Quantum Critical Phenomena in Superfluids and Superconductors

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ABSTRACT

We discuss some problems related to quantum critical phenomena in superfluids and superconductors.

In Ch. 1, we apply generalizations of hyperuniversality to quantum phase transitions at zero temperature. We find new universal amplitude combinations involving the superfluid density in Bose systems, as well as confirm and extend previous proposals for universal transport coefficients in two-dimensional superconducting films and magnetic-field-induced metal-insulator transitions.

In Ch. 2, we apply the double-dimensionality expansion of Dorogovstev to derive renormalization-group recursion relations for a Bose fluid in a random external potential. We find a nontrivial fixed point. The onset of mean-field behavior for dimensions $d > d_c = 4$ is unconventional, yielding discontinuous exponents. Including positive temperatures, we give a clear picture of various crossover regimes.

In Ch. 3, We give a detailed derivation of the critical thermodynamics of $O(n)$ spin models, correct to $O(\epsilon = 4 - d)$, using a generalization of the renormalization-group trajectory integral and noncritical matching technique introduced by Rudnick and Nelson. We especially emphasize the coexistence-curve behavior for spins with a continuous symmetry ($n \geq 2$), deriving detailed expressions for the renormalized spin-wave stiffness and longitudinal susceptibility.

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1. HYPERUNIVERSALITY IN QUANTUM CRITICAL PHENOMENA

1.1 Introduction

In recent years there has been great interest in the interplay between superconductivity, disorder, and dissipation. In the experiments that drew much attention to this problem, the sheet resistance of the granular films, prepared by depositing soft metals such as Sn, Pb, Ga, Al, and In on insulating substrates is measured as a function of the temperature and film thickness [1,2]. Remarkably, it has been observed that the normal state sheet resistance† of the film at the boundary between superconducting behavior and nonsuperconducting behavior at low temperatures is close to $R_Q = h/4e^2 \sim 6.45k\Omega$ and apparently independent of the material and the microscopic structure of the film (see Fig. 1). Global superconductivity is established as $T \rightarrow 0$ only for $R_N < R_Q$, whereas for $R_N > R_Q$, metallic or insulating behavior is found as $T \rightarrow 0$. In some films with R_N slightly bigger than R_Q, an intriguing reentrance behavior in the resistance curve is observed: There is a significant drop in $R(T)$ at $T \sim T_c^b$, but $R(T)$ rises again to approach some finite value at $T \ll T_c^b$.

There have been a large number of theoretical attempts to explain these

† The normal state resistance R_N is defined operationally as the resistance at some specified temperature well above the material's bulk critical temperature T_c^b.

observations, most of which are based on modeling the granular film as a regular array of small dissipative Josephson junctions [3,4]. However, none of them seem to be successful in explaining the apparent universality of the critical normal state sheet resistance.

A more recent experiment on an amorphous bismuth film with uniform microscopic disorder and no granular structure shows somewhat different behavior [5] (see Fig. 2). In this case neither flat tail nor re-entrance behavior in the resistance has been observed, and the transition seems to be a direct superconductor-insulator transition rather than a superconductor-normal metal-insulator transition. The $R(T)$ curve at the boundary between the two phases is approximately temperature-independent and again its value is close to R_Q.

Stimulated by this experiment, a new apporoach to apply the scaling theory of the superfluid-Bose insulator transition [6] to the superconductor-insulator transition has been proposed [7]. In this chapter we develop this approach further, and argue that the possible universality of the sheet resistance in the $T \to 0$ limit is a straightforward consequence of the generalized hyperuniversality (or "two-scale-factor universality") [8–11] in quantum critical phenomena. Later, in Chapters 2 and 3, we take a *microscopic* approach to critical phenomena and apply the renormalization-group ϵ expansion to various phase transitions.

In Sec. 1.2 we review the concept of hyperuniversality in classical critical phenomena. We show that the Nelson-Kosterlitz universal jump [12] for the superfluid density in the $2D$ XY model follows directly from this. In Sec. 1.3 we generalize hyperuniversality to quantum critical phenomena and derive the scaling form of the superfluid density. In Sec. 1.4 we apply quantum hyperuniversality to superconducting transitions, superfluid transitions, and metal-insulator transitions, deriving many experimentally measurable universal quantities including the universal critical conductivity. Finally, in Sec. 1.6 we discuss some of the more recent experiments from the scaling theory point of view and propose directions for future work.

1.2 Hyperuniversality in classical critical phenomena

It has long been recognized [8–10] that along with any critical-point scaling relation between exponents, there should be a corresponding universal relation between critical amplitudes. For example, from the equalities $\alpha = \alpha'$ and $\gamma = \gamma'$ for the specific-heat and susceptibility exponents, above and below the critical temperature T_c, follows the universality of the corresponding ratio of amplitudes A_+/A_- and Γ_+/Γ_- [10]. *Two-scale-factor universality*, or *hyper-universality*, is the name given to the generalization of these ideas to include the classical hyperscaling relation $d\nu = 2 - \alpha$, where d is the dimensionality and ν the correlation length exponent: $\xi \approx \xi_0^{\pm}|t|^{-\nu}$ for $t = (T - T_c)/T_c \gtrless 0$.

When hyperscaling is valid the singular parts of the free-energy density integrated over a correlation volume (measured in units of $\beta^{-1} = k_B T$), $\beta \xi^d f_{sing}$, are dimensionless constants when $t \to 0^{\pm}$. Hyperuniversality is the statement that these constants are universal [8,9]. Equivalently, the amplitude combinations $R_\xi^{\pm} = A_{\pm}(\xi_0^{\pm})^d$, where $C_{sing}/k_B \approx (A_{\pm}/\alpha)|t|^{-\alpha}$, are universal. For a renormalization-group proof, see Ref. 9. See also the recent review by Privman *et al.* [11] It follows also that in systems with a continuous symmetry (order-parameter dimensionality $n \geq 2$), for which $\xi_0^- \equiv \infty \equiv \Gamma_0^-$, one can extract an alternative diverging length $\xi_\Upsilon = (\Upsilon/k_B T)^{1/(2-d)} \approx \xi_0^\Upsilon |t|^{-\nu}$ below T_c ($d > 2$), with the corresponding universal ratio ξ_0^Υ/ξ_0^+. These are the famous spin-wave divergences. See, e.g., Ref. 13 for a detailed discussion. See also Ch. 3 for explicit calculations within the ϵ expansion formalism. Here $\Upsilon \approx \Upsilon_0 |t|^v$ is the helicity modulus, related to the superfluid density via $\rho_s = (m/\hbar)^2 \Upsilon$, where m is the particle mass (or Cooper pair mass in superconductors), while the corresponding exponent relation is the Josephson hyperscaling relation $v = (d-2)\nu$.

Hyperuniversality may also be applied to finite-size systems [14]. If, for simplicity, one considers a classical cubically shaped system of volume L^d, finite-size hyperuniversality states that at the *bulk* critical temperature, $T = T_c$, $\lim_{L\to\infty} \beta_c L^d f_{sing}$ is a universal constant, depending in general only on the sample shape (here assumed cubic) and boundary conditions. In particular, for applications to the superfluid density, if one chooses periodic boundary conditions in the first $(d-1)$ dimensions, and imposes an order-parameter phase-angle twist of θ across the final dimension [13], one should find

$$\lim_{L\to\infty} \beta_c L^d f_{sing}(\theta) = F(\theta) \ , \tag{1}$$

where $F(\theta)$ is a universal function. Since we may assume that all boundary-condition dependence of the free energy is contained in its singular part [14], the helicity modulus at T_c is then given by

$$\begin{aligned}\beta_c \Upsilon(T_c) &= \beta_c \frac{2}{\theta^2} \lim_{L\to\infty} L^2 [f_{sing}(\theta) - f_{sing}(0)] \\ &= \lim_{L\to\infty} \frac{2L^{2-d}}{\theta^2}[F(\theta) - F(0)] \ .\end{aligned} \tag{2}$$

Clearly, for $d > 2$, this correctly predicts $\Upsilon_c \equiv \Upsilon(T_c) = 0$, while in $d = 2$ one finds $F(\theta) = F(0) + F_2\theta^2$ for $|\theta| \leq \pi$, and $\beta_c \Upsilon_c = 2F_2$ is a *universal number.* This yields automatically the Nelson-Kosterlitz universal jump for the superfluid density in two-dimensions: Since one knows from detailed calculations in this case that $\beta_c \Upsilon_c = 2/\pi$, we predict that $F_2 = 1/\pi$. A caution is necessary here: The number F_2 is associated with a given fixed point, while in two dimensions one tends to have lines, or even higher-dimensional surfaces of fixed points with (perhaps several) associated marginal variables. The value of F_2, and in general, of any other universal quantity, will vary along these fixed surfaces and will be specified uniquely only if all marginal variables are specified. For the Kosterlitz-Thouless transition the unique fixed point describing $T = T_c$ is specified by the marginality of the vortex degrees of freedom at $\beta_c \Upsilon_c = 2/\pi$, or exponent $\eta = 1/4$ [15].

In dimensions $d > 2$ the function $F(\theta)$ is no longer quadratic in θ. If one assumes, as is very likely, that θ-boundary conditions in a finite system of size L are essentially equivalent to a uniform order-parameter twist with wave vector $k_0 = \theta/L$ in an infinite system, appropriate scaling of k_0 with ξ (see, e.g., Eq. (5) below) predicts that $f_{sing}(k_0) - f_{sing}(0) \propto k_0^d$ at T_c with a universal coefficient. This yields $F(\theta) = F(0) + F_d|\theta|^d$, $|\theta| \leq \pi$.

1.3 Hyperuniversality in quantum critical phenomena

We now generalize hyperuniversality to the case of quantum critical phenomena at zero temperature. The generalization itself is very straightforward, but the applications and consequences are very deep. We will recover the universal transport coefficients of Ref. 7 in a concise and unified way. We make new predictions for universal ratios involving ρ_s in two- and three-dimensional Bose systems. Applying the same ideas to metal-insulator transitions yields new predictions for universal transport coefficients that are consistent with those of previous work.

Consider then a continuous phase transition at temperature $T = 0$, as a function of a parameter such as the magnetic field H, the particle mass density ρ, or the chemical potential μ, which we denote generically by the dimensionless quantity δ. We assume that $\delta = 0$ defines the critical point, while $\delta > 0$ denotes the disordered phase and $\delta < 0$ denotes the ordered phase. Quantum fluctuations must dominate the critical instability because of the absence of the thermal fluctuation modes at $T = 0$. Unlike the classical case, statics and dynamics are no longer separable, since even the equilibrium properties depend on the details of the quantum dynamics [16]. At $T = 0$ and small $|\delta|$, one defines two correlation lengths, for definiteness, via the rate of the exponential decay of the Matsubara Green function: (a) The usual spatial correlation length $\xi \approx \xi_0^{\pm}|\delta|^{-\nu}$ and (b) the temporal correlation length $\xi_\tau \approx \xi_{\tau,0}^{\pm}|\delta|^{-\nu_\tau}$. We take ξ_τ to have the same units as β, i.e., inverse energy. The combination $\hbar\xi_\tau$ then has units of time. When $n \geq 2$, we have $\xi_0^- \equiv \infty \equiv \xi_{\tau,0}^-$ and Υ is again required: see below. The dynamical exponent z is defined as the ratio $z = \nu_\tau/\nu$. At positive temperatures the imaginary temporal extent, $0 \leq \tau \leq \beta$, of the system is finite. Thus ξ_τ can never diverge and the quantum dynamics cannot affect the static critical behavior: This is the usual statement of irrelevancy of quantum mechanics at finite temperatures. Only at $T = 0$ may ξ_τ diverge. When it does, the usual classical hyperscaling argument must be modified. Since the free-energy density is $f = -(\beta V)^{-1} ln(Z)$, Z being the partition function and V the volume, where now *both* β and V diverge in the thermodynamic limit, the natural hyperscaling ansatz is that $f_{sing} \sim \xi^{-d}\xi_\tau^{-1} \sim |\delta|^{2-\alpha}$. This yields the generalized hyperscaling relation $2-\alpha = (d+z)\nu$. The corresponding amplitude relation is that $\xi^d\xi_\tau f_{sing}$ should be universal when $|\delta| \to 0$. Equivalently

$$R_\tau^{\pm} = \xi_{\tau,0}^{\pm}(\xi_0^{\pm})^d A_{\pm} \qquad (3)$$

are universal amplitude combinations. Equation (3) is the basic result, which may be derived more formally within the renormalization-group framework by a straightforward generalization of the Appendix of Ref. 9 (see Appendix). The Josephson hyperscaling relation is generalized similarly: One finds [6] $v = (d-2+z)\nu$. The related ordered-phase diverging length is now

$$\xi_\Upsilon(\delta) = [\xi_\tau(-\delta)\Upsilon(\delta)]^{1/(2-d)} \approx \xi_0^\Upsilon |\delta|^{-\nu}, \delta \to 0^- \tag{4}$$

(note that ξ_τ replaces β in the finite temperature result), and $R_\Upsilon \equiv \xi_0^\Upsilon/\xi_0^+$ is universal. In two dimensions Eq. (4) is problematical, and a better approach is to use Υ to define a divergent temporal scale: $\xi_\tau^\Upsilon(\delta) \equiv \xi(-\delta)^{2-d}\Upsilon(\delta)^{-1} \approx \xi_{\tau,0}^\Upsilon|\delta|^{-z\nu}$, $\delta \to 0^-$, and $R_\Upsilon^\tau \equiv \xi_{\tau,0}^\Upsilon/\xi_{\tau,0}^+$ is universal. This definition does not run into any problems in two dimensions. If the ordered phase has a propagating mode, such as second-, or higher-order, sound in ^{4}He, spatial and temporal scales may be related to one another through the speed of sound, and often the exponent z may be determined explicitly [6]. We do not address this issue here, however.

All of the results to follow can be based on various universal scaling forms for the superfluid density, and quantities derived from them. We begin with the scaling of the singular part of the free-energy density in the presence of an imposed order-parameter twist with wave vector k_0 [6,13]:

$$f_{sing} \approx A|\delta|^{2-\alpha}\Phi_\pm(Bk_0|\delta|^{-\nu}) \ , \tag{5}$$

from which one derives

$$\Upsilon(\delta) = \lim_{k_0 \to 0} \frac{\partial^2 f_{sing}}{\partial k_0^2} = AB^2 |\delta|^{2-\alpha-2\nu} \Phi''_\pm(0) \ . \qquad (6)$$

Here Φ'' denotes a second derivative with respect to the argument. A convenient normalization is to choose $\Phi_-(0) = \Phi''_-(0) = 1$, making $\Phi_\pm(x)$ universal. Clearly $\Phi''_+(0) = 0$, and, with the standard definition [8–10] $A_\pm = -A\alpha(1-\alpha)(2-\alpha)\Phi_\pm(0)$, one has $A_+/A_- = \Phi_+(0)/\Phi_-(0)$. Note that at $T = 0$ we define α and $A_\pm$ via $-\partial^2 f_{sing}/\partial\delta^2 \approx (A_\pm/\alpha)|\delta|^{-\alpha}$. Universality requires that $B|\delta|^{-\nu}$ be universally related to ξ, in the present case finite only for $\delta > 0$. Thus $R_B = B/\xi_0^+$ is universal. Hyperuniversality implies that $R_\tau \equiv A(\xi_0^+)^d \xi_{\tau,0}^+$ is universal. One then has $R_\Upsilon = [R_\tau R_B^2 \Phi''_-(0)]^{1/(2-d)}$ and $R_\Upsilon^\tau = R_\Upsilon^{d-2}$, which are indeed universal.

Equation (6) can be extended in various ways. Of interest here are the extensions to small, but finite, temperature and frequency. We define the frequency-dependent superfluid density in terms of the temporal Fourier transform of the usual momentum-momentum (or current-current) correlation function [7]. The general scaling form we expect is

$$\Upsilon_{sing}(\delta, T, \omega) = AB^2|\delta|^{2-\alpha-2\nu} Y_\pm(C\hbar\omega|\delta|^{-z\nu}, D\beta^{-1}|\delta|^{-z\nu}) \ , \qquad (7)$$

where we also allow for a *regular* contribution to Υ, which, however, must *vanish* at $\omega = 0$ (see below). Universality requires that $R_C \equiv C/\xi_{\tau,0}^+$ and $R_D \equiv D/\xi_{\tau,0}^+$ be universal, and clearly $Y_\pm(0,0) = \Phi''_\pm(0)$.

1.4 Applications

1.4.1 Superconducting transitions

As a first application of Eq. (7), we consider the bosonic models of amorphous and granular superconductors [3,4,7] in which Cooper pairs are treated as conserved particles obeying Bose statistics, and unpaired electrons are either ignored or included as an effective harmonic oscillator heat bath [17]. See Ref. 7 for some discussion of the validity of these simplified models near the critical point. The frequency-dependent conductivity of these models is simply given by $\sigma(\delta,T,\omega) = (4e^2/\hbar)\Upsilon(\delta,T,-i\omega)/(-i\hbar\omega)$, where $2e$ is the Cooper pair charge. Consider now approaching the critical point at $\delta = 0$, $\omega = 0$, $T = 0$ along some path in the (δ, ω, T) space in such a way that $x = C\hbar\omega|\delta|^{-z\nu}$, and $y = D\beta^{-1}|\delta|^{-z\nu}$ approach some fixed values x_0, y_0 ($x_0 = 0$ or ∞ and $y_0 = \infty$ are probably the most useful experimentally). In order to get a reliable value of the critical conductance, it is essential to analyze experimental data using these scaling variables. One finds then

$$(\hbar/4e^2)\xi(|\delta|)^{d-2}\sigma_{sing} \to R_\sigma(x_0,y_0) \equiv R_\tau R_B^2 R_C Y_\pm(-ix_0,y_0)/(-ix_0) \qquad (8)$$

so that in particular, in $d = 2$ the limiting value of $(\hbar/4e^2)\sigma_{sing}$ is itself universal. A tacit assumption here is that no logarithmic factors appear: These are expected at the critical dimensions for the transition. For most applications of (8), the lower critical dimension is $d_< = 1$, while the upper critical dimension is at least $d_> = 4$. Hence no problems are expected in $d = 2$.

When the $\delta > 0$ phase is an insulator, as when the model does not include a heat bath (i.e., is purely bosonic), any analytic nonuniversal background conductivity must vanish when $\omega = 0$, independent of δ. In this case one may drop the subscript on σ_{sing} in (8). When the model includes a heat bath, which probably corresponds more closely to experimental reality at least in the case of granular films, the $\delta > 0$ phase may be a metal, and may possess an analytic background conductivity $\sigma_0(\delta, T, \omega) = \sigma_{0,0} + \sigma_{0,1}\delta + \sigma_{0,2}\omega + \cdots$, in which $\sigma_{0,i}(T)$ are nonuniversal. In $d = 2$ one will find $(\hbar/4e^2)\sigma \rightarrow R_\sigma(x_0, y_0) + (\hbar/4e^2)\sigma_{0,0}$, in place of (8). Often $\sigma_{0,0}$ is found to be very small, and hence, since $R_\sigma(x_0, y_0)$ is expected to be of order unity [7], one may simply ignore its existence. In general, one must take the difference between limits for two different values of $x_0 = x_1, x_2$ and $y_0 = y_1, y_2$ to obtain the universal result $R_\sigma(x_1, y_1) - R_\sigma(x_2, y_2)$.

1.4.2 Superfluid transitions

As a second application of (7), we consider the recent scaling theory of the superfluid to Bose glass transition in disordered boson systems [6]. The results are equally applicable to the previous nondissipative models of amorphous and granular superconductors, though Υ is much harder to measure experimentally in these cases. We consider (7) with $\omega \equiv 0$ but $T > 0$. We now assume, as is often the case, that there is a line of finite temperature transitions, $T_c(\delta)$, ending at the special point $T = 0$, $\delta = 0$. The scaling form (7) then requires that

$$k_B T_c(\delta) = (y_c/D)|\delta|^{z\nu} \ , \tag{9}$$

where y_c is the universal value of the scaling function argument at which $Y_\pm(0,y)$ displays the finite temperature singularity. Thus

$$\Upsilon(T=0,\delta)/k_B T_c(\delta) \approx [R_\tau R_B^2 R_D Y_\pm(0,0)/y_c]\xi(T=0,|\delta|)^{2-d}, \tag{10}$$

so that in $d=2$, $\beta_c(\delta)\Upsilon(0,\delta)$ is a universal number in the limit $\delta \to 0^-$ and $T_c(\delta) \to 0$. It follows also that $\Upsilon(0,\delta) \propto T_c(\delta)^{(d+z-2)/z}$, with a nonuniversal coefficient of proportionality when $d \neq 2$. The exponent was predicted in Ref. 6, but the possibility of universal ratios was not examined.

It has again been assumed that $d=2$ is not a critical dimension for the $T=0$ transition. For a *clean* interacting Bose gas, $d=2$ is the *upper* critical dimension [6], and for the continuum problem the transition takes place at zero density, ρ. For this case, in the limit where $\ln\ln(m/\rho a^2) \gg 1$ (probably an experimentally inaccessible limit), one finds [18]

$$\Upsilon(T=0,\rho)/k_B T_c(\rho) \approx \frac{1}{2\pi}\ln\ln(m/\rho a^2) \ , \tag{11}$$

where a is the atomic hard core diameter. Thus in order to obtain a universal ratio (in this case $1/2\pi$), the double logarithm should be divided out as well.

As a final point, it is also possible to construct more complicated universal amplitude combinations in three-dimensional Bose systems, an example of which is

$$\lim_{\delta \to 0^-} \hbar C_s(T=0,\delta)[\Upsilon(T=0,\delta)]^2/[k_B T_c(\delta)]^2 \ , \tag{12}$$

where C_s is the fourth sound speed [6]. All of the input parameters are in principle experimentally measurable.

1.4.3 Metal-insulator transitions

As our final application we speculate briefly about applying hyperscaling theory to the metal-insulator transition. Since there is no superfluid density in this case, we study the behavior of the current-current correlation function directly. We assume, without justification, that hyperscaling is indeed valid, and hence that the current-current correlation function scales in the same way the superfluid density would. Thus (7,8) are still valid, with the appropriate generalizations of the notions of correlation length and time. Thus in $d = 2$ one again expects a universal limiting conductance at the critical point. Since σ is finite on both sides of the transition, presumably zero on the localized side, this may be rephrased as a prediction for a universal jump, $[\sigma(\delta = 0^-) - \sigma(\delta = 0^+)]$, of the static conductivity. We again emphasize that $d = 2$ should not be critical, so the results do not apply to the standard Anderson transition. Models with strong spin-orbit scattering, however, may show the predicted behavior. The metallic conductivity here is believed to be infinite so rather than a universal jump, one should observe a universal critical conductivity, as in the superconducting case [19]. Similar arguments apply to the diagonal and Hall conductivities at the transition between plateaus in the quantum Hall effects [7,20]. On the insulating side of the transition, one may look at the dielectric constant $\epsilon(\omega) = 1 + 4\pi i\sigma(\omega)/\omega$, which, when combined with the correlation lengths, yields the hyperuniversal combination $\lim_{\delta\to 0^+}(1/e^2)\xi^{d-2}\xi_\tau^{-1}\epsilon_{sing}(\omega = 0)$. Thus ϵ_{sing} diverges as $|\delta|^{-\lambda}$ with $\lambda = (2 + z - d)\nu$.

In many cases the magnetic field H is a thermodynamically relevant perturbation at the $H = 0$ metal-insulator transition, since it breaks the symmetry between positive and negative winding numbers in the coherent backscattering picture of localization. The relevant length scale is set by $\sqrt{\Phi_0/H}$, where $\Phi_0 = \hbar c/e$ is the flux quantum. This quantity should then appear scaled by ξ as a *third* argument, $z = G|\delta|^{-\nu}\sqrt{H/\Phi_0}$, in (7), with $R_G \equiv G/\xi_0^+$ universal. Various further universal ratios may now be defined. A simple example is to consider

$$\sigma(\delta = 0, T = 0, H) \approx (e^2/\hbar)R_\tau R_B^2 R_C R_G^{d-2} Y_\infty (H/\Phi_0)^{(d-2)/2} \ , \qquad (13)$$

where $Y_\infty = \lim_{z\to\infty} z^{2-d} Y_\pm(0,0,z)$ which gives $\sigma \propto H^{(d-2)/2}$ with a *universal coefficient*. Finally, the transition at small H must take place at a universal value, z_c, of the argument of $Y_\pm(0,0,z)$. This yields $\delta_c(H) \propto H^{1/2\nu}$. The constant of proportionality is nonuniversal, but this relation gives an experimental means of extracting the exponent ν. These results have been derived within a much less general framework in Ref. 21. We note that the result $|\delta_c(H)| \propto H^{1/2\nu}$ should be useful for superconductor-insulator transitions, too. Some data on magnetic-field dependence in the thin-film experiments are available [1].

1.5 Discussion

In this section we briefly review the results of some of the more recent experiments on the superconductor-insulator transition and compare them with the predictions of the hyperscaling theory. We also suggest directions for future work.

As for the transition tuned by the film thickness, there have been measurements on amorphous films of materials other than bismuth, and somewhat surprisingly, the measured critical resistance is not the same as R_Q and no universality is observed [5,22]. A possible explanation for the apparent nonuniversality is that in these experiments, sufficiently low temperatures to reach the quantum critical regime have not yet been achieved. This explanation is supported by the mean-field-like behavior of the resistance data on the superconducting side. Data [Fig. 2] show that the mean-field transition temperature T_c^0 drops rapidly as the critical point is approached from the superconducting side. Since the non-mean-field quantum scaling behavior is expected only when $T < T_c^0$, the critical regime shrinks rapidly ($|\delta| \sim T^{1/z\nu} < (T_c^0)^{1/z\nu}$).

More reliable measurements have been made on the superconductor-insulator transition across the upper critical field at $T = 0$ in amorphous indium oxide films by Hebard and Paalanen [23]. They find the universality of

$R_\sigma(0,\infty)$ [Eq. (8)] by a full scaling analysis of the dc conductivity. In fact, their data, combined with a measurement of $\xi^{\pm}_{\tau,0}$, can be used to derive the universal function $R_\sigma(0,y)$. However, the critical Hall conductivity is observed to depend on the strength of the disorder in the film and is therefore nonuniversal [24]. This observation is clearly inconsistent with the scaling theory prediction and needs to be understood.

A superconductor-insulator transition has also been studied in artificially built Josephson junction arrays by tuning the ratio of the Josephson coupling energy to the electrostatic charging energy [25]. This transition is of considerable interest, because for these systems, it is possible to calculate the universal quantities using the ϵ expansion or $1/N$ expansion and compare them with experimental data. Furthermore, the arrays may show many new universality classes that can also be studied by the renormalization-group method [26,27].

More careful experiments on the apparent superconductor-metal transition in granular films may be interesting, too. The quantum critical regime is expected to be wider in granular films than in amorphous films, since the mean-field transition temperature of the granular film at the onset of superconductivity is observed to be finite and relatively large. In this case, one may have to take the strong smearing effects of the transition into account to deduce the critical conductance, which we predict to be nonuniversal, from experimental data.

Finally, it may be possible to test the predictions in Sec. 1.4.2 in some ongoing experiments on the superfluid-Bose insulator transition [28].

Appendix Proof of quantum hyperuniversality

We consider a renormalization-group transformation for the free energy of $(d+1)$-dimensional classical systems with anisotropic correlations*, where the length scales of d "spatial" dimensions are changed by a factor b and the length scale of the remaining "temporal" dimension is changed by a factor b^z. With each renormalization-group iteration, a constant contribution to the free energy is generated. The total free energy density f can be calculated by summing these contributions over many iterations:

$$f = \sum_{m=0}^{\infty} b^{-(d+z)m} G(m) \ , \tag{A1}$$

where $G(m)$ is the constant term generated at the m'th iteration. For convenience, we consider a sequence of infinitesimal transformations with $b = e^{\delta}$, $0 < \delta \ll 1$, and $l = m\delta$. Then we get

$$f = \int_0^{\infty} dl e^{-(d+z)l} G_0(l) \ , \tag{A2}$$

where

$$G_0(l) = \partial G / \partial b |_{b=1} \ . \tag{A3}$$

$G_o(l)$ depends on the scaling fields $g_i(l) = e^{\lambda_i l} g_i$, among which, we assume, $g_\delta(l)$ is the only relevant scaling field and $\lambda_\delta = 1/\nu$. We note that the correlation length in the spatial direction ξ and the correlation length in the temporal direction ξ_τ satisfy

$$\xi(l) = e^{-l} \xi(l=0) = e^{-l} \xi \tag{A4}$$

* These include d-dimensional quantum systems as special cases.

and

$$\xi_\tau(l) = e^{-zl}\xi_\tau(l=0) = e^{-zl}\xi_\tau \; . \tag{A5}$$

For δ slightly bigger than zero, we now define $\bar{l}$ such that $[\xi(\bar{l})]^d\xi_\tau(\bar{l}) = 1$; i.e.,

$$\xi^d\xi_\tau = e^{(d+z)\bar{l}} \; . \tag{A6}$$

We also choose $\tilde{l}$ to be a value of l sufficiently large, so that for $l \geq \tilde{l}$, all irrelevant variables have died away and are negligible. We further assume that δ is sufficiently small so that $\bar{l} \gg \tilde{l}$. Since one has

$$g_\delta(l) = g_\delta(\bar{l})e^{-(\bar{l}-l)/\nu} \; , \tag{A7}$$

for $l \gtrsim \tilde{l}$, $G_0(l)$ is actually a function of $(l-\bar{l})$, and furthermore, apart from an additive nonuniversal constant (which does not contribute to the singular part of the free energy density f_s), $G_0(l) = \phi(l-\bar{l})$ is a universal function characteristic of the renormalization-group flow along the relevant trajectory. Expanding $G_0(l)$ in powers of $e^{-(\bar{l}-l)/\nu}$, we get

$$G_0(l) = \phi(l-\bar{l}) = \sum_{n=0}^{\infty} \phi_n e^{-n(\bar{l}-l)/\nu} \; , \tag{A8}$$

where ϕ_n's are universal. We now divide the integral into three parts,

$$\int_0^\infty = \int_0^{\tilde{l}} + \int_{\tilde{l}}^{\bar{l}} + \int_{\bar{l}}^\infty \; .$$

The first integral is nonuniversal, but analytic in δ. The second integral is

$$\int_{\tilde{l}}^{\bar{l}} dl e^{-(d+z)l} G_0(l) = \sum_{n=0}^{\infty} \phi_n \frac{e^{-(d+z-n/\nu)\tilde{l}} e^{-n\bar{l}/\nu} - e^{-(d+z)\bar{l}}}{d+z-n/\nu} \; . \tag{A9}$$

The terms proportional to $e^{-n\bar{l}/\nu}$ are regular, since $e^{-n\bar{l}/\nu} = (\xi^d \xi_\tau)^{-n/(d+z)\nu} \propto \delta^n$. The last integral is of the form

$$\int_{\bar{l}}^{\infty} dl e^{-(d+z)l} G_0(l) = a e^{-(d+z)\bar{l}} \ , \tag{A10}$$

where

$$a = \int_0^{\infty} dl' e^{-(d+z)l'} \phi(l') \tag{A11}$$

is universal. Therefore

$$f_s = a_+ e^{-(d+z)\bar{l}} = a_+ \xi^{-d} \xi_\tau^{-1} \ , \tag{A12}$$

where

$$a_+ = a - \sum_{n=0}^{\infty} \frac{\phi_n}{d + z - n/\nu} \tag{A13}$$

is universal. Thus $(f_s \xi^d \xi_\tau)_{\delta \to 0^+}$ is universal.

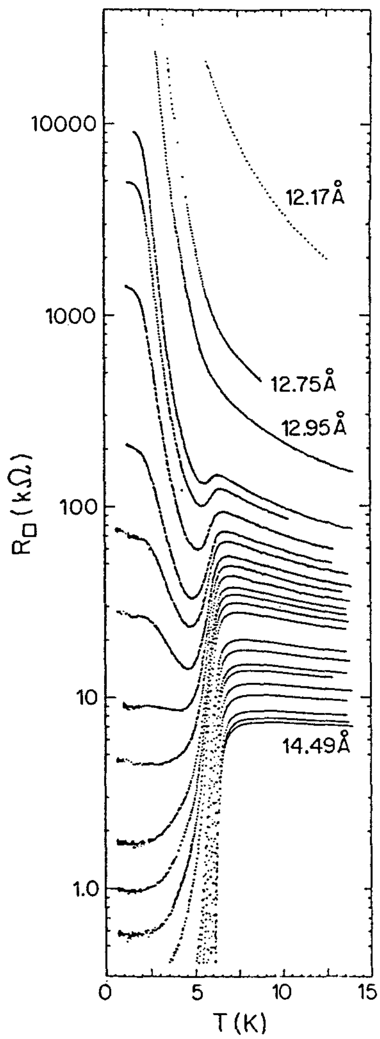

Fig. 1 Evolution of the temperature dependence of the sheet resistance $R(T)$ with thickness for a Ga film [1].

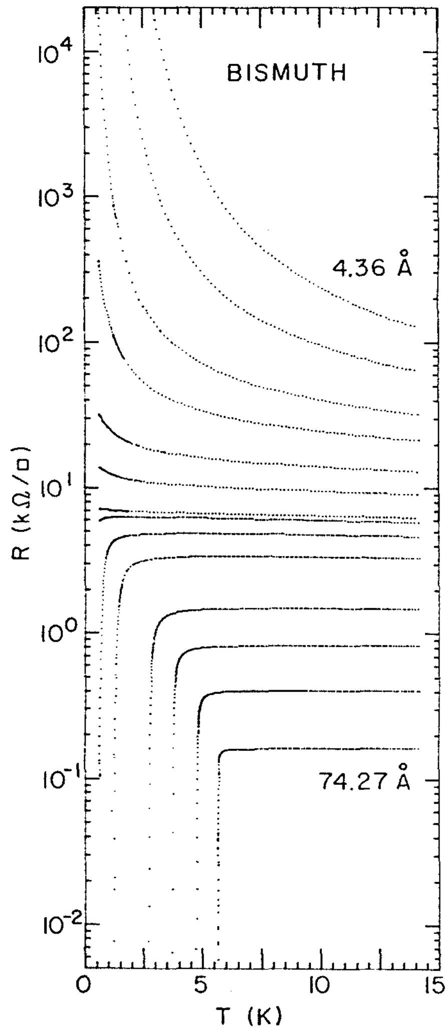

Fig. 2 Evolution of the temperature dependence of the sheet resistance $R(T)$ with thickness for a Bi film deposited onto Ge [5].

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2. DIMENSIONALITY EXPANSION FOR THE DIRTY-BOSON PROBLEM

2.1 Introduction

Over the past decade, enormous effort has been invested in trying to understand random-electron systems at zero temperature. In contrast, until recently, the corresponding boson problem has remained essentially unaddressed. This is in spite of the many experimental realizations of such systems, for example, ^{4}He adsorbed in various porous random media [1,2]. Of particular interest is the nature of the insulator to superfluid *onset* transition as the boson density n is increased through some critical density n_c at $T = 0$, and also how this onset transition affects the finite temperature λ transitions at fixed densities $n > n_c$.

The system that has received the most attention is ^{4}He adsorbed in porous Vycor glass [1,3,4], ironically, for the reason that it primarily displays the behavior characteristic of a pure nonrandom Bose fluid [3]. In fact, for very low coverages, $n - n_c \ll a^{-3}$, where a is the range of interactions, a crossover to *ideal* Bose-gas critical behavior is observed.

The reasons for the apparent invisibility of disorder in Vycor were explained qualitatively in Ref. 4 on the basis of scaling arguments, and the process of spinodal decomposition by which Vycor is made. However, a true quantitative understanding of the nature of the onset transition was still lacking. In this chapter we fill this gap by analyzing a model of bosons in a random external potential using the double-dimensionality expansion of Dorogovstev [5,6], deriving lowest-order renormalization-group recursion relations. The resulting fixed-point structure clearly elucidates the relation between onset at $T = 0$ and scaling near T_λ. In particular, for very weak disorder, there is a *range* of coverages over which the pure crossover to ideal gas behavior should be observed. Only at very low coverages is the $T = 0$ disorder-dominated onset regime encountered, and should deviations from pure behavior become visible. The Vycor experiments [1] have probably not yet entered this regime.

Once inside the random onset region, various predictions can be made [7,8,9]. For example, the temperature can be treated within a finite-size scaling formalism, and this allows the prediction of various exponents, such as that which gives T_λ as a function of $n - n_c$. The scaling forms also predict universal *shapes* for constant density profiles when properly normalized and plotted versus T/T_λ. The lack of universal shape in the Vycor data is further evidence that random onset has not yet been observed.

The work of Refs. 7 and 8 has come a long way toward understanding the nature of the zero-temperature onset transition. What is still lacking is a quantitative understanding of the transition in higher dimensions. In particular, one would like to have a dimensionality expansion, analogous to the ϵ expansion for classical spin systems, about the upper critical dimension d_c above which the mean-field theory is valid. In the rest of this chapter we will describe our attempt to find a proper dimensionality expansion.

2.2 Model and recursion relations

We start from the functional integral representation of the grand canonical partition function for the interacting Bose gas in a random potential [10]:

$$Z_G \propto \int D\psi D\psi^* e^{-S} \,. \tag{1}$$

The Euclidean action S is

$$\begin{aligned} S = \int d^dx \int_0^\beta d\tau [\frac{1}{\Gamma}\psi^*(\mathbf{x},\tau)\frac{\partial\psi(\mathbf{x},\tau)}{\partial\tau} + |\nabla\psi(\mathbf{x},\tau)|^2 + r|\psi(\mathbf{x},\tau)|^2 \\ + w(\mathbf{x})|\psi(\mathbf{x},\tau)|^2 + v|\psi(\mathbf{x},\tau)|^4] \,, \end{aligned} \tag{2}$$

where $2v$ is the on-site soft-core repulsion, $-r = \mu$ is the chemical potential, $w(\mathbf{x})$ is the random external single-particle potential, and $\beta = (k_BT)^{-1}$. Γ is introduced to control the strength of the quantum fluctuations and is initially equal to 1. Units have been chosen so that $\hbar = 2m = 1$, and an underlying spatial lattice with spacing $a_0 \approx a$ is assumed. Equivalently, a momentum space cutoff $k_\Lambda \sim \pi/a_0$ is imposed. The classical complex field $\psi(\mathbf{x},\tau)$ replaces the usual Bose-field operator, and the quantum-mechanical nature of the system is embodied in the extra imaginary time variable, τ. The *linear* time derivative $\psi^*\partial\psi/\partial\tau$ is characteristic of the Bose fluid.* For ease of later comparison to $O(n)$ spin models, it is convenient to generalize ψ to an m-component complex

* A second derivative term arises in the spin-$\frac{1}{2}$ Ising model in a transverse field [12], and in granular superconductor models with particle-hole symmetry

vector ψ_i, with $i = 1, \cdots, m$ and helium corresponding to $m = 1$. One expects the correspondence $n = 2m$.

When dealing with quenched disorder it is convenient to use the well-known replica trick [11] to derive an effective action in which the random external potential has been integrated out. The final form with which we work is

$$\begin{aligned} S_{eff} = \sum_{i=1}^{m}\sum_{\alpha=1}^{p}\int d^dx \int_0^\beta d\tau [\frac{1}{\Gamma}\psi^*_{\alpha i}(\mathbf{x},\tau)\frac{\partial\psi_{\alpha i}(\mathbf{x},\tau)}{\partial\tau} \\ + |\nabla\psi_{\alpha i}(\mathbf{x},\tau)|^2 + r|\psi_{\alpha i}(\mathbf{x},\tau)|^2] \\ + \sum_{i,j=1}^{m}\sum_{\alpha=1}^{p}\int d^dx \int_0^\beta d\tau v|\psi_{\alpha i}(\mathbf{x},\tau)|^2|\psi_{\alpha j}(\mathbf{x},\tau)|^2 \\ - \sum_{i,j=1}^{m}\sum_{\alpha,\alpha'=1}^{p}\int d^dx \int_0^\beta d\tau \int_0^\beta d\tau' g|\psi_{\alpha i}(\mathbf{x},\tau)|^2|\psi_{\alpha' j}(\mathbf{x},\tau')|^2 \; . \end{aligned} \tag{3}$$

The randomness has been taken as Gaussian, δ-function correlated, with amplitude $2g$:

$$\langle\langle w(\mathbf{x})\rangle\rangle = 0 \tag{4a}$$

$$\langle\langle w(\mathbf{x})w(\mathbf{x}')\rangle\rangle = 2g\delta(\mathbf{x}-\mathbf{x}') \; , \tag{4b}$$

where $\langle\langle\cdots\rangle\rangle$ denotes an average over the quenched-disorder probability distribution. The indices α, α' label the p identical replicas, with the formal limit $p \to 0$ to be taken at the end.

---

[8]. In these two cases time and space are symmetric and the d-dimensional quantum critical behavior is just that of the corresponding $(d+1)$-dimensional classical system. In general, however, the simple "add-a-dimension" rule fails [13].

What makes S_{eff} more difficult to analyze than the more standard classical spin models [11] is the fact that the interreplica coupling, proportional to g in Eq. (3), although local in space, is infinitely ranged in time. Boyanovsky and Cardy [6], extending earlier work of Dorogovstev [5], have solved this problem for the spin-$\frac{1}{2}$ Ising version of Eq. (3) [12]. They used field-theoretic techniques to generate a double expansion in the variables ϵ_d, the number of "temporal" dimensions along which the interreplica coupling has infinite range, and $\epsilon = 4 - D$, where $D = d + \epsilon_d$ is the total dimensionality. The actual physical situation corresponds to $\epsilon_d = 1$. In this section we will carry out the analogous calculation to first order in ϵ and ϵ_d for the Bose gas using standard momentum-shell renormalization-group techniques.

Before discussing the double ϵ, ϵ_d expansion, however, we present the details of the momentum-shell renormalization-group calculation when $\epsilon_d = 1$ to motivate the double expansion and to point out the difficulty we encounter in carrying it out. We also show that our recursion relations reduce to those of the $2m$-component classical random-bond spin model in the classical limit.

For convenience, we work in the Fourier-transformed space:

$$
\begin{aligned}
S_{eff} =& \sum_{\alpha} \sum_{i} \sum_{\mathbf{k}} \sum_{\omega} \beta(\frac{-i\omega}{\Gamma} + k^2 + r)|\phi_{\alpha i}(\mathbf{k},\omega)|^2 \\
&+ v\frac{\beta}{V} \sum_{\alpha} \sum_{i,j} \sum_{\mathbf{k},\mathbf{k}',\mathbf{k}''} \sum_{\omega,\omega',\omega''} \phi_{\alpha i}(\mathbf{k},\omega)\phi^*_{\alpha i}(\mathbf{k}',\omega') \\
&\qquad\qquad \times \phi_{\alpha j}(\mathbf{k}'',\omega'')\phi^*_{\alpha j}(\mathbf{k}-\mathbf{k}'+\mathbf{k}'',\omega-\omega'+\omega'') \\
&- g\frac{\beta^2}{V} \sum_{\alpha,\alpha'} \sum_{i,j} \sum_{\mathbf{k},\mathbf{k}',\mathbf{k}''} \sum_{\omega,\omega'} \phi_{\alpha i}(\mathbf{k},\omega)\phi^*_{\alpha i}(\mathbf{k}',\omega) \\
&\qquad\qquad \times \phi_{\alpha' j}(\mathbf{k}'',\omega')\phi^*_{\alpha' j}(\mathbf{k}-\mathbf{k}'+\mathbf{k}'',\omega')\ ,
\end{aligned}
\tag{5}
$$

where

$$
\phi_{\alpha i}(\mathbf{k},\omega) = \frac{1}{\beta\sqrt{V}} \sum_{\mathbf{k}} \sum_{\omega} e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega\tau)}\psi_{\alpha i}(\mathbf{x},\tau)\ . \tag{6}
$$

The bare propagator is

$$
\langle \phi_{\alpha i}(\mathbf{k},\omega)\phi^*_{\alpha' j}(\mathbf{k}',\omega')\rangle = \delta_{\alpha\alpha'}\delta_{ij}\delta_{\mathbf{k}\mathbf{k}'}\delta_{\omega\omega'} \frac{1}{\beta(\frac{-i\omega}{\Gamma} + k^2 + r)}\ . \tag{7}
$$

The first step in the momentum-shell renormalization-group analysis involves integrating out the components of the fields $\psi_{\alpha i}$ with wave numbers $\mathbf{k}$ in a shell $k_\Lambda/b < k < k_\Lambda$, and all frequencies order by order in the perturbation expansion in terms of v and g represented by the vertices in Fig. 1. $b = e^l > 1$ is the rescaling parameter, eventually to be taken infinitesimally close to 1. After the calculation, we will let the number of replicas go to zero. Therefore diagrams with an internal free-replica index will vanish and can be removed from the start. To one-loop order, there are only two such diagrams (see Fig. 2). Diagrams contributing to the renormalization of the propagator, v, and g to one-loop order are shown in Figs. 3, 4, and 5, and their corresponding algebraic expressions are listed in Tables 1, 2, and 3.

The propagator renormalization is tricky and needs careful consideration. To show the difficulty, we write down the explicit result of the propagator renormalization to first order in v and g:

$$\sum_{\alpha}\sum_{i}\sum_{k<k_\Lambda e^{-l}}\sum_{\omega}|\phi_{\alpha i}(\mathbf{k},\omega)|^2\beta[\frac{-i\omega}{\Gamma}+k^2+r + 2(m+1)vk_\Lambda^d K_d\frac{\Gamma}{e^{\beta\Gamma(k_\Lambda^2+r)}-1}l - 2gk_\Lambda^d K_d\frac{1}{\frac{-i\omega}{\Gamma}+k_\Lambda^2+r}l] , \quad (8)$$

where $K_d = 2/(4\pi)^{d/2}\Gamma(\frac{d}{2})$ is $(2\pi)^{-d}$ times the area of the unit sphere in d dimensions and $0 < l \ll 1$. Since we are summing over all the boson Matsubara frequencies, it seems that we cannot expand $(-i\omega/\Gamma + k_\Lambda^2 + r)^{-1}$ in terms of ω, and we need to solve the full nonlinear functional renormalization-group equation for the propagator. Here we introduce a frequency cutoff ω_Λ (or temporal lattice spacing $\tau_0 \sim \pi/\omega_\Lambda$) and make an assumption that it is equivalent to solving the functional renormalization-group equation exactly. This assumption seems difficult to justify, discrete time being a somewhat unnatural concept, but we have found no way of obtaining a sensible solution without it. The underlying problem seems to be in properly accounting for the analyticity properties of the boson Green's functions: Answers depend on whether frequency contour integrals are closed in the upper or lower half plane. In the more standard cases in which S_{eff} is even in ω, the upper and lower half planes are identical, and these problems do not arise. We can only speculate that a more careful higher-order calculation may provide some mechanism for an effective frequency cutoff, thereby eliminating the need for putting it in by hand. Having the cutoff ω_Λ, we can now use the Taylor expansion:

$$\frac{1}{\frac{-i\omega}{\Gamma} + k_\Lambda^2 + r} \sim \frac{1}{k_\Lambda^2 + r} + \frac{i\omega}{\Gamma(k_\Lambda^2 + r)^2} \ . \tag{9}$$

Higher-order terms are ignored since they are irrelevant in the renormalization-group sense.

The second step of the renormalization process is the rescaling of frequency, momentum, field, and temperature:

$$\begin{gathered} k \to k/e^l, \ \omega \to \omega/e^{zl}, \\ \phi \to \phi e^{(\zeta+\frac{d}{2})l}, \ \beta \to \beta e^{zl}, \end{gathered} \tag{10}$$

where z is the dynamical exponent. Combining the results obtained so far, we get the recursion relations for β, Γ, r, v, and g, to one-loop order:

$$\frac{d\beta}{dl} = -z\beta \tag{11a}$$

$$\frac{d\Gamma}{dl} = -(d+2\zeta)\Gamma - \frac{2gk_\Lambda^d K_d}{(k_\Lambda^2 + r)^2} \tag{11b}$$

$$\begin{aligned} \frac{dr}{dl} &= (d+z+2\zeta)r + 2(m+1)vk_\Lambda^d K_d \frac{\Gamma}{\exp[\beta\Gamma(k_\Lambda^2+r)] - 1} \\ &\quad - \frac{2gk_\Lambda^d K_d}{k_\Lambda^2 + r} \end{aligned} \tag{11c}$$

$$\begin{aligned} \frac{dv}{dl} &= (d+z+4\zeta)v - \{\frac{(m+3)\beta\Gamma^2}{2\sinh^2[\frac{1}{2}\beta\Gamma(k_\Lambda^2+r)]} \\ &\quad + \frac{\Gamma}{k_\Lambda^2+r}\coth[\frac{1}{2}\beta\Gamma(k_\Lambda^2+r)]\}v^2 k_\Lambda^d K_d + \frac{12vgk_\Lambda^d K_d}{(k_\Lambda^2+r)^2} \end{aligned} \tag{11d}$$

$$\begin{aligned} \frac{dg}{dl} &= (d+2z+4\zeta)g + \frac{8g^2 k_\Lambda^d K_d}{(k_\Lambda^2+r)^2} \\ &\quad - \frac{(m+1)vgk_\Lambda^d K_d \beta\Gamma^2}{\sinh^2[\frac{1}{2}\beta\Gamma(k_\Lambda^2+r)]} \ . \end{aligned} \tag{11e}$$

In the initial stage of renormalization, exponents ζ and z are adjusted so that the coefficients of k^2 and $-i\omega$ terms remain constant. From the first condition we get

$$2\zeta + d \approx 2 - z \ . \tag{12}$$

The flow equation for β shows that it flows toward $\beta = 0$ unless $\beta = \infty$ or $z = 0$. When β reaches 1 we keep it fixed and allow Γ to vary instead, yielding the plausible classical result $z = 0$. This coefficient then goes to zero as $l \to \infty$ and suppresses all but the $\omega = 0$ Matsubara frequency. The recursion relations, in this limit, reduce precisely to the usual ϵ_d-independent classical random-bond spin recursion relations, with the identification $n = 2m$ [11]:

$$d\tilde{r}/dl = 2\tilde{r} + 2(m+1)\tilde{v}/(1+\tilde{r}) - 2\tilde{g}/(1+\tilde{r}) \tag{13a}$$

$$d\tilde{v}/dl = (4-d)\tilde{v} - 2(m+4)\tilde{v}^2/(1+\tilde{r})^2 + 12\tilde{v}\tilde{g}/(1+\tilde{r})^2 \tag{13b}$$

$$d\tilde{g}/dl = (4-d)\tilde{g} + 8\tilde{g}^2/(1+\tilde{r})^2 - 4(m+1)\tilde{v}\tilde{g}/(1+\tilde{r})^2 \ , \tag{13c}$$

where $\tilde{r} = r/k_\Lambda^2$, $\tilde{v} = K_d v$, and $\tilde{g} = K_d g$. For $m < 2$ they possess an $O(\tilde{\epsilon} = 4-d)$ random fixed point R at $\tilde{v}^* = \tilde{\epsilon}/4(2m-1)$ and $\tilde{g}^* = (2-m)\tilde{\epsilon}/8(2m-1)$. In fact, for $d = 3$ ($\tilde{\epsilon} = 1$) and $n = 2$, the best estimates yield a negative specific-heat exponent $\alpha < 0$, and hence, by the Harris criterion [11] randomness should be irrelevant for $m \gtrsim 1$. The $O(\tilde{\epsilon})$ results therefore give a misleading picture of the flows for $m = 1$. Qualitatively correct flows can be obtained by taking $m \gtrsim 2$ in the $O(\tilde{\epsilon})$ recursion relations, which yield only a pure fixed point [at $\tilde{g}^* = 0$ and $\tilde{v}^* = \tilde{\epsilon}/2(m+4)$] which is stable against disorder. Of course, *quantitative* estimates (for exponents, etc.) can be obtained only by going to higher order in $\tilde{\epsilon}$.

We now consider the extreme quantum limit, $\beta = \infty$. In this case, we keep $\Gamma = 1$ and therefore, from Eq. (11b),

$$2\zeta + d = -\frac{2gk_\Lambda^d K_d}{(k_\Lambda^2 + r)^2} \ . \tag{14}$$

Substituting this into Eqs. (11d) and (11e), we get

$$\begin{aligned} \frac{dv}{dl} &= (2-d)v + O(v^2, g^2, vg) \\ \frac{dg}{dl} &= (4-d)g + O(v^2, g^2, vg) \ . \end{aligned} \tag{15}$$

We find that within the perturbation theory, v and g cannot be handled at the same time. This motivates the introduction of another small parameter, ϵ_d, the number of temporal dimensions. In this case, Eqs. (11) and (12) become

$$d\Gamma/dl = -(d + \epsilon_d z - z + 2\zeta)\Gamma + \cdots \tag{16a}$$

$$dr/dl = (d + \epsilon_d z + 2\zeta)r + \cdots \tag{16b}$$

$$dv/dl = (d + \epsilon_d z + 4\zeta)v + \cdots \tag{16c}$$

$$dg/dl = (d + 2\epsilon_d z + 4\zeta)g + \cdots \tag{16d}$$

$$2\zeta + d \approx 2 - \epsilon_d z \ . \tag{17}$$

One can show that both v and g are slow variables using (16a) and (17):

$$z = 2 + \cdots \tag{18a}$$

$$dr/dl = 2r + \cdots \tag{18b}$$

$$dv/dl = (\epsilon - \epsilon_d)v + \cdots \tag{18c}$$

$$dg/dl = (\epsilon + \epsilon_d)g + \cdots \ . \tag{18d}$$

We also mention that in the presence of the cutoff ω_Λ, the most natural way to do the renormalization-group transformation is to integrate out the high momentum-frequency shell with $k_\Lambda/e^l < k < k_\Lambda$ or $\omega_\Lambda/e^{zl} < \omega < \omega_\Lambda$.

The question arises at this point of how to continue analytically the frequency sums away from $\epsilon_d = 1$. We do this simply by replacing the term $\psi^*\partial\psi/\partial\tau$ by $\sum_{\mu=1}^{\epsilon_d}\psi^*\partial\psi/\partial\tau_\mu$, but with the further restriction that ψ contains only those frequency components for which $\omega_\mu \geq 0$ for all μ or $\omega_\mu \leq 0$ for all μ.* To first order in ϵ_d, it is sufficient to evaluate all frequency sums at $\epsilon_d = 0$ so that only $\omega = 0$ contributes. Hence, the lowest-order recursion relations require

---

* This insures that sums of the form $S[f] = \sum_{\{\omega_\mu\}} f(\sum_\mu \omega_\mu)$, where $\omega_\mu = 2\pi n_\mu/\beta$, converge properly if f decreases sufficiently rapidly at $\pm\infty$, and yields the correct result when $\epsilon_d = 0$ or 1. Analytic continuation is achieved by Laplace-transforming the sum to yield

$$S[f] = \int_0^\infty dt[\tilde{f}_+(t) + \tilde{f}_-(t)]/[1 - \exp(-2\pi t/\beta)]^{\epsilon_d} - f(0),$$

where

$$f(\omega) = \int_0^\infty dt \tilde{f}_\pm(t)\exp(\mp\omega t)$$

for $\pm\omega > 0$.

the existence of an analytic continuation, but are completely insensitive to its form. This, presumably, is why numerical values for the exponents obtained to $O(\epsilon_d)$ converge so poorly at $\epsilon_d = 1$.

The final recursion relations, to lowest nontrivial order in ϵ and ϵ_d, are

$$d\tilde{r}/dl = 2\tilde{r} + 2(m+1)\tilde{v}/(1+\tilde{r}) - 2\tilde{g}/(1+\tilde{r}) + O(\tilde{v}^2, \tilde{g}^2, \tilde{v}\tilde{g}) \tag{19a}$$

$$d\tilde{v}/dl = (\epsilon - \epsilon_d)\tilde{v} - 2(m+4)\tilde{v}^2 + 12\tilde{v}\tilde{g} + O(\tilde{v}^3, \cdots) \tag{19b}$$

$$d\tilde{g}/dl = (\epsilon + \epsilon_d)\tilde{g} + 8\tilde{g}^2 - 4(m+1)\tilde{v}\tilde{g} + O(\tilde{v}^3, \cdots) \tag{19c}$$

$$z = 2 + 2\tilde{g} + O(\tilde{v}^2, \cdots)\ . \tag{19d}$$

The fixed points and their eigenvalues are calculated and summarized in Table 4. As in Ref. 5, the eigenvalues associated with small deviations of $\tilde{v}$ and $\tilde{g}$ from their random-fixed-point values are complex, with a negative real part, and hence, can give rise to oscillatory corrections to scaling. The correlation-length exponent of the stable random fixed point is calculated easily:

$$\begin{aligned} \nu^{-1} &= 2 - 2(m+1)\tilde{v}^* + 2\tilde{g}^* \\ &= 2 - \frac{3m\epsilon + 7(m+4)\epsilon_d}{4(2m-1)}\ . \end{aligned} \tag{20}$$

2.3 Discussion

In this section we discuss some implications of the recursion relations we have obtained. In Fig. 6 we plot the flows in the critical hyperspace, defined by (19b) and (19c). For $d < 4$ ($\epsilon + \epsilon_d > 0$), the Gaussian fixed-point G_0 at

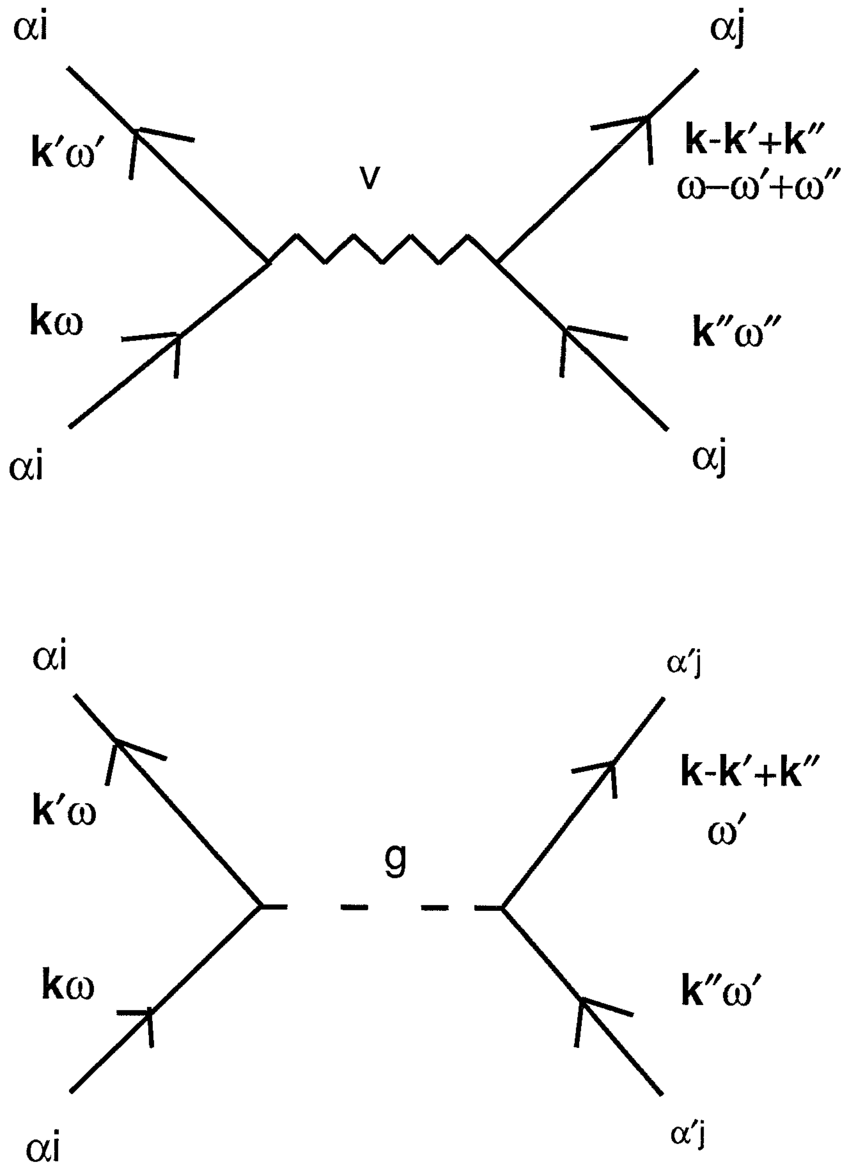
$\tilde{v} = \tilde{g} = 0$, is unstable to the random fixed point, and the flows are plotted in Fig. 6(a). For $d \gtrsim 4$ $(\epsilon + \epsilon_d \lesssim 0)$, *both* fixed points are stable, and a separatrix S divides the basins of attraction. This is shown in Fig. 6(b). Which of the two governs the critical behavior depends on the strength of the randomness. As d increases further, the separatrix moves upward, and for sufficiently large d it intersects the random fixed point, which then becomes unstable. In all cases (except $\epsilon = \epsilon_d = 0$) there are two distinct fixed points with different eigenvalues.

We see then that the transition to the mean-field theory as d increases through $d_c = 4$ is quite unconventional. The exponents change *discontinuously* to their mean-field values, whereas the conventional mechanism, involving the coalescence of the two fixed points, yields *continuous* exponents. In fact, it can be seen on very general grounds that precisely this kind of behavior must occur. A recent theorem of Chayes *et al.* [14] states that in any d-dimensional system with spatially uncorrelated disorder, one has the bound $\nu \geq 2/d$. The mean-field value is $\nu_{MF} = 1/2$, which immediately implies that $d_c \geq 4$. However, one also has the generalized Josephson hyperscaling relation [7,8] $\zeta = (d + z - 2)\nu$ for the superfluid density exponent ζ, valid for $d < d_c$. The mean-field value is $\zeta_{MF} = 1$. However, if $d_c \geq 4$ and $z > 0$ (in fact, scaling arguments yield $z = d$ [7,8]), this relation implies $\zeta > 1$ for d approaching d_c from below; i.e., a discontinuous change in ζ must occur as d passes through d_c. For the metal-insulator transition in noninteracting Fermi systems, this problem is apparently avoided by having $d_c = \infty$ [15]. For classical systems, one has $z = 0$, so that one can have $\nu = 1/2$ and $\zeta = 1$ in $d_c = 4$ without contradiction.

The present method gives the dynamical exponent z as a nontrivial expansion in ϵ and ϵ_d. As mentioned, various scaling arguments predict the exact equality $z = d$ for $\epsilon_d = 1$ [7,8]. It is not at all clear how such a precise resummation of the series might come about. A better understanding would be obtained if the scaling arguments could be carried out for general ϵ_d; however, they seem rather special to $\epsilon_d = 1$, and we have found no way to generalize them.

In Fig. 7 we show a schematic plot of the flows in the three-dimensional critical hyperspace defined by the variables $\tilde{v}$, $\tilde{g}$, and T. The picture is only schematic because of the different renormalization procedures used in different regions of parameter space. On this diagram various possible behaviors are shown, depending on the relative sizes of the starting parameters $\tilde{v}_0$, $\tilde{g}_0$, and T_λ. Since the pictured flows lie in the critical hyperspace, T_λ is in fact the physical transition temperature, while $\tilde{v}_0$ and $\tilde{g}_0$ are fixed by the atomic properties of ^{4}He and by the random medium, respectively.

Suppose that $\tilde{g}_0 = 0$: Then one is in the pure limit, a case that has been treated in great detail elsewhere [3]. When $T_\lambda^{(d-2)/2}\tilde{v}_0 \ll 1$, one explores the crossover to ideal gas behavior. This is described by flows which, because of the smallness of T_λ, closely approach the $T = 0$ Gaussian fixed point G_0 before collapsing down into the neighborhood of the classical Gaussian fixed point G, from which they cross over to the critical fixed point C [3,17]. Now suppose that there exists a range of T_λ for which $T_\lambda^{(d-2)/2}\tilde{v}_0 \ll 1 \ll T_\lambda^{(4-d)/2}/\tilde{g}_0$ [4]. Within this range of T_λ, the onset fixed point R_0 will essentially play no role. The flows will be dominated by pure crossover, being pulled down into the classical plane where randomness is irrelevant before R_0 has a chance to act. This range of T_λ corresponds to the range of coverages, alluded to in the Introduction, over which pure weakly interacting Bose-gas behavior is seen [3]. Only for $T_\lambda^{(4-d)/2}/\tilde{g}_0 \lesssim 1$ do the flows spend enough time near the $T = 0$ plane to be attracted towards R_0. For $T_\lambda^{(4-d)/2}/\tilde{g}_0 \ll 1$, the flows are dominated completely by a direct crossover from R_0 to C. The scaling form that results from this crossover (essentially finite-size scaling in $1/T$) predicts, as mentioned earlier, constant density profiles with a universal shape [7,8]. This shape is determined by the asymptotic trajectory that connects R_0 to C, and could, in principle, be calculated within the ϵ, ϵ_d formalism. Although the Vycor data [1] do not enter this true region of random onset, this regime should be much more accessible in materials that are more strongly random.



Incoming lines: ϕ

Outgoing lines: ϕ^*

Fig. 1 Vertices

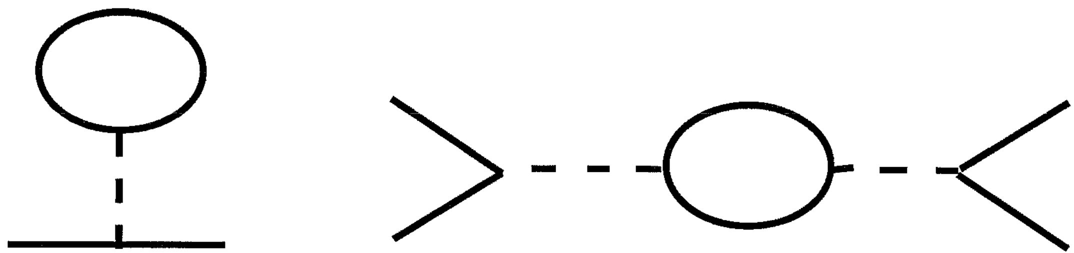
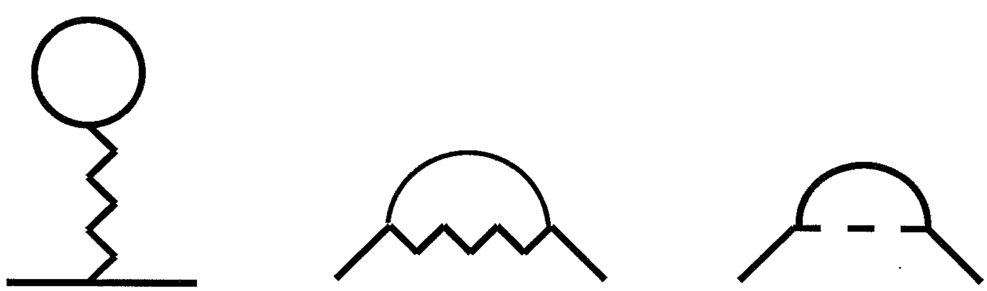


Fig. 2



(1) (2) (3)

Fig. 3 Propagator renormalization

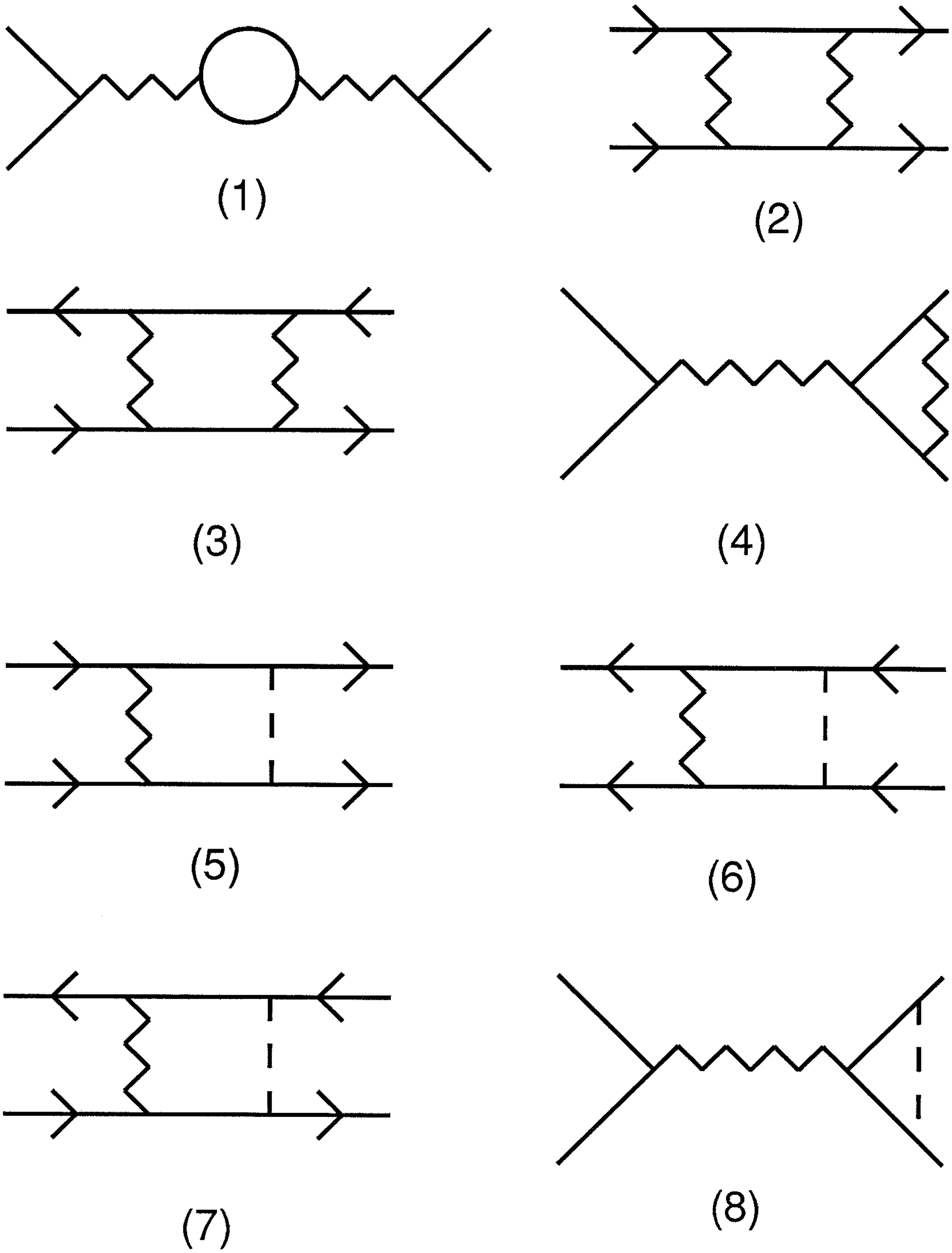


Fig. 4 v renormalization

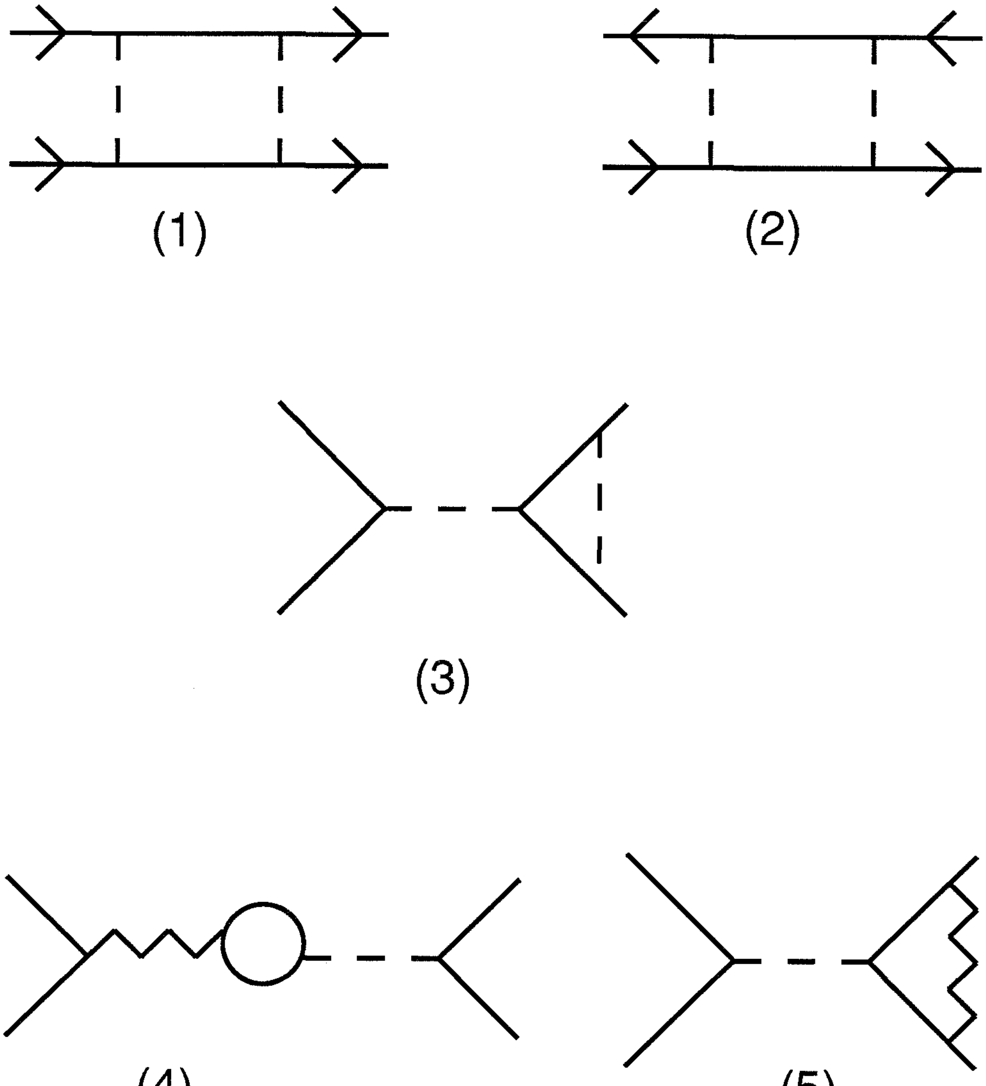


Fig. 5 g renormalization

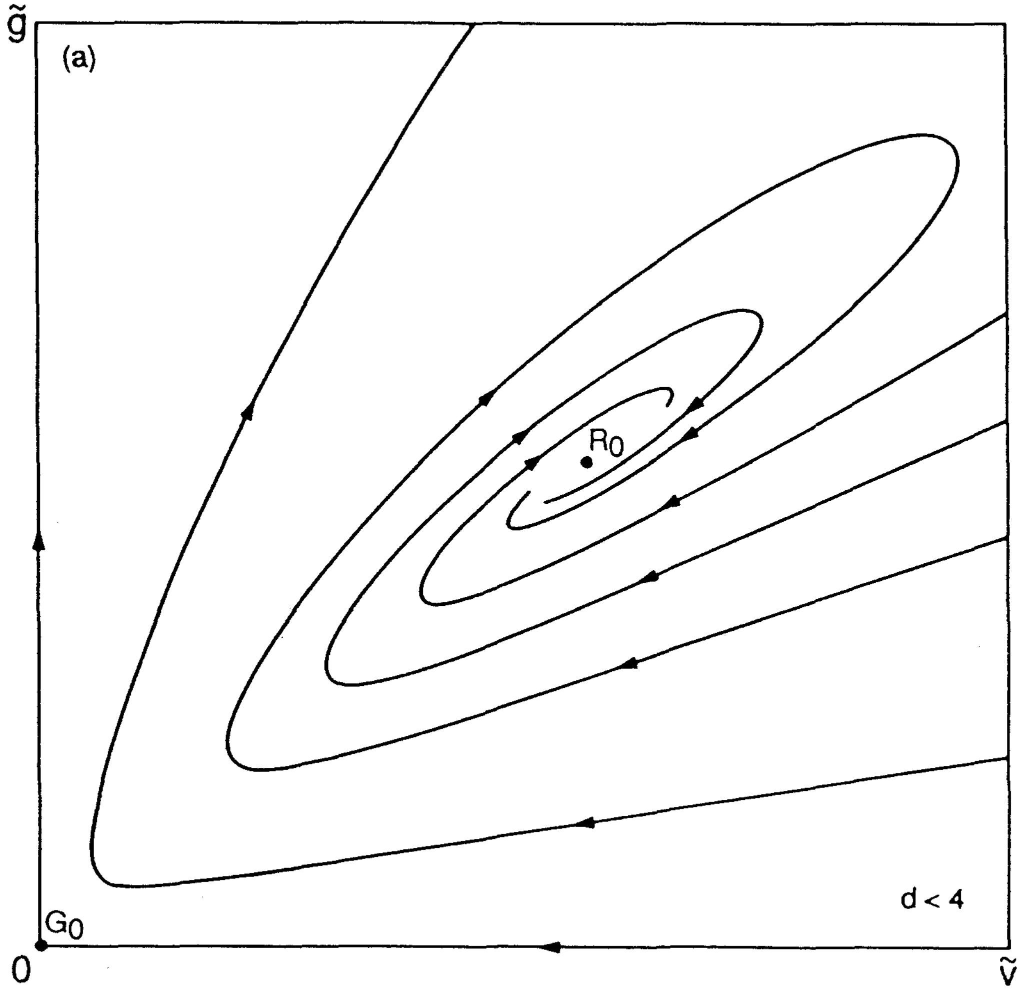
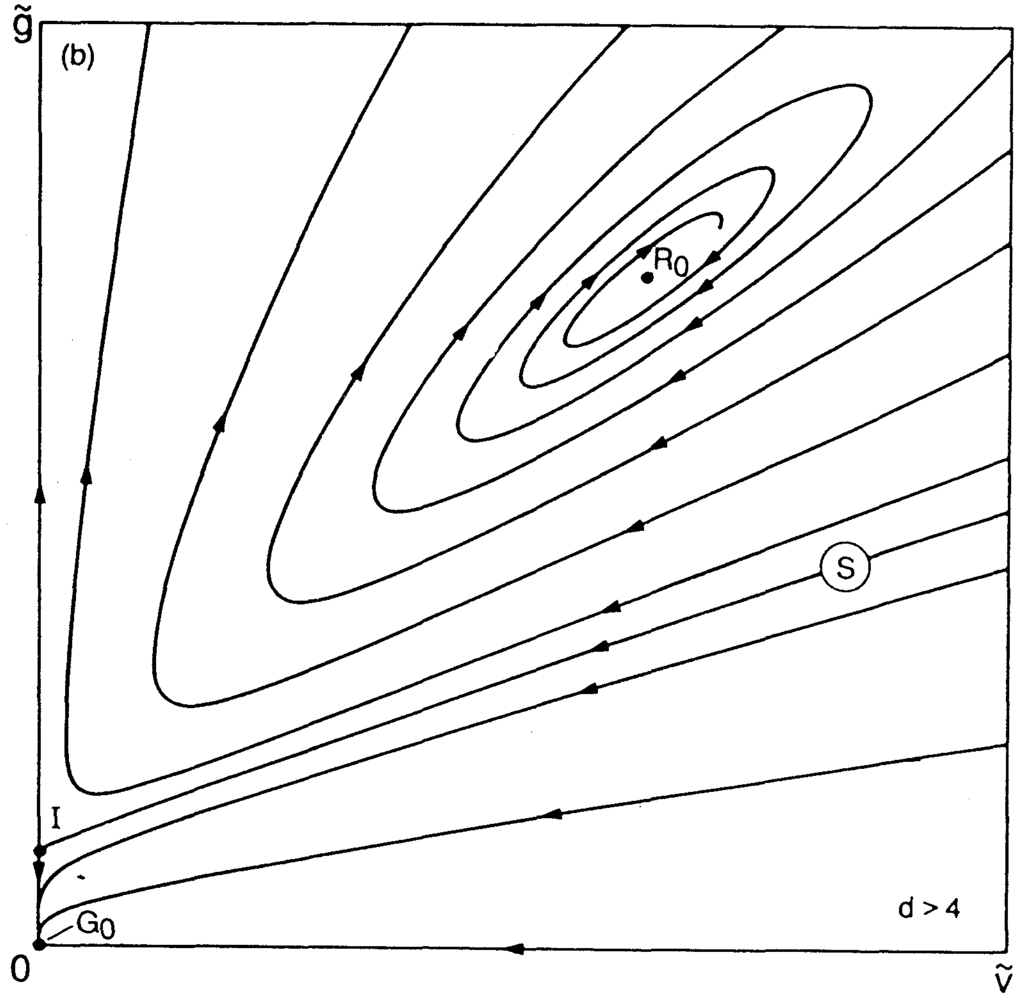


Fig. 6 (a), (b) (See the next page.) Renormalization-group flows in the $T = 0$ critical hyperplane for (a) $d < 4$ ($\epsilon/\epsilon_d = -0.5$) and (b) $d > 4$ ($\epsilon/\epsilon_d = -2$). The onset of the mean-field theory occurs by way of a separatrix S, which separates the basins of attraction for the zero-temperature Gaussian G_0, and randomR_0 fixed points.



g̃
(b)
R0
S
I
G0
d > 4
0
ṽ

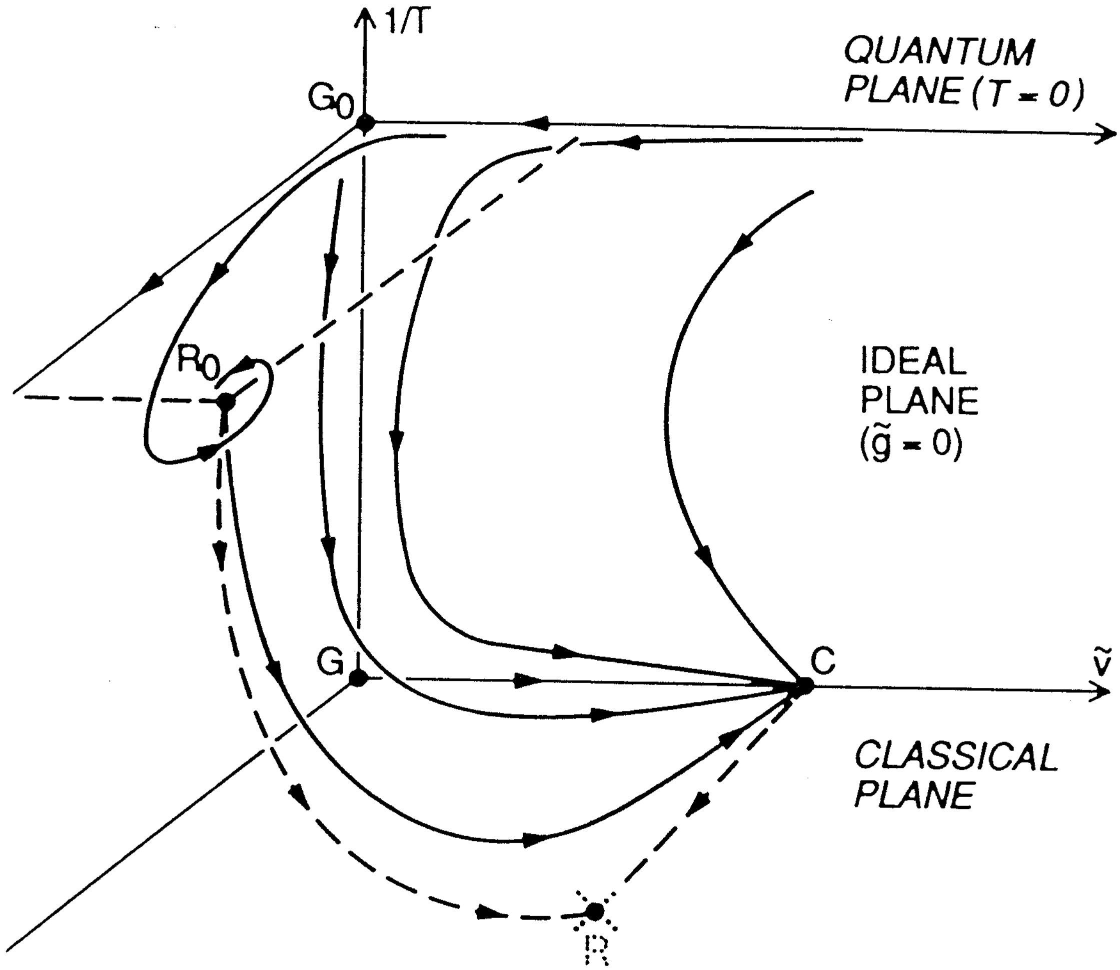


Fig. 7 Schematic plot of renormalization-group flows for $d < 4$, including finite temperatures. Nonrandom flows remain in the $\tilde{g} = 0$ plane, and crossover to ideal gas behavior involves flows that pass close to G_0, G, and finally C. The random onset regime involves flows that pass close to R_0 before collapsing into the classical plane. Depending on the sign of the specific-heat exponent, α, a classical random fixed point exists ($\alpha > 0$, dashed line) or does not exist ($\alpha < 0$, solid lines). The latter case probably holds for helium.

Table 1 Propagator renormalization

(1) $2mJv$ (2) $2Jv$ (3) $-2Hg$

$$
\begin{aligned}
J &= \frac{1}{\beta V} \sum_{\mathbf{k}}{}' \sum_{\omega} \frac{1}{\frac{-i\omega}{\Gamma} + k^2 + r} \\
&= \frac{\Gamma k_\Lambda^d K_d}{\exp[\beta\Gamma(k_\Lambda^2 + r)] - 1} l \\
H &= \frac{1}{V} \sum_{\mathbf{k}}{}' \frac{1}{\frac{-i\omega}{\Gamma} + k^2 + r} \\
&= \frac{k_\Lambda^d K_d}{\frac{-i\omega}{\Gamma} + k_\Lambda^2 + r} l
\end{aligned}
$$

where

$$
\sum_{\mathbf{k}}{}' = \sum_{k_\Lambda e^{-l} < k < k_\Lambda}
$$

$$
K_d = 2/(4\pi)^{d/2}\Gamma(\frac{1}{2}d)
$$

$$
0 < l \ll 1
$$

Table 2 v renormalization

(1) $-2mAv^2$ (2) $-2Bv^2$ (3) $-2Av^2$

(4) $-4Av^2$ (5) $2Cvg$ (6) $2Cvg$

(7) $4Cvg$ (8) $4Cvg$

Table 3 g renormalization

(1) $2Cg^2$ (2) $2Cg^2$ (3) $4Cg^2$

(4) $-4mAvg$ (5) $-4Avg$

$$A = \frac{1}{\beta V}\sum_{\mathbf{k}}{}' \sum_{\omega} \frac{1}{(\frac{-i\omega}{\Gamma} + k^2 + r)^2}$$
$$= \frac{k_\Lambda^d K_d \beta \Gamma^2}{4\sinh^2[\frac{1}{2}\beta\Gamma(k_\Lambda^2 + r)]} l$$

$$B = \frac{1}{\beta V}\sum_{\mathbf{k}}{}' \sum_{\omega} \frac{1}{(\frac{-i\omega}{\Gamma} + k^2 + r)(\frac{i\omega}{\Gamma} + k^2 + r)}$$
$$= \frac{k_\Lambda^d K_d \Gamma \coth[\frac{1}{2}\beta\Gamma(k_\Lambda^2 + r)]}{2(k_\Lambda^2 + r)} l$$

$$C = \frac{1}{V}\sum_{\mathbf{k}}{}' \frac{1}{(k^2 + r)^2}$$
$$= \frac{k_\Lambda^d K_d}{(k_\Lambda^2 + r)^2} l$$

Table 4 Fixed points and eigenvalues

Fixed point	v^*	g^*	Eigenvalues
Gaussian	0	0	$\lambda_v = \epsilon - \epsilon_d, \quad \lambda_g = \epsilon + \epsilon_d$
unphysical	0	$-\frac{\epsilon+\epsilon_d}{8}$	$\lambda_v = -\frac{(\epsilon+5\epsilon_d)}{2}, \quad \lambda_g = -(\epsilon + \epsilon_d)$
pure	$\frac{\epsilon-\epsilon_d}{2(m+4)}$	0	$\lambda_v = -(\epsilon - \epsilon_d), \quad \lambda_g = \frac{(2-m)\epsilon+3(m+2)\epsilon_d}{m+4}$
random	$\frac{\epsilon+5\epsilon_d}{4(2m-1)}$	$\frac{(2-m)\epsilon+3(m+2)\epsilon_d}{8(2m-1)}$	$\lambda_{1,2} = \frac{1}{4(2m-1)} \left(-\hat{A} \pm \sqrt{\hat{A}^2 - \hat{B}}\right)$

$$\hat{A} = 3m\epsilon + (8-m)\epsilon_d$$
$$\hat{B} = 8(2m-1)(\epsilon + 5\epsilon_d)[(2-m)\epsilon + 3(m+2)\epsilon_d]$$

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3. SPIN-WAVE SINGULARITIES: FREE ENERGY AND EQUATION OF STATE IN $O(n)$ SPIN MODELS NEAR T_c

3.1 Introduction

The low-temperature, ordered, phase of ferromagnets with a continuous symmetry (spin dimensionality $n > 1$) exhibits coexistence-curve singularities in various thermodynamic functions. For example, the longitudinal susceptibility, $\chi_L = \partial M/\partial h$, where M is the magnetization and h the external magnetic field, diverges as $h \to 0$:

$$\chi_L \sim h^{-\epsilon/2}, \quad h \to 0 \ , \quad T < T_c \ , \tag{1.1}$$

where $\epsilon = 4 - d$, and d, which we henceforth take to be in the range $2 < d < 4$, is the spatial dimensionality. This divergence is a direct consequence of the slow, power-law decay of correlations in the ordered phase. In particular, the longitudinal pair correlation function decays as (see, e.g., Ref. 9)

$$G_L(\mathbf{r}) \equiv \langle \mathbf{s}(\mathbf{r}) \cdot \hat{\mathbf{M}} \ \mathbf{s}(\mathbf{0}) \cdot \hat{\mathbf{M}} \rangle \sim |\mathbf{r}|^{-2(d-2)}, \quad h = 0 \ , \quad |\mathbf{r}| \to \infty \ , \tag{1.2}$$

where $\mathbf{s}(\mathbf{r})$ is the spin at site $\mathbf{r}$. The fact that $\chi_L(h = 0) = \infty$ for $d < 4$ follows directly from the spatial integral of (1.2).

The coexistence-curve divergence [Eq. (1.1)] should be contrasted with that occurring *at* $T = T_c$:

$$\chi_L \sim h^{1/\delta - 1} \ , \tag{1.3}$$

where $\delta > 1$ is the critical exponent which describes the vanishing of $M \sim h^{1/\delta}$ at T_c. Accompanying (1.3) is the analogous power-law decay of critical correlations

$$G_L(\mathbf{r}) \sim 1/|\mathbf{r}|^{d-2+\eta}, \quad h = 0, \quad T = T_c, \quad |\mathbf{r}| \to \infty, \tag{1.4}$$

which defines the critical exponent η.

In this chapter we reexamine the connection between coexistence-curve singularities, such as (1.1) and (1.2), and their critical-point counterparts, such as (1.3) and (1.4). Using straightforward renormalization-group recursion-relation techniques, we rederive various thermodynamic functions, valid throughout the critical regime, both above and below T_c, with and without an applied field. These functions properly exhibit each set of singularities in the appropriate limit. We also exhibit the full free-energy function, which, to our knowledge, has not been fully analyzed previously.

We will use the original trajectory integral and noncritical matching technique of Rudnick and Nelson [1], circumventing – by means of simple spin-wave theory – the difficulties these authors encountered near the coexistence curve. Some of the results presented here have been derived to various levels of completeness by other authors. In our eyes, however, their derivations, which often involve very sophisticated field-theoretic techniques, seem enormously complicated [2,3]. The most complete discussion has been given by Nicoll and Chang [3]. They used a more involved version of the trajectory integral technique;

many of their intermediate expressions bear a strong (presumably noncoincidental) resemblance to our own, but we have not attempted a detailed comparison.* We find precisely their result for the equation of state, but our result for the free energy (which we believe to be correct) differs in some details.

We view our work as a final demonstration of the simplicity and utility of the Rudnick and Nelson [1] technique. Our main claim to originality is in supplying the one ingredient missing from the original discussion, namely, the present understanding of spin waves in the ordered phase of vector ferromagnets.

The outline of the rest of this chapter is as follows: In Sec. 3.2 we recapitulate the model, its recursion relations, and their solutions. In Sec. 3.3 we derive the equation of state by combining the results of spin-wave theory with those of Sec. 3.2. Asymptotic scaling equations are derived and the corresponding *parametric* forms are exhibited for general n – these contain spin-wave singularities in the angular variable θ. In Sec. 3.4 we calculate the free energy and demonstrate consistency by deriving from it the correct equation of state. Finally, in Sec. 3.5 we give a short rederivation of the helicity modulus (or superfluid

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* D. R. Nelson [4] has attempted to account for the coexistence-curve singularities using a graphical resummation technique. Though his answers correct previous problems with negative susceptibilities [5], they disagree with those of Ref. 3 and the present chapter. We have made no effort at comparison here, either. See, however, Ref. 3 for some comments.

density), as well as explore various quantities derived from the free energy such as the entropy (or density, depending on how the variable r is interpreted) and specific heat. Appendix A contains some details of the recursion relation solutions; Appendix B gives some insight into the linear spin-wave approximation; in Appendix C some spin-wave integrals are evaluated.

3.2 Model and recursion relations

We work with the Landau-Ginzburg-Wilson continuous-spin Hamiltonian

$$H_{LGW} = \int d^d x [\frac{R_0^2}{2}|\nabla \mathbf{s}|^2 + \frac{1}{2} r|\mathbf{s}|^2 + u|\mathbf{s}|^4 - \mathbf{h}\cdot\mathbf{s}] \ , \tag{2.1}$$

where $-\infty < s^\alpha < \infty$, and $1 \leq \alpha \leq n$, where n is the spin dimensionality. An underlying lattice, with lattice spacing a or, equivalently, a momentum space cutoff $k_\Lambda \sim \pi/a$, must be assumed in order for the partition function

$$Z_{LGW} = \int DSe^{-H_{LGW}}, \tag{2.2}$$

defined as a functional integral over all spin configurations, to be well defined.

The model (2.1) undergoes a ferromagnetic phase transition as the temperaturelike variable r decreases through a critical value $r_c(u)$ in zero external field $h = 0$. In the mean-field theory, defined here as the limit in which the coefficient R_0^2 of $|\nabla \mathbf{s}|^2$ tends to infinity (so that fluctuations are effectively suppressed), one has $r_c(u) \equiv 0$. The *spontaneous magnetization*

$$\mathbf{M} \equiv \frac{1}{V}\int d^d x \ \mathbf{s}(\mathbf{x}) = \langle \mathbf{s}(\mathbf{x}) \rangle \tag{2.3}$$

becomes nonzero below r_c, increasing with a characteristic exponent β:

$$M \equiv |\mathbf{M}| \sim |t|^{\beta} \tag{2.4}$$

for small $t = r - r_c < 0$. In the mean-field theory we have

$$M_{MF} = (-r/4u)^{1/2} \qquad \Rightarrow \qquad \beta_{MF} = \frac{1}{2} \ . \tag{2.5}$$

When $M > 0$ (i.e., when $r < r_c$ or $h > 0$), it is convenient to expand the fluctuations in the spin variable around the uniform magnetization $\mathbf{M}$. If we define

$$\begin{aligned} \sigma(\mathbf{x}) &= (\mathbf{s}(\mathbf{x}) - \mathbf{M}) \cdot \hat{\mathbf{M}}, \qquad \hat{\mathbf{M}} \equiv \mathbf{M}/M \\ \mathbf{s}^{\perp}(\mathbf{x}) &= \mathbf{s}(\mathbf{x}) - \sigma(\mathbf{x})\hat{\mathbf{M}}, \end{aligned} \tag{2.6}$$

then (2.1) can be rewritten in the form

$$H_{LGW} = H_0 + H_1 + H_2 + H_3 + H_4 \ , \tag{2.7}$$

where

$$\begin{aligned} H_0 &= \Big[\frac{1}{2} r M^2 + u M^4 - h M\Big] V \\ H_1 &= -\int d^d x \tilde{h} \sigma \\ H_2 &= \frac{1}{2} \int d^d x \Big[R_0^2 |\nabla \mathbf{s}^{\perp}|^2 + R_0^2 |\nabla \sigma|^2 + r_T |\mathbf{s}^{\perp}|^2 + r_L \sigma^2\Big] \\ H_3 &= \int d^d x \Big[w_1 \sigma |\mathbf{s}^{\perp}|^2 + w_2 \sigma^3\Big] \\ H_4 &= \int d^d x \Big[u_1 \sigma^4 + 2u_2 |\mathbf{s}^{\perp}|^2 \sigma^2 + u_3 |\mathbf{s}^{\perp}|^4\Big] \ , \end{aligned} \tag{2.8}$$

in which we have defined

$$\begin{aligned} \tilde{h} &= h - rM - 4uM^3, \\ r_L &= r + 12uM^2, \qquad r_T = r + 4uM^2, \\ w_1 &= w_2 = 4uM, \qquad u_1 = u_2 = u_3 = u \ . \end{aligned} \tag{2.9}$$

The first term H_0 should be recognized as the Landau mean-field free energy, which yields (2.5) when $h = 0$.

Rudnick and Nelson [1] have derived differential recursion relations to one-loop order for the Hamiltonian (2.7), with $R_0 = 1$ and $k_\Lambda = 1$. We reproduce them here in more detail and with minor misprints corrected:

$$\begin{aligned}\frac{d\tilde{h}}{dl} =& (3 - \frac{1}{2}\epsilon)\tilde{h} - w_1 K_4 (n-1)/(1+r_T) - 3w_2 K_4/(1+r_L) \\ &+ O(uw, w^3) \end{aligned} \tag{2.10}$$

$$\begin{aligned}\frac{dr_L}{dl} =& 2r_L + 12K_4 u_1/(1+r_L) + 4(n-1)u_2 K_4/(1+r_T) \\ &- 18K_4 w_2^2/(1+r_L)^2 - 2(n-1)K_4 w_1^2/(1+r_T)^2 \\ &+ O(u^2, uw^2, w^4) \end{aligned} \tag{2.11}$$

$$\begin{aligned}\frac{dr_T}{dl} =& 2r_T + 4(n+1)u_3 K_4/(1+r_T) + 4K_4 u_2/(1+r_L) \\ &- 4K_4 w_1^2/(1+r_L)(1+r_T) + O(u^2, uw^2, w^4) \end{aligned} \tag{2.12}$$

$$\begin{aligned}\frac{dw_1}{dl} =& (1 + \frac{1}{2}\epsilon)w_1 - 4(n+1)w_1 u_3 K_4/(1+r_T)^2 \\ &- 16w_1 u_2 K_4/(1+r_L)(1+r_T) - 12w_2 u_2 K_4/(1+r_L)^2 \\ &+ 12w_1^2 w_2 K_4/(1+r_L)^2(1+r_T) + 4w_1^3 K_4/(1+r_T)^2(1+r_L) \\ &+ O(wu^2, w^3 u, w^5) \end{aligned} \tag{2.13}$$

$$\begin{aligned}\frac{dw_2}{dl} =& (1 + \frac{1}{2}\epsilon)w_2 - 4(n-1)w_1 u_2 K_4/(1+r_T)^2 - 36w_2 u_1 K_4/(1+r_L)^2 \\ &+ 36K_4 w_2^3/(1+r_L)^3 + \frac{4}{3}K_4 w_1^3 (n-1)/(1+r_T)^3 \\ &+ O(wu^2, w^3 u, w^5) \end{aligned} \tag{2.14}$$

$$\begin{aligned}
\frac{du_1}{dl} =&\epsilon u_1 - 36u_1^2K_4/(1+r_L)^2 - 4(n-1)u_2^2K_4/(1+r_T)^2 \\
&+ 216K_4u_1w_2^2/(1+r_L)^3 + 8K_4u_2w_1^2(n-1)/(1+r_T)^3 \\
&- 54w_2^4K_4/(1+r_L)^4 - \frac{2}{3}(n-1)w_1^4K_4/(1+r_T)^4 \\
&+ O(u^3, u^2w^2, uw^4, w^6) \qquad (2.15)
\end{aligned}$$

$$\begin{aligned}
\frac{du_2}{dl} =&\epsilon u_2 - 16K_4u_2^2/(1+r_L)(1+r_T) - 12u_1u_2K_4/(1+r_L)^2 \\
&- 4(n+1)u_2u_3K_4/(1+r_T)^2 + 4K_4u_3w_1^2(n+1)/(1+r_T)^3 \\
&+ 36K_4u_2w_2^2/(1+r_L)^3 + 12u_1w_1^2K_4/(1+r_L)^2(1+r_T) \\
&+ 4K_4u_2w_1^2/(1+r_T)^2(1+r_L) + 48K_4u_2w_1w_2/(1+r_L)^2(1+r_T) \\
&+ 16K_4u_2w_1^2/(1+r_T)(1+r_L)^2 - 4w_1^4K_4/(1+r_T)^3(1+r_L) \\
&- 36K_4w_1^2w_2^2/(1+r_T)(1+r_L)^3 - 12w_1^3w_2K_4/(1+r_L)^2(1+r_T)^2 \\
&+ O(u^3, u^2w^2, uw^4, w^6) \qquad (2.16)
\end{aligned}$$

$$\begin{aligned}
\frac{du_3}{dl} =&\epsilon u_3 - 4(n+7)u_3^2K_4/(1+r_T)^2 - 4u_2^2K_4/(1+r_L)^2 \\
&+ 24K_4u_3w_1^2/(1+r_T)^2(1+r_L) + 8K_4u_2w_1^2/(1+r_L)^2(1+r_T) \\
&- 4w_1^4K_4/(1+r_L)^2(1+r_T)^2 + O(u^3, u^2w^2, uw^4, w^6)\ . \qquad (2.17)
\end{aligned}$$

All terms in these recursion relations are evaluated at rescaling parameter l, with $l = 0$ corresponding to the unrenormalized parameters. Rudnick and Nelson did not distinguish between u_1, u_2 and u_3, and ignored a number of the terms on the right-hand sides of (2.13)-(2.17). In addition, the recursion relations for R_0, which we do not display, turn out to contain terms of $O(w^2)$ which could violate the assumption that $R_0 \equiv 1$ to $O(\epsilon)$ when $M > 0$. It happens that these differences do not affect the solutions to the recursion relations to $O(\epsilon)$. The reason for this is quite simple: Initially, at $l = 0$, we have $M \leq O(1)$ and $r, u \leq O(\epsilon)$, implying that $w \leq O(u) \leq O(\epsilon)$. Hence, initially, terms of order w^3 in (2.13) and (2.14) and terms of order $w^2 u$ and w^4 in (2.15)-(2.17) are much smaller – by relative factors of u or u^2 – than the other terms in (2.13)-(2.17). However, the recursion relations will be integrated out to a value $l = l^*$, defined in such a way that $r_L(l^*) = O(1)$. It will be shown below that at $l = l^*$ we will still have $u \leq O(\epsilon)$, but that $M = O(1/\sqrt{u}) \geq O(1/\sqrt{\epsilon})$, and hence $w = O(\sqrt{u}) \leq O(\sqrt{\epsilon})$. Thus, near $l = l^*$, one will have $u \sim w^2$. The terms with higher powers of w will therefore be of the *same* order as the others, and thus nominally should be kept. In fact, when $l \leq l^*$ it is readily apparent that only the *first* terms on the right-hand sides of (2.11), (2.14), and (2.15) need to be kept; the rest are smaller by relative order u. Initially, it would seem that the first term *always* dominates. However, since we are dealing with *exponential* growth, this turns out not to be the case. To complete the argument one must show that the neglected terms give small contributions on a *logarithmic* scale. It will turn out that in the regime in which $w = O(u)$, the next-to-leading terms are important, while the rest can indeed be neglected: In the regime $w \gg u$, as stated above, only the leading term is important.

To see this, the interval $[0, l^*]$ is divided into two parts: one, $[0, l_\theta]$, in which $r_L \leq O(\epsilon^\theta)$ and the other, $[l_\theta, l^*]$, in which $r_L \geq O(\epsilon^\theta)$, where $0 < \theta < 1$ is an arbitrary exponent (we require only that $\epsilon^\theta \gg \epsilon$). Let us estimate $l^* - l_\theta$: Assuming that in the range $l \geq l_\theta$ only the leading terms need to be kept, we find

$$\left.\begin{aligned} r_L(l) &\sim r_L(l_\theta)e^{2(l-l_\theta)} \\ w_i(l) &\sim w_i(l_\theta)e^{(1-\frac{1}{2}\epsilon)(l-l_\theta)}, \ i=1,2 \end{aligned}\right\} l \geq l_\theta \tag{2.18}$$

which implies [using $w_i(l^*) = O(\sqrt{u(l^*)})$]

$$\begin{aligned} e^{(l^*-l_\theta)} &= O(\epsilon^{-\theta/2}) \\ w_i(l_\theta) &= O(\sqrt{u(l^*)\epsilon^\theta}), \ i=1,2 \ . \end{aligned} \tag{2.19}$$

Substituting these estimates back into the higher-order terms in the recursion relations, we find, self-consistently, that the relative corrections to (2.18) are $O(u(l^*)/\epsilon^\theta, u(l^*)/\epsilon^{1-\theta})$ for r_L, and $O(u(l^*), u(l^*)\epsilon^\theta)$ for w_i. Since $u(l^*) \leq O(\epsilon)$, these corrections are indeed much smaller than unity as long as $\epsilon \ll 1$. Finally, we note that

$$u(l) - u(l_\theta) = O(\epsilon u(l^*), u(l^*)^2)(l - l_\theta) \ , \tag{2.20}$$

so that

$$u(l^*) = u(l_\theta)[1 + O(\epsilon \ln(\epsilon), u(l^*)\ln(\epsilon))] \ , \tag{2.21}$$

implying that $u(l)$ is essentially constant in the interval $[l_\theta, l^*]$.

The above considerations imply that to leading nontrivial order in ϵ, and for $r_L \leq O(1)$, we need only work with the following reduced recursion relations:

$$\begin{aligned} \frac{d\tilde{h}}{dl} =& (3 - \frac{1}{2}\epsilon)\tilde{h} - wK_4(n-1)/(1+r_T) - 3wK_4/(1+r_L) \qquad (2.22) \\ \frac{dr_L}{dl} =& 2r_L + 12K_4u/(1+r_L) + 4(n-1)K_4u/(1+r_T) \end{aligned}$$

$$- 18K_4w^2/(1+r_L)^2 - 2(n-1)K_4w^2/(1+r_T)^2 \qquad (2.23)$$

$$\frac{dr_T}{dl} = 2r_T + 4(n+1)uK_4/(1+r_T) + 4K_4u/(1+r_L)$$

$$- 4K_4w^2/(1+r_L)(1+r_T) \qquad (2.24)$$

$$\frac{dw}{dl} = (1+\frac{1}{2}\epsilon)w - 4(n+8)wuK_4 \qquad (2.25)$$

$$\frac{du}{dl} = \epsilon u - 4(n+8)u^2K_4 \ , \qquad (2.26)$$

where, to this order, we have $R_0 \equiv 1$, $w_1 = w_2 \equiv w$, and $u_1 = u_2 = u_3 \equiv u$. A further simplification is achieved when we note that the solution to (2.25), recalling (2.9), is $w(l) = 4M(0)e^{(1-\frac{1}{2}\epsilon)l}u(l)$. Since the renormalization-group transformation used here is quasi-linear, the renormalization of the magnetization is given precisely by the spin-rescaling factor, $\exp \frac{1}{2}\int_0^l (\eta(l') + d - 2)dl'$. To the present order we have $\eta = 0$, so that

$$M(l) = M(0)e^{\frac{1}{2}(d-2)l} = M(0)e^{(1-\frac{1}{2}\epsilon)l} \ , \qquad (2.27)$$

which then yields

$$w(l) = 4u(l)M(l) \ . \qquad (2.28)$$

The solution to (2.26) is straightforward and yields

$$u(l) = u(0)e^{\epsilon l}/Q(l), \quad \text{where} \quad Q(l) \equiv 1 - \tilde{u} + \tilde{u}e^{\epsilon l}$$

$$\text{and} \quad \tilde{u} \equiv u(0)/u^* = 4(n+8)K_4u(0)/\epsilon \ , \qquad (2.29)$$

where $u^* = \epsilon/4(n+8)K_4$ is the nontrivial fixed-point solution for u. The solutions for $\tilde{h}$, r_L, and r_T are more complicated. We will illustrate the general methodology in Appendix A, but quote only the answers here. (Rudnick and Nelson [1] have done most of their explicit calculations only when $M = h = 0$. We feel it worthwhile to outline the $M > 0$ calculation in more detail, as some subtleties do appear.) One finds, to lowest nontrivial order in ϵ,

$$\begin{aligned} r_L(l) =& T_L(l) - 2(n+2)K_4 u(l) + 6K_4 u(l) T_L(l) \ln(1+T_L(l)) \\ &+ 2(n-1)K_4 u(l) T_T(l) \ln(1+T_T(l)) \\ &+ 144 K_4 u(l)^2 M(l)^2 [\ln(1+T_L(l)) + \frac{T_L(l)}{1+T_L(l)}] \\ &+ 16(n-1)K_4 u(l)^2 M(l)^2 [\ln(1+T_T(l)) + \frac{T_T(l)}{1+T_T(l)}] \end{aligned} \tag{2.30}$$

$$\begin{aligned} r_T(l) =& T_T(l) - 2(n+2)K_4 u(l) + 6K_4 u(l) T_L(l) \ln(1+T_L(l)) \\ &+ 2(n-1)K_4 u(l) T_T(l) \ln(1+T_T(l)) \end{aligned} \tag{2.31}$$

$$\begin{aligned} \tilde{h}(l) =& h(l) - t(l)M(l) - 4u(l)M(l)^3 \\ &+ 2(n+2)K_4 u(l) M(l) - 2(n-1)K_4 u(l) M(l) T_T(l) \ln(1+T_T(l)) \\ &- 6K_4 u(l) M(l) T_L(l) \ln(1+T_L(l)) \ , \end{aligned} \tag{2.32}$$

where

$$\begin{aligned} T_L(l) &= t(l) + 12u(l)M(l)^2 \\ T_T(l) &= t(l) + 4u(l)M(l)^2 \\ t(l) &= t(0)e^{2l}/Q(l)^{\frac{n+2}{n+8}} \\ t(0) &\equiv r(0) + 2(n+2)K_4 u(0) + O(\epsilon u, u^2) \\ h(l) &\equiv h(0)e^{(3+\frac{1}{2}\epsilon)l} \ . \end{aligned} \tag{2.33}$$

The variable $t(0)$ is precisely $r - r_c(u)$.

3.3 Spin waves and the equation of state

From Eq. (2.27), the magnetization is given by

$$M(0) = M(l^*)e^{-(1-\frac{1}{2}\epsilon)l^*} \ . \tag{3.1}$$

The calculation of the renormalized magnetization $M(l^*)$ will be done within the linear spin-wave approximation, which, as we shall demonstrate, is valid precisely in the limit $u(l^*) \ll 1$, $r_L(l^*) = 0(1)$.

The linear spin-wave approximation is simply the quadratic fluctuation correction to the Landau mean-field solution. The only inputs are therefore the two orthogonal curvatures of the Landau free-energy surface at the mean-field minimum. Note that when the external field vanishes, the transverse curvature must vanish because of the continuous global spherical symmetry of the spins. This requirement will provide a consistency check on the renormalization-group calculation. From (2.8) (with $R_0 = 1$, $u_1 = u_2 = u_3 \equiv u$, and $w_1 = w_2 \equiv w$), it is easy to see that the minimum occurs for $\mathbf{s}^\perp = 0$ and $\sigma = \bar{M}(l) - M(l)$, where $\bar{M}$ satisfies

$$(\tilde{h} + r_L M - 8uM^3) = (r_L - 12uM^2)\bar{M} + 4u\bar{M}^3 \ . \tag{3.2}$$

It should be emphasized that $\bar{M}(l) \neq M(l)$, although when $l = l^*$ they differ only by terms of order w; but even then their *external field* dependence (which comes from wave numbers $k < e^{-l^*}$) is very different. This difference is crucial to the derivation of the correct equation of state (see below). When expanded around this minimum, the Hamiltonian takes the form

$$\begin{aligned} H(l) = \int d^d x \Big\{ &\frac{1}{2}|\nabla \mathbf{s}^{\perp}|^2 + \frac{1}{2}|\nabla\tilde{\sigma}|^2 + \frac{1}{2}\Big[r_T - 4u(M^2 - \bar{M}^2)\Big]|\mathbf{s}^{\perp}|^2 \\ &+ \frac{1}{2}\Big[r_L - 12u(M^2 - \bar{M}^2)\Big]\tilde{\sigma}^2 + 4u\bar{M}(\tilde{\sigma}^3 + \tilde{\sigma}|\mathbf{s}^{\perp}|^2) \\ &+ u(\tilde{\sigma}^2 + |\mathbf{s}^{\perp}|^2)^2\Big\} + e^{dl}\Big[\frac{1}{2}r(0)M(0)^2 + u(0)M(0)^4 - h(0)M(0)\Big]V \\ &+ \Big[\frac{1}{2}r_L(\bar{M} - M)^2 + 4uM(\bar{M} - M)^3 + u(\bar{M} - M)^4 - \tilde{h}(\bar{M} - M)\Big]V , \end{aligned} \tag{3.3}$$

where $\tilde{\sigma} = \sigma + (M - \bar{M})$. The square of transverse and longitudinal curvatures are, respectively, $\kappa_T(l)^2 \equiv r_T - 4u(M^2 - \bar{M}^2)$, and $\kappa_L(l)^2 \equiv r_L - 12u(M^2 - \bar{M}^2)$. Using (2.30)-(2.33) and (3.2), we find

$$\begin{aligned} \kappa_T^2 &= \frac{h(l)}{M(l)} + \left[\frac{1}{\bar{M}(l)} - \frac{1}{M(l)}\right]\left[\tilde{h}(l) + r_L(l)M(l) - 8u(l)M(l)^3\right] \\ &= \frac{h(l)}{\bar{M}(l)} + \left[\frac{M(l)}{\bar{M}(l)} - 1\right]\left[r_L(l) - r_T(l) - 8u(l)M(l)^2\right] . \end{aligned} \tag{3.4}$$

The second term in both cases is $O(u(l)^2)$, and hence is beyond the resolution of the present calculation. Therefore, to the order we are working, we may take

$$\kappa_T^2(l) = h(l)/M(l) , \tag{3.5}$$

which indeed vanishes in the ordered phase when $h(0) = 0$. It should be noted that because of the spherical symmetry, the *transverse* susceptibility χ_T is always given *exactly* by $M(l)/h(l)$, although (3.5), which is essentially the mean-field inverse susceptibility, is only approximate since fluctuations with $k < e^{-l}$ have not yet been accounted for. The longitudinal curvature is given by

$$\kappa_L^2 = r_L + 3(\kappa_T^2 - r_T) , \tag{3.6}$$

which is always nonzero, and yields essentially the mean-field inverse longitudinal susceptibility χ_L^{-1}. The true inverse longitudinal susceptibility *vanishes* at $h = 0$ because of spin-wave effects (see below), though not as rapidly as χ_T^{-1}. Again, this is strictly an effect of wave numbers with $k < e^{-l}$.

We now proceed to calculate the equation of state. Since the field $\sigma(\mathbf{x})$ was defined to have zero mean, $<\sigma(\mathbf{x})>= 0$, we will have

$$< \tilde{\sigma}(\mathbf{x}) >= M(l) - \bar{M}(l) \ . \tag{3.7}$$

But the left-hand side, to lowest order in the fluctuations around the mean-field minimum, is just

$$< \tilde{\sigma}(\mathbf{x}) >= - < \tilde{\sigma}(\mathbf{x}) \int d^d y \ 4u\bar{M}[\tilde{\sigma}(\mathbf{y})^3 + \tilde{\sigma}(\mathbf{y})|\mathbf{s}^{\perp}(\mathbf{y})|^2] >_0 \ ,$$

where $<>_0$ indicates an average with respect to the quadratic Hamiltonian

$$H_0 = \frac{1}{2}\int d^d x \left[|\nabla \mathbf{s}^{\perp}|^2 + |\nabla\tilde{\sigma}|^2 + \kappa_T^2|\mathbf{s}^{\perp}|^2 + \kappa_L^2\sigma^2\right] \ . \tag{3.8}$$

In Appendix B we give more insight into this form of perturbation theory. Combining (3.7) and (3.8), we find

$$\begin{aligned} M(l) - \bar{M}(l) &= -4u(l)\bar{M}(l)\int d^d y < \tilde{\sigma}(\mathbf{x})\tilde{\sigma}(\mathbf{y}) >_0 \\ &\qquad \times [3 < \tilde{\sigma}(\mathbf{y})^2 >_0 + (n-1) < |\mathbf{s}^{\perp}(\mathbf{y})|^2 >_0] \qquad (3.9) \\ &= \frac{-4u(l)\bar{M}(l)}{\kappa_L^2(l)}\int_q \Big[\frac{3}{q^2+\kappa_L^2(l)} + \frac{n-1}{q^2+\kappa_T^2(l)}\Big] \ , \end{aligned}$$

where $\int_q \equiv \int \frac{d^d q}{(2\pi)^d}$. The integrals in (3.9) are straightforward and are evaluated in Appendix C. The result is

$$M - \bar{M} = -\frac{\bar{M}}{\kappa_L^2}\{6K_4 u[1 - \kappa_L^2 \ln(1+\kappa_L^2) + \kappa_L^2 \ln(\kappa_L^2)] \\ + 2(n-1)K_4 u[1 - \kappa_T^2 \ln(1+\kappa_T^2) - \frac{2}{\epsilon}\kappa_T^2(\frac{\epsilon\pi/2}{\sin(\epsilon\pi/2)}(\kappa_T^2)^{-\epsilon/2} - 1)]\} \; . \tag{3.10}$$

Now it is straightforward to show that to first order in $\frac{(M-\bar{M})}{M}$, (3.2) can be written in the form

$$\tilde{h} \approx r_L(\bar{M} - M) \approx \kappa_L^2(\bar{M} - M) \; . \tag{3.11}$$

To the same order, M and $\bar{M}$ are interchangeable on the right-hand side of (3.10); one must, however, be careful to use (3.5) for κ_T^2 in the last, singular term of (3.10). Using (2.32) for $\tilde{h}$, (3.10) becomes, after a large number of cancellations,

$$\frac{h}{M} = T_T + 6K_4 u T_L \ln(T_L) + 2(n-1)K_4 u(\frac{-2}{\epsilon})\frac{h}{M}\left[(\frac{h}{M})^{-\epsilon/2} - 1\right] \; , \tag{3.12}$$

where all quantities are evaluated at $l = l^*$. Note that we have, correctly to the order we are working, used (T_L, T_T) and (r_L, r_T) interchangeably in the $O(u)$ terms on the right-hand side. We may substitute the solutions from Sec. 3.2 into (3.12) to get the complete equation of state:

$$h_0/M_0 = [t_0 Q^{\frac{6}{n+8}} + 4u_0 M_0^2]/Q - \frac{(n-1)}{(n+8)}\tilde{u}\frac{h_0}{M_0}\frac{1}{Q}[(\frac{M_0}{h_0})^{\epsilon/2} - e^{\epsilon l}] \\ + 6K_4 u_0 e^{(\epsilon-2)l^*} T_L \ln(T_L)/Q \; , \tag{3.13}$$

where the subscript zero denotes the *unrenormalized* initial values of the parameters and Q and $\tilde{u}$ were defined in (2.29) and (2.33). In order to compare with results of Ref. 3 we define

$$P = 1 + \tilde{u}\frac{n-1}{n+8}\Big[(\frac{M_0}{h_0})^{\epsilon/2} - 1\Big] + \frac{9}{n+8}(Q-1), \qquad (3.14)$$

and define the matching parameter l^* by $T_L(l^*) = 1$ [*not* $T_T(l^*) = O(1)$, as assumed in Ref. 1]; i.e.,

$$1/S = [t_0 Q^{6/(n+8)} + 12u_0 M_0^2]/Q \ , \qquad (3.15)$$

where $S = e^{2l^*}$, which implies that

$$Q = 1 + \tilde{u}(S^{\epsilon/2} - 1) \ . \qquad (3.16)$$

The equation of state then reads

$$\frac{h_0}{M_0} = (t_0 Q^{\frac{6}{n+8}} + 4u_0 M_0^2)/P \ . \qquad (3.17)$$

Equations (3.14)-(3.17) constitute the full equation of state, valid throughout the critical region. These expressions agree precisely with those derived by Nicoll and Chang [3] (see their Sec. III, with the correspondences $\Gamma_1 = h_0$, $Y = P^{-1}$, $Y_2 = Q^{-1}$, $\bar{\Gamma}_2 = S^{-1}$, $\lambda_4 = \epsilon$, $\lambda_2 = 2$, and $u_4 = 12K_4 u$). See also Ref. 6, Sec. VI for some applications of these equations.

As an application of (3.17), let us derive the low-field magnetization for $t_0 < 0$ ($T < T_c$) and identify the spin-wave singularity in the susceptibility. If

M_0 remains positive when $h_0 \to 0$, we see from (3.14) that P is dominated by

$$P \approx \tilde{u}\frac{n-1}{n+8}(\frac{M_0}{h_0})^{\epsilon/2} \ , \ h_0 \to 0 \ , \ t_0 < 0 \ , \tag{3.18}$$

so that the equation of state reads

$$\tilde{u}\frac{n-1}{n+8}(\frac{h_0}{M_0})^{1-\epsilon/2} \approx t_0 Q^{\frac{6}{n+8}} + 4u_0 M_0^2 \ . \tag{3.19}$$

The zero-field magnetization is therefore given by

$$M_0(h_0 = 0) = (-t_0/4u_0)^{\frac{1}{2}} Q^{\frac{3}{n+8}}, \ t_0 < 0 \ , \tag{3.20}$$

while (3.15) and (3.16) yield

$$Q = 1 - \tilde{u} + \tilde{u}(-2t_0)^{-\epsilon/2} Q^{\frac{\epsilon}{2}\frac{n+2}{n+8}}, \ h_0 = 0, \ t_0 < 0 \ . \tag{3.21}$$

Writing for small h_0, $M_0 = M_0(h_0 = 0) + \delta M_0$, and denoting the solution to (3.21) by $Q_-(\tilde{u}, t_0)$, we find

$$\delta M_0 = \tilde{u}\frac{n-1}{n+8}\frac{1}{8u_0 M_0(0)}\left(\frac{h_0}{M_0(0)}\right)^{1-\frac{\epsilon}{2}} \left\{ \frac{1 - \frac{\epsilon}{2}\frac{n+11}{n+8}(1 - \frac{1-\tilde{u}}{Q_-})}{1 - \frac{\epsilon}{2}\frac{n+2}{n+8}(1 - \frac{1-\tilde{u}}{Q_-})} \right\} , \tag{3.22}$$

so that the small-field susceptibility $\chi = \partial \delta M_0 / \partial h_0$ diverges as

$$\chi \approx \tilde{u}\frac{n-1}{n+8}(1 - \frac{\epsilon}{2})\frac{C(Q_-)}{8u_0 M_0(0)^2}(\frac{h_0}{M_0(0)})^{-\epsilon/2}, \ h_0 \to 0, \ t_0 < 0 \ , \tag{3.23}$$

where $C(Q_-)$ is the expression in braces in (3.22). This expression gives the precise $O(\epsilon)$ amplitude of the spin-wave singularity right up to T_c. In Ref. 6 it is shown how to cast such expressions into scaling form in order to best illustrate the various crossovers involved. A simple example will suffice here. For simplicity we take $\tilde{u} = 1$ (i.e., $u = u^*$; see Ref. 6 for the method treating the general case – which involves the introduction of a second scaling variable $\sim u_0/t_0^{\frac{\epsilon}{2}}$). Define the scaling combinations

$$\begin{aligned} \tau \equiv& t_0/(4u_0M_0^2)^{1/2\beta},\ \beta = \frac{1}{2}(1-\frac{1}{2}\epsilon)/(1-\frac{1}{2}\epsilon(n+2)/(n+8)) \\ \hat{Q} \equiv& (4u_0M_0^2)^{\omega_Q}Q,\ \omega_Q = \epsilon/(2-\epsilon) = (4-d)/(d-2)\ . \end{aligned} \tag{3.24}$$

One then finds that $\hat{Q}$ satisfies

$$\hat{Q}^{\frac{\epsilon-2}{\epsilon}} = 3 + \tau\hat{Q}^{\frac{6}{n+8}}, \qquad \tilde{u} = 1\ . \tag{3.25}$$

This represents the Griffiths scaling form for $\hat{Q}$, valid for all $\tau \geq \tau_{coex}$, negative and positive, where $\tau_{coex} = -2^{6\epsilon/(n+8)(2-\epsilon)}$ represents the coexistence curve. Proceeding now to the equation of state, we define the scaling variable for h_0:

$$\zeta^2 \equiv 4u_0h_0^2/(4u_0M_0^2)^{\delta},\ \delta = (6-\epsilon)/(2-\epsilon)\ . \tag{3.26}$$

The relation between ζ and τ then becomes

$$\zeta = (1+\tau\hat{Q}^{\frac{6}{n+8}})/(\frac{9}{n+8}\hat{Q} + \frac{n-1}{n+8}\zeta^{-\epsilon/2})\ , \tag{3.27}$$

which represents the equation of state in Griffiths scaled form. This should be solved to yield $\zeta = Z(\tau)$, or

$$h_0 = DM_0^{\delta}Z(ct_0/M_0)\ , \qquad u_0 = u^*\ , \tag{3.28}$$

where $D = (4u^*)^{-2/(6-\epsilon)}$ and $c = (4u^*)^{-1/2\beta}$. When $t_0 = \tau = 0$, this yields $h_0 = DZ_0M_0^{\delta}$, the usual definition of the exponent δ. The constant $Z_0 = Z(0)$

satisfies $Z_0(\frac{9}{n+8}3^{\epsilon/(2-\epsilon)} + \frac{n+1}{n+8}Z_0^{-\epsilon/2}) = 1$. In the opposite limit $\tau \to \tau_{coex}$, a careful analysis of (3.27) yields precisely (3.22) and (3.23) with $\tilde{u} = 1$. Thus (3.27) and (3.28) are a succinct way of representing the crossover between the spin-wave "fixed point" at $\tau = \tau_{coex}$ and the critical fixed point at $t_0 = h_0 = 0$ [the value of τ is not specified in this limit, depending along which particular path in the (t_0, h_0) plane one approaches the critical point].

Equations (3.24)-(3.27) may also be cast in so-called parametric form [7]. Here one uses a polar-coordinate-type representation in the (t_0, h_0) plane to simplify expressions. One writes

$$h_0 = R^{\beta\delta}h(\theta) \ , \ t_0 = Rt(\theta) \ , \ M_0 = R^{\beta}m(\theta) \ , \qquad (3.29)$$

where $R \geq 0$ is the radial variable and $-1 \leq \theta \leq 1$ is the angular variable. Since the system is symmetric under $h_0 \to -h_0$, a convenient normalization is obtained by taking the positive t_0 axis as $\theta = 0$, the coexistence curve as $\theta = \pm 1$ (where $h_0 \to \pm 0$, respectively), and the positive and negative h_0 axes as $\theta = \pm\theta_0$, respectively, where $0 < \theta_0 < 1$ is chosen for convenience. The utility of this representation is apparent when one realizes that in the Ising case $n = 1$, Eqs. (3.14)-(3.17) (with $\tilde{u} = 1$) correspond *exactly* to the choices

$$h(\theta) = \theta(1-\theta^2)/\sqrt{8u^*} \ , \ t(\theta) = (1 - \frac{3}{2}\theta^2) \ , \ m(\theta) = \theta/\sqrt{8u^*} \ , \qquad (3.30)$$

with $\theta_0^2 = 2/3$ and the exponents β, δ displayed in (3.24) and (3.26) and evaluated at $n = 1$. One also finds the correspondence $Q = R^{\beta(3-\delta)}$. This is the so-called *linear model.*

For $n > 1$, life cannot be so simple. A spin-wave singularity, roughly of the form $(1-\theta^2)^{\frac{1}{2}\epsilon}$, must appear in either $h(\theta)$ or $m(\theta)$ when $\theta \to \pm 1$. In fact, this is apparent in Ref. 5, where this term appears as an unexponentiated logarithm in their Eq. (25). The authors, however, were unable to interpret this term unambiguously. To see how (3.30) must be modified when $n > 1$, it is simplest to leave the functions $t(\theta)$ and $m(\theta)$ untouched [in fact, one may always choose $m(\theta)$ linear [7]]. This yields (3.15) and (3.16) (when $\tilde{u} = 1$) for general n, with the same relation between Q and R. We now modify $h(\theta)$ to obtain the equation of state (3.14) and (3.17) for general n. If we write

$$h(\theta) = \theta(1-\theta^2)/\sqrt{8u^*}\;\tilde{h}(\theta) \; , \tag{3.31}$$

we find that consistency yields the equation

$$\tilde{h}(\theta) = \nu_n(1-\theta^2)^{-\epsilon/2}\tilde{h}(\theta)^{\epsilon/2} + (1-\nu_n) \; , \tag{3.32}$$

where $\nu_n = (n-1)/(n+8)$ vanishes when $n = 1$. For $\theta^2 \to 1$ one finds

$$\tilde{h}(\theta) \approx \nu_n^{\frac{1}{1-\epsilon/2}}(1-\theta^2)^{-\frac{\epsilon/2}{1-\epsilon/2}} + O(1) \; ; \tag{3.33}$$

hence

$$h(\theta) \approx 2^{-\frac{1}{2}}\nu_n^{\frac{-1}{1-\epsilon/2}}(1-\theta^2)^{\frac{1}{1-\epsilon/2}}, \; \theta^2 \to 1 \; . \tag{3.34}$$

It is apparent that the linear model misses all of the essential physics of the coexistence-curve behavior when $n > 1$. If the spin-wave singularities are expanded naively in powers of ϵ, as in Ref. 5, one encounters a term proportional to $\epsilon(1-\theta^2)\ln(1-\theta^2)$, which drives h_0 unphysically negative as $\theta \to 1$, yielding, for example, a negative-going susceptibility.

3.4 Free energy

In this section, we compute the free energy. From Ref. 1 we have, to one-loop order,

$$F(r_0, u_0, h_0) = \int_0^l e^{-dl'} G_0(l')dl' + e^{-dl} F(r(l), u(l), h(l)) \ , \tag{4.1}$$

where

$$G_0(l) = \frac{1}{2} K_d [\ln(1 + r_L(l)) + (n-1)\ln(1 + r_T(l)) - \frac{2}{d} n] \ , \tag{4.2}$$

and we have dropped a trivial constant proportional to $\ln(2\pi)$ from the total free energy. The trajectory integral is a tedious, but straightforward, application of the techniques used in Appendix A. Most of the answer has already been given in Ref. 1. We begin by writing $\ln(1+r) = [\ln(1+r) - r + \frac{1}{2}r^2] + r - \frac{1}{2}r^2$, the idea being, once again, to isolate the small r from the large r behavior. The first (bracketed) term contains the large $r = O(1)$ dependence, the second contains the small r dependence, while the last is slowly varying and must be integrated exactly. One finds, then,

$$\int_0^l e^{-dl'} G_0(l')dl' = I_1(l) + I_2(l) - I_2(l=0) \ , \tag{4.3}$$

where

$$I_1(l) = -\frac{1}{4} K_d \int_0^l e^{-dl'} [r_L(l')^2 + (n-1) r_T(l')^2] dl' \tag{4.4}$$

$$I_2(l) = \frac{K_d}{2d}(n-1)e^{-dl}[(r_T(l)^2 - 1)\ln(1 + r_T(l)) - \frac{1}{2}r_T(l)^2 - \frac{2}{d-2}r_T(l) + 2/d] + \frac{K_d}{2d}e^{-dl}[(r_L(l)^2 - 1)\ln(1 + r_L(l)) - \frac{1}{2}r_L(l)^2 - \frac{2}{d-2}r_L(l) + 2/d] \ . \tag{4.5}$$

The expression (4.5) for $I_2(l)$ differs in some details from that given in Ref. 1 — the major difference being the $-\frac{1}{2}r^2$ terms which are needed for later cancellation. All other differences disappear when one takes, correct to order ϵ, $d = 4$ in the various coefficients on the right-hand side of (4.5). The remaining integral $I_1(l)$ is evaluated by using $r_L \approx T_L$ and $r_T \approx T_T$, substituting (2.27) and (2.33), then performing the integral exactly. (This is possible since the entire integrand is then slowly varying, a function only of $e^{\epsilon l}$.) The result is

$$I_1(l) = \frac{t_0^2}{16u_0}\frac{n}{n-4}[Q(l)^{\frac{4-n}{n+8}} - 1] + \frac{1}{2}t_0 M_0^2[Q(l)^{-\frac{n+2}{n+8}} - 1] + u_0 M_0^4[Q(l)^{-1} - 1] \ . \tag{4.6}$$

To complete the calculation, we require $F(l)$. This is given by the first fluctuation correction using the Hamiltonian (3.3). We find $e^{-dl}F(l) = F_0 + F_1(l) + F_2(l)$, where

$$F_0 = \frac{1}{2}r_0 M_0^2 + u_0 M_0^4 - h_0 M_0 \tag{4.7}$$

$$F_1(l) = e^{-dl}[\frac{1}{2}r_L(l)(\bar{M}(l) - M(l))^2 + 4u(l)M(l)(\bar{M}(l) - M(l))^3 + u(l)(\bar{M}(l) - M(l))^4 - \tilde{h}(l)(\bar{M}(l) - M(l))] \tag{4.8}$$

$$F_2(l) = \frac{1}{2}e^{-dl}\int_q [\ln\left(\frac{1}{q^2 + \kappa_L^2}\right) + (n-1)\ln\left(\frac{1}{q^2 + \kappa_T^2}\right)] \ . \tag{4.9}$$

Note that by (3.2), $\bar{M}(l)$ minimizes $F_1(l)$; i.e., $\partial F_1(l)/\partial \bar{M}(l) = 0$. The integral

$F_2(l)$ can be evaluated to give $F_2(l) = \bar{F}_2(l) + F_{SW}(l)$, where (see Appendix C)

$$\begin{aligned}\bar{F}_2(l) =& - e^{-dl}\frac{(n-1)K_d}{2d}[(\kappa_T^4 - 1)\ln(1+\kappa_T^2) - \frac{1}{2}\kappa_T^4 - \frac{2}{d-2}\kappa_T^2 + \frac{2}{d}] \\ & - e^{-dl}\frac{K_d}{2d}[(\kappa_L^4 - 1)\ln(1+\kappa_L^2) - \frac{1}{2}\kappa_L^4 - \frac{2}{d-2}\kappa_L^2 + \frac{2}{d}]\end{aligned} \tag{4.10}$$

$$\begin{aligned}F_{SW}(l) =&(n-1)\frac{K_d}{2d}\left(\frac{-2}{\epsilon}\right)\kappa_T^4[\kappa_T^{-\epsilon}\frac{4}{d}\frac{\pi\epsilon/2}{\sin(\pi\epsilon/2)} - 1]e^{-dl} \\ & + \frac{K_d}{2d}e^{-dl}\kappa_L^4[\ln(\kappa_L^2) - \frac{1}{2}] \ .\end{aligned} \tag{4.11}$$

We have displayed the exact $\kappa_T^{4-\epsilon}$ singularity in (4.11) for completeness. This is the spin-wave contribution to F. Note the similarity between $I_2(l)$ and $\bar{F}_2(l)$. We take advantage of this similarity by expanding $I_2 + \bar{F}_2$ in the small differences, $\kappa_L^2 - r_L = 12u(\bar{M}^2 - M^2)$ and $\kappa_T^2 - r_T = 4u(\bar{M}^2 - M^2)$:

$$\begin{aligned}I_2 + \bar{F}_2 \approx& -\frac{1}{2}e^{-dl}(\bar{M}^2 - M^2)\{-2(n+2)K_4u + 6K_4uT_L\ln(1+T_L) \\ & + 2(n-1)K_4uT_T\ln(1+T_T)\} \\ \approx& -M(\bar{M} - M)e^{-dl}\{r_T - T_T\} \ ,\end{aligned} \tag{4.12}$$

where we have used (r_L, T_L, κ_L^2) and (r_T, T_T, κ_T^2) interchangeably inside the terms of order u, and set $d = 4$ in the various coefficients on the right-hand side. The last line follows from (2.31) and (2.33). We now combine (4.12) and (4.8), and use (3.11) to evaluate $\bar{M} - M$ (correct to the order we are working):

$$\begin{aligned}I_2 + \bar{F}_2 + F_1 \approx& e^{-dl}\frac{1}{T_L}[\frac{1}{2}\tilde{h}^2 - \tilde{h}^2 - \tilde{h}M(r_T - T_T)] \\ =& -e^{-dl}(M^2/2T_L)[(h/M - T_T)^2 - (r_T - T_T)^2]\end{aligned} \tag{4.13}$$

where we have used $\tilde{h} = h - r_T M$ [see (2.31) and (2.32)]. The expressions for $F_0, I_1(l)$, and $I_2(l=0)$ combine to yield

$$\begin{aligned}F_0 - I_2(l=0) + I_1(l) =& A_{reg} + (t_0^2/16u_0)\frac{n}{n-4}(Q^{\frac{4-n}{n+8}} - 1) \\ & - h_0M_0 + (\frac{1}{2}t_0M_0^2Q^{6/(n+8)} + u_0M_0^4)/Q \ ,\end{aligned} \tag{4.14}$$

where

$$A_{reg} = (nK_d/2d)[(1 - r_0^2)\ln(1 + r_0) + \frac{1}{2}r_0^2 + \frac{2}{d-2}r_0 - \frac{2}{d}] \tag{4.15}$$

is the regular part of the free energy.

It is tempting to replace κ_T^2 by h/M in (4.11), according to (3.4) and (3.5). Unfortunately, this leads to incorrect results: we will ultimately be interested in deriving the correct equation of state from our free energy, and although $\kappa_T^2 \approx h/M$, the equality breaks down under differentiation. The correct replacement is actually $h/\bar{M}$ (see below) or, preferably, preserve the exact value $r_T - 4u(M^2 - \bar{M}^2)$ until after differentiation. One may then safely interchange M and $\bar{M}$. Let us define $\bar{M}_0 \equiv e^{-(1-\frac{\epsilon}{2})l}\bar{M}$, and [cf. (3.14)]

$$\tilde{P} = 1 + \frac{9}{n+8}(Q-1) + \frac{n-1}{n+8}\tilde{u}[\frac{4}{d}(h_0/\bar{M}_0)^{-\frac{\epsilon}{2}} - 1] \; . \tag{4.16}$$

The total free energy then reads

$$\begin{aligned} A \equiv F + h_0 M_0 = A_{reg} &+ \frac{t_0^2}{16u_0}\frac{4}{n-4}[Q^{\frac{4-n}{n+8}} - \frac{n}{4}] + \frac{1}{16u_0}(\frac{h_0}{\bar{M}_0})^2(Q - \tilde{P}) \\ &+ R^2/16u_0 Q + (K_4/8)e^{-dl}\kappa_L^4[\ln(\kappa_L^2) - \frac{1}{2}] \\ &- (SM_0^2/2T_L)[(h_0/M_0 - R/Q)^2 - (r_T/S - R/Q)^2] \; , \end{aligned} \tag{4.17}$$

where $S = e^{2l}$ [see (3.16)], and we have defined, for convenience,

$$R = t_0 Q^{6/(n+8)} + 4u_0 M_0^2 \; . \tag{4.18}$$

Equation (4.17) deserves some comment: Strictly speaking, F should depend only on h_0, while A should depend only on M_0. Thus one should actually write

$$F = F(r_0, u_0, h_0, M_0(h_0)), \;\; A = A(r_0, u_0, h_0(M_0), M_0) \; . \tag{4.19}$$

The function $M_0(h_0)$, or equivalently, $h_0(M_0)$, is determined by minimization:

$$\frac{\partial F}{\partial M_0}|_{M_0(h_0)} = 0 \qquad \text{or} \qquad \frac{\partial A}{\partial h_0}|_{h_0(M_0)} = 0 \; , \tag{4.20}$$

where the partials denote derivatives with respect to the fourth and third arguments of F and of A, respectively, in (4.19). Alternatively, the usual thermodynamic relations imply

$$\begin{aligned} \frac{dF}{dh_0} &\equiv \frac{\partial F}{\partial h_0} + \frac{\partial F}{\partial M_0}\frac{dM_0}{dh_0} = -M_0(h_0) \\ \frac{dA}{dM_0} &\equiv \frac{\partial A}{\partial M_0} + \frac{\partial A}{\partial h_0}\frac{dh_0}{dM_0} = h_0(M_0) \; . \end{aligned} \tag{4.21}$$

However, by (4.18), the second terms vanish, yielding simply

$$\frac{\partial F}{\partial h_0} = -M_0(h_0) \qquad \text{or} \qquad \frac{\partial A}{\partial M_0} = h_0(M_0) \; . \tag{4.22}$$

Note that since $A = F + h_0 M_0$, the second half of (4.20) is equivalent to the first half of (4.22), and vice versa. This still leaves two entirely distinct ways of calculating the equation of state. The main test of the free energy (4.17) is that it should yield the same result [Eqs. (3.14)-(3.17)] by either route. We begin by verifying the h_0 derivative. This is quite simple since the only explicit h_0 dependence is in κ_T^2 and κ_L^2 through $\bar{M}$. From (3.2) we find (for fixed l)

$$\frac{\partial \bar{M}}{\partial h} = \frac{1}{\kappa_L^2} \; , \tag{4.23}$$

and hence

$$\frac{\partial \kappa_T^2}{\partial h} = \frac{8u\bar{M}}{\kappa_L^2}, \; \frac{\partial \kappa_L^2}{\partial h} = \frac{24u\bar{M}}{\kappa_L^2} \; . \tag{4.24}$$

It is now straightforward to derive

$$0 = \frac{\partial A}{\partial h_0} = \frac{SM_0}{Q\kappa_L^2}\frac{h_0}{M_0}(Q-P) - (SM_0/T_L)(h_0/M_0 - R/Q)$$
$$+ (K_d/4)\kappa_L^2 \ln(\kappa_L^2)(24u\bar{M}_0/\kappa_L^2)\ . \tag{4.25}$$

Setting $T_L = 1 \approx \kappa_L^2$ [which yields (3.15)], the last term vanishes and (4.25) can be manipulated into the form

$$h_0/M_0 = R/P\ , \tag{4.26}$$

which, using (4.18), corresponds precisely to (3.17), the correct equation of state.

The derivative with respect to M is somewhat more tedious. Let us define

$$\begin{aligned} V = {} & 432K_4u^2[\ln(1+T_L) + T_L/(1+T_L)] \\ & + 48(n-1)K_4u^2[\ln(1+T_T) + T_T/(1+T_T)] \\ & + 3456u^3M^2K_4[1/(1+T_L) + 1/(1+T_L)^2] \\ & + 128(n-1)K_4u^3M^2[1/(1+T_T) + 1/(1+T_T)^2] \\ W = {} & 144K_4u^2[\ln(1+T_L) + T_L/(1+T_L)] \\ & + 16(n-1)K_4u^2[\ln(1+T_T) + T_T/(1+T_T)]\ . \end{aligned} \tag{4.27}$$

Note that $\frac{\partial}{\partial M}(WM^3) = VM^2$, and that both V and W are $O(u^2)$. It is then straightforward to show that

$$\begin{aligned} \frac{\partial \bar{M}}{\partial M} = {} & M(M-\bar{M})V/\kappa_L^2 \\ \frac{\partial r_L}{\partial M} = {} & (24u+V)M;\ \frac{\partial \kappa_L^2}{\partial M} = VM[1 + 24u\bar{M}(M-\bar{M})/\kappa_L^2] \\ \frac{\partial r_T}{\partial M} = {} & (8u+W)M;\ \frac{\partial \kappa_T^2}{\partial M} = M[W + V8u\bar{M}(M-\bar{M})/\kappa_L^2]\ . \end{aligned} \tag{4.28}$$

Thus $\partial \bar{M}/\partial M = O(u^2)$; i.e., $\bar{M}$ essentially does not depend on M. This can be seen more directly from (3.2): To the extent that $r_L - 12uM^2$ and $r_T - 4uM^2$ are approximately M-independent, the coefficients in (3.2) are M-independent. Similarly, κ_L^2 and κ_T^2 are weakly M-dependent, while r_L and r_T are dominated by the M dependence of T_L and T_T. To lowest order, then, we have

$$\frac{\partial A}{\partial M_0} \approx \frac{R}{8u_0 Q} 8u_0 M_0 + \frac{S}{T_L}(\frac{R}{Q} + \frac{8u_0 M_0^2}{Q})(h_0 - RM_0/Q) \ . \tag{4.29}$$

Using $T_L = S(R + 8u_0 M_0^2)/Q$, essentially everything cancels, and we are left with

$$\frac{\partial A}{\partial M_0} \approx h_0 \ , \tag{4.30}$$

which is the correct answer. It is apparent then that the equation of state, (4.26), can emerge from $\partial A/\partial M_0$ only in higher order. However, keeping the higher-order terms in (4.28) is not sufficient — many other terms of the same order will arise from higher Feynman graphs and from better approximate solutions to the recursion relations. This problem seems to be a general feature of loop expansions: Different paths leading to the same physical quantity may require different orders in perturbation theory to achieve equivalent results.

Having demonstrated satisfactory consistency of our free energy, we compare it to that derived by Nicoll and Chang [3]. Their result can be written in the form (see Ref. 6, Sec. VI)

$$A_{NC} - A_{reg} = \frac{t_0^2}{16u_0}\frac{4}{n-4}(Q^{\frac{4-n}{n+8}} - \frac{n}{4}) + R^2/16u_0 P \ , \tag{4.31}$$

with (3.14), (3.15), (3.16), and (4.18) defining P, Q, R, and S, respectively. They state that the equation of state should be derived by differentiating only the explicit M_0 dependence in (4.31), i.e., that which appears in R. This indeed yields (4.26). However, one should demonstrate that derivatives, with respect to the implicit M_0 dependence, do not contribute further terms. The simplest way to compare (4.31) to (4.17) is to use the equation of state to substitute for h_0/M_0 in (4.17), and to use M_0 and $\bar{M}_0$ interchangeably. This yields

$$\begin{aligned} A =& A_{reg} + \frac{t_0^2}{16u_0}\frac{4}{n-4}[Q^{\frac{4-n}{n+8}} - \frac{n}{4}] - \frac{K_d}{16}S^{-(2-\frac{1}{2}\epsilon)} \\ &+ \frac{R^2}{16u_0}[\frac{Q}{P^2} - \frac{\tilde{P}}{P^2} + \frac{1}{Q} - \frac{1}{(1+R/8u_0M_0^2)}(\frac{Q}{P^2} - \frac{2}{P} + \frac{1}{Q})] \,, \end{aligned} \tag{4.32}$$

where we have dropped the term proportional to $(r_T/S - R/Q)^2$, since it is apparently of higher order in u and in any case is approximately independent of both M_0 and h_0. The last term in (4.32) can be rewritten:

$$\frac{R^2}{16u_0\tilde{P}}[1 + \frac{\tilde{P}R/P^2}{R+8u_0M_0^2}\frac{1}{Q}(P-Q)^2 - (\frac{P-\tilde{P}}{P})^2] \,. \tag{4.33}$$

To the extent that $\tilde{P} \simeq P$ [compare (3.14) and (4.16)] and $R \ll 1$, (4.33) reproduces the last term in the Nicoll-Chang result [Eq. (4.31)]. It is easy to confirm that $(P - \tilde{P})/P$ is always $O(\epsilon)$, while $P - Q$ is roughly $O(\epsilon)$, unless $(h_0/M_0)^{\epsilon/2}$ is small — however, in this case the prefactor $R/P = h_0/M_0$ is small, so the whole term is always small. A better approximation is to take $\tilde{P}$ in place of P in (4.31). This is so because, for complete consistency, the equation of state should be derived not only from the M_0 dependence in R, but also from that in P. One finds [recall that $h_0/\bar{M}_0$ in (4.16) has been replaced by R/P]

$$\frac{\partial \tilde{P}}{\partial M_0} = \frac{4}{d}\frac{\partial P}{\partial M_0} = \frac{-16u_0 M_0}{R}\frac{(\tilde{P}-P)P}{P-\frac{1}{2}d(\tilde{P}-P)}\ , \tag{4.34}$$

so that neglecting the M_0 dependence of P in (4.31) entails errors of relative order $\frac{\tilde{P}-P}{P} = O(\epsilon)$ in the equation of state. Alternatively, if one takes $R^2/\tilde{P}$ in (4.31), the extra factor of $4/d$ cancels the error term linear in $\frac{\tilde{P}-P}{P}$, leaving errors only of $O([(\tilde{P}-P)/P]^2)$.

Finally, recall the $\kappa_L^4[\ln(\kappa_L^2) - \frac{1}{2}]$ term. This term is constructed so as to vanish when differentiated at fixed l and then evaluated at $\kappa_L^2 = 1$. However, if $\kappa_L^2 = 1$ is imposed *before* differentiation, this term, which then takes the value $\frac{1}{16}K_d S^{-(2-\frac{1}{2}\epsilon)}$, serves to maintain the identity $\partial A/\partial l^* = 0$ — cancelling contributions from the now M_0- and h_0- dependent functions Q and S appearing elsewhere in the free energy. This term is also crucial for correct evaluation of other derivatives, such as the entropy (or density, depending on how thermodynamic variable r is identified) $-(\partial A/\partial t_0)_{M_0}$. Therefore, the lack of this term in the Nicoll-Chang free energy represents a definite discrepancy with our own expression. For the reasons given above, we believe our expression to be the correct one.

In summary, then, the correct form for the free energy, closest in spirit to that of Nicoll and Chang, reads

$$\tilde{A}_{NC} - A_{reg} = \frac{t_0^2}{16u_0}\frac{4}{n-4}[Q^{\frac{4-n}{n+8}} - \frac{n}{4}] + R^2/16u_0\tilde{P} - \frac{1}{16}K_d S^{-(2-\frac{1}{2}\epsilon)}\ , \tag{4.35}$$

with the equation of state to be derived by differentiation with respect to the explicit M_0 dependence in R [according to (4.18)] and in $\tilde{P}$ [according to (4.34)]. By construction [explicitly verified for the free energy (4.17)], all other implicit M_0 dependence — embodied in the choice of the matching scale $l = l^*$ [i.e., Eq. (3.15)] — will cancel out under differentiation. In Sec. 3.5 we explore the effects of the extra $S^{-(2-\frac{1}{2}\epsilon)}$ term on quantities derived from the free energy. In particular, we reexamine the derivation of the helicity modulus Υ at constant density in Ref. 6. We also give an enormously simplified rederivation of the helicity modulus which agrees precisely with the expression calculated by Rudnick and Jasnow [8]. Our approach relies on the identification of Υ with the small $\mathbf{k}$ behavior of the Green's function [9], rather than on the method of comparing free energies for periodic and antiperiodic boundary conditions [8].

3.5 Helicity modulus, density and specific heat

3.5.1 Helicity modulus

In the ordered phase (in zero external field), the Green's function

$$G(\mathbf{k}) = \langle |\mathbf{s}_\mathbf{k}|^2 \rangle = \langle |\mathbf{s}_\mathbf{k} \cdot \hat{\mathbf{M}}|^2 \rangle + \langle |\mathbf{s}_\mathbf{k}^\perp|^2 \rangle \tag{5.1}$$

has the small-$\mathbf{k}$ behavior

$$G(\mathbf{k}) = |\mathbf{M}_0|^2 \delta(\mathbf{k}) + b_T/|\mathbf{k}|^2 + b_L/|\mathbf{k}|^\epsilon + O(1) \ , \tag{5.2}$$

where b_L is related to the amplitude of the divergence of the longitudinal susceptibility [Eq. (1.1)] and b_T is the amplitude of the transverse spin-wave singularity. One has the exact correspondence

$$b_T = (n-1)|\mathbf{M}_0|^2 k_B T/\Upsilon \ , \tag{5.3}$$

where Υ is the *helicity modulus*, which is related to the superfluid density via

$$\rho_s = (m/\hbar)^2 \Upsilon \ , \tag{5.4}$$

m being the mass of a ^{4}He atom.* One may also define Υ in terms of an integral over a current-current correlation [8,9] function (which involves an average over a four-spin operator, rather than a two-spin operator). The latter is more closely related to the definition of Υ in terms of the free-energy increment due to "twist" boundary conditions [8,11]. We concentrate on the former definition, which may be restated as

$$\Upsilon/k_B T|\mathbf{M}_0|^2 = \lim_{k\to 0} \frac{1}{k^2 G_\perp(\mathbf{k})} \ , \tag{5.5}$$

where $G_\perp(\mathbf{k}) = \frac{1}{n-1}\langle|\mathbf{s}_\mathbf{k}^\perp|^2\rangle$ is the transverse part of the Green's function. Since the renormalization-group transformation used to generate the recursion relations in Sec. 3.2 is quasi-linear, the small-$\mathbf{k}$ part of the Green's function transforms exactly as

$$dG_\perp/dl = -[2-\eta(l)]G_\perp \ . \tag{5.6}$$

To $O(\epsilon)$ one has $\eta(l) \equiv 0$, which yields

$$G_\perp(\mathbf{k}, l=0) = e^{2l^*} G_\perp(\mathbf{k}e^{l^*}, l^*) \tag{5.7}$$

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* See Ref. 9 for a nice discussion. The result has, of course, been known in the helium literature for several dacades [10].

and hence

$$\Upsilon/k_BTM_0^2 = \lim_{k\to 0} 1/k^2 G_\perp(\mathbf{k}, l^*) = \Upsilon(l^*)/k_BTM(l^*)^2 \ , \tag{5.8}$$

so that

$$\Upsilon = e^{-(d-2)l^*}\Upsilon(l^*) \ . \tag{5.9}$$

Thus we need only calculate $\Upsilon(l^*)$, which to $O(\epsilon)$ involves only the lowest-order spin-wave corrections to Υ. From the Hamiltonian (3.3) we find, to $O(u(l^*))$,

$$\begin{aligned} G_\perp(\mathbf{k})^{-1} =& k^2 + \kappa_T^2 + 4(n+1)u(l^*)I_T + 4u(l^*)I_L \\ & - \frac{1}{2}w(l^*)^2[4(n-1)I_T + 6I_L]/\kappa_L^2 \\ & - 4w(l^*)^2 I_{LT}(\mathbf{k}) + O(u(l^*)^2) \ , \end{aligned} \tag{5.10}$$

where

$$\begin{aligned} I_L &= \int_q \frac{1}{q^2+\kappa_L^2}, \quad I_T = \int_q \frac{1}{q^2+\kappa_T^2} \\ I_{LT}(\mathbf{k}) &= \int_q \frac{1}{(\mathbf{k}+\mathbf{q})^2+\kappa_L^2}\,\frac{1}{q^2+\kappa_T^2} \ . \end{aligned} \tag{5.11}$$

On the coexistence curve the right-hand side of (5.10) must vanish at $k=0$. This determines κ_T^2. Setting $k=0$, we find

$$\begin{aligned} (\kappa_T^2)_{\text{coex}} =& [-4(n+1)u(l^*) + 2(n-1)w(l^*)^2]I_T \\ & + [-4u(l^*) + 3w(l^*)^2/\kappa_L^2]I_L + 4w(l^*)^2 I_{LT}(k=0) \ . \end{aligned} \tag{5.12}$$

Hence the only contribution to $\Upsilon(l^*)$ comes from the k-dependence of $I_{LT}(\mathbf{k})$:

$$\Upsilon(l^*)/k_BTM(l^*)^2 = 1 - \lim_{k\to 0} 4w(l^*)^2\frac{1}{k^2}[I_{LT}(\mathbf{k}) - I_{LT}(0)] \ . \tag{5.13}$$

At this point one encounters problems with the sharp cutoff we have been using: The domain of integration for $I_{LT}(\mathbf{k})$ in (5.11) is defined as the region of $|\mathbf{q}| < 1$

such that $|\mathbf{k}+\mathbf{q}| < 1$ as well, i.e., the intersection between two hyperspheres whose centers are separated by $\mathbf{k}$. This yields a contribution to $I_{LT}(\mathbf{k}) \propto |\mathbf{k}|$, and hence the limit in (5.13) yields a divergent result. This problem is solved in the original physical model by imposing proper periodicity at the boundaries of the Brillouin zone (i.e., Umklapp processes in the Fourier space representation of the us^4 interaction). The spherical Brillouin zone we use here complicates matters since it cannot be repeated periodically via translation by reciprocal lattice vectors. We instead solve the problem by fiat: Since $I_{LT}(\mathbf{k}) - I_{LT}(0)$ is well defined if the cutoff is allowed to diverge to infinity, we define $\Upsilon(l^*)$ from the leading k^2 dependence of this cutoff-less expression. The result we will then derive agrees with that of Rudnick and Jasnow [8] (who encountered precisely this problem, and solved it in this same way) and with a field-theoretic derivation of universal amplitude ratios involving Υ (see below) [12].

To the requisite order, (5.13) may be evaluated with $\kappa_T^2 = 0$ on the right-hand side. We find

$$\lim_{k\to 0} \frac{1}{k^2}[I_{LT}(\mathbf{k}) - I_{LT}(0)] = \int_q \frac{1}{q^2} \frac{(\epsilon/d)q^2 - \kappa_L^2}{(q^2+\kappa_L^2)^2} \; . \tag{5.14}$$

To $O(\epsilon)$ we may also take $d = 4$ in this integral so that

$$\Upsilon(l^*)/k_BT = M(l^*)^2 + \frac{1}{4}K_4 + O(u(l^*)) \; , \tag{5.15}$$

where we have used the matching conditions $\kappa_L(l^*) \approx 1$ and $8u(l^*)M(l^*) \approx 1$ in the second term on the right-hand side. The final result is then

$$\Upsilon/k_BT = M_0^2 + e^{-(d-2)l^*}\frac{1}{4}K_4 + O(\epsilon^2) \; , \tag{5.16}$$

which corresponds precisely to the result of Rudnick and Jasnow [8].

A very similar calculation for the helicity modulus was carried out in Appendix B of Ref. 12 in the context of verifying two-scale-factor universality. There it was shown that the ratio $\xi_\Upsilon(-t_0)/\xi(t_0)$ tends to a universal constant as $t_0 \to 0^+$. Here ξ is the usual correlation length defined by the exponential decay of the spin-spin correlation function above T_c, while $\xi_\Upsilon = (\Upsilon/k_B T)^{-1/(d-2)}$ is the natural hydrodynamic length which diverges as T_c is approached from below. Universality of this ratio is a consequence of hyperscaling, and is therefore valid for $2 < d < 4$.

3.5.2 Density

We define the density ρ_0 via

$$\rho_0 = (\partial A/\partial r_0)_{M_0} = (\partial F/\partial r_0)_{h_0} \ . \tag{5.17}$$

We call this a density since in the problem of superfluidity in a dilute Bose gas $r_0 \propto -\mu$ is related to the chemical potential, and ρ_0 is related via a multiplicative temperature-dependent factor to the boson density ρ [6]. We carry out the above derivative on the free-energy expression (4.17) [or (4.35)] at fixed l, then set $l = l^*$. For simplicity, we will take $h_0 = 0$. At fixed l, Q is r_0-independent. For $t_0 < 0$ we also have $R = 0$ (coexistence curve). As mentioned earlier, the last term in (4.17) is designed to vanish under differentiation. The only t_0 (hence r_0) dependence that contributes to ρ_0 in the end comes from the first two terms in (4.17). Hence

$$\rho_0 = \rho_{0,reg} + \frac{t_0}{8u_0}\,\frac{4}{n-4}\left[Q_-^{\frac{4-n}{n+8}} - \frac{n}{4}\right] \ ,\ h_0 = 0\ ,\ t_0 < 0\ , \tag{5.18}$$

where Q_- satisfies (3.21) and

$$\rho_{0,reg} = (nK_4/4)[1 - r_0 \ln(1 + r_0)]\ . \tag{5.19}$$

This should be compared to Eq. (6.35) in Ref. 6 which is far more complicated. The extra complexity is a direct result of the missing $S^{-(2-\frac{1}{2}\epsilon)}$ term in (4.31) which would otherwise serve to *cancel* the extra terms. The numerical difference, however, is probably very small. Correcting the subsequent equations in Ref. 6 is very simple. In particular, the coefficient of $\dot{Q}^{\frac{4-n}{n+8}}$ in Eq. (6.48) and of $Z(\Xi)^{1/5}$ in Eq. (6.65) should simply be set to unity.

For $t_0 > 0$ one needs an expression for the limit $\kappa_T^2 \approx h/M$ when $h \to 0$; i.e., $\chi(l)^{-1}$ the inverse susceptibility. From the equation of state, or by direct diagramatic evaluation, one finds

$$\chi^{-1}(l=0) = e^{-2l}\chi(l)^{-1} = t_0 Q^{-\frac{n+2}{n+8}}[1 - 2(n+2)K_4 u(l)\ln(t(l))], \qquad h_0 = 0,\ t_0 > 0\ . \tag{5.20}$$

With the matching condition $t(l^*) = 1$, one determines $Q \equiv Q_+$ via

$$Q_+ = 1 - \tilde{u} + \tilde{u} t_0^{-\frac{\epsilon}{2}} Q_+^{\frac{\epsilon}{2}\frac{n+2}{n+8}} \tag{5.21}$$

[cf. (3.21)], and only the first term in (5.20) survives. From these equations one finds that $P = Q_+$ when $h_0 = 0$ and $t_0 > 0$. Hence only the fourth term in

(4.17) contributes any further t_0 dependence, and one finds

$$\rho_0 = \rho_{0,reg} + \frac{t_0}{8u_0}\frac{n}{n-4}\left[Q_+^{\frac{4-n}{n+8}} - 1\right], \; h_0 = 0, \; t_0 > 0 \; . \tag{5.22}$$

The compressibility (more commonly interpreted as the specific heat) is then given by

$$\begin{aligned}
\kappa_0 &= -\partial\rho_0/\partial t_0 \\
&= \kappa_{0,reg} + \frac{1}{2(4-n)u_0}\left[Q_-^{\frac{4-n}{n+8}}\left[\frac{1-\frac{\epsilon}{2}\frac{6}{n+8}(1-(1-\tilde{u})Q_-^{-1})}{1-\frac{\epsilon}{2}\frac{n+2}{n+8}(1-(1-\tilde{u})Q_-^{-1})}\right] - \frac{n}{4}\right], \\
&\qquad h_0 = 0, \; t_0 < 0 \qquad (5.23) \\
&= \kappa_{0,reg} + \frac{n}{8(4-n)u_0}\left[Q_+^{\frac{4-n}{n+8}}\left[\frac{1-\frac{\epsilon}{2}\frac{6}{n+8}(1-(1-\tilde{u})Q_+^{-1})}{1-\frac{\epsilon}{2}\frac{n+2}{n+8}(1-(1-\tilde{u})Q_+^{-1})}\right] - 1\right], \\
&\qquad h_0 = 0, \; t_0 > 0
\end{aligned}$$

where

$$\kappa_{0,reg} = (nK_4/4)[\ln(1+r_0) + r_0/(1+r_0)] \; . \tag{5.24}$$

It is worth commenting that (5.23) yields the universal specific heat amplitude ratio [12] correct only to *zeroth* order in ϵ. This is because the exponent $\alpha = \frac{\epsilon}{2}\frac{4-n}{n+8}/\left(1-\frac{\epsilon}{2}\frac{n+2}{n+8}\right)$ is $O(\epsilon)$. One needs the specific heat correct to $O(\epsilon^2)$ to obtain the universal ratio correctly to $O(\epsilon)$.

3.5.3 Specific heat

The specific heat at fixed X is given by

$$C = T\left(\frac{\partial S}{\partial T}\right)_X = -T\left(\frac{\partial}{\partial T}\left(\frac{\partial F}{\partial T}\right)_H\right)_X , \tag{5.25}$$

where X represents some thermodynamic constraint, for example, fixed density ρ. Evaluation of (5.25) requires knowledge of the implicit T-dependence of

t_0, u_0, etc., which depends on the particular path taken to arrive at the effective s^4 model, as well as on the precise nature of the constraint X. We shall exhibit the calculation for the case of the dilute Bose gas, where the constraint is that of fixed density (Sec. 3.5.2) and the temperature dependence enters via

$$r_0 = -R_d\beta\mu, \;\; u_0 = U_d\beta v_0/\Lambda_T^d, \;\; k_\Lambda = \Gamma_d/\Lambda_T \tag{5.26}$$

(see Eqs. (5.25), (5.27), and (6.4) in Ref. 6). Here $R_d = 4\pi/\Gamma_d^2$, $U_d = 8\pi^2/\Gamma_d^\epsilon$, and $\Gamma_d = 2\sqrt{\pi}(\frac{1}{2}(d-2)\Gamma(\frac{d}{2})\zeta(\frac{d}{2}))^{\frac{1}{d-2}}$ are dimensionless constants, $\beta = 1/k_BT$, $\Lambda_T = h/(2\pi m k_B T)^{1/2}$ is the thermal de Broglie wavelength, and m, v_0 are ^{4}He atomic parameters. Finally, the number density is

$$\rho = (8\pi/n\Gamma_d^2)k_\Lambda^d \rho_0 \ , \tag{5.27}$$

where $\rho_0 = (\partial A/\partial r_0)$ was calculated in Sec. 3.5.2. The free energy $A \equiv A_{Bose}$ which appears in (5.25) differs also by a temperature-dependent factor and an additive term from $A \equiv A_{spin}$ calculated in Sec. 3.4 due to spin and space rescaling: One finds

$$A_{Bose} = \frac{1}{\beta}k_\Lambda^d[A_{spin} - \frac{nK_d}{2d}\ln(R_d)] \ . \tag{5.28}$$

For completeness we also exhibit the relation between the Bose and spin order parameters and conjugate fields — determined by the spin and volume rescaling factors Eq. (6.2) in Ref. 6:

$$\begin{aligned} \beta H_{Bose} =& k_\Lambda^{d/2} R_d^{-1/2} h_0 \\ M_{Bose} =& k_\Lambda^{d/2} R_d^{1/2} M_0 \ . \end{aligned} \tag{5.29}$$

Setting $h_0 = 0$, we calculate the specific heat $C_\pm$ for $T \gtrless T_c$. As mentioned earlier, only the first two terms in (4.17) contribute for $T < T_c$, while only the first, second, and fourth contribute for $T > T_c$. One finds

$$S_\pm = k_\Lambda^d k_B \left\{ \rho_0(t_0 - d(n+2)K_d u_0) + \frac{(d-2)t_0}{4u_0}(\rho_0 - \rho_{0,reg}) \right.$$
$$\left. + \frac{(d-2)t_0^2}{8(n+8)u_0^2} \; Q_\pm^{\frac{4-n}{n+8}}(1 - Q_\pm^{-1}) - \frac{d+2}{2}(A_{spin} - \frac{nK_d}{2d}\ln(R_d)) \right\} . \tag{5.30}$$

The constraint may be put in the form [6]

$$\frac{n}{2(d-2)} K_d n \bar{t} = \rho_0 - \rho_{0,reg}, \; \bar{t} \equiv (T_c/T)^{d/2} - 1 \; , \tag{5.31}$$

where $T_c(\rho) \approx T_c^0(\rho)$ is the transition temperature at given density ρ, and $T_c^0(\rho)$ is the ideal gas transition temperature defined by $\rho\Lambda_{T_c}^d = \zeta(\frac{1}{2}d)$. It is easy to see that for $t_0 \to 0$, (5.31) yields $t_0 \sim \bar{t}^{\frac{1}{1-\alpha}}$, so long as $\alpha > 0$ (to order ϵ, this requires $n < 4$, which we henceforth assume). One sees therefore that the most singular parts of the entropy at constant density are the terms *linear* in t_0: $S_\pm = S_{reg} + S_{\pm,sing}$, with

$$\begin{aligned} S_{\pm,sing} =& \frac{nd\Gamma_d^2}{16\pi} k_B \rho t_0 + O(t_0^{2-\alpha}/u_0, t_0^2/u_0, t_0\bar{t}, \cdots) \\ S_{reg} =& \frac{nK_d}{2d}\frac{d+2}{2} k_\Lambda^d k_B \left[\frac{2}{d} + \ln(R_d) + O(u_0)\right] . \end{aligned} \tag{5.32}$$

One finds then $C_\pm = C_{reg} + C_{\pm,sing}$, with

$$\begin{aligned} C_{\pm,sing} =& \frac{-\frac{(nd)^2\Gamma_d^2 K_d}{8\pi(d-2)} k_B u_0 \rho [1 + O(\bar{t}, \bar{t}^{\frac{1}{1-\alpha}})]}{D_\pm(Q_\pm)} \\ C_{reg} =& \frac{d}{2} S_{reg} = \frac{n(d^2-4)\Gamma_d^2}{32\pi} k_B \rho \left[\frac{2}{d} + \ln(R_d) + O(u_0, \bar{t}\,)\right] \end{aligned} \tag{5.33}$$

and

$$D_+(Q_+) = \frac{n}{4-n}\left[Q_+^{\frac{4-n}{n+8}}\frac{1-\frac{\epsilon}{2}\frac{6}{n+8}[1-(1-\tilde{u})Q_+^{-1}]}{1-\frac{\epsilon}{2}\frac{n+2}{n+8}[1-(1-\tilde{u})Q_+^{-1}]} - \frac{n}{4}\right]$$
$$D_-(Q_-) = \frac{4}{4-n}\left[Q_-^{\frac{4-n}{n+8}}\frac{1-\frac{\epsilon}{2}\frac{6}{n+8}[1-(1-\tilde{u})Q_-^{-1}]}{1-\frac{\epsilon}{2}\frac{n+2}{n+8}[1-(1-\tilde{u})Q_-^{-1}]} - 1\right] . \tag{5.34}$$

Note the resemblence to the *inverse* of the unconstrained specific heat (5.23). The functions $Q_\pm(\bar{t}\ )$ are determined via the constraint equation (5.3) [6]. It is easy to see that (5.33) yields the usual Fisher-renormalized [13] specific-heat exponent $\alpha' = -\alpha/(1-\alpha)$. Similarly the universal amplitude ratio $r_c = C_+/C_-$ is renormalized via $r_c' = r_c^{\frac{-1}{1-\alpha}}$.

Appendix A. Details of recursion relation solutions

We outline here in somewhat more detail the solutions to the recursion relations (2.22)-(2.26). The solution for $u(l)$ is elementary and is given by (2.29). The solution for $w(l)$ follows immediately via (2.27) and (2.28). The solutions for $r_T(l)$ and $r_L(l)$ are more complicated. Following Rudnick and Nelson [1] we begin by analyzing the *simplified* recursion relations, valid for $r, u \leq O(\epsilon)$:

$$\frac{dr_L}{dl} = (2 - 12K_4u)r_L - 4(n-1)K_4ur_T + 4(n+2)K_4u \qquad (A1)$$

$$\frac{dr_T}{dl} = (2 - 4(n+1)K_4u)r_T - 4K_4ur_L + 4(n+2)K_4u \ . \qquad (A2)$$

Diagonalization of the first two terms in each equation yields two eigencombinations $r_1 = \frac{1}{n}(r_L + (n-1)r_T)$ and $r_2 = \frac{1}{n}(r_L - r_T)$ with solutions

$$r_1(l) = r_1(0)e^{2l}/Q(l)^{\frac{n+2}{n+8}} \qquad (A3)$$

$$r_2(l) = r_2(0)e^{2l}/Q(l)^{\frac{2}{n+8}} \ . \qquad (A4)$$

These are now used to generate the full solutions. The first step involves converting the recursion relations to integral equations. One finds in a straightforward way

$$
\begin{aligned}
r_1(l) =& r_1(0)e^{2l}/Q(l)^{\frac{n+2}{n+8}} + (e^{2l}/Q(l)^{\frac{n+2}{n+8}}) \int_0^l dl' e^{-2l'} Q(l')^{\frac{n+2}{n+8}} \\
& \times \{4(n+2)K_4 u(l') + (4(n+2)K_4/n) u(l') r_L(l')^2/(1+r_L(l')) \\
& + (4(n-1)(n+2)K_4/n) u(l') r_T(l')^2/(1+r_T(l')) \\
& - (18K_4/n) w(l')^2/(1+r_L(l'))^2 - (2(n-1)K_4/n) w(l')^2/(1+r_T(l'))^2 \\
& - (4(n-1)K_4/n) w(l')^2/(1+r_L(l'))(1+r_T(l'))\}
\end{aligned} \tag{A5}
$$

$$
\begin{aligned}
r_2(l) =& r_2(0)e^{2l}/Q(l)^{\frac{2}{n+8}} + (e^{2l}/Q(l)^{\frac{2}{n+8}}) \int_0^l dl' e^{-2l'} Q(l')^{\frac{2}{n+8}} \\
& \times \{(8K_4/n) u(l') r_T(l')^2/(1+r_T(l')) - (8K_4/n) u(l') r_L(l')^2/(1+r_L(l')) \\
& -(4K_4/n) w(l')^2/(1+r_L(l'))(1+r_T(l')) + (18K_4/n) w(l')^2/(1+r_L(l'))^2 \\
& + (2(n-1)K_4/n) w(l')^2/(1+r_T(l'))^2\} .
\end{aligned} \tag{A6}
$$

The basic technique used to evaluate the remaining integrals is to divide each term into a slowly varying piece, a function only of $e^{\epsilon l}$, and a rapidly varying piece. An integration by parts is then performed, putting the derivative on the slowly varying piece, which then becomes smaller by a factor of ϵ. The remaining integral can then be dropped. One must also take into consideration which region of integration contributes most to the integral. For example, $r_L(l)$ and $r_T(l)$ are small, of order ϵ, for most of the interval $0 \le l \le l^*$, becoming large, of order unity, over only the last part of the interval, during which slowly varying functions, such as $u(l)$, change only by $O(\epsilon^2)$. It was precisely arguments such as these that led to the reduced set of recursion relations (2.22)-(2.26), and must be used here again to further simplify the analysis. Finally, if the entire integrand is slowly varying, the integral is performed exactly: usually such terms involve only rational functions of $e^{\epsilon l}$.

To illustrate, the combinations $e^{-2l'} r_L(l')$, $e^{-2l'} r_T(l')$, and $e^{-2l'} w^2(l')$ are slowly varying, as are $Q(l')$ and $u(l')$. Thus, for example,

$$\int_0^l dl' \left[e^{-2l'} u(l') Q(l')^{\frac{n+2}{n+8}} r_L(l') \right] \frac{r_L(l')}{1+r_L(l')} \approx \left[\int^{l'} \frac{r_L(l'')}{1+r_L(l'')} dl'' \right] e^{-2l'} u(l') Q(l')^{\frac{n+2}{n+8}} r_L(l') \Bigg|_0^l , \quad (A7)$$

where the integral remaining after the integration by parts, with the derivative on the slowly varying part, has been dropped. The integral of the function $r_L/(1+r_L)$ is performed by realizing that the important contribution comes from the region $r_L \gg \epsilon$. In this region we may approximate $r_L(l'') \approx r_L(l') e^{-2(l'-l'')}$, $l'' < l'$. This yields

$$\int^{l'} \frac{r_L(l'')}{1+r_L(l'')} dl'' \approx \frac{1}{2} \ln(1 + r_L(l')) + c , \quad (A8)$$

where c is an arbitrary constant of integration, which we take to be zero. The result of (A7) is then

$$e^{-2l} u(l) Q(l)^{\frac{n+2}{n+8}} r_L(l) \ln(1 + r_L(l)) + O(u^2, \epsilon u) . \quad (A9)$$

Similarly we have

$$\int_0^l dl' \left[e^{-2l} u(l') Q(l')^{\frac{n+2}{n+8}} r_T(l') \right] \frac{r_T(l')}{1+r_T(l')} = e^{-2l} u(l) Q(l)^{\frac{n+2}{n+8}} \times r_T(l) \ln(1 + r_T(l)) + O(u^2, \epsilon u) . \quad (A10)$$

The w^2 integrals are evaluated by first ignoring the $1/(1+r)^2$ denominators, yielding a slowly varying integrand, which can then be treated exactly. The remainder is then evaluated via integration by parts: once again the major contribution comes from the region $r_L \gg \epsilon$, and the same approximations are made that led to (A8). The result is

$$\begin{aligned}\int_0^l & e^{-2l'} Q(l')^{\frac{n+2}{n+8}} w(l')^2 \left[1 - \frac{r_L(2+r_L)}{(1+r_L)^2}\right] \\ &= \frac{2}{3K_4} e^{-2l} Q(l)^{\frac{n+2}{n+8}} \left[u(l)M(l)^2 - e^{2l} Q(l)^{-\frac{n+2}{n+8}} u_0 M_0^2\right] \\ &- \frac{1}{2} w(l)^2 e^{-2l} Q(l)^{\frac{n+2}{n+8}} \left[\ln(1+r_L) + \frac{r_L}{1+r_L}\right] + O(u^2, \epsilon u) \ .\end{aligned} \tag{A11}$$

Analysis of all other terms is essentially the same. We quote only the final results for $r_1(l)$ and $r_2(l)$:

$$\begin{aligned} r_1(l) = & e^{2l} Q(l)^{-\frac{n+2}{n+8}} [r_1(0) + \frac{4(n+2)}{n} u_0 M_0^2 - 2(n+2)K_4 u_0 + O(\epsilon u, u^2)] \\ & - \frac{4(n+2)}{n} u(l) M(l)^2 - 2(n+2) K_4 u(l) \\ & + \frac{2(n+2)K_4}{n} u(l) r_L \ln(1+r_L) \\ & + \frac{2(n-1)(n+2)K_4}{n} u(l) r_T \ln(1+r_T) \\ & + \frac{9K_4}{n} w(l)^2 [\ln(1+r_L) + \frac{r_L}{1+r_L}] \\ & + \frac{(n-1)K_4}{n} w(l)^2 [\ln(1+r_T) + \frac{r_T}{1+r_T}] \\ & + \frac{2(n-1)K_4}{n} w(l)^2 \left[\frac{r_L}{r_L - r_T} \ln(1+r_L) + \frac{r_T}{r_T - r_L} \ln(1+r_T)\right] \ , \end{aligned} \tag{A12}$$

$$\begin{aligned} r_2(l) = & e^{2l} Q(l)^{-\frac{2}{n+8}} [r_2(0) + \frac{8}{n} u_0 M_0^2 + O(u^2, \epsilon u)] \\ & - \frac{8}{n} u(l) M(l)^2 + \frac{4K_4}{n} u(l) [r_T \ln(1+r_T) - r_L \ln(1+r_L)] \\ & + \frac{2K_4}{n} w(l)^2 [\frac{r_L}{r_L - r_T} \ln(1+r_L) + \frac{r_T}{r_T - r_L} \ln(1+r_T)] \end{aligned}$$

$$
\begin{aligned}
&- \frac{9K_4}{n} w(l)^2 [\ln(1+r_L) + \frac{r_L}{1+r_L}] \\
&+ \frac{(n-1)K_4}{n} [\ln(1+r_T) + \frac{r_T}{1+r_T}] \, .
\end{aligned}
\tag{A13}
$$

These expressions are now used to calculate r_L and r_T via $r_L = r_1 + (n-1)r_2$ and $r_T = r_1 - r_2$. After a number of cancellations, we find

$$
\begin{aligned}
r_L(l) = & [r_0 + 2(n+2)K_4 u_0 + O(\epsilon u_0, u_0^2)] e^{2l} / Q(l)^{\frac{n+2}{n+8}} \\
& + (n-1) O(\epsilon u_0, u_0^2) e^{2l} / Q(l)^{2/(n+8)} \\
& + 12 u(l) M(l)^2 - 2(n+2) K_4 u(l) \\
& + 6 K_4 u(l) r_L \ln(1+r_L) + 2(n-1) K_4 u(l) r_T \ln(1+r_T) \\
& + 9 K_4 w(l)^2 \left[\ln(1+r_L) + \frac{r_L}{1+r_L} \right] \\
& + (n-1) K_4 w(l)^2 \left[\ln(1+r_T) + \frac{r_T}{1+r_T} \right]
\end{aligned}
\tag{A14}
$$

$$
\begin{aligned}
r_T(l) = & [r_0 + 2(n+2)K_4 u_0 + O(\epsilon u_0, u_0^2)] e^{2l} / Q(l)^{\frac{n+2}{n+8}} + O(\epsilon u_0, u_0^2) e^{2l} / Q(l)^{\frac{2}{n+8}} \\
& + 4 u(l) M(l)^2 - 2(n+2) K_4 u(l) r \\
& + 2 K_4 u(l) r_L \ln(1+r_L) + 2(n+1) K_4 u(l) r_T \ln(1+r_T) \\
& + 2 K_4 w(l)^2 \left[\frac{r_L}{r_L - r_T} \ln(1+r_L) + \frac{r_T}{r_T - r_L} \ln(1+r_T) \right] \, .
\end{aligned}
\tag{A15}
$$

Defining $t(l) = [r_0 + 2(n+2)K_4 u_0 + O(\epsilon u_0, u_0^2)] e^{2l} / Q(l)^{\frac{n+2}{n+8}}$, $T_L(l) = t(l) + 12u(l)M(l)^2$, and $T_T(l) = t(l) + 4u(l)M(l)^2$, then substituting T_L and T_T for r_L and r_T in the terms of $O(u, w^2)$ on the right-hand sides of (A14), (A15) yields the final results (2.30), (2.31) [note that $w^2/(T_L - T_T) = 2u$].

The solution for $\tilde{h}(l)$ is now straightforward. The integral equation corresponding to (2.22) is

$$\tilde{h}(l) = \tilde{h}_0 e^{(3-\frac{1}{2}\epsilon)l} - e^{(3-\frac{1}{2}\epsilon)l} \int_0^l dl' e^{-(3-\frac{1}{2}\epsilon)l'} \left[\frac{(n-1)K_4 w(l')}{1+r_T} + \frac{3K_4 w(l')}{1+r_L} \right] . \tag{A16}$$

By writing $\frac{1}{1+r} = 1 - r + \frac{r^2}{1+r}$, we again can isolate the various asymptotic regions. The term linear in r is slowly varying, and can be integrated exactly once T is substituted for r. The $r^2/(1+r)$ term is handled in the same way as (A8) was. The final results [Eq. (2.32)] then follow in a straightforward way.

Appendix B. Validity of the linear spin-wave approximation

Since there is some confusion in the early literature [1,4] as to how to handle the vanishing transverse "mass" κ_T on the ordered-phase coexistence curve, we feel it worthwhile to indicate here the region of validity of the linear spin-wave theory.

Consider first a model with fixed-length spins $|\mathbf{s}_i| = 1$ at temperature T:

$$\bar{H}_1 = -\frac{J}{T} \sum_{<ij>} |\mathbf{s}_i - \mathbf{s}_j|^2 . \tag{B1}$$

At low temperatures, $J/T \gg 1$, it is appropriate to expand $\mathbf{s}_i$ around the uniform state, $\mathbf{s}_i = \hat{\mathbf{M}}$ for all i, where $\hat{\mathbf{M}}$ is a unit vector. One writes $\mathbf{s}_i = \sqrt{1 - |\mathbf{s}_i^{\perp}|^2}\hat{\mathbf{M}} + \mathbf{s}_i^{\perp}$, where $\mathbf{s}_i^{\perp} \cdot \hat{\mathbf{M}} = 0$. Keeping terms to quadratic order, one finds

$$\bar{H}_1 \approx -\frac{J}{T} \sum_{<ij>} |\mathbf{s}_i^{\perp} - \mathbf{s}_i^{\perp}|^2 . \tag{B2}$$

For small T/J we may treat $\mathbf{s}_i^\perp$ as extended $(n-1)$-dimensional spins, so that (B2) is just a Gaussian model. The change in magnetization is then

$$\Delta M = 1 - \left\langle \sqrt{1 - |\mathbf{s}_i^\perp|^2} \right\rangle \approx \frac{1}{2} \langle |\mathbf{s}_i^\perp|^2 \rangle \approx \frac{(n-1)T}{2J} \int\limits_{0<|\mathbf{q}|<1} \frac{1}{q^2} \, , \qquad (B3)$$

which yields $\Delta M \approx [(n-1)K_d/2(d-2)](T/J)$. Self-consistency requires $\Delta M \ll 1$, which is satisfied so long as $d > 2$ and $T/J \ll 1$.

The above calculation demonstrates that fluctuations are small, even though $\kappa_T = 0$, in the simplest case when $\kappa_L = \infty$. The only requirement is that the coefficient J/T of the gradient-squared term be large.

Let us now include longitudinal fluctuations via a spin weighting term W:

$$\bar{H}_W = -\frac{J}{T} \sum_{<ij>} |\mathbf{s}_i - \mathbf{s}_j|^2 - \frac{1}{\delta T} \sum_i W(|\mathbf{s}_i|^2 - 1) \, . \qquad (B4)$$

We assume $W'(0) = 0$ and $\frac{1}{2}W''(0) = 1$. We will be interested mainly in the case $W(x) = x^2$. Apparently we recover the case of fixed spins when $\delta \to 0$. It seems clear then that we may treat longitudinal fluctuations in the quadratic approximation around the minimum at $|\mathbf{s}_i| = 1$ so long as $\delta T \ll 1$.

Consider then the us^4 model:

$$\bar{H}_4 = -\frac{R_0^2}{a^2} \sum_{<i,j>} |\mathbf{s}_i - \mathbf{s}_j|^2 - \sum_i \left[\frac{1}{2} r |\mathbf{s}_i|^2 + u |\mathbf{s}_i|^4 \right] \, . \qquad (B5)$$

By rescaling the spin via

$$\tilde{\mathbf{s}}_i = (4u/|r|)^{\frac{1}{2}} \mathbf{s}_i, \qquad (B6)$$

which serves only to add a constant to the free energy, we find (for $r < 0$)

$$\bar{H}_4 = -\frac{R_0^2}{a^2}\frac{|r|}{4u}\sum_{<ij>}|\mathbf{s}_i - \mathbf{s}_j|^2 - \frac{r^2}{16u}\sum_i(|\mathbf{s}_i|^2 - 1)^2 \ . \qquad (B7)$$

Comparing with (B4) we see that $w(x) = x^2$, $\delta T = 16u/r^2$ and $T/J = \frac{a^2}{R_0^2}\frac{4u}{|r|}$. By the above arguments the linear spin-wave theory will be correct so long as $4a^2u/R_0^2|r| \ll 1$ and $16u/r^2 \ll 1$. In particular, if $r = O(1)$ and $a/R_0 = O(1)$, we require $u \ll 1$ which is obviously satisfied in our calculation, so long as $\epsilon \ll 1$. Alternatively, if we assume $u = O(\epsilon)$, then we require r, $r^2 \gg O(\epsilon)$; i.e., $r \gg O(\epsilon^{1/2})$.

Appendix C. Spin-wave integrals

In this appendix we consider the integral

$$\int_q \frac{1}{q^2 + r} = \frac{1}{2}K_d k_\Lambda^{d-2}\int_0^1 \frac{w^{\frac{d-2}{2}}dw}{w + x} \equiv K_d k_\Lambda^{d-2} I_d(x) \ , \qquad (C1)$$

where $x = r/q_\Lambda^2$. Of particular interest is the nature of the singularity when $x \to 0$. We assume, as usual, $2 < d \le 4$. We write

$$\begin{aligned} I_d(x) =& I_d(0) - x^{\frac{d-2}{2}}\int_0^\infty \frac{dw}{1+w}w^{\frac{d-4}{2}} + x\int_1^\infty \frac{dw}{x+w}w^{\frac{d-4}{2}} \\ =& \frac{2}{d-2} - B(\frac{d-2}{2}, \frac{4-d}{2})x^{\frac{d-2}{2}} + \frac{2}{4-d}x - x^2\int_1^\infty \frac{dw}{x+w}w^{\frac{d-6}{2}} \ . \end{aligned} \qquad (C2)$$

The last term now has a well-defined Taylor expansion around $x = 0$ for all $d < 6$. $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$ is the β function. This exhibits the exact $x \to 0$ nonanalyticity:

$$I_d^{sing}(x) = -\frac{2x}{\epsilon}\left[\frac{\epsilon\pi/2}{\sin(\epsilon\pi/2)}x^{-\epsilon/2} - 1\right] \ , \qquad (C3)$$

which is valid for $2 < d < 6$, and yields the correct $x\ln(x)$ behavior *in* $d = 4$. For small ϵ we may evaluate the remaining terms *in* $d = 4$ and approximate $(\epsilon\pi/2)/\sin(\epsilon\pi/2) \simeq 1$:

$$I_d(x) \approx 1 - \frac{2}{\epsilon}x[x^{-\epsilon/2} - 1] - x\ln(1+x) \ , \qquad \epsilon \ll 1 \ . \tag{C4}$$

Furthermore when $x = O(1)$ the singular term may be simplified to yield

$$I_d(x) \approx 1 + x\ln(x) - x\ln(1+x) \ , \qquad \epsilon \ll 1 \ , \ x = O(1) \ . \tag{C5}$$

Free energy integrals involve the function

$$\int_q \ln(q^2 + r) \approx \frac{K_d}{d}k_\Lambda^d \ln(k_\Lambda^2) + \frac{1}{2}K_d k_\Lambda^d \tilde{I}_d(x) \ , \tag{C6}$$

where

$$\tilde{I}_d(x) = \int_0^1 w^{\frac{d-2}{2}} \ln(w + x)dw \ . \tag{C7}$$

Obviously, $\tilde{I}'_d(x) = I_d(x)$. Using $\tilde{I}_d(0) = -4/d^2$, we may therefore simply integrate (C2) with respect to x to find $\tilde{I}_d(x)$. However, a simpler method is to integrate (C7) by parts to obtain

$$\tilde{I}_d(x) = \frac{2}{d}\ln(1+x) - \frac{2}{d}I_{d+2}(x) \ , \tag{C8}$$

which yields

$$\tilde{I}_d(x) = \frac{2}{d}\ln(1+x) - \frac{4}{d^2} + \frac{4}{d(d-2)}x - \frac{2}{d}x^3 \int_1^\infty \frac{dw}{x+w}w^{\frac{d-6}{2}} + \tilde{I}_d^{sing}(x) \tag{C9}$$

where

$$\tilde{I}_d^{sing}(x) = -\frac{4}{\epsilon d}x^2\left[\frac{\epsilon\pi/2}{\sin(\epsilon\pi/2)}x^{-\epsilon/2} - 1\right] \ . \tag{C10}$$

The formulas analogous to (C4) and (C5) are

$$\begin{aligned}\tilde{I}_d(x) \approx& \frac{1}{2}(1-x^2)\ln(1+x) - \frac{1}{4} + \frac{1}{2}x - \frac{2}{\epsilon}x^2\left[x^{-\frac{\epsilon}{2}} - 1\right], \qquad \epsilon \ll 1 \\ \approx& \frac{1}{2}(1-x^2)\ln(1+x) - \frac{1}{4} + \frac{1}{2}x + x^2\ln(x), \qquad \epsilon \ll 1, x = O(1)\ .\end{aligned} \tag{C11}$$

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