STABILITY THEORY OF LINEAR AND NONLINEAR STOCHASTIC DIFFERENCE SYSTEMS

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ABSTRACT

In this report the stability of linear and nonlinear stochastic difference systems is considered. Explicit criteria for stability are derived. An algorithm is developed for computing the moments of linear stochastic systems when a certain Lie-algebraic condition is satisfied. The relationship between various stability definitions is explored.

CONTENTS

Chapter 1

INTRODUCTION

An accurate mathematical model of a dynamic system in electrical, mechanical, or control engineering often requires the consideration of stochastic elements. Physical systems with random parameters are usually modeled by stochastic differential equations, and an extensive study of such equations was made in the sixties. However, the advent of many modern day sampled data control systems has necessitated a study of stochastic difference equations. A stochastic difference system is one in which one or more variables can change stochastically at discrete instants of time. Stochastic difference systems are the stochastic versions of deterministic discrete time systems. The class of stochastic difference systems includes most modern industrial and military control systems, for they invariably include some elements whose inputs or outputs are discrete in time. Examples of such elements are digital computers, pulsed radar units, and coding units in most communication systems. One of the most important qualitative properties of stochastic difference systems is the stability of such systems. The purpose of this report is to present criteria for stability of linear and nonlinear stochastic difference systems. We also attempt to initiate a systematic study of the stability of stochastic difference systems.

1.1 PROBLEM STATEMENT

Consider a stochastic difference system whose behavior is described by the equation

$$\underline{x}_{k+1} = A_k(\omega)\underline{x}_k + \underline{f}(\underline{x}_k,k) \quad (1)$$

where k is a nonnegative integer, $\underline{x}_k \in \mathbb{R}^n$ for all k, and $A_k(\omega)$ is a $n \times n$ stochastic matrix. By this we mean that the elements of $A_k(\omega)$ are stochastic variables which can be continuous or discrete, with $\omega \in \Omega$, where Ω is the sample description space of the stochastic variables. We assume that $A_k(\omega)$ are independent stochastic matrices, whose distribution may depend on the state k. The nonlinear term $\underline{f}(\underline{x}_k,k)$ satisfies $f(0,k) = 0$ for all k. For simplicity, we shall consider only the stability of (1) about the double point $\underline{x} = 0$. For generality we assume that the initial state $\underline{x}_0$ is excited by a certain noise process

$$P(\omega) \ , \quad \omega \in \Omega_1 \quad (2)$$

which is statistically independent of the $A_k(\omega)$. The homogeneous system corresponding to (1) is the linear system

$$\underline{x}_{k+1} = A_k(\omega)\underline{x}_k \quad (3)$$

with the nonlinear term $\underline{f}(\underline{x}_k,k)$ omitted. For the class of systems (1) the following stability definitions are discussed in the present report.

Definition 1

The equilibrium solution of the stochastic difference system (1) is said to be stable in p^{th} moments if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|E(x_1^{p_1}(k)\, x_2^{p_2}(k) \cdots x_n^{p_n}(k)| < \varepsilon \quad \text{whenever} \quad \|\underline{x}_0\| < \delta$$

for all k and for any set of nonnegative integers $p_1,p_2,\cdots,p_n$ such that

$p_1 + p_2 + \cdots + p_n = p$, where $\underline{x}_k^T = (x_1(k), x_2(k), \cdots, x_n(k))$. The system (1) is asymptotically stable in p^{th} moments if moreover

$$\lim_{k\to\infty} E(x_1^{p_1}(k)\, x_2^{p_2}(k) \cdots x_n^{p_n}(k) = 0$$

Definition 2

The equilibrium solution of (1) is stable in probability if given $\varepsilon, \varepsilon' > 0$, there exists $\delta > 0$ such that $\|x_o\| < \delta$ implies

$$P(\|\underline{x}_k\| > \varepsilon') < \varepsilon$$

for all k. The equilibrium solution of (1) is asymptotically stable in probability if it is stable in probability and if there exists $\delta' > 0$ such that $\|\underline{x}_o\| < \delta'$ implies

$$\lim_{k\to\infty} P(\|\underline{x}_k\| > \varepsilon) = 0$$

Definition 3

The stochastic system (1) is p^{th} mean stable for $p > 0$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$E(\|\underline{x}_k\|^p) < \varepsilon \quad \text{whenever} \quad \|\underline{x}_o\| < \delta$$

for all k. The system (1) is p^{th} mean asyptotically state if moreover

$$\lim_{k\to\infty} E(\|\underline{x}_k\|^p) = 0$$

It is clear from the definitions that p^{th} moment asymptotic stability expresses a convergence to zero property of the state vector only if p is an even integer. Notice that while we investigate the p^{th} mean stability of a system for any positive real p, we generally consider moment stability of integral orders only. We shall show that the definitions that we adopted constitute a logical development of stability theory by demonstrating that they lead to a class of well structured properties [1].

1.2 MATHEMATICAL PRELIMINARIES

In this section we shall investigate the relationship between various stability definitions. Properties peculiar to linear systems will be discussed in the next chapter. The following two lemmas yield some qualitative information on the relationship between mean stability and moment stability.

Lemma 1.

The p^{th} mean (asymptotic) stability is, for any p, at least as strong as p^{th} moment (asymptotic) stability. For even integers p, mean (asymptotic) stability and moment (asymptotic) stability are equivalent.

Proof.

For every $p > 0$ and any norm

$$|E(x_1^{p_1}(k)\, x_2^{p_2}(k) \cdots x_n^{p_n}(k))| \le E(|x_1^{p_1}(k)\, x_2^{p_2}(k) \cdots x_n^{p_n}(k)|) \le E(\|\underline{x}_k\|^p) \tag{4}$$

for $p_1 + \cdots + p_n = p$, proving the first part of Lemma 1.

Let $p = 2r$ be an even integer and $\|\underline{x}\|_2$ be the Euclidean norm of the vector

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{5}$$

defined by

$$\|\underline{x}\|_2 = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2} \tag{6}$$

Then

$$E(\|\underline{x}_k\|^p) = E[(x_1^2(k) + x_2^2(k) + \cdots + x_n^2(k))^r]$$

$$= \sum_{r_1+r_2+\cdots+r_n=r} \frac{r!}{r_1!r_2!\cdots r_n!}$$

$$\times E(x_1^{2r_1}(k)\, x_2^{2r_2}(k)\cdots x_n^{2r_n}(k)) \tag{7}$$

by using the multinomial formula. The second part of Lemma 1 is thus established for Euclidean norm. But any two norms in a finite dimensional linear space are equivalent [2], hence (7) holds for any norm and the proof is complete.

Lemma 2.

The p^{th} absolute moment (asymptotic) stability of the absolute moments

$$E(|x_1^{p_1}(k)\, x_2^{p_2}(k)\cdots x_n^{p_n}(k)|), \quad p_1 + p_2 + \cdots + p_n = p$$

is equivalent to p^{th} mean (asymptotic) stability for any integer p .

Proof.

Let

$$\|\underline{x}\|_1 = \sum_{i=1}^{n} |x_i| \tag{8}$$

be the absolute sum norm of the vector (5). Then

$$E(\|\underline{x}_k\|_1^p) = E((\sum_{i=1}^{n} |x_i|)^p)$$

$$= \sum_{p_1+p_2+\cdots+p_n=p} \frac{p!}{p_1!p_2!\cdots p_n!}$$

$$\times E(|x_1^{p_1}(k)\, x_2^{p_2}(k) \cdots x_n^{p_n}(k)|) \qquad (9)$$

Lemma 2 is thus true for the absolute sum norm. The general case follows from the fact that any two norms in a finite dimensional linear space are equivalent.

The following lemma relates mean stability of different orders.

Lemma 3.

If a stochastic difference system is p_1th mean (asymptotically) stable, then it is p_2th mean (asymptotically) stable for any p_2 satisfying $0 < p_2 \leq p_1$

Proof.

We need only consider $0 < p_2 < p_1$. Let

$$r = \frac{p_1}{p_2} > 1$$

Applying Hölder's inequality on a space of normalized unit measure [2]

$$E(\|\underline{x}_k\|^{p_2}) \leq E(\|\underline{x}_k\|^{p_2 r})^{\frac{1}{r}}\, E(1^{\frac{r}{r-1}})^{1-\frac{1}{r}} = E(\|\underline{x}_k\|^{p_1})^{p_2/p_1} \qquad (10)$$

Hence p_1th mean (asymptotic) stability implies p_2th mean (asymptotic) stability.

The following theorem shows that mean stability is much stronger than stability in probability.

Theorem 1

If the stochastic difference system (1) is p^{th} mean (asymptotically) stable for any $p > 0$, then it is (asymptotically) stable in probability.

Proof.

We first establish the following inequality

$$P(\|\underline{x}_n\| \geq \varepsilon) \leq \frac{1}{\varepsilon^p} E(\|\underline{x}_n\|^p) \tag{11}$$

for all $p, \varepsilon > 0$. Let $f(x)$ be the probability density of $\|\underline{x}_n\|^p$. Using the fact that $\|\underline{x}_n\| \geq 0$,

$$\begin{aligned} E(\|\underline{x}_n\|^p) &= \int_0^\infty x\, f(x)\, dx \\ &\geq \int_\alpha^\infty x\, f(x)\, dx \\ &\geq \alpha P(\|\underline{x}_n\|^p \geq \alpha) \end{aligned}$$

for each $\alpha > 0$. Choose $\alpha = \varepsilon^p$. It follows that

$$P(\|\underline{x}_n\| \geq \varepsilon) = P(\|\underline{x}_n\|^p \geq \varepsilon^p) \leq \frac{1}{\varepsilon^p} E(\|\underline{x}_n\|^p)$$

and (11) is established. Suppose ε, ε', are given. We first choose $r > 0$ satisfying

$$\left(\frac{1}{\varepsilon'}\right)^p r < \varepsilon$$

By mean stability there exists a $\delta > 0$ such that

$$E(\|\underline{x}_n\|^p) < r \quad \text{whenever} \quad \|\underline{x}_0\| < \delta$$

Hence,

$$\|\underline{x}_0\| < \delta$$

implies that

$$P(\|\underline{x}_n\| > \varepsilon') \leq P(\|\underline{x}_n\| \geq \frac{1}{2}\varepsilon')$$

$$\leq (\frac{2}{\varepsilon'})^p E(\|\underline{x}_n\|^p)$$

$$< \varepsilon$$

for all n. Thus system (1) is stable in probability. From (11), it is obvious that

$$\lim_{n\to\infty} P(\|\underline{x}_n\| > \varepsilon) = 0$$

if

$$\lim_{n\to\infty} E(\|\underline{x}_n\|^p) = 0$$

Hence asymptotic p^{th} mean stability for any $p > 0$ implies asymptotic stability in probability.

By our previous discussion, we have the following relationship between moment stability and stability in probability.

Corollary to Theorem 1

If the stochastic difference system (1) is (asymptotically) stable in even order moments, then it is (asymptotically) stable in probability. In particular, stability in second moments implies stability in probability.

The results that we have established should, with an appropriate choice of definitions and modifications in the proofs, apply in the continuous case to stochastic differential systems.

Chapter 2

LINEAR STOCHASTIC SYSTEMS

Since linear systems are easy to solve and study, the investigation of the linear part of a problem is often a first step, to be followed by the study of the relation between motions in a nonlinear system and in its linear model. In many nonlinear problems, linearization produces a satisfactory approximate solution, and several averaging methods are available for dealing with nonlinear stochastic problems.

2.1 MOMENT STABILITY CRITERIA

The homogeneous system (3) may be regarded as the linearized version of the nonlinear stochastic difference system (1). A solution of (3) satisfying

$$X_{n+1} = A_n X_n \qquad X_o = I \tag{12}$$

is called the fundamental solution. It is clear that the fundamental solution of (3) satisfies

$$X_n = A_{n-1}A_{n-2}\cdots A_2A_1A_0 \qquad X_m = A_{m-1}\cdots A_nX_n\,, \quad m > n \tag{13}$$

From (13), we immediately have the following result:

Lemma 4

The linear stochastic difference system (3) is p^{th} mean stable if and only if there exists a constant C such that

$$E(\|A_nA_{n-1}\cdots A_2A_1A_0\|^p) \leq C \tag{14}$$

for all n. It is p^{th} mean asymptotically stable if and only if

$$\lim_{n\to\infty} E(\|A_nA_{n-1}\cdots A_2A_1A_0\|^p) = 0$$

The system (3) is said to be strongly p^{th} mean stable [3,4] if there exists a constant C such that

$$E(\|A_{m-1}\cdots A_n\|^p) \leq C \tag{15}$$

for all m,n such that m > n. It is strongly p^{th} mean asymptotically stable [3,4] if

$$\lim_{m\to\infty} E(\|A_{m-1}\cdots A_n\|^p) = 0 \tag{16}$$

for all m,n such that m > n. Observe that since A_k are independent stochastic matrices, it follows from (13) that

$$E(\|X_m\|^p) \leq E(\|A_{m-1}\cdots A_n\|^p)\, E(\|X_n\|^p) \;, \; m > n \tag{17}$$

As we shall see in the next chapter, strong stability is a very useful concept in nonlinear analysis.

We have seen that mean stability implies stability of moments of the same order, and that stability of even order moments implies stability in probability. We would like to investigate moment stability of (3). Let us assume for the rest of this chapter that A_k are independent stochastic matrices with a common distribution. Let

$$\underline{y}_k = \begin{pmatrix} E(x_1^p(k)) \\ E(x_1^{p-1}(k)\, x_2(k)) \\ \vdots \\ E(x_1^{p_1}(k)\, x_2^{p_2}(k) \cdots x_n^{p_n}(k)) \\ \vdots \\ E(x_n^p(k)) \end{pmatrix} , \quad p_1 + p_2 + \cdots + p_n = p \qquad (18)$$

be a $\binom{n+p-1}{n-1}$ vector of the p^{th} moments of $\underline{x}_k$. Then

$$\underline{y}_{k+1} = E(A_{[p]})\, \underline{y}_k \qquad (19)$$

where $E(A_{[p]}) = E(A_{k[p]})$ for all k and $A_{k[p]}$ is the p^{th} Kronecker power [5] of A_k. If A_k has eigenvalues $\lambda_1(k)$, $\lambda_2(k)$, $\cdots$ $\lambda_n(k)$, then the eigenvalues of $A_{k[p]}$ are the $\binom{n+p-1}{n-1}$ values (counting multiplicities) given by

$$\lambda_1^{p_1}(k)\, \lambda_2^{p_2}(k) \cdots \lambda_n^{p_n}(k) \quad , \quad p_1 + p_2 + \cdots + p_n = p$$

Let us consider an arbitrarily fixed k and focus our attention on

$$E(\lambda_1^{p_1}(k)\, \lambda_2^{p_2}(k) \cdots \lambda_n^{p_n}(k)), \quad p_1 + p_2 + \cdots + p_n = p \qquad (20)$$

By considering (19), the following result is apparent.

Theorem 2

A necessary and sufficient condition for the stability of the p^{th} moments of the stochastic difference system (3) is that the quantities in (20) should be less than or equal to one in magnitude if $E(A_{k[p]})$ is nondefective (i.e., has a full complement of ordinary eigenvectors) and that the quantities in (20) should be strictly less than one if $E(A_{k[p]})$ is defective (i.e., does not have a full complement of ordinary eigenvectors). A necessary and sufficient condition for the asymptotic stability of the p^{th} moments of (3) is that the expressions in (20) should be less than one in magnitude.

To reduce the number of constraint conditions and to avoid the computation of the expectation of mixed eigenvalues in Theorem 2, we shall make use of the following result.

Lemma 5

Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be positive real numbers such that $\sum_{i=1}^{n} \alpha_i = 1$. If $f_1, f_2, \cdots, f_n$ are Lebesgue integrable functions, then

$$\left|\int f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_n^{\alpha_n}\right| \leq \prod_{i=1}^{n} \left(\int |f_i|\right)^{\alpha_i} \tag{21}$$

Proof

This is a generalized Hölder's inequality. See [2].

Theorem 3

The stochastic difference system (3) is asymptotically stable in p^{th} moments if for any arbitrarily fixed k

$$E(|\lambda_i(k)|^p) < 1$$

for all i.

Proof

A little manipulation with (21) shows that

$$\left|E(\lambda_1^{p_1}(k)\, \lambda_2^{p_2}(k) \cdots \lambda_n^{p_n}(k))\right| \leq \prod_{i=1}^{n} E(|\lambda_i(k)|^p)^{p_i/p} \tag{22}$$

for all $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = p$, proving the theorem.

2.2 COMPUTATION OF MOMENTS

Sometimes we may want to compute the exact p^{th} moments of (3). The problem of how the moments of (3) can be computed was first proposed by Bellman and Bertram [5,6]. A quadrature solution for this problem has not been discovered; however, we shall give an algorithm for computing the exact p^{th} moments of (3) when $L(A_k(\omega): \omega \in \Omega)$ is a solvable Lie algebra [7] for any arbitrarily fixed k. Our present discussion represents another attempt to bring Lie theory into the domain of Control Analysis [8]. A subspace L of the space of square matrices of order n constitutes a Lie algebra if for all A,B in L the commutator product $[A,B] = AB - BA$ belongs to L. We use the notation $L(A_1, A_2, \cdots, A_p)$ to denote the Lie algebra generated by the matrices $A_1, A_2, \cdots, A_p$. This is also the smallest Lie algebra containing $A_1, \cdots, A_p$. We associate with any Lie algebra L its derived series defined inductively as follows:

$$L^{(o)} = L$$

$$L^{(n+1)} = \{[S,T]: S,T \in L^{(n)}\} \quad , \qquad n \geq 0$$

L is solvable if $L^{(n)} = \{0\}$ for some n. The class of solvable Lie algebras includes the class of pairwise commuting matrices when $L^{(1)} = \{0\}$. An interesting property for solvability is expressed in the following.

<u>Lemma 6</u>

A matrix Lie algebra L is solvable if and only if there exists a nonsingular matrix S such that $S^{-1}AS$ is upper triangular for all $A \in L$.

<u>Proof</u>

See [7]. The important thing is that the proof presented there is a constructive one.

Since the A_k are random matrices with a common distribution, the Lie algebra generated by one of the matrices is the same as the one generated by all of them,

$$L(A_k(\omega): \omega \in \Omega) = L(A_o(\omega_o), A_1(\omega_1), A_2(\omega_2), \ldots \mid \omega_o, \omega_1, \omega_2 \ldots \in \Omega)$$

The assumption that $L(A_k(\omega): \omega \in \Omega)$ is solvable for an arbitrarily fixed k implies the existence of a constant matrix S such that $S^{-1}A_jS$ is upper triangular for all j. Let

$$\underline{x}_k = S\underline{z}_k \tag{23}$$

$$B_k = S^{-1}A_kS \tag{24}$$

Then (3) can be written as

$$\underline{z}_{k+1} = B_k\underline{z}_k \tag{25}$$

where B_k are upper triangular and have a common distribution. It follows that if $\underline{v}_k$ is the $\binom{n+p-1}{n-1}$ column vector of the p^{th} moments of $\underline{z}_k$, we have

$$\begin{aligned} \underline{v}_k &= E(B_{k-1[p]})\ \underline{v}_{k-1} \\ &= E(B_{k-1[p]})\ E(B_{k-2[p]}) \cdots E(B_{1[p]})\ \underline{v}_1 \\ &= \{E(B_{1[p]})\}^k\ \underline{v}_o \end{aligned}$$

and therefore

$$\underline{y}_k = S_{[p]}\{E(B_{1[p]})\}^k\ S^{-1}_{[p]}\ \underline{y}_o \tag{26}$$

where $\underline{y}_k$ is defined in (18). Since $E(B_{1[p]})$ is upper triangular, explicit expressions for the p^{th} moments of $\underline{x}_k$ in terms of the initial moments of $\underline{x}_0$ can be obtained by back-substitution. This algorithm can be easily adopted on computers by using an algebraic manipulator. In the 2 x 2 case where the algebra is more transparent, if

$$B_1 = \begin{pmatrix} \lambda_1 & a_{12} \\ 0 & \lambda_2 \end{pmatrix} \tag{27}$$

then

$$B_{1[p]} = \begin{pmatrix} \lambda_1^p & p\lambda_1^{p-1} a_{12} \cdots \\ & \lambda_1^{p-1} \lambda_2 \cdots \\ 0 & \ddots \end{pmatrix} \tag{28}$$

Let

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

be the triangularizing matrix, and

$$S = P_{11}P_{22} - P_{12}P_{21}$$

Then the following formulas for mean squares have been obtained [3]

$$E(x_1^2(k)) = A_1 E(x_1^2(0)) + A_2 E(x_2^2(0)) + A_3 E(x_1(0)x_2(0)) \tag{29}$$

$$E(x_2^2(k)) = B_1 E(x_1^2(0)) + B_2 E(x_2^2(0)) + B_3 E(x_1(0)x_2(0)) \tag{30}$$

$$E(x_1(k)x_2(k)) = C_1 E(x_1^2(0)) + C_2 E(x_2^2(0)) + C_3 E(x_1(0)x_2(0)) \tag{31}$$

where

$$A_1 = \frac{1}{S^2} \{P_{11}^2 P_{22}^2 \; E(\lambda_1^2)^k + P_{21}^2 \alpha - P_{21} P_{22} \beta\} \tag{32}$$

$$A_2 = \frac{1}{S^2} \{P_{11}^2 P_{12}^2 \; E(\lambda_1^2)^k + P_{11}^2 \alpha - P_{11} P_{12} \beta\} \tag{33}$$

$$A_3 = \frac{1}{S^2}\{-2P_{12}P_{22}P_{11}^2 \; E(\lambda_1^2)^k - 2P_{11}P_{21}\alpha + (P_{11}P_{22} + P_{12}P_{21})\beta\} \quad (34)$$

$$B_1 = \frac{1}{S^2}\{P_{21}^2 P_{22}^2 \; E(\lambda_1^2)^k + P_{21}^2\gamma - P_{21}P_{22}\delta\} \quad (35)$$

$$B_2 = \frac{1}{S^2}\{P_{12}^2 P_{21}^2 \; E(\lambda_1^2)^k + P_{11}^2\gamma - P_{11}P_{12}\delta\} \quad (36)$$

$$B_3 = \frac{1}{S^2}\{-2P_{12}P_{22}P_{21}^2 \; E(\lambda_1^2)^k - 2P_{11}P_{21}\gamma + (P_{11}P_{22} + P_{12}P_{21})\delta\} \quad (37)$$

$$C_1 = \frac{1}{S^2}\{P_{11}P_{21}P_{22}^2 \; E(\lambda_1^2)^k + P_{21}^2\varepsilon - 2P_{21}P_{22}\eta\} \quad (38)$$

$$C_2 = \frac{1}{S^2}\{P_{11}P_{21}P_{12}^2 \; E(\lambda_1^2)^k + P_{11}^2\varepsilon - P_{11}P_{12}\eta\} \quad (39)$$

$$C_3 = \frac{1}{S^2}\{2P_{11}P_{12}P_{21}P_{22} \; E(\lambda_1^2)^k - 2P_{11}P_{21}\varepsilon + (P_{11}P_{22} + P_{12}P_{21})\eta\} \quad (40)$$

$$\begin{aligned}\alpha = {} & P_{12}^2 E(\lambda_2^2)^k + 2P_{11}P_{12}E(a_{12}\lambda_2) \sum_{j=0}^{k-1} E(\lambda_1\lambda_2)^{k-1-j} \; E(\lambda_2^2)^j \\ & + P_{11}^2\{2E(\lambda_1 a_{12})E(a_{12}\lambda_2) \sum_{j=0}^{k-1} E(\lambda_1^2)^{k-1-j} \sum_{\gamma=0}^{j-1} E(\lambda_1\lambda_2)^{j-1-\gamma} E(\lambda_2^2)^\gamma \\ & + E(a_{12}^2) \sum_{j=0}^{k-1} E(\lambda_1^2)^{k-1-j} \; E(\lambda_2^2)^j\}\end{aligned} \quad (41)$$

$$\beta = 2P_{11}P_{12}E(\lambda_1\lambda_2)^k + 2P_{11}^2 E(\lambda_1 a_{12}) \sum_{j=0}^{k-1} E(\lambda_1^2)^{k-1-j} \; E(\lambda_1\lambda_2)^j \quad (42)$$

$$\begin{aligned}\gamma = {} & P_{22}^2 E(\lambda_2^2)^k + 2P_{21}P_{22}E(a_{12}\lambda_2) \sum_{j=0}^{k-1} E(\lambda_1\lambda_2)^{k-1-j} \; E(\lambda_2^2)^j \\ & + P_{21}^2\{2E(\lambda_1 a_{12})E(a_{12}\lambda_2) \sum_{j=0}^{k-1} E(\lambda_1^2)^{k-1-j} \sum_{\gamma=0}^{j-1} E(\lambda_1\lambda_2)^{j-1-\gamma} \; E(\lambda_2^2)^\gamma \\ & + E(a_{12}^2) \sum_{j=0}^{k-1} E(\lambda_1^2)^{k-1-j} \; E(\lambda_2^2)^j\}\end{aligned} \quad (43)$$

$$\delta = 2P_{21}P_{22}E(\lambda_1\lambda_2)^k + 2P_{21}^2 E(\lambda_1 a_{12}) \sum_{j=0}^{k-1} E(\lambda_1^2)^{k-1-j} \; E(\lambda_1\lambda_2)^j \quad (44)$$

$$\varepsilon = P_{12}P_{22}E(\lambda_2^2)^k + (P_{11}P_{22} + P_{12}P_{21})E(a_{12}\lambda_2) \sum_{j=0}^{k-1} E(\lambda_1\lambda_2)^{k-1-j}\ E(\lambda_2^2)^j$$
$$+ P_{11}P_{21}\{2E(\lambda_1 a_{12})E(a_{12}\lambda_2) \sum_{j=0}^{k-1} E(\lambda_1^2)^{k-1-j} \sum_{\gamma=0}^{j-1} E(\lambda_1\lambda_2)^{j-1-\gamma}E(\lambda_2^2)^{\gamma}$$
$$+ E(a_{12}^2) \sum_{j=0}^{k-1} E(\lambda_1^2)^{k-1-j}\ E(\lambda_2^2)^j\} \tag{45}$$

$$\eta = (P_{11}P_{22} + P_{12}P_{21})E(\lambda_1\lambda_2)^k + 2P_{11}P_{21}E(\lambda_1 a_{12}) \sum_{j=0}^{k-1} E(\lambda_1^2)^{k-1-j}E(\lambda_1\lambda_2)^j \tag{46}$$

Let us illustrate the previous discussion by an example.

Example 1

Consider the stochastic difference system (3) with $\Omega = \{\omega_1,\omega_2\}$ and

$$P(\omega_1) = P(\omega_2) = \frac{1}{2}$$

$$A_k(\omega_1) = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

$$A_k(\omega_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for all k. If λ_1,λ_2 are the eigenvalues of A_k, it is obvious that $|E(\lambda_i)| = 1$ and $|E(\lambda_i\lambda_j)| = 1$ for i,j = 1,2. In addition

$$E(A) = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$$

$$E(A_{[2]}) = \frac{1}{2}\begin{pmatrix} 5 & -4 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

are both defective. Hence by Theorem 2 the first and second moments are not stable. It follows that the stochastic system under consideration is not p^{th} mean stable for $p \geq 2$. Since $\{A_k\}$ generates a solvable Lie algebra, the moments can be readily calculated, with the triangularizing matrix given by

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Using (29)-(46), the following results are obtained

$$E(x_1(k)) = \frac{k+2}{2} E(x_1(0)) - \frac{k}{2} E(x_2(0)) \tag{47}$$

$$E(x_2(k)) = \frac{k}{2} E(x_1(0)) - \frac{k-2}{2} E(x_2(0)) \tag{48}$$

$$\begin{aligned} E(x_1^2(k)) &= \frac{1}{4} (k+1)(k+4)\ E(x_1^2(0)) \\ &+ \frac{1}{4} k(k+1)\ E(x_2^2(0)) \\ &- \frac{1}{2} k(k+3)\ E(x_1(0)\ x_2(0)) \end{aligned} \tag{49}$$

$$\begin{aligned} E(x_1(k)x_2(k)) &= \frac{1}{4} k(k+3)\ E(x_1^2(0)) \\ &+ \frac{1}{4} k(k-1)\ E(x_2^2(0)) \\ &- (k+2)(k-1)\ E(x_1(0)\ x_2(0)) \end{aligned} \tag{50}$$

$$\begin{aligned} E(x_2^2(k)) &= \frac{1}{4} k(k+1)\ E(x_1^2(0)) \\ &+ \frac{1}{4} (k^2-3k+4)\ E(x_2^2(0)) \\ &- \frac{1}{2} k(k-1)\ E(x_1(0)\ x_2(0)) \end{aligned} \tag{51}$$

Notice that the expectation of the initial state is taken with respect to $P(\omega)$ over Ω_1 in (2). It is easy to see that

$$E(\|\underline{x}_k\|) = O(k)$$

$$E(\|\underline{x}_k\|^2) = O(k^2)$$

For linear stochastic differential systems many investigators believe that the stability of moments of a certain order implies the stability of all lower order moments [9]. A discussion of the extent to which this is true will appear elsewhere. For stochastic difference systems, the stability of moments of a certain order need not imply the stability of lower order moments. However, it can be shown that the stability of moments of a certain even order implies the stability of all lower even order moments.

Example 2

Consider system (3) with sample space $\Omega = \{\omega_1, \omega_2\}$. Suppose that

$$P(\omega_1) = P(\omega_2) = \frac{1}{2}$$

$$A_k(\omega_1) = -A_k(\omega_2)$$

$$= \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$$

for all k. It is easy to check that

$$E(\lambda_1^2) = E(\lambda_2^2) = E(\lambda_1\lambda_2) = 4$$

$$E(\lambda_1^3) = E(\lambda_2^3) = E(\lambda_1^2\lambda_2) = E(\lambda_1\lambda_2^2) = 0$$

Hence although the third order moments are asymptotically stable, the second order moments grow without bound.

A discussion of the stability of some specific stochastic difference systems is contained in [10]. All results of this chapter can be extended to complex stochastic difference systems at the expense of increased mathematical complexity.

Chapter 3

STOCHASTIC STABILITY OF NONLINEAR SYSTEMS

All real physical systems are nonlinear. Linear behavior of a system can only be expected over a limited range. In order to describe the response of control systems with accuracy, a study of the stochastic stability of nonlinear systems is essential. The nonlinear system (1) can, with an appropriate choice of the nonlinear correction term $\underline{f}(\underline{x},n)$, serve as a realistic model for many real life sampled data control systems. We assume that the linearized system (3) corresponding to (1) is stable, and we would like to know how the nonlinear term $\underline{f}(\underline{x},n)$ can affect the stability of the system. A survey of the present discussion is contained in [11].

3.1 STABILITY IN FIRST MEAN

Since first mean stability is one of the most commonly examined stability definitions, we shall establish in this section criteria for the stability in first mean of system (1). The following auxiliary results will be needed.

Lemma 7

Let $\theta(i)$ and $\psi(i)$ be two nonnegative sequences and $\rho(i)$ be a positive sequence satisfying

$$\theta(n) \leq \rho(n)[C + \sum_{i=0}^{n-1} \psi(i)\, \theta(i)] \tag{52}$$

for some $C > 0$ and for $n \geq 0$. Then with the customary notation that $\prod_{i=m}^{n} \theta(i) = 1$ for $m > n$, the following inequalities hold

(a) $$\theta(n) \leq C\rho(n) \prod_{i=0}^{n-1} (1+\psi(i)) \tag{53}$$

for all n if $\rho(i)$ are bounded above by 1

(b) $$\theta(n) \leq C\left(\prod_{i=0}^{n} \rho(i)\right)\left(\prod_{i=0}^{n-1} (1+\psi(i))\right) \tag{54}$$

for all n if $\rho(i)$ are bounded below by 1.

<u>Proof</u>

If $0 < \rho(i) \leq 1$, then from (52)

$$\frac{\theta(n)}{\rho(n)} \leq C + \sum_{i=0}^{n-1} \psi(i)\theta(i) \leq C + \sum_{i=0}^{n-1} \psi(i) \frac{\theta(i)}{\rho(i)} \tag{55}$$

Hence

$$1 + \frac{\psi(n) \frac{\theta(n)}{\rho(n)}}{C + \sum_{i=0}^{n-1} \psi(i) \frac{\theta(i)}{\rho(i)}} \leq 1 + \psi(n)$$

giving

$$C + \sum_{i=0}^{n} \psi(i) \frac{\theta(i)}{\rho(i)} \leq (1+\psi(n))\left(C + \sum_{i=0}^{n-1} \psi(i) \frac{\theta(i)}{\rho(i)}\right)$$

By iteration

$$C + \sum_{i=0}^{n} \psi(i) \frac{\theta(i)}{\rho(i)} \leq \prod_{i=1}^{n} (1+\psi(i))\left(C+\psi(0) \frac{\theta(0)}{\rho(0)}\right)$$

But from (52)

$$\theta(0) \leq C\rho(0)$$

yielding

$$\theta(n) \leq \rho(n)(C + \sum_{i=0}^{n-1} \psi(i) \frac{\theta(i)}{\rho(i)}) \leq C\rho(n) \prod_{i=0}^{n-1} (1+\psi(i))$$

If $\rho(i) \geq 1$, then from (52)

$$\frac{\theta(n)}{\rho(n)(C + \sum_{i=0}^{n-1} \psi(i)\,\theta(i))} \leq 1$$

giving

$$\frac{C + \sum_{i=0}^{n} \psi(i)\,\theta(i)}{\rho(n)(C + \sum_{i=0}^{n-1} \psi(i)\,\theta(i))} \leq 1 + \frac{\psi(n)\,\theta(n)}{\rho(n)(C + \sum_{i=0}^{n-1} \psi(i)\,\theta(i))}$$

$$\leq 1 + \psi(n)$$

Hence

$$C + \sum_{i=0}^{n} \psi(i)\,\theta(i) \leq (1+\psi(n))\,\rho(n)(C + \sum_{i=0}^{n-1} \psi(i)\,\theta(i))$$

By iteration

$$C + \sum_{i=0}^{n} \psi(i)\,\theta(i) \leq (\prod_{i=1}^{n} (1+\psi(i)))(C+\psi(0)\,\theta(0))(\prod_{i=1}^{n} \rho(i))$$

But from (52)

$$\theta(0) \leq C\,\rho(0)$$

giving

$$C + \psi(0)\,\theta(0) \leq C(1 + \rho(0)\,\psi(0)) \leq C\,\rho(0)(1+\psi(0))$$

Thus

$$\theta(n) \leq \rho(n)(C + \sum_{i=0}^{n-1} \psi(i)\theta(i)) \leq C(\prod_{i=0}^{n-1} \rho(i))(\prod_{i=0}^{n-1} (1+\psi(i)))$$

The proof is therefore complete.

We consider Lemma 7 to be a generalization of the discrete form of the classical Bellman-Gronwall lemma [12].

Lemma 8

If $a_i \geq 0$ for all i, then the product

$$\prod_{i=1}^{n} (1 + a_i) \tag{56}$$

and the series

$$\sum_{i=1}^{n} a_i \tag{57}$$

converge or diverge together.

Proof

See [13].

The following assertion has been established [3].

Theorem 4

Given the stochastic difference system (1); if

(i) The corresponding homogeneous system (3) is strongly first mean asymptotically stable and the solutions approach zero sufficiently fast;

(ii) $\underline{f}(\underline{x},n)$ satisfies the nonlinearity condition: there exists a sufficiently small constant L such that

$$E(\|\underline{f}(\underline{x},n)\|) \leq LE(\|\underline{x}\|) \quad \text{for all } n \tag{58}$$

then the system (41) is first mean asymptotically stable.

Proof

The solution of (1) can be written as

$$\underline{x}_n = \prod_{j=0}^{n-1} A_j \underline{x}_o + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} A_j \right) \underline{f}(\underline{x}_i, i) \tag{59}$$

for all n. Recalling (16) and assuming that the decay is at least geometric in condition (i), so that

$$E(\|A_{m-1} \cdots A_n\|) \leq C\delta^{m-n}, \qquad m > n \tag{60}$$

where C is a constant and $0 < \delta < 1$. Without loss of generality, we assume that $C \geq 1$. By (59) and (60), we have

$$E(\|\underline{x}_n\|) \leq C\delta^n E(\|\underline{x}_o\|) + \sum_{i=0}^{n-1} C\delta^{n-i-1} L E(\|\underline{x}_i\|) \tag{61}$$

for all n. Notice that we have used the relations

$$E\left(\left\| \prod_{j=0}^{n-1} A_j \right\| \|\underline{x}_o\| \right) = E\left(\left\| \prod_{j=0}^{n-1} A_j \right\| \right) E(\|\underline{x}_o\|) \tag{62}$$

$$E\left(\left\| \prod_{j=i+1}^{n-1} A_j \right\| \|\underline{f}(\underline{x}_i, i)\| \right) = E\left(\left\| \prod_{j=i+1}^{n-1} A_j \right\| \right) E(\|\underline{f}(\underline{x}_i, i)\|) \tag{63}$$

since $P(\omega)$ and $A_0, A_1, A_2, \cdots, A_{n-1}$ are independent. Multiplying both sides of (61) by δ^{-n} and applying Lemma 7 with $\rho(i) \equiv 1$,

$$E(\|\underline{x}_n\|)\,\delta^{-n} \leq C E(\|\underline{x}_o\|) \prod_{i=0}^{n-1} \left(1 + \frac{LC}{\delta}\right) \quad \text{for all } n \tag{64}$$

yielding

$$E(\|\underline{x}_n\|) \leq C E(\|\underline{x}_o\|)(\delta + LC)^n \qquad \text{for all } n \tag{65}$$

If L is sufficiently small, then

$$\delta + LC < 1$$

and so

$$\lim_{n\to\infty} (\delta + LC)^n = 0$$

Hence

$$E(\|\underline{x}_n\|) \to 0 \quad \text{as} \quad n \to \infty$$

proving first mean asymptotic stability.

At this point we would like to interpret physically condition (ii) of Theorem 4. In most real-life sampled data control systems, the inherent nonlinearity is such that within the normal operating range the nonlinear term $\underline{f}(\underline{x},n)$ generally decreases in magnitude as $\underline{x}$ decreases in magnitude. This fact is reflected in condition (ii), which is typical cone condition. It means that the expected value of the norm of the nonlinear term must lie within a half cone whose apex angle depends on L.

Now we have shown that under a certain realistic assumption on the nonlinear term $\underline{f}(\underline{x},n)$ the asymptotic stability of (1) can be deduced from the strongly asymptotic stability of the corresponding linearized system (3). Suppose the homogeneous system (3) is only strongly p^{th} mean stable (15), we would like to know the kind of assumption that we can impose on $\underline{f}(\underline{x},n)$ to make the nonlinear system (1) p^{th} mean stable. We shall find that a variable cone condition is sufficient [3]. The following result has been obtained.

<u>Theorem 5</u>

Given the stochastic difference system (1); if

(i) The corresponding homogeneous system (3) is strongly first mean stable;

(ii) $\underline{f}(\underline{x},n)$ satisfies the following nonlinearity condition: for every n there exists a nonnegative number $B(n)$ such that

$$E(\|\underline{f}(\underline{x},n)\|) \leq B(n)\, E(\|\underline{x}\|) \tag{66}$$

then the system (1) is first mean stable if the series $\sum_{i=0}^{n} B(i)$ converges.

Proof

The solution of (1) is as given by (59)

$$\underline{x}_n = \prod_{j=0}^{n-1} A_j \underline{x}_o + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} A_j \right) \underline{f}(\underline{x}_i, i)$$

By (15), it follows that

$$E(\|\underline{x}_n\|) \leq CE(\|\underline{x}_o\|) + \sum_{i=0}^{n-1} CB(i)\, E(\|\underline{x}_i\|)$$

Applying Lemma 7

$$E(\|\underline{x}_n\|) \leq CE(\|\underline{x}_o\|) \prod_{i=0}^{n-1} (1 + CB(i)) \tag{67}$$

for all n. Since $\sum_{i=0}^{\infty} B(i) < \infty$, it follows by using Lemma 8 that the system (1) is first mean stable.

Example 3

Consider system (1) with $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and arbitrarily specified $P(\omega_i)$, $i = 1,2,3$. Let

$$A_k(\omega_1) = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{6} \end{pmatrix}$$

$$A_k(\omega_2) = \begin{pmatrix} -\frac{1}{9} & 0 \\ \frac{4}{5} & \frac{8}{9} \end{pmatrix}$$

$$A_k(\omega_3) = \begin{pmatrix} \frac{1}{8} & -\frac{1}{4} \\ \frac{2}{3} & \frac{1}{2} \end{pmatrix}$$

for all k

$$\underline{f}(\underline{x},n) = \frac{M}{2n^2} \begin{pmatrix} \frac{1}{2} x_1^9 e^{-x_1^4 - x_2^{16}} \\ x_1 + x_2 \end{pmatrix}, \ n \geq 1 \tag{68}$$

where M is a constant and

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{69}$$

Let

$$\|\underline{x}\| = |x_1| + |x_2| \tag{70}$$

be the simple absolute value norm. Then for the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

the natural norm induced by (70) is the maximum absolute column sum

$$\|A\| = \max_{k=1,2} (|a_{1k}| + |a_{2k}|) \tag{71}$$

Using these norms, it is easy to see that the linearized system in the present case is strongly first mean asymptotically stable and that (60) is satisfied. Moreover,

$$\|\underline{f}(\underline{x},n)\| \leq \frac{M}{2n^2} \left(|x_1| \, \left|\frac{1}{2} x_1^8 \, e^{-x_k^4}\right| + \|\underline{x}\| \right)$$

$$\leq \frac{M}{2n^2} \left(|x_1| + \|\underline{x}\| \right)$$

$$\leq \frac{M}{n^2} \|\underline{x}\|$$

where we have made use of the fact that for nonnegative real number x

$$\frac{1}{2} x^2 \, e^{-x} \leq 1$$

Thus $\underline{f}(\underline{x},n)$ satisfies the nonlinearity condition (58) when n is sufficiently large. If the nonlinearity condition holds when $n \geq N$, then by shifting index the vector $\underline{x}_N$ can be taken as the initial state. Hence the nonlinear system in this example is first mean asymptotically stable, as can be checked by direct calculations. It follows that the system is also stable in probability.

3.2 GENERAL CONSIDERATIONS

Naturally, we would like to extend the results of the previous section. Specifically, we would like to generalize Theorem 4 to arbitrary p. By Lemma 3 any such generalization is expected to introduce more restrictive assumptions. If a system is p^{th} mean stable, then evidently the greater the value of p, the more moderate the behavior of the samples of the system becomes, and at the same time the output process is squeezed more and more. The following generalization has been established.

Theorem 6

Given the stochastic difference system (1) and any $0 < p < \infty$; if

(i) the corresponding linearized system (3) is strongly p^{th} mean asymptotically stable, and the solutions converge in p^{th} mean to zero sufficiently fast;

(ii) $\underline{f}(\underline{x},n)$ satisfies the nonlinearity condition: there exist an integer N and a sufficiently small constant L such that

$$\begin{aligned} &E(\|\underline{f}(\underline{x},n)\|^p) \leq LE(\|\underline{x}\|^p) \quad , \quad n > N \\ &E(\|\underline{f}(\underline{x},n)\|^p) < \infty \quad , \quad 0 \leq n \leq N \end{aligned} \tag{72}$$

then the system (1) is p^{th} mean asymptotically stable.

Let us first explain the conditions of Theorem 6. In condition (i) we require that (16) be satisfied. To specify the rate of convergence to zero, it is required that the solutions of the strongly stable linearized system (3) satisfy

$$E(\|A_{m-1}\cdots A_n\|^p) \leq \alpha_{mn}(p)\ \delta^{m-n} \quad , \quad m > n \geq 0 \tag{73}$$

$$\alpha_{ij} = O(\frac{1}{i^p}) \quad , \quad i \geq 1 \quad , \quad i > j \geq 0 \tag{74}$$

where $0 < \delta < 1$. Since α_{ij} need only be defined for $0 \leq j < i$, one easy way to check (74) in practical calculations is to show that the double sequence α_{ij} can be dominated in the following way

$$\begin{aligned} \alpha_{ij} &\leq C\beta_i \\ \beta_i &= O(\frac{1}{i^p}) \end{aligned} \tag{75}$$

where C is a constant. Condition (ii) means that the p^{th} mean of the nonlinear term lies within a cone whose apex angle is determined by L.

Proof of Theorem 6

First assume that N = 0 in (72). The general solution of (1) can be written as

$$\underline{x}_n = B_o\underline{x}_o + \sum_{i=0}^{n-1} B_{i+1}\, \underline{f}(\underline{x}_i, i) \tag{76}$$

for all $n \geq 0$, and $B_i = \prod_{j=1}^{n-1} A_j$. Hence

$$||\underline{x}_n||^p \leq (||B_o\, \underline{x}_o|| + \sum_{i=0}^{n-1} ||B_{i+1}\, \underline{f}(\underline{x}_i, i)||)^p$$

$$\leq (n+1)^p (\max_{0\leq i\leq n-1}(||B_o\underline{x}_o||\, ,\, ||B_{i+1}\, \underline{f}(\underline{x}_i, i)||)^p$$

$$\leq (n+1)^p(||B_o||^p\, ||\underline{x}_o||^p + \sum_{i=0}^{n-1} ||B_{i+1}||^p\, ||\underline{f}(\underline{x}_i, i)||^p) \tag{77}$$

for all n. As previously explained,

$$E(||B_o||^p\, ||\underline{x}_o||^p) = E(||B_o||^p)\, E(||\underline{x}_o||^p) \tag{78}$$

$$E(||B_{i+1}||^p\, ||\underline{f}(\underline{x}_i, i)||^p) = E(||B_{i+1}||^p)\, E(||\underline{f}(\underline{x}_i, i)||^p) \tag{79}$$

Using (72), (75), (78), and (79)

$$E(||\underline{x}_n||^p) \leq (n+1)^p\{\alpha_{n,o}\delta^n E(||\underline{x}_o||^p) + \sum_{i=0}^{n-1} L\alpha_{n,i+1}\delta^{n-i-1} E(||\underline{x}_i||^p)\}$$

$$\leq C(n+1)^p\, \beta_n\{\delta^n E(||\underline{x}_o||^p) + \sum_{i=0}^{n-1} L\delta^{n-i-1}\, E(||\underline{x}_i||^p)\} \tag{80}$$

Now

$$\beta_n \leq M \frac{1}{n^p}, \quad n \geq 1$$

Without loss of generality, we can choose the constant M such that

$$M C \geq 1$$

It follows that

$$E(\|\underline{x}_n\|^p) \leq (1+\frac{1}{n})^p \{MC\delta^n E(\|\underline{x}_o\|^p) + \sum_{i=0}^{n-1} M C L \ \delta^{n-i-1} E(\|\underline{x}_i\|^p)\} \quad (81)$$

for $n \geq 1$. Define a sequence $\rho(i) \geq 1$ by

$$\rho(i) = \begin{cases} (1 + \frac{1}{i})^p & , \quad i \geq 1 \\ 1 & , \quad i = 0 \end{cases}$$

Using (81), we have

$$E(\|\underline{x}_n\|^p) \leq \rho(n) (M C \ \delta^n E(\|\underline{x}_o\|^p) + \sum_{i=0}^{n-1} M C L \, \delta^{n-i-1} E(\|x_i\|^p)) \quad (82)$$

for all $n \geq 0$. Multiplying both sides of (82) by δ^{-n} and applying Lemma 7,

$$E(\|\underline{x}_n\|^p) \ \delta^{-n} \leq MC \prod_{i=0}^{n} \rho(i) \ E(\|\underline{x}_o\|^p) \prod_{i=0}^{n-1} (1 + \frac{MCL}{\delta})$$

for all $n \geq 0$, or

$$E(\|\underline{x}_n\|^p) \leq MC \prod_{i=0}^{n} \rho(i) \ E(\|\underline{x}_o\|^p)(\delta + M C L)^n$$

for all $n \geq 0$. If L is sufficiently small, then

$$\delta + MCL = \gamma < 1$$

Thus

$$E(\|\underline{x}_n\|^p) \leq M C E(\|\underline{x}_o\|^p) \prod_{j=1}^{n} (1 + \frac{1}{j})^p \gamma^n \tag{83}$$

for all n. Although $\prod_{j=1}^{n} (1 + \frac{1}{j})^p$ is a divergent infinite product, the product $\prod_{j=1}^{n} [(1+\frac{1}{j})^p \gamma]$ converges to zero (or diverges to zero. For terminology, cf. [13]). To see this, choose $0 < \varepsilon < 1$ such that

$$\gamma < \varepsilon^p$$

There exist an integer N and $0 < t < 1$ such that $n > N$ implies

$$[(1 + \frac{1}{n})\varepsilon]^p < t < 1 \tag{84}$$

Thus

$$\prod_{j=1}^{n} [(1 + \frac{1}{j})^p \gamma] < \prod_{j=1}^{n} [(1 + \frac{1}{j})\varepsilon]^p$$

$$= \prod_{j=1}^{N} [(1 + \frac{1}{j})\varepsilon]^p \prod_{j=N+1}^{n} [(1 + \frac{1}{j})\varepsilon]^p$$

$$< t^{n-N} \prod_{j=1}^{N} [(1 + \frac{1}{j})\varepsilon]^p$$

implying

$$\lim_{n\to\infty} \prod_{j=1}^{n} [(1 + \frac{1}{j})^p \gamma] = 0$$

Hence

$$\lim_{n\to\infty} E(\|\underline{x}_n\|^p) = 0, \tag{85}$$

proving p^{th} mean asymptotic stability for $N = 0$. If $N \geq 1$, then by the assumption

$$E(\|\underline{f}(\underline{x},n)\|^p) < \infty$$

for $0 \leq n \leq N$, we can shift index and take $\underline{x}_N$ as the initial vector by using the transformation

$$\underline{y}_k = \underline{x}_{k+N}$$

$$\underline{B}_k = A_{k+N}$$

or we may apply the previous analysis directly to

$$\underline{x}_{n+N} = \prod_{j=N}^{n+N-1} A_j \underline{x}_N + \sum_{i=N}^{n+N-1} \left(\prod_{j=i+N+1}^{n+N-1} A_j \right) \underline{f}(\underline{x}_i, i) \ , \quad n \geq 0 \tag{86}$$

This completes the proof.

We have mentioned that the cone condition is a realistic condition. In our previous analysis we required the apex angle as determined by L to be small. This restriction should be removed if our results are to be of practical use. We shall establish that if, instead of a decay as described by (73,74), the following inequalities are satisfied:

$$E(\|A_{m-1} \cdots A_n\|^p) \leq \alpha_{m,n(p)}, \quad m > n \geq 0 \tag{87}$$

$$\alpha_{ij}(p) \leq \beta_i t_j \tag{88}$$

where β_i and t_j are positive sequences satisfying

$$\sum_j t_j < \infty$$
$$\beta_i = o\left(\frac{1}{(i+1)^p}\right) \tag{89}$$

then the apex angle of the cone as determined by L in condition (ii) can be arbitrarily enlarged and the vertex of the same cone can be freely translated along the horizontal axis. If $E(\|A_n\|^p) = O(1/n^p)$ and $p > 1$, then (87)-(89) obviously hold.

Theorem 7

Given the stochastic difference system (1) and any $0 < p < \infty$; if

(i) The corresponding homogeneous system (3) is strongly p^{th} mean asymptotically stable and the solutions converge in p^{th} mean to zero with a rate, for example, exceeding that in (87)-(89);

(ii) $\underline{f}(\underline{x},n)$ satisfies the nonlinearity condition: there exist an integer N and arbitrary constants L_1, L_2 such that

$$\begin{aligned} &E(\|\underline{f}(\underline{x},n)\|^p) \leq L_1 + L_2(E(\|\underline{x}\|^p)\ , \qquad n > N \\ &E(\|\underline{f}(\underline{x},n)\|^p) < \infty \qquad , \qquad 0 \leq n \leq N \end{aligned} \tag{90}$$

then the system (1) is p^{th} mean asymptotically stable.

Proof

We can assume without loss of generality that $N = 0$ in (90). It is easily seen that (77) holds in the present case

$$\|\underline{x}_n\|^p \leq (n+1)^p\{\|B_0\underline{x}_0\|^p + \sum_{i=0}^{n-1} \|B_{i+1}\|^p \ \|\underline{f}(\underline{x}_i,k)\|^p\}$$

for all $n \geq 0$. Using (87)-(90),

$$E(\|\underline{x}_n\|^p) \leq (n+1)^p\{\beta_n t_0 E(\|\underline{x}_0\|^p) + \sum_{i=0}^{n-1} L_1\beta_n t_{i+1} + \sum_{i=0}^{n-1} L_2\beta_n t_{i+1} E(\|\underline{x}_i\|^p)\} \tag{91}$$

Let

$$\sum_i t_{i+1} = \eta$$

By (89), we can choose a constant $K < 1$, satisfying

$$0 < K(n+1)^p \beta_n \leq 1 \quad , \quad K < t_0 \quad , \qquad n \geq 1$$

It follows from (91) that

$$E(\|\underline{x}_n\|^p) \leq K(n+1)^p\beta_n\{\frac{1}{K}t_oE(\|\underline{x}_o\|^p) + L_1\eta + \sum_{i=0}^{n-1}\frac{L_2t_{i+1}}{K}E(\|\underline{x}_i\|^p)\} \tag{92}$$

Define a sequence $0 < \rho(i) \leq 1$ by

$$\rho(i) = \begin{cases} K(i+1)^p\ \beta_i & , \quad i \geq 1 \\ 1 & , \quad i = 0 \end{cases}$$

Thus

$$E(\|\underline{x}_n\|^p) \leq \rho(n)\{\frac{t_o}{K}E(\|\underline{x}_o\|^p) + L_1\xi + \sum_{i=0}^{n-1}\frac{L_2t_{i+1}}{K}E(\|\underline{x}_i\|^p)\} \tag{93}$$

for all $n \geq 0$, where $\xi = \eta/K$. Applying Lemma 7

$$E(\|\underline{x}_n\|^p) \leq \rho(n)(\frac{t_o}{K}E(\|\underline{x}_o\|^p) + L_1\xi)\prod_{i=0}^{n-1}(1 + \frac{L_2t_{i+1}}{K}) \tag{94}$$

Since

$$\sum_i t_{i+1} = \eta < \infty$$

the infinite product

$$\prod_{i=0}^{\infty}(1 + \frac{L_2t_{i+1}}{K})$$

is bounded by Lemma 8. But

$$\lim_{n\to\infty}\rho(n) = 0$$

Hence

$$\lim_{n\to\infty}E(\|\underline{x}_n\|^p) = 0$$

proving p^{th} mean asymptotic stability.

It is surprising to find that (87)-(89) are strong enough to allow for a "projection vector" type or a "combination cone" nonlinearity condition in $\underline{f}(\underline{x},n)$. The following generalization of Theorem 7 is useful in practical calculations.

Theorem 8

The assertion of Theorem 7 holds if condition (ii) is replaced by (ii)^1 $\underline{f}(\underline{x},n)$ satifies the following nonlinearity condition: There exist an integer N,r real numbers $0 < p_1, p_2, \cdots, p_r \leq p$, and r+1 constants $L, L_1, L_2, \cdots, L_r$ such that

$$\begin{aligned} &E(\|\underline{f}(\underline{x},n)\|^p) \leq L + \sum_{i=1}^{r} L_i E(\|\underline{x}\|^{p_i}) \quad , \quad n > N \\ &E(\|\underline{f}(\underline{x},n)\|^p) < \infty \quad , \quad 0 \leq n \leq N \end{aligned} \tag{95}$$

Proof

As before, assume N = 0 in (95). For $0 < p_i \leq p$

$$\|\underline{x}\|^{p_i} \leq 1 + \|\underline{x}\|^p \quad , \quad 1 = 1,2,\cdots,r$$

Hence

$$E(\|\underline{f}(\underline{x},n)\|^p) \leq L + \sum_{i=1}^{r} L_i E(\|\underline{x}\|^{p_i}) \leq (L + \sum_{i=1}^{r} L_i) + (\sum_{i=1}^{r} L_i) E(\|\underline{x}\|^p) \tag{96}$$

and this is condition (ii) of Theorem 7. The result follows.

A generalization of Theorem 5 is the following statement.

Theorem 9

Given the stochastic difference system (1) and any $0 < \rho < \infty$; if

(i) The corresponding linearized system (3) is strongly p^{th} mean stable;

(ii) $\underline{f}(\underline{x},n)$ satisfies the following nonlinearity condition: for every n there exists a nonnegative number B(n) such that

$$E(\|\underline{f}(\underline{x},n)\|^p) \leq B(n)\ E(\|\underline{x}\|^p) \tag{97}$$

and B(n) can be dominated by

$$B(n) \leq \beta_n t_n \tag{98}$$

in such a way that the sequence $s(n) = n^p \sum_{i=0}^{n-1} \beta_i$ is bounded and the series $\sum_{i=0}^{n-1} t_i$ converges, then the system (1) is p^{th} mean stable.

Proof

The general solution of system (1) is given by (76)

$$\underline{x}_n = B_o \underline{x}_o + \sum_{i=0}^{n-1} B_{i+1}\ \underline{f}(\underline{x}_i, i)$$

for all n, with $B_i = \prod_{j=1}^{n-1} A_j$. Hence

$$\|\underline{x}_n\|^p \leqq \|B_o \underline{x}_o\|^p + n^p \sum_{i=0}^{n-1} \|B_{i+1}\ \underline{f}(\underline{x}_i, i)\|^p \tag{99}$$

for all $n \geq 0$. Using (15), (97), and (98)

$$\begin{aligned} E(\|\underline{x}_n\|^p) &\leq CE(\|\underline{x}_o\|^p) + Cn^p \sum_{i=0}^{n-1} B(i)\ E(\|\underline{x}_i\|^p) \\ &\leq CE(\|\underline{x}_o\|^p) + Cn^p \sum_{i=0}^{n-1} \beta_i t_i\ E(\|\underline{x}_i\|^p) \\ &\leq CE(\|\underline{x}_o\|^p) + Cn^p \sum_{i=0}^{n-1} \beta_i \sum_{i=0}^{n-1} t_i E(\|\underline{x}_i\|^p) \end{aligned} \tag{100}$$

Since the monotonic sequence

$$s(n) = n^p \sum_{i=0}^{n-1} \beta_i$$

is bounded, there exists a constant K such that

$$s(n) \leq K$$

for all n. Applying Lemma 7, with $\rho(i) \equiv 1$,

$$E(\|\underline{x}_n\|^p) \leq CE(\|\underline{x}_0\|^p) \prod_{i=0}^{n-1} (1 + CK t_i) \tag{101}$$

for all n. Since $\sum_{i=0}^{\infty} t_i < \infty$, it follows from Lemma 8 that the system is p^{th} mean stable. The proof is complete.

Example 4

Consider system (1) with $\Omega = \{\omega_1, \omega_2\}$ and arbitrarily specified $P(\omega_i)$, $i = 1,2$. Let

$$A_k(\omega_1) = \frac{1}{k^2}\begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{6} \end{pmatrix}$$

$$A_k(\omega_2) = \frac{1}{k^2}\begin{pmatrix} \frac{1}{8} & -\frac{1}{4} \\ \frac{2}{3} & \frac{1}{2} \end{pmatrix} \qquad \text{for all } k$$

$$\underline{f}(\underline{x},n) = \begin{pmatrix} A + C\, x_1^{1/2}\, x_2^{1/3} \\ \frac{1}{2} B\, x_1^9\, e^{-x_1^4 - x_2^{16}} \end{pmatrix} \tag{102}$$

where A,B,C are constants, $\underline{x} \in \mathbb{R}^2$ is defined by (69). Using the norms in (70), (71), it can be readily checked that the linearized system is strongly second mean asymptotically stable and that (87)-(89) are satisfied. Clearly

$$E(\|\underline{f}(\underline{x},n)\|^2) \leq E((|A| + |B|\,\|\underline{x}\| + |C|\ \ \|\underline{x}\|^{5/6})^2)$$

$$= A^2 + B^2\ E(\|\underline{x}\|^2) + C^2\ E(\|\underline{x}\|^{5/3})$$

$$+ 2|AB|\ E(\|\underline{x}\|) + 2|BC|\ E(\|\underline{x}\|^{11/6}) \qquad (103)$$

Hence by Theorem 8 the nonlinear system in this example is second mean asymptotically stable. It follows that the system is also p^{th} mean asymptotically stable for $p \leq 2$.

Criteria for the mean stability of some special systems can be found in [3].

Chapter 4

DIRECTIONS FOR FURTHER RESEARCH

In previous chapters the stability of linear and nonlinear stochastic difference systems has been investigated. Since the theory of difference equations has not been adequately developed, many problems on discrete systems still await solution. We shall discuss in this chapter a number of unsolved or partially solved problems on the stability of stochastic systems.

4.1 STABILITY OF STOCHASTIC TRANSFORMATIONS

Consider a stochastic point mapping of the form

$$\underline{x}_{k+1} = \underline{f}_k(\underline{x}_k) \quad , \qquad k \geq 0 \tag{104}$$

where the state vector $\underline{x}_k \in \mathbb{R}^n$ for all k, and $\{\underline{f}_k\}_{k=0}$ constitutes a sequence of independent stochastic transformations having a common underlying sample space Ω. By this we mean that for each $\omega \in \Omega$ and each k, $\underline{f}_k[\omega]$ is a deterministic recurrent operator defined on a domain in $\mathbb{R}^n$. We assume for generality that the initial state $\underline{x}_o$ is excited by a noise process $P(\omega)$, $\omega \in \Omega_1$, which is statistically independent of the $\underline{f}_k$. Equation (104) plays a prominent role in stochastic control, crystal lattice dynamics, pharmacokinetics, mathematical economics and biomathematics. It also represents the discrete version of a class of stochastic differential equations which are of great physical importance, and whose solution behavior is not at all well understood. The stochastic equation (104) was first considered by Bellman [14,15], where he derived the asymptotic behavior of $E(\underline{f}_k)$ for large k when $\underline{f}_k$ are analytic and have a common Bernoulli distribution, and when the Abel-Schröder functional relation can be applied to $\underline{f}_k$. The question is how the stability of (104) can be investigated in the general case.

An equilibrium point $\underline{a}$ of (104) is a double point [16] satisfying

$$\underline{a} = \underline{f}_k(\underline{a}) \qquad (105)$$

for all k and for all $\omega \in \Omega$. Let $\underline{a}$ be a double point of (104). Provided all second-order partial derivatives of $\underline{f}_k$ exist in a neighborhood of $\underline{a}$, equation (104) can be written as

$$\underline{x}_{k+1} = \underline{a} + A_k(\underline{x}_k - \underline{a}) + R_2(\underline{x}_k - a) \qquad (106)$$

where $A_k = \left.\frac{\partial \underline{f}_k}{\partial \underline{x}}\right|_{\underline{x}=\underline{a}}$ and $R_2(\underline{x}) = O(\underline{x}^2)$. The linear operator A_k is a $n \times n$ stochastic matrix, the elements of which are stochastic variables. From (106) it is obvious that we can always assume, by a translation of the coordinate system, that the origin is an equilibrium point. It can now be seen that all definitions and properties established in Chapter 1 apply to the stochastic transformation (104). We have recently investigated the stability [17] of the transformation (104). For the linear transformation the investigation is essentially equivalent to that presented in Chapter 2. For the nonlinear case we have established that the stochastic transformation (104) is p^{th} mean asymptotically stable if the corresponding linearized transformation is strongly p^{th} mean asymptotically stable and the solutions of which converge in p^{th} mean to zero sufficiently fast.

The control and stability analysis of the general implicit stochastic transformation

$$\underline{x}_{k+1} = \underline{f}_k(\underline{x}_k, \underline{x}_{k+1})$$
$$\underline{x}_k \in \mathbb{R}^n \tag{107}$$

remains an open problem. The nonautonomous stochastic transformation corresponding to (104) has not been fully investigated [17]. The general stability analysis of stochastic transformations possessing a more complicated form has not been attempted [15], although transformations with certain specific properties have been considered [18]. Before concluding this section we would like to mention the problem of studying the behavior of various functions of (104), such as the determinant or the characteristic roots of the linear transformation. For a discussion of this problem, see [19,20]. For some problems involving linear stochastic transformations arising from physics, see [21,22].

4.2 STOCHASTIC LIAPUNOV FUNCTIONS

The stability analysis of nonlinear stochastic transformations can often be facilitated by using discrete analogues of differential inequalities [17,23]. However, this method becomes involved for transformations assuming complicated structures. Naturally, we would like to know whether the direct method of Liapunov [24,25] can be used in (104) or (107). This gives rise to the idea of a stochastic Liapunov function. Let us consider the following linear stochastic transformation

$$\underline{x}_{n+1} = A(\omega)\ \underline{x}_n$$
$$\omega \in \Omega \tag{108}$$

Suppose there exists a positive definite stochastic matrix P, such that

$$A^T(\omega)\ P(\omega)\ A(\omega) - P(\omega) = -Q(\omega) \tag{109}$$

where $Q(\omega)$ is positive definite for every $\omega \in \Omega$. Then using the positive definite function

$$V_n = \underline{x}_n^T P\, \underline{x}_n \tag{110}$$

we have

$$\begin{aligned} V_{n+1} - V_n &= (A\underline{x}_n)^T P (A\underline{x}_n) - \underline{x}_n^T P\, \underline{x}_n \\ &= \underline{x}_n^T (A^T PA - P)\, \underline{x}_n < 0 \end{aligned}$$

whenever $\underline{x}_n \neq 0$. Hence

$$V_n \to 0 \quad \text{as} \quad n \to \infty$$

and the transformation is asymptotically stable at $\underline{x} = 0$ [12,26]. The function constructed in (110) is called a stochastic Liapunov function, with a probability structure induced by A. It is clear that the idea of employing stochastic Liapunov functions can be generalized. The problem has not been previously studied. Of obvious interest is the probability structure of stochastic Liapunov functions. We would also like to know how stochastic Liapunov functions can be constructed [27, 28]. A report on this investigation will appear elsewhere.

It should be noted that the term stochastic Liapunov function has been used in some recent works to mean a different thing [29,30], as we now explain. Given the Ito differential equation [31,32]

$$d\underline{x} = \underline{a}(t,\underline{x})dt + \underline{b}(t,\underline{x})d\underline{z} \tag{111}$$

where $\underline{x}$, $\underline{a} \in \mathbb{R}^n$, $\underline{b}$ is a $n \times n$ matrix, and $\underline{z}$ is a vector of independent Brownian processes, with $E(dz_i dz_j) = \delta_{ij} dt$, define

$$B_{ij}(t,\underline{x}) = \sum_{k=1}^{n} b_{ik}(t,\underline{x})\, b_{jk}(t,\underline{x}) \tag{112}$$

$$L = \sum_{i=1}^{n} a_i(t,\underline{x}) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} B_{ij}(t,\underline{x}) \frac{\partial^2}{\partial x_i \partial x_j} \tag{113}$$

Then a positive definite function $V(\underline{x})$ may be called a stochastic Liapunov function if $LV(\underline{x}) \leq 0$ in a certain region. Thus $V(\underline{x})$ defined in this way is a deterministic function, its study is part of the theory of partial differential equations [33,34]. An interesting question is the possibility of employing the stochastic Liapunov functions as defined by us to treat Ito differential equations.

4.3 DISCRETIZATION OF STOCHASTIC DIFFERENTIAL EQUATIONS

Various special methods have been devised for the study of the stability of stochastic differential equations [35,36]. We would like to have methods that can be applied to large classes of stochastic differential equations. For linear stochastic systems excited by additive or multiplicative colored noise, the corresponding differential equation may be discretized and the stability of the resulting stochastic difference equation investigated. It can be shown that if the point ∞ is an ordinary or a regular singular point, then the leading behaviors of solutions to the difference equation and its differential equation analogue are the same [37,38]. Hence the stability theory of difference equations as developed by us may contribute to the stability analysis of stochastic differential equations. For nonlinear systems, the previous method applies when there is a nonlinear transformation reducing either the differential or difference equation to linear form. Work along this line is in progress.

We would like to point out that in the literature a complicated stochastic differential equation $R\underline{x} = 0$ is usually studied by approximating it by a manageable simple equation $S\underline{x} = 0$ for which $\|R - S\|$ is small [24,39]. This approach is hampered by the lack of efficient systematic methods for constructing the mapping S. Our approach focuses on the correspondence relationships between differential and difference equations and can be easily applied in real calculations. However, our present approach requires modification when applied to Ito differential equation (111), for in this case the solutions of the difference equation obtained by discretization do not in general converge to the solutions of the differential equation [40,41]. This is because Ito calculus is a self-consistent theory and is not an extension or limit of ordinary calculus [40,42]. One approximate technique for the solution of the stochastic differential equation (111) is the iterative or the numerical solution of the associated Fokker-Planck equation [32,43]. This belongs to the theory of partial differential equations. For a discussion of various approximate methods, see [32].

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