On the Development of Turbulence

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Abstract

The stability of two-dimensional parallel flows of an incompressible fluid is investigated, based upon a study of the equation of Orr and Sommerfeld along the lines initiated by Heisenberg. The theory of Heisenberg is carefully examined and further developed to obtain several general and specific results on hydrodynamic stability. Most of the disputes in the existing theories are clearly brought out and carefully settled. It is further shown that all symmetrical and all boundary-layer types of velocity distributions are unstable above a certain minimum critical Reynolds number, whose approximate value can be easily calculated from equations (12.24) and (12.25) respectively. General characteristics of the curve of neutral stability are obtained (Fig. 9). Complete numerical calculations of this curve have been carried through for the plane Poiseuille flow and the Blasius flow. In the first case, the minimum critical Reynolds number is found to be 16000, based upon the maximum velocity and the width of the channel. In the second case, the number is 400, based upon the free stream velocity and the displacement thickness of the boundary layer. Physical interpretations of the results obtained are given, based upon the conservation of vorticity in a perfect fluid and its diffusion by viscous forces. Indications are also given to connect the stability theory with Taylor's theory of transition to turbulence. It is hoped that this work may remove all the doubts of applying the theory of small oscillations to the treatment of hydrodynamic stability using Navier-Stokes equations for an incompressible fluid.

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PRELIMINARY CONSIDERATIONS

I. Introduction

The study of the stability of laminar motion and its transition to turbulence dates back to the time of Helmholtz and Reynolds,[38] and has already attracted great attention at the end of the last century.* From that time onwards, the subject has not only become a major problem for research workers in hydrodynamics, but has also attracted the attention of such great theoretical physicists as Lord Rayleigh,[35-37] Lord Kelvin,[17,18] Lorentz,[21] Sommerfeld[51] and Heisenberg,[12]—just to mention a few names,— whose chief interest is not essentially limited to the study of mechanics itself. Although numerous contributions have since been made, the subject has remained one of considerable dispute, as can be seen from the two general lectures given by Taylor[63] and by Synge[56] as late as 1938. Still more recently, there appeared the work of Görtler[6,7] and of Thomas[64].

All the research works on the stability of laminar motions have the following final aims.

1) The first aim is to determine whether a given flow (or a given class of flows) is ultimately unstable for sufficiently large Reynolds numbers. For this purpose, it is desirable to obtain some simple general criterion, which will give a rapid classification of velocity profiles regarding their stability.

2) The second purpose is to determine the minimum critical Reynolds number at which instability begins. It is often easier to

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* In 1888, the problem was proposed as the subject for the Adams Prize Essay by Rayleigh and Stokes.

find sufficient conditions for stability, than to find the condition for passage from stability to instability.

3) Finally, we want to understand the physical mechanism underlying the phenomena by giving theoretical interpretations and experimental confirmations of the results obtained from mathematical analysis.

Although numerous attempts have been made in these directions, especially for the apparently simplest cases of parallel flows in two dimensions, our knowledge is still very meagre. The classical case of plane Poiseuille motion has remained an unsettled problem,* and no satisfactory general results have been reached regarding the stability of a real fluid. The best-known general criterion is that of Rayleigh and Tollmien, classifying profiles according to the occurrence of a flex** with respect to the stability of a fluid at infinite Reynolds numbers. However, the significance of their results has been too much exaggerated and often misunderstood, and no physical interpretation has ever been given. The present work offers such an interpretation, but also shows that the results can give only little indication regarding the instability of a real (viscous) fluid. This will be discussed more in detail below.

The chief aim of the present work is to try to answer the three questions mentioned above for two-dimensional parallel flows. The following results have been obtained.

1) It is shown that all velocity distributions of the symmetrical type and of the boundary-layer type are unstable for sufficiently

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* cf. Synge's lecture

** Following Professors E.H. Moore and H. Bateman, we shall use the word "flex" for "point of inflection."

large (but finite) values of the Reynolds number. (The plane Poiseuille motion is included as a special case)

2) A very simple method is obtained by which one can calculate the minimum Reynolds number marking the beginning of instability with very little numerical labor.

3) The tendency toward instability of a profile with a point of inflection is interpreted by considering the distribution of vorticity. The effect of viscosity is considered as diffusing the disturbance from the "critical layer" inside the fluid and from the solid boundary. A very simple quantity is thereby derived which serves as a measure of the effect of viscosity. This can also be easily connected with the general mathematical theory.

As numerical examples, we have worked out the curve of neutral stability for the Poiseuille case and the Blasius case. Comparisons with existing results are discussed. The relation between instability and transition to turbulence is discussed in the last section of this paper.

Since the present results differ markedly from customary beliefs, it is necessary to trace the history of the existing lines of thought in order to give recognition to earlier ideas and results of which the present work makes use, and to analyse all the results in disagreement with the present conclusions. The review of literature is not intended to be exhaustive; we only try to cite all the necessary references. A more complete bibliography is given by Bateman[2].

Before coming to the complete survey, we shall make a few more remarks regarding the criterion of stability of Rayleigh[35] and Tollmien[67].

Remarks on the criterion of Rayleigh and Tollmien

The work of Rayleigh and of Tollmien tends to give the impression that the occurrence of a flex in the profile is the decisive factor in the determination of instability not only in the case of an inviscid fluid* but also in the case of a viscous fluid**. However, the present investigation shows that this is by no means the case. When instability first occurs, as one increases the Reynolds number, viscous forces still play a dominant role, and the main characteristics of the behavior of the fluid with respect to a disturbance do not depend upon the occurrence of a flex in the velocity curve. Indeed, it is physically improbable that a slight change of pressure gradient in the case of a boundary layer,—which may cause a passage from a velocity curve without a flex to one with a flex—should cause a radical change in the essential characteristics of stability. As we shall see later, the instability of a boundary layer depends more on the outside free stream than on the occurrence of a point of inflection. It might be argued that the free stream is analogous to a point of inflection in that a vanishing curvature is involved; but even if this is admitted, we must still note that the essential features in this case are not obtained from an analysis neglecting the effect of viscosity. Indeed, from inviscid analysis, it is concluded that a boundary layer with zero or favorable pressure gradient is stable, except for the very trivial type of

* By this we mean the limiting case of infinitely large Reynolds numbers.
** See Taylor's discussion on p. 308 of reference (63).

disturbance with infinite wave-length and zero phase velocity. The present investigation shows that <u>all</u> boundary-layer profiles can be unstable, and exhibits results in agreement with the physical suggestion just discussed.

The importance of a flex is further belittled by considering an example where its occurrence does not imply the possibility of a neutral disturbance in an inviscid fluid, even of the trivial type (section 7). It is also to be noted that Tollmien's proof of the instability of a profile with a flex has been carried through only for the symmetrical and the boundary-layer types. These profiles are shown by the present work to be unstable under the influence of viscosity, whether a flex occurs or not.

In spite of all these points against the decisive nature of a flex, it must be admitted that its occurrence certainly makes the motion comparatively unstable. This can be expected from Rayleigh's original results, and can be seen more clearly from the interpretation of the mechanism of "inertial instability" given in sections 9, 10. However, these results must not be taken to indicate any <u>decisive</u> physical significance of a flex. The essential features of instability can only be obtained through considerations of the effect of viscosity.

There is another point which is usually misunderstood in connection with the stability of an inviscid fluid. It is often concluded from Rayleigh's analysis that there is no possible disturbance of any kind at infinite Reynolds numbers if the profile has no flex.*

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* Tollmien (67), p. 88

This has aroused suspicion regarding the validity of Rayleigh's analysis, because it does not appear reasonable physically.* However, the conclusion is purely a mathematical misunderstanding. Damped disturbances are not excluded by Rayleigh's analysis.

This point is very closely related to the long-disputed question of the crossing substitution.** In the present investigation, it is shown that such a process (in its proper sense) is <u>not</u> necessary in obtaining all the stability characteristics. It is used only when we want to calculate the <u>form</u> of a <u>damped</u> disturbance, —a rather insignificant thing to do. A damped disturbance is shown in the present investigation to have <u>two</u> inner friction layers. Only the form of the damped disturbance in between these layers is to be calculated by using a "crossing substitution." In the usual theories, the existence of <u>two</u> inner friction layers is not even recognized.

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* Frederichs, (101), p. 209

** The explanation and discussion of this question will be given in section 5.

II. Historical Survey of Existing Theories

There seem to be two schools of thought in regard to the cause of transition from steady to turbulent conditions. One school has thought that transition is due to a definite instability of the flow, i.e., to a condition in which infinitesimal disturbances will grow exponentially. Another regards the motion in most cases as definitely stable for infinitesimal disturbances but liable to be made turbulent by suitable disturbances of finite magnitude or by a large enough pressure gradient. Both schools however, agree that the fluid can be considered as incompressible and that its motion is governed by the Navier-Stokes equations of motion. Since the agreement between theory and experiment has not been very satisfactory, it has also been proposed that the cause of transition must be traced back to the effect of compressibility or to the possible failure of the Navier-Stokes equations. The present work tends to confirm the simplest point of view that the motion in most cases is definitely unstable for infinitesimal disturbances governed by the Navier-Stokes equations for an incompressible fluid.

The theory of finite disturbances dates back to Reynolds[38] and Kelvin[19]. It was developed by Schiller, Taylor and others.* Mathematical investigations of such finite disturbances are mainly based on considerations of energy or of the square of vorticity of the disturbance, because the solution of the non-linear equations satisfied by the disturbance is extremely difficult. We shall dis-

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* See Taylor's lecture (63) for references to the works on finite disturbances.

cuss this line of thought briefly at the end of this paper together with our own results. For more details, the reader is referred to the lecture of Taylor(63), and the papers of Synge(55), and Thomas(64).

For small disturbances, the use of positive definite integrals of energy and vorticity of the disturbance has also been extensively used. These considerations have been discussed by Orr(29), Lorentz(21), von Kármán(15), Synge(56,57) and others. For excellent accounts of this phase of the theory, the reader is referred to the works of Noether(27), von Kármán(15), Prandtl(34), and Synge(57). Some more references are cited at the end of this paper. As is now well-known, this method can only give sufficient conditions for stability. Also, since all disturbances are usually allowed, including those which do not satisfy the hydrodynamic equations of motion, a larger viscous decay is required to insure stability than when these disturbances are excluded. Consequently, the limit of stability is always found to be much lower than the experimental values. However, from these considerations Synge(57) has arrived at a very convenient form for a sufficient condition for stability of two-dimensional parallel flow with respect to two-dimensional disturbances. This will be found to be very useful for the discussions in section 12.

To get more concrete results, we have to solve the linearized equations satisfied by the disturbance. The most successful case appeared to be Taylor's treatment of Couette flow(60) between concentric cylinders. His work was verified by the experiments carried out by himself(60,62) and by others(20). A rigorous mathematical investigation in this connection was made by Faxen(4).

In fact, it is now known that his analysis is a typical case of the stability of a fluid motion where the centrifugal force plays a dominant part. Such cases were first considered by Lord Rayleigh[37], who gave a condition for the stability of an inviscid fluid. Mathematical proof of a sufficient condition of stability of Couette flow was recently given by Synge[58]. Extension of Taylor's work to the boundary layer over a curved wall was carried out by Görtler[6,7], who used numerical methods with success.

While the investigation of curved flows was uneventful, the investigation of axially symmetrical flows was not extensive. The Poiseuille flow in a circular pipe was studied by Sexl[47,48] with a conclusion of stability. Prandtl gave some discussions of the possible cause of instability in his article in Durand's Aerodynamic Theory.

The most extensive and most eventful discussion of hydrodynamic stability seems to be the treatment of parallel flows by attempting to solve the eigen-value problem associated with the linearized equations governing the disturbance.

This line of development can be easily traced along the work of Helmholtz, Lord Rayleigh[35,36], Orr[29], Sommerfeld[51], von Mises[23][24], Hopf[14], Prandtl[33], Tietjens[65], Heisenberg[12], Tollmien[66,68], and Schlichting[44,46]. Other contributions are those of Noether[28], Solberg[50], Southwell[52], Squire[53], Goldstein[5], and Pekeris[31,32].

For convenience, the theory considers two-dimensional wavy disturbances propagating along the direction of the main flow. Squire has shown that three-dimensional wavy disturbances are more stable than two-dimensional ones. However, Prandtl still mentions the possibility of greater instability of three-dimensional disturbances in his article appearing after Squire's paper.

The first study of two-dimensional hydrodynamic stability was made by Helmholtz. He proved the instability of wavy disturbances over the surface of discontinuity of two parallel streams of different velocities. Later, Rayleigh[35] realized that Helmholtz's approximation was not good enough for bringing out the main features of a flow with continuous velocity distributions. He therefore made an improved approximation consisting of several linear profiles joined up continuously. The vorticity distribution then has constant values in several layers, but has a discontinuity in passing from one layer to another. Investigations with continuous vorticity distributions were also made. Rayleigh's work was mainly concerned with an inviscid fluid, but he also realized the importance of the inner friction layer[35]. Two main results were obtained. The first is that instability (in an inviscid fluid) can only occur with velocity distributions having a point of inflection, and that in the case of neutrally stable disturbance, the inner friction layer is unavoidable. The second is obtained from the analysis of broken linear profiles; it substantiates the first result by demonstrating definite instability of broken linear velocity distributions of the type shown in Fig. 1,(a), and only stability in the other cases. Rayleigh[35] supported his result by obtaining the condition determining stability in the approximate form

$$\int_{y_1}^{y_2} dy\,(w-c)^{-2} = 0\,, \tag{2.1}$$

where $w(y)$ is the velocity distribution, y_1, y_2 are the coordinates of the solid boundaries, and c is a constant, whose real part represents the wave velocity and whose imaginary part gives damping or amplification.

Meanwhile, the exact analysis of linear velocity distributions including the effect of viscosity was given by von Mises[23,24], and Hopf[14] and was also studied by Rayleigh[36]. The results indicate only stability. Prandtl and Tietjens[65] applied Rayleigh's method of approximation to the stability of the boundary layer, taking account of the effect of viscosity. In such an approximation, the inner friction layer mentioned above for continuous vorticity distributions was left out. The result of Tietjens did not give a minimum critical Reynolds number.

It was Heisenberg[12] who first studied the stability of a variable continuous vorticity distribution with success. As a particular example, he demonstrated that the plane Poiseuille flow was unstable for sufficiently large Reynolds numbers. Also, using the same equation (2.1) with which Rayleigh supported his approximation with linear profiles, Heisenberg pointed out the fallacy in Rayleigh's method. The essential point is that the corners in the velocity profile introduce extraneous roots for the above equation for c. Consequently, the results of this type of analysis depend upon the manner in which the velocity distribution is approximated.

Heisenberg's numerical computation was, however, incomplete and very rough, and his theory was not generally accepted. Better known are the results of Tollmien and Schlichting. They studied the cases of Blasius(66) and plane Couette flow(41), using essentially Heisenberg's theory. The former case was pursued very much in detail. For the latter case, Schlichting followed the idea of Prandtl, asserting that the instability may be attributed to the initial unsteady distribution prior to the formation of the linear profile. Indeed, the same kind of idea was also suggested by Prandtl to account for the instability of Poiseuille flow by ascribing it to the entrance section where the profile is not yet parabolic(33). This problem will be discussed in detail later, (section 14.)

For an inviscid fluid, Tollmien has also proved the instability of boundary-layer and symmetrical profiles with a point of inflection(67). For a viscous fluid, the present investigation shows that instability depends upon the general type of these profiles rather than on the appearance of the point of inflection. The inner friction layer plays a dominant role in determining the instability. Attempts to interpret this point physically are given by Prandtl(34) and in the present paper.

PART I GENERAL THEORY

3. General formulation of the problem

We shall now formulate the problem of the stability of two-dimensional parallel flows mathematically. In the first place, we note that if the steady motion is strictly two-dimensional and parallel, the velocity distribution must be either linear or parabolic (if body forces are absent). We then have

1) the plane Couette flow, or

2) the plane Poiseuille flow, or

3) a combination of these two flows.

The problem would then be very restricted.

However, there is a large number of cases where the flow is essentially parallel to one direction. These are the cases where the boundary-layer consideration is permissible. The following are important special cases belonging to this class:

4) inlet flow between parallel walls, flow in a slightly convergent or divergent channel,

5) flow along a flat plate,

6) wake behind a cylindrical body, jet from a narrow slit.

Whether these flows can be properly considered as belonging to the same class as the above three is a question of some controversy. Taylor has criticized Tollmien's work with the boundary layer on this ground[63]. In the Appendix of this paper, we shall try to demonstrate that this treatment is generally permissible, but that the interpretation of the results must be taken up with care. A full discussion of Tollmien's work will also be found there.

In considering the stability of the main flow, we superpose upon it a hydrodynamically possible small disturbance, and consider its behavior. The disturbance is small in the sense that the inertial forces corresponding to the disturbance alone are negligible and that its behavior is unaltered when its amplitude is (say) doubled or halved. It is then simplest to consider separate harmonic components with respect to time, which may be damped, neutral, or self-excited. By considering disturbances which are also spacially periodic both in the direction of flow and in the direction perpendicular to the plane of symmetry of the main motion, Squire(53) was able to show that two-dimensional disturbances are less stable than three-dimensional disturbances. Hence, important features of the stability problem can be obtained by considering two-dimensional disturbances alone. This is an essential difference between the stability of a parallel flow and of a curved flow. In the latter case, three-dimensional disturbances are of utmost importance.

The consideration of periodic disturbances alone is again a question of some controversy. Justification has been attempted and objection has been raised. We shall see later that at least the difficulties raised are chiefly due to a misinterpretation of the mathematical results.

Admitting that we can consider two-dimensional disturbances alone, we have a much simplified physical picture at hand. If the effect of viscosity is negligible, we have the well-known fact of conservation of vorticity for two-dimensional motions. Actually, the stability problem is found to depend both on the inertial forces and on the viscous forces. However, the effect of viscosity is also well-known to be one of diffusion of vorticity. Thus, important results can be expected from considerations of vorticity transfer.

Since the question of stability occurs only at very large Reynolds numbers, it seems reasonable to neglect the effect of viscosity completely at first and then consider its effect later. The first step of the investigation may be termed "inertial stability"; the second step may be termed "viscous stability", particularly when viscosity plays a dominant role. These concepts will prove useful later.

Let us now proceed to the mathematical formulation of the problem. We shall give a complete derivation of the stability equations so that we can see how to settle the disputes about the approximations in considering velocity distributions of the boundary-layer type.

Admitting Squire's work as a proper indication that only two-dimensional disturbances need be considered, we may conveniently consider the equation of vorticity

(3.1) $$\Delta\psi_t + \psi_y \Delta\psi_x - \psi_x \Delta\psi_y = \nu\Delta\Delta\psi,$$

with the velocity components

(3.2) $$u = \psi_y = \frac{\partial\psi}{\partial y}, \quad v = -\psi_x = -\frac{\partial\psi}{\partial x},$$

and the vorticity

(3.3) $$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -(\psi_{xx} + \psi_{yy}) = -\Delta\psi.$$

As usual, ν is the kinematical viscosity in (3.1). We may add that Squire's original proof was intended for flow bounded between two parallel walls. There is no difficulty in seeing that the proof holds also for a fluid extending to infinity, because the essential boundary conditions for the disturbances are the same.

Let us put

$$\psi = \Psi(x,y) + \psi'(x,y,t), \tag{3.4}$$

where $\Psi(x,y)$ represents the steady main flow and $\psi'(x,y,t)$ represents the disturbance. Main flows which vary but slowly with time can also be treated this way, but we shall restrict ourselves to steady flows in order to fix our ideas.

If we substitute (3.4) into (3.1), the terms corresponding to the main flow will cancel each other out. If we now drop the terms quadratic in $\psi'(x,y,t)$ and its derivatives, we have the equation

$$\Delta\psi_t' + \Psi_y\Delta\psi_x' - \Psi_x\Delta\psi_y' + \psi_y'\Delta\Psi_x - \psi_x'\Delta\Psi_y = \nu\Delta\Delta\psi'. \tag{3.5}$$

We shall now assume the flow to be mainly parallel to the x-axis. Using the boundary-layer approximation, we should drop the x-derivative of any quantity connected with the mean flow compared with its y-derivative. But for the disturbance we would expect ψ_x' and ψ_y' to be of the same order of magnitude. This will be verified a posteriori in the specific examples. Further discussions will be found in the appendix. With these considerations, (3.5) reduces to

$$\Delta\psi_t' + \Psi_y\Delta\psi_x' - \psi_x'\frac{\partial^3\Psi_y}{\partial y^3} = \nu\Delta\Delta\psi'. \tag{3.6}$$

Now we shall make an approximation of the same order by taking for $w = \Psi_y$ and $\frac{\partial^2 w}{\partial y^2} = \frac{\partial^3\Psi}{\partial y^3}$ their local values at a given value x_0 of x. Then we may write

$$\Delta \psi'_t + w(y)\, \Delta \psi'_x - w''(y)\, \psi'_x = \nu \Delta \Delta \psi'. \tag{3.7}$$

For the boundary conditions, we shall also consider the local boundaries. The problem is then essentially simplified, and can be treated similarly to plane Couette and Poiseuille flows. We consider a main flow between two parallel planes $y = y_1$ and $y = y_2$ with a more or less arbitrary distribution of velocity w(y). Then the disturbance $\psi'(x,y,t)$ must be found as a solution of (3.7) satisfying the conditions $u' = v' = 0$ over the boundaries.

The usual way of dealing with the solution of (3.7) subject to given boundary conditions is to consider periodic disturbances. We shall refer all velocities to a characteristic velocity U and all lengths to a characteristic length ℓ, and define the Reynolds number $R = U\ell/\nu$. The two-dimensional periodic disturbance of a field of flow in which the main flow is w(y) may be represented by the stream function $\psi' = \varphi(y)\, e^{i\alpha(x-ct)}$, and the linearized differential equation for $\varphi(y)$ is

$$(w-c)(\varphi'' - \alpha^2 \varphi) - w''\varphi = -\frac{i}{\alpha R}\left(\varphi^{iv} - 2\alpha^2\varphi'' + \alpha^4\varphi\right), \tag{3.8}$$

as can be easily obtained from (3.7). We shall take α to be always real and positive, while c may be complex:

$$c = c_r + i c_i . \tag{3.9}$$

If we consider a flow limited between the planes $y = y_1$ and $y = y_2$, the equation (3.8) is to be solved under the boundary conditions

$$\varphi(y_1) = 0, \quad \varphi(y_2) = 0, \quad \varphi'(y_1) = 0, \quad \varphi'(y_2) = 0. \tag{3.10}$$

Let us now forget about the physical problem and consider the differential equation (3.8) as a linear differential equation of the fourth order in the complex y-plane. To be sure, the function $w(y)$ is defined only for real values of y between y_1 and y_2. We can of course, consider it as defined for other values of y by analytical continuation. We shall assume that the function thus defined is holomorphic in every finite region with which we shall be concerned.

The equation (3.8) then has every point in the region under consideration as a regular point, and its coefficients are also entire functions of the parameters c, α, and αR (regarded as complex variables). By a well-known theorem in the theory of differential equations, the solutions of (3.8) are analytic functions of the variable y and of the parameters c, α, and αR, being in fact entire functions of the parameters.

These simple general analytical considerations appear to have escaped notice from earlier investigators. In sections 4, 5 of this paper, we shall find this type of consideration very important in settling the controversies about the question of convergence of the series used in the actual solution of the equations (3.8) and (3.14).

Let us denote a fundamental system of solutions of (3.8) by $\varphi_1(y)$, $\varphi_2(y)$, $\varphi_3(y)$, $\varphi_4(y)$, the dependence upon the parameters c, α, αR being understood. The condition (3.10) will give rise to the secular equation

$$(3.11)\quad F(c,\alpha,\alpha R) = \begin{vmatrix} \varphi_1(y_1) & \varphi_2(y_1) & \varphi_3(y_1) & \varphi_4(y_1) \\ \varphi_1(y_2) & \varphi_2(y_2) & \varphi_3(y_2) & \varphi_4(y_2) \\ \varphi_1'(y_1) & \varphi_2'(y_1) & \varphi_3'(y_1) & \varphi_4'(y_1) \\ \varphi_1'(y_2) & \varphi_2'(y_2) & \varphi_3'(y_2) & \varphi_4'(y_2) \end{vmatrix} = 0 .$$

Since the function $F(c,\alpha,\alpha R)$ is an entire function of the variables c, α, αR, we can solve for c, obtaining

$$c = c\ (\alpha, R). \tag{3.12}$$

There may be several branches of the solution, or there may be none as in the case where $F(c,\alpha,\alpha R)$ is (say) $\exp(\alpha Rc)$. In general, we would expect the solution to be unique, or we may consider only one branch of the solution.

When α and R are later taken to be real and positive, it is convenient to separate (3.12) into its real and imaginary parts. Thus,

$$\begin{cases} c_r = c_r(\alpha, R), \\ c_i = c_i(\alpha, R). \end{cases} \tag{3.13}$$

It is customary to plot curves of constant c_i or αc_i in the α-R plane. The curve $c_i = 0$ gives the limit of stability.

We are particularly interested in the case where the Reynolds number is very large. The study of this case is complicated by the fact that the functions $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ involved have essential singularities at the infinite point of the αR plane. From the differential equation (3.8) itself, we see that when $\alpha R \to \infty$, we have the equation

$$(w - c)(\varphi'' - \alpha^2\varphi) - w''\varphi = 0, \tag{3.14}$$

which is only of the second order. Thus, two solutions of (3.8) are lost. From detailed mathematical investigations, we shall find later that two linear independent solutions of (3.8), say φ_1 and φ_2, will satisfy (3.14) in the limit of infinite αR, except in the neighborhood of the point where $w = c$. The other two linearly independent solutions φ_3 and φ_4 are highly oscillating for large αR and would therefore disappear in the limit of infinite αR. Furthermore, we shall see that φ_3 and φ_4 can be so chosen that if $\varphi_3(y_1) \gg \varphi_4(y_1)$, then $\varphi_3(y_2) \ll \varphi_4(y_2)$, with corresponding relations for their derivatives. It then appears plausible that the limiting form of (3.11) for infinite αR is

$$\begin{vmatrix} \varphi_1(y_1) & \varphi_2(y_1) \\ \varphi_1(y_2) & \varphi_2(y_2) \end{vmatrix} = 0, \tag{3.15}$$

with $\varphi_1(y)$, $\varphi_2(y)$ satisfying (3.14). This point will be fully discussed in section 6.

The condition (3.15) states that we are looking for a solution $\varphi(y)$ of (3.14) satisfying the boundary conditions

$$\varphi(y_1) = 0, \qquad \varphi(y_2) = 0, \tag{3.16}$$

with the other two conditions of (3.10) relaxed. Physically, this means that we allow a slipping along the walls $y = y_1$ and $y = y_2$. For very large Reynolds numbers, only a very thin layer of fluid will stick to the solid, and we have naturally an apparent slipping. These points will be taken up again more carefully after a thorough mathematical investigation of the solutions.

4. Solution of Orr-Sommerfled's equation by methods of successive approximations

The stability equation of Orr and Sommerfeld

$$(w-c)(\varphi''-\alpha^2\varphi)-w''\varphi=-\frac{i}{\alpha R}\left(\varphi^{iv}-2\alpha^2\varphi''+\alpha^4\varphi\right) \tag{4.1}$$

has a fundamental system of four solutions, which are analytic functions of y (wherever w(y) is analytic) and which are entire functions of α , c, and αR. In order to obtain useful solutions, it is usual to expand the solutions as power series of a suitable small parameter, say, $1/\alpha R$. However, since $1/\alpha R$ occurs with the highest derivative in (4.1), the study of such an expansion becomes very complicated. It will be done later.

a) Solution by convergent series. An alternative method* is to choose a small parameter ε related to $1/\alpha R$ and first make a change of variable (y_0 being so far an arbitrary point)

$$y-y_0=\varepsilon\eta, \quad \varphi(y)=\chi(\eta), \tag{4.2}$$

so that (4.1) becomes

$$(w-c)(\chi''-\alpha^2\varepsilon^2\chi)-\varepsilon^2w''\chi=-\frac{i}{\alpha R\varepsilon^2}\left(\chi^{iv}-2\alpha^2\varepsilon^2\chi''+\alpha^4\varepsilon^4\chi\right), \tag{4.3}$$

where

$$\begin{cases} w-c=(w_0-c)+w_0'\cdot(\varepsilon\eta)+\frac{w_0''}{2}\cdot(\varepsilon\eta)^2+\cdots, \\ w''=w_0''+w_0'''\cdot(\varepsilon\eta)+\frac{w_0^{iv}}{2}\cdot(\varepsilon\eta)^2+\cdots. \end{cases} \tag{4.4}$$

The solution is then obtained in the form

$$\varphi(y)=\chi(\eta)=\chi^{(0)}(\eta)+\varepsilon\chi^{(1)}(\eta)+\varepsilon^2\chi^{(2)}(\eta)+\cdots, \tag{4.5}$$

and the differential equations for the approximations of successive

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* This method was first used by Heisenberg, loc. cit.(12), p. 588.

orders can be obtained by substituting (4.4) and (4.5) into (4.3) and equating all the coefficients of the various powers of ε to zero.

If we take y_0 to be the point where $w = c$, the proper parameter to be chosen is

$$\varepsilon = (\alpha R)^{-\frac{1}{3}}. \tag{4.6}$$

The differential equations for the functions $\chi^{(0)}(\eta), \chi^{(1)}(\eta), \chi^{(2)}(\eta), \cdots$ are as follows:

$$\begin{cases} \varepsilon^0 & w_0'\eta\chi^{(0)''} + i\chi^{(0)iv} = 0, \\ \varepsilon^n & w_0'\eta\chi^{(n)''} + i\chi^{(n)iv} = L_{n-1}(\chi), \qquad n \geq 1, \end{cases} \tag{4.7}$$

where $L_{n-1}(\chi)$ is a linear combination of $\chi^{(0)}, \chi^{(1)}, \cdots, \chi^{(n-1)}$ and their derivatives. In particular,

$$L_0(\chi) = w_0''\left(\chi^{(0)} - \frac{\eta^2}{2}\chi^{(0)''}\right). \tag{4.8}$$

We note that the homogeneous part is the same for all the differential equations of (4.7). Hence, if we can solve the initial approximation, the rest can all be obtained by quadratures. Indeed, the first equation of (4.7) is Stokes' equation* for $\chi^{(0)''}$ whose solution can be readily expressed in terms of Bessel's functions of the order 1/3. Thus, we have the four particular integrals**

---

* cf. The exact treatment of (4.1) by Hopf[(14)] and Rayleigh[(36)] for the case $w'' = 0$.

** Note that $\chi_3^{(0)}$ and $\chi_4^{(0)}$ and also $\chi_i^{(n)}$ have no branch point at $\eta = 0$. The order of the solutions $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ agrees with Tollmien's notation. They are $\{\varphi_3, \varphi_4, \varphi_1, \varphi_2\}$ in Heisenberg's notation. Heisenberg gave the solutions φ_3 and φ_4 in the form $\varphi_{1,2} = (w-c)\int \frac{1}{\eta} H_{\frac{2}{3}}^{(1),(2)}\left[\frac{2}{3}(i\alpha_0\eta)^{\frac{3}{2}}\right] d\eta$ (p. 589 and Eq. (19a), p. 591). It can be easily verified that these are the same as $\chi_{3,4}^{(0)}$ up to a constant factor $w_0'\varepsilon$ and to the proper order of approximation. Note that throughout Heisenberg's paper, i is to be replaced by -i in order to conform to our notation. This can be seen from a comparison of our equation (4.1) with his equation (7a). The difference arises from a difference of notation in the stream function $\psi'(x,y,t)$.

$$
(4.9) \qquad \begin{cases} \chi_1^{(0)} = \eta , \\ \chi_2^{(0)} = 1 , \\ \chi_3^{(0)} = \int_{+\infty}^{\eta} d\eta \int_{+\infty}^{\eta} d\eta \, \eta^{\frac{1}{2}} H_{\frac{1}{3}}^{(1)} \left[\frac{2}{3} (i\alpha_0 \eta)^{\frac{3}{2}} \right] , \\ \chi_4^{(0)} = \int_{-\infty}^{\eta} d\eta \int_{-\infty}^{\eta} d\eta \, \eta^{\frac{1}{2}} H_{\frac{1}{3}}^{(2)} \left[\frac{2}{3} (i\alpha_0 \eta)^{\frac{3}{2}} \right] , \end{cases}
$$

for the first equation of (4.7), where

$$
(4.10) \qquad \alpha_0 = (w_0')^{\frac{1}{3}} .
$$

The higher approximations are given by

$$
(4.11) \qquad \chi_i^{(n)} = \frac{\pi}{6} \int^{\eta} d\eta \int^{\eta} d\eta \left\{ \chi_4^{(0)''} \int^{\eta} d\eta \, \chi_3^{(0)''} L_{n-1}(\chi) - \chi_3^{(0)''} \int^{\eta} d\eta \, \chi_4^{(0)''} L_{n-1}(\chi) \right\} , \quad i = 1, 2, 3, 4 .
$$

These are the explicit formulae for finding the approximations of various orders. In actual calculations, only the initial approximation (4.9) is required.

Furthermore, the series (4.5) is convergent provided ε is restricted so that the series (4.4) are convergent. For then the differential equation (4.3) for $\chi(\eta)$, when normalized, has analytic functions of the parameter ε as its coefficients. Hence, a fundamental system of its solutions consists of four analytic functions of ε.

It should be mentioned that if y_0 is not taken at the particular point for which $w = c$, the proper parameter to be taken is $(\alpha R)^{-\frac{1}{2}}$ instead of $(\alpha R)^{-\frac{1}{3}}$. In this case, all the approximations can be expressed in terms of elementary transcendental functions. However, it is not found particularly advantageous to do so, because the study of crossing substitution would not be easy. Also, the method is then too much different from those used by earlier investigators to allow an easy comparison of the results.

b) Solution by asymptotic series. Although the previous method is theoretically complete, it is usually more convenient to use asymptotic series for numerical purposes. Heisenberg has given two asymptotic methods, each of which gives only two particular solutions of (4.1). These methods will now be described and investigated mathematically more in detail, because Heisenberg's work has received some criticism in this connection.*

The first of these methods is to develop $\varphi(y)$ in powers of $(\alpha R)^{-1}$. We put

$$\varphi(y) = \varphi^{(0)}(y) + \frac{1}{\alpha R}\varphi^{(1)}(y) + \frac{1}{(\alpha R)^2}\varphi^{(2)}(y) + \cdots \tag{4.12}$$

and substitute into (4.1). Comparing corresponding powers of $(\alpha R)^{-1}$, we have the following differential equations

$$\left\{\begin{array}{l} (w-c)(\varphi^{(0)''} - \alpha^2\varphi^{(0)}) - w''\varphi^{(0)} = 0, \\ (w-c)(\varphi^{(k)''} - \alpha^2\varphi^{(k)}) - w''\varphi^{(k)} = -i\left\{\varphi^{(k-1)iv} - 2\alpha^2\varphi^{(k-1)''} + \alpha^4\varphi^{(k-1)}\right\}, \\ \qquad\qquad (k \geq 1). \end{array}\right. \tag{4.13}$$

The initial approximation satisfies the inviscid equation and can be solved by developing $\varphi^{(0)}$ in powers of α^2. Indeed, two particular integrals of (4.13) are

$$\left\{\begin{array}{l} \varphi_1^{(0)} = (w-c)\left\{h_0(y) + \alpha^2 h_2(y) + \alpha^4 h_4(y) + \cdots\right\}, \\ \varphi_2^{(0)} = (w-c)\left\{h_1(y) + \alpha^2 h_3(y) + \alpha^4 h_5(y) + \cdots\right\}, \end{array}\right. \tag{4.14}$$

---

* Tollmien, loc. cit., 1929, p. 43

where

$$
h_0(y) = 1, \qquad h_{2n+2}(y) = \int_{y_1}^{y} dy\,(w-c)^{-2} \int_{y_1}^{y} dy\,(w-c)^{2}\, h_{2n}(y), \quad n \geq 0,
$$

$$
k_1(y) = \int_{y_1}^{y} dy\,(w-c)^{-2}, \qquad k_{2n+3}(y) = \int_{y_1}^{y} dy\,(w-c)^{-2} \int_{y_1}^{y} dy\,(w-c)^{2}\, k_{2n+1}(y), \quad n \geq 0. \tag{4.15}
$$

The point y_1 might have been any fixed point instead of one of the end points; but it is found convenient to take it this way.

Having found two particular integrals for $\varphi^{(0)}$, we can obtain the higher approximations by quadratures. In actual calculations, this is not necessary.

From the general nature of the equation (4.1), $\varphi(y)$ is an entire function of αR. Hence, the infinite point of the αR-plane is a singular point, unless $\varphi(y)$ is independent of αR. Consequently, the series (4.11) is asymptotic, unless $\varphi(y)$ is a polynomial of $(\alpha R)^{-1}$. We note also that (4.13) is of the second order, so that only two solutions are obtained by this method. The solutions of (4.13) are entire functions of α^2 and hence the series (4.14) are uniformly convergent for any finite region of the complex α^2-plane, for a fixed value of y, except when y is the singular point y_0 of the differential equation (4.13).*

In fact, the differential equation (4.13) has a logarithmic singularity at the point y_0. This point is however an ordinary point in the exact equation (4.1), and the singularity is introduced purely by the method of asymptotic integration. However, the

---

* This can also be seen from the series itself. So long as it is possible to lead a path of finite length from y_1 to y on which $w - c \neq 0$, the general terms $\alpha^{2n} h_{2n}$ and $\alpha^{2n+1} k_{2n+1}$ of the two series are bounded by $A(\alpha M)^{2n}/(2n)!$ and $B(\alpha M)^{2n+1}/(2n+1)!$ respectively (A, B, M being (suitably) fixed constants), and hence the series converge like the cosine and the sine series respectively. Heisenberg did not prove the convergence of these series, but stated that their convergence can be hoped to be sufficiently rapid for α^2 of the order of unity (loc. cit., 1924, pp. 584, 587). This was made a point of criticism by Tollmien (loc. cit., 1929, p. 43).

appearance of this singularity gives a serious ambiguity in the determination of the correct path leading from y_1 to y in order that (4.14) may give valid approximations to integrals of (4.1) <u>all along the path.</u>* The proper way to settle this question is to compare the solutions (4.14) with the <u>asymptotic expansions of the regular solutions</u> obtained by the previous method. This will be done later after we have described the second asymptotic method of Heisenberg for the other <u>two</u> particular integrals; for the same kind of problem also arises there.

To obtain <u>two</u> other integrals of (4.1) in asymptotic forms, let us make the transformation

$$\varphi(y) = \exp\left\{\int g(y)\,dy\right\}. \tag{4.16}$$

Then, we obtain the non-linear differential equation

$$(w-c)\left\{g^2+g'-\alpha^2\right\} - w'' = -\frac{i}{\alpha R}\left\{g^4+6g^2g'+3g'^2+4gg''+g'''-2\alpha^2(g^2+g')+\alpha^4\right\} \tag{4.17}$$

for the function g(y). We try to solve this by putting

$$g(y) = \sqrt{\alpha R}\, g_0(y) + g_1(y) + \frac{1}{\sqrt{\alpha R}}\, g_2(y) + \cdots. \tag{4.18}$$

Then, we obtain the set of equations

$$\left\{\begin{array}{l} (w-c)\,g_0^2 = -i\,g_0^4, \qquad (w-c)(g_0'+2g_0g_1) = -i\left(4g_0^3g_1+6g_0^2g_0'\right), \\ (w-c)\left(g_1'+g_1^2+2g_0g_2-\alpha^2\right) - w'' = \cancel{-i\left(4g_0^3g_2+6g_1^2+2g_0g_1g_0'+3g_0'^2-2\alpha^2g_0^2\right)}, \\ \qquad -i\left\{3(g_0')^2+4g_0g_0''+12g_0g_1g_0'+6g_0^2g_1'+4g_0^3g_2+6g_0^2g_1^2-2\alpha^2g_0^2\right\} \\ \qquad \cdots \end{array}\right.$$

* Considerable dispute has arisen in this connection. Note that it is impossible to dispense with this difficulty be remarking that the two different determinations will differ only by a constant multiple of a particular integral. If we draw two paths from y_1 to y and obtain such a difference in the solution, it is evident that the asymptotic solution cannot be valid on both paths, because the <u>exact</u> equation (4.1) has <u>no</u> singular point at $y = y_0$ and hence its solution must be single-valued. Although a mistake here would not cause serious difficulties so far as the numerical evaluation of the eigen-value problem is concerned, it does lead to misunderstanding and confusion elsewhere. Even after Heisenberg and Tollmien have analyzed this problem in some detail, they still take the very misleading step of taking the complex conjugate of the inviscid equation (Heisenberg, loc. cit., 1924, p. 596; Tollmien, loc. cit., 1935, p. 88). This point will be discussed more fully later.

Hence, we can obtain the successive approximations without integration. Thus,

(4.10) $$g_0 = \pm\sqrt{i(w-c)}\,, \qquad g_1 = -\frac{5g_0'}{2g_0}\,, \;\cdots.$$

For definiteness, we define

(4.20) $$\arg(i) = \frac{\pi}{2}\,, \quad \arg(w-c) = 0, \quad \text{for } w-c > 0.$$

For negative values of w-c, we cannot decide, without further investigation, whether $\arg(w-c) = \pi$ or $\arg(w-c) = -\pi$. The point y_0 where w = c appeared in the previous asymptotic solution as a logarithmic branch point; here it is an algebraic branch point. The determination of correct path should follow the same criterion as the other two integrals, that (4.18) gives two asymptotic solutions of the exact equation (4.1), all along the path. The correct path might be expected to be the same as the previous path. All these will be discussed in the next section.

After such a question is settled, substitution of (4.19) into (4.16) and (4.17) gives the two asymptotic solutions

(4.21) $$\begin{cases} \varphi_3(y) = (w-c)^{-\frac{5}{4}} \exp\left\{-\int_{y_0}^{y}\sqrt{i\alpha R(w-c)\,dy}\right\}, \\ \varphi_4(y) = (w-c)^{-\frac{5}{4}} \exp\left\{+\int_{y_0}^{y}\sqrt{i\alpha R(w-c)\,dy}\right\}, \end{cases}$$

where factors of the order $\exp(\alpha R)^{-\frac{1}{2}} = 1 + O(\alpha R)^{-\frac{1}{2}}$ are taken as unity.

5. Analytical properties of the solutions.

Having thus obtained four asymptotic solutions of the equation (4.1), we must try to correlate them with the four solutions (4.9) and (4.11), and above all to study the correct determination of path around the artificial singularity introduced by the asymptotic methods. For this purpose, we consider the asymptotic expansions of the four regular solutions obtained by the first method and transform them back into the independent variable y.

Let us recall that the asymptotic expansions of the Hankel functions $H^{(1),(2)}_{\frac{1}{3}}(\xi)$ are given by*

$$(5.1)\quad \begin{cases} H^{(1)}_{\frac{1}{3}}(\xi) \sim \left(\frac{2}{\pi\xi}\right)^{\frac{1}{2}} \exp\left\{i\left(\xi-\frac{5\pi}{12}\right)\right\}\cdot\left\{1+\sum_{r=1}^{\infty}\frac{(-)^r\left(\frac{1}{3},r\right)}{(2i\xi)^r}\right\}, & -\pi<\arg\xi<2\pi, \\ H^{(2)}_{\frac{1}{3}}(\xi) \sim \left(\frac{2}{\pi\xi}\right)^{\frac{1}{2}} \exp\left\{-i\left(\xi-\frac{5\pi}{12}\right)\right\}\cdot\left\{1+\sum_{r=1}^{\infty}\frac{\left(\frac{1}{3},r\right)}{(2i\xi)^r}\right\}, & -2\pi<\arg\xi<\pi. \end{cases}$$

If we put $\xi=\frac{2}{3}(i\alpha_0\eta)^{\frac{3}{2}}$, then (5.1) becomes

$$(5.2)\quad \begin{cases} H^{(1)}_{\frac{1}{3}}\left[\frac{2}{3}(i\alpha_0\eta)^{\frac{3}{2}}\right] \sim \left(\frac{3}{\pi}\right)^{\frac{1}{2}}(i\alpha_0\eta)^{-\frac{3}{4}}\exp\left\{\frac{2}{3}(\alpha_0\eta)^{\frac{3}{2}}e^{\frac{5\pi}{4}i}-\frac{5\pi}{12}\right\}\left\{1+O\left(\eta^{-\frac{3}{2}}\right)\right\}, & -\frac{7\pi}{6}<\arg(\alpha_0\eta)<\frac{5\pi}{6}, \\ H^{(2)}_{\frac{1}{3}}\left[\frac{2}{3}(i\alpha_0\eta)^{\frac{3}{2}}\right] \sim \left(\frac{3}{\pi}\right)^{\frac{1}{2}}(i\alpha_0\eta)^{-\frac{3}{4}}\exp\left\{\frac{2}{3}(\alpha_0\eta)^{\frac{3}{2}}e^{\frac{\pi}{4}i}+\frac{5\pi}{12}\right\}\left\{1+O\left(\eta^{-\frac{3}{2}}\right)\right\}, & -\frac{11\pi}{6}<\arg(\alpha_0\eta)<\frac{\pi}{6}. \end{cases}$$

With the help of these formulae and taking the legitimate process of integrating the asymptotic expansions term by term, we obtain

$$(5.3)\quad \begin{cases} \chi_1^{(0)}+\varepsilon\chi_1^{(1)} = \eta+\frac{\varepsilon w_0''}{w_0'}\eta^2 = \frac{1}{w_0'\varepsilon}\left(w_0'y+\frac{w_0''}{2}y^2\right), \\ \chi_2^{(0)}+\varepsilon\chi_2^{(1)} = 1+\varepsilon\frac{w_0''}{w_0'}\eta\log\eta \sim 1+\frac{w_0''}{w_0'}y\log y, \\ \chi_3^{(0)} \sim \text{const.}\,\eta^{-\frac{5}{4}}\exp\left\{\frac{2}{3}(\alpha_0\eta)^{\frac{3}{2}}e^{\frac{5\pi}{4}i}\right\} = \text{const.}\,(y-y_0)^{-\frac{5}{4}}\exp\left\{-\int_{y_0}^{y}\sqrt{i\alpha R w_0'(y-y_0)}\,dy\right\}, \\ \chi_4^{(0)} \sim \text{const.}\,\eta^{-\frac{5}{4}}\exp\left\{\frac{2}{3}(\alpha_0\eta)^{\frac{3}{2}}e^{\frac{\pi}{4}i}\right\} = \text{Const.}\,(y-y_0)^{-\frac{5}{4}}\exp\left\{\int_{y_0}^{y}\sqrt{i\alpha R w_0'(y-y_0)}\,dy\right\}. \end{cases}$$

* cf., e.g., G.N. Watson, "Theory of Bessel Functions", Cambridge, (1922), p. 198.

These formulae can be easily seen to agree with the four asymptotic solutions (4.13) and (4.21) to the proper order of approximation, if we replace y_1 by y_0 in $\varphi_1^{(0)}$ (which is permissible).

In evaluating the asymptotic expressions (5.3), the argument of $\alpha_0\eta$ must satisfy both requirements specified in (5.2), i.e.,

$$-\frac{7\pi}{6} < \arg(\alpha_0\eta) < \frac{\pi}{6}. \tag{5.4}$$

In this range, the asymptotic solutions (4.13) and (4.21) hold.*
Having thus established the range of validity of these solutions, it is no longer necessary to make further comparisons of the two methods of solution.

At least three plans are now possible for further numerical work. First, we may use the four solutions obtained in the approximate form (4.9). Secondly, we may use the four asymptotic solutions (4.14) and (4.21). Thirdly, we may approximate $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ by the four functions $\{\varphi_1^{(0)}, \varphi_2^{(0)}, \chi_3^{(0)}, \chi_4^{(0)}\}$ given by (4.14) and (4.9). The first method is very similar to the method used by Hopf and Tietjens for linear velocity distributions, where the exact solutions are given by functions of the general nature as those in (4.9). For curved velocity distributions, the functions $\chi_1^{(0)}, \chi_2^{(0)}$ do not give φ_1 and φ_2 with sufficient accuracy, and this plan is not good. The second plan was used by Heisenberg in his investigation of the stability of the Poiseuile flow; but he also realized that it served

*Cf. Heisenberg, loc. cit., p. 591. Notice a difference of notation.

only part of his purpose, and he stated that the third plan should be used.* Tollmien substantially adopted the third plan for his investigation of the stability of the boundary layer, although he did not point out the connection of his method with Heisenberg's work. Instead of the expressions (4.14) for φ_1 and φ_2, he used solutions in the forms of power series in y. These solutions are easily manageable only for linear and parabolic velocity distributions. Accordingly, he tried to approximate the Blasius profile with such profiles. Since such approximations are not good enough in the neighborhood of the point $y = y_0$, where $w = c$, his discussion becomes very complicated. In the present work, we base our calculations upon the use of (4.14). It will be seen that our method can be applied to any profile with good accuracy. A comparison with Tollmien's method will be discussed in the Appendix.

It may be added that the adoption of the third plan leaves an error of the order of $(\alpha R)^{-1}$ in φ_1 and φ_2, and an error of the order of $(\alpha R)^{-\frac{1}{3}}$ in φ_3 and φ_4. These errors can be reduced by including the higher approximations. In practice, this is hardly necessary. A detailed discussion of numerical accuracy will be found in the Appendix.

Further discussions. Having thus established the range of validity of the asymptotic solutions, we shall try to settle a few questions of considerable dispute, namely, (a) the "crossing substi-

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*Loc. cit., p. 604.

tution", (b) the inner friction layers, and (c) the complex conjugate of the inviscid solution.

a) <u>The "crossing substitution"</u>. From previous discussions, it is evident that if we pass from $y > R\ell(y_0)$ to $y < R\ell(y_0)$ along a path <u>below</u> the point y_0, we are always in a region of the y-plane where the above asymptotic solutions hold, and no further investigation is necessary. In fact, if $c_i > 0$ (and is small) and $R\ell(w_0') > 0$, the point y_0 is above the real axis, and the asymptotic solutions are <u>valid along the real axis of y</u>. In the case of real c, the point y_0 is on the real axis, and there is <u>one point on the real axis</u> where the asymptotic solutions fail to be valid. In the case $c_i < 0$, and $R\ell(w_0') < 0$, the point y_0 is below the real axis, and the lines $\arg\{\alpha_0(y-y_0)\} = -\frac{7\pi}{6}, \frac{\pi}{6}$ intersect the real axis in two points y_f', y_f'' ($y_1 < y_f' < y_f'' < y_2$) *. Thus, the asymptotic expressions (4.14) and (4.21) represent one solution for $y_1 \leq y < y_f'$ and $y_f'' < y < y_2$, but not the same solution for $y_f' < y < y_f''$. It is necessary to obtain a suitable <u>"crossing substitution"</u> in order to obtain the correct solutions for $\frac{\pi}{6} < \arg\{\alpha_0(y-y_0)\} < \frac{5\pi}{6}$ (i.e., in <u>crossing</u> the lines $\arg\{\alpha_0(y-y_0)\} = -\frac{7\pi}{6}, \frac{\pi}{6}$). For this purpose, we must obtain the asymptotic expansion of the Hankel function $H^{(2)}_{\frac{1}{3}}\left[\frac{2}{3}(i\alpha_0\eta)^{\frac{3}{2}}\right]$ proper to that region. The analytical expression would then be quite different from that given in (5.2). Thus, in crossing the <u>two points</u> y_f' <u>and</u> y_f'' <u>of the real axis</u>, the asymptotic solutions fail to be

* The whole theory must be modified for extremely highly damped solutions for which $y_f' < y_1 < y_2 < y_f''$.

analytic expressions. However, it is to be noted that the failure of the asymptotic solutions along the real axis does not exclude their use in the investigation of the boundary value problems to be considered below, so long as we are concerned only with the eigen-value problem. It is only necessary that these solutions should be valid in a connected region containing the end-points y_1 and y_2. The calculation of the amplitude distribution of the disturbance (the eigen-function) in the neighborhood of the inner friction layers, however, is to be made with the regular solutions. Or, we can calculate the eigen-function for $y_f' < y < y_f''$ by using a proper "crossing substitution". Since we are chiefly concerned with the eigen-value problem, we shall not go into further details.

In order to make the situation still clearer, let us see what would happen if we try to obtain our solutions for $y < \mathcal{R}\ell\,(y_0)$ by going along a path above the point y_0. For simplicity, let us take the case of real c with $w_0' > 0$, and consider the asymptotic expressions $\varphi_3^{(0)}$ and $\varphi_4^{(0)}$ given by (5.3). We have (A,B being arbitrary constants)

$$\begin{cases} \varphi_3^{(0)} \sim A\eta^{-\frac{5}{4}} \exp\left\{\frac{2}{3}(\alpha_0\eta)^{\frac{3}{2}} e^{\frac{5\pi}{4}i}\right\} & \eta > 0, \\ \varphi_3^{(0)} \sim A|\eta|^{-\frac{5}{4}} e^{\frac{5\pi}{4}i} \exp\left\{\frac{2}{3}(\alpha_0|\eta|)^{\frac{3}{2}} e^{-\frac{\pi}{4}i}\right\} & \eta < 0; \end{cases}$$

$$\begin{cases} \varphi_4^{(0)} \sim B\eta^{-\frac{5}{4}} \exp\left\{\frac{2}{3}(\alpha_0\eta)^{\frac{3}{2}} e^{\frac{\pi}{4}i}\right\} & \eta > 0, \\ \varphi_4^{(0)} \sim B|\eta|^{-\frac{5}{4}} e^{\frac{5\pi}{4}i} \exp\left\{\frac{2}{3}(\alpha_0|\eta|)^{\frac{3}{2}} e^{-\frac{5\pi}{4}i}\right\} & \eta < 0. \end{cases}$$

These are obtained by taking a path below the point y_0. If we had taken the other path, then $\arg(\eta) = \pi$ for $\eta < 0$, and we would have the functions $\tilde{\varphi}_3^{(0)}$ and $\tilde{\varphi}_4^{(0)}$, which agrees with φ_3 and φ_4 for $\eta > 0$, but are defined by

$$\begin{cases} \tilde{\varphi}_3^{(0)} \sim A|\eta|^{-\frac{5}{4}} e^{-\frac{5\pi}{4}i} \exp\left\{\frac{2}{3}(\alpha_0|\eta|)^{\frac{3}{2}} e^{\frac{3\pi}{4}i}\right\}, \\ \tilde{\varphi}_4^{(0)} \sim B|\eta|^{-\frac{5}{4}} e^{-\frac{5\pi}{4}i} \exp\left\{\frac{2}{3}(\alpha_0|\eta|)^{\frac{3}{2}} e^{\frac{7\pi}{4}i}\right\}, \end{cases}$$

for $\eta < 0$. Thus, if A and B are taken to be the same, we have

$$\tilde{\varphi}_3^{(0)} = -i\,\varphi_4^{(0)}, \qquad \tilde{\varphi}_4^{(0)} = -i\,\varphi_3^{(0)}, \qquad \text{for } \eta < 0.$$

Hence, if we took $\tilde{\varphi}_3^{(0)}$ and $\tilde{\varphi}_4^{(0)}$ to be the proper determinations, we would have to make the following "crossing substitution" corresponding to a passage from $\eta > 0$ to $\eta < 0$:

$$\begin{cases} \varphi_3: & \tilde{\varphi}_3^{(0)} \to i\,\tilde{\varphi}_4^{(0)} \\ \varphi_4: & \tilde{\varphi}_4^{(0)} \to i\,\tilde{\varphi}_3^{(0)} \end{cases}$$

If we note that $\tilde{\varphi}_3^{(0)} \ll \tilde{\varphi}_4^{(0)}$ both for $w-c > 0$ and for $w-c < 0$, we would also have the following equivalent change:

$$\begin{cases} \varphi_3: & \tilde{\varphi}_3^{(0)} \to \tilde{\varphi}_3^{(0)} + i\,\tilde{\varphi}_4^{(0)}, \\ \varphi_4: & \tilde{\varphi}_3^{(0)} - i\,\tilde{\varphi}_4^{(0)} \to \tilde{\varphi}_3^{(0)}, \end{cases}$$

which may be compared with equation (16), p. 589 of Heisenberg's paper. In making the comparison, note his definition of the angle of w-c, (p. 585), and the difference of notation in the fundamental equation of stability. Note also that he is having in mind the case where $w_0' < 0$. It seems that Heisenberg has made the situation unnecessarily complicated by taking an unsuitable path.

The study of crossing substitution whould also be compared with the work of Jeffreys(102) and the W-K-B method(103) in quantum mechanics.* In those cases, a differential equation of the form $\varepsilon^3 \varphi'' + q(y)\varphi = 0$ is considered. If this equation is treated by the method of section 4 by writing $\varphi = \chi^{(0)}(\eta) + \varepsilon\chi^{(1)}(\eta) + \cdots$, $y - y_0 = \varepsilon\eta$, and $q(y) = q_0'\varepsilon\eta + \frac{q_0''}{2}\varepsilon^2\eta^2 + \cdots$, the equation for $\chi^{(0)}(\eta)$ is $\chi^{(0)''} + q_0'\eta\chi^{(0)} = 0$ as compared with (4.7) $\chi^{(0)iv} - i\eta w_0'\chi^{(0)''} = 0$. It is evident that our η corresponds to $+i\eta$ in their case. Kramers has shown that the cuts in their asymptotic expansions are the lines $\arg(\eta) = \pm\pi/3$. Thus, in our case, the cuts should be $\arg(\eta) = \frac{\pi}{6}, \frac{5\pi}{6}$. This agrees with our previous discussions. An important difference is the following. In their case, the two boundary points on the real axis are separated into two regions of the complex plane by the cuts, so that a "crossing substitution" is absolutely necessary. In our case, the two boundary points on the real axis belong to the same region, and a "crossing substitution" is superfluous, so far as the eigen-value problem is concerned.

b) There is also a very significant physical interpretation associated with the "crossing substitution" of the asymptotic solutions. The initial approximations $\varphi_1^{(0)}$ and $\varphi_2^{(0)}$ satisfy the inviscid equation. Hence, if $c_i > 0$, these solutions hold throughout the part

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* I am indebted to Professor P. S. Epstein for calling my attention to this comparison.

(y_1, y_2) of the real axis, the effect of viscosity is entirely negligible inside the fluid for sufficiently large Reynolds numbers. If $c_i \leq 0$, the inviscid solution can never hold all along the real axis, and hence the effect of viscosity inside the fluid is not negligible, however large the Reynolds number may be. The singularity of the asymptotic solutions means a very rapid change of velocity within a small distance so that the effect of viscosity is no longer negligible there. Physically, such a point on the real axis corresponds to a layer of fluid where the viscous forces play an important role. The existence of such an inner friction-layer at $y = y_0$ where $w = c$ (real) was first noticed by Lord Rayleigh.[35]

Referring to the foregoing discussions, we see that there are two inner friction layers for the damped oscillations, one for the neutral oscillations, and none for the self-excited oscillations.

In the neutral case, the first term of (4.1) disappears at the critical layer $w = c$. The equation then represents a balancing of vorticity transferred by the disturbance and that diffused away by the effect of viscosity. It is therefore understandable that the effect of viscosity must be predominant there. In the other two cases, $w - c$ never vanishes in the fluid, there is the vorticity carried by the main flow (relative to an observer moving with the phase velocity c_r); and there is always the change of vorticity due to amplification or damping. In the case of amplified oscillations, these two effects can be in equilibrium with the transfer of vorticity due to the disturbance, and the effect of viscosity is completely negligible at very large Reynolds numbers. In the case of damped oscillations, these effects presumably never balance out each other, thus resulting in the formation of two critical layers, where the effect of viscosity is not negligible.

c) The complex conjugate of the solution $\varphi(y)$. It is often argued,* that if $\varphi(y)$ is a solution of the inviscid equation with an eigen-value c, then $\bar{\varphi}(y)$ is another solution with the eigen-value $\bar{c}$, satisfying the same boundary conditions on the real axis. Thus, to each damped solution, there is always a corresponding amplified solution, and vice versa. This argument is in direct contradiction to the foregoing discussions, because an amplified solution and a damped solution have entirely different characteristics with respect to inner friction layers. It appears therefore, that $\bar{\varphi}(y)$ should still represent a solution of the same nature as $\varphi(y)$.

This can be seen more clearly from an examination of the complete equation (4.1). If we take the complex conjugate of that equation, and write y for $\bar{y}$ (which is essentially done in the usual argument), we have

$$(5.5) \quad \{w(y)-\bar{c}\}\{\bar{\varphi}''-\alpha^2\bar{\varphi}\}-w''(y)\,\bar{\varphi} = \frac{i}{\alpha R}\{\bar{\varphi}^{iv}-2\alpha^2\bar{\varphi}''+\alpha^4\bar{\varphi}\}.$$

The complete stream function $\bar{\psi}'(x,y,t)$ satisfying this equation is

$$\bar{\psi}'(x,y,t) = \bar{\varphi}(y)\, e^{-i\alpha(x-\bar{c}t)}.$$

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* Heisenberg, loc. cit. p. 596; Tollmien, loc. cit., 1935, p. 88. The failure of such an argument would indicate that Heisenberg's classification of velocity profiles on p. 597 of his paper is untenable.

Thus, if $Im(c) < 0$, $Im(\bar{c}) > 0$, we still have a damped solution. This should also hold for the inviscid equation, since it is regarded as a limiting case of the viscous equation. From the inviscid equation itself, there is no way of telling whether $Im(\bar{c}) > 0$ corresponds to damping or to amplification.

In fact, the asymptotic solutions of equation (5.5) (including the inviscid solutions) holds for

$$-\frac{\pi}{6} < \arg\{\bar{\alpha}_0(y-y_0)\} < \frac{7\pi}{6}, \qquad w(y_0) = \bar{c}. \tag{5.6}$$

Thus, we have a solution $\bar{\varphi}(y)$, valid in a region which is quite improper for an asymptotic solution of (4.1). (Compare (5.4) and (5.6).) Hence, it is not legitimate to conclude that a solution of a different nature can be obtained by taking the complex conjugate. The influence of these discussions upon the usual conclusions regarding inertial stability will be discussed fully in the next part of the paper.

6. The boundary value problems

So far, the solutions are discussed with little reference to the boundary conditions to be satisfied. The boundary conditions are essentially that the velocities of disturbance should vanish on the solid boundaries, and also at infinity, if the field of flow extends to infinity. However, it is often convenient to use equivalent boundary conditions for certain types of velocity distributions.

In order not to be lost in too much generalities, we shall limit ourselves to three classes of velocity distributions (as specified below and shown in the figure), and select our fundamental interval (y_1, y_2) so that

$$w'(y) \geq 0, \quad \text{for} \quad y_1 < y < y_2 .$$

We shall define our characteristic length so that $y_2 - y_1 = 1$, and let $\varphi_1(y; c, \alpha, \alpha R)$, $\varphi_2(y; c, \alpha, \alpha R)$, $\varphi_3(y; c, \alpha, \alpha R)$, $\varphi_4(y; c, \alpha, \alpha R)$ be a fundamental system of solutions of (4.1) arranged in the order discussed above.

Case (1) Flow between solid walls in relative motion. In this case, the boundary conditions are given by

$$\varphi(y_1) = \varphi'(y_1) = \varphi(y_2) = \varphi'(y_2) = 0 , \tag{6.1}$$

because the velocity of the disturbance should vanish on both the solid boundaries. The determinantal equation corresponding to these conditions is

$$(6.2) \qquad F_1(\alpha, c, \alpha R) = \begin{vmatrix} \varphi_{11} & \varphi_{21} & \varphi_{31} & \varphi_{41} \\ \varphi_{12} & \varphi_{22} & \varphi_{32} & \varphi_{42} \\ \varphi'_{11} & \varphi'_{21} & \varphi'_{31} & \varphi'_{41} \\ \varphi'_{12} & \varphi'_{22} & \varphi'_{32} & \varphi'_{42} \end{vmatrix} = 0,$$

where φ_{11}, φ'_{11}, ... stand for $\varphi_1(y_1)$, $\varphi'_1(y_1)$, ... In this and all later discussions, a subscript 1 or 2 attached to a function of y shall denote the value of that function at $y = y_1$ or $y = y_2$ respectively.

<u>Case (2) Symmretical flow between solid walls at rest.</u> In this case, it is easily seen from (4.1) that the disturbance can be separated into two independent parts, one symmetrical with respect to the line $y = y_2$ and the other antisymmetrical. (a) If $\varphi(y)$ is a symmetrical function (antisymmetrical disturbance), then the conditions

$$(6.3) \qquad \varphi(y_1) = \varphi'(y_1) = \varphi'(y_2) = \varphi'''(y_2) = 0$$

hold, and we have the determinantal equation

$$(6.4) \qquad F_2(\alpha, c, \alpha R) = \begin{vmatrix} \varphi_{11} & \varphi_{21} & \varphi_{31} & \varphi_{41} \\ \varphi'_{11} & \varphi'_{21} & \varphi'_{31} & \varphi'_{41} \\ \varphi'_{12} & \varphi'_{22} & \varphi'_{32} & \varphi'_{42} \\ \varphi'''_{12} & \varphi'''_{22} & \varphi'''_{32} & \varphi'''_{42} \end{vmatrix} = 0.$$

(b) If $\varphi(y)$ is an odd function of $y-y_2$ (symmetrical disturbance), then the boundary conditions are

$$(6.5) \qquad \varphi(y_1) = \varphi'(y_1) = \varphi(y_2) = \varphi''(y_2) = 0,$$

and we have the relation

$$F_3(\alpha, c, \alpha R) = \begin{vmatrix} \varphi_{11} & \varphi_{21} & \varphi_{31} & \varphi_{41} \\ \varphi_{12} & \varphi_{22} & \varphi_{32} & \varphi_{42} \\ \varphi_{11}' & \varphi_{21}' & \varphi_{31}' & \varphi_{41}' \\ \varphi_{12}'' & \varphi_{22}'' & \varphi_{32}'' & \varphi_{42}'' \end{vmatrix} = 0 . \tag{6.6}$$

<u>Case (3) Flow of the boundary-layer type.</u> In this case, the point y_2 is taken to correspond to the "<u>edge</u>" of the boundary layer, beyond which the velocity is substantially constant. The boundary conditions to be satisfied at y_1 are the usual ones:

$$\varphi(y_1) = \varphi'(y_1) \tag{6.7}$$

The boundary conditions for y becoming infinite are to be replaced as follows.* Since the particular integral φ_4 becomes infinite as y becomes infinite, our boundary condition requires that φ is a linear combination of $\varphi_1, \varphi_2, \varphi_3$ alone. Thus,

$$\varphi = C_1\varphi_1 + C_2\varphi_2 + C_3\varphi_3 , \tag{6.8}$$

where C_1, C_2, C_3 are constants of integration. Also, the integral φ_3 has practically no contribution for $y > y_2$ so that we expect $\varphi(y)$ to satisfy the inviscid equation for $y > y_2$. Here $w'' = 0$, and hence two particular integrals are $e^{\pm \alpha y}$. The condition that $\varphi \to 0$ as $y \to \infty$ excludes the integral $e^{+\alpha y}$. Hence, φ must be proportional to $e^{-\alpha y}$ for $y > y_2$. This may be expressed as

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* Cf. Tietjens (65) and Tollmien (66)

(6.9) $$\varphi' + \alpha\varphi = 0 \quad \text{for} \quad y \geq y_2$$

Hence, we have the determinantal equation

(6.10) $$F_4(\alpha, c, \alpha R) = \begin{vmatrix} \varphi_{11} & \varphi_{21} & \varphi_{31} \\ \varphi_{12}' + \alpha\varphi_{12} & \varphi_{22}' + \alpha\varphi_{22} & 0 \\ \varphi_{11}' & \varphi_{21}' & \varphi_{31}' \end{vmatrix} = 0.$$

We note that the point y_2 can be replaced by any value of $y > y_2$. This is equivalent to the fact that the thickness of the boundary layer cannot be definitely defined. The larger this thickness is taken, the more accurate the results should be.*

The functions F_1, F_2, F_3, and F_4 are entire functions of the parameters α, c, and R.

Reduction of the equations for large values of αR

The equations (6.2), (6.4), and (6.6) can be substantially simplified for large values of αR. Referring to (4.21), we see that

$$\varphi_3(y) = A(y)e^{-Y}, \qquad \varphi_4(y) = B(y)e^{Y}$$

where A(y) and B(y) are of the order of unity, and Y is defined by

$$Y = \int_{y_0}^{y} \sqrt{i\alpha R(w-c)}\, dy$$

Hence, we have the following relations, giving the order of magnitude of certain quantities:

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* In later calculation of the Blasius case, we shall take a thickness of the boundary layer about 1.19 times that used by Tollmien.

$$
(6.11)\quad \begin{cases}
\dfrac{\varphi_{31}'}{\varphi_{31}} = -\sqrt{i\alpha R(w_1-c)} + \dfrac{A_1'}{A_1} \\
\dfrac{\varphi_{32}}{\varphi_{31}} = \dfrac{A_2}{A_1}\, e^{-P} \\
\dfrac{\varphi_{32}'}{\varphi_{31}} = \left\{-\sqrt{i\alpha R(w_2-c)}\,\dfrac{A_2}{A_1} + \dfrac{A_2'}{A_1}\right\} e^{-P} \\
\dfrac{\varphi_{32}''}{\varphi_{31}} = \left\{i\alpha R(w_2-c)\,\dfrac{A_2}{A_1} + O\left(\sqrt{\alpha R}\right)\right\} e^{-P} \\
\dfrac{\varphi_{32}'''}{\varphi_{31}} = \left\{-\left[i\alpha R(w_2-c)\right]^{\frac{3}{2}}\dfrac{A_2}{A_1} + O(\alpha R)\right\} e^{-P}
\end{cases}
\qquad P = \int_{y_1}^{y_2}\sqrt{i\alpha R(w-c)}\,dy,
$$

$$
(6.12)\quad \begin{cases}
\dfrac{\varphi_{41}'}{\varphi_{41}} = \sqrt{i\alpha R(w_1-c)} + \dfrac{B_1'}{B_1}, \\
\dfrac{\varphi_{42}}{\varphi_{41}} = \dfrac{B_2}{B_1}\, e^{P}, \\
\dfrac{\varphi_{42}'}{\varphi_{41}} = \left\{\sqrt{i\alpha R(w_2-c)}\,\dfrac{B_2}{B_1} + \dfrac{B_2'}{B_1}\right\} e^{P}, \\
\dfrac{\varphi_{42}''}{\varphi_{41}} = \left\{i\alpha R(w_2-c)\,\dfrac{B_2}{B_1} + O\left(\sqrt{\alpha R}\right)\right\} e^{P}, \\
\dfrac{\varphi_{42}'''}{\varphi_{41}} = \left\{\left[i\alpha R(w_2-c)\right]^{\frac{3}{2}}\dfrac{B_2}{B_1} + O(\alpha R)\right\} e^{P}.
\end{cases}
$$

It then appears that the sign of the real part of $P = \int_{y_1}^{y_2}\sqrt{i\alpha R(w-c)}\,dy$ is of consequence. It can be verified that it is always positive when $c_i > 0$. For then the path of integration can be taken along the real axis of y, and we have $-\pi < \arg(w-c) < 0$; consequently, $-\frac{\pi}{4} < \arg(P) < \frac{\pi}{4}$. With reference to (4.21), (6.11) and (6.12), we see that the condition $P = n\pi i$, n = integer, express the fact that $\varphi_{31}'\varphi_{42}' = \varphi_{41}'\varphi_{32}'$, when terms of the order $(\alpha R)^{-\frac{1}{2}}$ are neglected. This is the corrected form of the first solution of

Heisenberg as expressed by the condition (27) on p. 596 of his paper. Heisenberg also stated that such a condition can only be satisfied for damped solutions. In fact, from the condition just obtained for $c_i > 0$, we see that Rl(P) can be negative only for highly damped solutions, for which the whole discussion must be modified. (Cf. footnote on p. 31, section 5).

Neglecting quantities of the orders e^{-P} and $(\alpha R)^{-1}$ against quantities of the order of unity, we have the following simplifications of the equations (6.2), (6.4), and (6.6) for the cases (1) and (2).

Case (1) For flow between solid walls in relative motion, we have

$$\frac{f_1(\alpha,c)}{f_3(\alpha,c)} = \frac{\varphi_{31}}{\varphi_{31}'} + \frac{\varphi_{42}}{\varphi_{42}'} \quad \frac{f_2(\alpha,c)}{f_4(\alpha,c)}. \tag{6.13}$$

Case (2a) For anti-symmetrical disturbance in symmetrical flow between solid walls, we have

$$\frac{f_2(\alpha,c)}{f_4(\alpha,c)} = \frac{\varphi_{31}}{\varphi_{31}'}. \tag{6.14}$$

Case (2b) For symmetrical disturbance in symmetrical flow between solid walls, we have

$$\frac{f_1(\alpha,c)}{f_3(\alpha,c)} = \frac{\varphi_{31}}{\varphi_{31}'}. \tag{6.15}$$

In these equations, the functions $f_1(\alpha,c)$, $f_2(\alpha,c)$, $f_3(\alpha,c)$, and $f_4(\alpha,c)$ are defined as follows:

$$f_1(\alpha,c) = \begin{vmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{vmatrix}, \qquad f_2(\alpha,c) = \begin{vmatrix} \varphi_{11} & \varphi_{12}' \\ \varphi_{21} & \varphi_{22}' \end{vmatrix}; \tag{6.16}$$

$$f_3(\alpha,c) = \begin{vmatrix} \varphi_{11}' & \varphi_{12} \\ \varphi_{21}' & \varphi_{22} \end{vmatrix}, \qquad f_4(\alpha,c) = \begin{vmatrix} \varphi_{11}' & \varphi_{12}' \\ \varphi_{21}' & \varphi_{22}' \end{vmatrix}.$$

These functions depend only on α and c, because we may take the inviscid solutions for φ_1 and φ_2, which are accurate up to the order of $(\alpha R)^{-1}$.

Case (3) For flow in the boundary layer along a flat plate, we can reduce (6.10) to

$$\frac{f_2+\alpha f_1}{f_4+\alpha f_3} = \frac{\varphi_{31}}{\varphi_{31}'}, \tag{6.17}$$

if we also replace φ_1 and φ_2 by their inviscid approximations. The equations (6.13), (6.14), (6.15), and (6.17) are the final equations based on which the stability investigations are to be made.

The "inviscid case". *In the limit* $\alpha R \to \infty$, *the above* equations reduce to

(6.18) $f_1(\alpha, c) = 0$, for case (1) and case (2b),

(6.19) $f_2(\alpha, c) = 0$, for case (2a),

(6.20) $f_2 + \alpha f = 0$, for case (3).

Mathematically, these are equivalent to the solution of the inviscid equation

$$(w - c)(\varphi'' - \alpha^2 \varphi) - w''\varphi = 0 \tag{6.21}$$

subjected to one of the following three sets of boundary conditions

$$\varphi(y_1) = \varphi(y_2) = 0, \quad \varphi(y_1) = \varphi'(y_2) = 0, \quad \varphi(y_1) = \varphi'(y_2) + \alpha\varphi(y_2) = 0. \tag{6.22}$$

We have thus arrived at the conclusion that some asymptotic behavior of the stability conditions can be obtained by neglecting the effect of viscosity (provided proper care is given to the inner friction layer). This was tacitly assumed in the work of Rayleigh and others, while Heisenberg pointed out that a proof was necessary in accordance with some remarks of Oseen[30] (loc. cit., p. 583); he also virtually gave the proof.

In the next part of the paper, we shall therefore consider the simpler problem of an inviscid fluid. After a thorough investigation of that problem, we shall investigate the effect of viscosity by considering the equations (6.13), (6.14), (6.15), and (6.17) in greater detail. Numerical calculations of the stability limit based upon those equations will also be carried out for certain important special cases. For these purposes, the evaluation of the six functions $f_1(\alpha, c)$, $f_2(\alpha, c)$, $f_3(\alpha, c)$, $f_4(\alpha, c)$, $\varphi_{31}/\varphi'_{31}$, $\varphi_{42}/\varphi'_{42}$ is necessary. We shall discuss this briefly here.

1) <u>Evaluation of</u> $f_1(\alpha, c)$, $f_2(\alpha, c)$, $f_3(\alpha, c)$, $f_4(\alpha, c)$. These quantities are related to the inviscid solutions $\varphi_1^{(0)}$, $\varphi_2^{(0)}$ given by (4.14) with the path of integration subjected to the condition (5.4). Hence, we have

$$\varphi_{11} = -c, \quad \varphi_{21} = 0, \quad \varphi'_{11} = w'_1, \quad \varphi'_{21} = -\frac{1}{c}; \tag{6.23}$$

$$\begin{cases} \varphi_{12} = (1-c)\sum_{n=0}^{\infty} \alpha^{2n} H_{2n}(c), \quad \varphi_{22} = (1-c)\sum_{n=0}^{\infty} \alpha^{2n} K_{2n+1}(c), \\ \varphi'_{12} = (1-c)^{-1}\sum_{n=1}^{\infty} \alpha^{2n} H_{2n-1}(c) + (1-c)^{-1} w'_2\, \varphi_{12}, \\ \varphi'_{22} = (1-c)^{-1}\sum_{n=0}^{\infty} \alpha^{2n} K_{2n}(c) + (1-c)^{-1} w'_2\, \varphi_{22}; \end{cases} \tag{6.24}$$

where

$$\begin{cases} H_{2n}(c) = h_{2n}(y_2), \quad H_{2n-1}(c) = (1-c)^{+2}\, h'_{2n}(y_2), \\ K_{2n+1}(c) = k_{2n+1}(y_2), \quad K_{2n}(c) = (1-c)^2\, k'_{2n+1}(y_2), \end{cases} \tag{6.25}$$

are functions of c alone. In the above evaluations, we have put $w_1 = 0$, in accordance with the actual conditions of all the cases considered. We have also chosen the characteristic velocity so that $w_2 = 1$. Referring to (6.23), we have

$$\text{(6.26)} \qquad \begin{cases} f_1(\alpha,c) = -c\,\varphi_{22}, & f_2(\alpha,c) = -c\,\varphi_{22}', \\ f_3(\alpha,c) = w_1'\varphi_{22} + \frac{1}{c}\varphi_{12}, & f_4(\alpha,c) = w_1'\varphi_{22}' + \frac{1}{c}\varphi_{12}'. \end{cases}$$

It is found convenient to transform (6.24) into

$$\text{(6.27)} \qquad \begin{aligned} \varphi_{12} &= (1-c)(1-\alpha^2 H_2)^{-1}\Big(1-\sum_{n=2}^{\infty}\alpha^{2n}M_{2n}\Big), \\ \varphi_{22} &= K_1\varphi_{12} - (1-c)\sum_{n=1}^{\infty}\alpha^{2n}N_{2n+1}, \\ \varphi_{12}' &= (1-c)^{-1}(1-\alpha^2 H_2)^{-1}\Big(\alpha^2 H_1 - \sum_{n=2}^{\infty}\alpha^{2n}M_{2n-1}\Big) + (1-c)^{-1}w_2'\varphi_{12}, \\ \varphi_{22}' &= K_1\varphi_{12}' + (1-c)^{-1}\Big(1-\sum_{n=1}^{\infty}\alpha^{2n}N_{2n}\Big) + (1-c)^{-1}w_2'[\varphi_{22}, -K_1\varphi_{12}] \end{aligned}$$

where the functions $M_n(c)$ and $N_n(c)$ are defined by

$$\text{(6.28)} \qquad \begin{cases} -M_n = H_n - H_2 H_{n-2} & n \geq 3, \\ -N_n = K_n - K_1 H_{n-1} & n \geq 2. \end{cases}$$

The principal advantage of such transformations is to bring out the dominant terms of the functions φ_{12}, φ_{12}', φ_{22}, φ_{22}'. For the terms in M's and N's are usually negligible, (particularly for small values of c, with which we are usually concerned), while in (6.24), all the terms in the series are of considerable importance. This point will be more fully discussed in the appendix.

2) Evaluation of $\varphi_{31}/\varphi_{31}'$ and $\varphi_{42}/\varphi_{42}'$. These quantities are related to the highly oscillating integrals φ_3 and φ_4. For extremely large values of αR so that $(\alpha R)^{\frac{1}{3}}c$, $(\alpha R)^{\frac{1}{3}}(1-c) \gg 1$, the approxi-

This agrees with (6.29) if $\alpha_0^3(y_1-y_0)=w_0'(y_1-y_0)$ is replaced by $-c$.

The equations (6.23) — (6.33) are the explicit formulae useful for the following investigations of (6.13) — (6.20). These results may be compared with the earlier ones of Heisenberg and Tollmien.

Table I

The Functions $F(z)$ and $\mathfrak{F}(z)$

z	$F_r(z)$	$F_i(z)$	$\mathfrak{F}_r(z)$	$\mathfrak{F}_i(z)$
1.0	0.920	-0.389	0.506	-2.47
1.5	0.696	-0.198	2.305	-1.500
2.0	0.603	-0.074	2.18	-0.364
2.5	0.543	0.043	2.17	0.204
3.0	0.467	0.172	1.701	0.548
3.5	0.337	0.323	1.220	0.594
4.0	0.115	0.313	1.004	0.355
4.5	-0.009	0.180	0.960	0.1714
5.0	0.001	0.073	0.996	0.0726
5.5	0.035	0.044	1.000	0.0458

PART II STABILITY IN AN INVISCID FLUID

2. <u>Sufficient conditions for the existence of a disturbance.</u> So far, the sufficient conditions are known only for symmetrical and for boundary-layer velocity distributions. The results may be stated as follows. (a) There is always the neutral disturbance given by $c = 0$, $\alpha = 0$, $\varphi(y) = w(y)$. (b) If $w''(y_s) = 0$, $y_1 < y_s < y_2$, there is a neutral disturbance with $c = w(y_s)$; furthermore, if $w'''(y_s) \neq 0$, self-excited disturbances also exist.

<u>Discussion.</u> Let us first take up the question of a damped disturbance. In the proof of Lord Rayleigh, the solution is taken to be valid all along the real axis. Hence, in accordance with the discussion of section 5, such considerations do not include damped disturbances. However, Rayleigh and Tollmien did not distinguish between an amplified disturbance and a damped disturbance, because they regarded them as complex conjugates. As pointed out in section 5, this is not permissible. In fact, if we accept the original conclusions of Rayleigh and Tollmien, a profile without a flex could not execute any kind of disturbance. This can hardly be reconciled with our intuition regarding the state of affairs in a real fluid at infinitely large Reynolds numbers.* According to the present interpretation, only damped solutions can exist. Such a conclusion will also be borne out by the following calculations for a viscous fluid (see figures in Part III). It is to be noted that the neutral and the self-excited disturbances existing for $w''(y_s) = 0$ are free from the

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* This is the objection of Friedrichs, loc. cit. (101), p. 209. It must also be noted that the non-linear terms are not negligible in the case of an <u>ideal</u> fluid. We shall consistently restrict the magnitude of our disturbances so that the effect of viscosity is always more important than the effect of non-linearity.

effect of viscosity inside the fluid, because the neutral solution is also regular at $y = y_s$ where $w = c$. Hence, we may conclude that <u>disturbances essentially free from the effect of viscosity inside a fluid can only exist for velocity distributions with a flex.</u>

Next, consider the neutral solution $c = 0$, $\alpha = 0$, $\varphi = w$. This seems to be a trivial one; actually, it is quite significant. Although its existence can be seen very easily from (6.21) and (6.22), its significance can be more clearly seen by deriving it from (6.19) and (6.20). Referring to (6.26), we see that these conditions are reduced to

(7.1) $$\varphi_{22}' = 0, \qquad \text{and} \qquad \varphi_{22}' + \alpha\,\varphi_{22} = 0.$$

If we note that in (6.27)

$$K_1(c) = \int_{y_1}^{y_2} \frac{dy}{(w-c)^2} = -\frac{1}{w_1' c} - \frac{w_0''}{w_0'^3}(\log c + i\pi) + O(1),$$

and that $M_n(c)$, $N_n(c)$ approach finite values as $c \to 0$, we see that

(7.2) $$\begin{cases} \lim\limits_{\alpha, c \to 0} \varphi_{22}' = 0, & \text{if } \lim\limits_{\alpha, c \to 0} \dfrac{\alpha^2 H_1(0)}{w_1' c} = 1, \\[2ex] \lim\limits_{\alpha, c \to 0} (\varphi_{22}' + \alpha\,\varphi_{22}) = 0, & \text{if } \lim\limits_{\alpha, c \to 0} \dfrac{w_1' c}{\alpha} = 1. \end{cases}$$

Hence, the above-mentioned neutral solution exist for these cases. The conditions in (7.2) will be found to be significant in later investigations of a viscous fluid. (Cf. (12.8) and (12.11))

Tollmien's proof for the existence of a self-excited disturbance requires the condition $w'''(y_s) \neq 0$. This condition can be removed by a more thorough mathematical investigation of the problem.

Such a discussion will be given in the next section. The proof to be given is also more rigorous (which seems to be desirable for an existence proof).

At first sight, the above sufficient conditions for the existence of a neutral or amplified disturbance might be expected to hold also for other types of flow, e.g., when the walls are in relative motion. However, this is not true. If we look for the solution $\alpha = 0$, $c = 0$ in this case, (with $w' > 0$), by considering the root of (cf. (6.18))

(7.3) $$\varphi_{22} = 0,$$

we find that the relation $\lim_{\alpha, c \to 0} \varphi_{22} = 0$ corresponding to (7.2) is not satisfied. Indeed, it is very difficult to discuss the solution of $\varphi_{22} = 0$. Also, the flex in the velocity profile does not play a very significant role in this case. To see this, consider the velocity distribution

$$w(y) = A + B \sin y, \qquad y_1 \leq y \leq y_2,$$

which has a flex at $y = 0$, if $y_1 < 0 < y_2$. According to the above necessary conditions, the only possible neutral disturbance is the one with $c = A$. Then the equation (6.21) reduces to

$$\varphi'' + (1 - \alpha^2)\varphi = 0 .$$

It has the solution

$$\varphi(y) = C \sin\left\{\sqrt{1-\alpha^2}\,(y - y_1)\right\},$$

which vanishes at $y = y_1$. If $\varphi(y_2)$ is also required to vanish, we must have

$$\sqrt{1-\alpha^2}\,(y_2-y_1) = n\pi, \qquad (n = \text{integer}),$$

and hence

$$\alpha^2 = 1 - \left(\frac{n\pi}{y_2-y_1}\right)^2.$$

Thus, if $y_2 - y_1 < \pi$, there is no possible neutral disturbance; if $y_2 - y_1 = \pi$, there is the one with $\alpha = 0$; if $2\pi > y_2 - y_1 > \pi$, there is one with $\alpha \neq 0$; in general, if $(m+1)\pi > (y_2-y_1) > m\pi$, there are m neutral disturbances with $\alpha^2 \neq 0$. In the last case, there are also m points of inflection in the velocity profile. Thus, the mathematical theory of inertial instability is not yet closed.

Heisenberg's classification of velocity distributions.

Heisenberg attempted the case of flow between solid walls in relative motion with the condition that Rl(w-c) vanishes only once for $y_1 \leq y \leq y_2$ (p. 592). Regarding α^2 as small, he approximated the condition (7.3) by $K_1(c) = 0$. He then classified the profile into four classes: (i) those for which $K_1(c) = 0$ has a complex root, (ii) those for which $K_1(c) = 0$ has a real root, (iii) those for which the real part of $K_1(c)$ vanishes for a certain real value of c, (iv) those for which neither of the above three cases is true. Heisenberg concludes that the first class is unstable, the second generally unstable, the rest stable.

In discussing the validity of these conclusions, the following point must be borne in mind. If we can show that a certain type of disturbance exists for $\alpha^2 = 0$ and $\alpha R \to \infty$, it also exists for sufficiently large values of αR and sufficiently small values of α^2. However, the non-existence of a certain type of disturbance for $\alpha^2 = 0$ and $\alpha R \to \infty$ does not exclude the possibility of its existence for finite values of α^2 and αR. It appears therefore that we can only expect to conclude the instability of a velocity distribution by discussing the roots of $K_1(c) = 0$. Thus, apart from some flaws in Heisenberg's mathematical deductions, only the first two classes can have any decisive significance.

If $K_1(c)$ has a root with a positive imaginary part, the motion is unstable. If $K_1(c)$ has a real root, the motion would be unstable when the effect of viscosity is considered. This is shown by Heisenberg and will be studied more fully in a generalized form in section 11. However, if $K_1(c)$ has a root with a negative imaginary part, we cannot conclude the instability of the flow by taking the complex conjugate of $K_1(c) = 0$ (as Heisenberg did). For if $\int_C dy(w-c)^{-2} = 0$, then (cf. Fig. 5)

$$\int_{C'} dy(w-c)^{-2} = -2\pi i R_0 ,$$

where R_0 is the residue of $(w-c)^{-2}$ at y_0. In fact, $R_0 = -w_0''/w_0'^3$. Now $K_1(\bar{c})$ is the complex conjugate of $\int_{C'} dy(w-c)^{-2}$. Hence,

$$K_1(\bar{c}) = 2\pi i R_0 = -2\pi i \bar{w}_0''/\bar{w}_0'^3 ,$$

which does not vanish unless $w''_0 = 0$. Hence, the equation $K_1(c) = 0$ tells us nothing about the existence of the root $\bar{c}$ or any other root with a positive imaginary part.

Thus, Heisenberg's attempt is not as successful as Tollmien's work, which at least brings out the characteristic properties of symmetrical and boundary-layer distributions. A complete classification of velocity distributions, however, is not yet existent.

Approximation using broken linear profiles. Some investigations of Lord Rayleigh were carried out by approximating the velocity profile with straight-line segments. With this approximation, the solutions of (6.21) can be expressed in terms of elementary functions. Lord Rayleigh also tried to verify his conclusions by considering the roots of $K_1(c) = 0$, using the same approximation for velocity. However, the results of his investigations is doubtful, because the number of roots obtained for c is equal to the number of corners chosen in the approximation. This was demonstrated by Heisenberg to be inherent to the method of approximation. The general idea is as follows. As discussed above, the stability condition (7.3) may be approximated by $K_1(c) = 0$. Although Rayleigh's approximation may be made very close so far as the velocity distribution is concerned, the approximations to $(w-c)^{-2}$ is always bad in the neighborhood of the corners. Consequently, the integral $K_1(c)$ is not properly approximated. In fact, a continuous broken profile w(y) does not allow itself to be continued analytically to the complex y-plane without introducing discontinuities (cuts). It thus appears that all results deduced from the consideration of broken profiles must be regarded with caution. The same criticism applies to Tietjen's work with the viscous fluid. His analysis failed to give a minimum Reynolds number below which all small disturbances are damped out.

8. Rigorous proof and extension of Tollmien's result for the existence of unstable modes of oscillation.

In this section, we want to give a rigorous proof of the existence of amplified solutions of (6.21) satisfying the second and the third boundary conditions of (6.22) when the velocity profile $w(y)$ has a flex at $y = y_s$, i.e.,

$$w''(y_s) = 0. \tag{8.1}$$

The idea of the proof is essentially the same as that used by Tollmien, but the method is rigorous. It has the further advantage of enabling us to extend the results to cover cases where $w'''(y_s) = 0$, — a condition which has to be excluded by Tollmien.

According to previous results, the neutral disturbance must have a phase velocity c equal to

$$c_s = w_s = w(y_s). \tag{8.2}$$

Let the corresponding value of α be denoted by α_s. The essential idea of the proof is (1) to show that there exist eigen-values of c and $\alpha > 0$ in the neighborhood of the values c_s and α_s such that the imaginary part of c does not vanish, and then (2) to show that the imaginary part is actually positive. The first statement can be expected and can be readily established, if we can show that the left-hand sides of (6.18) — (6.20) are analytic functions $f(\alpha,c)$ of the two variables α and c in the neighborhoods of α_s and c_s. For if this is true, we can always solve $f(\alpha,c) = 0$ for c as an analytic function of α, (there may be more than one branch), by the implicit function theorem. Hence, there is at least one value of c

corresponding to every real value of α in the neighborhood of $\alpha = \alpha_s$. Furthermore, by (8.2), this value of c, being unequal to c_s, cannot be real, and the first part of our result is established.

To prove the analyticity of $f(\alpha, c)$ seems to be a trivial problem. Nevertheless, we shall find below that it is impossible to establish it in the neighborhood of $(\alpha, c) = (0, 0)$. The chief problem in the proof is to overcome the difficulty caused by the singular point of the differential equation (6.21).

If $w-c \neq 0$, we can write (6.21) as

$$\varphi'' - \alpha^2 \varphi - \frac{w''}{w-c} \varphi = 0. \tag{8.3}$$

Let us now consider a simply-connected region R of the y-plane which encloses the points $y = y_1$ and $y = y_2$, but excludes the point y_s, the passage from y_1 to y_2 being taken in the lower half of the y-plane. Consider also a neighborhood S of y_s, mutually exclusive with the region R. Let us regard the relation

$$c = w(y) \tag{8.4}$$

as mapping the regions R and S into two regions R' and S' of the c-plane (Fig. 6). If the mapping is one-to-one, (as can be expected if $w'(y) \neq 0$ for $y_1 < y < y_2$), these regions will also be mutually exclusive. Then, if we restrict y to R and c to S', the coefficients of (8.3) are analytic functions of the independent variable y and the parameters α and c. Hence, a fundamental system of solutions of (8.3), $\varphi_1(y; \alpha, c)$ and $\varphi_2(y; \alpha, c)$, are analytic functions of the three variables y, α, and c. We understand that y is restricted to the region R, c is restricted to the region S', while α may be in any finite region enclosing α_s. Thus, (for example),

(8.5)
$$f_2(\alpha, c) \equiv \begin{vmatrix} \varphi_1(y_1; \alpha, c) & \varphi_2(y_1; \alpha, c) \\ \varphi_1'(y_2; \alpha, c) & \varphi_2'(y_2; \alpha, c) \end{vmatrix}$$

is an analytic function of the variables α and c. Hence, the result.

We note that in the neighborhood of $(\alpha, c) = (0,0)$, the above reasoning fails. The region R (which has to enclose the point $y = y_1$) and the region S (which has to enclose the point where $w = c = 0$) cannot be taken to be mutually exclusive. In fact, $f(\alpha, c)$ presumably has a singular point at the point $c = 0$ (a logarithmic branch point). We shall discuss this case a little more at the end of this section.

Let us proceed to show that there actually exist values of $c = c(\alpha^2)$ with a positive imaginary part corresponding to positive real values of α. This is necessary because the usual argument of taking complex conjugates has been shown to be invalid. For this purpose, consider the power series $(\lambda = \alpha^2)$

(8.6)
$$c = c_s + \left(\frac{dc}{d\lambda}\right)_s (\lambda - \lambda_s) + \frac{1}{2!}\left(\frac{d^2c}{d\lambda^2}\right)_s (\lambda - \lambda_s)^2 + \cdots$$

Since λ is restricted to real values, the important point to be shown is that the first of the derivatives in (8.6) for which the imaginary part does not vanish is of an odd order. Then, by taking values of λ slightly greater or smaller than λ_s, we can always make $c_i > 0$. For these values of c and α^2, we can continue our solution $\varphi(y)$ analytically so that it is valid along the real axis between y_1 and y_2, thus obtaining an inviscid solution.

Let us now consider the equation (8.3) writing λ for α^2. We have

(8.7)
$$L(\varphi) \equiv \varphi'' - \lambda\varphi - \frac{w''}{w-c}\varphi = 0.$$

Let φ be an eigen-function with λ, c as the corresponding eigen-values.

Then

(8.8) $$L(\varphi_\lambda) \equiv \varphi_\lambda'' - \alpha^2 \varphi_\lambda - \frac{w''}{w-c}\varphi_\lambda = \left\{1 + \frac{w''}{(w-c)^2}\frac{dc}{d\lambda}\right\}\varphi,$$

where

$$\varphi_\lambda = \frac{\partial \varphi}{\partial \lambda} + \frac{\partial \varphi}{\partial c}\frac{dc}{d\lambda}.$$

We distinguish two cases: (1) the point $y = y_s$ is a simple root of $w''(y) = 0$, (2) the point $y = y_s$ is a multiple root of $w''(y) = 0$.

In the first case, $w_s''' \neq 0$. In the limit $\lambda \to \lambda_s$, $c \to c_s$ the equations (8.7) and (8.8) become

(8.9) $$L_s(\varphi_s) \equiv \varphi_s'' - \lambda_s \varphi_s - \frac{w''}{w-c_s}\varphi_s = 0.$$

(8.10) $$L_s(\varphi_{\lambda s}) \equiv \varphi_{\lambda s}'' - \lambda_s \varphi_{\lambda s} - \frac{w''}{w-c_s}\varphi_{\lambda s} = \left\{1 + \frac{w''}{(w-c_s)^2}\left(\frac{dc}{d\lambda}\right)_s\right\}\varphi_s.$$

From these, we deduce that

$$\varphi_s L_s(\varphi_{\lambda s}) - \varphi_{\lambda s} L_s(\varphi_s) = \frac{d}{dy}\left\{\varphi_s \varphi_{\lambda s}' - \varphi_{\lambda s}\varphi_s'\right\} = \left\{1 + \frac{w''}{(w-c_s)^2}\left(\frac{dc}{d\lambda}\right)_s\right\}\varphi_s^2.$$

Now, φ_λ satisfies the same boundary conditions as φ does, because those conditions are satisfied by φ for each pair of values of λ and c, and φ is an analytic function of them. Hence, integrating $\left\{\varphi_s L_s(\varphi_{\lambda s}) - \varphi_{\lambda s} L_s(\varphi_s)\right\}$ between the limits (y_1, y_2), we have

$$\left(\frac{dc}{d\lambda}\right)_s \int_{y_1}^{y_2} \frac{w''}{(w-c_s)^2}\varphi_s^2\,dy + \int_{y_1}^{y_2}\varphi_s^2\,dy = 0,$$

or

(8.11) $$\left(\frac{dc}{d\lambda}\right)_s = -\frac{\int_{y_1}^{y_2}\varphi_s^2\,dy}{\int_{y_1}^{y_2}\frac{w''}{(w-c_s)^2}\varphi_s^2\,dy}.$$

The denominator of the above expression is equal to

$$\int_{y_1-y_s}^{y_2-y_s} (w_s''' y + \tfrac{1}{2} w_s^{iv} y^2 + \cdots)(w_s' y + \tfrac{1}{2} w_s'' y^2 + \cdots)^{-2} (\varphi_{ss}^2 + 2\varphi_{ss}\varphi_{ss}' y + \cdots)\, dy$$

$$= \frac{w_s'''}{w_s'^2} \varphi_{ss}^2 \int_{y_1-y_s}^{y_2-y_s} \left\{ \frac{1}{y} + A_0 + A_1 y + \cdots \right\} dy,$$

where φ_{ss} is the value of φ_s at $y = y_s$, and A_0, A_1, A_2, ... are real. Hence, the imaginary part of the above expression is $\pi \varphi_{ss}^2 w_s''' / w_s'^2$. Since $\varphi_s(y)$ is real and φ_{ss} does not vanish,* we have arrived at the required result. The above argument is a more rigorous formulation of Tollmien's work.

In case $w''(y)$ has a multiple root at $y = y_s$, the proof of Tollmien does not hold, but the above method can still be carried through. The restriction must be made, however, that the tangent drawn through the point of inflection must actually cross the velocity curve. Then, the first of the derivatives $w^{(iv)}(y_s)$, $w^{(v)}(y_s)$, ... which does not vanish is of an odd order, and y_s is a root of $w''(y)$ of odd multiplicity. If we differentiate the equation (8.7) n times with respect to λ, we have the following equation for each value of n:

(8.12) $$L(\varphi_{\lambda^{(n)}}) = n\,\varphi_{\lambda^{(n-1)}} + \sum_{r=1}^{n} C_r^n\, r!\, \frac{w''}{(w-c)^{r+1}}\, c^{(r)} \varphi_{\lambda}^{(n-r)}.$$

Let $w''(y)$ have the root y_s up to the multiplicity $2m + 1$, $m > 0$. Then, the equation (8.10) is regular in a neighborhood of $y = y_s$, and the value of $\left(\frac{dc}{d\lambda}\right)_s$ as given by (8.11) is real. Let us consider the boundary value problem of the differential equation

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* Tollmien (67), p. 92.

(8.10), requiring $\varphi_{\lambda s}$ to satisfy the same boundary conditions as φ_s. The solution can be obtained from $\varphi_\lambda(y)$ by making $\lambda \to \lambda_s$, and is moreover real along the real axis, by a direct consideration of (8.10).

Continuing the same argument with equations of the type (8.12) with n = 2, 3, ..., 2m and $c \to c_s$, $\lambda \to \lambda_s$, we find that

$$\varphi_{\lambda^{(2)}s}\,,\ \varphi_{\lambda^{(3)}s}\,,\ \cdots,\ \varphi_{\lambda^{(2m)}s}\ ;\quad \left(\frac{d^2c}{d\lambda^2}\right)_s,\ \left(\frac{d^3c}{d\lambda^3}\right)_s,\ \cdots,\left(\frac{d^{2m}c}{d\lambda^{2m}}\right)_s$$

are all real. Finally, for n = 2m + 1, we obtain the relation

(8.13) $$(2m+1)\int_{y_1}^{y_2}\varphi_s\,\varphi_{\lambda^{(2m)}s}\,dy+\sum_{r=1}^{2m}C_r^{2m+1}\,r!\left(\frac{d^rc}{d\lambda^r}\right)_s\int_{y_1}^{y_2}\frac{w''}{(w-c)^{r+1}}\,\varphi_s\,\varphi_{\lambda^{(2m+1-r)}s}\,dy$$
$$+(2m+1)!\left(\frac{d^{2m+1}c}{d\lambda^{2m+1}}\right)_s\int_{y_1}^{y_2}\frac{w''}{(w-c)^{2m+2}}\,\varphi_s^2\,dy=0.$$

Analogous to the equation preceding (8.11), it can be easily seen that the integral $\int_{y_1}^{y_2}w''\varphi_s^2(w-c)^{-(2m+2)}dy$ has the imaginary part $\frac{w_s^{(2m+3)}\,\varphi_{ss}^2}{(2m+1)!\,(w_s')^{2m+2}}$ while the other terms in (8.13) are all real.

Thus, $\left(\frac{d^{2m+1}c}{d\lambda^{2m+1}}\right)_s$ has a non-vanishing imaginary part. This is the result desired.

The proof of the existence of amplified solutions near the neutral solution c = 0, $\alpha = 0$ cannot be so easily formulated into a rigorous form. From the solutions (4.14), it is very easy to obtain the solution φ_I which approaches the eigen-solution $\varphi = w(y)$ as $c \to 0$, $\alpha^2 \to 0$, with $\alpha^2 = O(c)$. The solution is

(8.14) $$\varphi_I=-c\,w_1'\,(w-c)\int_{y_1}^{y}(w-c)^{-2}dy\left\{1+\alpha^2\int_{y_1}^{y}dy\,(w-c)^2\int_{y_1}^{y}dy\,(w-c)^{-2}+\cdots\right\}.$$

As can be easily verified from (6.21) the condition for φ_I to be an eigenfunction is

(8.15) $$c\,w_1'+\lambda\int_{y_1}^{y_2}(w-c)\,\varphi_I\,dy=0.$$

From this, it follows that

$$\left(\frac{dc}{d\lambda}\right)_0 = \frac{1}{w_1'}\int_{y_1}^{y_2} w^2 dy, \tag{8.16}$$

and that the imaginary part of $\left(\frac{d^2c}{d\lambda^2}\right)_0$ is $2\pi\left(\frac{dc}{d\lambda}\right)_0^2 \frac{w_1''}{w_1'^2}$, which is positive if there is a flex in the velocity profile ($w''_1 > 0$). However, the real part of $\left(\frac{d^2c}{d\lambda^2}\right)_0$ becomes logarithmically infinite, and hence the argument is not rigorous. Also, it does not seem easy to make suitable modifications and extensions in case $w''(y_1) = 0$. It should be remarked that Tollmien's proof is not essentially different from the argument just given.

Similar considerations can be applied to boundary-layer profiles, and similar results can be obtained confirming and extending Tollmien's original results.

9. Physical interpretation of inertial instability.*

The fact that the instability of a two-dimensional parallel laminar flow is so closely connected with the occurrence of a point of inflection in the velocity profile demands a physical interpretation. Since the equation (6.21) is essentially the vorticity equation, we would expect $w'' = 0$ to indicate a maximum or minimum of the vorticity $-w'$ of the main flow. This is actually where the explanation is to be found.

Since we have neglected the effect of viscosity, a fluid element maintains its vorticity throughout the motion. From this point of view, a two-dimensional parallel flow may be regarded as the motion of a large number of vortex filaments under the action of each other. Filaments of equal vorticity are arranged in the same layer, and the whole flow is built up of a collection of such layers.

The following physical interpretation is based upon the fact that a fluid element is accelerated in such a field, if it is associated with an excess or a defect of vorticity. These considerations were originally developed by Professor von Kármán** for the interpretation of the failure of the simple vorticity-transfer theory of fully developed turbulence as applied to the case of parallel Couette flow. The idea is developed in greater mathematical detail here in this

* After the whole manuscript had been finished (Dec. 31, 1943), I read (Jan. 5, 1943) the two-page note of Lord Kelvin, commenting on Rayleigh's work concerning the flex. I was rather surprised to find that Kelvin had pointed out the relevant facts used in the following physical interpretations as "surprising". However, he did not make any attempt to follow the detailed mechanism.

**Cf. discussions of the vorticity transfer theory of turbulence in his general lecture at the Fourth International Congress for Applied Mechanics (16). Some developments in that direction were also made by Professor C.B. Millikan (not published).

section and the next. It will be noticed that the consideration is essentially two-dimensional, and hence is even more suitable here than for fully developed turbulence, where the fluctuations are presumably three-dimensional. An alternative interpretation of the results of Rayleigh and Tollmien, but still based upon vorticity considerations, is also given, with the purpose of demonstrating the role of the viscous forces.

Let us imagine a disturbance of the flow such that an element of fluid E_1 of the layer L_1 is interchanged with an element E_2 of a neighboring layer L_2. For definiteness, let us suppose that the layer L_2 has a higher vorticity than the layer L_1 in the undisturbed state. Since E_1 preserves its vorticity, it will appear to have a defect of vorticity when it is in L_2. Similarly, E_2 appears to have an excess of vorticity.

Let us fix our attention on one of them, say, E_2. It will be shown below that a fluid element with an excess of vorticity is accelerated in the direction of the positive y-axis with an acceleration $\frac{1}{\Gamma}\iint \{v'(x,y)\}^2 \zeta_0'(y)\,dx\,dy$, where $\zeta_0'(y)$ is the gradient of vorticity of the main flow, $v'(x,y)$ is the component of the disturbing velocity perpendicular to the direction of flow, and Γ is the total strength of the vortex filaments corresponding to the disturbance. It can thus be easily seen that E_2 is accelerated toward a region of higher vorticity, if the gradient of vorticity does not change sign anywhere in the fluid. Then, E_2 is accelerated toward L_2. A similar consideration holds for the element E_1. Thus, in either case, <u>the fluid</u>

element is returned to the layer where it belonged (by the acceleration due to its interaction with other vortex filaments). The motion is therefore stable when the gradient of vorticity does not vanish.

When there is an extremum of vorticity, an interchange of fluid elements on opposite sides of the extremem does not give rise to an excess or a defect of vorticity. Furthermore, the gradient of vorticity vanishes there, and has opposite signs on opposite sides of that layer. It can be easily seen from the above acceleration formula that the restoring force mentioned above is largely impaired in such a case. Thus, an exchange of fluid elements is not as strongly forced back by the action discussed above. Such an exchange forms a real disturbance because there is an exchange of momentum. Thus, a disturbance may tend to persist and perhaps to augment. The motion is not necessary stable.

The above discussion is based on very general considerations and does not depend on the consideration of a periodic wavy disturbance used in the mathematical analysis. We shall now support the above argument by considering a neutral wavy disturbance, with the understanding that if such a disturbance can persist, (except for the exceptional case of infinite wave-length and zero phase velocity), the motion is presumably unstable. From these considerations, the importance of viscosity in the inner friction layer will also be brought out.

Let us consider an observer moving with the phase velocity of a neutral wavy disturbance. He will observe a stationary pattern of the flow. (See Fig. 7). Closed stream lines are inevitable unless the disturbance has no v-component of velocity in the critical layer $w = c$, for the flow on opposite sides of the critical layer are in opposite directions relative to the observer. It appears unlikely that the v-component of the disturbance should be zero throughout that layer. Indeed, it has been shown to be impossible mathematically*. Thus, whenever a neutral disturbance persists, it involves a steady exchange of fluid elements on opposite sides of the critical layer.

If the effect of viscosity is to be negligible, fluid elements on the same stream line must have the same vorticity. If the gradient of vorticity of the main flow is zero or small near the critical point, it is easy to compensate this small difference of vorticity by the vorticity of the superposed flow, while the "scale" of disturbance (as measured in order of magnitude by $u' / \frac{\partial u'}{\partial y}$) remain the same as that of the main flow. It is thus not impossible to find a neutral disturbance for which the effect of viscosity is negligible. The motion may be inertially unstable.

On the other hand, if the gradient of vorticity of the main flow is finite, the superposed small disturbance must also give a finite gradient of vorticity. This means that the "scale" of the disturbance must be very small in the critical layer. The diffusion of vorticity by the effect of viscosity is then inevitable. It is thus impossible to find a neutral disturbance for which the effect of viscosity is negligible. The motion is inertially stable.

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* This follows at once from Rayleigh's original results, if we apply it to the region between this layer and the solid wall.

Acceleration of vortices in a non-uniform field of vorticity.

In the foregoing physical interpretation of inertial instability, we have considered the acceleration of an element of fluid in a two-dimensional parallel flow when this element of fluid does not have the same vorticity as befitting the layer where it belongs. We are now going to derive the explicit formula for the acceleration. The derivation shall be made in two different ways: (1) by kinematical considerations (using vorticity theorems), and (2) by considerations of the pressure gradient (section 10). In either method, we shall consider a perfect fluid in accordance with the concept of inertial instability.

Let us consider a two-dimensional flow between two solid walls, which shall be taken to be $y = \pm b$. Let the velocity components of the main flow be

(9.1) $$\begin{cases} U = w(y), \\ V = 0, \end{cases}$$

and those of the secondary flow be

(9.2) $$\begin{cases} u' = u'(x,y), \\ v' = v'(x,y). \end{cases}$$

The distribution of vorticity of the main flow is

(9.3) $$\zeta_0 = \zeta_0(y) = -w'(y),$$

and that of the secondary flow is

(9.4) $$\zeta' = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} .$$

The latter distribution shall approximate a vortex "at" the point (ξ_0, η_0). Thus, if ζ' has the same sign (opposite signs) as ζ_0, we have essentially a small element of fluid having an excess (a defect) of vorticity near the point (ξ_0, η_0).

The stream function for the secondary flow is

$$\psi'(x,y) = -\frac{1}{2\pi}\iint \zeta'(\xi,\eta)\, G(x,y;\xi,\eta)\, d\xi\, d\eta, \tag{9.5}$$

with

$$\begin{cases} u'(x,y) = -\dfrac{\partial\psi'}{\partial y} = \dfrac{1}{2\pi}\displaystyle\iint \zeta'(\xi,\eta)\frac{\partial}{\partial y} G(x,y;\xi,\eta)\, d\xi\, d\eta, \\ v'(x,y) = \dfrac{\partial\psi'}{\partial x} = -\dfrac{1}{2\pi}\displaystyle\iint \zeta'(\xi,\eta)\frac{\partial}{\partial x} G(x,y;\xi,\eta)\, d\xi\, d\eta. \end{cases} \tag{9.6}$$

In these expressions, the integrals are extended over the whole region between the planes. The function $G(x,y;\xi,\eta)$ is the Green's function of the first kind for the region under consideration. It is defined by the following conditions:

$$\begin{cases} \dfrac{\partial^2 G}{\partial x^2} + \dfrac{\partial^2 G}{\partial y^2} = 0 & \text{except at } (\xi,\eta), \\ G(x,y;\xi,\eta) \sim -\log\{(x-\xi)^2+(y-\eta)^2\}^{\frac{1}{2}} & \text{near } (\xi,\eta), \\ G(x,y;\xi,\eta) = 0 & \text{over the solid boundaries.} \end{cases} \tag{9.7}$$

As is well-known, it has the reciprocity property

$$G(x,y;\xi,\eta) = G(\xi,\eta;x,y). \tag{9.8}$$

For the case of a channel, it is given by the real part of

$$f(z) = -\left\{\log\sinh\frac{\pi}{4b}(z-z_0) - \log\cosh\frac{\pi}{4b}(z-\bar{z}_0)\right\},\quad z = x+iy,\ z_0=\xi+i\eta,\ \bar{z}_0=\xi-i\eta. \tag{9.9}$$

Let us now consider the behavior of a particular element of fluid at (ξ, η) having an excess (a defect) of vorticity corresponding to the secondary flow (9.5) and (9.6). It causes a distortion of the main vorticity distribution as indicated in the figure (Fig. 8). After a very small interval of time δt the vorticity at the point (x,y) is changed by the amount

$$\delta\zeta(x,y) = -v'(x,y)\,\delta t\cdot\zeta_0'(y), \tag{9.10}$$

because it is replaced by a fluid element from below, which retains its original vorticity. This change produces an effect at the "vortex", i.e., at the element of fluid under consideration. It can be easily seen that the effect is a small velocity with components

$$\left\{\begin{aligned} \delta u(\xi,\eta) &= -\frac{1}{2\pi}\iint \frac{\partial}{\partial\eta} G(x,y;\xi,\eta)\,\delta\zeta(x,y)\,dx\,dy, \\ \delta v(\xi,\eta) &= \frac{1}{2\pi}\iint \frac{\partial}{\partial\xi} G(x,y;\xi,\eta)\,\delta\zeta(x,y)\,dx\,dy, \end{aligned}\right. \tag{9.11}$$

the integrals being extended over the whole region between the planes. Dividing these quantities by δt and passing to the limit $\delta t\to 0$, we have the following components of acceleration at the point (ξ,η) :

$$\left\{\begin{aligned} a_x(\xi,\eta) &= -\frac{1}{2\pi}\iint \frac{\partial}{\partial\eta} G(x,y;\xi,\eta)\,v'(x,y)\,\zeta_0'(y)\,dx\,dy, \\ a_y(\xi,\eta) &= \frac{1}{2\pi}\iint \frac{\partial}{\partial\xi} G(x,y;\xi,\eta)\,v'(x,y)\,\zeta_0'(y)\,dx\,dy. \end{aligned}\right. \tag{9.12}$$

Let us first consider the y-component of this acceleration. From the special form in which x and ξ enter into the Green's function (cf. (9.9), we can also write

$$a_y(\xi,\eta) = -\frac{1}{2\pi}\iint v'(x,y)\,\zeta_0'(y)\,\frac{\partial}{\partial x} G(x,y;\xi,\eta)\,dx\,dy. \tag{9.13}$$

If we multiply this equation by $\zeta'(\xi,\eta)$ and integrate over the whole region, we have the final formula

(9.14) $$\iint a_y(\xi,\eta)\,\zeta'(\xi,\eta)\,d\xi\,d\eta = \iint \{v'(x,y)\}^2\,\zeta_0'(y)\,dx\,dy,$$

upon using (9.6). Before discussing its significance, let us first notice that

(9.15) $$\iint a_x(\xi,\eta)\,\zeta'(\xi,\eta)\,d\xi\,d\eta = 0,$$

if $\zeta'(\xi,\eta)$ is an even function of $\xi-\xi_0$, i.e., if the vorticity distribution $\zeta'(\xi,\eta)$ has a symmetry about the line $\xi=\xi_0$. For then $v'(x,y)$ is an odd function of $(x-\xi)$ and $a_x(\xi,\eta)$ is an odd function of $\xi-\xi_0$, because $G(x,y;\xi,\eta)$ is an even function of $x-\xi$. Hence, the conclusion.

If we recall that the vorticity $\zeta'(\xi,\eta)$ is spread over a small region, we may take $\Gamma=\iint\zeta'(\xi,\eta)\,d\xi\,d\eta$ as the strength of the "superposed vortex". If we divide the left-hand side of (9.14) and (9.15) by Γ, we may consider the results as giving the components of the "average acceleration". The x-component of acceleration vanishes; the sign of the y-component depends upon the sign of the superposed vortex and the sign of $\zeta_0'(y)$. This component of acceleration is the one used in the above physical considerations.

It should be mentioned that in considering the stability of a motion we are having a vortex pair. Although this makes it difficult to obtain a compact formula for the average accelerations of the individual vortices, a kinematical consideration as that given above shows that the general tendency is not changed. Furthermore, the two vortices will soon be separated, because they are situated in layers of different mean velocity.

Another point should be mentioned. If we notice the tendency for the main vorticity to be swung around the secondary vortex, there is an acceleration of every element of fluid toward the vortex. Whatever this acceleration may be, it is expected to be of minor importance, because the effect is spread out over the whole field. This point will be brought out clearly in the following section, where we shall study the whole phenomenon from the point of view of pressure forces. The acceleration will be identified with the negative of the pressure gradient divided by the density of the fluid, because the effect of viscosity has been neglected. In particular, the formula (9.14) will be verified.

10. Pressure forces correlated with vorticity fluctuations.

The differential equation for pressure. To calculate the pressure distribution from a given velocity distribution, we use a differential equation for pressure of Poisson's type obtained by taking the divergence of the equation of motion. Thus, if the equations of motion are*

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} \qquad (i=1,2,3), \tag{10.1}$$

and the equation of continuity is

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{10.2}$$

We have

$$\frac{\partial^2}{\partial x_i^2}\left(\frac{p}{\rho}\right) = -\sigma, \tag{10.3}$$

where

$$\sigma = \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} = e_{ij}e_{ij} - \omega_{ij}\omega_{ij} = 2\left\{\frac{\partial(u_1,u_2)}{\partial(x_1,x_2)} + \frac{\partial(u_2,u_3)}{\partial(x_2,x_3)} + \frac{\partial(u_3,u_1)}{\partial(x_3,x_1)}\right\}, \tag{10.4}$$

and

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad \omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right), \tag{10.5}$$

are the components of deformation and of rotation. If we integrate (10.3) under proper boundary conditions, σ being known at the <u>initial</u> instant, we obtain the <u>initial</u> distribution of pressure. The initial acceleration field is then obtained from (10.1) as the negative gradient of pressure, if we neglect the effect of viscosity.

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* The usual notation is used: x_i $(i=1,2,3)$ are the coordinates, u_i are the components of velocity, p is the pressure, and ρ is the density of the fluid. Summation over a repeated index is understood. For a discussion of this type, see Lichtenstein's book (ref. 104).

<u>Boundary conditions.</u> If we neglect the effect of viscosity, the boundary condition for solid walls is

(10.6) $$u_i n_i = 0,$$

where n_i is the outward-drawn normal of the boundary surface. If we multiply (10.1) with n_i, neglecting the effect of viscosity, we have

(10.7) $$-\frac{1}{\rho}\frac{\partial p}{\partial n} = V_0 \frac{\partial u_i}{\partial s} n_i ,$$

where V_0 is the velocity along a stream line on the boundary, and ds is an element of its arc. If we write

$$u_i = V_0 l_i , \qquad \frac{\partial u_i}{\partial s} = V_0 \frac{\partial l_i}{\partial s} + l_i \frac{\partial V_0}{\partial s} ,$$

where l_i are the direction cosines of the velocity over the boundary surface, we have

(10.8) $$-\frac{1}{\rho}\frac{\partial p}{\partial n} = \frac{V_0^2}{R} ,$$

where R is the radius of curvature of the stream line, $\frac{1}{R} = n_i \frac{\partial l_i}{\partial s}$. This relation expresses the balance of pressure and centrifugal force. With a given distribution of velocity, the right-hand side is known. We have thus a potential problem of the <u>second</u> kind for the pressure.

<u>Two-dimensional flow between solid walls.</u> Returning to the problem at hand, we have the very simple boundary condition

(10.9) $$\frac{\partial p}{\partial y} = 0 \quad \text{at} \quad y = \pm b .$$

Since the main motion is a two-dimensional parallel motion, we have

(10.10) $$u_1 = w(y) + u'(x,y), \; u_2 = v'(x,y), \; u_3 = 0,$$

where w(y) represents the main flow, and u' and v' give a secondary flow approximating a vortex. The equation (10.3) becomes

(10.11) $$\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)=-\sigma, \qquad \sigma = 2\left\{w'(y)\frac{\partial v'}{\partial x}-\frac{\partial(u',v')}{\partial(x,y)}\right\}.$$

We note that σ can be separated into two parts, $\sigma=\sigma_1+\sigma_2$,

(10.12) $$\sigma_1=-2\frac{\partial(u',v')}{\partial(x,y)}, \quad \sigma_2=-2\zeta_0\frac{\partial v'}{\partial x},$$

where σ_1 depends upon the structure of the secondary vortex itself, and σ_2 depends upon its interaction with the main flow. We shall also separate the pressure into two parts and require them to satisfy (10.9) separately:

(10.13) $$\begin{cases} p = p_1 + p_2, \qquad \sigma=\sigma_1+\sigma_2, \\ \frac{1}{\rho}\left(\frac{\partial^2 p_1}{\partial x^2}+\frac{\partial^2 p_1}{\partial y^2}\right)=-\sigma_1, \qquad \frac{1}{\rho}\left(\frac{\partial^2 p_2}{\partial x^2}+\frac{\partial^2 p_2}{\partial y^2}\right)=-\sigma_2, \\ \frac{\partial p_1}{\partial y}=0, \quad \frac{\partial p_2}{\partial y}=0, \qquad \text{at } y=\pm b. \end{cases}$$

We can reduce our problem to that of the <u>first</u> kind by looking for the acceleration $a_y(x,y)$ in the y-direction, $a_y = -\frac{1}{\rho}\frac{\partial p}{\partial y}$. If we differentiate (10.13) with respect to y, we have

(10.14) $$\begin{cases} a_y = \alpha_1+\alpha_2 \\ \frac{\partial^2\alpha_1}{\partial x^2}+\frac{\partial^2\alpha_1}{\partial y^2}=\frac{\partial\sigma_1}{\partial y}, \qquad \frac{\partial^2\alpha_2}{\partial x^2}+\frac{\partial^2\alpha_2}{\partial y^2}=\frac{\partial\sigma_2}{\partial y}, \\ \alpha_1=0, \quad \alpha_2=0 \qquad \text{at } y=\pm b. \end{cases}$$

with

The x-component of acceleration is zero from symmetry considerations.

<u>The field of acceleration.</u> The solutions of (10.14) are

$$(10.15)\qquad \begin{cases} \alpha_1(x,y) = -\frac{1}{2\pi}\iint G(x,y;\xi,\eta)\left(\frac{\partial\sigma_1}{\partial y}\right)_{\xi,\eta} d\xi\, d\eta, \\ \alpha_2(x,y) = -\frac{1}{2\pi}\iint G(x,y;\xi,\eta)\left(\frac{\partial\sigma_2}{\partial y}\right)_{\xi,\eta} d\xi\, d\eta, \end{cases}$$

where the integrals are extended over the whole region in between the planes. These formulae give the distribution of acceleration. Actually, it is more convenient to deal with the integrated quantities

$$(10.16)\qquad I_1 = \iint \alpha_1(x,y)\,\zeta_0(x,y)\,dx\,dy,$$

$$(10.17)\qquad I_2 = \iint \alpha_2(x,y)\,\zeta_0(x,y)\,dx\,dy,$$

$$(10.18)\qquad J_1 = \iint \alpha_1(x,y)\,\zeta'(x,y)\,dx\,dy,$$

$$(10.19)\qquad J_2 = \iint \alpha_2(x,y)\,\zeta'(x,y)\,dx\,dy.$$

The first two integrals correspond to the accelerations of the main flow by the secondary flow itself and by the interaction; the latter two quantities correspond to the accelerations of the secondary flow by itself and by the interaction. It can be verified (as will be done presently) that

$$(10.20)\qquad I_1 = -\iint v'^2 \frac{d\zeta_0}{dy}\,dx\,dy,$$

$$(10.21)\qquad I_2 = 0,$$

$$(10.22)\qquad J_1 = 0,$$

$$(10.23)\qquad J_2 = \iint v'^2 \frac{d\zeta_0}{dy}\,dx\,dy.$$

We note that (10.23) is essentially a reproduction of the formula (9.14), whose significance has been discussed in the last section.

The integral I_1 has a value equal and opposite to J_2. This is the above-mentioned acceleration, which is distributed among the fluid elements throughout the field. It is therefore relatively unimportant. Thus, all the statements made in the last section have been verified, if we can verify the equations (10.20) - (10.23).

<u>Verification of (10.20) - (10.23).</u> To verify these equations, let us first examine the behavior of the quantities u', v', $\frac{\partial p}{\partial x}$, $\frac{\partial p}{\partial y}$ for large values of x. From the expression (9.9) for the Green's function, we see that if ζ' vanishes sufficiently rapidly as x becomes infinite, we have

(10.24) $$u' = O\left(|x|^{-3}\right), \qquad v' = O\left(|x|^{-2}\right),$$

for large values of x. From the equations of motion, we then find that

(10.25) $$\frac{\partial p}{\partial x} = O(u') = O\left(|x|^{-3}\right), \quad \frac{\partial p}{\partial y} = O(v') = O\left(|x|^{-2}\right).$$

This will assure the convergence of the integrals involved and the validity of the steps taken in the following transformations.

In section 9, we have been mainly concerned with J_2. So, let us consider it first. Referring to (10.15) and (9.5), we see that

$$J_2 = \iint \psi'(\xi,\eta)\left(\frac{\partial\sigma_2}{\partial y}\right)_{\xi,\eta} d\xi\, d\eta .$$

If we now introduce the value of σ_2 as given by (10.12) and replace (ξ, η) by (x,y), we have

$$J_2 = -2\iint \psi'(x,y)\frac{\partial^2}{\partial x\,\partial y}(v'\zeta_0)\,dx\,dy .$$

On integrating by parts with respect to x,

$$J_2 = 2\iint v'\frac{\partial}{\partial y}(v'\zeta_0)\,dx\,dy = \iint\left\{\frac{\partial}{\partial y}(\zeta_0 v'^2) + v'^2\frac{d\zeta_0}{dy}\right\}dx\,dy.$$

The result (10.23) or (9.14) is thereby verified.

Similar calculations can be carried out for the integrals I_1, I_2, and J_1. Thus,

$$I_1 = \iint \alpha_1\zeta_0\,dx\,dy = \iint w(y)\frac{\partial\alpha_1}{\partial y}\,dx\,dy$$

$$= \iint w\left(\sigma_1 + \frac{1}{\rho}\frac{\partial^2 p_1}{\partial x^2}\right)dx\,dy, \qquad \text{by (10.13).}$$

If we note that

$$w\sigma_1 = -2w\frac{\partial(u',v')}{\partial(x,y)} = 2\frac{\partial}{\partial x}\left(wv'\frac{\partial u'}{\partial y}\right) - 2\frac{\partial}{\partial y}\left(wv'\frac{\partial u'}{\partial x}\right) - \frac{dw}{dy}\frac{\partial v'^2}{\partial y},$$

the above integral is easily transformed into the form (10.20). Following an exactly analogous process, we have

$$I_2 = \iint w\left\{\sigma_2 + \frac{1}{\rho}\frac{\partial^2 p_2}{\partial x^2}\right\}dx\,dy = 0,$$

when we make use of (10.12). The integral J_1 has also the significance that it is the effect of the solid boundaries upon a general flow consistent with (10.25). Using (10.15), (9.5), and (10.12), we have

$$J_1 = \iint \alpha_1\zeta'\,dx\,dy = \iint \psi'(\xi,\eta)\left(\frac{\partial\sigma_1}{\partial y}\right)_{\xi,\eta} d\xi\,d\eta$$

$$= -2\iint u'\frac{\partial(u',v')}{\partial(x,y)}\,dx\,dy.$$

If we note that

$$u' \frac{\partial(u', v')}{\partial(x, y)} = \frac{\partial}{\partial y}\left(u'v' \frac{\partial u'}{\partial x}\right) - \frac{\partial}{\partial x}\left(u'v' \frac{\partial u'}{\partial y}\right),$$

we see that $J_1 = 0$. The results (10.20) - (10.23) are thereby verified. We have thus completed the investigations indicated at the end of last section.

PART III STABILITY IN A VISCOUS FLUID

11. General considerations; Heisenberg's criterion.

The foregoing inviscid investigations of the stability characteristics of a flow lead to very useful information. In the first place, they enable us to visualize the effect of pressure forces very clearly. In the second place, the results can be used as a guide for attacking the stability problem in a real fluid, because instability is expected to occur only for sufficiently large Reynolds numbers. Thus, if we know the general characteristics of inertial stability for a given velocity distribution, some stability characteristics in a real fluid can be obtained by considering a modification of these results by the effect of viscosity. Such a consideration was first made by Heisenberg.[12] He demonstrated that the effect of viscosity is generally destabilizing at very large Reynolds numbers. There are, however, a few minor points to be supplemented in his discussion. We shall therefore discuss this problem in some detail in this section.

To get a better understanding of the stability problem, we would like to know the general shape of the limit curve of neutral stability, i.e., the curve $c_i(\alpha,R) = 0$. In fact, we would like to know the lowest Reynolds number, below which all small disturbances are damped out. These questions will be answered in section 12, where asymptotic forms (for large R) of the two branches of $\alpha(R)$ curve obtained by solving $c_i(\alpha,R) = 0$ are obtained. By combining these results with a condition of stability given by Synge, a knowledge of the general shape can be deduced. A simple rule is also derived by which the minimum critical Reynolds number can be very easily obtained from quantities involving very simple integral and differential expressions of $w(y)$. Very little numerical labor is involved for the calculation in any particular case.

Heisenberg also discussed the general shape of the curve $c_i(\alpha, R) = 0$. However, his argument does not appear to be very decisive, and some of his results are not well stated. He did not obtain the asymptotic forms of the $\alpha(R)$ curve as given below. He also tried to estimate the order of magnitude of the critical Reynolds number, but seemed to have given up any hope of making an approximate calculation in terms of simple differential and integral expressions of $w(y)$. (Loc. cit., p. 600).

After such general considerations, we shall apply our theory to the study of the stability of special velocity distributions, in order to obtain definite numerical results. In section 13, we shall calculate the neutral curve for two cases: (i) the Blasius case, (ii) the Poiseuille case. The significance of the results obtained, physical interpretations of the effect of viscosity, and prospects of further developments will be discussed in section 14.

<u>Heisenberg's criterion.</u> Let us restrict ourselves to cases where α and c do not approach zero as R goes to infinity along the neutral curve. Then, the approximations (6.29) are valid for sufficiently large values of R. By using (6.26), (6.24) and (6.29),

we can transform (6.13) into the following:

$$\sum_{n=0}^{\infty} \alpha^{2n} K_{2n+1}(c) = \frac{e^{-\frac{\pi}{4}i}}{\sqrt{\alpha R(1-c)^5}} \left\{ \sum_{n=0}^{\infty} \alpha^{2n} K_{2n}(c) + w_2'(1-c) \sum_{n=0}^{\infty} \alpha^{2n} K_{2n+1}(c) \right\}$$

$$+ \frac{e^{\frac{\pi}{4}i}}{\sqrt{\alpha R c^5}} \left\{ \sum_{n=0}^{\infty} \alpha^{2n} H_{2n}(c) + w_1' c \sum_{n=0}^{\infty} \alpha^{2n} K_{2n+1}(c) \right\}.$$

If we know that the motion is neutrally stable for $c = c_s$, $\alpha = \alpha_s$, we have

$$\sum_{n=0}^{\infty} \alpha_s^{2n} K_{2n+1}(c_s) = 0.$$

Hence, when $\alpha = \alpha_s$, and $c = c_s + \Delta c$, differing but slightly from c_s, we have

$$\text{(11.1)} \qquad \Delta c = \frac{e^{-\frac{\pi}{4}i}}{\sqrt{\alpha_s R(1-c_s)^5}} \frac{\sum_{n=0}^{\infty} \alpha_s^{2n} K_{2n}(c_s)}{\sum_{n=0}^{\infty} \alpha_s^{2n} K'_{2n+1}(c_s)} + \frac{e^{\frac{\pi}{4}i}}{\sqrt{\alpha_s R c_s^5}} \frac{\sum_{n=0}^{\infty} \alpha_s^{2n} H_{2n}(c_s)}{\sum_{n=0}^{\infty} \alpha_s^{2n} K'_{2n+1}(c_s)}.$$

Similar considerations of (6.14), (6.15), and (6.16) give respectively

$$\text{(11.2)} \qquad \Delta c = \frac{e^{\frac{\pi}{4}i}}{\sqrt{\alpha_s R c_s^5}} \frac{\sum_{n=0}^{\infty} \alpha_s^{2n} H_{2n+1}(c_s)}{\sum_{n=0}^{\infty} \alpha_s^{2n} K'_{2n+2}(c_s)},$$

$$\text{(11.3)} \qquad \Delta c = \frac{e^{\frac{\pi}{4}i}}{\sqrt{\alpha_s R c_s^5}} \frac{\sum_{n=0}^{\infty} \alpha_s^{2n} H_{2n}(c_s)}{\sum_{n=0}^{\infty} \alpha_s^{2n} K'_{2n+1}(c_s)},$$

$$\text{(11.4)} \qquad \Delta c = \frac{e^{\frac{\pi}{4}i}}{\sqrt{\alpha_s R c_s^5}} \frac{\sum_{n=1}^{\infty} \alpha_s^{2n} H_{2n-1}(c_s) + (1-c_s)^2 \sum_{n=0}^{\infty} \alpha_s^{2n+1} H_{2n}(c_s)}{\sum_{n=1}^{\infty} \alpha_s^{2n} K'_{2n}(c_s) + (1-c_s)^2 \sum_{n=0}^{\infty} \alpha_s^{2n+1} K'_{2n+1}(c_s)}.$$

In general, it is not very easy to determine whether $\Delta c_i > 0$ or < 0. However, when c_s and α_s are both small, but not zero, all the above equations reduce to

$$\text{(11.5)} \qquad \Delta c = \frac{e^{\frac{\pi}{4}i}}{\sqrt{\alpha_s R c_s^5}} \frac{1}{K_1'(c_s)},$$

after we make use of the reductions corresponding to (6.27) and (6.28). By using the explicit expression for $K_1(c)$ (equation between (7.1) and (7.2)), it can be easily verified that $K'_1(c)$ is approximately real and positive for small real values of c. Hence, in every case, $\Delta c_i > 0$. The disturbance with wave length $2\pi/\alpha_s$, neutrally stable in the inviscid case, is unstable when viscous forces are considered. This result was first obtained by Heisenberg, and may be formulated as follows:

Heisenberg's criterion. If a velocity profile has an "inviscid" neutral disturbance with non-vanishing wave number and phase velocity, the disturbance with the same wave number is unstable in the real fluid when the Reynolds number is sufficiently large.

In Heisenberg's original discussion, only the first type of motion is considered. His last equation on p. 597 of his paper is essentially our equation (11.1) with all the terms in α^2 dropped. Evidently, the above arguments hold only for α_s, $c_s \neq 0$. This was not explicitly stated by Heisenberg. It will be seen later (cf. fig. 7) that if the disturbance with $\alpha = 0$ has $c = 0$ for infinite R, then it has $c_i < 0$ for finite values of R.

Method of numerical calculation. To carry out the numerical calculation for any particular case, the following transformations of (6.14), (6.15), and (6.17) are convenient. The calculation of (6.13) is more complicated, but it is found possible to take (6.15) as a first approximation. Since we are not going to make any actual calculation for this case, we shall not go into further details with (6.13). All the other equations can be transformed into the form

(11.6) $$\mathfrak{F}(z) = \frac{(1+\lambda)(u+iv)}{1+\lambda(u+iv)}$$

with $\mathfrak{F}(z)$ defined by (6.32) and $\lambda = \lambda\,(c)$ defined by

(11.7) $$w_1'(y_1-y_0) = -c\,(1+\lambda),$$

so that it is usually very small. The quantities u and v are real functions of α and c, different for different cases. For (6.14), we have

(11.8) $$\begin{aligned} u+iv &= 1+w_1'c\,\varphi_{22}'/\varphi_{12}' \\ &= w_1'c\left(K_1+\frac{1}{w_1'c}\right)+\frac{w_1'c}{\alpha^2}(1-\alpha^2H_2)(1-\alpha^2H_2-\alpha^4N_4-\cdots)(H_1-\alpha^2M_3-\alpha^4M_5-\cdots)^{-1}, \end{aligned}$$

the second being derived by using (6.27). Similarly, for (6.15) and (6.17), we have, respectively,

(11.9) $$\begin{aligned} u+iv &= 1+w_1'c\,\varphi_{22}/\varphi_{12} \\ &= w_1'c\left(K_1+\frac{1}{w_1'c}\right)\mp w_1'c\,\alpha^2(1-\alpha^2H_2)(1-\alpha^4M_4-\alpha^6M_6+\cdots)^{-1}(N_3+\alpha^2N_5+\cdots), \end{aligned}$$

and

(11.10) $$\begin{aligned} u+iv &= 1+w_1'c\,(\varphi_{22}'+\alpha\varphi_{22})/(\varphi_{12}'+\alpha\varphi_{12}) \\ &= w_1'c\left(K_1+\frac{1}{w_1'c}\right)+\frac{w_1'c}{\alpha}\,\frac{(1-\alpha^2H_2)\{1-\alpha^2H_2-\alpha^4N_4-\cdots-(1-c)^2(\alpha^3N_3+\alpha^5N_5+\cdots)\}}{(1-c)^2(1-\alpha^4M_4-\alpha^6M_6+\cdots)+\alpha(H_1-\alpha^2M_3+\cdots)}. \end{aligned}$$

The equation (11.6) contains the two real equations

(11.11) $$\begin{cases} \mathfrak{F}_r(z) = (1+\lambda)\{u(1+\lambda u)+\lambda v^2\}\{(1+\lambda u)^2+(\lambda v)^2\}^{-1}, \\ \mathfrak{F}_i(z) = (1+\lambda)\,v\,\{(1+\lambda u)^2+(\lambda v)^2\}^{-1}, \end{cases}$$

for z, α, c. Thus, for each value of z, we can determine corresponding values of α and c. Finally, the Reynolds number is given by (cf. (6.30))

(11.12) $$\alpha R = \frac{1}{w_0'(1+\lambda)^3}\left(\frac{w_1' z}{c}\right)^3 .$$

The actual procedure of calculation will be described in the appendix; the numerical results for the two particular cases of Poiseuille and Blasius will be given in section 13. Before going to these details, let us first discuss some general characteristics of the neutral α(R) curve.

12. General characteristics of the curve of neutral stability.

The purpose of the investigation in this section is as follows. If we can show that a real $\alpha(R)$ curve exists for real values of c, we may conclude that the motion is unstable with respect to disturbances of certain wave lengths, because such a curve may be expected to separate regions of stability from regions of instability in the $\alpha - R$ plane. It is therefore desirable to have some general theorems regarding the existence of such a curve, and still better to know its general shape in case it exists. Such problems will now be solved for velocity distributions of the symmetrical type and of the boundary-layer type. For each of these cases, we first establish the asymptotic behavior of that curve for large values of R. Then, by using a sufficient condition for stability in a form given by Synge, the general shape of the neutral $\alpha(R)$ curve is established. A simple rule is then given by which the minimum critical Reynolds number can be approximately calculated for each of these cases.

Asymptotic behavior of the neutral $\alpha(R)$ curve. For large values of αR, we would generally expect z of (6.30) to be $\gg 1$, but it is also possible for z to approach a finite value or zero. We shall therefore discuss both possibilities.

For large values of z, we have approximately,

$$(12.1) \qquad \mathfrak{F}_r = 1, \quad \mathfrak{F}_i = \frac{1}{\sqrt{2z^3}} = \frac{w_1'}{\sqrt{2\alpha R c^3}},$$

where $\mathcal{F}_i$ is small. If we refer to (11.8) - (11.10), we see that the imaginary part v depends on that of $w'_1 \; c \; K_1(c)$ and those of the integrals H_1, H_2, M's and N's. If α and c are small, which will be verified <u>a posteriori</u> in certain cases, and assumed in others, we have only the contribution from the first term; thus,*

$$v = -\pi w_1' \frac{w w''}{w'^3}, \qquad \text{for } w = c. \tag{12.2}$$

By using (12.1) and (12.2), we see that the two equations of (11.11) can be approximated by

$$u = 1, \tag{12.3}$$

and

$$v = -\frac{\pi}{w_1'^2} c w_0'' = \frac{w_1'}{\sqrt{2\alpha R c^3}}. \tag{12.4}$$

These are the equations for determining a relation $\alpha(R)$, if we eliminate c between them.

In the case where $\alpha R c^3$ approaches a finite limit as $\alpha R \to \infty$, c must approach zero. Hence, from (11.11), $\mathcal{F}_i$ must also approach zero. From the curve for $\mathcal{F}_i(z)$, we see that this happens for z = 2.30, for which $\mathcal{F}_r = 2.28$. Then, using (11.12), (12.1), and (12.2), we have

$$\alpha R = \frac{w_1'^2 z^3}{c^3}, \qquad z = 2.30, \tag{12.5}$$

and

$$u = \mathcal{F}_r = 2.28. \tag{12.6}$$

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* In fact, the other terms never give considerable contributions to the imaginary part even for only moderately small values of α and c. This point will be discussed in the Appendix. The approximation (12.2) will be used for all later calculations.

The two types of relations (12.3), (12.4) and (12.5), (12.6) evidently correspond to two different branches of the $\alpha(R)$ curve. These conditions can be satisfied in cases (2a) and (3) of section 6 (cf. (11.8), (11.10)), but it appears to be difficult in case (2b) (cf. (11.9)).

Case (2a) Symmetrical velocity distribution with symmetrical $\varphi(y)$. We consider the cases where both α and c are small. By using (11.8) and noting that u takes on a finite value in either (12.3) or (12.6), we see that we must have

(12.7) $$u = \frac{w_1' c}{H_{10}\alpha^2}, \qquad H_{10} = H_1(0) = \int_{y_1}^{y_2} w^2\,dy,$$

i.e., c must approach zero as fast as α^2. The asymptotic behavior of the $\alpha(R)$ curves as given by (12.3)-(12.6) are as follows:

(12.8) $$R = \frac{w_1'^{\,11}}{2\pi^2 H_{10}^5 w_0''^{\,2}}\,\alpha^{-11}, \qquad c = \frac{H_{10}}{w_1'}\alpha^2, \qquad \text{(first branch)}$$

(12.9) $$R = \frac{w_1'^{\,5}}{H_{10}^3}\left(\frac{z}{\mathcal{F}_r}\right)^3 \alpha^{-7}, \qquad c = \frac{H_{10}}{w_1'}\mathcal{F}_r\,\alpha^2, \qquad \text{(second branch)},$$

where $\mathcal{F}_r$ = 2.28, z = 2.30 approximately. The first equation should be compared with (7.2).

Case (3) Boundary-layer profile. Here, the equation corresponding to (12.7) is (cf. (11.10))

(12.10) $$u = \frac{w_1' c}{\alpha},$$

i.e., c must approach zero as fast as α. Note that in the previous case, the relation (12.7) depends both on w'_1 and on $\int_{y_1}^{y_2} w^2\,dy$. Here, it depends only on the initial slope of the velocity curve w'_1.

The two branches of the $\alpha(R)$ curve for large values of R are

$$(12.11) \qquad R = \frac{w_1'''}{2\pi^2 w_0''^2}\,\alpha^{-6}, \qquad c = \frac{\alpha}{w_1'}, \qquad \text{(first branch)},$$

$$(12.12) \qquad R = w_1'^5 \left(\frac{z}{\mathcal{F}_r}\right)^3 \alpha^{-4}, \qquad c = \frac{\alpha}{w_1'}\,\mathcal{F}_r, \qquad \text{(second branch)},$$

where $\mathcal{F}_r = 2.28$, $z = 2.30$ approximately. The equation (12.11) should be compared with the second equation of (7.2).

<u>Effect of varying curvature in the curve of velocity distribution.</u> In either case, the second branch of our asymptotic curve depends very little upon the shape of the velocity profile, while the first branch depends very much upon it through the term w''_0. This fact will enable us to correlate our present results with the inviscid investigations of Rayleigh and Tollmien, as discussed in Part II.

In all the cases considered, we have $w'' < 0$ for $y < y_2$ but sufficiently near to it. If $w''(y)$ never vanishes for $y_1 < y < y_2$, we can replace w''_0 by w''_1 in the expressions (12.8) and (12.11). In general,

$$w_0'' = w_1'' + \frac{w_1'''}{w_1'}\,c + \left(\frac{w^{iv}}{2w_1'^2} - \frac{w_1''w_1'''}{2w_1'^3}\right)c^2 + \cdots$$

Now, for a flow which is essentially parallel, the boundary condition $\Delta\Delta\psi = 0$, which holds on the solid wall for all two-dimensional laminar flows, can be easily verified to be equivalent to $w'''_1 = 0$. Hence, we have

$$w_0'' = w_1'' + \frac{w_1^{iv}}{2w_1'^2}\,c^2 + \cdots .$$

Thus, if $w''_1 = 0$, but w'' does not vanish for $y_1 < y < y_2$, we have

(12.13) $$R = \frac{2w_1'^{19}}{\pi^2 H_{10}^9 (w_1^{iv})^2} \alpha^{-19}, \qquad \text{(for case (2a))},$$

(12.14) $$R = \frac{2w_1'^{19}}{\pi^2 (w_1^{iv})^2} \alpha^{-10}, \qquad \text{(for case (3))}.$$

In case the velocity profile shows a point of inflection,

$$w''_0 = 0 \quad \text{for } c = c_s.$$

Then, approximately,

(12.15) $$w_0'' = \frac{w_1^{iv}}{2w_1'^2}(c^2 - c_s^2)$$

We have, instead of (12.13) and (12.14),

(12.16) $$R = \frac{2w_1'^{19}}{\pi^2 H_{10}^9 (w_1^{iv})^2} \alpha^{-11} (\alpha^4 - \alpha_s^4)^{-2},$$

(12.17) $$R = \frac{2w_1'^{19}}{\pi^2 (w_1^{iv})^2} \alpha^{-6} (\alpha^2 - \alpha_s^2)^{-2}.$$

In all these approximations, we assume α_s and c_s to be so small that the previous approximations still hold. Thus, for either a symmetrical or a boundary-layer distribution with a flex, we have

(12.18) $$R \sim (\alpha - \alpha_s)^{-2}$$

as $R \to \infty$, $\alpha \to \alpha_s$, $c \to c_s$. This can be seen by using (12.15) in (12.8) and (12.11). The general characteristics obtained from our foregoing discussions are summarized in Table II, and are indicated by the solid lines in Fig. 9. Let us proceed to discuss the significance of these results.

Table II

Behavior of $R(\alpha)$ Curve for Large Values of R for Velocity Distributions With $w'' < 0$ for the Main Part of the Profile

	First branch			second branch
	$w''_1 < 0$	$w''_1 = 0$	$w''_1 > 0$	
Symmetrical profile	α^{-11}	α^{-19}	$(\alpha-\alpha_s)^{-2}$	α^{-7}
Boundary-layer profile	α^{-6}	α^{-10}	$(\alpha-\alpha_s)^{-2}$	α^{-4}

(i) It may be expected that the region between the two asymptotic branches of the curves represents a region of instability. Thus, every symmetrical or boundary-layer profile is unstable for sufficiently large values of the Reynolds number. This point will be substantiated below.

(ii) In the cases where $w''_1 > 0$, the two branches of curves approach the two different asymptotes $\alpha = 0$ and $\alpha = \alpha_s$, leaving a finite instable region for infinite Reynolds number. In the other cases, the two branches approach the same asymptote $\alpha = 0$, leaving only the possibility of a neutral disturbance of infinite wave-length at infinite Reynolds number. These results agree with those obtained from the considerations of an inviscid fluid in Part II of this paper.

The minimum critical Reynolds number and the minimum critical wave-length. Having demonstrated the instability of the symmetrical and the boundary-layer profiles, we want to answer the following questions. First, does there exist a minimum critical Reynolds number, below which the flow is stable for disturbances of all wave-lengths? If so, can we get an approximate estimate of its magnitude? Secondly, does there exist a minimum wave-length of the disturbance (maximum α) below which the flow is stable at all Reynolds numbers? If so, can we get an approximate estimate of its magnitude? We shall see that in trying to answer these questions, we can also roughly depict the complete $\alpha(R)$ curve, which separates stability from instability.

The existence of these minimum values can be most conveniently inferred from a condition of stability derived by Synge* from energy considerations. His condition reads

(12.17) $$(qR)^2 < (2\alpha^2+1)(4\alpha^4+1)/\alpha^2, \qquad q = \max|w'|.$$

This condition insures the existence of a minimum critical Reynolds number. It permits α to become infinite only for $R \to \infty$. But we know from our previous considerations that $\alpha \to 0$ as $R \to \infty$. Hence, we would expect that there exists a maximum value of α, above which there is always stability. The general shape of the neutral curve must therefore be of the general shape shown. The solid curves are drawn in accordance with (12.11), (12.12), and (12.17); the dotted curves are drawn merely to indicate the general shape of the

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* Synge, (56), eq. (11.23), p. 258. His λ is our α. The condition is originally stated for plane Couette motion and plane Poiseuille motion; but it is easily seen that it holds for a general velocity distribution with $q = \max |w'|$.

curve. It is evident that the region outside the curve is the region of stability, and the enclosed region is the region of instability. Similar conclusions have been reached by Heisenberg* but neither his arguments nor his results appear to be completely clear.

Having established the existence of the minimum critical value of R and the maximum critical value of α, we proceed to make an estimate of their magnitude. We shall see that our theory permits us to give a quite good approximation to the minimum value of αR (cf. (12.20)). Since this roughly corresponds to the minimum value of R and also the maximum value of α, (as will be clear from the individual examples given below), we can get a rough estimate of these values by making a rough estimate of α corresponding to the minimum value of αR.

Using the second equation of (11.11) and the approximation (12.2) for v, we have approximately

(12.18) $$\mathfrak{F}_i(z) = v(c) = -\pi w_1' \frac{w w''}{w'^3} .$$

If we recall that z is proportional to $c(\alpha R)^{\frac{1}{3}}$, this equation determines $(\alpha R)^{\frac{1}{3}}$ as a function of c. It can then be easily verified that the minimum value of $(\alpha R)^{\frac{1}{3}}$ occurs when

(12.19) $$z\,\mathfrak{F}_i'(z) = c\,v'(c) .$$

If this occurs at $z = z_o$, then we have approximately from (11.18)**

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* Heisenberg, loc. cit., p. 601.

** Cf. Heisenberg, loc. cit., eq. (29a), p. 602. He put $z_o \sim 1$.

$$\alpha R = w_1'^2 \left(\frac{z_0}{c}\right)^3. \qquad (12.20)$$

The point z_0 is roughly the value where $\mathfrak{F}_i(z)$ takes its maximum value, because c is small, so that (12.19) is approximately $\mathfrak{F}_i'(z) = 0$. The corresponding value of α can be approximately obtained by taking

$$u = \mathfrak{F}_r(z_0), \qquad (12.21)$$

in accordance with the first equation of (11.11), where u is given by the real part of (11.8) or (11.10), as the case may be.

Approximate rules. We now proceed to make very rough approximations in order to obtain convenient rules for estimating the minimum critical Reynolds number. We take $z_0 = 3.2$ where $\mathfrak{F}_i(z_0) = 0.60$, $\mathfrak{F}_i'(z_0) = 0.15$, $\mathfrak{F}_r(z_0) = 1.48$. With a consultation of the values of the integrals H(c), K(c), M(c), and N(c) given in the appendix, we may derive the following reasonable estimates of α:

$$\alpha^2 = \frac{w_1' c}{H_1(0)}, \quad H_1(0) = \int_{y_1}^{y_2} w^2 dy, \qquad (12.22)$$ for symmetric profiles,

$$\alpha = w_1' c, \qquad (12.23)$$ for boundary-layer profiles.

These values are usually lower than the exact values, (as will be seen clearly from the specific examples below). With an approximate allowance for these inaccuracies and taking round numbers, we get the following approximate rules for the minimum critical Reynolds number:

$$R = \frac{30 w_1'}{c^3}\sqrt{\frac{H_{10} w_1'}{c}}, \qquad (12.24)$$ for symmetrical profiles,

$$R = \frac{25 w_1'}{c^4}, \qquad (12.25)$$ for boundary-layer profiles.

To find the value of c from (12.18), it is convenient to plot its right-hand side together with $w(y)$ against y and read off the curve the value of the latter when the former is equal to 0.6.

The attached figures (Fig. 10) give the calculations for the Blasius case and the plane Poiseuille case. In the first case, the thickness of the boundary layer is taken so that the initial slope is $w'_1 = 2$. It is found that

$$(12.26) \qquad \begin{cases} R = 7940 & \text{for Poiseuille case,} \\ R\,\delta_1 = 468 & \text{for Blasius case.} \end{cases}$$

The quantity δ_1 is the displacement thickness

$$\delta_1 = \int_0^\infty (1-w)\,dy = 0.2867,$$

where y is measured from the solid wall. These values for the minimum critical Reynolds number agree fairly well with those obtained below from more elaborate numerical calculations.

When these estimates of the minimum values of R and the corresponding values of α (eqs. (12.22) -- (12.25)) are combined with the asymptotic behavior of the $\alpha(R)$ curves (eqs. (12.11) -- (12.12)), the curve of neutral stability in any case can be sketched with fair accuracy with very little labor.

The maximum value of α on the neutral curve cannot be very well estimated. It is usually somewhat higher than the values of α given by (12.22) and (12.23).

13. Stability of special velocity distributions.

We shall now apply our theory to some special cases in order to obtain some numerical results comparable with experiments. We take the Blasius case as a typical boundary-layer profile, and the plane Poiseuille motion as a typical symmetrical profile. In any case, the resultant curve of stability limit should have the general shape discussed in the last section. Only the results will be given here; the method of calculation will be discussed in the appendix.

Stability of plane Poiseuille flow. The velocity distribution of plane Poiseuille motion is given by

$$w(y) = 2y - y^2, \text{ with } w'_1 = 2,\ H_1(0) = \frac{8}{15}. \tag{13.1}$$

The two branches of the $\alpha(R)$ curve are given by (cf. (12.8), (12.9))

$$\begin{cases} R^{\frac{1}{3}} = 8.44\,(\alpha^2)^{-\frac{11}{6}}, & c = \frac{4}{15}\alpha^2\,; \\ R^{\frac{1}{3}} = 3.00\,(\alpha^2)^{-\frac{7}{6}}, & c = 0.608\,\alpha^2. \end{cases} \tag{13.2}$$

The numerical results are shown in the following table (Table III) and the attached figure (Fig. 11). The significance of the last column in the table will be explained in the next section. From the figure, we see that the minimum critical Reynolds number occurs at $R^{\frac{1}{3}} = 20$, or $R = 8000$, agreeing very well with our previous estimation.

Earlier results. The stability of plane Poiseuille flow has been a standard problem of hydrodynamic stability attempted by many authors. Comparatively recent works are those of Heisenberg,[12], Noether[28], Goldstein[5], Pekeris[31,32], and Synge[57]. The papers of Goldstein and Synge and one of the papers of Pekeris[31] give definite indication of stability at sufficiently low Reynolds numbers. Heisenberg's work is in general agreement with the present investigations. He only gave a very rough calculation, whose result is reproduced in the figure. It seems that his curve is

$$R^{\frac{1}{3}} = 13.4\,(\alpha^2)^{-\frac{11}{6}}.$$

This is different from our present result (13.2) by a numerical factor. Also, for the values of α for which his curve is drawn, the approximation used in deriving (13.2) is no longer legitimate. Noether's work is based upon a very good mathematical approach, which seems to promise further developments. However, in applying his method to particular examples, he neglected the terms in α^2 in the inviscid solutions. As is evident from previous discussions, this is bound to lead to the wrong conclusion that the plane Poiseuille flow is stable (as Noether actually did). Pekeris' second paper[32] is a numerical treatment of (4.1), replacing a derivative by the ratio of two finite differences. However, his results cannot be taken seriously, because he has virtually <u>neglected the inner friction layer.</u> In his approximation, he divided the half-width of the channel into (at most) four equal parts corresponding to $w = 0, 7/16, 3/4, 15/16, 1$. From the present work, we know that the inner friction layer occurs definitely for $c < 6/16$. We know also from our previous investigations that the function φ varies very rapidly in the neighborhood of the inner friction layer. Hence, it is not legitimate to replace φ' by $\Delta\varphi / \Delta y$ for the interval $(0, 1/4)$, y being measured from the solid wall here. Also, most of the combinations of values (α, R) he selected do not correspond to a strong instability. These values are marked with crosses in the figure.

Table III

Stability of Plane Poiseuille Flow

c	α^2	R	z	s
0.1	0.215	51.0	2.47	0.65
0.2	0.577	25.4	2.71	0.561
0.276	1.41	22.6	3.32	0.418
0.2	1.12	37.4	3.88	0.329
0.1	0.488	89.6	4.47	0.267

Stability of Blasius flow. For this case, we choose the boundary-layer thickness to be defined by

(13.3) $$\tilde{y} = \tilde{\delta} = 6\tilde{x}\sqrt{R_x}, \quad R_x = \tilde{u}_1 \tilde{x}/\nu,$$

where $\tilde{x}$, $\tilde{y}$ are the dimensional distances from the leading edge and the wall respectively, and $\tilde{u}_1$, ν are the dimensional free stream velocity and the kinematic viscosity respectively.* With this definition, the dimensional displacement thickness is

(13.4) $$\tilde{\delta}_1 = 0.2867\,\tilde{\delta}.$$

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* Cf. Goldstein(106), vol. I, p. 135.

Such a choice has the convenience that the initial part of the velocity curve can be very accurately represented by

(13.5) $w(y) = 2y - 3y^4$,

y being measured from the wall. Also, since the edge of the boundary layer is farther from the solid wall than that set by Tollmien and Schlichting, greater accuracy can be expected. To make it easy to compare with other results, all final values are presented in terms of

(13.6) $$\alpha_1 = \alpha\delta_1 , \quad R = R\delta_1$$

The two asymptotic branches of the $\alpha(R)$ curve are given by the following formulae (cf. (12.12) and (12.14)):

(13.7) $$R_1 = 2.20 \times 10^{-5}\, \alpha_1^{-10} , \qquad c = 1.74\, \alpha_1 ,$$

(13.8) $$R_1 = 0.0635\, \alpha_1^{-4} , \qquad c = 4.00\, \alpha_1 .$$

These formulae may be compared with those given by Tollmien.* The complete numerical result is shown in the following table (Table IV) and the attached figures (Fig.12, 12a). The minimum critical Reynolds number occurs at $R_1 = 400$, agreeing fairly well with our previous estimation and the earlier results of Tollmien and Schlichting.

Earlier results. The stability of the boundary layer has been calculated by Tollmien and later by Schlichting, approximating the velocity distribution by linear and parabolic distributions.

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* Loc. cit., (66), first paper, p. 42.

For the evaluation of the imaginary part corresponding to the inviscid solutions, they used the exact profile. The calculation of Schlichting is shown dotted in the figure. Tollmien's curve lies between the present curve and Schlichting's. Schlichting also calculated the amplification of the unstable disturbances, and the amplitude distribution and energy balance of the neutral disturbances. Since the neutral curve in his calculation is inexact, it might be desirable to repeat his calculations. Using the present scheme of calculation (as will be explained in the appendix), less numerical labor will be required than in Schlichting's original work.

Table IV

Stability of Blasius Flow

c	c_1	R_1	z	s
0.2	0.071	7510	2.35	0.696
0.3	0.139	1340	2.48	0.641
0.35	0.201	696	2.66	0.578
0.40	0.298	436	2.94	0.496
0.41	0.339	395	3.04	0.473
0.42	0.411	399	3.32	0.414
0.41	0.435	512	3.60	0.367
0.40	0.417	648	3.68	0.356
0.35	0.356	1435	4.03	0.310
0.3	0.287	3045	4.15	0.296
0.2	0.161	33800	5.10	0.218

CONCLUDING DISCUSSIONS

14. Physical discussion of results; prospect of further developments.

Let us now summarize all the results which have been obtained and discuss their physical significance. In the first place, we may conclude that all the inertial forces controlling the stability of two-dimensional parallel flows can be considered in terms of the distribution of vorticity. If the gradient of vorticity of the main flow does not vanish inside the fluid, then amplified disturbances cannot exist except through the effect of viscosity.

In fact, the effect of viscosity is in general destabilizing for very large Reynolds numbers. Thus, if a wavy disturbance of finite wave-length can exist neutrally for an inviscid fluid, it will be amplified through the effect of viscosity. Indeed, if the Reynolds number of a flow is continually decreased, a disturbance of finite wave-length, which is damped at very large Reynolds numbers, becomes amplified, unless the wave-length is so small as to cause excessive dissipation at any Reynolds number. For still smaller Reynolds numbers, the damping effect becomes predominant, and we have again a decay of the disturbance. However, for the particular disturbance of infinite wave-length (essentially a steady deviation), the effect of viscosity may be said to be always of the nature of a damping.

The effect of viscosity is essentially one of diffusion of vorticity. It can be seen more clearly from the following considerations: Let us imagine a disturbance originating from the inner friction layer where the phase velocity of the disturbance is equal to the velocity of the main flow. During one period $2\pi l/\alpha c U$ of the disturbance, the viscous forces will propagate it side-wise through a distance of the order $\sqrt{2\pi\nu l/\alpha c U} = l\sqrt{2\pi/\alpha Rc}$. It is significant to compare this distance with the distance between the inner friction layer and the solid boundary. For if they are nearly equal, it means that the effect of viscosity is dominant at least from the solid surface to that layer. This ratio is approximately

$$s = \sqrt{\frac{2\pi}{z^3}}\,, \tag{14.1}$$

with z defined by (11.6). This quantity may be regarded as a measure of the effect of viscosity. Its value is included in Tables III and IV. We notice that the value of s decreases from 0.7 to 0.5 as we follow the lower branch of the neutral curve of stability from infinite Reynolds number to the minimum critical Reynolds number. Then, as s decreases to zero, we are following the other branch of the neutral curve to infinite Reynolds numbers. Thus, (see Figs. 10, 11, 12) the lower branch is essentially controlled by the effect of viscosity; its effect is stabilizing, an increase of Reynolds number giving instability. On the other branch, the effect of viscosity on diffusion of vorticity is overwhelming in comparison with the effect of dissipation, its effect is destabilizing, an increase of Reynolds number giving stability. This destabilizing mechanism is essentially to shift the phase difference between the u and v components of the disturbance. It has been explained in some detail in Prandtl's article from the point of view of energy balance.

If we consider disturbances from the wall and from the inner friction layer, we may regard the region in between to be wholly governed by the effect of viscosity, if these disturbances meet after a period. Thus, it is not without significance that the minimum critical Reynolds number occurs for s = 1/2 approximately, which may be regarded as marking the passage from stabilizing effect to destabilizing effect of the viscous forces.

These discussions hold both for symmetrical velocity distributions and for boundary-layer distributions. In both cases, it has been demonstrated that the instability is essentially caused by the effect of viscosity. These velocity distributions are unstable whether a point of inflection occurs in the velocity profile or not. Thus, although the gradient of vorticity plays a part in controlling the stability of the flow, it is by no means the dominant factor, particularly at low Reynolds numbers. There is thus no reason to associate a point of inflection in the profile directly with instability. This removes Taylor's objection of instability theories based on von Doenhoff's experiments.* Even if the point of inflection in the velocity profile occurs in the leading part of the plate in his experiments, the flow there is definitely stable.

There is another objection raised by Taylor against Tollmien's work on the stability of the boundary layer. He questions whether the change of boundary-layer thickness should not have a drastic in-

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* Taylor, loc. cit. (63), p. 308

fluence. This point can only be settled experimentally. So far as mathematical considerations are concerned, it seems justifiable to consider a boundary layer as a parallel flow; the fractional variation of thickness is very small over a distance of one wave-length of the disturbance, and the error incurred is only a few per cent. A fuller discussion of all the errors involved in the theory will be given in the Appendix.

There is another point which should be settled by experimental investigations. Since the general impression was that the plane Poiseuille flow was stable, Prandtl had advanced the idea that instability occurred at the entrance flow where the velocity distribution is not yet parabolic. The present work certainly concludes that such entrance flows are unstable, if they can be considered as approximately parallel. It is hard to decide theoretically whether a well-developed turbulence has already been reached before the parabolic profile is established. This presumably depends upon the conditions of disturbance at the inlet. The question can be best settled experimentally.

Of the six types of problems mentioned in section 3, (1), (2), and (5) seem to be quite settled. The present work on the boundary layer checks Tollmien's result approximately, with a minimum critical Reynolds number $R\,\delta_1 = 400$. The minimum critical Reynolds number for plane Poiseuille flow is found to be 16000 based upon the width of the channel. These values are at least not in disagreement with the existing experimental results. It would be very interesting if experiments can be carried out to check the theoretical results so far obtained.

Since plane Couette motion is concluded to be stable while plane Poiseuille motion is concluded to be unstable, it seems interesting to investigate a combination of them to find out when does the instability begin as one varies the pressure gradient and the relative motion of the plates.

The stability of two-dimensional jets and wakes has never been investigated including the effect of viscosity. It seems that a study of the stability of two-dimensional wake might give us valuable information regarding the Kármán vortex street, — particularly regarding the minimum Reynolds number of its occurrence and the width of the street as compared with the size of the body.*

Transition to turbulence. The success of Taylor's theory of transition[61,63] to turbulence in the boundary-layer as caused by external turbulence seems to throw the instability theories at a disadvantage. However, Taylor's work can at most be regarded as only one phase of the problem, i.e., concerning cases where the external turbulence plays a dominant role. In fact, it is not impossible to construct a stability theory, taking account of the free turbulence outside the boundary-layer, in case this is the main cause of transition. The boundary condition $\varphi' + \alpha\varphi = 0$ at the edge of the boundary

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* This problem has been attempted by Heisenberg; see Goldstein's book (106).

layer signifies that the disturbance there has equal magnitudes in directions parallel and perpendicular to the wall. This can be easily reconciled with the nearly isotropic turbulence in the free stream. Of course, the theory can only be pushed to the point where <u>non-linear</u> effects <u>begin</u> to appear. Otherwise, we have to deal with a very difficult mathematical problem. It is possible that the beginning of non-linear effect is not far from the actual point of transition. Then the instability theory should give useful results regarding transition, which might be expected to check with experiment.

APPENDIX

Appendix

In the following paragraphs, we shall describe the methods with which the numerical calculations of section 13 are carried out. We shall then give a more thorough discussion of the numerical accuracy involved in the calculations. Special emphasis will be placed on the case of Blasius flow.

Rule of calculation. The calculations required in section 13 are as follows: (1) to find the values of α and z corresponding to each value of c by using equations (11.11), with u and v defined by (11.8) and (11.10); and (2) to calculate R from (11.12). To do this, we may take the following procedure. We first plot $\mathfrak{F}_i$ against $\mathfrak{F}_r$; then plot the corresponding right-hand side members of (11.11) in a similar manner in the same diagram. Noting that the latter are functions of α and c only, the plotting may be done by drawing curves of constant α (or constant c). The intersections of this set of curve with the ($\mathfrak{F}_r$, $\mathfrak{F}_i$) curve give the desired results.

This procedure is however, very laborious. A simpler method is as follows: As will be seen below, the imaginary parts of H's, M's and N's appearing in (11.8) and (11.10) are very small compared with that of $K_1(c)$, we can therefore use the approximation

(1) $$v = v(c) = - \pi w'_1 \frac{ww''}{w'^3} \quad \text{for } w = c.$$

The following steps are then taken:

(i) Calculation of αR. In this step, the auxiliary functions

$$\lambda(c),\ w'_o(c),\ v(c)$$

are required. These are tabulated in Tables (Va) and (VI) for the cases considered. Having calculated these functions, we can determine z and u for each value of c; R is then determined from (11.12). In actual practice, a procedure of successive approximations is used. Taking $\mathcal{F}_i^{(0)} \equiv \mathcal{F}_i(z^{(0)}) = v, \quad u^{(0)} = \mathcal{F}_r^{(0)} \equiv \mathcal{F}_r(z^{(0)})$, we have the successive approximations of u, z, $\mathcal{F}_r$, $\mathcal{F}_i$ given by

$$\begin{cases} \mathcal{F}_i^{(n+1)} = v\left\{1+\lambda\right\}\left\{(1+\lambda u^{(n)})^2 + (\lambda v)^2\right\}^{-1}, \\ u^{(n+1)} = \mathcal{F}_r^{(n+1)}\left\{(1+\lambda u^{(n)})^2 + (\lambda v)^2\right\}\left\{(1+\lambda)(1+\lambda u^{(n)})\right\}^{-1} + \lambda v\left(1+\lambda u^{(n)}\right)^{-1}. \end{cases}$$

In each approximation, $\mathcal{F}_r^{(k)}$ and $z^{(k)}$ are determined graphically from $\mathcal{F}_i^{(k)}$.

(ii) Calculation of α. For this purpose, the additional auxiliary functions

$$w_1' cRl = w'_1 cRl\left\{K_1(c) + \frac{1}{w_1' c}\right\}, \quad H_1(c),\ H_2(c),\ M_3(c),\ N_3(c),\ \ldots$$

are required. These are tabulated in Tables (Vb) and (VI) for the cases considered. The methods of evaluating these functions and their accuracy will be discussed below. For sufficient accuracy in the final results, only the real parts of H_2, M_3, N_3 are required, besides $w'_1 cRl$ and H_1. Having calculated these functions, we can determine the value of α from the real part of the equation (11.8) or (11.10). A similar method of successive approximations may be used by writing those equations in the forms

$$\begin{cases} \alpha^2 = \dfrac{w_1' c}{H_1(u - w_1' c Rl)}\left(1-\alpha^2 H_2\right)\dfrac{1-\alpha^2 H_2 - \alpha^4 N_4 - \cdots}{1-\alpha^2 P_2 - \alpha^4 P_4 - \cdots}, \quad P_{2n} = M_{2n+1}/H_1, \\ \alpha = \dfrac{w_1' c}{u - w_1' c Rl}\left(1-\alpha^2 H_2\right)\dfrac{1-\alpha^2 H_2 - \alpha^4 N_4 - \cdots - (1-c)^2(\alpha^3 N_3 + \alpha^5 N_5 + \cdots)}{(1-c)^2(1-\alpha^4 M_4 - \alpha^6 M_6 - \cdots) + \alpha(H_1 - \alpha^2 M_3 - \alpha^4 M_5 - \cdots)}. \end{cases}$$

An approximate value of α is put into the right-hand side to obtain an approximation of the higher order on the left-hand side. For the initial approximation, take $\alpha = 0$.

Table V

Auxiliary Functions for Calculating the Stability of Plane Poiseuille Flow

(a)

c	w'_o	λ	v
0	2.000	0	0
0.1	1.898	0.0263	0.186
0.2	1.789	0.0556	0.444
0.3	1.672	0.0889	0.815
0.4	1.548	0.1270	1.369

(b)

c	$w'_1 cR1$	H_1	H_2	M_3	N_3
0	0	0.533	0.218	0.060	0.193
0.1	0.101	0.410	0.195	0.042	0.222
0.2	0.154	0.307	0.169	0.028	0.254
0.3	0.191	0.223	0.141	0.020	0.292
0.4	0.223	0.160	0.111	0.015	0.345

Table VI

Auxiliary Functions for Calculating the Stability of Blasius Flow

c	v	$w'_1 cR1$	H_1	H_2	M_3	N_3
0	0	0	0.602	0.235	0.075	0.198
0.1	0.0069	0.0615	0.470	0.212	0.057	0.227
0.2	0.0635	0.207	0.357	0.186	0.043	0.259
0.3	0.202	0.458	0.265	0.158	0.035	0.297
0.4	0.523	0.858	0.192	0.128	0.030	0.350
0.5	1.195	1.423	0.139	0.092	0.030	0.415

In all the calculations, it is found accurate enough to take $w'_0 = w'_1$, $\lambda = 0$

Numerical accuracy of the calculations. The numerical accuracy of our calculation as based upon the final equations given in section 6 are limited by several factors:

(i) by neglecting quantities of the orders e^{-P} and $(\alpha R)^{-1}$ in the reduction of the determinantal equations of the boundary-value problems,

(ii) by using the inviscid solutions for φ_1 and φ_2 (error of the order $(\alpha R)^{-1}$),

(iii) by the approximations of the rapidly varying solutions φ_3 and φ_4 as discussed at the end of section 6,

(iv) by the boundary-layer approximation used in setting up the equation of stability (except in the cases of plane Couette and Poiseuille flows).

Finally, certain numerical approximations have to be used in the actual evaluation of the quantities u and v in equations (11.11). We shall now discuss these factors one after another.

The inaccuracy due to (i) and (ii) is negligible in all the cases considered, because αR is always sufficiently large. In connection with (iii), the situation is more complicated. The first approximation of the asymptotic solution should give an error of the order of $(\alpha R)^{-\frac{1}{2}}$; while the first approximation using Hankel functions should give an error of the order $(\alpha R)^{-\frac{1}{3}}$. It might therefore be thought that the asymptotic method should always give a better approximation. However, this is not the case. For the order of accuracy of the first method is based upon a fixed value of y, while that of the second is based upon a fixed value of η . Thus, if αR may be allowed to become very large while $y - y_o$ remains to be of the order of unity, the first method is definitely better. This is the case with the quantities $\varphi_{42}, \varphi_{42}'$. With φ_{31} and φ_{31}' , the situation is different. Here, $y_1 - y_o$ is always small. Except for one branch of the neutral curve for profiles with a flex, $y_1 - y_o$ goes to zero as αR becomes infinite. Because of the smallness of $y_1 - y_o$, the asymptotic solution (which fails to be accurate in the neighborhood of y_o) never gives a good approximation. This is why the other method has to be used in most of the calculations, and we are limited to an accuracy of $(\alpha R)^{-\frac{1}{3}}$. We note that the curvature of the velocity distribution does not come into this approximation. Thus, for better accuracy, a second approximation should be used, the error being then reduced to the order of $(\alpha R)^{-\frac{2}{3}}$. However, since the error in the method used is only a few per cent, and an improvement in accuracy would not alter the general conclusions, it does not seem worth while to improve the accuracy in the light of general interest. Indeed, the inaccuracy due to the other causes (to be discussed) is also of the same order of magnitude. Another support to the method used is that it does agree with the asymptotic method when z is large; there is only a small discrepancy (cf. eq. (6.33)).

The effect of the change of the thickness of the boundary layer may be taken to be more serious than a mere numerical inaccuracy. Taylor regarded this as invalidating the existing instability theory of the boundary layer. This question can best be settled experimentally.

For the present, we only want to know its effect upon our boundary value problem. An approximate estimate of this effect may be calculated by considering the change of the thickness of the boundary layer for one wave-length of the disturbance. This can be easily verified* to be $\pi(1.72)^2/\alpha_1 R_1$. For the lowest value of $\alpha_1 R_1$ involved in the calculations of section 13, this is about 6 per cent. Thus, the error is not large. Hence, in the physical interpretation of the results, we need only consider a change of Reynolds number as we pass down stream. One interesting point is the following: As the Reynolds number keeps on increasing, all disturbances finally become stable, if the linear theory holds throughout. Thus, the transition to turbulence depends upon the occurrence of the non-linear effect and hence must depend upon the amount of initial disturbance.

<u>Calculation of</u> $\varphi_{12}^{(0)}$, $\varphi_{22}^{(0)}$, <u>etc.</u> We shall now discuss the method by which these quantities are evaluated for the calculation of u and v in the equations (11.8) and (11.10). A discussion of the accuracy of the present method and of Tollmien's method of evaluating these quantities will also be made.

The original question is to evaluate the integrals $H_m(c)$ and $K_m(c)$ as occurring in (6.24). Various methods are possible for carrying out the calculation, including straightforward numerical integration. The method to be described is an attempt at a simplest one. With the transformations (6.28), we hope to bring out the dominant

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* Cf. Goldstein (106), last column of table of p. 157.

terms of the series (6.24), and the calculations of u and v according to (11.8) and (11.10) are based upon the use of the transformed series (6.27). We make the following approximations.

(i) The imaginary part v is chiefly given by that of the first term, namely, $w'_1 c(K_1 + \frac{1}{w'_1 c})$; this implies that the imaginary part due to $H_2(c)$, $M_3(c)$, $N_3(c)$, etc. are negligible.

(ii) The real part receives also little contribution from those of $H_2(c)$, $M_3(c)$, $N_3(c)$,... and hence these need be calculated only approximately.

(iii) The series are cut short, terms like N_4, M_4,... are entirely neglected. Let us proceed to justify these approximations.

The justification of (ii) and (iii) is based upon the following two facts. (a) the quantities in the series involved decrease roughly like $1/m!$, m being the number of integrations involved in defining a certain term. (b) For $\alpha < 1$, the terms also decrease as α^m . Thus, the accuracy is not very good for $\alpha > 1$, namely for low Reynolds numbers. But from a consultation of Tables V and VI, and the manner in which the integrals $H_2(c)$, $M_3(c)$, $N_3(c)$, .. enter (11.8) and (11.10), we see that an error of ten per cent in these integrals will cause a negligible error in the final results.

The justification of (i) needs more explanation. For definiteness, let us take $N_3(c)$ as an example. Now,

$$N_3(c) = \int_{y_1}^{y_2} dy\,(w-c)^{-2} \int_{y_1}^{y} dy\,(w-c)^2 \int_{y_1}^{y} dy\,(w-c)^{-2}.$$

This can be expressed as the sum of the following three integrals:

$$N_{31}(c) = K_1(c)\cdot\int_{y_1}^{y_0} dy\,(w-c)^2\int_y^{y_2} dy\,(w-c)^{-2},$$

$$N_{32}(c) = \int_{y_1}^{y_0} dy\,(w-c)^{-2}\int_{y_0}^{y} dy\,(w-c)^2\int_y^{y_2} dy\,(w-c)^{-2},$$

$$N_{33}(c) = \int_{y_0}^{y_2} dy\,(w-c)^{-2}\int_{y_0}^{y} dy\,(w-c)^2\int_y^{y_2} dy\,(w-c)^{-2}.$$

The third integral is real, because $y > y_0$. A further transformation of the last integration in N_{31} and N_{32} like

$$\int_y^{y_2} dy\,(w-c)^{-2} = K_1(c) - \int_{y_1}^{y} dy\,(w-c)^{-2}$$

gives

$$N_{31} = \left\{K_1(c)\right\}^2\int_{y_1}^{y_0} dy\,(w-c)^2 - K_1(c)\int_{y_1}^{y_0} dy\,(w-c)^2\int_{y_1}^{y} dy\,(w-c)^{-2},$$

$$N_{32} = K_1(c)\int_{y_1}^{y_0} dy\,(w-c)^{-2}\int_{y_0}^{y} dy\,(w-c)^2 - \int_{y_1}^{y_0} dy\,(w-c)^{-2}\int_{y_0}^{y} dy\,(w-c)^2\int_{y_1}^{y} dy\,(w-c)^{-2}.$$

Now, the last integral is real because $y < y_0$. Further, it can be easily verified that

$$\int_{y_1}^{y_0} dy\,(w-c)^2\int_{y_1}^{y} dy\,(w-c)^{-2} = \int_{y_1}^{y_0} dy\,(w-c)^{-2}\int_{y_0}^{y} dy\,(w-c)^2.$$

Hence, the only term which can contribute to the imaginary part of $N_3(c)$ is $\left\{K_1(c)\right\}^2\int_{y_1}^{y_0} dy\,(w-c)^2$. Now, c is usually small so that we may put

$$\int_{y_1}^{y_0} dy\,(w-c)^2 = \frac{w_1'^2}{3}(y_1-y_0)^3 = \frac{1}{3}\frac{(w_1'c)^3}{w_1'^4}.$$

Hence,

$$\left\{K_1(c)\right\}^2 \int_{y_1}^{y_0} dy\,(w-c)^2 = \left\{w_1' c\, K_1(c)\right\}^2 \frac{1}{3} \frac{(w_1' c)}{w_1'^{\,4}} .$$

Now we have approximately

$$w'_1 c K_1(c) = 1 - vi .$$

Substituting into the above expression, we obtain the imaginary part of $N_3(c)$ as $-2w'_1 cv/3w'_1{}^4$. This will give a contribution of approximately $-\frac{2}{3}\left\{\frac{\alpha c(1-c)}{w_1'}\right\}^2 v$ to the imaginary part of v. This is negligible, because the factor preceding v is at most of the order of 0.02 in our calculations. Thus, it is justifiable to neglect the contribution of N_3 to v. With the other terms, the approximation is even better; thus, the imaginary part of $H_2(c)$ is of the order of c^3 times that of $K_1(c)$, and that of $M_3(c)$ is of the order of c^6 times that of $K_1(c)$.

Having thus justified the approximations described above, the task is to evaluate $K_1(c)$, $H_1(c)$, $H_2(c)$, $M_3(c)$, and $N_3(c)$ with proper degree of accuracy. For parabolic distribution, these integrals can be evaluated exactly; the approximation (ii) is not necessary. Thus,

(2) $$H_1(c) = A$$

(3) $$K_1(c) = -\frac{1}{2a^2(1-a^2)} + \frac{\cancel{A}\,1}{\cancel{4}a^3}\left\{\log \frac{1+a}{1-a} + i\pi\right\},$$

(4) $$H_2(c) = \frac{1}{30}\,\frac{4a^2-3}{a^2} + \frac{A}{2a^3}\log(1+a) - \frac{2a^2}{15}\log a^2 - \frac{B}{4a^3}\left\{\log(1-a^2) - i\pi\right\},$$

(5) $$M_3(c) = \frac{B^2}{2a^2} K_1 + \frac{8a^2A}{15}\left\{\frac{1}{a+1} + \log\frac{a+1}{a}\right\} + \frac{1}{225}\left\{-\frac{54}{7} + \frac{108}{5}a^2 - \frac{38}{3}a^4 - 64a^6\right\},$$

$$(6) \qquad \mathcal{R}\ell\{N_3(c)\} = \frac{1}{24a^4}(1+3a^2) - \frac{1}{16a^5}(1-a^2)^2\log\frac{1+a}{1-a} + \frac{1}{16a^6}\left\{\left[\int_0^1 dx\left\{(a^2-x^2)\log\left|\frac{a+x}{a-x}\right|\right\}\right]^2 - B\pi^2\right\},$$

$$(7) \qquad \mathcal{I}m\{N_3(c)\} = \frac{\pi}{16a^6}\left\{2B\log\frac{1+a}{1-a} + \frac{32a}{15}\log\frac{1+a}{2a} - \frac{2a}{15}(1-a)(2+3a+18a^2+20a^3)\right\},$$

where

$$(8) \qquad a^2 = 1-c, \qquad A = a^4 - \frac{2a^2}{3} + \frac{1}{5}, \qquad B = A - \frac{8a^5}{15}.$$

These are the equations on which Table V is based, where only the real parts are given. For any other profile, the rule is as follows.

(i) Evaluate $K_1(c)$ with as much accuracy as possible. Usually, we break it into two parts

$$(9) \qquad K_1(c) = K_{11}(c) + K_{12}(c), \quad K_{11}(c) = \int_{y_1}^{y_j} dy\,(w-c)^{-2}, \quad K_{12}(c) = \int_{y_j}^{y_2} dy\,(w-c)^{-2},$$

where $y_1 < y_j < y_2$. The value of y_j is chosen so that $K_{11}(c)$ can be calculated with sufficient accuracy by developing w as a power series of (y_1-y_0), while $K_{12}(c)$ can be evaluated by developing the integrand as a power series of c/w.

(ii) Evaluate by numerical integration the quantities

$$(10) \qquad H_1(0) = \int_{y_1}^{y_2} w^2\,dy,$$

$$(11) \qquad H_1'(0) = -2\int_{y_1}^{y_2} w\,dy,$$

$$(12) \qquad H_2(0) = \int_{y_1}^{y_2} w^{-2}dy\int_{y_1}^{y} w^2\,dy$$

$$(13) \qquad M_3(0) = \int_{y_1}^{y_2} w^2 dy\int_{y}^{y_2} w^{-2}dy\int_{y_1}^{y} w^2\,dy,$$

$$(14) \qquad N_3(0) = \int_{y_1}^{y_2} w^{-2}dy\int_{y_1}^{y} w^2 dy\int_{y}^{y_2} w^{-2}\,dy.$$

(iii) The integral $H_1(c)$ is then given by

$$H_1(c) = H_1(0) + H'_1(0)\, c + c^2. \tag{15}$$

(iv) The real part of the integrals $H_2(c)$, $M_3(c)$, $N_3(c)$ are obtained by comparison with the corresponding quantities for parabolic distribution (Table V). Thus, for example,

(16) $\quad M_3(c) - M_3(0) = \left(\frac{w_1'}{2}\right)^2 \times$ corresponding quantity for parabolic distribution.

The idea of the last step is essentially to approximate the given profile with a parabolic one.

For the Blasius profile, (with $w'_1 = 2$, $y_j - y_1 = 0.4$), we obtain

$$(17)\quad \left\{ \begin{aligned} K_{11}(c) + \frac{1}{w_1'c} &= -0.5615 - 0.3937c - 1.543c^2 - 1.803c^3 - 1.368c^4 - 5.022c^5 + \cdots \\ &\quad + \frac{9}{8}\left(c^2 + \frac{21}{8}c^5 + \cdots\right)\left(\log\frac{0.8-c}{c} + i\pi\right), \\ K_{12}(c) &= 0.7080 + 1.3564c + 2.588c^2 + 3.860c^3 + 5.446c^4 + 7.445c^5 + \cdots, \\ K_1(c) + \frac{1}{w_1'c} &= 0.1465 + 1.2467c + 1.045c^2 + 2.039c^3 + 4.078c^4 + 2.423c^5 + \cdots \\ &\quad + \frac{9}{8}\left(c^2 + \frac{21}{8}c^5 + \cdots\right)\left(\log\frac{0.8-c}{c} + i\pi\right). \end{aligned} \right.$$

In evaluating these integrals, we take

$$(18)\quad \left\{ \begin{aligned} &w = 2(y-y_1) - 3(y-y_1)^4, && 0 \leqslant y-y_1 \leqslant 0.4, \\ &w = 1 - \left\{0.9 - (y-y_1)\right\}^2, && 0.4 < y-y_1 \leqslant 0.9, \\ &w = 1 && 0.9 \leqslant y-y_1 \leqslant 1. \end{aligned} \right.$$

For the integrals $H_1(0)$ and $H'_1(0)$, we make use of the known values*

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* Cf. Goldstein, (106), p. 136.

of the displacement thickness δ_1 and the momentum thickness δ_2 .

$$(19) \qquad \begin{cases} \delta_1 = \frac{1}{6} \ (1.7208) = 0.28673, \\ \\ \delta_2 = \frac{1}{12} \ (1.32824) = 0.11067. \end{cases}$$

Thus,

$$(20) \qquad H_1(0) = 1 - \delta_1 - \delta_2 = 0.6026, \ H'_1(0) = -2\ (1 - \delta_1\ ) = -1.4265.$$

The values of $H_2(0)$, $M_3(0)$, $N_3(0)$ as evaluated by numerical integration are given in the first row of Table VI. The rest of the table is constructed by following the procedure described above.

We see that the method of approximation developed above is purely a numerical one, and the calculation can be done without excessive labor. In any case, even if the above method does not give satisfactory results, suitable approximations can always be devised for the evaluation of the necessary integrals. This is the advantage of using Heisenberg's form of the inviscid solutions. In the method used by Tollmien, it is necessary that the profile may be approximated by linear and parabolic parts; otherwise, the numerical labor is excessive. A more serious criticism of Tollmien's method is the joining of the inviscid solutions at the point of junction of the two approximate profiles. Mathematically speaking, such a junction presents an essential singularity in the coefficients of the differential equation (3.8) or (3.14). Numerically speaking, serious difficulty would be expected when c is equal or even only very near to the velocity of junction, because the inviscid solution fails there. Tollmien did not reveal how he overcame this difficulty.*

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* Tollmien (66), footnote, p. 37.

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Legend

Fig. 1 - Broken profiles investigated by Lord Rayleigh.

Case (a) may be unstable; the other two cases are stable.

Fig. 2 - Diagrams showing the relative position of the real axis and the region of validity of the asymptotic solutions in each of the three cases, $c_i \gtreqless 0$.

Fig. 3 - The three types of velocity distributions.

Fig. 4 - The function $\mathcal{F}(z)$ shown in its real and imaginary parts (cf. Table I).

Fig. 5 - Paths around the critical point in the case $c_i < 0$.

Fig. 6 - The region of analyticity of the inviscid solutions.

Fig. 7 - Stream lines of a neutral disturbance as observed by an observer moving with the wave velocity.

Fig. 8 - Acceleration of vortices in a non-uniform field of vorticity.

Fig. 9 - General shapes of the curve of neutral stability.

Fig. 10 - Calculation of the minimum critical Reynolds numbers.

Fig. 11 - Stability of pressure flow through a channel.

——— Present calculations, with the extrapolation of an asymptotic branch.

- - - - - Heisenberg's calculation.

x Points investigated by Pekeris.

Fig. 12 - Stability of the boundary layer with zero pressure gradient.

——— Present calculation

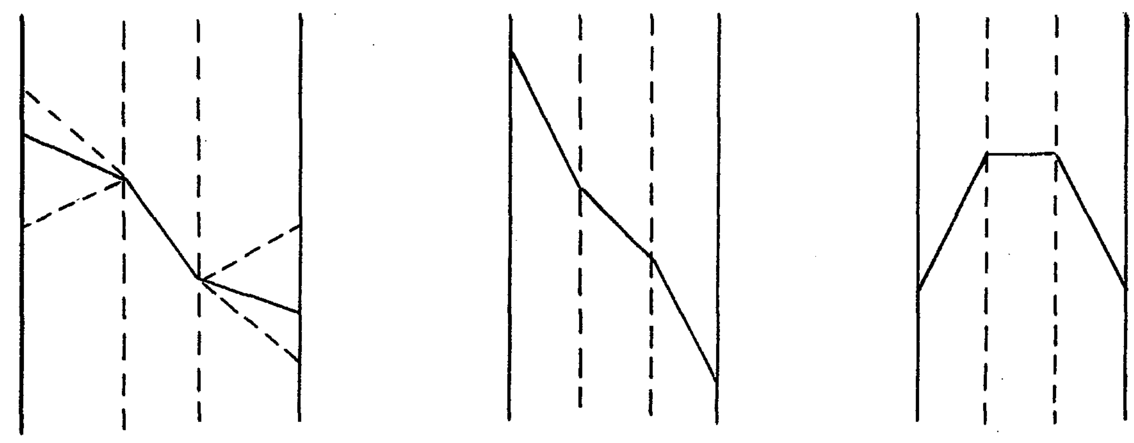
- - - - - - Schlichting's calculation

Fig. 12a - Stability of the boundary layer with zero pressure gradient (Semi-logarithmic scale).

——— Present calculation.

- - - - - Tollmien's calculation.

—·—· Schlichting's calculation.



(a) (b) (c)

Fig. 1.

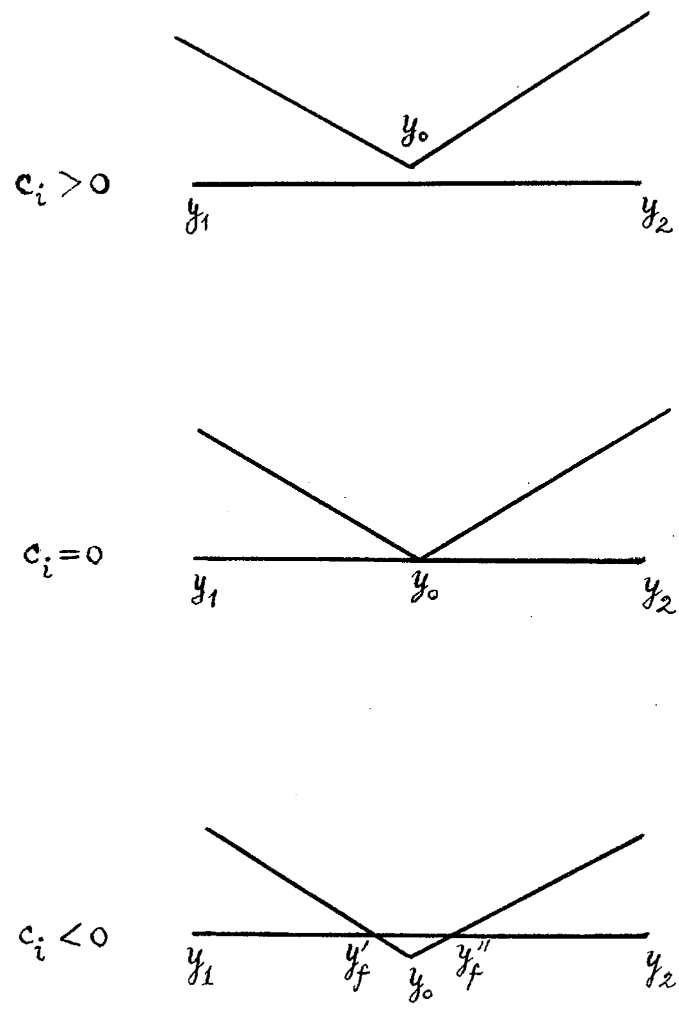


Fig. 2.

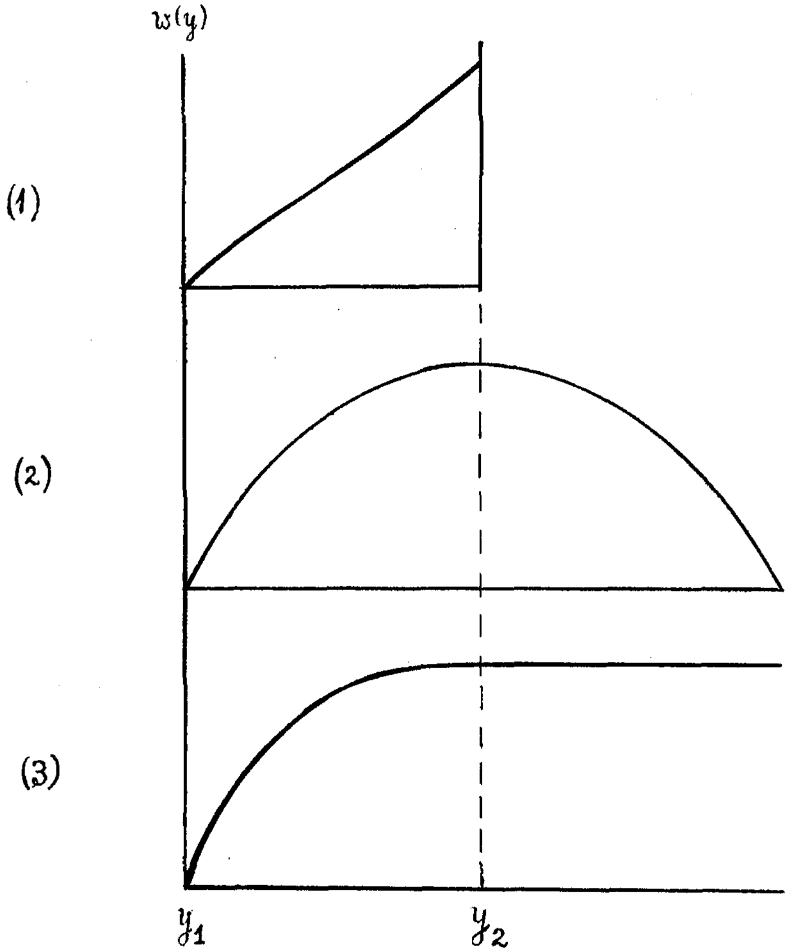
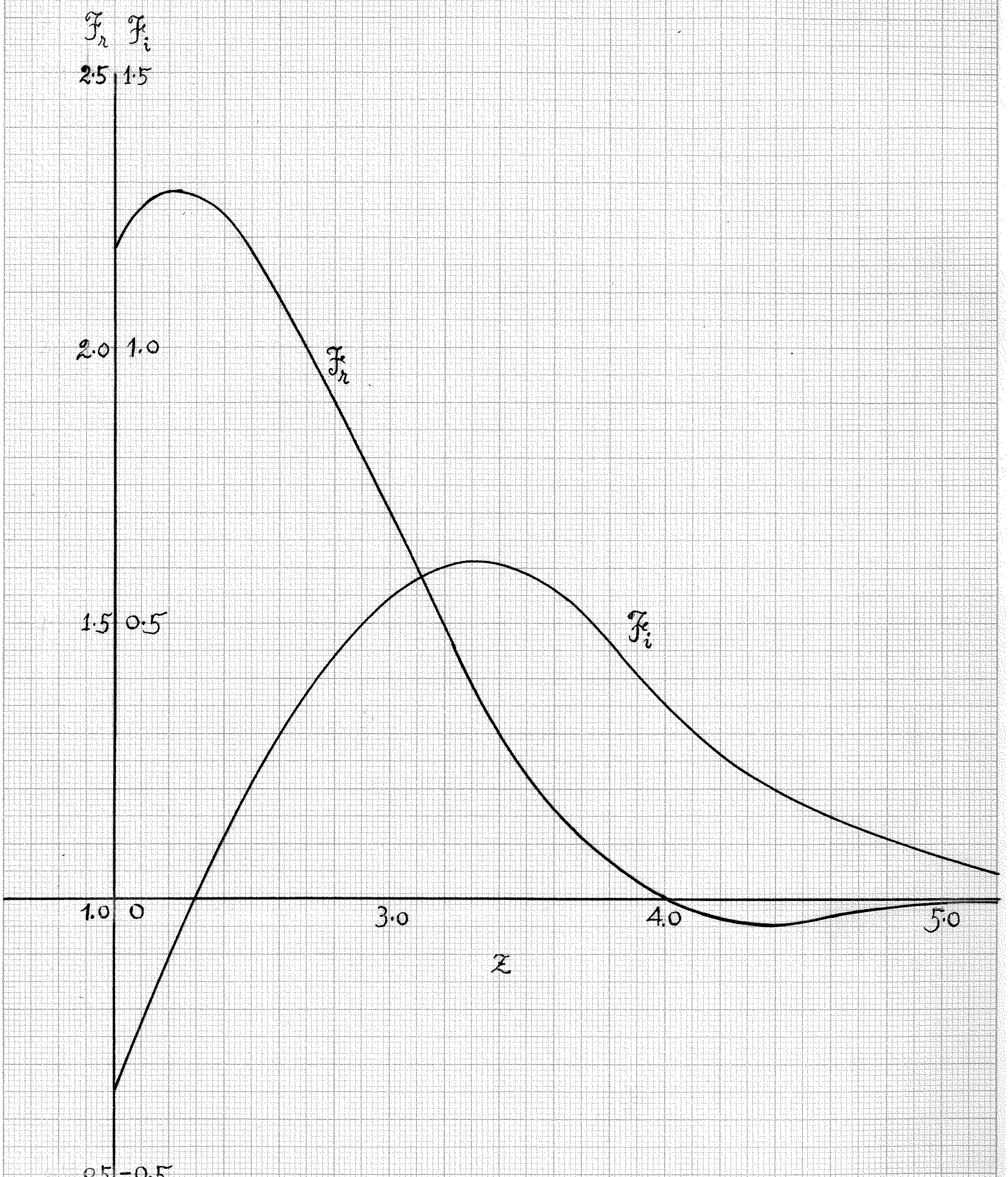


Fig. 3.



$\mathfrak{F}_r$ $\mathfrak{F}_i$
2.5 1.5
2.0 1.0
1.5 0.5
1.0 0
0.5 -0.5
3.0
4.0
5.0
z
$\mathfrak{F}_r$
$\mathfrak{F}_i$

Fig. 4.

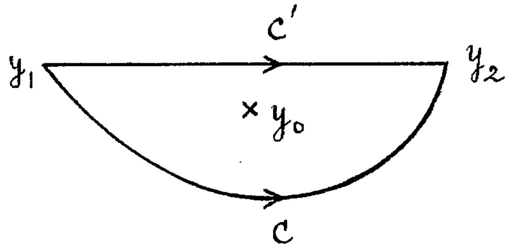


Fig. 5.

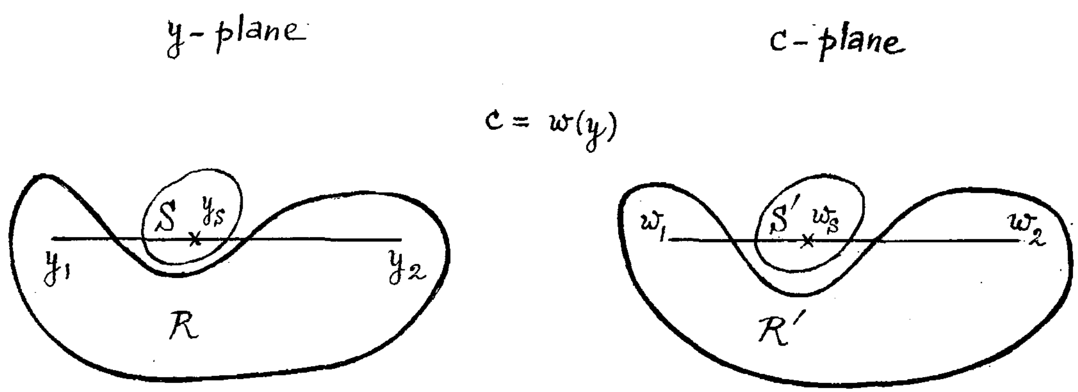


Fig. 6

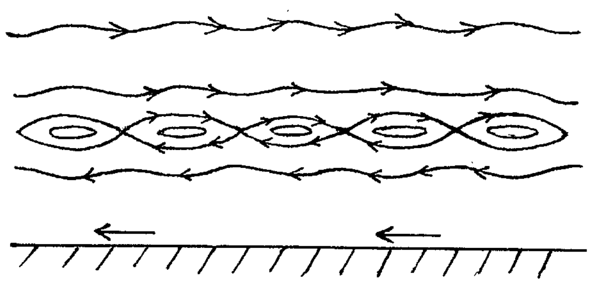


Fig. 7.

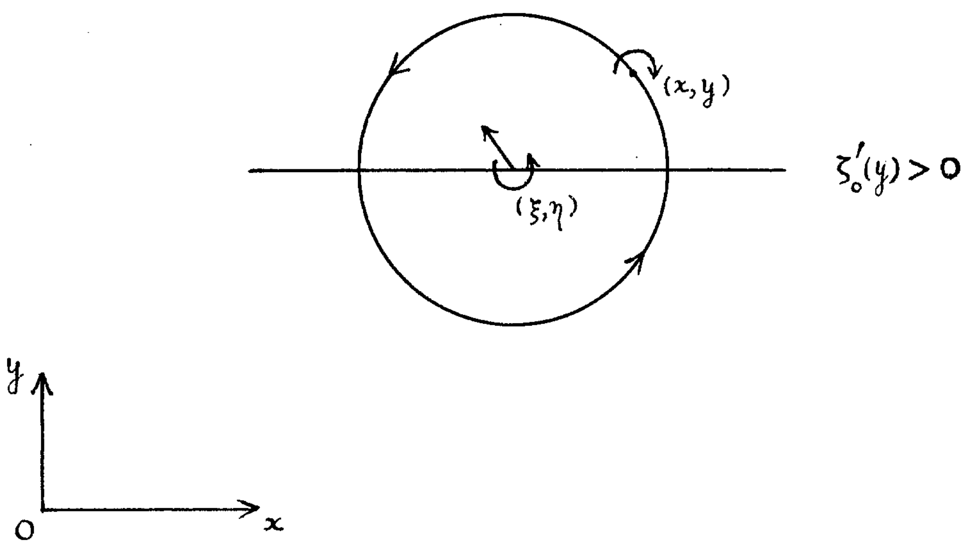


Fig. 8.

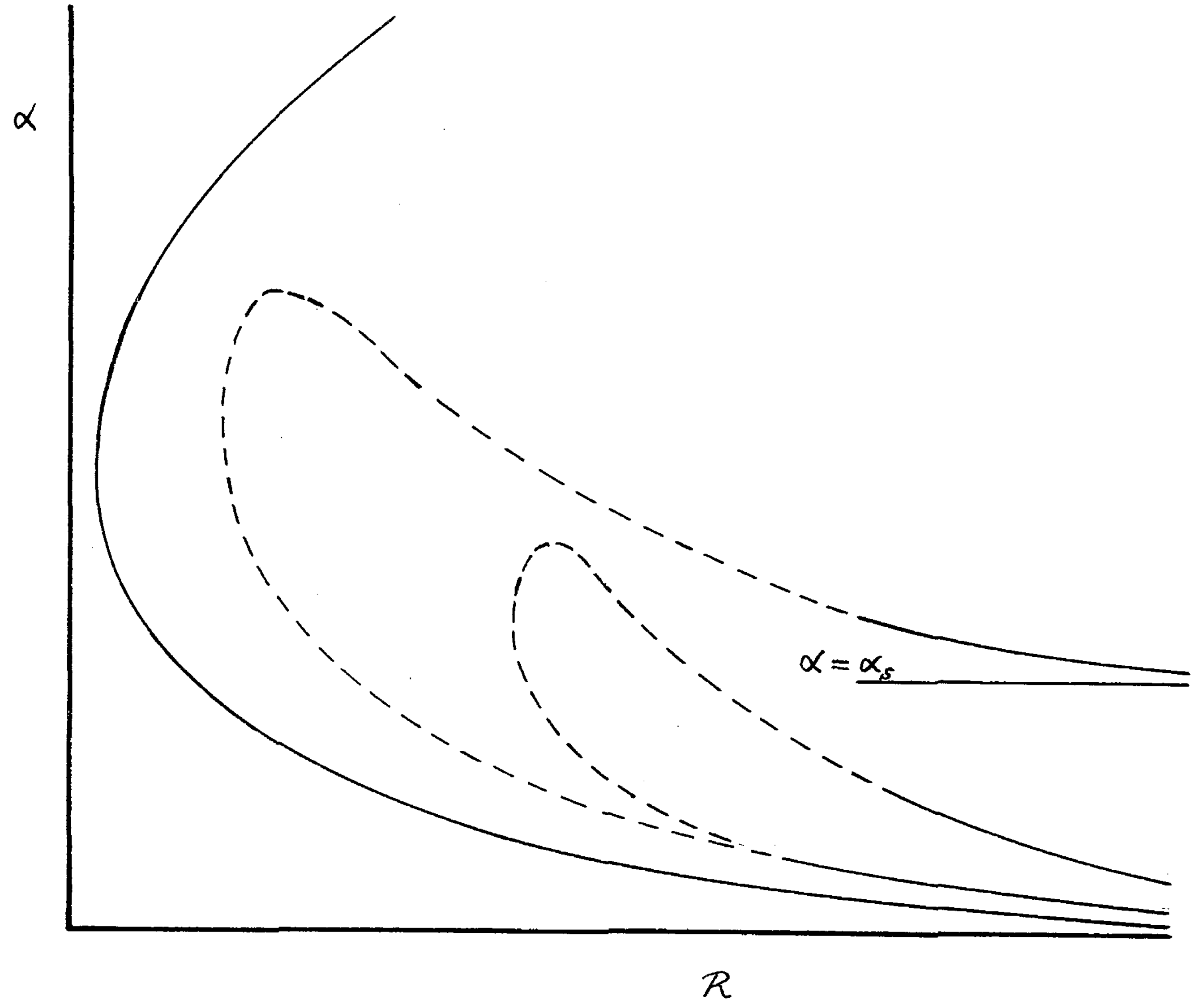


Fig. 9.

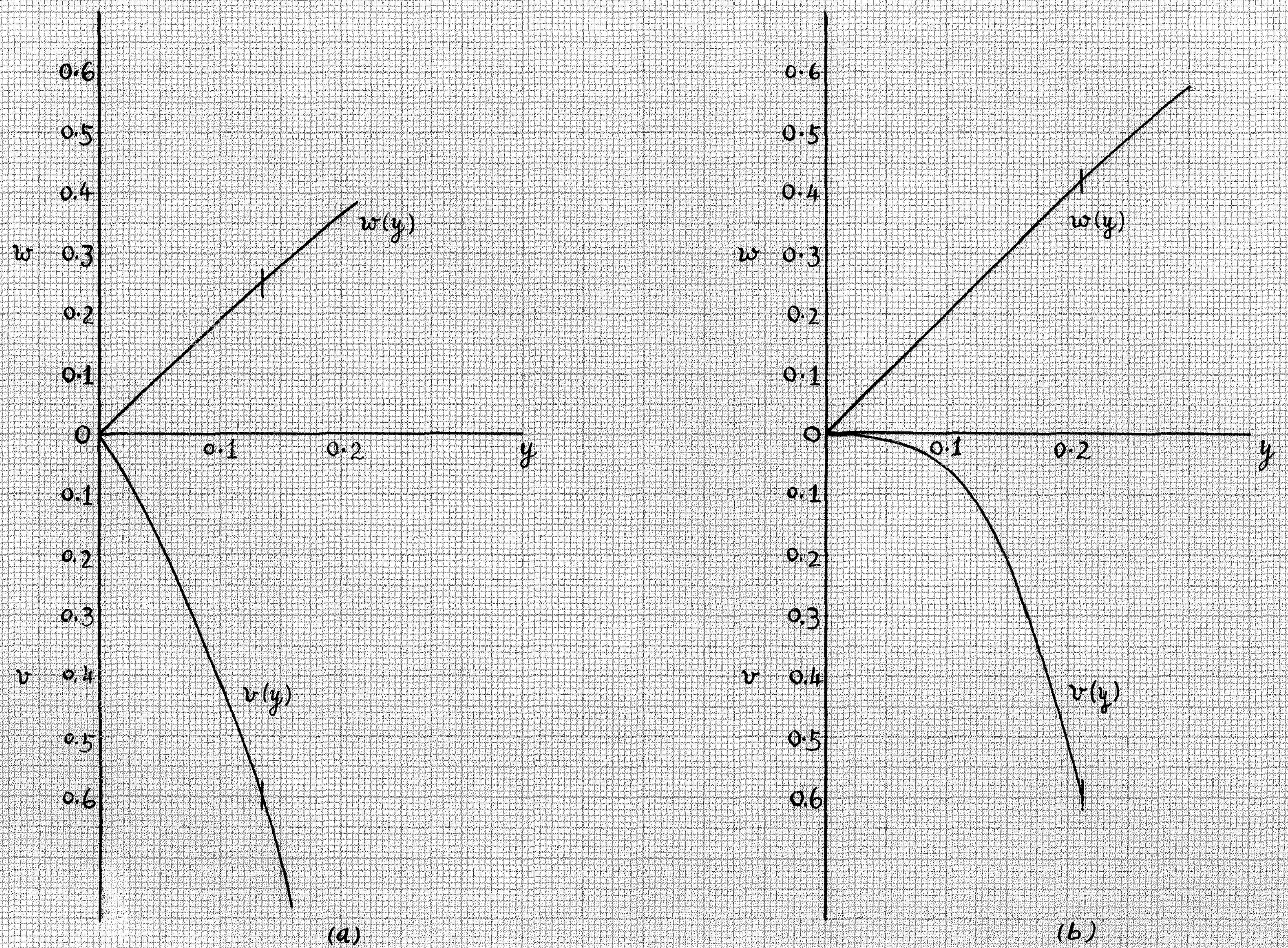
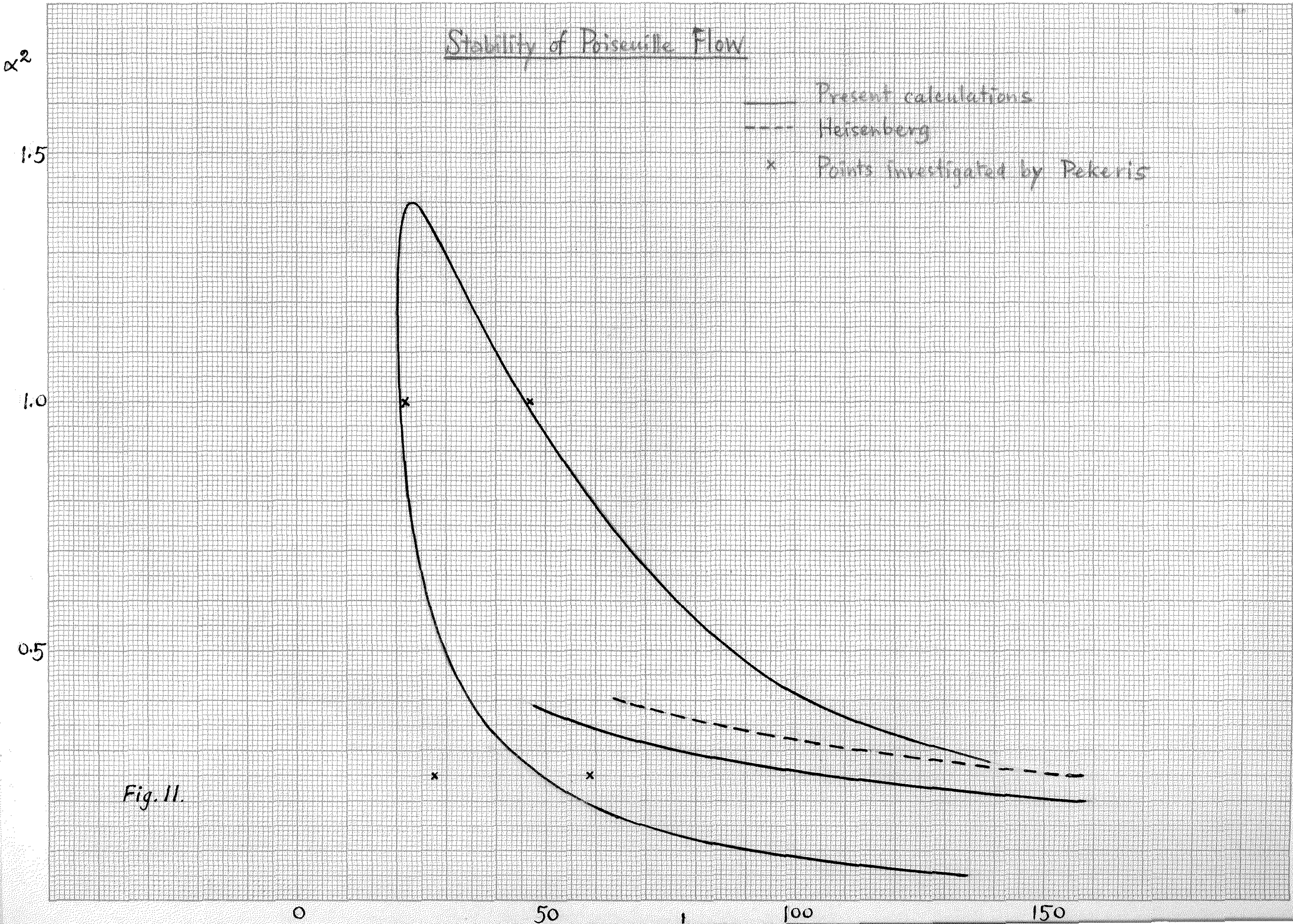


Fig. 10.

Fig. 11.



Stability of Poiseuille Flow
α^2
1.5
1.0
0.5
0
50
100
150
Present calculations
Heisenberg
Points investigated by Pekeris

Fig. 11.

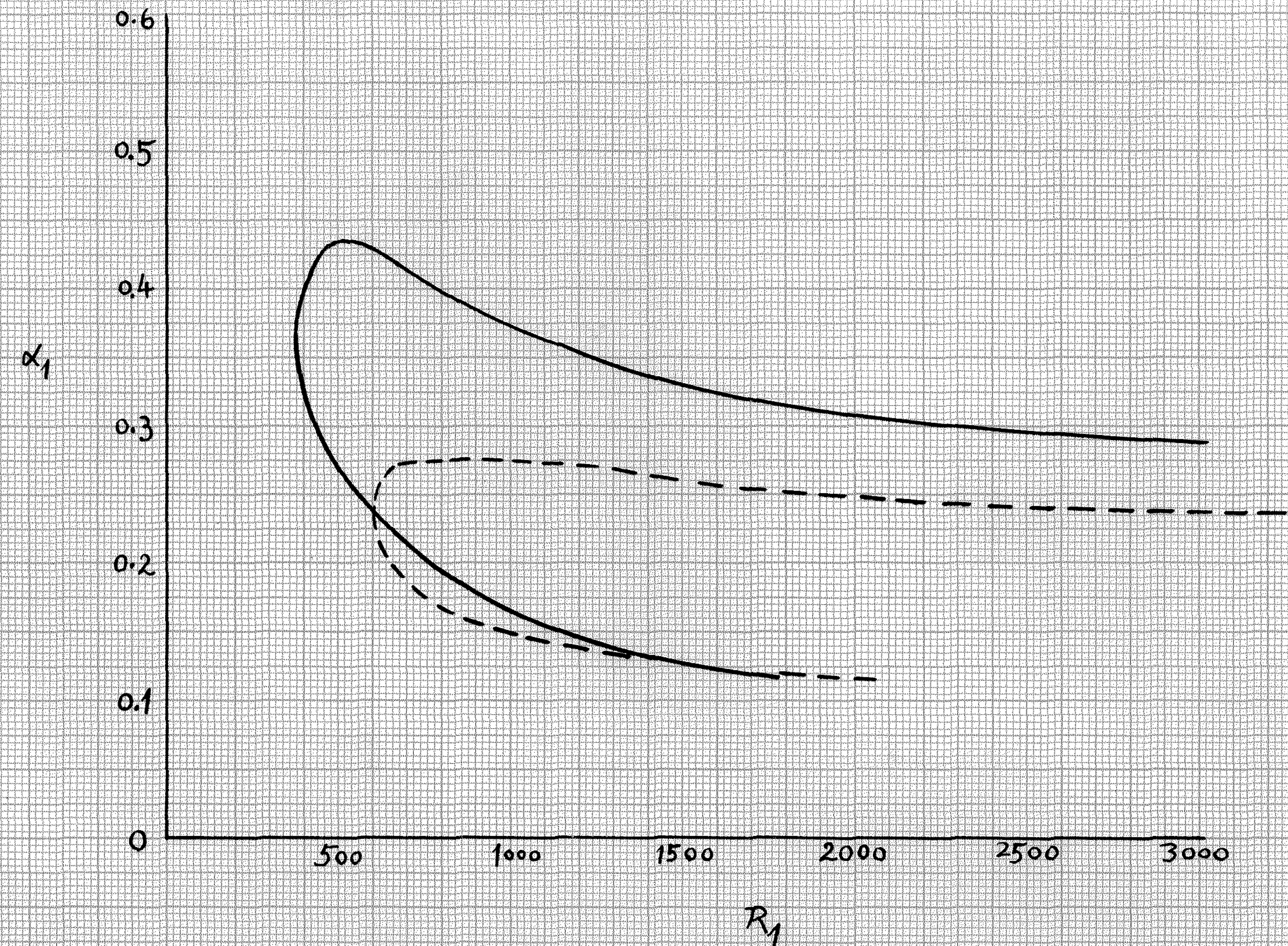
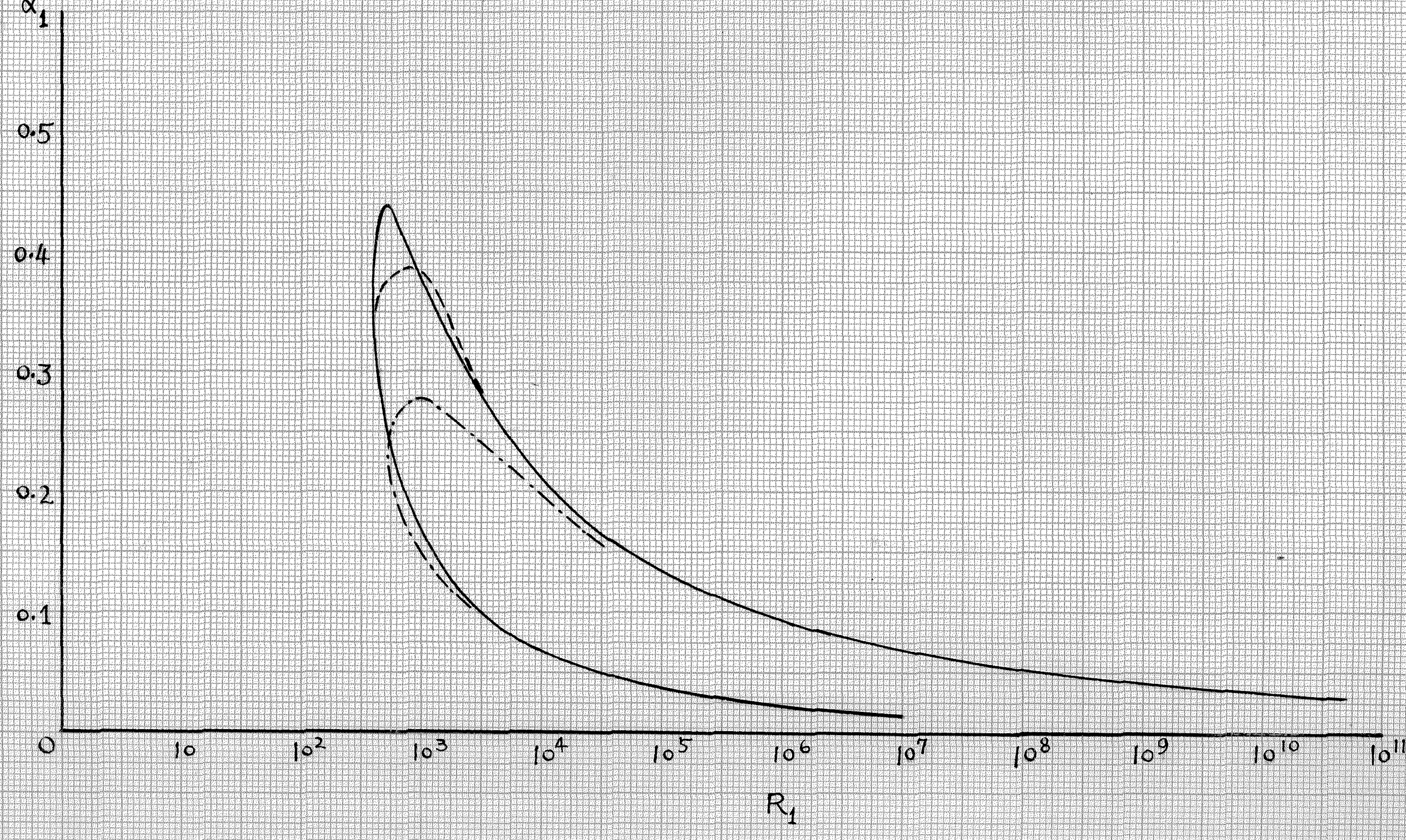


Fig. 12.



α_1
0.5
0.4
0.3
0.2
0.1
0
10
10^2
10^3
10^4
10^5
10^6
10^7
10^8
10^9
10^{10}
10^{11}
R_1

Fig. 12a