A MODIFICATION OF THE MOMENT DISTRIBUTION METHOD TO TAKE INTO ACCOUNT THE DEPTH OF MEMBERS

Thesis by

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In Partial Fulfillment of the Requirements for the Degree of Master of Science

California Institute of Technology

Pasadena, California

1934

ASSUMPTIONS

The derivation of the formulas is based on two main assumptions: (1) The moment of inertia of any beam is uniform center to center of the columns into which it frames, and the moment of inertia of any column is uniform center to center of the beams into which it frames. (2) The distribution of moment in any beam or column is as shown in figure 1. For the beam shown in this figure, the moment is zero at the center of adjacent columns, increasing linearly to a max-

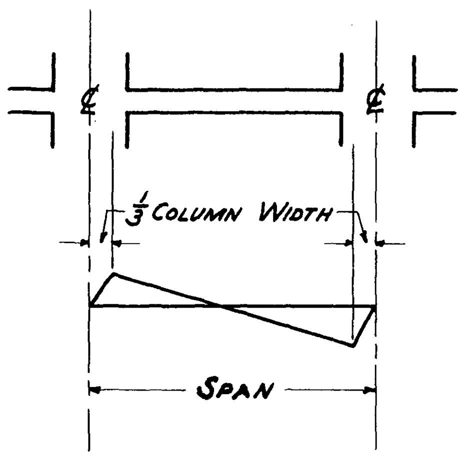

Figure 1

imum at a point one-third of the column width from the center of the column.

Assumption (2) was adopted after considering four cases, shown in figures 2, 3, 4, and 5.

In each figure, (a) represents a hub, composed of beams and columns, which is considered as a free body. The beams and columns have been cut at inflection points, and shear forces added; when necessary for equilibrium, direct forces in the beams and columns have also been added. In each figure, (b) represents the beams as a free body, the columns being replaced by reactions at the neutral axis of the beams sufficient to produce equilibrium. These reactions are placed one-third of the column width from the column center, for the following reasons. It is probable that the column reaction is zero at some point in the column, increasing in tension to one side of this point, increasing in compression to the other side. If the point of zero reaction is at the center line of the column, and if the reactions are assumed to increase uniformly to a maximum at the column face, the resultant of the distributed reaction on either side of the center will be one-third of the column width from the center. By making the latter assumptions, the analysis was greatly simplified,

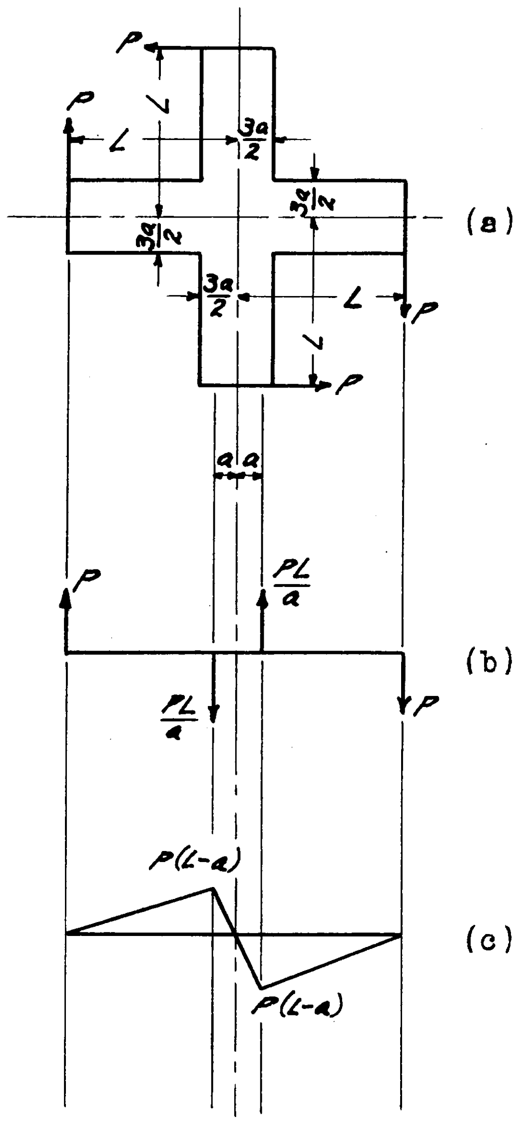

Figure 2

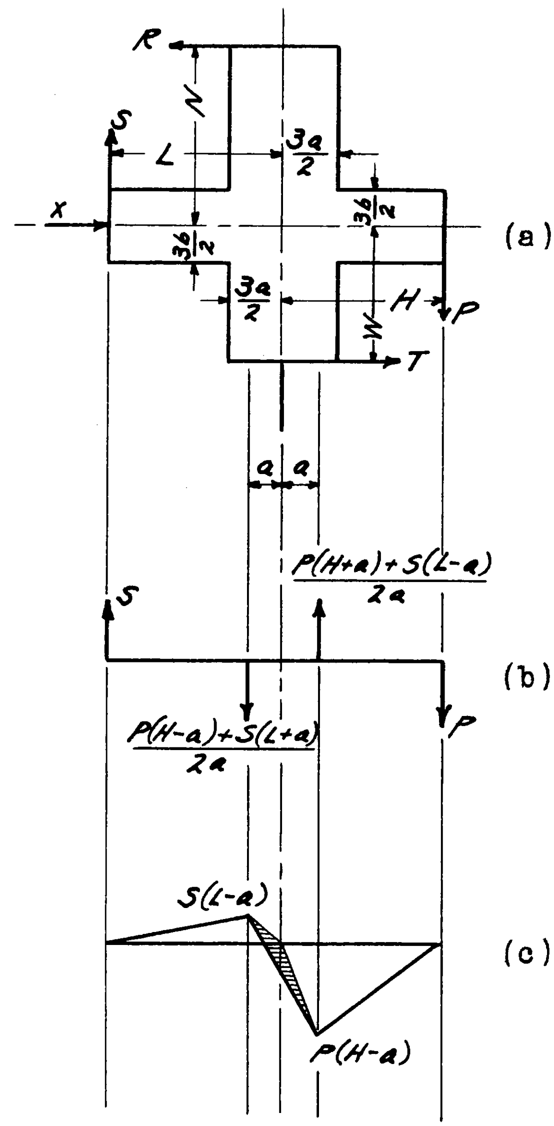

Figure 3

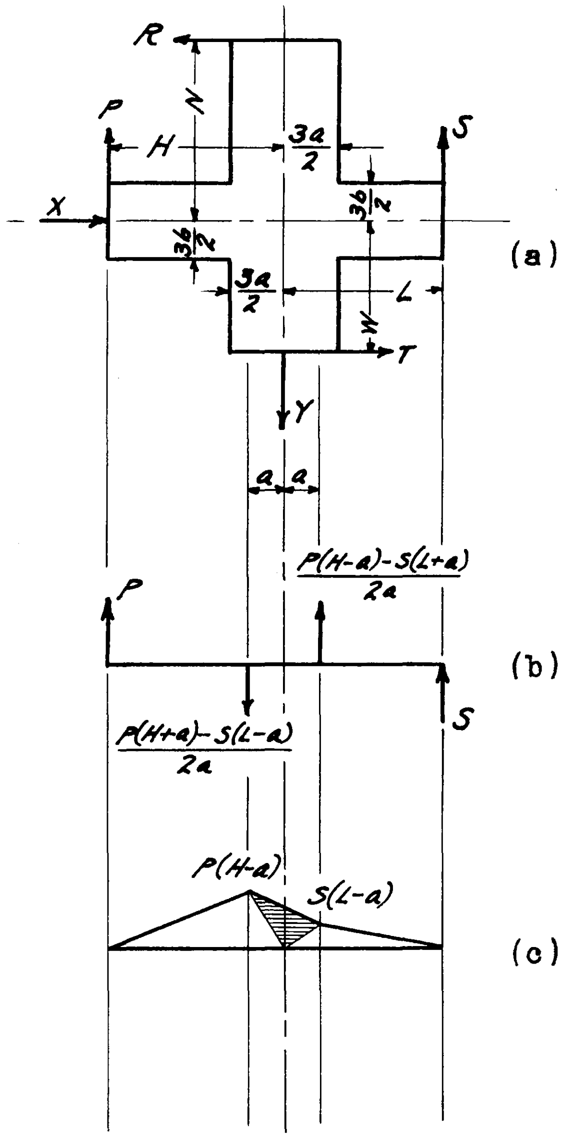

Figure 4

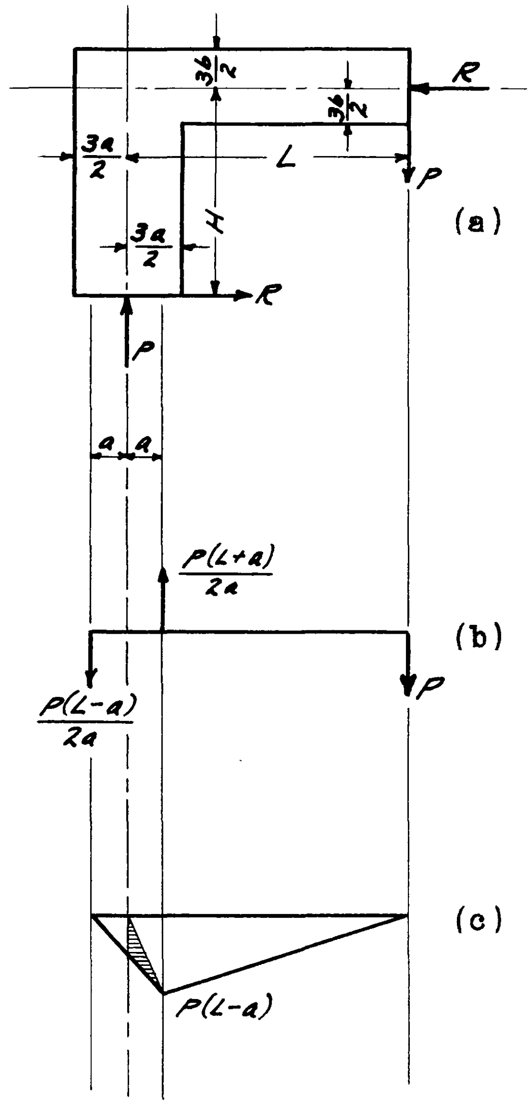

Figure 5

probably without greatly affecting its accuracy. In each figure, (c) represents the resulting moment diagram for the beams. The shaded portion of the diagrams shows the area which is neglected in the derivation of the formulas. It is seen that in the case of figure 2 the moment diagram is identical with the type assumed, while in the other cases, there is more or less deviation from the assumed type. Another possibility is when no inflection point occurs in the beam; in this case, the moment diagram near the hub would be similar to that shown in figure 4c.

This analysis does not consider deformations due to shear. It is believed that in all cases where the relative depth and span of the various beams and columns are such that shear can be neglected, the errors resulting from the basic assumptions will be negligible.

CASE I Columns of equal width at ends of span

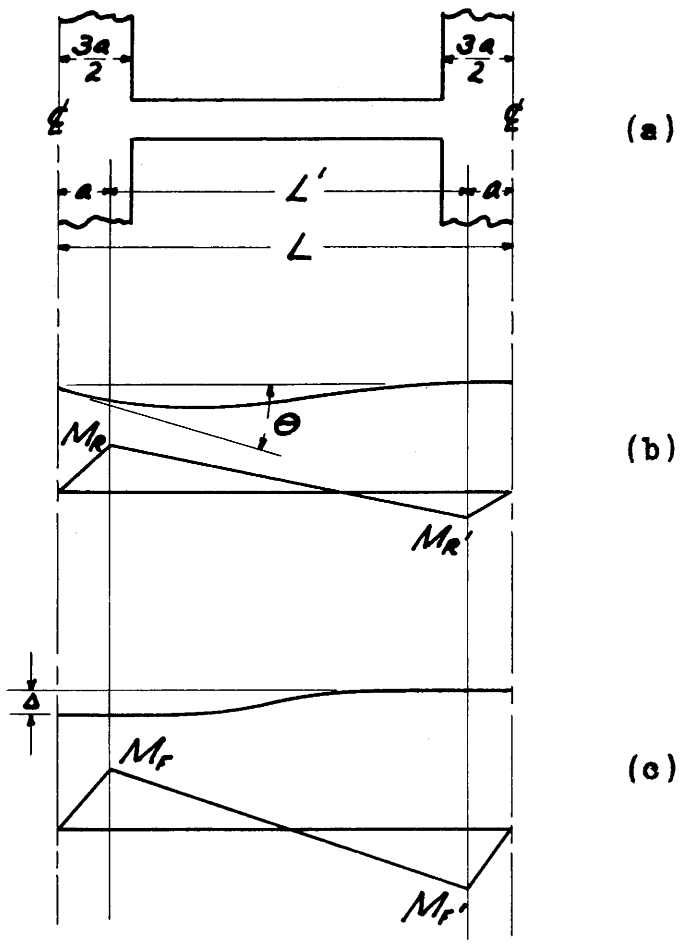

Figure 6

Figure 6a represents a beam of span L center to center of columns, the columns being of width 3a. Figure 6b represents the elastic curve and moment diagram for this beam when one end is fixed and the other rotates through an angle θ without deflecting vertically. The subscript R will be used to denote this case. Figure 6c represents the elastic curve and moment diagram for this beam when the ends deflect a distance Δ with respect to each other, but neither end rotates. The subscript F will be used to denote this case. M and H denote moment and shear, respectively, for the beam of figure 6; m and h denote like quantities for a beam of the same moment of inertia and the same span center to center of columns, but in which the depth of beams and columns is neglected (assumed to be zero). When considering the latter beam, the subscripts R and F will be used to denote the conditions described above for the beam of figure 6.

When $\theta \neq 0$, $\Delta = 0$ (figure 6b)

$$EI\theta = (M_R - M_{R'})\frac{a}{2} + (M_R - M_{R'})\frac{L'}{2}$$

$$= (M_R - M_{R'})\frac{L-a}{2} \qquad (1)$$

$$EI\Delta = 0 = M_R\left(\frac{a}{2}\right)\left(\frac{2a}{3}\right) + M_R\left(\frac{L'}{2}\right)\left(\frac{L'}{3} + a\right)$$

$$- M_{R'}\left(\frac{a}{2}\right)\left(\frac{4a}{3} + L'\right) - M_{R'}\left(\frac{L'}{2}\right)\left(\frac{2L'}{3} + a\right)$$

$$EI\Delta = 0 = \left(\frac{M_R}{6} - \frac{M_R'}{3}\right)(L)(L-a) \tag{2}$$

whence $M_R' = \frac{M_R}{2}$ (3)

From equations (1) and (3)

$$EI\theta = M_R\left(\frac{L-a}{4}\right) \tag{4}$$

For the same beam, neglecting depth of beam and columns

$$EI\theta = m_R\left(\frac{L}{4}\right) \tag{5}$$

Equating θ's for the two cases

$$M_R = m_R\left(\frac{L}{L-a}\right) \tag{6}$$

Also, from equations (3) and (6), noting that the carry-over factor in which depth is neglected is one-half

$$M_R' = m_R'\left(\frac{L}{L-a}\right) \tag{7}$$

Stiffness is defined as the moment required to produce unit rotation of the end of the member. Therefore, equating θ to unity in equations (4) and (5)

Stiffness $M_R = 4E\left(\frac{I}{L-a}\right)$ (8)

Stiffness $m_R = 4E\left(\frac{I}{L}\right)$ (9)

From equations (6), (8), and (9) it is seen that moments distributed in proportion to $\frac{I}{L}$ in the system in which the depth of the members is neglected is equivalent to distributing moments in the given system in proportion to $\frac{I}{L-a}$. The angle of rotation of both joints and of all members will be the same.

For the beam of figure 6b, shear is given by the expression

$$H_R = \frac{M_R + M_R'}{L'} = \frac{3M_R}{2L'} \tag{10}$$

For the same beam, neglecting depth of beam and columns

$$h_R = \frac{3m_R}{2L} \tag{11}$$

From equations (6), (10), and (11)

$$h_R = H_R \frac{L'(L-a)}{L^2} \tag{12}$$

When $\theta = 0, \Delta \neq 0$ (figure 6c) The equations for θ and Δ will be of the same form as equations (1) and (2), F being the subscript instead of R. From equation (1)

$$EI\theta = 0 = (M_F - M_{F'})\frac{L-a}{2} \tag{13}$$

whence $M_F = M_{F'}$ (14)

From equations (2) and (14)

$$EI\Delta = -\frac{M_F}{6}(L)(L-a) \tag{15}$$

For the same beam, neglecting depth of beam and columns

$$EI\Delta = -\frac{m_F}{6}(L^2) \tag{16}$$

Equating Δ's for the two cases

$$M_F = m_F\left(\frac{L}{L-a}\right) \tag{17}$$

When both ends of the beam rotate through an angle θ such that the tangents at the ends remain parallel, and when neither end is displaced vertically, the angle θ is given by the expression

$$\theta = \frac{\Delta}{L} = \frac{M_F}{6EI}(L-a) \tag{18}$$

whence $$M_F = 6E\theta\left(\frac{I}{L-a}\right) \tag{19}$$

For the same conditions, in the case where depth of beam and columns is neglected

$$m_F = 6E\theta\left(\frac{I}{L}\right) \tag{20}$$

From equations (17), (19), and (20) it is seen that moments in proportion to $\frac{I}{L}$ in the ends of the beam in which the depth is neglected are equivalent to moments in proportion to $\frac{I}{L-a}$ in the ends of the given beam. The angle of rotation of both ends of both beams will be the same.

Shears in the two equivalent beams being considered are given by the expressions

$$H_F = \frac{2M_F}{L'} \tag{21}$$

and $$h_F = \frac{2m_F}{L} \tag{22}$$

From equations (17), (21), and (22)

$$h_F = H_F \frac{L'(L-a)}{L^2} \tag{23}$$

The relative horizontal displacement of the two ends of all columns of any story must be equal. From equations (15), and (21)

$$H_F = 12E\Delta \frac{I}{LL'(L-a)} \tag{24}$$

When the value of H_F from (23) is substituted in (24), we get

$$h_F = 12E\Delta \frac{I}{L^3} \tag{25}$$

Therefore the total shear in any story is distributed among the columns of the story in proportion to $\frac{I}{LL'(L-a)}$. In each column, this value of shear, H_F, is changed to the equivalent value of shear, h_F, by multiplying by the factor $\frac{L'(L-a)}{L^2}$. This is equivalent to distributing shears among the columns in which the depth is neglected in proportion to $\frac{I}{L^3}$.

The values of fixed-end moment, distributed moment, and carry-over moment for members in which the depth is neglected have been shown to be related by the factor $\frac{L}{L-a}$ to the corresponding moments in the same members in which the depth is taken into account. Therefore, the true maximum moment in the end of a member can be determined at any time from the equation

$$M = (\sum m) \frac{L}{L-a} \tag{26}$$

where $\sum m$ is the sum of all moments, fixed-end, distributed, and carry-over, which have been placed at the end under consideration. Likewise the true value of shear in any member can be computed at any time from the equation

$$H = h \frac{L^2}{L'(L-a)} \tag{27}$$

where h is the shear in the member in which depth is neglected due to all moments, m, which have been added to both ends of the member.

CASE II Columns of unequal width at ends of span

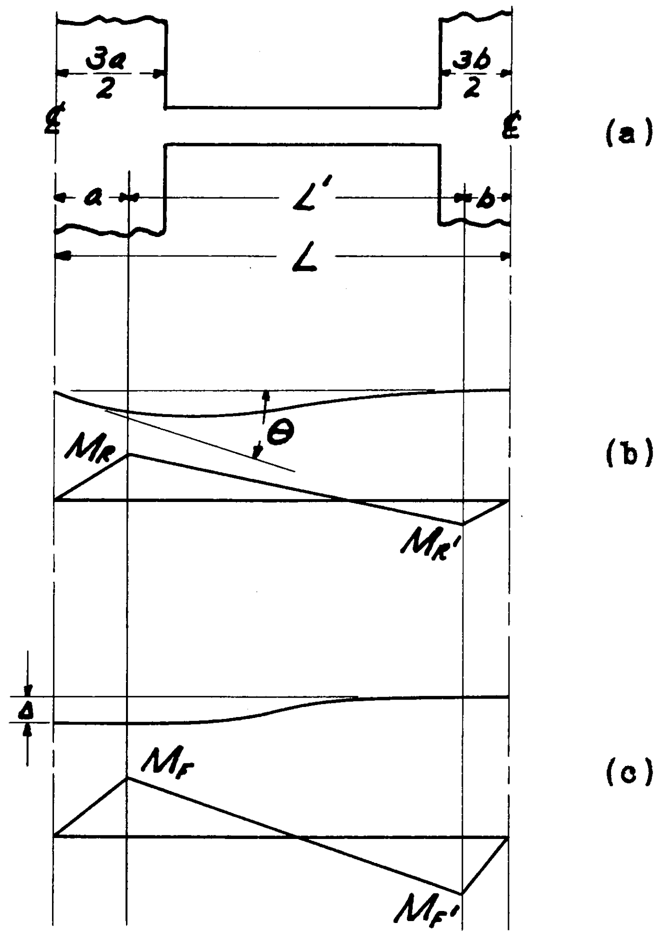

Figure 7

Figure 7 represents this case, the column widths being 3a and 3b. The description of this figure is similar to that of figure 6. It should be noted that moments designated by prime apply to the end at the column of width 3b. The same notation will be used in the derivation of formulas for this case.

When $\theta \neq 0, \Delta = 0$ (figure 7b)

$$EI\theta = M_R\left(\frac{a+L'}{2}\right) - M_{R'}\left(\frac{b+L'}{2}\right)$$

$$= M_R\left(\frac{L-b}{2}\right) - M_{R'}\left(\frac{L-a}{2}\right) \tag{28}$$

$$EI\Delta = 0 = M_R\left(\frac{a}{2}\right)\left(\frac{2a}{3}\right) + M_R\left(\frac{L'}{2}\right)\left(\frac{L'}{3}+a\right) - M_{R'}\left(\frac{L'}{2}\right)\left(\frac{2L'}{3}+a\right) - M_{R'}\left(\frac{b}{2}\right)\left(\frac{b}{3}+L'+a\right)$$

$$= \frac{M_R}{6}(L-b)(L+a-b) - \frac{M_{R'}}{6}(L-a)(2L+a-b) \tag{29}$$

whence $$M_{R'} = M_R \frac{(L-b)(L+a-b)}{(L-a)(2L+a-b)} \tag{30}$$

From equations (28) and (30)

$$EI\theta = \frac{M_R}{2}\left[\frac{L(L-b)}{2L+a-b}\right] \tag{31}$$

For the same beam, neglecting depth of beam and columns

$$EI\theta = m_R\left(\frac{L}{4}\right) \tag{5}$$

Equating θ's for the two cases

$$M_R = \frac{m_R}{2}\left(\frac{2L+a-b}{L-b}\right) \tag{32}$$

From equations (30) and (32), noting that the carry-over factor is one-half when the depth of beam and columns is neglected

$$M_{R'} = m_{R'}\left(\frac{L+a-b}{L-a}\right) \tag{33}$$

Equating θ to unity in equation (31)

Stiffness $M_R = 4E\left[\frac{I(2L+a-b)}{2L(L-b)}\right]$ (34)

From equations (9), (32), and (34) it is seen that moments distributed in proportion to $\frac{I}{L}$ in the system in which the depth of the members is neglected are equivalent to moments distributed in the given system in proportion to $\frac{I(2L+a-b)}{2L(L-b)}$. The angle of rotation of all members of both joints will be the same.

For the beam of figure 7b, shear is given by the expression

$$H_R = \frac{M_R + M_R'}{L'} \tag{35}$$

From equations (30), (32), and (35)

$$H_R = \frac{m_R}{2L'}\left(\frac{2L+a-b}{L-b} + \frac{L+a-b}{L-a}\right) \tag{36}$$

$$H_R = \frac{m_R'}{L'}\left(\frac{2L+a-b}{L-b} + \frac{L+a-b}{L-a}\right) \tag{37}$$

When $\theta = 0, \Delta \neq 0$ (figure 7c) The equations for θ and Δ will be of the same form as equations (28) and (29), F being the subscript instead of R. From equation (28)

$$EI\theta = 0 = M_F\left(\frac{L-b}{2}\right) - M_F'\left(\frac{L-a}{2}\right) \tag{38}$$

whence $M_F' = M_F\left(\frac{L-b}{L-a}\right)$ (39)

From equations (29) and (39)

$$EI\Delta = -\frac{M_F}{6}(L)(L-b) \tag{40}$$

For the same beam, neglecting depth of beam and columns

$$EI\Delta = -\frac{m_F}{6}(L^2) \tag{16}$$

Equating Δ's for the two cases

$$M_F = m_F\left(\frac{L}{L-b}\right) \tag{41}$$

From (39) and (41)

$$M_F' = m_F\left(\frac{L}{L-a}\right) \tag{42}$$

When both ends of the beam rotate through an angle θ such that the tangents at the ends of the beam remain parallel, and when neither end is displaced vertically, the angle θ is given by the expression

$$\theta = \frac{\Delta}{L} = \frac{M_F}{6EI}(L-b) \tag{43}$$

whence $$M_F = 6E\theta\left(\frac{I}{L-b}\right) \tag{44}$$

For the same conditions, in the case where depth of beam and columns is neglected

$$m_F = 6E\theta\left(\frac{I}{L}\right) \tag{20}$$

From equations (20), (42), and (44) it is seen that moments in proportion to $\frac{I}{L}$ in the ends of the beam in which the depth is neglected are equivalent to moments in proportion to $\frac{I}{L-b}$ in the ends of the given beam. The angle of rotation of both ends of both beams will be the same.

Shear in the two equivalent cases being considered is given by the expressions

$$H_F = \frac{M_F + M_F'}{L'} \tag{45}$$

$$h_F = \frac{2m_F}{L} \tag{22}$$

From (22), (41), (42), and (45)

$$h_F = H_F\left[\frac{2L'(L-a)(L-b)}{L^2(2L-a-b)}\right] \tag{46}$$

$$H_F = m_F\left[\frac{L(2L-a-b)}{L'(L-a)(L-b)}\right] \tag{47}$$

The relative horizontal displacement of the ends of all columns of any story must be equal. From equations (39), (40), and (45)

$$H_F = 12E\Delta\left[\frac{I(2L-a-b)}{2LL'(L-a)(L-b)}\right] \tag{48}$$

When the value of H_F from (46) is substituted in (48), we get

$$h_F = 12E\Delta\left(\frac{I}{L^3}\right) \tag{25}$$

Therefore, the total shear in any story is distributed among the columns of the story in proportion to $\frac{I(2L-a-b)}{2LL'(L-a)(L-b)}$. In each column, this value of shear, H_F, is changed to the equivalent value of shear, h_F, by multiplying by the factor $\frac{2L'(L-a)(L-b)}{L^2(2L-a-b)}$. This is equivalent to distributing shears among the columns in which the depth is neglected in proportion to $\frac{I}{L^3}$.

At the end of any member, the true total moment is the sum of three classes of moment: the fixed-end moments, M_F, added to the end in question; moments, M_R, distributed to the end in question from the joint of which it is a part; and moments carried over to the end in question from the other end of the member. Denote the latter moments by Ω_R' for the case where the depth is taken into account, by ω_R' for the case where the depth is neglected. Expressing the above as an equation

$$M = \sum M_F + \sum M_R + \sum \Omega_{R'} \qquad (49)$$

From (32), (33), (41), and (49)

$$M = (\Sigma m_F)\left(\frac{L}{L-b}\right) + (\Sigma m_R)\left[\frac{2L+a-b}{2(L-b)}\right] + (\Sigma \omega_{R'})\left(\frac{L+b-a}{L-b}\right) \qquad (50)$$

It should be noted in equation (50) that a corresponds to the end of the member at which the moment is being determined; b corresponds to the other end of the member.

Shear in any member consists of two parts: that due to fixed-end moments, and that due to distributed and carry-over moments. Using equations (36), (37), and (47), expressing shear as a function of moments at one end, only, of the member, we get

$$H = (\Sigma m_F)\left[\frac{L(2L-a-b)}{L'(L-a)(L-b)}\right] + (\Sigma m_R)\left[\frac{1}{2L'}\left(\frac{2L+a-b}{L-b} + \frac{L+a-b}{L-a}\right)\right]$$

$$+ (\Sigma \omega_R{}')\left[\frac{1}{L'}\left(\frac{2L+b-a}{L-a} + \frac{L+b-a}{L-b}\right)\right] \qquad (51)$$

It should be noted in equation (51) that a corresponds to the end of the member from which moment summations are taken; b corresponds to the other end of the member.

CASE III Pin-end conditions

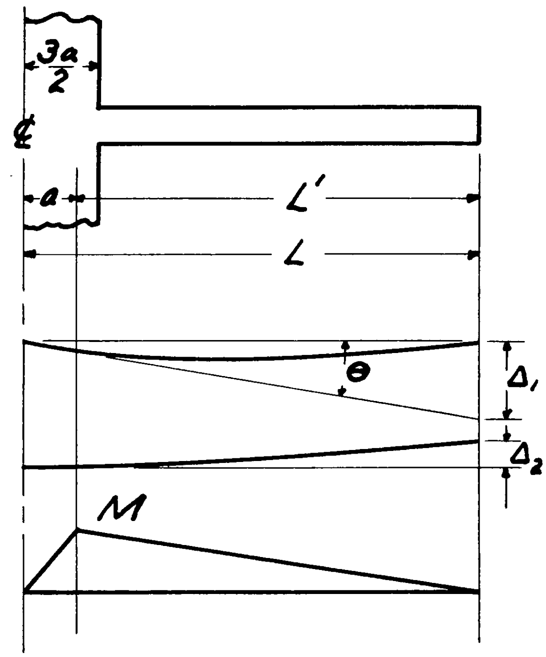

Figure 8

Figure 8 represents this case. As shown, the moment diagrams for the two types of deflection are the same. The same notation will be used in the derivation of formulas for this case.

When vertical deflection is zero

$$EI\Delta_1 = M_R\left(\frac{a}{2}\right)\left(\frac{a}{3}+L'\right) + M_R\left(\frac{L'}{2}\right)\left(\frac{2L'}{3}\right)$$

$$= \frac{M_R}{6}(L)(2L-a) \qquad (52)$$

$$EI\theta = EI\frac{\Delta_1}{L} = \frac{M_R}{6}(2L - a) \tag{53}$$

For the same beam, neglecting depth of beam and columns

$$EI\theta = \frac{m_R}{3}(L) \tag{54}$$

Equating θ's for the two cases

$$M_R = m_R\left(\frac{2L}{2L - a}\right) \tag{55}$$

Equating θ's to unity

$$\text{Stiffness} \quad M_R = 4E\left[\frac{3I}{2(2L-a)}\right] \tag{56}$$

$$\text{Stiffness} \quad m_R = 4E\left(\frac{3I}{4L}\right) \tag{57}$$

From equations (55), (56), and (57) it is seen that moments distributed in proportion to $\frac{3I}{4L}$ in the system in which the depth of the members is neglected are equivalent to moments distributed in the given system in proportion to $\frac{3I}{2(2L-a)}$. A comparison of equations (8), (9), (34), (56), and (57) shows that moments distributed to the various members at any joint in proportion to the ratios that have been established will produce equal rotation of the ends of all members of the joint, both in the system in which the depth of the members is taken into account, and in the equivalent system in which the depth of the members is neglected.

When the "a" end of the beam deflects vertically such that the tangent at that end remains parallel to its original position, we get, since the moment diagrams in this and the previous case are of the same shape

$$M_F = m_F \left(\frac{2L}{2L-a}\right) \tag{58}$$

Comparing (55) and (58) it is seen that at any time in the solution of a problem the true total moment will be given by the expression

$$M = (\Sigma m)\frac{2L}{2L-a} \tag{59}$$

Shear in the two equivalent cases being considered is given by the expressions

$$H = \frac{M}{L'} \tag{60}$$

$$h = \frac{m}{L} \tag{61}$$

From (59), (60), and (61)

$$H = \frac{m}{L'}\left(\frac{2L}{2L-a}\right) \tag{62}$$

$$h = H\left[\frac{L'(2L-a)}{2L^2}\right] \tag{63}$$

The relative horizontal displacement of the ends of all columns of any story must be equal. From equations (52) and (60)

$$H = 12E\Delta\left[\frac{I}{2LL'(2L-a)}\right] \tag{64}$$

When the value of H from (63) is substituted in (64), we get

$$h = 12E\Delta\left(\frac{I}{4L^3}\right) \tag{65}$$

Therefore, the total shear in any story is distributed among the columns of the story in proportion to $\frac{I}{2LL'(2L-a)}$. In each column, this value of shear, H, is changed to the equivalent value of shear, h, by multiplying by the factor $\frac{L'(2L-a)}{2L^2}$. A comparison of equations (24), (25), (48), (64), and (65) shows that shears distributed to the various types of members considered in proportion to the ratios established by these equations will produce equal relative deflections of the ends of the members.

When it is desired to rotate the end of the beam adjacent to the column through an angle θ, the relative vertical displacement of the two ends of the beam being zero, the moment required is, from equations (53) and (54)

$$M = 6E\theta\left(\frac{I}{2L-a}\right) \tag{66}$$

$$m = 6E\theta\left(\frac{I}{2L}\right) \tag{67}$$

From equations (55), (66), and (67) it is seen that moments in proportion to $\frac{I}{2L}$ in the end of the beam in which the depth is neglected are equivalent to moments in proportion to $\frac{I}{2L-a}$ in the end of the given beam. The angle of rotation of the end of both beams will be the same. A comparison of equations (19), (20), (44), (66), and (67) shows that moments in the ends of the various types of members considered in proportion to the ratios established by these equations will produce equal rotation of the ends of all members.

METHOD OF SOLUTION OF SIDESWAY PROBLEMS

The method outlined for solving problems involving sidesway in which it is necessary to consider the depths of the members in the computations is very similar to the Hardy Cross method of solving sidesway problems neglecting the depths of the members. The total shear in any story is first distributed to the various columns of the story in proportion to the constants previously derived. This shear in each column is changed to the shear in an equivalent column in which the depth is neglected by multiplying by a factor which is a constant for the column. The work then follows the usual Hardy Cross method in every particular. When the joints have been balanced, the true values of shear and moment in the various members are found from the equations previously established. Shear corrections, when necessary, can be made by a repitition of the above steps. When shears and moments have been properly balanced in the given bent, deflections can be calculated from the moments in the equivalent bent in which the depth of the members was neglected. This is seen from the fact that the entire method, as outlined in the preceding pages, is based on the balancing of deflections and rotations of the joints of the two equivalent bents.

RECAPITULATION OF FORMULAS

In calculating fixed-end moments in columns, total shear of any story is distributed to the various columns of the story in proportion to:

CASE I $\dfrac{I}{LL'(L-a)}$ (24)

CASE II $\dfrac{I(2L-a-b)}{2LL'(L-a)(L-b)}$ (48)

CASE III $\dfrac{I}{2LL'(2L-a)}$ (64)

True shear in any column is changed to shear in the equivalent column in which depth is neglected by multiplying by the following factor:

CASE I $\dfrac{L'(L-a)}{L^2}$ (23)

CASE II $\dfrac{2L'(L-a)(L-b)}{L^2(2L-a-b)}$ (46)

CASE III $\dfrac{L'(2L-a)}{2L^2}$ (63)

After a series of balance and carry-over, total shears are calculated by the following formulas:

CASE I $H = h\left[\dfrac{L^2}{L'(L-a)}\right]$ (27)

CASE II $$H = (\Sigma m_F)\left[\frac{L(2L-a-b)}{L'(L-a)(L-b)}\right] + (\Sigma m_R)\left[\frac{1}{2L'}\left(\frac{2L+a-b}{L-b} + \frac{L+a-b}{L-a}\right)\right]$$

$$+ (\Sigma w_R')\left[\frac{1}{L'}\left(\frac{2L+b-a}{L-a} + \frac{L+b-a}{L-b}\right)\right] \qquad (51)$$

CASE III $H = h\left[\frac{2L^2}{L'(2L-a)}\right]$ (63)

After a series of balance and carry-over, true total moments are calculated by the following formulas:

CASE I $M = (\Sigma m)\left(\frac{L}{L-a}\right)$ (26)

CASE II $M = (\Sigma m_F)\left(\frac{L}{L-b}\right) + (\Sigma m_R)\left(\frac{2L+a-b}{2[L-b]}\right) + (\Sigma w_R')\left(\frac{L+b-a}{L-b}\right)$ (50)

CASE III $M = (\Sigma m)\left(\frac{2L}{2L-a}\right)$ (59)