OPTIMAL FEEDBACK CONTROL OF SPATIAL XENON OSCILLATIONS IN A NUCLEAR REACTOR

Thesis by

Donald Martin Wiberg

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ABSTRACT

A kinetic model of spatial processes in a nuclear power reactor is formulated according to the state space approach. The model is very general, and may include the spatial effects of control rods, temperature, and almost any other deterministic spatial process that can be described by a finite set of partial differential equations. It is noted that any locally unstable process may induce a spatial instability, and that a spatial instability will occur only if a local process is unstable. The concept of a temperature coefficient of reactivity is extended to include spatial variations. A linearization about an operating point is performed, and the resulting linear equations are solved by using a non-interacting modal expansion. Some properties of this type of mode and applications to other physical processes are discussed. It is shown that non-interacting modes exist whenever any other modal expansion exists. An efficient computational procedure for exact numerical solution for the non-interacting modes is given in the case of spatial separability. Simple linear stability estimates for spatial processes are formed by variational techniques which permit evaluation of the effects of spatially varying parameters upon stability. A theorem presents mathematical proof of the linear controllability of any finite number of modes by very few control rods. The circumstances under which an infinite number of modes can be controlled are discussed. Finally, methods of optimal feedback

control are used for the analytical design of a spatial control system for minimum integrated quadratic loss, and detailed examples are given.

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CHAPTER I

INTRODUCTION

The specific problem studied here is the oscillation of the power density in a nuclear reactor, such that a "hot spot" moves from one region to another and back again in the course of time. This is illustrated in Fig. I.1. The reactor kinetic processes will be investigated mathematically with the goal of the thesis in mind; namely, to develop a method of controlling these oscillations with a feedback system.

The oscillations are due in part to the fission product Xe^{135}. The mathematical formulation of the problem can be generalized to include other effects, in addition to the xenon process. However, for clarity and utility, only the xenon problem will be fully analyzed, with many extensions indicated in the development.

The xenon process involves the decay chain that starts with the fission of the reactor fuel and results in the production of Xe^{135}. This decay chain is illustrated in Fig. I.2 for the case in which the fuel is U^{235}.(1)

Thermal fission yields of this decay chain, for some different reactor fuels, are summarized in Table I.1.(2)

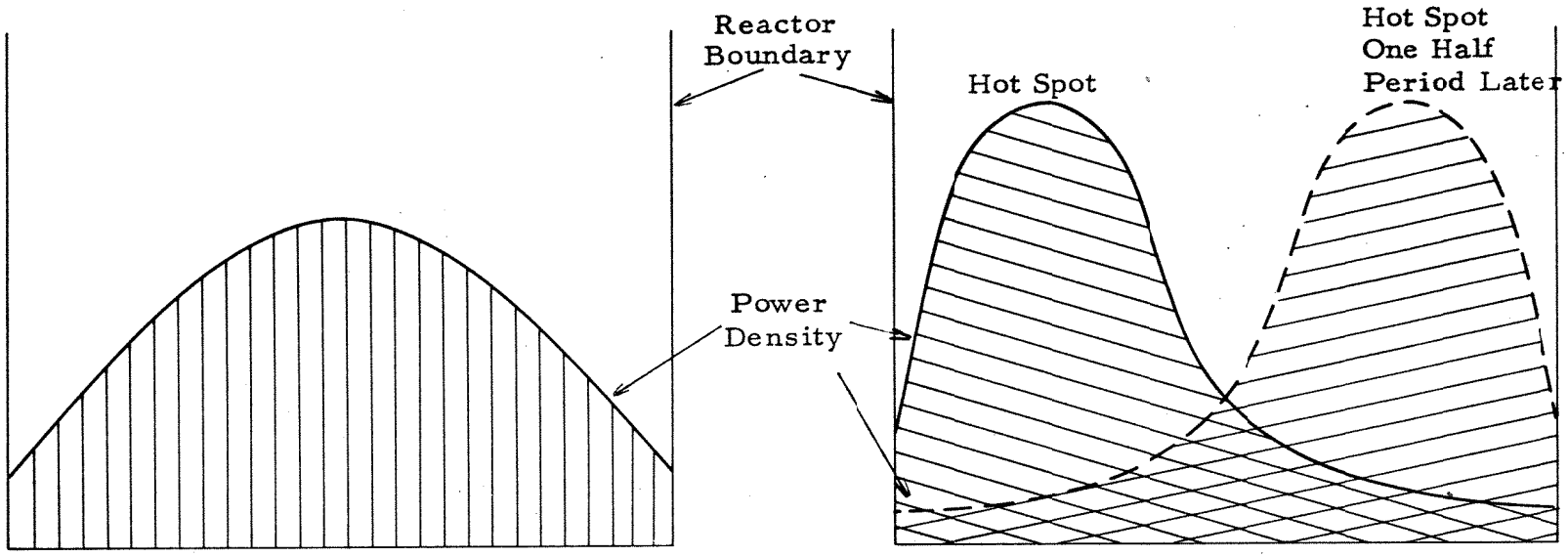


Fig. I. 1 Power Density in a Nuclear Slab Reactor

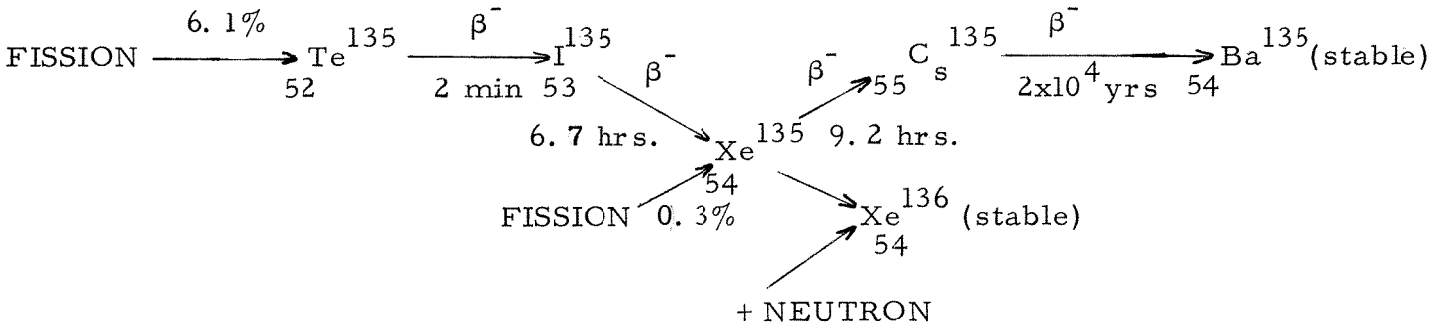


Fig. I.2 Xenon Decay Chain[1]

TABLE I. 1

Xenon Yields from Different Fuels[2]

Fissioning nuclide	U^{233}	U^{235}	Pu^{239}
Avg. yield of Te^{135} (atoms/fission)	0.051	0.061	0.055
Avg. yield of Xe^{135} (atoms/fission)	--	0.003	--

The Point Reactor Model

To examine the effect of this decay chain, first consider a point reactor model. A point reactor model represents the reactor as a single point in space by using average spatial values of the variables. Therefore, the average flux, the average xenon concentration, and the average iodine concentration are functions of time only in this case.

For a xenon unstable point reactor, an initial perturbation in the flux grows with time. Consider a reactor that has been operating at a steady state value of the flux for a long time. A steady state concentration of Xe^{135} is formed, because, if the flux is constant in time, then the rate of Xe^{135} born by decay from I^{135} and directly from fission is equal to the rate of Xe^{135} that is both decaying to Ce^{135} and transmuting to Xe^{136}. If a perturbation causes the flux to increase slightly above its steady state value, Xe^{135} will transmute to Xe^{136} at a faster rate. This means that the concentration of Xe^{135} will begin to decrease. The thermal neutron capture cross-section of Xe^{135} is as high as 3×10^6 barns, whereas the cross-section of Xe^{136} is less than five barns.[3] Therefore, the

composite absorption cross-section will decrease. Because the absorption will be less, the birth rate of neutrons will exceed the death rate. The cyclic nature of a nuclear reactor, therefore, will produce more neutrons, and these second generation neutrons will transmute even more Xe^{135}. In this manner, the initial flux perturbation will grow with time.

According to the literature,[4, 5, 6] a point reactor can become unstable only if the steady state value of the flux is higher than a certain threshold value. This threshold value depends on the parameters of the particular reactor. However, Schultz[5] indicates that, for U^{235} fueled reactors, "oscillations are not possible for fluxes below 3×10^{11} neutrons/cm^2-sec." Below this threshold, the stabilizing effect of the direct xenon yield from fission is more important than the destabilizing effect of the xenon decaying from iodine. Therefore, a high operating flux level is a necessary condition for a possible instability in a point reactor.

Spatial Effects

In a point reactor, the flux is controlled by the control rods. Even if the point reactor were unstable, the whole plant would be stable and capable of maintaining a steady state flux by adjustment of the rods because of the slow time constants involved in the xenon process. Thus, if a reactor can be considered as a point reactor, the instability due to xenon can be controlled by conventional techniques.

However, if the reactor has a spatial dimension many times the migration length, it may not be considered as a point reactor. The effects of a process occurring on one side of a large reactor may not influence processes on the other side to a great extent. A large reactor may be considered to be made up of a number of point reactors, coupled by the current of neutrons flowing between the points. Spatial xenon oscillations can occur only if the reactor is too large to be represented as a single spatial point.

To examine the spatial oscillation, the early paper (1957) of Henry and Germann[7] analyzed two coupled xenon unstable point reactors, separated by a moderating material. This is illustrated in Fig. I. 3.

The flux is proportional to the power density in the regions 1 and 3 because the fission cross-section is constant in each region. It is assumed that there is a control system keeping the total power of all three regions a constant, although the flux, or power, in an individual region is not constant.

A slight increase from equilibrium flux on one side gives rise to the unstable xenon process described for a point reactor. The control system keeps the total power level constant, forcing the flux level on the other side down. This induces the reverse process on the other side, where the flux level continually decreases. A steeper tilt in the flux occurs, being limited only by two effects. One limiting effect is the transmutation or burn out of most of the xenon on the

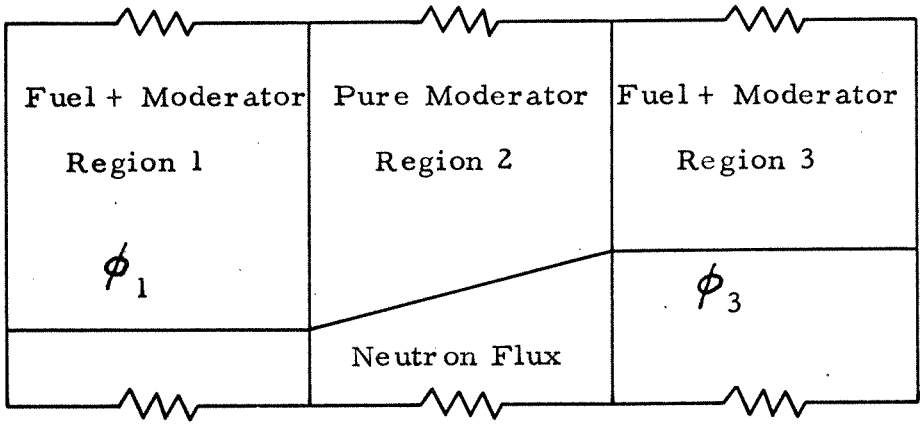


Fig. I. 3 Henry and Germann's Reactor Model

high flux side. The other limiting effect is a tilting steep enough to create a flux gradient in region 2 sufficient to support a current carrying all excess neutrons being produced on one side to compensate for the depletion on the other.

The flux remains in this tilted position for a number of hours. Then the high flux side will have created a concentration of I^{135} that is considerably greater than the original concentration. Since the decay constant of iodine to xenon is 6.7 hours, more xenon will be created, appearing after this delay period. Similarly, on the low flux side, less xenon will be created. This reverses the flux tilt eventually, and produces the side to side oscillation with a period of from fifteen to thirty hours. Thus the xenon process tends to be self limiting and produces the effect of a moving hot spot to the reactor operator.

The History of the Problem

The history of the spatial oscillations due to the fission product xenon is somewhat clouded by the security measures under which the design and operation of the early reactors were conducted. The problem was considered in the Savannah River reactor, and also in the PWR at Shippingport. The first unclassified paper on the subject appeared in 1956 by Ward,[8] under Canadian print. Ward described the problem, and performed an analysis to get a stability criterion.

The first U. S. papers were the one by Henry and Germann,[7] discussed on page 6, and one by Randall and St. John,[9] which was published about a year later.

Randall and St. John extended Ward's paper to include different reactor shapes, and showed that a simple stability criterion could be formed from the amount of material buckling that must be added.

These papers were of a theoretical nature. The first experimental report was that of Simpson and Rickover,[10] appearing in 1958, telling of the spatial oscillations of the PWR. They reported that the reactor operators could satisfactorily correct the oscillation by moving the shim rods.

These early studies triggered a relative torrent of papers.[11]-[44] As it was concluded that the oscillations should not be allowed to build up in the reactor, most of the authors studied the conditions under which oscillations might occur. In general, these conditions are large physical size and high flux levels, which now are in the range of present-day design specifications for power reactors that will be built in the very near future.

The Advantages of Feedback Control and Statement of the Problem

If a future reactor were to have a xenon oscillation, the heat generated in the hot spot could very possibly damage the reactor. Flux amplitudes as high as 170% of the steady state are possible.[43] Most designers presently try to design the reactor in such a manner that it is inherently stable. Usually, this is at the cost of some other design parameter, such as efficiency. This is the reason for a

general statement, appearing in Randall and St. John's paper:[9]

> "It would be more economical in almost any case to design for the detection and control of incipient oscillations than to design around them altogether."

Beckjord[22] and Fullmer[31] show that local control by the reactor operators can indeed damp out the oscillations by adjusting the shim rods, and Beckjord[13] experimentally verifies this. However, simply inserting a local rod into the hot spot may not be the most efficient way of damping the oscillation. This procedure has been likened to squeezing a balloon, in that if the hot spot is corrected in one place, it often pops up somewhere else. Furthermore, even slightly stable reactors can have a very bad transient response, such that it takes quite a while for the oscillation to die out. For these reasons, operator control alone may not be satisfactory, and study of automatic feedback control should be considered. Indeed, one can consider the operators as being a possible link in the feedback control.

It is the purpose of this thesis to study automatic feedback control of the spatial neutron distribution in a reactor. The problem is to find the feedback control laws governing a number of shim rods located spatially in the reactor, and to find the best number and positioning of these rods to give a satisfactory response so as to damp out xenon oscillations.

This proposed control system would have other beneficial effects. Firstly, such a control system would aid in controlling the long term changes in a reactor, e. g., making fuel burn out in a more uniform manner. Secondly, any spatial change in the flux due to a change in coolant flow would be smoothed out. Finally, any other spatially unstable process may be controlled also.

It should be noted that spatial oscillations may occur in any reactor in which any point, or local, unstable process is possible. There are other locally unstable processes, in addition to the xenon decay chain. Samarium-149 has a similar decay chain, but since its yield[2] and cross-section[3] are not as great as Xe^{135}, the effect is not quite as bad. The increase of fission cross-section of Pu^{239} with increasing temperatures in a certain range[3] give it a positive temperature coefficient. Many other processes can also give rise to a positive temperature coefficient, which leads to a local instability. A slight increase in flux will increase the temperature, which in turn increases the reactivity, thus giving rise to an instability. The analyses of these other locally unstable processes can be carried out in a manner similar to that used in analyzing the effect of xenon, and an increase in calculational complexity would be the only result of their inclusion in the presentation. Therefore, simply the xenon process will be considered in further analysis.

The advantages of the study of a method of controlling the spatial distribution of the flux have been enumerated. However, it must also be kept in mind that this is an initial study of the feedback

problem for a general reactor, and as such should not be taken as giving a cook book approach to the design of the system. Rather than evolving a step by step procedure for a specific design, the aim is to examine the properties of the mathematical description of a general reactor, and to find out the quantities of interest. Many simplifying assumptions are made, and where these assumptions are unjustified, the final form of the control laws will be affected.

In order to simplify the problem from the beginning, one major assumption is made. It will be assumed that the value of the neutron flux is known, by means of detection or perfect estimation, at all times and at all spatial points of the reactor. This assumption introduces errors of the order of magnitude of the estimation error in an imperfectly monitored reactor if the estimation is unbiased. Any results derived under the assumption of perfect estimation must be checked against the actual system used.

This assumption permits the analysis to be free of the actual devices used for detection. Any means of estimating flux in the reactor can be used, from spatially located neutron detectors to a highly complex calculation on a digital computer using inferential data. The results found from this analysis with perfect estimation will then show the best possible response, and the response using an imperfect estimator can be compared. The feedback system is only as good as the data it receives describing the controlled reactor.

To summarize, the problem may be stated in the following manner: given the flux distribution as a function of r and t, and a

mathematical description of a general reactor, find the error signals to be fed back to each one of N control rods located at points $r_1, r_2, \ldots r_N$, so as to give good response to transient spatial phenomena, and the xenon oscillation in particular.

CHAPTER II

THE REACTOR KINETIC STATE EQUATIONS

The Multigroup Flux Equations

Before attempting the feedback control problem, a mathematical investigation of the plant that is to be controlled must be undertaken. The plant in this case is a power reactor in which the fuel is stationary. Most generally, the neutron balance in such a reactor should be described by the Boltzmann, or transport theory, model. The neutron flux variable is not only dependent on space and time, but on neutron energy and direction of motion. However, in places where there are no sharp gradients in the flux, the Boltzmann equation may be adequately approximated by the multigroup diffusion equations, which depend only on space and time. For a derivation of the multigroup equations and their approximation to the Boltzmann equation, especially in the consideration of appropriate boundary conditions, see reference (45). The form of the equations may be illustrated here, for one fast and one slow group and one group of delayed neutrons:[29]

$$\begin{Bmatrix} \frac{1}{v_F} & 0 & 0 \\ 0 & \frac{1}{v_S} & 0 \\ 0 & 0 & 1 \end{Bmatrix} \frac{\partial}{\partial t} \begin{Bmatrix} \Phi_F \\ \Phi_S \\ C \end{Bmatrix} =$$

$$\begin{Bmatrix} \left[\nabla \cdot D_F \nabla - \left| \Sigma_{aF} + \Sigma_R \right| + \left| 1-\beta \right| \nu \Sigma_{fF} \right] \Phi_F + \left| 1-\beta \right| \nu \Sigma_{fS} \Phi_S + \lambda_c C \\ \Sigma_R \Phi_F + \left| \nabla \cdot D_S \nabla - \Sigma_{aS} \right| \Phi_S \\ \beta \nu \Sigma_{fF} \Phi_F + \beta \nu \Sigma_{fS} \Phi_S - \lambda_c C \end{Bmatrix} \quad \text{(II. 1)}$$

The boundary conditions are: Φ_F, Φ_S, $D_F \nabla \Phi_F$ and $D_S \nabla \Phi_S$ are continuous at all region interfaces, and Φ_F, Φ_S, and C vanish at the extrapolated boundaries. The functional dependencies of the coefficients will be treated in this chapter. For a discussion of the evaluation of the coefficients, see, for example, ref. (46).

Equation II. 1 defines as state variables the elements of the state vector, denoted as $\bar{Z}(r,t)$. Here, the state variables are $\Phi_F(r,t)$, $\Phi_S(r,t)$ and $C(r,t)$. Therefore, the dimension of the state vector for equation II. 1 is three, because there are three elements in the vector $\bar{Z}$. Knowledge of the three quantities for all r and t is necessary and sufficient to describe the reactor to the desired approximation. In addition to these three, more elements of the state vector can be included for additional flux groups and delayed neutron groups, and also for describing other processes (temperature, Sm concentration, etc.). Note that II. 1 is non-linear, because the coefficients depend on the state vector, in addition to r and t. The general form of II. 1, in matrix notation, is:

$$\mu(\bar{Z}, \bar{r}, t) \frac{\partial}{\partial t} \bar{Z}(\bar{r}, t) = \bar{\mathcal{f}}(\bar{Z}, \bar{r}, t), \qquad \text{(II. 2)}$$

Here the script $\bar{\mathcal{f}}$ is a vector operator, including operations in $\bar{r}$ such as ∇, but excluding such operations in time. This is the general formulation, since $\bar{Z}$ may be of any dimension. A rigorous discussion of the concept of a state variable is given in reference (47).

The state space formulation is given in a quite general form and additional phenomena may be included by increasing the dimension

of the state vector and adding more equations as rows of the matrix state equation. The analysis is presented this way so that the processes specifically considered here do not restrict the generality of the method of feedback control design. If the analytical treatment of any process does not suit the reader, the state equation may be altered and the method of design is still valid.

Furthermore, the generality of the state equation II. 2 is such that the equation is valid for a reactor with many regions. A "region" is a bounded volume of the reactor in which a coefficient of II. 2 suffers a step change on the interfaces. Inside a region, there are no discontinuities. As was mentioned, the variables and their currents are everywhere continuous in the reactor. Using this fact to form boundary conditions at the edge of each region, the state equation can be extended to cover reactors having a number of regions.

The Effect of Control Rods

By dividing the reactor up into regions, a description of the action of the control rods may be attempted. The control of the reactor described by II. 2 is accomplished by varying the positions of region boundaries. Thus, in a region of the reactor in which the control rods are present, the various cross-sections will be different from those in regions with no control rods. Movement of the control rods corresponds to movement of the region boundaries. The static case of multiregion description of the control rods has been done by H. L. Garabedian.[45, 49, 50] This technique is popularly called the

"window shade" method.

Unfortunately, the diffusion equations are not valid near a control rod. The large change in absorption cross-section creates a steep flux gradient, and then the diffusion equations are not as good an approximation of the Boltzmann equation, as mentioned previously.

A possible way of estimating the effect of the large change in the absorption cross-section is by patching up the multi-group equation by a change in the interface boundary conditions. Inonu[51] has had considerable success doing this in the calculation of non-escape probability for small reactors, and his "first-flight" approximation might very well apply to control rod boundaries also.

At the interface, $\bar{a}$, it is possible the region boundary conditions might be altered as in equations II. 3 and II. 4.

$$\Phi(\bar{a} -) = \alpha_1 \Phi(\bar{a}+) \tag{II. 3}$$

$$D(a -)\nabla \Phi (a -) = \alpha_2 D(a+)\nabla\Phi (a+) \tag{II. 4}$$

In these equations, the constants α_1 and α_2 could be found by experiment or by first flight considerations. Admittedly, the application to the description of a static control rod is mere speculation. However, there is a very close analogy between a highly absorbant control rod and the non-return of neutrons from a vacuum, the case discussed by Inonu.

By theory or by direct measurement, it is usually possible to find the effect of a static control rod. For small movements of the

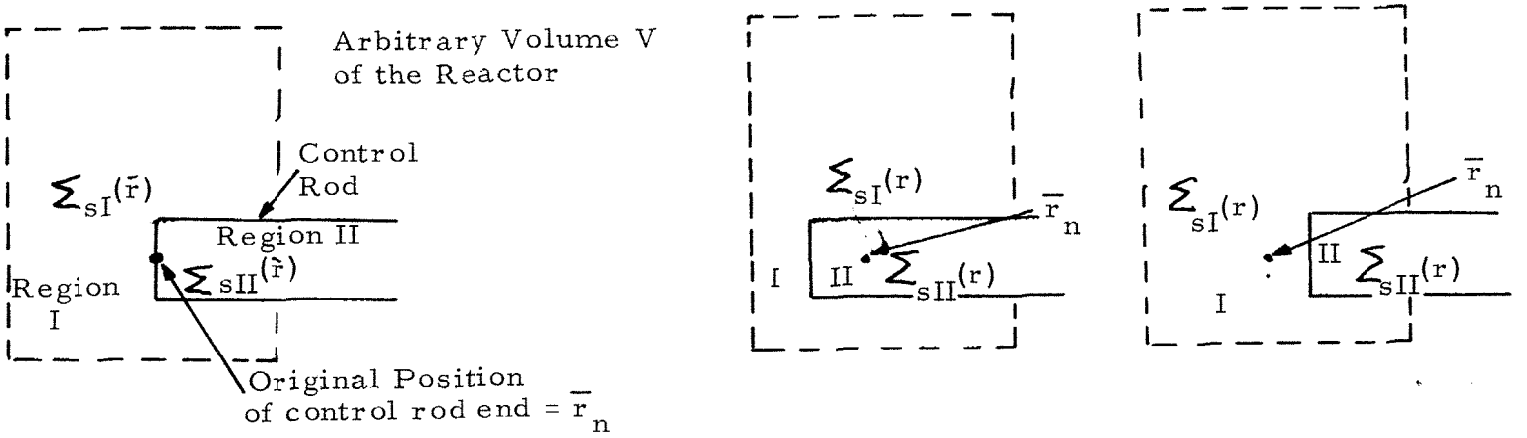
control rod, the effect of the discontinuity will remain essentially constant. It should then be possible to represent small static changes in the control rod positions by small changes in the absorption cross-sections, when the effect of the material discontinuity is known originally.

In light of these considerations, it is probable that a moving control rod may be expressed as a moving interface. Of course this is quite difficult to handle analytically, and further approximations must be made. The first approximation is that only Σ_{as} will be affected, because control rods have a large thermal absorption cross-section compared to the rest of the reactor, but hardly affect the other cross-sections. The second approximation is that, for small motions of a control rod, the thermal absorption cross-section may be represented as

$$\Sigma_{as}(\bar{r}, t) = \Sigma_{aso}(\bar{r}) + \sum_{n=1}^{N} u_n(t)\, \delta(\bar{r}-\bar{r}_n) \qquad \text{(II. 5)}$$

where $\Sigma_{aso}(\bar{r})$ is some mean value and $u_n(t)$ is the strength of a thermal absorption source or sink at the end, $\bar{r}_n$, of the n^{th} control rod.

Equation II. 5 may be justified by a perturbation analysis about the mean position of the considered motion of each control rod. At the unperturbed, or mean, static positions of the control rods, $\Sigma_{as}(\bar{r}, t)$ is equal to $\Sigma_{aso}(\bar{r}, t)$. This is illustrated in Fig. II. 1(a). Here, $\Sigma_{aso}(\bar{r}, t)$ is equal to Σ_{as1} in region 1 and Σ_{as2} in region 2.



(A)
Mean Position of the Rod

(B)
Rod Inserted Slightly

(C)
Rod Removed Slightly

Fig. II. 1 Illustration of the Control Rod Approximation

The cross-sections Σ_{as1} and Σ_{as2} may vary with position within their respective regions, but for clarity they will be considered constant since the extension to the non-constant case is obvious. Therefore, in the mean position, u_n is equal to zero.

When the control rod is displaced from its mean position, an absorption source or sink is created. If the rod is inserted slightly, a source is created and u_n is positive (see Fig. II. 1(b)). If the rod is removed slightly, a sink is created and u_n is negative (see Fig. II. 1(c)). Values of u_n may be calculated by the perturbation formulation II. 6[52] for $u_n << \Sigma_a V$.

$$\int_{V_1+V_2} \hat{\Phi}_o(\bar{r}) \left[\Sigma_{aso}(\bar{r}) + u_n \delta(\bar{r}-\bar{r}_n) \right] \Phi_o(\bar{r}) d^3r$$

$$= \int_{V_1} \hat{\Phi}_o(\bar{r}) \Sigma_{as1} \Phi_o(\bar{r}) d^3r + \int_{V_2} \hat{\Phi}_o(\bar{r}) \Sigma_{as2} \Phi_o(\bar{r}) d^3r \qquad \text{(II. 6)}$$

A more complex method of determining u_n may be used by extending Inonu's method to this case. However, in most practical cases experimental values probably should be used. Therefore, under the assumptions mentioned, equation II. 5 represents the effect of slightly moving the control rods in a reactor.

It has been shown that the value of $u_n(t)$ depends on the position of the control rod at time t. In general, u_n is not a linear function of control rod position. However, it should be noted that some sort

of gearing can make $u_n(t)$ almost any function of a control signal, and in particular $u_n(t)$ can be made a linear function of the control signal.

The Xenon Process

In addition to the control rods, the xenon process affects the time and space behavior of Σ_{as}. The thermal absorption cross-section of Xe^{135} is very large compared to that of the other members of the decay chain. Its other properties are similar, and therefore simply considering changes in Σ_{as} is a good approximation. If the control rods are at their mean position so that all $u_n \equiv 0$, then $\Sigma_{as} = \Sigma_{aso}(\bar{r})$. To calculate $\Sigma_{aso}(\bar{r})$, only the "clean" reactor is considered. $\Sigma_{aso}(\bar{r})$ lumps the thermal absorption cross-sections of all the other atomic species in the reactor except Xe^{135} and Sm^{149}.

The contribution of Xe^{135} to the absorption cross-section is readily found. Adding in the effect of the concentration of xenon atoms, N_x, the total thermal absorption cross-section of the reactor is given by II. 7 (with the control rods at their mean positions).

$$\Sigma_{as} = \Sigma_{aso}(r) + \sigma_{as}^{x} N_x(r, t) \qquad \text{(II. 7)}$$

This is the effect of xenon on the absorption cross-section, and can be related to the flux by inclusion in equation II. 1.

Equations for the effect of the flux on the xenon can also be written. The rate of change of xenon concentration is equal to the

rate of birth of xenon from iodine decay and from fission, less the rate of xenon decay to cesium and the rate of xenon burn off.

$$\frac{\partial N_x(\bar{r}, t)}{\partial t} = \lambda_I N_I(\bar{r}, t) + \gamma_x \Sigma_{fF}(r) \Phi_F(\bar{r}, t) + \gamma_x \Sigma_{fS}(r) \Phi_S(\bar{r}, t)$$

$$- \lambda_x N_x(\bar{r}, t) - \sigma_{as}^x N_x(\bar{r}, t) \Phi_S(\bar{r}, t) \qquad \text{(II. 8)}$$

The rate of change of iodine concentration is equal to the rate of birth from fission less the rate of decay to xenon.

$$\frac{\partial N_I(\bar{r}, t)}{\partial t} = \gamma_I \Sigma_{fF}(\bar{r}, t) \Phi_F(\bar{r}, t) + \gamma_I \Sigma_{fS}(\bar{r}, t) \Phi_S(r, t) - \lambda_I N_I(\bar{r}, t) \qquad \text{(II. 9)}$$

Equations II. 7, II. 8, and II. 9 describe the xenon process. Quite similar equations can be developed for samarium, also. Detailed derivation of both the xenon and samarium processes can be found in reference (1), pages 330-333.

It has been shown[53] that consideration of the two decay chains of Xe^{135} and Sm^{149} is a very good approximation to the total fission product poisoning. Hence, the effects of most of the important kinetic processes have been described, and the only other major process to be analyzed is the temperature variation.

The Effect of Temperature

The effect of temperature in reactor kinetics is exceedingly complex, and the effect may be quite different for different types of reactors. Not only may the individual reactor cooling systems be

different, but the effect on the reactor nuclear properties may also be of significance. A positive overall temperature coefficient of reactivity may induce a reactor instability, whereas a large negative overall coefficient may over-ride other destabilizing effects and insure a stable reactor.[55] Since the large reactors that are candidates for spatial control may be considered as a number of loosely coupled smaller reactors, each with its own effective temperature and fission product internal feedback, it is apparent that the spatial effects of temperature must be taken into account.

Very little work has been done on the kinetic spatial temperature distribution. Carnosa[69] found that the spatial effect on the fundamental is considerable. Guppy[40] found that the effect on the spatial transfer functions was large in the Calder Hall type of reactor. Since both of these references are recent, it seems that spatial temperature kinetic effects are of present research interest.

However, it is not the purpose of this thesis to attempt a general formulation of the kinetic temperature spatial equations. Any equations so found would be non-linear, and there is a possibility that they could not be linearized even over a small range. However, to indicate the effects of temperature variation on the properties, it will be assumed that each coefficient in the state equations can be expanded in a Taylor series about a mean temperature $T_o(r)$ such that the linear deviation term is sufficient to describe the temperature effect in a neighborhood of the mean temperature. For example, the macroscopic slow flux fission cross-section is assumed to be

expressible in the form of II. 10.

$$\Sigma_{fS}(\bar{r}, T) \approx \Sigma_{fS}(\bar{r}, T_o) + (T-T_o) \frac{\partial \Sigma_{fS}}{\partial T} (\bar{r}, T_o) \qquad \text{(II. 10)}$$

This is roughly equivalent to assuming a spatially varying temperature coefficient of reactivity, and reduces to this when Σ_a is uniform. Although the partial derivatives of these spatially varying coefficients have not been considered in the literature, it will be assumed here that they can be estimated. Therefore, to take into account the first order effects of temperature in the equations of other nuclear processes, the first two terms in the Taylor series expansion for each coefficient will be used in place of the temperature varying form of the coefficient.

This device will give an estimate of the effect of temperature on other nuclear processes. However, now there must be an equation for the temperature included in the development. Heat radiated from the reactor surfaces will be neglected, and just the useful heat will be considered, as given by Bonilla.[56] Also as stated there, the heating source will be assumed to follow the local flux density. All heat may be considered as removed by the cooling system. Since cooling systems are so different, only a very general relationship can be found. The reactor can be divided up into regions again, each of which contains a different material. If the material is solid, only heat conduction is possible. Therefore, in

solid bodies, the rate of change of internal energy is equal to the energy gained from fission less the heat conducted away.

$$\rho(\bar{r})\, c_T(\bar{r}) \frac{\partial T(r,t)}{\partial t} = \nabla \cdot \kappa(\bar{r}) \nabla T(\bar{r},t) + e \Sigma_{fF}(T,\bar{r}) \Phi_F(\bar{r},t) + e \Sigma_{fS}(T,\bar{r}) \Phi_S(\bar{r},t) \qquad \text{(II. 11)}$$

Equation II. 11 has boundary conditions similar to the neutron diffusion equations in that the temperature between solid bodies and the heat flux are both continuous at solid interfaces.

However, a problem arises when considering the coolant channels. The heat conduction is usually neglected within the fluid, and an average bulk fluid temperature is usually used to describe an average cross-sectional temperature. The fluid motion is also taken into account to get the coolant heat transfer equation. If x is the axial dimension down the coolant channel, then equation II. 12 is obeyed for the cross-sectional average temperature T(x, t).

$$\rho c_T \frac{\partial T(x,t)}{\partial t} = -\rho c_T v \frac{\partial T(x,t)}{\partial x} + h \left[T_{wall}(x,t) - T(x,t) \right] \qquad \text{(II. 12)}$$

In II. 11, $T_{wall}(x,t)$ is the wall temperature. The coolant inlet temperature gives the boundary condition T(0, t). Now the nature of the problem can be seen. The wall temperature varies with time as well as position. This gives rise to an inhomogeneous boundary condition at solid-fluid interfaces.

Approximations must be made, dependent upon the type of reactor. Guppy[40] assumes that the temperature distribution is

proportional to the spatial flux distribution. Another possible assumption is that the temperature deviation from the mean distribution, $T_o(r)$, is proportional to the change in coolant temperature deviation. The boundary condition would then be as expressed in II. 13.

$$-\mathcal{K}\frac{\partial \delta T(x,t)}{\partial \bar{n}}\bigg|_{wall} = h\left(\delta T_{wall}(x,t) - \delta T(x,t)\right) = h(1-a)\,\delta T_{wall}(x,t)$$

$$0 \leq a \leq 1 \qquad \text{(II. 13)}$$

In equation II. 12, a is the constant of proportionality. This is a homogeneous boundary condition. It will be assumed that the temperature distribution can be found as a function of time and space throughout the reactor to some satisfactory approximation by using these or other appropriate boundary conditions. It must be remembered that this procedure is merely to estimate the effect of temperature in the following analysis, and that each individual reactor design should be closely checked with experiment. Even if the temperature assumptions are violated, it may be of use to the control system designer to perform the feedback calculations omitting temperature effects. This would give a rough guide for the ultimate type of spatial control system that might be used. In the general case, equation II. 11 will be used to represent a temperature equation in the state equations of the system, and can be replaced by a more specific equation in any given case.

The Reactor Kinetic State Equation

Equation II. 1 gives the neutron distribution, equation II. 5 gives the effect of small control rod motions, equations II. 7, II. 8 and II. 9 describe the xenon process, and II. 11 gives the temperature in the reactor. Other time dependent processes occur in the reactor, but these are the most important when considering the spatial oscillations. One neglected process is fuel burn-up. This occurs over a matter of months, and may be considered to be constant over the period of oscillation. Other neglected processes can be included in the general formulation if necessary, but only those discussed will be specifically used in further examples. Therefore, equations II. 1, II. 5, II. 7, II. 8, II. 9, and II. 11 may be combined into a state space notation of the form II. 2. The combined equation, II. 14, is the state equation of a nuclear reactor, valid for small control rod motions and small temperature deviations.

Equation II. 14 does not take into account the statistical nature of reactor processes. However, Wigner[57] states:

> "That the results obtained on the basis of the transport equations are adequate seems to be a consequence not of the large number of neutrons which participate in any given process, but rather of the relatively large fluctuations of individual processes which cancel each other because some of these processes yield more, others less, neutrons than can be expected on the average."

Thus II. 14 is an equation for the averages of the state variables, and a truly descriptive equation would be an equation similar to II. 14, but whose coefficients would also be statistical functions of time.

$$
\begin{Bmatrix}
\frac{1}{v_F} + (T-T_o)\frac{\partial v_F^{-1}}{\partial T} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{v_S} + (T-T_o)\frac{\partial v_S^{-1}}{\partial T} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \rho(\bar{r})c(\bar{r})
\end{Bmatrix}
\frac{\partial}{\partial t}
\begin{Bmatrix}
\phi_F(\bar{r},t) \\ \phi_S(\bar{r},t) \\ C(\bar{r},t) \\ N_x(\bar{r},t) \\ N_I(\bar{r},t) \\ T(\bar{r},t)
\end{Bmatrix} =
$$

(II. 14)

$$
\begin{Bmatrix}
\left[\nabla\cdot(D_F(\bar{r})+(T-T_o)\frac{\partial D_F}{\partial T})\nabla - \Sigma_{aF}(\bar{r}) - (T-T_o)\frac{\partial \Sigma_{aF}}{\partial T} + (1-\beta)\nu(\Sigma_{fF}+(T-T_o)\frac{\partial \Sigma_{fF}}{\partial T})\right]\Phi_F(\bar{r},t) + (1-\beta)\nu(\Sigma_{fs}(\bar{r})+(T-T_o)\frac{\partial \Sigma_{fs}}{\partial T})\Phi_s(\bar{r},t) + \lambda_c C(\bar{r},t) \\
(\Sigma_R(\bar{r})+(T-T_o)\frac{\partial \Sigma_R}{\partial T})\Phi_F(\bar{r},t) + \left[\nabla\cdot(D_s(\bar{r})+(T-T_o)\frac{\partial D_s}{\partial T})\nabla - \Sigma_{aso}(\bar{r}) - (T-T_o)\frac{\partial \Sigma_{aso}}{\partial T} - \sigma_{as}^x N_x(\bar{r},t) - (T-T_o)\frac{\partial \sigma_{as}^x}{\partial T}N_x(\bar{r},t) - \sum_{n=1}^{N}(u_n(t)+(T-T_o)\frac{\partial u_n}{\partial T})\delta(\bar{r}-\bar{r}_n)\right]\Phi_s(\bar{r},t) \\
\beta\nu(\Sigma_{fF}(\bar{r})+(T-T_o)\frac{\partial \Sigma_{fF}}{\partial T})\Phi_F(\bar{r},t) + \beta\nu(\Sigma_{fs}(\bar{r})+(T-T_o)\frac{\partial \Sigma_{fs}}{\partial T})\Phi_s(\bar{r},t) - \lambda_c C(\bar{r},t) \\
\lambda_I N_I(\bar{r},t) + \gamma_x(\Sigma_{fF}(\bar{r})+(T-T_o)\frac{\partial \Sigma_{fF}}{\partial T})\Phi_F(\bar{r},t) + \gamma_x(\Sigma_{fs}(\bar{r})-(T-T_o)\frac{\partial \Sigma_{fs}}{\partial T})\Phi_s(\bar{r},t) - \lambda_x N_x(\bar{r},t) - (\sigma_{as}^x+(T-T_o)\frac{\partial \sigma_{as}^x}{\partial T})N_x(\bar{r},t)\Phi_s(\bar{r},t) \\
\gamma_I(\Sigma_{fF}(\bar{r})+(T-T_o)\frac{\partial \Sigma_{fF}}{\partial T})\Phi_F(\bar{r},t) + \gamma_I(\Sigma_{fs}(\bar{r})+(T-T_o)\frac{\partial \Sigma_{fs}}{\partial T})\Phi_s(\bar{r},t) - \lambda_I N_I(\bar{r},t) \\
e(\Sigma_{fF}(\bar{r})+(T-T_o)\frac{\partial \Sigma_{fF}}{\partial T})\Phi_F(\bar{r},t) + e(\Sigma_{fs}(\bar{r})+(T-T_o)\frac{\partial \Sigma_{fs}}{\partial T})\Phi_s(\bar{r},t) + \nabla\cdot\mathcal{K}(\bar{r})\nabla T(\bar{r},t)
\end{Bmatrix}
$$

However, for the most part, only the averages are the quantities of interest, and so in most cases equation II. 14 will be used to represent the complete reactor in the following development.

CHAPTER III

THE LINEARIZED STATE EQUATIONS

The state equation of the reactor has been described very generally by equation II. 14. This equation is so general and so complex that an attempt of the general solution is presently out of the question. However, it is possible to solve a linearized version of it, and it is the purpose of this chapter to derive this linearized equation.

The Steady State

To obtain the linearized equation, only small deviations of the state variables from some operating point will be considered. The deviations may be assumed small because it is the purpose of the control system to keep them small. Optimal performance will be required at the operating point considered. Satisfactory performance at all other operating points can usually be obtained by adjustment of the stability margin if necessary. The operating point to be considered will usually be some point near the maximum power output and at some mean time in the lifetime of the reactor, since power reactors are the prime candidates for spatial control. This operating point can be made a steady state solution of the equations II. 14 by using the corresponding position of the control rods to determine the steady state absorption, and also by specifying all coefficients to be evaluated at the conditions of the operating point.

It must be emphasized that all quantities of interest are to be

taken about the operating point, which is not the case in the usual zero power clean reactor calculations. Henceforth, it will be assumed that the control rod steady state position is defined to keep a reactor just critical in the presence of poisons, etc., at the desired running flux level. This implies that the "steady state" is the operating point about which are taken all the Taylor series expansions of the coefficients of the previous chapter. By defining the quantities and the steady state in this manner, it can be stated that the purpose of the spatial control system is to make all the deviations from the steady state equal to zero.

The preceding remarks permit definition of the steady state values of the state variables. These are denoted by affixing a zero as a subscript to the particular variable, and noting that by definition the steady state has no time variation. The variables of the steady state are then found by solution of equation III. 1, which is obtained by suppressing the time variation of the state equation II. 14.

$$
\left\{\begin{array}{l}0\\0\\0\\0\\0\\0\end{array}\right\} = \left\{\begin{array}{l}\left[\nabla\cdot D_F(T_o,\bar{r})\nabla - \Sigma_{aF}(T_o,\bar{r}) + (1-\beta)\nu\Sigma_{fF}(T_o,\bar{r})\right]\Phi_{Fo}(\bar{r}) + (1-\beta)\nu\Sigma_{fS}(T_o,\bar{r})\Phi_{So}(\bar{r}) \\ \qquad\qquad + \lambda_c C_o(\bar{r}) \\ \Sigma_R(T_o,\bar{r})\Phi_{Fo}(\bar{r}) + \left[\nabla\cdot D_S(T_o,\bar{r})\nabla - \Sigma_{aSo}(T_o,\bar{r}) - \sigma_{aS}^{x}(T_o)N_{xo}(\bar{r})\right]\Phi_{So}(\bar{r}) \\ \beta\nu\Sigma_{fF}(T_o,\bar{r})\Phi_{Fo}(\bar{r}) + \beta\nu\Sigma_{fS}(T_o,\bar{r})\Phi_{So}(\bar{r}) - \lambda_c C_o(\bar{r}) \\ \lambda_I N_{Io}(\bar{r}) + \gamma_x\Sigma_{fF}(T_o,\bar{r})\Phi_{Fo}(\bar{r}) + \gamma_x\Sigma_{fS}(T_o,\bar{r})\Phi_{So}(\bar{r}) - \lambda_x N_{xo}(\bar{r}) \\ \qquad\qquad - \sigma_{aS}^{x}(T_o)N_{xo}(\bar{r})\Phi_{So}(\bar{r}) \\ \gamma_I\Sigma_{fF}(T_o,\bar{r})\Phi_{Fo}(\bar{r}) + \gamma_I\Sigma_{fS}(T_o,\bar{r})\Phi_{So}(\bar{r}) - \lambda_I N_{Io}(\bar{r}) \\ e\Sigma_{fF}(T_o,\bar{r})\Phi_{Fo}(\bar{r}) + e\Sigma_{fS}(T_o,\bar{r})\Phi_{So}(\bar{r}) + \nabla\cdot\kappa(\bar{r})\nabla T_o(\bar{r})\end{array}\right. \qquad \text{(III. 1)}
$$

The general state equation, II. 2, may then be written symbolically as III. 2 in the steady state.

$$\bar{0} = \bar{f}(\bar{Z}_o(\bar{r}), \bar{r}) \qquad \text{(III. 2)}$$

It is necessary to assume that these equations can be solved by some means to some reasonable approximation. Possibly, the steady state can be measured in the particular reactor to be controlled, and this can be used as the solution. Knowing the solution of these non-linear spatial equations gives the desired state to which the control system is to return the variables.

The Linearization

From this point on, it will be assumed that the steady state is known, and all variables are the deviations from the steady state. The coefficients (including partial derivatives with respect to temperature) of the state equation II. 13 are to be evaluated at the local steady state temperature, and so vary with $\bar{r}$. The deviations of the variables from their steady state are defined in III. 3.

$$\phi_F(\bar{r},t) = \Phi_F(\bar{r},t) - \Phi_{F_o}(\bar{r})$$

$$\phi_S(\bar{r},t) = \Phi_S(\bar{r},t) - \Phi_{S_o}(\bar{r})$$

$$\begin{aligned} c(\bar{r},t) &= C(\bar{r},t) - C_o(\bar{r}) \\ x(\bar{r},t) &= N_x(\bar{r},t) - N_{x_o}(\bar{r}) \\ i(\bar{r},t) &= N_I(\bar{r},t) - N_{I_o}(\bar{r}) \\ \theta(\bar{r},t) &= T(\bar{r},t) - T_o(\bar{r}) \\ \bar{z}(\bar{r},t) &= \bar{Z}(r,t) - \bar{Z}_o(\bar{r}) \end{aligned} \qquad \text{(III. 3)}$$

These definitions are put into the state equation, II. 14, and the steady state equation III. 1 is subtracted from II. 14 to get equation III. 4.

It is the purpose of the control system to keep the deviations small. An optimal control system, furthermore, should attempt to keep the deviations as small as possible, and eventually return the unforced system to the steady state. Therefore, it will be assumed that deviations from the steady state are small for the system under control. The effect of large deviations should be investigated, but this can be done after a design has been suggested by study of the linearized equations. The methods of Gyftopoulos[58] might be extended to spatial systems so as to estimate the range of validity.

Since deviations will be considered small, the expansion of the reactor due to temperature will be small compared with the reactor size. Therefore, the boundary conditions on the

$$
\begin{Bmatrix}
\frac{1}{v_F} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{v_s} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \rho c_p
\end{Bmatrix}
\frac{\partial}{\partial T}
\begin{Bmatrix} \phi_F \\ \phi_s \\ c \\ x \\ i \\ \theta \end{Bmatrix}
=
\begin{Bmatrix}
\nabla\cdot D_F\nabla - \Sigma_{aF} + (1-\beta)\nu\Sigma_{fF} & (1-\beta)\nu\Sigma_{fs} & \lambda_c & 0 & 0 & \nabla\cdot(\nabla\Phi_{Fo})\frac{\partial D_F}{\partial T} - \frac{\partial \Sigma_{aF}}{\partial T}\Phi_{Fo} + (1-\beta)\nu\left(\frac{\partial \Sigma_{fF}}{\partial T}\Phi_{Fo} + \frac{\partial \Sigma_{fs}}{\partial T}\Phi_{so}\right) \\
\Sigma_R & \nabla\cdot D_s\nabla - \Sigma_{aso} - \sigma_{as}^x N_{x_o} & 0 & -\sigma_{as}^x\Phi_{so} & 0 & \nabla\cdot(\nabla\Phi_{so})\frac{\partial D_s}{\partial T} + \frac{\partial \Sigma_R}{\partial T}\Phi_{Fo} - \frac{\partial \Sigma_{aso}}{\partial T}\Phi_{so} - \frac{\partial \sigma_{as}^x}{\partial T}N_{x_o}\Phi_{so} \\
\beta\nu\Sigma_{fF} & \beta\nu\Sigma_{fs} & -\lambda_c & 0 & 0 & \beta\nu\left(\Phi_{Fo}\frac{\partial \Sigma_{fF}}{\partial T} + \Phi_{so}\frac{\partial \Sigma_{fs}}{\partial T}\right) \\
\gamma_x\Sigma_{fF} & \gamma_x\Sigma_{fs} - \sigma_{as}^x N_{x_o} & 0 & -(\lambda_x + \sigma_{as}^x\Phi_{so}) & \lambda_I & \gamma_x\left(\Phi_{Fo}\frac{\partial \Sigma_{fF}}{\partial T} + \Phi_{so}\frac{\partial \Sigma_{fs}}{\partial T}\right) - \frac{\partial \sigma_{as}^x}{\partial T}N_{x_o}\Phi_{so} \\
\gamma_I\Sigma_{fF} & \gamma_I\Sigma_{fs} & 0 & 0 & -\lambda_I & \gamma_I\left(\Phi_{Fo}\frac{\partial \Sigma_{fF}}{\partial T} + \Phi_{so}\frac{\partial \Sigma_{fs}}{\partial T}\right) \\
e\Sigma_{fF} & e\Sigma_{fs} & 0 & 0 & 0 & e\left(\Phi_{Fo}\frac{\partial \Sigma_{fF}}{\partial T} + \Phi_{so}\frac{\partial \Sigma_{fs}}{\partial T}\right) + \nabla\cdot\chi\nabla
\end{Bmatrix}
\begin{Bmatrix} \phi_F \\ \phi_s \\ c \\ x \\ i \\ \theta \end{Bmatrix}
+
$$

$$
\begin{Bmatrix}
0 & \cdots & 0 \\
-\Phi_{so}\delta(\bar{r}-\bar{r}_1) & \cdots & -\Phi_{so}\delta(\bar{r}-\bar{r}_N) \\
0 & \cdots & \\
0 & \cdots & \\
0 & \cdots & \\
0 & \cdots &
\end{Bmatrix}
\begin{Bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_N \end{Bmatrix}
+
\begin{Bmatrix}
\nabla\cdot\theta\frac{\partial D_F}{\partial T}\nabla\phi_F - \frac{\partial \Sigma_{aF}}{\partial T}\phi_F\theta + (1-\beta)\nu\frac{\partial \Sigma_{fF}}{\partial T}\phi_F\theta + (1-\beta)\nu\frac{\partial \Sigma_{fs}}{\partial T}\phi_s\theta \\
\frac{\partial \Sigma_R}{\partial T}\phi_F\theta + \nabla\cdot\theta\frac{\partial D_s}{\partial T}\nabla\phi_s - \frac{\partial \Sigma_{aso}}{\partial T}\phi_s\theta - \frac{\partial \sigma_{as}^x}{\partial T}\theta(N_{x_o}\phi_s + \Phi_{so}x + \phi_s^x) \\
\quad -\theta\beta\nu\left(\frac{\partial \Sigma_{fF}}{\partial T}\phi_F + \frac{\partial \Sigma_{fs}}{\partial T}\phi_s\right) \\
\gamma_x\theta\left(\frac{\partial \Sigma_{fF}}{\partial T}\phi_F + \frac{\partial \Sigma_{fs}}{\partial T}\phi_s\right) - \theta\frac{\partial \sigma_{as}^x}{\partial T}(N_{x_o}\phi_s + \Phi_{so}^x + \phi_s^x) \\
\quad \gamma_I\theta\left(\frac{\partial \Sigma_{fF}}{\partial T}\phi_F + \frac{\partial \Sigma_{fs}}{\partial T}\phi_s\right) \\
\quad e\theta\left(\frac{\partial \Sigma_{fF}}{\partial T}\phi_F + \frac{\partial \Sigma_{fs}}{\partial T}\phi_s\right)
\end{Bmatrix}
+
\begin{Bmatrix}
\theta\frac{\partial v_F^{-1}}{\partial T}\frac{\partial F}{\partial t} \\
\theta\frac{\partial v_s^{-1}}{\partial T}\frac{\partial \phi_s}{\partial t} - \sigma_{as}^x\phi_s x - \phi_s\sum_{n=1}^{N}u_n\delta(\bar{r}-\bar{r}_n) - (\Phi_{so}+\phi_s)\theta\sum_{n=1}^{N}\frac{\partial u_n}{\partial T}\delta(\bar{r}-\bar{r}_n) \\
0 \\
-\sigma_{as}^x\phi_s x \\
0 \\
0
\end{Bmatrix}
\qquad \text{(III. 4)}
$$

equation are assumed fixed.

For small deviations, it will be assumed that the reactor processes can be represented by only the linear part of equation III. 4. This means that the last two matrices on the right are assumed to contribute very little. It can be seen that one of the two matrices contains only temperature derivative terms times the temperature deviation times the deviations of other variables. Therefore, this matrix is an order of magnitude less than the coefficients of either the temperature deviation or the deviations of the other variables, if the temperature derivatives are small. This is usually the case, and so the matrix can be neglected.

The terms in the last matrix must be dismissed individually. Only if the rate of change of the flux is small will $\frac{\partial v^{-1}}{\partial T} \frac{\partial \phi}{\partial t}$ be small compared to the other coefficients of θ. However, the process of xenon instability is very slow, and so this will usually be true.

Also, since ϕ_s and x are both assumed small in comparison with their steady state values, the term $\sigma_{as}^{x} \phi_s x$ can be neglected. It is interesting to note that this term was retained in the work of Gyorey,[59] who found a "hard-spring" type of behavior in the phase plane analysis of the i, x, ϕ_s system. Hence, neglect of this term gives a pessimistic evaluation of the response to control.

Next, there is the term $\phi_s \sum_{n=1}^{N} u_n \delta(\bar{r}-\bar{r}_n)$. Most studies of the control rod would require only that u_n be small in comparison with the absorption operator. This is because these studies deal only with the steady state. Here, it must be required that the absorption

represented by this term be small compared to the production operator minus the absorption operator. This is indicative of the rate of change of flux and flux leakage, which must be much larger than this term so that the term may be neglected. This appears to be the most stringent condition of the linearization, and should be checked carefully. Hence, large control rod motions cannot be included in the analysis.

Finally, there is the term $\left[\Phi_{so} + \phi_s\right] \theta \sum_{n=1}^{N} \frac{\partial u_n}{\partial T} \delta(r - r_n)$. From the remarks in Chapter II concerning the calculation of u_n, it can be seen that u_n is equal to a mean value of Σ_{as} times a function of time. The remarks of the preceding paragraph require u_n to be small compared with the absorption operator, so this function of time must be small. Therefore, the rate of change of u_n with temperature is equal to that of Σ_{as} times a small function of time. This small function of time and the small deviation θ assure that the whole term may be neglected.

With the above assumptions, the linear part of equation III. 4 is rewritten in matrix notation as equation III. 5.

$$\mathcal{M}(r) \frac{\partial}{\partial t} \bar{z}(\bar{r}, t) = \mathcal{L}(\bar{r})\, \bar{z}(\bar{r}, t) + \mathcal{Q}(\bar{r})\, \bar{u}(t) \qquad \text{(III. 5)}$$

Equation III. 5 approximates the spatial behavior of a nuclear reactor near some steady state operating point, and will be used as the state equation of the controlled reactor.

Separation of the Power Control System

The proposed control system is a spatial control system, the purpose of which is to return all spatial deviations to a steady state. This is somewhat at cross-purposes with the power control system, the purpose of which is to follow the changes of the desired value of the steady state. The spatial control system is known formally as a regulator, whereas the power control system is a servomechanism. If all other modes are controlled by the spatial system, the power control system need only detect and influence the fundamental mode of the solution of the linearized equation of the deviations, III. 5. This permits minor variations in power due to the contributions of the higher modes, but these contributions should quickly be returned to zero and so will be ignored. Therefore, the reactor can be controlled by one feedback system detecting and controlling only the fundamental, and another feedback system detecting and controlling all the other modes.

Analytical design of the spatial control system can be performed without regard to the exact nature of the power control system if the fundamental interacts only weakly, or does not interact at all, with the harmonics. This interaction will be examined in detail later, but now it will be assumed weak at most. In general, the fundamental will be denoted $\bar{\psi}_o(\bar{r})$, and $\bar{z}_H(\bar{r}, t)$ will be defined as the state variable to be controlled by the spatial, or harmonic, control system. Thus $\bar{z}_H$ is written as in III. 6.

$$\bar{z}_H(\bar{r}, t) = \bar{z}(\bar{r}, t) - a_o(t)\, \psi_o(\bar{r}) \qquad \text{(III. 6)}$$

Therefore, $z_H(\bar{r}, t)$ is controlled by the harmonic control system, and $a_o(t)$, a scalar, is controlled by the power control system.

Since the harmonic control system may at most weakly influence the fundamental mode, the action of one harmonic system control rod must be forced to negate the effect of all the other harmonic system rods on the fundamental. The action of this one control rod may be found by substituting III. 6 in III. 5 and taking the inner product with respect to the adjoint of the fundamental.

$$\langle \hat{\psi}_o(\bar{r}), \mu(\bar{r}) \frac{\partial \vec{z}_H}{\partial T} \rangle + \dot{a}_o(t) \langle \hat{\psi}_o(\bar{r}), \mu(\bar{r})\, \bar{\psi}_o(\bar{r}) \rangle =$$

$$\langle \hat{\psi}_o(\bar{r}), \mathcal{L}(\bar{r}) \bar{z}_H \rangle + a_o(t) \langle \hat{\psi}_o(\bar{r}), \mathcal{L}(\bar{r})\, \psi_o(\bar{r}) \rangle + \langle \hat{\psi}_o(\bar{r}), \mathcal{U}(\bar{r}) \rangle \bar{u}(t)$$

(III. 7)

The first terms on either side are small with respect to the coefficients of a_o and $\dot{a}_o$ because of the weak interaction requirement, and so the effect of the control rods on the fundamental can approximately be negated if the last term in III. 7 is zero. Thus equation III. 8 is found by equating the last term of III. 7 to zero.

$$0 = \langle \hat{\Psi}_o(\bar{r}), \mathcal{G}(\bar{r}) \rangle \bar{u}(t) = \int_V \left[\hat{\Psi}_o^{*(1)}(\bar{r}) \ldots \hat{\Psi}_o^{*(6)}(\bar{r}) \right] \begin{Bmatrix} 0 & 0 \\ -\Phi_{so}(\bar{r}-\bar{r}_1) & \ldots -\Phi_{so}(\bar{r}-\bar{r}_N) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{Bmatrix} d^3r \begin{Bmatrix} u_1 \\ \cdot \\ \cdot \\ u_N \end{Bmatrix}$$

$$= - \sum_{n=1}^{N} \int_V \hat{\Psi}_o^{*(2)}(\bar{r}) \Phi_{so}(\bar{r}) \delta(\bar{r}-\bar{r}_n) d^3r \, u_n(t) \qquad \text{(III. 8)}$$

$$= - \sum_{n=1}^{N} \hat{\Psi}_o^{*(2)}(\bar{r}_n) \Phi_{so}(\bar{r}_n) \, u_n(t)$$

From equation III. 8, the motion of one control rod may be expressed in terms of the motion of all the rest.

$$u_N(t) = - \frac{\sum_{n=1}^{N-1} \hat{\Psi}_o^{*(2)}(\bar{r}_n) \Phi_{so}(\bar{r}_n) \, u_n(t)}{\Phi_{so}(r_N) \hat{\Psi}_o^{*(2)}(r_N)} \qquad \text{(III. 9)}$$

This condition is imposed on the spatial control system so that its performance will have little disturbing effect on the power control system. Conversely, it may be desirable to have the power control system primarily affect the fundamental, although this is not necessary. As will be shown later, this would reduce the input to the spatial control system.

Now $\bar{u}$ and $\mathcal{G}$ in III. 5 may be redefined to incorporate III. 9 , and will be denoted u_{N-1} and $\mathcal{G}_{N-1}$, as in III. 10.

$$\bar{u}_{N-1}(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_{N-1}(t) \end{pmatrix}$$

$$\mathcal{U}_{N-1}(\bar{r}) = \begin{Bmatrix} 0 & \cdots & 0 \\ -\Phi_{so}(\bar{r})\left(\delta(\bar{r}-\bar{r}_1) - \dfrac{\hat{\psi}^{*(2)}(\bar{r}_1)\,\Phi_{so}(\bar{r}_1)}{\hat{\psi}^{*(2)}(\bar{r}_N)\,\Phi_{so}(\bar{r}_N)}\delta(\bar{r}-\bar{r})\right) & \cdots & -\Phi_{so}(\bar{r})\left(\delta(\bar{r}-\bar{r}_{N-1}) - \dfrac{\hat{\psi}^{*(2)}(\bar{r}_{N-1})\,\Phi_{so}(\bar{r}_{N-1})}{\hat{\psi}^{*(2)}(\bar{r}_N)\,\Phi_{so}(\bar{r}_N)}\delta(\bar{r}-\bar{r}_{N-1})\right) \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{Bmatrix}$$

(III. 10)

Thus the state equation for the spatial control system is put into the form of III. 11, by using III. 5, III. 6, and the redefinition of $\bar{u}$ and $\mathcal{U}$ to $\bar{u}_{N-1}$ and $\mathcal{U}_{N-1}$.

$$\mu(\bar{r})\,\frac{\partial \bar{z}_H(\bar{r},t)}{\partial t}$$

$$= \mathcal{L}(\bar{r})\,\bar{z}_H(\bar{r},t) + \mathcal{U}_{N-1}(\bar{r})\,\bar{u}_{N-1}(t) - \Phi_{so}\,\delta(\bar{r}-\bar{r}_N)\left(u_N - \frac{\sum_{n-1}^{N-1}\hat{\psi}_o^*(\bar{r}_n)\,\Phi_{so}(\bar{r}_n)\,u_n}{\hat{\psi}_o^*(\bar{r}_N)\,\Phi_{so}(\bar{r}_N)}\right)\begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$+ a_o(t)\,\mathcal{L}(\bar{r})\,\bar{\psi}_o(\bar{r}) - \dot{a}_o(t)\,\mu(\bar{r})\,\bar{\psi}_o(\bar{r}) \qquad \text{(III. 11)}$$

Equation III. 11 is a linear equation consisting of a homogeneous and an inhomogeneous part. Since $u_{N-1}(t)$ is to be made up from measurements of $z_H(\bar{r},t)$ as a feedback, $u_{N-1}(t)$ is part of the homogeneous equation. The last three terms, however, are functions of space and time only and are not dependent on $z_H(\bar{r}, t)$. These terms may be considered as part of an input $\bar{f}(\bar{r}, t)$ that is discussed in detail in Chapter VI. Therefore, the state equation for the spatial control of a reactor has the final form of III. 12.

$$\mu(\bar{r}) \frac{\partial \bar{z}_H(\bar{r}, t)}{\partial t} = \mathcal{L}(\bar{r})\, \bar{z}_H(\bar{r},t) + \mathcal{U}_{N-1}(\bar{r})\, \bar{u}_{N-1}(t) + \bar{f}\,(\bar{r}, t) \qquad \text{(III. 12)}$$

Now that the linearized state equation III.12 has been found, a simpler form can be used to develop the rest of the theory and yet retain the essential features of just the xenon process. Only three elements will be retained in the vector state variable, the deviations of iodine and xenon precursors, and the deviations of a single flux group. Thus all neutron flux groups and delayed neutrons are averaged into one scalar equation with the coefficients

altered accordingly. Also the temperature processes are neglected, as are the fast leakage terms. These alterations retain the main effects, and considerably reduce the complexity of future notation. Accordingly, for purposes of preliminary investigation and for simplicity in later notation and development, equation III. 13 will be used as the state equation with the understanding that the theory also applies to a more detailed model.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{v} \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} i \\ x \\ \phi \end{pmatrix} = \begin{Bmatrix} -\lambda_I & 0 & \gamma_I \epsilon \Sigma_f \\ \lambda_I & -(\lambda_x + \sigma_a^x \Phi_o) & -\sigma_a^x N_{x_o} + \gamma_x \epsilon \Sigma_f \\ 0 & -\sigma_a^x \Phi_o & \left(\nabla \cdot D\nabla + \nu\epsilon\Sigma_f - \Sigma_a - \sigma_a^x N_{x_o}\right) \end{Bmatrix} \begin{pmatrix} i \\ x \\ \phi \end{pmatrix}$$

$$+ \begin{Bmatrix} 0 & & 0 \\ 0 & \cdots & 0 \\ -\Phi_o\left(\delta(\bar{r}-\bar{r}_1) - \frac{\psi_o^{(3)}(\bar{r}_1)}{\psi_o^{(3)}(\bar{r}_n)}\delta(\bar{r}-\bar{r}_N)\right) & & -\Phi_o\left(\delta(\bar{r}-\bar{r}_{N-1}) - \frac{\psi_o^{(3)}(\bar{r}_{N-1})}{\psi_o^{(3)}(\bar{r}_N)}\delta(\bar{r}-\bar{r}_{N-1})\right) \end{Bmatrix} \begin{Bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_{N-1} \end{Bmatrix}$$

(III. 13)

CHAPTER IV

TWO METHODS OF SOLUTION OF THE LINEARIZED STATE EQUATION

There are a number of possible methods of solution of the linearized state equation, III. 13. This chapter will consider two methods of modal expansion; expansion into clean reactor modes and expansion into a type of mode originated by S. Kaplan.[29]

Expansion Into Clean Reactor Modes

The linearized state equations are first solved by expansion into clean reactor modes. The equations of III. 13 may be written separately, instead of in matrix form.

$$\frac{1}{v}\frac{\partial \phi}{\partial t} = \nabla \cdot D\nabla\phi + \nu\epsilon\Sigma_f\phi - \Sigma_a\phi - \sigma_a^x N_{x_o}\phi - \sigma_a^x \Phi_o x - \sum_{n=1}^{N} u_n \Phi_o \delta(\bar{r}-\bar{r}_n) \qquad \text{(a)}$$

$$\frac{\partial i}{\partial t} = -\lambda_I i + \gamma_I \epsilon \Sigma_f \phi \qquad \text{(b)} \qquad \text{(III. 13)}$$

$$\frac{\partial x}{\partial t} = \lambda_I i - (\lambda_x + \sigma_a^x \Phi_o)x - \sigma_a^x N_{x_o}\phi + \gamma_x \epsilon \Sigma_f \phi \qquad \text{(c)}$$

The clean reactor equation is III. 13a, in which N_{x_o}, x, and u_n are set equal to zero. To get the mode shapes, let the scalar eigenfunctions $\theta_m(r)$ and the eigenvalues α_m be determined by IV. 1.

$$a_m \theta_m(\bar{r}) = \nabla \cdot D \nabla \theta_m(\bar{r}) + (\nu \epsilon \Sigma_f - \Sigma_a)\theta_m(\bar{r}) \qquad \text{(IV. 1)}$$

The equation consists solely of Hermitian operators, therefore the modes can form an orthonormal basis and IV. 2 holds.

$$\int_V \theta_m(\bar{r})\, \theta_k(\bar{r}) d^3 r = \langle \theta_m, \theta_k \rangle = \delta_{mk} \qquad \text{(IV. 2)}$$

The mode shapes $\theta_m(\bar{r})$ are those that are found when solving for the time behavior of the total flux in a clean reactor. However, here these modes are used in the expansions of the deviations that are measured from an operating point in a poisoned reactor. The assumed expansions are written in IV. 3.

$$\phi(\bar{r}, t) = \sum_{m=0}^{\infty} b_m(t)\, \theta_m(\bar{r})$$

$$i(\bar{r}, t) = \sum_{m=0}^{\infty} c_m(t)\, \theta_m(\bar{r}) \qquad \text{(IV. 3)}$$

$$x(\bar{r}, t) = \sum_{m=0}^{\infty} d_m(t)\, \theta_m(\bar{r})$$

To obtain a set of equations for the modal time coefficients, $b_m(t)$, $c_m(t)$, and $d_m(t)$, the expansions IV. 3 are substituted into III. 13 and the relationship IV. 1 is used, resulting in IV. 4.

$$\frac{1}{v}\sum_{m=0}^{\infty}\frac{db_m}{dt}\theta_m(\bar{r}) = \sum_{m=0}^{\infty} a_m b_m \theta_m(\bar{r}) - \sigma_a^x \sum_{m=0}^{\infty} b_m N_{x_o}(\bar{r})\,\theta_m(\bar{r})$$

$$- \sigma_a^x \sum_{m=0}^{\infty} d_m \Phi_o(\bar{r})\,\theta_m(\bar{r}) - \sum_{n=1}^{N} u_n \Phi_o \delta(\bar{r}-\bar{r}_n)$$

$$\sum_{m=0}^{\infty}\frac{dc_m}{dt}\theta_m(\bar{r}) = -\lambda_I \sum_{m=0}^{\infty} c_m \theta_m(\bar{r}) + \gamma_I \epsilon \sum_{m=0}^{\infty} b_m \Sigma_f(\bar{r})\theta_m(\bar{r}) \qquad \text{(IV. 4)}$$

$$\sum_{m=0}^{\infty}\frac{dd_m}{dt}\theta_m(\bar{r}) = \lambda_I \sum_{m=0}^{\infty} c_m \theta_m(\bar{r}) - \lambda_x \sum_{m=0}^{\infty} d_m \theta_m(\bar{r}) - \sigma_a^x \sum_{m=0}^{\infty} d_m \Phi_o(\bar{r})\theta_m(\bar{r})$$

$$+ \sum_{m=0}^{\infty} b_m \left(\gamma_x \epsilon \Sigma_f(\bar{r}) - \sigma_a^x N_{x_o}(\bar{r})\right)\theta_m(\bar{r})$$

Multiplying by $\theta_k(r)$ and integrating over the reactor volume gives the desired set of total differential equations for the modal time coefficients, after the orthogonality property IV. 2 is used:

$$\frac{1}{v}\dot{b}_k = a_k b_k - \sigma_a^x \sum_{m=0}^{\infty} b_m \int_V N_{x_o}\theta_m\theta_k d^3r - \sigma_a^x \sum_{m=0}^{\infty} d_m \int_V \Phi_o \theta_m \theta_k\, d^3r$$

$$- \sum_{n=1}^{N} u_n \int_V \Phi_o \theta_k \delta(r-r_n) d^3r$$

$$\dot{c}_k = -\lambda_I c_k + \gamma_I \epsilon \sum_{m=0}^{\infty} b_m \int_V \Sigma_f \theta_m \theta_k d^3r \qquad \text{(IV. 5)}$$

$$\dot{d}_k = \lambda_I c_k - \lambda_x d_k - \sigma_a^x \sum_{m=0}^{\infty} d_m \int_V \Phi_o \theta_m \theta_k d^3r + \sum_{m=0}^{\infty} b_m \int_V (\gamma_x \epsilon \Sigma_f - \sigma_a^x N_{x_o}) \theta_m \theta_k d^3r.$$

An exponential form in time is assumed in order to obtain the next equation IV.6, as is standard in the solution of systems of first order linear constant coefficient total differential equations. This shows that b_k, c_k, and d_k are linear combinations of exponentials with time constants ω, and are forced by the $u_n(t)$ term. The time constants are chosen such that IV. 6 will permit arbitrary selection of either b_k, c_k, or d_k.

$$\frac{1}{v}\omega b_k = a_k b_k - \sigma_a^x \sum_{m=0}^{\infty} b_m \int_V N_{x_o} \theta_m \theta_k dr - \sigma_a^x \sum_{m=0}^{\infty} d_m \int_V \Phi_o \theta_m \theta_k d^3r$$

$$\omega c_k = -\lambda_I c_k + \gamma_I \epsilon \sum_{m=0}^{\infty} b_m \int_V \Sigma_f \theta_m \theta_k d^3r \qquad \text{(IV. 6)}$$

$$\omega d_k = \lambda_I c_k - \lambda_x d_k - \sigma_a^x \sum_{m=0}^{\infty} d_m \int_V \Phi_o \theta_m \theta_k d^3r + \sum_{m=0}^{\infty} b_m \int_V (\gamma_x \epsilon \Sigma_f - \sigma_a^x N_{x_o}) \theta_m \theta_k d^3r$$

Note that unless Σ_f, Φ_o and N_{x_o} are flat spatially, the equation IV. 6 will contain infinite sums. Therefore, the modes "interact" with one another. Modal interaction will be seen to be

the main drawback of this method.

If the infinite sums are convergent, an approximate solution may be found by truncating the series at the M^{th} mode. This finite set of algebraic equations may then be solved, and the values of found can then be used to express the b_m and c_m in terms of d_m, which in turn is determined by the initial conditions. Having the modal time coefficients, the state vector is then of the form IV. 3. This is the method of separation into clean reactor modes.

Expansion into Kaplan Modes

Now the linearized equations are solved by expansion into Kaplan modes. In this method, the state equations are written in matrix form, as equation III. 13. For generality, Kaplan's form[29] is extended to include a spatial operator multiplying the time partial derivative of the state vector. This permits inclusion of spatially varying neutron velocities and, when the effect of temperature is included, spatially varying heat capacity. Equation III. 13 is repeated here for convenience.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{v} \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} i \\ x \\ \phi \end{pmatrix} = \begin{Bmatrix} -\lambda_I & 0 & \gamma_I \epsilon \Sigma_f \\ \lambda_I & -(\lambda_x + \sigma_a^x \Phi_o) & -\sigma_a^x N_{x_o} + \gamma_x \epsilon \Sigma_f \\ 0 & -\sigma_a^x \Phi_o & \nabla \cdot D\nabla + \nu\epsilon\Sigma_f - \Sigma_a - \sigma_a^x N_{x_o} \end{Bmatrix} \begin{pmatrix} i \\ x \\ \phi \end{pmatrix}$$

$$+ \begin{Bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ -\Phi_o \delta(\bar{r}-\bar{r}_1) & \cdots & -\Phi_o \delta(\bar{r}-\bar{r}_N) \end{Bmatrix} \begin{Bmatrix} u_1 \\ \vdots \\ u_N \end{Bmatrix} \qquad \text{(III. 13)}$$

The mode shapes are vectors in this case, and are defined by IV. 7.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{v} \end{pmatrix} \omega_m \begin{pmatrix} \psi_m^{(1)}(\bar{r}) \\ \psi_m^{(2)}(\bar{r}) \\ \psi_m^{(3)}(\bar{r}) \end{pmatrix}$$

$$= \begin{Bmatrix} -\lambda_I & 0 & \gamma_I \epsilon \Sigma_f \\ \lambda_I & -(\lambda_x + \sigma_a^x \Phi_o) & -\sigma_a^x N_{x_o} + \gamma_x \epsilon \Sigma_f \\ 0 & -\sigma_a^x \Phi_o & \nabla \cdot D\nabla + \nu\epsilon\Sigma_f - \Sigma_a - \sigma_a^x N_{x_o} \end{Bmatrix} \begin{Bmatrix} \psi_m^{(1)}(\bar{r}) \\ \psi_m^{(2)}(\bar{r}) \\ \psi_m^{(3)}(\bar{r}) \end{Bmatrix}$$

(IV. 7)

Equation IV. 7 is rewritten in matrix operator notation as IV. 8.

$$\omega_m \mu(\bar{r}) \bar{\psi}_m(\bar{r}) = \mathcal{L}(\bar{r}) \bar{\psi}_m(\bar{r}) \qquad \text{(IV. 8)}$$

Equation IV. 7 may be thought of as giving a triply infinite set of eigenvalues, *each infinite set due to the partial differential operator*, and three sets to correspond to the number of elements in the state vector.

As shown in Kaplan's paper,[29] a bi-orthonormal set of vectors

can be set up using the vectors of the adjoint equation, IV. 9.

$$\omega^*_k \mathcal{M}^+_{(\bar{r})} \hat{\bar{\Psi}}_k = \mathcal{L}^+_{(\bar{r})} \hat{\bar{\Psi}}_k \qquad \text{(IV. 9)}$$

In the particular case of IV. 7, all elements of $\mathcal{M}$ and $\mathcal{L}$ are real and Hermitian operators. Therefore $\mathcal{M}^+ = \mathcal{M}$ and $\mathcal{L}^+$ is the transpose of the matrix $\mathcal{L}$. Taking the inner product of IV. 8 with $\hat{\bar{\Psi}}_k$ and subtracting the inner product of the adjoint of IV. 9 with $\bar{\Psi}_m$ gives the bi-orthogonality condition IV. 10.

$$(\omega_m - \omega_k) \langle \hat{\bar{\Psi}}_k, \mathcal{M} \bar{\Psi}_m \rangle = 0 \qquad \text{(IV. 10)}$$

The assumption of distinct eigenvalues ($\omega_m \neq \omega_k$) then implies that each vector eigenfunction is bi-orthogonal to all the others. Also, if $\omega_m = \omega_k$ for $m \neq k$, often the vector eigenfunctions can be geometrically constructed to be bi-orthogonal by symmetry. It will be assumed that no other cases of repeated roots exist. Example VIII.A of Chapter VIII illustrates a case of geometrical and a case of non-geometrical multiplicity. Also it is assumed that $\langle \hat{\bar{\Psi}}_k, \mathcal{M} \bar{\Psi}_k \rangle \neq 0$ so that the set of $\{\omega_k\}$ is equivalent to the set of $\{\omega^*_k\}$. In general, any cases of multiplicity other than those due to symmetry may be said to be accidents of nature, and the multiplicity may be broken by considering a slight perturbation. Similarly, any case of $\langle \hat{\bar{\Psi}}_k, \mathcal{M} \bar{\Psi}_k \rangle = 0$ is a great coincidence, and again a slight perturbation will make the inner product non-zero.

Finally, each vector eigenfunction is determined to within an arbitrary constant a, as shown in IV. 11.

$$\omega_m \mu (a \bar{\varphi}_m) = a \omega_m \mu \bar{\varphi}_m = a \mathcal{L} \bar{\varphi}_m = \mathcal{L} (a \bar{\varphi}_m) \tag{IV. 11}$$

Set

$$a = \frac{1}{\sqrt{\langle \hat{\bar{\varphi}}_k, \mu \bar{\varphi}_k \rangle}} \tag{IV. 12}$$

Then, with the stated assumptions, the bi-orthonormality condition is given by IV. 13, in which δ_{km} is the Kronecker delta.

$$\langle \hat{\bar{\varphi}}_k, \mu \bar{\varphi}_m \rangle = \delta_{km} \tag{IV. 13}$$

Having a bi-orthonormal set of vectors (assumed complete) a solution is attempted by expansion in Kaplan modes. The state vector is thus assumed of the form IV. 14.

$$\bar{z}(r, t) = \sum_{m=0}^{\infty} a_m(t) \bar{\varphi}_m(\bar{r}) \tag{IV. 14}$$

This expansion can be used in the state equation III. 14.

$$\sum_{m=0}^{\infty} \frac{da_m(t)}{dt} \mu(\bar{r}) \bar{\varphi}_m(\bar{r}) = \sum_{m=0}^{\infty} a_m(t) \mathcal{L}(r) \bar{\varphi}_m(\bar{r}) + \mathcal{G}(\bar{r}) \bar{u}(t) \tag{IV. 15}$$

Use of relation IV. 8 puts the first term on the right in a more useful form.

$$\sum_{m=0}^{\infty} \frac{da_m}{dt} \mu \bar{\psi}_m = \sum_{m=0}^{\infty} \omega_m a_m \mu \bar{\psi}_m + \mathcal{G} \bar{u} \qquad \text{(IV. 16)}$$

Multiplication by the adjoint vector and integration over the reactor volume permits use of the bi-orthonormality condition IV. 13.

$$\frac{da_k}{dt} = \omega_k a_k + \langle \bar{\hat{\psi}}_k, \mathcal{G} \bar{u} \rangle \qquad \text{(IV. 17)}$$

This equation is analogous to IV. 5 of the clean mode expansion, except that there are no infinite sums. Therefore, there is no modal interaction between the Kaplan modes.

An explicit solution for each a_k can be obtained by solution of IV. 17.

$$a_k(t) = a_k(0) e^{\omega_k t} + \int_0^t e^{\omega_k (t-\tau)} \langle \bar{\hat{\psi}}_k, \mathcal{G} \bar{u}(\tau) \rangle \, d\tau \qquad \text{(IV. 18)}$$

Here

$$a_k(0) = \langle \bar{\hat{\psi}}_k, \bar{z}(\bar{r}, 0) \rangle \qquad \text{(IV. 19)}$$

Equation IV. 14, with IV. 8, IV. 9, IV. 18, and IV. 19 then give the solution for the state variable $\bar{z}(\bar{r}, t)$ under the action of control u(t). This is the method of separation into Kaplan modes.

Relations Between the Types of Modes

Substitution of an expansion for each Kaplan mode in terms of clean reactor modes yields a useful relationship between the modes.

Let

$$\overline{\psi}_m(\overline{r}) = \sum_{k=0}^{\infty} \begin{pmatrix} c_{km} \\ d_{km} \\ b_{km} \end{pmatrix} \theta_k(\overline{r}) \qquad \text{(IV. 20)}$$

where the c_{km}, d_{km}, and b_{km} are arbitrary constants to be found. Substitution into IV. 7, using relation IV. 2, yields equation IV. 21.

$$\omega_m \sum_{k=0}^{\infty} c_{km}\theta_k = -\lambda_I \sum_{k=0}^{\infty} c_{km}\theta_k + \gamma_I \epsilon \sum_{k=0}^{\infty} b_k \Sigma_f \theta_k \qquad \text{(IV. 21)}$$

$$\omega_m \sum_{k=0}^{\infty} d_{km}\theta_k = \lambda_I \sum_{k=0}^{\infty} c_{km}\theta_k - \lambda_x \sum_{k=0}^{\infty} d_{km}\theta_k - \sigma_a^x \sum_{k=0}^{\infty} d_{km}\Phi_o\theta_k$$

$$+ \sum_{k=0}^{\infty} b_{km}(\gamma_x \epsilon \Sigma_f - \sigma_a^x N_{x_o})\theta_k$$

$$\omega_m \frac{1}{v} \sum_{k=0}^{\infty} b_{km}\theta_k = \sum_{k=0}^{\infty} \alpha_k b_{km}\theta_k - \sigma_a^x \sum_{k=0}^{\infty} b_{km} N_{x_o}\theta_k - \sigma_a^x \sum_{k=0}^{\infty} d_{km}\Phi_o\theta_k$$

Multiplication by θ_j and integration over the volume of the reactor gives equation IV. 22 for the c_{km}, d_{km}, and b_{km}.

$$\omega_m c_{km} = -\lambda_I c_{km} + \gamma_I \epsilon \sum_{j=0}^{\infty} b_{jm} \int_V \Sigma_f \theta_j \theta_k \, d^3r \qquad \text{(IV. 22)}$$

$$\omega_m d_{km} = \lambda_I c_{km} - \lambda_x d_{km} - \sigma_a^x \sum_{j=0}^{\infty} d_{jm} \int_V \Phi_o \theta_j \theta_k \, d^3r$$

$$+ \sum_{j=0}^{\infty} b_{jm} \int_V (\gamma_x \epsilon \Sigma_f - \sigma_a^x N_{x_o}) \theta_j \theta_k \, d^3r$$

$$\omega_m \frac{1}{v} b_{km} = a_k b_{km} - \sigma_a^x \sum_{j=0}^{\infty} b_{jm} \int_V N_{x_o} \theta_j \theta_k \, d^3r - \sigma_a^x \sum_{j=0}^{\infty} d_{km} \int_V \Phi_o \theta_j \theta_k \, d^3r$$

Equation IV. 22 is exactly equivalent to equation IV. 6.

It can be seen from the relationship just shown that even if the infinite series are not convergent, the eigenvalues of the Kaplan modes satisfy the infinite set of equations formed out of the modal time coefficients of the clean reactor modes. If these series are convergent, the Kaplan modes can be expressed in terms of the clean reactor modes. By the same device the Kaplan modes may be expressed in terms of any other suitable expansion. Of course, the converse is not true. It may be concluded that whenever an expansion in terms of another type of mode exists, then the expansion in terms of the Kaplan modes exists.

Although existence is usually easy to ascertain, completeness is not. Expansion in clean reactor modes can usually be proven complete, but there is no guarantee that the Kaplan expansion is

complete because the conditions under which transformation IV. 20 is one to one are not known.

Most of the mathematical questions evolve basically from the fact that the matrix operator $\mathcal{L}(\bar{r})$ is not Hermitian. This is the reason that modal expansions of other than the Kaplan type are considered. The spectral theory of the clean reactor modes has been fairly well worked out, even in the case of the multigroup operator.[60,61] Therefore, in the cases of expansion in the clean reactor modes, the main question is the convergence of the "modal interaction" series. In each specific case, calculation of a number of these coefficients will usually give an idea of the convergence. If the coefficients converge, then there exists a firm mathematical basis from which the analysis can proceed.

However, in many cases it seems it would be advantageous to merely assume that the spectral theory (i.e., behavior of the eigenvalues, etc.) of the Kaplan type of modes is somewhat similar to the clean reactor modes. Then analysis of the physical problem would not be held up by the formal proof of existence, uniqueness, and completeness. These mathematical properties, therefore, will be assumed in the rest of the thesis.

In defense of this assumption, calculational results seem to indicate that the Kaplan mode spectral theory is close to the clean reactor mode spectral theory for the types of reactors considered. Only the terms involving the effect of xenon are negative among the off-diagonal terms of the linear operator. Because of this, the

results of Varga[61] for operators with only positive off-diagonal terms cannot be used. His results probably would apply even if a small negative term were introduced, but it is difficult to determine at exactly what magnitude of the negative term his conclusions would not apply. The introduction of the negative xenon terms appears not to have changed any conclusions of Varga in the examples of reactors studied here. The eigenvalue of the fundamental mode may be estimated to be always real and larger than other eigenvalues by using the estimation technique of Chapter VI. Furthermore, the estimates for the eigenvalues may be ordered. This is in agreement with the results of Varga for clean reactor modes, and gives some justification of the mathematical assumptions.

However, there are differences between the Kaplan and clean reactor modes. The clean reactor mode eigenvalues most probably tend to minus infinity with increasing order of mode. Example VIII. B shows a specific case where a set of Kaplan mode eigenvalues tend to a finite limit when there is no differential operator in a row.

In summary, the Kaplan modes and clean reactor modes seem to be somewhat similar, but the specific behavior of the Kaplan type of mode is not as well known as the clean reactor modes.

A Comparison of the Two Types of Modes

Now that some of the relationships between the types of modes have been discussed, a comparison of their suitability for the solution of the spatial control problem will be undertaken.

The clean reactor modes have some advantages. The spectral theory of the clean reactor modes has been worked out, and their completeness is really not in doubt. Their physical usefulness rests on the fact that, in the constant coefficient case, the modes are usually well known functions and are easily found. Also, these same functions can be pieced together at the interfaces to form modes in a multiregion reactor. Finally, they are independent of power level, and therefore do not have to be recalculated at each operating point.

However, the clean reactor modes also have a number of disadvantages. In addition to the question of convergence, the clean reactor modes also have two drawbacks for their possible use in the spatial control problem. The first is that the equations for the modal time coefficients can be of very high order, complicating their numerical solution. The second is that any control effort expended on one mode will disturb the behavior of all the other modes, due to modal interaction. There is a possibility that control action based on a finite number of modes will excite and even make unstable modes that are not controlled. This last consideration dictates the choice of the Kaplan modes, which will be used in the rest of the thesis despite their drawbacks.

The Kaplan modes do not interact, so feedback control of one

will not affect the stability of all the others. Unless controlled, each mode will decay with its own time constant, and the movement of the control rods will appear merely as a forcing function. This property dictates their choice, but there are other desirable properties.

This separability of modes means that the power control system and spatial control system can be completely separated, so that the spatial control system can be designed without regard to the specific characteristics of the power control system.

The simplicity of mathematical expression reduces computational labor in the determination of modal time coefficients. Also, the simple expressions obtained are useful from a theoretical standpoint, even if other modes must be used for calculational purposes.

Finally, there is a certain aesthetic appeal to using what Kaplan calls the "natural" modes. Only the Kaplan mode shapes are the mode shapes that can be excited separately, so that one could actually "see" these mode shapes in a reactor for small deviations.

However, there is one disadvantage of the Kaplan modes that has not yet been discussed. Besides their mathematical drawbacks, (i.e., the questions of existence, uniqueness, and completeness) the Kaplan modes are, in general, much harder to calculate than the clean reactor modes. The next chapter will discuss methods of calculation.

CHAPTER V

METHODS OF SOLUTION OF THE KAPLAN MODES

As yet, there is no general method for solution of the Kaplan modal equation, IV. 8. If the spatial control system is to be designed by the analysis presented here, the Kaplan mode eigenvalues and vector eigenfunctions must be found. This chapter discusses four methods of solution and gives some advantages and disadvantages of each method. Conceivably, there may be cases in which no method works, but for almost every practical case the Kaplan modes may be found to a good approximation.

Expansion of Each Kaplan Mode in Terms of Other Modes

As discussed in the previous chapter, if the modal interaction sum converges, it is always possible to expand each Kaplan mode as a series in the clean reactor modes. If another set of modes can be found whose shape is closer to the Kaplan modes, their convergence will be faster and so they should be used instead of the clean reactor modes. By clever selection of the expansion mode, expansion of each Kaplan mode may be the only method of obtaining solutions to the Kaplan modal equation.

This method has many drawbacks. The Kaplan modes can exist in cases where there is no convergence of another expansion. Computation is complicated by the many unknowns to be found. Also, Gyorey[32] found that truncation of the modal interaction series usually gives an estimate that is smaller than the true value of the

Kaplan eigenvalue, so that the stability margin for each mode is usually less than that computed. The converse effect is more useful in an analytical design. Finally, the most serious drawback is that the modal interaction series may converge very slowly. It becomes difficult to tell when enough modes have been used to represent each Kaplan mode. In this case, computational effort is such that almost any other method of finding the Kaplan modes would be better.

A Direct Finite Difference Model

A finite difference approximation to the Kaplan modal equation, IV. 8, may be constructed by standard procedures for a three-dimensional mesh of discrete points in space. There are many sophisticated methods of solution of the finite dimensional eigenvalue problem formed by these finite difference procedures. The interested reader is referred to the book[62] by Varga for specific numerical methods.

These numerical methods have one common failing. They must solve a boundary value problem, and therefore much computer storage is necessitated. If only a rough mesh is desired, or if a special method can be found to save storage space, the direct finite difference model can be used to obtain the Kaplan modes.

Mu Modes

The μ-mode approximation was introduced by Harris and Lacy.[21] Xenon is assumed to be 180 degrees out of phase with the slow flux, so that a constant, μ, can represent the effective

contribution of the xenon to the thermal absorption cross-section. This assumption gives exact results when the steady state flux is flat in a homogeneous reactor, giving no modal interaction. A complete discussion of the method and its assumptions is given by Ewen.[36]

Ewen found good agreement with the exact solution in a number of cases where the steady state flux is not flat. It seems hard to reconcile this with the results of Gyorey,[32] who found the effects of modal interaction induced by the steady state flux to be quite important when the flux is not flat. Furthermore, a private communication from S. Kaplan states that the μ modes are hard to calculate if obvious modal lines are not provided by symmetry. He also states that the assumption of flat flux leading to the xenon being exactly 180 degrees out of phase is difficult to evaluate.

Shooting

Whenever the Kaplan mode equation IV. 8 can be completely separated in space, a very efficient numerical computation scheme known as "shooting" can determine the Kaplan modes. It is known as shooting because the solution trajectory is adjusted at one boundary until it "hits" the other boundary.

The Kaplan equation must be altered a bit before the standard shooting technique is employed. The matrix equation IV. 8 must be reduced to a scalar partial differential equation. To illustrate the procedure, the equations for the elements, j, of the m^{th} vector eigenfunction $\bar{\psi}_m(\bar{r})$ are written out to correspond to the matrix

equation IV. 7.

$$-\lambda_I \psi_m^{(1)} + \gamma_I \epsilon\Sigma_f \psi_m^{(3)} = \omega_m \psi_m^{(1)}$$

$$\lambda_I \psi_m^{(1)} - (\lambda_x + \sigma_a^x \Phi_o) \psi_m^{(2)} + (\gamma_x \epsilon\Sigma_f - \sigma_a^x N_{x_o}) \psi_m^{(3)} = \omega_m \psi_m^{(2)} \qquad \text{(V. 1)}$$

$$-\sigma_a^x \Phi_o \psi_m^{(2)} + (\nabla \cdot D\nabla + \nu\epsilon\Sigma_f - \Sigma_a - \sigma_a^x N_{x_o}) \psi_m^{(3)} = \frac{1}{v} \omega_m \psi_m^{(3)}$$

The upper equations may be multiplied or divided by a non-zero function to obtain an expression for $\psi_m^{(2)}$ to put into the bottom equation.

$$\psi_m^{(1)} = \frac{\gamma_I \epsilon\Sigma_f}{\omega_m + \lambda_I} \psi_m^{(3)}$$

$$\psi_m^{(2)} = \left(\frac{\gamma_I \epsilon\Sigma_f}{\omega_m + \lambda_I} \lambda_I + \gamma_x \epsilon\Sigma_f - \sigma_a^x N_{x_o} \right) (\lambda_x + \omega_m + \sigma_a^x \Phi_o)^{-1} \psi_m^{(3)} \qquad \text{(V. 2)}$$

It is shown in Appendix B that $\psi_m^{(3)}$ is zero when ω_m is the negative of λ_I or $\lambda_x + \sigma_a^x \Phi_o$, and so equations V. 2 are valid. Putting the last equation of V. 2 into the last equation of V. 1 completes the reduction to a scalar equation, V. 3, for this case.

$$\left[\nabla \cdot D\nabla + \nu\epsilon\Sigma_f - \Sigma_a - \sigma_a^x N_{x_o} - \frac{\omega_m}{v} - \frac{\sigma_a^x \Phi_o}{\omega_m + \lambda_x + \sigma_a^x \Phi_o} \left(\frac{\gamma_I \epsilon\Sigma_f \lambda_I}{\omega_m + \lambda_I} + \gamma_x \epsilon\Sigma_f - \sigma_a^x N_{x_o} \right) \right] \psi_m^{(3)} = 0 \qquad \text{(V. 3)}$$

This may be referred to as the "characteristic" equation of the matrix operators $\mathcal{M}$ and $\mathcal{L}$. The characteristic equation should be derived by the technique of successive elimination just illustrated, although it corresponds to "taking the determinant" of the matrix operator $\mathcal{L} - \mathcal{M}\omega$.

The problem is now reduced to finding solutions $\psi_m^{(3)}$ and coefficients ω_m that solve V. 3 and fit the boundary conditions. If the reactor properties and N_{x_o} and Φ_o are spatially flat, the clean reactor modes are solutions to V. 3. Then $\nabla . D \nabla$ may be replaced by the geometric buckling term, $-DB_m^2$, and V. 3 is a cubic in ω_m. Three values of ω_m may be found from the cubic for each mode shape. Some of the values of ω_m may be complex, reaffirming the non-Hermitian character of the operators.

The method of shooting is employed when the reactor properties and N_{x_o} and Φ_o are not spatially flat. A (possibly complex) value of ω_m is guessed, perhaps from the technique given in the next chapter. The equation V. 3 is separated into equations such that each contains only a single independent variable. Then the solution trajectory is started from the initial conditions of zero value and arbitrary slope at one boundary for both real and complex parts of $\psi_m^{(3)}$. The trajectory is computed by a finite difference approximation of V. 3, using the guessed value of ω_m. At any region interface, the conditions of continuity and continuous current may be applied as the trajectory progresses across to the other side of the reactor. If the trajectory is non-zero when evaluated at the other

boundary, the value of ω_m guessed is wrong. Systematic guessing of the ω_m in the complex plane will determine a trajectory that hits the opposite boundary at a point where the trajectory is exactly zero. This trajectory is the eigenfunction that was sought, and the guessed value of ω_m is the true value of the eigenvalue. In this manner all the desired Kaplan eigenfunctions and eigenvalues may be found.

The sequence of guessed eigenvalues is usually very rapidly converging, since a plot in the complex plane of the trajectory values at the opposite boundary will usually give an indication of where the zero lies in the plane. The storage space required is on the order of the number of points approximating a linear distance in the reactor, and may be even less if a differential equation is used to generate values of N_{x_o} and $\underline{\Phi}_o$ at each point. Only a small number of arithmetic operations are needed, so that trajectory computation times can be on the order of a few seconds on a large computer. Thus it can be seen that this method gives a very quick way to evaluate the Kaplan modes for any multiregion reactor in which the Kaplan modes are spatially separable.

CHAPTER VI

A METHOD OF ESTIMATING THE EFFECTS OF SPATIAL VARIATION ON STABILITY

This chapter presents a method of estimation of the eigenvalues of the Kaplan modes by a variational procedure. Spatial stability and eigenvalue determination are interrelated, as inspection of equation IV. 18 shows that the reactor is strictly stable if and only if the real part of each eigenvalue is less than zero. Since many spatial processes may be put in the form of the state equation, this method gives a means of estimating the effects of spatial changes upon the stability of a broad class of physical processes. The method also provides a means of finding an initial estimate of the eigenvalues to start the shooting procedure described in the previous chapter. Finally, the need for a spatial control system may be determined by the use of the method to evaluate the spatial stability of the reactor.

It has been tacitly assumed in all papers dealing with reactor spatial stability[11-44] that investigation of the uncontrolled reactor is sufficient. This is valid only in cases of no interaction with the power control system, as was shown by equation III. 7. If the interaction is negligible, the question of the spatial stability of the hot poisoned reactor, as discussed qualitatively in the introduction, is equivalent to finding the Kaplan eigenvalue with the largest real part, as represented by equation IV. 7. If the interaction is not negligible, equation IV. 7 may be extended to include the dynamics of the particular power control system used. The method of stability

estimation can then be applied to the controlled as well as the uncontrolled model to determine its stability.

The variational method is applied to estimate the eigenvalues of equation IV. 7, although it is applicable to the general form IV. 8.

The Variational Estimate When D is Constant

First, consider the case when the diffusion coefficient, D, is constant. Then the scalar form of IV. 7, found in the previous chapter as V. 3, can be used to form a Lagrangian, as in VI. 1.

$$L = \int_V \chi_m^* \left[D\nabla^2 + \nu\epsilon\Sigma_f - \Sigma_a - \sigma_a^x N_{x_o} - \frac{\omega_m}{v} - \frac{\sigma_a^x \Phi_o}{\omega_m + \lambda_x + \sigma_a^x \Phi_o} \right.$$

$$\left. \left(\frac{\gamma_I \epsilon \Sigma_f \lambda_I}{\omega_m + \lambda_I} + \gamma_x \epsilon \Sigma_f - \sigma_a^x N_{x_o} \right) \right] \chi_m \, d^3r \qquad \text{(VI. 1)}$$

In equation VI. 1, the value of ω_m is to be regarded as a fixed complex constant. Then VI. 1 can be rewritten in the shorthand notation of VI. 2.

$$L = \int_V \left(\chi_m^* \; D\nabla^2 + \mathcal{M}(\bar{r}) \right) \chi_m \, d^3r \qquad \text{(VI. 2)}$$

Suppose an estimate, χ_m, can be formed to approximate the shape of $\psi_m^{(3)}$, so that ϵ is much less than one in equation VI. 3.

$$\chi_m(\bar{r}) = \psi_m^{(3)}(\bar{r}) + \epsilon \, \delta\chi_m(\bar{r}) \qquad \text{(VI. 3)}$$

Note that χ_m and $\psi_m^{(3)}$ are zero on the boundary of the reactor, so $\delta\chi_m$ is also zero on the reactor boundaries.

Now the stationarity of the Lagrangian, VI. 2, will be shown. The expression for χ_m, VI. 3, is used to expand the Lagrangian.

$$L = \int_V \psi_m^{(3)*}(D\nabla^2+\mathcal{M})\,\psi_m^{(3)}d^3r + \epsilon \int_V \delta\chi_m^*(D\nabla^2+\mathcal{M})\,\psi_m^{(3)}d^3r$$

$$+\epsilon\int_V \psi_m^{(3)*}(D\nabla^2+\mathcal{M})\,\delta\chi_m d^3r+\epsilon^2\int_V \delta\chi_m^*(D\nabla^2+\mathcal{M})\,\delta\chi_m d^3r \qquad \text{(VI. 4)}$$

The first two terms are zero by equation V. 3, and the third may be eliminated by using Green's theorem, VI. 5.

$$\int_V \psi_m^{(3)*}\nabla^2\delta\chi_m d^3r = \int_S(\psi_m^{(3)*}\nabla\delta\chi_m - \delta\chi_m\nabla\psi_m^{(3)*})d^2r$$

$$+\int_V \delta\chi_m\nabla^2\psi_m^{(3)*}d^3r \qquad \text{(VI. 5)}$$

Since both $\psi_m^{(3)}$ and $\delta\chi$ are zero on the reactor extrapolated surface, the surface term of VI. 5 is zero. Therefore, the third term of VI. 4 may be represented as in VI. 6.

$$\int_V \psi_m^{(3)*}(D\nabla^2+\mathcal{M})\,\delta\chi_m d^3r = \int_V \delta\chi_m(D\nabla^2+\mathcal{M})\,\psi_m^{(3)*}d^3r \qquad \text{(VI. 6)}$$

Since the set of $\{\omega_m^*\}$ is taken to be the set of $\{\omega_m\}$ from the discussion in Chapter IV, $\psi_m^{(3)*}$ is a solution of VI. 7.

$$(D\nabla^2+\mathcal{M})\psi_m^{(3)*} = 0 \qquad \text{(VI. 7)}$$

Therefore, the Lagrangian, VI. 1, is stationary at the mode shapes of the flux and its adjoint. If the shape χ_m is chosen such that ϵ is small, VI. 4 can be rewritten as VI. 8.

$$0 = \int_V \chi_m^* (D\nabla^2+\mathcal{M})\chi_m d^3r + O(\epsilon^2) \qquad \text{(VI. 8)}$$

If the real and imaginary part of the guessed shape χ_m is taken to be the unpoisoned reactor flux shape θ_m, and χ_m^* to be θ_m^+, then VI. 8 becomes VI. 9.

$$0 = a_m^2 - \int_V \left[\sigma_a^x N_{x_o} + \frac{\omega_m}{v} + \frac{\sigma_a^x \Phi_o}{\omega_m^+ \lambda_x^+ \sigma_a^x \Phi_o}\left(\frac{\gamma_I \epsilon \Sigma_f \lambda_I}{\omega_m^+ \lambda_I} + \gamma_x \epsilon \Sigma_f - \sigma_a^x N_{x_o}\right)\right]\theta_m^2 d^3r + O(\epsilon^2) \qquad \text{(VI. 9)}$$

This can be put in more familiar form if all reactor parameters are constant and VI. 9 is divided by Σ_a.

$$0 = k^\infty - 1 - L^2 B_m^2 - \ell\omega_m - \frac{\sigma_a^x}{\Sigma_a}\int_V \frac{N_{x_o}(\omega_m^+\lambda_x^+) + \epsilon\Sigma_f\Phi_o\left(\frac{\gamma_I\lambda_I}{\omega_m^+\lambda_I} + \gamma_x\right)}{\omega_m^+\lambda_x^+\sigma_a^x\Phi_o}\theta_m^2 d^3r + O(\epsilon^2) \qquad \text{(VI. 10)}$$

In the examples of Chapter VIII, the real and imaginary parts of ω_m to solve VI. 10 are merely guessed, and a very rapidly converging

trial and error procedure enables close determination of ω_m (see Table VIII. 3). Therefore, equation VI. 9 provides a useful means of estimating the effect of spatial variations on the kinetic behavior of reactor processes.

As further verification of the utility of VI. 9, the Lellouche xenon stability criterion,[34] used in the case of infinite flux, may be derived from VI. 9. In the limit of Φ_o tending to infinity, equation VI. 10 becomes VI. 11.

$$0 = k^{\infty} - 1 - L^2 B_m^2 - \ell\omega_m - \frac{k^{\infty}}{\nu}\left(\frac{\gamma_I \lambda_I}{\omega_m + \lambda_I} + \gamma_x\right) + O(\epsilon^2) \qquad \text{(VI. 11)}$$

Application of the Routh-Hurwitz[63] stability criterion to VI. 11 gives the conditions for stability, VI. 12.

$$k^{\infty} - 1 - L^2 B_m^2 - \frac{k^{\infty}}{\nu}\gamma_x + O(\epsilon^2) < \min\left(\ell\lambda_I, \frac{k^{\infty}}{\nu}\gamma_I\right) \qquad \text{(VI. 12)}$$

Equation VI. 12, without $O(\epsilon^2)$, may be recognized as the Lellouche stability criterion, which was obtained from the original non-linear state equations (that are linear in the limit of infinite flux).

An Explicit Expression for the Eigenvalues

An explicit expression for the eigenvalues may be obtained by choosing χ_m as in VI. 13.

$$\chi_m = (\omega_m + \lambda_x + \sigma_a^x \Phi_o)\theta_m(r) \qquad \text{(VI. 13)}$$

The term $\sigma_a^x \Phi_o$ usually dominates in the high flux power reactors considered, especially when ω_m is small. Therefore, VI. 13

probably overestimates the flattening that usually occurs in the flux upon the poisoning of the clean reactor. This overestimation increases the error, $O(\epsilon^2)$, in the Rayleigh product. However, the eigenvalues calculated from this approximate shape may provide some indication of the error involved, when compared with the eigenvalues obtained from VI. 9. The two calculations may bound the eigenvalue, because VI. 9 underestimates the flattening. Furthermore, the estimate obtained from this simpler calculation can start the iteration procedure needed to solve VI. 9. For the constant coefficient case, putting the approximate form, VI. 13, in the Lagrangian equation, VI. 8, gives the next equation, VI. 14.

$$\left[(\omega_m+\lambda_x)^2 + 2(\omega_m+\lambda_x)\,\sigma_a^x\,\overline{\Phi}_o + (\sigma_a^x)^2\,\overline{\Phi}_o^2\right]\left[k^\infty - 1 - L^2 B_m^2 - \ell\,\omega_m\right]$$

$$+ (\omega_m+\lambda_x)^2\,\frac{\sigma_a^x}{\Sigma_a}\,\overline{N}_{x_o} + \frac{(\sigma_a^x)^2}{\Sigma_a}\,(\omega_m+\lambda_x)\,\overline{N_{x_o}\Phi_o}$$

$$+ \sigma_a^x k^\infty\left[\gamma_x + \frac{\lambda_I}{\omega_m+\lambda_x}\,\gamma_I\right]\left[\omega_m\,\overline{\Phi}_o + \lambda_x\,\overline{\Phi}_o + \sigma_a^x\,\overline{\Phi}_o^2\right] \qquad \text{(VI. 14)}$$

$$+ \sigma_a^x L^2(\omega_m+\lambda_x)\,\overline{\nabla^2\Phi_o} + L^2(\sigma_a^x)^2\,\overline{\Phi_o\nabla^2\Phi_o} + O(\epsilon^2) = 0$$

The bar denotes spatial averaging of the quantity under the bar with respect to θ_m^2. Equation VI. 14 is a constant coefficient quartic equation in ω_m, which may be solved by standard procedures. The extraneous root introduced usually causes no difficulty, since it lies near $-\lambda_x - \sigma_a^x\Phi_o$. In the limit of infinite steady state flux, the stability

criterion becomes equation VI. 15 if $\nabla^2 \underline{\Phi}_o = -B_o^2 \underline{\Phi}_o$.

$$k^\infty - 1 - L^2 B_m^2 - L^2 B_o^2 - \frac{k^\infty \gamma_x}{\nu} + O(\epsilon^2) \leq \min(\ell \lambda_I, \frac{k^\infty \gamma_I}{\nu}) \qquad \text{(VI. 15)}$$

The appearance of the term $L^2 B_o^2$ is a serious error, and illustrates the breakdown of the assumed form VI. 13 in the limit of infinite flux. However, for low flux values, the criterion approaches that of the clean reactor and should be a good estimate. Also, the relative simplicity of VI. 14 usually makes the calculation of this estimate of the eigenvalues worth while.

The Variational Estimate When D is a Function of Space

Now, a somewhat more complicated Lagrangian is formed for the case when the diffusion coefficient D is a function of space. The Lagrangian formed by importance function and the flux has been found to be stationary in all cases by Lewins.[64-66] The equations for the linearized importance deviations may be developed in a manner similar to the development of the linearized flux deviations. Therefore, in the state space notation used previously, VI. 16 is stationary.

$$\int_V \bar{\chi}_m^+ \mathcal{L}(\bar{r})\, \bar{\chi}_m \, d^3 r = \omega_m \int_V \bar{\chi}_m^+ \mu(\bar{r})\, \bar{\chi}_m \, d^3 r + O(\epsilon^2) \qquad \text{(VI. 16)}$$

Here, $\bar{\chi}_m$ and $\bar{\chi}_m^+$ are the approximations to the vector eigenfunctions $\bar{\psi}_m$ and $\bar{\psi}_m^+$. The elements of $\bar{\chi}_m$ may be formed from the

approximation of the thermal flux by means of equations similar to V. 2. The elements are thus functions of ω_m, so that VI. 16 may be transcendental in ω_m. If this is not desirable, some elements of $\bar{\chi}_m$ may have to be guessed independently. This probably increases the magnitude of the error term. A further drawback is that VI. 16 contains many more terms than the Lagrangian, VI. 1, of the scalar equation.

Finally, one small brief mathematical point will be brought up in this estimation problem. If all elements of $\bar{\chi}_m$ must be guessed independently, the relationship VI. 17 is probably stationary in ω_m even in the case of multiple eigenvalues.

$$\omega_m = \frac{\int_V \bar{\chi}_m^+ \mathcal{L} \bar{\chi}_m \, d^3r}{\int_V \bar{\chi}_m^+ \mu \bar{\chi}_m \, d^3r} + O(\epsilon^2) \qquad \text{(VI. 17)}$$

In the finite difference case, Ostrowski[67] has shown this expression to be stationary if all elementary divisors are linear. He has also given an example of a non-stationary point if the elementary divisors are non-linear. But the assumption of the diagonalizeability of the finite difference operator implies linear elementary divisors,[68] so VI. 17 may be assumed stationary.

Relations With Reactor Transfer Functions

The transfer function of a uniform reactor, considering only the thermal flux, is equivalent to the spatial variational equation VI. 10 with σ_a^x and $O(\epsilon^2)$ set equal to zero and the fundamental

mode shape considered. If the effect of delayed neutrons had been included in the original model, IV. 7, then VI. 10 would yield the familiar inhour equation.[1] It might be said that the variational procedure yields a point reactor transfer function that incorporates the effect of spatial variations. The effect of spatial variations is currently the topic of much research interest.[69-71] The only drawback of the procedure is the transcendental nature of the expression for ω_n in most cases.

The analysis by Kaplan modes shows that the transfer function itself does not "vary in space". What really happens is that, in the non-constant coefficient case, the modal shapes are not exactly the same for each root of the inhour equation. The inhour equation then corresponds to the scalar equation for the Kaplan modes. The changes in shape of the modes are illustrated by Figures VIII, 4, 5, and 6, giving the shapes associated with different values of the Kaplan eigenvalues for the xenon case.

CHAPTER VII

ANALYTICAL DESIGN OF A SPATIAL FEEDBACK CONTROL SYSTEM

Three fundamental questions must be investigated before a spatial control system can be proposed. Is it possible to control the reactor with a given number of control rods? What types of input are encountered? What kind of response is desired from the control system? These questions will be discussed before a method of spatial control system design is presented.

Controllability

Prevalent opinion[22] seems to be that many regional control rods are necessary to control spatial oscillations in a reactor. Appendix C gives mathematical proof that the number of control rods needed is only equal to the maximum multiplicity of the Kaplan operator, plus those needed for power control. In light of the assumption that only geometrical multiplicities are permitted, this means that spatial instabilities in a reactor of one, two, or three spatial dimensions need be controlled by at most one, two, or three extra control rods, regardless of how many modes are unstable. The proof is given formally as a theorem in Appendix C. Although relegated to an appendix in order to enhance readability, this theorem is one of the major conclusions of the thesis. The theorem applies only to a linear model, and so is not valid for large inputs that necessitate use of a non-linear model.

It is assumed that there are only a finite number of unstable modes and perhaps a number of stable modes, all of which are controllable. This leaves an infinite number of stable modes uncontrolled. Since the uncontrolled modes can be chosen to be the more stable ones, their damping of any input is usually acceptable. By neglecting the infinite number of the more stable uncoupled Kaplan modes, the problem is reduced to that of controlling a finite number of modes. The controllability theorem gives the conditions under which any finite number of modes can be controlled.

The controllability theorem states that any finite number of modes of the reactor model III. 13 can be returned to zero by the action of a control if:

1) There are at least as many control rods as the maximum multiplicity of the spatial operator IV. 8.
2) All control rods are not on any possible nodes of a combination of modes having the same eigenvalue.

Note that although a reactor may be controllable with only a few control rods, this does not mean that response to an input cannot be improved by the addition of more control rods.

The Input

Excitations of the spatial control system usually come from two different types of processes. One process, an input, comes from changes in the operating level, following the commands of the reactor operators. Another type of excitation comes from the random

disturbances inherent within a reactor, as mentioned at the end of Chapter II.

Since the Kaplan modes permit complete uncoupling of the fundamental from the harmonics, the power control system will not excite the harmonics by inter-mode coupling. Furthermore, if condition III. 13 is met, small power adjustments (that keep the system nearly linear) will not provide an input. Therefore, only the larger, non-linear shifts in the operating level will affect the spatial control system. These shifts in the operating level are brought about suddenly by the attempts of the system to compensate for a large load change or a change in the steady state reactor properties. These shifts in operating level usually occur infrequently, and so may be approximated by a step input into the system.

The random disturbances inherent in the reactor processes come from the random variations of the reactor parameters. These random parameter variations are usually uncorrelated.[72]

Since most of the excitations are of these two types, the spatial control system should be designed to give optimal response to step function inputs and white noise parametric excitation, in addition to acceptable response to other excitations.

The Criterion

To evaluate the effectiveness of a control system, some number should be determined to measure the controlled response to a class of inputs. Since the purpose of the reactor spatial control system is to

return the flux deviation to zero, a measure of the flux deviation over time is a good criterion by which the control system design can be evaluated. This flux measurement can be taken in many ways. For example, one way is to calculate the expected number of deviations occurring that are greater than a certain value. Of course, both positive and negative values must count, or a control system that continually made the negative deviations very large would be considered very good. Therefore, a measurement of the flux deviations must lead to a positive definite criterion, such as $\left|\phi(\bar{r}, t)\right| \leq f(\bar{r}, t)$ or $\min \int_V \int_{t_o}^{t_1} \left[\phi(\bar{r}, t)\right]^{2a} f(\bar{r}, t) dt\, d^3r$, where $f(\bar{r}, t)$ is greater than zero and a is a positive integer.

However, it is not enough to limit the deviations of the flux. As is known from the section on controllability, the motions of a few control rods can essentially return the flux deviations to zero. By making the rod motions large enough, the deviations can be returned to zero almost immediately. Large and quick control rod motions are not permitted in the practical case of an operating reactor. The control rod motion should be limited so that the reactivity never exceeds prompt critical. Furthermore, one of the assumptions of the linearization was that of small control rod motions. For these reasons, a positive definite measurement must be used for control rod motion. Similarly, these may be of the form of $\left|u_n(t)\right| \leq f_n(t)$ or $\min \int_{t_o}^{t_1} \left[u_n(t)\right]^{2a} X$ $f_n(t)dt$. Therefore, a criterion to measure the effectiveness of a reactor spatial control system must usually take into account the control rod motion, as well as the flux deviation.

In summary, the conditions under which control is possible, the types of input, and the types of criteria have been discussed. The design of any spatial control system must take these factors into account. The next section gives a method for design of a spatial control system that is optimal for a certain type of criterion and input.

Feedback Control Design for Minimum Integrated Quadratic Loss

This section presents a method of analytical design of a reactor spatial control system that is optimal, in the sense that it minimizes the integrated quadratic loss for any initial condition of the state variable as an input. A criterion formed from a quadratic of the flux deviation plus a quadratic of the control rod motion forms a quadratic loss matrix. The system minimizing the integral of the quadratic loss over the reactor volume and response time is considered optimal.

The selection of initial conditions as an input and integrated quadratic loss as a criterion may not be the best for every reactor spatial control system. The methods and results obtained by the study of the control system resulting from these selections, however, may serve as a guide to the design, or as a measure of the effectiveness, of any other system designed by any other means. The usefulness of these input and criterion selections stems from their ability to yield a very simple constant linear feedback, in addition to the fact that there are physical foundations for their selection.

The two most commonly occurring excitations of the control system, step function inputs and uncorrelated parameter variations, are equivalent to the initial condition input under a quadratic loss criterion. If all quantities are related to the steady state resulting as $t \rightarrow \infty$ after the application of a step, the problem is "turned around" so that the old steady state conditions become initial conditions for the new state, as the "height" of the step. In this manner, a step function input may be treated as an initial condition input. Also, for the finite mode case, it has been shown[73, 74] that the feedback calculated for uncorrelated parameter variation is exactly that calculated in the initial condition input case for the quadratic loss criterion. Since the step function input and parameter variation are equivalent to initial condition input, feedback calculated from the initial condition input will be optimal for most excitations encountered in the reactor.

In addition to being a criterion that has some physical foundations, the quadratic criterion yields a linear system as a feedback. A linear system that is strictly stable for any initial value is bounded for all bounded inputs. Therefore, this feedback gives a stable and probably acceptable response for any input small enough to preserve the linearity of the system.

Furthermore, the quadratic criterion can be formed quite generally, so that the addition of weighting functions can change its character quite a bit. First, any part or function of the state

variable can be included in the response measurement by changing variables to form the quadratic, as in VII. 1.

$$\bar{y}(\bar{r}, t) = \mathcal{H}(\bar{r})\, \bar{z}(\bar{r}, t) \tag{VII. 1}$$

Specifically, in this case the deviation of the flux is probably the quantity of interest.

$$\phi(\bar{r}, t) = (0 \quad 0 \quad 1) \begin{pmatrix} i(\bar{r}, t) \\ x(\bar{r}, t) \\ \phi(\bar{r}, t) \end{pmatrix} \tag{VII. 2}$$

Next a general weighting function in space can be incorporated into the quadratic by defining a symmetric matrix that is a positive definite function of $\bar{r}$, denoted $\frac{1}{2}\mathcal{Q}(\bar{r})$. Then the integrated quadratic measure of the response can be represented as VII. 3.

$$\frac{1}{2}\int_{t_o}^{t_1}\int_{V} \bar{z}^{T}\mathcal{H}^{T}\mathcal{Q}\mathcal{H}\bar{z}\, d^{3}\bar{r}\, dt \tag{VII. 3}$$

It is possible to form a linear, time-varying feedback if $\mathcal{H}$ and $\mathcal{Q}$ are extended to include time varying functions.[75]

Similarly, a quadratic in the control effort can be formed by defining a constant coefficient symmetric positive definite matrix $\frac{1}{2}R$. The elements in R are chosen according to the relative motion desired in each control rod. Then the measure of the control rod motion is VII. 4.

$$\frac{1}{2}\int_{t_o}^{t_1} \bar{u}^T R \bar{u}\, dt \quad , \tag{VII. 4}$$

The two measures, VII. 3 and VII. 4, are added together and their sum is a criterion of the "optimality" of the system. In general, if better response is desired in a given feedback system, it can be obtained only at the expense of increased control rod travel. Conversely, control rod travel may be reduced only if poorer response is acceptable. This is the reason for using the sum of the two measures as the criterion of optimality.

Since each measure is determined to an arbitrary multiplicative constant, one "constant of proportionality" must be determined to judge the importance of flux deviation compared to control rod motion. This essentially determines the stability margin of the system. The constant of proportionality may be looked upon as the price one is willing to pay for better response in terms of control rod motion. This constant may be determined by trial and error to give the minimum control rod motion and maximum stability margin.

It should be noted here that the criterion must converge. One control system cannot be compared with another if they both yield an *infinite criterion. This is an important and subtle point, especially if* $t_1 \rightarrow \infty$. Since the power control system must be free to respond to the command of the operators, the deviation of the fundamental must not enter into the criterion. Therefore, the criterion must be made up of the deviation of the harmonics and of those control rods which are

driven by the spatial control system.

In light of the above remarks, the problem may be stated mathematically in terms of the calculus of variations. Find the infimum of VII. 5 subject to the constraint VII. 6.

$$= \frac{1}{2}\int_{t_o}^{t_1}\int_{V} \bar{z}_H^T \mathcal{H}^T \mathcal{Q} \mathcal{H} \bar{z}_H \, d^3\bar{r}\, dt + \frac{1}{2}\int_{t_o}^{t_1} \bar{u}_{N-1}^T R \bar{u}_{N-1} dt \qquad \text{(VII. 5)}$$

Here, VII. 6 is written for any linear model of the reactor.

$$\mathcal{M}\dot{\bar{z}}_H = \mathcal{L}\bar{z}_H + \mathcal{G}_{N-1}\bar{u}_{N-1} \qquad \text{(VII. 6)}$$

If the state variable $\bar{z}$ is representable as the infinite sum of Kaplan modes, a truncation to a finite number of modes may be immediately put in the form of present theory[75-81] by integrating the spatial terms to arrive at a constant coefficient time dependent problem. However, to investigate the conditions under which the finite mode solution tends to that of the infinite mode solution, the problem will be treated in the infinite case for a while longer.

Accordingly, define a Lagrange multiplier $\bar{\xi}_H(\bar{r}, t)$, a real vector function of space and time. This may be formally identified as the importance, in the sense of Lewins.[65] From VII. 6, the inner product with $\bar{\xi}_H$ is zero.

$$\int_{t_o}^{t_1}\int_{V} \bar{\xi}_H^T (\mathcal{L}\bar{z}_H + \mathcal{G}_{N-1}\bar{u}_{N-1} - \mathcal{M}\dot{\bar{z}}_H) d^3r\, dt = 0 \qquad \text{(VII. 7)}$$

Since VII. 7 is equal to zero, it may be added to VII. 5 without changing its value.

$$\mathcal{V} = \int_{t_o}^{t_1} \left(\frac{1}{2} \bar{u}_{N-1}^T R \bar{u}_{N-1} + \int_V \bar{\xi}_H^T \mathcal{G}_{N-1} d^3r \, \bar{u}_{N-1} \right) dt$$

$$+ \int_{t_o}^{t_1} \int_V \left(\frac{1}{2} \bar{z}^T \mathcal{H}^T Q \mathcal{H} \bar{z} + \bar{\xi}_H^T \mathcal{L} z_H - \bar{\xi}_H^T \mu \dot{z} \right) d^3r \, dt \qquad \text{(VII. 8)}$$

The first integral is a minimum if the gradient of the integrand with respect to $\bar{u}_{N-1}$ is zero.

$$R \bar{u}_{N-1} + \int_V \bar{\xi}_H^T \mathcal{G}_{N-1} d^3r = \bar{0} \qquad \text{(VII. 9)}$$

Since R is positive definite, it can be inverted.

$$\bar{u}_{N-1} = -R^{-1} \int_V \xi_H^T \mathcal{G}_{N-1} d^3r \qquad \text{(VII. 10)}$$

Equation VII. 10 gives the optimal control in terms of the adjoint. The criterion $\mathcal{V}$ will then be a minimum if the adjoint is picked such that the second integral on the right of VII. 8 is a minimum. To calculate the minimum of this integral, the boundary conditions must be such that it is equal to the adjoint integral.

$$\int_{t_o}^{t_1} \int_V \left(\bar{\xi}_H^T \mathcal{L} \bar{z}_H - \bar{\xi}_H^T \mathcal{M} \dot{\bar{z}} \right) d^3r\,dt = \int_{t_o}^{t_1} \int_V \left(\bar{z}_H^T \mathcal{L}^T \bar{\xi}_H + \bar{z}_H^T \mathcal{M}^T \dot{\bar{\xi}}_H \right) d^3r\,dt \tag{VII. 11}$$

Using certain physical arguments, Lewins[66] ascribes a physical meaning to the adjoint in this case, calling it the importance. If the physical arguments of Lewins can be accepted, then VII. 11 is true for any diffusion coefficient that is a positive function of space. If his arguments are not acceptable, the diffusion coefficients must be a constant and Green's theorem invoked to show VII. 11. This is quite similar to the variational procedure undergone in Chapter VI (see especially equation VI. 5), except that the reactor model may have many diffusion coefficients. Using VII. 11, the gradient of the second integrand of VII. 8 with respect to $\bar{z}_H$ is set equal to zero to assume a minimum for the criterion.

$$\mathcal{M}^T \dot{\bar{\xi}}_H = - \mathcal{L}^T \bar{\xi}_H - \mathcal{H}^T \mathcal{Q} \mathcal{H} \bar{z} \tag{VII. 12}$$

Equation VII. 10 can be put into the system equation, VII. 6, to form another equation dependent only on $\bar{\xi}$ and $\bar{z}$.

$$\mathcal{M} \dot{\bar{z}} = \mathcal{L} \bar{z} - \mathcal{G}_R^{-1} \int_V \bar{\xi}_H^T \mathcal{G}_{N-1}\, d^3r \tag{VII. 13}$$

Equations VII. 12 and 13 form a distributed parameter analogy to Kalman's "canonical equations".[75]

Having derived the canonical equations, their solution reverts to calculating the feedback for a finite number of Kaplan modes. Note,

however, that the equations developed for a finite number of modes from the canonical equations, VII. 12 and 13, give the same system as if the spatial dependence had been integrated out initially. This is true whether or not the diffusion coefficients are functions of space, and so lends credence to Lewins' physical arguments.

To apply specifically the methods of Kalman[75] for determining the feedback, the canonical equations must be reduced to a finite set of equivalent real, constant coefficient total differential equations in the modal time coefficients. Although these modal coefficients can be those of the clean reactor modes, for reasons previously stated the Kaplan modes will be used.

It has been shown in Chapter IV how the state equation may be reduced to an infinite set of complex time dependent equations for the Kaplan modal coefficients. The expansion for the state variable and the importance are written as VII. 14.

$$\bar{z}_H(\bar{r},t) = \sum_{m=0}^{\infty} a_m(t)\, \bar{\psi}_m(\bar{r})$$

$$\bar{\xi}_H(\bar{r},t) = \sum_{m=0}^{\infty} b^*_m(t)\, \hat{\bar{\psi}}_m(\bar{r}) \qquad \text{(VII. 14)}$$

According to the procedures described in Chapter IV, the canonical equations for the modal time coefficients are given as VII. 15.

$$\dot{a}_m = \omega_m a_m + \langle \hat{\bar{\psi}}_m, \mathcal{G} \rangle R^{-1} \langle \sum_{n=0}^{\infty} b^* \hat{\bar{\psi}}, \mathcal{G} \rangle$$

$$\dot{b}^*_m = -\omega_m {}^* b^*_m - \langle \bar{\psi}_m, \mathcal{H}^T Q \mathcal{H} \sum_{n=0}^{\infty} a_n \bar{\psi}_n \rangle \qquad \text{(VII. 15)}$$

When these equations are truncated to M modes, they may always be put into real form by separation into real and imaginary parts, as described in Appendix C. Then the equations VII. 15 are denoted as in VII. 16.

$$\dot{\bar{a}} = F \bar{a} - G R^{-1} G^T \bar{b}$$

$$\dot{\bar{b}} = -F^T \bar{b} - Q \bar{a} \qquad \text{(VII. 16)}$$

In VII. 16, the following definitions have been used:

$$\bar{a} = \begin{pmatrix} \text{Re}\{a_1\} \text{ or } \text{Im}\{a_1\} \\ \vdots \\ \text{Re}\{a_M\} \text{ or } \text{Im}\{a_M\} \end{pmatrix} \quad \bar{b} = \begin{pmatrix} \text{Re}\{b^*_1\} \text{ or } \text{Im}\{b^*_1\} \\ \vdots \\ \text{Re}\{b^*_M\} \text{ or } \text{Im}\{b^*_M\} \end{pmatrix}.$$

where

$$\bar{z}(r, t) = \sum_{m=0}^{M} a_m(t) \bar{\psi}_m(\bar{r}) \quad \text{and} \quad \bar{\xi}(\bar{r}, t) = \sum_{m=0}^{M} b^*_m(t) \hat{\bar{\psi}}_m(\bar{r}),$$

$$G_{mn} = \left\{ \langle \hat{\bar{\psi}}_m, \mathcal{G}_n \rangle \right\}, \quad F = \text{direct sum of } \omega_m I \text{ or } \begin{pmatrix} \rho_m & -\sigma_m \\ \sigma_m & \rho_m \end{pmatrix}$$

where $\omega_m = \rho_m + i \sigma_m$, according to Appendix C, and

$$Q_{mj} = \{ < \bar{\psi}_m, \mathcal{H}^T \mathcal{Q} \mathcal{H} \bar{\psi}_j > \}$$

for m, j=1, . . . , M, the number of modes considered, and n=1, . . . , N, the number of control rods.

Having the form of VII. 16, the procedure of Kalman[75] can be used. Define an M x M time variable matrix P as in VII. 17.

$$\bar{b}(t) = P(t)\, \bar{a}(t) \tag{VII. 17}$$

The validity of this substitution and the proof that P(t) is positive definite and symmetric are given in Kalman's paper. Putting VII. 17 into VII. 16 gives VII. 18.

$$\begin{aligned} \dot{\bar{a}} &= F\bar{a} - GR^{-1} G^T P\bar{a} \\ \dot{P}\,\bar{a} + P\,\dot{\bar{a}} &= -F^T P\bar{a} - Q\bar{a} \end{aligned} \tag{VII. 18}$$

The top expression for $\dot{\bar{a}}$ can be substituted into the bottom expression.

$$-\dot{P}\,\bar{a} = PF\bar{a} + F^T P\bar{a} - PGR^{-1} G^T P\bar{a} + Q\bar{a} \tag{VII. 19}$$

This is true for all $\bar{a}$, and so a matrix equation for P is obtained.

$$-\dot{P} = PF + F^T P - PGR^{-1} G^T P + Q \tag{VII. 20}$$

Kalman has shown that the unique, stable solution of this matrix Riccati equation backwards in time is the desired solution. Therefore, one may always compute a positive definite symmetric matrix solution to VII. 20 by computing, on a computer, the solution to the initial value problem VII. 21.

$$\dot{P} = PF + F^T P - PGR^{-1} G^T P + Q \qquad \text{(VII. 21)}$$

with P(0) = O. The steady state solution to the initial value problem VII. 21 is then the constant coefficient matrix that determines the constant linear feedback to form $\bar{u}(t)$ as in VII. 22.

$$\bar{u}(t) = -R^{-1} \int_{\text{reactor } V} \bar{\xi}_H^T \mathcal{Y}_{N-1} d^3r = -R^{-1} \int_{\text{reactor } V} \left[\sum_{m=1}^{M} b^*(t) \hat{\bar{\psi}}_m(\bar{r}) \right]^T \mathcal{Y}_{N-1} d^3r \qquad \text{(VII. 22)}$$

Here, the b*(t) are found by VII. 17.

In summary, Kalman[75] has given a procedure for determining the constant coefficient optimal feedback control of the finite mode model. This procedure is quite useful as a guide for suggesting the form of spatial control systems, as is shown in the examples in the next chapter. Its usefulness stems from the fact that only a few modes in a reactor really need to be controlled, if the control system does not interact with the uncontrolled modes.

A Discussion of the Infinite Mode Feedback

It is of interest to know the conditions under which the solution for the finite mode case tends to the solution in the infinite mode case. Knowing these conditions will give an indication of the usefulness of including more and more modes into the finite mode design. If the sum of the remaining modes, $\sum_{n=M+1}^{\infty} a_m(t) \bar{\psi}_m(\bar{r})$, tends to zero for M tending to infinity, then there is justification for neglecting the higher modes because they contribute nothing to the measure of the

deviation and need no separate control.

It can be shown that the sum of the remaining modes tends to zero under certain conditions. If $\mathrm{Re}\{\omega_m\} = \rho_m \leq -c_1 m^{c_2}$ for all $m > M$, where c_1 and c_2 are any constants such that $c_1 > 0$ and $c_2 > 1$, then for any bounded control or input, the limit of the sum of uncontrolled modes, $\lim_{M\to\infty} \sum_{m=M+1}^{\infty} a_m(t)\,\bar{\psi}_m(\bar{r})$, equals zero for all $t > 0$.

To demonstrate this, each term in the sum will be shown to be smaller in absolute value than a term in a sum that tends to zero. Equation VII. 23 is obtained using the Schwartz inequality.

$$\sum_{m=M+1}^{\infty} a_m(t)\,\bar{\psi}_m(\bar{r}) \leq \left|\psi_m(\bar{r})\right|_{\max} \sum_{m=M+1}^{\infty} \left|a_m(t)\right| \qquad \text{(VII. 23)}$$

From IV. 18 is obtained an expression for $a_m(t)$, where $\bar{u}(\tau)$ may contain a continuous input as well as a control.

$$a_m(t) = a_m(0)e^{\omega_m t} + \langle \hat{\bar{\psi}}_m, \mathcal{Q} \rangle \int_0^t e^{\omega_m(t-\tau)}\,\bar{u}(\tau)\,d\tau \qquad \text{(IV. 18)}$$

Equation IV. 18 is put into VII. 23.

$$\sum_{m=M+1}^{\infty} \left|a_m(t)\right| \leq \left|a_m(0)\right|_{\max} \sum_{m=M+1}^{\infty} e^{\rho_m t}$$

$$+ \left|u_i\right|_{\max} \left|\langle \hat{\bar{\psi}}_m, \mathcal{Q} \rangle_j\right|_{\max} \sum_{m=M+1}^{\infty} e^{\rho_m t} \int_0^t e^{-\rho_m \tau}\,d\tau \qquad \text{(VII. 24)}$$

Note the u_i are bounded by hypothesis and the $a_m(0)$ and $\bar{\psi}_m(\bar{r})$ are bounded because $\bar{z}(r, 0)$ is bounded. The integral in VII. 24 may be evaluated easily.

$$e^{\rho_m t}\int_0^t e^{-\rho_m \tau}\, d\tau = \frac{1-e^{\rho_m t}}{-\rho_m} < -\frac{1}{\rho_m} \qquad \text{(VII. 25)}$$

This last inequality follows because ρ_m is hypothesized negative for all $m > M$. Using VII. 25, VII. 24 is rewritten as VII. 26.

$$\sum_{m=M+1}^{\infty} |a_m(t)| \leq |a_m(0)|_{max} \sum_{m=M+1}^{\infty} e^{\rho_m t}$$

$$+ |u_i|_{max} |\langle \hat{\bar{\psi}}_m, \vartheta \rangle_j|_{max} \sum_{m=M+1}^{\infty} \frac{1}{-\rho_m} \qquad \text{(VII. 26)}$$

The hypothesized inequality for the asymptotic form of ρ_m is now used.

$$\sum_{m=M+1}^{\infty} |a_m(t)| \leq |a_m(0)|_{max} \sum_{m=M+1}^{\infty} e^{-tc_1 m^{c_2}}$$

$$+ |u_i|_{max} |\langle \hat{\bar{\psi}}_m, \vartheta \rangle_j|_{max} \sum_{m=M+1}^{\infty} \frac{1}{c_1 m^{c_2}} \qquad \text{(VII. 27)}$$

For all $t > 0$, a suitable positive constant c_3 can be chosen such that VII. 28 holds.

$$\sum_{m=M+1}^{\infty} |a_m(t)| \leq c_3 \sum_{m=M+1}^{\infty} m^{-c_2} \leq c_3 \int_M^{\infty} \frac{d\zeta}{\zeta^{c_2}} = \frac{c_3}{(1-c_2)M^{c_2-1}} \qquad \text{(VII. 28)}$$

Since $\lim_{M \to \infty} \frac{c_3}{(1-c_2)M^{c_2-1}} = 0$ for $c_2 > 1$, and the $\overline{\psi}_m(\overline{r})$ are bounded, VII 28 may be used in VII. 23 to prove the hypothesis.

The asymptotic behavior of the eigenvalues of the Sturm-Liouville problem satisfies the hypothesized behavior $\text{Re}\{\omega_m\} \leq -c_1 m^{c_2}$ for all eigenfunctions except those over infinite or semi-infinite domains.[82] In the infinite or semi-infinite cases, the constant c_2 is equal to one, and not greater than one as required. However, all regions in the reactor optimization problem are considered bounded, so this restriction is of no great importance.

The restriction that is of great importance is that there can be only a finite number of eigenvalues whose real parts are greater than any real number. As was mentioned in Chapter IV, it is likely that any matrix differential operator that has a row with no operator on the diagonal will have a set of eigenvalues tending to some finite number. This is the case in the xenon oscillation problem. Physically, any motion of the control rods must disturb the xenon distribution, which cannot be damped out immediately by quick flux readjustments. Therefore, there are an infinite set of modes that do contribute to the criterion, despite the fact that they are stable. In light of this, it seems to be futile to try to improve performance by controlling more than a certain number of modes. Any truly optimal feedback that

would satisfy the canonical equations should include an infinite number of modes for this case. From the remarks following the controllability theorem in Appendix C, it would seem that point control rods would be inadequate for this, and perhaps something like a spatially distributed poison would be necessary. The Euler equations and feedback can be derived for this case, $\overline{u}(\overline{r}, t)$, but this is of no practical importance at present to reactor control system designers.

CHAPTER VIII

EXAMPLES

Example A: Types of Multiplicity Encountered in a Reactor

A geometrical multiplicity is encountered in the solution for the clean reactor modes in cylindrical geometry for the case of constant coefficients.

$$\alpha_m \theta_m(\bar{r}) = (D\nabla^2 + \nu\epsilon\Sigma_f - \Sigma_a)\,\theta_m(\bar{r}) \qquad \text{(IV. 1)}$$

Solutions to IV. 1 in this example are given in terms of sines and cosines in the azimuthal direction. For each mode containing a sine term, there is a corresponding mode with a cosine term that gives the same value of α. Spatial symmetry suggests a rotation of 90° to construct an orthogonal mode, so this is considered a geometrical multiplicity.

A non-geometrical multiplicity might arise in the case of a cylindrical reflected reactor, with regionally homogeneous compositions in the core and reflector. In a multi-region reactor like that pictured in Figure VIII. 1, the second harmonic of the flux deviation in one region may match the interface conditions of the first harmonic of the flux deviation in a neighboring region. Although it is doubtful this can happen, the possibility is still an open question, especially in reactors with a large number of regions.

This improbable circumstance might give rise to the two distinct modes pictured in Figure VIII. 1. These two different modes have the same eigenvalue, as can be seen from the symmetry about

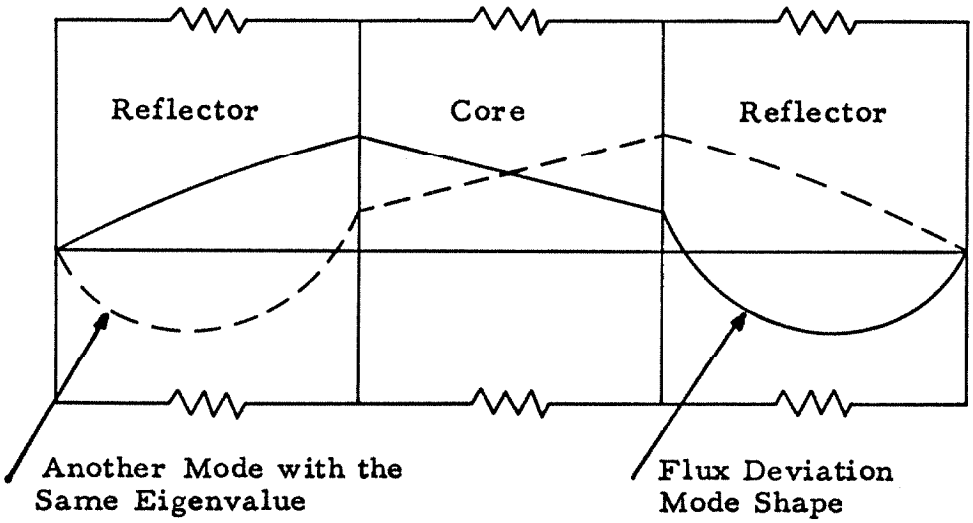


Fig. VIII.1 Illustration of a Non-Geometrical Multiplicity

the center line of the reactor. Whether it is possible to construct two orthogonal modes from the modes illustrated is not known. This is an illustration of non-geometrical multiplicity, because the modes are in one linear dimension and thus a shift of coordinates will not produce an orthogonal mode.

Note, however, that any asymmetric perturbation in materials properties or the placing of the boundaries will break the multiplicity. It is safe to say that any non-geometrical multiplicity would be a remarkable coincidence.

Example B: Asymptotic Eigenvalue Behavior of Kaplan Modes

Consider the Kaplan modal equation VIII. 1 in linear geometry, where a, b, and c are constants.

$$\omega_m \begin{pmatrix} \psi_m^{(1)} \\ \psi_m^{(2)} \end{pmatrix} = \begin{pmatrix} a & b \\ c & \frac{d^2}{dr^2} \end{pmatrix} \begin{pmatrix} \psi_m^{(1)} \\ \psi_m^{(2)} \end{pmatrix} \qquad \text{(VIII. 1)}$$

If the boundary conditions are $\bar{\psi}(0) = \bar{\psi}(h) = 0$, then $\bar{\psi}_m(r)$ is proportional to $\sin \frac{n\pi r}{h}$, and the eigenvalues are the solution of the characteristic equation VIII. 2.

$$(\omega_m - a)\left[\omega_m + \left(\frac{n\pi}{h}\right)^2\right] - bc = 0 \qquad \text{(VIII. 2)}$$

The solutions are shown as VIII. 3.

$$\omega_m = \frac{1}{2}\left[a - \left(\frac{n\pi}{h}\right)^2\right]\left\{1 \pm \sqrt{1 + \frac{4a\left(\frac{n\pi}{h}\right)^2 + 4bc}{\left[a - \left(\frac{n\pi}{h}\right)^2\right]^2}}\right\} \qquad \text{(VIII. 3)}$$

The limit as n tends to infinity of the positive square root shows that one set of ω_m tends to negative infinity. In the case of the negative square root, further analysis is needed. The square root is expanded in VIII. 4.

$$\omega_m^- = \frac{1}{2}\left[a - \left(\frac{n\pi}{h}\right)^2\right]\left[1 - 1 - 2\frac{a\left(\frac{n\pi}{h}\right)^2 + bc}{\left[a - \left(\frac{n\pi}{h}\right)^2\right]^2} + O\left(\frac{1}{n^4}\right)\right] \qquad \text{(VIII.4)}$$

The limit as n tends to infinity may now be taken.

$$\lim_{n\to\infty} \omega_m^- = \lim_{n\to\infty} \frac{a\left(\frac{n\pi}{h}\right)^2 + bc}{\left(\frac{n\pi}{h}\right)^2 - a} = a \qquad \text{(VIII. 5)}$$

The point is that especially when there is no differential operator in a row of the matrix operator, as in VIII. 1, then it is possible that there is an infinite set of eigenvalues tending to some finite limit.

Example C: Spatial Feedback Control of a Reactor With Uniform Poisoning

Example C illustrates the method of spatial feedback control for a very simple model. One averaged value represents all energy groups of flux and all the delayed neutron groups, and all other effects are ignored. Therefore, this model cannot exhibit spatial xenon oscillation, and the spatial control is probably of no practical importance in this case.

The model is analogous to a homogeneous slab reactor, whose spatial variatons about an operating point are to be controlled. The state equation, II. 14, is a linear constant coefficient scalar equation in this case.

$$\frac{1}{v}\frac{\partial \Phi}{\partial t} = D\frac{\partial^2 \Phi}{\partial r^2} + (\nu\epsilon\Sigma_f - \Sigma_a)\Phi - \sum_{n=1}^{N} \Phi u_n \delta(r-r_n) \qquad \text{(VIII. 6)}$$

The steady state equation, III. 1, is given as VIII. 7 for this case.

$$0 = D\frac{d^2}{dr^2}\Phi_o + (\nu\epsilon\Sigma_f - \Sigma_a)\Phi_o \qquad \text{(VIII. 7)}$$

For a slab of width h, the steady state flux is then VIII. 8.

$$\Phi_o = \Phi_{o\,max} \sin\frac{\pi r}{h} \qquad \text{(VIII. 8)}$$

Criticality then requires VIII. 9.

$$0 = -D\left(\frac{\pi}{h}\right)^2 + \nu\epsilon\Sigma_f - \Sigma_a \qquad \text{(VIII. 9)}$$

The linearized kinetic equation of the flux variation, III. 5, is then VIII. 10.

$$\frac{1}{v}\frac{\partial \phi}{\partial t} = D\frac{\partial^2 \phi}{\partial r^2} + (\nu\epsilon\Sigma_f - \Sigma_a)\phi - \Phi_{o\,max}\sin\frac{\pi r}{h}\sum_{n=1}^{N} u_n(t)\,\delta(r-r_n) \qquad \text{(VIII. 10)}$$

Solutions are of the form VIII. 11.

$$\phi = \sqrt{\frac{2}{h}}\sum_{m=0}^{\infty} a_m(t)\sin\frac{(m+1)\pi r}{h} \qquad \text{(VIII. 11)}$$

Therefore, the deviation of the harmonics, III. 6, is expressed as VIII. 12.

$$\phi_H = \phi - \sqrt{\frac{2}{h}}\,a_o(t)\sin\frac{\pi r}{h} \qquad \text{(VIII. 12)}$$

If the spatial control system is not to interfere with the power control system, condition III. 9 must be imposed.

$$u_N = -\left[\Phi_{o\,max}\sin^2\frac{\pi r_N}{h}\right]^{-1}\sum_{n=1}^{N-1}\Phi_{o\,max}\sin^2\frac{\pi r_n}{h}\,u_n \qquad \text{(VIII. 13)}$$

Substituting VIII. 12 and 13 into VIII. 10 and using the criticality condition VIII. 9 to eliminate the fundamental mode, the state equation III. 12 can be written in the form of VIII. 14.

$$\frac{1}{v}\frac{\partial \phi_H}{\partial t} = D\frac{\partial^2}{\partial r^2}\phi_H + (\nu\epsilon\Sigma_f - \Sigma_a)\phi_H$$

$$- \mathcal{E}_{o\,max} \sin\frac{\pi r}{h} \sum_{n=1}^{N-1} \left[\delta(r-r_n) - \sin^2\frac{\pi r_n}{h} \sin^{-2}\frac{\pi r_N}{h}\,\delta(r-r_N)\right] u_n \qquad \text{(VIII. 14)}$$

This is the system equation, as indicated in the chapter on computation of the feedback, VII. 6. In order to compute the optimal feedback to control the system, the quadratic loss criterion must be decided upon. For simplicity, only the square of the harmonic flux deviation and the square of each control rod deviation will be used. This gives the criterion VIII. 15, corresponding to the general case VII. 5.

$$\mathcal{V} = \frac{1}{2}\int_0^\infty \left[\int_0^h \phi_H^2\, dr + a \sum_{n=1}^{N} u_n^2\right] dt \qquad \text{(VIII. 15)}$$

Selection of this criterion implies that $\mathcal{H} = 1$ and $\mathcal{Q} = 1$ in VII. 3, and that R is a times an N x N unit matrix.

In the criterion VIII. 15, a is the constant of proportionality representing the price one is willing to pay for flux deviation in terms of control rod movement. The constant a should be chosen after all calculations are completed, repeating system design for different values of a to get the desired balance between system response and control rod action. However, for this example, one value of a will be chosen from an initial rough estimate, and will be used throughout the example. To provide an estimate, it will be conjectured that the

contribution of deviation term at its maximum in the criterion should equal the contribution of the control energy term at its maximum.

At its maximum, suppose the deviation is to be no more than ten per cent of the steady state flux.

$$\phi_{H\,max} = 0.1\,\overline{\Phi}_{o\,max} \tag{VIII. 16}$$

At the maximum of the control energy term, suppose the control reactivity is to be no more than ten per cent of the total material reactivity, so that the time-varying absorption term in equation III. 4 can be ignored.

$$\left[\phi_H \sum_{n=1}^{N} u_n(t)\,\delta(r-r_n)\right]_{max} = 0.1(\nu\epsilon\Sigma_f - \Sigma_a)\phi_H \tag{VIII. 17}$$

Multiply VIII. 17 by the adjoint and integrate over the reactor.

$$\left[\sum_{n=1}^{N} u_n(t)\,\phi_H^2(r_n)\right]_{max} = 0.1(\nu\epsilon\Sigma_f - \Sigma_a)\int_0^h \phi_H^2\,dr \tag{VIII. 18}$$

For simplicity, assume the flux deviation is constant in VIII. 18.

$$\left[\sum_{n=1}^{N} u_n(t)\right]_{max} = 0.1(\nu\epsilon\Sigma_f - \Sigma_a)h \tag{VIII. 19}$$

Also for simplicity, assume that each maximum is equal and occurs at the same time.

$$u_{n\,max} = 0.1h(\nu\epsilon\Sigma_f - \Sigma_a)\,N^{-1} \tag{VIII. 20}$$

By the conjecture that the maximum of the flux deviation term equals the maximum of the control energy term, relations VIII. 16 and 20 can

be used to find a.

$$\int_0^h (0.1\,\Phi_{o\,max})^2 dr = a\,N\left[0.1h(\nu\epsilon\Sigma_f - \Sigma_a)N^{-1}\right]^2 \qquad \text{(VIII. 21)}$$

This rough calculation determines a.

$$a = \frac{N}{h}\left(\frac{\Phi_{o\,max}}{\nu\epsilon\Sigma_f - \Sigma_a}\right)^2 \qquad \text{(VIII. 22)}$$

The value of a can now be incorporated into the criterion, VIII. 15. In addition, the constraint on the N^{th} control rod, VIII. 13, must be included in the criterion.

$$\mathcal{V} = \frac{1}{2}\int_0^\infty \left\{ \int_0^h \phi_H^2\, dr + \frac{N}{h}\left(\frac{\Phi_{o\,max}}{\nu\epsilon\Sigma_f - \Sigma_a}\right)^2 \left[\sum_{n=1}^{N-1} u_n^2 + \sin^{-4}\frac{\pi r_N}{h}\left(\sum_{n=1}^{N-1} u_n \sin^2 \frac{\pi r_n}{h} \right)^2 \right] \right\} dt$$

(VIII. 23)

From equations VIII. 14 and VIII. 23, the matrix notation used in VII. 5 and 6 may be evaluated.

$$\bar{z}_H = \phi_H,\quad \mathcal{H} = 1,\quad \mathcal{Q} = 1,\quad \bar{u}_{N-1}^T = (u_1 \cdots u_{N-1}),$$

$$R_{N-1} = \frac{N}{h}\left(\frac{\Phi_{o\,max}}{\nu\epsilon\Sigma_f - \Sigma_a}\right)^2 \left\{ \begin{array}{ccc} 1+\dfrac{\sin^4\frac{\pi r_1}{h}}{\sin^4\frac{\pi r_N}{h}} & \cdots & \dfrac{\sin^2\frac{\pi r_1}{h}\,\sin^2\frac{\pi r_{N-1}}{h}}{\sin^4\frac{\pi r_N}{h}} \\ \cdots & & \cdots \\ \dfrac{\sin^2\frac{\pi r_1}{h}\,\sin^2\frac{\pi r_{N-1}}{h}}{\sin^4\frac{\pi r_N}{h}} & \cdots & 1+\dfrac{\sin^4\frac{\pi r_{N-1}}{h}}{\sin^4\frac{\pi r_N}{h}} \end{array} \right\},$$

(VIII. 24)

$$\mu = \frac{1}{v}, \quad \mathcal{L} = D\frac{d^2}{dr^2} + \nu\epsilon\Sigma_f - \Sigma_a,$$

$$\mathcal{Q}_{N-1} = -\Phi_{o\,max}\sin\frac{\pi r}{h}\left(\delta(r-r_1) - \frac{\sin^2\frac{\pi r_1}{h}}{\sin^2\frac{\pi r_N}{h}}\delta(r-r_N)\cdots\delta(r-r_{N-1})\right.$$

$$\left. - \frac{\sin^2\frac{\pi r_{N-1}}{h}}{\sin^2\frac{\pi r_N}{h}}\delta(r-r_N)\right)$$

Using the definitions VIII. 24, the canonical equations VII. 12 and 13 can be found.

$$\frac{1}{v}\frac{\partial \xi_H}{\partial t} = -\left(D\frac{\partial^2}{\partial r^2} + \nu\epsilon\Sigma_f - \Sigma_a\right)\xi_H - \phi_H \qquad \text{(VIII. 25)}$$

$$\frac{1}{v}\frac{\partial \phi_H}{\partial t} = \left(D\frac{\partial^2}{\partial r^2} + \nu\epsilon\Sigma_f - \Sigma_a\right)\phi_H - \mathcal{Q}_{N-1}R_{N-1}^{-1}\int_0^h \mathcal{Q}_{N-1}^T\xi_H\,dr$$

(VIII. 26)

These canonical equations, VIII. 25 and 26, can be solved for any finite number of control rods, N, and approximated by any finite number of modes, M. For simplicity, only the first two harmonics will be considered, so M=2. To control the reactor, it will be assumed that the minimum number of rods will be used in this example. By the controllability theorem, since the reactor is one dimensional, one extra rod in addition to the power control rod is needed. This means N=2. Henceforth, in this example, the feedback for two control rods controlling the fundamental and two harmonics will be computed by the methods of Chapter VII.

From VII. 14, the approximate forms for ϕ_H and ξ_H are given by VIII. 27, and note that b* equals b in this case only because $\mathcal{A}$ and $\mathcal{M}$ are Hermitian.

$$\phi_H = \sqrt{\frac{2v}{h}} \left[a_1(t) \sin \frac{2\pi r}{h} + a_2(t) \sin \frac{3\pi r}{h}\right]$$

$$\xi_H = \sqrt{\frac{2v}{h}} \left[b_1(t) \sin \frac{2\pi r}{h} + b_2(t) \sin \frac{3\pi r}{h}\right] \qquad \text{(VIII. 27)}$$

If VIII. 27 is used to express the solutions of the canonical equations VIII. 25 and 26, multiplication by $\sqrt{\frac{2v}{h}} \sin \frac{2\pi r}{h}$ or $\sqrt{\frac{2v}{h}} \sin \frac{3\pi r}{h}$ and integration over r from 0 to h gives an equation for the modal time coefficients corresponding to VII. 15.

$$\dot{a}_1 = \omega_1 a_1 - \frac{v(\nu\epsilon\Sigma_f - \Sigma_a)^2 \sin^2\frac{\pi r_1}{h} \sin^4\frac{\pi r_2}{h}}{\sin^4\frac{\pi r_1}{h} + \sin^4\frac{\pi r_2}{h}} s_1(s_1 b_1 + s_2 b_2)$$

$$\dot{a}_2 = \omega_2 a_2 - \frac{v(\nu\epsilon\Sigma_f - \Sigma_a)^2 \sin^2\frac{\pi r_1}{h} \sin^4\frac{\pi r_2}{h}}{\sin^4\frac{\pi r_1}{h} + \sin^4\frac{\pi r_2}{h}} s_2(s_1 b_1 + s_2 b_2) \qquad \text{(VIII. 28)}$$

$$\dot{b}_1 = -\omega_1 b_1 - v a_1$$

$$\dot{b}_2 = -\omega_2 b_2 - v a_2$$

In VIII. 28, ω_m and s_m are defined by VIII. 29.

$$\omega_m = v\left[\nu\epsilon\Sigma_f - \Sigma_a - D(m+1)^2\left(\frac{\pi}{h}\right)^2\right]$$

$$s_m = \sin\frac{(m+1)\pi r_1}{h} - \sin^{-1}\frac{\pi r_2}{h}\sin\frac{\pi r_1}{h}\sin\frac{(m+1)\pi r_2}{h} \qquad \text{(VIII. 29)}$$

Using the material constants of Table VIII. 1, a reactor width of 250 cm, and control rods at .25 h and .6 h, equations VIII. 28 are put in the numerical matrix form VIII. 30, corresponding to VII. 16.

$$\begin{pmatrix}\dot{a}_1\\ \dot{a}_2\end{pmatrix} = \begin{pmatrix}-1.024 & 0\\ 0 & -2.304\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix} - \begin{pmatrix}0.01034 & 0.00823\\ 0.00823 & 0.00656\end{pmatrix}\begin{pmatrix}b_1\\ b_2\end{pmatrix}$$

$$\begin{pmatrix}\dot{b}_1\\ \dot{b}_2\end{pmatrix} = -\begin{pmatrix}-1.024 & 0\\ 0 & -2.304\end{pmatrix}\begin{pmatrix}b_1\\ b_2\end{pmatrix} - \begin{pmatrix}10 & 0\\ 0 & 10\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix} \qquad \text{(VIII. 30)}$$

TABLE VIII-1

Material Constants for Example 1

migration length	$L^2 = \frac{D}{\Sigma_a} = 160\ cm^2$
I^{135} atoms/fission	$\gamma_I = 0.061$
neutrons/fission	$\nu = 2.5$
iodine decay constant	$\lambda_I = 2.9 \times 10^{-5}$ sec
xenon decay constant	$\lambda_x = 2.1 \times 10^{-5}$ sec
xenon microscopic absorption cross section	$\sigma_a^x = 3 \times 10^{-18}\ cm^2$
resonance escape probability	$p = 1$
non-fast leakage probability	$g^{th} = 1$
fast fission factor	$\epsilon = 1$
Xe^{135} atoms/fission	$\gamma_x = 0$
average neutron lifetime	$\ell = 1/\Sigma_a v = 0.1$ sec
infinite homogeneous multiplication constant	$k^\infty = \frac{\nu \epsilon \Sigma_f}{\Sigma_a} = 1.0256$
maximum steady state flux density	$\Phi_{o\ max} = 2.0 \times 10^{14}$ neutrons/cm^2-sec

Using VIII. 30 and VII. 19, the feedback matrix P may be found by solving VIII. 31 backwards in time from an "initial" condition of $P(t_1) = 0$.

$$-\begin{pmatrix} \dot{P}_{11} & \dot{P}_{12} \\ \dot{P}_{12} & \dot{P}_{22} \end{pmatrix} = \begin{pmatrix} -1.024 & 0 \\ 0 & -2.304 \end{pmatrix} \begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix} + \begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix} \begin{pmatrix} -1.024 & 0 \\ 0 & -2.304 \end{pmatrix}$$

$$- \begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix} \begin{pmatrix} 0.01034 & 0.00823 \\ 0.00823 & 0.00656 \end{pmatrix} \begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix} + \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} \qquad \text{(VIII. 31)}$$

Equation VIII. 31 can be solved easily on a computer, noting the equilibrium solution after the computer has proceeded a while. This is feasible for systems of fairly high order, because of the large storage space of modern computers. However, instead of solving VIII. 31, the low dimensionality of VIII. 30 makes hand computation feasible. Equation VIII. 30 may be rewritten to find the eigenvalues of the system.

$$\frac{d}{dt} \begin{Bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{Bmatrix} = \begin{Bmatrix} -1.024 & 0 & -0.01034 & -0.00823 \\ 0 & -2.304 & -0.00823 & -0.00656 \\ -10 & 0 & 1.024 & 0 \\ 0 & -10 & 0 & 2.304 \end{Bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{Bmatrix} \qquad \text{(VIII. 32)}$$

This may be solved for the eigenvalues.

$$\lambda_{1,2} = +(1.024 + 0.02857)$$
$$\lambda_{3,4} = \pm(2.304 + 0.01453) \qquad \text{(VIII. 33)}$$

Stability requires the minus sign, and so the closed loop poles are $-(1.024 + 0.02857)$ and $-(2.304 + 0.01453)$. This small improvement of the closed loop response over the open loop is due to the fact that the control energy is weighted quite heavily, and that the open loop poles are stable.

To find the feedback, let

$$\begin{Bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{Bmatrix} = \begin{Bmatrix} A\, e^{-1.05257t} + B\, e^{-2.31853t} \\ C\, e^{-1.05257t} + D\, e^{-2.31853t} \\ E\, e^{-1.05257t} + F\, e^{-2.31853t} \\ G\, e^{-1.05257t} + H\, e^{-2.31853t} \end{Bmatrix} \qquad \text{(VIII. 34)}$$

Then, to find C, E, and G, put VIII. 34 into VIII. 32.

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} +0.02857 & 0 & -0.01034 & -0.00823 \\ 0 & -1.23143 & -0.00823 & -0.00656 \\ -10 & 0 & 2.09657 & 0 \\ 0 & -10 & 0 & 3.37657 \end{bmatrix} \begin{Bmatrix} A \\ C \\ E \\ G \end{Bmatrix} \qquad \text{(VIII. 35)}$$

Using the values found from VIII. 35 and the corresponding operations on B, D, F, and H, equation VIII. 34 can be rewritten in the form of equation VIII. 36.

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -.0314 & 70.5 \end{pmatrix} \begin{pmatrix} A\, e^{-1.07257t} \\ B\, e^{-2.31853t} \end{pmatrix} \qquad \text{(VIII. 36)}$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 4.77 & 3.00 \\ -.0930 & 154.0 \end{pmatrix} \begin{pmatrix} A\, e^{-1.07257t} \\ B\, e^{-2.31853t} \end{pmatrix}$$

Since $\bar{b} = P\bar{a}$, P can be found from VIII. 36.

$$P = \begin{pmatrix} 4.77 & 3.00 \\ -.0930 & 154.0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -.0314 & 70.5 \end{pmatrix}^{-1} = \begin{pmatrix} 4.77 & -.0251 \\ -.0245 & 2.18 \end{pmatrix} \quad \text{(VIII. 37)}$$

The existence of a finite unique stable solution to VIII. 31 guarantees that the inverse matrix in VIII. 37 always exists. Note also that A and B are determined by the initial conditions $a_1(0)$ and $a_2(0)$ and that VIII. 37 is valid for any A and B, so that the feedback is optimal for any initial conditions.

As a check, VIII. 31 was solved on a computer, and the values so obtained agree favorably with VIII. 37. *

$$P_{\text{computer soln.}} = \begin{pmatrix} 4.769 & -.0251 \\ -.0251 & 2.164 \end{pmatrix} \quad \text{(VIII. 38)}$$

Having P, the feedback for both control rods can be computed. From VII. 10, the value for $u_1(t)$ can be obtained.

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* After performing the computations, a small error was discovered. The open loop poles of this example should be $-.768\ \text{sec}^{-1}$ and $-2.048\ \text{sec}^{-1}$ instead of $-1.024\ \text{sec}^{-1}$ and $-2.304\ \text{sec}^{-1}$. This change is minor and no conclusions are altered, and the method is illustrated equally as well with either set of poles. This can be shown easily in the scalar case. The feedback equation is $0 = 2PF + Q - R^{-1}G^2P^2$. Since R is large and F is negative, the positive definite solution for P is $-Q/2F$, so P changes linearly with the small change in F.

$$u_1(t) = -R^{-1}\int_0^h G^+ \xi_H dr$$

$$-\frac{h(\nu\epsilon\Sigma_f - \Sigma_a)^2 \sin^4\frac{\pi r_2}{h}}{\Sigma_a \Phi_{o\,max}(\sin^4\frac{\pi r_1}{h} + \sin^4\frac{\pi r_2}{h})} \sin\frac{\pi r_1}{h}(s_1 b_1 + s_2 b_2) \qquad \text{(VIII. 39)}$$

$$=\frac{h(\nu\epsilon\Sigma_f - \Sigma_a)^2 \sin^4\frac{\pi r_2}{h}}{\Sigma_a \Phi_{o\,max}(\sin^4\frac{\pi r_1}{h} + \sin^4\frac{\pi r_2}{h})} \sin\frac{\pi r_1}{h}(s_1 \quad s_2)\, P\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Using the values of the example, u_1 can be found numerically.

$$u_1(t) = \frac{\Sigma_a h}{\Phi_{o\,max}} (-.000483 \quad -.000173)\begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix} \qquad \text{(VIII. 40)}$$

The action of the other control rod is found from the power control constraint VIII. 13.

$$u_2(t) = \frac{\Sigma_{aso} h}{\Phi_{o\,max}} (.000267 \quad .0000956) \begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix} \qquad \text{(VIII. 41)}$$

This is the feedback to each control rod that was to be found. Note that, because of the homogeneity of the reactor, if a device were set up to divide the feedback by $\Phi_{o\,max}$, the control system would be optimal for all operating levels. This is not true in general, but provides another hint for better overall performance.

In the derivation of the reactor model, it was assumed that $u << (\nu\epsilon\Sigma_f - \Sigma_a)h$, or $u/\Sigma_a h << k^{\infty}-1 = .0256$ for the example. Since $a_m << \Phi_{o\,max}$, because the flux deviation is much less than the operating flux, the feedback formed from VIII. 40 and 41 certainly satisfies the assumption.

The determination of a_1 and a_2 is performed by the detection system. It is assumed that $\Phi(r,t)$ is known at all points, so that $\phi(r,t)$ can be detected by subtracting the steady state, $\Phi_o(r)$. From this, the inner products with respect to $\sin(m\pi r/h)$ will give the $a_m(t)$.

If the flux is detected only at three points, the feedback may be constructed from the output of the three detectors and the system compared with that of perfect estimation. Using three detection points, r_1, r_2, and r_3, the relationship VIII. 42 can approximately represent a_0, a_1, and a_2.

$$\begin{Bmatrix} \Phi(r_1,t) - \Phi_o(r_1) \\ \Phi(r_2,t) - \Phi_o(r_2) \\ \Phi(r_3,t) - \Phi_o(r_3) \end{Bmatrix} = \begin{Bmatrix} \sin\frac{\pi r_1}{h} & \sin\frac{2\pi r_1}{h} & \sin\frac{3\pi r_1}{h} \\ \sin\frac{\pi r_2}{h} & \sin\frac{2\pi r_2}{h} & \sin\frac{3\pi r_2}{h} \\ \sin\frac{\pi r_3}{h} & \sin\frac{2\pi r_3}{h} & \sin\frac{3\pi r_3}{h} \end{Bmatrix} \begin{Bmatrix} a_0(t) \\ a_1(t) \\ a_2(t) \end{Bmatrix} \quad \text{(VIII. 42)}$$

The inversion of the matrix will give the a_m in terms of the flux deviations at the three points.

The location of the detectors can influence the gain of the feedback. If the detectors are located favorably, this feedback gain

may be used to decrease the value of the criterion. For detectors located at $r_1 = .25h$, $r_2 = .33h$, and $r_3 = .5h$, the feedback is given by VIII. 43.

$$\frac{u_1(t)}{\Sigma_a h} = .00075 \frac{\phi(.25h, t)}{\Phi_{o\,max}} - .00140 \frac{\phi(.33h, t)}{\Phi_{o\,max}} + .00070 \frac{\phi(.5h, t)}{\Phi_{o\,max}} \qquad \text{(VIII. 43)}$$

The effect of moving the detector at . 33h to . 75h is given by VIII. 44.

$$\frac{u_1(t)}{\Sigma_a h} = -.00030 \frac{\phi(.25h, t)}{\Phi_{o\,max}} + .00018 \frac{\phi(.75h, t)}{\Phi_{o\,max}} + .000086 \frac{\phi(.5h, t)}{\Phi_{o\,max}} \qquad \text{(VIII. 44)}$$

This decreases the gain significantly while achieving the same response. Therefore, the same percentage flux deviation at the first three detection points will give rise to much more control rod action than at the second set of detection points. Thus, the second set are chosen to give a possible final feedback configuration as illustrated in Figure VIII. 2.

Example D: Positioning of the Control Rods

Example C has provided a simple, hand calculable example of reactor spatial feedback control. The small change in the poles of the system has pointed out that stable modes are hardly worth while putting into the calculation when the penalty for moving a control rod is so great.

However, the formulation of Example C can be used to provide a guide to the placing of control rods for maximum effectiveness

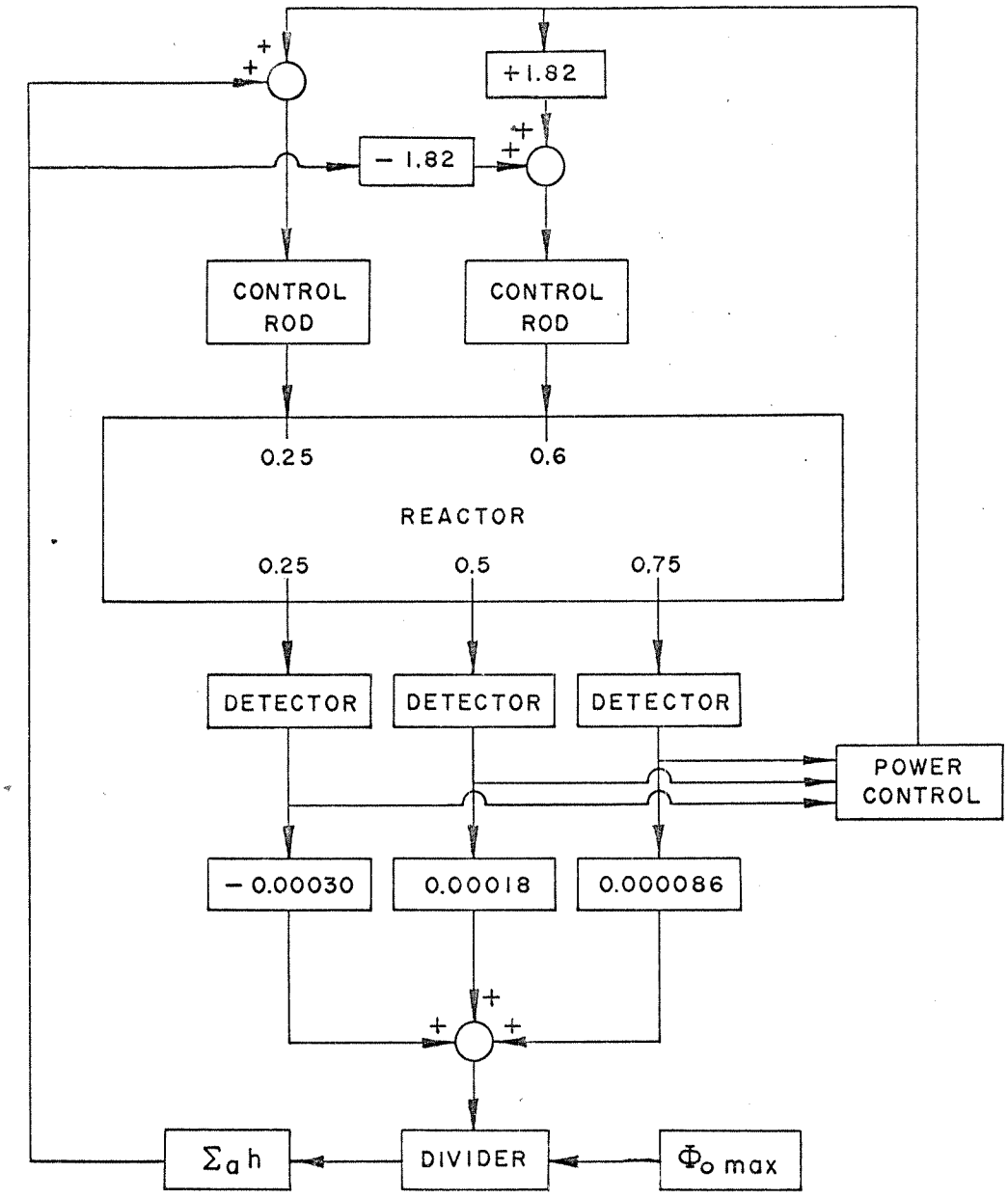

Fig. VIII.2 Spatial Feedback Control for Example C

when the open loop poles are unstable. Instead of the stable poles at -1.024 and -2.304, suppose that by some means these poles were shifted to the values +0.163 and +0.0221. These unstable poles are one hundred times the two most unstable poles of Example E. Using these values, control rod placing was investigated using 7090 Routine No. 1 (see Appendix D) to solve VIII.31 for different control rod positions. Table VIII.2 gives the results of the effect of different control rod positions, r_1 and r_2, on the resulting closed loop system poles. The control effort is also minimum at the minimum deviation, corresponding to the smallest value of the closed loop poles. It appears that the best response is obtained at approximately $r_1 = 0.2h$ and $r_2 = 0.4h$, or their symmetrical values of 0.6h and 0.8h. These are the control rod positions chosen for Example E. It is to be noted that in this case there is a distinct advantage to control with rods on the same side of the reactor, which was not obvious before attempting an analysis of this type.

TABLE VIII. 2

Control Rod Effectiveness

The formulation and material properties of Example C are used, except the open loop poles are shifted to 0.163 sec^{-1} and 0.0221 sec^{-1}.

r_1/h	r_2/h	Larger closed loop pole-sec^{-1}	Smaller closed loop pole-sec^{-1}	r_1/h	r_2/h
0.1	0.2	-.0155	-.0350	0.9	0.8
0.1	0.3	-.0155	-.0805	0.9	0.7
0.1	0.4	-.0151	-.1197	0.9	0.6
0.1	0.5	-.0144	-.1407	0.9	0.5
0.1	0.6	-.0129	-.1412	0.9	0.4
0.1	0.7	-.0097	-.1307	0.9	0.3
0.1	0.8	-.0045	-.1228	0.9	0.2
0.2	0.3	-.0154	-.1452	0.8	0.7
0.2	0.4	-.0149	-.2860	0.8	0.6
0.2	0.5	-.0139	-.3343	0.8	0.5
0.2	0.6	-.0116	-.3681	0.8	0.4
0.2	0.7	-.0068	-.2334	0.8	0.3
0.3	0.4	-.0143	-.2197	0.7	0.6
0.3	0.5	-.0125	-.4153	0.7	0.5
0.3	0.6	-.0081	-.3943	0.7	0.4
0.4	0.5	-.0087	-.1768	0.6	0.5

EXAMPLE E

Example E takes into account the xenon and iodine distributions in a homogeneous slab reactor, in addition to the thermal flux group that was considered in Example D. The price of the added complexity is that a digital computer must be used in the calculations. However, this example illustrates the occurrence and control of a spatial instability due to xenon. The solution for the Kaplan modes by means of shooting, the complications introduced by a non-Hermitian operator, and the method of obtaining the feedback for the state variable from only flux measurements are included in this example.

The homogeneous reactor in this example also has the material constants given in Table VIII. 1. However, it was decided to make the reactor of such a size that two clean mode harmonics are unstable at the operating point. For the given material constants, this size is 988 centimeters.

To initiate the design procedure, the steady state operating point must be determined. For the example reactor, the scalar steady state equation corresponding to III. 1 for xenon, iodine and one flux group is given as VIII. 45.

$$0 = L^2 \frac{d^2\Phi_o}{dr^2} + \left(k\text{-}1\text{-} \frac{\gamma_I}{\nu} k \frac{\sigma_a^x \Phi_o}{\sigma_a^x \Phi_o + \lambda_x} \right) \Phi_o \qquad \text{(VIII. 45)}$$

This equation is obtained by dividing the scalar steady state equation by the constant Σ_a and setting γ_x equal to zero, so that prompt xenon is neglected. It may be recognized as the usual steady state flux

equation, with the non-linear term arising from the non-uniform poisoning of xenon.

A value of k=1.0256 gives the flux shape of Figure VIII.3, when the peak flux is 2×10^{14} neutrons/cm^2 sec in the center of the reactor. The solution to VIII.45 depicted in Figure VIII.3 was calculated on the Caltech IBM 7090 by Routine No. 2 of Appendix D. This solution is compared with the clean reactor sinusoidal shape. The values of k and reactor size chosen are such that the third harmonic is marginally stable by the Lellouche criterion (see Chapter VI). Note that k as defined in this example is the infinite multiplication constant for the clean controlled reactor, so $k=\nu\epsilon\Sigma_f\Sigma_{aso}^{-1}$. The comparatively low value is due to the inclusion of the steady state control rod position in Σ_{aso}, so that the value of k is determined to be that which maintains the steady state flux, Φ_o, in equation VIII.45.

After the steady state flux has been determined, the Kaplan modes must be found for the linearized equation for the deviations about the steady state, V.1. Since there is only one spatial variable, the shooting technique of Chapter V can be used. To provide an initial estimate to start the shooting, the estimation methods of Chapter VI are used. The explicit form of the estimate is not very good because of the high flux, and so eigenvalues are calculated for only one clean mode shape for illustration. The transcendental estimate, however, provides excellent agreement with the true values. In Table VIII.3, these estimates and an estimate obtained from the clean mode expansion neglecting modal interaction are compared with the true

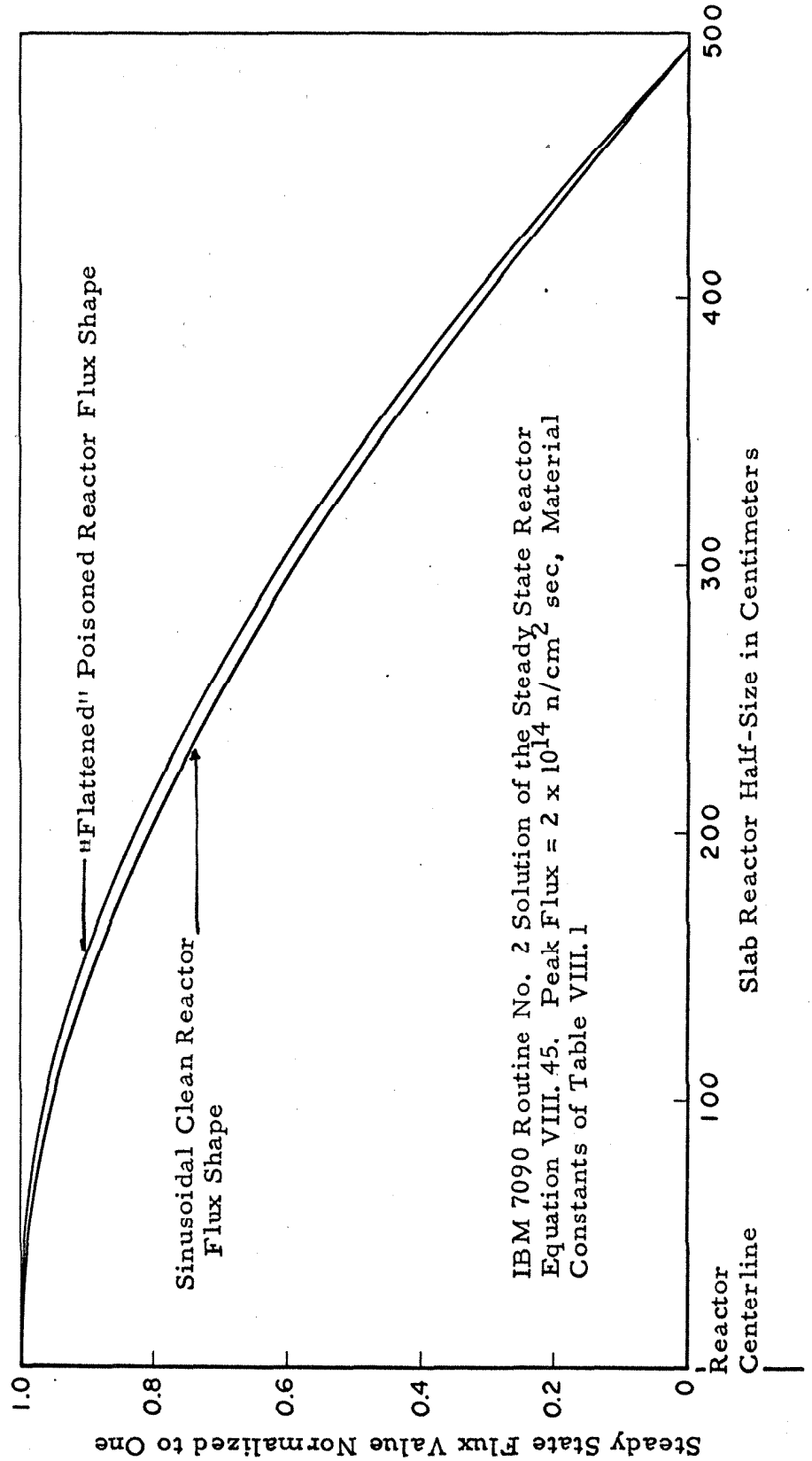

Fig. VIII. 3 Steady State Solution for the Poisoned Reactor, Example E.

TABLE VIII. 3

Comparison of Kaplan Mode Eigenvalues for Example E

Clean Mode Shape	Clean Mode Estimate	Explicit Variational Technique	Transcendental Variational Technique	True Eigenvalue
	Values are dimensionless: ω_n/λ_I			
Fundamental	--	--	366	374
	--	--	.046	.043
	--	--	-401	-392
First Harmonic	65.1	98.5	54.3	56.28
	.328	.368	.334	.3306
	--	--	-1,335*	-1,638
Second Harmonic	12.0	--	7.42	7.63
	.138	--	2.02	2.00
	--	--	-4,000*	-4,340
Third Harmonic	-.882+.863j	--	-1.039+3.391j	-1.067+3.475j
	-.882-.863j	--	-1.039-3.391j	-1.067-3.475j
	--	--	-7,900*	-8,182

* Desk calculated, ignoring integral terms.

eigenvalues obtained from the shooting technique.

The integrations involving the real and imaginary part of the estimate can be performed in seconds on the 7090 by Routine No. 3 in Appendix D. This gives a very rapidly converging estimate for the eigenvalues, and is useful to start (and also check) the search in the complex plane by the shooting technique.

The shooting technique can be performed rapidly on the 7090, also, by Routine No. 4 that solves equation V. 3 for the material constants of this example. The resulting harmonic mode shapes are given as Figures VIII. 4, 5, and 6. The fundamental has the shape of a sinusoid to within one percent for all three eigenvalues corresponding to the clean mode fundamental. However, the harmonics differ significantly from a sinusoid, and also differ significantly from one another. This characteristic suggests the remarks about spatial transfer functions in Chapter VI.

Because the spatial shape of the flux components of the Kaplan modes corresponding to the fundamental clean mode are so similar, they are all approximately bi-orthogonal to the modes controlled by the spatial control system. Therefore, these modes are omitted from the analysis and will be controlled by the power control system.

Having the Kaplan harmonic modes, the analytical design of the spatial control system can proceed. Only two control rods will be used in the control of the reactor. From Example D, the positions chosen are at 0. 2h and 0. 4h. From the control rod constraint, III. 9, the motion of one rod is expressed in terms of the motion of the other.

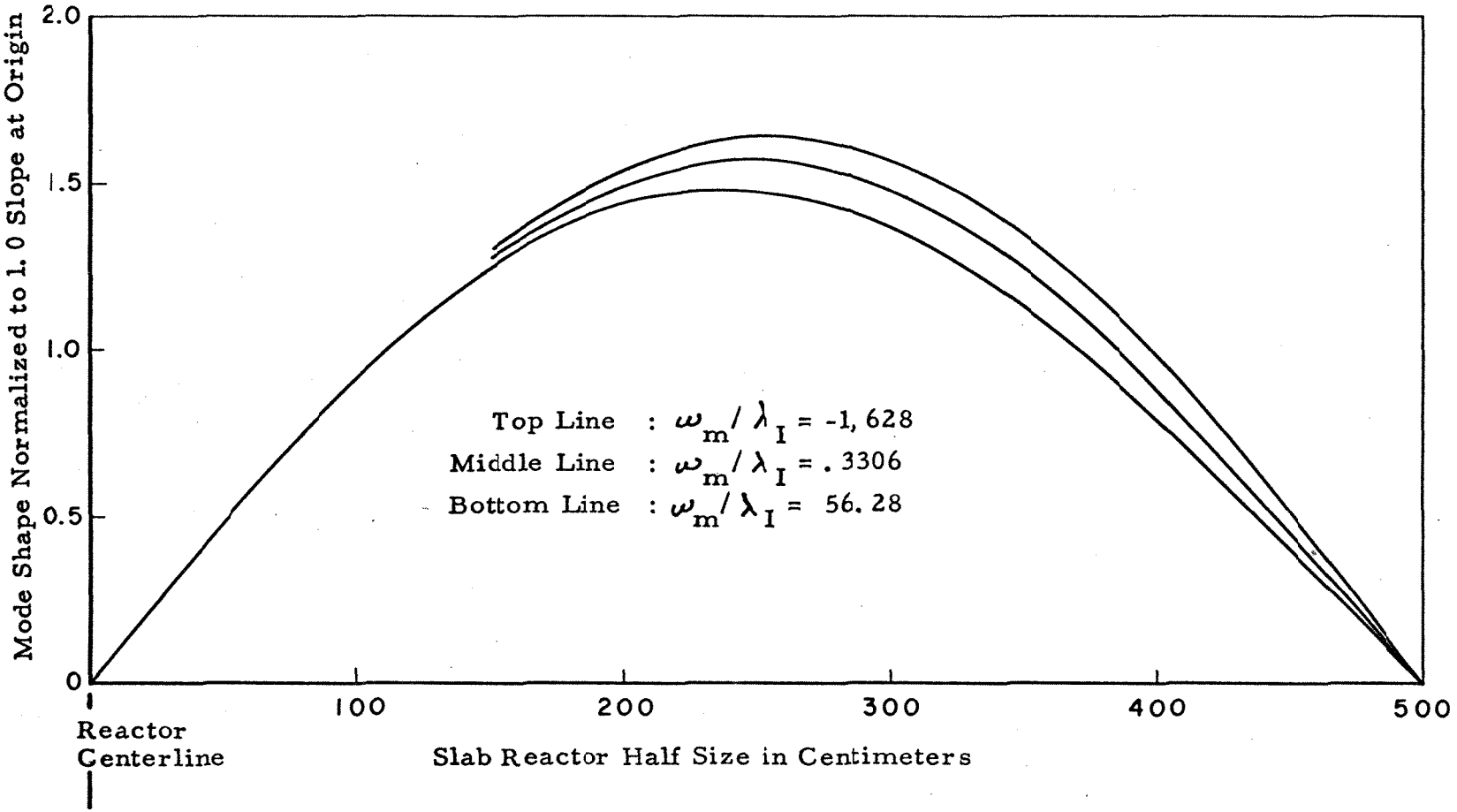

Fig. VIII. 4 Kaplan Modes Corresponding to the First Harmonic of the Flux Deviation.

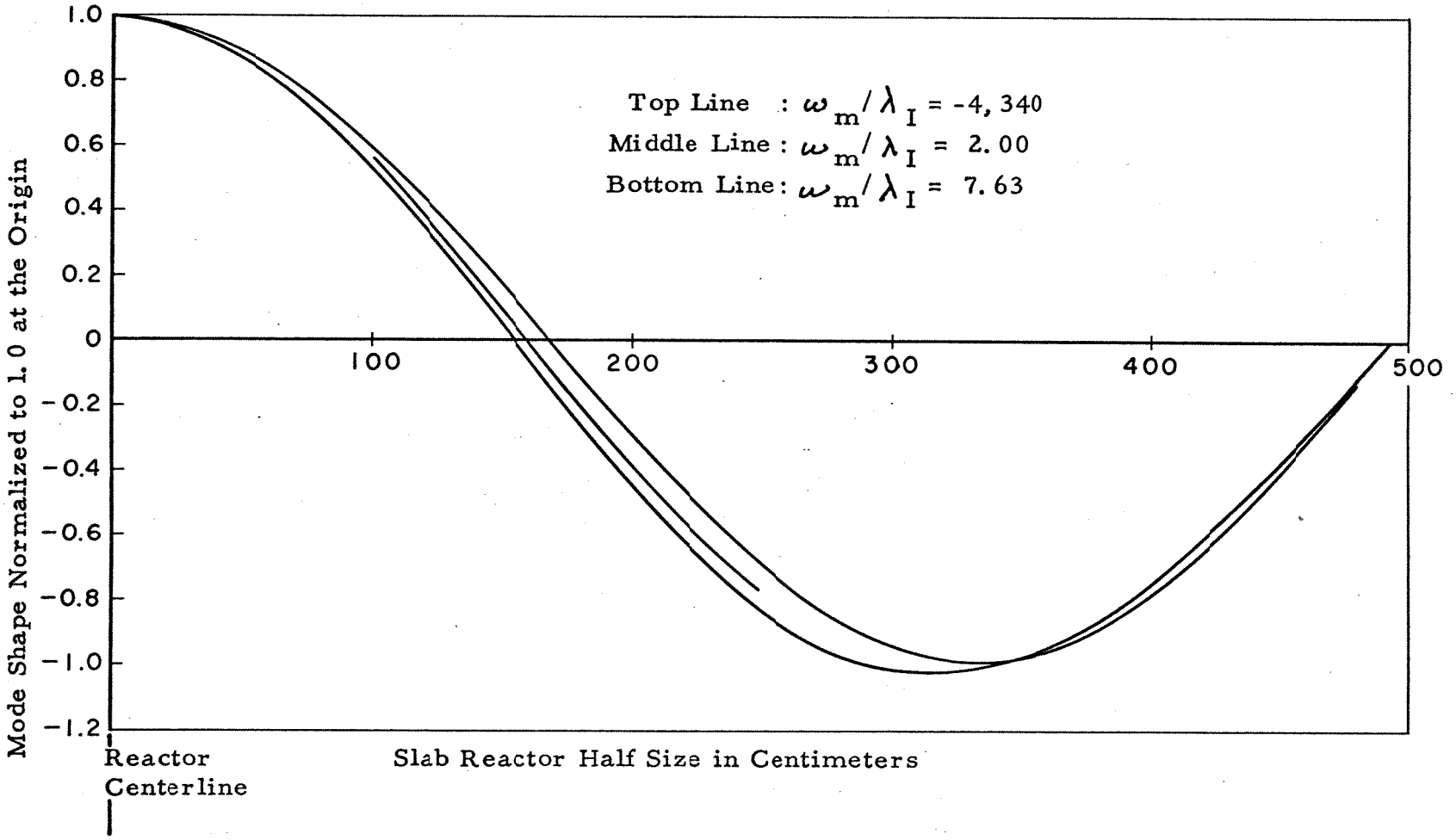

Fig. VIII. 5 Kaplan Modes Corresponding to the Second Harmonic of the Flux Deviation

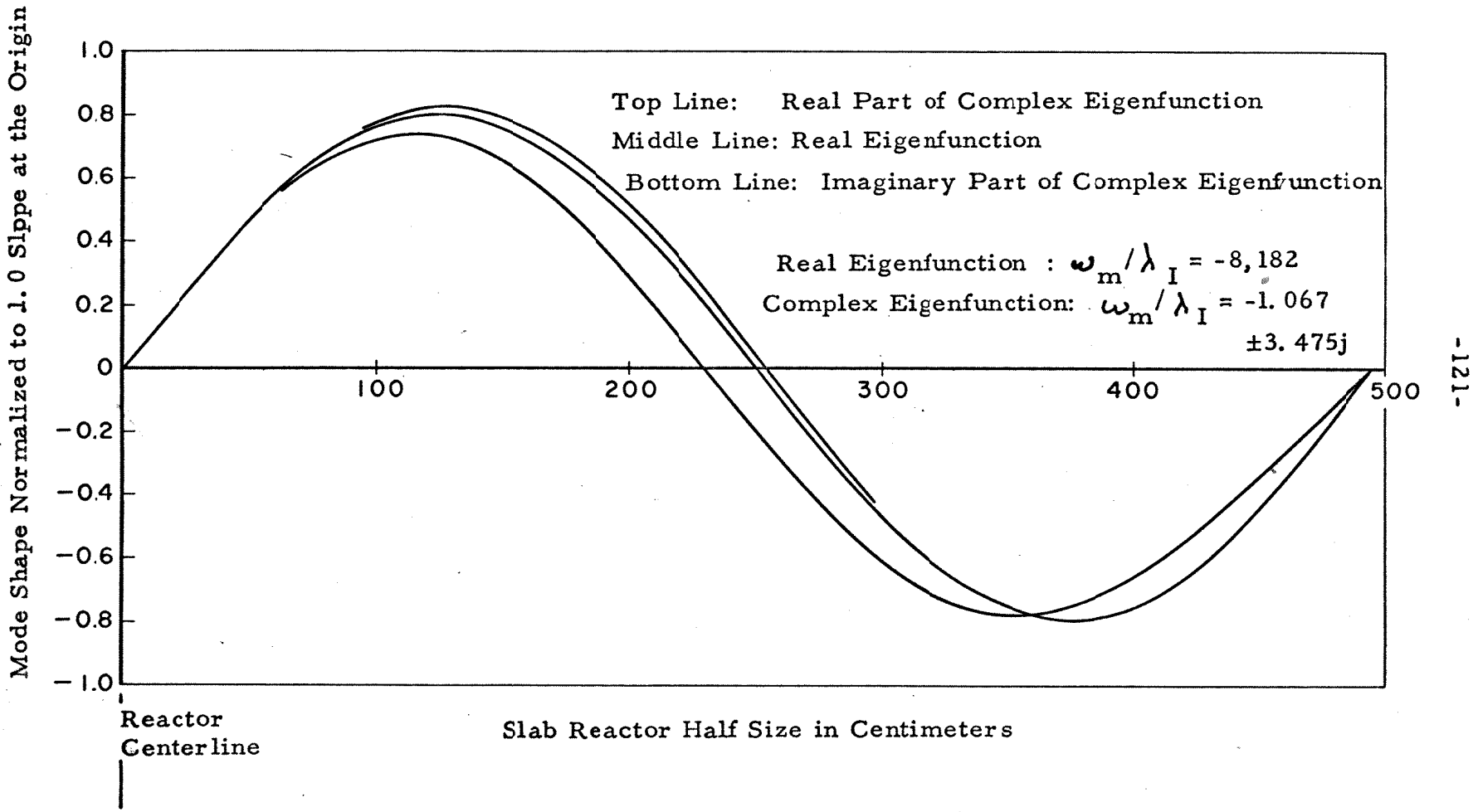

Fig. VIII. 6 Kaplan Modes Corresponding to the Third Harmonic of the Flux Deviation

$$u_2(t) = -\psi_o^{(3)}(r_1)\,\Phi_o(r_1)\,u_1(t) / \psi_o^{(3)}(r_2)\,\Phi_o(r_2) = -2.56u_1(t) \qquad \text{(VIII. 46)}$$

The reactor model III. 15 can be used with condition VIII. 46 to form the homogeneous reactor harmonic state equation corresponding to VII. 6.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \ell \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} i \\ x \\ \phi_H \end{pmatrix}$$

$$= \begin{Bmatrix} -\lambda_I & 0 & \gamma_I \epsilon \Sigma_f \\ \lambda_I & -(\sigma_a^x \Phi_o(r) + \lambda_x) & -\sigma_a^x N_{x_o}(r) \\ 0 & -\Sigma_{aso}^{-1} \sigma_a^x \Phi_o(r) & L^2 \frac{\partial^2}{\partial r^2} + k^\infty - 1 - \Sigma_{aso}^{-1} \sigma_a^x N_{x_o} \end{Bmatrix} \begin{pmatrix} i \\ x \\ \phi_H \end{pmatrix}$$

$$+ \begin{Bmatrix} 0 \\ 0 \\ -\Sigma_{aso}^{-1} \Phi_o(r) \{ \delta(r-0.4h) - 2.56\,\delta(r-0.2h) \} \end{Bmatrix} u_1(t) \qquad \text{(VIII. 47)}$$

Having the harmonic state equation VIII. 47, a criterion must be decided upon. In this example, the flux deviation will be weighted, because deviations in the center are more important than deviations at the edge. This is true because total power density is assumed temperature limited, and power density is assumed proportional to the total flux. Since the total flux is the sum of the steady state flux and the flux deviations, and the steady state flux is highest in the center, it follows that larger deviations can be permitted towards the sides than in the center.

A rough method of weighting the spatial importance of the deviations might be decided upon in the following manner. Suppose the reactor can withstand only a twenty percent increase in flux over the maximum value in the center. A total flux value equal to $1.2\Phi_{o\,max}$ anywhere in the reactor is as detrimental as that value occurring at any other point in the reactor. Therefore, a flux deviation equal to $1.2\,\Phi_{o\,max} - \Phi_o(r)$ will give this value of total flux. The weighting function of the deviation can then be represented as in equation VIII. 48, so that deviations in the center will be weighted six times as heavily as those at the very edge.

$$\alpha = \frac{1}{\left[1.2 - \Phi_o(r)\,\Phi_{o\,max}^{-1}\right]^2} \qquad \text{(VIII. 48)}$$

The same assumptions as in Example C lead to the same choice of measure of control rod effort. Therefore, the criterion can be given by VIII. 49.

$$\mathcal{V} = \frac{1}{2}\int_0^{\infty}\left[\int_0^{h} \frac{\phi_H^{2}(r,t)}{\left[1.2 - \Phi_o(r)\Phi_{o\,max}^{-1}\right]^2}\,dr + \frac{1}{h}\left(\frac{\Phi_{o\,max}}{\Sigma_{aso}(k-1)}\right)^2\left[1+(2.56)^2\right]u_1^{2}(t)\right]dt \qquad \text{(VIII. 49)}$$

The canonical equations VIII. 12 can now be formed, using equations VIII. 47 and VIII. 49.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \ell \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} i \\ x \\ \phi \end{pmatrix}_H = \begin{Bmatrix} -\lambda_I & 0 & \gamma_I \in \Sigma_f \\ \lambda_I & -(\sigma_a^x \Phi_o + \lambda_x) & -\sigma_a^x N_{x_o} \\ 0 & -\Sigma_{aso}^{-1} \sigma_a^x \Phi_o & L \frac{\partial^2}{\partial r^2} + k - 1 - \frac{\sigma_a^x N_{x_o}}{\Sigma_{aso}} \end{Bmatrix} \begin{Bmatrix} i \\ x \\ \phi \end{Bmatrix}_H$$

$$- \begin{Bmatrix} 0 \\ 0 \\ -\Sigma_{aso}^{-1} \Phi_o \left| \delta(r-r_1) - 2.56\, \delta(r-r_2) \right| \end{Bmatrix}$$

$$\frac{h \Sigma_{aso}^2 (k-1)^2}{\Phi_{o\,max}^2 \left[1 + (2.56)^2 \right]} \int_0^h \left\{ 0 \; 0 - \Sigma_{aso}^{-1} \Phi_o \left| \delta(r-r_1) - 2.56\, \delta(r-r_2) \right| \right\} \begin{Bmatrix} \xi_i \\ \xi_x \\ \xi_\phi \end{Bmatrix}_H dr$$

(VIII. 50)

$$\begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \ell \end{Bmatrix} \frac{\partial}{\partial t} \begin{Bmatrix} \xi_i \\ \xi_x \\ \xi_\phi \end{Bmatrix}_H = - \begin{Bmatrix} -\lambda_I & \lambda_I & 0 \\ 0 & -(\sigma_a^x \Phi_o + \lambda_x) & -\Sigma_{aso}^{-1} \sigma_a^x \Phi_o \\ \gamma_I \in \Sigma_f & -\sigma_a^x N_{x_o} & L \frac{\partial^2}{\partial r^2} + k - 1 - \frac{\sigma_a^x N_{x_o}}{\Sigma_{aso}} \end{Bmatrix} \begin{Bmatrix} \xi_i \\ \xi_x \\ \xi_\phi \end{Bmatrix}_H$$

$$- \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \frac{1}{\left| 1.2 - \Phi_o(r) \Phi_{o\,max}^{-1} \right|^2} \{ 0 \; 0 \; 1 \} \begin{Bmatrix} i \\ x \\ \phi \end{Bmatrix}_H$$

As an approximate solution to VIII. 50, only the four unstable harmonics drawn in Figures VIII. 4 and 5 will be considered, so as to decrease the computational complexity. The modal expansion VII. 14 is then used to obtain VIII. 51, corresponding to the complex

equations for modal time coefficients, VII. 15.

$$\dot{a}_m = \omega_m a_m$$

$$-\frac{h(k-1)}{1+(2.56)^2}\left(\frac{\Phi_o(.4h)}{\Phi_{o\,max}}\right)\frac{\psi_m^{(3)}(.4h)-2.56\psi_m^{(3)}(.2h)}{\sqrt{\langle\hat{\bar{\psi}}_m,\mu\bar{\psi}_m\rangle}}\sum_{n=1}^{4}\frac{\psi_n^{(3)}(.4h)-2.56\psi_n^{(3)}(.2h)}{\sqrt{\langle\hat{\bar{\psi}}_n,\mu\bar{\psi}_n\rangle}}b_n^* \qquad \text{(VIII. 51)}$$

$$\dot{b}_m^* = -\omega_m b_m^* - \sum_{n=1}^{4}\frac{a_n}{\sqrt{\langle\hat{\bar{\psi}}_m,\mu\bar{\psi}_m\rangle}\sqrt{\langle\hat{\bar{\psi}}_n,\mu\bar{\psi}_n\rangle}}\int_0^h \frac{\psi_m^{(3)}(r)\,\psi_n^{(3)}(r)}{\left[1.2-\Phi_o(r)\Phi_{o\,max}^{-1}\right]^2}\,dr$$

The integrals needed to evaluate VIII. 51 can be computed by 7090 Routine No. 5, in Appendix D. The results are tabulated as Table VIII. 4 for this example. Note that for m equal to two or four, corresponding to the less unstable modes of VIII. 3 and 4, the inner product is negative. Since the square root of the inner product is used to normalize the vector eigenfunction, the normalized vector eigenfunctions two and four are purely imaginary. Therefore, the modal coefficients corresponding to these vector eigenfunctions are purely imaginary, and the equivalent real matrix form of VIII. 51 is then VIII. 52.

TABLE VIII. 4

Evaluation of Numerical Integrations

m	$\lambda_I \langle \bar{\psi}_m, \mu \bar{\psi}_m \rangle$ in meters
1	.0288
2	-4. 205
3	.0712
4	-. 2640

m	n	$\int_0^h \frac{\psi_m^{(3)}(r)\psi_n^{(3)}(r)dr}{\left[1.2-\Phi_o(r)\Phi_{o\,max}^{-1}\right]^2}$ in meters
1	1	19. 55
1	2	20. 44
1	3	-2. 811
1	4	-2. 181
2	2	21. 40
2	3	-3. 282
2	4	-2. 640
3	3	9. 139
3	4	9. 080
4	4	9. 048

$$\lambda_I \begin{Bmatrix} a_1 \\ |a_2| \\ a_3 \\ |a_4| \end{Bmatrix} = \begin{bmatrix} 56.28 & 0 & 0 & 0 \\ 0 & .3306 & 0 & 0 \\ 0 & 0 & 7.63 & 0 \\ 0 & 0 & 0 & 2.00 \end{bmatrix} \begin{Bmatrix} a_1 \\ |a_2| \\ a_3 \\ |a_4| \end{Bmatrix}$$

$$- \begin{bmatrix} .124 & -.0118 & -.132 & .0674 \\ -.0118 & .00115 & .0127 & -.00654 \\ -.132 & .0127 & .143 & -.0732 \\ .0674 & -.00654 & -.0732 & .0376 \end{bmatrix} \begin{Bmatrix} b_1 \\ |b_2| \\ b_3 \\ |b_4| \end{Bmatrix}$$

(VIII. 52)

$$\lambda_I \begin{Bmatrix} b_1 \\ |b_2| \\ b_3 \\ |b_4| \end{Bmatrix} = - \begin{bmatrix} 56.28 & 0 & 0 & 0 \\ 0 & .3306 & 0 & 0 \\ 0 & 0 & 7.63 & 0 \\ 0 & 0 & 0 & 2.00 \end{bmatrix} \begin{Bmatrix} b_1 \\ |b_2| \\ b_3 \\ |b_4| \end{Bmatrix}$$

$$- \begin{bmatrix} 680.0 & 58.7 & -62.0 & -25.0 \\ 56.7 & 5.1 & -6.0 & -2.51 \\ -62.0 & -6.0 & 128.0 & 66.2 \\ -25.0 & -2.51 & 66.2 & 34.3 \end{bmatrix} \begin{Bmatrix} a_1 \\ |a_2| \\ a_3 \\ |a_4| \end{Bmatrix}$$

Note that both cross-coupling matrices are non-negative definite.

The feedback matrix equation corresponding to VII. 21 was solved by 7090 Routine No. 6, Appendix D.

$$P = \begin{Bmatrix} 1868.56 & 244.69 & 727.69 & 426.70 \\ 244.69 & 1390.05 & 387.68 & 710.10 \\ 727.69 & 387.64 & 672.89 & 570.74 \\ 426.70 & 710.10 & 570.74 & 737.32 \end{Bmatrix} \qquad \text{(VIII. 53)}$$

Note this matrix is positive definite, as required. It gives closed loop poles at $-.002779\lambda_I$, $-2.959\lambda_I$, $-7.43\lambda_I$, and $-57.25\lambda_I$.

Using the P matrix, the control rod motions can be found as functions of the modal coefficients.

$$\frac{u_1 \Phi_{o\,max}}{\Sigma_{aso}\, h} = \frac{(k-1)^2 \Phi_o(.4h)}{\left[1+(2.56)^2\right]\Phi_{o\,max}} \sum_{m=1}^{4} \frac{\psi_m^{(3)}(.4h) - 2.56\psi_m(.2h)}{\sqrt{\vec{\psi}_m, \mu\vec{\psi}_m}} \left\{ \begin{pmatrix} 1&0&0&0 \\ 0&i&0&0 \\ 0&0&1&0 \\ 0&0&0&i \end{pmatrix} P \begin{pmatrix} a_1 \\ |a_2| \\ a_3 \\ |a_4| \end{pmatrix} \right\}^{(m)} \qquad \text{(VIII. 54)}$$

The control rod motion at 0.2h can be found by using VIII. 46. Equation VIII. 54 corresponds to VII. 22, Using the values of this example, VIII. 54 is evaluated as VIII. 55.

$$\frac{u_1 \Phi_{o\,max}}{\Sigma_{aso}\, h\sqrt{\lambda_I}} = -.6897a_1 + .1713|a_2| - .0279a_3 + .1385|a_4| \qquad \text{(VIII. 55)}$$

This is the general form of the feedback for any detection system. The a_m can be evaluated by taking the inner product of $\vec{\psi}_m$ with the state vector. The state vector can be formed from the knowledge of the flux by taking first and second derivatives and using the knowledge of the closed loop system poles. This procedure is illustrated for the

case of three detectors positioned at .25h, .5h, and .75h. Actually, it would be better to use more detectors and have an estimation scheme to reduce the effects of the disregarded harmonics, but this will not be pursued here to keep the calculations as simple as possible.

The signals detected are $\Phi(r_j, t)$, $\dot{\Phi}(r_j, t)$ and $\ddot{\Phi}(r_j, t)$ for j=1, 2, 3. The steady state flux is the setting at which the reactor is desired to run, and so $\Phi_o(r_j)$ is known. Therefore, equations VIII. 56 give the error signals.

$$\Phi(r_j, t) - \Phi_o(r_j) = \sum_{m=1}^{6} a_m(t) \frac{\psi_m^{(3)}(r_j)}{\sqrt{\langle \hat{\psi}_m, \mu \psi_m \rangle}} + \sum_{k=1}^{3} a_{ok}(t) \frac{\psi_o^{(3)}(r_j)}{\sqrt{\langle \hat{\psi}_o, \mu \psi_o \rangle}} \qquad j=1, 2, 3$$

$$\dot{\Phi}(r_j, t) = \sum_{m=1}^{6} \dot{a}_m(t) \frac{\psi_m^{(3)}(r_j)}{\sqrt{\langle \hat{\psi}_m, \mu \psi_m \rangle}} + \sum_{k=1}^{3} \dot{a}_{ok}(t) \frac{\psi_o^{(3)}(r_j)}{\sqrt{\langle \hat{\psi}_o, \mu \psi_o \rangle}} \qquad j=1, 2, 3 \qquad \text{(VIII. 56)}$$

$$\ddot{\Phi}(r_j, t) = \sum_{m=1}^{6} \ddot{a}_m(t) \frac{\psi_m^{(3)}(r_j)}{\sqrt{\langle \hat{\psi}_m, \mu \psi_m \rangle}} + \sum_{k=1}^{3} \ddot{a}_{ok}(t) \frac{\psi_o^{(3)}(r_j)}{\sqrt{\langle \hat{\psi}_o, \mu \psi_o \rangle}} \qquad j=1, 2, 3$$

The derivatives of the modal time coefficients can be calculated from the closed loop system matrix, VIII. 57. The closed loop power system will be assumed the same as the open loop, for purposes of illustration,

although in general this will not be true.

$$\begin{Bmatrix} \ddot{a}_{01} \\ \ddot{a}_{02} \\ \ddot{a}_{03} \\ \ddot{a}_{1} \\ |\ddot{a}_{2}| \\ \ddot{a}_{3} \\ |\ddot{a}_{4}| \\ \ddot{a}_{5} \\ \ddot{a}_{6} \end{Bmatrix} = \lambda_I \begin{bmatrix} 374 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .043 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -392 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -105.24 & -10.63 & -35.31 & -18.89 & 0 & 0 \\ 0 & 0 & 0 & 15.32 & 1.39 & 3.33 & 1.79 & 0 & 0 \\ 0 & 0 & 0 & 170.72 & 11.19 & 44.32 & 19.66 & 0 & 0 \\ 0 & 0 & 0 & -87.12 & -5.72 & -18.72 & -8.06 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1,638 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4,340 \end{bmatrix} \begin{Bmatrix} a_{01} \\ a_{02} \\ a_{03} \\ a_{1} \\ |a_{2}| \\ a_{3} \\ |a_{4}| \\ a_{5} \\ a_{6} \end{Bmatrix}$$

(VII. 57)

Using VIII. 57 and its derivative, the modal time coefficient first and second derivatives may be expressed in terms of the modal time coefficients themselves. Therefore, the equation for the detected deviations, VIII. 56, can be expressed in terms of the modal time coefficients only. This equation can then be inverted by 7090 Routine No. 7, Appendix D, to obtain the modal time coefficients as functions of the detector readings and their first and second derivatives.

$$\sqrt{\lambda_I}\begin{Bmatrix} a_1 \\ |a_2| \\ a_3 \\ |a_4| \end{Bmatrix} = \begin{bmatrix} 2.11 & -2.11 & 0 & 14.21 & -14.21 & 0 & 7.88 & -7.88 & 0 \\ -23.35 & 23.35 & 0 & -161.12 & 161.12 & 0 & -89.59 & 89.59 & 0 \\ -5.82 & 8.86 & -2.15 & -64.48 & 27.00 & 26.53 & -29.61 & 20.72 & 6.14 \\ 11.46 & -17.89 & 4.55 & 128.91 & -54.03 & -53.02 & 59.23 & -41.38 & 12.29 \\ -.01 & .01 & 0 & -.04 & .04 & 0 & .09 & -.09 & 0 \\ .02 & -.04 & -.01 & .27 & -.11 & -.12 & .11 & .11 & 0 \\ -1.73 & 2.52 & -.56 & -17.69 & 8.92 & 7.55 & -7.22 & 7.41 & 3.31 \\ 2.64 & -3.26 & 1.44 & 25.71 & -11.21 & -10.38 & 9.47 & 10.87 & -5.76 \\ -1.46 & 2.43 & -.68 & -17.87 & 6.42 & 6.87 & -6.77 & 6.62 & 3.38 \end{bmatrix} \begin{Bmatrix} \phi(.25h, t) \\ \phi(.75h, t) \\ \phi(.5h, t) \\ \phi'(.25h, t) \\ \phi'(.75h, t) \\ \phi'(.5h, t) \\ \phi''(.25h, t) \\ \phi''(.75h, t) \\ \phi''(.5h, t) \end{Bmatrix}$$

$$\frac{1}{\sqrt{K\hat{\bar{\psi}}_m, \mu\bar{\psi}_m>}}\begin{Bmatrix} a_5 \\ a_6 \\ a_{01} \\ a_{02} \\ a_{03} \end{Bmatrix}$$

(VIII. 58)

The primes stand for differentiation with respect to $\lambda_I t \times 10^{+3}$.

Using the first four rows of VIII. 58, the feedback VIII. 55 can be evaluated.

$$\frac{u_1 \Phi_{o\,max}}{\Sigma_{aso}\,h} = -3.70\phi(.25h) + 2.66\phi(.75h) + .688\,\phi(.5h)$$
$$-17.76\,\phi'(.25h) + 29.60\,\phi'(.75h) - 8.06\,\phi'(.5h) \qquad \text{(VIII. 59)}$$
$$-11.76\,\phi''(.25h) + 14.47\,\phi''(.75h) - 1.87\,\phi''(.75h)$$

This expression and the expression for u_2, VIII. 46, can be used to design a reactor feedback control system similar to that depicted in Figure VIII. 2 for Example C. Large filters should be added to eliminate the fast transients.

Finally, note that a step function input of $0.02\,\Phi_{o\,max}$ probably would satisfy the mathematical requirement $u << \Sigma_{aso}\,h$. The effect of larger inputs and different operating points should be investigated. However, it is hoped that an analytical investigation similar to this example can provide insight into this very complex feedback control problem.

CHAPTER IX

CONCLUSIONS AND SUGGESTED FURTHER WORK

Conclusions

The problem of spatially dependent reactor kinetics has been attacked in a very general manner, extending the usual point reactor equations to spatially dependent equations. This has been done not only to attain a more detailed description of reactor kinetic processes, but also to control them. Although the thesis has been primarily concerned with the immediately practical problem of controlling xenon oscillations, methods for attacking the description and control of most reactor spatial processes have been investigated.

The state space formulation, known to be very effective in the description of kinetic processes in many fields, has been consistently applied to spatial nuclear processes. The form II. 2 permits the inclusion of the spatial effects of control rods, temperature, and other deterministic processes within a reactor with stationary fuel. Specifically, the formulation can include the spatial effects of a locally unstable process. It was shown that any locally unstable process can lead to a spatial instability, and this possibility can be investigated by the methods used in the thesis. Also, the concept of the temperature coefficient of reactivity was extended so that spatial temperature effects can be investigated by their inclusion in the general model.

An additional benefit of using the general state space formulation is that the methods used in nuclear reactor kinetics may also

be used for any physical system that happens to fit the formulation. One practical case of this has already been found. In fluid mechanics, the Benard problem of the linearization of the equations of a fluid heated from below is usually treated by a Fourier transformation in space.[83] If the heating is uneven so that the adverse temperature gradient $\beta(\bar{r})$ is a function of space, the technique of Kaplan modes and the resultant stability criteria may be applied to the equations expressed as IX. 1, instead of the Fourier transform method.

$$\begin{Bmatrix} 1 & 0 & 0 \\ 0 & \nabla^2 & 0 \\ 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} \rho \\ \frac{dr_z}{dt} \\ T \end{Bmatrix} = \begin{Bmatrix} \nu & 0 & 0 \\ 0 & \nu\nabla^4 & g\alpha\left(\frac{\partial^2}{\partial r_x^2} + \frac{\partial^2}{\partial r_y^2}\right) \\ 0 & \beta(\bar{r}) & \kappa \end{Bmatrix} \begin{Bmatrix} \rho \\ \frac{dr_z}{dt} \\ T \end{Bmatrix} \qquad \text{(IX. 1)}$$

From this it can be seen that the technique of Kaplan modes, and spatial control, can be applied to many other problems of general scientific interest.

However, the linearized state space approach does have a drawback, in that the steady state condition must be defined very carefully. The steady state as defined in Chapter III is not of the type most reactor engineers can use with familiarity.

The linearized state space equation as applied to the reactor has many advantages to offset this. The Kaplan modes can be applied to the analysis of the linearized equation, especially as generalized to include another spatial operator, μ, before the $\partial\tilde{z}/\partial t$. The use of Kaplan modes gives more insight into the assumptions made in deriving the point reactor equations. In the usual point reactor calculations, only the fundamental shape is considered.

Although the power control system may be driven by information of the reactor total power, the control rods are assumed to act on the fundamental only. The Kaplan modal analysis has shown that the control rod action excites the harmonics unless the control rods are specifically controlled to excite only the fundamental. The harmonics are assumed to die off quickly in the point reactor analysis, and if they do not, spatial effects should be analyzed.

The use of the Kaplan modal expansion has been justified by pointing out the relations between it and other expansions. Examples of the behavior have been given to stimulate further research in this mathematical problem, and a new method for their exact solution has been given.

This method of reduction to a scalar equation permits the use of variational techniques to estimate stability. The new method has been shown to agree with that of Lellouche as a special case. The only drawback is the transcendental nature of the solution, which has been shown to be no problem when used with modern computational methods. In an effort to eliminate this drawback, substitution of the clean mode shape times a function containing the steady state flux was used as a trial function in the variational estimate. However, this gives an error which increases with increasing flux and better methods are available in the high flux range.

The transcendental estimate is shown to be the point reactor transfer function that includes the effect of spatial variations. However, the Kaplan modal solution makes it obvious that transfer

functions do not "vary in space".

The use of Kaplan modes permit the reactor to be described by a finite number of modes. Because of this, the controllability theorem could be proven. This indicates the feasibility of controlling spatial shapes with a very few control rods, leading to a practical solution of the xenon oscillation problem.

Furthermore, the methods of modern control theory may be applied to the finite number of Kaplan modes. The technique of Kalman gives a method of optimal analytical design. This reduces the control problem to its barest mathematical essentials, and provides a way to judge the effect of design variations in system parameters. The technique of Kalman is shown to be a good starting point for practical control system design.

The assumption of perfect estimation permitted the analytical design of the system to be free of the method of estimation. Thus, methods of estimation can be compared.

Finally, a discussion of the control of the infinite mode system was given. This problem is shown to be very hard, but a few conclusions can be drawn. Perhaps a few considerations have been shown for possible further work in this area.

Suggestions for Further Work

Possible further work on the infinite mode control problem has been indicated. Another very fertile area seems to be the mathematics of the Kaplan modes. Although they have been shown to be a reasonable expansion, there is need of an answer to the rigorous

mathematical questions of the conditions under which existence, convergence, and completeness are assured. The behavior of the eigenvalues is an especially interesting question.

The operators in the state equations have been restricted to a diffusion type. The variational representation of dissipative processes has been given by Lewins[68] by "physical" arguments. This representation has been substantiated somewhat by the remarks in Chapter VII. However, mathematical rigor is lacking. Future mathematical research may find the circumstances under which the boundary conditions are such that equation VII. 11 is valid.

It may be possible to extend the type of operators beyond the diffusion type. What if a general partial differential operator is permitted? Perhaps even integral operators may be investigated, because integral operators also may give an eigenvalue problem.

From the reactor engineering standpoint, a better model of the kinetic behavior of the control rod is desired. In the present model, only very small motions can be permitted. The control theory is able to handle any linear model for the control rods, and perhaps a better linear model can be found. Inonu's theory should be checked against experiment for control rods, in addition to the fuel cell experiments presented in his paper. In addition to better spatial kinetic models for control rods, better spatial kinetic models for the temperature need to be found.

Finally, a non-linear stability theory and control system should be studied so as to predict behavior at all operating points, and the

behavior in moving from one operating point to another.

APPENDIX A

NOTATION

Operations:

A = The vector with components $A^{(i)}$

A^{+} = Complex conjugate transpose of A

A^{T} = Transpose of A

A^{*} = Complex conjugate of A

$\overline{A}$ = Solution of adjoint equation for A

A, B = Integration over the reactor volume of the scalar product of the vector $\overline{B}$ and the complex conjugate of the vector $\overline{A}$

$O(\ )$ = Order of

$|\ |$ = Absolute value of

Re = Real part of

Im = Imaginary part of

Subscripts:

F = Fast flux group

H = Harmonic part of

o = Steady state value of

S = Slow flux group

Script Letters (denoting operations dependent on $\overline{r}$)

$\mathcal{f}$ = Vector containing spatial operations

$\mathcal{G}$ = Control coefficient matrix

$\mathcal{H}$ = Output matrix

$\mathcal{L}$ = Matrix containing spatial operations

$\mathcal{M}$ = Shorthand for non-operator part of VI. 1 in brackets

$\mathcal{Q}$ = Matrix measure of flux deviation

$\mathcal{U}$ = Matrix containing spatial operations

$\mathcal{V}$ = The control system criterion functional

Capital Latin Letters:

B_m = Geometric buckling

C = Neutron precursor density

D = Diffusion coefficient

F = Plant matrix of the modal decomposition of $\mathcal{L}$ and $\mathcal{U}$

G = Control coefficient matrix of the modal decomposition of $\mathcal{U}$

I = Unit matrix

J_m = Number of multiplicities of ω_m

L = Lagrangian

M = Number of modes

N = Number of control rods

N_I = Iodine concentration

N_x = Xenon concentration

P = Feedback matrix

Q = Measure matrix of the modal decomposition of $\mathcal{H}\mathcal{Q}\mathcal{H}$

R = A measure matrix of $\bar{u}$

S = A matrix defined in Appendix C

T = Temperature, also a matrix defined in Appendix C

U = Unitary matrix

V = Reactor volume, also a matrix defined in Appendix C

Z = State vector

Small Latin Letters:

a = Modal expansion coefficient

b = Modal expansion coefficient

c = Modal expansion coefficient, delayed neutron precursor deviation

c_T = Thermal heat capacity

d = Modal expansion coefficient

d^3r = Volume derivative

e = Energy of fission, also base of natural logarithm

f = Input

h = Film temperature coefficient, also linear dimension

i = Iodine concentration deviation

j = $\sqrt{-1}$, also a dummy integer

k = Multiplication constant

ℓ = Neutron lifetime

m = Dummy integer

n = Dummy integer

r = Spatial dimension

t = Time dimension

u = Control rod effective absorption change at $\bar{r}_n$

v = Neutron velocity, also coolant velocity

x = Xenon deviation

z = State variable deviation

Greek Letters:

a_m = Clean mode eigenvalue

β = Fraction of delayed neutrons

γ_I = Fraction of iodine atoms per fission

γ_x = Fraction of xenon atoms per fission

δ = Dirac δ function, Kronecker δ, virtual displacement

∇ = Gradient

∇^2 = Laplacian operator

ϵ = Fast fission factor, also a small number

ζ = Riemann ζ variable

θ = Clean mode shape

κ = Heat conduction constant

λ_c = Decay constant of neutron precursors

λ_I = Decay constant of iodine

λ_x = Decay constant of xenon

ν = Number of neutrons per fission

ξ = Adjoint variable

ρ = Density

ρ_m = Real part of ω_m

Σ_a = Absorption macroscopic cross-section

Σ_f = Fission macroscopic cross-section

Σ_R = Removal macroscopic cross-section

σ_a^x = Absorption microscopic cross-section

σ_m = Imaginary part of ω_m

τ = Dummy time variable of integration

Φ = Neutron flux

ϕ = Neutron flux deviation

χ = Arbitrary function

ψ_m = Kaplan mode

ω_m = Eigenvalue of a Kaplan mode

APPENDIX B

PROOF OF THE VALIDITY OF THE SCALAR REPRESENTATION

The scalar representation, V. 3, of the Kaplan operator matrix, V.1, is valid if the scalar $\psi_m^{(3)}$ is zero at the singular points introduced. The singular points are introduced in the process of reduction to a scalar equation upon division by functions that have a zero in the range of the independent variable considered. The range considered is the interior of the volume of the reactor. The boundary is not considered because the function $\bar{\psi}_m$ is required to be zero on the boundary.

In the case of the derivation of the specific scalar equation V. 3, it must be shown that $\psi^{(3)}$ is zero whenever the expressions $\omega_m + \lambda_I$ and $\omega_m + \lambda_x + \sigma_a^x \Phi_o$ are zero. This will be shown first in multiplying media, i.e., where Σ_f is non-zero. Then all material constants in the Kaplan operator matrix are greater than zero.

$$\omega_m \psi_m^{(1)} = -\lambda_I \psi_m^{(1)} + \gamma_I \epsilon \Sigma_f \psi_m^{(3)} \qquad \text{(a)}$$

$$\omega_m \psi_m^{(2)} = \lambda_I \psi_m^{(1)} - (\lambda_x + \sigma_a^x \Phi_o)\psi_m^{(2)} + \left(\gamma_x - \frac{(\gamma_I + \gamma_x)\sigma_a^x \Phi_o}{\lambda_x + \sigma_a^x \Phi_o}\right)\psi_m^{(3)} \qquad \text{(b) (V.1)}$$

$$\frac{\omega_m}{v} \psi_m^{(3)} = -\sigma_a^x \Phi_o \psi_m^{(2)} + (\nabla \cdot D \nabla + \nu\epsilon\Sigma_f - \Sigma_a - \sigma_a^x N_{x_o})\psi_m^{(3)} \qquad \text{(c)}$$

If $\omega_m + \lambda_I$ is zero, then $\psi_m^{(3)}$ is zero by V.1a. If $\omega_m + \lambda_x + \sigma_a^x \Phi_o$ is zero at a point, then V.1a and b may be combined to form equation B.1 at that point.

$$0 = \frac{\lambda_x \sigma_a^x \Phi_o (\gamma_I - \gamma_x) + \gamma_x \lambda_x (\lambda_I - \lambda_x) + \gamma_I \lambda_I \lambda_x + \gamma_I (\sigma_a^x \Phi_o)^2}{\lambda_x + \sigma_a^x \Phi_o} \epsilon \Sigma_f \psi_m^{(3)} \quad \text{(B. 1)}$$

Since $\gamma_I > \gamma_x$ and $\lambda_I > \lambda_x$, the numerator of the fraction is greater than zero. The denominator is always positive, and so $\psi_m^{(3)}$ is zero. Therefore, the scalar representation is valid in multiplying media.

In regions where Σ_f equals zero, the xenon and iodine concentrations are zero and then equation V. 1 reduces to B. 2.

$$\frac{\omega_m}{v} \psi_m^{(3)} = (\nabla \cdot D \nabla - \Sigma_a) \psi_m^{(3)} \quad \text{(B. 2)}$$

Since $\nabla . D \nabla$ represents a leakage term, B. 2 implies the inequality B. 3.

$$-\omega_m > \Sigma_a v \quad \text{(B. 3)}$$

But $\Sigma_a v$ is the inverse of the neutron lifetime, and so if $\omega_m + \lambda_I$ or $\omega_m + \lambda_x + \sigma_a^x \Phi_o$ are zero, then B. 4 must be true.

$$\begin{aligned} &\ell \lambda_I > 1 && \text{(a)} \\ &\ell (\lambda_x + \sigma_a^x \Phi_o) > 1 && \text{(b)} \end{aligned} \quad \text{(B. 4)}$$

In a practical case, neither of these inequalities is satisfied, and so the only solution to B. 2 is $\psi_m^{(3)}$ identically zero. Therefore, the scalar representation V. 3 is valid everywhere in the region of interest.

Similar arguments for the reduction of the general matrix operator to a scalar equation can usually be made, which implies the variational methods of Chapter VI are valid. The shooting method of solution to the Kaplan equations can be applied directly to the matrix form. Although this is more cumbersome, it is valid in any case.

APPENDIX C

PROOF OF CONTROLLABILITY

The proof of the controllability theorem rests on a criterion stated by Pontryagin[84] and clarified by Kalman.[75] It concerns complete controllability, which means that every state at any time can eventually be returned to zero. Kalman's statement is that "A (linear) constant plant is completely controllable if and only if . . . (criterion C. 1 holds)."

$$\text{rank}\left[G, FG, \ldots, F^{M-1}G\right] = M \qquad \text{(C. 1)}$$

Here, F is the M x M real constant coefficient plant matrix and G is the M x N control coefficient matrix for the plant described by C. 2.

$$\frac{d\bar{z}(t)}{dt} = F\bar{z}(t) + G\bar{u}(t) \qquad \text{(C. 2)}$$

Therefore, the composite matrix in equation C. 1 has M rows and MN columns.

Before giving the theorem, it is necessary to show that the modes of the state equation, III. 5, can be put in an equivalent real form. In general, the Kaplan eigenvalues and vector eigenfunctions are complex. The Pontryagin-Kalman criterion may only be applied to real forms. The equivalence to a real form is shown by a lemma.

Lemma: If the Kaplan modes exist and form a complete set, the state equation, III. 5, can be put in the equivalent form of an infinite set of real, time dependent total differential equations that may be coupled only in pairs.

Proof: It has been shown in Chapter IV how the state equation may be reduced to an infinite set of complex time dependent total differential equations for the Kaplan mode modal coefficients. The expansion of the state variable in Kaplan modes is rewritten here.

$$\bar{z}(\bar{r}, t) = \sum_{m=0}^{\infty} a_m(t)\, \bar{\psi}_m(\bar{r}) \qquad \text{(IV. 14)}$$

A typical equation for the modal coefficients is written as C. 3, for N control rods.

$$\dot{a}_m = \omega_m a_m + \sum_{n=1}^{N} \langle \hat{\bar{\psi}}_m , \mathcal{G}_n \rangle\, u_n \qquad \text{(C. 3)}$$

Note that the assumption that $\mathcal{L}$ is able to be diagonalized is implicit in this, implying only geometrical multiplicities.

If ω_m and $\bar{\psi}_m$ are complex, then their complex conjugate is guaranteed to be another eigenvalue and vector eigenfunction by the reality of the linear operators. Taking the complex conjugate of the modal equation IV. 8 gives equation C. 4.

$$\mathcal{L}^* \bar{\psi}_m^* = \omega_m^* \mathcal{M}^* \bar{\psi}_m^* \qquad \text{(C. 4)}$$

Since $\mathcal{L}$ and $\mathcal{M}$ are purely real, they are identical to their complex conjugate.

$$\mathcal{L} \bar{\psi}_m^* = \omega_m^* \mathcal{M} \bar{\psi}_m^* \qquad \text{(C. 5)}$$

Therefore, if ω_m and ψ_m are complex, then their complex conjugates are also a solution to the eigenvalue problem.

Also, note that if ω_m is real, then $\bar{\psi}_m$ is a real vector times

an arbitrary constant. Since $\langle \hat{\Psi}_m, \mu \bar{\Psi}_m \rangle = 1$, the normalization guarantees that $\bar{\Psi}_m$ can be chosen purely real or purely imaginary for the case ω_m real, depending on whether $\langle \hat{\Psi}_m, \mu \bar{\Psi}_m \rangle$ is positive or negative when computed with all real elements.

The state variable is real, so the terms in the Kaplan expansion must be real for all time and for any initial condition. This determines that the $a_m(t)$ are purely real or imaginary, corresponding to the cases in which $\bar{\Psi}_m$ is purely real or imaginary, respectively. Also, the $a_m(t)$ of a complex $\bar{\Psi}_m$ must be the complex conjugate of the $a_k(t)$ of $\bar{\Psi}_m^*$.

Therefore, the infinite set of equations C. 3 may be broken up in three ways.

1. ω_m real, $\bar{\Psi}_m$ purely real.

$$\mathrm{Re}\{\dot{a}_m\} = \omega_m \, \mathrm{Re}\{a_m\} + \sum_{n=1}^{N} \langle \hat{\Psi}_m, \mathcal{L}_n \rangle \, u_n \qquad \text{(C. 6)}$$

2. ω_m real, $\bar{\Psi}_m$ purely imaginary.

$$\mathrm{Im}\{a_m\} = \omega_m \, \mathrm{Im}\{a_m\} + \sum_{n=1}^{N} \langle \hat{\Psi}_m, \mathcal{L}_n \rangle \, u_n \qquad \text{(C. 7)}$$

3. ω_m complex $= \rho_m + i\sigma_m$. (C. 8)

The pair of equations for $\bar{\Psi}_m$ and $\bar{\Psi}_m^*$ must be used.

$$\mathrm{Re}\{\dot{a}_m\} + i\,\mathrm{Im}\{\dot{a}_m\} = (\rho_m + i\sigma_m)(\mathrm{Re}\{a_m\} + i\,\mathrm{Im}\{a_m\}) + \sum_{n=1}^{N} (\langle \mathrm{Re}\{\hat{\bar{\psi}}_m\}, \mathcal{G}_n \rangle + i \langle \mathrm{Im}\{\hat{\bar{\psi}}_m\}, \mathcal{G}_n \rangle) u_n \tag{C.9}$$

$$\mathrm{Re}\{\dot{\bar{a}}_m\} - i\,\mathrm{Im}\{\dot{a}_m\} = (\rho_m - i\sigma_m)(\mathrm{Re}\{a_m\} - i\,\mathrm{Im}\{a_m\}) + \sum_{n=1}^{N} (\langle \mathrm{Re}\{\hat{\bar{\psi}}_m\}, \mathcal{G}_n \rangle - i \langle \mathrm{Im}\{\hat{\bar{\psi}}_m\}, \mathcal{G}_n \rangle) u_n$$

Both equations of C. 9 may be seen to be equivalent to C. 10.

$$\begin{pmatrix} \mathrm{Re}\{\dot{a}_m\} \\ \mathrm{Im}\{\dot{a}_m\} \end{pmatrix} = \begin{pmatrix} \rho_m & -\sigma_m \\ +\sigma_m & \rho_m \end{pmatrix} \begin{pmatrix} \mathrm{Re}\{a_m\} \\ \mathrm{Im}\{a_m\} \end{pmatrix} + \sum_{n=1}^{N} \begin{pmatrix} \langle \mathrm{Re}\{\hat{\bar{\psi}}_m\}, \mathcal{G}_n \rangle \\ \langle \mathrm{Im}\{\hat{\bar{\psi}}_m\}, \mathcal{G}_n \rangle \end{pmatrix} u_n \tag{C.10}$$

The infinite set of complex equations C. 3 is equivalent to the set of equations for the real and imaginary parts of the modal coefficients C. 6, C. 7, and C. 10. This set of equations may be put in the infinite dimensional matrix form C. 11.

$$\dot{\bar{a}} = F\bar{a} + G\bar{u} \tag{C.11}$$

In C. 11, $\bar{a}$ is the vector made up of the real and/or imaginary parts of $a_m(t)$, $\bar{u}$ is still the N dimensional control vector, and G is defined as in C. 12.

$$G_{ij} = \{ < \hat{\vec{\psi}}_i , \vec{\psi}_j > \} \quad j=1, 2, \ldots, N; \quad i=1, 2, \ldots \infty \tag{C. 12}$$

The plant matrix, F, is the direct sum of ω_m or $\begin{pmatrix} \beta_m & -\sigma_m \\ \sigma_m & \beta_m \end{pmatrix}$. Therefore, F is almost diagonal except for the 2 x 2 blocks corresponding to the complex eigenvalues. Each eigenvalue, ω_m, is repeated J_m times. The sets of equations C. 6, C. 7, and C. 10 may be put in decreasing order of the real parts of the eigenvalues, so that the most unstable modes appear first. Since equations C. 6, C. 7, and C. 10 have a one to one correspondence with the equations C. 3, equation C. 11 is equivalent to C. 3 and the lemma is shown to be true.

The matrixes F and G are now truncated at some finite dimension, M, so that the controllability of a finite number of the most unstable modes may be investigated. If there are K distinct eigenvalues, the relationships C. 13 hold.

$$\sum_{i=1}^{K} J_m = M \quad , \quad K \leq M \tag{C. 13}$$

Having proven the lemma and described the finite plant, the controllability theorem can now be proven.

Theorem: Any finite number of modes of the reactor model C. 11 are controllable if:

1) There are at least as many control rods as the maximum multiplicity, max J_m.

2) All control rods are not on any possible nodes of a combination of modes having the same eigenvalue.

Proof: Set $u_n \equiv 0$ for $n > \max J_m$. This is equivalent to taking $N = \max J_m$. The $u_n \equiv 0$ should be chosen so that condition 2 is not violated. Then the Pontryagin-Kalman criterion can be applied.

$$\text{rank}\left[G, FG, \ldots, F^{M-1}G\right] = M \Leftrightarrow \text{controllability} \qquad \text{(C. 1)}$$

The matrix F may be put back into diagonal form by the transformation C. 14.

$$UFU^{-1} = \Omega \qquad \text{(C. 14)}$$

Here, the matrix U is a direct sum of unit matrices and unitary matrices of the form $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$. The unit matrices occur whenever F has ω_m on the diagonal, and the unitary matrices occur in U where the $\begin{pmatrix} \rho_m & -\sigma_m \\ \sigma_m & \rho_m \end{pmatrix}$ occur in F. It can be seen that Ω is a diagonal matrix with the eigenvalues of the Kaplan modes on the diagonal. Therefore, using C. 14, the criterion C. 1 may be rewritten as C. 15.

$$\text{rank } U^{-1}\left[UG, \Omega UG, \ldots, \Omega^{M-1}UG\right] = M \Leftrightarrow \text{controllability} \qquad \text{(C. 15)}$$

Since U is non singular, the rank of the matrix in brackets determines the rank of the product. This matrix is written out as C. 16.

$$\left\{\begin{vmatrix} c_{11} \cdots c_{1N} \\ \vdots \\ c_{J_1 1} \cdots c_{J_1 N} \\ \vdots \\ c_{M1} \cdots c_{MN} \end{vmatrix} \cdots \begin{vmatrix} \omega_1^{M-1} c_{11} \cdots \omega_1^{M-1} c_{1N} \\ \vdots \\ \omega_1^{M-1} c_{J_1 1} \cdots \omega_1^{M-1} c_{J_1 N} \\ \vdots \\ \omega_K^{M-1} c_{M1} \cdots \omega_K^{M-1} c_{MN} \end{vmatrix}\right\} \qquad \text{(C. 16)}$$

In C. 16, c_{ij} is defined as $\{UG\}_{ij}$. (C. 17)

The rank of a matrix is not changed by the addition of more rows equal to the rows already in the matrix. Add as many rows as necessary to make each distinct ω_m appear as many times as the maximum multiplicity, $\max J_m \equiv N$. Then consider the KN x KN matrix formed by deleting the last MN-KN columns, as shown in C.18.

$$S = \left\{ \begin{array}{c|c|c|c} \overbrace{\phantom{c_{11}\cdots c_{IN}}}^{N} & & & \\ \underbrace{\begin{matrix} c_{11} \cdots c_{IN} \\ c_{11} \cdots c_{IN} \end{matrix}}_{N-J_1} & \cdots & \begin{matrix} \omega_1^{K-1} c_{11} \cdots \omega_1^{K-1} c_{IN} \\ \omega_1^{K-1} c_{11} \cdots \omega_1^{K-1} c_{IN} \end{matrix} & \\ \begin{matrix} c_{J_1 1} \cdots c_{J_1 N} \\ \vdots \\ c_{M1} \cdots c_{MN} \end{matrix} & & \begin{matrix} \omega_1^{K-1} c_{J_1 1} \cdots \omega_1^{K-1} c_{J_1 N} \\ \vdots \\ \omega_K^{K-1} c_{M.} \cdots \omega_K^{K-1} c_{MN} \end{matrix} & \end{array} \right\} \qquad \text{(C.18)}$$

But S may be recognized as the product of two matrices, as shown in VI.9.

$$S = \left\{ \begin{array}{c|c} \begin{matrix} c_{11} \cdots c_{IN} \\ c_{11} \cdots c_{IN} \\ c_{J_1 N} \cdots c_{J_1 N} \end{matrix} & \begin{matrix} N \\ O \end{matrix} \\ \begin{matrix} \overleftarrow{\qquad N \qquad} \\ O \quad N \end{matrix} & \begin{matrix} \overleftarrow{\qquad N \qquad} \\ c_{M1} \cdots c_{MN} \end{matrix} \end{array} \right\} \left\{ \begin{array}{cccc} I & \omega_1 I & \cdots & \omega_1^{K-1} I \\ \cdots & \cdots & \cdots & \cdots \\ I & \omega_K I & \cdots & \omega_K^{K-1} I \end{array} \right\} \qquad \text{(C.19)}$$

$$= TV.$$

However, V may be recognized as a rearrangement of a direct sum of N Vandermonde matrices, K x K, made up of the K distinct roots. Therefore, V is non-singular and the rank of S is that of T. But the

rank of T is equal to the sum of the ranks of the N x N blocks on the diagonal. Since these blocks each have $N-J_m$ identical rows, the maximum rank each block can have is J_m. Therefore, the maximum rank of T is $\sum_{m=1}^{K} J_m$, which equals M by C. 13. If and only if each block on the diagonal of T has rank J_m will the rank of S equal M, implying controllability. A typical block is written out as C. 20.

$$C = \begin{Bmatrix} c_{m1} & c_{m2} & \cdots & c_{mN} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mN} \\ \vdots & \vdots & & \vdots \\ c_{J_m 1} & c_{J_m 2} & \cdots & c_{J_m N} \end{Bmatrix} \qquad \text{(C. 20)}$$

But each $c_{ij} = UG_{ij}$ from C. 17, so that a $J_m \times J_m$ submatrix of C is non-singular only if a corresponding $J_m \times J_m$ submatrix of G is non-singular. The $J_n \times J_n$ submatrix of G is non-singular if C. 21 holds.

$$0 \neq \det_{J_m \times J_m} G = \det_{J_m \times J_m} \{ \langle \bar{\psi}, \mathcal{Y} \rangle \} = \det_{J_m \times J_m} \{ \Phi_0(r_j)\, \psi_i^{(3)}(r_j) \}$$

(C. 21)

The last matrix, whose $J_m \times J_m$ determinant is to be taken in VI. 21, is the product of the matrix of Kaplan mode slow flux elements evaluated at the control rod positions times a diagonal matrix of the steady state flux evaluated at the control rod positions. This diagonal matrix is always non-singular since $\Phi_0(\bar{r}_j) > 0$ at any interior point in the reactor. Therefore, the only way any $J_m \times J_m$ determinant can be zero is that a linear combination of $\psi_i^{(3)}(\bar{r}_j)$ equal zero, or

that all control rod position ends are on possible nodes of the vector eigenfunctions belonging to the same eigenvalue. This violates condition 2, and so the theorem is proven.

This requirement that all control rods not be on a possible node cannot be met in the infinite case. Surely one of the infinite number of modes can be found such that it has nodes on the finite number of control rod ends. Therefore, it is doubtful that a model with an infinite number of unstable modes could be controlled by a finite number of control rods.

Note that the proof of the theorem rested on the existence and completeness of Kaplan modes. If the node shapes "interacted", it would not have been possible to consider only a finite number of them, so that the theorem could not have been formulated.

Finally, the form of the $\mathcal{G}$ matrix has been assumed to be $\mathcal{G}_n = \Phi_o(\bar{r})\,\delta(\bar{r}-\bar{r}_n)$ in the row corresponding to the thermal flux, and zero elsewhere. If this is not the form, then condition 2 of the theorem becomes, instead, that the $J_m \times J_m$ determinant of $\langle \Phi, \mathcal{G} \rangle$ is non-zero. In Example D, two rods placed symmetrically about the center of the reactor cannot control the second harmonic because they are constrained by the power control system requirement that changes $\mathcal{G}$ to $\mathcal{G}_{N-1}$. Therefore, even though the rods are not on any nodes, the $J_m \times J_m$ determinant of $\langle \hat{\Phi}, \mathcal{G}_{N-1} \rangle$ is zero when the rods are symmetrical about the reactor center line.

APPENDIX D

FORTRAN IV PROGRAMS USED IN THE EXAMPLES

In each of the first six separate programs, the DEQ subroutine from the Caltech subroutine library was used to solve a set of nonlinear total differential equations with given initial conditions. The DEQ subroutine uses a Runge-Kutta technique to start the solution and goes to an Adams-Bashforth method once it has started. It has built-in error control, and has proven satisfactory in past operation. For further information, a copy of DEQ is in the Caltech Physics Library.[85]

The seventh program uses EIG 1, from the IBM SHARE library, to determine the eigenvalues of the closed loop matrix, and MATINV, on CIT tape, to invert the detection matrix.

7090 Routine No.1: Feedback Calculation For Various Control Rod Positions of Example D with Unstable Open Loop Poles

```
      DIMENSION PHI(3),PHIDOT(3)
   10 READ (5,500) R1,R2                    R1,R2 ARE CONTROL ROD POSITIONS
  500 FORMAT (2F10.2)
      WRITE (6,600) R1,R2
      PHI(1) = 0.0
      PHI(2) = 0.0
      PHI(3) = 0.0
      R = 0.0                               R=TIME
      DR = 200.0
      EPSLON = .01
      C11 = .0001308*SIN(3.1415926*R1)**2/(1.0+(SIN(3.1415926*R1)/
     1  SIN(3.1415926*R2))**4)
      C22 = SIN(6.2831852*R1) - SIN(3.1415926*R1)*SIN(6.2831852*R2)/
     1   SIN(3.1415926*R2)
      C33 =SIN(9.4247778*R1) - SIN(3.1415926*R1)*SIN(9.4247778*R2)/
     1   SIN(3.1415926*R2)
      WRITE (6,615) C11, C22,C33
      F11 = -C11*C22**2
      F22 = -C11*C33**2
      F12 = -C11*C22*C33
      WRITE (6,620) F11,F22,F12
      CALL DEQ(K,3,R,PHI,PHIDOT,DR,EPSLON)
      GO TO (100,200,200,300),K
C
C   DERIVATIVE BOX  - GENERATES FUNCTION
C
  100 CONTINUE      NOTE- TIME SCALE = REAL TIME X 100
      PHIDOT(1)=.0003260*PHI(1) + .1 - C11*(C22*PHI(1)+C33*PHI(3))**2
      PHIDOT(2)=.0000442*PHI(2) + .1 - C11*(C22*PHI(3)+C33*PHI(2))**2
      PHIDOT(3)=.0001851*PHI(3)      - C11*(C22**2*PHI(1)*PHI(3)+
     1  C22*C33*(PHI(1)*PHI(2)+PHI(3)**2) +C33**2*PHI(2)*PHI(3))
      CALL DEQ2
  200 CONTINUE
C
C  PRINT RESULTS,TEST FOR END OF INTEGRATION
C
      WRITE (6,610) R,PHI(1),PHI(2),PHI(3)
  610 FORMAT (4X4HR = E15.8,4X9HPHI(1) = E15.8,4X9HPHI(2) = E15.8,
     1 4X9HPHI(3) = E15.8)
      IF (R .GT.50000.0) GO TO 20
      CALL DEQ1
      GO TO 200
   20 A11 = .0001630 + F11*PHI(1) + F12*PHI(3)
      A22 =  .0000221 + F12*PHI(3) + F22*PHI(2)
      A12 = F11*PHI(3) + F12*PHI(2)
      A21 = F12*PHI(1) + F22*PHI(3)
      B = 0.5*(A11+A22)
      SQUARE = SQRT(B**2 + A21*A12 - A11*A22)
      WRITE (6,650) B,SQUARE
      EIGEN1 = B+SQUARE                     } CLOSED LOOP EIGENVALUES
      EIGEN2 = B- SQUARE
      WRITE (6,660) EIGEN1,EIGEN2
      Z1 =      -C11/SIN(3.1415926*R1)*(C22*PHI(1)+C33*PHI(3))
      Z2 =      -C11/SIN(3.1415926*R1)*(C22*PHI(3)+C33*PHI(2))
      EFORT1 = Z1 + Z1*SIN(3.1415926*R1)**4/SIN(3.1415926*R2)**4
                                            CONTROL EFFORT. =
                                            EFFORT1^2 + EFFORT2 * a2^2
```

```
      EFORT2 = Z2 + Z2*SIN(3.1415926*R1)**4/SIN(3.1415926*R2)**4
      WRITE (6,670) Z1,Z2,EFORT1,EFORT2
      GO TO 10
C
C  ERROR ON RETURN FROM DEQ
C
  300 CONTINUE
      WRITE (6,511)
      GO TO 10
  600 FORMAT (10X5HR1 = E15.8,10X5HR2 = E15.8//)
  615 FORMAT (10X6HC11 = E15.8,10X6HC22 = E15.8,10X6HC33 = E15.8//)
  620 FORMAT (10X6HF11 = E15.8,10X6HF22 = E15.8,10X6HF12 = E15.8//)
  650 FORMAT (15X4HB = E15.8,10X9HSQUARE = E15.8////)
  660 FORMAT (15X9HEIGEN1 = E15.8,15X9HEIGEN2 = E15.8//)
  670 FORMAT (5X5HZ1 = E15.8,5X5HZ2 = E15.8,5X9HEFORT1 = E15.8,
     1                                  5X9HEFORT2 = E15.8////)
  511 FORMAT (//10X21HERROR RETURN FROM DEQ)
      END
```

7090 Routine No. 2: Steady State Flux of Example E

```
      DIMENSION PHI(2), PHIDOT(2)
   10 READ 15,500) ALPHA                    .0016*ALPHA = k∞-1
  500 FORMAT (F10.4)
      WRITE (6,512) ALPHA
      PHI(1)=1.0
      PHI(2)=0.0
      R=0.0     METERS - SLAB REACTOR HALF DIMENSION
      DR=.05
      EPSLON=.0001     TRUNCATION ERROR CONTROL. DIVIDES INTO SUBINTERVALS
                                                 IF NOT MET.
      CALL DEQ(K,2,R,PHI,PHIDOT,DR,EPSLON)
      GO TO (100,200,200,300),K
C
C  DERIVATIVE BOX - GENERATES FUNCTION
C
  100 CONTINUE
      PHIDOT(1)=PHI(2)
      PHIDOT(2)=(-ALPHA+(1.525+.0244*ALPHA)*PHI(1)/(.035+PHI(1)))*PHI(1)
      CALL DEQ2
  200 CONTINUE
C
C  PRINT RESULTS, TEST FOR END OF INTEGRATION
C
      WRITE (6,510) R,PHI(1), PHI(2)
  510 FORMAT (10X4HR = E15.8,10X9HPHI(1) = E15.8,10X9HPHI(2) = E15.8)
      IF(R .GT. 20.0) STOP
      IF(PHI(1) .LE. 0.0) GO TO 10
      CALL DEQ1
      GO TO 200
C  ERROR ON RETURN FROM DEQ
C
  300 CONTINUE
      WRITE (6,511)
      GO TO 10
  512 FORMAT(1H1 40X8HALPHA = E15.8//)
  511 FORMAT(//10X21HERROR RETURN FROM DEQ)
       END
```

7090 Routine No. 3: Estimation of Kaplan Mode Eigenvalues

```
 10 READ (5,500) P                                    HARMONIC = SIN P X.
 15 READ (5,500) R,S                                  R = Re{ω_n}, S = Im{ω_n}
    IF (R.EQ.0.0.AND.S .EQ. 0.0) GO TO 10
500 FORMAT (2F10.2)
    REAL =0.0                                         REAL = REAL PART OF RAYLEIGH PROD
    UNREAL = 0.0                                      UNREAL = UNREAL " " " "
    DX = 0.02
    X = 0.04
    SUM = 0.0
    TUM = 0.0
    RS = (1.0 + R)**2+S**2
    RL = .725 +R
    DO 50 I = 1,77
    SX = 20.7*SIN(X)
    V = SIN(P*X)**2*SX/((SX+RL)**2+S**2)
    SUM=SUM+V*((RL*(SX+RL)+S**2)/(SX+.725)+((SX+RL)*(1.0+R)-S**2)/RS)
    TUM=TUM+V*(SX/(SX+.725)-(SX+1.725+2.0*R)/RS)
    X=X+DX
 50 CONTINUE
    REAL = 16.0-P**2-.0018*R-00.4*SUM
    UNREAL =S*(.0018+.4*TUM)
    WRITE (6,600) P,R,S,REAL,UNREAL
600 FORMAT(/2X4HP = F10.4,5X4HR = F10.5,5X4HS = F10.4,5X7HREAL =
   1E15.8,5X9HUNREAL = F15.8)
    GO TO 15
     END
```

7090 Routine No. 4: Calculation of Complex Kaplan Eigenfunctions

```
    DIMENSION PHI(6), PHIDOT(6)
 10 READ (5,500) RHO, SIGMA
500 FORMAT (2F10.4)
    WRITE (6,512) RHO, SIGMA
    PHI(1)=1.0
    PHI(2)=0.0
    PHI(3) = 0.0
    PHI(4)=1.0
    PHI(5)=0.0
    PHI(6)=1.0
    R=0.0
    DR=.05
    EPSLON=.0001
    CALL DEQ(K,6,R,PHI,PHIDOT,DR,EPSLON)
    GO TO (100,200,200,300),K

 DERIVATIVE BOX - GENERATES FUNCTION

100 CONTINUE
    PHIDOT(1)=PHI(2)
    PHIDOT(2) = (-1.6+1.525*1.0256*PHI(1)/(.035+PHI(1)))*PHI(1)
    PHIDOT(3)=PHI(4)
    PHIDOT(4)=(.00018125*RHO-1.6+32.36*PHI(1)*(((0.72414+RHO)*
   1(20.69*PHI(1)+0.74214+RHO)+SIGMA**2)/((0.72414+20.69*PHI(1))*
   2((0.72414+20.69*PHI(1)+RHO)**2+SIGMA**2))+((20.69*PHI(1)+
   30.72414+RHO)*(1.0+RHO)-SIGMA**2)/(((20.69*PHI(1)+0.72414+RHO)**2
   4+SIGMA**2)*((1.0+RHO)**2+SIGMA**2))))*PHI(3)
   5-SIGMA*(.00018125+(32.36*PHI(1)/((20.69*PHI(1)+0.72414+RHO)**2
   6+SIGMA**2))*(20.69*PHI(1)/(0.72414+20.69*PHI(1))-(20.69*PHI(1)
   7+1.72414+2.0*RHO)/((1.0+RHO)**2+SIGMA**2)))*PHI(5)
    PHIDOT(5)=PHI(6)
    PHIDOT(6)=SIGMA*(.00018125+(32.36*PHI(1)/((20.69*PHI(1)
   1+0.72414+RHO)**2+SIGMA**2))*(20.69*PHI(1)/(0.72414+20.69*PHI(1))
   2-(20.69*PHI(1)+1.72414+2.0*RHO)/((1.0+RHO)**2+SIGMA**2)))*PHI(3)
   3+(.00018125*RHO-1.6+32.36*PHI(1)*(((0.72414+RHO)*(20.69*PHI(1)
   4+0.72414+RHO)+SIGMA**2)/((0.72414+20.69*PHI(1))*((20.69*PHI(1)
   5+0.72414+RHO)**2+SIGMA**2))+((20.69*PHI(1)+0.72414+RHO)*(1.0+RHO)
   6-SIGMA**2)/(((20.69*PHI(1)+0.72414+RHO)**2+SIGMA**2)*
   7((1.0+RHO)**2+SIGMA**2))))*PHI(5)
    CALL DEQ2
200 CONTINUE

 PRINT RESULTS, TEST FOR END OF INTEGRATION

    WRITE (6,510) R,PHI(3), PHI(5)
510 FORMAT (10X4HR = E15.8,10X9HPHI(3) = E15.8,10X9HPHI(5) = E15.8)
    IF(R .GT. 10.0) STOP
    IF(PHI(1) .LE. 0.0) GO TO 10
    CALL DEQ1
    GO TO 200
 ERROR ON RETURN FROM DEQ

300 CONTINUE
    WRITE (6,511)
    GO TO 10
512 FORMAT (1H1 30X6HRHO = F10.4, 5X8HSIGMA = F10.4//)
511 FORMAT(//10X21HERROR RETURN FROM DEQ)
     END
```

STEADY STATE ϕ_1

$\mathrm{Re}\{\psi_m^{(3)}(r)\}$

$\mathrm{Im}\{\psi_m^{(3)}(r)\}$

7090 Routine No. 5: Calculation of the Coefficients of Table VIII.4

```
      DIMENSION PHI(10),PHIDOT(10)
      PHI(1)=1.0
      PHI(2)=0.0
      PHI(3)=0.0
      PHI(4)=1.0
      PHI(5)=0.0
      PHI(6)=1.0
      PHI(7)=1.0
      PHI(8)=0.0
      PHI(9)=1.0
      PHI(10)=0.0
      ALFA33=0.0
      ALFA35=0.0
      ALFA37=0.0
      ALFA39=0.0
      ALFA55=0.0
      ALFA57=0.0
      ALFA59=0.0
      ALFA77=0.0
      ALFA79=0.0
      ALFA99=0.0
      ETA33=0.0
      ETA55=0.0
      ETA77=0.0
      ETA99=0.0
      R=0.0
      DR=.05
      EPSLON=.0001
      CALL DEQ(K,10,R,PHI,PHIDOT,DR,EPSLON)
      GO TO (100,200,200,300),K
```

$$\alpha_{nm} = \int_0^h \frac{\psi_{n\frac{1}{2}} \psi_{m\frac{1}{2}} dr}{[1.2 - \phi_{(r)}/\phi_{max}]^2}$$

$$\eta_{nm} = \langle \psi^{+}_{n-\frac{1}{2}}, \mu \psi_{n-\frac{1}{2}} \rangle$$

```
 DERIVATIVE BOX - GENERATES FUNCTION

100 CONTINUE
      PHIDOT(1)=PHI(2)
      PHIDOT(2) = (-1.6+1.525*1.0256*PHI(1)/(.035+PHI(1)))*PHI(1)
      PHIDOT(3)=PHI(4)
      PHIDOT(4)=(.00018125*56.28-1.6+89.463*PHI(1)*(1.0+14.3*PHI(1)
     1+1.1905*56.28+0.69048*56.28**2)/((1.0+28.6*PHI(1))*
     2(1.0+28.6*PHI(1)+1.38095*56.28)*(1.0+56.28)))*PHI(3)
      PHIDOT(5)=PHI(6)
      PHIDOT(6)=(.00018125*.3306-1.6+89.463*PHI(1)*(1.0+14.3*PHI(1)
     1+1.1905*.3306+0.69048*.3306**2)/((1.0+28.6*PHI(1))*
     2(1.0+28.6*PHI(1)+1.38095*.3306)*(1.0+.3306)))*PHI(5)
      PHIDOT(7)=PHI(8)
      PHIDOT(8)=(.00018125*7.630-1.6+89.463*PHI(1)*(1.0+14.3*PHI(1)
     1+1.1905*7.630+0.69048*7.630**2)/((1.0+28.6*PHI(1))*
     2(1.0+28.6*PHI(1)+1.38095*7.630)*(1.0+7.630)))*PHI(7)
      PHIDOT(9)=PHI(10)
      PHIDOT(10)=(.00018125*2.00-1.6+89.463*PHI(1)*(1.0+14.3*PHI(1)
     1+1.1905*2.000+0.69048*2.000**2)/((1.0+28.6*PHI(1))*
     2(1.0+28.6*PHI(1)+1.38095*2.000)*(1.0+2.000)))*PHI(9)
      CALL DEQ2
200 CONTINUE
```

CALCULATES $\phi_{(r)}/\phi_{max}$

CALCULATES $\psi^{(3)}_{1}(r)$

$\psi^{(3)}_{2}(r)$

$\psi^{(3)}_{3}(r)$

$\psi^{(3)}_{4}(r)$

```
PRINT RESULTS, TEST FOR END OF INTEGRATION

    WRITE (6,510) R,PHI(3), PHI(5)
510 FORMAT (10X4HR = E15.8,10X9HPHI(3) = E15.8,10X9HPHI(5) = E15.8)
    SX=20.69*PHI(1)
    ALFA33=ALFA33+ DR*(PHI(3)*PHI(3)/(1.2-PHI(1)**2))
    ALFA35=ALFA35+ DR*(PHI(3)*PHI(5)/(1.2-PHI(1)**2))
    ALFA37=ALFA37+ DR*(PHI(3)*PHI(7)/(1.2-PHI(1)**2))
    ALFA39=ALFA39+ DR*(PHI(3)*PHI(9)/(1.2-PHI(1)**2))
    ALFA55=ALFA55+ DR*(PHI(5)*PHI(5)/(1.2-PHI(1)**2))
    ALFA57=ALFA57+ DR*(PHI(5)*PHI(7)/(1.2-PHI(1)**2))
    ALFA59=ALFA59+ DR*(PHI(5)*PHI(9)/(1.2-PHI(1)**2))
    ALFA77=ALFA77+ DR*(PHI(7)*PHI(7)/(1.2-PHI(1)**2))
    ALFA79=ALFA79+ DR*(PHI(7)*PHI(9)/(1.2-PHI(1)**2))
    ALFA99=ALFA99+ DR*(PHI(9)*PHI(9)/(1.2-PHI(1)**2))
    ETA33=ETA33+ DR*(.000116+SX/(SX+57.0)*((SX/(SX+.72414)-
   1.01746)/(SX+57.00)-.000305)*PHI(3)**2)
    ETA55=ETA55+ DR*(.000116+SX/(SX+1.055)*((SX/(SX+.72414)-
   1.75154)/(SX+1.055)-.387344)*PHI(5)**2)
    ETA77=ETA77+ DR*(.000116+SX/(SX+8.354)*((SX/(SX+.72414)-
   1.11587)/(SX+8.354)-.013425)*PHI(7)**2)
    ETA99=ETA99+ DR*(.000116+SX/(SX+2.724)*((SX/(SX+.72414)-
   1.333333)/(SX+2.724)-.111111)*PHI(9)**2)
    WRITE (6,513)ALFA33,ALFA35,ALFA37,ALFA39,ALFA55,ALFA57,ALFA59,
   1ALFA77,ALFA79,ALFA99,ETA33,ETA55,ETA77,ETA99
513 FORMAT (/2X9HALFA33 = E15.8,2X9HALFA35 = E15.8,2X9HALFA37 = E15.8,
   12X9HALFA39 = E15.8,2X9HALFA55 = E15.8/,2X9HALFA57 = E15.8,
   22X9HALFA59 = E15.8,2X9HALFA77 = E15.8,2X9HALFA79 = E15.8,
   32X9HALFA99 = E15.8/,2X8HETA33 = E15.8,2X8HETA55 = E15.8,
   42X8HETA77 = E15.8,2X8HETA99 = E15.8//)
    IF(R .GT. 10.0) STOP
    IF(PHI(1) .LE. 0.0)STOP
    CALL DEQ1
    GO TO 200
ERROR ON RETURN FROM DEQ

300 CONTINUE
    WRITE (6,511)
511 FORMAT(//10X21HERROR RETURN FROM DEQ)
    STOP
     END
```

INTEGRATIONS.

NOTE: DR > .05 - IF SUBROUTINE DEQ DIVIDES INTO SMALLER STEPS, THE ADDITION OF THESE STEPS MUST BE SUBTRACTED FROM THE FINAL RESULT.

7090 Routine No. 6: Feedback Calculation for Example E

```
      DIMENSION PHI(10),PHIDOT(10)
      PHI(1)  = 1868.5
      PHI(2)  = 244.7
      PHI(3)  = 727.6
      PHI(4)  = 426.7
      PHI(5)  = 1390.2
      PHI(6)  = 387.7
      PHI(7)  = 710.2
      PHI(8)  = 672.9
      PHI(9)  = 570.7
      PHI(10) = 737.3
      DR = .002
      EPSLON = .001
      R = 0.0
      A11 = .124
      A12 = -.0118
      A13 = -.132
      A14 =.0674
      A22 =.00115
      A23 =.0127
      A24 =-.00654
      A33 =.143
      A34 =-.0732
      A44 =.0376
      CALL DEQ(K,10,R,PHI,PHIDOT,DR,EPSLON)
      GO TO (100,200,200,300),K
C
C   DERIVATIVE BOX  - GENERATES FUNCTION
C
  100 CONTINUE
      PHIDOT(1) =680.0+112.56    *PHI(1)
     1-PHI(1)*(A11*PHI(1)+A12*PHI(2)+A13*PHI(3)+A14*PHI(4 ))
     2-PHI(2)*(A12*PHI(1)+A22*PHI(2)+A23*PHI(3)+A24*PHI(4 ))
     3-PHI(3)*(A13*PHI(1)+A23*PHI(2)+A33*PHI(3)+A34*PHI(4 ))
     4-PHI(4)*(A14*PHI(1)+A24*PHI(2)+A34*PHI(3)+A44*PHI(4 ))
      PHIDOT(2) =58.7    +PHI(2)*(56.28+.3306)
     1-PHI(1)*(A11*PHI(2)+A12*PHI(5)+A13*PHI(6)+A14*PHI(7 ))
     2-PHI(2)*(A12*PHI(2)+A22*PHI(5)+A23*PHI(6)+A24*PHI(7 ))
     3-PHI(3)*(A13*PHI(2)+A23*PHI(5)+A33*PHI(6)+A34*PHI(7 ))
     4-PHI(4)*(A14*PHI(2)+A24*PHI(5)+A34*PHI(6)+A44*PHI(7 ))
      PHIDOT(3) =-62.0   +PHI(3)    *(56.28+7.63)
     1-PHI(1)*(A11*PHI(3)+A12*PHI(6)+A13*PHI(8)+A14*PHI(9 ))
     2-PHI(2)*(A12*PHI(3)+A22*PHI(6)+A23*PHI(8)+A24*PHI(9 ))
     3-PHI(3)*(A13*PHI(3)+A23*PHI(6)+A33*PHI(8)+A34*PHI(9 ))
     4-PHI(4)*(A14*PHI(3)+A24*PHI(6)+A34*PHI(8)+A44*PHI(9 ))
      PHIDOT(4) =-25.0   +PHI(4)*    (56.28+2.0)
     1-PHI(1)*(A11*PHI(4)+A12*PHI(7)+A13*PHI(9)+A14*PHI(10))
     2-PHI(2)*(A12*PHI(4)+A22*PHI(7)+A23*PHI(9)+A24*PHI(10))
     3-PHI(3)*(A13*PHI(4)+A23*PHI(7)+A33*PHI(9)+A34*PHI(10))
     4-PHI(4)*(A14*PHI(4)+A24*PHI(7)+A34*PHI(9)+A44*PHI(10))
      PHIDOT(5) =5.10+.6612      *PHI(5)
     1-PHI(2)*(A11*PHI(2)+A12*PHI(5)+A13*PHI(6)+A14*PHI(7 ))
     2-PHI(5)*(A12*PHI(2)+A22*PHI(5)+A23*PHI(6)+A24*PHI(7 ))
     3-PHI(6)*(A13*PHI(2)+A23*PHI(5)+A33*PHI(6)+A34*PHI(7 ))
     4-PHI(7)*(A14*PHI(2)+A24*PHI(5)+A34*PHI(6)+A44*PHI(7 ))
```

STARTED AT $\vec{0}$ WITH MUCH LARGER DR = .02 TO SPEED UP CONVERGENCE - THIS IS 2ND OF TWO CALCULATIONS

IT IS NECESSARY TO WRITE OUT THE PHI(J) BECAUSE OF THE SUBROUTINE.

```
      PHIDOT(6) =-6.0    +PHI(6)    *(.3306+7.63)
     1-PHI(2)*(A11*PHI(3)+A12*PHI(6)+A13*PHI(8)+A14*PHI(9 ))
     2-PHI(5)*(A12*PHI(3)+A22*PHI(6)+A23*PHI(8)+A24*PHI(9 ))
     3-PHI(6)*(A13*PHI(3)+A23*PHI(6)+A33*PHI(8)+A34*PHI(9 ))
     4-PHI(7)*(A14*PHI(3)+A24*PHI(6)+A34*PHI(8)+A44*PHI(9 ))
      PHIDOT(7) =-2.51   +PHI(7)    *(.3306+2.0)
     1-PHI(2)*(A11*PHI(4)+A12*PHI(7)+A13*PHI(9)+A14*PHI(10))
     2-PHI(5)*(A12*PHI(4)+A22*PHI(7)+A23*PHI(9)+A24*PHI(10))
     3-PHI(6)*(A13*PHI(4)+A23*PHI(7)+A33*PHI(9)+A34*PHI(10))
     4-PHI(7)*(A14*PHI(4)+A24*PHI(7)+A34*PHI(9)+A44*PHI(10))
      PHIDOT(8) =128.0+15.26     *PHI(8)
     1-PHI(3)*(A11*PHI(3)+A12*PHI(6)+A13*PHI(8)+A14*PHI(9 ))
     2-PHI(6)*(A12*PHI(3)+A22*PHI(6)+A23*PHI(8)+A24*PHI(9 ))
     3-PHI(8)*(A13*PHI(3)+A23*PHI(6)+A33*PHI(8)+A34*PHI( 9))
     4-PHI(9)*(A14*PHI(3)+A24*PHI(6)+A34*PHI(8)+A44*PHI(9 ))
      PHIDOT(9) =66.2    +PHI(9)    *(7.63+2.0)
     1-PHI(3)*(A11*PHI(4)+A12*PHI(7)+A13*PHI(9)+A14*PHI(10))
     2-PHI(6)*(A12*PHI(4)+A22*PHI(7)+A23*PHI(9)+A24*PHI(10))
     3-PHI(8)*(A13*PHI(4)+A23*PHI(7)+A33*PHI(9)+A34*PHI(10))
     4-PHI(9)*(A14*PHI(4)+A24*PHI(7)+A34*PHI(9)+A44*PHI(10))
      PHIDOT(10)=34.3+4.00*PHI(10)
     1-PHI(4)*(A11*PHI(4)+A12*PHI(7)+A13*PHI(9)+A14*PHI(10))
     2-PHI(7)*(A12*PHI(4)+A22*PHI(7)+A23*PHI(9)+A24*PHI(10))
     3-PHI(9)*(A13*PHI(4)+A23*PHI(7)+A33*PHI(9)+A34*PHI(10))
     4-PHI(10)*(A14*PHI(4)+A24*PHI(7)+A34*PHI(9)+A44*PHI(10))
      CALL DEQ2
  200 CONTINUE

  PRINT RESULTS,TEST FOR END OF INTEGRATION

      WRITE (6,610) R,PHI(1),PHI(2),PHI(3),PHI(4),PHI(5),PHI(6),PHI(7)
     1 ,PHI(8),PHI(9),PHI(10)
  610 FORMAT (20X4HR = E15.8,/1X7HPHI(1)=E15.8,1X7HPHI(2)=E15.8,
     1 1X7HPHI(3)=E15.8,1X7HPHI(4)=E15.8,1X7HPHI(5)=E15.8,/
     2 1X7HPHI(6)=E15.8,1X7HPHI(7)=E15.8,1X7HPHI(8)=E15.8,
     3 1X7HPHI(9)=E15.8,1X8HPHI(10)=E15.8/)
      IF (R .GT. 1.0    ) GO TO 20
      CALL DEQ1
      GO TO 200

  ERROR ON RETURN FROM DEQ

  300 CONTINUE
      WRITE (6,511)
  511 FORMAT (//10X21HERROR RETURN FROM DEQ)
       STOP
       END
```

7090 Routine No. 7: Calculation of the Closed Loop Poles And Inversion of the Detection Matrix

```
      DIMENSION A(101,103),NC(100),RTR(100),B(50,50),C(50)
      READ (5,500)((A(I,J),J=1,4),I=1,4)

  500 FORMAT (4F8.2)
      CALL EIG1(A,4,4,RTR,RTI)
      READ (5,501)  ((B(I,J),J=1,9),I=1,9)

  501 FORMAT(9F8.2)
      CALL MATINV(B,9,C,0,DETERM)
      WRITE(6,650) DETERM
      WRITE (6,651)((B(I,J),J=1,9),I=1,9)

  650 FORMAT (1H110X14HDETERMINANT = E15.8//)
  651 FORMAT (9E14.5)
      STOP
       END
```

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