NUCLEAR SIZE CORRECTIONS TO THE HYPERFINE STRUCTURE OF HYDROGEN

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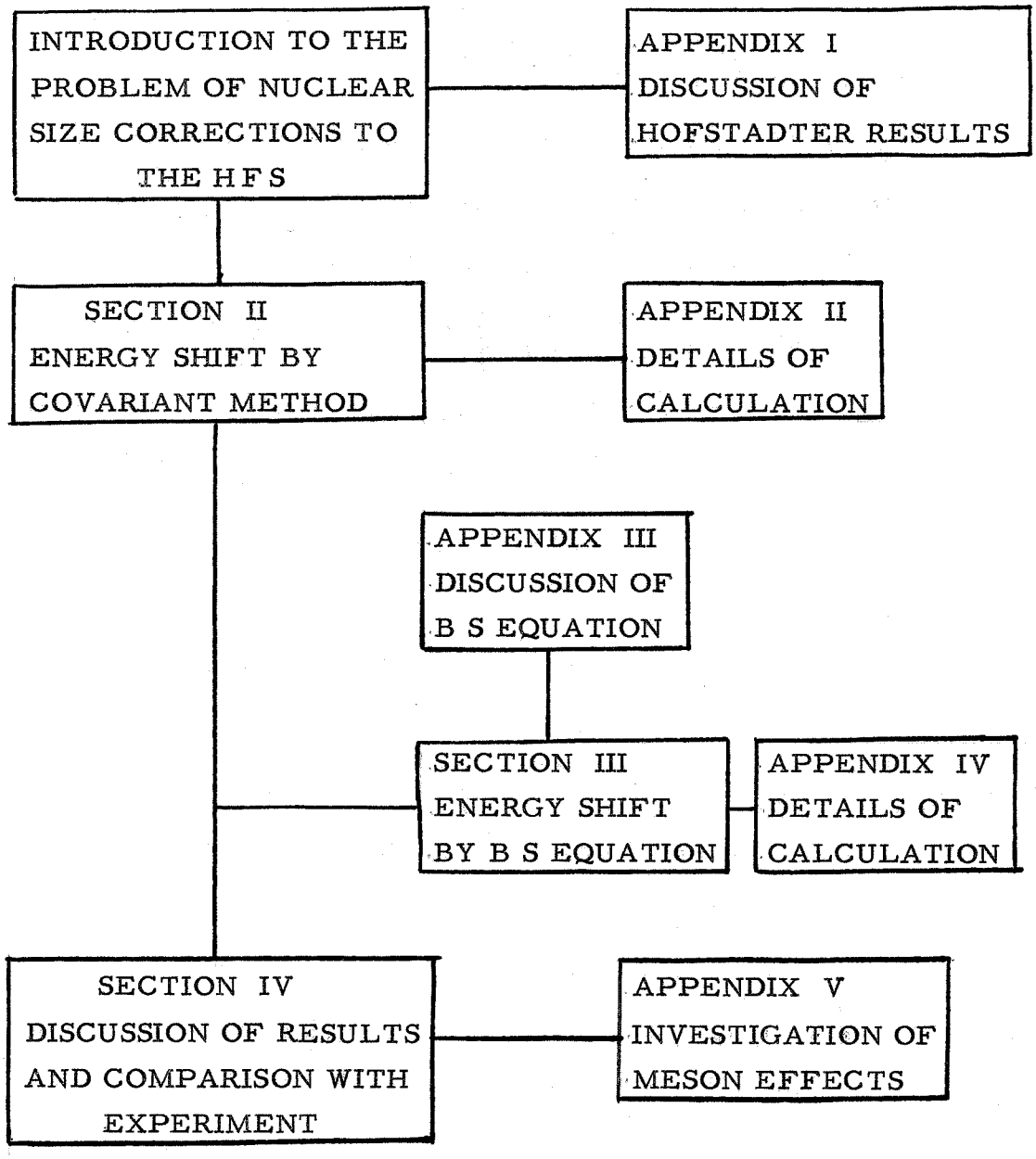
SUMMARY

This thesis is a calculation of the energy shift in the hyperfine structure of atomic H due to the finite size of the proton. We use experimental data of Hofstadter to determine the size of the proton and we find that the measured value of the hyperfine structure is in slight disagreement with our predicted value. Possible reasons for this are investigated.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Introduction

Measurements of the hyperfine shift (h f s) in atomic hydrogen have given the splitting as:[1]

$$\Delta\nu = 1420,\ 405.73 \pm 0.05 \text{ kc/sec}$$

This great precision is useless unless a theoretical expression of comparable accuracy can be found. The present expression is most conveniently written as a series of corrections to the Fermi formula:

$$\Delta\nu = \left\{\frac{16\,a^2 c\,R_\infty}{3}\,\frac{M_p}{M_e}\,(\mu_e)^2\right\}\left(1+\frac{m}{M}\right)^{-3}\left(1+\frac{3}{2}a^2\right)\left[1-\left(\frac{5}{2}-\ell n\,2\right)a^2\right]R$$

$$= \left\{\frac{2\pi a\mu_p}{3mM}\,\sigma_e\cdot\sigma_p\,|\varphi(0)|^2\right\}\left(1+\frac{m}{M}\right)^{-3}\left(1+\frac{3}{2}a^2\right)\left[1-\left(\frac{5}{2}-\ell n\,2\right)a^2\right]R \quad (1)$$

where the expression in curly brackets is the Fermi formula, M_p/M_e is the ratio of the magnetic moments of the proton and electron, μ_e is the electron moment in Bohr magnetons and μ_p is the proton moment in nuclear magnetons. The factor $(1 + m/M)^{-3}$ is a recoil correction which gives the reduced mass change in $\varphi(0)$ (since we are here assuming that φ is the wave function for an infinitely heavy, coulombic proton.) The factor $(1 + 3/2\ a^2)$ is a relativistic correction for the motion of the electron. It may be derived by assuming that the proton is a fixed point charge plus a static magnetic moment, $M_p\sigma_p$, and finding the perturbation solution of the single-electron Dirac equation. The factor $1 - (5/2 - \ell n\,2)\,a^2)$ gives electrodynamic corrections to the electron magnetic moment; it is not the anomalous moment correction (which is included in μ_e) but represents a form factor for the electron-photon

interaction.[2] The factor R includes the remaining corrections of (roughly) this order. These are all proportional to $\alpha\, m/M$ and may thus be viewed as retardation effects due to the nuclear motion. For a proton which interacts with photons by means of the Lagrangian:

$$L_\mu^o = \gamma_\mu + \frac{\mu}{4M}\left(\not{k}\gamma_\mu - \gamma_\mu \not{k}\right) \tag{2}$$

($\mu \tilde{=} 1.79$) the factor R has already been calculated to be [3]*

$$R = \left\{1 - \frac{\alpha m}{\pi M \mu_p}\left[3 \ln \frac{M}{m} - \frac{(\mu_p - 1)^2}{4}\left\{+\frac{1}{2} - 9 \ln \frac{2k}{M} + 3 \ln \frac{M}{m}\right\}\right]\right\} \tag{3}$$

where k is a cut-off momentum, introduced to get sensible results from divergent terms in the perturbation series. These divergencies arise only because of the anomalous term in the interaction, 2, and presumably they are not present in a correct theory in which the anomalous moments are due to mesonic clouds around a core. Effectively, the cut off, k, corresponds to some kind of "smearing" of the anomalous moment. The problem of improving the result, 3, of Newcomb and Salpeter (NS) is to estimate the correct interaction of a nucleon with the electromagnetic field. We shall refer to the interaction 2, in which no cut off is employed, as that of a "point proton".

Since all the multiplicative factors in 1, (c, R_∞, μ_e, $\frac{M_p}{M_e}$) except α are known to a probable error of about one part in 10^6 [4] and since α is known only to 4 parts in 10^6, if the small correction R could be given, the hfs measurements would provide a more precise value of α. More important, it would be an independent value of α, since the

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* We call the reader's attention to the distinction between the anomalous proton moment, μ, and the total proton moment, $\mu_p = 1 + \mu$

presently accepted value [5] is weighted heavily in favor of fine structure measurements in deuterium. These are inherently less accurate than the atomic beam measurement of the hfs, since several data must be combined. However, to this order, there are no theoretical uncertainties in the interpretation of the deuterium results while the effects of proton shape on the hfs are estimated at several ppm.

The object of this thesis is to calculate the factor R by employing an empirical form factor for the proton which has been found to fit the Hofstadter experiments on high energy, elastic electron-proton scattering. The interpretation of Hofstadter's work is discussed briefly in Appendix I. Since Hofstadter's results are an empirical function they may be approximated by several analytic forms and the numerical results must not depend too strongly upon the particular choice of representation, if they are to have any meaning. The form we choose is:

$$L_\mu = L^o_\mu \, F(k^2) \tag{4}$$

$$F(k^2) = \frac{\Lambda^4}{(k^2 - \Lambda^2)^2} \; ; \quad \frac{\Lambda}{M} = 0.91 \tag{5}$$

where $k^2 = \omega^2 - \vec{k}^2$, the "mass" (squared) of the photon. Equation 4 distributes <u>both</u> the charge and the magnetic moment. This is important since any smearing of the coulomb potential will result in a change in the value of the wave function at the origin, $\varphi(0)$. As is shown by a straight forward, non-relativistic argument, [6] when both the charge <u>and</u> moment are distributed the resulting shift is proportional to a convolution of the two form factors, magnetic and electric.

It is not the sum of separate terms describing a spread charge and point moment and point charge and spread moment. This can be understood by imagining the proton to be a point charge and point moment, separated by some variable distance, both of which move around to give the "smearing" effect of "finite size". The electron wave function is centered on the point charge as this charge moves around. The hfs is determined by the value of the wave function at the point moment; thus the average separation distance of charge and moment is the important quantity; not the distance of the moment from the center-of-mass of the proton. This is expressed as a correlation of the two distributions, electric and magnetic and for independent distributions involves a convolution integral. Since the product of the two form factors is involved, we may view the process as the exchange of two photons by the electron and proton. We shall find that relativistically the basic process is also a two-photon exchange.

In Section II we evaluate the energy shift as two Feynman diagrams. Section III arrives at the same results by the use of the Bethe Salpeter equation. Section IV discusses the results and limitations of this estimate of the hfs corrections.

The material discussed in this thesis has appeared in two articles[7] in somewhat more condensed form.

Section II The Covariant Calculation of the Shift in the hfs

We shall now evaluate the hfs for a proton with a Hofstadter form factor (Hff). By this we mean that the transition operator for the absorbtion of a photon of polarization μ and momentum $\not{k}$ is given by:

$$\Lambda_\mu = e\left\{\gamma_\mu + \frac{\mu}{4M}\left(\not{k}\gamma_\mu - \gamma_\mu \not{k}\right)\right\} F(k^2) = e\,\Gamma_\mu(\not{k})\,F(k^2) \qquad (6)$$

$$F(k^2) = \frac{\Lambda^4}{(\Lambda^2 - k^2)^2} \quad \text{where} \quad \frac{\Lambda}{M} = 0.91 \quad \text{and} \quad \mu \cong 1.79 \qquad (7)$$

The usual electrodynamics of a point proton corresponds to $F = 1$ but here;

$$F = 1 + 2\frac{k^2}{\Lambda^2} + \text{---} \qquad (8)$$

Thus there are departures from a point proton for $k^2 \sim \Lambda^2 \sim M^2$. As was discussed in Appendix I, although this choice is indicated by the results of Hofstadter's experiments, it is not unique and it represents only an empirical fit to the data.

Previous calculations have given all corrections to the hfs up to and including order $\alpha\, m/M$ for a point proton.[3] We shall compute the effect of a Hff only to this order. Higher order corrections would involve much more work and, as we shall see in Section IV, would not be justified on the basis of present knowledge. Rather than find all the energy shifts due to Λ_μ of equation (6), it is more convenient to find just the difference in the hfs due to the use of the Hff, F, instead of 1. This can be done by writing equation (6) in the form:

$$\Lambda_\mu = e\Gamma_\mu + e\Gamma_\mu (F - 1) \tag{9}$$

The work of NS [3] then gives the shift R, of equation (1), due to the term $e\Gamma_\mu$ in 9 and we must find the additional effect of the term $e\Gamma_\mu (F - 1)$

Since the particles are in a bound state and therefore not on the mass shell (i.e., $E^2 - p^2 \neq m^2$), it is not immediately clear how to find the shift in energy in terms of the interaction 9. We shall simplify the problem by assuming that the electron and proton are free particles, almost at rest and thus neglecting the binding effects. As we shall see, this is possible because the correction term, $e\Gamma_\mu (F - 1)$, has an extremely short range compared to the dimensions of the hydrogen atom. Thus the momenta involved are all so much higher than the Bohr momentum (αm) that the neglect of the binding causes no error to order $\alpha m/M$. For the first term of 9, $e\Gamma_\mu$, this would not be a good approximation, however, as we have noted above, its value is already known for the hfs.

The calculation of an energy shift for free particles is given by the methods of Feynman.[8] After a long time T, the wave function is given by*

$$\psi(T) = S\psi(0) = e^{i\Delta E T}\psi$$

$$(\psi | S\psi) = e^{i\Delta E T} \tag{10}$$

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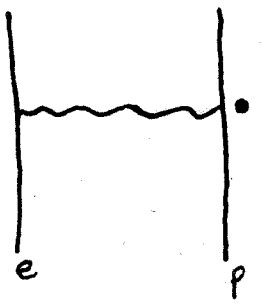
* The lifetime of the triplet state is so much longer than the inverse energy difference, $(E_{3_s} - E_{1_s})^{-1}$, that the decay may be neglected and the state treated as if it were stable.

By developing the S matrix in diagrams, we obtain the series expansion of the exponential on the left side of 10.

The lowest order correction is shown in Figure 1.



• Indicates that Γ_μ (F-1) acts at a proton vertex, no dot indicates Γ_μ alone.

Figure 1

Although this graph, without the dot, gives rise to the hfs itself, it gives nothing in our calculation since the momentum of the photon can be only of the order of the Bohr momentum, αm. For such a momentum, $F - 1 \approx \alpha^2 m^2 / M^2$ so that the correction is negligible compared to the $\alpha m/M$ we seek.

Other single photon exchange graphs such as Figure 2 are just radiative corrections to either the electron (2a) or proton (2b) vertex.

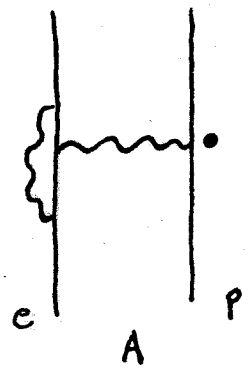
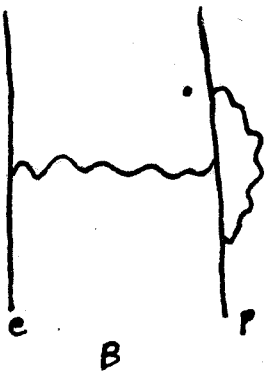


Figure 2



For all these terms, the <u>difference</u> of such corrections from their values for a point proton will be of order $\alpha^2 m^2/M^2$ for photon momenta $\sim \alpha m$. Thus they may be omitted. (Figure 2b is questionable on other grounds; it represents a vertex correction for the proton and therefore gives a contribution to the anomalous moment. But we have explicitly used the measured anomalous moment, μ, in 6 and thus already included all orders of vertex corrections).

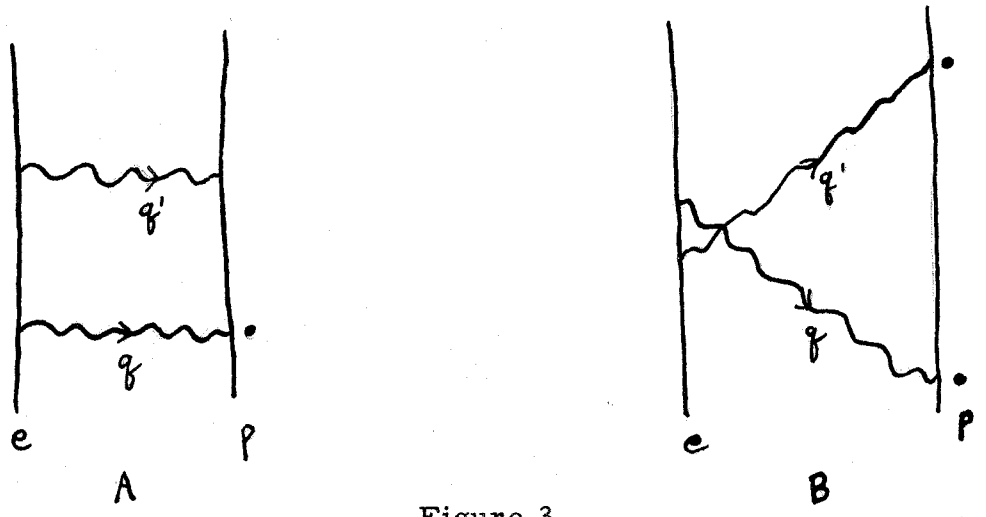


Figure 3

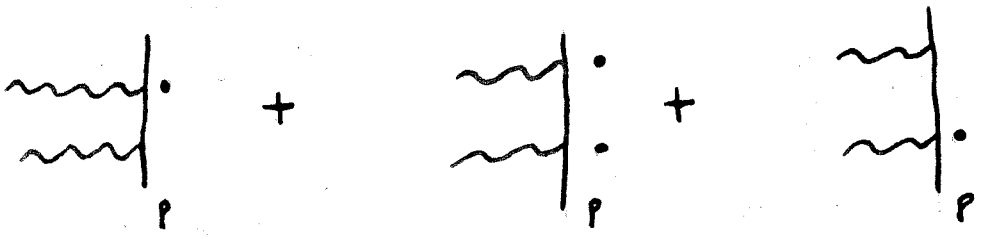
Graphs such as Figure 3 can contribute to the hfs since both photons together transfer very small momentum, thus leaving the final state at rest, but the momentum of each photon is rather large so that the Hff is appreciably different from 1. We find that the shift due to 3a is:

$$\langle i \Delta E \rangle = \frac{4e^4}{(2\pi)^2} \int \frac{d4q}{q^2 q'^2} \; (\bar{u} M_{\mu\nu} u) \; (\bar{v} N_{\mu\nu} v) \left\{ F(q^2) \; F(q'^2) - 1 \right\} \qquad (11)$$

$$M_{\mu\nu} = \Gamma_\mu (q') \; \frac{1}{\not p_p + \not q - M} \; \Gamma_\nu (q) \qquad (12)$$

$$N_{\mu\nu} = \gamma_\mu \frac{1}{\not p_e - \not q - m} \gamma_\nu \tag{13}$$

(The notation is that of reference 8 except that $d^4q = dq_0\, dq_1\, dq_2\, dq_3$ u is a proton spinor and v an electron spinor.) The term $\left\{ F(q^2)\, F(q'^2) - 1 \right\}$ arises when 3a is summed over the three possible combinations of dots,



and is clearly ΔE (Hff) - ΔE (point), as required. By comparison with the work of NS the order of 3a is $\alpha m/M$ for a point proton; therefore the factor $\left\{ F(q^2)\, F(q'^2) - 1 \right\}$ must be of order 1 if we are to get a similar contribution from 11. This means that $q'^2 \approx q^2 \approx M^2 \approx \Lambda^2$. We now note that if we were to include bound-state effects, the three-momenta $\vec{p}_e$ and $\vec{p}_p$ would be of the order of the Bohr momentum, αm, and the energies, $p_{4,e}$ and $p_{4,p}$ of the order of the rest masses. Since we require only the zeroth power of m/M in an expansion of 11, we may neglect $\vec{p}_e$, m and $\vec{p}_p$ in 12 and 13 and put $p_p = M$, $q' = -q$. Thus the use of free particles, at rest, is correct to order $\alpha m/M$.

The hfs will be affected only by

$$\langle 3s \,|\, \Delta E \,|\, 3s \rangle - \langle 1s \,|\, \Delta E \,|\, 1s \rangle \equiv \delta E \tag{14}$$

To evaluate this, we introduce projection operators for the triplet and singlet, positive energy, states,

$$p_{3s}^{+} = \frac{3 + \sigma\cdot\bar{\sigma}}{4} \quad \frac{1+\gamma_t}{2} \quad \frac{1+\bar{\gamma}_t}{2}$$

$$p_{1s}^{+} = \frac{1 - \sigma\cdot\bar{\sigma}}{4} \quad \frac{1+\gamma_t}{2} \quad \frac{1+\bar{\gamma}_t}{2}$$

and sum over all polarizations (three triplet and one singlet). We use the notation that a superscript bar refers to an operator on the proton coordinates.

$$i\,\delta E = \sum_{\substack{u,\,v \\ \text{polarizations,} \\ \text{energy}}} u^*\bar{\beta}\, v^*\beta \quad i\Delta E\left\{\frac{P_{3s}^{+}}{3} - P_{1s}^{+}\right\} \quad uv$$

$$= \frac{4e^4}{2\pi^2}\int \frac{d^4q}{q^4}\left[F^2(q^2) - 1\right] \text{trace}\left\{M_{\mu\nu}\ \frac{1+\bar{\gamma}_t}{2}\right.$$

$$\left.\times\ N_{\mu\nu}\ \frac{1+\gamma_t}{2}\ \frac{\sigma\cdot\bar{\sigma}}{3}\right\}$$

Evaluation is simplified by the observation that since the initial and final three-momenta are zero and since the energy is a scalar under rotations, $\sigma\cdot\bar{\sigma}$ may be replaced by $3\sigma_z\bar{\sigma}_z = -3\gamma_x\gamma_y\bar{\gamma}_x\bar{\gamma}_y$. We now introduce dimensions such that the proton has unit mass.

$$i\,\delta E = \frac{4e^4}{(2\pi)^2 M^2}\int \frac{d^4k}{k^4} \quad \frac{-8\eta\left[F^2(k^2) - 1\right]}{k^2(2\omega + k^2)}$$

$$q = Mk \qquad F(k^2) = \frac{a^4}{(a^2 - k^2)^2} \qquad a = \frac{\Lambda}{M}$$

$$\eta = -\frac{M^2(2\omega + k^2)\,k^2}{8}\ \mathrm{tr}\left\{ M_{\mu\nu}\ \frac{1+\bar{\gamma}_t}{2}\ \bar{\delta}_z \right\}\ \mathrm{tr}\left\{ N_{\mu\nu}\ \frac{1+\gamma_t}{2}\ \delta_z \right\}$$

$$= \left(\frac{\omega^2}{3} + \frac{2}{3}k^2\right) - \frac{\mu}{2}\left(2k^2\omega + \frac{4\omega^2}{3} - \frac{4}{3}k^2\right) - \frac{\mu^2}{4} \times \left(\frac{2}{3}\omega k^2 - \frac{2}{3}\omega^3 - \frac{\omega^2 k^2}{3} - \frac{2}{3}k^4\right) \qquad (15)$$

η has been simplified by replacing quantities like k_z^2 by their spherical averages $\frac{\omega^2 - k^2}{3}$. The individual components of the traces are given in Appendix II.

The contribution of graph 3b differs only in the order of the photons along the electron line, to the extent that we may neglect the electron's mass, it gives exactly the same shift as 3a. (This is elaborated in Appendix II). The entire calculation has now been done for one proton and one electron per unit volume; in the hydrogen atom there are one proton and $|\varphi(0)|^2$ electrons per unit volume at this proton. Therefore we should multiply by $|\varphi|^2$ to find the atomic energy shift δE_A

$$i\,\delta E_A = \frac{8e^4\,|\varphi|^2}{(2\pi)^2\,M^2}\int \frac{d^4k}{k^6}\ \frac{-8\eta\,(F^2-1)}{(k^2+2\omega)}$$

$$= -(hfs)\left(\frac{a\,m}{\mu_p\,M}\right)\int \frac{d^4k\ \eta\ (F^2-1)}{k^6\,(k^2+2\omega)} \qquad (16)$$

$$(hfs) = \frac{2\pi a \mu_p\,\delta\cdot\bar{\delta}\ |\varphi(0)|^2}{3\,m\,M} \qquad (17)$$

The dimensionless integral appearing in 16 is evaluated in Appendix II.

Section III Calculation by the Bethe-Salpeter Equation

We shall now compute the hfs correction by the use of the Bethe-Salpeter (BS) equation. This method is more complicated than that of Section II; however, it explicitly includes bound state effects and, in principle, gives all the corrections to the hfs. Thus a reduction of this equation is a complete check on our previous work. NS[3] have already derived equation (1) including an explicit expression for R in the case of a point proton. We shall compute the change in equation (1) for a proton with a Hff. In other words, we shall compute the difference of the answer that NS would have gotten for a Hofstadter proton (interaction of equation (6)) and the answer they obtained for a point proton. Just as in Part II, the calculation of this difference is very much easier than either term alone would be. Appendix III gives a brief discussion of the BS equation and the perturbation formulae which we shall employ.*

The kernel, G, of the BS equation requires the interaction operator for the exchange of a photon of momentum k (absorbed by the proton). According to equation (6) this is:

$$G(k_\nu) = \frac{e^2}{2\pi^2} \gamma_4 \bar{\gamma}_4 \frac{\gamma_\mu \bar{\Gamma}_\mu}{k^2} (k_\nu) F(k^2) \qquad (18)$$

where the bars refer to operators for proton coordinates. Using

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* With minor changes, we use the notation of NS; it is the same as that of Section II.

gauge invariance, NS write this in the form:

$$G = \frac{e^2}{2\pi^2}\left\{ \overset{C}{-\frac{1}{K^2}} \; \overset{Q}{-\frac{\bar{A}_4}{K^2}} \; \overset{D}{-\sum_{1,2}\frac{a_i \bar{a}_i}{k^2}} \; \overset{P}{+\frac{a_i \bar{A}_i}{k^2}} \right\}\left[(F-1)+1\right] \quad (19)$$

where

$$A_\nu = \frac{\mu}{4M}\,\vec{\gamma}_4\,(\bar{k}\,\vec{\gamma}_\nu - \vec{\gamma}_\nu\,k) \quad (20)$$

$$k = (\omega, K) \qquad \vec{a} = \gamma_0\vec{\gamma}$$

and the sum on i is over the two orthogonal spatial directions, perpendicular to K. The various pieces of the interaction have been labeled C, Q, D and P in agreement with NS.

For reasons which will appear shortly, it is most convenient to consider that piece of the kernel of the BS equation corresponding to the irreducible diagram of Figure 4. Momentarily, let us forget about the presence of F and look at the result for a point proton. This is written:

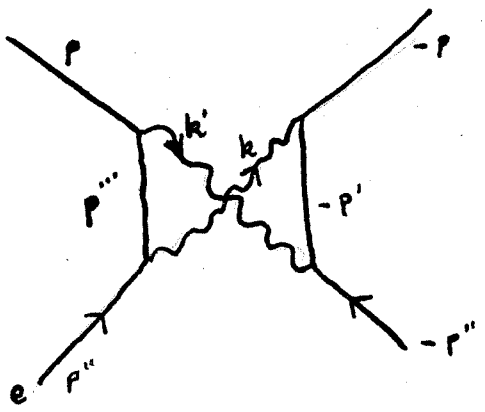


Figure 4

$p''' = p'' - k$

$k = p' - p$

$k' = p'' - p'$

$p = (\epsilon, P)$

$p'' = (\epsilon'', P'')$

$k^2 = (\omega^2 - K^2 + i\epsilon)$

$k'^2 = (\omega'^2 - K'^2 + i\epsilon')$

$$G^{x} = \beta\bar{\beta}\left\{\frac{e^2}{2\pi^2}\right\}^2 \frac{1}{2\pi i}\int \frac{d^4k}{k'^2\,k^2}\,\gamma_\nu \frac{1}{\mu_r W - H(P''') + \epsilon'' - \omega + i\delta\beta}$$

$$\gamma_\mu \overline{\Gamma}_\mu(k) \frac{1}{\bar{\mu}_r W - \bar{H}(P') - \epsilon - \omega + i\delta\bar{\beta}} \overline{\Gamma}_\nu(-k') \qquad (21)$$

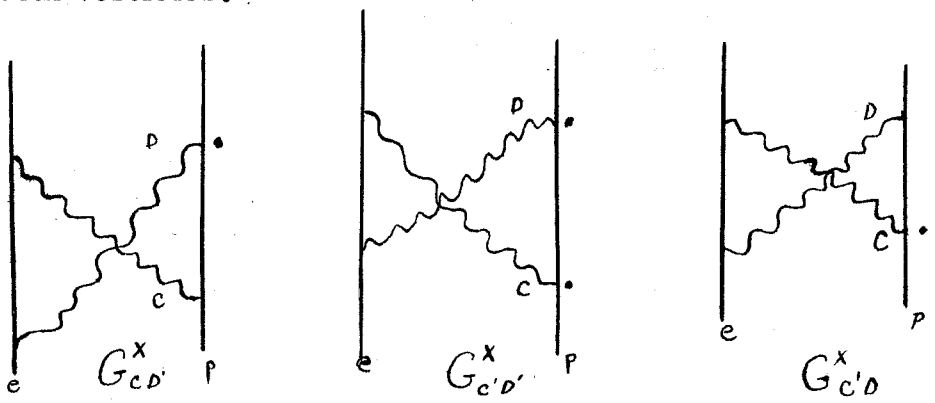
where W is the total energy of the bound state, $\mu_r = m/(m + M)$ and $\bar{\mu}_r = M/(m + M)$ are the reduced masses. H and $\bar{H}$ are Dirac single particle Hamiltonions for electron and proton, respectively. (See equation (16) of appendix III). The notation ^{x} refers to the crossed diagram, figure 4. If we break each proton vertex up into the four terms given in 19 then G^{x} is a sum of sixteen terms. We label these with two subscripts indicating the nature of the photon involved. Thus G^{x}_{CD} is given by

$$G^{x}_{CD} = 2\left\{\frac{e^2}{2\pi^2}\right\}^2 \frac{1}{2\pi i}\int \frac{d^4k}{k^2\,K'^2} \sum_{i=1,2} a_i$$

$$\frac{1}{\mu_r W - H(P''') + \epsilon'' - \omega + i\delta\beta} \frac{1}{\bar{\mu}_r W - \bar{H}(P') - \epsilon - \omega + i\delta\bar{\beta}} \bar{a}_i \qquad (22)$$

The other terms, G^{x}_{QD}, G^{x}_{PD}, G^{x}_{CP}, G^{x}_{QP}, G^{x}_{DD} and G^{x}_{PP} are included in Appendix IV. The energy shift due to an interchange of subscripts is always equal to the unexchanged form, i.e., $G^{x}_{CD} = G^{x}_{DC}$. Equation (22) and the appropriate G's of Appendix IV have therefore been multiplied by a factor of 2 to include this and the order of the subscripts no longer matters. The terms G^{x}_{CC}, G^{x}_{CQ} and G^{x}_{QQ} do not split triplet and singlet states and so are of no interest in this problem.

When we calculate the additional energy shift caused by the (F - 1) in equation (19), each term of the NS calculation is split into three terms corresponding to this factor at either vertex separately or at both verticies.



G^x for the interaction 22, • indicates (F - 1)

Figure 5

We use a prime to denote which photon carries the factor (F - 1). We now find the energy shift due to G^x by formula 28 of Appendix III. For the term $G^x_{CD'}$ this becomes:

$$\Delta E^x_{CD'} = 2\left\{\frac{e^2}{2\pi^2}\right\}^2 \frac{-1}{2\pi i}\int \frac{d^4p''d^4p'd^4p}{k^2\,K'^2}\sum_i \tilde{\psi}(p)$$

$$a_i \quad \frac{1}{\mu_4 W - H''' + \epsilon'' - \omega + i\delta\beta} \quad \frac{1}{\bar{\mu}_4 W - \bar{H}' - \epsilon - \omega + i\delta\bar{\beta}} \quad \bar{a}_i$$

$$\psi(p'')\left\{F(k^2) - 1\right\} \qquad (23)$$

NS discuss a case similar to this, namely one in which the factor F-1 is replaced by 1.* Then it is permissible to replace ψ and $\tilde{\psi}$ by ψ_{++} and $\tilde{\psi}_{++}$, the large components of ψ. This is possible because we shall want ψ only for $|p| \sim \alpha m$ and here $\psi(p) = \psi_{++} + 0(\alpha)$.

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* Reference (3) p 1150 following equation (26) and (27).

$$\psi_{++}(p) = \frac{E - E(P) - \bar{E}(P)}{\left[\mu_r W - E(P) + \epsilon + i\delta\right]\left[\bar{\mu}_r W - \bar{E}(P) - \epsilon + i\delta\right]} \varphi_{++}(P) \tag{24}$$

$\varphi_{++}(P)$ is defined in Appendix IV and $p = (\epsilon, P)$.

In order to do the integral (23) we insert in the numerator the energy factor

$$\left[\Lambda_+(p''') + \Lambda_-(p''')\right]\left[\bar{\Lambda}_+(p') + \bar{\Lambda}_-(p')\right] \equiv 1 \tag{25}$$

This energy projection factor (see equation 16 of Appendix III) splits things up into more terms but enables us to replace the Dirac Hamiltonian H in the denominator of (23) by a number $\pm E(p) = \pm\sqrt{P^2 + M^2}$, and similiarly for $\bar{H}$.

This results in four terms which we label $\Delta E^{x++}_{CD'}$, ΔE^{x+-}_{CD-}, $\Delta E^{x-+}_{CD'}$ and $\Delta E^{x--}_{CD'}$. For example

$$\Delta E^{x++}_{CD'} = -\frac{1}{\pi i}\left\{\frac{e^2}{2\pi^2}\right\}^2 \int \frac{d^4p''d^4p'd^4p}{K'^2\, k^2}\left\{F(k^2) - 1\right\}$$

$$\tilde{\psi}_{++}(p) \sum_i a_i \Lambda'''_+ \bar{\Lambda}'_+ \bar{a}_i$$

$$\frac{1}{(\mu_r W - E''' - \omega + \epsilon'' + i\delta)(\bar{\mu}_r W - \bar{E}' - \omega - \epsilon + i\delta)} \psi_{++}(p'') \tag{26}$$

Again the work of NS shows that if F-1 is replaced by 1 then only the contribution from the region $|k| \gtrsim m$; p, $p'' \sim \alpha m$ need be retained to find the energy shift to the required order, (hfs $\alpha m/M$). In our case, the factor F-1 forces $k \sim M$ and therefore to the same order, we need retain only

the zeroth order term of an expansion in powers of m/M, again taking $p, p'' \lesssim \alpha m$. Thus $k \sim p' \sim M$ and $k' \sim -k$. We may therefore, neglect p and p'' in the denominator of (26) and factor the integral into three separate integrals. For the integrals involving the wave function, we find

$$\int \psi_{++}(p, \epsilon)\ d^4 p = \int \varphi_{++}\ d^3 p = (2\pi)^{3/2} \varphi_{++}(0)$$

where $\varphi_{++}(0)$ is the wave function in coordinate space, evaluated at the origin. It is sufficiently accurate to replace $\varphi_{++}(0)$ by $u_e\ u_p\ \varphi_s(0)$, free particle, zero momentum, positive energy, spinors times the non-relativistic Schrödinger wave function at the origin. Then we find

$$\Delta E^{x++}_{CD'} = -\frac{(2\pi)^3}{\pi i} \left(\frac{e^2}{2\pi^2}\right)^2 |\varphi_s(0)|^2 \int \frac{d^4k}{k^2 K^2} (F-1)$$

$$\frac{\left\langle \sum_i a_i\ \Lambda_+(-k)\ \bar{\Lambda}_+(k)\ \bar{a}_i \right\rangle}{(\mu_r W - E(k) - \omega + i\delta)(\bar{\mu}_r W - \bar{E}(k) - \omega + i\delta)} \qquad (27)$$

where we have written $\langle \cdots \rangle$ for the expectation value of the spin operators in the numerator. We replace this expectation value by its spherical average over K, since the denominator depends only on K^2.

$$\frac{1}{4\pi} \int d\Omega_K \left\langle \sum_i a_i\ \Lambda_+(-k)\ \bar{\Lambda}_+(k)\ \bar{\alpha}_i \right\rangle = \sigma \cdot \bar{\sigma}\ \frac{K^2}{6E(K)\ \bar{E}(K)} \qquad (28)$$

The resulting shift is:

$$\Delta E^{x++}_{CD'} = \left[\left(\frac{e^2}{2\pi^2}\right)^2 (2\pi)^3 \left|\varphi\right|^2 \frac{4\pi\,\delta\cdot\bar{\delta}}{12\,M^2}\right] Op_1 \; T^{x++}_{CD}$$

$$= \frac{a\,m}{\pi\,\mu_p\,M} \left(h\,f\,s\right) Op_1 \; T^{x++}_{CD} \qquad (29)$$

where

$$Op_1 \equiv -\left[1 - \Lambda^2 \frac{\partial}{\partial \Lambda^2}\right] \qquad Op_1 \; \frac{k^2}{k^2 - \Lambda^2} = \left\{F(k^2) - 1\right\} \qquad (30)$$

$$T^{x++}_{CD} = \frac{12\,M^2}{4\pi\,\delta\cdot\bar{\delta}} \int \frac{d^4k \left\langle \sum \alpha_i \Lambda_+ \bar{\Lambda}_+ \bar{\alpha}_i \right\rangle}{(\mu_r W - E - \omega + i\delta)} \frac{\left(\frac{-2}{2\pi i}\right)}{K^2(k^2-\Lambda^2)} \frac{1}{(\bar{\mu}_r W - \bar{E} - \omega + i\delta)} \qquad (31)$$

$$Op_1 \; T^{x++}_{CD} = \left(\frac{2\pi^2}{e^2}\right)^2 \frac{12\,M^2}{4\pi\,\delta\cdot\bar{\delta}} \left\langle G^{x++}_{CD'} \right\rangle \qquad (32)$$

We have expressed the result in terms of a dimensionless function T^{x++}_{CD} in order to simplify subsequent work. Inserting (28) in (31) and performing the contour integral gives:

$$T^{x++}_{CD} = -\int_0^\infty \frac{\chi^2 d\chi}{\sqrt{\chi^2 + (m/M)^2}\sqrt{\chi^2+1}\sqrt{\chi^2+a^2}} \cdot \frac{1}{\left\{\mu_r \frac{W}{M} - \sqrt{\chi^2+(m/M)^2} - \sqrt{\chi^2+a^2}\right\}\left\{\bar{\mu}_r \frac{W}{M} - \sqrt{\chi^2+1} - \sqrt{\chi^2+a^2}\right\}} \qquad (33)$$

where $a = \Lambda/M$ and, as discussed above, we may neglect m/M, μ_r and put $W/M = 1$.

It is now clear how to proceed. By Equation (29) we express all energy shifts as $(\alpha m / \pi \mu_p M)$ hfs times a dimensionless integral and by (32) this integral is essentially just the expectation value of the kernel G^x. It is convenient to employ the identity

$$\left[F(k^2) - 1 \right]^2 = Op_2 \left\{ \frac{k^2}{k^2 - \Lambda^2} \right\} \tag{35}$$

$$Op_2 = \left[1 - \Lambda^2 \frac{\partial}{\partial \Lambda^2} + \frac{\Lambda^4}{6} \left(\frac{\partial}{\partial \Lambda^2} \right)^2 \right] \left[1 + \Lambda^2 \frac{\partial}{\partial \Lambda^2} \right] \tag{36}$$

Thus for a given term, say $\Delta E^{x++}_{CD'} + \Delta E^{x++}_{C'D} + \Delta E^{x++}_{C'D'}$ there is only one term proportional to $(2\,Op_1 + Op_2)\; T^{x++}_{CD}$. We have anticipated this in Equation (32) and dropped primed subscripts. We find:

$$(2\,Op_1 + Op_2) = Op \tag{37}$$

where Op is defined in Appendix II, Equation (II A:8) The total energy shift is then given by

$$\Delta E^x = \frac{\alpha m}{\pi \mu_p M} (hfs)\; Op \left\{ T^x_{CD} + T^x_{QD} + T^x_{CP} + T^x_{QP} + T^x_{DD} + T^x_{DP} + T^x_{PP} \right\} \tag{38}$$

where

$$T^x_{CD} = \left\{ T^{x++}_{CD} + T^{x+-}_{CD} + T^{x-+}_{CD} + T^{x--}_{CD} \right\} \tag{39}$$

Expressions for the expectation values of the Dirac matricies in the numerators of the G^x's, averaged over the solid angles of K, will be found in Appendix IV. We also give there all the $T^x_{\alpha\beta}$

which result from combining the various energy projections as in Equation (39). As comparison with Equation (34) suggests, these are somewhat simpler than the individual components, $T^{x\,ij}_{\alpha\beta}$.

The next term to consider is that corresponding to the exchange of a single photon. Here, to get the required accuracy, we must employ the iterated perturbation formula (IIIA:29) of Appendix III. This formula, in diagrams, looks like two successive exchanges of a single photon. (This uncrossed diagram is denoted by a superscript ^{o})

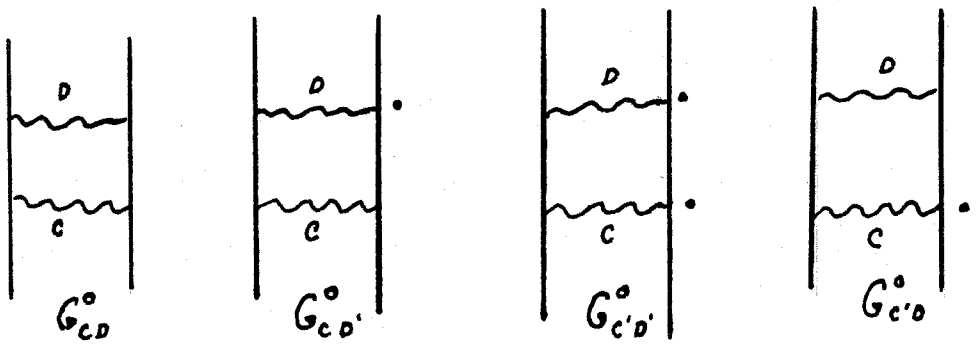


Figure 6 Diagrams for the iteration perturbation formula

Although the kernel of the B S Equation is made of irreducible graphs, the actual calculation of an energy shift introduces the reducible ones too. Thus the S matrix expansion of Part II and the present use of the B S Equation are actually only different notations for computing the same diagrams. The B S Equation allows us to estimate bound state effects because the wave functions appear in the integrands while the S matrix approach immediately reduces the entire answer to a few integrals.

We have labeled the uncrossed diagrams (see Figure 6) by analogy with the crossed ones. Just as in the crossed case, there are three terms corresponding to different positions of $\{F(k^2) - 1\}$.

For our sample term:

$$\Delta E^{o}_{CD'} = \frac{-2}{2\pi i}\left(\frac{e^2}{2\pi^2}\right)^2 \int \frac{d^4p\, d^4p'\, d^4p''}{K'^2\, k^2}\ \psi^*(p)\left\{F(k^2)-1\right\}$$

$$\frac{1}{(\mu_r W - H(P') + \epsilon + \omega + i\delta\beta)}\ \frac{1}{(\bar{\mu}_r W - \bar{H}(P') - \epsilon - \omega + i\delta\bar{\beta})} \sum_i \alpha_i \bar{\alpha}_i \psi(p'') \tag{40}$$

where $k = p' - p$, $k' = p'' - p'$ and i is orthogonal to K, as before. The factor 2 includes $\Delta E^{o}_{D'C} = \Delta E^{o}_{CD'}$. Evaluation of this diagram proceeds in exactly the same manner as before. We substitute φ_{++} for ψ by (24) and insert projection factors in the numerator as we did in (26). The same regions of the variables contribute as in (26) so that by analogous reasoning we arrive at:

$$\Delta E^{o++}_{CD'} = -\frac{(2\pi)^3}{\pi i}\left(\frac{e^2}{2\pi^2}\right)^2 \left|\varphi_s\right|^2 \int \frac{d^4k}{k^2 K^2}\left\{F(k^2)-1\right\}$$

$$\frac{\left\langle \sum_i \alpha_i \bar{\alpha}_i \Lambda_+^{(K)} \bar{\Lambda}_+^{(K)} \right\rangle}{(\mu_r W - E + \epsilon + i\delta)(\bar{\mu}_r W - \bar{E} - \epsilon + i\delta)} \tag{41}$$

This numerator has the same expectation value as given in (28) so that in terms of (29) we may write:

$$\mathrm{Op}_1\ T^{o++}_{CD} = \frac{-M^2}{(2\pi)(\pi i)} \int \frac{K^2\, d^4k \left\{F(k^2)-1\right\}}{E(K)\bar{E}(K) K^2 k^2 (\mu_r W - \epsilon + i\delta)(\bar{\mu}_r W - \bar{E} - \epsilon + i\delta)} \tag{42}$$

This term is equal to $\mathrm{Op}_1\ T^{x-+}_{CD}$ since by putting the electron into a

negative energy state, the crossed graph electron denominator becomes $(\mu_r W - E(K) - \omega - i\delta) \cong (E - \omega - i\delta) = -(-E + \omega + i\delta)$, to the order required. But the numerator of T_{CD}^{x-+} is just the negative of that for T_{CD}^{o++} so that the two terms are equal.

This same symmetry holds for all other uncrossed terms as well. The large component reduction of the Dirac operators from the uncrossed numerators is given in Appendix IV. From here it can be seen that when the electron mass is neglected, the crossed numerator for $+\rho$ $(-\rho)$ is the negative of the numerator for $-\rho$ $(+\rho)$ where $\rho = \pm$ and refers to the protonic energy state. But in each case the denominators are also opposite in sign since they are the same denominators as in ΔE_{CD}. The entire result of the uncrossed terms is then a factor of two in the energy shift.

The problem remaining is to show that the answer of this section is the same as that of Section II. This may be done by writing those results as (see Equations (16), IIA:7-11)

$$\Delta E = (hfs)\ \frac{a\,m}{\pi \mu_p M}\ Op\ J$$

$$J = A + \mu B + \mu^2 H + \mu^2 \ell \tag{43}$$

$$A = \frac{3}{i\pi^2}\int \frac{d^4k}{k^4(k^2-a^2)(k^2+2\omega)}\left\{\frac{2}{3} - \frac{\omega}{6}\right\}k^2 \tag{44}$$

$$B = \frac{3}{i\pi^2}\int \frac{d^4k}{k^4(k^2-a^2)(k^2+2\omega)}\ \frac{2}{3}\{1-\omega\}\,k^2 \tag{45}$$

$$H = \frac{3}{i\pi^2}\int \frac{d^4k}{k^4(k^2-a^2)(k^2+2\omega)}\left\{\frac{-\omega k^2}{2}\right\} \tag{46}$$

$$\ell = \frac{3}{i\pi^2}\int \frac{d^4k}{k^4(k^2-a^2)(k^2+2\omega)} \qquad (+\tfrac{1}{12})(2\omega+k^2)(2k^2+\omega^2) \quad (47)$$

We integrate A, B and H by contour integration instead of combining denominators as in Part II. The logarithmic term ℓ, was already treated this way in Appendix II. We find:

$$A = -4\int_0^\infty \chi^2 d\chi \left\{ \frac{\chi-4}{4\chi^2 a^2} + \frac{4-\sqrt{\chi^2+a^2}}{2a^2\sqrt{\chi^2+a^2}\left(a^2+2\sqrt{\chi^2+a^2}\right)} \right.$$

$$\left. + \frac{5-\sqrt{\chi^2+1}}{\left(2-a^2-2\sqrt{\chi^2+1}\right)\left(4\sqrt{\chi^2+1}\right)\left(1-\sqrt{\chi^2+1}\right)} \right\}$$

$$B = -16\int_0^\infty \chi^2 d\chi \left\{ \frac{\chi-1}{4\chi^2 a^2} + \frac{1-\sqrt{\chi^2+a^2}}{2a^2\sqrt{\chi^2+a^2}\left(a^2+2\sqrt{\chi^2+a^2}\right)} \right.$$

$$\left. + \frac{2-\sqrt{\chi^2+1}}{\left(2-a^2-2\sqrt{\chi^2+1}\right)4\sqrt{\chi^2+1}\left(1-\sqrt{\chi^2+1}\right)} \right\}$$

$$H = -12\int_0^\infty \chi^2 d\chi \left\{ \frac{\chi}{4\chi^2 a^2} + \frac{\sqrt{\chi^2+a^2}}{2a^2\sqrt{\chi^2+a^2}\left(a^2+2\sqrt{\chi^2+a^2}\right)} \right.$$

$$\left. + \frac{1-\sqrt{\chi^2+1}}{\left(2-a^2-2\sqrt{\chi^2+1}\right)4\sqrt{\chi^2+1}\left(1-\sqrt{\chi^2+1}\right)} \right\}$$

It is now a matter of considerable algebra to show that the integrands are equal:

$$\frac{A}{2} = T^{x}_{CD} + T^{x}_{DD}$$

$$\frac{B}{2} = T^{x}_{QD} + T^{x}_{CP} + T^{x}_{DP}$$

$$\frac{H + \ell}{2} = T^{x}_{QP} + T^{x}_{PP}$$

Part IV Results and Discussion

We give Λ the value $0.91\,M$ and by equations (IIA:16-20) find a recoil correction of

$$\Delta E = (hfs)\ \frac{\alpha m}{\mu_p \pi M}\left\{-73 \pm 6 + \frac{9}{4}\mu^2 \ln\frac{\Lambda_o}{M}\right\} \tag{51}$$

where the uncertainty is that present in Hofstadter's results.

Figure (7) shows a plot of this correction for various values of Λ/M (omitting the cutoff term, $\ln \Lambda_o/M$). While we do not regard Λ as variable, it is interesting to see what the Λ-dependence is. The curve is fitted quite accurately by the form (constant/Λ). The correction, (51), is to be added to the NS correction of $\left\{-5 - \frac{9}{4}\mu^2 \ln\frac{\Lambda_o}{M}\right\}$ so that we get finally:

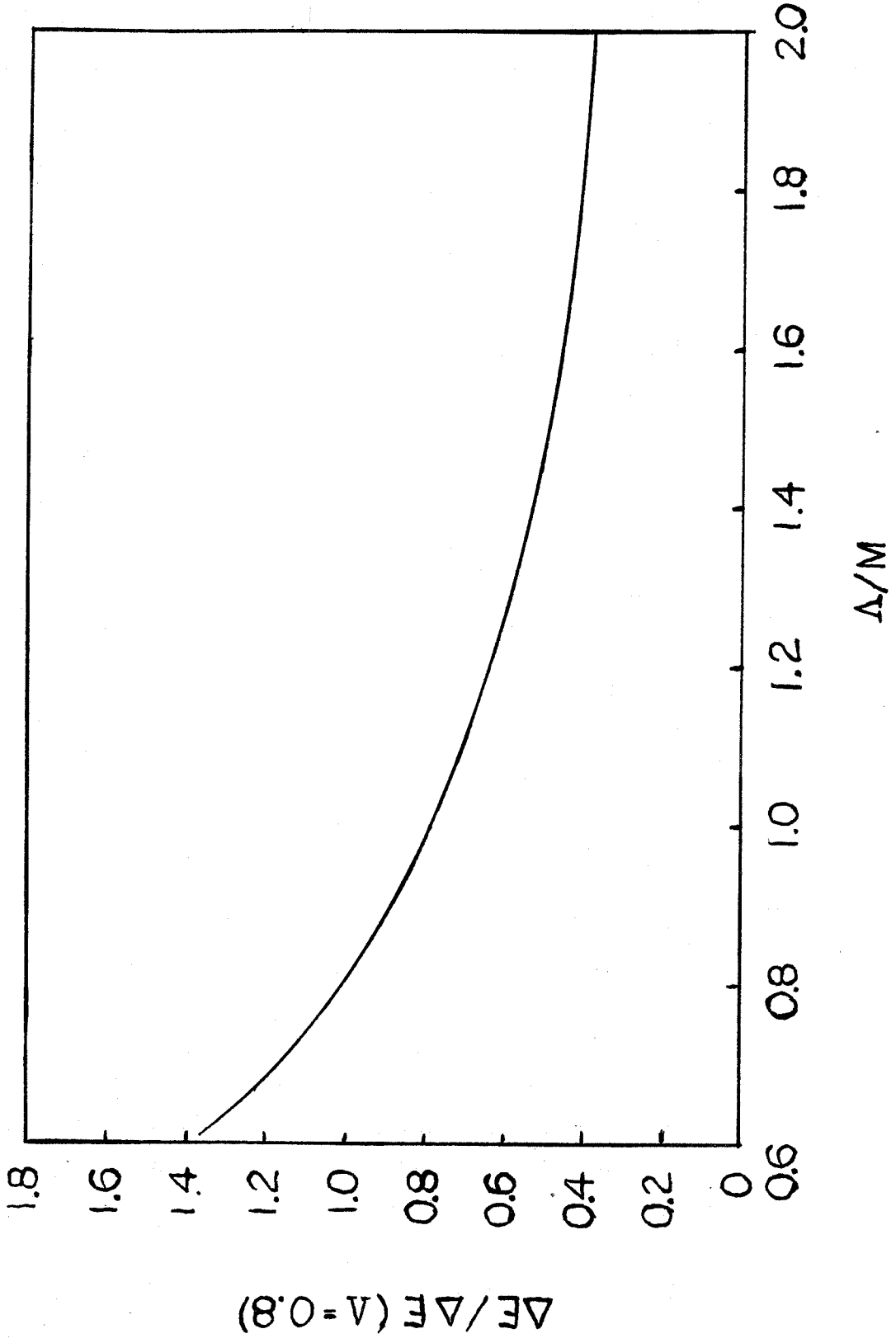
$$R = 1 - 35 \times 10^{-6} \pm 3 \times 10^{-6} \tag{52}$$

where R goes into Equation (1). As discussed in the introduction, α^2 is the least accurately known factor so it is best to write (1) including experimental results, as:

$$\alpha^{-2} = \frac{16\, c\, R_\infty\, (\mu_e)^2\, M_p}{3\,\Delta\nu\qquad M_e}\left(1 + \frac{m}{M}\right)^{-3}\left(1 + \frac{3\alpha^2}{2}\right)\left\{1 - \left(\frac{5}{2} - \ln 2\right)\alpha^2\right\} R$$

$$\alpha^{-1} = \left\{137.0391 \pm 0.0001\right\}\ R^{1/2}$$

$$\alpha^{-1} = 137.0367 \pm 0.0001 \tag{53}$$


ΔE/ΔE (Λ=0.8)
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
Λ/M

If we use the currently accepted value[4] of α^{-1} (from the fine structure measurements in deuterium) to find an experimental R then we get:

$$\alpha^{-1} = 137.0390 \pm 0.0006$$

$$R = 1 - 1.4 \times 10^{-6} \pm 9 \times 10^{-6} \qquad (54)$$

which disagrees with the calculated value by more than a probable error.*

First there is the possibility that the results depend upon the analytic form of the Hff. We have investigated this by using a different form factor:

$$F_2(q^2) = \frac{\Lambda_1^2 \Lambda_2^2}{\Lambda_1^2 - \Lambda_2^2} \left\{ \frac{1}{q^2 - \Lambda_1^2} - \frac{1}{q^2 - \Lambda_2^2} \right\} \qquad (55)$$

$$\Lambda_1^2 = 0.52\, M^2 \qquad \Lambda_2^2 = 2.06\, M^2 \qquad (56)$$

The mean square radius is the same as that of the original Hff and, of course, F(0) = 1. The result is a shift of

$$\Delta E = (hfs)\, \frac{\alpha m}{\pi \mu_p M} \left\{ -74 + \frac{9}{4}\mu^2 \ell n \frac{\Lambda_0}{M} \right\} \qquad (56)$$

---

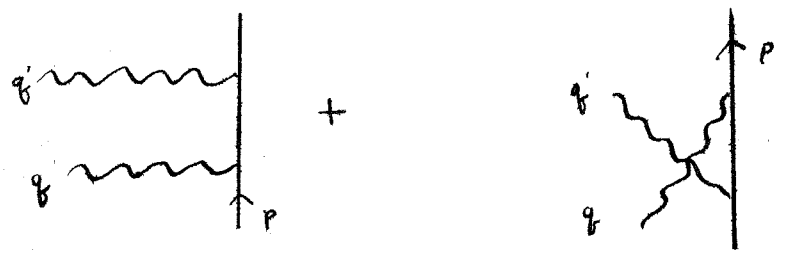
* The anomalous moment of the electron, μ_e, enters both the hfs Equation (1) and the Lamb measurements in practically the same way. (See Reference (9), p. 353, Equation (22-22A). Therefore, although the numerical values of α from the two methods are both changed by the recent recalculation of μ_e, the relative disagreement remains. In the results given here we have used the new value of the anomalous moment given in (10).

This is in extremely close agreement with (51)*.

Next, we look at the structure of the matrix element for the energy shift given by Equation (11) (and the crossed term). The proton part of this matrix element is

$$F(q)\ F(q')\left\{\Gamma_\mu(\not{q}')\ \frac{1}{\not{p}_p+\not{q}-M}\ \Gamma_\nu(\not{q})+\Gamma_\nu(\not{q})\ \frac{1}{\not{p}_p+\not{q}'-M}\ \Gamma_\mu(\not{q}')\right\}$$

which is given in diagrams as



This is exactly the Compton scattering of a photon except that the "photon" involved has a "mass", i.e., $q^2 = 0$ in the integral 11. Thus what we really want is to replace $M_{\mu\nu}(k, k')$ by the laboratory S matrix element for the forward, spin-flip, scattering of a virtual

---

* The use of $\{F_2^2(k^2)-1\} = \frac{\lambda_1^2\lambda_2^2}{\lambda_1^2-\lambda_2^2}$

$$\times\left\{\frac{1}{\lambda_1^2}\left[\frac{\partial}{\partial a^2}-\left(\frac{1}{\lambda_1^2}+\frac{2}{\lambda_1^2-\lambda_2^2}\right)\right]_{a^2=\lambda_1^2}-\frac{1}{\lambda_2^2}\left[\frac{\partial}{\partial a^2}-\left(\frac{1}{\lambda_2^2}-\frac{2}{\lambda_1^2-\lambda_2^2}\right)\right]_{a^2=\lambda_2^2}\right\}\frac{k^2}{k^2-a^2}$$

where $\lambda_1 = \Lambda_1/M$ and $\lambda_2 = \Lambda_2/M$ enables us to express the shift from this form factor in terms of an operator times the integral J which is given by IIA:14.

photon of mass k^2. (It should be spin-flip because of the factor σ_z in 15, involved in the energy difference of singlet and triplet; it is the matrix element in the laboratory since the initial proton is at rest and the photon is scattered in the forward direction because the final proton is at rest.) This is quite reasonable since the electron "probes" the charge distribution with "hot" (shortwave length) photons. Only such photons can "see" the difference between a point and distributed charge. Such photons lead to large recoil for the electron and, because its average momentum in H is very low, every time the electron emits a hot photon almost at once, it must emit another of opposite momentum. This means that the energy shift we seek is fundamentally a double photon exchange processes. This is in contrast to the Hofstadter experiments where the two photon processes are a small correction (see Appendix I). Although the single photon vertex must have the form $F(q^2)\,\Gamma_\mu(q)$, the double vertex can have more complicated forms. This is easily seen from the diagrams of figure 8a and b.

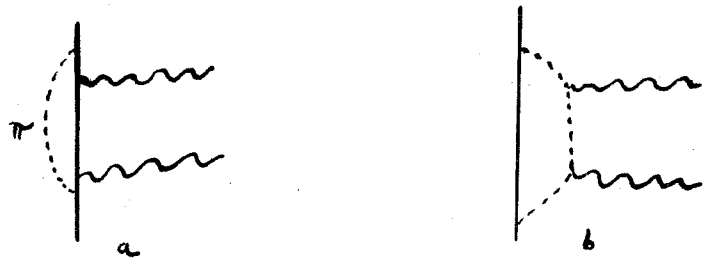


Figure 8 Meson corrections to the "Compton Vertex"

The meson which links the entire photon exchange process complicates the matrix element. In general, the sum of all such graphs

will not even be expressible as two verticies connected by a propagator. To the extent that the processes represented by figure 8 are "small", we can regard 16 as giving the correct value of the hfs but there is no a priori basis for their omission. Since such diagrams involve strongly interacting particles, we cannot use perturbation theory reliably. In the following appendix (Appendix V) we give a fairly crude dispersion theoretic estimate of the hfs shift due to such a diagram. We find that under "reasonable assumptions" about the photomeson production cross section for virtual (massive) photons, that the additional hfs shift is of the order of 1-5 ppm and therefore that it is unimportant unless the corrections are carried to a higher order of accuracy. A more detailed summary of these results is given there.

We now look at the experimental error given by Lamb. His results for the fine structure splitting in deuterium are: [11]

$$\Delta E_D = 10971.59 \pm 0.20 \text{ Mc/sec}$$

DuMond, Cohen and Crowe[4] take this result but with an uncertainty of 0.10 Mc/sec. If we use the Lamb value for the uncertainty we find in place of 54,

$$R = 1 - 1.4 \times 10^{-6} \pm 18 \times 10^{-6}$$

Thus the disagreement is 34 ± 21 ppm. The "unexplained difference" of 14 ppm is just at the edge of both theory and experiment.

The major source of uncertainty is the meson corrections. There are also higher order electrodynamic and recoil corrections which could give a few ppm. On the basis of the estimate of the meson corrections of Appendix V the author feels that although some corrections to the result of equation (51) are quite probable, it is rather unlikely that these total more than about 10 ppm. The remaining difference is then insignificant unless confirmed by highly accurate measurements.

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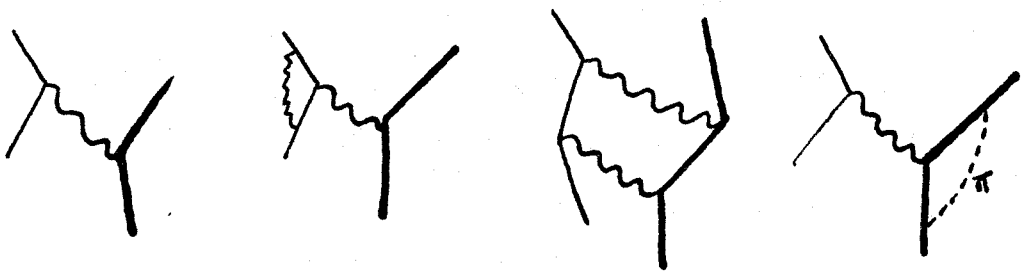
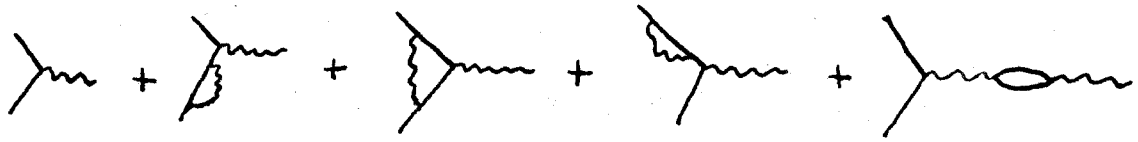
Appendix I - The Measurement of the Proton Form Factor by High Energy Electron-Proton Scattering.

In perturbation theory the scattering of a high energy electron by a proton looks like the exchange of one or more photons and radiative electromagnetic and mesonic corrections to these processes.



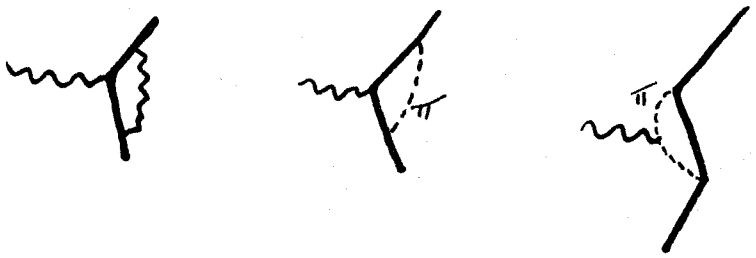
We concentrate first on single photon exchange but include radiative corrections at the vertices. For the electron the corrections lead to a "renormalized vertex" containing the anomalous moment.



$$= \Lambda_\mu = e\,\gamma_\mu + \mu(k^2)\,(\gamma_\mu \not{k} - \not{k}\,\gamma_\mu) \qquad \text{(IA:1)}$$

These corrections must be included at the energies of the Hofstadter experiments because they give relative effects of about 10%. [12]

For the proton there are corrections of the form.



The calculation of these is beyond present methods; however, their net effect is measured with the Stanford accelerator for momentum transfers up to about 1 Bev/c.

All corrections to the vertex must take the form:*

$$e f_1(k^2)\ \gamma_\mu + \frac{\mu}{M} f_2(k^2)\ (\gamma_\mu \not{k} - \not{k} \gamma_\mu) \qquad \text{(IA:2)}$$

where f_1 and f_2 are real functions of $k^2 = \omega^2 - \vec{k}^2$ and $f_1(0) = f_2(0) = 1$. The functions f_1 and f_2 are essentially the Fourier transforms of the charge and anomalous moment distributions. An especially simple form for the distribution is given by the "exponential model":

$$f = \frac{1}{\left[1 - \frac{(k a)^2}{12}\right]^2} = \text{Fourier transform of} \left\{ \frac{12\sqrt{3}}{4\pi a^3} e^{-\sqrt{12}\frac{r}{a}} \right\} \qquad \text{(IA:3)}$$

where a is the r.m.s. radius. With the choice $f_1 = f_2 = f$, $a = 0.8 \times 10^{-13}$ cm., Hofstadter has been able to fit his experimental data.[12] Of course, it is also possible to fit the data with several other analytic functions and examples are given in the above reference. The m.s. radius†, which is 6 times the coefficient of k^2 in a power series expansion of $f(k^2)$, turns out to be about the same, regardless of the form used for f. We shall not place any great reliance on the form of the f's and our results are

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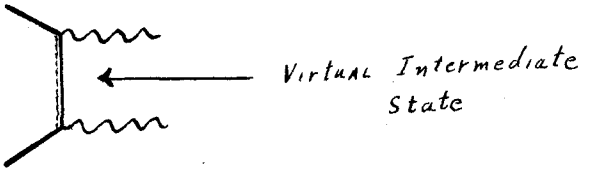
* The uniqueness of this form follows from gauge and Lorentz invariance.

† mean square radius

realistic only if they do not depend too strongly on such details as the higher moments. It is also possible to obtain a fit using different radii for the charge and moment terms (f_1 and f_2);however, we do not consider this in the absence of data clearly indicating the necessity of such a choice.

The remaining corrections which might conceivably complicate the interpretation of the high energy scattering results of Hofstadter are due to the exchange of two or more photons between the electron and proton. The effect that this would have is uncertain since the intermediate state in a two-photon exchange is virtual. This makes the form factor for each vertex depend upon the extent to which the mass of the intermediate state departs from that of a free proton.



In other words, for a two-photon exchange, there are more invariants available with which to construct a transition matrix element and therefore we are not limited to anything so simple as IA:2. In particular, there is no requirement that the double vertex should be given by the iteration of two single verticies, at each of which IA:2 acts, although this may be a useful approximation. Indeed, the existence of a resonance in photomeson production suggests that there will be some sort of a resonance in the Compton scattering for real photons at about 300 Mev laboratory energy and probably for photons not too far off the mass shell ($k^2 \approx 0$) there will also be such a resonance. We shall have to analyze this point later in connection

with the hfs corrections (Appendix V). Drell and others [13] have set an upper limit of about 1% (at laboratory energies up to 1 Bev) to the effect these corrections would have on the Hofstadter data. With the present experimental uncertainty, these corrections may be ignored.

Appendix II Miscellaneous Details of the Covariant Calculation

The components of the traces (dimensionless units, M = 1)

Let

$$B_{\mu\nu} = 2\ \mathrm{trace}\left\{\gamma_\mu \frac{1}{\not k}\ \gamma_\nu\ \frac{1+\gamma_t}{2}\ \gamma_x\gamma_y\right\} \tag{IIA1}$$

$$\begin{aligned} &B_{\mu\nu} = 0 \text{ if } \mu = \nu && B_{xy} = \frac{4\omega}{k^2} \\ &B_{\mu\nu} = -B_{\nu\mu} && B_{xt} = \frac{4k_y}{k^2} \\ &B_{z\nu} = 0 && B_{ty} = \frac{4k_x}{k^2} \end{aligned} \tag{IIA2}$$

Let

$$A_{\mu\nu} = 2\ \mathrm{trace}\left\{\Gamma_\mu(-k)\frac{1}{\gamma_t + \not k - 1}\ \Gamma_\nu(k)\ \frac{1+\gamma_t}{2}\ \gamma_x\gamma_y\right\} \tag{IIA3}$$

$$\Gamma_\mu = \gamma_\mu + \frac{\mu}{4}\left(\gamma_\mu \not k - \not k \gamma_\mu\right) \tag{IIA4}$$

$$\begin{aligned} A_{xy} = -A_{yx} = \frac{1}{k^2 + 2\omega}\ &4\omega - \frac{\mu}{2}\left[8k^2 + 4(k_x^2 + k_y^2)\right] \\ &+ \mu^2\left[-\omega k^2 + 2(k^2 - \omega^2) + 2(k_x^2 + k_y^2)\right] \end{aligned} \tag{IIA5}$$

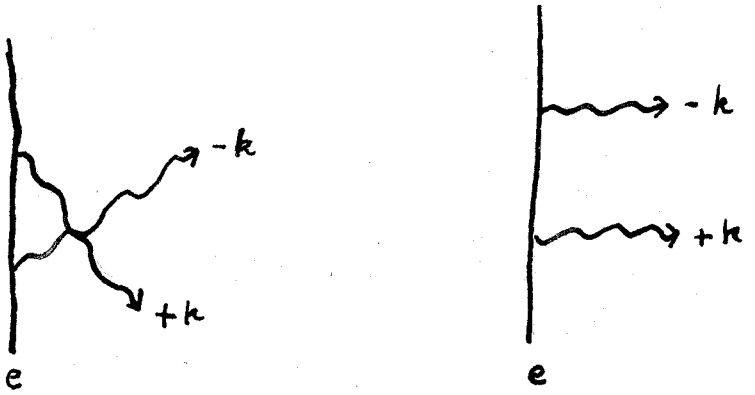
$$A_{xt} = -A_{tx} = \frac{1}{k^2 + 2\omega}\ 4k_y\left[1 + \frac{\mu}{2}(\omega - 2) + \frac{\mu^2 k^2}{4}\right]$$

$$A_{yt} = -A_{ty} = \frac{-1}{k^2 + 2\omega}\ 4k_x\left[1 + \frac{\mu}{2}(\omega - 2) + \frac{\mu^2 k^2}{4}\right]$$

η of equation 15 is given by;

$$\frac{16\eta}{k^2\ (k^2 + 2\omega)} = A_{\mu\nu}B_{\mu\nu} \tag{IIA6}$$

The equality of the crossed (×) and uncrossed (°) diagrams when m and $\not p_e$ are neglected.



$$B^{\circ}_{\mu\nu} = \text{trace } \gamma_\mu \not{k} \gamma_\nu (1 + \gamma_t)\gamma_x\gamma_y \frac{1}{k^2}$$

$$B^{x}_{\mu\nu} = \text{trace } \gamma_\nu \frac{(-\not{k})}{k^2} \gamma_\mu (1 + \gamma_t)\gamma_x\gamma_y = B^{\circ}_{\mu\nu}$$

by comparison with equations IIA 2. The operator $A_{\mu\nu}$ is unchanged.

The evaluation of the integral in equation 16

We write

$$F^2(k^2) - 1 = Op \frac{k^2}{k^2 - a^2} \qquad \text{IIA7}$$

where

$$Op = \left\{ -1 + a^2 \frac{\partial}{\partial a^2} - \frac{a^4}{2}\left(\frac{\partial}{\partial a^2}\right)^2 + \frac{a^6}{6}\left(\frac{\partial}{\partial a^2}\right)^3 \right\} \qquad \text{IIA8}$$

Thus

$$\int \frac{d^4k\ \eta\ (F^2 - 1)}{k^6(k^2 + 2\omega)} = Op\ J \qquad \text{IIA9}$$

$$J = \int \frac{d^4k\ \eta}{k^4(k^2+2\omega)(k^2-a^2)} \qquad \text{IIA10}$$

Now write η as

$$\eta = \left[(1 + \mu)/3 - (1 + \mu)^2/2\right]\omega k^2 + \frac{2}{3}(1 + \mu)k^2 + \frac{\mu^2}{4}(2\omega + k^2)\left(\frac{2}{3}k^2 + \frac{\omega^2}{3}\right)$$

$$+ \left(\frac{1}{3} - \frac{2}{3}\mu\right)\frac{\omega}{2}(2\omega + k^2) \qquad \text{IIA11}$$

In IIA11 the last term is odd in ω and so gives zero. The first two terms are done in section A of the appendix of reference 8.

$$\frac{1}{(2\pi)^2}\int \frac{d^4k\ (\omega,1)}{k^2\ (k^2-a^2)(k^2+2\omega)} = \frac{1}{8i}\int_0^1 \frac{dx\left[-(1-x^2),\ +2(1+x)\right]}{(1-x)^2+a^2x} \qquad \text{(IIA12)}$$

The third term in IIA11 is integrated first by residues on ω and then over solid angles giving;

$$\frac{1}{(2\pi)^2}\int \frac{d^4k\ (\frac{2}{3}k^2+\frac{\omega^2}{3})}{k^4\ (k^2-a^2)} = \frac{1}{3ia^2}\int_0^{\Lambda_0/M} dK\left[\frac{K^3}{a^2}+\frac{5K}{2}\ \frac{-K^4}{\sqrt{K^2+a^2}\ a^2}\ \frac{-3K^2}{\sqrt{K^2+a^2}}\right] \qquad \text{(IIA13)}$$

Since the final integral in IIA13 is logarithmically infinite, we have cut it off in momentum space at $\frac{\Lambda_0}{M}$. The reason that such an infinity appears is that the interaction of NS (F = 1) is divergent while ours (F = Hff) is not. The difference between the two, δE_A, must then be divergent but when this correction is added to that of NS the result will be finite, since the effective interaction will then be that of equation 6. In order to make the cancellation unambiguous, we have employed the same type of a cut-off which NS do, namely, integration up to a definite value of the three-momentum in the center-of-mass Lorentz frame.

When we combine all these results, we get; (primes indicate differentiation with respect to a^2)

$$J = \frac{\pi^2}{2}\left\{-\int_0^1 \frac{2D(1+x) - C(1-x^2)}{(1-x)^2 + a^2 x}\,dx + \frac{\mu^2}{3}\left[-\frac{17}{16} + \frac{9}{4}\ln\frac{2\Lambda_0}{\Lambda}\right]\right\}$$

$$= \frac{\pi^2}{2}\left\{-C + (2D + 2C - a^2 C)\ln a - \left[2(D-C) + (2+a^2)\left(D + C - \frac{a^2 C}{2}\right)\right] I + \frac{\mu^2}{3}\left[-\frac{17}{16} + \frac{9}{4}\ln\frac{2\Lambda_0}{\Lambda}\right]\right\} \qquad \text{(IIA14)}$$

$$C = \frac{1+\mu}{3} - \frac{(1+\mu)^2}{2} \qquad D = \frac{2}{3}(1+\mu) \qquad \text{(IIA15)}$$

$$-\mathrm{Op}\, J = \frac{\pi^2}{2}\left\{-C + 4D\,\mathrm{Op} I + 2(D+C)\left[-\ln a + \frac{11}{2} - \frac{a^8}{12} I'''\right] - C\left[\frac{a^2}{6} - \frac{a^8 I''}{4} - \frac{a^8(a^2-2) I'''}{12}\right] - \frac{\mu^2}{3}\left[-1 - \frac{9}{4}\ln\frac{2\Lambda_0}{M} + \frac{9}{4}\ln a\right]\right\} \qquad \text{(IIA16)}$$

where

$$I = I(a^2) = \frac{2}{\sqrt{4a^2 - a^4}} \tan^{-1}\sqrt{\frac{4-a^2}{a^2}} = \frac{2/a}{\sqrt{4-a^2}} \cos^{-1}\left(\frac{a}{2}\right) \qquad \text{(IIA17)}$$

$$\mathrm{Op}\, I = \frac{-228 + 146a^2 - 26a^4}{6(4-a^2)^3} + \frac{-140 + 120a^2 - 36a^4 + 4a^6}{(4-a^2)^3} I \qquad \text{(IIA18)}$$

$$I'' = \frac{3(2-a^2)}{(4a^2 - a^4)} + \frac{2a^4 - 8a^2 + 12}{(4a^2 - a^4)} I \qquad \text{(IIA19)}$$

$$I''' = \frac{-11a^4 + 44a^2 - 60}{(4a^2 - a^4)} + \frac{6a^6 - 36a^4 + 108a^2 - 120}{(4a^2 - a^4)^3} I \qquad \text{(IIA20)}$$

Appendix III The Bethe-Salpeter Equation

Since the BS equation may not be familiar to the reader, we include in this appendix an elementary discussion of its properties and the perturbation formulae we shall use. Most of this material will be found in articles by Bethe and Salpeter.[14, 15]

First, consider the Dirac equation for a central field

$$\varphi(P) = \frac{1}{\alpha \cdot P + \beta m - W} \int V(K)\ \varphi(P-K)\ d^3K \qquad \text{(IIIA:1)}$$

The terms may be given the interpretation

Amplititude to find momentum P present = $\sum_{\text{all possible } K}$

Amplititude to propagate as a free particle, momentum P

X Amplititude to find momentum K in the potential

X Amplititude to find momentum P-K in the wave function

(Actually, in a bound state of energy $W < m$, the propagator is not free.) In other words, one more scattering in the potential does not alter the wave function.

If we were considering the scattering of a free particle by the potential we would have the same equation, with an inhomogeneous term, representing the possibility of no scattering.

$$\varphi = \varphi_i + \frac{1}{\alpha \cdot P + \beta (m - i\underset{\sim}{\epsilon}) - W} \int V(K)\ \varphi(P-K)\ d^3K \qquad \text{(IIIA:2)}$$

where $W \doteq m$ and φ_i is the incident wave. For a bound state, this term is not present because, after a long time, the unscattered

waves spread out in space and "diffuse away" while the scattered waves are held in, near the origin. Graphically, each equation represents the ladder diagrams Figure 9, for, when we solve by iteration with $\varphi^{(0)} = \varphi_i$, we get just such a series.

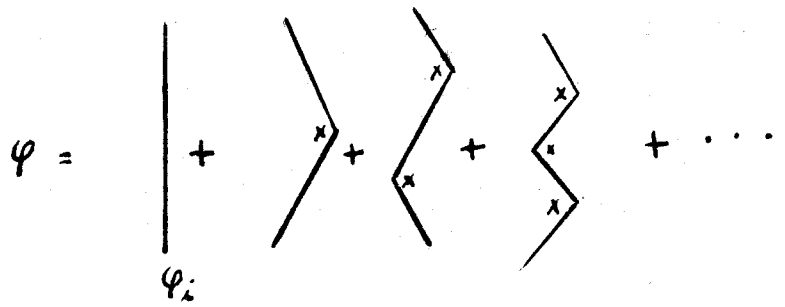


Figure 9 The first term is not present for a bound state.

In the presence of a quantized electromagnetic field, we know that the scattering by an external potential is correctly given by a sum over diagrams as follows

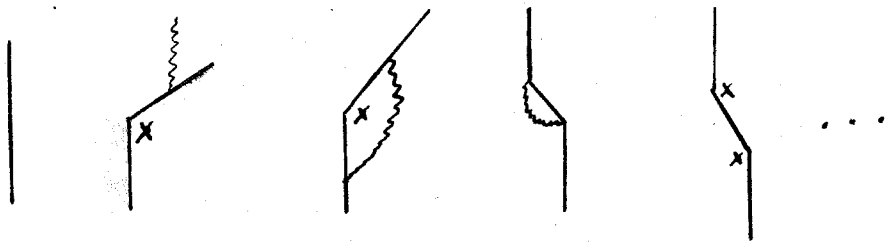


Figure 10 Scattering with radiative corrections

Thus we expect that the integral equation which the final state wave function satisfies is:

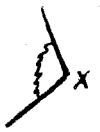
$$\varphi = \varphi_i + \frac{1}{\alpha \cdot P + \beta(m - i\epsilon) - W} \int I(K, P, W)\, \varphi(P-K)\, d^3K \qquad \text{(IIIA:3)}$$

where I is some kernel, which, when (IIIA:3) is iterated, will generate the entire series of diagrams of Figure 10. The correct

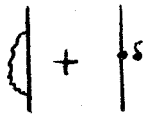
choice for I is then the sum of the "irreducible graphs" because, upon iteration, the term $\left[\alpha P + \beta(m - i \underset{\sim}{\epsilon}) - W\right]^{-1}$ will become the propagator in all diagrams with two or more parts connected by a bare electron line. Clearly, iteration will not provide anything but such reducible diagrams. By analogy with the central potential problem of equations (IIIA:1 and 2), the bound state equation simply omits the φ_i in (IIIA:3) and puts $W = m - E$ (binding).

The interaction I is constructed according to the usual rules[8]. For the following diagrams we have:

$$I_1 = V(K)$$



$$I_2 = e^2 \int \beta \gamma_\mu \frac{1}{\not p - \not k' - m} V(K) \frac{1}{\not p - \not k - \not k' - m} \gamma_\mu \, d^4k'$$



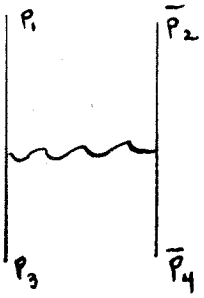
$$I_3 = e^2 \int \beta \gamma_\mu \frac{}{\not p - \not k' - m} \gamma_\mu \, d^4k' - \delta m$$

$$I_{Total} = \sum_i I_i \qquad \text{and} \qquad \not p \equiv W \gamma_o - \vec{\gamma} \cdot P$$

We generalize this to two particles by analogy, the wave function satisfies a scattering equation without the inhomogeneous term.

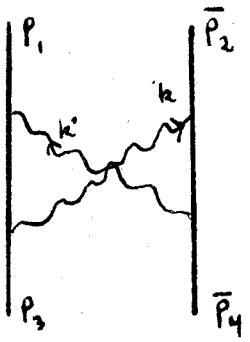
$$\psi = \frac{1/(2\pi i)}{(\not p_1 - m)(\not \bar p_2 - M)} \int I(p_1 \bar p_2 : p_3 \bar p_4) \, \Psi(p_3, \bar p_4) \, d^4p_3 \, d^4p_4 \qquad \text{(IIIA:4)}$$

The bar refers to the heavy particle operators and ψ is a sixteen component wave function of the eight variables $p_{1\mu}$, $\bar p_{2\mu}$. I is again a sum over irreducible diagrams.



$$I_1 = e^2 \frac{\bar{\Gamma}_\mu (p_3 - p_1)\, \gamma_\mu}{(p_1 - p_3)^2} \qquad \text{(IIIA:5)}$$

where $\bar{\Gamma}_\mu(k) = \bar{\gamma}_\mu + \frac{\mu}{2M}(\bar{k}\bar{\gamma}_\mu - \bar{\gamma}_\mu \bar{k})$ is the proton vertex operator.



$$I_2 = e^4 \int \bar{\Gamma}_\mu(k) \frac{1}{\bar{p}_4 - \bar{k}' - M} \bar{\Gamma}_\nu(-k') \frac{1}{k^2 k'^2} \gamma_\nu \frac{1}{\bar{p}_3 - \bar{k} - m} \gamma_\mu \, d^4k \qquad \text{(IIIA:6)}$$

$$\bar{p}_4 = k' - k + \bar{p}_2$$

$$p_1 = k' - k + p_3$$

We confine ourselves to the Lorentz frame where

$$p + \bar{p} = P = (W, 000) \qquad \text{(IIIA:7)}$$

$$W = m + M - E(\text{binding}) \qquad \text{(IIIA:8)}$$

and we define:

$$q = \bar{\mu}_r p - \mu_r \bar{p} \qquad \text{(IIIA:9)}$$

$$\bar{\mu}_r = \frac{m}{m + M} \qquad \mu_r = 1 - \bar{\mu}_r \qquad \text{(IIIA:10)}$$

$$p = \mu_r P + q \qquad \bar{p} = \bar{\mu}_r P - q$$

Omitting the trivial exp (i W $\frac{t_1 + t_2}{2}$) in the wave function, we find that (IIIA:4) may be written:

$$\psi(P\ ,\ q) = \frac{-1/(2\pi i)}{(\mu_r P + \not{q} - m + i\underset{\sim}{\delta})\,(\bar{\mu}_r \bar{P} - \bar{\not{q}} - M + i\underset{\sim}{\delta})} \int G\,(P,\ q,\ q')\ \psi(q - q')\ d^4q' \qquad \text{(IIIA:11)}$$

where $G = I\,(\mu_r P + q,\ \bar{\mu}_r P - q\ ;\ \mu_r P + q - q',\ \bar{\mu}_r P - q + q')$ (IIIA:12)

This is the bound state equation first given by Bethe and Salpeter.[14] A formal derivation is due to Gell-Mann and Low.[16]

In general, the BS equation is difficult to solve since the kernel is an infinite sum of singular terms. Even if the kernel is approximated by the first few terms, as is usually done, it is still a non-linear function of the eigenvalue, W, so that some sort of a perturbation method must be used. Another difficulty is that since complete covariance has been kept, we have a "relative energy" variable of the two particles and its conjugate, the relative time. Because classical, covariant, two body problems are not well understood, we are unable to attach any obvious physical meaning to this variable. What we think of as the usual wave function is the covariant wave function taken at zero relative time. This clearly corresponds to:*

---

* We now change notation and use W for total energy momentum and p = (ϵ, P) for the relative energy momentum. This will cause no confusion since from now on, all our four vectors will refer to the relative coordinate and we imagine the total energy momentum as fixed at W.

$$\varphi(P) = \int \psi(P, \epsilon)\, d\epsilon \qquad \text{(IIIA:13)}$$

Both of these difficulties may be sidestepped in practical problems by splitting G into two parts.

$$G = G_o + G_\Delta \qquad \text{(IIIA:14)}$$

$$G_o = \frac{e^2 \beta \bar{\beta}}{Q^2} \qquad \text{(IIIA:15)}$$

The instantaneous part G_o is a function of only the three-momentum transfer and the remainder G_Δ is the additional terms that we are using to approximate the sum over all irreducible diagrams. G_o, being the coulomb potential, gives most of the binding and a rather good approximation to the wave function. G_Δ may then be included as a perturbation.

We develop now the perturbation methods, applicable to the hfs problem. For the fine structure or other problems, there are difficulties which require other techniques as discussed by Salpeter, especially Section IV of reference 15.

First we introduce some definitions:

$$\begin{aligned} H(P) &= \alpha\cdot P + \beta m & \bar{H}(P) &= -\bar{\alpha}\cdot P + \bar{\beta} M \\ E(P) &= \sqrt{P^2 + m^2} & \bar{E}(P) &= \sqrt{P^2 + M^2} \\ \Lambda_{\pm}(P) &= \frac{E(P) \pm H(P)}{2E(P)} & \bar{\Lambda}_{\pm}(P) &= \frac{\bar{E}(P) \pm \bar{H}(P)}{2\bar{E}(P)} \end{aligned} \qquad \text{(IIIA:16)}$$

$$\left\{\Lambda_+(P) + \Lambda_-(P)\right\}\left\{\bar{\Lambda}_+(P) + \bar{\Lambda}_-(P)\right\} = 1$$

The Λ's are projection operators, by using them we may replace H (P) by $\pm$ E (P) and likewise for $\bar{H}$ (P). Now let us multiply

equation (IIIA:11) (for the case $G_\Delta = 0$) by $\beta\,\bar{\beta}$ and abbreviate it as follows:

$$\psi = \frac{-1/(2\pi i)}{F(W, P, \epsilon)} \mathcal{G}_0 \psi \qquad \text{(IIIA:17)}$$

using $\mathcal{G}$ as a symbol for the integral operator and

$$F = (\mu_r W - H(P) - \epsilon + i\,\underset{\sim}{\delta}\,\beta)\,(\bar{\mu}_r W - \bar{H}(P) + \epsilon + i\,\underset{\sim}{\delta}\,\bar{\beta}) \qquad \text{(IIIA:18)}$$

$$\mathcal{G}_0 \psi = e^2 \int \frac{d^4 q'}{Q'^2}\, \psi(q-q') \qquad \text{(IIIA:19)}$$

We assume that when the term $\mathcal{G}_\Delta$ is added, W undergoes a small shift and becomes $W + \delta$ and ψ becomes $\psi + R$. Thus*

$$\psi + R = \frac{-1/(2\pi i)}{F(W + \delta, P, \epsilon)} (\mathcal{G}_0 + \mathcal{G}_\Delta)(\psi + R) \qquad \text{(IIIA:20)}$$

Now let us mutliply by the projection operators of IIIA:16 and integrate out the relative energy, ϵ.

$$\int \psi + R\, d\epsilon = \varphi + r = \frac{-1}{2\pi i} \int \frac{\{\Lambda_+\bar{\Lambda}_+ + \Lambda_-\bar{\Lambda}_+ + \Lambda_+\bar{\Lambda}_- + \Lambda_-\bar{\Lambda}_-\}}{F}$$

$$\{\mathcal{G}_0 + \mathcal{G}_\Delta\}\,(\psi + R)\, d\epsilon$$

$$= \frac{(\Lambda_+\bar{\Lambda}_+ - \Lambda_-\bar{\Lambda}_-)}{\delta + W - H(P) - \bar{H}(P)} \mathcal{G}_0(\varphi + r) - \frac{1}{2\pi i} \int d\epsilon\, \frac{1}{F}\, \mathcal{G}_\Delta (\psi + R) \qquad \text{(IIIA:21)}$$

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*In this appendix we have explicitly shown $\underset{\sim}{\delta}$ and $\underset{\sim}{\epsilon}$ which serve to define the contour. In the usual way we want the limit as $\underset{\sim}{\delta}, \underset{\sim}{\epsilon} \longrightarrow 0$. Of course these quantities have no connection with the perturbation energy shift, δ, or the fourth component of p, ϵ. Elsewhere (section III and appendix IV) we omit the subscript $\sim$ since δ appears only as a "contour fixer".

where we use IIIA:13 and $\int R\, d\epsilon = r$. The poles are such that for an instantaneous kernel, G_o, only the $\Lambda_+\bar{\Lambda}_+$ and $\Lambda_-\bar{\Lambda}_-$ projections are non-zero in the three dimensional wave function . That is:

$$\varphi = \frac{(\Lambda_+\bar{\Lambda}_+ - \Lambda_-\bar{\Lambda}_-)}{(W - H - \bar{H})} G_o \varphi \qquad \text{(IIIA:22)}$$

By comparison with IIIA:19 we see that the dependence of ψ on ϵ is trivial:

$$\psi = -\frac{1}{2\pi i} \frac{1}{F(W, P, \epsilon)} G_o \varphi \qquad \text{(IIIA:23)}$$

Now let us multiply IIIA:21 by $\varphi^* (\delta + W - H - \bar{H}) (\Lambda_+\bar{\Lambda}_+ - \Lambda_-\bar{\Lambda}_-)$ and integrate over P. The normalization condition is:

$$\int \varphi^* (\Lambda_+\bar{\Lambda}_+ - \Lambda_-\bar{\Lambda}_-) \varphi = 1 \qquad \text{(IIIA:24)}$$

Thus:

$$\int \varphi^* (\delta + W - H - \bar{H}) (\Lambda_+\bar{\Lambda}_+ - \Lambda_-\bar{\Lambda}_-) (\varphi + r)$$
$$= \int \varphi^* (\Lambda_+\bar{\Lambda}_+ + \Lambda_-\bar{\Lambda}_-) G_o (\varphi + r) - \frac{1}{2\pi i} \int \varphi^* (\delta + W - H - \bar{H})$$
$$(\Lambda_+\bar{\Lambda}_+ - \Lambda_-\bar{\Lambda}_-) \frac{d\epsilon}{F} G_\Delta (\psi + R) \qquad \text{(IIIA:25)}$$

Now since the operators in IIIA:22 are hermitian, for any function, Z

$$\int \varphi^* (W - H - \bar{H}) (\Lambda_+\bar{\Lambda}_+ - \Lambda_-\bar{\Lambda}_-) Z = \int \varphi^* G_o Z \qquad \text{(IIIA:26)}$$

clearly $(\Lambda_+\bar{\Lambda}_+ + \Lambda_-\bar{\Lambda}_-) \varphi = \varphi$

Then $\delta \int \varphi^* (\Lambda_+\bar{\Lambda}_+ - \Lambda_-\bar{\Lambda}_-) (\varphi + r)$

$$= -\frac{1}{2\pi i} \int \varphi^* (\Lambda_+\bar{\Lambda}_+ - \Lambda_-\bar{\Lambda}_-) (\delta + W - H - \bar{H})$$
$$\frac{d\epsilon}{F} G_\Delta (\psi + R) \qquad \text{(IIIA:27)}$$

We now neglect quantities of the "second order of smallness" in IIIA:27. Thus δ on the right side and δ in F are dropped. We shall retain R however. On the left side we neglect r since it gives only a small change in normalization. Using IIIA:26 then gives:

$$\delta = \frac{-1}{2\pi i} \int \varphi^* \mathcal{G}_o \frac{d\epsilon}{F} \mathcal{G}_\Delta (\psi + R)$$

and by IIIA:23 we get

$$\delta = \int \tilde{\psi} \mathcal{G}_\Delta (\psi + R) \qquad \text{(IIIA:28)}$$

Where $\tilde{\psi}$ is defined as the transpose of IIIA:23 with φ replaced by φ^* and no other terms complex conjugated. If we take the iteration of IIIA:20 as an approximation to R we get

$$\delta = \int \tilde{\psi} \mathcal{G}_\Delta \frac{-1/2\pi i}{F} (\mathcal{G}_o + \mathcal{G}_\Delta) \psi \qquad \text{(IIIA:29)}$$

This is our perturbation formula. It is really a "first order" formula in spite of the retention of the term R. This is because the kernel of $\mathcal{G}_\Delta$ is usually highly singular. Such behavior modifies the high-momentum components of the wave function which, in turn, give the major contribution to the integral IIIA:28. Thus R is retained in IIIA:28 however smaller terms such as δr and δ^2 may be dropped because they depend less critically on the high-momentum "tail" of the wave function.

Appendix IV Details of the BS Method

The G^x operators for a point proton. Momenta are the same as in Section III, equation 22. i and j run over the two mutually orthogonal directions which are normal to K and K' respectively.

$$G^x_{QD} = \left[2\left(\frac{e^2}{2\pi^2}\right)^2\right]\int\frac{d^4k}{2\pi i\ k^2}\ \frac{1}{K'^2}\sum_i \alpha_i \left(\frac{1}{\mu_r W - H''' + \varepsilon'' - \omega + i\delta\beta}\right)\ \bar{A}_4(K')$$

$$\left\{\frac{1}{\bar{\mu}_r W - \bar{H}' - \omega - \varepsilon + i\delta\,\bar{\beta}}\right\}\ \bar{\alpha}_i$$

$$G^x_{CP} = \left[\cdots\right]\int\frac{d^4k}{2\pi i\ k^2}\ \frac{1}{K'^2}\ \sum_i \alpha_i\ \left(\cdots\right)\ \left\{\cdots\right\}\ \bar{A}_i(k)$$

$$G^x_{QP} = \left[\cdots\right]\int\frac{d^4k}{2\pi i\ k^2}\ \frac{1}{K'^2}\ \sum_i \alpha_i\ \left(\cdots\right)\ \bar{A}_4(K')\ \left\{\cdots\right\}\ \bar{A}_i(k)$$

$$G^x_{DD} = \left[\cdots\right]\int\frac{d^4k}{2\pi i\ k^2}\ \frac{1}{k'^2}\sum_{ij} \alpha_i\ \left(\cdots\right)\ \alpha_j\bar{\alpha}_j\ \left\{\cdots\right\}\ \bar{\alpha}_i$$

$$G^x_{DP} = \left[\cdots\right]\int\frac{d^4k}{2\pi i\ k^2}\ \frac{1}{k'^2}\sum_{ij} \alpha_i\ \left(\cdots\right)\ \alpha_j\bar{\alpha}_j\ \left\{\cdots\right\}\ \bar{A}_i(k)$$

$$G^x_{PP} = \left[\cdots\right]\int\frac{d^4k}{2\pi i\ k^2}\ \frac{1}{k'^2}\sum_{ij} \alpha_i\ \left(\cdots\right)\ \alpha_j\bar{A}_j(k')\ \left\{\cdots\right\}\ \bar{A}_i(k)$$

The expectation values for the Dirac operators between rest spinors, averaged over solid angles of the vector K. We have defined (see Appendix III);

$$\Lambda_\xi(P) = \frac{E(P) + \xi(\alpha\cdot P + \beta m)}{2E(P)} \qquad \bar{\Lambda}_\eta(P) = \frac{\bar{E}(P) + \eta(-\bar{\alpha}\cdot P + \beta M)}{2\,\bar{E}(P)}$$

Where ξ, η, are + or − , Our notation is ^{x} is crossed and ^{o} is uncrossed.

Term	Operator	Reduced Operator
CD^{0}	$\left\langle \sum_{i} \alpha_i \bar{\alpha}_i \Lambda_{\xi}(K)\ \bar{\Lambda}_{\eta}(K) \right\rangle$	$\xi\eta\,\sigma\cdot\bar{\sigma}\,\frac{K^2}{6E(K)\bar{E}(K)}$
CD^{x}	$\left\langle \sum_{i} \alpha_i \Lambda_{\xi}(-K)\ \bar{\Lambda}_{\eta}(K) \bar{\alpha}_i \right\rangle$	$\xi\eta\,\sigma\cdot\bar{\sigma}\,\frac{K^2}{6E(K)\bar{E}(K)}$
DD^{0}	$\left\langle \sum_{ij} \alpha_i \bar{\alpha}_i \Lambda_{\xi}(K)\ \bar{\Lambda}_{\eta}(K)\ \alpha_j \bar{\alpha}_j \right\rangle$	$-\,\sigma\cdot\bar{\sigma}\,\frac{(E(K)-\xi m)(\bar{E}(K)-\eta M)}{6E(K)\bar{E}(K)}$
DD^{x}	$\left\langle \sum_{ij} \alpha_i \bar{\alpha}_j \Lambda_{\xi}(-K)\ \bar{\Lambda}_{\eta}(K) \alpha_j \bar{\alpha}_i \right\rangle$	$\sigma\cdot\bar{\sigma}\,\frac{(E(K)-\xi m)(\bar{E}(K)-\eta M)}{6E(K)\bar{E}(K)}$
QD^{0}	$\frac{\mu}{2M}\left\langle \sum_{i} \alpha_i \bar{\alpha}_i \Lambda_{\xi}(K)\ \bar{\Lambda}_{\eta}(K)\ \overline{\beta\alpha}\cdot K \right\rangle$	$\frac{\mu}{2M}\,\sigma\cdot\bar{\sigma}\,\frac{K^2(E(K)-\eta M)\xi}{6E(K)\bar{E}(K)}$
QD^{x}	$\frac{\mu}{2M}\left\langle \sum_{i} \bar{\alpha}_i \Lambda_{\xi}(-K)\ \bar{\Lambda}_{\eta}(K) \alpha_i \overline{\beta\alpha}\cdot K \right\rangle$	$\frac{\mu}{2M}\,\sigma\cdot\bar{\sigma}\,\frac{K^2(E(K)-\eta M)\xi}{6E(K)\bar{E}(K)}$
CP^{0}	$\frac{\mu}{2M}\left\langle \sum_{i} \alpha_i \bar{\alpha}_i \overline{\beta K} \Lambda_{\xi}(K)\ \bar{\Lambda}_{\eta}(K) \right\rangle$	$-\frac{\mu}{2M}\,\sigma\cdot\bar{\sigma}\,\frac{\xi K^2(\bar{E}(K)+\eta M+\eta\omega)}{6E(K)\bar{E}(K)}$
CP^{x}	$\frac{\mu}{2M}\left\langle \sum_{i} \bar{\alpha}_i \overline{\beta K} \Lambda_{\xi}(-K)\ \bar{\Lambda}_{\eta}(K)\ \alpha_i \right\rangle$	$-\frac{\mu}{2M}\,\sigma\cdot\bar{\sigma}\,\frac{\xi K^2(\bar{E}(K)+\eta M+\eta\omega)}{6E(K)\bar{E}(K)}$
QP^{0}	$\left(\frac{\mu}{2M}\right)^2\left\langle \sum_{i} \alpha_i \bar{\alpha}_i \overline{\beta K} \Lambda_{\xi}(K)\ \bar{\Lambda}_{\eta}(K) \overline{\beta\alpha}\cdot K \right\rangle$	$-\left(\frac{\mu}{2M}\right)^2\frac{\sigma\cdot\bar{\sigma}\,\xi K^2(\omega+\eta\bar{E}+M)(\bar{E}-\eta M)}{6E\bar{E}}$
QP^{x}	$\left(\frac{\mu}{2M}\right)^2\left\langle \sum_{i} \bar{\alpha}_i \overline{\beta K} \Lambda_{\xi}(-K)\ \bar{\Lambda}_{\eta}(K) \alpha_i \overline{\beta\alpha}\cdot K \right\rangle$	$-\left(\frac{\mu}{2M}\right)^2\frac{\sigma\cdot\bar{\sigma}\,\xi K^2(\omega+\eta\bar{E}+M)(\bar{E}-\eta M)}{6E\bar{E}}$
PD^{0}	$\frac{\mu}{2M}\left\langle \sum_{ij} \alpha_i \bar{\alpha}_i \overline{\beta K} \Lambda_{\xi}(K)\ \bar{\Lambda}_{\eta}(K) \alpha_j \bar{\alpha}_j \right\rangle$	$\frac{\mu}{2M}\,\frac{\sigma\cdot\bar{\sigma}}{6E\bar{E}}\,(E-\xi m)(\bar{E}-\eta M)(\omega+\eta\bar{E}+M)$

Continued

Term	Operator	Reduced Operator
PD^X	$\frac{\mu}{2M}\left\langle \sum_{ij} \bar{\alpha}_j \bar{K} \alpha_i \Lambda_\xi(-K)\, \bar{\Lambda}_\eta(K) \bar{\alpha}_i \alpha_j \right\rangle$	$-\frac{\mu}{2M}\frac{\sigma\cdot\bar{\sigma}}{6E\bar{E}}(E-\xi m)(\bar{E}-\eta M)(\omega+\eta\bar{E}+M)$
PP^O	$\left(\frac{\mu}{2M}\right)^2\left\langle \sum_{ij} \alpha_i \bar{\alpha}_i \bar{K} \Lambda_\xi(K)\, \bar{\Lambda}_\eta(K) \alpha_j \bar{\alpha}_j \bar{K} \right\rangle$	$\left(\frac{\mu}{2M}\right)^2\frac{\sigma\cdot\bar{\sigma}}{6E\bar{E}}(E-\xi m)(E-\eta M)(\omega+\eta E+M)^2$
PP^X	$\left(\frac{\mu}{2M}\right)^2\left\langle \sum_{ij} \alpha_i \bar{\alpha}_j \bar{K} \Lambda_\xi(-K)\, \bar{\Lambda}_\eta(K) \alpha_j \bar{\alpha}_i \bar{K} \right\rangle$	$-\left(\frac{\mu}{2M}\right)^2\frac{\sigma\cdot\bar{\sigma}}{6E\bar{E}}(E-\xi m)(E-\eta M)(\omega+\eta E+M)^2$

Where we have used;

$$E = E(K) = \sqrt{K^2+m^2} \qquad \bar{E} = \bar{E}(K) = \sqrt{K^2+M^2}$$

Expressions for $T^X_{\alpha\beta}$

$$a = \frac{\Lambda}{M} \qquad b^2 = a^2 - \frac{a^4}{4} \qquad \mu = \mu_p - 1 \approx 1.79$$

$$T^X_{CD} = 2a^2\int_0^\infty \frac{x\,dx}{\sqrt{x^2+a^2}\,(x^2+b^2)\left(x+\sqrt{x^2+a^2}\right)} - \int_0^\infty \frac{(a^2-2)\,dx}{(x^2+b^2)\sqrt{x^2+1}}$$

$$+\int_0^\infty \frac{2\,dx}{(x^2+b^2)\left(x+\sqrt{x^2+1}\right)}$$

$$T^X_{QD} = \frac{\mu}{2}T^X_{CD} - \frac{\mu}{2}\int_0^\infty \frac{(4-a^2)\,dx}{(x^2+b^2)\left(x+\sqrt{x^2+1}\right)}$$

$$+\int_0^\infty \frac{\mu a^2 x\,dx}{(x^2+b^2)\sqrt{x^2+a^2}\left(x+\sqrt{x^2+a^2}\right)} - \mu\int \frac{x\,dx}{x^2+b^2}\left(1-\frac{2x^2}{\sqrt{x^2+a^2}\left(x+\sqrt{x^2+a^2}\right)}\right)$$

$$T^X_{CP} = T^X_{CD}$$

$$T^{\chi}_{QP} = \mu T^{\chi}_{QD} - 2\mu^2 \int_0^{\Lambda_0/M} \frac{x\,dx}{(x+\sqrt{x^2+a^2})\sqrt{x^2+a^2}}$$

$$T^{\chi}_{DD} = -\int_0^{\infty} \frac{x^2\,dx}{(x+\sqrt{x^2+a^2})(x^2+b^2)} + \frac{1}{2}\int_0^{\infty} \frac{x\,dx}{x^2+b^2}$$

$$+\frac{1}{4}\int_0^{\infty} \frac{(2-a^2)\,x\,dx}{(x^2+b^2)\sqrt{x^2+1}\,(x+\sqrt{x^2+1})}$$

$$T^{\chi}_{DP} = 2\mu\, T^{\chi}_{DD}$$

$$T^{\chi}_{PP} = \mu^2\, T^{\chi}_{DD} - \frac{\mu^2}{a^2}\int_0^{\Lambda_0/M} dx \left\{\frac{x}{2} - \frac{x^2}{x+\sqrt{x^2+a^2}}\right\}$$

APPENDIX V

The Meson Corrections to the hfs

This appendix describes a rough calculation of the effect of meson corrections on the energy shift found in part IV. It estimates the extent to which the two-photon form factor differs from the product of two single-photon form factors.

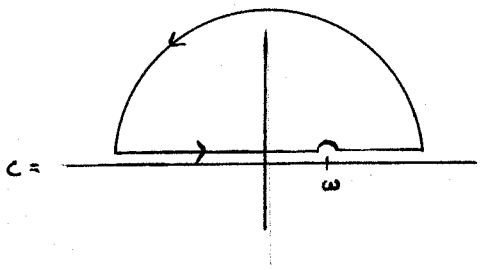
We write the unitarity relation on the S matrix as:

$$(f|S|i) = \delta_{fi} + \delta^4(p_f - p_i)(f|M|i)$$

$$(i|M|f)^* + (f|M|i) = -\sum_n (n|M|f)^*(n|M|i)\delta^4(n-i) \qquad \text{(VA.1)}$$

Let us suppose, for the moment, that we know all the $(n|M|i)$ and $(n|M|f)$. Then, by performing the sum in VA.1, we obtain $(f|M^+ + M|i)$. This is essentially the "absorptive part" of $(f|M|i)$. (For example, in a photon scattering problem, if we write $(f|M|i) = if(\omega)\vec{e}\cdot\vec{e}'$ where e and e' are the initial and final polarization vectors for the photon, then the left hand side of VA.1 is just $-2\vec{e}\cdot\vec{e}'$ Im $f(\omega)$.) We now assume that $f(\omega)$ satisfies some kind of a dispersion relation:

$$1/2\pi i \int_C d\omega'/\omega-\omega' \; f(\omega') = 0 \qquad \text{(VA.2)}$$



That is, $f(\omega)$ is analytic in the complex ω plane and vanishes fast enough at ∞ so that only the portion of the contour along the real axis need be retained. Then, knowing Im $f(\omega)$ for all real ω, we may use VA.2 to find $f(\omega)$ and hence (fMi).

$$f(\omega) = 1/\pi \int_{-\infty}^{\infty} \overline{d\omega'} \; \text{Im } f(\omega') \Big/ \omega-\omega'-i\epsilon \qquad \text{(VA.3)}$$

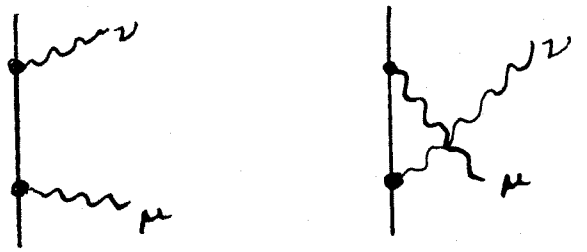
In parts II and IV we found that the energy shift was given by the M-matrix element for i as one virtual photon and one real proton and f the same particles, after they had scattered elastically. Actually, it was sufficient to know only the forward direction, spin flip scattering in the labratory frame (proton at rest). We approximated this by $(f|M^{(2)}|i)\ F^2(k^2)$ where $M^{(2)}$ is the Born approximation (point proton) to Compton scattering and F is the Hff. We restate our final result:

$$E = (\alpha m/\pi\mu_p M)(hfs)\ 3i/32\pi^2 \int d^4k/k^4 \qquad (A^o_{\mu\nu} + A^x_{\mu\nu})B^o_{\mu\nu} \qquad \text{(VA.4)}$$

where o denotes "uncrossed" and x crossed and where:

$$(A^o_{\mu\nu} + A^x_{\mu\nu}) = \overline{2\pi\omega M_p/\alpha i} \text{ trace } \overline{M^{(2)}_{\mu\nu}(1+\gamma_t)i\gamma_x\gamma_y} \qquad \text{(VA.4')}$$

$A^o_{\mu\nu}$ and $B^o_{\mu\nu}$ are given in appendix II. $M^{(2)}_{\mu\nu}$ is the M matrix element for Compton scattering of a virtual photon by the proton and M_p is the proton mass, introduced so as to make A dimensionless. Although in section II we simply regarded $M^{(2)}$ as the sum of the two diagrams of figure 11, we can also understand $M^{(2)}$ as coming from formulae VA.1 and VA.3 if we take the state n as a single proton (or as a photon and a proton).



Ball indicates Hff

Figure 11

The effective diagram is shown in figure 12a and the details are given in section C of this appendix and in reference 17.

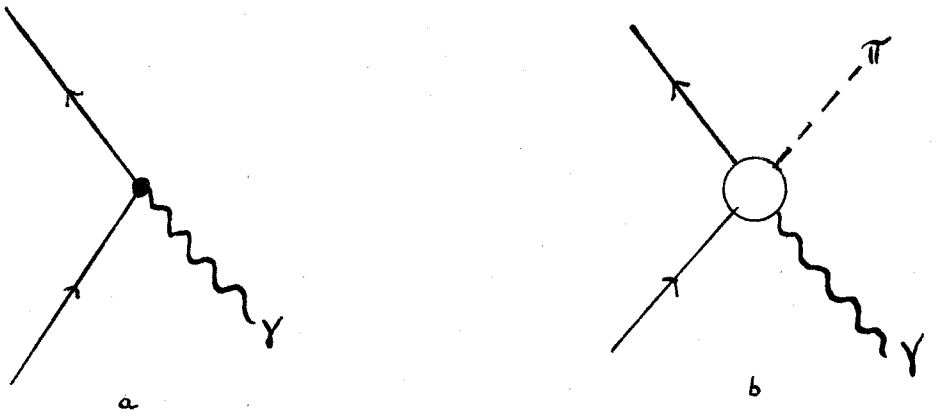


Figure 12

The process is unphysical since no real photon can ever be absorbed by a real proton, however by formally including it and other "forbidden" processes, we get the perturbation series result. Of course, the proton does not exist in the isolated atmosphere of a self-consistent and self-contained electrodynamics. Thus with a photon and a proton, we can make a meson (fig. 12b) and clearly this should be included in the sum over intermediate states in VA.1. So far we have completely neglected most of the dynamic effects of this intermediate state and (by the addition of the Hff and anomalous moment) taken into account only the term in VA.1 where n is a real proton. If perturbation theory worked, we would simply include another diagram such as figure 13; however, since the coupling is too strong for this, we shall try to use dispersion methods to introduce experimental results for figure 12b and thus guess the value of figure 13.

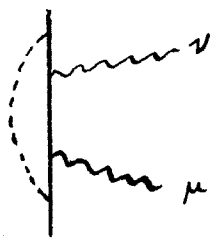


Figure 13

We shall choose an empirical (nMi) for the process $\gamma + p \rightarrow n + \pi$ and find the resulting (fMi) by assuming a dispersion integral. We may then compute the additional shift in H by VA.4.

There are several rather weak points in this method which must be mentioned. First, the experiments are only over a limited range and we shall extrapolate them outside this region. In fact, photons of non-zero rest mass are produced experimentally[18] by the scattering of high energy electrons (see fig. 14) and thus are "space-like." We require time-like masses in VA.4 (and space-like ones).

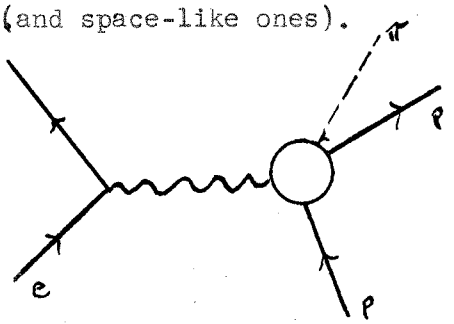


Figure 14

Panofsky's Pion Production by Virtual Photons

Second, a photon with enough energy can create much more than a single π, yet we shall omit such processes in the sum VA.1. We shall consider only the (3/2,3/2) resonance since it is well-measured and dominates the low- and middle-energy cross sections. At high energies this will clearly be wrong; however, as we shall argue later, it is not unreasonable that these corrections are small for the hfs problem. The last difficulty is the number and value of possible "subtractions." On the basis of various examples, it has been conjectured that a scattering amplititude, $f(\omega)$, may not satisfy VA.3, but that $\overline{f(\omega) - f(o)}$ or $\overline{f(\omega) - f(o) - \omega f'(o)}$ does. If we apply VA.3 to the first expression we find in place of VA.3,

$$f(\omega) = f(o) + \omega/\pi \int [\mathrm{Im} f(\omega')d\omega'] / (\omega'+i\epsilon)(\omega-\omega'-i\epsilon) \qquad (VA.5)$$

This is a "subtracted" dispersion relation. We know of no way, at present, of fixing the form of the dispersion relation or the values of their constants, f(o), etc. We therefore compute the energy shift in two cases, with and without a subtraction, and note that the results are the same for our purposes. This disposes of the problem in a very crude way and our answer is not to be regarded as more than an estimate.

In spite of the fact that we talk of "dispersion relations," it would probably be fairer to regard this calculation as a perturbation estimate of the contribution of an "isobar graph," containing a 3/2, 3/2 particle (fig. 15), rather than as an application of dispersion relations. We have used the language of dispersion relations because we believe it is simpler than the formalism of spin-3/2 particles.

This calculation is divided into several parts and follows very closely this outline.

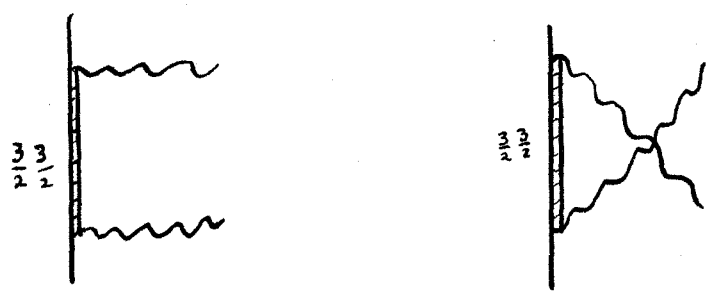


Figure 15

The Isobar Model

A. The Expression for (nMi)

At first, we shall assume only real photons are involved and later remove this restriction. Gell-Mann and Watson[19] have fitted the

total cross section for (3/2, 3/2) photoproduction by the expression:

$$\sigma = 3\pi/K^2 \ \Gamma_\gamma \ \Gamma / \{(W-W_o)^2+\Gamma^2/4\} \qquad \text{(VA.6)}$$

where K is the photon momentum, W the total energy in the center of mass and $W_o \approx 1239$ Mev. We shall neglect the width, Γ, of this cross section and put

$$\sigma = 6\pi^2/K^2 \ \Gamma_\gamma \ \delta(W-W_o) \qquad \text{(VA.7)}$$

We take Γ_γ as constant and equal to 0.1 Mev. Assuming that all the (3/2, 3/2) production is due to the absorption of M1 radiation, the form of M must be:

$$(nMi) = \left\{ 2\overline{Kxe\cdot Q} + i\overline{\sigma\cdot(Kxe)xQ} \right\} f(W)/|K|\,|Q| \qquad \text{(VA.8)}$$

where e is the photon polarization vector and Q the meson momentum. Now the cross section is related to M by:

$$\sigma = (2\pi)^2 \ E_1/W \ \sum_n \ \overline{\sum_i} \ |(nMi)|^2 \delta(n-i) = 4(2\pi)^3 \ E_1E_2q_o|Q|/W^2 |f(W)|^2 \qquad \text{(VA.9)}$$

E_1 and E_2 are initial and final nucleon energies; q_o, the meson energy Thus:

$$|f(W)|^2 = \{W^2/q_o \ Q \ E_1E_2\}(3/16\pi) \ \Gamma_\gamma \ \delta(W-W_o)/K^2 \qquad \text{(VA.10)}$$

Both electric quadrupole and longitudinal quadrupole (for virtual photons) can also lead to the (3/2,3/2) state; we ignore the first possibility because the static-theory coefficient of the M1 term, $(\mu_p - \mu_n)$, is much larger than that of the E2 (pair) term, and we neglect the second because no reliable measurements are available. We shall now apply VA.8 and 10 to virtual photons by multiplying by a Hff.

$$(nMi) = \frac{F(k^2)}{|K||Q|}\left\{2[\vec{K}xe\cdot\vec{Q}] + i\sigma\cdot[(Kxe)x\vec{Q}]\right\}\sqrt{\frac{W^2\Gamma_\gamma}{q_0 Q E_1 E_2}\,\frac{\delta(W-W_0)}{K^2}\,\frac{3}{16\pi}} \qquad (VA.11)$$

Multiplication by F is certainly correct for the so called "bound state term" in photoproduction (see reference 20) but it is not required as a factor in the entire matrix element. The experiments of Panofsky and Alton on high energy electro-pion production[18] in the neighborhood of the (3/2,3/2) resonance can be fitted rather well by such a factor. We shall also investigate the effect of omitting the Hff.**

B. The Expression for (fMi) + (iMf)*

We choose

$$[(f|M|i) = i\,A(W,k^2)(\vec{e}_\perp\cdot\vec{e}'_\perp) + B(W,k^2)\,[\vec{\sigma}\cdot\vec{e}_\perp x\vec{e}'_\perp] \qquad (VA.12)$$

where $e_\perp$ and $e'_\perp$ are the transverse parts of the polarization vectors. This choice of invariants will turn out to be sufficient since we require only the forward scattering and we are limiting ourselves to M1 production (which is clearly transverse). Using VA.11 and

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** M. Gell-Mann has suggested the following approach as a means of including relativistic retardation effects in the determination of (nMi). First, compute the lowest order Born approximation to pion production in the straight γ_5 theory. Then expand this in the center of mass in angular momentum and isotopic spin eigenstates. Take the M1:3/2,3/2 term and give it an empirical resonance coefficient which matches the experimental cross section or at least fits the data near the resonance. The nucleon propagation demoninator from the Born approximation will give the retardation. On the other hand, the expansion in Legendre polynomials, which is required in order to separate the M1 3/2 piece, does not converge in the neighborhood of the pole (where the propagator vanishes). The coefficients in the expansion turn out to be multiple-valued for complex ω. Although recent work[21] indicates that dispersion relations may be possible for a given state of angular momentum such as this, we shall not use them.

$$\sum_n \delta^4(n-i) = |Q| q_o E_2/W \, d\Omega_Q \sum_{\text{spins of } n} \qquad \text{(VA.13)}$$

in VA.1 we find

$$\text{Im } B = F^2 \, 3/8 \, W/E_1 \, \overline{[} \Gamma_\gamma \, \delta(W-W_o) \overline{]}/K^2 = 1/2 \text{ Im } A \qquad \text{(VA.14)}$$

This answer is clearly independent of the unknown phase of VA.11 because of the structure of VA.1. We understand in VA.14 that all quantities are expressed in terms of W and k^2 as implied by VA.12. We wish to apply a dispersion relation to find the real parts of A and B; however we know their imaginary parts only in the C.M. Dispersion relations have been postulated to hold in the laboratory and we shall employ them in this frame. The transformation of the scattering amplitude, VA.14, to the laboratory frame is immediate because all that is required is a relative translation along the direction of K. Since $e_\perp$ and $e'_\perp$ are orthogonal to K, they are the same in both systems. Clearly $\overline{[} e_\perp \times e'_\perp \cdot \sigma \overline{]}$ selects that component of spin parallel to K. Spin in the direction of motion is unchanged by a Lorentz transformation (in the same direction). Thus

$$\overline{[} \vec{e}_\perp \times \vec{e}'_\perp \cdot \sigma \overline{]}_{lab} = \overline{[} \vec{e}_\perp \times \vec{e}'_\perp \cdot \sigma \overline{]}_{cm}$$

and we then find:

$$(fMi)_{lab} = i(\vec{e}_\perp \cdot \vec{e}'_\perp) A_{lab}(\omega_L, k^2) = \overline{[} \vec{e}_\perp \times \vec{e}'_\perp \cdot \sigma \overline{]} B_{lab}(\omega_L, k^2) \qquad \text{(VA.15)}$$

$$\begin{aligned} \text{Im } B_{lab} &= t \text{ Im } B_{cm} \\ \text{Im } A_{lab} &= t \text{ Im } A_{cm} \end{aligned} \qquad \text{(VA.16)}$$

$$t(\omega_L, k^2) = E_1^{cm} \omega^{cm} / M \omega_L \qquad \text{(VA.17)}$$

Let us take W_o as the cm resonance energy, W_μ as the total energy-momentum of the system and ω_L as the frequency of the photon in the laboratory system. We now express everything in terms of ω_L and k^2, the invariant photon mass.

$$W_{cm} = \sqrt{2\omega_L M + M^2 + k^2} \qquad \text{(VA.18)}$$

$$\omega_{cm} = (\omega_L M + k^2)/W_{cm} \qquad \text{(VA.19)}$$

$$E_1^{cm} = (\omega_L M + k^2)/W_{cm} \qquad \text{(VA.20)}$$

W_4, lab $= \omega_L + M$ $\qquad\qquad$ $\vec{K}^2_{cm} = (\omega_L^2 - k^2)M^2/W^2_{cm}$

We shall apply dispersion relations to $\omega_L B$ and not to B directly since we wish to remove the singularity in the M-matrix at $\omega = 0$ which is due to the normalization of the wave function. Using VA.14 and VA.16-20 we get:

$$\begin{aligned} \text{Im}\ \omega_L B_L &= 3/8\ F^2\ \Gamma_\gamma\ \delta(\omega_L - \omega_L^*)\ \frac{W_o^2}{M^2}\ \frac{\omega^*_{cm}}{\vec{K}^2_{cm}} \\ &= 3/8\ F^2\ \Gamma_\gamma\ \delta(\omega_L - \omega_L^*)\ \frac{W_o^2}{M^2}\ \frac{\sqrt{2\omega_L M + M^2 + k^2}\ (\omega_L M + k^2)}{M^2(\omega_L^2 - k^2)} \end{aligned} \qquad \text{(VA.21)}$$

where ω_L^* is the laboratory frequency corresponding to resonance in the cm.

$$\omega_L^* = \omega_L^*(k^2) = (W_o^2 - M^2 - k^2)/2M \qquad \text{(VA.22)}$$

We have made use of

$$\delta(W - W_o) = \delta(\omega_L - \omega_L^*)\left[d\omega_L/dW\Big|_{\omega^*}\right] = \delta(\omega_L - \omega_L^*)\ W_o/M$$

We do not understand either the meaning of the pole at $\vec{K}_L^2 = 0$ in VA.21 or the dependence on k^2 and ω^2 for $k^2 < 0$. Since the entire factor depends only weakly on k^2 and ω_L, away from the singularity, we shall put it equal to its value at resonance, $\omega = \omega_L^*$, for real photons, $k^2 = 0$. This simplifies VA.21 to:

$$\mathrm{Im}(\omega_L B_L) = \beta\, F^2\, \delta(\omega_L - \omega_L^*) \qquad \text{(VA.23)}$$

$$\beta = 3/4\ \Gamma_\gamma W_o^3/M^2(W_o^2 - M^2) \approx 2.5 \times 10^{-4}$$

The smallness of the dimensionless constant β is due to the narrowness of the width, Γ_γ, when measured in units of $\omega_{cm}^*(o)$.

C. The Dispersion Integrals

Our dispersion integrals are supposed to hold for fixed k^2 and variable ω_L. (Since we shall work entirely in the laboratory system from now on, we shall drop the subscripts L and cm.) We shall apply the following two relations to ωB.

$$\mathrm{Re}(\omega B) = 2\omega/\pi\ P\int_o^\infty \frac{\mathrm{Im}(\omega' B)}{\omega^2 - \omega'^2}\, d\omega' \quad \text{no subtraction} \qquad \text{(VA.24)}$$

$$\mathrm{Re}(\omega B) = \omega C + 2\omega^3/\pi\ P\int_o^\infty \frac{\mathrm{Im}(\omega' B)}{\omega'^2(\omega^2 - \omega'^2)}\, d\omega' \quad \text{subtracted} \qquad \text{(VA.25)}$$

The subtracted case has been given a constant, C, equal to zero. This implies that for low energies, the scattering is given by the Born approximation. The integrals have been expressed over the range o to ∞ on ω, the symmetry implied for the $\mathrm{Re}(\omega B)$ under the crossing substitution $\omega \to -\omega$ is $\mathrm{Re}[-\omega\ B(-\omega, k^2)] = -\mathrm{Re}[\omega\ B(\omega, k^2)]$. This follows from the invariance of the M-matrix under crossing and agrees with the Born approximation. Using VA.23 in VA.24 and 25 we get:

$$\mathrm{Re}(\omega B) = \left(2\omega\beta F^2/\pi\right) 1/\omega^2-\omega^{*2} \qquad \text{No subtraction} \qquad \text{(VA.26)}$$

$$\mathrm{Re}(\omega B) = \left(2\omega^3\beta F^2/\pi\right) \left[1/\omega^*(k^2)\right]^2 \; 1/\omega^2-\omega^{*2} \qquad \text{subtracted} \qquad \text{(VA.27)}$$

where β is defined in VA.23 and $\omega^* = \omega^*(k^2)$ in VA.22. The denominator, $\omega^2-\omega^{*2}$ is just $-\left[\bar{W}_o^2 -(M-\omega)^2 -\underline{K}^2\right]\left[\bar{W}_o^2 -(M+\omega)^2 -\underline{K}^2\right]/4M^2$ and corresponds to the combination of two propagators for an isobar of mass W_o which we would get from the diagrams of figure 15. For comparison, if we allow only the intermediate state n = proton, then the diagram 12a gives an imaginary part proportional to $\delta(\omega \pm k^2/2M)$ and the dispersion integrals give the propagators:

$$\frac{1}{\omega^2-\omega^{*2}} = \frac{(2M)^2}{(2M\omega-k^2)(2M\omega+k^2)^2}$$

With the correct proportionality constant for the imaginary part, i.e. $(nMi) = e^2(\bar{u}_n \left|\Gamma_\mu F\right| u_i)$ and subtractions, this does indeed give the lowest Born approximation to forward scattering for the interaction $e^2\Gamma_\mu F$. We therefore argue that the poles in $\omega^2-\omega^{*2}$ are to be taken in the sense of the poles in the Born approximation, $W \to W-i\epsilon$, when the amplitudes VA.26 and 27 are used in VA.4. As has been discussed elsewhere[17,22], these Feynman amplitudes no longer obey a dispersion relation based on VA.2 since they have poles in the upper half plane. This simply corresponds to a different analytic continuation of the scattering amplitude and seems clearly indicated, since if there really were an isobar, it would come into VA.4 (in perturbation theory) with "Feynman poles." The dispersion integral then generates the real part from an "incorrect" imaginary part which we modify later. The other pole in VA.27 we omit by evaluating it at $k^2 = 0$. Thus:

$$\mathrm{Re}(\omega B) = \frac{2\omega^3 \beta F^2}{\pi \frac{W_o^2 - M^2}{2M}} \frac{1}{\omega^2 - \omega^{*2}} \qquad \text{subtracted} \qquad \text{(VA.27')}$$

This ignores more kinematics but, since the integrals are being cut off by the propagation denominator this suppressed factor effectively does little more than modify the Hff, F^2.

D. Expression for the Energy Shift

Suppose that the entire (virtual photon) Compton scattering matrix element is M. To order α^2 we write:

$$M = M^{(2)} + M_\pi \qquad \text{(VA.28)}$$

where $M^{(2)}$ is the scattering given by assuming only the graphs of figure 11 and M_π is the additional part of M arising from the isobar graphs of figure 15. Similarly, we suppress the subscripts μ, ν and write $A_{\mu\nu}$ of equation VA.4 as

$$A = A^{(2)} + A_\pi \qquad \text{(VA.29)}$$

Using VA.4' we have:

$$A = (A^{(2)o} + A^{(2)x}) + \frac{2\pi\omega M_p}{i\alpha} \mathrm{trace}\left\{ \not{e}'_\perp \not{x} e_\perp \cdot \not{\sigma} \, (1+\gamma_t) i\gamma_x\gamma_y \right\} \qquad \text{(VA.30)}$$

The factor $(1+\gamma_t)/2$ just insures positive energies; we have already taken this into account by considering only such matrix elements. Since the proton is always at rest we may take the trace of $\not{e}'_\perp \not{x} e_\perp \cdot \not{\sigma}$ directly. We get:

$$\mathrm{Trace}\left\{ \not{e}'_\perp \not{x} e_\perp \cdot \not{\sigma} i\gamma_x\gamma_y \right\}_{\substack{e'=\nu \\ e=\mu}} = 4\vec{K}_z^2/\vec{K}^2 \; \epsilon(\mu, \nu, z) \qquad \text{(VA.31)}$$

where $\epsilon(ijk)$ is the antisymmetric 3 x 3 tensor and μ and ν run from 1 to 3. Because $B^{o}_{\mu\nu}$ (see equation VA.4 and Appendix II) is a gauge invariant, the time component need not be considered in a purely transverse matrix element such as VA.31 and VA.4 is then the contraction of μ, ν over space indices alone. Thus:

$$\sum_{\mu\nu} (A^{o}_{\mu\nu} + A^{x}_{\mu\nu}) B^{o}_{\mu\nu} = \frac{32\eta F^2}{k^2(k^2+2\omega)} + \frac{32i}{3k^2} \frac{2\pi\omega^2 M_p B}{i\alpha} \qquad \text{(VA.32)}$$

When substituted into VA.4, the first term gives the energy shift due to $M^{(2)}$ which we have already evaluated in part IV. The second term is an additive correction giving meson changes in the double vertex, in addition to those already included in M_E by the use of the Hff. We find an additional energy shift of:

$$E = \frac{\alpha m}{\pi\mu_p M} (hfs)\ i/\pi^2 \int \frac{d^4k}{k^4} \frac{1}{k^2} \frac{(2\pi\omega^2 M_p B)}{\alpha} \qquad \text{(VA.33)}$$

Measuring the energy shift in units of $(\alpha m/\pi\mu_p M)(hfs)$ and using VA.27' gives: (using $\Delta\mathcal{E}$ for the shift in these units)

$$\Delta\mathcal{E} = i/\pi^2 \int \frac{d^4k}{k^6} \frac{2\beta F^2}{\pi\alpha\, \omega^{*2}_{(o)}} \frac{2\pi\,\omega^4}{\omega^2-\omega^{*2}}$$

$$= \frac{4\beta}{\alpha\, \omega^{*2}_{(o)}} \frac{i}{\pi^2} \int \frac{d^4k}{k^6} F^2 \frac{\omega^3}{\omega+\omega^*} \qquad \text{(VA.34)}$$

We have again changed to a dimensionless momentum, k, such that the mass of the proton is unity. This means:

$$2\omega\,(k^2) = 2\omega^* = (W_o/M)^2 - 1 - k^2$$

$$2\omega^*(o) = (W_o/M)^2 - 1 \qquad \text{(VA.35)}$$

Using VA.26 we find that in the case of no subtractions:

$$\Delta\mathcal{E} = i/\pi^2 \int \frac{d^4k}{k^6} \frac{2\omega\beta F^2}{\pi\alpha} \frac{2\pi\,\omega}{\omega^2-\omega^{*2}}$$
$$= \frac{4\beta i}{\alpha\pi^2} \int \frac{d^4k}{k^6} \frac{\omega}{\omega+\omega^*} F^2 \qquad \text{(VA.36)}$$

This integral is logarithmically divergent because of the strong singularity at $k^2 = 0$! If we had retained the electron rest mass, this singularity would be absent because the electron propagator, $(k^2+2\omega m/M)^{-1}$ would replace one of the k^{-2} in VA.36. We therefore evaluate the no subtraction case as:

$$\Delta\mathcal{E} = \frac{2\beta i}{\alpha\pi^2} \int \frac{d^4k}{k^4} \frac{\omega F^2}{\omega+\omega^*} \left\{ \frac{1}{k^2-2m/M\,\omega} + \frac{1}{k^2+2m/M\,\omega} \right\} \qquad \text{(VA.37)}$$

We may remark that the device of using the electron rest mass is a rather unphysical convenience. It means that although the meson effects are strong at low frequencies (as opposed to the one subtraction case), the presence of an electron rest mass will make them finite. Intuitively, it seems clear that at low energies the Hff should characterize the proton completely and additional meson corrections should be quite small. Thus the divergence of VA.36 is perhaps an indication that one subtraction is necessary. Nevertheless, we shall consider VA.37 as one estimate of the meson corrections to the results of section IV. This result is probably far too large because of excessive low energy contributions.

E. The Results

We have placed upper bounds on the energy shifts VA.34 and

VA.37 as follows. By inspection of the singularities in the integrands, it is readily shown that the path of integration in the ω-plane may be rotated 90° to integration along $i\omega$ (fig. 16). Then the Hff is always less than or equal to 1 so that if we set it equal to 1 we obtain an upper bound on the energy shift. The integration corresponding to VA.34 without F^2 is straightforward. VA.37 is more difficult when done analytically because various terms tend to cancel. We have therefore estimated it numerically. Our results are:*

No Subtraction $|\Delta E| \leq 2.6$

Subtracted $|\Delta E| \approx 1/2$

where the units are $(\alpha m/\pi\mu_p M)$ hfs. As shown in part D of this appendix these shifts are to be added to the shift of 73 (same units) found in section IV. Hence these additional meson corrections may be safely ignored because our evaluation of the integrals of part III was accurate to only about 1%.

We now summarize our conclusions on the validity of this whole calculation. The energy shift of part II equation 16 is the most important term. The result obtained for the energy shift, from this term, does not depend appreciably on the approximate analytic form we have chosen for the Hff, except in so far as it is experimentally determined. We need only assume that the singularities of the true structure factor lie as shown in figure 16. For then, instead of evaluating the integral for ΔE (equation 16, part II) by integrating along the real ω axis, we can rotate the path of integration through 90 degrees in the complex ω plane so as to integrate along $i\omega$. When this is done, the resulting

* We have not been able to find an independent check on the sign of the energy shift and therefore state only the magnitude.

metric is positive definite, the structure factor is needed now only for negative k^2 and over 90% of the contribution to the integral comes from values of $-k^2$ in the range zero to 0.6 M^2 where $F(k^2)$ is directly measured.

The other question is what to do about higher mass intermediate states. There are two possibilities. If we postulate that all amplitudes for photoproduction by a virtual photon contain a factor $F(k^2)$ then we are justified in neglecting these higher-mass diagrams. The same rotation will be possible and the corrections will come from low-mass, low-energy photons. The contributions in this range are then correctly given by the low-mass intermediate states. Even the contribution of the(3/2, 3/2)resonance is small compared with that of the bare proton. Higher states are expected to be further damped. On the other hand, virtual photons may not act through $F(k^2)$ in all processes. In this appendix we have calculated the case of a single meson, single nucleon intermediate state, without using a Hff and found that even so, these corrections change our answer by only about 1%. We cannot rule out the unlikely possibility that higher mass diagrams do contribute a significant amount to the energy shift.

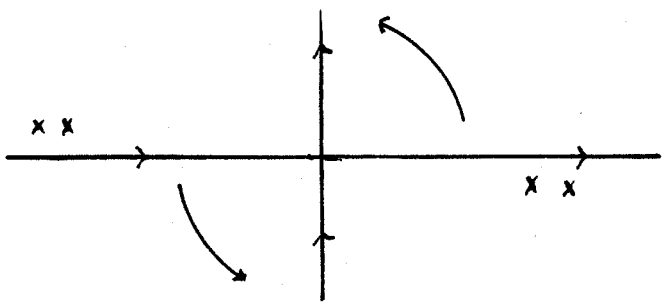

Figure 16

Assumed location of singularities in true structure factor