THE ANALYSIS OF STRESSES IN A THIN CYLINDRICAL SHELL OF CIRCULAR CROSS-SECTION

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In Partial Fulfilment of the

Requirements for the Degree of

Doctor of Philosophy

California Institute of Technology,

Pasadena, California,

1940

ACKNOWLEDGMENT

The author takes this opportunity to express his gratitude to Professor R. R. Martel for his advice and encouragement, so valuable in carrying out this work.

SUMMARY

In Part I., the equations of equilibrium and expressions for the strain components are set up, for a thin shell of a general shape, by the use of the methods of vector analysis. The simplicity of the vector method of approach to this problem is shown.

In Part II., the theory developed in Part I. is particularized for the case of a circular cylindrical shell. Expressions for force components are obtained in terms of the deformation components, and three equations of equilibrium in terms of the three deformation components are derived.

In Part III., the expression for strain energy for a circular cylindrical shell is set up, and, by means of the Principle of Virtual Work, equilibrium equations and boundary conditions for a general type of boundary are deduced. The boundary conditions are then particularized for the cases in which the boundaries are parametric curves.

In Part IV., an exact solution of the differential equations derived in Parts II. and III. is obtained. This solution is applicable to circular cylindrical shells of the type which occurs in the problem of the design and construction of barrel roofs.

In Part V., the exact solution of Part IV., is carried out numerically for two particular cases.

In Part VI., an account of several approximate methods of solution of this problem is given. A brief summary of some of the

literature on this subject is included. The thesis is concluded with a rather detailed account of several of the methods by which the author attempted to obtain short and approximate solutions of this problem.

PART I.

In Part I., we shall derive, by the use of the methods of vector analysis, the equations of equilibrium and formulas for the strain components for a general shell. Consider any shell which we shall describe by means of the coordinates of the points of its middle surface, and its thickness normal to that surface. The middle surface may be any general surface having no singularities. We shall assume the thickness to be constant throughout the shell, although this assumption is not essential to the development of the theory.

Under the action of external loads, the shell will deform slightly. The following basic assumption regarding the deformations will be made; namely, that lines which are normal to the middle surface before deformation remain normal to the deformed middle after deformation, and suffer no extension. It is also assumed that the deformations are continuous functions of the coordinates of the middle surface.

We shall take the equation of the undeformed middle surface to be given in terms of two independent parameters α and β , in vector form:

$$\underline{r} = \underline{r}(\alpha, \beta) \qquad (1)$$

Throughout this thesis, a short line or bar underneath a symbol will be used to indicate a vector. It will be assumed that the curves α = const. and β = const. are lines of curvature of the undeformed middle surface. The reason for making this assumption will be referred to later in connection with the determination of the strain components.

Consider an element of the undeformed shell bounded by the two faces of the shell and four plane surfaces which are normal to the middle surface and which intersect the middle surface in the parametric curves $\alpha = \alpha_0$, $\alpha = \alpha_0 + d\alpha$, $\beta = \beta_0$, and $\beta = \beta_0 + d\beta$. We shall assume the shell faces to be free of force; however, over the plane surfaces described above forces due to internal stress must exist. The element must be held in equilibrium under the action of these forces and of any external loads which may be present. The external loads are assumed to act in any direction, their points of application being in the middle surface. At each point of the undeformed middle surface consider a triad of orthogonal axes directed as follows: the x axis tangent to the curve β = const., the y axis tangent to the curve α = const., the directions being those, respectively, of increasing α and β ; and the z axis normal to the middle surface. Let the positive directions of these axes define unit vectors i, j, and k, respectively, and let the direction of positive z be such that k = i x j. At each point of the shell there will be, in general, six components of stress, designated by σ_x, σ_y, σ_z, τ_{yz}, τ_{zx}, and τ_{xy}. The assumptions that the shell faces are free of force and that normals to the middle surface of the shell suffer no extension require that $\sigma_z = 0$. The forces which act on the four plane surfaces of the element under consideration will be given by the integrals of these stress components over the area of the corresponding faces of the element. The commonly accepted sign convention for the stress components, $\sigma_x \ldots \tau_{xy}$, will be adhered to; see Timoshenko, "Theory of Elasticity", page 4.

The force and moment components which act on two faces of the element are shown in Figure 1, the arrows indicating the assumed positive directions of these components. Let R_α and R_β be the radii of curvature of the middle surface along the curves β = const., and α = const., respectively, the shell being in the undeformed state. When the z coordinate of the center of curvature is positive, we consider the corresponding radius of curvature to be positive; hence, for the element shown in Figure 1, we must attach negative signs to both radii of curvature. Let dx and dy be the arc lengths of the intersection of the middle surface with the two plane surfaces of the element on which the forces are shown acting. Then the force N_x is given by the following formula:

$$N_x = \int_0^{dy} dy \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_x \left(1 - \frac{z}{R_\beta}\right) dz \quad ; \quad \frac{\partial N_x}{\partial y} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_x \left(1 - \frac{z}{R_\beta}\right) dz$$

where t is the shell thickness, and is assumed to be constant over the arc length dy. If we redefine N_x as the force per unit distance along the y axis, we may replace $\frac{\partial N_x}{\partial y}$ in the above equation by N_x. Using a similar definition for the other force and moment components; i.e., force or moment per unit length of arc of the parametric curves, we have the following formulae for these components:

On face α = const.

$$N_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_x \left(1 - \frac{z}{R_\beta}\right) dz$$

$$N_{xy} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{xy} \left(1 - \frac{z}{R_\beta}\right) dz$$

$$M_x = -\int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_x \left(1 - \frac{z}{R_\beta}\right) z\, dz$$

On face β = const.

$$N_y = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_y \left(1 - \frac{z}{R_\alpha}\right) dz$$

$$N_{yx} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{yx} \left(1 - \frac{z}{R_\alpha}\right) dz$$

$$M_y = -\int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_y \left(1 - \frac{z}{R_\alpha}\right) z\, dz \qquad (2)$$

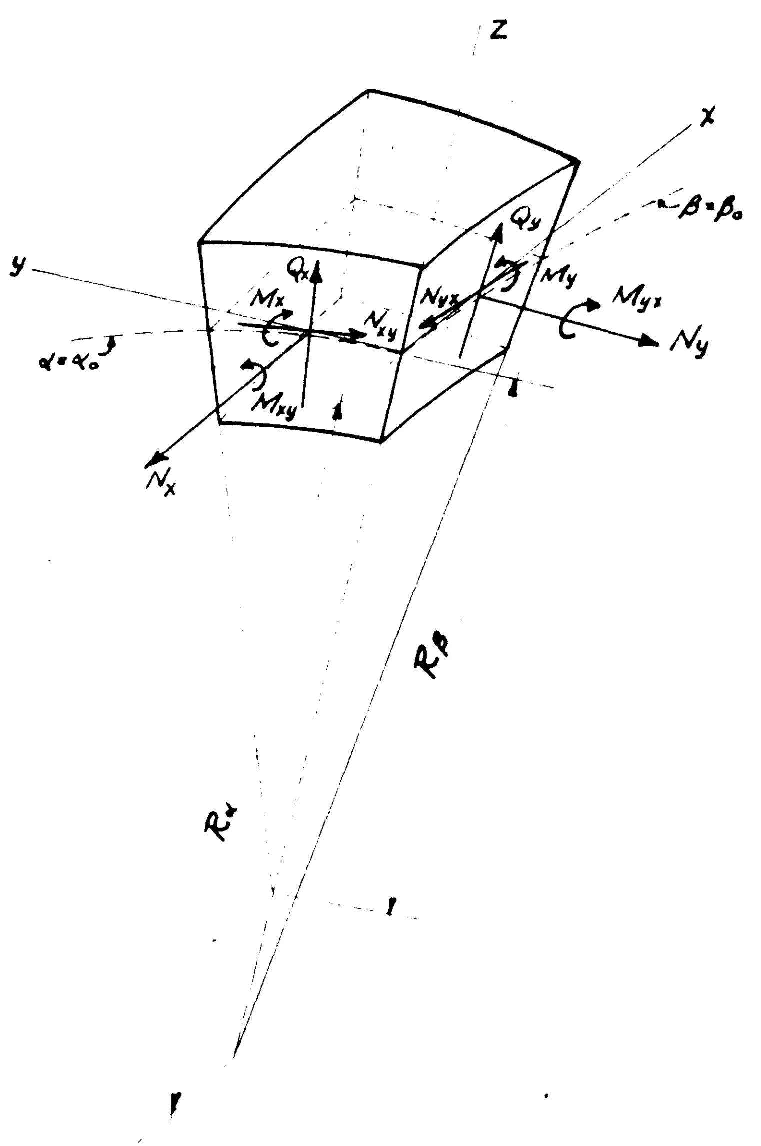


FIGURE 1

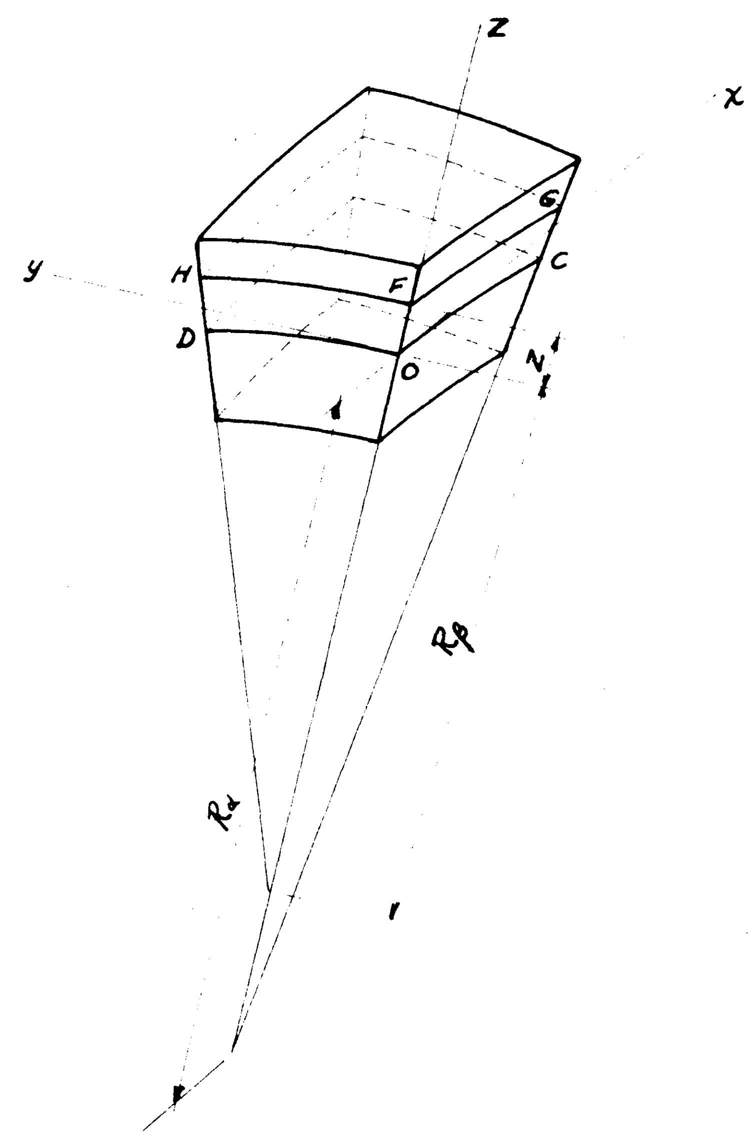


FIGURE 2

$$M_{xy} = -\int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{xy}\left(1-\frac{z}{R_\beta}\right) z\,dz \qquad M_{yx} = -\int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{yx}\left(1-\frac{z}{R_\alpha}\right) z\,dz$$

$$Q_x = -\int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{xz}\left(1-\frac{z}{R_\beta}\right) dz \qquad Q_y = -\int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{yz}\left(1-\frac{z}{R_\alpha}\right) dz$$

The negative signs appearing in the formulae for M_x .... Q_y are obtained from consideration of the assumed positive directions of these components as indicated in Figure 1, and the adopted sign convention for the stresses σ_x .... τ_{xy}.

In setting up the equations of equilibrium, we will assume these forces and moments to be localized along certain tangents and normals to the middle surface. Consider an element of middle surface of dimensions dx and dy corresponding, respectively, to variations $d\alpha$ and $d\beta$ of the parameters of the surface. Let axes x,y,z, as previously described, be set up with origin at the center of this element. Then if $\alpha = \alpha_o$, and $\beta = \beta_o$ are the parameters at the origin, the sides of the element will have the parameters $\alpha = \alpha_o \pm \frac{d\alpha}{2}$, and $\beta = \beta_o \pm \frac{d\beta}{2}$. At the origin, the forces and moments $N_x \ldots Q_y$ act, being directed along the axes x,y,z; i.e., in the directions of the unit vectors $\underline{i}$, $\underline{j}$, and $\underline{k}$. On the four faces of the element, there will be acting forces and moments which differ slightly from those at the origin, and which will be directed in slightly different directions from the unit vectors at the origin. Thus, for example, on the face $\alpha = \alpha_o - \frac{d\alpha}{2}$, the normal force will have the magnitude $N_x dy - \frac{\partial (N_x dy)}{\partial \alpha}\frac{d\alpha}{2}$ and it will have the direction of the x axis set up at the mid-point of the side $\alpha_o - \frac{d\alpha}{2}$; i.e., the direction of the vector $-(\underline{i} - \frac{\partial \underline{i}}{\partial \alpha}\frac{d\alpha}{2})$. The other forces and moments will have magnitudes and directions determined in a similar manner. Under the action of these forces and moments, together with the applied forces, the element must be in equilibrium. We now sum the various forces and moments in the three coordinate directions to obtain the six equations of equilibrium. However, before proceeding to set up these equations, it will be necessary to obtain geometrical properties of the middle surface in order that the variations with respect to α and β of the various quantities involved may be obtained.

Consider the undeformed surface whose vector equation is $\underline{r} = \underline{r}(\alpha, \beta)$. In terms of the unit vectors $\underline{i}, \underline{j}, \underline{k}$, this equation of the surface is expressed as follows:

$$\underline{r} = x(\alpha, \beta)\underline{i} + y(\alpha, \beta)\underline{j} + z(\alpha, \beta)\underline{k} \tag{3}$$

In this part of the thesis, the subscripts 1 and 2 will be used exclusively to denote differentiations with respect to α and β, respectively. Then, by definition, $\underline{r}_1 = \frac{\partial r}{\partial \alpha}$; $\underline{r}_2 = \frac{\partial r}{\partial \beta}$; $\underline{r}_{11} = \frac{\partial^2 r}{\partial \alpha^2}$; $\underline{r}_{12} = \frac{\partial^2 r}{\partial \alpha \partial \beta}$; and $\underline{r}_{22} = \frac{\partial^2 r}{\partial \beta^2}$. The following results are easily derived; see Weatherburn "Differential Geometry", Chapters 3, 4, and 5.

$$\underline{i} = E^{-\frac{1}{2}}\underline{r}_1 \; ; \; \underline{j} = G^{-\frac{1}{2}}\underline{r}_2 \; ; \; \underline{k} = \frac{r_1 \times r_2}{H} \; ; \; dx = \sqrt{E}\, d\alpha \; ; \; dy = \sqrt{G}\, d\beta$$

$$\text{where } E = \underline{r}_1^2 \; ; \; G = \underline{r}_2^2 \; ; \; F = \underline{r}_1 \cdot \underline{r}_2 \; ; \; H^2 = EG - F^2 \tag{4}$$

Define: $L = \underline{k} \cdot \underline{r}_{11}$; $S = \underline{k} \cdot \underline{r}_{12}$; $P = \underline{k} \cdot \underline{r}_{22}$.

Weatherburn derives the following formulae for the derivatives of the unit vectors:

$$\underline{i}_1 = \frac{L}{\sqrt{E}}\underline{k} - \frac{E_2}{2H}\underline{j} \qquad \underline{i}_2 = \frac{S}{\sqrt{E}}\underline{k} + \frac{G_1}{2H}\underline{j}$$

$$\underline{j}_1 = \frac{S}{\sqrt{G}}\underline{k} + \frac{E_2}{2H}\underline{i} \qquad \underline{j}_2 = \frac{P}{\sqrt{G}}\underline{k} - \frac{G_1}{2H}\underline{i}$$

$$\underline{k}_1 = \frac{\sqrt{E}}{H^2}(FS - GL)\ \ \underline{i} + \frac{\sqrt{G}}{H^2}(FL - ES)\ \ \underline{j} \tag{5}$$

$$\underline{k}_2 = \frac{\sqrt{E}}{H^2}(FP - GS)\ \ \underline{i} + \frac{\sqrt{G}}{H^2}(FS - EP)\ \ \underline{j}$$

We have made the assumption that the parametric curves are lines of curvature of the middle surface; they are therefore orthogonal also. These assumptions are satisfied by taking $F = S = 0$ in the above formulae. Defining two new quantities A and B by the equations $A^2 = E$ and $B^2 = G$, the formulae (5) may be simplified to read as follows:

$$\underline{i} = \frac{1}{A}\underline{r}_1\ ;\quad \underline{j} = \frac{1}{B}\underline{r}_2\ ;\quad \underline{k} = \frac{\underline{r}_1 \times \underline{r}_2}{AB}\ ;\ dx = A\,d\alpha;\ dy = B\,d\beta$$

where $A^2 = \underline{r}_1^2$; and $B^2 = \underline{r}_2^2$.

$$\underline{i}_1 = \frac{L}{A}\underline{k} - \frac{A_2}{B}\underline{j} \qquad\qquad \underline{i}_2 = \frac{B_1}{A}\underline{j}$$

$$\underline{j}_1 = \frac{A_2}{B}\underline{i} \qquad\qquad \underline{j}_2 = \frac{P}{B}\underline{k} - \frac{B_1}{A}\underline{i} \tag{6}$$

$$\underline{k}_1 = -\frac{L}{A}\underline{i} \qquad\qquad \underline{k}_2 = -\frac{P}{B}\underline{j}$$

We now make use of these results to set up the equilibrium equations. On the face $\alpha = \alpha_0 - \frac{d\alpha}{2}$, the following force acts:

$$-\left[N_x dy - (N_x dy)_1 \frac{d\alpha}{2}\right]\left[\overline{\underline{i} - \underline{i}_1 \frac{d\alpha}{2}}\right] - \left[N_{xy} dy - (N_{xy} dy)_1 \frac{d\alpha}{2}\right]\left[\overline{\underline{j} - \underline{j}_1 \frac{d\alpha}{2}}\right] + \left[Q_x dy - (Q_x dy)_1 \frac{d\alpha}{2}\right]\left[\overline{\underline{k} - \underline{k}_1 \frac{d\alpha}{2}}\right]$$

On the face $\alpha = \alpha_0 + \frac{d\alpha}{2}$, the following force acts:

$$\left[N_x dy + (N_x dy)_1 \frac{d\alpha}{2}\right]\left[\overline{\underline{i} + \underline{i}_1 \frac{d\alpha}{2}}\right] + \left[N_{xy} dy + (N_{xy} dy)_1 \frac{d\alpha}{2}\right]\left[\overline{\underline{j} + \underline{j}_1 \frac{d\alpha}{2}}\right] - \left[Q_x dy + (Q_x dy)_1 \frac{d\alpha}{2}\right]\left[\overline{\underline{k} + \underline{k}_1 \frac{d\alpha}{2}}\right]$$

In these two expressions, the bar over a vector is used to indicate a unit vector in the direction of the vector in question. Thus $\overline{\underline{i} \pm \underline{i}_1 \frac{d\alpha}{2}}$ is a unit vector in the direction of the vector $\underline{i} \pm \underline{i}_1 \frac{d\alpha}{2}$.

Expressions of the following type for the unit vectors will be used to simplify the above two force expressions:

$$\overline{\underline{i} \pm \underline{i}_{,1}\frac{d\alpha}{2}} = \frac{\underline{i} \pm \underline{i}_{,1}\frac{d\alpha}{2}}{\sqrt{1 + (\underline{i}_{,1}\frac{d\alpha}{2})^2}} = \left[\underline{i} \pm \underline{i}_{,1}\frac{d\alpha}{2}\right]\left[1 + \frac{1}{2}(\underline{i}_{,1}\frac{d\alpha}{2})^2 + \cdots\right] = \underline{i}\left[1 + \frac{1}{2}(\underline{i}_{,1}\frac{d\alpha}{2})^2\right] \pm \underline{i}_{,1}\frac{d\alpha}{2} \quad (7)$$

where terms of order higher than two in $d\alpha$ are neglected. Adding the above two force expressions, making use of Equation (7), replacing dy by B $d\beta$, and neglecting terms of order higher than two in the differentials $d\alpha$ and $d\beta$, we obtain the following expression for the net force acting on the two faces $\alpha = \alpha_0 - \frac{d\alpha}{2}$, and $\alpha = \alpha_0 + \frac{d\alpha}{2}$:

$$\left\{\left[(N_x B)_{,1}\, \underline{i} + N_x B\, \underline{i}_{,1}\right] + \left[(N_{xy} B)_{,1}\, \underline{j} + N_{xy} B\, \underline{j}_{,1}\right] - \left[(Q_x B)_{,1}\, \underline{k} + Q_x B\, \underline{k}_{,1}\right]\right\} d\alpha\, d\beta$$

This may be simplified by use of Equations (6) to read:

$$\left\{\left[(N_x B)_{,1} + A_{,2} N_{xy} + \frac{LB}{A} Q_x\right] \underline{i} + \left[(N_{xy} B)_{,1} - A_{,2} N_x\right] \underline{j} - \left[(Q_x B)_{,1} - \frac{LB}{A} N_x\right] \underline{k}\right\} d\alpha\, d\beta \quad (8)$$

In a similar manner, it may be shown that the net force acting on the two faces $\beta = \beta_0 - \frac{d\beta}{2}$, and $\beta = \beta_0 + \frac{d\beta}{2}$ is given by the following expression:

$$\left\{\left[(N_{yx} A)_{,2} - B_{,1} N_y\right] \underline{i} + \left[(N_y A)_{,2} + B_{,1} N_{yx} + \frac{PA}{B} Q_y\right] \underline{j} - \left[(Q_y A)_{,2} - \frac{PA}{B} N_y\right] \underline{k}\right\} d\alpha\, d\beta \quad (9)$$

Let the external force acting on an element of area of the middle surface be given by the expression:

$$(X\underline{i} + Y\underline{j} - Z\underline{k})\, dx dy = (X\underline{i} + Y\underline{j} - Z\underline{k})\, AB\, d\alpha\, d\beta \quad (10)$$

The sum of expressions (8), (9), and (10) must vanish if the element is in equilibrium. Adding these three expressions, rearranging and equating to zero the coefficients of the unit vectors, we obtain the three force equations of equilibrium:

$$
\begin{aligned}
&(N_x B)_{,1} + (N_{yx} A)_{,2} + A_{,2} N_{xy} - B_{,1} N_y + \frac{LB}{A} Q_x + ABX = 0 \\
&(N_{xy} B)_{,1} + (N_y A)_{,2} - A_{,2} N_x + B_{,1} N_{yx} + \frac{PA}{B} Q_y + ABY = 0 \qquad (11) \\
&(Q_x B)_{,1} + (Q_y A)_{,2} - \frac{LB}{A} N_x - \frac{PA}{B} N_y + ABZ = 0
\end{aligned}
$$

We proceed in the same way to obtain the three moment equations of equilibrium. The various moment components are localized in vectors, the directions of which are given by the well-known right hand rule. In writing the expressions for moment on the various faces of the element, we use the vector equation $\underline{M} = \underline{r} \times \underline{F}$ where $\underline{M}$ is the moment of a force $\underline{F}$ about the origin of vectors, and $\underline{r}$ is the vector from the origin to any point on the line of action of $\underline{F}$. The moment due to forces acting on the face $\alpha = \alpha_0 - \frac{d\alpha}{2}$ is given by the expression:

$$
\begin{aligned}
&-\left[M_{xy} dy - (M_{xy} dy)_{,1} \frac{d\alpha}{2}\right]\left[\overline{\underline{i} - \underline{i}_{,1} \frac{d\alpha}{2}}\right] + \left[M_x dy - (M_x dy)_{,1} \frac{d\alpha}{2}\right]\left[\overline{\underline{j} - \underline{j}_{,1} \frac{d\alpha}{2}}\right] \\
&\quad + \left[-\underline{i} \frac{dx}{2}\right] \times \left[Q_x dy - (Q_x dy)_{,1} \frac{d\alpha}{2}\right]\left[\overline{\underline{k} - \underline{k}_{,1} \frac{d\alpha}{2}}\right] \\
&\quad + \left[-\underline{i} \frac{dx}{2}\right] \times \left\{-\left[N_{xy} dy - (N_{xy} dy)_{,1} \frac{d\alpha}{2}\right]\left[\overline{\underline{j} - \underline{j}_{,1} \frac{d\alpha}{2}}\right]\right\}
\end{aligned}
$$

Similarly, the moment due to forces acting on the face $\alpha = \alpha_0 + \frac{d\alpha}{2}$ is given by the expression:

$$
\begin{aligned}
&\left[M_{xy} dy + (M_{xy} dy)_{,1} \frac{d\alpha}{2}\right]\left[\overline{\underline{i} + \underline{i}_{,1} \frac{d\alpha}{2}}\right] - \left[M_x dy + (M_x dy)_{,1} \frac{d\alpha}{2}\right]\left[\overline{\underline{j} + \underline{j}_{,1} \frac{d\alpha}{2}}\right] \\
&\quad + \left[\underline{i} \frac{dx}{2}\right] \times \left\{-\left[Q_x dy + (Q_x dy)_{,1} \frac{d\alpha}{2}\right]\left[\overline{\underline{k} + \underline{k}_{,1} \frac{d\alpha}{2}}\right]\right\} \\
&\quad + \left[\underline{i} \frac{dx}{2}\right] \times \left[N_{xy} dy + (N_{xy} dy)_{,1} \frac{d\alpha}{2}\right]\left[\overline{\underline{j} + \underline{j}_{,1} \frac{d\alpha}{2}}\right]
\end{aligned}
$$

Combining these two expressions as in the case of forces, and using the equation $\underline{i} \times \underline{j} = \underline{k}$, we obtain the following expression for the net moment due to the forces acting on the faces $\alpha = \alpha_0 - \frac{d\alpha}{2}$, and $\alpha = \alpha_0 + \frac{d\alpha}{2}$:

$$\left\{\left[(M_{xy}B)_1 - A_2 M_x\right] \underline{i} - \left[(M_x B)_1 + A_2 M_{xy} - Q_x AB\right] \underline{j} + \left[\frac{LB}{A} M_{xy} + N_{xy} AB\right] \underline{k}\right\} d\alpha\, d\beta \qquad (12)$$

In a similar manner, it may be shown that the net moment due to the forces which act on the faces $\beta = \beta_0 - \frac{d\beta}{2}$, and $\beta = \beta_0 + \frac{d\beta}{2}$ is given by the following expression:

$$\left\{\left[(M_y A)_2 + B_1 M_{yx} - Q_y AB\right] \underline{i} - \left[(M_{yx} A)_2 - B_1 M_y\right] \underline{j} + \left[-N_{yx} AB - M_{yx} \frac{PA}{B}\right] \underline{k}\right\} d\alpha\, d\beta \qquad (13)$$

We assume that the moment of the external forces is zero (or, that the moment of the external forces is of the order neglected in setting up the above moment expressions). The total moment of all forces acting on the element of surface is given by the sum of the two expressions (12) and (13), and this sum must vanish if the element is in equilibrium. Equating to zero the coefficients of the unit vectors in this sum, we obtain the three moment equations of equilibrium:

$$\begin{aligned}
&(M_{xy} B)_1 + (M_y A)_2 + B_1 M_{yx} - A_2 M_x - Q_y AB = 0 \\
&(M_x B)_1 + (M_{yx} A)_2 - B_1 M_y + A_2 M_{xy} - Q_x AB = 0 \qquad (14) \\
&\frac{LB}{A} M_{xy} - \frac{PA}{B} M_{yx} + N_{xy} AB - N_{yx} AB = 0
\end{aligned}$$

Equations (11) and (14) are the equations of equilibrium of the shell element and are based on the assumption that the effect of deformation of the shell on the equilibrium of the element is negligible. The constants A, B, L, and P in these equations are to be calculated for the undeformed shell. It is these equations which will be used later in this thesis for the investigation of stresses in a cylindrical shell.

We shall now derive equilibrium equations in which the effect of deformation on the equilibrium of the element is taken into account. However, before proceeding to set up such equations, it will be necessary to consider the geometry of the deformed middle surface.

Let the deformation of the middle surface be a vector point function of the coordinates of the undeformed middle surface, so that the deformation vector is given by the equation:

$$\underline{s} = \underline{s}\,(\alpha, \beta) \tag{15}$$

where $\underline{s}$ is assumed to be small enough that squares and products of $\underline{s}$ and its derivatives may be neglected. The position vector of a point on the deformed middle surface is given by the equation:

$$\underline{r}' = \underline{r} + \underline{s} \tag{16}$$

where $\underline{r}$ is the position vector of the undeformed middle surface given by Equations (1) and (3). The parametric curves α = const. and β = const. of the deformed surface will be no longer orthogonal, but will meet each other at an angle χ which will differ slightly from a right angle. Let the deformation $\underline{s}$ be expressed in terms of components parallel to the axes x, y, z, of the undeformed middle surface, as follows:

$$\underline{s} = u(\alpha,\beta)\underline{i} + v(\alpha,\beta)\underline{j} + w(\alpha,\beta)\underline{k} \tag{17}$$

In terms of the unit vectors $\underline{i}$, $\underline{j}$, $\underline{k}$, we may calculate the derivatives $\underline{r}'_1$, $\underline{r}'_2$, $\underline{r}'_{11}$, $\underline{r}'_{12}$, and $\underline{r}'_{22}$. In making these calculations, the derivatives of the unit vectors $\underline{i}$, $\underline{j}$, and $\underline{k}$ must be considered. These derivatives are given by Equations (6). We now define a new set of unit vectors referred to the deformed middle surface. Let $\underline{a}$ be a unit vector tangent to the curve β = const. and directed according to increasing α; let $\underline{b}$ be a unit vector tangent to the curve α = const. and directed according to increasing β; and let $\underline{n}$ be a unit vector normal to the deformed surface and directed according to the vector $\underline{a} \times \underline{b}$. Thus the unit vectors $\underline{i}$, $\underline{j}$, $\underline{k}$ before deformation go into the unit vectors $\underline{a}$, $\underline{b}$, $\underline{n}$ after deformation. Having the derivatives of $\underline{r}'$, we may calculate the following quantities:

$$E = A^2 = \underline{r}'^2_1 \; ; \; F = \underline{r}'_1 \cdot \underline{r}'_2 \; ; \; G = B^2 = \underline{r}'^2_2 \; ; \; H^2 = EG - F^2$$

$$\underline{a} = \frac{1}{A}\underline{r}'_1 \; ; \; \underline{b} = \frac{1}{B}\underline{r}'_2 \; ; \; \underline{n} = \frac{\underline{r}'_1 \times \underline{r}'_2}{H} \tag{18}$$

$$L = \underline{n} \cdot \underline{r}'_{11} \; ; \; S = \underline{n} \cdot \underline{r}'_{12} \; ; \; P = \underline{n} \cdot \underline{r}'_{22}$$

It is to be noted that all vector quantities involved in Equations (18) are expressed in terms of the unit vectors $\underline{i}$, $\underline{j}$, $\underline{k}$. The angle χ at which the parametric curves on the deformed surface intersect will be given by the following expressions, the first of which will be taken as the definition of the quantity θ:

$$\theta = \cos\chi = \underline{a} \bullet \underline{b} = \frac{F}{A\,B} \;;\quad \sin\chi = \frac{H}{A\,B} \tag{19}$$

Since the angle χ differs only slightly from a right angle, we may use the approximation $\sin\chi = 1$; the validity of this approximation is subject to verification in any particular case, and will be verified later for the case of the circular cylinder. Using Equations (18), we may define and calculate the following quantities:

$$\begin{aligned}
\ell &= \frac{1}{2H^2}(GE_1 - 2FF_1 + FE_2) \;; & \Gamma &= \frac{1}{H^2}(FS - GL) \\
\lambda &= \frac{1}{2H^2}(2EF_1 - EE_2 - FE_1) \;; & \Lambda &= \frac{1}{H^2}(FL - ES) \\
m &= \frac{1}{2H^2}(GE_2 - FG_1) \;; & C &= \frac{1}{H^2}(FP - GS) \\
\mu &= \frac{1}{2H^2}(EG_1 - FE_2) \;; & D &= \frac{1}{H^2}(FS - EP)
\end{aligned} \tag{20}$$

In terms of these quantities, the derivatives of the unit vectors $\underline{a}$, $\underline{b}$, and $\underline{n}$ may be expressed as follows: (See Weatherburn, "Differential Geometry", pages 61 and 90.)

$$\begin{aligned}
\underline{a}_1 &= \frac{L}{A}\,\underline{n} + \left(\ell - \frac{A_1}{A}\right)\underline{a} + \lambda\frac{B}{A}\,\underline{b} \\
\underline{a}_2 &= \frac{S}{A}\,\underline{n} + \left(m - \frac{A_2}{A}\right)\underline{a} + \mu\frac{B}{A}\,\underline{b} \\
\underline{n}_1 &= A\Gamma\,\underline{a} + B\Lambda\,\underline{b} \\
\underline{n}_2 &= AC\,\underline{a} + BD\,\underline{b}
\end{aligned} \tag{21}$$

Let us now define at each point of the surface a new triad of orthogonal axes x', y', z' directed as follows: the x' axis along the unit vector $\underline{a}$; the z' axis along the unit vector $\underline{n}$; and the y' axis perpendicular to the x'z' plane, and so directed that the angle between the positive y' axis and the unit vector $\underline{b}$ is a small acute angle. Let unit vectors along the axes x', y', z' be $\underline{i}'$, $\underline{j}'$, $\underline{k}'$, respectively. Let us define, also, a unit vector $\underline{c}$ in the x'y' plane which is perpendicular to the unit vector $\underline{b}$ and so directed that the angle between $\underline{c}$ and $\underline{a}$ is a small acute angle. The following relations are seen to exist between the unit vectors $\underline{a}$, $\underline{b}$, $\underline{c}$, $\underline{n}$, $\underline{i}'$, $\underline{j}'$, and $\underline{k}'$:

$$\begin{aligned} &\underline{a} = \underline{i}' \ ; \quad \underline{b} = \cos\chi\, \underline{i}' + \sin\chi\, \underline{j}' \cong \theta\underline{i}' + \underline{j}' \\ &\underline{n} = \underline{k}' \ ; \quad \underline{c} = \sin\chi\, \underline{i}' - \cos\chi\, \underline{j}' \cong \underline{i}' - \theta\underline{j}' \\ &\underline{c} \times \underline{b} = \underline{k}' \ ; \quad \underline{a} \times \underline{b} = \sin\chi\, \underline{k}' \cong \underline{k}' \end{aligned} \tag{22}$$

We now simplify the Equations (21) for the derivatives of the unit vectors $\underline{a}$ and $\underline{n}$, and also obtain expressions for the derivatives of $\underline{b}$, $\underline{c}$ and $\underline{j}'$. Substituting Equations (22) in the first two of Equations (21), and making use of Equations (18) and (20), we obtain:

$$\underline{a}_1 = \underline{i}'_1 = \frac{L}{A}\,\underline{k}' + \lambda\frac{B}{A}\,\underline{j}' \ ; \quad \underline{a}_2 = \underline{i}'_2 = \frac{S}{A}\,\underline{k}' + \mu\frac{B}{A}\,\underline{j}' \tag{23}$$

Substituting Equations (22) in the last two of Equations (21), we obtain:

$$\begin{aligned} \underline{n}_1 = \underline{k}'_1 &= (A\Gamma + \theta B\Lambda)\underline{i}' + B\Lambda\,\underline{j}' \\ \underline{n}_2 = \underline{k}'_2 &= (AC + \theta BD)\underline{i}' + BD\,\underline{j}' \end{aligned} \tag{24}$$

Since $\underline{a}$ and $\underline{n}$ are vectors which are everywhere orthogonal, the equation $\underline{a} \cdot \underline{n} = 0$ must be identically true. Differentiating, we obtain two further identities, as follows:

$$\underline{a}_1 \cdot \underline{n} + \underline{a} \cdot \underline{n}_1 = 0 \; ; \quad \underline{a}_2 \cdot \underline{n} + \underline{a} \cdot \underline{n}_2 = 0$$

Substituting Equations (22), (23), and (24) in these two identities, and noting that the dot product of two different vectors of the triad $\underline{i}'$, $\underline{j}'$, $\underline{k}'$ vanish due to their orthogonality, we obtain the following two relations:

$$- \frac{L}{A} = A\Gamma + \Theta B\Lambda \; ; \quad - \frac{S}{A} = AC + \Theta BD \qquad (25)$$

Differentiating the identity $\underline{k}' \times \underline{i}' = \underline{j}'$, we obtain the following expressions for the derivatives of $\underline{j}'$:

$$\underline{j}'_1 = \underline{k}'_1 \times \underline{i}' + \underline{k}' \times \underline{i}'_1 \; ; \quad \underline{j}'_2 = \underline{k}'_2 \times \underline{i}' + \underline{k}' \times \underline{i}'_2$$

Substituting Equations (22), (23), (24), and (25) in these expressions, we obtain:

$$\underline{j}'_1 = - \lambda \frac{B}{A} \underline{i}' - B\Lambda \underline{k}' \; ; \quad \underline{j}'_2 = - \mu \frac{B}{A} \underline{i}' - BD \underline{k}' \qquad (26)$$

The derivatives of $\underline{b}$ and $\underline{c}$ may now be obtained by differentiating Equations (22) as follows:

$$\underline{b}_1 = \Theta_1 \underline{i}' + \Theta \underline{i}'_1 + \underline{j}'_1 \; ; \quad \underline{b}_2 = \Theta_2 \underline{i}' + \Theta \underline{i}'_2 + \underline{j}'_2$$

$$\underline{c}_1 = \underline{i}'_1 - \Theta_1 \underline{j}' - \Theta \underline{j}'_1 \; ; \quad \underline{c}_2 = \underline{i}'_2 - \Theta_2 \underline{j}' - \Theta \underline{j}'_2$$

Substituting Equations (23) and (26) in these expressions, we obtain the following equations:

$$\begin{aligned}
\underline{b}_1 &= (\theta_1 - \lambda \tfrac{B}{A})\underline{i}' + \theta\lambda \tfrac{B}{A}\underline{j}' + (\theta \tfrac{L}{A} - B\Lambda)\underline{k}' \\
\underline{b}_2 &= (\theta_2 - \mu \tfrac{B}{A})\underline{i}' + \theta\mu \tfrac{B}{A}\underline{j}' + (\theta \tfrac{S}{A} - BD)\underline{k}' \\
\underline{c}_1 &= \theta\lambda \tfrac{B}{A}\underline{i}' - (\theta_1 - \lambda \tfrac{B}{A})\underline{j}' + (\tfrac{L}{A} + \theta B\Lambda)\underline{k}' \\
\underline{c}_2 &= \theta\mu \tfrac{B}{A}\underline{i}' - (\theta_2 - \mu \tfrac{B}{A})\underline{j}' + (\tfrac{S}{A} + \theta BD)\underline{k}'
\end{aligned} \qquad (27)$$

We now rewrite the results of Equations (23), (24), (25), (26), and (27) in the following final form:

$$\begin{aligned}
\underline{i}'_1 &= \underline{a}_1 = r_\alpha \underline{j}' - q_\alpha \underline{k}'. & \underline{i}'_2 &= \underline{a}_2 = r_\beta \underline{j}' - q_\beta \underline{k}' \\
\underline{j}'_1 &= p_\alpha \underline{k}' - r_\alpha \underline{i}' & \underline{j}'_2 &= p_\beta \underline{k}' - r_\beta \underline{i}' \\
\underline{k}'_1 &= \underline{n}_1 = q_\alpha \underline{i}' - p_\alpha \underline{j}' & \underline{k}'_2 &= \underline{n}_2 = q_\beta \underline{i}' - p_\beta \underline{j}' \\
\underline{b}_1 &= \xi_\alpha \underline{i}' + \eta_\alpha \underline{j}' + \zeta_\alpha \underline{k}' & \underline{b}_2 &= \xi_\beta \underline{i}' + \eta_\beta \underline{j}' + \zeta_\beta \underline{k}' \\
\underline{c}_1 &= \eta_\alpha \underline{i}' - \xi_\alpha \underline{j}' + \psi_\alpha \underline{k}' & \underline{c}_2 &= \eta_\beta \underline{i}' - \xi_\beta \underline{j}' + \psi_\beta \underline{k}'
\end{aligned} \qquad (28)$$

where

$$\begin{aligned}
p_\alpha &= -B\Lambda & p_\beta &= -BD \\
q_\alpha &= -\frac{L}{A} & q_\beta &= -\frac{S}{A} \\
r_\alpha &= \lambda \frac{B}{A} & r_\beta &= \mu \frac{B}{A} \\
\xi_\alpha &= \theta_1 - r_\alpha & \xi_\beta &= \theta_2 - r_\beta
\end{aligned} \qquad (29)$$

$$\eta_\alpha = \theta r_\alpha \qquad \eta_\beta = \theta r_\beta$$

$$\zeta_\alpha = p_\alpha - \theta q_\alpha \qquad \zeta_\beta = p_\beta - \theta q_\beta$$

$$\psi_\alpha = -\theta p_\alpha - q_\alpha \qquad \psi_\beta = -\theta p_\beta - q_\beta$$

Using these results, we may rewrite the equations of equilibrium, taking into account the effect of deformation of the surface on the directions of forces and moments. Previously, all forces and moments were assumed to be directed according to the vectors $\underline{i}$, $\underline{j}$, and $\underline{k}$. In the present case, forces and moments on faces $\alpha = \alpha_o \pm \frac{d\alpha}{2}$ of the shell element are assumed to be directed according to the vectors $\underline{b}$, $\underline{c}$, and $\underline{k}'$; forces and moments on faces $\beta = \beta_o \pm \frac{d\beta}{2}$ are assumed to be directed according to the vectors $\underline{i}'$, $\underline{j}'$, and $\underline{k}'$. We proceed as before. On the face $\alpha = \alpha_o - \frac{d\alpha}{2}$ of the element of shell, the following force acts:

$$-\left[N_x dy - (N_x dy)_{,}\frac{d\alpha}{2}\right]\left[\overline{\underline{c} - \underline{c}_{,}\frac{d\alpha}{2}}\right] - \left[N_{xy} dy - (N_{xy} dy)_{,}\frac{d\alpha}{2}\right]\left[\overline{\underline{b} - \underline{b}_{,}\frac{d\alpha}{2}}\right]$$

$$+\left[Q_x dy - (Q_x dy)_{,}\frac{d\alpha}{2}\right]\left[\overline{\underline{k}' - \underline{k}'_{,}\frac{d\alpha}{2}}\right]$$

Similarly, on the face $\alpha = \alpha_o + \frac{d\alpha}{2}$, the following force acts:

$$\left[N_x dy + (N_x dy)_{,}\frac{d\alpha}{2}\right]\left[\overline{\underline{c} + \underline{c}_{,}\frac{d\alpha}{2}}\right] + \left[N_{xy} dy + (N_{xy} dy)_{,}\frac{d\alpha}{2}\right]\left[\overline{\underline{b} + \underline{b}_{,}\frac{d\alpha}{2}}\right]$$

$$-\left[Q_x dy + (Q_x dy)_{,}\frac{d\alpha}{2}\right]\left[\overline{\underline{k}' + \underline{k}'_{,}\frac{d\alpha}{2}}\right]$$

Adding these two expressions, considering expressions similar to Equation (7) for the unit vectors $\overline{\underline{c} \pm \underline{c}_{,}\frac{d\alpha}{2}}$, etc., replacing dy by

$B\,d\beta$, and neglecting terms of order higher than two in $d\alpha$ and $d\beta$, we obtain the following expression for the net force acting on the two faces $\alpha = \alpha_0 - \frac{d\alpha}{2}$ and $\alpha = \alpha_0 + \frac{d\alpha}{2}$:

$$\left\{\left[(N_x B)_{,1}\,\underline{c} + N_x B\underline{c}_{,1}\right] + \left[(N_{xy} B)_{,1}\,\underline{b} + N_{xy} B\underline{b}_{,1}\right] - \left[(Q_x B)_{,1}\,\underline{k}' + Q_x B\underline{k}'_{,1}\right]\right\} d\alpha\, d\beta$$

By use of Equations (22) and (23), this expression may be rewritten as follows:

$$\left\{\left[(N_x B)_{,1} + \Theta(N_{xy}B)_{,1} + \eta_\alpha BN_x + \xi_\alpha BN_{xy} - q_\alpha BQ_x\right]\underline{i}' \right.$$
$$+ \left[(N_{xy}B)_{,1} - \Theta(N_x B)_{,1} - \xi_\alpha BN_x + \eta_\alpha BN_{xy} + p_\alpha BQ_x\right]\underline{j}' \qquad (30)$$
$$\left. - \left[(Q_x B)_{,1} - \psi_\alpha BN_x - \zeta_\alpha BN_{xy}\right]\underline{k}'\right\} d\alpha\, d\beta$$

In a similar manner, by assuming the forces to be directed according to the vectors $\underline{i}'$, $\underline{j}'$, $\underline{k}'$, it may be shown that the net force acting on the two faces $\beta = \beta_0 - \frac{d\beta}{2}$ and $\beta = \beta_0 + \frac{d\beta}{2}$ is given by the following expression:

$$\left\{\left[(N_{yx}A)_{,2} - r_\beta AN_y - q_\beta AQ_y\right]\underline{i}' + \left[(N_y A)_{,2} + r_\beta AN_{yx} + p_\beta AQ_y\right]\underline{j}'\right.$$
$$\left. - \left[(Q_y A)_{,2} + q_\beta AN_{yx} - p_\beta AN_y\right]\underline{k}'\right\} d\alpha\, d\beta \quad (31)$$

Let the external load on the element of middle surface be given by the expression:

$$(X'\underline{i}' + Y'\underline{j}' - Z'\underline{k}')dxdy = (X'\underline{i}' + Y'\underline{j}' - Z'\underline{k}')\,AB\,d\alpha\, d\beta \qquad (32)$$

As before, we obtain the three force equations of equilibrium by adding expressions (30), (31), and (32), and equating to zero the coefficients of the unit vectors:

$$(N_x B)_1 + (N_{yx}A)_2 + \theta(N_{xy}B)_1 + \eta_\alpha BN_x + \xi_\alpha BN_{xy} - r_\beta AN_y - q_\alpha BQ_x - q_\beta AQ_y + ABX' = 0$$

$$(N_{xy}B)_1 + (N_y A)_2 - \theta(N_x B)_1 - \xi_\alpha BN_x + r_\beta AN_{yx} + \eta_\alpha BN_{xy} + p_\alpha BQ_x + p_\beta AQ_y + ABY' = 0 \quad (33)$$

$$(Q_x B)_1 + (Q_y A)_2 - \psi_\alpha BN_x + q_\beta AN_{yx} - \zeta_\alpha BN_{xy} - p_\beta AN_y + ABZ' = 0$$

The moments are treated similarly. The moment of the forces acting on the face $\alpha = \alpha_0 - \frac{d\alpha}{2}$ of the element is given by the following expression:

$$-\left[M_{xy}dy - (M_{xy}dy)_1\frac{d\alpha}{2}\right]\left[\overline{\underline{c} - \underline{c}_1\frac{d\alpha}{2}}\right] + \left[M_x dy - (M_x dy)_1\frac{d\alpha}{2}\right]\left[\overline{\underline{b} - \underline{b}_1\frac{d\alpha}{2}}\right]$$

$$+\left[-\underline{i}'\frac{dx}{2}\right]\times\left[Q_x dy - (Q_x dy)_1\frac{d\alpha}{2}\right]\left[\overline{\underline{k}' - \underline{k}'_1\frac{d\alpha}{2}}\right] + \left[-\underline{i}'\frac{dx}{2}\right]\times\left\{-\left[N_{xy}dy - (N_{xy}dy)_1\frac{d\alpha}{2}\right]\left[\overline{\underline{b} - \underline{b}_1\frac{d\alpha}{2}}\right]\right\}$$

The moment of the forces acting on the face $\alpha = \alpha_0 + \frac{d\alpha}{2}$ is given by the following expression:

$$\left[M_{xy}dy + (M_{xy}dy)_1\frac{d\alpha}{2}\right]\left[\overline{\underline{c} + \underline{c}_1\frac{d\alpha}{2}}\right] - \left[M_x dy + (M_x dy)_1\frac{d\alpha}{2}\right]\left[\overline{\underline{b} + \underline{b}_1\frac{d\alpha}{2}}\right]$$

$$+\left[\underline{i}'\frac{dx}{2}\right]\times\left\{-\left[Q_x dy + (Q_x dy)_1\frac{d\alpha}{2}\right]\left[\overline{\underline{k}' + \underline{k}'_1\frac{d\alpha}{2}}\right]\right\} + \left[\underline{i}'\frac{dx}{2}\right]\times\left[N_{xy}dy + (N_{xy}dy)_1\frac{d\alpha}{2}\right]\left[\overline{\underline{b} + \underline{b}_1\frac{d\alpha}{2}}\right]$$

Adding these two expressions and simplifying, as in the case of forces, by making use of Equations (22) and (28), we obtain the following expression for the net moment of the forces acting on the two faces $\alpha = \alpha_0 - \frac{d\alpha}{2}$ and $\alpha = \alpha_0 + \frac{d\alpha}{2}$:

$$\left\{\left[(M_{xy}B)_1 - \theta(M_x B)_1 + \eta_\alpha BM_{xy} - \xi_\alpha BM_x\right]\underline{i}' - \left[(M_x B)_1 + \theta(M_{xy}B)_1 + \xi_\alpha BM_{xy} + \eta_\alpha BM_x - ABQ_x\right]\underline{j}'\right.$$

$$\left. + \left[ABN_{xy} + \psi_\alpha BM_{xy} - \zeta_\alpha BM_x\right]\underline{k}'\right\} d\alpha \; d\beta \quad (34)$$

In like manner, we obtain an expression for the net moment of the forces acting on the faces $\beta = \beta_0 - \frac{d\beta}{2}$ and $\beta = \beta_0 + \frac{d\beta}{2}$, as follows:

$$\left\{\left[(M_y A)_2 + r_\beta AM_{yx} - ABQ_y\right]\underline{i}' - \left[(M_{yx} A)_2 - r_\beta AM_y - \theta ABQ_y\right]\underline{j}' + \left[-ABN_{yx} - q_\beta AM_y - p_\beta AM_{yx}\right]\underline{k}'\right\} d\alpha \; d\beta \qquad (35)$$

We assume, as before, that the moment of the external forces is zero. Adding expressions (34) and (35), and equating to zero the coefficients of the unit vectors, we obtain the three moment equations of equilibrium:

$$\begin{aligned}
&(M_{xy}B)_1 + (M_y A)_2 - \theta(M_x B)_1 + r_\beta AM_{yx} - \xi_\alpha BM_x + \eta_\alpha BM_{xy} - ABQ_y = 0 \\
&(M_x B)_1 + (M_{yx}A)_2 + \theta(M_{xy}B)_1 + \eta_\alpha BM_x + \xi_\alpha BM_{xy} - r_\beta AM_y - ABQ_x - \theta ABQ_y = 0 \qquad (36) \\
&ABN_{xy} - ABN_{yx} - p_\beta AM_{yx} - \zeta_\alpha BM_x + \psi_\alpha BM_{xy} - q_\beta AM_y = 0
\end{aligned}$$

Equations (33) and (36) are equations of equilibrium which take into consideration the effect of deformation of the shell on the equilibrium of the element. The constants appearing in these equations are to be calculated for the deformed shell.

In writing Equations (33) and (36), we have assumed that the normal stresses σ_x and σ_y rotate in direction as the element deforms so that they are applied always normal to the faces of the element. (See page 17). A second possible assumption would be that these stresses act always parallel to the parametric curves; i.e., after deformation they become inclined to the faces on which they act. The shear stresses are assumed to act in the faces of the element in both cases. On the basis of this second assumption, the forces N_x, N_{xy}, M_x, M_{xy}, and

Q_x acting on the faces α = const. will be directed according to the unit vectors a, b, j', c, and k', respectively; while the forces N_y, N_{yx}, M_y, M_{yx}, and Q_y which act on the faces β = const. will be directed according to the unit vectors b, a, c, j', and k', respectively. Proceeding exactly as in the previous case, we could set up another set of equilibrium equations, applicable under conditions similar to those under which Equations (33) and (36) are applicable. Which of these two methods of approximation is nearer the actual truth is difficult to say. A. E. H. Love gives a set of equilibrium equations ("Mathematical Theory of Elasticity", Fourth Edition, page 535) derived on still a third assumption; namely, that all forces and moments on the faces of the deformed element are directed according to the unit vectors i', j', and k'. Physically, it seems that either of the first two assumptions is more reasonable than the latter; however, it is probable that the results will be in close agreement regardless of the particular assumption used in deriving the equations. As no use will be made of these equations in this thesis, we shall not attempt to go further into the merits of these assumptions. Our main purpose has been to demonstrate the ease with which the equations of equilibrium may be set up, regardless of the particular assumption used.

In order to solve either Equations (11) and (14) or Equations (33) and (36), we first express all forces and moments involved in terms of the deformation components, u, v, w, and their derivatives with respect to α and β. In general, the strain at any point is determined by six quantities ϵ_x, ϵ_y, ϵ_z, γ_{yz}, γ_{zx}, and γ_{xy}. The basic assumption regarding the deformation of the shell, as stated on page 1, requires that ϵ_z, γ_{yz}, and γ_{zx} shall be zero. Likewise, as previously noted, the stress σ_z must vanish. Hooke's law for isotropic materials then gives the following relations between the non-vanishing stress and strain components:

$$
\begin{aligned}
\epsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y) & \sigma_x &= \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y) \\
\epsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x) & \sigma_y &= \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x) \\
\gamma_{xy} &= \frac{2(1+\nu)}{E}\tau_{xy} & \tau_{xy} &= \frac{E}{1-\nu^2}\left(\frac{1-\nu}{2}\right)\gamma_{xy}
\end{aligned}
\tag{37}
$$

In order to express the force and moment components in terms of deformation components, we proceed as follows: The strains ϵ_x, ϵ_y, and γ_{xy}, as functions of z, the coordinate varying over the shell thickness, are determined from the geometry of the shell and substituted in the above equations of Hooke's law. The resulting expressions for the stress components, as functions of z, are substituted in Equations (2) and integrated over the shell thickness, to give the desired relations between the deformation components and the force and moment components.

As stated above, the basic assumption regarding the deformation of the shell requires that the strain components γ_{yz} and γ_{zx} shall vanish. This, of course, is inconsistent with the presence of shearing stresses τ_{yz} and τ_{zx}. Actually, strains γ_{yz} and γ_{zx} must exist, but on the assumption that they are small compared to the other strains, they may be neglected. This is equivalent to neglecting the strain energy due to the strains γ_{yz} and γ_{zx}, in comparison to the strain energy due to the other strains. Also, it corresponds to the usual assumption of elementary beam theory, that "plane sections remain plane". When the ratios of shell thickness to radius of curvature, and shell thickness to arc length of middle surface between boundaries, are small, this assumption, in all probability, will give a good approximation; in any case, the assumption is subject to experimental verification.

We now turn to the derivation of expressions for ϵ_x , ϵ_y , and γ_{xy} as functions of the coordinate z. Consider, again, the element of shell described on page 2; this element is shown again in Figure 2. In this figure, OCD represents the middle surface of the shell and FGH represents a surface parallel to the middle surface and at distance z from it. According to our basic assumption, the surface FGH remains parallel, after deformation, to the deformed middle surface, and at precisely the same distance z from it. We proceed to write expressions for the position vectors of the points O, C, D, F, G, and H, both before and after deformation. The following notation will be used: the letter, underlined, which designates a point will be used to indicate the position vector of that point; two letters underlined, for example $\underline{FH}$, will indicate the vector directed from the first named point to the second, in the example, the vector from the point F to the point H; and the subscript o shall be used to distinguish quantities relating to the undeformed shell from the corresponding quantities relating to the deformed shell. The unit vectors involved have been described previously. The point O_o on the undeformed middle surface will be taken as the origin of the position vectors of all the points mentioned above. If $\underline{s}$ is the deformation vector at the origin, the position vectors of the various points may be written as follows:

$$\underline{O}_o = 0$$

$$\underline{O} = \underline{s}$$

$$\underline{C}_o = R_{\alpha_o} \sin\frac{A_o d\alpha}{R_{\alpha_o}} \underline{i} - R_{\alpha_o}\left(1 - \cos\frac{A_o d\alpha}{R_{\alpha_o}}\right)\underline{k} = A_o\, d\alpha\ \underline{i}$$

$$\underline{C} = R_\alpha \sin\frac{A\, d\alpha}{R_\alpha} \underline{a} - R_\alpha\left(1 - \cos\frac{A\, d\alpha}{R_\alpha}\right)\underline{n} + \underline{s} = A\, d\alpha\ \underline{a} + \underline{s}$$

where the sine and cosine are expanded in series and terms of the first order in $d\alpha$, only, are retained. Similarly:

$$\underline{D}_o = B_o\, d\beta\, \underline{j}$$

$$\underline{D} = B\, d\beta\, \underline{b} + \underline{s}$$

$$\underline{F}_o = z\, \underline{k}$$

$$\underline{F} = \underline{s} + z\underline{n}$$

$$\underline{G}_o = \underline{C}_o + z\, \overline{(\underline{k} + \underline{k}_{,} d\alpha)} = z\, \underline{k} + (A_o \underline{i} + z\, \underline{k}_{,})d\alpha$$

where an expression similar to Equation (7) for the unit vector $\overline{\underline{k} + \underline{k}_{,} d\alpha}$ is used and only terms of the first degree in $d\alpha$ are retained.

Similarly:

$$\underline{G} = \underline{C} + z\,\overline{(\underline{n} + \underline{n}_1 d\alpha)} = z\,\underline{n} + (A\,\underline{a} + z\,\underline{n}_1)\,d\alpha + \underline{s}$$

$$\underline{H}_o = \underline{D}_o + z\,\overline{(\underline{k} + \underline{k}_2 d\beta)} = z\,\underline{k} + (B_o\,\underline{j} + z\,\underline{k}_2)\,d\beta$$

$$\underline{H} = \underline{D} + z\,\overline{(\underline{n} + \underline{n}_2 d\beta)} = z\,\underline{n} + (B\,\underline{b} + z\,\underline{n}_2)\,d\beta + \underline{s}$$

By subtraction, we obtain the following vectors:

$$\underline{O_oC_o} = A_o\,d\alpha\,\underline{i} \qquad \underline{O_oD_o} = B_o\,d\beta\,\underline{j}$$

$$\underline{OC} = A\,d\alpha\,\underline{a} \qquad \underline{OD} = B\,d\beta\,\underline{b} \tag{38}$$

$$\underline{F_oG_o} = (A_o\underline{i} + z\,\underline{k}_1)d\alpha \qquad \underline{F_oH_o} = (B_o\underline{j} + z\,\underline{k}_2)d\beta$$

$$\underline{FG} = (A\,\underline{a} + z\,\underline{n}_1)d\alpha \qquad \underline{FH} = (B\,\underline{b} + z\,\underline{n}_2)d\beta$$

Using these expressions, we may write equations expressing the strain components as functions of z. It is here that the assumption that the parametric curves are lines of curvature of the undeformed middle surface is of importance. For this is the condition that the faces OFHD and OFGC, before deformation, be plane and normal to each other. (See Weatherburn "Differential Geometry", page 66.) When the latter condition is fulfilled, the strains may be expressed by the following formulas, where use is made of Equations (38):

$$\epsilon_x = \frac{|\underline{FG}| - |\underline{F_oG_o}|}{|\underline{F_oG_o}|} = \frac{|A\,\underline{a} + z\,\underline{n}_1| - |A_o\,\underline{i} + z\,\underline{k}_1|}{|A_o\,\underline{i} + z\,\underline{k}_1|}$$

$$\epsilon_y = \frac{|\underline{FH}| - |\underline{F_oH_o}|}{|\underline{F_oH_o}|} = \frac{|B\,\underline{b} + z\,\underline{n}_2| - |B_o\underline{j} + z\,\underline{k}_2|}{|B_o\underline{j} + z\,\underline{k}_2|} \tag{39}$$

$$\gamma_{xy} = \frac{(\underline{FG}) \bullet (\underline{FH})}{|\underline{FG}| \quad |\underline{FH}|} = \frac{(A\,\underline{a} + z\,\underline{n}_1) \bullet (B\,\underline{b} + z\,\underline{n}_2)}{|A\,\underline{a} + z\,\underline{n}_1| \quad |B\,\underline{b} + z\,\underline{n}_2|}$$

Similarly, the strain components of the middle surface (z = 0) are given by the following equations:

$$\epsilon_{x_{z=0}} = \frac{|\underline{OC}| - |\underline{O_oC_o}|}{|\underline{O_oC_o}|} = \frac{A - A_o}{A_o}$$

$$\epsilon_{y_{z=0}} = \frac{|\underline{OD}| - |\underline{O_oD_o}|}{|\underline{O_oD_o}|} = \frac{B - B_o}{B_o} \qquad (40)$$

$$\gamma_{xy_{z=0}} = \frac{(\underline{OC}) \bullet (\underline{OD})}{|\underline{OC}| \quad |\underline{OD}|} = \underline{a} \bullet \underline{b} = \frac{F}{A\,B}$$

In order that the integrals of Equations (2) may be completely evaluated, it will be necessary to obtain expressions for the curvatures $\frac{1}{R_\alpha}$ and $\frac{1}{R_\beta}$. These curvatures are given by the following equations: (See Weatherburn "Differential Geometry", page 62.)

$$\frac{1}{R_{\alpha_o}} = \frac{L_o}{A_o^2} \quad ; \quad \frac{1}{R_{\beta_o}} = \frac{P_o}{B_o^2} \qquad \text{(undeformed shell)}$$

$$\frac{1}{R_\alpha} = \frac{L}{A^2} \quad ; \quad \frac{1}{R_\beta} = \frac{P}{B^2} \qquad \text{(deformed shell)} \qquad (41)$$

For the undeformed shell, these quantities are the principal curvatures, the parametric curves being lines of curvature. For the deformed shell, however, the parametric curves are no longer lines of curvature, and Equations (41) give the normal curvatures in the directions of the parametric curves, not the principal curvatures.

We now substitute Equations (39) and (41) in Equations (2) and carry out the integrations with respect to z. There will result expressions for the force and moment components in terms of the deformation components of the middle surface, and of their derivatives. These equations are substituted in the equations of equilibrium (11) and (14) or (33) and (36). The first two of Equations (14) (or (36)) are solved for Q_x and Q_y, and these expressions are then substituted in Equations (11) (or (33)). The latter equations are thus reduced to three simultaneous partial differential equations in the three components of deformation, u, v, w, of the middle surface.

PART II.

We shall now apply the theory developed in Part I. to set up the equations for a shell whose middle surface has the form of a circular cylinder. We shall take as parameters describing a point which moves on the middle surface, the distance x from the point to a fixed plane normal to the axis of the cylinder, measured along the generator through the point, and the angle ϕ between a fixed axial plane and the axial plane through the point; in the equations of Part I., we replace α by x and β by ϕ. It will be shown later that the parametric curves, so defined, are lines of curvature of the middle surface. (See page 30 .) Let the radius of the undeformed middle surface be a. Consider a set of fixed, right-handed, orthogonal axes, X, Y, Z, such that the X axis coincides with the axis of the cylinder. Let ϕ be the angle between the XY plane and a moving axial plane, measured positive about the X axis according to the right-hand rule. Then if $\underline{I}$, $\underline{J}$, $\underline{K}$, are unit vectors in the directions of these axes, the equation of the undeformed middle surface may be written in the following form:

$$\underline{r} = x\,\underline{I} + a \cos\phi\,\underline{J} + a \sin\phi\,\underline{K} \tag{42}$$

As before, we shall denote differentiations with respect to x and ϕ by the subscripts 1 and 2, respectively. Differentiating Equation (42), we obtain:

$$\begin{aligned} &\underline{r}_1 = \underline{I} \qquad && \underline{r}_2 = -a \sin\phi\,\underline{J} + a \cos\phi\,\underline{K} \\ &\underline{r}_{11} = \underline{r}_{12} = 0 \qquad && \underline{r}_{22} = -a \cos\phi\,\underline{J} - a \sin\phi\,\underline{K} \end{aligned} \tag{43}$$

Substituting Equations (43) in Equations (4) and (6), we may calculate the following quantities:

$$A = \sqrt{\underline{r}_1^2} = 1 \ ; \quad B = \sqrt{\underline{r}_2^2} = a \ ; \quad F = \underline{r}_1 \cdot \underline{r}_2 = 0 \ ; \quad H = \sqrt{A^2B^2 - F^2} = a$$

$$\underline{i} = \frac{1}{A}\underline{r}_1 = \underline{I} \ ; \quad \underline{j} = \frac{1}{B}\underline{r}_2 = -\sin\phi\,\underline{J} + \cos\phi\,\underline{K} \ ; \qquad (44)$$

$$\underline{k} = \frac{\underline{r}_1 \times \underline{r}_2}{H} = -\cos\phi\,\underline{J} - \sin\phi\,\underline{K}$$

$$L = \underline{k} \cdot \underline{r}_{11} = 0 \ ; \quad S = \underline{k} \cdot \underline{r}_{12} = 0 \ ; \quad P = \underline{k} \cdot \underline{r}_{22} = a$$

Since A and B are constants, their derivatives vanish, and we have the following expressions for the derivatives of the unit vectors, from Equations (6):

$$\underline{i}_1 = 0 \ ; \quad \underline{j}_1 = 0 \ ; \quad \underline{k}_1 = 0 \qquad (45)$$

$$\underline{i}_2 = 0 \ ; \quad \underline{j}_2 = \underline{k} \ ; \quad \underline{k}_2 = -\underline{j}$$

From the above results, it is seen that the unit vector $\underline{k}$ is normal to the surface and directed inward. It will be convenient to redefine the vector $\underline{k}$ so that it is directed outward. The vector $\underline{j}$ will be left unchanged; i.e., directed according to increasing ϕ. The vector $\underline{i}$, therefore, must be reversed in sense, in order that the triad of unit vectors $\underline{i}$, $\underline{j}$, $\underline{k}$ may form a right-handed system of vectors. The only reason for making this change is to make the results which are to be obtained agree more closely with results which have been obtained by other writers. It seems to be the usual practice to take w, the radial component of deformation, as positive when it is in the outward direction; in order to adhere to this convention, we direct $\underline{k}$ outward also. Otherwise, it would be necessary to change the sign of w throughout all results.

In terms of the new system of unit vectors, defined in the preceding paragraph, Equations (42) and (45) may be rewritten as follows:

$$\underline{r} = x\,\underline{i} + a\,\underline{k}$$

$$\underline{i}_{,1} = \underline{i}_{,2} = \underline{j}_{,1} = \underline{k}_{,1} = 0 \qquad (46)$$

$$\underline{j}_{,2} = -\underline{k} \; ; \quad \underline{k}_{,2} = \underline{j}$$

Using these equations to recalculate the quantities of Equations (4) and (6), we find that:

$$A = 1 \; ; \; B = a \; ; \; H = a \; ; \; P = -a \; ; \; F = L = S = 0 \qquad (47)$$

These are identical with the results of Equation (44), except that P has changed in sign. It is to be noted that the quantities F and S are both zero. This is the condition that the parametric curves be the lines of curvature of the surface.

Using the results of Equations (46) and (47) in Equations (11) and (14), we obtain the following equilibrium equations, valid on the assumption that the effect of deformation on the equilibrium of the element is negligible:

$$a(N_x)_{,1} + (N_{\phi x})_{,2} + aX = 0$$

$$a(N_{x\phi})_{,1} + (N_\phi)_{,2} - Q_\phi + aY = 0$$

$$a(Q_x)_{,1} + (Q_\phi)_{,2} + N_\phi + aZ = 0 \qquad (48)$$

$$a(M_{x\phi})_{,1} + (M_\phi)_{,2} - aQ_\phi = 0$$

$$a(M_x)_{,1} + (M_{\phi x})_{,2} - aQ_x = 0$$

$$a(N_{x\phi} - N_{\phi x}) + M_{\phi x} = 0$$

We now consider the deformed surface, and write the deformation vector of Equation (17):

$$\underline{s} = u(x,\phi)\underline{i} + v(x,\phi)\underline{j} + w(x,\phi)\underline{k} \qquad (49)$$

where the unit vectors are those used in Equations (46). Adding the first of Equations (46) and (49), we obtain the vector equation of the deformed middle surface, as follows:

$$\underline{r}' = \underline{r} + \underline{s} = (x + u)\underline{i} + v\underline{j} + (a + w)\underline{k} \qquad (50)$$

Differentiating Equation (50), and making use of Equations (46) for the derivatives of the unit vectors, we obtain the following results:

$$\underline{r}'_{,1} = (1 + u_{,1})\underline{i} + v_{,1}\underline{j} + w_{,1}\underline{k}$$

$$\underline{r}'_{,2} = u_{,2}\underline{i} + (a + w + v_{,2})\underline{j} + (w_{,2} - v)\underline{k}$$

$$\underline{r}'_{,11} = u_{,11}\underline{i} + v_{,11}\underline{j} + w_{,11}\underline{k} \qquad (51)$$

$$\underline{r}'_{,12} = u_{,12}\underline{i} + (v_{,12} + w_{,1})\underline{j} + (w_{,12} - v_{,1})\underline{k}$$

$$\underline{r}'_{,22} = u_{,22}\underline{i} + (v_{,22} - v + 2w_{,2})\underline{j} + (-a - 2v_{,2} + w_{,22} - w)\underline{k}$$

We now proceed to calculate the quantities of Equations (18), (19), (20), and (29). In making these calculations, it is assumed that the deformation components, u, v, w, are small quantities, so that terms of degree higher than one in these quantities and their derivatives may be neglected. Frequent use, in which advantage is taken of this approximation, will be made of the binomial theorem.

From Equations (18):

$$E = \underline{r}_1'^2 = 1 + 2u_1 \ ; \quad A = \sqrt{E} = 1 + u_1$$

$$G = \underline{r}_2'^2 = a^2(1 + 2\frac{v_2}{a} + 2\frac{w}{a}) \ ; \quad B = \sqrt{G} = a(1 + \frac{v_2}{a} + \frac{w}{a})$$

$$F = \underline{r}_1' \cdot \underline{r}_2' = u_2 + av_1$$

$$H^2 = EG - F^2 = a^2(1 + 2u_1 + 2\frac{v_2}{a} + 2\frac{w}{a})$$

$$H = a(1 + u_1 + \frac{v_2}{a} + \frac{w}{a})$$

$$\underline{a} = \frac{1}{A}\underline{r}_1' = \underline{i} + v_1\underline{j} + w_1\underline{k} \tag{52}$$

$$\underline{b} = \frac{1}{B}\underline{r}_2' = \frac{u_2}{a}\underline{i} + \underline{j} + (\frac{w_2}{a} - \frac{v}{a})\underline{k}$$

$$\underline{n} = \frac{\underline{r}_1' \times \underline{r}_2'}{H} = -w_1\underline{i} + (\frac{v}{a} - \frac{w_2}{a})\underline{j} + \underline{k}$$

$$L = \underline{n} \cdot \underline{r}_{11}' = w_{11}$$

$$S = \underline{n} \cdot \underline{r}_{12}' = w_{12} - v_1$$

$$P = \underline{n} \cdot \underline{r}_{22}' = -a(1 + 2\frac{v_2}{a} + \frac{w}{a} - \frac{w_{22}}{a})$$

Also: $A_2 = u_{12}$; $B_1 = v_{12} + w_1$

From Equations (19):

$$\theta = \cos \chi = \frac{F}{AB} = \frac{u_2}{a} + v_1 \tag{53}$$

$$\sin \chi = \frac{H}{AB} \cong 1$$

$$\theta_1 = \frac{u_{12}}{a} + v_{11} \;;\quad \theta_2 = \frac{u_{22}}{a} + v_{12}$$

The result, $\sin \chi = 1$, verifies, for the circular cylinder, the validity of the approximation noted on page 13 just below Equation (19).

From Equations (20):

$$\lambda = \frac{1}{2H^2}(2EF_1 - EE_2 - FE_1) = \frac{V_{11}}{a}$$

$$\mu = \frac{1}{2H^2}(EG_1 - FE_2) = \frac{1}{a}(V_{12} + W_1)$$

$$\Gamma = \frac{1}{H^2}(FS - GL) = -w_{11} \tag{54}$$

$$\Lambda = \frac{1}{H^2}(FL - ES) = \frac{1}{a^2}(v_1 - w_{12})$$

$$C = \frac{1}{H^2}(FP - GS) = -\left(\frac{u_2}{a} + w_{12}\right)$$

$$D = \frac{1}{H^2}(FS - EP) = \frac{1}{a}\left(1 - \frac{W}{a} - \frac{W_{22}}{a}\right)$$

From Equations (29):

$$p_x = -B\Lambda = -\frac{1}{a}(v_1 - w_{12}) \qquad p_\phi = -BD = -\left(1 + \frac{V_2}{a} - \frac{W_{22}}{a}\right)$$

$$q_x = -\frac{L}{A} = -w_{11} \qquad q_\phi = -\frac{S}{A} = v_1 - w_{12}$$

$$r_x = \lambda \frac{B}{A} = v_{11} \qquad r_\phi = \mu \frac{B}{A} = v_{12} + w_1$$

$$\xi_x = \theta_1 - r_x = \frac{u_{12}}{a} \qquad \xi_\phi = \theta_2 - r_\phi = \frac{u_{22}}{a} - w_1 \tag{55}$$

$$\eta_x = \theta r_x = 0 \qquad \eta_\phi = \theta r_\phi = 0$$

$$\zeta_x = p_x - \theta q_x = -\frac{1}{a}(v_1 - w_{12}) \qquad \zeta_\phi = p_\phi - \theta q_\phi = -(1 + \frac{v_2}{a} - \frac{w_{22}}{a})$$

$$\psi_x = -\theta p_x - q_x = w_{11} \qquad \psi_\phi = -\theta p_\phi - q_\phi = \frac{u_2}{a} + w_{12}$$

Using the results of Equations (52), (53), (54), and (55), in Equations (33) and (36), we obtain the following equilibrium equations which take into consideration the effect of deformation on the equilibrium of the shell element:

$$\begin{aligned}
& a(N_x)_1 (1 + \frac{v_2}{a} + \frac{w}{a}) + (N_{\phi x})_2 (1 + u_1) + (N_{x\phi})_1 (u_2 + a v_1) \\
& \quad + (N_x - N_\phi)(v_{12} + w_1) + (N_{\phi x} + N_{x\phi}) u_{12} + a w_{11} Q_x - (v_1 - w_{12}) Q_\phi \\
& \quad\quad + a(1 + u_1 + \frac{v_2}{a} + \frac{w}{a}) X' = 0 \\
& a(N_{x\phi})_1 (1 + \frac{v_2}{a} + \frac{w}{a}) + (N_\phi)_2 (1 + u_1) - (N_x)_1 (u_2 + a v_1) \\
& \quad + (N_{\phi x} + N_{x\phi})(v_{12} + w_1) - (N_x - N_\phi) u_{12} - (1 + u_1 + \frac{v_2}{a} - \frac{w_{22}}{a}) Q_\phi \\
& \quad\quad - (v_1 - w_{12}) Q_x + a(1 + u_1 + \frac{v_2}{a} + \frac{w}{a}) Y' = 0 \\
& a(Q_x)_1 (1 + \frac{v_2}{a} + \frac{w}{a}) + (Q_\phi)_2 (1 + u_1) + (v_{12} + w_1) Q_x + u_{12} Q_\phi - a w_{11} N_x \\
& \quad + (1 + u_1 + \frac{v_2}{a} - \frac{w_{22}}{a}) N_\phi + (N_{\phi x} + N_{x\phi})(v_1 - w_{12}) \\
& \quad\quad + a(1 + u_1 + \frac{v_2}{a} + \frac{w}{a}) Z' = 0 \\
& a(M_{x\phi})_1 (1 + \frac{v_2}{a} + \frac{w}{a}) + (M_\phi)_2 (1 + u_1) - (M_x)_1 (u_2 + a v_1) \qquad (56) \\
& \quad + (M_{\phi x} + M_{x\phi})(v_{12} + w_1) - (M_x - M_\phi) u_{12} - a(1 + u_1 + \frac{v_2}{a} + \frac{w}{a}) Q_\phi = 0
\end{aligned}$$

$$a(M_x)_{,1}\left(1 + \frac{v_2}{a} + \frac{w}{a}\right) + (M_{\phi x})_{,2}(1 + u_{,1}) + (M_{x\phi})_{,1}(u_{,2} + av_{,1})$$
$$+ (M_x - M_\phi)(v_{,12} + w_{,1}) + (M_{\phi x} + M_{x\phi})u_{,12} - (u_{,2} + av_{,1})Q_\phi$$
$$- a\left(1 + u_{,1} + \frac{v_2}{a} + \frac{w}{a}\right)Q_x = 0$$

$$a\left(1 + u_{,1} + \frac{v_2}{a} + \frac{w}{a}\right)(N_{x\phi} - N_{\phi x}) + \left(1 + u_{,1} + \frac{v_2}{a} - \frac{w_{22}}{a}\right)M_{\phi x}$$
$$+ aw_{,11}M_{x\phi} + (M_x - M_\phi)(v_{,1} - w_{,12}) = 0$$

Similar equations are easily derived on the basis of any of the possible assumptions mentioned on page 21.

When all terms involving the product of a force or moment component by a deformation component are neglected, Equations (56) reduce to Equations (48). It will be seen later that the froce and moment components are expressible, approximately, as linear functions of the deformation components and their derivatives. When such expressions are substituted in the two sets of equilibrium equations, the terms of degree higher than one in the deformation components are neglected, it is seen that Equations (56) again reduce to Equations (48). The utility of Equations (56) occurs in the solution of stability problems; however, no further use of these equations will be made in this thesis.

We now turn to the calculation of the strain components. Substituting Equations (52) and (54) in the last two of Equations (21), we obtain the following expressions:

$$\underline{n}_{,1} = A\Gamma\,\underline{a} + B\Lambda\,\underline{b} = -w_{,11}\underline{a} + \frac{1}{a}(v_{,1} - w_{,12})\underline{b}$$

$$\underline{n}_2 = AC\,\underline{a} + BD\,\underline{b} = -(\frac{U_2}{a} + w_{/2})\underline{a} + (1 + \frac{V_2}{a} - \frac{W_{22}}{a})\underline{b}$$

Using these results, we obtain the following expressions for the various vectors which occur in Equations (39):

$$A\underline{a} + z\underline{n}_1 = (1 + u_{/} - zw_{//})\underline{a} + \frac{z}{a}(v_{/} - w_{/2})\underline{b}$$

$$B\underline{b} + z\underline{n}_2 = -z(\frac{U_2}{a} + w_{/2})\underline{a} + a\left[1 + \frac{V_2}{a} + \frac{W}{a} + \frac{z}{a}(1 + \frac{V_2}{a} - \frac{W_{22}}{a})\right]\underline{b}$$

$$= -z(\frac{U_2}{a} + w_{/2})\underline{a} + (a + z)(1 + \frac{V_2}{a} + \frac{W}{a+z} - \frac{z}{a}\cdot\frac{W_{22}}{a+z})\underline{b}$$

$$A_o\,\underline{i} + z\underline{k}_1 = \underline{i}$$

$$B_o\,\underline{j} + z\underline{k}_2 = (a + z)\underline{j}$$

Using expressions of the type, $|\underline{e}| = \sqrt{\underline{e}^2}$, for the absolute value of a vector, we obtain the following expressions for the absolute values of the vectors of Equations (39), where, as before, terms of degree higher than one in the deformation components are neglected:

$$|A\underline{a} + z\underline{n}_1| = 1 + u_{/} - zw_{//}$$

$$|B\underline{b} + z\underline{n}_2| = (a + z)(1 + \frac{V_2}{a} + \frac{W}{a+z} - \frac{z}{a}\cdot\frac{W_{22}}{a+z})$$

$$|A_o\underline{i} + z\underline{k}_1| = 1$$

$$|B_o\underline{j} + z\underline{k}_2| = a + z \tag{57}$$

$$(A\underline{a} + z\underline{n}_1)\bullet(B\underline{b} + z\underline{n}_2) = u_2 + \frac{V_1}{a}(a + z)^2 - \frac{z}{a}(2a + z)w_{/2}$$

$$|A\underline{a} + z\underline{n}_1||B\underline{b} + z\underline{n}_2| = (a + z)(1 + u_{/} - zw_{//} + \frac{V_2}{a} + \frac{W}{a+z} - \frac{z}{a}\cdot\frac{W_{22}}{a+z})$$

Substituting Equations (57) in Equations (39), we obtain the following equations for the strain components:

$$\epsilon_x = u_{,1} - zw_{,11}$$

$$\epsilon_\phi = \frac{v_{,2}}{a} + \frac{w}{a+z} - \frac{z}{a} \cdot \frac{w_{,22}}{a+z} \qquad (58)$$

$$\gamma_{x\phi} = \frac{u_{,2}}{a+z} + \frac{v_{,1}}{a}(a + z) - \frac{z}{a}\left(\frac{2a+z}{a+z}\right)w_{,12}$$

Substituting Equations (47) and (52) in Equations (41), we obtain the following expressions for the curvature of the parametric curves:

$$\frac{1}{R_{x_0}} = 0 \quad ; \quad \frac{1}{R_{\phi_0}} = -\frac{1}{a} \qquad (59)$$

$$\frac{1}{R_x} = w_{,11} \quad ; \quad \frac{1}{R_\phi} = -\frac{1}{a}\left(1 - \frac{w}{a} - \frac{w_{,22}}{a}\right)$$

It was remarked on page 26, that the quantities $\frac{1}{R_x}$ and $\frac{1}{R_\phi}$ for the deformed surface are normal curvatures in the directions of the parametric curves. However, it may be shown by calculation of the principle curvatures (see Weatherburn, "Differential Geometry", page 69) that the latter are equal to the curvatures of Equations (59) to the degree of approximation used in writing these equations; i.e., when terms of degree higher than one in the deformation components are neglected.

We now calculate the force and moment components which are given by Equations (2). The stress components are first obtained by substituting Equations (58) into the second group of Equations (37).

The resulting expressions for the stress components, together with the curvature expressions from Equations (59), are then substituted in Equations (2), quantities of degree higher than one in the deformation components are neglected, and the integration with respect to z is carried out. In carrying out the integration, we make the assumption that $t \ll a$; whence, we may expand the quantity $(1 + \frac{z}{a})^{-1}$ by the binomial theorem, retaining only the first four terms of the expansion, thus obtaining the approximate result:

$$(1 + \frac{z}{a})^{-1} \cong 1 - \frac{z}{a} + (\frac{z}{a})^2 - (\frac{z}{a})^3 \tag{60}$$

In this way, we obtain the following expression for N_x:

$$N_x = \frac{E}{1-\nu^2} \int_{-\frac{t}{2}}^{\frac{t}{2}} (\epsilon_x + \nu\epsilon_\phi)(1 - \frac{z}{R_\phi}) dz = \frac{E}{1-\nu^2} \left[(u_{,1} + \nu \frac{v_{,2}}{a} + \nu \frac{w}{a}) t - \frac{t^3}{12 a} w_{,11} \right]$$

A slightly different notation for derivatives will simplify the writing of this and similar expressions. We shall denote by a prime the derivative with respect to x, multiplied by the radius a; and a dot shall denote the derivative with respect to ϕ, thus:

$$(\quad)' \equiv a(\quad)_{,1} \quad ; \quad (\quad)^{\bullet} \equiv (\quad)_{,2} \tag{61}$$

We shall introduce, also, the following notation:

$$D = \frac{Et}{1-\nu^2} \; ; \quad K = \frac{Et^3}{12(1-\nu^2)} \; ; \quad k = \frac{K}{Da^2} = \frac{t^2}{12 a^2} \tag{62}$$

Using the notation of Equations (61) and (62), we obtain the following expressions for the force and moment components from Equations (2):

$$
\begin{aligned}
N_x &= \frac{D}{a}(u' + \nu \dot{v} + \nu w) - \frac{K}{a^3} w'' = \frac{D}{a}(u' + \nu \dot{v} + \nu w - k w'') \\
N_\phi &= \frac{D}{a}(\dot{v} + w + \nu u') + \frac{K}{a^3}(\ddot{w} + w) = \frac{D}{a}(\dot{v} + w + \nu u' + k\,[\ddot{w} + w]) \\
N_{x\phi} &= \frac{D}{a}\left(\frac{1-\nu}{2}\right)(\dot{u} + v') + \frac{K}{a^3}\left(\frac{1-\nu}{2}\right)(v' - \dot{w}') = \frac{D}{a}\left(\frac{1-\nu}{2}\right)(\dot{u} + v' + k\,[v' - \dot{w}']) \\
N_{\phi x} &= \frac{D}{a}\left(\frac{1-\nu}{2}\right)(\dot{u} + v') + \frac{K}{a^3}\left(\frac{1-\nu}{2}\right)(\dot{u} + \dot{w}') = \frac{D}{a}\left(\frac{1-\nu}{2}\right)(\dot{u} + v' + k\,[\dot{u} + \dot{w}']) \\
M_x &= \frac{K}{a^2}(w'' - u' + \nu \ddot{w} - \nu \dot{v}) = Dk(w'' - u' + \nu \ddot{w} - \nu \dot{v}) \qquad (63) \\
M_\phi &= \frac{K}{a^2}(\ddot{w} + w + \nu w'') = Dk(\ddot{w} + w + \nu w'') \\
M_{x\phi} &= \frac{K}{a^2}(1 - \nu)(\dot{w}' - v') = Dk(1 - \nu)(\dot{w}' - v') \\
M_{\phi x} &= \frac{K}{a^2}(1 - \nu)\left(\dot{w}' + \frac{\dot{u}}{2} - \frac{v'}{2}\right) = Dk(1 - \nu)\left(\dot{w}' + \frac{\dot{u}}{2} - \frac{v'}{2}\right)
\end{aligned}
$$

The eight Equations (63) together with the six equilibrium Equations (48), constitute a system of fourteen simultaneous differential equations in thirteen dependent variables, the ten force and moment components and the three deformation components. This apparent excess of equations over dependent variables is accounted for by noticing that, if the third, fourth, and last of Equations (63) are introduced into the last of Equations (48), the result is an identity. We therefore have, in reality, only thirteen independent differential equations, and we take these to be the first five of Equations (48) and Equations (63). From these thirteen equations, we eliminate the ten force and moment components to obtain three equations in the three deformation components.

We proceed as follows. The fourth and fifth of Equations (48) are solved for Q_ϕ and Q_x, respectively, and these results are substituted in the first three of Equations (48), giving the following set of equations:

$$
\begin{aligned}
& N_x' + \dot{N}_{\phi x} + aX = 0 \\
& N_{x\phi}' + \dot{N}_\phi - \frac{1}{a}\left(\dot{M}_\phi + M_{x\phi}'\right) + aY = 0 \qquad (64) \\
& \frac{1}{a}\left(\ddot{M}_\phi + M_{x\phi}^{\prime\cdot} + M_{\phi x}^{\prime\cdot} + M_x''\right) + N_\phi + aZ = 0
\end{aligned}
$$

Substituting Equations (63) in Equations (64), we obtain the following three equations in the deformation components:

$$
\begin{aligned}
& u'' + \frac{1-\nu}{2}\ddot{u} + \frac{1+\nu}{2}\dot{v}' + \nu w' + k\left(\frac{1-\nu}{2}\ddot{u} + \frac{1-\nu}{2}w^{\prime\cdot\cdot} - w'''\right) + \frac{a^2}{D}X = 0 \\
& \frac{1+\nu}{2}\dot{u}' + \frac{1-\nu}{2}v'' + \ddot{v} + \dot{w} + k\left(\frac{3[1-\nu]}{2}v'' - \frac{3-\nu}{2}w^{\prime\prime\cdot}\right) + \frac{a^2}{D}Y = 0 \qquad (65) \\
& \nu u' + \dot{v} + w + k\left(\frac{1-\nu}{2}u^{\prime\cdot\cdot} - u''' - \frac{3-\nu}{2}v^{\prime\prime\cdot} + w^{IV} + 2w^{\prime\prime\cdot\cdot} + \ddddot{w} + 2\ddot{w} + w\right) + \frac{a^2}{D}Z = 0
\end{aligned}
$$

These are the differential equations of the circular cylindrical shell in the form in which they will be used in later parts of this thesis.

PART III.

In this Part, we shall set up the general expression for the potential energy of strain for the case of the circular cylindrical shell and, by means of the principle of virtual work, deduce, again, the differential Equations (65), together with the conditions which must be satisfied by their solution at the boundary edges of the shell. The general expression of the principle of virtual work, for the case under consideration, may be written as follows:

$$\delta\left\{\iiint\left[V-(X_z u_z+Y_z v_z-Z_z w_z)\right](1+\frac{z}{a})dz\, a d\phi\, dx-\iint(\bar{X}_z u_z+\bar{Y}_z v_z+\bar{Z}_z w_z)dz\, ds'\right\}=0 \quad (66)$$

where V is the strain energy per unit volume, X_z, Y_z, Z_z, are body forces per unit volume, and $\bar{X}_z$, $\bar{Y}_z$, $\bar{Z}_z$, are edge forces per unit of area of edge surface, the subscript z being used to emphasize the fact that the corresponding quantity is a function of z as well as of x and ϕ. The edge surface is assumed to be a developable surface whose generators are the normals to the middle surface. For the sake of brevity, we rewrite Equation (66) as follows:

$$\delta\left\{\iiint(V-B)(1+\frac{z}{a})dz\, a d\phi\, dx-\iint F\, dz\, ds'\right\}=0 \quad (67)$$

In these equations, ds' represents the element of arc length of a curve, in the edge surface, which is parallel to the arc ds of the intersection of the edge surface and middle surface, ds and ds', both being included between the same two neighboring generators of the edge surface. Let R_s be the radius of normal curvature of the middle surface along its curve of intersection with the boundary surface. Then ds and ds' are related approximately as follows:

$$ds' = ds\ (1 - \frac{z}{R_s}) \tag{68}$$

In writing this equation, we make the assumption that the actual edge surface may be replaced by a series of plane surfaces whose lines of intersection are the generators of the actual edge surface at neighboring points of the boundary curve. By the boundary curve is meant the curve of intersection of the boundary surface and middle surface.

Consider the triad of moving orthogonal axes x, y, z, as described on page 2, which are related to the undeformed middle surface. Let ψ be the angle between the positive x axis and the outward drawn normal to the boundary curve, the normal being drawn in the tangent plane to the middle surface; i.e., in the xy plane. Then from Euler's theorem on normal curvature, we have the following expression for $\frac{1}{R_s}$:

$$\frac{1}{R_s} = -\frac{1}{a}\cos^2\psi \tag{69}$$

where use is made of Equation (59). Using Equations (68) and (69), and taking the variation sign under the integrals, Equation (67) may be rewritten as follows:

$$\iiint(\delta V - \delta B)(1 + \frac{z}{a})dz\ ad\phi\ dx - \iint \delta F\ (1 + \frac{z}{a}\cos^2\psi)dz\ ds = 0 \tag{70}$$

or, more briefly:

$$I_1 - I_2 - I_3 = 0$$

$$I_1 \equiv \iiint \delta V\ (1 + \frac{z}{a})dz\ ad\phi\ dx \quad ; \quad I_2 \equiv \iiint \delta B\ (1 + \frac{z}{a})dz\ ad\phi\ dx \tag{71}$$

$$I_3 = \iint \delta F \, (1 + \tfrac{z}{a}\cos^2\psi)\, dz\, ds$$

Let us now consider I_3, which may be written out as follows:

$$I_3 = \iint (\overline{X}_z \delta u_z + \overline{Y}_z \ \delta v_z + \overline{Z}_z \ \delta w_z)(1 + \tfrac{z}{a}\cos^2\psi)\, dz\, ds \qquad (72)$$

where it is assumed, of course, that the applied forces and stresses remain constant during the virtual displacement. The quantities u_z, v_z, w_z, may be expressed in terms of the deformation components u, v, w, of the middle surface in the following manner. From Equation (52), we have the following expression for the unit normal to the deformed middle surface:

$$\underline{n} = -\frac{w'}{a}\underline{i} + \left(\frac{v}{a} - \frac{w^{\bullet}}{a}\right)\underline{j} + \underline{k}$$

where the notation of Equation (61) has been introduced. If $\underline{s}$ is the deformation vector of a point of the strained middle surface, as given by Equation (49), then the deformation vector $\underline{s}_z$ of the corresponding point at a distance z from the middle surface is given by the equation:

$$\underline{s}_z = \underline{s} + z\underline{n} - z\underline{k} = \left(u - z\frac{w'}{a}\right)\underline{i} + \left[v\left(1 + \frac{z}{a}\right) - z\frac{w^{\bullet}}{a}\right]\underline{j} + w\underline{k} \qquad (73)$$

where the undeformed position vector, $z\underline{k}$, has been subtracted from the deformed position vector to give the deformation vector. The coefficients of the unit vectors in Equation (73) are the deformation components u_z, v_z, w_z, of Equations (66) and (72). Introducing Equation (73) into Equation (72), the latter may be rewritten as follows:

$$I_3 = \iint \left\{ \bar{X}_z \left(\delta u - z\frac{\delta w'}{a}\right) + \bar{Y}_z \left(\delta v\left[1 + \frac{z}{a}\right] - z\frac{\delta w^{\cdot}}{a}\right) + \bar{Z}_z \delta w \right\} \left(1 + \frac{z}{a}\cos^2\psi\right) dz\, ds \quad (74)$$

We introduce the following definitions of boundary forces and moments:

$$\bar{N}_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \bar{X}_z \left(1 + \frac{z}{a}\cos^2\psi\right) dz \qquad \bar{N}_\phi = \int_{-\frac{t}{2}}^{\frac{t}{2}} \bar{Y}_z \left(1 + \frac{z}{a}\cos^2\psi\right) dz$$

$$\bar{M}_x = -\int_{-\frac{t}{2}}^{\frac{t}{2}} \bar{X}_z\, z \left(1 + \frac{z}{a}\cos^2\psi\right) dz \qquad \bar{M}_\phi = -\int_{-\frac{t}{2}}^{\frac{t}{2}} \bar{Y}_z\, z \left(1 + \frac{z}{a}\cos^2\psi\right) dz \quad (75)$$

$$\bar{Q}_{x\phi} = -\int_{-\frac{t}{2}}^{\frac{t}{2}} \bar{Z}_z \left(1 + \frac{z}{a}\cos^2\psi\right) dz$$

Substituting these expressions in Equation (74), the latter may be rewritten as follows:

$$I_3 = \int \left[\bar{N}_x \delta u + \left(\bar{N}_\phi - \frac{\bar{M}_\phi}{a}\right) \delta v - \bar{Q}_{x\phi} \delta w + \frac{\bar{M}_x}{a} \delta w' + \frac{\bar{M}_\phi}{a} \delta w^{\cdot} \right] ds \quad (76)$$

Where the integration is carried out around the closed boundary curve. We now re-express the forces and moments appearing in this equation in terms on forces and moments acting parallel and normal to the boundary curve.

Consider a small triangular element of area of the middle surface bounded by an arc ds of the boundary curve and arcs dx and $ad\phi$ of parametric curves. This surface element, together with the normals to the middle surface along its boundary, determine a small element of volume of the shell. As noted on page 42, we assume that the surface of this element which is part of the boundary surface of the shell may be approximated by a plane surface. Now the forces and moments of Equation (75) act on this surface in the directions of the parametric curves. Let us resolve them in directions parallel to and normal to the boundary arc ds. Consider a set of moving orthogonal axes n, s, z, at each point of the boundary curve, and directed as follows: the z axis coinciding with the z axis of the set of axes x, y, z, described on page 2; the n axis coinciding with the normal to the boundary curve, positive when directed outward; and the s axis tangent to the boundary curve and so directed that when the x axis is revolved through the angle ψ about the z axis, going into the n axis, the y axis goes into the s axis. We shall denote by $\underline{n}$ and $\underline{s}$ unit vectors parallel to the n and s axes, respectively. Then $\underline{n}$ and $\underline{s}$ are related to $\underline{i}$ and $\underline{j}$ by the following equations:

$$\underline{n} = \underline{i}\cos\psi + \underline{j}\sin\psi \qquad \underline{i} = \underline{n}\cos\psi - \underline{s}\sin\psi \tag{77}$$

$$\underline{s} = -\underline{i}\sin\psi + \underline{j}\cos\psi \qquad \underline{j} = \underline{n}\sin\psi + \underline{s}\cos\psi$$

From the definitions of Equations (75), it is seen that the force and moment vectors acting on the boundary face may be written as follows:

$$\underline{F} = \overline{N}_x\underline{i} + \overline{N}_\phi\underline{j} - \overline{Q}_{x\phi}\underline{k} \tag{78}$$

$$\underline{M} = \overline{M}_\phi\underline{i} - \overline{M}_x\underline{j}$$

where the right-hand rule has been used in determining the signs in the second of the above equations. Substituting Equations (77) in Equations (78), these vectors may be re-expressed in terms of the unit vectors $\underline{n}$, $\underline{s}$, $\underline{k}$, as follows:

$$\underline{F} = \overline{N}_n \underline{n} + \overline{N}_s \underline{s} - \overline{Q}_{ns} \underline{k} \tag{79}$$

$$\underline{M} = \overline{M}_s \underline{n} - \overline{M}_n \underline{s}$$

where

$$\overline{N}_n = \overline{N}_x \cos\psi + \overline{N}_\phi \sin\psi \qquad \overline{M}_n = \overline{M}_x \cos\psi + \overline{M}_\phi \sin\psi \tag{80}$$

$$\overline{N}_s = -\overline{N}_x \sin\psi + \overline{N}_\phi \cos\psi \qquad \overline{M}_s = -\overline{M}_x \sin\psi + \overline{M}_\phi \cos\psi$$

$$\overline{Q}_{ns} = \overline{Q}_{x\phi}$$

Equations (80) may be inverted to give:

$$\overline{N}_x = \overline{N}_n \cos\psi - \overline{N}_s \sin\psi \qquad \overline{M}_x = \overline{M}_n \cos\psi - \overline{M}_s \sin\psi \tag{81}$$

$$\overline{N}_\phi = \overline{N}_n \sin\psi + \overline{N}_s \cos\psi \qquad \overline{M}_\phi = \overline{M}_n \sin\psi + \overline{M}_s \cos\psi$$

$$\overline{Q}_{x\phi} = \overline{Q}_{ns}$$

Corresponding to the variations δu and δv of the deformation components in the directions of the axes x and y, we define variations δn and δs of the deformation components in the directions of the n and s axes. The variation components in the two systems will be related exactly as the forces, as given by the first two of Equations (80) and (81). We may thus write the following equations between the variation components:

$$\delta n = \delta u \cos\psi + \delta v \sin\psi \qquad \delta u = \delta n \cos\psi - \delta s \sin\psi \tag{82}$$

$$\delta s = -\delta u \sin\psi + \delta v \cos\psi \qquad \delta v = \delta n \sin\psi + \delta s \cos\psi$$

The directional derivatives of δw in the directions of the n and s axes may be written as follows:

$$\delta w^{(n)} \equiv \frac{\partial \delta w}{\partial n} = \frac{\partial \delta w}{\partial x}\frac{dx}{dn} + \frac{\partial \delta w}{a\partial \phi}\frac{a\,d\phi}{dn} = \frac{1}{a}(\delta w' \cos\psi + \delta w^{\cdot} \sin\psi) \tag{83}$$

$$\delta w^{(s)} \equiv \frac{\partial \delta w}{\partial s} = \frac{\partial \delta w}{\partial x}\frac{dx}{ds} + \frac{\partial \delta w}{a\partial \phi}\frac{a\,d\phi}{ds} = \frac{1}{a}(-\delta w' \sin\psi + \delta w^{\cdot} \cos\psi)$$

and these equations may be inverted to give:

$$\delta w' = a(\delta w^{(n)} \cos\psi - \delta w^{(s)} \sin\psi) \tag{84}$$

$$\delta w^{\cdot} = a(\delta w^{(n)} \sin\psi + \delta w^{(s)} \cos\psi)$$

We now substitute Equations (81), (82), and (84) in Equation (76). Rearranging and collecting terms according to the variation components, we obtain:

$$I_3 = \int \Big[(\bar{N}_n - \frac{\bar{M}_n}{a}\sin^2\psi - \frac{\bar{M}_s}{a}\sin\psi\cos\psi)\delta n + (\bar{N}_s - \frac{\bar{M}_s}{a}\cos^2\psi - \frac{\bar{M}_n}{a}\sin\psi\cos\psi)\delta s \tag{85}$$
$$- \bar{Q}_{ns}\,\delta w + \bar{M}_n \delta w^{(n)} + \bar{M}_s \delta w^{(s)} \Big]\, ds$$

The following identity may be written:

$$\bar{M}_s \delta w^{(s)} \equiv \bar{M}_s \frac{\partial \delta w}{\partial s} = \frac{\partial}{\partial s}(\bar{M}_s \delta w) - \frac{\partial \bar{M}_s}{\partial s}\,\delta w \tag{86}$$

If Equation (86) is substituted in Equation (85), one of the terms obtained will be $\int \frac{\partial}{\partial s}(\bar{M}_s \delta w)ds$, which vanishes when integrated around the closed boundary curve, on the assumption that the quantity $\bar{M}_s \delta w$ is single-valued and continuous on the boundary. Thus Equation (85) may be rewritten as follows:

$$I_3 = \int\Big[(\bar{N}_n - \frac{\bar{M}_n}{a}\sin^2\psi - \frac{\bar{M}_s}{a}\sin\psi\cos\psi)\delta n + (\bar{N}_s - \frac{\bar{M}_s}{a}\cos^2\psi - \frac{\bar{M}_n}{a}\sin\psi\cos\psi)\delta s \tag{87}$$
$$- (\bar{Q}_{ns} + \frac{\partial\bar{M}_s}{\partial s})\delta w + \bar{M}_n \delta w^{(n)}\Big] ds$$

We now return to Equation (71) and consider the integral I_2. We have made the assumption that the loads are applied to the shell on the middle surface only, see page 2. Then in Equations (66) and (71) X_z, Y_z, Z_z, are to be replaced by X, Y, Z; and hence, at the same time, u_z, v_z, w_z, may be replaced by u, v, w, the latter quantities being the deformation components of the middle surface. I_2 may, therefore, be rewritten in the following form:

$$I_2 = \iint (X\,\delta u + Y\,\delta v - Z\,\delta w)\, a d\phi\, dx \tag{88}$$

Let us now evaluate the integral I_1. The variation of the strain energy per unit of volume is given by the following expressions: (See Timoshenko, "Theory of Elasticity", page 139):

$$\delta V = \sigma_x\,\delta\epsilon_x + \sigma_\phi\,\delta\epsilon_\phi + \tau_{x\phi}\,\delta\gamma_{x\phi} \tag{89}$$

$$\delta V = \delta\Big[\frac{E}{2(1-\nu^2)}(\epsilon_x^2 + \epsilon_\phi^2 + 2\nu\,\epsilon_x\,\epsilon_\phi + \frac{1-\nu}{2}\,\gamma_{x\phi}^2)\Big]$$

since, as it has been pointed out on page 22, ϵ_z, γ_{xz}, and $\gamma_{\phi z}$ are assumed to be zero.

The first of these expressions emphasizes the fact that, during the virtual displacement, the stresses are assumed to remain constant, the strains only undergoing small variations due to the virtual displacement. We now substitute Equations (58) for the strain components in the second of Equations (89), multiply both sides of the latter by the quantity $1 + \frac{z}{a}$, and integrate with respect to z over the shell thickness. In carrying out the integration, we shall make frequent use of the assumption of Equation (60). It will be convenient to treat the four terms of Equation (89) separately. We thus obtain the following equations, where the notation is that of Equations (61) and (62):

$$\frac{E}{2(1-\nu^2)}\int_{-\frac{t}{2}}^{\frac{t}{2}} \epsilon_x^2 (1 + \frac{z}{a})dz = \frac{D}{2a^2}\left[u'^2 + k(w''^2 - 2u'w'')\right]$$

$$\frac{E}{2(1-\nu^2)}\int_{-\frac{t}{2}}^{\frac{t}{2}} \epsilon_\phi^2 (1 + \frac{z}{a})dz = \frac{D}{2a^2}\left[v^{\cdot 2} + w^2 + 2v^\cdot w + k(w^{\cdot\cdot 2} + w^2 + 2ww^{\cdot\cdot})\right] \qquad (90)$$

$$\frac{E}{2(1-\nu^2)}\int_{-\frac{t}{2}}^{\frac{t}{2}} \epsilon_x\epsilon_\phi (1 + \frac{z}{a})dz = \frac{D}{2a^2}\left[u'v^\cdot + u'w + k(w^{\cdot\cdot}w'' - v^\cdot w'')\right]$$

$$\frac{E}{2(1-\nu^2)}\int_{-\frac{t}{2}}^{\frac{t}{2}} \gamma_{x\phi}^2 (1 + \frac{z}{a})dz = \frac{D}{2a^2}\left[u^{\cdot 2} + v'^2 + 2u^\cdot v' + k(u^{\cdot 2} + 3v'^2 + 4w'^{\cdot 2} + 2u^\cdot w'^\cdot - 6v'w'^\cdot)\right]$$

Combining Equations (90) according to Equation (89), we obtain the following expression for the strain energy of the shell per unit of area of the middle surface:

$$V = \frac{D}{2a^2}\Big\{u'^2 + v^{\cdot 2} + 2v^\cdot w + (1 + k)w^2 + 2\nu(u'v^\cdot + u'w)$$

$$+ \frac{1-\nu}{2}\left[(1 + k)u^{\cdot 2} + (1 + 3k)v'^2 + 2u^\cdot v'\right] \qquad (91)$$

$$+k\left[w''^2 - 2u'w'' + w^{\cdot\cdot 2} + 2ww^{\cdot\cdot} + 2\nu(w^{\cdot\cdot}w'' - v^\cdot w'') + \frac{1-\nu}{2}(4w'^{\cdot 2} + 2u^\cdot w'^\cdot - 6v'w'^\cdot)\right]\Big\}$$

Forming the variation of Equation (91) and integrating over the middle surface of the shell, we obtain the following expression for I_1:

$$I_1 = \frac{D}{a^2}\iint\Big\{u'\delta u' + v^\cdot\delta v^\cdot + w\,\delta v^\cdot + v^\cdot\delta w + (1 + k)w\,\delta w + \nu(u'\delta v^\cdot + v^\cdot\delta u' + u'\delta w + w\,\delta u')$$

$$+ \frac{1-\nu}{2}\left[(1 + k)u^\cdot\delta u^\cdot + (1 + 3k)v'\delta v' + u^\cdot\delta v' + v'\delta u^\cdot\right] + k\Big[w''\delta w'' - w''\delta u' \qquad (92)$$

$$- u'\delta w'' + w^{\cdot\cdot}\delta w^{\cdot\cdot} + w^{\cdot\cdot}\delta w + w\,\delta w^{\cdot\cdot} + \nu(w^{\cdot\cdot}\delta w'' + w''\delta w^{\cdot\cdot} - v^{\cdot}\delta w'' - w''\delta v^{\cdot})$$

$$+ \frac{1-\nu}{2}\,(4w'^{\cdot}\,\delta w'^{\cdot} + w'^{\cdot}\delta u^{\cdot} + u^{\cdot}\,\delta w'^{\cdot} - 3w'^{\cdot}\,\delta v' - 3v'\delta w'^{\cdot})\Big]\Big\}\ a d\phi\ dx$$

Equation (92) is now transformed by means of the following identities:

(a) $u'\,\delta u' = (u'\delta u)' - u''\,\delta u$

(b) $v^{\cdot}\,\delta v^{\cdot} = (v^{\cdot}\delta v)^{\cdot} - v^{\cdot\cdot}\,\delta v$

(c) $w\,\delta v^{\cdot} = (w\,\delta v)^{\cdot} - w^{\cdot}\,\delta v$

(d) $u'\,\delta v^{\cdot} = (u'\delta v)^{\cdot} - u'^{\cdot}\,\delta v$

(e) $v^{\cdot}\,\delta u' = (v^{\cdot}\delta u)' - v'^{\cdot}\,\delta u$

(f) $w\,\delta u' = (w\,\delta u)' - w'\,\delta u$

(g) $u^{\cdot}\,\delta u^{\cdot} = (u^{\cdot}\delta u)^{\cdot} - u^{\cdot\cdot}\,\delta u$

(h) $v'\,\delta v' = (v'\delta v)' - v''\,\delta v$

(i) $u^{\cdot}\,\delta v' = (u^{\cdot}\delta v)' - u'^{\cdot}\,\delta v$

(j) $v'\,\delta u^{\cdot} = (v'\delta u)^{\cdot} - v'^{\cdot}\,\delta u$

(k) $w''\,\delta w'' = (w''\delta w')' - (w'''\delta w)' + w^{iv}\,\delta w$ (93)

(l) $w''\,\delta u' = (w''\delta u)' - w'''\,\delta u$

(m) $u'\,\delta w'' = (u'\,\delta w')' - (u''\,\delta w)' + u'''\,\delta w$

(n) $w^{\cdot\cdot}\,\delta w^{\cdot\cdot} = (w^{\cdot\cdot}\,\delta w^{\cdot})^{\cdot} - (w^{\cdot\cdot\cdot}\delta w)^{\cdot} + w^{\cdot\cdot\cdot\cdot}\,\delta w$

$$\text{(o)} \qquad w\,\delta \ddot{w} = (w\,\delta \dot{w})^{\cdot} - (\dot{w}\,\delta w)^{\cdot} + \ddot{w}\,\delta w$$

$$\text{(p)} \qquad \ddot{w}\,\delta w'' = (\ddot{w}\,\delta \dot{w}')' - (\ddot{w}'\,\delta w)' + \ddot{w}''\,\delta w$$

$$\text{(q)} \qquad w''\,\delta \ddot{w} = (w''\,\delta \dot{w})^{\cdot} - (\dot{w}''\,\delta w)^{\cdot} + \ddot{w}''\,\delta w$$

$$\text{(r)} \qquad \dot{v}\,\delta w'' = (\dot{v}\,\delta \dot{w}')' - (\dot{v}'\,\delta w)' + \dot{v}''\,\delta w$$

$$\text{(s)} \qquad w''\,\delta \dot{v} = (w''\,\delta v)^{\cdot} - \dot{w}''\,\delta v$$

$$\text{(t)} \qquad \dot{w}'\,\delta \dot{u} = (\dot{w}'\,\delta u)^{\cdot} - \ddot{w}'\,\delta u$$

$$\text{(u)} \qquad \dot{u}\,\delta \dot{w}' = (\dot{u}\,\delta w')^{\cdot} - (\ddot{u}\,\delta w)' + \ddot{u}'\,\delta w$$

$$\text{(v)} \qquad \dot{w}'\,\delta v' = (\dot{w}'\,\delta v)' - \dot{w}''\,\delta v$$

$$\text{(w)} \qquad \dot{w}'\,\delta \dot{w}' = (\dot{w}'\,\delta \dot{w})' - (\dot{w}''\,\delta w)^{\cdot} + \ddot{w}''\,\delta w$$

$$= (\dot{w}'\,\delta w')^{\cdot} - (\ddot{w}'\,\delta w)' + \ddot{w}''\,\delta w$$

$$\text{(x)} \qquad v'\,\delta \dot{w}' = (v'\,\delta \dot{w})' - (v''\,\delta w)^{\cdot} + \dot{v}''\,\delta w$$

$$= (v'\,\delta w')^{\cdot} - (\dot{v}'\,\delta w)' + \dot{v}''\,\delta w$$

Equations (93) are now substituted in Equation (92); in making this substitution both forms of w) and the first form of x) are each used twice, and the second form of x) is used once. Equation (92) is thus transformed into the following:

$$I_1 = \frac{D}{a^2}\iint\Big\{-\Big[u'' + \frac{1-\nu}{2}\ddot{u} + \frac{1+\nu}{2}\dot{v}' + \nu w' + k\Big(\frac{1-\nu}{2}\ddot{u} + \frac{1-\nu}{2}\ddot{w}' - w'''\Big)\Big]\delta u$$

$$-\Big[\frac{1+\nu}{2}\dot{u}' + \frac{1-\nu}{2}v'' + \ddot{v} + \dot{w} + k\Big(\frac{3[1-\nu]}{2}v'' - \frac{3-\nu}{2}\dot{w}''\Big)\Big]\delta v$$

$$+\Big[\nu u' + \dot{v} + w + k\Big(\frac{1-\nu}{2}\ddot{u}' - u''' - \frac{3-\nu}{2}\dot{v}'' + w^{IV} + 2\ddot{w}'' + \overset{::}{w} + 2\ddot{w} + w\Big)\Big]\delta w\Big\}\,a\,d\phi\,dx$$

$$+\frac{D}{a^2}\iint\Big\{(u'+\nu v^{\cdot}+\nu w-kw'')\delta u+\frac{1-\nu}{2}\left[u^{\cdot}+(1+3k)v'-3kw'^{\cdot}\right]\delta v$$

$$-k\left[w'''-u''+\nu w'^{\cdot\cdot}-\nu v'^{\cdot}+\frac{1-\nu}{2}(2w'^{\cdot\cdot}+u^{\cdot\cdot}-v'^{\cdot})\right]\delta w \qquad (94)$$

$$+k(w''-u'+\nu w^{\cdot\cdot}-\nu v^{\cdot})\delta w'+k(1-\nu)(w'^{\cdot}-v^{\cdot\prime})\delta w^{\cdot}\Big\}' a d\phi\, dx$$

$$+\frac{D}{a^2}\iint\Big\{\frac{1-\nu}{2}\left[(1+k)u^{\cdot}+v'+kw'^{\cdot}\right]\delta u+(\nu u'+v^{\cdot}+w-k\nu w'')\delta v$$

$$-k\left[w^{\cdot\cdot\cdot}+w^{\cdot}+\nu w''^{\cdot}+(1-\nu)(w''^{\cdot}-v'')\right]\delta w+k(w^{\cdot\cdot}+w+\nu w'')\delta w^{\cdot}$$

$$+k(1-\nu)(w'^{\cdot}+\frac{u^{\cdot}}{2}-\frac{v'}{2})\delta w'\Big\}^{\cdot} a d\phi\, dx$$

For the sake of brevity, let the three integrals in this expression be denoted by I_4, I_5, and I_6, so that:

$$I_1 = I_4 + I_5 + I_6 \qquad (95)$$

it being assumed that the terms of Equations (94) and (95) are written in the same order.

By the use of Equations (63) and the fourth and fifth of Equations (48), the following relations may be verified:

$$Q_\phi = \frac{1}{a}(M'_{x\phi}+M^{\cdot}_{\phi}) = \frac{Dk}{a}\left[w^{\cdot\cdot\cdot}+w^{\cdot}+w''^{\cdot}-(1-\nu)v''\right]$$

$$Q_x = \frac{1}{a}(M'_x+M^{\cdot}_{\phi x}) = \frac{Dk}{a}(w'''-u''+w'^{\cdot\cdot}+\frac{1-\nu}{2}u^{\cdot\cdot}-\frac{1+\nu}{2}v'^{\cdot}) \qquad (96)$$

$$N_\phi - \frac{M_\phi}{a} = \frac{D}{a}(v^{\cdot}+w+\nu u'-k\nu w'')$$

$$N_{x\phi} - \frac{M_{x\phi}}{a} = \frac{D}{a}(\frac{1-\nu}{2})\left[u^{\cdot}+(1+3k)v'-3kw'^{\cdot}\right]$$

Using Equations (63) and (96), the sum of I_5 and I_6 may be written as follows:

$$I_5 + I_6 = \frac{1}{a}\iint\left\{\left[N_x\,\delta u + (N_{x\phi} - \frac{M_{x\phi}}{a})\delta v - Q_x\,\delta w + \frac{M_x}{a}\,\delta w' + \frac{M_{x\phi}}{a}\,\delta \dot{w}\right]' \right. \tag{97}$$
$$\left. + \left[N_{\phi x}\,\delta u + (N_\phi - \frac{M_\phi}{a})\delta v - Q_\phi \delta w + \frac{M_{\phi x}}{a}\,\delta w' + \frac{M_\phi}{a}\,\delta \dot{w}\right]^{\boldsymbol{\cdot}}\right\} a d\phi\; dx$$

We may transform the double integral of Equation (97) into a line integral around the boundary curve in the following manner: Consider the double integral of the function $\frac{\partial f}{\partial x}$ taken over the area A of the surface of a circular cylinder, A being bounded by a closed curve S. By carrying out the integration with respect to x, the following equations may be written:

$$\iint_A \frac{\partial f}{\partial x}\, dx\; ad\phi = \int_{\phi_1}^{\phi_2} f\Big]_{x_1(\phi)}^{x_2(\phi)} ad\phi = \int_S f\,\cos\psi\, ds \tag{98a}$$

where ψ is the angle defined on page 42, $x = x_1(\phi)$ and $x = x_2(\phi)$ are the equations of the boundary S written as functions of ϕ, and the integration is carried out in the positive direction around S, this direction being defined as the direction of the unit vector $\underline{s}$ described on page 45. Similarly, by integrating with respect to ϕ, we may write:

$$\iint_A \frac{\partial f}{\partial \phi}\, ad\phi\; dx = \int_{x_1}^{x_2} f\Big]_{\phi_1(x)}^{\phi_2(x)} a\; dx = \int_S af\,\sin\psi\, ds \tag{98b}$$

where $\phi = \phi_1(x)$ and $\phi = \phi_2(x)$ are the equations of the boundary S written as functions of x. Using Equations (98), we may rewrite Equation (97) in the following form:

$$I_5 + I_6 = \int\left\{\left[N_x\delta u + (N_{x\phi} - \frac{M_{x\phi}}{a})\,\delta v - Q_x\,\delta w + \frac{M_x}{a}\,\delta w' + \frac{M_{x\phi}}{a}\,\delta \dot{w}\right]\cos\psi \right. \tag{99}$$
$$\left. + \left[N_{\phi x}\delta u + (N_\phi - \frac{M_\phi}{a})\,\delta v - Q_\phi\,\delta w + \frac{M_{\phi x}}{a}\,\delta w' + \frac{M_\phi}{a}\,\delta \dot{w}\right]\sin\psi\right\} ds$$

where the integration is around the closed boundary curve in the positive sense.

Let us consider, again, a small triangular element of the type described on page 44, but not necessarily having one side in the boundary curve; the side ds may be any small arc on the middle surface. We shall define the forces and moments acting on the face which passes through the arc ds exactly as in Equations (75), but we shall distinguish these quantities from those of Equations (75) by using two bars. Those quantities designated by two bars are to be considered as expressed in terms of the deformation components by means of Equations (63) and (96), while those quantities designated by a single bar are expressed in terms of boundary forces which are assumed to be known. Now let us write the equations of equilibrium of this triangular element. On the faces which pass through arcs of parametric curves, forces and moments given by Equations (2) act; on the face which passes through the arc ds we take forces and moments to be given in the form of Equations (75); loading forces applied to the middle surface will be infinitesimals of a higher order than the forces mentioned above, and so may be neglected. Summing the forces and moments in the directions of the axes x, y, z, and taking into consideration the lengths of the sides of the element of middle surface, we obtain the following equations of equilibrium:

$$\begin{aligned} \bar{\bar{N}}_x &= N_x \cos\psi + N_{\phi x} \sin\psi & \bar{\bar{M}}_x &= M_x \cos\psi + M_{\phi x} \sin\psi \\ \bar{\bar{N}}_\phi &= N_{x\phi} \cos\psi + N_\phi \sin\psi & \bar{\bar{M}}_\phi &= M_{x\phi} \cos\psi + M_\phi \sin\psi \\ \bar{\bar{Q}}_{x\phi} &= Q_x \cos\psi + Q_\phi \sin\psi \end{aligned} \qquad (100)$$

where, as indicated above, the right sides of these equations are to be expressed in terms of the deformation components by means of Equations (63) and (96). Substituting Equations (100) in Equation (99), and rearranging and combining terms, we obtain:

$$I_5 + I_6 = \int \Big[\bar{\bar{N}}_x \, \delta u + (\bar{\bar{N}}_\phi - \frac{\bar{\bar{M}}_\phi}{a}) \, \delta v - \bar{\bar{Q}}_{x\phi} \, \delta w + \frac{\bar{\bar{M}}_x}{a} \, \delta w' + \frac{\bar{\bar{M}}_\phi}{a} \, \delta w^{\cdot} \Big] ds \quad (101)$$

We note that Equations (76) and (101) are identical in form, one being expressed in terms of single bar forces while the other is expressed in terms of double bar forces. However, the two types of forces must transform, due to a rotation of axes, in exactly the same way. Hence, by the same line of reasoning that Equation (76) was transformed into Equation (87), we may transform Equation (101) to the following:

$$I_5 + I_6 = \int \Big[(\bar{\bar{N}}_n - \frac{\bar{\bar{M}}_n}{a} \sin^2\psi - \frac{\bar{\bar{M}}_s}{a} \sin\psi \cos\psi) \delta n + (\bar{\bar{N}}_s - \frac{\bar{\bar{M}}_s}{a} \cos^2\psi - \frac{\bar{\bar{M}}_n}{a} \sin\psi \cos\psi) \, \delta s \quad (102)$$

$$- (\bar{\bar{Q}}_{ns} + \frac{\partial \bar{\bar{M}}_s}{\partial s}) \delta w - \bar{\bar{M}}_n \delta w^{(n)} \Big] ds$$

We now combine Equations (87), (88), (94) and (102) according to Equation (71):

$$I_1 - I_2 - I_3 \equiv (I_4 - I_2) + (I_5 + I_6 - I_3) = 0 \quad (103)$$

The term $(I_4 - I_2)$ is a double integral taken over the area of the middle surface, while the term $(I_5 + I_6 - I_3)$ is a line integral taken around its boundary. In order that Equation (103) may hold true in general, we must demand that its two terms shall vanish separately:

$$I_4 - I_2 = 0 \quad ; \quad I_5 + I_6 - I_3 = 0 \tag{104}$$

Consider the first of these equations. When $I_4 - I_2$ is written out in full according to Equations (88) and (94), a double integral is obtained whose integrand is composed of three terms, each of which is itself composed of a coefficient multiplying the variation of one of the deformation components. The latter variations are perfectly arbitrary. Hence the coefficients of the variations must vanish separately in order that the first of Equations (104) may be valid in general. We are thus led, again, to the differential Equations (65).

The second of Equations (104) gives the boundary conditions which must be satisfied by the solution of Equations (65). Before writing this equation out in full, let us define the following quantities:

$$\begin{aligned}
\bar{P}_n &= \bar{N}_n - \frac{\bar{M}_n}{a}\sin^2\psi - \frac{\bar{M}_s}{a}\sin\psi\cos\psi \\
\bar{P}_s &= \bar{N}_s - \frac{\bar{M}_s}{a}\cos^2\psi - \frac{\bar{M}_n}{a}\sin\psi\cos\psi \\
\bar{S} &= \bar{Q}_{ns} + \frac{\partial \bar{M}_s}{\partial s}
\end{aligned} \tag{105}$$

with the quantities $\bar{\bar{P}}_n$, $\bar{\bar{P}}_s$, and $\bar{\bar{S}}$ defined similarly. Forming the second of Equations (104) by combining Equations (87) and (102) and making use of the above definitions, we obtain:

$$0 = \int \left[(\bar{\bar{P}}_n - \bar{P}_n)\delta n + (\bar{\bar{P}}_s - \bar{P}_s)\delta s - (\bar{\bar{S}} - \bar{S})\delta w + (\bar{\bar{M}}_n - \bar{M}_n)\delta w^{(n)} \right] ds \tag{106}$$

Since Equation (106) must hold identically, we demand that the four terms of the integrand vanish separately. The latter may vanish due either to the vanishing of coefficient of the variation component or to the vanishing of the variation component itself. We are led thus to four boundary conditions, each taking one of two alternative forms:

$$
\begin{aligned}
&\text{either} \quad n = 0 \qquad &&\text{or} \quad \bar{\bar{P}}_n = \bar{P}_n \\
&\quad " \qquad s = 0 \qquad &&\quad " \quad \bar{\bar{P}}_s = \bar{P}_s \qquad (107)\\
&\quad " \qquad w = 0 \qquad &&\quad " \quad \bar{\bar{S}} = \bar{S} \\
&\quad " \qquad w^{(n)} = 0 \qquad &&\quad " \quad \bar{\bar{M}}_n = \bar{M}_n
\end{aligned}
$$

where in the first two of these equations n and s are used to denote the components of deformation in the directions of the axes of n and s as described on page 45. The first alternatives, where the deformation components vanish, are the conditions along a boundary which is perfectly fixed; the second alternatives apply along a boundary which is only partially restrained or free. In the case of a free edge, the boundary forces $\bar{P}_n \ldots \bar{M}_n$ are to be taken as zero. The second set of conditions apply also to the important cases of two separate shells joined together along a boundary, and of a shell joined along a boundary to a stiffening beam. In these cases, we equate the forces $\bar{\bar{P}}_n \ldots \bar{\bar{M}}_n$ for one shell to edge forces $\bar{P}_n \ldots \bar{M}_n$ which must be expressed in terms of forces $\bar{\bar{P}}_n \ldots \bar{\bar{M}}_n$ obtained from the second shell or from the stiffening beam. Thus the true boundary conditions are conditions concerning boundary forces rather than boundary displacements in all cases except that of fixity.

Frequently, the second alternative conditions of Equations (107) will all apply along a boundary. In this case, the first two conditions may be simplified by introducing into the first two, the last condition. We thus obtain for this special case:

$$\bar{\bar{N}}_n - \frac{\bar{\bar{M}}_s}{a}\sin\psi\cos\psi = \bar{N}_n - \frac{\bar{M}_s}{a}\sin\psi\cos\psi$$

$$\bar{\bar{N}}_s - \frac{\bar{\bar{M}}_s}{a}\cos^2\psi = \bar{N}_s - \frac{\bar{M}_s}{a}\cos^2\psi \qquad (108)$$

$$\bar{\bar{S}} = \bar{S}$$

$$\bar{\bar{M}}_n = \bar{M}_n$$

Let us consider the form of the boundary conditions of Equations (107) and (108) for the particular cases where the boundaries coincide with parametric curves. For a boundary along which x = const., $\sin\psi = 0$ and $\cos\psi = \pm 1$, and the conditions of Equations (107) become:

either	$u = 0$	or	$\bar{\bar{N}}_x = \bar{N}_x$	
"	$v = 0$	"	$\bar{\bar{P}}_{x\phi} = \bar{P}_{x\phi}$	(109)
"	$w = 0$	"	$\bar{\bar{S}}_x = \bar{S}_x$	
"	$w' = 0$	"	$\bar{\bar{M}}_x = \bar{M}_x$	

For a boundary along which ϕ = const., $\sin\psi = \pm 1$ and $\cos\psi = 0$, and the conditions of Equations (107) become:

either	$v = 0$	or	$\bar{\bar{P}}_\phi = \bar{P}_\phi$	
"	$u = 0$	"	$\bar{\bar{N}}_{\phi x} = \bar{N}_{\phi x}$	(110)

either $w = 0$ or $\bar{\bar{S}}_\phi = \bar{S}_\phi$

" $w^{\bullet} = 0$ " $\bar{\bar{M}}_\phi = \bar{M}_\phi$

Several of the quantities used in writing Equations (109) and (110) are defined as follows:

$$\bar{P}_{x\phi} = \bar{N}_{x\phi} - \frac{\bar{\mathcal{M}}_{x\phi}}{a} \qquad \bar{S}_x = \bar{Q}_x + \frac{\dot{\bar{\mathcal{M}}}_{x\phi}}{a} \tag{111}$$

$$\bar{P}_\phi = \bar{N}_\phi - \frac{\bar{\mathcal{M}}_\phi}{a} \qquad \bar{S}_\phi = \bar{Q}_\phi + \frac{\bar{\mathcal{M}}'_{\phi x}}{a}$$

the corresponding double bar quantities being similarly defined. If, along the boundary ϕ = const., all of the second alternatives of Equations (110) apply, the latter may be simplified to the following set of conditions:

$$\bar{\bar{N}}_\phi = \bar{N}_\phi \quad ; \quad \bar{\bar{N}}_{\phi x} = \bar{N}_{\phi x} \quad ; \quad \bar{\bar{S}}_\phi = \bar{S}_\phi \quad ; \quad \bar{\bar{M}}_\phi = \bar{M}_\phi \tag{112}$$

In obtaining Equation (87) from Equation (86), the integral $\int \frac{\partial}{\partial s}(\bar{M}_s\, \delta w)ds$ was neglected due to the fact that it vanishes when integrated around the closed boundary curve if the quantity $\bar{M}_s\, \delta w$ is single valued and continuous on the boundary. Similarly, in transforming Equation (101) into Equation (102), the integral $\int \frac{\partial}{\partial s}(\bar{\bar{M}}_s\, \delta w)ds$ was neglected. If the conditions under which these integrals vanish do not hold, the following integral must be added to the right side of Equation (106):

$$\int \frac{\partial}{\partial s}\left[(\bar{\bar{M}}_s - \bar{M}_s)\delta w\right] ds$$

We must demand, again, that all terms of the resulting equation vanish separately. We thus obtain the condition that the above integral must vanish, as well as the conditions previously obtained. In the case of a boundary curve which is not a continuously differentiable curve on the cylindrical middle surface, this last condition leads to the concentrated forces which have been observed at corners of the boundary curve. As an example, consider the case where the boundary consists of two generators $\phi = \phi_1$ and $\phi = \phi_2$, and two circular arcs lying in planes normal to the cylinder axis and having the parameters $x = x_1$ and $x = x_2$. Carrying out the integration, we are led to the following condition:

$$\begin{aligned}
&\left\{\bar{\bar{M}}_{\phi x}(x_2\phi_2) - \bar{\bar{M}}_{x\phi}(x_2\phi_2) - \left[\bar{M}_{\phi x}(x_2\phi_2) - \bar{M}_{x\phi}(x_2\phi_2)\right]\right\} \delta w(x_2\phi_2) \\
+&\left\{\bar{\bar{M}}_{\phi x}(x_1\phi_1) - \bar{\bar{M}}_{x\phi}(x_1\phi_1) - \left[\bar{M}_{\phi x}(x_1\phi_1) - \bar{M}_{x\phi}(x_1\phi_1)\right]\right\} \delta w(x_1\phi_1) \\
+&\left\{\bar{\bar{M}}_{x\phi}(x_2\phi_1) - \bar{\bar{M}}_{\phi x}(x_2\phi_1) - \left[\bar{M}_{x\phi}(x_2\phi_1) - \bar{M}_{\phi x}(x_2\phi_1)\right]\right\} \delta w(x_2\phi_1) \\
+&\left\{\bar{\bar{M}}_{x\phi}(x_1\phi_2) - \bar{\bar{M}}_{\phi x}(x_1\phi_2) - \left[\bar{M}_{x\phi}(x_1\phi_2) - \bar{M}_{\phi x}(x_1\phi_2)\right]\right\} \delta w(x_1\phi_2) = 0
\end{aligned}$$

Again, we demand that the terms of this equation vanish separately, and each may vanish due to the vanishing of either of its two factors. If the variation of w vanishes at the corners, the non-vanishing of the curly brackets leads to the observed concentrated forces at the corners. If the curly brackets vanish, then the corners of the shell tend to deflect radially. We would arrive, of course, at similar conclusions for any other type of boundary curve which is not a single continuously differentiable curve on the middle surface.

In the remainder of this thesis, we shall be concerned mainly with the problem of a cylindrical shell supported along two edges x = const. and perfectly free along two edges ϕ = const.. The boundary conditions which apply to this problem are those of Equations (109) and (112), the boundary (single bar) forces vanishing in the latter equations.

PART IV.

An important field of application of the theory which has been developed in the preceding parts of this paper is in the design and construction of barrel type roofs of reinforced concrete. The main purpose of the investigations set forth herein was to attempt to obtain approximate methods of solution of the equations of the cylindrical shell which would be accurate enough to be applicable to the design of structures of this type, and which would be easier to apply than the more exact methods. In order to judge the accuracy of any approximate solution, an exact solution must be known, and we shall take the solution of the Equations (65) to be the exact solution of our problem. In even the simplest of practical cases, the numerical solution of these equations is extremely laborious, the time factor alone making the use of this theory impractical for purposes of design. In this part of the thesis, we will set up a procedure for solving Equations (65) which is applicable to the problem at hand, and in Part V. we will solve these equations numerically in two particular cases in order to have the "exact" solution in these cases for the purpose of comparison with the results of approximate methods.

Consider a cylindrical shell which is supported along two boundaries x = const. and either partially restrained or free along two generators ϕ = const.. It is seen that in most barrel roof applications, the shell would come into this general classification.

We consider the solution of Equations (65) having a shell of this type in mind. This solution has been set up by Dischinger (Beton und Eisen, Aug. and Sept., 1935) in a form applicable to any type of loading. His solution, however, may be considerably simplified if both the loading and the shell are symmetrical about an axial plane. As both types of solution are of some importance in roof design, we will first set up the solution in the form given by Dischinger, and will then simplify it by consideration of symmetry.

We are mainly interested in this problem as it applies to structures of reinforced concrete. As Poisson's ratio is small for this material, we shall make the simplifying assumption that $\nu = 0$. That the simplification thereby introduced is considerable will be obvious upon attempting a similar solution without this simplification; and this may be considered, at the present time, as sufficient reason in itself for making the assumption. Placing $\nu = 0$ in Equations (65) the latter become:

$$
\begin{aligned}
&u^{\prime\prime} + \tfrac{1}{2}u^{\cdot\cdot} + \tfrac{1}{2}v^{\prime\cdot} + k\left(\tfrac{1}{2}u^{\cdot\cdot} + \tfrac{1}{2}w^{\prime\cdot\cdot} - w^{\prime\prime\prime}\right) = -\tfrac{a^2}{D}X \\
&\tfrac{1}{2}u^{\prime\cdot} + \tfrac{1}{2}v^{\prime\prime} + v^{\cdot\cdot} + w^{\cdot} + k\left(\tfrac{3}{2}v^{\prime\prime} - \tfrac{3}{2}w^{\prime\prime\cdot}\right) = -\tfrac{a^2}{D}Y \qquad (113) \\
&v^{\cdot} + w + k\left(\tfrac{1}{2}u^{\prime\cdot\cdot} - u^{\prime\prime\prime} - \tfrac{3}{2}v^{\prime\prime\cdot} + w^{IV} + 2w^{\prime\prime\cdot\cdot} + w^{\cdot\cdot\cdot\cdot} + 2w^{\cdot\cdot} + w\right) = -\tfrac{a^2}{D}Z
\end{aligned}
$$

The obtaining of particular integrals of these equations will present little difficulty in most cases; the chief difficulties arise in the determination of the complementary function and the subsequent evaluation of the arbitrary constants from the boundary conditions.

Let us fix definitely the boundaries of the shell under consideration. Let the shell be supported along two edges x = const. whose values we take to be 0 and ℓ; the length of the generators of the shell is therefore equal to ℓ. Let the axis of the shell be placed horizontally and let it coincide with a fixed x axis. The angle ϕ will be measured from some axial plane of reference, and will be reckoned positive when clockwise as viewed from looking from the positive end of the x axis toward the origin. Considering the moving x, y, z, axes, described on page 2, it is seen that, with ϕ measured as above, the positive direction along the fixed and moving x axes will be the same, the positive y axis will be directed according to increasing ϕ, and the positive z axis will be directed outward. These axes, then, are identical with those defining the unit vectors used in Equations (46). As always, we reckon the deformation components as positive when in the positive directions along the moving axes. The position of the axial plane of reference for ϕ will vary, depending on whether or not certain conditions of symmetry exist. If the shell and loading are both symmetrical with respect to a vertical axial plane, it will be convenient to measure ϕ from this plane; and the two boundary generators will be taken to correspond to the parameter values $\phi = \pm \phi_o$. If either the shell or the loading or both are unsymmetrical with respect to the vertical axial plane, it will be convenient to measure ϕ from one edge; and we will take the boundary generators to have the parameter values $\phi = 0$ and $\phi = \phi_o$.

We shall now obtain particular integrals of Equations (113). The load components X, Y, Z, will be expressible, under very general conditions, in double Fourier series valid over the middle surface of the shell. For the unsymmetrical case, with the boundary generators having the values specified above, we may write these series in either of the following forms:

$$X = \sum_{r,n} X_{r,n} \cos m\phi \cos \lambda \frac{x}{a} \qquad \text{or} \qquad X = \sum_{r,n} X_{r,n} \sin m\phi \cos \lambda \frac{x}{a}$$

$$Y = \sum_{r,n} Y_{r,n} \sin m\phi \sin \lambda \frac{x}{a} \qquad \qquad Y = \sum_{r,n} Y_{r,n} \cos m\phi \sin \lambda \frac{x}{a} \qquad (114)$$

$$Z = \sum_{r,n} Z_{r,n} \cos m\phi \sin \lambda \frac{x}{a} \qquad \qquad Z = \sum_{r,n} Z_{r,n} \sin m\phi \sin \lambda \frac{x}{a}$$

where

$$m = \frac{r\pi}{\phi_o} \; ; \quad \lambda = \frac{n\pi a}{\ell} \; ; \quad \text{r and n are integers.} \qquad (115)$$

Two other sets of possibilities, which are of some practical importance, also exist, in which the functions $\cos \lambda \frac{x}{a}$ and $\sin \lambda \frac{x}{a}$ in Equations (114) are interchanged. Corresponding to the load series in these forms, we assume the following series for the deformation components:

$$u = \sum_{r,n} u_{r,n} \cos m\phi \cos \lambda \frac{x}{a} \qquad \text{or} \qquad u = \sum_{r,n} u_{r,n} \sin m\phi \cos \lambda \frac{x}{a}$$

$$v = \sum_{r,n} v_{r,n} \sin m\phi \sin \lambda \frac{x}{a} \qquad \qquad v = \sum_{r,n} v_{r,n} \cos m\phi \sin \lambda \frac{x}{a} \qquad (116)$$

$$w = \sum_{r,n} w_{r,n} \cos m\phi \sin \lambda \frac{x}{a} \qquad \qquad w = \sum_{r,n} w_{r,n} \sin m\phi \sin \lambda \frac{x}{a}$$

where, again, we may interchange $\cos \lambda \frac{x}{a}$ and $\sin \lambda \frac{x}{a}$ to obtain series corresponding to the other two possibilities mentioned above. It is to be noted that, in each case, the series to be assumed for the deformation components u, v, w, have the same form as the series by which the load components X, Y, Z, respectively, are expressed. In the symmetrical case, the boundary generators having the values of ϕ previously specified, we take the series to have the first of the alternative forms of Equations (114) and (116); the second forms do not satisfy the assumed conditions of symmetry.

If any of the corresponding series for the load and deformation components are substituted in the complete Equations (113), the latter are reduced to algebraic equations for the determination of the constants $u_{r,n}$, $v_{r,n}$, and $w_{r,n}$. We shall carry out the substitution using only the first alternatives of Equations (114) and (116). Substituting the latter into Equations (113), collecting terms according to the various functions of ϕ, and x, and equating to zero the coefficient of each of these functions, we obtain the following system of three equations for the determination of the constants $u_{r,n}$, $v_{r,n}$, $w_{r,n}$:

$$
\begin{aligned}
&u_{r,n}\left[-\lambda^2-\frac{m^2}{2}(1+k)\right]+v_{r,n}\left[\frac{\lambda m}{2}\right]+w_{r,n}\left[k\lambda\left(\lambda^2-\frac{m^2}{2}\right)\right]=-\frac{a^2}{D}X_{r,n}\\
&u_{r,n}\left[\frac{\lambda m}{2}\right]+v_{r,n}\left[-m-\frac{\lambda^2}{2}(1+3k)\right]+w_{r,n}\left[-m\left(1+\frac{3}{2}k\lambda^2\right)\right]=-\frac{a^2}{D}Y_{r,n} \qquad (117)\\
&u_{r,n}\left[-k\lambda\left(\lambda^2-\frac{m^2}{2}\right)\right]+v_{r,n}\left[m\left(1+\frac{3}{2}k\lambda^2\right)\right]+w_{r,n}\left[1+k(\lambda^4+2\lambda^2m^2+m^4-2m^2+1)\right]=-\frac{a^2}{D}Z_{r,n}
\end{aligned}
$$

These equations are easily solved in any numerical case, and give the required particular integrals of Equations (113).

Before turning to the complementary function, let us consider the effect on Equations (117) of neglecting all terms containing k.

Neglecting these terms in the above equations is equivalent to neglecting the corresponding terms in k in the differential Equations (113). Thus from Equations (117) we obtain:

$$\begin{aligned} u_{r,n}\left(-\lambda^2 - \frac{m^2}{2}\right) + v_{r,n}\left(\frac{\lambda m}{2}\right) &= -\frac{a^2}{D} X_{r,n} \\ u_{r,n}\left(\frac{\lambda m}{2}\right) + v_{r,n}\left(-m^2 - \frac{\lambda^2}{2}\right) + w_{r,n}(-m) &= -\frac{a^2}{D} Y_{r,n} \qquad (118) \\ v_{r,n}(m) + w_{r,n} &= -\frac{a^2}{D} Z_{r,n} \end{aligned}$$

whose solution gives us a particular integral of the equations:

$$\begin{aligned} u'' + \frac{1}{2} u^{\cdot\cdot} + \frac{1}{2} v'^{\cdot} &= -\frac{a^2}{D} X \\ \frac{1}{2} u'^{\cdot} + \frac{1}{2} v'' + v^{\cdot\cdot} + w^{\cdot} &= -\frac{a^2}{D} Y \qquad (119) \\ v^{\cdot} + w &= -\frac{a^2}{D} Z \end{aligned}$$

Using Equations (63), in which ν and k have been placed equal to zero, for the force and moment components, Equations (119) may be written as follows:

$$\begin{aligned} N_x' + N_{\phi x}^{\cdot} &= -aX \\ N_{x\phi}' + N_{\phi}^{\cdot} &= -aY \qquad (120) \\ N_{\phi} &= -aZ \end{aligned}$$

These are the membrance theory equations which, of course, are obtainable directly from Equations (48) by neglecting all moments. Now, in most practical applications, k will be very small; we shall take

$k = 10^{-5}$ in the examples of Part V.. Hence, the coefficients in Equations (117) and (118) will differ only slightly, and since the roots u_{rn}, v_{rn}, w_{rn}, of Equations (117) must be continuous functions of the coefficients, if the coefficients are themselves continuous, we see that the roots of Equations (117) and (118) will also differ only slightly. Thus, in most practical cases, particular integrals of Equations (119) will be good approximations to particular integrals of the exact Equations (113). It may be argued, therefore, that solutions of Equations (120) will also be good approximations for particular integrals of Equations (113). This is important for several reasons. The particular integrals of Equations (120) are easily obtained by direct integration of these equations in succession. This has been carried out quite completely by Dischinger (Proceedings of the International Association for Bridge and Structural Engineering, Vol. 4, 1936, page 227.) These integrals may be made to satisfy a variety of boundary conditions on the support edges $x = 0$ and $x = \ell$, conditions which are often hard to fulfill by the series previously assumed. Particular integrals obtained in this way will have to be developed in Fourier series in x in order that the boundary condition equations along the edge generators may be solved. The reason for this will be apparent from the form in which the complementary function is set up; see Equations (150) and (159). Whether or not there will be any considerable saving of time arising out of the use of the approximate integrals will depend largely on the difficulty of obtaining such developments. Let us consider the physical basis for taking solutions of Equations (120) as particular integrals of Equations (113). On the basis of the membrane theory alone a stress distribution is obtained which violates certain of the boundary conditions along the two free generators. These conditions cannot be satisfied by the membrane stresses alone, but require the presence of bending moments and normal shears Q. We now superimpose certain stresses on the membrane stresses, and require that the sums thus obtained shall satisfy all necessary boundary conditions. The stresses to be superimposed on the membrane stresses are determined by the solution of the more exact equations of equilibrium, in which we may now place all load terms equal to zero. Thus we obtain primary stresses by solution of Equations (120), and secondary or correction stresses by solution of the homogeneous Equations (113). Both of these methods of obtaining particular integrals will be used in Part VI..

We shall now obtain the complementary function of the Equations (113). To do this, we assume solutions of the homogeneous equations in the form of double series of the following alternate types:

$$u = \sum_n E \; e^{m\phi} \cos \lambda \frac{x}{a} \qquad \text{or} \qquad u = \sum_n E e^{m\phi} \sin \lambda \frac{x}{a}$$

$$v = \sum_n F \; e^{m\phi} \sin \lambda \frac{x}{a} \qquad\qquad v = \sum_n F e^{m\phi} \cos \lambda \frac{x}{a} \qquad (121)$$

$$w = \sum_n G \; e^{m\phi} \sin \lambda \frac{x}{a} \qquad\qquad w = \sum_n G e^{m\phi} \cos \lambda \frac{x}{a}$$

The complementary function obtained from the first of these assumptions will be added to particular integrals in the form of either of Equations (116), to give the general solution of the differential equations; however, if the second of the above assumptions is used, the complementary function so obtained must be combined with particular integrals obtained by the use of either of the other two possibilities mentioned on page 65 immediately below Equations (116). The general solutions thus obtained will be in the form of Fourier series in x but not in ϕ.

In making the assumptions of Equations (121) (and likewise, in assuming particular integrals of the form (116)) the conditions satisfied by the solution on the boundaries x = 0 and x = ℓ become definitely fixed. Thus, considering the first set of assumptions (121), it is seen that, at $x = 0, \ell$; $u' = v = w = 0$, while u, v', and w', do not vanish. From the first of Equations (63) we see that several of these conditions are equivalent to $N_x = 0$. For the second set of assumptions, $u = v' = w' = 0$ along the edges $x = 0, \ell$, while u', v, and w, do not vanish along these edges. Neither set of conditions represent fully the conditions of perfect fixity, of perfect freedom, or of a pin support. We shall now consider solutions derived from the first of assumptions (121), which, of the two, most nearly approximate pin-end conditions; mathematically, solutions obtained from the second set of assumptions will differ only in detail.

We now substitute the first of expressions (121) into Equations (113) in which the right side is set equal to zero. We thus obtain the following algebraic equations:

$$E\left[-\lambda^2+\frac{m^2}{2}(1+k)\right]+F\left[\frac{\lambda m}{2}\right]+G\left[k(\lambda^3+\frac{\lambda m^2}{2})\right]=0$$

$$E\left[-\frac{\lambda m}{2}\right]+F\left[m^2-\frac{\lambda^2}{2}(1+3k)\right]+G\left[m(1+\frac{3}{2}k\lambda^2)\right]=0 \qquad (122)$$

$$E\left[-k\left(\lambda^3+\frac{\lambda m^2}{2}\right)\right]+F\left[m\left(1+\frac{3}{2}k\lambda^2\right)\right]+G\left[1+k(m^4+2m^2+1-2\lambda^2m^2+\lambda^4)\right]=0$$

In order that these equations may be satisfied by values of E, F, and G, not all zero, it is necessary that the determinant of the coefficients of E, F, and G, in these equations shall vanish. Forming this determinant and expanding, we obtain the following equation for the determination of m:

$$m^8+m^6(2-4\lambda^2)+m^4(1-8\lambda^2+6\lambda^4)+m^2(-4\lambda^6+6\lambda^4-4\lambda^2)+4\lambda^4+\lambda^8+\frac{\lambda^4}{k}=0 \qquad (123)$$

For each value of λ, corresponding to each interger value of n in Equation (115), there will exist, in general, eight values of m, the eight roots of this equation, for each of which the assumed solution, Equations (121), will actually be valid. Obtaining the roots of this equation is one of the principal difficulties encountered in the exact solution of our problem. Dischinger, in the paper cited above (Beton und Eisen) has obtained curves for these roots. However, in one point checked by the author, their accuracy is questionable. It is probable that small errors in the determination of the roots of Equation (123) will not greatly affect the accuracy of the final solution, although this has not been definitely shown; if so, the use of Dischinger's curves will considerably shorten the labor of the exact solution. Until it has been demonstrated that Dischinger's curves are sufficiently accurate, it is probably desirable, at least for an "exact" solution, to compute exactly the needed roots. This is especially true since the time required to compute these roots is small when compared to the total time required for carrying out the entire solution of the problem. We will, therefore, set up a procedure for the solution of Equation (123) which is quite easy to apply if a computing machine is available. The solution will be formulated without any attempt being made to justify the various steps or prove the results; reference is made to Burnside and Panton, "Theory of Equations", Vol. 1, Chapter 6, for the latter.

To establish a procedure for the solution of Equation (123), we note, first, that this equation may be considered to be of the fourth degree in m^2. Substituting $x = m^2$ in Equation (123), the latter may be rewritten as follows:

$$x^4 + 4ax^3 + 6bx^2 + 4cx + d = 0$$

where:

$$\begin{aligned} 4a &= 2 - 4\lambda^2 \\ 6b &= 1 - 8\lambda^2 + 6\lambda^4 \\ 4c &= -4\lambda^6 + 6\lambda^4 - 4\lambda^2 \\ d &= 4\lambda^4 + \lambda^8 + \frac{\lambda^4}{k} \end{aligned} \qquad (124)$$

Making the change in variable $x = z - a$, we obtain the reduced quartic:

$$z^4 + pz^2 + qz + r = 0$$

where:

$$p = 6(b - a^2)$$

$$q = 4(c - 3ab + 2a^3) \tag{125}$$

$$r = d - 4ac + 6a^2b - 3a^4$$

From this equation we obtain the cubic resolvant:

$$v^3 + 3\ell v^2 + 3mv + n = 0$$

where:

$$\ell = \frac{2}{3}p$$

$$m = \frac{1}{3}(p^2 - 4r) \tag{126}$$

$$n = -q^2$$

Making the change of variable $v = y - \ell$, we obtain the reduced cubic resolvant:

$$y^3 + sy + t = 0$$

where:

$$s = 3(m - \ell^2) \tag{127}$$

$$t = n - 3\ell m + 2\ell^3$$

Let Θ be defined by the equation:

$$\sin 3\Theta = \frac{3t}{s\sqrt{\frac{4}{3}(-s)}} \tag{128}$$

We may now write the solution of Equation (127) in the following form:

$$y_1 = -\sqrt{\tfrac{4}{3}(-s)}\sin\Theta \;;\quad y_2 = -\sqrt{\tfrac{4}{3}(-s)}\sin(\tfrac{\pi}{3} - \Theta) \;;\quad y_3 = +\sqrt{\tfrac{4}{3}(-s)}\sin(\tfrac{\pi}{3} + \Theta) \tag{129}$$

The identity $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$ will prove useful in determining $\sin \theta$ from Equation (128). In most cases of practice, $\sin \theta$ will be small, so that it may be obtained from this identity by a method of successive approximations, the first approximation being, of course, to neglect $\sin^3 \theta$ in comparison to $\sin \theta$. The solution of the cubic resolvant, Equation (126), is now given by the following:

$$v_1 = y_1 - \ell \;;\quad v_2 = y_2 - \ell \;;\quad v_3 = y_3 - \ell \qquad (130)$$

We now extract the square roots of these quantities, assigning to each a sign such that the product:

$$\sqrt{v_1 \; v_2 \; v_3} = -q \qquad (131)$$

<u>With the signs so determined</u>, let the square roots of these quantities be written as:

$$\sqrt{v_1} \;,\quad \sqrt{v_2} \;,\quad \sqrt{v_3} \qquad (132)$$

In terms of these roots, we may now write the solution of the reduced quartic, Equation (125), in the following form:

$$z_1 = \tfrac{1}{2}(\sqrt{v_1} + \sqrt{v_2} + \sqrt{v_3}) \qquad z_3 = \tfrac{1}{2}(\sqrt{v_1} - \sqrt{v_2} - \sqrt{v_3}) \qquad (133)$$

$$z_2 = \tfrac{1}{2}(-\sqrt{v_1} - \sqrt{v_2} + \sqrt{v_3}) \qquad z_4 = \tfrac{1}{2}(-\sqrt{v_1} + \sqrt{v_2} - \sqrt{v_3})$$

The solution of the given quartic may now be written as:

$$x_1 = z_1 - a \;;\quad x_2 = z_2 - a \;;\quad x_3 = z_3 - a \;;\quad x_4 = z_4 - a \qquad (134)$$

The eight roots of Equation (123) are now obtained by extracting the square roots of these quantities, and attaching a double sign to each.

In the cases of practice, it will be found that some of the roots v_1, v_2, v_3, will be negative. The quantities z_1, z_2, z_3, z_4, will, therefore, be complex and it will be found, moreover, that these quantities are paired off into two pairs of conjugate complex numbers. The quantities x_1, x_2, x_3, x_4, will likewise occur as two pairs of conjugate complex numbers. In extracting their roots, the following identities are useful:

$$\sqrt{a + ib} = \pm\left\{\sqrt{\tfrac{1}{2}(\sqrt{a^2 + b^2} + a)} + i\sqrt{\tfrac{1}{2}(\sqrt{a^2 + b^2} - a)}\right\} \qquad (135)$$

$$\sqrt{a - ib} = \pm\left\{\sqrt{\tfrac{1}{2}(\sqrt{a^2 + b^2} + a)} - i\sqrt{\tfrac{1}{2}(\sqrt{a^2 + b^2} - a)}\right\}$$

We thus obtain the eight roots of Equation (123) which may be written in the following form:

$$m_1 = -m_5 = \alpha_1 + i\beta_1 \qquad m_3 = -m_7 = \alpha_2 + i\beta_2 \qquad (136)$$

$$m_2 = -m_6 = \alpha_1 - i\beta_1 \qquad m_4 = -m_8 = \alpha_2 - i\beta_2$$

For a numerical example in which this method of solution is carried through completely, see Table I. in Part V.

For each value of λ, and for each value of m as determined from Equation (123) for that value of λ, the Equations (122) will form a consistent set, and we may use any two of them to solve for the ratios between the constants E, F, G. Corresponding to the eight values of m, there will be eight arbitrary constants, which we may take to be the E's, and eight ratios, ψ and χ, relating the E's to the F's and G's. Thus, let E_r, r = 1 ...... 8, be the arbitrary constants corresponding to the roots m_r. Then F_r and G_r will be given by equations of the type:

$$F_r = \psi_r E_r \quad ; \quad G_r = \chi_r E_r \tag{137}$$

We use the first two of Equations (122) to solve for the ratios ψ_r and χ_r. By the use of determinants or otherwise, they are easily obtained, giving:

$$\psi_r = \frac{F_r}{E_r} = \frac{-m_r^3[1 + k(1 + 2\lambda^2)] + 2\lambda^2 m_r^2 (1 + k\lambda^2)}{-m_r^4 k\lambda + m_r^2 \lambda + k\lambda^5} \tag{138}$$

$$\chi_r = \frac{G_r}{E_r} = \frac{m_r^4(1 + k) - 2m_r^2\lambda^2(1 + k) + \lambda^4(1 + 3k)}{-m_r^4 k\lambda + \lambda m_r^2 + k\lambda^5}$$

Having the roots m_r and the ratios ψ_r and χ_r, the solution of the homogeneous differential Equations (113) may be written in the following form:

$$u = \sum_\lambda \sum_{r=1}^{8} E_r e^{m_r\phi} \cos\lambda\frac{x}{a} \; ; \quad v = \sum_\lambda \sum_{r=1}^{8} \psi_r E_r e^{m_r\phi} \sin\lambda\frac{x}{a} \; ; \quad w = \sum_\lambda \sum_{r=1}^{8} \chi_r E_r e^{m_r\phi} \sin\lambda\frac{x}{a} \tag{139}$$

or, written out more completely:

$$u = \sum_\lambda \cos\lambda\frac{x}{a}\Big\{e^{\alpha_1\phi}(E_1 e^{i\beta_1\phi} + E_2 e^{-i\beta_1\phi}) + e^{\alpha_2\phi}(E_3 e^{i\beta_2\phi} + E_4 e^{-i\beta_2\phi}) + e^{-\alpha_1\phi}(E_5 e^{-i\beta_1\phi} + E_6 e^{i\beta_1\phi})$$
$$+ e^{-\alpha_2\phi}(E_7 e^{-i\beta_2\phi} + E_8 e^{i\beta_2\phi})\Big\}$$

$$v = \sum_\lambda \sin\lambda\frac{x}{a}\Big\{e^{\alpha_1\phi}(\psi_1 E_1 e^{i\beta_1\phi} + \psi_2 E_2 e^{-i\beta_1\phi}) + \dots\dots\dots\dots\dots\Big\} \tag{140}$$

$$w = \sum_\lambda \sin\lambda\frac{x}{a}\Big\{e^{\alpha_1\phi}(\chi_1 E_1 e^{i\beta_1\phi} + \chi_2 E_2 e^{-i\beta_1\phi}) + \dots\dots\dots\dots\dots\Big\}$$

These solutions involve complex quantities; it will be convenient to transform them so that they will involve only real numbers.

From the expressions (136) for the roots of the determinental Equation (123), the following relations are obvious:

$$
\begin{array}{llll}
m_1 = \overline{m}_2 & m_1^2 = m_5^2 & m_1^3 = -m_5^3 & m_1^4 = m_5^4 \\
m_3 = \overline{m}_4 & m_2^2 = m_6^2 & m_2^3 = -m_6^3 & m_2^4 = m_6^4 \\
m_5 = \overline{m}_6 & m_3^2 = m_7^2 & m_3^3 = -m_7^3 & m_3^4 = m_7^4 \\
m_7 = \overline{m}_8 & m_4^2 = m_8^2 & m_4^3 = -m_8^3 & m_4^4 = m_8^4
\end{array}
\qquad (141)
$$

where the bar over a quantity is used to indicate the conjugate. From Equations (138) and (141), the following relations may be written:

$$
\begin{array}{llll}
\psi_5 = -\psi_1 & \chi_5 = \chi_1 & \psi_1 = \overline{\psi}_2 & \chi_1 = \overline{\chi}_2 \\
\psi_6 = -\psi_2 & \chi_6 = \chi_2 & \psi_3 = \overline{\psi}_4 & \chi_3 = \overline{\chi}_4 \\
\psi_7 = -\psi_3 & \chi_7 = \chi_3 & \psi_5 = \overline{\psi}_6 & \chi_5 = \overline{\chi}_6 \\
\psi_8 = -\psi_4 & \chi_8 = \chi_4 & \psi_7 = \overline{\psi}_8 & \chi_7 = \overline{\chi}_8
\end{array}
\qquad (142)
$$

We define new constants as follows:

$$
\begin{array}{ll}
A_1' = E_1 + E_2 & A_5' = E_6 + E_5 \\
A_2' = i(E_1 - E_2) & A_6' = i(E_6 - E_5) \\
A_3' = E_3 + E_4 & A_7' = E_8 + E_7 \\
A_4' = i(E_3 - E_4) & A_8' = i(E_8 - E_7)
\end{array}
\qquad (143)
$$

with new constants B' and C' being similarly defined in terms of the old constants F and G, respectively. Inverting the two equations defining A_1' and A_2', and substituting the results in the two equations defining B_1' and B_2', we obtain the following relations:

$$B_1' = R(\psi_1)A_1' + I(\psi_1)A_2' \quad ; \qquad B_2' = -I(\psi_1)A_1' + R(\psi_1)A_2'$$

In this equation, we use the notation, since $\psi_1 = \bar{\psi}_2$:

$$R(\psi_1) = \frac{\psi_1 + \psi_2}{2} = \text{real part of } \psi_1 \tag{144}$$

$$I(\psi_1) = \frac{\psi_1 - \psi_2}{2i} = \text{imaginary part of } \psi_1$$

In a similar manner, and making further use of Equations (142), we may write the following relations between the constants A_r', B_r', C_r':

$$\begin{aligned}
B_1' &= R(\psi_1)A_1' + I(\psi_1)A_2' & C_1' &= R(\chi_1)A_1' + I(\chi_1)A_2' \\
B_2' &= -I(\psi_1)A_1' + R(\psi_1)A_2' & C_2' &= -I(\chi_1)A_1' + R(\chi_1)A_2' \\
B_3' &= R(\psi_3)A_3' + I(\psi_3)A_4' & C_3' &= R(\chi_3)A_3' + I(\chi_3)A_4' \\
B_4' &= -I(\psi_3)A_3' + R(\psi_3)A_4' & C_4' &= -I(\chi_3)A_3' + R(\chi_3)A_4' \\
B_5' &= -\left\{R(\psi_1)A_5' - I(\psi_1)A_6'\right\} & C_5' &= R(\chi_1)A_5' - I(\chi_1)A_6' \\
B_6' &= -\left\{I(\psi_1)A_5' + R(\psi_1)A_6'\right\} & C_6' &= I(\chi_1)A_5' + R(\chi_1)A_6' \\
B_7' &= -\left\{R(\psi_3)A_7' - I(\psi_3)A_8'\right\} & C_7' &= R(\chi_3)A_7' - I(\chi_3)A_8' \\
B_8' &= -\left\{I(\psi_3)A_7' + R(\psi_3)A_8'\right\} & C_8' &= I(\chi_3)A_7' + R(\chi_3)A_8'
\end{aligned} \tag{145}$$

We now rewrite Equations (140) in terms of the new constants A_r', B_r', C_r'. To do this we make use of the following identity:

$$M\,e^{im\phi} + N\,e^{-im\phi} = (M + N)\cos m\phi + i(M - N)\sin m\phi$$

Using relations of this type, together with the definitions of Equations (143), in Equations (140), the latter become:

$$u = \sum_{\lambda} \cos\lambda\frac{x}{a}\Big\{e^{\alpha_1\phi}(A_1'\cos\beta_1\phi + A_2'\sin\beta_1\phi) + e^{\alpha_2\phi}(A_3'\cos\beta_2\phi + A_4'\sin\beta_2\phi)$$
$$+ e^{-\alpha_1\phi}(A_5'\cos\beta_1\phi + A_6'\sin\beta_1\phi) + e^{-\alpha_2\phi}(A_7'\cos\beta_2\phi + A_8'\sin\beta_2\phi)\Big\} \qquad (146)$$
$$v = \sum_{\lambda} \sin\lambda\frac{x}{a}\Big\{e^{\alpha_1\phi}(B_1'\cos\beta_1\phi + B_2'\sin\beta_1\phi) + \ldots\ldots\ldots\ldots\Big\}$$
$$w = \sum_{\lambda} \sin\lambda\frac{x}{a}\Big\{e^{\alpha_1\phi}(C_1'\cos\beta_1\phi + C_2'\sin\beta_1\phi) + \ldots\ldots\ldots\ldots\Big\}$$

where the constants are related by Equations (145). Equations (146) are the general solution in real form of the homogeneous Equations (113).

It will be convenient to express this solution, Equations (146), in two alternate forms, their use depending on whether or not symmetry conditions exist. Consider the case where either the shell or the loading or both are unsymmetrical with respect to the vertical axial plane. As previously indicated, we take the boundary generators to have the parameter values $\phi = 0$ and $\phi = \phi_o$, and we define the angle ω by the equation:

$$\omega = \phi_o - \phi \qquad (147)$$

The angle ϕ increases with motion inward from one edge of the shell, while the angle ω increases with motion inward from the other edge. We shall define new unprimed constants by the following equations:

$$
\begin{aligned}
A_1' &= e^{-\alpha_1\phi_0}(A_1\cos\beta_1\phi_0 + A_2\sin\beta_1\phi_0) \qquad & A_5' &\equiv A_5 \\
A_2' &= e^{-\alpha_1\phi_0}(A_1\sin\beta_1\phi_0 - A_2\cos\beta_1\phi_0) \qquad & A_6' &\equiv A_6 \\
A_3' &= e^{-\alpha_2\phi_0}(A_3\cos\beta_2\phi_0 + A_4\sin\beta_2\phi_0) \qquad & A_7' &\equiv A_7 \\
A_4' &= e^{-\alpha_2\phi_0}(A_3\sin\beta_2\phi_0 - A_4\cos\beta_2\phi_0) \qquad & A_8' &\equiv A_8
\end{aligned}
\tag{148}
$$

the new constants B_r and C_r being defined by similar equations in terms of the old constants B_r' and C_r', respectively. We now determine the relations existing between the constants just defined. Substituting the expressions for A_1', A_2', B_1', and B_2', from Equations (148) in the first two of Equations (145), we obtain, after rearranging the terms:

$$
e^{-\alpha_1\phi_0}(B_1\cos\beta_1\phi_0 + B_2\sin\beta_1\phi_0) = e^{-\alpha_1\phi_0}\Big[\cos\beta_1\phi_0\left\{R(\psi_1)A_1 - I(\psi_1)A_2\right\} + \sin\beta_1\phi_0\left\{I(\psi_1)A_1 + R(\psi_1)A_2\right\}\Big]
$$

$$
e^{-\alpha_1\phi_0}(B_1\sin\beta_1\phi_0 - B_2\cos\beta_1\phi_0) = e^{-\alpha_1\phi_0}\Big[\sin\beta_1\phi_0\left\{R(\psi_1)A_1 - I(\psi_1)A_2\right\} - \cos\beta_1\phi_0\left\{I(\psi_1)A_1 + R(\psi_1)A_2\right\}\Big]
$$

These two equations are satisfied if:

$$
B_1 = R(\psi_1)A_1 - I(\psi_1)A_2 \qquad\qquad B_2 = I(\psi_1)A_1 + R(\psi_1)A_2
$$

Proceeding in a similar manner, the following relations are seen to hold between the unprimed constants:

$$
\begin{aligned}
B_1 &= R(\psi_1)A_1 - I(\psi_1)A_2 & C_1 &= R(\chi_1)A_1 - I(\chi_1)A_2 \\
B_2 &= I(\psi_1)A_1 + R(\psi_1)A_2 & C_2 &= I(\chi_1)A_1 + R(\chi_1)A_2 \\
B_3 &= R(\psi_3)A_3 - I(\psi_3)A_4 & C_3 &= R(\chi_3)A_3 - I(\chi_3)A_4 \\
B_4 &= I(\psi_3)A_3 + R(\psi_3)A_4 & C_4 &= I(\chi_3)A_3 + R(\chi_3)A_4 \\
B_5 &= -\left\{R(\psi_1)A_5 - I(\psi_1)A_6\right\} & C_5 &= R(\chi_1)A_5 - I(\chi_1)A_6 \\
B_6 &= -\left\{I(\psi_1)A_5 + R(\psi_1)A_6\right\} & C_6 &= I(\chi_1)A_5 + R(\chi_1)A_6 \\
B_7 &= -\left\{R(\psi_3)A_7 - I(\psi_3)A_8\right\} & C_7 &= R(\chi_3)A_7 - I(\chi_3)A_8 \\
B_8 &= -\left\{I(\psi_3)A_7 + R(\psi_3)A_8\right\} & C_8 &= I(\chi_3)A_7 + R(\chi_3)A_8
\end{aligned}
\tag{149}
$$

Substituting the first two of Equations (148) into the expression $e^{\alpha_1\phi}(A_1' \cos\beta_1\phi + A_2' \sin\beta_1\phi)$ which appears in the expression for u of Equations (146), we obtain, after rearranging the terms and using known trigonometric identities:

$$e^{\alpha_1\phi}(A_1'\cos\beta_1\phi + A_2'\sin\beta_1\phi) = e^{-\alpha_1\omega}(A_1\cos\beta_1\omega + A_2\sin\beta_1\omega)$$

The remaining terms of Equations (146) involving the constants with subscripts 1 to 4 may be similarly transformed, and we thus obtain the solution in the following form:

$$
\begin{aligned}
u = \sum_\lambda \cos\lambda\frac{\chi}{a}\Big\{ & e^{-\alpha_1\omega}(A_1\cos\beta_1\omega + A_2\sin\beta_1\omega) + e^{-\alpha_2\omega}(A_3\cos\beta_2\omega + A_4\sin\beta_2\omega) \\
& + e^{-\alpha_1\phi}(A_5\cos\beta_1\phi + A_6\sin\beta_1\phi) + e^{-\alpha_2\phi}(A_7\cos\beta_2\phi + A_8\sin\beta_2\phi)\Big\}
\end{aligned}
\tag{150}
$$

$$v = \sum_{\lambda} \sin\lambda\frac{x}{a}\left\{e^{-\alpha_1\omega}(B_1\cos\beta_1\omega + B_2\sin\beta_1\omega) + \cdots\cdots\cdots\right\}$$

$$w = \sum_{\lambda} \sin\lambda\frac{x}{a}\left\{e^{-\alpha_1\omega}(C_1\cos\beta_1\omega + C_2\sin\beta_1\omega) + \cdots\cdots\cdots\right\}$$

where the constants are related by Equations (149). It is seen that this solution has the form of four damped waves, two proceeding inward from each of the two boundary generators.

In order to set up the boundary condition Equations (110) or (112), it will be necessary to obtain expressions for the derivatives of the solutions (150). Differentiations with respect to x present no difficulties. Let us consider the derivative with respect to ϕ of the first and third terms in the expression for u of Equations (150). Let $f_1(\phi) \equiv e^{-\alpha_1\omega}(A_1\cos\beta_1\omega + A_2\sin\beta_1\omega)$; $f_2(\phi) \equiv e^{-\alpha_1\phi}(A_5\cos\beta_1\phi + A_6\sin\beta_1\phi)$. Differentiating these expressions with respect to ϕ, we obtain:

$$f_1' = e^{-\alpha_1\omega}(A_1^{(1)}\cos\beta_1\omega + A_2^{(1)}\sin\beta_1\omega)\ ;\ f_2' = -e^{-\alpha_1\phi}(A_5^{(1)}\cos\beta_1\phi + A_6^{(1)}\sin\beta_1\phi)$$

where: $A_1^{(1)} = \alpha_1 A_1 - \beta_1 A_2$ where: $A_5^{(1)} = \alpha_1 A_5 - \beta_1 A_6$

$A_2^{(1)} = \beta_1 A_1 + \alpha_1 A_2$ $A_6^{(1)} = \beta_1 A_5 + \alpha_1 A_6$

Second differentiations yield the following:

$$f_1'' = e^{-\alpha_1\omega}(A_1^{(2)}\cos\beta_1\omega + A_2^{(2)}\sin\beta_1\omega)\ ;\ f_2'' = (-1)^2\, e^{-\alpha_1\phi}(A_5^{(2)}\cos\beta_1\phi + A_6^{(2)}\sin\beta_1\phi)$$

where: $A_1^{(2)} = \alpha_1 A_1^{(1)} - \beta_1 A_2^{(1)}$ where: $A_5^{(2)} = \alpha_1 A_5^{(1)} - \beta_1 A_6^{(1)}$

$A_2^{(2)} = \beta_1 A_1^{(1)} + \alpha_1 A_2^{(1)}$ $A_6^{(2)} = \beta_1 A_5^{(1)} + \alpha_1 A_6^{(1)}$

In general, the nth derivatives will be given by:

$$f_1^{(n)} = e^{-\alpha_1\omega}(A_1^{(n)}\cos\beta_1\omega + A_2^{(n)}\sin\beta_1\omega) \;;\; f_2^{(n)} = (-1)^n\, e^{-\alpha_1\phi}(A_5^{(n)}\cos\beta_1\phi + A_6^{(n)}\sin\beta_1\phi)$$

where: $A_1^{(n)} = \alpha_1 A_1^{(n-1)} - \beta_1 A_2^{(n-1)}$ where: $A_5^{(n)} = \alpha_1 A_6^{(n-1)} - \beta_1 A_6^{(n-1)}$ (151)

$A_2^{(n)} = \beta_1 A_1^{(n-1)} + \alpha_1 A_2^{(n-1)}$ $A_6^{(n)} = \beta_1 A_5^{(n-1)} + \alpha_1 A_6^{(n-1)}$

Using expressions of this type, we may write the derivative of u with respect to ϕ as follows:

$$u^{\bullet} = \sum_{\lambda} \cos\lambda\frac{\chi}{a}\Big\{e^{-\alpha_1\omega}(A_1^{(1)}\cos\beta_1\omega + A_2^{(1)}\sin\beta_1\omega) + e^{-\alpha_2\omega}(A_3^{(1)}\cos\beta_2\omega + A_4^{(1)}\sin\beta_2\omega) \quad (152)$$
$$- e^{-\alpha_1\phi}(A_5^{(1)}\cos\beta_1\phi + A_6^{(1)}\sin\beta_1\phi) - e^{-\alpha_2\phi}(A_7^{(1)}\cos\beta_2\phi + A_8^{(1)}\sin\beta_2\phi)\Big\}$$

The other derivatives with respect to ϕ of the solutions (150) are written in a similar manner.

We are now in a position to set up the boundary condition Equations (110) or (112). In general, there will be four conditions to be satisfied on each of the two edges $\phi = 0$ and $\phi = \phi_0$, leading to eight equations for the determination of the eight constants $A_1 \ldots\ldots A_8$. These equations may be broken into two sets of four equations each, each set involving only four constants, in the following manner. We note that the constants A_1 to A_4 will be multiplied always by factors $e^{-\alpha_1\omega}$ or $e^{-\alpha_2\omega}$, while the constants A_5 to A_8 will always be multiplied by factors $e^{-\alpha_1\phi}$ or $e^{-\alpha_2\phi}$, When $\phi = 0$, $e^{-\alpha_1\phi}$ and $e^{-\alpha_2\phi}$ are unity while $e^{-\alpha_1\omega}$ and $e^{-\alpha_2\omega}$ will usually have values quite small by comparison. We may then, approximately, evaluate the constants A_5 to A_8 from the four boundary conditions at $\phi = 0$, neglecting all terms in the other constants due to the presence of the small exponential factors.

Similarly, the constants A_1 to A_4 may be evaluated approximately from the conditions which hold at $\phi = \phi_0$. The primary reason for introducing the unprimed constants of Equations (148), rather than using the solution in the form of Equations (146), was to be able to split the eight boundary equations into two sets of four each, thus avoiding the extremely cumbersome problem of solving eight simultaneous algebraic equations.

The solution thus obtained is essentially that given by Dischinger in the paper already cited (Beton und Eisen), and is applicable, of course, to any shell and loading of the type discussed, whether or not conditions of symmetry exist. If symmetry of the shell and load exists with respect to the vertical axial plane, the numerical solution of the problem may be considerably simplified. We shall now transform the solution of Equations (146) on the assumption that such symmetry does exist.

Equations (146) may be rearranged in the following form:

$$u = \sum_{\lambda} \cos\lambda\frac{x}{a}\Big\{\cos\beta_1\phi(A_1' e^{\alpha_1\phi} + A_5' e^{-\alpha_1\phi}) + \sin\beta_1\phi(A_2' e^{\alpha_1\phi} + A_6' e^{-\alpha_1\phi})$$

$$+ \cos\beta_2\phi(A_3' e^{\alpha_2\phi} + A_7' e^{-\alpha_2\phi}) + \sin\beta_2\phi(A_4' e^{\alpha_2\phi} + A_8' e^{-\alpha_2\phi})\Big\} \qquad (153)$$

$$v = \sum_{\lambda} \sin\lambda\frac{x}{a}\Big\{\cos\beta_1\phi(B_1' e^{\alpha_1\phi} + B_5' e^{-\alpha_1\phi}) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\Big\}$$

$$w = \sum_{\lambda} \sin\lambda\frac{x}{a}\Big\{\cos\beta_1\phi(C_1' e^{\alpha_1\phi} + C_5' e^{-\alpha_1\phi}) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\Big\}$$

We now define a new set of unprimed constants in the following manner:

$$A_1 = A_1' + A_5' \qquad A_3 = A_2' + A_6' \qquad A_5 = A_3' + A_7' \qquad A_7 = A_4' + A_8' \qquad (154)$$

$$A_2 = A_1' - A_5' \qquad A_4 = A_2' - A_6' \qquad A_6 = A_3' - A_7' \qquad A_8 = A_4' - A_8'$$

the unprimed constants B_r and C_r being defined by similar equations in terms, respectively, of the primed constants B_r' and C_r' . If in the expression for B_1 we substitute Equations (145), and then make use of the above expressions for A_1 and A_2, we obtain:

$$B_1 = B_1' + B_5' = R(\psi_1)(A_1' - A_5') + I(\psi_1)(A_2' + A_6') = R(\psi_1)A_2 + I(\psi_1)A_3$$

In a similar manner, we may obtain the following relations between the unprimed constants A_r, B_r, C_r:

$$\begin{aligned}
B_1 &= R(\psi_1)A_2 + I(\psi_1)A_3 & C_1 &= R(\chi_1)A_1 + I(\chi_1)A_4 \\
B_2 &= R(\psi_1)A_1 + I(\psi_1)A_4 & C_2 &= R(\chi_1)A_2 + I(\chi_1)A_3 \\
B_3 &= -I(\psi_1)A_1 + R(\psi_1)A_4 & C_3 &= -I(\chi_1)A_2 + R(\chi_1)A_3 \\
B_4 &= -I(\psi_1)A_2 + R(\psi_1)A_3 & C_4 &= -I(\chi_1)A_1 + R(\chi_1)A_4 \qquad (155) \\
B_5 &= R(\psi_3)A_6 + I(\psi_3)A_7 & C_5 &= R(\chi_3)A_5 + I(\chi_3)A_8 \\
B_6 &= R(\psi_3)A_5 + I(\psi_3)A_8 & C_6 &= R(\chi_3)A_6 + I(\chi_3)A_7 \\
B_7 &= -I(\psi_3)A_5 + R(\psi_3)A_8 & C_7 &= -I(\chi_3)A_6 + R(\chi_3)A_7 \\
B_8 &= -I(\psi_3)A_6 + R(\psi_3)A_7 & C_8 &= -I(\chi_3)A_5 + R(\chi_3)A_8
\end{aligned}$$

If we now invert the Equations (154), to express the primed constants in terms of the unprimed ones, and use these results, together with identities of the type:

$$M\, e^{\alpha\phi} + N\, e^{-\alpha\phi} = (M + N)\cosh \alpha\phi + (M - N)\sinh\alpha\phi$$

in Equations (153), the latter may be rewritten as follows:

$$u = \sum_{\lambda} \cos\lambda\frac{x}{a}\left\{A_1 E_1(\phi) + A_2 O_1(\phi) + A_3 O_2(\phi) + A_4 E_2(\phi) + A_5 E_3(\phi) + A_6 O_3(\phi) + A_7 O_4(\phi) + A_8 E_4(\phi)\right\} \quad (156)$$

$$v = \sum_{\lambda} \sin\lambda\frac{x}{a}\left\{B_1 E_1(\phi) + \ldots\ldots\ldots\ldots\ldots\right\}$$

$$w = \sum_{\lambda} \sin\lambda\frac{x}{a}\left\{C_1 E_1(\phi) + \ldots\ldots\ldots\ldots\ldots\right\}$$

where the functions $E_1(\phi) \ldots\ldots O_4(\phi)$ are defined as follows:

$$E_1(\phi) = \cosh\alpha_1\phi \cos\beta_1\phi \qquad O_1(\phi) = \sinh\alpha_1\phi \cos\beta_1\phi$$

$$E_2(\phi) = \sinh\alpha_1\phi \sin\beta_1\phi \qquad O_2(\phi) = \cosh\alpha_1\phi \sin\beta_1\phi \quad (157)$$

$$E_3(\phi) = \cosh\alpha_2\phi \cos\beta_2\phi \qquad O_3(\phi) = \sinh\alpha_2\phi \cos\beta_2\phi$$

$$E_4(\phi) = \sinh\alpha_2\phi \sin\beta_2\phi \qquad O_4(\phi) = \cosh\alpha_2\phi \sin\beta_2\phi$$

the significance of the notation E and O being that the functions so represented are even and odd functions, respectively, of ϕ.

Thus far, the solution is perfectly general; but we now make use of the assumed conditions of symmetry. If both the load and the shell are symmetrical with respect to the vertical axial plane, then it is plain that the deformation components u and w must be even functions of ϕ, while v must be an odd function. Using these facts in Equations (156), we see at once that:

$$A_2 = A_3 = A_6 = A_7 = B_1 = B_4 = B_5 = B_8 = C_2 = C_3 = C_6 = C_7 = 0 \quad (158)$$

It is easily checked that these equations are consistent with the relations of Equations (155). The solution, Equations (156), may, therefore, be rewritten in the following form, where for convenience we have changed the subscript numbers of the constants:

$$u = \sum_{\lambda} \cos\lambda\frac{x}{a}\left\{A_1E_1(\phi) + A_2E_2(\phi) + A_3E_3(\phi) + A_4E_4(\phi)\right\}$$

$$v = \sum_{\lambda} \sin\lambda\frac{x}{a}\left\{B_1O_1(\phi) + B_2O_2(\phi) + B_3O_3(\phi) + B_4O_4(\phi)\right\} \qquad (159)$$

$$w = \sum_{\lambda} \sin\lambda\frac{x}{a}\left\{C_1E_1(\phi) + C_2E_2(\phi) + C_3E_3(\phi) + C_4E_4(\phi)\right\}$$

where

$$B_1 = R(\psi_1)A_1 + I(\psi_1)A_2 \qquad C_1 = R(\chi_1)A_1 + I(\chi_1)A_2$$

$$B_2 = -I(\psi_1)A_1 + R(\psi_1)A_2 \qquad C_2 = -I(\chi_1)A_1 + R(\chi_1)A_2 \qquad (160)$$

$$B_3 = R(\psi_3)A_3 + I(\psi_3)A_4 \qquad C_3 = R(\chi_3)A_3 + I(\chi_3)A_4$$

$$B_4 = -I(\psi_3)A_3 + R(\psi_3)A_4 \qquad C_4 = -I(\chi_3)A_3 + R(\chi_3)A_4$$

As before, in order to set up the boundary conditions, we require expressions for the derivatives of solution (159). The following identities will prove useful in carrying out the differentiations with respect to ϕ:

$$\dot{E}_1(\phi) = \alpha_1O_1(\phi) - \beta_1O_2(\phi) \qquad \dot{O}_1(\phi) = \alpha_1E_1(\phi) - \beta_1E_2(\phi)$$

$$\dot{E}_2(\phi) = \beta_1O_1(\phi) + \alpha_1O_2(\phi) \qquad \dot{O}_2(\phi) = \beta_1E_1(\phi) + \alpha_1E_2(\phi) \qquad (161)$$

$$\dot{E}_3(\phi) = \alpha_2O_3(\phi) - \beta_2O_4(\phi) \qquad \dot{O}_3(\phi) = \alpha_2E_3(\phi) - \beta_2E_4(\phi)$$

$$\dot{E}_4(\phi) = \beta_2O_3(\phi) + \alpha_2O_4(\phi) \qquad \dot{O}_4(\phi) = \beta_2E_3(\phi) + \alpha_2E_4(\phi)$$

By means of these relations, all higher derivatives with respect to ϕ of the functions $E_1(\phi)$ .....$O_4(\phi)$ may be expressed in terms of these functions themselves. Using these relations, we obtain, for example, derivatives of the function u of Equations (159) as follows:

$$\dot{u} = \sum_{\lambda} \cos \lambda \frac{x}{a} \left\{ A_1^{(1)} O_1(\phi) + A_2^{(1)} O_2(\phi) + A_3^{(1)} O_3(\phi) + A_4^{(1)} O_4(\phi) \right\} \tag{162}$$

$$\ddot{u} = \sum_{\lambda} \cos \lambda \frac{x}{a} \left\{ A_1^{(2)} E_1(\phi) + A_2^{(2)} E_2(\phi) + A_3^{(2)} E_3(\phi) + A_4^{(2)} E_4(\phi) \right\}$$

where the extension to the nth derivative is obvious. The relations between the various constants appearing in the successive derivatives are easily verified to be the following:

$$A_1^{(n)} = \alpha_1 A_1^{(n-1)} + \beta_1 A_2^{(n-1)} \qquad A_3^{(n)} = \alpha_2 A_3^{(n-1)} + \beta_2 A_4^{(n-1)} \tag{163}$$

$$A_2^{(n)} = -\beta_1 A_1^{(n-1)} + \alpha_1 A_2^{(n-1)} \qquad A_4^{(n)} = -\beta_2 A_3^{(n-1)} + \alpha_2 A_4^{(n-1)}$$

The derivatives with respect to ϕ of v and w may be written in a similar way. It is to be noted, of course, that the successive derivatives of these functions are alternately even and odd functions of ϕ. In the expression for any of the deformation components or any of their derivatives the constants having the subscripts 1 and 2 are very simply related, and the same is true for the constants having the subscripts 3 and 4. Hence, in carrying out the calculation of these quantities, it is necessary to calculate only the constants having subscripts 1 and 3, it being then possible to write by inspection the other two constants.

The boundary condition Equations (110) or (112) may now be set up, and will lead to four equations, from the conditions holding at the generator $\phi = \phi_o$ (say), for the determination of the four constants $A_1 \ldots A_4$. Due to the assumed conditions of symmetry, the boundary conditions at the generator $\phi = -\phi_o$ will, of course, lead to the same four equations.

The advantages of the solution in the symmetrical form just given over that in Dischinger's more general form, are that, firstly, when the former solution applies, it is possible to place the computations in somewhat more compact tabular forms, and, secondly, the functions $E_1(\phi) \ldots\ldots O_4(\phi)$ are more easily computed than the corresponding functions of ϕ required by Dischinger's solution. Further, the solution in the symmetrical form does not involve the approximation, necessary in the unsymmetrical case, required to split the boundary condition equations into two sets of four equations each; this, however, is of small practical importance in most cases.

PART V.

We shall now carry out the calculations in detail for a particular case. We shall assume a symmetrical shell in the form of a complete half cylinder and we shall take the loading to be dead load. We shall, therefore, be able to use the solution as set up in the symmetrical form. Let p be the dead weight of the shell per unit of area of the middle surface, the shell thickness being everywhere uniform. Measuring ϕ from the vertical plane of symmetry, the loading may be expressed by the following relations:

$$X = 0 \quad ; \quad Y = p \sin \phi \quad ; \quad Z = p \cos \phi \tag{164}$$

Also, the two boundary generators will have the parameter values $\phi = \pm \frac{\pi}{2}$. We shall assume these boundary generators to be entirely free of force; the boundary conditions to be satisfied along these edges are therefore (see Equations (112)):

$$\overline{\overline{N}}_\phi = \overline{\overline{N}}_{\phi x} = \overline{\overline{S}}_\phi = \overline{\overline{M}}_\phi = 0 \tag{165}$$

further, the span length ℓ, the radius a, and the thickness t must be assumed; however, rather than making specific assumptions for these quantities, it will be convenient to assume values of the quantities D, k, and λ, as defined by Equations (62) and (115). For the numerical case to be worked out, we shall assume, therefore, $\lambda = 0.4$ and $k = 0.00001$, and the solution will be expressed in terms of the quantity D. Taking n to be one in Equation (115), it is seen that $\lambda = 0.4$ corresponds to a ratio of span to radius of approximately 8 to 1.

We first obtain particular integrals. We develop the load functions, Equations (164), in series according to the first group of Equations (114), obtaining:

$$X = 0 \quad ; \quad Y = \frac{4pa}{\ell} \sin \phi \sum_{n=1,3..}^{\infty} \frac{1}{\lambda} \sin \lambda \frac{x}{a} \quad ; \quad Z = \frac{4pa}{\ell} \cos \phi \sum_{n=1,3..}^{\infty} \frac{1}{\lambda} \sin \lambda \frac{x}{a} \qquad (166)$$

where use is made of the second of Equations (115). It is to be noted that Equations (166) contain series in x only; all of the terms of these series correspond to the single value m = 1, the terms vanishing for all other values of m. To obtain the complete solution of our problem, we must carry out the solution as previously outlined for each value of λ corresponding to each odd integer value of n. However, if the series involved are quickly convergent, their first terms will give an approximation to the true results. We shall, therefore, consider only the first terms in the series (166), and in these terms take λ to have the value 0.4. Corresponding to these load terms, the deformation components are given by the first terms of the series of the first set of Equations (116). The coefficients of the latter terms are now obtained from Equations (117). Substituting $\lambda = 0.4$, m = 1, and k = 0.00001 in these equations, and using the notation:

$$S = \frac{4pa^3}{D\ell} \qquad (167)$$

We obtain the following equations for the determination of $u_{//}$, $v_{//}$, and $w_{//}$:

$$u_{//}(-.660005) + v_{//}(.2) + w_{//}(-.00000136) = 0$$

$$u_{//}(.2) + v_{//}(-1.0800024) + w_{//}(-1.0000024) = -2.5 \ S \qquad (168)$$

$$u_{//} (.00000136) + v_{//} (1.0000024) + w_{//} (1.000003456) = -2.5 \quad S$$

These equations are easily solved by successive elimination of the variables, giving the following particular integrals:

$$u_{//} = 78.1221315 \quad S \quad \cos\phi \cos .4 \frac{x}{a}$$

$$v_{//} = 257.803217 \quad S \quad \sin\phi \sin .4 \frac{x}{a} \qquad (169)$$

$$w_{//} = -260.303042 \quad S \quad \cos\phi \sin .4 \frac{x}{a}$$

It is to be noted that, if k had been neglected in writing Equations (168), the results would have been very close to those given above.

We now calculate the complementary function. Substituting $\lambda = 0.4$ and $k = 0.00001$ in the determinantal Equation (123), we obtain:

$$m^8 + 1.36\, m^6 - .1264\, m^4 - .502784\, m^2 + 2560.10306 = 0 \qquad (170)$$

The solution of this equation is given in Table I. The numbers in the left column of this table correspond to the equation numbers of the text of pages 72 and 73. The high degree of accuracy necessary in carrying out the solution is illustrated by the value of $v_{/}$ in this table; this quantity is given by the difference of the numbers .546667685 and .546666667. The eight roots of Equation (170) are given in the last lines of Table I.

We now calculate the ratios ψ_r and χ_r. Substituting the values of λ and k in Equations (138), we obtain the following expressions for these ratios:

TABLE I.

Calculation of the Roots of the

Determinantal Equation (123)

$\lambda = 0.4$ $k = 0.00001$

No.	Equation
123	$m^8 + 1.36m^6 - .1264m^4 - .502784m^2 + 2560.10306 = 0$
124	$x^4 + 4(.34)x^3 + 6(-.0210666667)x^2 + 4(-.125696)x + 2560.10306 = 0$
125	$z^4 + (-.82)z^2 + (-.1024)z + 2560.21931 = 0$
126	$v^3 + 3(-.546666667)v^2 + 3(-3413.40161)v + (-.01048576) = 0$
127	$y^3 + (-10241.1014)y + (-5598.31587) = 0$
128	$\sin 3\theta = .0140342525$
	$\sin \theta = .00467822067$ $\cos \theta = .999889051$
129	$y_1 = -.546667685$; $y_2 = -100.913765$; $y_3 = 101.460432$
130	$v_1 = -.000001018$; $v_2 = -100.367098$; $v_3 = 102.007099$
132	$\sqrt{v_1} = -.00100895986\, i$; $\sqrt{v_2} = 10.0183380\, i$; $\sqrt{v} = 10.0998564$
131	$\sqrt{v_1 v_2 v_3} = .102090367 \cong -(-.1024) = -q$ (See Equation 125)
133	$z_1 = 5.0499282 + 5.0086645\, i$; $z_3 = -5.0499282 - 5.0096735\, i$
	$z_2 = 5.0499282 - 5.0086645\, i$; $z_4 = -5.0499282 + 5.0096735\, i$
134	$x_1 = 4.7099282 + 5.0086645\, i$; $x_3 = -5.3899282 - 5.0096735\, i$
	$x_2 = 4.7099282 - 5.0086645\, i$; $x_4 = -5.3899282 + 5.0096735\, i$
136	$m_1 = -m_5 = 2.40678799 + 1.04052882\, i$
	$m_2 = -m_6 = 2.40678799 - 1.04052882\, i$
	$m_3 = -m_7 = .992122666 + 2.52472485\, i$
	$m_4 = -m_8 = .992122666 - 2.52472485\, i$

$$\psi_r = \frac{.320000512\ m_r - (-1.0000132)m_r^3}{-.000004\ m_r^4 - .4\ m_r^2 - .0000001024} \tag{171}$$

$$\chi_r = \frac{1.00001\ m_r^4 - (-.3200032)m_r^2 - .025600768}{-.000004\ m_r^4 - .4\ m_r^2 - .0000001024}$$

It is evident from Equations (160) that we need calculate only the values of ψ_1, ψ_3, χ_1, and χ_3 corresponding to the roots m_1 and m_3. Substituting the values of these roots from Table I. into Equations (171), we obtain the following values for the ratios appearing in Equations (160):

$$\begin{aligned} R(\psi_1) &= -5.73713336 & R(\chi_1) &= 10.9811970 \\ I(\psi_1) &= -2.72284622 & I(\chi_1) &= 12.5161446 \\ R(\psi_3) &= -2.37202059 & R(\chi_3) &= -14.2811919 \\ I(\psi_3) &= -6.58614089 & I(\chi_3) &= 12.5169976 \end{aligned} \tag{172}$$

The calculations involved in obtaining these results from Equations (171) are rather tedious and are most easily carried out by means of a table, the form of which is obvious from the form of Equations (171).

The following equations will prove useful in raising the complex roots m_r to the powers required by Equations (171):

$$m = \alpha + i\beta$$

$$m^2 = (\alpha^2 - \beta^2) + i(2\alpha\beta)$$

$$m^3 = \left[\alpha(\alpha^2 - \beta^2) - \beta(2\alpha\beta)\right] + i\left[\beta(\alpha^2 - \beta^2) + \alpha(2\alpha\beta)\right]$$

$$m^4 = \left[(\alpha^2-\beta^2)^2 - (2\alpha\beta)^2\right] + i\left[2(\alpha^2-\beta^2)(2\alpha\beta)\right]$$

Again, the computations here involved are best carried out in tabular form.

We are now in a position to write the general solution in terms of the four arbitrary constants $A_1 \ldots A_4$. It will be convenient to write, at the same time, the expressions for the derivatives required for setting up the boundary condition equations. To determine which derivatives are required, we proceed as follows. The general solution is obtained as the sum of the particular integrals (116) and the solution, Equations (159), of the homogeneous equation, and may be expressed in the following form:

$$u = \sum_\lambda U(\phi)\cos\lambda\frac{x}{a} \; ; \quad v = \sum_\lambda V(\phi)\sin\lambda\frac{x}{a} \; ; \quad w = \sum_\lambda W(\phi)\sin\lambda\frac{x}{a} \qquad (173)$$

where the functions U,V,W, are functions of ϕ only. If these expressions are substituted in Equations (63),(96), and (111), (in which we take $\nu = 0$) and the differentiations with respect to x are carried out, the following equations may be written:

$$\begin{aligned}
N_\phi &= \frac{D}{a}\sum_\lambda \sin\lambda\frac{x}{a}\left[V^\bullet + (1+k)W + kW^{\bullet\bullet}\right] \\
N_{\phi x} &= \frac{D}{2a}\sum_\lambda \cos\lambda\frac{x}{a}\left[(1+k)U^\bullet + \lambda V + k\lambda W^\bullet\right] \\
M_\phi &= Dk\sum_\lambda \sin\lambda\frac{x}{a}\,(W + W^{\bullet\bullet}) \qquad (174) \\
S_\phi &= \frac{Dk}{a}\sum_\lambda \sin\lambda\frac{x}{a}\left[(1-2\lambda^2)W^\bullet + W^{\bullet\bullet\bullet} - \frac{\lambda}{2}U^\bullet + \frac{3}{2}\lambda^2 V\right] \\
N_x &= \frac{D}{a}\sum_\lambda \sin\lambda\frac{x}{a}\left[\lambda(-U + k\lambda W)\right]
\end{aligned}$$

of which functions the first four are involved in the boundary conditions (165). To set up the boundary condition equations, we, therefore, require, beside the functions U,V, and W, the derivatives $U^{\cdot}$, $V^{\cdot}$, $W^{\cdot}$, $W^{\cdot\cdot}$, and $W^{\cdot\cdot\cdot}$. Table II. shows the calculation of these quantities in terms of the constants $A_1 \ldots A_4$; these calculations involve the use of Equations (160) and (163). It is to be noted that the coefficients with subscripts 1 and 3 must be calculated, and that the coefficients with subscripts 2 and 4 may then be written by inspection.

We now calculate the boundary functions of Equations (165) and (174) for the particular case at hand. Substituting $\lambda = 0.4$, $k = 0.00001$ in the latter equations, we obtain the first terms of the series as follows:

$$\begin{aligned}
N_\phi &= \frac{D}{a} \sin .4\frac{x}{a}(V^{\cdot} + 1.00001\, W + .00001\, W^{\cdot\cdot}) \\
N_{\phi x} &= \frac{D}{2a} \cos .4\frac{x}{a}(1.00001\, U^{\cdot} + .4\, V + .000004\, W^{\cdot}) \\
M_\phi &= Dk \sin .4\frac{x}{a}(W + W^{\cdot\cdot}) \\
S_\phi &= \frac{Dk}{a} \sin .4\frac{x}{a}(.68\, W^{\cdot} + W^{\cdot\cdot\cdot} - .2\, U^{\cdot} + .24\, V) \\
N_x &= \frac{D}{a} \sin .4\frac{x}{a}(-.4\, U + .0000016\, W)
\end{aligned} \tag{175}$$

The derivatives of Table II. are now combined according to Equations (175). Again, in making these calculations, it is only necessary to compute the first and third coefficients, the other two then being

TABLE II

	1ST TERM		2ND TERM		3RD TERM		4TH TERM		LOAD TERM	
U	A_1	E_1	A_2	E_2	A_3	E_3	A_4	E_4	78.1221315	$\cos\phi$
U˙	$\alpha_1 A_1 + \beta_1 A_2$ $2.40678799 A_1 + 1.04052882 A_2$	O_1	$-1.04052882 A_1$ $+2.40678799 A_2$	O_2	$\alpha_2 A_3 + \beta_2 A_4$ $.992122666 A_3 + 2.52472485 A_4$	O_3	$-2.52472485 A_3$ $+.992122666 A_4$	O_4	-78.1221315	$\sin\phi$
V	$R(\psi_1)A_1 + I(\psi_1)A_2$ $-5.73713336 A_1 - 2.72284622 A_2$	O_1	$2.72284622 A_1$ $-5.73713336 A_2$	O_2	$R(\psi_3)A_3 + I(\psi_3)A_4$ $-2.37202059 A_3 - 6.58614089 A_4$	O_3	$6.58614089 A_3$ $-2.37202059 A_4$	O_4	257.803217	$\sin\phi$
V˙	$\alpha_1 B_1 = -13.8080637 A_1 - 6.35331358 A_2$ $\beta_1 B_2 = 2.83319996 A_1 - 5.96965261 A_2$ $-10.9748637 A_1 - 12.5229662 A_2$	E_1	$12.5229662 A_1$ $-10.9748637 A_2$	E_2	$\alpha_2 B_3 = -2.35333539 A_3 - 6.53425966 A_4$ $\beta_2 B_4 = 16.6281936 A_3 - 5.98869933 A_4$ $14.2748582 A_3 - 12.5229590 A_4$	E_3	$12.5229590 A_3$ $+14.2748582 A_4$	E_4	257.803217	$\cos\phi$
W	$R(\chi_1)A_1 + I(\chi_1)A_2$ $10.9811970 A_1 + 12.5161446 A_2$	E_1	$-12.5161446 A_1$ $+10.9811970 A_2$	E_2	$R(\chi_3)A_3 + I(\chi_3)A_4$ $-14.2811919 A_3 + 12.5169976 A_4$	E_3	$-12.5169976 A_3$ $-14.2811919 A_4$	E_4	-260.303042	$\cos\phi$
W˙	$\alpha_1 C_1 = 26.4294131 A_1 + 30.1237065 A_2$ $\beta_1 C_2 = -13.0234092 A_1 + 11.4262520 A_2$ $13.4060039 A_1 + 41.5499585 A_2$	O_1	$-41.5499585 A_1$ $+13.4060039 A_2$	O_2	$\alpha_2 C_3 = -14.1686942 A_3 + 12.4183970 A_4$ $\beta_2 C_4 = -31.6019749 A_3 - 36.0560801 A_4$ $-45.7706691 A_3 - 23.6376831 A_4$	O_3	$23.6376831 A_3$ $-45.7706691 A_4$	O_4	260.303042	$\sin\phi$
W˙˙	$\alpha_1 C_1' = 32.2654092 A_1 + 100.001941 A_2$ $\beta_1 C_2' = -43.2339293 A_1 + 13.9493334 A_2$ $-10.9685201 A_1 + 113.951274 A_2$	E_1	$-113.951274 A_1$ $-10.9685201 A_2$	E_2	$\alpha_2 C_3' = -45.4101183 A_3 - 23.4514812 A_4$ $\beta_2 C_4' = 59.6786459 A_3 - 115.558346 A_4$ $14.2685276 A_3 - 139.009827 A_4$	E_3	$139.009827 A_3$ $+14.2685276 A_4$	E_4	260.303042	$\cos\phi$
W˙˙˙	$\alpha_1 C_1'' = -26.3989024 A_1 + 274.236558 A_2$ $\beta_1 C_2'' = -118.569585 A_1 - 11.4130613 A_2$ $-144.968487 A_1 + 262.843497 A_2$	O_1	$-262.843497 A_1$ $-144.968487 A_2$	O_2	$\alpha_2 C_3'' = 14.1561296 A_3 - 137.914800 A_4$ $\beta_2 C_4'' = 350.961565 A_3 + 36.0241062 A_4$ $365.117695 A_3 - 101.890694 A_4$	O_3	$101.890694 A_3$ $+365.117695 A_4$	O_4	-260.303042	$\sin\phi$

written by inspection. In this way, the following expressions are obtained for the force and moment components of Equations (175):

$$
\begin{aligned}
N_\phi = \frac{D}{a} \sin .4\frac{x}{a} \Big[& (.0063334\, A_1 - .0055569\, A_2) E_1(\phi) \\
& + (.0055569\, A_1 + .0063334\, A_2) E_2(\phi) \\
& + (-.0063338\, A_3 - .0072263\, A_4) E_3(\phi) \\
& + (.0072263\, A_3 - .0063338\, A_4) E_4(\phi) - 2.499825\, S \cos\phi \Big] \\
N_{\phi x} = \frac{D}{2a} \cos .4\frac{x}{a} \Big[& (.11201234\, A_1 - .04843306\, A_2) O_1(\phi) \\
& + (.04843306\, A_1 + .11201234\, A_2) O_2(\phi) \\
& + (.043141268\, A_3 - .10980081\, A_4) O_3(\phi) \\
& + (.10980081\, A_3 + .043141268\, A_4) O_4(\phi) + 24.999415\, S \sin\phi \Big] \\
M_\phi = Dk \sin .4\frac{x}{a} \Big[& (.0126769\, A_1 + 126.467419\, A_2) E_1(\phi) \\
& + (-126.467419\, A_1 + .0126769\, A_2) E_2(\phi) \\
& + (-.0126643\, A_3 - 126.492829\, A_4) E_3(\phi) \\
& + (126.492829\, A_3 - .0126643\, A_4) E_4(\phi) \Big] \\
S_\phi = \frac{Dk}{a} \sin .4\frac{x}{a} \Big[& (-137.710674\, A_1 + 290.235880\, A_2) O_1(\phi) \\
& + (-290.235880\, A_1 - 137.710674\, A_2) O_2(\phi) \\
& + (333.225930\, A_3 - 120.049938\, A_4) O_3(\phi) \\
& + (120.049938\, A_3 + 333.225930\, A_4) O_4(\phi) - 5.799775\, S \sin\phi \Big]
\end{aligned}
\tag{176}
$$

$$N_x = \frac{D}{a} \sin .4 \frac{x}{a} \Big[(-.399982430\ A_1 + .0000200258314\ A_2)E_1(\phi)$$

$$+ (-.0000200258314\ A_1 - .399982430\ A_2)E_2(\phi)$$

$$+ (-.400022850\ A_3 + .0000200271962\ A_4)E_3(\phi)$$

$$+ (-.0000200271962\ A_3 - .400022850\ A_4)E_4(\phi) - 31.2492691\ S \cos\phi \Big]$$

The calculations involved in obtaining the above expressions are most easily carried out by means of a table, the form of which is obvious from inspection of Equations (175).

In order to set up the boundary equations, we require values of the functions $E_1(\phi)\ldots\ldots O_4(\phi)$ at $\phi = \frac{\pi}{2}$. In order to plot the final results we will require the values of these functions at points between 0 and $\frac{\pi}{2}$ as well. From Equations (157), by a rather tedious set of calculations, which, again, is best carried out in tabular form, we obtain the results shown in Table III.

The quantities $E_1(\frac{\pi}{2})\ldots\ldots O_4(\frac{\pi}{2})$ are now introduced into the first four of expressions (176). Introducing also the values $\sin\frac{\pi}{2} = 1$ and $\cos\frac{\pi}{2} = 0$, we obtain the expressions for the boundary functions of Equations (165) at $\phi = \frac{\pi}{2}$ in terms of the four constants $A_1 \ldots A_4$. By Equations (165), these functions must vanish. We thus obtain the following four boundary condition equations:

$$.112676028\ A_1 + .146248348\ A_2 - .0013718676\ A_3 + .0227333435\ A_4 = 0$$

$$.90405351\ A_1 + 2.51946567\ A_2 - .266503143\ A_3 + .090748455\ A_4 = -24.999415\ S$$

$$-2765.51008\ A_1 - 176.202763\ A_2 - 210.810893\ A_3 + 213.164174\ A_4 = 0 \qquad (177)$$

TABLE III.

ϕ	$E_1(\phi)$	$E_2(\phi)$
0°	1.00000000	0.00000000
30°	1.62914435	0.84027661
60°	3.08779385	5.47566311
75°	2.42385557	11.4003906
82.5°	1.16066981	15.6210208
90°	-1.39545800	21.8672320
	$E_3(\phi)$	$E_4(\phi)$
0°	1.00000000	0.00000000
30°	0.28031328	0.52648830
60°	-1.39717976	0.59020036
75°	-1.94271745	-0.27566469
82.5°	-1.94289773	-0.93210011
90°	-1.68502094	-1.66675245
	$O_1(\phi)$	$O_2(\phi)$
0°	0.00000000	0.00000000
30°	1.38659517	0.98726141
60°	3.04810483	5.54696108
75°	2.41497972	11.4422908
82.5°	1.15840413	15.6515734
90°	-1.39400683	21.8899958
	$O_3(\phi)$	$O_4(\phi)$
0°	0.00000000	0.00000000
30°	0.13379170	1.10307043
60°	-1.08626774	0.75912776
75°	-1.67342676	-0.32002512
82.5°	-1.73183590	-1.04569677
90°	-1.54206659	-1.82126557

$$-6161.29257\ A_1 - 3419.07688\ A_2 - 732.499393\ A_3 - 421.767914\ A_4 = 5.799775\ S$$

These equations may be solved by successive elimination of the variables, or otherwise, to give the following results:

$$\begin{aligned} A_1 &= 9.79345811\ S \\ A_2 &= -19.8190429\ S \\ A_3 &= -34.1524917\ S \\ A_4 &= 76.8985704\ S \end{aligned} \qquad (178)$$

Using these results in the expressions of Table II. and Equations (176), we obtain finally the following expressions:

$$u = S \cos .4\frac{x}{a}\Big[9.79345811\ E_1(\phi) - 19.8190429\ E_2(\phi) - 34.1524917\ E_3(\phi) + 76.8985704\ E_4(\phi) + 78.1221315 \cos\phi\Big]$$

$$v = S \sin .4\frac{x}{a}\Big[-2.2221692\ O_1(\phi) + 140.370572\ O_2(\phi) - 425.454405\ O_3(\phi) - 407.338114\ O_4(\phi) + 257.803217 \sin\phi\Big]$$

$$w = S \sin .4\frac{x}{a}\Big[-140.514114\ E_1(\phi) - 340.213152\ E_2(\phi) + 1450.27751\ E_3(\phi) - 670.716583\ E_4(\phi) - 260.303042 \cos\phi\Big] \qquad (179)$$

$$N_\phi = \frac{DS}{a} \sin .4\frac{x}{a}\Big[.172158327\ E_1(\phi) - .071100659\ E_2(\phi) - .339377087\ E_3(\phi) - .733856316\ E_4(\phi) - 2.499825 \cos\phi\Big]$$

$$N_x = \frac{DS}{a} \sin 4\frac{x}{a}\Big[-3.91760806\ E_1(\phi) + 7.92707282\ E_2(\phi)$$

$$+ 13.6633172\ E_3(\phi) - 30.7605013\ E_4(\phi) - 31.2492691 \cos\phi\Big]$$

$$N_{\phi x} = \frac{DS}{2a} \cos 4\frac{x}{a}\Big[2.05688505\ O_1(\phi) - 1.74565023\ O_2(\phi) - 9.91690712\ O_3(\phi)$$

$$- .43246942\ O_4(\phi) + 24.999415 \sin\phi\Big]$$

By introducing the functions of Table III. into these equations, the first terms of the series for the various force and deformation components may be calculated.

A similar set of calculations was performed using the values $k = 0.00001$ (as before) and $\lambda = 4.0$. The final results of these and the previous calculations are collected in Table IV.

In this table, the numbers of the last six columns are to be multiplied by the factor in the second column to give the first term of the series for the quantity designated in the first column. The tabulated numbers represent functions of ϕ (the functions $W(\phi)$ and $V(\phi)$ of Equations (173), and corresponding functions in the case of the forces) which may be considered as the amplitudes of the first terms of the Fourier series in x for the various force and deformation components; they therefore give the distribution of these quantities, as functions of ϕ, at those points where $\sin\lambda\frac{x}{a}$ and $\cos\lambda\frac{x}{a}$ have the value unity; i.e., at the center of the span in the case of the sine factor, or at the supports in the case of the cosine factor. In Figures 3 and 4, the tabulated functions of ϕ representing

TABLE IV.

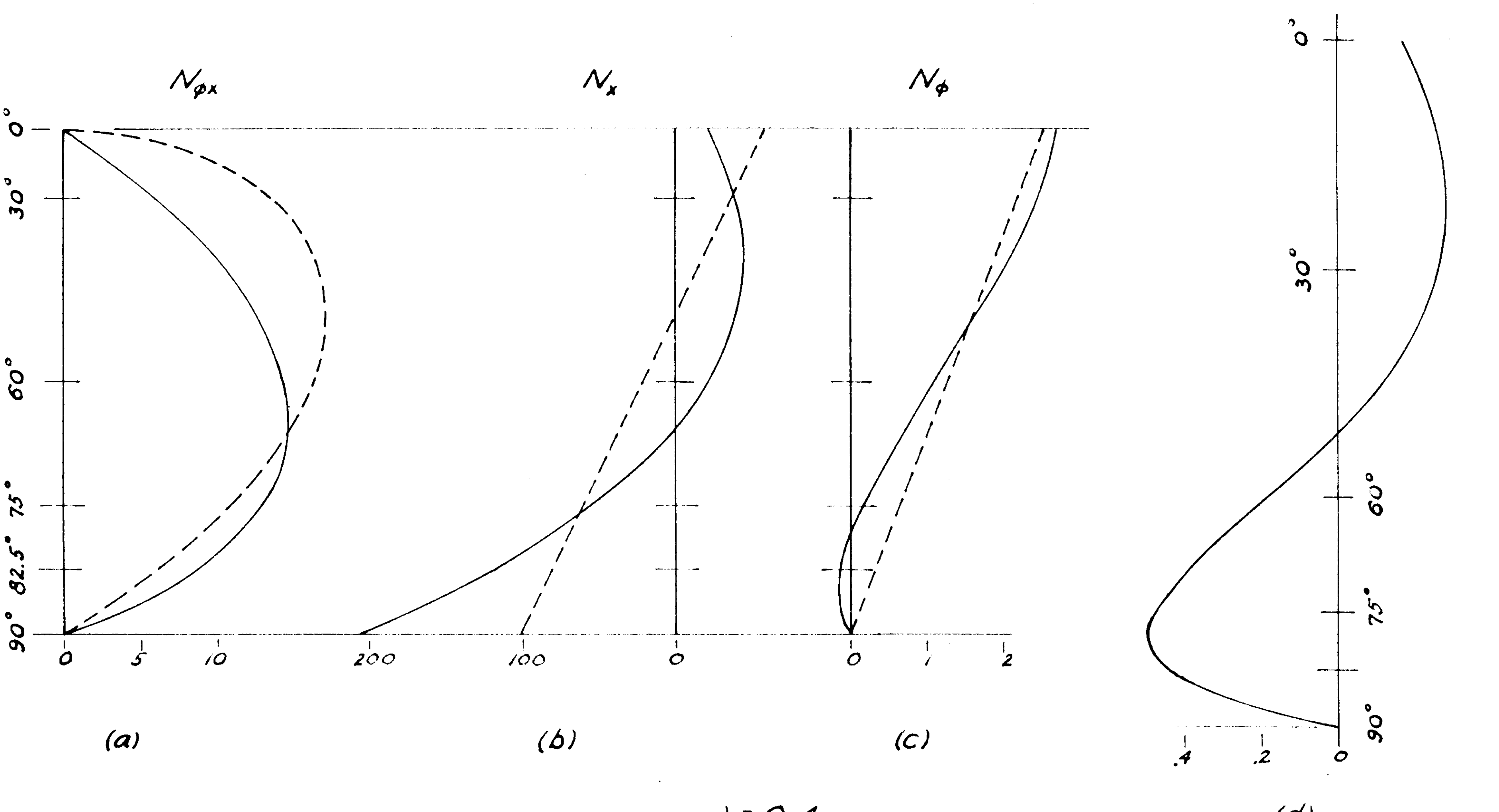
$\lambda = 0.4$

	Factor	0°	30°	60°	75°	82.5°	90°
w	$S \sin .4\frac{x}{a}$	1049	-686.8	-4849	-6919	-7704	-8569
v	$S \sin .4\frac{x}{a}$	0	-241.8	1148	2692	3613	4732
N_x	$\frac{DS}{a} \sin .4\frac{x}{a}$	-21.5	-39.1	-21.6	54.7	117.3	207.1
N_ϕ	$\frac{DS}{a} \sin .4\frac{x}{a}$	-2.667	-2.426	-1.067	-.179	.106	0
$N_{\phi x}$	$\frac{DS}{a} \cos .4\frac{x}{a}$	0	5.912	14.34	12.94	8.736	0

$\lambda = 4.0$

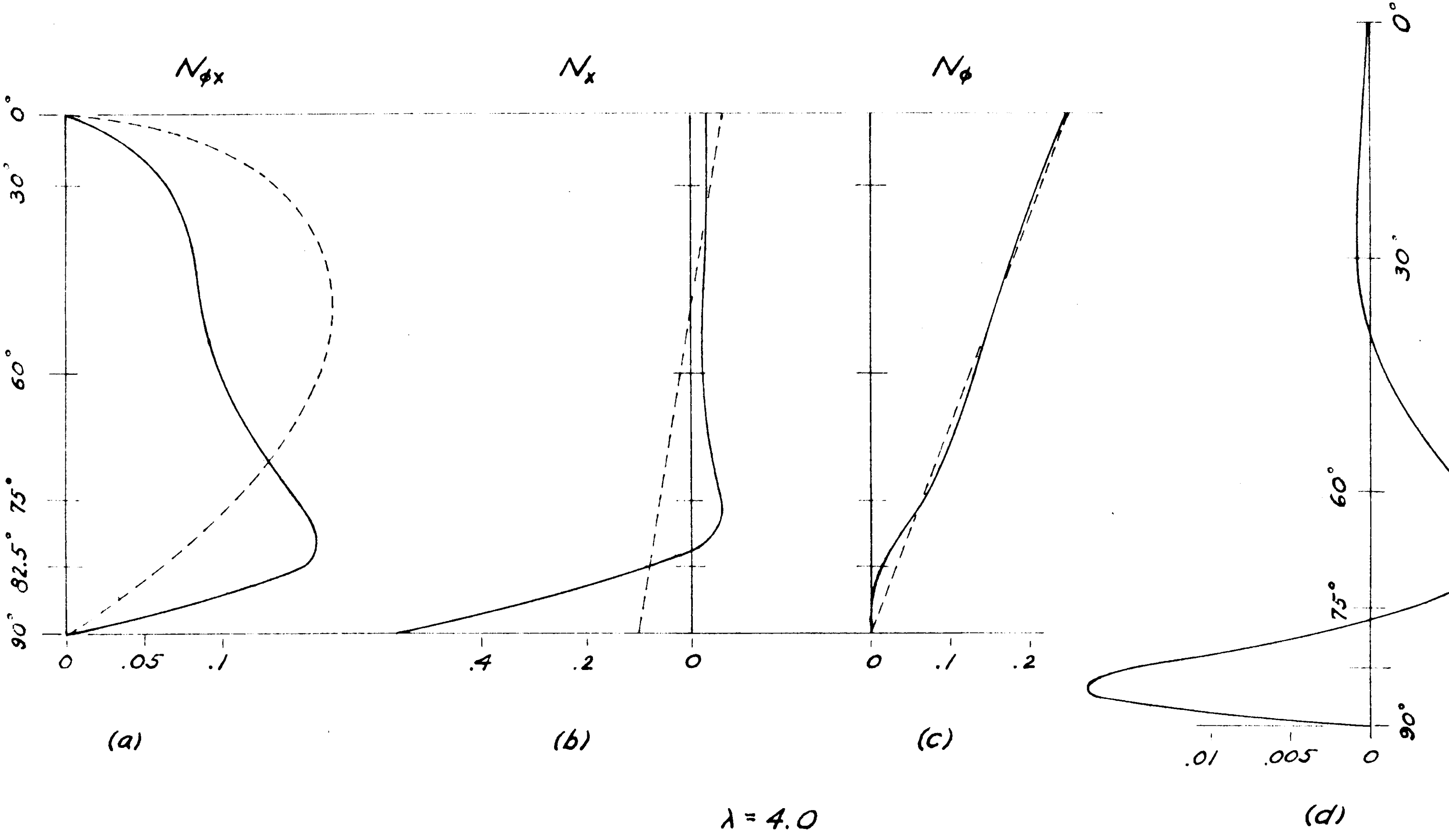
	Factor	0°	30°	60°	75°	82.5°	90°
w	$S \sin 4\frac{x}{a}$	-.307	-.290	-.105	-.656	-1.088	-1.436
v	$S \sin 4\frac{x}{a}$	0	.0347	.0370	.0934	.2025	.3659
N_x	$\frac{DS}{a} \sin 4\frac{x}{a}$	-.0302	-.0286	-.0219	-.0543	.0828	.5718
N_ϕ	$\frac{DS}{a} \sin 4\frac{x}{a}$	-.2492	-.2150	-.1308	-.0665	-.0176	0
$N_{\phi x}$	$\frac{DS}{a} \cos 4\frac{x}{a}$	0	.0646	.0975	.1472	.1495	0

the force components have been plotted for the two different values of λ. The full lines in (a), (b), and (c) of these figures show these functions plotted to a convenient scale against values of $\cos \phi$. In (a) and (b) the dotted lines represent, to the same scale as the full lines, the distribution of stress calculated by the use of the elementary beam theory. This theory gives the following formulas for the axial stresses at the center of the span and the shearing stresses at the supports, assuming simply supported end conditions:



$\lambda = 0.4$

FIGURE 3



$\lambda = 4.0$

FIGURE 4

$$N_x = \frac{2\pi}{a^2[\pi^2-8]} \, M \, (\tfrac{2}{\pi} - \cos\phi) = (\frac{2\pi}{a^2[\pi^2-8]})(\frac{\pi a p \ell^2}{8})(\tfrac{2}{\pi} - \cos\phi)$$

$$= \frac{DS}{a}(\frac{\pi^2}{16[\pi^2-8]})(\frac{\ell}{a})^3 (\tfrac{2}{\pi} - \cos\phi) \qquad (180)$$

$$N_{\phi x} = \frac{2\pi}{a[\pi^2-8]} \, V \, (\sin\phi - \tfrac{2\phi}{\pi}) = (\frac{2\pi}{a[\pi^2-8]})(\frac{\pi a p \ell}{2})(\sin\phi - \tfrac{2\phi}{\pi})$$

$$= \frac{DS}{a}(\frac{\pi^2}{4[\pi^2-8]})(\frac{\ell}{a})^2 (\sin\phi - \tfrac{2\phi}{\pi})$$

where M is the bending moment at the span center and V is the shear at the support. In (c) the dotted line represents, to the same scale as the full line, the coefficient of the first term of the Fourier series in x for the stress N_ϕ calculated according to the third of Equations (120) of the membrane theory; i.e., it represents the coefficient of the first term of the following series:

$$N_\phi = -\, pa \cos\phi = -\frac{DS}{a}\cos\phi \sum_{n=1,3\ldots}^{\infty} \frac{1}{\lambda}\sin\lambda\frac{x}{a} \qquad (181)$$

In adjusting the scales between the dotted and full lines in these figures, it was assumed that the quantities of Table IV. represent the first terms of the series, so that the ratio $\frac{a}{\ell}$ was taken as $\frac{0.4}{\pi}$ and $\frac{4.0}{\pi}$ in calculating the dotted curves from Equations (180) and (181) when λ was equal, respectively, to 0.4 and 4.0. In (d), the difference between the full and dotted curves of figures (c) is plotted against values of the angle ϕ; these curves may be regarded as the curves of the correction to be applied to the membrane stress N_ϕ to give the true stress N_ϕ. The approximate curves, shown dotted in these figures, are to be used later in attempting to obtain approximate methods of solution of this problem.

Some idea of the quickness of convergence of the series when $\frac{a}{\ell} = \frac{0.4}{\pi}$ may be gained from consideration of the results of Table IV. When $\frac{a}{\ell} = \frac{0.4}{\pi}$, the values of λ for successive terms of the series, corresponding to n = 1,3,5,7,9,11, etc., will be, respectively, 0.4, 1.2, 2.0, 2.8, 3.6, and 4.4, etc.. It is reasonable to believe that the solutions are continuous functions of λ. Further, it will be noted that the general shape of the curves for $\lambda = 0.4$ over the entire range $0° < \phi < 90°$ is very similar to the shape of the curves for $\lambda = 4.0$ over the range $0° < \phi < 30°$ (approximately), the magnitudes, of course, being smaller in the latter case. Expressed otherwise, having the curves for $\lambda = 4.0$, the general shape of the curves for $\lambda = 0.4$ is indicated approximately by the shape of that part of the former curves when ϕ takes values between 0° and 30°, where this range is then magnified so that it corresponds to a variation of ϕ from 0° to 90° for the latter case. Then, due to the continuity of the solution as a function of λ , the general shape of the curves for any other value of λ between 0.4 and 4.0 should be indicated by the shape of some portion of the curves for $\lambda = 4.0$, and the magnitudes should lie between the corresponding magnitudes for $\lambda = 0.4$ and 4.0; also, it is reasonable to believe that the curves for $\lambda = 4.0$ bear similar relations to the curves for still larger values of λ. It is thus probable that the values obtained for $\lambda = 4.0$ closely approximate the corresponding values for $\lambda = 3.6$ and 4.4, being less than the former, greater than the latter, and, therefore, that they indicate approximately the order of magnitude of the fifth or sixth term of the series when $\frac{a}{\ell} = \frac{0.4}{\pi}$. Considering Table IV. from this point of view, it is seen that the series for w, v, N_x, and $N_{\phi x}$, are probably quite rapidly convergent, while the convergence of the series for N_ϕ is much slower. The latter is to be expected, however, since the Fourier series in x for the membrane theory solution for N_ϕ has terms of the order of $\frac{1}{\lambda}$, and the corresponding terms of the exact solution do not differ greatly from these.

It is interesting, and perhaps important, to note that at the boundary generators, for both cases computed, the deflection is radially inward and downward. This seems rather surprising, as physical intuition would seem to indicate that this deflection should be radially outward, rather than inward, and downward.

PART VI.

(a) In this Part, we shall consider several approximate methods of obtaining solutions of this problem. However, before going into the methods used by the author, it may be desirable to give a brief review of some of the methods used by other writers on this subject. Finsterwalder has obtained a solution on the assumption that the moments M_x, $M_{x\phi}$, and $M_{\phi x}$, and the shear Q_x are negligible, and that all terms involving k in Equations (63) for N_x, N_ϕ, $N_{\phi x}$, and $N_{x\phi}$ may be neglected. (See Finsterwalder, Proceedings International Association for Bridge and Structural Engineering, Vol. I., 1932, page 127; Schorer, Transactions A.S.C.E., Vol. 62, 1936, page 767; Flügge, Statik und Dynamik der Schalen, Berlin, J. Springer, page 137.) On the basis of these assumptions, the forces N_x, N_ϕ, $N_{\phi x}$, $N_{x\phi}$, and Q_ϕ, and the deformations u, v, and w, are all expressible in terms of the ring moment M_ϕ, and a single partial differential equation of the eighth order in this quantity is obtained. The solution of this equation involves the solution of an eighth degree algebraic equation of the same type as Equation (123), and the final solution is obtained in the form of series of the type of Equations (150). Although the approximations involved in this solution are considerable, it is doubtful if the labor is less, to any great extent, than that of the exact solution. Finsterwalder's solution does not involve the troublesome computation of the ratios ψ_r and χ_r of Equations (138) and (172); however, aside from this, the computations involved are almost identical in the two solutions. Jakobsen (See Jakobsen, "Travaux", January, 1938) has placed Finsterwalder's solution in a somewhat more compact form, but again it is doubtful if any great saving of labor is involved. It is possible to obtain a solution in the same form as Finsterwalder's in which the moments M_x, $M_{x\phi}$, and $M_{\phi x}$, are not completely neglected. The latter quantities may be expressed, approximately, in terms of w only, as will be shown later; and in a manner similar to that used by Flügge (See reference above) the differential Equations (64) may be reduced to a single differential equation of the eighth order in w. Again, the labor involved is certainly comparable to that of the exact solution, so this method will not be pursued further in this thesis. An interesting variation of the solution in the latter form has been given by Jakobsen (See Jakobsen, Der Bauingenieur, July 28, 1939, Vol. 20, page 394) in which the exact equations are solved by a method of iteration; as before, there seems to be no appreciable shortening of the computations.

(b) Let us consider several approximations which appear possible from the exact solutions of Part V.. In the formulas of Equations (63) and (175) for the force components, very small errors would be introduced by neglecting all terms in k. Let us see the physical significance of this approximation. In calculating the results of Equations (63), we retained all powers up to the third of the quantity $\frac{z}{a}$. Let us recalculate the quantities of Equations (63) in the following manner. We substitute expressions for the stress components into Equations (2), in which the factors $(1 + \frac{z}{a})$ of the integrands are taken as unity on the assumption that $\frac{z}{a}$ is small compared to one. In the resulting integrands, we expand $(1+\frac{z}{a})^{-1}$ by the binomial theorem and retain only terms up to the first power in $\frac{z}{a}$. Carrying out the integrations, we obtain in this way:

$$N_x = \frac{D}{a}(u' + \nu v^{\cdot} + \nu w) \qquad M_x = Dk(w'' + \nu w^{\cdot\cdot} + \nu w)$$

$$N_\phi = \frac{D}{a}(v^{\cdot} + w + \nu u') \qquad M_\phi = Dk(w^{\cdot\cdot} + w + \nu w'') \tag{182}$$

$$N_{x\phi} = N_{\phi x} = \frac{D}{a}\left(\frac{1-\nu}{2}\right)(u^{\cdot} + v') \qquad M_{x\phi} = M_{\phi x} = Dk(1-\nu)(w'^{\cdot} + \frac{u^{\cdot}}{2} - \frac{v'}{2})$$

It is seen that the first three of these results are given by Equations (63) when k is neglected. The formulas for M_x and $M_{x\phi}$ differ somewhat from the corresponding results of Equations (63) while the formulas for M_ϕ and $M_{\phi x}$ are identical in the two sets of equations. The above expression for M_x will be useful because it expresses this quantity in terms of w only. Concerning the torsion moments $M_{x\phi}$ and $M_{\phi x}$, we may, conveniently, either neglect them completely, as Finsterwalder does, or use the approximation that $\frac{1}{2}(u^{\cdot} - v')$ is small compared to $w'^{\cdot}$ and so write:

$$M_{x\phi} = M_{\phi x} = Dk(1-\nu)w'^{\cdot} \tag{183}$$

the advantage of the latter equation over the corresponding one of Equations (182) being that these moments are now expressed in terms of the single quantity w. Which of these two approximations is the better cannot be said off-hand, and both will be used later in this thesis.

(c) Perhaps the most obvious method by which to attempt an approximate solution is to assume that the stresses N_x and $N_{\phi x}$ are given by the elementary beam theory, see Equations (180). These results may be used as a first approximation in solving the differential equations by a method of successive approximations. We start with the equilibrium equations in the form of Equations (48). Between the second and third of these, we eliminate first Q_ϕ and then N_ϕ, thus obtaining the following two equations, where the notation of Equations (61) has been introduced:

$$N_\phi^{\cdot\cdot} + N_\phi = -a(Z + Y^{\cdot}) - Q_x' - N_{x\phi}'^{\cdot} \tag{184}$$

$$Q_\phi^{\cdot\cdot} + Q_\phi + Q_x'^{\cdot} = a(Y - Z^{\cdot}) + N_{x\phi}'$$

We shall make the assumption, referred to above, that the torsion moments $M_{x\phi}$ and $M_{\phi x}$ are small and may be neglected. Then from the fourth and fifth of Equations (48) we have:

$$Q_\phi = \frac{1}{a} M_\phi^{\cdot} \quad ; \quad Q_x = \frac{1}{a} M_x' \tag{185}$$

Substituting Equations (185) in Equations (184), we obtain:

$$N_\phi^{\cdot\cdot} + N_\phi = -a(Z + Y^{\cdot}) - \frac{1}{a} M_x'' - N_{x\phi}'^{\cdot} \tag{186}$$

$$\frac{1}{a}(M_\phi^{\cdot\cdot\cdot} + M_\phi^{\cdot} + M_x''^{\cdot}) = a(Y - Z^{\cdot}) + N_{x\phi}' \tag{187}$$

We shall confine our attention to a shell of the type considered in Part V, the loading being dead load, as given by Equations (164) when ϕ is measured from the vertical axial plane of symmetry. We may integrate Equation (187) with respect to ϕ from ϕ to $\frac{\pi}{2}$, obtaining:

$$\frac{1}{a}(M_\phi^{\cdot\cdot} + M_\phi + M_x'') = -\int_\phi^{\frac{\pi}{2}} (aY + N_{x\phi}')\,d\phi - aZ \tag{188}$$

The constant of integration which normally would appear on the left side of this equation may be shown to vanish as follows. This constant has the form $-\frac{1}{a}(M_\phi^{\cdot\cdot} + M_\phi + M_x'')_{\phi=\frac{\pi}{2}}$. On the boundary generator $M_\phi \equiv 0$ is one of the boundary conditions to be satisfied; whence, the second term of this constant must vanish. Also, by the third of Equations (48) and Equation (185):

$$M_\phi^{\cdot\cdot} + M_x'' = a(Q_\phi^{\cdot} + Q_x') = -a(N_\phi + aZ)$$

Again, boundary conditions require that $N_\phi = 0$ along the generator $\phi = \frac{\pi}{2}$; also, along this generator, Z vanishes for dead load, by Equations (164). Hence, the sum of the first and third terms of the constant must vanish also. Equations (186) and (188) may also be obtained by consideration of the equilibrium of an element of shell included between two planes $x = x_o$ and $x = x_o + dx$ normal to the cylinder axis, and two axial planes, one passing through the boundary generator $\phi = \frac{\pi}{2}$, the other having the variable parameter ϕ.

We now make use of Equations (182) in which, as before, we take $\nu = 0$. Substituting the fourth and fifth of these relations in Equations (186) and (188), we obtain:

$$N_\phi^{\cdot\cdot} + N_\phi = -a(Z + Y^{\cdot}) - \frac{Dk}{a} w^{IV} - N_{x\phi}'^{\cdot} \tag{189}$$

$$w^{::} + 2w^{\cdot\cdot} + w + w^{IV} = -\frac{a}{Dk}\left[\int_\phi^{\frac{\pi}{2}} (aY + N_{x\phi}')\,d\phi + aZ\right] \tag{190}$$

We may now proceed by a method of successive approximations. Assuming the load to be dead load, given by Equations (164), we take the results of elementary beam theory as a first approximation. Assuming a simply supported beam, the shearing force is given by:

$$V = \pi\, ap\left(\frac{\ell}{2} - x\right)$$

Using this result in the second of Equations (180) we obtain the first approximation for $N_{\phi x}$ and $N_{x\phi}$. Placing this in the right side of Equation (190) and using Equation (164), it is seen that the right side of Equation (190) becomes an entirely known function of ϕ only. We may now solve this equation by assuming a solution in the form of a Fourier series in x, $w = \sum_{\lambda} W(\phi)\sin\lambda\frac{x}{a}$, and determining $W(\phi)$ from the resulting ordinary differential equation. The right side of Equation (190) must bedeveloped in a corresponding Fourier series in x. $W(\phi)$ will involve four arbitrary constants which may be reduced to two by use of the condition that w must be an even function of ϕ. The remaining two constants are evaluated from the conditions that $M_{\phi} = Q_{\phi} = 0$ at the boundary generator $\phi = \frac{\pi}{2}$.

The solution for w, thus obtained, is now placed, along with the load Equations (164) and the assumed first approximation for $N_{x\phi}$, in the right side of Equation (189) which is thus reduced to a known function of ϕ. This equation may be solved by assuming a solution of the form $N_{\phi} = \sum_{\lambda} N_{\phi}(\phi)\sin\lambda\frac{x}{a}$, the right side of the equation being developed in a similar series, and determining the function $N_{\phi}(\phi)$ from the resulting ordinary differential equation. This solution introduces two arbitrary constants, one of which is determined by the condition that N_ϕ be an even function of ϕ. It would be supposed that the second constant could be evaluated from the condition that $N_\phi = 0$ at $\phi = \frac{\pi}{2}$; however, on setting up the equation representing this condition, it is found to be independent of this constant, and, furthermore, to be identical with the equation previously used expressing the condition that $M_\phi = 0$ at $\phi = \frac{\pi}{2}$. This constant, as yet not evaluated, is carried along in the solution, and will be determined later.

From the second of Equations (182), we may write the following expression for v:

$$v^{\cdot} = \frac{a}{D} N_\phi - w \tag{191}$$

Substituting the solutions just obtained for N_ϕ and w in this equation, we may integrate, obtaining v as a Fourier series of the form $v = \sum_\lambda V(\phi)\sin\lambda\frac{x}{a}$. This integration introduces one constant which is determined by the condition that v must be an odd function of ϕ.

We now substitute the first and third of Equations (182), in which we take $\nu = 0$, in the first of Equations (48), obtaining:

$$u'' + \frac{1}{2} u^{\cdot\cdot} = -\frac{1}{2} v'^{\cdot} \tag{192}$$

where the load component X is taken as zero in accordance with Equations (164). Using the solution for v just obtained in the right member of this equation, we may solve it, obtaining u as a series of the form $u = \sum_\lambda U(\phi)\cos\lambda\frac{x}{a}$. This integration introduces two arbitrary constants, one of which is determined by the condition that u must be an even function of ϕ. The other constant, together with the constant remaining from the solution of Equation (189) must now be evaluated. To do this, we use the conditions that $N_{\phi x} = M_{\phi x} = 0$ at $\phi = \frac{\pi}{2}$, $N_{\phi x}$ and $M_{\phi x}$ being given by the third and sixth of Equations (182) in which $\nu = 0$. The validity of the last condition is somewhat questionable, as $M_{\phi x}$ was neglected at the outset of the solution. However, it seems, physically, to be the most reasonable condition to use, inasmuch as the usual boundary condition concerning N_ϕ does not yield an independent equation.

The solutions for u and v thus obtained are now substituted in the first and third of Equations (182), to give expressions for N_x and $N_{x\phi}$ which may be regarded as second approximations to these stresses. This process of successive solution of the Equations (189) to (192) may now be repeated using these solutions to obtain third approximations, and so on; and it is believed that the process should converge eventually to the true solution. While the labor involved in this method of obtaining the second approximation is less than that in the exact solution, it is still considerable; and to obtain a third approximation in the way described would make the approximate solution even more difficult than the exact. Thus, for the solution to be of any practical use, it is necessary that the second approximation be sufficiently close to the exact solution to be usable. Judged from this point of view, it will be seen that this method of approximation is entirely unsatisfactory; for this reason, it is not considered necessary to give the solution in more detail. Complete computations using this method were carried out for the case, considered in Part V., $\lambda = 4.0$ and $k = 0.00001$, some of the results of which are given in Table V. In this table, under first approximation, are tabulated values of $N_{x\phi}$ at the support and N_x at the span center, as calculated from Equations (180), assuming simple support. Under second approximation, are given the first terms of the series for $N_{x\phi}$ and N_x. The numbers tabulated represent amplitudes of the first terms of the Fourier series in x, and give the values of $N_{x\phi}$ and N_x , at the support and span center, respectively, where the functions $\cos\lambda\frac{x}{a}$ and $\sin\lambda\frac{x}{a}$, respectively, become unity.

TABLE V.

	Factor	30°	45°	60°
	First Approximation			
$N_{x\phi}$	$\frac{DS}{a}$	.1357	.1688	.1625
N_x	$\frac{DS}{a}$	-.03668	-.01128	.02184
	Second Approximation			
$N_{x\phi}$	$\frac{DS}{a}\cos 4\frac{x}{a}$	2.65	11.6	23.8
N_x	$\frac{DS}{a}\sin 4\frac{x}{a}$	1030	836	588

Assuming that the series obtained in this way are quickly convergent, the results given as first and second approximations should be nearly equal, in order that the solution may be usable. As this is far from the case, we are led to conclude that the assumed first approximation is not sufficiently close to the exact result to produce quick convergence of the solution by this method of successive approximations. Due to the extreme difference between first and second approximations, as shown by Table V., it was not considered worthwhile to carry out the calculations for other cases, although a comparison of the results of Figures 3a and 4a indicates that the first approximation would be somewhat nearer the exact solution for smaller values of λ . It is possible that these large differences may be due in large part to the use of the boundary condition $M_{\phi x} = 0$. This question could possibly be avoided by integrating the equations in a manner similar to that to be described in Sections (d) and (f). However, in view of the rather poor results obtained by the method of the latter sections, it was not considered worthwhile to recalculate in this way, using beam theory as a first approximation.

(d) We shall now consider another method of approximation which gives results that seem more promising, but which still cannot be considered as satisfactory. We shall assume in this method that the ring stress N_ϕ is given as a first approximation by the membrane theory; an examination of Figures 3c and 4c gives an indication of the error involved in this assumption. We shall also use the approximation that particular integrals of the differential equations are given by the solution of the membrane theory equations. This approximation has been discussed on page 68. We start with the differential equations of equilibrium in the form of Equations (64), particular integrals of which we take to be given by solutions of the membrane theory Equations (120); these will be called the primary solution. Upon the primary solution we will superimpose the secondary, or correction solution, obtained from the homogeneous Equations (113).

Let us obtain, first, the primary, or membrane theory solution. As mentioned on page 68, solutions of the membrane equations, for various conditions at the support, have been given quite completely by Dischinger. We shall merely state the desired results, referring the reader to Dischinger's paper for the derivations. Assuming the conditions of simple support, and dead load, Equations (164), Dischinger obtains, by successive integration of the membrane theory equations, the following solution:

$$
\begin{aligned}
N_\phi &= -pa \cos\phi \\
N_{x\phi} &= p \sin\phi \, (\ell - 2x) \\
N_x &= \frac{p}{a} \cos\phi \, (x^2 - \ell x) \\
u &= \frac{p}{12Da} \cos\phi \, (4x^3 - 6\ell x^2 + \ell^3) \\
v &= \frac{2p}{Da^2} \sin\phi \left[a^2 (\ell x - x^2) + \frac{1}{24}(x^4 - 2\ell x^3 + \ell^3 x) \right] \\
w &= -\frac{2p}{Da^2} \cos\phi \left[\frac{a^4}{2} + a^2(\ell x - x^2) + \frac{1}{24}(x^4 - 2\ell x^3 + \ell^3 x) \right]
\end{aligned}
\tag{193}
$$

These solutions may be developed in Fourier series in x as follows:

$$
N_\phi = -\frac{DS}{a} \cos\phi \sum_{n=1,3\cdots}^{\infty} \frac{1}{\lambda} \sin \lambda \frac{x}{a}
$$

$$
N_{x\phi} = \frac{DS}{a} (2 \sin\phi) \sum_{n=1,3\cdots}^{\infty} \frac{1}{\lambda^2} \cos \lambda \frac{x}{a}
$$

$$
\begin{aligned}
N_x &= -\frac{DS}{a}(2\cos\phi)\sum_{n=1,3\ldots}^{\infty}\frac{1}{\lambda^3}\sin\lambda\frac{x}{a} \\
u &= S\,(2\cos\phi)\sum_{n=1,3\ldots}^{\infty}\frac{1}{\lambda^4}\cos\lambda\frac{x}{a} \\
v &= S\,(2\sin\phi)\sum_{n=1,3\ldots}^{\infty}\frac{2\lambda^2+1}{\lambda^5}\sin\lambda\frac{x}{a} \\
w &= -S\cos\phi\sum_{n=1,3\ldots}^{\infty}\frac{\lambda^4+4\lambda^2+2}{\lambda^5}\sin\lambda\frac{x}{a}
\end{aligned}
\tag{194}
$$

where the notation of Equations (115) and (167) is used. We shall take Equations (194) as the primary solution of our problem, and we now seek solutions of the homogeneous Equations (64).

We shall now make use of the assumption, previously stated, that, as a first approximation, the true stress N_ϕ is given by the membrane theory without any correction. Then in the third of Equations (64), we may take N_ϕ equal to zero. Taking, also, the load terms equal to zero, these equations for the determination of the correction stresses become:

$$
\begin{aligned}
N_x' + \dot{N}_{\phi x} &= 0 \\
N_{x\phi}' + \dot{N}_\phi &= \frac{1}{a}(\dot{M}_\phi + M_{x\phi}') \\
\ddot{M}_\phi + \dot{M}_{x\phi}' + \dot{M}_{\phi x}' + M_x'' &= 0
\end{aligned}
\tag{195}
$$

We now express the force and moment components appearing in these equations by means of the first five of Equations (182) and Equation (183), thus introducing the assumptions involved in writing the latter equations into our solution. Equations (195)may thus be rewritten in the following form:

$$u'' + \frac{1}{2} u^{\cdot\cdot} + \frac{1}{2} v'^{\cdot} = 0$$

$$\frac{1}{2} u'^{\cdot} + \frac{1}{2} v'' + v^{\cdot\cdot} = -w^{\cdot} + k(w^{\cdot\cdot\cdot} + w^{\cdot} + w''^{\cdot}) \tag{196}$$

$$w^{::} + w^{\cdot\cdot} + 2w''^{\cdot\cdot} + w^{IV} = 0$$

We assume solutions of these equations in the form of Fourier series as follows:

$$u = \sum_{\lambda} U(\phi) \cos\lambda\frac{x}{a} \;\; ; \;\; v = \sum_{\lambda} V(\phi) \sin\lambda\frac{x}{a} \;\; ; \;\; w = \sum_{\lambda} W(\phi) \sin\lambda\frac{x}{a} \tag{197}$$

the functions U, V, and W, being functions of ϕ only. Substituting Equations (197) in Equations (196), we obtain the following ordinary differential equations for the determination of U, V, and W:

$$\frac{1}{2}U^{\cdot\cdot} - \lambda^2 U + \frac{\lambda}{2} V^{\cdot} = 0$$

$$-\frac{\lambda}{2}U^{\cdot} - \frac{\lambda^2}{2} V + V^{\cdot\cdot} = -W^{\cdot} + k(W^{\cdot\cdot\cdot} + W^{\cdot}[1 - \lambda^2]) \tag{198}$$

$$W^{::} + W^{\cdot\cdot}(1 - 2\lambda^2) + \lambda^4 W = 0$$

Equations (198) are easily solved in succession. We first solve the third equation. Making use of the condition that W must be an even function of ϕ, the solution of this equation may be written in the following form:

$$W = A\,E_1(\phi) + B\,E_2(\phi) \tag{199}$$

where A and B are arbitrary constants, and $E_1(\phi)$ and $E_2(\phi)$ are defined as follows:

$$\underline{\text{If } \lambda < \tfrac{1}{2}:} \quad E_1(\phi) = \cos\alpha\phi \quad \text{where } \alpha = \sqrt{\tfrac{1}{2} - \lambda^2 + \sqrt{\tfrac{1}{4} - \lambda^2}}$$

$$E_2(\phi) = \cos\beta\phi \quad \text{where } \beta = \sqrt{\tfrac{1}{2} - \lambda^2 - \sqrt{\tfrac{1}{4} - \lambda^2}} \qquad (200)$$

$$\underline{\text{If } \lambda > \tfrac{1}{2}:} \quad E_1(\phi) = \cos\frac{\phi}{2}\cosh\alpha\phi \quad \text{where } \alpha = \sqrt{\lambda^2 - \tfrac{1}{4}}$$

$$E_2(\phi) = \sin\frac{\phi}{2}\sinh\alpha\phi$$

It will be convenient, here, to define odd functions as follows:

$$\underline{\text{If } \lambda < \tfrac{1}{2}:} \quad O_1(\phi) = \sin\alpha\phi \qquad \underline{\text{If } \lambda > \tfrac{1}{2}:} \quad O_1(\phi) = \cos\frac{\phi}{2}\sinh\alpha\phi \qquad (201)$$

$$O_2(\phi) = \sin\beta\phi \qquad\qquad O_2(\phi) = \sin\frac{\phi}{2}\cosh\alpha\phi$$

the quantities α and β of Equations (201) being defined exactly as in Equations (200) for the corresponding cases. Substituting the solution given by Equation (199) in the right member of the second of Equations (198), the first two of the latter equations may be solved simultaneously for U and V. These solutions are easily obtained and may be shown to have the following form, where account is taken of the fact that U must be an even function of ϕ and V an odd function:

$$U = M\,E_1(\phi) + N\,E_2(\phi) + \frac{1}{\lambda}(H + 4J)E_3(\phi) + J\,E_4(\phi) \qquad (202)$$

$$V = P\,O_1(\phi) + Q\,O_2(\phi) + \frac{1}{\lambda}(H + J)\,O_3(\phi) + J\,O_4(\phi)$$

H and J are arbitrary constants, and the remaining quantities, as yet undefined, are given by the following equations:

$$E_3(\phi) = \cosh\lambda\phi \qquad O_3(\phi) = \sinh\lambda\phi \qquad (203)$$

$$E_4(\phi) = \phi\sinh\lambda\phi \qquad O_4(\phi) = \phi\cosh\lambda\phi$$

and the constants M, N, P, and Q, are given in terms of the arbitrary constants A and B by the following sets of equations:

<u>If $\lambda < \frac{1}{2}$:</u>

$$M(-\alpha^2 - 2\lambda^2) + \alpha\lambda P = 0 \tag{204a}$$

$$\alpha\lambda M + P(-2\alpha^2 - \lambda^2) = 2A\alpha\left[1 - k(\beta^2 + \lambda^2)\right]$$

$$N(-\beta^2 - 2\lambda^2) + \beta\lambda Q = 0$$

$$\beta\lambda N + Q(-2\beta^2 - \lambda^2) = 2B\beta\left[1 - k(\alpha^2 + \lambda^2)\right]$$

where α and β are defined in Equation (200) for the case $\lambda < \frac{1}{2}$.

<u>If $\lambda > \frac{1}{2}$:</u>

$$-M(2\lambda^2 + 1) + 2\alpha N + 2\alpha\lambda P + \lambda Q = 0 \tag{204b}$$

$$-2\alpha M - N(2\lambda^2 + 1) - \lambda P + 2\alpha\lambda Q = 0$$

$$-2\alpha\lambda M - \lambda N + 2P(\lambda^2 - 1) + 4\alpha Q = -4\left[A\alpha + B(\tfrac{1}{2} - k\lambda^2)\right]$$

$$\lambda M - 2\alpha\lambda N - 4\alpha P + 2Q(\lambda^2 - 1) = -4\left[B\alpha - A(\tfrac{1}{2} - k\lambda^2)\right]$$

where α is defined by Equation (200) for the case $\lambda > \frac{1}{2}$.

By combining the solutinns (199) and (202) in the series of Equations (197), we obtain the solution of Equations (196). These may be differentiated and combined according to Equations (182) and (183) to give the force and moment components. To the secondary, or correction, solution thus obtained we add the membrane theory solution, Equations (194), to give the final results which are as follows:

$$N_\phi = \frac{D}{a}\sum \sin\lambda\frac{x}{a}\left\{F\,E_1(\phi) + G\,E_2(\phi) + (H+2J)\,E_3(\phi) + \lambda J\,E_4(\phi) - \frac{S}{\lambda}\cos\phi\right\}$$

$$N_{x\phi} = \frac{D}{2a}\sum \cos\lambda\frac{x}{a}\left\{\Gamma\,O_1(\phi) + \Lambda\,O_2(\phi) + 2(H+3J)\,O_3(\phi) + 2\lambda J\,O_4(\phi) + \frac{4S}{\lambda^2}\sin\phi\right\}$$

$$N_x = -\frac{D}{a}\sum \sin\lambda\frac{x}{a}\left\{\lambda M E_1(\phi) + \lambda N\,E_2(\phi) + (H+4J)\,E_3(\phi) + \lambda J\,E_4(\phi) + \frac{2S}{\lambda^3}\cos\phi\right\} \quad (205)$$

$$u = \sum \cos\lambda\frac{x}{a}\left\{M\,E_1(\phi) + N\,E_2(\phi) + \frac{1}{\lambda}(H+4J)\,E_3(\phi) + J\,E_4(\phi) + \frac{2S}{\lambda^4}\cos\phi\right\}$$

$$v = \sum \sin\lambda\frac{x}{a}\left\{P\,O_1(\phi) + Q\,O_2(\phi) + \frac{1}{\lambda}(H+J)\,O_3(\phi) + J\,O_4(\phi) + 2S\,\frac{2\lambda^2+1}{\lambda^5}\sin\phi\right\}$$

$$w = \sum \sin\lambda\frac{x}{a}\left\{A\,E_1(\phi) + B\,E_2(\phi) - S\,\frac{\lambda^4+4\lambda^2+2}{\lambda^5}\cos\phi\right\}$$

In these equations, all summations are over values of λ corresponding to all odd integer values of n, and the constants F, G, Γ, and Λ, are defined in terms of the previously given constants as follows:

<u>If $\lambda < \frac{1}{2}$:</u>

$$F = \alpha P + A \qquad \Gamma = \lambda P - \alpha M$$

$$G = \beta Q + B \qquad \Lambda = \lambda Q - \beta N$$

<u>If $\lambda > \frac{1}{2}$:</u>

(206)

$$F = \alpha P + \frac{Q}{2} + A \qquad \Gamma = \alpha M + \frac{N}{2} + \lambda P$$

$$G = \alpha Q - \frac{P}{2} + B \qquad \Lambda = \alpha N - \frac{M}{2} + \lambda Q$$

α and β being defined in these equations exactly as in Equation (200) for the corresponding cases.

The arbitrary constants A, B, H, and J, are now determined by the conditions of Equations (165) at the boundary $\phi = \frac{\pi}{2}$. Using the definitions of Equations (182) and (183) in Equations (111) and (185), we obtain:

$$S_\phi = \frac{Dk}{a}\left(w^{\cdots} + w^{\cdot} + 2\, w^{''\cdot}\right) \tag{207}$$

Substituting the last of Equations (205) in the expressions for M_ϕ and S_ϕ, and placing $\phi = \frac{\pi}{2}$, we obtain the following equations for A and B:

If $\lambda < \frac{1}{2}$:

$$A\ (1-\alpha^2)\, E_1(\tfrac{\pi}{2}) + B\ (1-\beta^2)\, E_2(\tfrac{\pi}{2}) = 0$$

$$A\ \beta\, O_1(\tfrac{\pi}{2}) + B\,\alpha\, O_2(\tfrac{\pi}{2}) = -4S\,\frac{2\lambda^2+1}{\lambda^5}$$

If $\lambda > \frac{1}{2}$: (208)

$$A\left[(\lambda^2+\tfrac{1}{2})E_1(\tfrac{\pi}{2}) - \alpha E_2(\tfrac{\pi}{2})\right] + B\left[\alpha E_1(\tfrac{\pi}{2}) + (\lambda^2+\tfrac{1}{2})E_2(\tfrac{\pi}{2})\right] = 0$$

$$A\left[\alpha\, O_1(\tfrac{\pi}{2}) + \tfrac{1}{2}\, O_2(\tfrac{\pi}{2})\right] - B\left[\tfrac{1}{2}\, O_1(\tfrac{\pi}{2}) - \alpha\, O_2(\tfrac{\pi}{2})\right] = -4S\,\frac{2\lambda^2+1}{\lambda^5}$$

α and β of Equations (208) being defined as in Equation (200) for the corresponding cases. Placing $\phi = \frac{\pi}{2}$ in the first two of Equations (205) and equating to zero, we obtain the following equations for H and J:

$$H\, E_3(\tfrac{\pi}{2}) + J\left[2\, E_3(\tfrac{\pi}{2}) + \lambda\, E_4(\tfrac{\pi}{2})\right] = -F\, E_1(\tfrac{\pi}{2}) - G\, E_2(\tfrac{\pi}{2}) \tag{209}$$

$$2H\, O_3(\tfrac{\pi}{2}) + 2J\left[3O_3(\tfrac{\pi}{2}) + \lambda\, O_4(\tfrac{\pi}{2})\right] = -\Gamma\, O_1(\tfrac{\pi}{2}) - \Lambda\, O_2(\tfrac{\pi}{2}) - \frac{4S}{\lambda^2}$$

We thus have the complete solution on the basis of the stated assumptions, and it is seen to be in a form making numerical computations relatively easy. We might consider the solution (205) as a second approximation to be used in resolving the equations to obtain a third approximation, and so on. However, after the second approximation, the computations become too complicated to be of practical use. Therefore, as before, we must compare the solution in the form of Equations (205) with the exact solution, and with the first approximation, to judge its usefulness.

The solution (205) was computed numerically for the two cases discussed in Part V. of this thesis, ($k = 0.00001$, $\lambda = 0.4$ and 4.0). The results of these calculations for the stresses N_ϕ and N_x are shown in Table VI. This table is arranged exactly as Table IV., see page 103.

TABLE VI.

	Factor	0°	30°	60°	75°	82.5°	90°
				$\lambda = 0.4$			
N_ϕ	$\frac{DS}{a}\sin .4\frac{x}{a}$	-13.85	-10.69	-3.70	-.84	-.08	0
N_x	$\frac{DS}{a}\sin .4\frac{x}{a}$	-169.1	-109.8	60.0	179.6	246.5	317.4
				$\lambda = 4.0$			
N_ϕ	$\frac{DS}{a}\sin 4\frac{x}{a}$	-.2480	-.2098	-.0946	-.0192	.0066	0
N_x	$\frac{DS}{a}\sin 4\frac{x}{a}$	-.0329	-.0314	-.0191	.0309	.1049	.2581

In Figures 5 and 6, these results have been plotted in full lines against values of cos ϕ Also, the exact results of Table IV., and the assumed distribution of N_ϕ according to the membrane theory are shown by dotted lines.

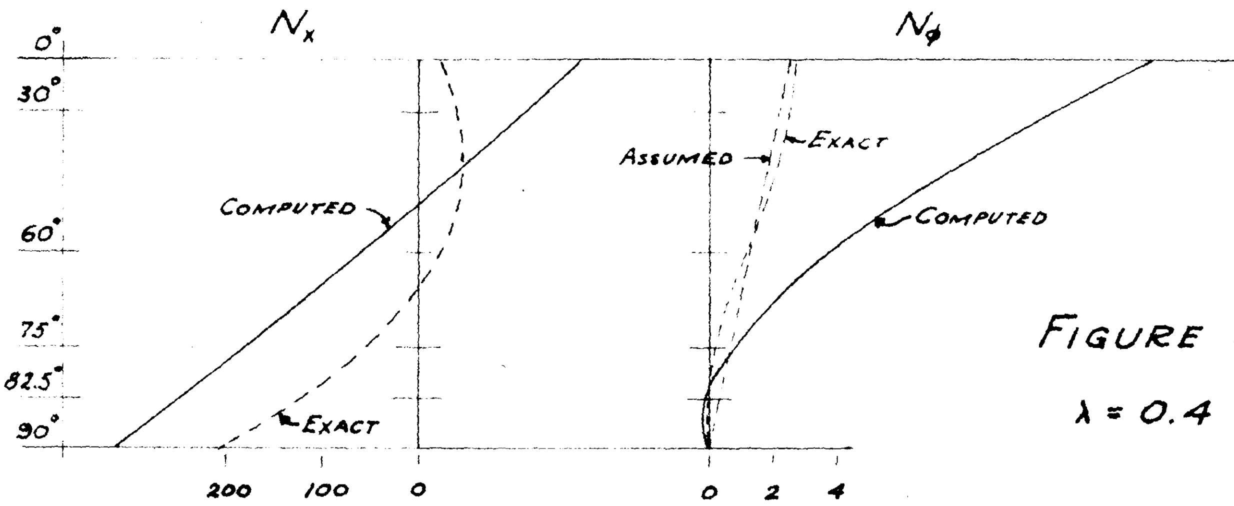


FIGURE 5

λ = 0.4

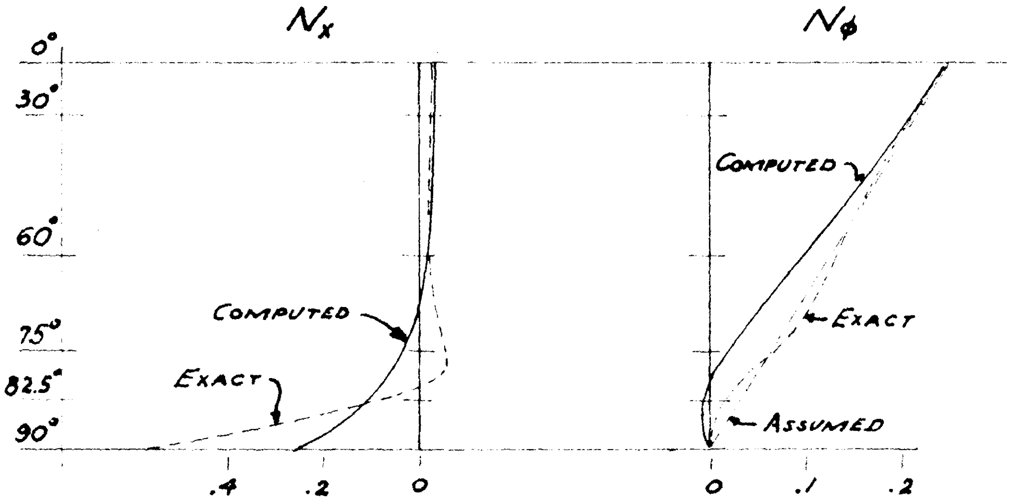


FIGURE 6

λ = 4.0

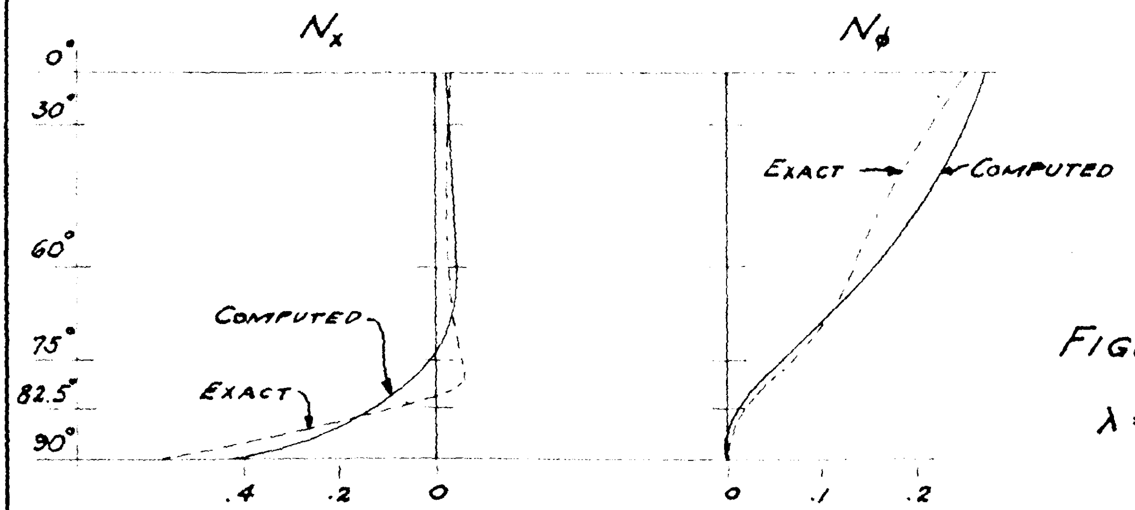


FIGURE 7

λ = 4.0

It is evident from these figures that the approximate curves cannot be considered satisfactory for design purposes. The extreme sensitiveness of the solution of this problem is illustrated by these curves. For $\lambda = 4.0$, the assumed and exact distributions of N_ϕ are very close over the range $0 < \phi < 75°$; while the deviation in the range $75° < \phi < 90°$ is much greater, especially if this deviation is considered in terms of the ratio of the assumed to the actual distribution. The resulting second approximation stresses agree fairly well with the exact over the somewhat smaller range $0 < \phi < 60°$, the deviations becoming serious between 60° and 90°. For $\lambda = 0.4$, the agreement between assumed and actual values of N_ϕ would seem, at first sight, to be satisfactory, at least for $\phi < 45°$. However, the error in the computed second approximation is seen to be very large for almost all values of ϕ. Furthermore, the family resemblances between the second approximation curves for the two values of λ are by no means as marked as they are in the case of the exact curves.

(e) This method of approximation, especially as it applies to relatively short spans (corresponding to large values of λ) seems to be the quite promising; and several attempts were made to devise methods of correcting the computed second approximation. However, one essential requirement of any method of correction must be that it be quite easy to apply; else, the approximate solution will become comparable in difficulty to the exact, defeating its own purpose. It was assumed originally in setting up the approximate solution, that the correction to the membrane stress N_ϕ could be taken as zero. Instead of this, we may assume various types of expressions for this correction. One possible method is to assume it to be given by a Fourier series in ϕ; however, there seems to be no easy way of evaluating the constants in such a series. From the shape of the curves of Figures 3d and 4d, it is evident that the larger terms of the series will not be the first terms. Assuming one or two terms of a Fourier series led to no better results. In the first place, there seems to be no definite way of foretelling which terms to assume, and, secondly, having assumed any such term, the problem of evaluating the necessary constant becomes quite troublesome.

Another method of correction which cannot be considered practical but which leads to some improvement in the results will be briefly described. The curves of Figures 3d and 4d are seen to be similar in form to damped waves. It was found that the first arch near 90° of the curve of Figure 4d could be closely approximated by a function of the type:

$$A\left\{e^{-7\theta}\sin 14(\pi - \theta) + e^{-7(\pi-\theta)}\sin 14\,\theta\right\}$$

where θ is the angle measured inward from one of the boundary generators, and is related to the coordinate ϕ by the equation $\theta = \frac{\pi}{2} + \phi$. A is an amplitude factor which must be determined by the subsequent solution. Using a correction of this type, second approximation curves were computed for the case $\lambda = 4.0$ and some of the results are shown in Table VII. This table is arranged exactly as Table IV, see page 103.

TABLE VII.

	Factor	0°	$\lambda = 4.0$ 30°	60°	75°	82.5°	90°
N_ϕ	$\frac{DS}{a} \sin 4\frac{x}{a}$	-.2703	-.2474	-.1547	-.0587	-.0113	0
N_x	$\frac{DS}{a} \sin 4\frac{x}{a}$	-.0257	-.0273	-.0449	.0042	.1259	.4255

These results, along with the exact results for comparison, are plotted in Figure 7 against values of cos ϕ. The agreement is seen to be much improved but for several reasons this method cannot be considered satisfactory. Firstly, this type of correction is rather difficult to apply, the computations being about double the amount involved in the solution given by Equations (205). Secondly, the parameter A is difficult to determine, the author being forced to use a trial and error method which depends on the previous knowledge of the exact solution. Thirdly, in the general case, the constants 7 and 14 appearing in the correction expression used would have to be replaced by variable parameters, their values being determined by the subsequent solution, or otherwise; and there seems to be no definite and easy way to determine these parameters without some foreknowledge of the exact solution. Due to the second and third points just mentioned, the accuracy of the solution carried out in this manner will always be doubtful unless checked by other means.

(f) A slight variation of the method of Section (d) leads to slightly better results. The methods are identical except for the way in which the loads are brought into the analysis. Instead of taking particular integrals according to Dischinger's theory, we proceed as follows. According to the membrane theory, the ring stress is given by the following formula:

$$N_\phi = -pa \cos \phi = -\frac{DS}{a} \cos \phi \sum_{n=1,3\cdots}^{\infty} \frac{1}{\lambda} \sin \lambda \frac{x}{a}$$

As a first approximation, we will assume N_ϕ to be given, not by this formula, but by the following slightly different one:

$$N_\phi = -\frac{DS}{a} \cos \phi \sum_{n=1,3\cdots}^{\infty} \frac{\sigma}{\lambda} \sin \lambda \frac{x}{a} \qquad (210)$$

where the values of the constants σ depend on the value of λ, and are to be determined later in the solution. Thus, this method is practically equivalent to assuming a correction to the membrane stress N_ϕ given by the first term of a Fourier series, as discussed in the first paragraph of Section (e). We now consider the complete Equations (64) and substitute Equation (210) in the third of these equations, thus obtaining for dead load:

$$N_x' + N_{\phi x}^{\bullet} = 0 \qquad (211)$$

$$N_{x\phi}' + N_\phi^{\bullet} = \frac{1}{a}(M_\phi^{\bullet} + M_{x\phi}') - pa \sin \phi = \frac{1}{a}(M_\phi^{\bullet} + M_{x\phi}') - \frac{DS}{a} \sin \phi \sum_{n=1,3\cdots}^{\infty} \frac{1}{\lambda} \sin \lambda \frac{x}{a}$$

$$M_\phi^{\bullet\bullet} + M_{x\phi}'^{\bullet} + M_{\phi x}'^{\bullet} + M_x'' = -aN_\phi - pa^2 \cos \phi = -DS \cos \phi \sum_{n=1,3\cdots}^{\infty} \frac{1-\sigma}{\lambda} \sin \lambda \frac{x}{a}$$

We now solve Equations (211) in a manner similar to that in which Equations (195) were solved. The complementary functions of Equations (211) will be identical in form with the complementary functions of Equations (195). We then need to solve Equations (211) only for particular integrals arising from the two series appearing in the second and third equations. Proceeding as before, by assuming solutions in the form of Equations (197), we obtain the following equations corresponding to Equations (198):

$$\frac{1}{2}U^{\cdot\cdot} - \lambda^2 U + \frac{\lambda}{2} V^{\cdot} = 0$$

$$-\frac{\lambda}{2}U^{\cdot} - \frac{\lambda^2}{2} V + V^{\cdot\cdot} = -W^{\cdot} + k(W^{\cdot\cdot\cdot} + W^{\cdot}[1 - \lambda^2]) - \frac{S}{\lambda}\sin\phi \qquad (212)$$

$$W^{\cdot\cdot\cdot\cdot} + W^{\cdot\cdot}(1 - 2\lambda^2) + \lambda^4 W = -\frac{\tau S}{k\lambda}\cos\phi$$

where we have introduced the notation:

$$\tau = 1 - \sigma \qquad (213)$$

We now solve the third equation, place this solution in the second equation, and then solve the first two equations simultaneously. In this way, we obtain the following results:

$$N_\phi = \frac{D}{a}\sum \sin\lambda\frac{x}{a}\left\{F\,E_1(\phi) + G\,E_2(\phi) + (H + 2J)\,E_3(\phi) + \lambda J\,E_4(\phi) + SC\cos\phi\right\}$$

$$N_{x\phi} = \frac{D}{2a}\sum \cos\lambda\frac{x}{a}\left\{\Gamma\,O_1(\phi) + \Lambda\,O_2(\phi) + 2(H + 3J)\,O_3(\phi) + 2\lambda J\,O_4(\phi) + \frac{2\lambda^3 T}{(1+\lambda^2)^2}\sin\phi\right\}$$

$$N_x = -\frac{D}{a}\sum \sin\lambda\frac{x}{a}\left\{\lambda M E_1(\phi) + \lambda N E_2(\phi) + (H + 4J)\,E_3(\phi) + \lambda J\,E_4(\phi) + \frac{\lambda^2 T}{(1+\lambda^2)^2}\cos\phi\right\}$$

$$u = \sum \cos\lambda\frac{x}{a}\left\{M\,E_1(\phi) + N\,E_2(\phi) + \frac{1}{\lambda}(H + 4J)\,E_3(\phi) + J\,E_4(\phi) + \frac{\lambda T}{(1+\lambda^2)^2}\cos\phi\right\} \qquad (214)$$

$$v = \sum \sin\lambda\frac{x}{a}\left\{P\,O_1(\phi) + Q\,O_2(\phi) + \frac{1}{\lambda}(H + J)\,O_3(\phi) + J\,O_4(\phi) + \frac{(1+2\lambda^2)T}{(1+\lambda^2)^2}\sin\phi\right\}$$

$$w = \sum \sin\lambda\frac{x}{a}\left\{A\,E_1(\phi) + B\,E_2(\phi) - \frac{\tau S}{k\lambda^3(\lambda^2+2)}\cos\phi\right\}$$

In these equations, we have introduced the following notation:

$$C = \frac{1 + 2\lambda^2}{\lambda(1+\lambda^2)^2} + \frac{\tau}{k\lambda^3(\lambda^2+2)}\left\{\frac{(1+k\lambda^2)(1+2\lambda^2)}{(1+\lambda^2)^2} - 1\right\} \qquad (215)$$

$$T = \frac{S}{\lambda}\left\{1 + \frac{\tau(1+k\lambda^2)}{k\lambda^2(\lambda^2+2)}\right\}$$

τ has been defined in Equation (213) above, and the remaining notation of Equations (214) is defined exactly as in Equations (200), (201), (203), (204), and (206). The boundary condition equation are slightly different from Equations (208) and (209). The equations for the determination of A and B are as follows:

If $\lambda < \frac{1}{2}$:

$$A\,(1-\alpha^2)\,E_1(\tfrac{\pi}{2}) + B\,(1-\beta^2)\,E_2(\tfrac{\pi}{2}) = 0 \qquad (216a)$$

$$A\,\beta O_1(\tfrac{\pi}{2}) + B\,\alpha\,O_2(\tfrac{\pi}{2}) = -\frac{2\tau S}{k\lambda^3(\lambda^2+2)}$$

If $\lambda > \frac{1}{2}$:

$$A\left[(\lambda^2+\tfrac{1}{2})E_1(\tfrac{\pi}{2}) - \alpha E_2(\tfrac{\pi}{2})\right] + B\left[\alpha E_1(\tfrac{\pi}{2}) + (\lambda^2+\tfrac{1}{2})E_2(\tfrac{\pi}{2})\right] = 0 \qquad (216b)$$

$$A\left[\alpha\,O_1(\tfrac{\pi}{2}) + \tfrac{1}{2}O_2(\tfrac{\pi}{2})\right] - B\left[\tfrac{1}{2}O_1(\tfrac{\pi}{2}) - \alpha O_2(\tfrac{\pi}{2})\right] = -\frac{2\tau S}{k\lambda^3(\lambda^2+2)}$$

The equations for the determination of H and J are as follows:

$$H\,E_3(\tfrac{\pi}{2}) + J\left[2\,E_3(\tfrac{\pi}{2}) + \lambda\,E_4(\tfrac{\pi}{2})\right] = -F\,E_1(\tfrac{\pi}{2}) - G\,E_2(\tfrac{\pi}{2}) \qquad (217)$$

$$2H\,O_3(\tfrac{\pi}{2}) + 2J\left[3\,O_3(\tfrac{\pi}{2}) + \lambda\,O_4(\tfrac{\pi}{2})\right] = -\Gamma\,O_1(\tfrac{\pi}{2}) - \Lambda\,O_2(\tfrac{\pi}{2}) - \frac{2\lambda^3 T}{(1+\lambda^2)^2}$$

By means of these equations, all quantities will be determined in terms of the constant τ. We may now evaluate τ by making the expression for N_ϕ of Equation (214) agree at some particular value of ϕ with the assumed value of N_ϕ as given by Equation (210). The best particular value of ϕ to use is a matter of judgment, or, knowing the exact solution, a trial and error method may be used to make the calculated and exact solutions as nearly alike as possible. If the assumed and calculated values of N_ϕ are made to agree for the particular value $\phi = 0$, we obtain the following equation from which τ may be determined:

$$F\,E_1(0) + G\,E_2(0) + (H + 2J)\,E_3(0) + \lambda J\,E_4(0) + SC = -\frac{1-\tau}{\lambda}\,S$$

With the value of τ obtained from this equation, the values of the constants may be completely determined.

The author has computed the first term of this solution for the values of λ and k previously used, and some of the results are tabulated in Table VIII., which is arranged exactly as Table IV., see page 103.

TABLE VIII.

	Factor	0°	30°	60°	75°	82.5°	90°
			$\lambda = 0.4$				
N_ϕ	$\frac{DS}{a}\sin .4\frac{x}{a}$	-2.500	-1.767	-.2643	.1936	.2029	0
N_x	$\frac{DS}{a}\sin .4\frac{x}{a}$	-51.73	-33.59	18.34	54.94	75.40	97.08
			$\lambda = 4.0$				
N_ϕ	$\frac{DS}{a}\sin 4\frac{x}{a}$	-.2491	-.2141	-.1047	-.0267	.0029	0
N_x	$\frac{DS}{a}\sin 4\frac{x}{a}$	-.0315	-.0303	-.0230	.0267	.1089	.2878

In Figures 8 and 9, these results are shown by full lines, plotted against values of cos ϕ. Also, the exact results, the assumed distribution of N_ϕ , and N_x calculated from elementary beam theory, Equation (180), for $\lambda = 0.4$, are shown by dotted lines for comparison.

It is evident from these figures that the approximate curves cannot be considered as satisfactory. However, it may be noted that the ratio of the values of N_x at the free edges given by the exact solution, Table IV., to that given by the above approximate solution is very close to 2 for both values of λ used. The calculated curves err on the side of safety from the exact curves at most critical points except along the free edges. It may be possible, therefore, to use this method of calculation, later revising the results by multiplying the stress N_x at the free edges by 2, (or some constant) and then sketching by eye the distribution of N_x in the region of the free edges. However, before such a semi-empirical method could be used with safety, it would be necessary to carry through the exact solution for other values of λ and k.

It will be noted that the solutions of Sections (d) and (f) are in close agreement for $\lambda = 4.0$, while the disagreement is considerable for $\lambda = 0.4$. For both values of λ , the two methods make use of almost identical assumptions for the distribution of N_ϕ; yet the results computed by the two methods differ considerably for the smaller value of λ . This is probably due to the following reasons. In the solution of this Section, givenby Equations (214), the particular integrals satisfy the first set of boundary conditions at the

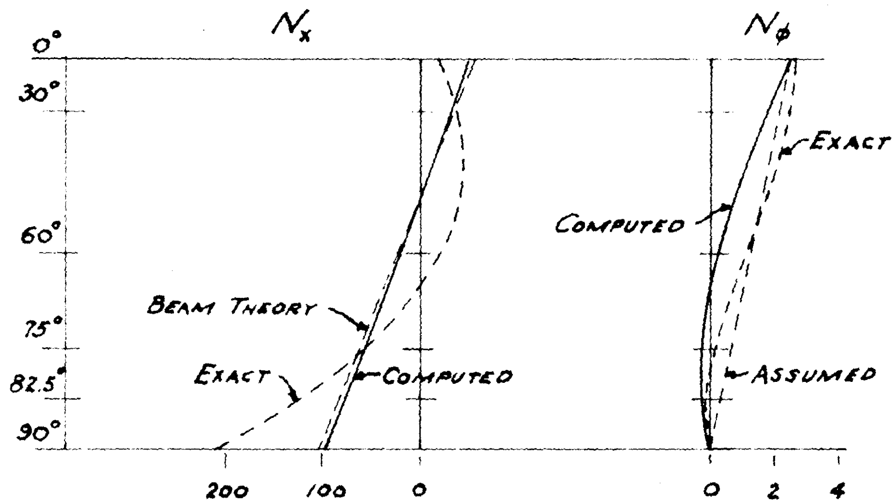


FIGURE 8

$\lambda = 0.4$

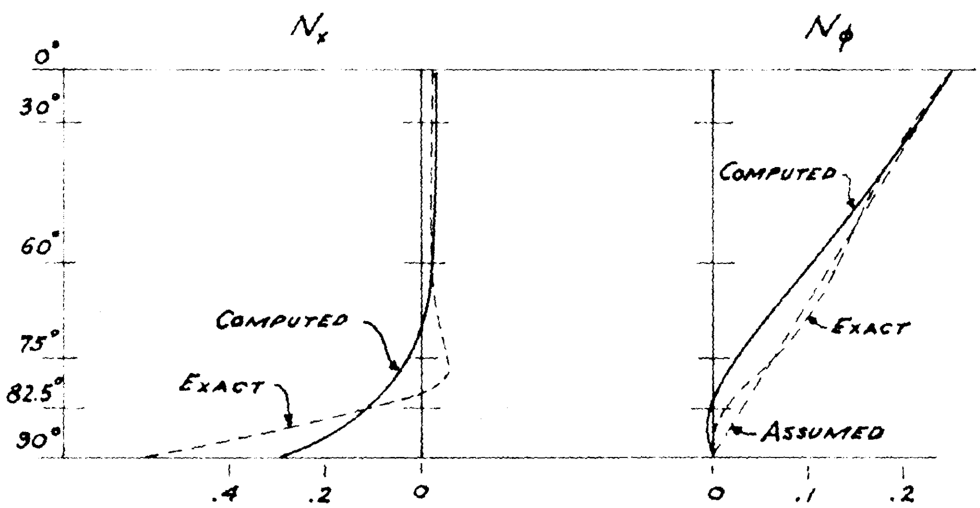


FIGURE 9

$\lambda = 4.0$

supporting edges which were discussed on page 70. Dischinger's particular integrals satisfy somewhat different conditions at the supporting edges, as an inspection of Equations (193) shows. From these expressions, it is seen that for $x = 0, \ell$ we have $N_x = v = 0$, but w does not vanish; while the particular integrals included in Equations (214) satisfy the conditions $N_x = v = w = 0$ at $x = 0, \ell$. It is probable that for short deep spans (corresponding to large values of λ) the effect of radial deformations at the supports is small, while this effect is much more important for the longer spans (corresponding to small values of λ). It should be noted that the particular integrals contained in Equations (214) are exact, while those in Equations (205) are approximate. It may be that the approximation introduced in this way is much better for the larger values of λ. However, it is believed that the discrepancy between the solutions for $\lambda = 0.4$ obtained by the two approximate methods is due principally to the incorrectness of the initial assumptions for the distribution of N_ϕ rather than to the approximations introduced by the use of Dischinger's particular integrals.

It is interesting to note that both approximate methods lead to an almost linear distribution of N_x for the value of $\lambda = 0.4$. See Figures 5 and 8. Also, the distribution of N_x obtained by the method of this Section is in very close agreement with the results of the elementary beam theory, Equation (180).

(g) The only type of problem investigated in this work has been that of a shell in the form of a complete half cylinder, loaded with dead load, and completely free along the boundary generators. Another problem of a similar nature which is of considerable importance in roof design, is that of two or more shells supported at the ends as in the examples considered herein, and joined together rigidly along the boundary generators. In the paper by Finsterwalder previously cited, (Proceedings International Association for Bridge and Structural Engineering, Vol. I, 1932, page 127) an example of this type has been computed quite completely using Finsterwalder's approximate method. An examination of Finsterwalder's results shows that the computed ring stress N_ϕ is very nearly linear if considered as a function of cos ϕ. It is possible, therefore, that a solution by the methods of Sections (d) and (f) would give more satisfactory results when applied to a shell of the type considered by Finsterwalder; due to lack of time, it was impossible for the author to pursue this possibility further.

The investigations described in Part VI. of this thesis have not led to the discovery of any entirely satisfactory approximate method, and one is led to conclude that the best method, so far available, for the calculation of cylindrical shells of the type considered is the exact method. It is hoped that a perusal of these pages may help future investigators in this field to avoid a large amount of fruitless computation through a knowledge of the devices described in this thesis, or that it may suggest to the reader some possible method of attack on the problem which has been overlooked by the author. If either possibility occurs, the main purpose of this somewhat detailed report will have been fulfilled.