 \leq h \leq p$ as defined by (3.12), then

$$y_{p+1} = y_0 + Q(y_0, A, y_0) + \sum_{r=1}^{2^p-1} \{Q(y_0, A, \Gamma_r^{(p)}(A)) + Q(\Gamma_r^{(p)}(A), A, y_0)\} +$$

$$+ \sum_{r=2}^{2^{p+1}-2} \sum_{\substack{l+s=K \\ 1 \leq l, s \leq 2^p-1}} Q(\Gamma_l^{(p)}(A), \Gamma_s^{(p)}(A)).$$

Hence

$$\Gamma_{K+1}^{(p+1)}(A) = Q(y_0, A, \Gamma_K^{(p)}(A)) + Q(\Gamma_K^{(p)}(A), A, y_0) +$$

$$+ \sum_{\substack{l+s=K \\ 1 \leq l, s \leq K-1 < 2^p-1}} Q(\Gamma_l^{(p)}(A), A, \Gamma_s^{(p)}(A)),$$

and by (3.11)

$$\Gamma_{K+1}^{(p+1)}(A) = Q(y_0, A, \Omega_K(A)) + Q(\Omega_K(A), A, y_0) +$$

$$+ \sum_{\substack{l+s=K \\ l,s \geqslant 1}} Q(\Omega_l(A), A, \Omega_s(A)) = \Omega_{K+1}(A).$$

Thus the second induction is complete, giving the conclusion that $\Gamma_{K+1}^{(n)}(A) = \Omega_{K+1}(A)$ for all n, $n \geq K+1$, or that $\Gamma_{K+1}^{(K+1)}(A) = \Omega_{K+1}(A)$ is unique. Also by (3.12)

$$\Omega_{K+1}(A) = Q(y_0, A, \Omega_K(A)) + Q(\Omega_K(A), A, y_0) + \sum_{\substack{l+s=K \\ l,s \geq 1}} Q(\Omega_l(A), A, \Omega_s(A)),$$

thereby completing the first induction with the conclusion that the lemma is satisfied for all m. Hence lemma is proved.

The next lemma will be concerned with the norm of the homogeneous polynomial, $\Gamma_i^{(n)}$, of degree i with respect to both the complete normed linear ring R and the Banach space B.

Lemma 3.3 $\|\Gamma_i^{(n)}(A)\|_R \leq |t-t_0|^i m^{2i} \|y_0\|^{i+1} \|A\|^i$

and $\|\Gamma_i^{(n)}(A)\| \leq h^i m^{2i} \|y_0\|^{i+1} \|A\|^i$

for $n = 0, 1, 2, \cdots$ and $0 \leq i \leq 2^n - 1$, and when $|t-t_0| \leq h$.

Proof: By lemma 3.1

$$\|y_0\|_R = \|\Gamma_0^{(0)}(A)\|_R = \|y_0\| = \|\Gamma_0^{(n)}(A) = \|\Gamma_0^{(n)}(A)\| = \|\Gamma_0^{(n)}(A)\|_R \text{ all } n.$$

Since $y_1 = y_0 + Q(y_0, A, y_0)$ then $\Gamma_1^{(1)}(A) = Q(y_0, A, y_0)$ and we have by (3.6)

$$\|\Gamma_1^{(1)}(A)\|_R = \|Q(y_0, A, y_0)\|_R \leq m^2 \|A\| \left|\int_{t_0}^{t} \|y_0\|_R^2 ds\right| = m^2 |t-t_0| \|A\| \|y_0\|^2$$

and by (3.7)

$$\|\Gamma_1^{(1)}(A)\| \leq m^2 \|A\| \|y_0\|^2 h$$

Thus the lemma is satisfied for $n=0$ and $n=1$.

Now for purposes of induction assume that lemma is true for all $n \leq K$. Then by Lemma 3.1 $y_K = \sum_{i=0}^{2^K-1} \Gamma_i^{(K)}(A)$.

and by (3.4)

$$y_{K+1} = y_0 + Q(y_K, A, y_K) = y_0 + \sum_{j=0}^{2^{K+1}-2} \sum_{\substack{r+s=j \\ 0 \leq r,s \leq 2^K-1}} Q(\Gamma_r^{(K)}(A), A, \Gamma_s^{(K)}(A)).$$

Hence

$$(3.13) \quad \Gamma_i^{(K+1)}(A) = \sum_{\substack{r+s=i-1 \\ 0 \leq r,s \leq 2^K-1}} Q(\Gamma_r^{(K)}(A), A, \Gamma_s^{(K)}(A))$$

for $1 \leq i \leq 2^{K+1}-1$. Now by (3.13), our induction hypothesis and (3.6) we have

$$\|\Gamma_i^{(K+1)}(A)\|_R \leq \sum_{\substack{r+s=i-1 \\ 0\leq r,s\leq 2^K-1}} \|Q(\Gamma_r^{(K)}(A), A, \Gamma_s^{(K)}(A)\|_R$$

$$\leq m^2\|A\| \sum_{\substack{r+s=i-1 \\ 0\leq r,s\leq 2^K-1}} \left|\int_{t_0}^t \|\Gamma_r^{(K)}(A)\|_R \|\Gamma_s^{(K)}(A)\|_R\, ds\right|$$

$$\leq m^2\|A\| \sum_{\substack{r+s=i-1 \\ 0\leq r,s\leq 2^K-1}} \left|\int_{t_0}^t |\sigma-t_0|^r m^{2r}\|y_0\|^{r+1}\|A\|^r|\sigma-t_0|^s m^{2s}\|y_0\|^{s+1}\|A\|^s d\sigma\right|$$

$$= m^{2i}\|A\|^i\|y_0\|^{i+1} \sum_{\substack{r+s=i-1 \\ 0\leq r,s\leq 2^K-1}} \left|\int_{t_0}^t |\sigma-t_0|^{r+s} d\sigma\right|$$

$$= m^{2i}\|A\|^i\|y_0\|^{i+1} \sum_{\substack{r+s=i-1 \\ 0\leq r,s\leq 2^K-1}} \frac{|t-t_0|^{r+s+1}}{r+s+1}$$

$$\leq m^{2i}\|A\|^i\|y_0\|^{i+1}(i)\frac{|t-t_0|^i}{i} = m^{2i}\|A\|^i\|y_0\|^{i+1}|t-t_0|^i$$

for $1 \leq i \leq 2^{K+1}$ and $|t-t_0| \leq h$.

Thus $\|\Gamma_i^{(K+1)}(A)\|_R \leq |t-t_0|^i m^{2i}\|y_0\|^{i+1}\|A\|^i$ when $0 \leq i \leq 2^{K+1}$ and $|t-t_0| \leq h$. But $\|\Gamma_i^{(K+1)}(A)\|_R \leq h^i m^{2i}\|y_0\|^{i+1}\|A\|^i$ for $0 \leq i \leq 2^{K+1}$, independently of t for $|t-t_0|\leq h$ hence $\|\Gamma_i^{(K+1)}(A)\| \leq m^{2i}\|A\|^i\|y_0\|^{i+1}h^i$ for $0 \leq i \leq 2^{K+1}$, and our induction is complete. Thus Lemma is true.

In the previous lemmas it is to be understood that $y_n \equiv y_n[A/t]$ and $\Omega_n(A) \equiv \Omega_n[A/t]$, and that both are contained in R for each t in interval, $|t-t_0| \leq h$, or contained in B for each $A(t) \in B$, following (3.1) and (3.3). With this in mind we shall prove the following theorem:

Theorem 3.1 The unique continuous solution to (1.16) on the interval $|t-t_0| \leq h$ of E to R is given by the uniformly convergent series

(3.14) $\quad y[A/t] = y_0 + \sum_{i=1}^{\infty} \Omega_i[A/t]$

when $\|A\| < \dfrac{1}{m^2 h\|y_0\|}$, where m is the modulus of the product in R, and $A(t)$ is continuous over interval $|t-t_0| \leq h$. Considered as a functional of $A(t)$ on B to B, (3.14) is a regular power series of homogeneous polynomials, $\Omega_i[A|t]$ in $A(t)$ of degree i as defined by Lemma 3.2, with radius of analyticity $r \geq \frac{1}{m^2 h \|y_0\|}$.

Proof: By Lemmas 3.2 and 3.3

$$\|\Omega_i[A|t]\| \leq h^i m^{2i} \|y_0\|^{i+1} \|A\|^i,$$

and from (6)

$$m(\Omega_i) = \sup_{A \in B} \frac{\|\Omega_i[A|t]\|}{\|A\|^i} \leq h^i m^{2i} \|y_0\|^{i+1}$$

is the bound to the modulus of the homogeneous polynomial $\Omega_i[A|t]$. But now

$$\sum_{i=0}^{\infty} m(\Omega_i)\lambda^i \leq \sum h^i m^{2i} \|y_0\|^{i+1} \lambda^i$$

converges absolutely for $\lambda < \frac{1}{hm^2\|y_0\|}$. Thus r, the radius of analyticity is $\geq \frac{1}{hm^2\|y_0\|} > 0$, and the right of (3.14) is a regular power series in B. Thus the last part of the theorem is proved.

But again we have by Lemma 3.3 that the right of (3.14) converges absolutely with respect to the norm $\| \ \|_R$, independent of t when $|t-t_0| \leq h$ and $\|A\| < \frac{1}{hm^2\|y_0\|}$. Hence if we call the n-th partial sum of the right of (3.14), $y_n'(t)$, then by the completeness of R and the preceding sentence there exists $y(t)$ such that $\|y(t) - y_n'(t)\|_R \to 0$, uniformly for $|t-t_0| \leq h$ and $\|A\| < \frac{1}{hm^2\|y_0\|}$. From this it is clear that $y(t)$ is a continuous function of t when $|t-t_0| \leq h$ and $\|A\| < \frac{1}{hm^2\|y_0\|}$.

Now by Lemmas 3.1, 3.2 and 3.3 and (3.4)

$$y_n = \sum_{i=1}^{n} \Omega_i(A) + \sum_{i=n+1}^{2^n-1} \Gamma_i^{(n)}(A)$$

so that

$$\|y_n(t)-y_n'(t)\|_R = \Big\|\sum_{i=n+1}^{2^n-1}\Gamma_i^{(n)}(A)\Big\|_R \leq \sum_{i=n+1}^{2^n-1}\|\Gamma_i^{(n)}(A)\|_R \leq$$

$$\leq \sum_{i=n+1}^{2^n-1} h^i m^{2i}\|y_0\|^{i+1}\|A\|^i \leq \frac{h^{n+1}m^{2(n+1)}\|y_0\|^{n+1}\|A\|^{n+1}}{1-hm^2\|y_0\|\,\|A\|}$$

when $|t-t_0|\leq h$ and $\|A\|<\dfrac{1}{hm^2\|y_0\|}$. Thus $\|y_n(t)-y_n'(t)\|_R\to 0$ uniformly when $|t-t_0|\leq h$ and $\|A\|<\dfrac{1}{hm^2\|y_0\|}$. Hence we have

(3.15) $\quad y(t)=\lim_{n\to\infty} y_n(t)=\lim_{n\to\infty} y_n'(t)$

continuous when $|t-t_0|\leq h$ and $\|A\|<\dfrac{1}{hm^2\|y_0\|}$ with both limits being uniform with respect to the norm, $\|\ \|_R$.

Now the sequence of functions $y_n(t)$ of (1.16) corresponds to the sequence $y_n(t)$, used in the existence proof of the general differential equation of Theorem 1.1, where $f(t,y(t))$ corresponds to $y(t)A(t)y(t)$. To show that $y(t)$ given by (3.15), satisfies (1.16) when $|t-t_0|\leq h$ and $\|A\|<\dfrac{1}{hm^2\|y_0\|}$, we need to show that (1.13) is satisfied. Suppose that $\|A\|\leq N<\dfrac{1}{hm^2\|y_0\|}$, then we have by (3.15)

$$\Big\|\int_{t_0}^{t}[y(s)A(s)y(s)-y_{n-1}(s)A(s)y_{n-1}(s)]ds\Big\|$$

$$\leq\Big|\int_{t_0}^{t}m^2\|A\|(\|y\|+\|y_n\|)\|y(s)-y_n(s)\|_R\,ds\Big|$$

$$\leq\frac{2m^2N}{1-hm^2N\|y_0\|}\,\epsilon_n h$$

where ϵ_n is independent of t and tends to zero as n tends to infinity. This is the desired result. Thus $y[A/t]\equiv y(t)$, as given by (3.14), satisfies the differential equation (1.16) when $|t-t_0|\leq h$ and $\|A\|<\dfrac{1}{m^2h\|y_0\|}$.

For the uniqueness suppose that $\|A\|\leq N<\dfrac{1}{m^2h\|y_0\|}$, and further that there exists another continuous function $Y(t)$, distinct from $y(t)$ which satisfies (1.16) with $Y(t_0)=y_0$ under the conditions of the present theorem. By the continuity of $Y(t)$ there exists M such that $\|Y(t)\|_R\leq\|Y\|\leq M$ for $|t-t_0|\leq h$ and $\|A\|\leq N\leq\dfrac{1}{m^2h\|y_0\|}$. Thus

(3.16)
$$\|Y(t)-y(t)\| = \left\| \int_{t_0}^{t} [Y(s)A(s)Y(s) - y(s)A(s)y(s)]\,ds \right\|$$
$$\leq m^2 \|A\| (\|Y\| + \|y\|) \left| \int_{t_0}^{t} \|Y(s)-y(s)\|_R\,ds \right|$$
$$\leq m^2 N \left(M + \frac{1}{1-hm^2N\|y_0\|}\right) \left| \int_{t_0}^{t} \|Y(s)-y(s)\|_R\,ds \right|$$
$$= K \left| \int_{t_0}^{t} \|Y(s)-y(s)\|_R\,ds \right|$$
where $K = m^2N\left(M + \frac{1}{1-hm^2N\|y_0\|}\right)$ and $|t-t_0| \leq h$.

From (3.16) we have by the successive substitution of the left side into the right side
$$\|Y(t)-y(t)\|_R \leq K \left| \int_{t_0}^{t}\int_{t_0}^{t_1}\cdots\int_{t_0}^{t_{n-2}}\int_{t_0}^{t_{n-1}} \|Y(t_n)-y(t_n)\|_R\,dt_n\,dt_{n-1}\cdots dt_1 \right|$$
$$\leq \frac{K^n \|Y-y\| h^n}{n!} \qquad (n=1,2,3,\dots)$$
which implies

(3.17) $\|Y-y\| \leq \frac{K^n h^n}{n!}\|Y-y\| \qquad (n=1,2,3,\cdots)$

where $K = m^2N\left(M + \frac{1}{1-m^2N\|y_0\|}\right)$. But now by assumption $Y(t)$ is distinct from $y(t)$, thus $\|Y-y\| = \max_{|t-t_0|\leq h} \|Y(t)-y(t)\|_R = c > 0$ and hence from (3.17)
$$1 \leq \frac{K^n h^n}{n!} .$$
But this is a contradiction for there exists n_0 such that $\frac{K^{n_0}h^{n_0}}{n_0!} < 1$. Hence $\|Y(t)-y(t)\|_R \leq \|Y-y\| = 0$, implying that $Y(t) = y(t)$; therefore the solution (3.14) is unique. Thus the theorem is proved.

The following theorem is concerned with the Fréchet differentiability of $y[A/t]$, the solution of (1.16), with respect to $A(t)$.

<u>Theorem 3.2</u> The unique solution $y[A/t]$ of
$$\frac{dy}{dt} = y\,A(t)\,y\ , \quad y(t_0) = y_0$$
under the hypotheses of Theorem 3.1, as a function of $A \in B$ to B has successive Fréchet differentials of all orders for each A in the sphere $\|A\| < \frac{1}{m^2 h \|y_0\|}$ and can be given by a term by term Fréchet differentiation of the power series (3.14).

<u>Proof</u>: The theorem follows immediately by Theorem 3.1 and the theorem in (6).

It is evident that Theorem 3.1 is a much stronger existence theorem for the solution to (1.16) than was Theorem 1.2 for clearly

$$\frac{b}{hm^2(\|y_0\|+b)^2} < \frac{1}{hm^2\|y_0\|} \qquad \text{any } b>0$$

The explicit solution $y[A/t]$ of (1.16) is now in order; in fact one method would be to construct it inductively by the use of (3.10) and (3.14). However, we shall postpone this problem to a later chapter. (See Chapter VI).

In the next two chapters we will discuss a few of the problems related to the functional equation (2.3), introduced in Chapter II.

Chapter IV

A Functional Equation, Involving Endomorphisms

The motivation for this chapter is the study made in Chapter II and (7), pg. 16. Consider a linear bounded transformation of T for a real Banach space B to itself, or for $x \in B$ then $T(x) \in B$. The norm of T in the space of transformations will be $\|T\| = \sup_{\|x\| \leq 1} \|T(x)\|$, the uniform topology. Following (4), pg. 33-34, we will now state the following definitions and theorems, concerning rings of endomorphisms:

Definition 4.1 A linear bounded transformation on a Banach space B to itself is called an endomorphism of B. If T, T_1, and T_2 are endomorphisms of B, then $T_1 + T_2$, aT for a real number, and $T_1 T_2$ are defined for all $x \in B$ by

$$(T_1 + T_2)(x) = T_1(x) + T_2(x), \quad (aT)(x) = aT(x)$$

$$(T_1 T_2)(x) = T_1[T_2(x)].$$

Theorem 4.1 If T, T_1, and T_2 are endomorphisms of B, so are $T_1 + T_2$, aT, and $T_1 T_2$. Further

$$\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|, \quad \|aT\| = |a| \, \|T\|$$

$$\|T_1 T_2\| \leq \|T_1\| \, \|T_2\|.$$

Theorem 4.2 The set of all endomorphisms of B forms a complete normed linear ring $R(B)$ under the uniform topology. It is normally non-commutative and the identity transformation may be adjoined as the unit element.

Although $R(B)$ possesses a unit element it is sometimes more convenient to speak of the quasi-inverse, following (8), and the reverse, following (4), pg. 455 than the inverse. We shall state some of the known results of these concepts, which are needed for present purposes.

Definition 4.2 If $x \circ y = x + y + xy = 0$, then y is the right quasi inverse of x for x and y in a ring R. If $x \circ y = y \circ x = 0$

then x is quasi-regular and y is the quasi-inverse. The quasi-inverse of x will be denoted by $\bar{x}$.

<u>Definition 4.3</u> If the cross product $a \times b = a + b - ab = 0$, then b is the right reverse of a for a and b in a ring R. If $a \times b = b \times a = 0$, then a is reversible and b is the reverse of a . The reverse of a will be denoted by b^{-} .

<u>Theorem 4.3</u> If x has a right quasi-inverse y as well as a left quasi-inverse z, then $y = z$ and x has a unique quasi-inverse.

<u>Theorem 4.4</u> If x has a right reverse y as well as a left reverse z, then $y = z$ and x has a unique reverse.

The relationships of the reverse or the quasi-inverse to inverses in $R(B)$ is clear if they exist. The next theorem will show the existence of reverses and quasi inverses in $R(B)$.

<u>Theorem 4.5</u> Every element T of $R(B)$ with $\|T\| < 1$ has a unique reverse given by

(41) $$T^{-} = -\sum_{n=1}^{\infty} T^{n}$$

and a unique quasi inverse given by

(4.2) $$\bar{T} = \sum_{n=1}^{\infty} (-1)^{n} T^{n}$$

Both $\bar{T}$ and T^{-} are analytic functions of T of radius 1 .

<u>Proof</u>: From (4.1) $T^{-} = -\sum_{n=1}^{\infty} T^{n}$, thus

(4.3) $$\|T^{-}\| < \|T\| \sum_{n=0}^{\infty} \|T\|^{n} = \|T\| (1 - \|T\|)^{-1}$$

when $\|T\| < 1$. Hence $\|T^{-}\|$ is bounded and T^{-} as given by (4.1) is an analytic function of T of radius 1 . By the completeness of $R(B)$ it is clear that T^{-}, as defined by (4.1), is an element of $R(B)$ for all T in $R(B)$ with $\|T\| < 1$. Also

$$T \times T^{-} = T^{-} \times T = T - \sum_{n=1}^{\infty} T^{n} + \sum_{n=2}^{\infty} T^{n} = 0 .$$

Thus by Theorem 4.4, the first part of the theorem follows. The second part follows in the same manner.

Since there is almost complete duality in the concepts of the quasi inverse and reverse, it will be convenient at this point to give relationship between the two ideas in $R(B)$.

Lemma 4.1 For T in $R(B)$ with $\|T\| < 1$

$$\bar{T} = -(-T)^{-}$$

and

$$T^{-} = -(-\bar{T}) .$$

Proof: The lemma follows clearly from Theorem 4.5 and a direct substitution in the defining relations of $\bar{T}$ and T^{-}, given in Definition 4. 2 and 4.3, respectively.

Until now some of the properties of the elements of $R(B)$ have been characterized, enough so, that the solutions of the functional equations $y = f + T(y)$ and $f = y + T(y)$, where $f \in B$ and $T \in R(B)$, can be obtained.

Theorem 4.6 If $f \in B$ and $T \in R(B)$ with $\|T\| < 1$, then the unique solution of the functional equation,

(4.4) $\quad y = f + T(y)$

is given by

(4.5) $\quad y = f - T^{-}(f)$

where T^{-} is the reverse of T. The unique solution of

(4.6) $\quad f = y + T(y)$

is given by

(4.7) $\quad y = f + \bar{T}(f)$

where $\bar{T}$ is the quasi inverse of T. The expressions, given by (4. 5) and (4.7) are analytic functions of T.

Proof: By Theorem 4.5 $T^{-}(f)$ is a linear bounded transformation of f on B to B when $\|T\| < 1$. Hence by (4.3),

$$\|y\| \leq \|f\| + \|T^{-}\| \|f\| \leq (1 - \|T\|)^{-1} \|f\|,$$

when $\|T\| < 1$, showing by the completeness of B that y exists in B, as expressed in (4.5), and is an analytic function of T of radius 1. By substituting (4.4) in the right side of (4.4),

(4.8) $\quad y = f + T(f - T^{-}(f)) = f + (T - TT^{-})(f)$

But $T \times T^{-} = T + T^{-} - TT^{-} = 0$, or $T - TT^{-} = -T^{-}$.

Thus 4.8 becomes $y = f - T^{-}(f)$, showing that (4.5) identically satisfies (4.4).

To obtain the uniqueness suppose that y and y' are two solutions, satisfying (4.4). Then

$$y = f + T(y)$$
$$y' = f + T(y')$$

from which we have

(4.9) $\quad y - y' = T(y - y')$.

Now by taking the norm of (4.9) we obtain

(4.10) $\quad \|y - y'\| \leq \|T\| \|y - y'\|$

Now suppose that y' were different from y, then $\|y - y'\| > 0$ and (4.10) gives

$$1 \leq \|T\|$$

But by hypothesis this is a contradiction. Hence $\|y - y'\| = 0$ or $y = y'$ and the solution of (4.4) is unique. Thus the first part of the theorem is proved. The rest of the theorem follows by Lemma 4.1 or a proof similar to the above proof.

We will next determine the reverse and the quasi-inverse of the sum of two elements in $\mathcal{R}(B)$. If T_1, and T_2 are in $\mathcal{R}(B)$, then by Theorem 4.5 the unique reverse of T_1+T_2 is given by

(4.11) $$(T_1+T_2)^- = -\sum_{n=1}^{\infty}(T_1+T_2)^n, \quad \|T_1+T_2\|<1,$$

the unique quasi-inverse by

(4.12) $$(\overline{T_1+T_2}) = \sum_{n=1}^{\infty}(-1)^n(T_1+T_2)^n, \quad \|T_1+T_2\|<1.$$

he next two lemmas will be concerned with the finding of two other expressions for (4.11) under a slightly stronger condition; one in terms of T_1^- and T_2^- and the other in terms of T_1^- and T_2 or T_1 and T_2^-.

<u>Lemma 4.2</u> If $A, B \in \mathcal{R}(B)$ with $\|A\|<1$ and $\|B\|<1-\|A\|$ and A^- and B^- are the reverses of A and B, respectively, then the unique reverse of $A+B$ is

(4.13) $$(A+B)^- = (A^- + B^- - A^-B^- - B^-A^-) + \\ + \sum_{i=1}^{\infty}[(A^-B^-)^i(A^- - A^-B^-) + (B^-A^-)^i(B^- - B^-A^-)]$$

<u>Proof</u>: The existence and uniqueness of $(A+B)^-$ is guaranteed by Theorem 4.5. Further-more by (4.3) it is clear that the right of (4.13) is convergent and hence bounded, leaving us only to show that the right of (4.13) identically satisfies $(A\times B)\times(A\times B)^-=0$. Since $A\times A^-=0$ and $B\times B^-=0$, we have the relations,

(4.14) $$B+B^- = BB^- \text{ and } A+A^- = AA^-.$$

Further since $(A+B)\times(A+B)^-=0$,

4.15) $$(A+B)^- = (A+B)(A+B)^- - (A+B).$$

f one substitutes the right side of (4.13) into the right of (4.15), the right of (4.15) becomes

$$(A+B)(A^- + B^- - A^-B^- - B^-A^-) + \\ + (A+B)\sum_{i=1}^{\infty}[(A^-B^-)^i(A^- - A^-B^-) + (B^-A^-)^i(B^- - B^-A^-)] - (A+B),$$

which becomes by expanding and using (4.14),

$$(A+B)^- - AB^-A^-B^- - BA^-B^-A^- + \sum_{i=1}^{\infty} AB^-(A^-B^-)^i(A^- - A^-B^-)$$
$$+ \sum_{i=1}^{\infty} A(B^-A^-)^i(B^- - B^-A^-) + \sum_{i=1}^{\infty} B(A^-B^-)^i(A^- - A^-B^-)$$
$$+ \sum_{i=1}^{\infty} (BA^-)(B^-A^-)^i(B^- - B^-A^-)$$

where $(A+B)^-$ designates the right of (4.13). We need now to show that the sum of the remaining terms is zero, then the right of (4.13) will identically satisfy $(A+B)\times(A+B)^- = 0$.

Clearly, the remaining terms are equal to

$$-AB^-(A^-B^-) - BA^-(B^-A^-) + \sum_{i=1}^{\infty} AB^-(A^-B^-)^i A^- - \sum_{i=2}^{\infty} AB^-(A^-B^-)^i +$$
$$+ \sum_{i=1}^{\infty} A(B^-A^-)^i B^- - \sum_{i=2}^{\infty} A(B^-A^-)^i + \sum_{i=1}^{\infty} B(A^-B^-)^i A^- -$$
$$- \sum_{i=2}^{\infty} B(A^-B^-)^i + \sum_{i=1}^{\infty} (BA^-)(B^-A^-)^i B^- - \sum_{i=2}^{\infty} (BA^-)(B^-A^-)^i$$

which by further reduction becomes

$$\sum_{i=2}^{\infty} A(B^-A^-)^i - \sum_{i=1}^{\infty} AB^-(A^-B^-)^i + \sum_{i=1}^{\infty} AB^-(A^-B^-)^i -$$
$$- \sum_{i=2}^{\infty} A(B^-A^-)^i + \sum_{i=1}^{\infty} BA^-(B^-A^-)^i - \sum_{i=2}^{\infty} B(A^-B^-)^i +$$
$$+ \sum_{i=2}^{\infty} B(A^-B^-)^i - \sum_{i=1}^{\infty} (BA^-)(B^-A^-)^i = 0$$

Hence lemma is proved.

Lemma 4.3 If $A, B \in \mathcal{R}(B)$ with $\|A\| < 1$ and $\|B\| < 1 - \|A\|$, and A^- is the reverse of A, the unique reverse of $A+B$ is

$$(A+B)^- = A^- + \sum_{i=1}^{\infty} [(B - A^-B^-)^i A^- - (B - A^-B)^i]. \tag{4.16}$$

Proof: The existence and uniqueness of $(A+B)^-$ and the existence of the right of (4.16) in $\mathcal{R}(B)$ follows as in Lemma 4.2. If we substitute the right of (4. 16) in the right of (4.15), the right of (4.15) is

$$(A+B)\{A^- + \sum_{i=1}^{\infty}[(B-A^-B)^i A^- - (B-A^-B)^i]\} - (A+B) =$$

$$= AA^- + BA^- + \sum_{i=1}^{\infty}[A(B-A^-B)^i A^- - A(B-A^-B)^i] +$$

$$+ \sum_{i=1}^{\infty}[B(B-A^-B)^i A - B(B-A^-B)^i] - A - B =$$

$$= A^- + BA^- + A(B-A^-B)A^- - A(B-A^-B) - B + \sum_{i=1}^{\infty}[A(B-A^-B)^{i+1}A^- -$$

$$- A(B-A^-B)^{i+1} + B(B-A^-B)^i A^- - B(B-A^-B)^i] =$$

$$= A^- + BA^- - B + ABA^- - ABA^- - A^-BA^- + AB - AB - A^-B +$$

$$+ \sum_{i=1}^{\infty}[(AB - AB - A^-B)(B-A^-B)^i A^- - (AB - AB - A^-B)(B-A^-B)^i +$$

$$+ B(B-A^-B)^i A - B(B-A^-B)^i] =$$

$$= A^- + (B-A^-B)A^- - (B-A^-B) +$$

$$+ \sum_{i=1}^{\infty}[-A^-B(B-A^-B)^i A^- + A^-B(B-A^-B)^i + B(B-A^-B)^i A^- - B(B-A^-B)^i] =$$

$$= A^- + (B-A^-B)A^- - (B-A^-B) +$$

$$+ \sum_{i=1}^{\infty}[(B-A^-B)^{i+1}A^- - (B-A^-B)^{i+1}] =$$

$$= A^- + \sum_{i=1}^{\infty}[(B-A^-B)^i A^- - (B-A^-B)^i]$$

which is the left of (4.15) as defined by the right of (4.16). Hence lemma is proved.

Now by combining (4.11), (4.12), Lemmas 4.2 and 4.3 and using Lemma 4.1, we have the following theorem.

<u>Theorem 4.7</u> If T_1, $T_2 \in \mathcal{R}(B)$ with $\|T_1\| < 1$ and $\|T_2\| < 1 - \|T_1\|$ and T_1^- and T_2^- are the reverses of T_1 and T_2, respectively, the unique reverse of $T_1 + T_2$ is

$$(T_1 + T_2)^- = -\sum_{n=1}^{\infty}(T_1 + T_2)^n =$$

$$= (T_1^- + T_2^- - T_1^- T_2^- - T_2^- T_1^-) +$$

$$+ \sum_{i=1}^{\infty}[(T_1^- T_2^-)^i (T_1^- - T_1^- T_2^-) + (T_2^- T_1^-)^i (T_2^- - T_2^- T_1^-)] =$$

$$= T_1^- + \sum_{i=1}^{\infty} [(T_2 - T_1^- T_2)^i T_1^- - (T_2 - T_1^- T_2)^i] =$$

$$= T_2^- + \sum_{i=1}^{\infty} [(T_1 - T_2^- T_1)^i T_2^- - (T_1 - T_2^- T_1)^i].$$

The unique quasi inverse of $T_1 + T_2$ is

$$(\overline{T_1 + T_2}) = \sum_{n=1}^{\infty} (-1)^n (T_1 + T_2)^n =$$

$$= \bar{T}_1 + \sum_{i=1}^{\infty} \{(-1)^i (T_2 + \bar{T}_1 T_2)^i \bar{T}_1 + (-1)^i (T_2 + \bar{T}_1 T_2)^i\} =$$

$$= (\bar{T}_1 + \bar{T}_2 + \bar{T}_1 \bar{T}_2 + \bar{T}_2 \bar{T}_1) +$$

$$+ \sum_{i=1}^{\infty} [(\bar{T}_1 \bar{T}_2)^i (\bar{T}_1 + \bar{T}_1 \bar{T}_2) + (\bar{T}_2 \bar{T}_1)^i (\bar{T}_2 + \bar{T}_2 \bar{T}_1)] =$$

$$= \bar{T}_2 + \sum_{i=1}^{\infty} \{(-1)^i (T_1 + \bar{T}_2 T_1)^i \bar{T}_2 + (-1) (T_1 + \bar{T}_2 T_1)^i\}$$

where $\bar{T}_1$ and $\bar{T}_2$ are the quasi inverses of T_1 and T_2 respectively.

Now since $\bar{T}$ in Theorem 4.5 was an analytic function of of radius 1, we can show this dependence on T by writing

(4.17) $\qquad \bar{T} = \bar{T}[T], \quad \|T\| < 1.$

urthermore by (6) the Fréchet differential $\delta \bar{T}[T]$ of $\bar{T}[T]$ with increment δT exists for T in $\mathcal{R}(B)$ with $\|T\| < 1$. The next lemma will give the determination of $\delta \bar{T}[T]$.

<u>Lemma 4.4</u> For $T \in \mathcal{R}(B)$ with $\|T\| < 1$ and $\bar{T}[T]$, given by (4.17),

(4.18) $\qquad \delta \bar{T}[T] = -\delta T - \delta T \bar{T} - \bar{T} \delta T - \bar{T} \delta T \bar{T}$

and

(4.19) $\qquad \delta T^-[T] = -\delta T + \delta T T^- + T^- \delta T - T^- \delta T T^-$

when $\delta T \in \mathcal{R}(B)$

<u>Proof</u>: By the statement, preceding the lemma, the existence of the Fréchet differential of $\bar{T}[T]$ is guaranteed. The

implicit Fréchet differential of

$$T \circ \bar{T} = T + \bar{T} + T\bar{T} = 0$$

with increment δT is

$$\delta T + \delta \bar{T} + T\delta\bar{T} + \delta\bar{T} = 0$$

or then

$$\delta\bar{T} + \bar{T}\delta T = -\delta T - \delta T\bar{T} .$$

From this we have

$$\bar{T} \circ (\delta\bar{T} + \bar{T}\delta\bar{T}) = \bar{T} \circ (-\delta T - \delta T\bar{T})$$

which upon expanding becomes

$$\delta\bar{T} + (T \circ \bar{T})\delta T = -\delta T - \delta T\bar{T} - \bar{T}\delta T - \bar{T}\delta T\bar{T} .$$

Since $(T \circ \bar{T}) = 0$ we thus have

$$\delta\bar{T} = \delta\bar{T}[T] = -\delta T - \delta T\bar{T} - \bar{T}\delta T - \bar{T}\delta T\bar{T}$$

which is the required result. (4.19) may be obtained from a similar proof or by the use of Lemma 4.1 in (4.18). Hence lemma is established.

Theorem 4.6 stated that the unique solution of (4.6) was given by (4.7) where the right of (4.6) was an analytic function of T with radius 1 . Rewriting (4.7) to show the functional dependence of y on T we have

(4.20) $\quad y[T] = f + \bar{T}[T](f), \quad \|T\| < 1 ,$

where f is a fixed element of B . By virtue of (4.20) and (3) the Fréchet differential $\delta y[T]$ with increment $\delta T \in R(B)$ exists for each T in $R(B)$ with $\|T\| < 1$, and likewise, each succeeding differential with increment δT . These Fréchet differentials will be given in the following theorem.

<u>Theorem 4.8</u> For $T \in \mathcal{R}(B)$ with $\|T\| < 1$ the n-th Fréchet differential with increment δT of $y[T]$, the solution of $f = y + T(y)$ with $f \in B$, is given by

(4.21) $\delta^n y[T] = (-1)^n n! (\delta T + \bar{T}\delta T)^n (y), \quad (n = 1, 2, 3, \cdots),$

and the n-th Fréchet differential with increment δT of $y[T]$, the solution of $y = f + T(y)$ with $f \in B$, is given by

4.22) $\delta^n y[T] = n! (\delta T - T^{-}\delta T)^n (y) \quad (n = 1, 2, 3, \ldots)$

<u>Proof:</u> By the preceding paragraph $\delta y[T]$ exists as a Fréchet differential, and by (4.20) the solution of $f + y + T(y)$ is

$$y[T] = f + \bar{T}[T](f) .$$

Thus $\delta y[T] = \delta\bar{T}[T](f)$ so that by Lemma 4.4

$$\begin{aligned} \delta y[T] &= (-\delta T - \delta T\bar{T} - \bar{T}\delta T - \bar{T}\delta T\bar{T})(f) \\ &= (-\delta T - \delta T\bar{T} - \bar{T}\delta T - \bar{T}\delta T\bar{T})(y) + (-\delta T - \delta T\bar{T} - \bar{T}\delta T - \bar{T}\delta T\bar{T})T(y) \\ &= (-\delta T - \delta T(T \circ \bar{T}) - \bar{T}\delta T - \bar{T}\delta T(T \circ \bar{T}))(y) \\ &= -(\delta T + \bar{T}\delta T)(y) \end{aligned}$$

which is (4.21) for $n = 1$. Now for purposes of induction let us assume (4.21) true for all intergers less than $(n+1)$, then

$$\begin{aligned} \delta^n y[T] &= (-1)^n n! (\delta T + \bar{T}\delta T)^n (y) \\ &= (-1)^n n! [(\delta T + \bar{T}\delta T)^n + (\delta T + \bar{T}\delta T)^n \bar{T}](f) \end{aligned}$$

By taking the next Fréchet differential with increment δT, we have

$$\delta^{n+1} y[T] = (-1)^n n! (\delta T + \bar{T}\delta T)^{n-1} [n\delta\bar{T}\delta T + n\delta\bar{T}\delta T\bar{T} + (\delta T + \bar{T}\delta T)\delta\bar{T}](f)$$

and by using (4.10) one obtains after simplification

$$\begin{aligned} \delta^{n+1} y[T] &= (-1)^{n+1} (n+1)! (\delta T + \bar{T}\delta T)^{n-1} [(\delta T + \bar{T}\delta T)^2 + (\delta T + \bar{T}\delta T)^2 \bar{T}](f) \\ &= (-1)^{n+1} (n+1)! (\delta T + \bar{T}\delta T)^n \{(\delta T + \bar{T}\delta T)(y) + (\delta T + \bar{T}\delta T)(T \circ \bar{T})(y)\} \\ &= (-1)^{n+1} (n+1)! (\delta T + \bar{T}\delta T)^{n+1} (y) \end{aligned}$$

thereby completing induction and establishing (4.21). (4.22) may be shown in the same manner. Hence theorem is proved.

The following theorem, concerning differential systems, can be proved by the same methods used to prove Theorems 1 and 2 of (4).

Theorem 4.9 For $T \in R(B)$ with $\|T\| < 1$ the differential system

$$\delta\lambda[T] = -(\delta T + \lambda\delta T + \delta T\lambda + \lambda\delta T\lambda) \; , \; \lambda[0] = 0,$$

has a unique analytic solution of radius 1 given by

$$\lambda[T] = \sum_{n=1}^{\infty} (-1)^n T^n = \bar{T},$$

the quasi inverse of T ; the solution of

$$\delta\lambda[T] = -\delta T + \lambda\delta T + \delta T\lambda - \lambda\delta T\lambda, \; \lambda[0] = 0,$$

is given by

$$\lambda[T] = -\sum_{n=1}^{\infty} T^n = T^-,$$

the reverse of T in $R(B)$. For $f \in B$, the unique analytic solution of radius 1 of

$$\delta y[T] = -(\delta T + \bar{T}\delta T)(y), \quad y[0] = f,$$

is given by

$$y = f + \bar{T}(f),$$

and the solution of the differential system,

$$\delta y[T] = (\delta T - T^-\delta T)(y), \quad y[0] = f,$$

is given by

$$y = f - T^-(f).$$

With the aid of Theorems 4.7 and 4.8 the following theorem is proved:

Theorem 4.10 For any given T_0 in $R(B)$ with $\|T_0\| < 1$, the analytic solution $y[T]$ of (4.4) or (4.6) can be expanded in a generalized Taylor's series of successive Fréchet differentials with equal increments δT valid for $\delta T \in R(B)$ with $\|\delta T\| < 1 - \|T_0\|$,

$$y[T_0+\delta T] = y[T_0] + \sum_{i=1}^{\infty} \frac{1}{i!}[\delta^i y[T]]_{T=T_0}$$

with values in B .

Proof: From the last expansion of Theorem 4.7 and Theorem 4.6 the unique solution of

$$f = y + (T_0+\delta T)(y)$$

is given by

(4.23) $$y = y[T_0+\delta T] = f + (\overline{T_0+\delta T})(f)$$

$$= f + \bar{T}_0(f) + \sum_{i=1}^{\infty} (-1)^i(\delta T + \bar{T}_0\delta T)^i(f+\bar{T}_0(f))$$

when $\|T_0\| < 1$ and $\|\delta T\| < 1 - \|T_0\|$ By (4.20)

$y[T_0] = f + \bar{T}_0(f)$ and by (4.21)

$$[\delta^n y[T]]_{T=T_0} = (-1)^n n!(\delta T + \bar{T}_0\delta T)^n(y[T_0])$$

$$= (-1)^n n!(\delta T + \bar{T}_0\delta T)^n(f+\bar{T}_0(f)).$$

Thus (4.23) becomes

$$y[T_0+\delta T] = y[T_0] + \sum_{i=1}^{\infty} \frac{1}{i!}[\delta^i y[T]]_{T=T_0}$$

which is the desired result. The corresponding result for the solution of (4.4) is obtained in a similar manner. Hence theorem is proved.

The results of this chapter will be used with slight modifications in the next two chapters. The functional equations studied in this chapter clearly correspond to the Fredholm integral equation.

Chapter V

A Further Study of the Functional Equation $y = f + K \odot y + y \otimes H$

Although, as we shall see in Chapter VI, we will manage to avoid the use of the integral equation (2.2) in obtaining the Fréchet differential of the solution of (1.16) as a functional of $A(t)$, it is of some interest to study further the related functional equation (2.3), first, as an application of the methods of Chapter IV, and, secondly, as a means for finding Fréchet differentials of the solutions of other, more difficult, non linear abstract differential equations. For the present we will treat (2.3) under Assumptions, 2.1, 2.2, 2.3, and 2.4.

The functional equation that we are considering is

$$y = f + K \odot y + y \otimes H$$

where $K \odot y$ and $y \otimes H$ satisfy the assumptions of Chapter II. Heuristically, $H \in \mathcal{R}_H$ is written on the right of the operation $\otimes$ in $y \otimes H$ to suggest a non-commutativity and non-associativity with respect to the operation $\odot$ in the function $K \odot y$ as was the case in Theorem 2.2. If we let

$$(5.1) \qquad T_K(y) = K \odot y \quad , \quad T_H(y) = y \otimes H ,$$

then from definition 4.1 it is clear that $T_K(y)$ and $T_H(y)$ are endomorphisms of $\mathcal{B}$. Consider the following definition.

<u>Definition 5.1</u> If T_K, T_{K_1}, and T_{K_2} with $K_i \in \mathcal{R}_K$ are endomorphisms of $\mathcal{B}$, as defined by (5.1), then $T_{K_1} + T_{K_2}$, aT for "a", a real number, and T_{K_1}, T_{K_2} are defined for all $y \in \mathcal{B}$ by

$$(T_{K_1} T_{K_2})(y) = T_{K_1}(T_{K_2}(y)), \; (T_{K_1} + T_{K_2})(y) = T_{K_1}(y) + T_{K_2}(y)$$

$$(aT_K)(y) = aT_K(y) .$$

Similarly, for $H_i \in R_H$

$$(T_{H_1} T_{H_2})(y) = T_{H_1}(T_{H_2}(y)), (T_{H_1} + T_{H_2})(y) = T_{H_1}(y) + T_{H_2}(y)$$

$$(aT_H)(y) = aT_H(y).$$

Moreover if $K \in R_K$ and $H \in R_H$, then $T_K + T_H$ and $T_K T_H$ are defined for all $y \in B$ by

$$(T_K + T_H)(y) = T_K(y) + T_H(y), \quad T_K T_H(y) = T_K(T_H(y)).$$

From Definition 5.1 it is evident that T_K and T_H and more generally any multinomial of elements, T_{K_i} and T_{H_i} with $K_i \in R_K$ and $H_i \in R_H$, will be contained in the ring $\mathcal{R}(B)$, the complete ring of all endomorphisms of B. Hence we may state the following lemma:

<u>Lemma 5.1</u> Under Assumptions 2.1, 2.2, 2.3, and 2.4 any multinomial of elements, T_{K_i} and T_{H_i} with $K_i \in R_K$ and $H_i \in R_H$, the endomorphisms defined by (5.1) and Definition 5.1, is contained in $\mathcal{R}(B)$, the complete normed linear ring of all endomorphisms of B as defined by Theorem 4.1.

We shall prove now the following two lemmas.

<u>Lemma 5.2</u> For $K_i \in R_K$ and $H_i \in R_H$ under (5.1) and Definition 5.2 the following is true:

$$aT_{K_1} = T_{aK_1}, \quad T_{K_1} + T_{K_2} = T_{K_1 + K_2}, \quad T_{K_1} T_{K_2} = T_{K_1 K_2}$$

and

$$aT_{H_1} = T_{aH_1}, \quad T_{H_1} + T_{H_2} = T_{H_1 + H_2}, \quad T_{H_1} T_{H_2} = T_{H_1 H_2}$$

<u>Proof</u>: Clearly,

$$(aT_K)(y) = aT_K(y) = a(K \circ y) = (aK) \circ y = T_{aK}(y)$$

$$(T_{K_1} + T_{K_2})(y) = T_{K_1}(y) + T_{K_2}(y) = K_1 \circ y + K_2 \circ y = (K_1 + K_2) \circ y = T_{K_1 + K_2}(y)$$

and

$$(T_{K_1} T_{K_2})(y) = T_{K_1}(T_{K_2}(y)) = T_{K_1}(K_2 \circ y) = K_1 \circ (K_2 \circ y) = (K_1 K_2) \circ y = T_{K_1 K_2}(y)$$

Thus the first part of lemma is proved. The second part follows in a similar manner.

Lemma 5.3 If $K \in R_K$ and $H \in R_H$, the endomorphism, $(T_K + T_H)^n$ of B satisfies

$$\|(T_K + T_H)^n\| \leq \frac{M^n(\|K\| + \|H\|)^n}{n!}$$

where the norm on the left is the norm of the uniform topology and M is given by assumption 2.4.

Proof: Now

$$\|(T_K+T_H)^n\| = \|T_K^n + T_K^{n-1}T_H + \dots T_H^n\|$$

$$\leq \|T_K^n\| + \|T_K^{n-1}T_H\| + \dots + \|T_H^n\|.$$

Thus by assumption 2.4

$$\|(T_K+T_H)^n\| \leq \frac{M^n}{n!}(\|K\|^n + \|K\|^{n-1}\|H\| + \dots + \|H\|^n)$$

$$\leq \frac{M^n}{n!}(\|K\| + \|H\|)^n.$$

Thus lemma is proved.

With the aid of the above lemmas we can now prove the following theorem:

Theorem 5.1 The unique solution of the functional equation

$$y = f + K \odot y + y \otimes H$$

where $K \odot y$ and $y \otimes H$ are bilinear functions, satisfying Assumptions 2.1, 2.2, 2.3, 2.4 with $K \in R_K$ and $H \in R_H$ is given by

$$y[K,H] = f - (T_K + T_H)^-(f)$$

$T_K(y) = K \odot y$ and $T_H(y) = y \otimes H$ are endomorphisms of B for each $K \in R_K$ and $H \in R_H$ under Definition 5.1. $(T_K + T_H)^-$ is the unique reverse of $T_K + T_H$ in the ring $R(B)$ of all endomorphisms of B under the uniform topology. $-(T_K + T_H)^-$ has the following expansions:

$$-(T_K+T_H)^- = \sum_{i=1}^{\infty}(T_K+T_H)^i$$

$$= T_K + \sum_{i=1}^{\infty}(T_H + T_K T_H)^i T_K + (T_H + T_K T_H)^i$$

$$= T_h + \sum_{i=1}^{\infty}(T_K + T_h T_K)^i T_h + (T_K + T_h T_K)^i$$

$$= T_K + T_h + T_K T_h + T_h T_K + \sum_{i=1}^{\infty}[(T_K T_h)^i(T_K + T_K T_h) + (T_h T_K)^i(T_h + T_h T_K)]$$

where $k = \sum_{i=1}^{\infty} K^i$ and $h = \sum_{i=1}^{\infty} H^i$. Moreover, the solution $y[K, H]$ will be an entire analytic function of each variable, K and H, separately, and an entire analytic function of the pair (K, H) on the product space $R_K R_H$ to B when the norm in $R_K R_H$ is

$$\|(K,H)\| = \|K\| + \|H\| .$$

Finally, $y[K, H]$ will be Fréchet differentiable with respect to each variable separately, and with respect to the pair (K, H) when $y[K, H]$ is defined on $R_K R_H$ to B.

Proof: It is clear from Lemma 5.3 that $T_K + T_H$ is quasi-nilpotent for all $K \in R_K$ and $H \in R_H$, and hence that $T_K + T_H$ is reversible. By the proof of Theorem 4.5 and Lemma 5.1

$$(5.2) \qquad (T_K + T_H)^- = -\sum_{n=1}^{\infty} (T_K + T_H)^n$$

is the unique reverse of $T_K + T_H$, and it is certainly an element of $\mathcal{R}(B)$ since by Lemma 5.3

$$(5.3) \qquad \|(T_K + T_H)^-\| \leq e^{M(\|K\| + \|H\|)} - 1 .$$

Now by Lemma 5.2 and Assumption 2.3 we have

$$(5.4) \qquad T_K^- = -T_k \quad \text{and} \quad T_H^- = -T_h$$

where $k = \sum_{i=1}^{\infty} K^i$ and $h = \sum_{i=1}^{\infty} H^i$. Hence by (5.2) (5.3) Assumption 2.4 and the algebraic proofs of Lemmas 4.2 and 4.3 we have

$$\begin{aligned}
-(T_K+T_H)^- &= \sum_{i=1}^{\infty} (T_K + T_H)^i \\
&= T_k + \sum_{i=1}^{\infty} (T_H + T_k T_H)^i T_k + (T_H + T_k T_H)^i \\
&= T_h + \sum_{i=1}^{\infty} (T_K + T_h T_K)^i T_h + (T_K + T_h T_K)^i \\
&= T_k + T_h + T_k T_h + T_h T_k + \sum_{i=1}^{\infty} \left[(T_k T_h)^i (T_k + T_k T_h) + (T_h T_k)^i (T_h + T_h T_k) \right]
\end{aligned}$$

where $K = \sum_{i=1}^{\infty} K$ and $h = \sum_{i=1}^{\infty} H^{i}$, existing in $R(B)$ for all $K \in R_K$ and $H \in R_H$.

From the proof of Theorem 4.6, the uniqueness proof of Theorem 2.1, and (5.2) the unique solution of

$$y = f + K \odot y + y \circledast H = f + (T_K + T_H)(y)$$

is

(5.4) $\quad [K, H] = f - (T_K + T_H)^{-}(f)$

for all $K \in R_K$ and $H \in R_H$. From (5.3) and (5.4), clearly

(5.5) $\quad \| y[K, H] \| \leq \| f \| \, e^{M(\|K\| + \|H\|)}$.

(5.5) shows us that $y[K, H]$ is an entire analytic function of K and H, separately. Hence by (6) $y = [K, H]$, considered as a function of $K \in R_K$ and $H \in R_H$, is réchet differentiable with respect to each variable, separately. Also from (5.5) if $y[K, H]$ is considered as a function of $(K, H) \in R_K R_H$, the product space of R_K and R_H, then $y[K, H]$ is an entire analytic function of (K, H) and is Fréchet differentiable with respect to the pair (K, H). The norm in $R_K R_H$ is in this case $\|(K, H)\| = \|K\| + \|H\|$. With the above the theorem is now proved.

It is not difficult to see that the last expansion of Theorem 5.1 for the solution of (2.3) is the same solution obtained in Theorem 2.1, but in different notation. It is evident that the above theorem may be generalized to obtain the solution of

$$y = f + \sum_{i=1}^{M} K_i \odot_i y$$

where $K_i \theta_i y$ are distinct bilinear functions, satisfying assumptions similar to those of Chapter II with $K_i \in R_{K_i}$ a complete normed linear ring.

Theorem 5.1 furnishes a possible solution of (2.1), giving $\delta y[A/t]$ when $y(t)$ is known. In the same way Fréchet differentials may be obtained from other abstract differential equations, either by Theorem 5.1 or its above mentioned generalization. However as we mentioned earlier we shall circumvent this method in the next chapter by obtaining $y[A/t]$ of (1.16) explicitly.

Before concluding this chapter it may be mentioned that attempts were made to answer the following question: Under what condition is R_K algebraically isomorphic and topologically homeomorphic to a ring of endomorphisms of B, a Banach space, under the uniform topology? We were only able to answer this question under very restrictive assumptions, so much so that illustrative examples were difficult to find. Hence we shall leave this chapter for the next where we will be concerned with the solution of the Riccati equation, introduced in Chapter I, Theorem 1.2.

Chapter VI

The Solution to the Abstract Riccati Equation and the Associated Differential System in Fréchet Differentials

To begin with let us prove a lemma on reverses, defined by Definition 4.4.

Lemma 6.1 If R is a complete normed linear ring with the product satisfying,

(6.1) $$\|xy\| \leq m\|x\|\|y\|$$

where m is the modulus of the product or bilinear function xy for x and $y \in R$, then $x \in R$ has a unique reverse x^- if $\|x\| < \frac{1}{m}$. This reverse satisfies

(6.2) $$\|x^-\| \leq \frac{\|x\|}{1-m\|x\|}$$

Furthermore for x and $x' \in R$, satisfying

$$(\|x\|, \|x'\|) \leq \rho < \frac{1}{m}$$

the following inequality will hold:

(6.3) $$\|x^- - x'^-\| \leq \frac{\|x-x'\|}{(1-m\rho)^2}$$

x^- and x'^- being the unique reverses of x and x', respectively.

Proof: If $\|x\| < \frac{1}{m}$, then $x^- = -\sum_{n=1}^{\infty} x^n$ satisfies $x + x^- - xx^- = 0$ and $x + x^- - x^-x = 0$, and

$$\|x^-\| \leq \sum_{n=1}^{\infty} m^{n-1}\|x\|^n = \frac{\|x\|}{1-m\|x\|} \quad \text{for } \|x\| < \frac{1}{m}.$$

Hence under (6.1) the first part of the lemma and (6.2) is satisfied. Now for $\|x\|, \|x'\| \leq \rho < \frac{1}{m}$, x^- and x'^- are the unique reverses of x and x', respectively. Thus $x + x^- - xx^- = 0$ and $x' + x'^- - x'x'^- = 0$. From this we obtain

$$x^- - x'^- = (x'-x) + (x-x')x^- + x'(x^- - x'^-)$$

and

$$\|x^- - x'^-\| \le \|x' - x\| + m\|x^-\|\,\|x - x'\| + m\|x'\|\,\|x^- - x'^-\|.$$

Hence we have

$$(6.4) \qquad \|x^- - x'^-\| \le \|x' - x\| \frac{(1 + m\|x^-\|)}{1 - m\|x'\|} .$$

By (6.2), (6.4) and the fact that $(\|x\|, \|x'\|) \le \rho < \frac{1}{m}$ we have finally,

$$\|x^- - x'^-\| \le \|x' - x\| \frac{(1 + \frac{m\rho}{1-m})}{(1 - m\rho)} = \frac{\|x' - x\|}{(1 - m\rho)^2}$$

which is the required result. Hence lemma is proved.

The next theorem will be concerned with the solution of the Riccati differential equation.

$$\frac{dy}{dt} = y\,A(t)\,y \quad , \quad y(t_0) = y_0$$

or (1.16) whose existence and general expansion has already been discussed in Theorems 1.2 and 3.1.

<u>Theorem 6.1</u> If R is a complete normed linear ring without unit element and E is the space of real numbers then for $A(t)$ on E to R, continuous over interval $|t - t_0| \le h$, the unique continuous solution over $|t - t_0| \le h$ on ER to R of

$$(6.5) \qquad \frac{dy}{dt} = y\,A(t)\,y \quad , \quad y(t_0) = y_0 ,$$

is given by

$$y(t) = y_0 - y_0 \left(\int_{t_0}^{t} A(s)\,y_0\,ds\right)^-$$

when $\|A\| < \dfrac{1}{m^2 h \|y_0\|}$ where the norm, $\|\ \|$, is with respect to the Banach space, B, of continuous functions on the interval $|t - t_0| \le h$ to R, $\left(\int_{t_0}^{t} A(s)\,y_0\,ds\right)^-$ is the reverse of $\int_{t_0}^{t} A(s)\,y_0\,ds$, and m is the modulus of the product xy in R.

<u>Proof</u>: It is possible to prove this theorem by the method suggested in the second to last paragraph of Chapter III,

but we shall use, however, the following more expedient method: In the following we shall denote the norm with respect to $\mathcal{R}$ by $\| \|_R$. When $\|A\| < \frac{1}{m^2 h \|y_0\|}$,

$$\left\| \int_{t_0}^{t} A(s) y_0 \, ds \right\|_R \leq m \left| \int_{t_0}^{t} \|A(s)\|_R \|y_0\| \, ds \right|$$
$$\leq m \left| \int_{t_0}^{t} ds \right| \|A\| \|y_0\| \leq m h \|A\| \|y_0\|$$
$$< m h \left(\frac{1}{m^2 h \|y_0\|} \right) \|y_0\| < \frac{1}{m}$$

Hence by lemma 6.1 $\left(\int_{t_0}^{t} A(s) y_0 \, ds \right)^-$, the reverse of $\int_{t_0}^{t} A(s) y_0 \, ds$ exists. By 6.3 and the differentiability of $\int_{t_0}^{t} A(s) y_0 \, ds$ it is clear that $\left(\int_{t_0}^{t} A(s) y_0 \, ds \right)$ is continuous and differentiable with respect to t when $|t - t_0| \leq h$.

Now if $z(t) = y_0 - y_0 \left(\int_{t_0}^{t} A(s) y_0 \, ds \right)^-$, then

$$z(t) \times \left(\int_{t_0}^{t} A(s) y_0 \, ds \right) = \left[y_0 - y_0 \left(\int_{t_0}^{t} A(s) y_0 \, ds \right)^- \right] \times \left(\int_{t_0}^{t} A(s) y_0 \, ds \right)$$

where $\times$ is the cross product, defined in Definition 4.4. Expanding we have

$$z(t) + \int_{t_0}^{t} A(s) y_0 \, ds - z(t) \int_{t_0}^{t} A(s) y_0 \, ds =$$
$$= y_0 + \int_{t_0}^{t} A(s) y_0 \, ds - y_0 \left(\int_{t_0}^{t} A(s) y_0 \, ds \right) + y_0 \left(\int_{t_0}^{t} A(s) y_0 \, ds \right) \left(\int_{t_0}^{t} A(s) y_0 \, ds \right)^-$$

or finally,

$$z(t) - z(t) \int_{t_0}^{t} A(s) y_0 \, ds = y_0 - y_0 \left[\left(\int_{t_0}^{t} A(s) y_0 \, ds \right) \times \left(\int_{t_0}^{t} A(s) y_0 \, ds \right)^- \right] = = y_0 \tag{6.6}$$

By differentiating (6.6) we obtain

$$z'(t) - z'(t) \int_{t_0}^{t} A(s) y_0 \, ds - z(t) A(t) y_0 = 0 \tag{6.7}$$

From (6.7) we have

$$\left(z'(t) - z'(t) \int_{t_0}^{t} A(s) y_0 \, ds \right) \times \left(\int_{t_0}^{t} A(s) y_0 \, ds \right)^- = z(t) \times \left(\int_{t_0}^{t} A(s) y_0 \, ds \right)^-$$

or

$$z'(t) - z'(t) \int_{t_0}^{t} A(s) y_0 \, ds + \left(\int_{t_0}^{t} A(s) y_0 \, ds \right)^- - z'(t) \left(\int_{t_0}^{t} A(s) y_0 \, ds \right)^- + z'(t) \left(\int_{t_0}^{t} A(s) y_0 \, ds \right) \left(\int_{t_0}^{t} A(s) y_0 \, ds \right)^- =$$
$$= z(t) A(t) y_0 + \left(\int_{t_0}^{t} A(s) y_0 \, ds \right)^- - z(t) A(t) y_0 \left(\int_{t_0}^{t} A(s) y_0 \, ds \right)^-$$

so that

$$z'(t) - z'(t) \left[\left(\int_{t_0}^{t} A(s) y_0 \, ds \right) \times \left(\int_{t_0}^{t} A(s) y_0 \, ds \right)^- \right] =$$
$$= z(t) A(t) \left[y_0 - y_0 \left(\int_{t_0}^{t} A(s) y_0 \, ds \right)^- \right].$$

Thus $z'(t) = z(t)A(t)[y_0 - y_0(\int_{t_0}^{t} A(s)\,y_0\,ds)^-] =$
$= z(t)\,A(t)\,z(t),$

showing that $y_0 - y_0(\int_{t_0}^{t} A(s)\,y_0\,ds)^-$ satisfies the differential equation (6.5). Hence by Theorem 3.1 $y(t) = y_0 - y_0(\int_{t_0}^{t} A(s)\,y_0\,ds)^-$ is the unique continuous solution under the hypotheses of our theorem. Thus theorem is proved.

The solution, $y(t) = y_0 - y_0(\int_{t_0}^{t} A(s)\,y_0\,ds)^-$, given by the above theorem, is, when expanded,

(6.8) $$y(t) = y_0 + y_0 \sum_{n=1}^{\infty} [\int_{t_0}^{t} A(s)\,y_0\,ds]^n$$

valid when $\|A\| < \frac{1}{m^2 h \|y_0\|}$. Clearly from (6.8) $y[A|t]$ as a function of A on B to B is representable as a power series in A with radius of analyticity $\geq \frac{1}{m^2 h \|y_0\|}$. This agrees with Theorem 3.1, and further by Theorem 3.2 or (6) we know that the Fréchet differential of $y[A|t]$ will exist when $\|A\| < \frac{1}{m^2 h \|y_0\|}$. This fact will be used in the following theorem.

<u>Theorem 6.2</u> The Fréchet differential of $y[A|t]$, the solution of (6.5), is given by

(6.9) $$\delta y[A|t] = y(\int_{t_0}^{t} \delta A(s)\,ds)y.$$

The n-th Fréchet differential of $y[A|t]$ with equal increments $\delta A(s)$ is given by

(6.10) $$\delta^n y[A|t] = y[(\int_{t_0}^{t} \delta A(s)\,ds)\,y]^n.$$

(6.9) and (6.10) hold when $\|A\| < \frac{1}{m^2 h \|y_0\|}$ and $\delta A(s) \in B$, the Banach space of continuous ring valued functions over interval $|t - t_0| \leq h$.

<u>Proof</u>: By Theorem 6.1 we have
$$y[A|t] = y_0 + y_0(\int_{t_0}^{t} A(s)\,y_0\,ds)^-$$

when $\|A\| < \frac{1}{m^2 h \|y_0\|}$, so that by the same device, used to obtain (6. 6),

(6.11) $y[A|t] - y[A|t]\int_{t_0}^{t} A(s)\, y_0\, ds = y_0$.

Using Theorem 3. 2 and taking the Fréchet differential of (6.11) with respect to A , one obtains

(6.12) $\delta y[A|t] - \delta y[A|t]\int_{t_0}^{t} A(s)\, y_0\, ds = y[A|t]\left(\int_{t_0}^{t} \delta A(s)\, y_0\, ds\right)$.

Hence by (6.12)

$$(\delta y[A|t]) - \delta y[A|t]\int_{t_0}^{t} A(s)\, y_0\, ds) \times \left(\int_{t_0}^{t} A(s) y_0\, ds\right)^{-} =$$
$$= \left[y[A|t]\left(\int_{t_0}^{t} \delta A(s)\, y_0\, ds\right)\right] \times \left(\int_{t_0}^{t} \delta A(s)\, y_0\, ds\right)^{-},$$

and by expanding

$$\delta y[A|t] - \delta y[A|t]\left[\left(\int_{t_0}^{t} A(s)\, y_0\, ds\right) \times \left(\int_{t_0}^{t} A(s)\, y_0\, ds\right)^{-}\right] =$$
$$= y[A|t]\left[\left(\int_{t_0}^{t} \delta A(s)\, y_0\, ds\right) - \left(\int_{t_0}^{t} \delta A(s)\, y_0\, ds\right)\left(\int_{t_0}^{t} A(s)\, y_0\, ds\right)^{-}\right].$$

Thus $\delta y[A|t] = y[A|t]\left(\int_{t_0}^{t} \delta A(s)\, ds\right)\left[y_0 - y_0\left(\int_{t_0}^{t} A(s)\, y_0\, ds\right)^{-}\right] =$

$$= y[A|t]\left(\int_{t_0}^{t} \delta A(s)\, ds\right) y[A|t] .$$

Hence (6.9) is proved. (6.10) is now true for $n=1$ so let us assume for purposes of induction that (6.10) is true for all integers $\leq n$. Then

$$\delta^n y[A|t] = n!\, y[A|t]\left[\left(\int_{t_0}^{t} A(s)\, ds\right) y[A|t]\right]^n .$$

By taking the Fréchet differential with increment $\delta A(s)$, we obtain

(6.12)
$$\delta^{n+1} y[A|t] = n!\,\Big\{\delta y[A|t]\left[\left(\int_{t_0}^{t} A(s)\, ds\right) y[A|t]\right]^n +$$
$$+ y[A|t]\left(\int_{t_0}^{t} \delta A(s)\, ds\right)\delta y[A|t]\left[\int_{t_0}^{t} \delta A(s)\, ds\, y[A|t]\right]^{n-1} + \dots +$$
$$+ y[A|t]\left[\left(\int_{t_0}^{t} \delta A(s)\, ds\right) y[A|t]\right]^{n-1}\left(\int_{t_0}^{t} \delta A(s)\, ds\right)\delta y[A|t]\Big\} .$$

By (6.9), (6.12) becomes

$$\delta^{n+1} y[A|t] = n!\,\Big\{y[A|t]\left(\int_{t_0}^{t} \delta A(s)\, ds\right) y[A|t]\left[\left(\int_{t_0}^{t} \delta A(s)\, ds\right) y[A|t]\right]^n +$$
$$+ y[A|t]\left(\int_{t_0}^{t} \delta A(s)\, ds\right) y[A|t]\left(\int_{t_0}^{t} \delta A(s)\, ds\right) y[A|t]\left[\left(\int_{t_0}^{t} \delta A(s)\, ds\right) y[A|t]\right]^{n-1} +$$
$$+ \dots +$$
$$+ y[A|t]\left[\left(\int_{t_0}^{t} A(s)\, ds\right) y[A|t]\right]^{n-1}\left(\int_{t_0}^{t} \delta A(s)\, ds\right) y[A|t]\left(\int_{t_0}^{t} \delta A(s)\, ds\right) y[A|t]\Big\} =$$
$$= n!\,(n+1)\, y[A|t]\left[\left(\int_{t_0}^{t} \delta A(s)\, ds\right) y[A|t]\right]^{n+1} =$$
$$= (n+1)!\, y[A|t]\left[\left(\int_{t_0}^{t} \delta A(s)\, ds\right) y[A|t]\right]^{n+1},$$

thereby completing induction, showingthat (6.10) is true for all n. Hence theorem is proved.

<u>Theorem 6.3</u> If $A, \delta A \in B$, the Banach space of continuous ring valued functions over $|t-t_0| \leq h$, the differential system $\delta y[A/t] = y[A/t](\int_{t_0}^t \delta A(s)\,ds)\,y[A/t]$

(6.13) $\quad y[0/t] = y_0$

has a unique ~~ala~~ anal ytic solution of radius $\geq \frac{1}{m^2 h \|y_0\|}$ given by

$$y[A/t] = y_0 + y_0 \sum_{n=1}^{\infty} [(\int_{t_0}^t A(s)\,ds)\,y_0]^n$$

$$= y_0 - y_0 [\int_{t_0}^t A(s)ds\, y_0]^{-},$$

the unique solution of the differential equation (6.5).

<u>Proof</u>: The proof follows along the same lines used to prove Theorem 1 of (7). We want to find the necessary and sufficient conditions that the analytic function

(6.14) $\quad y[A/t] = y_0 + \sum_{i=1}^{\infty} \Omega_i[A/t]$

satisfy the differential system (6.13) when $\|A\| < \frac{1}{m^2 \|y_0\| h}$.

In (6.14) $\Omega_i[A/t]$ is a homogeneous polynomial of degree i on B to B, or $\Omega[A/t]$ is a continuous function of A such that

(a) $\Omega_i[\lambda A/t] = \lambda^i \Omega_i[A/t]$ all real λ and $A \in B$

(b) $\Omega_i[A+\lambda B/t] = \sum_{r=0}^{i} \lambda^r \Phi_r[A, B/t]$ for all real λ and all $A, B \in B$

But we know by (5) that a homogeneous polynomial in a Banach space has a unique polar, or that there exists a unique i-linear function $\omega_i[A_1, A_2, \ldots, A_i/t]$ such that $\omega_i[A, A, \ldots A/t] = \Omega_i[A/t]$.

Furthermore, the Fréchet differential of the homogeneous polynomial $\Omega_i[A/t]$ exists and is given by

(6.15) $\quad \delta\Omega_i[A/t] = i\,\omega_i[A, A, \ldots A, \delta A/t]$

By (6.14) $y_0 + \sum_{n=1}^{\infty} \Omega_n[A/t]$ has a radius of analyticity of

$\frac{1}{m^2h\,\|y_0\|}$, thus by (6) we can use (6.15) to obtain the term by term Fréchet differential of (6.14) within this sphere. Hence

(6.16) $\delta y[A|t] = \sum_{i=1}^{\infty} i\omega_i[A, A, \cdots A, \delta A|t]$ when $\|A\| \leq \frac{1}{m^2h\,\|y_0\|}$.

To obtain a necessary condition that (6.13) hold, we have by (6.16) and (6.14)

(6.17)

$$\begin{aligned}
\delta y[A|t] &= \sum_{i=1}^{\infty} i\omega_i[A, A, \ldots A, \delta A|t] = \\
&= y[A|t]\left(\int_{t_0}^{t}\delta A(s)\,ds\right)y[A|t] = \\
&= \left[y_0 + \sum_{i=1}^{\infty}\Omega_i[A|t]\right]\left(\int_{t_0}^{t}\delta A(s)\,ds\right)\left[y_0 + \sum_{i=1}^{\infty}\Omega_i[A|t]\right] = \\
&= y_0\left(\int_{t_0}^{t}\delta A(s)\,ds\right)y_0 + \Omega_1[A|t]\left(\int_{t_0}^{t}\delta A(s)\,ds\right)y_0 + y_0\left(\int_{t_0}^{t}\delta A(s)\,ds\right)\Omega_1[A|t] + \\
&\quad + \sum_{n=2}^{\infty}\left\{y_0\left(\int_{t_0}^{t}\delta A(s)\,ds\right)\Omega_n[A|t] + \Omega_n[A|t]\left(\int_{t_0}^{t}\delta A(s)\,ds\right)y_0\right\} + \\
&\quad + \sum_{\substack{i+j=n\\ i,j\geq 1}}\Omega_i[A|t]\left(\int_{t_0}^{t}\delta A(s)\,ds\right)\Omega_j[A|t], \quad \text{when } \|A\| \leq \frac{1}{m^2h\,\|y_0\|}
\end{aligned}$$

Now if we let $\delta A(s) = A(s)$ in (6.17) and equate coefficients, it is clear by (6.15) that

(6.18)
$$\begin{aligned}
\omega_1[A|t] &= y_0\int_{t_0}^{t}A(s)\,y_0\,ds = \Omega_1[A|t] \\
2\omega_2[A|t] &= \Omega_1[A|t]\int_{t_0}^{t}A(s)\,y_0\,ds + y_0\int_{t_0}^{t}A(s)\,ds\,\Omega_1[A|t] = \\
&= 2y_0\left(\int_{t_0}^{t}A(s)\,y_0\,ds\right)^2 = 2\Omega_2[A|t]
\end{aligned}$$

or

(6.19) $\Omega_2[A|t] = y_0\left(\int_{t_0}^{t}A(s)\,y_0\,ds\right)^2,$

and in general

(6.20)
$$\begin{aligned}
(n+1)\,\omega_{n+1}[A, A, \ldots A|t] &= y_0\left(\int_{t_0}^{t}A(s)\,ds\right)\Omega_n[A|t] \\
&+ \Omega_n[A|t]\int_{t_0}^{t}A(s)\,y_0\,ds + \sum_{\substack{i+j=n\\ i,j\geq 1}}\Omega_i[A|t]\left(\int_{t_0}^{t}A(s)\,ds\right)\Omega_j[A|t]
\end{aligned}$$

when $\|A\| \leq \frac{1}{m^2h\,\|y_0\|}$. We thus see that a necessary condition for (6.20) to hold is that for $n \geq 2$

(6.21)
$$\begin{aligned}
\Omega_{n+1}[A|t] = \frac{1}{n+1}\Big\{ & y_0\left(\int_{t_0}^{t}A(s)\,ds\right)\Omega_n[A|t] + \Omega_n[A|t]\left(\int_{t_0}^{t}A(s)\,y_0\,ds\right) + \\
& + \sum_{\substack{i+j=n\\ i,j\geq 1}}\Omega_i[A|t]\left(\int_{t_0}^{t}A(s)\,ds\right)\Omega_j[A|t]\Big\}
\end{aligned}$$

when $\|A\| < \frac{1}{m^2 h \|y_0\|}$. Now since $\Omega_n[A|t] = y_0[\int_{t_0}^t A(s) y_0\,ds]^n$ for $n = 1$ and 2 by (6.18) and (6.19), let us assume for purposes of induction that it is satisfied for all integers $\le n$. Then

(6.22) $$\Omega_n[A|t] = y_0[\int_{t_0}^t A(s) y_0\,ds]^n,$$

and by (6.21)

$$\Omega_{n+1}[A|t] = \frac{1}{n+1}\{y_0(\int_{t_0}^t A(s)\,ds)y_0[\int_{t_0}^t A(s)\,y_0\,ds]^n + y_0[\int_{t_0}^t A(s)y_0\,ds]^n(\int_{t_0}^t A(s)y_0\,ds) +$$
$$+ \sum_{\substack{i+j=n \\ i,j\ge 1}} y_0[\int_{t_0}^t A(s)\,y_0\,ds]^i(\int_{t_0}^t A(s)\,y_0\,ds)[\int_{t_0}^t A(s)\,y_0\,ds]^j\} =$$
$$= y_0[\int_{t_0}^t A(s)\,y_0\,ds]^{n+1},$$

thereby completing induction. Furthermore from Theorem 6.1 and (6.8) we know that

$$y_0 + \sum_{n=1}^{\infty} y[\int_{t_0}^t A(s)\,y_0\,ds]^n$$

is a regular power series in A when $\|A\| < \frac{1}{m^2 h\|y_0\|}$. Thus the necessary condition that (6.14) satisfy the differential system (6.13) is

(6.23) $$\Omega_i[A|t] = y_0[\int_{t_0}^t A(s)\,y_0\,ds]^i \quad, \quad (i = 1, 2, 3, \cdots),$$

when $\|A\| < \frac{1}{m^2 h \|y_0\|}$. Now condition (6.23) is also sufficient for by (6.15),

$$(n+1)\ \omega_{n+1}[A, A, \cdots A, \delta A|t] = \delta\Omega_{n+1}[A|t] =$$
$$= y_0(\int_{t_0}^t \delta A(s)\,ds)\,y_0[\int_{t_0}^t A(s)\,y_0\,ds]^n + y_0[\int_{t_0}^t A(s)\,y_0\,ds](\int_{t_0}^t \delta A(s)\,ds)y_0[\int_{t_0}^t A(s)y_0\,ds]^{n-1} +$$
$$+ \cdots + y_0[\int_{t_0}^t A(s)\,y_0\,ds]^{n-1}(\int_{t_0}^t \delta A(s)\,ds)\,y_0[\int_{t_0}^t A(s)\,y_0\,ds] +$$
$$+ y_0[\int_{t_0}^t A(s)\,y_0\,ds]^n(\int_{t_0}^t \delta A(s)\,ds)\,y_0 =$$
$$= y_0(\int_{t_0}^t \delta A(s)\,ds)\Omega_n[A|t] + \Omega_1[A|t](\int_{t_0}^t \delta A(s)\,ds)\Omega_{n-1}[A|t] +$$
$$+ \Omega_{n-1}[A|t](\int_{t_0}^t \delta A(s)\,ds)\Omega_1[A|t] + \Omega_n[A|t](\int_{t_0}^t \delta A(s)\,ds)\,y_0$$

when $\|A\| < \frac{1}{m^2 h\|y_0\|}$. Hence (6.20) holds for $n \ge 2$, and by the term by term Fréchet differentiability of $y_0 + y_0 \sum_{n=1}^{\infty} [\int_{t_0}^{t} A(s) y_0 ds]^n$ with respect to A for $\|A\| \leq \frac{1}{m^2 h \|y_0\|}$ the theorem follows.

The next lemma will be useful in obtaining a Taylor expansion to the solution of the differential equation, discussed in Theorem 6.1.

Lemma 6.2 If R is a complete normed linear ring with m, the modulus of a product in R, then

(6.24) $$(a+x)^- = a^- + \sum_{i=1}^{\infty} [(x-ax)^i a^- - (x-a^-x)^i]$$

is the unique reverse of the element $a+x \in R$ when the following condition is satisfied:

(6.25) If $\|A\| \leq \frac{q}{m}$, $0 \leq q < 1$, the norm of x must be $\|x\| < \frac{1-q}{m}$.

Proof: By Lemma 6.1 if $\|a\| \leq \frac{q}{m}$, then

(6.26) $$\|a^-\| < \frac{\|a\|}{1 - m\|a\|} \leq \frac{q}{(1-q)m}.$$

Hence,
$$\|a^- + \sum_{i=1}^{\infty} [(x-a^-x)^i a^- - (x-a^-x)^i\| \leq$$
$$\leq \|a^-\| + \|x\| (1+m\|a^-\|)^2 \sum_{n=0}^{\infty} m^i \|x\|^i (1+m\|a^-\|)^i \leq$$
$$\leq \frac{q}{(1-q)m} + \frac{\|x\|}{(1-q)^2} \sum_{n=0}^{\infty} [\frac{m\|x\|}{1-q}]^i \quad by \ (6.26).$$

But this converges when $\|x\| < \frac{1-q}{m}$, thus the right of (6.24) exists under the condition, (6.25). That the right of (6.24) represents the unique reverse of $a+x$ follows algebraically as in the proof of Lemma 4.6. Hence Lemma is proved.

Theorem 6.4 For $A_0 \in B$, the Banach space of R-valued continuous functions, $A(t)$, over interval $|t-t_0| \leq h$, and $\|A_0\| < \frac{1}{m^2 h \|y_0\|}$, the analytic solution $y[A|t]$ of (6.13) can be expanded in a generalized Taylor series of successive Fréchet differentials with equal increments, δA, valid for $\delta A \in B$ and $\|\delta A\| < \frac{1}{m^2 h \|y_0\|} - \|A_0\|$;

(6. 27) $\quad y[A_0 + \delta A|t] = y_0[A|t] + \sum_{i=1}^{\infty} \frac{1}{i!}[\delta^i y[A|t]]_{A=A_0}$.

Proof: Now let

(6.28) $\quad \|\int_{t_0}^{t} A_0(s)\, y_0\, ds\|_R \le m\, \|A_0\|\, \|y_0\|\, h = \frac{q}{m}$

then

(6.29) $\quad \|\int_{t_0}^{t} \delta A(s)\, y_0\, ds\|_R \le m h\, \|y_0\|\, \|\delta A\| <$

$$< m h\, \|y_0\| \left(\frac{1}{m^2 h \|y_0\|} - \|A_0\| \right) =$$

$$= \frac{1 - m^2 \|A_0\|\, \|y_0\| h}{m} = \frac{1-q}{m} .$$

By (6.28), (6.29) and the fact that $\|A_0\| < \frac{1}{m^2 h \|y_0\|}$, we have from Lemma 6.2 and Theorem 6.3,

$$y[A_0 + \delta A|t] = y_0 - y_0\left(\int_{t_0}^{t} A_0(s)\, y_0\, ds + \int_{t_0}^{t} \delta A(s)\, y_0\, ds\right)^{-} =$$

$$= y_0 - y_0\left(\int_{t_0}^{t} A_0(s) y_0\, ds\right) - y_0 \sum_{i=1}^{\infty} \Big[\{(\int_{t_0}^{t} \delta A(s) y_0\, ds) - (\int_{t_0}^{t} A_0(s) y_0\, ds)^{-} (\int_{t_0}^{t} \delta A(s) y_0\, ds)\}^i (\int_{t_0}^{t} A_0(s) y_0\, ds)^{-} -$$

$$- \{\int_{t_0}^{t} \delta A(s) y_0\, ds - (\int_{t_0}^{t} A_0(s) y_0\, ds)^{-} (\int_{t_0}^{t} \delta A(s) y_0\, ds)\}^i \Big] =$$

$$= y[A_0|t] - \sum_{i=1}^{\infty} \Big(y_0 \{[\int_{t_0}^{t} \delta A(s)\, ds - (\int_{t_0}^{t} A_0(s) y_0\, ds)^{-} (\int_{t_0}^{t} \delta A(s)\, ds)] y_0\}^i (\int_{t_0}^{t} A_0(s) y_0)^{-} -$$

$$- y_0 \{[\int_{t_0}^{t} \delta A(s)\, ds - (\int_{t_0}^{t} A_0(s) y_0\, ds)^{-} (\int_{t_0}^{t} \delta A(s)\, ds)] y_0\}^i \Big) =$$

$$= y[A_0|t] + \sum_{i=1}^{\infty} \Big(\{y_0 [\int_{t_0}^{t} \delta A(s)\, ds - (\int_{t_0}^{t} A(s) y_0\, ds)^{-} (\int_{t_0}^{t} \delta A(s)\, ds)]\}^i y_0 -$$

$$- \{y_0 [\int_{t_0}^{t} \delta A(s)\, ds - (\int_{t_0}^{t} A_0(s) y_0\, ds)^{-} (\int_{t_0}^{t} \delta A(s)\, ds)]\}^i y_0 (\int_{t_0}^{t} A_0(s) y_0\, ds)^{-} \Big) =$$

$$= y[A_0|t] + \sum_{i=1}^{\infty} \{[y_0 - y_0(\int_{t_0}^{t} A_0(s) y_0\, ds)^{-}](\int_{t_0}^{t} \delta A(s)\, ds)\}^i [y_0 - y_0(\int_{t_0}^{t} A_0(s) y_0\, ds)^{-}] =$$

$$= y[A_0|t] + \sum_{i=1}^{\infty} [y[A_0|t](\int_{t_0}^{t} \delta A(s)\, ds)]^i\, y[A_0|t] =$$

$$= y[A_0|t] + \sum_{i=1}^{\infty} y[A_0|t] [(\int_{t_0}^{t} \delta A(s)\, ds)\, y[A_0|t]]^i .$$

But by Theorem 6.2

$$[\delta^i y[A|t]]_{A=A_0} = i!\, y[A_0|t] [(\int_{t_0}^{t} \delta A(s)\, ds)\, y[A|t]]^i ,$$

hence

$$y[A_0 + \delta A|t] = y[A_0|t] + \sum_{i=1}^{\infty} \frac{\{\delta^i y[A|t]\}_{A=A_0}}{i!}$$

which is (6.28). Hence theorem is proved.

The next theorem is concerned with the Fréchet differential of $y[y_0|t]$, the solution of (6.5), holding $A(t)$ fixed.

Theorem 6.5 The Fréchet differential of $y[y_0|t]$, the solution of (6.5), holding $A(t)$ fixed, with increment δy_0 is given by

(6.30) $$\delta y[y_0|t] = \delta y_0 - \delta y_0 \left(\int_{t_0}^{t} A(s) y_0\, ds\right)^{-} + y[y_0|t]\int_{t_0}^{t} A(s)\,ds\left[\delta y_0 - \delta y_0\left(\int_{t_0}^{t} A(s) y_0\, ds\right)^{-}\right].$$

The n-th Fréchet differential of $y[y_0|t]$ with equal increments δy_0 is given by

(6.31) $$\delta^n y[y_0|t] = n!\,\delta y[y_0|t]\left\{\left(\int_{t_0}^{t} A(s)\,ds\right)\left[\delta y_0 - \delta y_0\left(\int_{t_0}^{t} A(s) y_0\, ds\right)^{-}\right]\right\}^{n-1}$$

where $\delta y[y_0|t]$ is given by (6.30). (6.30) and (6.31) hold when $\|y_0\| < \dfrac{1}{m^2 h\|A\|}$ and $\delta y_0 \in B$, the Banach space of continuous ring valued functions over intervals, $|t-t_0| \le h$.

Proof: Now $$y[y_0|t] = y_0 - y_0\left(\int_{t_0}^{t} A(s) y_0\, ds\right)^{-} = y_0 + y_0 \sum_{n=1}^{\infty}\left(\int_{t_0}^{t} A(s) y_0\, ds\right)^{n}$$

is clearly a power series in y_0 with radius of analyticity $\ge \dfrac{1}{m^2 h\|A\|}$. Hence by (6) the Fréchet differential of $y[y_0|t]$ exists when $\|y_0\| < \dfrac{1}{m^2 h\|A\|}$.

From (6.6)

$$y[y_0|t] - y[y_0|t]\int_{t_0}^{t} A(s) y_0\, ds = y_0 .$$

Taking the Fréchet differential we have

$$\delta y[y_0|t] - \delta y[y_0|t]\int_{t_0}^{t} A(s) y_0\, ds - y[y_0|t]\int_{t_0}^{t} A(s)\,\delta y_0\, ds = \delta y_0$$

or

(6.31) $$\delta y[y_0|t] - \delta y[y_0|t]\int_{t_0}^{t} A(s) y_0\, ds = \delta y_0 + y[y_0|t]\int_{t_0}^{t} A(s)\,\delta y_0\, ds .$$

Hence from (6.31)

$$\Big(\delta y[y_0|t]-\delta y[y_0|t]\int_{t_0}^{t}A(s)y_0\,ds\Big)\times\Big(\int_{t_0}^{t}A(s)y_0\,ds\Big)^{-}=$$

$$=\Big(\delta y_0+y[y_0|t]\int_{t_0}^{t}A(s)\delta y_0\,ds\Big)\times\Big(\int_{t_0}^{t}A(s)y_0\,ds\Big)^{-}$$

or $$\delta y[y_0|t]-\delta y[y_0|t]\int_{t_0}^{t}A(s)y_0\,ds+\Big(\int_{t_0}^{t}A(s)ds\Big)^{-}-\delta y[y_0|t]\Big(\int_{t_0}^{t}A(s)y_0\,ds\Big)^{-}+$$

$$+\delta y[y_0|t]\Big(\int_{t_0}^{t}A(s)y_0\,ds\Big)\Big(\int_{t_0}^{t}A(s)y_0\,ds\Big)^{-}=$$

$$=\delta y_0+y[y_0|t]\int_{t_0}^{t}A(s)\delta y_0\,ds+\Big(\int_{t_0}^{t}A(s)y_0\,ds\Big)^{-}-\delta y_0\Big(\int_{t_0}^{t}A(s)y_0\,ds\Big)^{-}-$$

$$-\delta y[y_0|t]\Big(\int_{t_0}^{t}A(s)y_0\,ds\Big)\Big(\int_{t_0}^{t}A(s)y_0\,ds\Big)^{-}.$$

Upon simplification we have

$$\delta y[y_0|t]=\delta y_0-\delta y_0\Big(\int_{t_0}^{t}A(s)y_0\,ds\Big)^{-}+y[y_0|t]\int_{t_0}^{t}A(s)ds\Big[\delta y_0-\delta y_0\Big(\int_{t_0}^{t}A(s)y_0\,ds\Big)^{-}\Big]$$

which is (6.30). By taking the Fréchet differential of (6.31) with increment δy_0 we obtain

$$\delta^2 y[y_0|t]-\delta^2 y[y_0|t]\int_{t_0}^{t}A(s)y_0\,ds=2\delta y[y_0|t]\Big(\int_{t_0}^{t}A(s)ds\Big)\delta y_0 .$$

suppose for purposes of induction that

$$\delta^i y[y_0|t]-\delta^i y[y_0|t]\int_{t_0}^{t}A(s)y_0\,ds=i\,\delta^{i-1}y[y_0|t]\int_{t_0}^{t}A(s)\delta y_0\,ds$$

for all $i\leq n$. Then

$$\delta^n y[y_0|t]-\delta^n y[y_0|t]\int_{t_0}^{t}A(s)y_0\,ds=n\,\delta^{n-1}y[y_0|t]\int_{t_0}^{t}A(s)\delta y_0\,ds,$$

and the Fréchet differential with increment δy_0 is

$$\delta^{n+1}y[y_0|t]-\delta^{n+1}y[y_0|t]\int_{t_0}^{t}A(s)y_0\,ds-\delta^n y[y_0|t]\int_{t_0}^{t}A(s)\delta y_0\,ds=$$

$$=n\,\delta^n y[y_0|t]\int_{t_0}^{t}A(s)\delta y_0\,ds$$

or

$$\delta^{n+1}y[y_0|t]-\delta^{n+1}y[y_0|t]\int_{t_0}^{t}A(s)y_0\,ds=(n+1)\,\delta^n y[y_0|t]\int_{t_0}^{t}A(s)\delta y_0\,ds$$

completing induction. Thus

(6.32) $$\delta^n y[y_0|t]-\delta^n y[y_0|t]\int_{t_0}^{t}A(s)y_0\,ds=n\,\delta^{n-1}y[y_0|t]\int_{t_0}^{t}A(s)\delta y_0\,ds$$

for $(n=2,3,4,\ldots)$. By (6.32)

$$\Big(\delta^n y[y_0|t]-\delta^n y[y_0|t]\int_{t_0}^{t}A(s)y_0\,ds\Big)\times\Big(\int_{t_0}^{t}A(s)y_0\,ds\Big)^{-}=$$

$$=(n+1)\Big(\delta^n y[y_0|t]\int_{t_0}^{t}A(s)\delta y_0\,ds\Big)\times\Big(\int_{t_0}^{t}A(s)y_0\,ds\Big)^{-}.$$

By expanding and simplifying we obtain

(6.33) $$\delta^n y[y_0|t]=n\,\delta^{n-1}y[y_0|t]\Big\{\Big(\int_{t_0}^{t}A(s)ds\Big[\delta y_0-\delta y_0\Big(\int_{t_0}^{t}A(s)y_0\,ds\Big)^{-}\Big]\Big\}$$

for $(n=2,3,4,\ldots)$. From (6.33) we can obtain

(6.31) by an evident induction. Hence theorem is proved.

The next step in this chapter will be to generalize the differential equation of Theorem 6.1 to

$$\frac{dy(t)}{dt} = T(y(t), A(t), y(t)) \quad \text{when} \quad y(t_0) = y_0$$

where the product, $y\,A(t)\,y$, is replaced by the trilinear function, $T(y, A(t), y)$, to which suitable assumptions will now be attached (See (7)).

Assumption 6.1 B_1 and B_2 are Banach spaces such that $y_i \in B_1$, $A \in B_2$ and $T(y_1, A, y_2)$ is a trilinear function (additive and continuous in each of the three variables) on $B_1\,B_2\,B_1$ to B_1.

Assumption 6.2 $T(T(y_1, A_2, y_2), A_1, y_3) = T(y_1, A_2, T(y_2, A_1, y_3))$ for all $y_i \in B$ and $A_i \in B_2$.

Two immediate consequences of the above assumptions are:

Lemma 6.3 There exists an M such that

$$\|T(y_1, A_1, y_2)\| \leq M \|A_1\| \|y_1\| \|y_2\|$$

all $y_i \in B_1$, $A_1 \in B_2$.

Lemma 6.4 If $y_1(t)$, $y_2(t)$, $A_1(t)$ are differentiable functions of t in interval $|t - t_0| \leq h$, then $T(y_1(t), A_1(t), y_2(t))$ is differentiable over the same interval, and, explicitly,

$$\frac{dT}{dt}(y_1(t), A_1(t), y_2(t)) = T\left(\frac{dy_1(t)}{dt}, A_1(t), y_2(t)\right) + T\left(y_1(t), \frac{dA_1(t)}{dt}, y_2(t)\right) + T\left(y_1(t), A_1(t), \frac{dy_2(t)}{dt}\right).$$

Definition 6.1 $T^n(y_1, A_1, y)$ is the n-th iteration of the linear function $T(y_1, A_1, y)$ of y where $T^1(y_1, A_1, y) = T(y_1, A_1, y)$.

From Definition 6.1 it follows that

(6.34) $\quad T^n(y_1, A_1, y) = T^i(y_1, A_1, T^j(y_1, A_1, y))$

all i, j such that $i + j = n$. We shall now prove the following two lemmas:

Lemma 6.5 $\quad T^n(y_1, A_1, T(y_1, A_2, y)) = T(T^n(y_1, A_1, y_1), A_2, y)$,

$(n=1,2,3\cdots)$ when $y_1, y \in B_1$ and $A_1, A_2 \in B_2$.

Proof: Let us first prove

(6.35) $T^{i+1}(y_1, A_1, y) = T(T^i(y_1, A_1, y_1), A_1, y), \quad (i=1,2,3,\cdots)$

for $y_1, y \in B_1$ and $A_1 \in B_2$. By Assumption 6.2 and Definition 6.1 (6.35) is true for $n=1$. Now let us assume for purposes of induction that it is true for $i \leq n$, then for $z \in B_1$,

$$T^n(y_1, A_1, T(y_1, A, z)) = T(T^n(y_1, A_1, y_1) A_1, z),$$

and by letting $z = T(y_1, A_1, y)$ we obtain

(6.36) $T^{n+1}(y_1, A_1, T(y_1, A_1, y)) = T(T^n(y_1, A_1, y_1), A_1, T(y_1, A_1, y_1))$.

The left of (6.36) is by (6.34),

$$T^{n+2}(y_1, A_1, y),$$

and the right side by Assumption 6.2, induction hypothesis, and (6.34),

$$T(T^{n+1}(y_1, A_1, y_1), A_1, y).$$

The (6.36) becomes

$$T^{n+2}(y_1, A_1, y) = T(T^{n+1}(y_1, A_1, y_1), A_1, y),$$

thereby completing induction, showing (6.35) to be true. Since (6.35) is true we have

(6.37) $T^n(y_1, A_1, z) = T(T^{n-1}(y_1, A_1, y_1), A_1, z) \quad , \quad n=(2,3,4,\cdots)$

for $y_1, z \in B_1$ and $A_1 \in B_2$. Now let $z = T(y_1, A_2, y)$, then (6.37) gives by Assumption 6.2 and (6.35),

$$\begin{aligned} T^n(y_1, A_1, T(y_1, A_2, y)) &= T(T^{n-1}(y_1, A_1, y_1), A_1, T(y_1, A_2, y)) \\ &= T(T(T^{n-1}(y_1, A_1, y_1), A_1, y_1), A_2, y) \\ &= T(T^n(y_1, A_1, y_1), A_2, y) \end{aligned}$$

for $(n=2,3,4,\ldots)$. But this is true for $n=1$ by Assumption 6.2, hence lemma is proved.

Lemma 6.6 $\|T^i(y_1, A_1, y)\| \leq M^i \|y_1\|^i \|A_1\|^i \|y\|$,

$(i = 1, 2, 3, \dots)$, $y_1, y \in B_1$, and $A_1 \in B_2$.

Proof: The lemma is true by Lemma 6.3 for $i=1$, so let us assume that it is true for $i \leq n$ then

(6.38) $\|T^n(y_1, A_1, y)\| \leq M^n \|y_1\|^n \|A_1\|^n \|y\|$.

By (6.34)

$$T^{n+1}(y_1, A_1, y) = T(y_1, A_1, T^n(y_1, A_1, y)).$$

Hence by Lemma 6.3 and (6.38)

$$\|T^{n+1}(y_1, A_1, y)\| \leq M \|y_1\| \|A_1\| \|T^n(y_1, A_1, y)\|$$
$$\leq M^{n+1} \|y_1\|^{n+1} \|A_1\|^{n+1} \|y\|.$$

Thus induction is completed and lemma is proved.

Now if we let

(6.39) $T_{y_1, A_1}(y) = T(y_1, A_1, y)$

and $T_{y_1,A_1} + T_{y_2,A_2}$, $T_{y_1,A_1} T_{y_2,A_2}$ and $a T_{y_1,A_1}$ be defined by

(6.40) $(T_{y_1,A_1} + T_{y_2,A_2})(y) = T_{y_1,A_1}(y) + T_{y_2,A_2}(y) =$

$$= T(y_1, A_1, y) + T(y_2, A_2, y),$$

$$(T_{y_1,A_1} T_{y_2,A_2})(y) = T_{y_1,A_1}[T_{y_2,A_2}(y)] = T(y_1, A_1, T(y_2, A_2, y)),$$

$$(a T_{y_1,A_1})(y) = a T_{y_1,A_1}(y) = a T(y_1, A_1, y),$$

for all $y_i \in B_1$, $A_i \in B_2$, then T_{y_1,A_1} by Definition 4.1 is an endomorphism for all $y_1 \in B_1$, $A_1 \in B_2$, contained in the complete normed linear ring of all endomorphisms, $R(B_1)$, of B_1. We are now in a position to prove the following theorem:

Theorem 6.6 If $A(t)$ with values in B_2 is continuous over interval $|t-t_0| \leq h$, then the unique continuous solution with values in B_1 of

(6.41) $\frac{dy}{dt} = T(y, A(t), y)$, $y(t_0) = y_0$, where T satisfies Assumptions 6.1 and 6.2, is given by

(6.42) $$y(t) = [I - T_{y_0, \int_{t_0}^t A(s)ds}]^{-1}(y_0)$$
$$= y_0 + \sum_{n=1}^{\infty} T^n(y_0, \int_{t_0}^t A(s)\,ds, y_0)$$

when $\|A\|_2 < \dfrac{1}{Mh\|y_0\|}$ and where the norm $\|\ \|_2$ is with respect to the Banach space of continuous functions on the interval, $|t-t_0| \leq h$, to B_2, M is given by Lemma 6.3, and I is the identity transformation on B_1 to B_1.

<u>Proof</u>: By Lemma 6.6 and hypothesis

$$\|T^i(y_0, \int_{t_0}^t A(s)ds, y)\| \leq M^i\|y_0\|^i\|\int_{t_0}^t A(s)ds\|^i\|y\| \leq$$
$$\leq M^i\|y_0\|^i h^i\|A\|_2^i\|y\| <$$
$$< \|y\| .$$

Hence by (6.34), (6.40) and Definition 6.1

(6.43)
$$\|T^i_{y_0, \int_{t_0}^t A(s)ds}\| = \sup_{\|y\| \leq 1} \|T^i(y_0, \int_{t_0}^t A(s)ds, y)\| < 1 .$$

From (6.43) and the completeness of $\mathcal{R}(B_1)$ we thus have that $[I - T_{y_0, \int_{t_0}^t A(s)ds}]^{-1}$ exists in $\mathcal{R}(B_1)$ and is given uniquely as follows:

(6.44) $$[I - T_{y_0, \int_{t_0}^t A(s)ds}]^{-1} = I + \sum_{n=1}^{\infty} T^n_{y_0, \int_{t_0}^t A(s)ds}$$

when $\|A\|_2 < \dfrac{1}{Mh\|y_0\|}$.

By Lemma 6.4, (6.39) $T_{y_0, \int_{t_0}^t A(s)ds}(y_0)$ is differentiable. Hence by (6.44) and the fourth paragraph of Chapter I it is clear that $[I - T_{y_0, \int_{t_0}^t A(s)ds}]^{-1}(y_0)$ is differentiable (see also (4), pg. 45). Now let

(6.45) $$z(t) = [I - T_{y_0, \int_{t_0}^t A(s)ds}]^{-1}(y_0) .$$

Then by operating on the left of (6.45) with $[I - T_{y_0, \int_{t_0}^t A(s)ds}]$ we obtain

(6.46) $$z(t) - T_{y_0, \int_{t_0}^t A(s)ds}(z(t)) = y_0 .$$

Differentiating (6.46), using Lemma 6.4,

$$z'(t) - T_{y_0, A(t)}(z(t)) - T_{y_0, \int_{t_0}^t A(s)ds}(z'(t)) = 0 .$$

Thus by (6.44)

$$(6.47)\quad z'(t) = ([I - T_{y_0,\int_{t_0}^t A(s)ds}]^{-1} T_{y_0,A(t)})(z(t)) =$$
$$= (I + \sum_{n=1}^{\infty} T^n_{y_0,\int_{t_0}^t A(s)ds})(T_{y_0,A(t)})(z(t)) =$$
$$= T_{y_0,A(t)}(z(t)) + \sum_{n=1}^{\infty}(T^n_{y_0,\int_{t_0}^t A(s)ds})(T_{y_0,A(t)})(z(t)) =$$
$$= T(y_0, A(t), z(t)) + \sum_{n=1}^{\infty} T^n(y_0, \int_{t_0}^t A(s)ds, T(y_0, A(t), z(t))).$$

By Lemma 6.5, (6.47), and the trilinearity of $T(y_1, A_1, y_2)$ we thus have

$$z'(t) = T(y, A(t), z(t)) + \sum_{n=1}^{\infty} T(T^n(y_0, \int_{t_0}^t A(s)ds, y_0), A(t), z(t)) =$$
$$= T_{y_0,A(t)}(z(t)) + T(\sum_{n=1}^{\infty} T^n(y_0, \int_{t_0}^t A(s)ds, y_0), A(t), z(t)) =$$
$$= T_{y_0,A(t)}(z(t)) + T_{\sum_{n=1}^{\infty} T^n_{y_0,\int_{t_0}^t A(s)ds}(y_0), A(t)}(z(t)) =$$
$$= T_{(I+\sum_{n=1}^{\infty} T^n_{y_0,\int_{t_0}^t A(s)ds})(y_0), A(t)}(z(t)) =$$
$$= T_{(I - T_{y_0,\int_{t_0}^t A(s)ds})^{-1}(y_0), A(t)}(z(t)) = T_{z(t),A(t)}(z(t)) =$$
$$= T(z(t), A(t), z(t)).$$

Thus $[I - T_{y_0,\int_{t_0}^t A(s)ds}]^{-1}(y_0) = y_0 + \sum_{n=1}^{\infty} T^n(y_0, \int_{t_0}^t A(s)ds, y_0)$ satisfies (6.42) when $\|A\|_2 < \frac{1}{Mh\|y_0\|}$. The uniqueness by a similar proof to the one given in Theorem 3.1. Hence theorem is proved.

The analogy in the proof of Theorem 6.6 to that of Theorem 6.1 is evident, moreover Theorem 6.1 is a special case of Theorem 6.6. We shall now state the theorems which are the analogs of Theorems 6.2, 6.3, 6.4, and 6.5. Their proofs will follow along the same lines as before.

<u>Theorem 6.7</u> The Fréchet differential of $y[A/t]$ with increment $\delta A(s)$ of the solution to (6.41) is given by

$$\delta y[A/t] = T(y, \int_{t_0}^t \delta A(s)ds, y).$$

The n-th Fréchet differential of $y[A/t]$ with equal increments $\delta A(s)$ is given by

$$\delta^n y[A/t] = T^n(y, \int_{t_0}^t \delta A(s)ds, y).$$

These expressions hold when $\|A\|_2 < \frac{1}{Mh\|y_0\|}$ and $\delta A(s)$ is in the Banach space of B_2- valued continuous functions over $|t-t_0| \leq h$.

Theorem 6.8 If $y \in B_1$ and $A \in B_2$ where $\|A\|_2 \leq \frac{1}{Mh\|y_0\|}$, the differential system

$$\delta y[A|t] = T(y[A|t], \int_{t_0}^{t} \delta A(s)\,ds, y[A|t])$$
$$y[0|t] = y_0$$

has a unique analytic solution of radius $\geq \frac{1}{Mh\|y_0\|}$, given by (6.42).

Theorem 6.9 For $A_0 \in B_2'$, the Banach space of B_2- valued functions, $A(t)$, over interval, $|t-t_0| \leq h$, and $\|A_0\|_2 \leq \frac{1}{Mh\|y_0\|}$, the analytic solution $y[A|t]$ of (6.41) can be expanded in a generalized Taylor series of successive Fréchet differentials with equal increments, δA, valid for $\delta A \in B_2'$ and $\|\delta A\|_2 < \frac{1}{Mh\|y_0\|} - \|A_0\|_2$;

$$y[A_0+\delta A|t] = y[A|t] + \sum_{i=0}^{\infty} \frac{1}{i!}[\delta^i y[A|t]]_{A=A_0}$$

Theorem 6.10 The Fréchet differential of $y[A|t]$, the solution of the differential equation (6.41) holding $A(t)$ fixed, with increment δy_0 is given by

(6.47) $$\delta y[y_0|t] = (I - T_{y_0, \int_{t_0}^{t} A(s)\,ds})^{-1}(y_0 + T(\delta y_0, \int_{t_0}^{t} A(s)\,ds, y[y_0|t])).$$

The n-th Fréchet differential of $y[y_0|t]$ with equal increments δy_0 is given by

(6.48) $$\delta^n y[y_0|t] = n!\,T^{n-1}((I - T_{y_0, \int_{t_0}^{t} A(s)ds})^{-1}\delta y_0, \int_{t_0}^{t} A(s)\,ds, \delta y[y_0|t])$$

$(n = 2, 3, 4, \cdots)$, where $\delta y[y_0|t]$ is given by (6.47).

(6.47) and (6.48) hold when $\|y_0\| < \frac{1}{Mh\|A\|_2}$ and $\delta y_0 \in B_1'$, the Banach space of continuous B_1- valued functions over interval, $|t-t_0| \leq h$.

The work in Chapter III suggests another type of problem, the solution of the functional equation, (6.49) $y = f + Q(y, A, y)$,

where $Q(y_1, A_1, y_2)$ is a trilinear function(additive and continuous in each of the three variables) on $B_1 B_2 B_1$ to B_1. We shall not devote much time to this problem, but if $Q(y_1, A, y_2)$ satisfies Assumptions 1, 2, and 4 of (7), it is possible to show by the methods of Chapter III or directly that the following is true: The unique analytic solution of (6.49) is given by

(6.50) $y = f + \sum_{n=1}^{\infty} \frac{(2n)!}{(n+1)!n!} Q^n(f, A, f)$,

when $\|A\| < \frac{1}{4M\|f\|}$ and where M is determined as in Lemma 6.3 and $Q^n(f, A, f)$ is the n-th iteration of $Q(f, A, z)$ of z evaluated at $z = f$. W hen Assumption 3 of (7) is satisfied, y, given by (6.50), will be an entire analytic function of A.

There are other differential equations which may be treated by methods similar to those we have used and suggested. It will not, however, be our present purpose to delve into this further. The next and concluding chapter will concern itself with a few illustrative examples of the theory we have so far discussed.

Chapter VII

Examples and Applications

The first example of this chapter will be a specialization of the differential equation (6.5). If $A(t) = a$, a constant in R , and R contains the unit element, e with modulus, $m=1$, then by Theorem 6.1, the unique solution of

$$(7.1) \qquad \frac{dy}{dt} = yay \quad , \quad y(0) = e \quad , \quad |t| < h$$

is given by

$$(7.2) \qquad y(t) = e - e\left(\int_{t_0}^{t} ae\,dt\right)^{-}$$

$$= e(e-ta)^{-1} = (e-ta)^{-1}$$

when $\|a\| < \frac{1}{h}$. By the proof of Theorem 6.1 it is clear that the solution, (7.2) of (7.1) may be extended to any value of t and $a \in R$ such that $(e-ta)^{-1}$ exists (see (4) pg. 95). Hence by the definition of the resolvent ([4], pg. 97), the inverse of $\lambda e - a$ when it exists and denoted by $R(\lambda ; a)$, we can state the following theorem.

__Theorem 7.1__ For $a \in R$, a constant, with unit, e , the unique solution of

$$\frac{dy}{dt} = yay, \quad y(0) = e$$

is given by the resolvent function

$$y(t) = \frac{1}{t} R\left(\frac{1}{t}, a\right) = (e-ta)^{-1},$$

valid whenever $(e-ta)^{-1}$ exists.

In a like manner we obtain from Theorem 6.3:

__Theorem 7.2__ If $a \in R, \; |t| < h$, and $\|a\| < \frac{1}{h}$, the differential system

$$\delta y[a/t] = t\,y[a/t]\,\delta a\,y[a/t]$$
$$y[0/t] = e$$

has a unique analytic solution of radius $\geq \frac{1}{h}$ given by

$$y[a/t] = (e-ta)^{-1} = \frac{1}{t} R\left(\frac{1}{t}; a\right).$$

Another example, which illustrates the usage of Theorem 6.1 as applied to matrix integral-differential equations, is the following:

(7.3) $$\frac{\partial y^l_j}{\partial t}(\rho,\xi,t) = \int_a^b\int_a^b y^l_m(\rho,\rho_1,t)A^m_2(\rho_1,\rho_2,t)y^2_j(\rho_2,\xi,t)\,d\rho_2\,d\rho_1$$

$y^l_j(\rho,\xi,t_0) = y^l_{0j}(\rho,\xi)$ for ρ,ξ such that $a \leq \rho,\xi \leq b$,

where $A^m_2(\rho_1,\rho_2,t)$ are real continuous functions over $a \leq \rho_1,\ \rho_2 \leq b,\ |t-t_0| \leq h,$ when $1 \leq m, 2 \leq h$. The norm with respect to the ring, R , is

(7.4) $$\| A^m_2(\rho_1,\rho_2,t)\|_R = \max_{\substack{a \leq \rho_1,\ \rho_2 \leq b \\ 1 \leq m,\ 2 \leq h}} |A^m_2(\rho_1,\rho_2,t)|$$

and with respect to B ,

(7.5) $$\| A^m_2(\rho_1,\rho_2,t)\| = \max_{|t-t_0|\leq h} \| A^m_2(\rho_1,\rho_2,t)\|_R .$$

If we denote the "product" in R ,

$$\int_a^b A^m_2(\rho,\rho_1,t)B^2_j(\rho_1,\xi,t)\,d\rho_1$$

by

(7.6) $$(A^m_2(t)) * (B^m_2(t))$$

it is easily seen that

(7.7) $$\|(A^m_2(t)) * (B^m_2(t))\|_R \leq n(b-a)\,\|(A^m_2(t))\|_R\,\|(B^m_2(t))\|_R$$

From (7.7) the modulus, m , of the product is $\leq n^2(b-a)$. Thus from (6.4), (6.5), (6.6), (6.7) and Theorem 6.1, the unique solution of (7.3) is

(7.8) $$y^l_j(\rho,\xi,t) = y^l_{0j}(\rho,\xi) + \sum_{n=1}^{\infty}(y^l_{0j}) * \left([\textstyle\int_{t_0}^t (A^l_j(s))ds] * (y^l_{0j})\right)^{n*}$$

$$= y^l_{0j}(\rho,\xi) +$$

$$+ \int_a^b\int_a^b y^l_{0m}(\rho,\rho_1)\left[\int_{t_0}^t A^m_2(\rho_1,\rho_2|s)\,ds\right]y^2_{0j}(\rho_2,\xi)\,d\rho_2\,d\rho_1 + \cdots,$$

valid when $\| A^m_2(\rho_1,\rho_2,t)\| \leq \frac{1}{\{n(b-a)\}^2 h \,\| y^l_{0j}(\rho,\xi)\|}$, where $(\quad)^{n*}$ denotes the n-th power in R . From

Theorem 6.3 we have also that (7.8) is the unique analytic ~~solution~~ functional of A_2^m, radius $\geqslant \frac{1}{\{n(b-a)\}^2 h \| y_{0j}^i(\rho,\xi)\|}$, satisfying the differential system,

$$\delta y_j^i[A_2^m/\rho,\xi,t] = \int_a^b\int_a^b y_m^{\ell}(\rho,\rho_1,t)\,[\int_{t_0}^t \delta A_2^m(\rho_1,\rho_2,s)\,ds]\,y_j^{\ell}(\rho_2,\xi,t)\,d\rho_2\,d\rho_1$$

$$y_j^i[0/\rho,\xi,t] = y_{0j}^i(\rho,\xi) .$$

There are other examples we could give to illustrate Theorem 6.1, for instance the replacement of the Fredholm compositions in (7.3) by Volterra compositions, but we shall leave this to give an example which illustrates Theorem 6.6. We shall discuss briefly a rectangular matrix differential equation.

The differential equation in question is

(7.9) $$\frac{dy_\alpha^i}{dt} = y_\beta^i A_j^\beta(t)\, y_\alpha^j \;, \quad y_\alpha^i(t_0) = y_{0\alpha}^i$$

where $1 \leqslant i \leqslant n,\ 1 \leqslant \alpha \leqslant m$, and $A_j^\beta(t)$ are continuous functions of t over interval $|t-t_0| \leqslant h$. It is clear that $T(y_1,A_1,y) \equiv y_{1\beta}^i(t)\,A_{1j}^\beta(t)\,y_\alpha^j(t)$ satisfies Assumptions 6.1 and 6.2 with B_1, the Banach space of continuous matrices, $y_\beta^i(t)$, over interval $|t-t_0| \leqslant h$, and B_2, the Banach space of continuous matrices, $A_j^\alpha(t)$, over $|t-t_0| \leqslant h$. The norm for B_1 is

(7.10) $$\| y_\beta^i(t) \| = \max_{\substack{1\leqslant i \leqslant n \\ 1 \leqslant \beta \leqslant m}} |y_\beta^i(t)|$$

and for B_2

(7.11) $$\| A_j^\alpha(t)\| = \max_{\substack{1\leqslant i \leqslant n \\ 1 \leqslant \beta \leqslant m}} |A_j^\alpha(t)| .$$

From (7.10) and (7.11),

(7.12) $$\| y_{1\beta}^i(t)\, A_{1j}^\beta(t)\, y_\alpha^j(t)\| \leqslant mn \|y_{1\beta}^i\| \, \|A_{1j}^\beta(t)\| \, \|y_\alpha^j(t)\| .$$

Thus from (7.12) and Theorem 6.6 the unique solution of (7.9) is

(7.13) $$y_\alpha^i(t) = y_{0\alpha}^i + y_{0\beta}^i \sum_{n=1}^\infty \left(\int_{t_0}^t A_j^\beta(s)\, y_{0\alpha}^j\, ds\right)^{n*},$$

valid when $\|A_j^\beta(s)\|_2 < \frac{1}{mnh\|y_{0\alpha}^i\|}$, where $(\quad)^*$ denotes the n-th power of matrix in bracket.

From Theorem 6.7 we conclude that (7.13) is the unique analytic functional of $A_j^\beta(t)$, radius $\geqslant \frac{1}{mn\|y^i_{0\alpha}\|}$, satisfying the differential system

$$\delta y^i_\alpha[A_j^\beta|t] = y^i_\beta[A_j^\beta|t]\left(\int_{t_0}^t \delta A_j^\beta(s)\,ds\right) y^j_\alpha[A_j^\beta|t]$$
$$y^i_\alpha[0|t] = y^i_{0\alpha}\,.$$

An interesting special case of (7.9) is when $\alpha = 1$, or

(7.14) $$\frac{dy^i}{dt} = y^i A_j(t) y^j,\quad y^i(t_0) = y^i_0\,,\quad (i = 1, 2, \ldots, n).$$

From (7.13) the solution of (7.14) is easily seen to be

(7.15) $$y^i(t) = \frac{y^i_0}{1 - \int_{t_0}^t A_j(s) y^j_0\, ds}$$

when $\|A_j(s)\|_2 < \frac{1}{nh\|y^i_0\|}$. By direct substitution of (7.15) in (7.14) it is seen that (7.15) is the solution of (7.14) whenever the inverse of $1 - \int_{t_0}^t A_j(s) y^j_0\, ds$ exists.

In conclusion we give *two* examples of the functional equation, (6.49). The first is

(7.16) $$Y = F + YAY$$

where F is an $m \times n$ matrix and A is an $n \times m$ matrix. The solution of (7.16) for Y is by (6.50)

$$Y = F + \sum_{n=1}^{\infty} \frac{(2n)!}{(n+1)!\,n!} (FA)^n F$$

when $\|A\| < \frac{1}{4mh\|F\|}$, the norms given by (7.10) and (7.11).

The final example under the conditions and notation of the first paragraph of section 6 of (7) is the integral equation,

$$y(t,r) = f(t,r) + \int_r^t y(t,s) A(s) y(s,r)\, ds\,.$$

The unique solution of this by (6.50) is given by the entire analytic functional of $A(s)$,

$$y(t,r) = f(t,r) + \sum_{i=1}^{\infty} \frac{(2n)!}{(n+1)!\,n!} I^{*i}(f,A) * f$$
$$= f(t,r) + \int_r^t f(t,s) A(s) f(s,r)\, ds + \ldots$$

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