A COLORING PROBLEM RELATED TO KÖNIG'S THEOREM

Thesis by

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ABSTRACT

A connection is shown between König's Theorem on 0-1 matrices and theorems giving sufficient conditions, in terms of certain forbidden subgraphs, for a graph G to have chromatic number equal to the maximum number of vertices in any clique of G. A conjecture is proposed which would, if true, give the best possible such theorem. Three special cases of this conjecture are proved, and König's Theorem is shown to be an easy corollary of any one of them.

INTRODUCTION

The chromatic number of a graph G is always bounded below by the maximum number of vertices in any complete subgraph of G. It is of some interest to find conditions under which these two numbers are actually equal.

In Section 1 we discuss this problem in a general way and point out its connection with two classical combinatorial theorems: The König Theorem on 0-1 matrices and the Dilworth Theorem on partially ordered sets. A conjecture is proposed which would in a sense give a best-possible solution.

In Section 2 we prove two rather general lemmas, which are the basic tools for the proofs of the succeeding section.

In Section 3 we prove three theorems, all of which are very special cases of the conjecture in Section 1, and which are generalizations of the above-mentioned theorem of König.

The modest nature of these generalizations is indicated in Section 4.

SECTION 1

The term *graph* will always be used to denote an undirected graph in which there is at most one edge joining any given pair of vertices and no edge joining any vertex to itself. (For definitions not given here see Ore (6).) If two vertices are joined by an edge they are said simply to be joined.

A graph G is called *complete* if every pair of distinct vertices in G is joined.

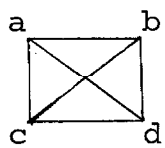
A graph is called *edge-empty* if its edge set is empty. A graph H will be called a *subgraph* of the graph G if its vertex set is a subset of the vertex set of G, and its edge set is the set of edges of G whose endpoints lie in the vertex set of H. The graph G is said to *contain* the graph H.

The union $\bigcup_\alpha H_\alpha$ of a collection $\{H_\alpha\}$ of subgraphs of G is defined to be the subgraph whose vertex set is the union of the vertex sets of the H_α. The operations $\bigcap$, $-$ are defined similarly. We write $H = \Sigma_\alpha K_\alpha$ to mean $H = \bigcup_\alpha K_\alpha$ and $K_\alpha \cap K_\beta = \emptyset$ for $\alpha \neq \beta$.

We denote by $|G|$ the cardinality of the vertex set of G.

If G is any graph we denote by G* the graph whose vertex set is the vertex set of G and in which two distinct vertices are joined if and only if they are not joined in G.

We shall often denote particular graphs geometrically, representing the vertices by dots and the edges by arcs joining pairs of dots. For example,



denotes the complete graph with vertices a, b, c, d.

A complete subgraph of G is called a clique of G.

An edge-empty subgraph of G is called a <u>color</u> of G.

The vertex set of a color is called an <u>independent set</u>.

If two vertices form an independent set they are said to be <u>independent</u>.

Thus if H is a clique of G, H* is a color of G*, and vice versa.

If C is a clique and x a vertex not in C, x is said to be joined to C if it is joined to every vertex of C. If C and D are disjoint cliques, C is said to be joined to D if every vertex of C is joined to every vertex of D.

If $G = \sum_{\alpha \epsilon A} H_\alpha$, where each H_α is a color, and A has cardinality k, we say that G is k-colorable. If $m = \min \{k \mid G \text{ is k-colorable}\}$, we say that G is m-chromatic, and we call m the <u>chromatic number</u> $\chi(G)$.

Very little is known about the function $\chi(G)$ in general. One fact is however obvious. If we let $\gamma(G) = \sup \{k \mid G \text{ contains a clique with k vertices}\}$, then clearly we must have $\chi(G) \geqslant \gamma(G)$. It is natural to ask the question: for which graphs G does the identity

$$\text{(A)} \qquad \chi(G) = \gamma(G)$$

hold?

Two well-known combinatorial theorems can be interpreted as saying that the identity (A) holds for certain graphs G:

<u>Theorem of König</u> (5): Let A be a matrix of zeroes and ones. We call a row or column of A a <u>line</u>. Then the minimum number of lines containing all the ones of A is equal to the maximum number of ones of A no two on a line.

If we let the ones in the matrix A be the vertices of a graph G, and let two vertices be joined if and only if they do not lie on a line, then the König Theorem states that the graph G so defined satisfies (A).

<u>Theorem of Dilworth</u> (2): If P is a partially ordered set, and the maximum number of pairwise non-comparable elements of P is k, then P is the union of k chains.

Here we take the elements of P as the vertices of the graph G, and join two vertices if and only if they are not comparable in P; then again the Theorem states that $\chi(G) = \gamma(G)$.

Note that in both these cases the graph relation was defined as the <u>negation</u> of the "natural" relation. This suggests the following "complementary" reformulation:

We define

$$c(G) = \chi(G^*),$$

$$\beta(G) = \gamma(G^*),$$

and ask the question: when does the identity

$$(A^*) \qquad c(G) = \beta(G)$$

hold?

In general the two problems are of course equivalent, but in particular cases one or the other formulation may be the more convenient.

Theorems giving sufficient conditions for the identity (A*) to hold have been proved by Hajnal and Surányi (4), and by Gallai (3). A simple proof of the theorem of Gallai, which is the more general of the two, will be given in the next section.

If $\gamma(G) = 1$, then (A) holds trivally (as indeed does (A*)). The answer in the case $\gamma(G) = 2$ is also well-known and simple. First we need a few definitions and lemmas.

If a and b are distinct vertices of the graph G, a <u>path</u> of length n from a to b is a sequence $c_0 = a$, c_1, $c_2, \ldots, c_n = b$ of distinct vertices of G, such that c_i is joined to c_{i+1}, $i = 0, 1, \ldots, n-1$.

A graph G with n vertices is called a <u>circuit</u> of length n if for some numbering $x_1, x_2, \ldots, x_n$ of the vertices of G, x_i is joined to x_j if $i-j \equiv \pm 1 \pmod{n}$. If

$i-j \equiv \pm 1 \pmod{n}$, x_i and x_j are called _consecutive_ vertices of the circuit. The circuit G is called _irreducible_ if x_i is joined to x_j only if $i-j \equiv \pm 1 \pmod{n}$.

Lemma 1.1: Every circuit of odd length contains an irreducible circuit of odd length.

Proof: By induction, it is sufficient to show that every circuit of odd length which is not irreducible contains a shorter circuit of odd length.

Let then G be a non-irreducible circuit of length 2n+1, with vertices $x_1, x_2, \ldots, x_{2n+1}$, where x_i is joined to x_j whenever $i-j \equiv \pm 1 \pmod{2n+1}$. Since G is not irreducible, some x_i, which we may take to be x_1, is joined to some x_j, where $j \neq 2, 2n+1$. If j is odd, then $x_1, x_2, x_3 \ldots, x_j$ form a circuit of (odd) length $j < 2n+1$. If j is even, then $x_j, x_{j+1}, \ldots, x_{2n+1}, x_1$ form a circuit of (odd) length $2n-j+3 < 2n+1$.

Lemma 1.2: If a and b are distinct vertices of the graph G, and there is a path of even length from a to b and also a path of odd length, then G contains a circuit of odd length.

Proof: Choose a and b in G such that $c_0 = a, c_1, c_2 \ldots, c_m = b$ and $d_0 = a, d_1, d_2, \ldots, d_n = b$ are paths, where m+n is odd, and where if e and f are distinct vertices of G, and there is a path of length r from e to f and a path of length s from e to f with r+s odd, then $r+s \geqslant m+n$. Let i be the least positive integer for which $c_i = d_j$ for some j. If i+j is odd, then $c_0, c_1, \ldots, c_i, d_{j-1}, d_{j-2}, \ldots, d_1$ form a circuit of odd length. If on the other hand i+j is even, then c_i is joined to b by a path of length m-i-1 and also by a path of length n-j-1. But m-i-1 + n-j-1 = m+n - (i+j) - 2, which is odd and $< m+n$, contrary to the choice of a and b.

It is now quite easy to prove the

<u>Two-Color Theorem</u>: The graph G is 2-colorable if and only if it contains no irreducible circuit of odd length.

Proof: Suppose $x_1, x_2, \ldots, x_{2n+1}$ form a circuit of odd length is G, and that G is the sum of two colors A and B, where $x_1 \epsilon A$. Then $x_2 \epsilon B$, $x_3 \epsilon A$, and so on so that $x_{2n+1} \epsilon A$, a contradiction. The condition is therefore necessary.

Now let G contain no irreducible circuit of odd length. We may certainly assume that G is connected. By Lemma 1.1 it contains no circuit of odd length, whence by Lemma 1.2, for a and b distinct vertices of G, either all paths from a to b have even length or they all have odd length. Define aRb to mean a = b or there is a path from a to b of even length. R is obviously an equivalence relation. By Lemma 1.2, the equivalence classes are colors. Finally, there are only two equivalence classes, for if all paths from a to b are of odd length, and all paths from b to c are of odd length, then since G is connected, there is a path of even length from a to c, whence aRc. Thus G is 2-colorable.

This theorem gives us the simplest examples of graphs for which the identity (A) does not hold. For if G is an irreducible circuit of odd length > 3, then $\gamma(G) = 2$, but $\chi(G) = 3$.

Let P_n be the irreducible circuit of length n, and consider now the graph P_{2n+1}^*, $n > 1$. We have $\gamma(P_{2n+1}^*) = n$. A color in P_{2n+1}^* can contain at most 2 vertices, since $\gamma(P_{2n+1}) = 2$. If P_{2n+1}^* were n-colorable, we would have therefore $2n+1 = |P_{2n+1}^*| < 2n$, a contradiction, Hence $\chi(P_{2n+1}^*) > \gamma(P_{2n+1}^*)$.

<u>Conjecture</u>: The graphs P_{2n+1}, P_{2n+1}^*, $n > 1$ are precisely the minimal counterexamples to the identity (A) in the case when $\gamma(G)$ is finite. That is, if neither G nor G* contains an irreducible circuit of odd length > 3, then $\chi(G) = \gamma(G)$, if $\gamma(G)$ is finite.

In a sense, asking for minimal counterexamples to (A) rather than for <u>all</u> counterexamples seems to be the "right" question. For if G is <u>any</u> graph, then by adjoining to G a complete graph with $\chi(G)$ vertices, we obtain a graph $\overline{G}$ with $\chi(\overline{G}) = \gamma(\overline{G})$. Thus every graph is a subgraph of a graph in which (A) holds, so that the problem of determining when (A) holds in general is no easier than the problem of determining all k-chromatic graphs for every k.

A theorem of deBruijn and Erdős (1) states that a graph every finite subgraph of which is k-colorable, where k is fixed and finite, is itself k-colorable. It follows that for $\gamma(G)$ finite, any minimal counterexample to (A) must be finite.

SECTION 2

Lemma 2.1: Let G be a minimal counterexample to (A*), where $\beta(G)$ is finite. By the remark at the end of Section 1, G is necessarily finite. Let p be any vertex of G. We have

$G-\{p\} = C_1+C_2+\ldots+C_\beta$, where each C_i is a clique. Let

$R_i = \{x \in C_i \mid x \text{ is joined to } p\}$. Let $N_i = C_i - R_i$, and let

$R = \Sigma R_i$, $N = \Sigma N_i$. Then

(1) $\beta(N) < \beta$.

(2) For i and j distinct integers between 1 and β inclusive, there exists an independent set $\{x_k\}$, $x_k \in N_k$, $k \neq i$, such that x_j is not joined to C_i, and no x_k is joined both to C_i and to every other vertex which is joined to C_i.

Proof: (1) If $\beta(N) \geqslant \beta$, then $\beta(G) \geqslant \beta(N + \{p\}) \geqslant \beta + 1$.

(2) If any vertices of N are joined to C_i, we add them to C_i, forming an enlarged clique C_i'. If any further vertex of $N-N_i$ is joined to C_i', we add it to C_i' forming a new clique C_i''. We continue in this way until no further vertices may be added. Let the resulting cliques be $\overline{C}_1, \overline{C}_2, \ldots, \overline{C}_\beta$; $\overline{N}_1, \overline{N}_2, \ldots, \overline{N}_\beta$. If $\beta(N-\overline{N}_i) < \beta-1$, then $c(G) \leqslant c(\overline{C}_i) + c(G-\overline{C}_i) = 1 + \beta-1 = \beta$. Hence $\beta(N-\overline{N}_i) = \beta-1$, and we may choose $\beta-1$ independent vertices x_k, $x_k \in \overline{N}_k$, $k \neq i$; where x_j is not joined to C_i, and where no x_k is joined to $\overline{C}_i$. But every vertex of $\overline{C}_i - C_i$ is joined to C_i, so the

lemma is proved.

In view of (1) of Lemma 2.1, we may write

$N = K_1 + K_2 \ldots + K_{\beta-1}$, where each K_i is a clique. For $x \in N$, we define $K(x)$ by $K(x) = K_i$, $x \in K_i$.

<u>Lemma 2.2</u>: Let G satisfy the hypothesis of Lemma 2.1. Let $I \cup J = B = \{1, 2, \ldots, \beta\}$. Let $\{x(i)\}$, $x(i) \in N_i$, $i \in I$, and $\{y(j)\}$, $y(j) \in N_j$, $j \in J$, be sets of $|I|$ and $|J|$ independent vertices respectively. Then for some $i \in I-J$ and some $j \in J-I$, there exists a sequence $n_0 = i, n_1, n_2, \ldots, n_k = j$ of distinct integers in B such that $K(x(n_s)) = K(y(n_{s+1}))$, $s = 0, 1, \ldots, k-1$.

Proof: For any $i \in I$ there is at most one $j \in J$ satisfying $y(j) \in K(x(i))$, since $K(x(i))$ is a clique and the $y(j)$ are independent. This j when it exists we call $\alpha(i)$. If for $i, i' \in I$, $\alpha(i)$ and $\alpha(i')$ exist and are equal, then $i = i'$. For if $\alpha(i) = \alpha(i')$, then $y(\alpha(i)) = y(\alpha(i')) \in K(x(i)) \cap K(x(i'))$, whence $x(i) = x(i')$ and $i = i'$.

If either I-J or J-I is empty, then by (1) of Lemma 2.1 there is nothing to prove. For any $i \in I-J$ consider the sequence $\alpha(i), \alpha^2(i), \alpha^3(i), \ldots$. Since the terms of this sequence are distinct, it must terminate, say in $\alpha^{m(i)}(i)$. This can happen in exactly two ways:

(1) $\alpha^{m(i)}(i) \in J-I$. In this case the sequence $n_s = \alpha^s(i)$ has the desired property.

(2) $\alpha^{m(i)}(i) \in I$, but there is no $y(j) \in K(x(\alpha^{m(i)}(i)))$. Suppose that this case holds for every $i \in I-J$. Now if for $i, i' \in I-J$, $i \neq i'$, $\alpha^{m(i)}(i) = \alpha^{m(i')}(i')$, and $m(i) \geqslant m(i')$, it follows that $i' = \alpha^{m(i)-m(i')}(i)$. But if $m(i) = m(i')$, then $i = i'$, and if $m(i) > m(i')$, then $\alpha^{m(i)-m(i')}(i) \in J$. In either case we have a contradiction. It follows then that there are $|I-J|$ distinct K_s to which no $y(j)$ belongs. But then the $|J|$ $y(j)$ belong to at most $\beta-1-|I-J| = |J|-1$ of the K_s, contradicting the independence of the $y(j)$. It follows that (1) must hold for at least one $i \in I-J$, and the lemma is proved.

<u>Corollary 1</u>: A set of vertices $\{r_i\}$, $r_i \in R_i$, $i \in I$ will be called <u>strongly independent</u> if there exist $\beta - |I|$ vertices $x(k) \in N_k$, $k \notin I$ such that $\{r_i\} \cup \{x(k)\}$ is an independent set.

Let G satisfy the hypothesis of Lemma 2.1, and let it contain no irreducible circuit of odd length > 3. If $\{r_i\}$, $r_i \in R_i$, $i \in I$, and $\{s_j\}$, $s_j \in R_j$, $j \in J$ are strongly independent sets, where $I \cap J = \emptyset$, then for some $i \in J$, $j \in J$, r_i is joined to s_j.

Proof: Let $I' = B-I$, $J' = B-J$. Since $I \cap J = \emptyset$, $I' \cup J' = B$. Since $\{r_i\}$ and $\{s_j\}$ are strongly independent sets, there exist independent sets $\{x(k)\}$, $x(k) \in N_k$, $k \in I'$ and $\{y(k)\}$, $y(k) \in N_k$, $k \in J'$ such that no r_i is joined to any $x(k)$ and no s_j is joined to any $y(k)$. By Lemma 2.2 there exist

$i' \in I'-J'$, $j' \in J'-I'$ and a sequence $n_1, n_2, \ldots, n_k$ such that $x(i'), y(n_1), x(n_1), \ldots, y(n_k), x(n_k), y(j')$ is a path. But then $p, s_{i'}, x(i'), y(n_1), x(n_1), \ldots, y(n_k), x(n_k), y(j'), r_{j'}$ form a circuit of odd length > 3. Since no such circuit can be irreducible, this must contain a circuit of length 3, by Lemma 1,1; but this is possible only if $r_{j'}$ is joined to $s_{i'}$.

Corollary 2 (Gallai): If every irreducible circuit in G which is contained in a circuit of odd length has length 3, then $c(G) = \beta(G)$, if $\beta(G)$ is finite.

Proof: Let G be a minimal counterexample. Then G certainly satisfies the hypothesis of Lemma 2.1. Let $r_i \in R_i$, $r_j \in R_j$, $i \neq j$. By Lemma 2.1 there exist independent sets $\{x(k)\}$, $x(k) \in N_k$, $k \neq i$, and $\{y(k)\}$, $y(k) \in N_k$, $k \neq j$. By Lemma 2.2 there exists a sequence $n_1, n_2, \ldots, n_k$ such that $y(i), x(n_1), y(n_1), \ldots, x(n_k), y(n_k), x(j)$ is a path, whence $p, r_i, y(i), x(n_1), \ldots, y(n_k), x(j), r_j$ form a circuit of odd length. This circuit contains an irreducible circuit containing p and r_i, which by hypothesis must have length 3. The only possible such circuit has to consist of p, r_i, and r_j. Thus r_i is joined to r_j. It follows that R is a clique, whence $c(G) \leqslant c(R) + c(N) = 1 + \beta - 1 = \beta$.

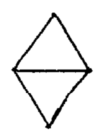
SECTION 3

We continue to use the notation of Section 2.

Theorem 3.1: Let G be a graph satisfying the following two conditions:

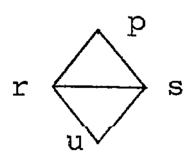
(a) If S is a circuit of odd length in G, then some vertex of S is joined to two consecutive vertices of S.

(b) G does not contain the graph



Then $c(G) = \beta(G)$, if $\beta(G)$ is finite.

Proof: Let G be a minimal counterexample. Suppose that for some i R_i contains two distinct vertices r, s. Since $\beta > 1$ (obviously), $N_i \neq \emptyset$, by (2) of Lemma 2.1. Let $u \in N_i$. Then G contains the subgraph



contrary to hypothesis. Hence $|R_i| \leqslant 1$ for all i. If $R_i \neq \emptyset$, let $r_i \in R_i$. Let R_i and R_j be distinct and non-empty, and suppose r_i is not joined to r_j. By Lemma 2.1 there exist independent sets $\{x(k)\}$, $x(k) \in N_k$, $k \neq i$ and $\{y(k)\}$, $y(k) \in N_k$, $k \neq j$. By Lemma 2.2 there exists a sequence $n_0 = i, n_1, \ldots, n_m = j$ such that $K(y(n_s)) = K(x(n_{s+1}))$ for $s = 0, 1, \ldots, m-1$. Therefore p, r_i, $y(i)$, $x(n_1), \ldots,$ $x(n_{m-1})$, $y(n_{m-1})$, $x(j)$, r_j form a circuit of odd length > 3.

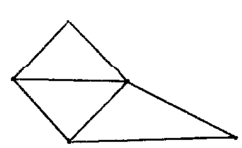
The only vertices of this circuit which can possibly be joined to two consecutive vertices are r_i and r_j. Suppose r_i is, and consider the first such consecutive pair in the sequence $y(i), x(n_1),\ldots,x(j)$. This pair cannot be of the form $x(n_s)$, $y(n_s)$, for then the circuit $\{r_i, y(i),\ldots, x(n_s)\}$ would violate condition (a) It follows that r_i is joined to two vertices of some K_s, and therefore to all K_s, by hypothesis (b). Let $R' = \{r\varepsilon R | r$ is joined to all of K_s for some s for which $|K_s| \geqslant 2$. Now if r_i and r_j are both joined to all of K_s, where $|K_s| \geqslant 2$, then r_i is joined to r_j, again by condition (b). Hence $c(R' + N) = \beta-1$. But we have seen that $c(R-R') = 1$. Therefore $c(G) \leqslant c(R'+N) + c(R-R') = \beta -1 + 1 = \beta$.

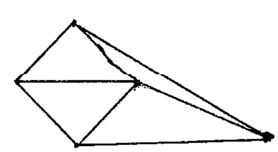
Almost certainly condition (a) can be replaced by the weaker condition that G contain no irreducible circuit of odd length > 3.

<u>Theorem 3.2</u>: Let G be a graph satisfying the following two conditions:

(a) If S is a circuit of odd length in G, then some 3 consecutive vertices of S form a clique.

(b) G contains neither of the following subgraphs:





Then $c(G) = \beta(G)$, if $\beta(G)$ is finite.

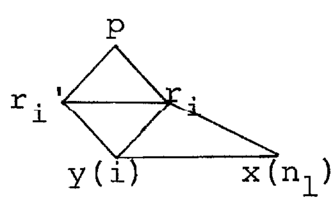
Proof: Again we let G be a minimal counterexample. Let R_i and R_j be distinct and non-empty, and let $r_i \epsilon R_i$, $r_j \epsilon R_j$. By Lemma 2.1 there exist independent sets $\{x(k)\}$, $x(k) \epsilon N_k$, $k \neq i$, and $\{y(k)\}$, $y(k) \epsilon N_k$, $k \neq j$, such that no $x(k)$ is joined both to C_i and to every other vertex which is joined to C_i, and no $y(k)$ is joined both to C_j and to every other vertex which is joined to C_j. By Lemma 2.2 there is a path $y(i)$, $x(n_1)$, $y(n_1)$, $x(n_2)$, $y(n_2)$,...$x(n_s)$, $y(n_s)$, $x(j)$. Thus p, r_i, $y(i)$, $x(n_1)$,...,$y(n_s)$, $x(j)$, r_j form a circuit of odd length. Applying condition (a), we find there are 3 possibilities:

(1) r_i is joined to $x(n_1)$.

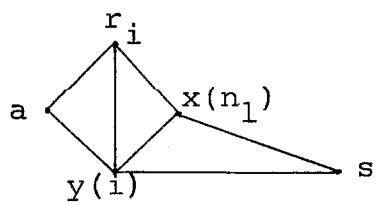
(2) r_j is joined to $y(n_s)$.

(3) r_i is joined to r_j.

In case (1), let r_i' be any further vertex of R. We have

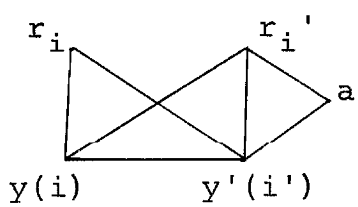


By condition (b), therefore, r_i' must be joined to $x(n_1)$, since p is not joined to y(i). Thus $x(n_1)$ is joined to R_i. Let r_i now be <u>any</u> vertex of R_i. By the choice of the x(k), there is a vertex a which is joined to r_i and to y(i) but not to $x(n_1)$. Let s be any further vertex of K(y(i)). We have



Since a is not joined to $x(n_1)$, r_i must be joined to s. Thus R_i is joined to K(y(i)).

For a further pair $R_{i'}$, $R_{j'}$, let $\{x'(k)\}$, $\{y'(k)\}$ be the associated independent sets. Suppose case (1) holds here also, and that K(y(i)) = K(y'(i')). Then y(i) is joined to $R_{i'}$. $y(i) \epsilon N_i$, and is one of a set of $\beta-1$ independent vertices in $N-N_j$. If $i' \neq j$, one of this set must lie in $N_{i'}$. If on the other hand i' = j, then y(i) is not joined both to $C_{i'}$ and to every other vertex which is joined to $C_{i'}$. Hence in any case there is a vertex a which is joined to y'(i') and to $R_{i'}$ but not to y(i). Thus for $r_i \epsilon R_i$, $r_{i'} \epsilon R_{i'}$ we have



Therefore r_i is joined to $r_{i'}$. If we now let R' be the union of those R_i for which case (1) holds for some R_j, it follows that $c(R' + N) = \beta-1$. But we have shown that for r, s $\epsilon R-R'$, r is joined to s. Therefore $c(G) \leqslant c(R'+N) + c(R-R') + \{p\}) = \beta-1 + 1 = \beta$, and the proof is complete.

It seems likely that condition (a) alone is enough to guarantee $c(G) = \beta(G)$, and also $\chi(C) = \gamma(C)$, but certainly much more powerful methods would be needed to prove this.

<u>Theorem 3.3</u>: Let G be a graph satisfying the following two conditions:

(a) Neither G nor G* contains an irreducible circuit of odd length > 3.

(b) No vertex of G is joined to 3 independent vertices of G. Then $c(G) = \beta(G)$, if $\beta(G)$ is finite.

We prove this theorem via a series of lemmas. We assume throughout that G is a minimal counterexample to Theorem 3.3.

<u>Lemma 3.3.1</u>: For any x in G the set of vertices joined to x is the sum of 2 cliques.

Proof: Let H be the subgraph formed by the set of vertices joined to x, and consider the graph H*. By condition (a), H* contains no irreducible circuit of odd length >3, and by condition (b), it contains no circuit of length 3. Therefore, by the Two-Color Theorem, H* is the sum of two colors, whence H is the sum of two cliques.

<u>Lemma 3.3.2</u>: For any r in R the set of vertices in N which are joined to r is a clique.

Proof: If r is joined to x_1 and x_2 in N and x_1 is not joined to x_2, then p, x_1, and x_2 are 3 independent vertices all joined to r, contradicting condition (b).

<u>Lemma 3.3.3</u>: If R_i, R_j, R_k are distinct, $r_i \epsilon R_i$, $r_j \epsilon R_j$, $r_k \epsilon R_k$, and r_i is joined to r_j, and r_j is joined to r_k, then either

(I) r_i is joined to r_k, or

(II) $\beta = 3$ and every vertex of N_j is joined either to N_i or to N_k.

Proof: If r_i is not joined to r_k, and x is any vertex of N_j, then x must be joined either to r_i or to r_k, since otherwise r_i, x, r_k would be 3 independent vertices all joined to r_j. But then every vertex of N_j is joined either to N_i or to N_k, whence $\beta(N_i + N_j + N_k) = 2$, and $\beta = 3$, by Lemma 2.1.

If a single vertex of R forms a strongly independent set, we call it <u>strong</u>; otherwise we call it <u>weak</u>. By Corollary 1 of Lemma 2.2, the strong vertices form a clique. We let S_i be the set of strong vertices in R_i, W_i the set of weak vertices in R_i, $S = \Sigma S_i$.

<u>Lemma 3.3.4</u>: Let W_i and W_j be distinct and non-empty. Then one of the following must hold:

(1) W_i is joined to W_j.

(2) There exists a vertex x_i in N_i which is joined to N_j but not to C_j, and such that every vertex of R_j not joined to x_i is joined to R_i; furthermore S_j is joined to R_i.

(3) There exists a vertex x_j in N_j which is joined to N_i but not to C_i, and such that every vertex of R_i not joined to x_j is joined to R_j; furthermore S_i is joined to R_j.

Proof: Let W_i and W_j be distinct and non-empty, and assume none of (1), (2), (3) holds. Choose $\beta -1$ independent vertices $y(k)$, $y(k) \epsilon N_k$, $k \neq j$, such that $y(i)$ is not joined to C_j; if $S_j \neq \emptyset$, then $y(k)$ may be chosen independent of any preassigned $s_j \epsilon S_j$. Clearly none of the $y(k)$ is joined to C_i (by Lemma 3.3.2). If necessary enlarge C_i to a new clique $\overline{C}_i$ by adding vertices of $N-N_i$ in such a way that no vertex of $N-\overline{C}_i$ is joined to $\overline{C}_i$. Choose $\beta-1$ independent vertices $x(k)$, $x(k) \epsilon N_k - \overline{C}_i$, $k \neq i$. If $S_i \neq \emptyset$, the $x(k)$ may be chosen independent of any preassigned $s_i \epsilon S_i$. Choose $w_i \epsilon W_i$, $w_j \epsilon W_j$ such that w_i is not joined to w_j. Since w_i is weak it is joined to some $x(n_1)$, which by Lemma 3.3.2 is unique, and we assume for now that $n_1 \neq j$. Then $y(n_1)$ exists and is different from $x(n_1)$, since w_i is joined to $x(n_1)$ but not to $y(n_1)$. w_i is also not joined to $x(k)$, $k \neq n_1$. Thus the set $\{y(n_1)\} \cup \{x(k),\ k \neq n_1\}$ is not an independent set (since w_i is weak), and $y(n_1)$ is joined to some $x(n_2)$, where $n_2 \neq n_1$. If $n_2 = j$, we stop. Otherwise $y(n_2)$ exists and is different from $x(n_2)$, since $y(n_1)$ is joined to $x(n_2)$ but not to $y(n_2)$. The set $\{y(n_1),\ y(n_2)\} \cup \{x(k),\ k \neq n_1,\ n_2\}$ is not an independent set, but $y(n_1)$

is joined to $x(n_1)$ and to $x(n_2)$ and therefore not to any $x(k)$, $k \neq n_1, n_2$. Thus $y(n_2)$ is joined to some $x(n_3)$, $n_3 \neq n_1, n_2$. Again if $n_3 = j$, we stop. Otherwise we continue, repeating the same argument, and obtain in this way a path

$$w_i, x(n_1), y(n_1), x(n_2), y(n_2), \ldots, x(n_{s-1}), y(n_{s-1}), x(n_s) = x(j).$$

Similarly we can construct a path

$$w_j, y(m_1), x(m_1), y(m_2), x(m_2), \ldots, y(m_{t-1}), x(m_{t-1}), y(m_t) = y(i),$$

where we assume $m_1 \neq i$. Suppose $m_1 = n_k$. Then $y(n_k)$ is joined to $x(n_k)$ and $x(n_{k+1})$ and also to $x(j)$. This is possible only if $n_{k+1} = j$, that is if $k = s-1$. But then

$$p, w_i, x(n_1), y(n_1), \ldots, x(n_{s-1}), y(n_{s-1}), w_j$$

form an irreducible circuit of odd length > 3, a contradiction. Thus $m_1 \neq n_k$. If $m_2 = n_k$, then $y(n_k)$ is joined to $x(n_k)$, $x(n_{k+1})$, and $x(m_1)$, which is impossible. Repeating this argument, we find that $m_h \neq n_k$ for all h, k.

Now $x(n_1)$ is joined to w_i and therefore to $y(i)$. If the s_i defined above exists, then $y(i)$ is joined to s_i, $x(n_1)$, and $x(m_{t-1})$, which is impossible, since s_i, $x(n_1)$, and $x(m_{t-1})$ are independent. Thus $S_i = S_j = \emptyset$. $x(n_1)$ is joined to $\overline{N}_i$ but not to $\overline{C}_i$. Hence there exists $r_i \varepsilon R_i$ such that $x(n_1)$ is not joined to r_i. Since $y(i)$ is joined to $x(n_1)$, $x(m_{t-1})$, and r_i, these cannot be independent, whence r_i is joined to $x(m_{t-1})$. But now p, $r_i, x(m_{t-1})$, $y(m_{t-1}), \ldots, x(m_1)$, $y(m_1)$, w_j form a circuit of odd length

> 3. This circuit cannot be irreducible, so r_i must be joined to w_j. But r_i is also joined to w_i and to $x(m_{t-1})$; and w_i, $x(m_{t-1})$, and w_j are independent.

Therefore the assumptions $n_1 \neq j$ and $m_1 \neq i$ cannot both be valid. Suppose $n_1 = j$. Then $x(j)$ is joined to w_i and therefore to $\overline{N}_i$. It is not joined to $\overline{C}_i$, so there exists $r_i \epsilon R_i$ such that $x(j)$ is not joined to r_i. In particular we may take $r_i = s_i$ if s_i exists. r_i is not joined to R_j, by assumption. Choose $r_j \epsilon R_j$ such that r_i is not joined to r_j. Not every vertex of $\overline{N}_i$ can be joined to r_j, for then every vertex of $\overline{N}_i$ would be joined to every vertex of $\overline{N}_j$, contrary to the fact that $\beta > 2$. Choose $y \epsilon \overline{N}_i$ such that y is not joined to r_j. Then p, r_i, y, $x(j)$, r_j form an irreducible circuit of length 5. The assumption $m_1 = i$ leads to an exactly similar contradiction.

Lemma 3.3.5: Assume (II) of Lemma 3.3.3 does not hold. Let R_i, R_j, R_k be distinct and non-empty, and let $S_j = S_k = \emptyset$. Then some two of R_i, R_j, R_k are joined.

Proof: Assume first that $S_i = \emptyset$, and that no two of W_i, W_j, W_k are joined. Then with suitable renumbering we have for some $w_i \epsilon W_i, w_j \epsilon W_j$, w_i joined to W_j and w_j joined to W_k, by the preceding lemma. But then w_i is joined to w_j, which is joined to W_k, so w_i is joined to W_k; and W_j is joined to w_i, which is joined to W_k, so W_j is joined to W_k, contrary to assumption.

We may therefore assume $S_i \neq \emptyset$. Suppose first that $W_i \neq \emptyset$, and that W_j is not joined to W_k. Then, with appropriate renumbering, there exists $w_j \epsilon W_j$ joined to W_k. If W_i is joined to W_j, then W_i is joined to w_j, which is joined to W_k, so W_i is joined to W_k; but then W_k, which is joined to W_i, which is joined to W_j, is joined to W_j. Similarly, if S_i is joined to W_j, then it is joined to W_k and W_k is joined to W_j. The remaining possibility is that there exists $w_j' \epsilon W_j$ joined to R_i. If W_i is joined to W_k, then w_j' is joined to W_i, which is joined to W_k, so w_j' is joined to W_k; but w_j' is also joined to R_i, so R_i is joined to W_k. Similarly, if S_i is joined to W_k, then w_j', which is joined to S_i, is joined to W_k and W_k is joined to R_i. Thus we may assume that there exists $w_k \epsilon W_k$ joined to R_i. But W_k is joined to w_j, so this implies that R_i is joined to w_j and therefore to W_k.

Finally, suppose $W_i = \emptyset$, W_j not joined to W_k. Then, with suitable renumbering, there exists $x \epsilon N_k$ joined to N_j but not to W_j, such that the vertices of W_j not joined to x are joined to W_k. Let W_j' be this set of vertices, and let $W_j'' = W_j - W_j'$. Let $w_j \epsilon W_j''$, $w_k \epsilon W_k$ be independent. Then any $s \epsilon S_i$ is joined either to w_j or to w_k. If it is joined to w_k, then it is joined to W_j' and therefore to W_k. Suppose on the other hand s is joined to w_j. w_j is joined to x, but x is not joined to s, for then x would be joined to N_i, and N_i would be joined to N_j. Therefore every $w_j' \epsilon W_j'$ is joined either to x or to s, but by definition no vertex of W_j' is joined to x. Thus s is joined to W_j' and therefore to W_k, whence S_i is joined to W_k. This completes the proof of the lemma.

We can now prove Theorem 3.3 under the additional assumption that case(II) of Lemma 3.3.3 does not hold. We know from Lemma 3.3.1 that R = A + B, where A and B are cliques. If A and B can be chosen so that $S \subseteq A$, we are finished, since then $c(G) \leqslant c(A) + c(G-A) = 1+\beta-1 = \beta$. Suppose there exist S_i and S_j distinct and non-empty. Since $\beta > 2$, we may assume that A intersects some further clique R_k. If $S_i + S_j \subseteq B$, then every strong vertex in A will be joined to B. Thus we may assume that A intersects S_j. Then every $s \epsilon S_i \cap B$ is joined to A. It follows that every strong vertex in B is joined to A. Therefore all S_i but one, say S_1, must be empty. But now by Lemma 3.3.5 some two of R_i, R_j, R_k are joined for every i, j, k distinct, whence R = A + B where A and B are cliques and $S \subseteq A$.

We have thus reduced Theorem 3.3 to the case where $\beta = 3$ and every vertex of N_2 is joined either to N_1 or to N_3. If any vertices of N_1 are joined to C_2, we add them to C_2, forming new cliques N_1', N_2', N_3. Every vertex of N_2' is still joined either to N_1' or to N_3. Write $N_2' = N_{21}' + N_{23}'$, where N_{21}' is joined to N_1' and N_{23}' is joined to N_3. Suppose some $x \epsilon N_1'$ is joined to N_{23}'. Then N_{23}' is joined to x, to R_2, and to N_3, where R_2 is non-empty by Lemma 3.3.4. No vertex of R_2 can be joined to N_3, since then N_2 would be joined to N_3. Hence x is joined to R_2. But this implies that x is joined to N_2', which is impossible by the construction of N_1' and N_2'. Therefore, by Lemma 3.3.4, either W_1 is joined to W_2 or S_2 is joined to R_2. Similarly, either W_2 is joined to W_3 or S_3 is joined to R_2. Finally, if any vertices of N_2 are joined to C_1, we add them to C_1; if any of the remaining vertices of N_2 are joined to C_3, we add them to C_3, forming new cliques $\overline{N}_1$, $\overline{N}_2$, $\overline{N}_3$. Suppose some vertex $x \in \overline{N}_1$ is joined to $\overline{N}_3$. x is joined also to R_1 and to $\overline{N}_{21}$. No vertex of $\overline{N}_{21}$ can be joined to $\overline{R}_1$, since it is already joined to $\overline{N}_1$. Therefore some vertex of R_1 is joined to $\overline{N}_3$, and $\overline{N}_3$ is joined to $\overline{N}_1$, which is impossible. Similarly, no vertex of $\overline{N}_3$ can be joined to $\overline{N}_1$. It follows that W_1 is joined to W_3. If W_1 is joined to W_2, it follows from Lemma 3.3 that either W_2 is joined to W_3, in which case we are finished, or every vertex of $\overline{N}_1$ is joined either to $\overline{N}_2$ or to $\overline{N}_3$. But no vertex of $\overline{N}_1$ is joined to $\overline{N}_3$. On the other hand not every vertex of $\overline{N}_1$ can be joined to $\overline{N}_2$. Similarly, if W_2 is joined to W_3, we find that W_1 is joined to W_2. The only remaining possibility is that S_1 is joined to R_2 and S_3 is joined to R_2. But then R is the sum of the two cliques $S_1+R_2+S_3$ and W_1+W_3, and the proof is complete.

SECTION 4

Let $A = (a_{ij})$ be a matrix of zeroes and ones. We take as the vertices of a graph G those ordered pairs (i,j) such that $a_{ij} = 1$. We join the distinct pairs (i_1,j_1), (i_2,j_2) by an edge if and only if $i_1 = i_2$ or $j_1 = j_2$. We note that the resulting graph G satisfies all the conditions of Theorems 3.1, 3.2, and 3.3:

For let (i_1,j_1), $(i_2,j_2),\ldots,(i_{2n+1}, j_{2n+1})$ form a circuit, and suppose no three consecutive vertices of this circuit form a clique. Since (i_1,j_1) is joined to (i_2,j_2) we may assume that $i_1 = i_2$. Since (i_3,j_3) is joined to (i_2,j_2) but not to (i_1,j_1), we have $j_2 = j_3$. Similarly, $i_3 = i_4$, $j_4 = j_5,\ldots,j_{2n} = j_{2n+1}$. But then $i_{2n+1} = i_1 = i_2$, a contradiction. Thus (a) of Theorem 3.2 is satisfied, as therefore is (a) of Theorem 3.1.

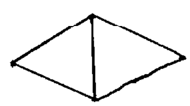
If (i_1,j_1), (i_2,j_2), and (i_3,j_3) form a clique, with say $i_1 = i_2$, then we must have $i_1 = i_2 = i_3$. If (i_2,j_2) (i_3,j_3), and (i_4,j_4) also form a clique, then $i_1=i_2=i_3=i_4$, and (i_1,j_1), (i_2,j_2), (i_3,j_3), and (i_4,j_4) form a clique. Thus condition (b) of Theorem 3.1 is satisfied, as therefore are condition (b) of Theorem 3.2 and condition (a) of Theorem 3.3.

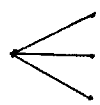
Finally, if (i_1,j_1) is joined to (i_2,j_2), (i_3,j_3) and (i_4,j_4), then we must have, say, $i_1 = i_2$, $i_1 = i_3$, whence $i_2 = i_3$. Thus condition (b) of Theorem 3.3 is satisfied.

Thus each of Theorems 3.1, 3.2, and 3.3 gives a different generalization of König's Theorem. We now prove a converse result.

Theorem 4: Let G be a graph containing none of the following subgraphs:

(a) An irreducible circuit of odd length > 3.

(b) 

(c) 

Then there exist sets A and B and a function $f(x) = (a(x), b(x))$, $a(x) \in A$, $b(x) \in B$, such that distinct vertices x and y in G are joined if and only if $a(x) = a(y)$ or $b(x) = b(y)$.

Proof: Let M(G) be the graph whose vertex set is the set of maximal cliques of G, and in which C_1 is joined to C_2 if and only if $C_1 \cap C_2 \neq \emptyset$. We show that M(G) contains no irreducible odd circuits.

Since the subgraph (b) does not occur in G, any two distinct maximal cliques intersect in at most one vertex. Suppose C_1, C_2, C_3 form a clique in M(G). If $C_1 \cap C_2 \cap C_3 \neq \emptyset$, let x belong to this intersection, and let $c_1 \in C_1 - \{x\}$,

$c_2 \epsilon C_2 - \{x\}$, $c_3 \epsilon C_3 - \{x\}$. Since subgraph (b) does not appear in G, c_1, c_2, c_3 are distinct and independent. But they are all joined to x, which is a contradiction. Thus we must have $C_1 \cap C_2 = \{x_{12}\}$, $C_1 \cap C_3 = \{x_{13}\}$, $C_2 \cap C_3 = \{x_{23}\}$, where x_{12}, x_{13}, x_{23} are distinct. Any further vertex a_1 of C_1 would be joined to x_{12} and to x_{13} and therefore to x_{23}. But then a_1 would be joined to all of C_3, which is impossible. Similarly C_2 contains only x_{12} and x_{23}, and C_3 contains only x_{13} and x_{23}. But then $C_1 \cup C_2 \cup C_3$ is a clique. It follows that M(G) contains no circuit of length 3.

Let now C_1, C_2, $C_3, \ldots, C_{2n+1}$ form an irreducible circuit, where $n > 1$, and let $x_i \epsilon C_i \cap C_{i+1}$. Then x_1, $x_2, \ldots,$ x_{2n+1} is a circuit of odd length in G. It cannot be irreducible, so we must have, say, x_2 joined to x_i, $i \neq 1,3$. x_1 cannot be joined to x_3, since it is not joined to all of C_3. x_i cannot be joined to x_1, since it is not joined to all of C_1. Similarly x_i cannot be joined to x_3. But then x_1, x_i, x_3 are independent vertices all joined to x_2, which is impossible, since subgraph (c) does not appear in G. It follows that M(G) contains no odd circuit, whence $M(G) = P + Q$, where P and Q are colors.

Let now $A = P \cup G$, $B = Q \cup G$, and define

$$a(x) = \begin{cases} C, & \text{if } x \epsilon C \epsilon P \\ x, & \text{if } x \text{ belongs to no } C \epsilon P \end{cases}$$

$$b(x) = \begin{cases} C, & \text{if } x \in C \in Q \\ x, & \text{if } x \text{ belongs to no } C \in Q \end{cases}$$

The function $f(x) = (a(x), b(x))$ is certainly one-one. If $a(x) = a(y)$ or $b(x) = b(y)$, and $x \neq y$, then x and y must belong to the same clique, whence x is joined to y. If on the other hand x is joined to y, then x and y belong to the same maximal clique, which is unique and must belong either to P or to Q, whence $a(x) = a(y)$ or $b(x) = b(y)$. This completes the proof.

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